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VAN WETENSCHAPPEN
-:- TE AMSTERDAM -:-

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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday May 28, 1910.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 28 Mei 1910, Dl. XIX).

CONTENTS.

Physics. — *"Note on the interpretation of spectroheliograph results and of line-shifts, and on anomalous scattering of light."* By Prof. W. H. JULIUS.

(Communicated in the meeting of April 29, 1910).

The puzzling character of solar problems is well illustrated by the fact, that the images obtained with the spectroheliograph give rise to widely different explanations, and that it seems impossible as yet to answer in a satisfactory way even the fundamental question: what is the principal cause of the very unequal distribution of different kinds of light over the sun's disk?

HALE and ELLERMAN, in a paper "On the nature of the hydrogen flocculi and their structure at different levels in the solar atmosphere" [1]) reject the hypothesis advanced by W. J. S. LOCKYER, that the dark hydrogen flocculi indicate regions where there is a deficiency of hydrogen. They also refute DESLANDRES' argument, according to which those dark flocculi are not mainly due to a particular distribution of the emissive or absorbing power of hydrogen, but to a simple instrumental cause, an inherent defect of the spectroheliograph. In their own opinion, the best way to account for the observed phenomena is the hypothesis, that the dark hydrogen flocculi are produced by increased absorption (probably resulting from greater depth and decreased temperature of the hydrogen gas in these regions of the solar atmosphere), while the bright flocculi represent regions of increased radiation. Finally HALE and ELLERMAN state, that the results obtained in the high dispersion work with the hydrogen lines are also in accord with certain inferences which I deduced [2]) from the hypothesis first advanced in 1904 [3]), that the distribution of the light in photographs, taken with the spectroheliograph, is mainly caused by anomalous dispersion. They wish to defer a general discussion of the effects of anomalous refraction in the solar atmosphere until many more observations have been made; but a preliminary survey of the results already obtained induces them to believe that the principal phenomena of the dark hydrogen flocculi may be explained more satisfactorily as absorption effects, and that the evidence can hardly be considered favourable to my theory.

In the present note I wish not to combat the absorption hypo-

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1) HALE and ELLERMAN, Proc. Roy. Soc. **83**, 177, January 1910.

2) JULIUS, Anomalous refraction phenomena investigated with the spectroheliograph. Proc. Roy. Acad. Amst. XI, p. 213 (1908).

3) JULIUS, Spectroheliograph results explained by anomalous dispersion. Proc. Roy. Acad. Amst. VII, p. 140, (1904). Astroph. Journ. **21**, 278, (1905).

thesis proposed by HALE and ELLERMAN, but only to show, that their objections to an explanation on the basis of anomalous dispersion are easily refuted, and that the results so far obtained are by no means less favourable to the theory which ascribes the flocculi in the main to anomalous dispersion, than to that which explains them as mere absorption effects.

The intensity and width of the hydrogen lines, especially of H_α, differ greatly in different regions of the sun. If the widening of these lines is caused by increased absorption only, there is no reason to expect them to be asymmetrical (except perhaps by local displacements in consequence of motion in the line of sight). If, on the other hand, we are chiefly dealing with dispersion bands[1]), enveloping the real absorption lines, so that the widening results from the fact that the strongly refracted waves bordering the central lines, have their origin on the average in less luminous regions — then, at first sight, it seems as if a marked and variable asymmetry must be the general appearance. Indeed, when comparing waves at equal distances from the centre of the line on the red and violet sides, we must find the rays curved in opposite directions by the same density gradients of the solar atmosphere HALE and ELLERMAN think it improbable that equal amounts of light would reach the observer in both cases, and therefore conclude that, if anomalous dispersion were the principal cause, the spectroheliograph should, as a rule, give very different images when set on the one or the other side of the same line.

They tried the effect of photographing the flocculi simultaneously with light from opposite sides of the H_α line at equal distances from the centre. In general the two images proved to be almost identical in their principal features, though small differences of detail were often visible. In the case of what they call "eruptive phenomena" the images were very unlike, as the distortion of the H_α line would lead one to expect. It stands to reason that, if such distortions are satisfactorily explained on the basis of the DOPPLER effect, a corresponding explanation can be given of the unlike parts in the images just mentioned.

These are the considerations, adduced by HALE and ELLERMAN in support of their conclusion, that the results hitherto obtained are unfavourable to the anomalous dispersion theory.

On closer examination, however, the consequences of anomalous dispersion turn out to be in harmony with the observed phenomena.

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[1]) JULIUS, Proc. Roy. Acad. Amst. VII, p. 134—140; IX, p. 343—359; Astroph. Journ. **21**, p. 271—291 (1905), **25**, p. 95—115 (1907).

It will indeed prove very probable that R-light and V-light [1]) selected at equal distances from H_α should, in spite of their opposite curvature, produce images which are in general almost identical in their principal features, and that differences of detail should chiefly appear in much disturbed regions, where steep density gradients occur.

In a paper on "Regular consequences of irregular refraction in the sun" [2]) I attempted to obtain a general idea of the optical effect which local condensations and rarefactions in the solar atmosphere must produce, if only the incurvation of rays is taken into account.[3]) The result was as follows.

Let us first consider light for which the refracting power ($n-1=R\Delta$) of the solar atmosphere has a certain *positive* value. Somewhere on the central part of the disk we imagine in the gaseous envelope a region of any shape, only satisfying the condition that, from the outline inward, the density of the gases either diminishes or increases continuously, so that the region includes either a minimum or a maximum of density. In both cases the region will show a dark rim. If in these two cases the density gradients, though opposite in sign, were equal in magnitude, the optical images, presented by the rarefaction or the condensation, would be almost identical in their principal features. This is due to the fact, that the light, transmitted by our region, comes from a source, extending nearly symmetrically round the line of sight. As soon as the latter condition is not fulfilled, if e. g. some of the rays, before entering our region, had already suffered strong deviation in a neighbouring very marked density gradient, the symmetry of the apparent source of light would be disturbed, and then the aspect of the rarefaction might sensibly differ from that of the condensation of the same shape.

Let us now consider light for which the refracting power of the solar atmosphere is equal in absolute magnitude, but *negative*. Such

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[1]) By R-light and V-light will be denoted waves on the red and violet sides of absorption lines within the limits where anomalous dispersion is perceptible.

[2]) Julius, Proc. Roy. Acad. Amsterdam XII, 266, 1909; Memorie della Società degli Spettroscopisti italiani **38**, 173, 1909; Physikalische Zeitschrift **11**, 56, 1910.

[3]) It may be well here to remark, that in the paper referred to as well as in former publications on anomalous dispersion I never thought of denying the probable effects of selective radiation, absorption, scattering, radial motion, pressure, radio activity, magnetism; but because in solar literature full attention is generally paid to most of these subjects, whereas refraction and anomalous dispersion are little noticed, I wished to consider the latter agencies separately, and to inquire which solar phenomena may be produced or influenced by them. The object in view was not a theory of the sun, but a study of the cosmical consequences of anomalous dispersion.

waves behave in a rarefaction just like the other waves, first considered, would do in the condensation that would be obtained by reversing the gradients. The optical effect is generally the same in its principal features.

Consequently, confining our attention to the central parts of the disk, and excluding the much disturbed regions, we must expect to find only small difference between spectroheliograph images taken with R-light and V-light selected at the proper distances from an absorption line.

Hale and Ellerman admit that the small differences frequently observed when we compare images given by opposite sides of H_α are, perhaps, due to anomalous refraction; I see no reason why the same principle should be inactive in the production of the remaining, almost identical, parts of the images.

As we approach the limb, the conditions of refraction are however modified.

When seen projected on the disk at a sufficient distance from the centre, a region with a minimum and a region with a maximum of density will appear different. With R-light the rarefaction shows *dark* on the side *opposite* the centre of the disk, and may be *brighter* than the surroundings on the side *facing* the centre, whereas the condensation shows *dark* on the side *facing* the centre, and may come out *bright* on the *opposite* side. With V-light these effects are the reverse, rarefaction and condensation optically changing parts [1]). So we have reason to expect that between spectroheliograms, taken with light from the red and violet sides of a line, some systematic differences of detail — increasing as we proceed from the centre toward the limb, and relating to distribution of brightness rather than to structure — will be observed.

It will prove necessary, however, to check the latter expectation, because there is a physical law, not hitherto considered in our argument, which tends to efface the differences just mentioned, and to promote similarity of the corresponding R-light and V-light images all over the disk. I mean the fact, discovered by Rayleigh, that the light is *scattered* by the molecules of a transmitting medium.

Effects of scattering on the character of the total radiation transmitted by stellar atmospheres were first considered by Schuster in a most interesting article: "Radiation through a foggy atmosphere" [2]). It would lie beyond the scope of the present note to discuss the general bearing of the remarkable results, there described, upon

[1]) Proc. Roy. Acad. Amsterdam XII, p. 269 and 274—276 (1909).

[2]) Schuster, Astroph. Journ. **21**, 1—22, 1905.

conclusions deduced from the anomalous dispersion theory. One point, however, which may prove very important with respect to the explanation of spectroheliograph results, requires our special notice, viz., that scattering is a *selective* process. This peculiarity was alluded to on p. 17 of SCHUSTER's paper, but not further considered there.

Indeed, if we accept RAYLEIGH's formula, the coefficient of scattering, called s in SCHUSTER's paper, depends not only on the number N of scattering particles per unit volume, and on the wave-length λ of the light under consideration, but also on the index of refraction n of the medium:

$$s = \frac{32\pi^3 (n-1)^2}{3N\lambda^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The terms "anomalous dispersion" and "anomalous refraction" were until now used indiscriminately. We shall in future distinguish between the two expressions. By anomalous dispersion we denote the general property of matter, that its refracting power $\pm (n-1)$ varies rapidly as we approach an absorption line. This property, of course, subsists even when the density of the medium is perfectly uniform, and the propagation of light in it rectilinear. Whenever the density is not uniform, it may cause very different deviations of neighbouring waves. That effect of anomalous dispersion — which I exclusively studied in former papers on the subject [1]) — will be called *anomalous refraction*. Another effect, dependent on the same property, and now considered for the first time, is *anomalous scattering*.

Equation (1) shows, that the coefficient of scattering passes through a sharp maximum in the neighbourhood of every value of λ which corresponds to an absorption line, because there the factor $(n-1)^2$ increases rapidly as we approach the line from either side. In the nearest vicinity of the absorption lines of a mixture of gases RAYLEIGH's formula is perhaps not rigorously applicable, but we may use it as a first approximation.

Even absolutely monochromatic absorption would thus, in an extensive atmosphere, give rise to a line of a certain width. If a group of neighbouring waves are absorbed, the width of the resulting

[1]) I am very much indebted to Prof. LORENTZ, who, at my request, was kind enough to subject my preceding work on the consequences of anomalous dispersion to a thorough criticism. According to him the weak side of my conclusions was, that I had not duly noticed the diminution of the light by scattering. I intend to discuss this important point more fully on a later occasion. The resulting new aspect of the anomalous dispersion problem will render necessary certain modifications of the theory (e.g. regarding the explanation of prominences), and may thus perhaps serve to reconcile opposite opinions on this matter.

dark line will always exceed that of the spectral region of real absorption. Every absorption line of a stellar atmosphere is, therefore, enveloped in what we may call a *dispersion band*, because it depends upon anomalous dispersion. In an atmosphere of perfectly uniform density, the dispersion band would be caused by anomalous *scattering* only; but if irregular density gradients occur, anomalous *refraction* adds to the effect in two ways: 1 by directing back toward the luminous surface some of the strongly refracted rays [1]), and, 2, by lengthening the paths along which the beams are subject to loss of intensity by scattering.

These notions may gain clearness if we imagine ourselves to be placed somewhere in the solar atmosphere, looking outward. Then a spectroscope, if directed on the "solar sky", would show us the Fraunhofer lines bright on a less luminous ground, not only on account of luminescence or of selective temperature radiation, but also because the scattering is more intense in the vicinity of absorption lines than in blank parts of the spectrum. The energy which thus returns to the sun by the *scattering* process, is wanting in the Fraunhofer spectrum as seen on earth. Besides, the irregular density gradients of the solar atmosphere would give rise to "mirage" on a large scale, also of a selective character. Distorted images of parts of the brilliant solar surface would appear everywhere in the sky, different in shape and extension for kinds of light that are differently refracted. This is the portion which anomalous *refraction* contributes to the returning energy, and withdraws from the radiation leaving the sun.

Applying our ideas on the combined consequences of anomalous scattering and refraction to the interpretation of spectroheliograph results, we must remember:

1. that anomalous scattering darkens the solar spectrum almost equally on both sides of a strong absorption line [2]), thus reducing the differences which photographs made with R-light and V-light at equal distances from the same line would have shown, if anomalous refraction were the only agent;

2. that the width of a certain Fraunhofer line would be a minimum at points of the sun's image corresponding to regions of uniform

[1]) This process was more fully treated of in my paper on "Regular consequences of irregular refraction in the sun", Proc. Roy. Acad. Amst. XII, p. 279.

[2]) It will be mentioned farther on, that especially the weaker Fraunhofer lines are asymmetrical by anomalous dispersion. So long as spectroheliograms are only made with light from the domain of strong lines, we may, in interpreting them, neglect that systematic asymmetry.

density and composition in the solar atmosphere, because there anomalous scattering would be the only cause of the dispersion band;

3. that the same line will be wider and, in general, darker in the spectrum of regions where irregular gradients disturb the rectilinear propagation of the light. (In this way we explain the varying width of H_γ, as shown in Fig. 2 of Pl. I, Proc. Roy. Soc. Vol. 83 p. 189. If, therefore, the camera slit of the spectroheliograph is set e.g. between the centre and the edge of H_α, but nearer to the edge, the dark flocculi indicate regions, where density gradients with large components perpendicular to the line of sight are in evidence. Almost the same structure must be revealed, if the camera slit is set on H_β or H_γ, provided the distance from the centre of these lines be taken smaller than with H_α, in order to catch waves that are refracted to the same degree as those in the former case. This connection between spectroheliograms obtained with different hydrogen lines was predicted in my paper Proc. Roy. Acad. Amst. XI, on p. 221, and afterward found confirmed by Hale and Ellerman); [1])

4. that gradients of exceptional magnitude and extension may (by refraction) produce marked irregularities in the distribution of the light within the range of a dispersion band;

5. that the composition of the solar atmosphere very probably varies with the level, but that convection currents tend to efface local differences of composition as well as of temperature.

If these statements are kept in mind, it will be found possible to explain, on the basis of anomalous dispersion, at least as many particulars of the spectroheliograms, as were explained by Hale and Ellerman on the basis of their temperature and absorption hypothesis. We will not, on this occasion, enter into a comparison of the advantages of both points of view, the principal aim of the present paper only being to prevent a premature criticism of either of them.

With a similar object in view we shall now consider another important solar phenomenon — systematic displacements of Fraunhofer lines — which was also explained according to two entirely different theories.

I showed elsewhere [2]) that anomalous refraction by irregular density gradients causes the Fraunhofer lines to be asymmetrical, the narrower ones generally to a higher degree than the wider ones, thus producing an apparent displacement of the lines toward the red. The displa-

1) The optical effect produced by the systematized density gradients near vortices requires special treatment.

2) See the paper on "Regular consequences etc." referred to above.

cement must increase when we pass from the centre of the disk to the limb. These effects depend upon the rule, that the refracting power of the mixture of gases, constituting the solar atmosphere, is on the average greater on the red side of an absorption line than on the violet side. Anomalous scattering also being determined by the values of the refracting power on both sides of the absorption lines, it co-operates in producing those systematic displacements.

From a recent remarkable investigation of the displacements of the spectrum lines at the sun's limb, by W. S. ADAMS [1]), it appears that out of a total of 470 lines only one or two are shifted unmistakably toward the violet; the other lines all show displacements to the red, ranging from 0,000 to 0,014 Angström. The various characteristics of the list of these lines will have to be studied in detail from the point of view of anomalous dispersion. I must defer that inquiry to a later date, and now confine myself to a few remarks on prominent statements made in ADAMS' paper.

ADAMS concludes that pressure is the effective agent in producing the displacements observed. He evidently paid very little attention to the possibility of explaining these phenomena by anomalous dispersion, for although he refers to the explanation which I recently published in the Memorie della Società degli Spettroscopisti Italiani, and rejects it, the clue of my argument entirely escaped his notice. Indeed, he writes:

"According to his (JULIUS') point of view the photospheric light is anomalously refracted in the vicinity of the absorption lines produced by the metallic vapours, and, since in general the density-gradient decreases outward, the widening will be upon the red side of the lines producing the observed displacements. The fact that the sodium lines D_1 and D_2 are not displaced, although they show the largest amount of anomalous dispersion of any which have been investigated for this effect, is rather strongly opposed to this view".

In the first place I do not quite understand, why the decrease of the density-*gradient* should be material to the case. This, however, may be a lapse; probably the author intended to say: "since in general the density decreases outward." But then the inference expressed in the sentence as a whole is erroneous. A little reflection will easily show, that in parts of the disk near the limb the regular radial density gradient assists R-light and hinders V-light in curving from the photosphere toward the observer. The result would be an apparent displacement of the dark line to the *violet*, not to the red. The radial

[1]) ADAMS, Contrib. from the Mount Wilson Solar Obs. No. 43; Astroph. Journ. **31**, p. 30—61, 1910.

gradient, therefore, if it is of any importance in this matter, counteracts the effective agent which produces the observed shifts toward the red.

The principal point overlooked by ADAMS is that, according to my explanation, the effective agent in producing the phenomenon is the general asymmetry of the dispersion bands enveloping the absorption lines of the solar spectrum. It does not depend upon the incurvation which rays undergo in the regular radial density gradient of the solar atmosphere, but is caused by anomalous scattering, and refraction in irregular gradients, combined with the fact that the refracting power of the mixture of gases is on the average greater for R-light than for V-light.

If we keep this in mind, we shall have a useful base for investigating the relationship between anomalous dispersion and the results of ADAMS' measurements. That a simple comparison of GEISLER's observations on anomalous dispersion of metallic vapours in the arc with displacements at the limb — as given by ADAMS [1]) — could not possibly serve the purpose of finding such a relationship, is evident; for the amount of that part of the displacement which is due to anomalous dispersion, is determined by the degree of asymmetry of the Fraunhofer line under consideration; and this asymmetry is not a mere property of the corresponding element itself, revealable in laboratory experiments, but depends upon the concentration with which that element is represented in the solar atmosphere. No shade of proportionality between the results of those two investigations could be expected. So it is not at all opposed to our view, that the winged lines of sodium and calcium are little or not displaced at the limb, although they show strong anomalous dispersion. On the contrary, that result might have been foreseen; for if the wide wings are really owing to that cause, the wave-length corresponding to the zero value of the refracting power of the mixture, which always lies on the violet side of a Fraunhofer line, must be at a rather great distance from the absorbed waves [2]), thus making the asymmetry of the dispersion band imperceptible. The central part of the line, the true absorption line, cannot be displaced by anomalous dispersion.

A peculiar feature of our explanation is, that both very strong and very weak anomalous dispersion make the displacements small, whereas intermediate values give larger displacements. Indeed, with decreasing width of the dispersion band, its asymmetry increases;

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1) ADAMS, l. c. p. 28.

2) Cf. fig. 8 on Plate I, Proc. Roy. Acad. Amst. XII p. 282. 1909.

but the resulting apparent displacement can never surpass half the width of the line. (Whenever greater shifts are observed, pressure or magnetism or Doppler-effect certainly come into play).

The largest displacements observed by ADAMS occur with many lines of iron and nickel. From the point of view of our hypothesis this means, that near these lines the amount of anomalous dispersion of the mixture is most suitable for producing the phenomenon, neither too great, nor too small. Considerably smaller are the displacements for titanium, vanadium, and scandium - perhaps because those elements are less in evidence in the mixture of gases. That those iron lines, which are most strengthened at the limb, show smaller displacements than the average iron lines, also perfectly fits our point of view, for their asymmetry must be less conspicuous on account of their greater width. That the lines of the elements of very high atomic weight, such as lanthanum and cerium, show very small displacements, is easily accounted for if we assume their vapours to be extremely rare in the solar atmosphere. This explanation is certainly not less simple than the one proposed by ADAMS on p. 17 and 18 of his paper, [1]) where he has to find a way out of the discrepancy to which in this case the pressure hypothesis seems to lead.

Various other characteristics of ADAMS' interesting list of displacements (e. g. the special behaviour of the enhanced lines as a class) will be discussed on a later occasion, together with his equally valuable observations of the spectrum of sun-spots.

Geophysics. "*On the determination of the epicentre of earth-quakes by means of records at a single station*". By Dr. C. BRAAK. (Communicated by Dr. J. P. VAN DER STOK).

(Communicated in the meeting of April 29, 1909).

In working out seismograms of the WIECHERT-seismograph I was repeatedly struck by the fact that the azimuth of the epicentre could be determined with satisfactory results from the two components of the motion of the ground.

As informations relative to other stations are generally received at Batavia some time after the occurrence of earth-quakes, I have often used this method to come to a preliminary determination of the epicentre from the Batavia seismograms only. In this way e.g. informations concerning the *Korintji* earth-quake of June 4, 1909 could be

[1]) Astroph. Journ. 31, 46—47, 1910.

given when from Singapore and Banka telegrams about experienced earth-quakes were received.

The most important quakes of 1909 have now been worked out with a view of ascertaining the accuracy to be obtained in determining the azimuth.

In the mean time GALITZIN [1]) has applied the same method to records obtained by means of two seismographs set up at right angles to each other, the records being strongly magnified and the damping such as to make the vibrations aperiodic; in this way very satisfactory results were obtained.

As has been remarked above, the WIECHERT-seismograph may be used with success for the same inquiry.

The two components must be independent of each other, a condition that can be fulfilled by an accurate adjustment.

It is no inconvenience that the damping ratio is only 5 : 1 if we take this circumstance duly into account when calculating the period of vibration.

The most serious difficulty is experienced by the small magnification so that only earth-quakes of large amplitudes have been worked out.

This difficulty can be overcome to some extent by measuring out not only the first, but also some of the next deflections, in so far as the two components show a perfect congruity.

When the free periods of the two pendulums are equal, or nearly equal, then we may, as a first approximation, apply the same calculation to these values as to the first deflection.

In order to calculate the true motion of the ground we have used the formula given by WIECHERT

$$W = \frac{V}{\sqrt{\left\{1-\left(\frac{T}{T_0}\right)^2\right\}^2 + 4\left(\frac{T_0}{2\pi\tau}\right)^2\left(\frac{T}{T_0}\right)^2}}$$

where

$$\left(\frac{T_0}{2\pi\tau}\right)^2 = \frac{(\textit{log nat. } \varepsilon)^2}{\pi^2 + (\textit{log nat. } \varepsilon)^2}$$

and

V = indicator-magnification.
T_0 = free period.
T = period of the motion of the ground.
ε = damping ratio.

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[1]) Fürst B. GALITZIN. Zur Frage der Bestimmung des Azimuts des Epicentrums eines Bebens Bull. de l'Acad. Imp. d. Sc. de St. Pétersbourg, 1909.

The value V was determined by direct measuring; this value as deduced by means of small weights being put on the pendulum and by determination of the period of vibration is hardly accurate owing to the sluggishness of the instrument, unless the weights remain on the cylinder for at least half an hour. During the whole year V has been equal

for the E—W component to 234,

for the N—S component to 186.

For T_0 and ε monthly mean values have been used as given in the following table.

	T_{0E}	T_{0N}	ε_E	ε_N
Jan.	10.3	10.2	4.8	5.1
Febr.	11.0	11.1	5.6	5.1
March	11.0	10.8	4.9	4.3
April	10.7	10.9	4.9	4.5
May	10.2	10.2	4.3	4.0
June	10.4	10.4	5.7	5.3
July	10.0	10.4	5.7	5.3
Aug.	10.1	10.3	5.8	5.5
Sept.	10.0	10.1	5.4	5.5
Oct.	9.6	9.8	5.4	4.4
Nov.	9.7	10.1	4.5	4.4
Dec.	9.6	10.0	4.3	5.0

In the first place a number of quakes will be treated where the amplitude of the first deflection was large enough to admit of an accurate measurement.

In the monthly bulletin these quakes are mentioned under the numbers 29, 58, 124, 211, 237, 242 and 262; copies of the diagrams are given in the adjoining plate.

The seismograph is set up so that the pointer moves downwards in the diagram when the pendulum has an E. and S. motion relative to the frame, which therefore occurs in the case of an impulse from W. or N.

After a slowly increasing motion, as observable on many diagrams, a small zigzag motion seems to indicate, at least on some diagrams, the arrival of the true impulse. The E.—W. component of N°. 124 clearly shows that the sharp turning corresponds with the beginning of the true first impulse.

In treating the first corresponding waves in either component, the distances of subsequent extreme positions being measured out, the following deflections were found.

No.	Date 1909		Deflections n mM.	$\frac{W_E}{W_N}$	Azimuth 1st deflection	Mean Azimuth	True Azimuth
29	3. II	E—W	7.5 25.6 43.9 53.7 56.7	1.24	N 54 W	N 56 W	N 66 W
		N—S	4.4 13.8 26.2 26.0 31.4				
58	13. III	E—W	2.3 9.6 6.0 7.2 8.0	1.23	N 32 E	N 33 E	N 35 E
		N—S	2.9 5.5 5.9 9.9 12.3				
124	4. VI	E—W	63.2	1.24	N 48 W		N 51 W
		N—S	46.0				
211	28. IX	E—W	4.2 8.4 10.2 14.4	1.25	S 75 W	S 72 W	S 73 W
		N—S	0.9 2.4 3.2 4.3				
237	30. X	E—W	3.0 14.7	1.22	S 74 O	S 76 O	S 76 O
		N—S	0.7 2.6				
242	10. XI	E—W	3.9 15.6 15.9 9.8	1.28	N 27 E	N 27 E	N 32 E
		N—S	5.9 22.7 23.7 15.2				
262	10. XII	E—W	[4.0] 14.0 20.0 22.4 19.2 13.9	1.31	N 81 E	N 71 E	N 65 E
		N—S	[0.5] 3.4 5.3 6.2 5.4 4.1				

In the fourth column the quotient is given of the magnifications of either component; it appears to be subject to small fluctuations only; the influence of period of vibration and damping is small and, consequently, inaccuracies in these quantities can exercise only an unimportant influence.

The azimuth was calculated separately from the first deflections and from all measurements taken together (column 6). If we disregard N°. 262, in which case the first deflection has obviously been disturbed, either method is found to lead to identical results within the limits of the preciseness of the observations. The latter method, which is serviceable also when the deflections are small, will in future be applied exclusively.

The true azimuth has been deduced, partially from earth-quake-informations from the Archipelago, partially by means of the records taken at Batavia, Manila, Zi-ka-wei and Osaka:

N°. 29, from the Lampong-districts, according to earth-quake-informations.

N°. 58, near Tokio, according to the papers.

N°. 124, in Korintji, according to earth-quake-informations.

N°. 211, in S.-Bantam, according to earth-quake-informations.

N°. 237, in the Carpentaria-gulf, according to records at Batavia, Manila and Zi-ka-wei.

N°. 242, near Kioe-sioe, according to records at Batavia, Manila, Zi-ka-wei and Osaka.

N°. 262, near the Caroline-islands, according to records at Batavia, Manila and Osaka.

Owing to incomplete informations, inaccurate data concerning the time of the first and second forerunners, and the total failing of indications about the time of the second forerunner, both methods often leave much to be desired. In calculating the distances from the available data the curves given by Wiechert and Zöppritz were used.

It may be noticed that the quadrant, whence the vibrations travel, may, in by far the most cases, be indicated without ambiguity. It is an exception when the first impulse comes from the direction of the epicentre; in by far the most cases it came from the opposite side.

In this way the azimuth has been determined for a number of other earth-quakes; the results are given in the following table.

The deviations between the calculated and the true azimuths can be put only partially on account of errors in the azimuth as calculated; for tremors which have travelled over a great distance the deviations are small; for the quakes N°. 4, 27, 52, 114, 203 and 243, all coming from places situated not far off, the deviations are great, but they are, without doubt, principally due to incomplete informations concerning the true focus, and the locus of the epicentre, as calculated from the diagrams, is more reliable than that based on informations received.

N⁰.	Data		Epicentre	Longitude E. from Gr.	Latitude	Azimuth calculated	Azimuth true
4	2	I	Preanger	107°.6	7°.1 S	S 25° E	S 35° E
13	15	I	W. fr. Mindanas	132.2	5.4 N	N 65 E	N 65 E
19	23	I	Luristan	50.0	32.5 N	N 43 W	N 50 W
27	31	I	Cheribon	108.1	7.2 S	S 19 E	S 51 E
28	2	II	Celebes	119.7	0.7 S	N 63 E	N 67 E
29	3	II	S.-Sumatra	105.1	5.2 S	N 56 W	N 60 W
52	8	III	Banjoemas	109.2	7.2 S	S 26 E	S 66 E
56	13	III	E. from Tokio	140.7	34.5 N	N 33 E	N 35 E
58	13	III	E. from Tokio	140.7	34.5 N	N 33 E	N 35 E
59	17	III	Celebes	121.6	1.0 N	N 63 E	N 64 E
61	18	III	Celebes	121.5	1.2 S	N 69 E	N 71 E
85	15	IV	N. fr. Formosa	124.0	30.0 N	N 29 E	N 21 E
114	17	V	S.-Sumatra	103.2	4.0 S	N 75 W	N 60 W
120	31	V	Harafoera-see	135.0	10.8 S	S 78 E	S 80 E
124	4	VI	Korintji	101.6	2.1 S	N 48 W	N 51 W
143	15	VI	Preanger	107.4	7.0 S	S 33 E	S 40 E
155	8	VII	Samarkand	67.0	40.0 N	N 37 W	N 34 W
176	14	VIII	Japan	139.0	36.5 N	N 38 E	N 34 E
198	11	IX	Philippines	124.8	10.0 N	N 53 E	N 48 E
203	17	IX	S.-Sumatra	103.1	4.1 S	N 70 W	N 60 W
210	27	IX	Preanger	107.1	6.9 S	S 22 E	S 20 E
211	28	IX	Bantam	106.0	6.4 S	S 72 W	S 73 W
237	30	VIII	G. of Carpentaria	138.3	12.0 S	S 76 E	S 76 E
242	10	XI	W. fr. Kioe-sioe	1[illegible]1.0	31.0 N	N 27 E	N 32 E
243	10	XI	Cheribon	108.2	7.1 S	S 31 E	S 56 E
261	10	XII	Ceram	128.5	3.3 S	0	N 82 E
262	10	XII	Carolines	146.0	11.5 N	N 70 E	N 65 E

Geophysics. — *"On the semi-diurnal lunar tide as deduced from records of the astatic seismograph at Batavia".* By Dr. C BRAAK. (Communicated by Dr. J. P. VAN DER STOK).

(Communicated in the meeting of April 29, 1910).

1. Since the beginning of December 1908 an astatic seismograph of the WIECHERT pattern with a mass of about 1000 KG. has been in working order at Batavia. This instrument being very sensitive to change of inclination, it was rational to inquire whether the tidal motion of the earth, investigated by HECKER [1]), could be deduced from the records, because tidal forces at Batavia are considerably stronger than at Potsdam.

A preliminary inquiry having afforded satisfactory results, the records obtained during the half year July to December were used for this investigation which, for the present, is restricted to the E-W component of the principal semi-diurnal linear tide.

The seismograph consists of a cylindrical heavy load of 900 KG. weight, resting on a foot of 70 cm. length, and is pivoted on four steel lamels by means of which a Cardanic freedom of motion is ensured. The masonry column is free from the floor and rests on a broad foundation. The building is situated with its longer dimensions in the N-S-direction and is sheltered from the sun's rays by galleries on the east and west side.

The influence of the undulatory motion of the ground due to the sun's heat is evident in the diagrams, principally in those of the E-W-component; this disturbance is so strong that only the records at the hours 7 p.m. to 8 a.m. can be used for this inquiry; on days of strong disturbance even some of these records had to be omitted.

As there is no basis-line, the distances between the records at 7^h and 10^h, 8^h and 11^h etc. were measured out and, in order to eliminate, at least partially, variations of the zero-point and other disturbances of long duration, the differences were taken between the distances at 13^h to 10^h and 10^h to 7^h, which differences, but for a small correction, correspond to the tidal motion at 10^h with twice the amplitude. The tidal motion is evident in these data even at first sight.

2. For the half year July to December 1909 these data were

[1]) O. HECKER. Beobachtungen an Horizontalpendeln über die Deformation des Erdkörpers unter dem Einflusz von Sonne und Mond. Veröffentl. des Kön. Preuss. Geod. Inst. Neue Folge n⁰. 32. 1907.

arranged according to lunar hours. The records for the months April, May and June might have been used too, but they have been discarded because the determination of the sensitiveness of the instrument during these months seemed to be insufficienttly accurate.

As a control the data were arranged separately for each hour; in this way eight independent series of numbers were obtained for 10^h p.m. to 5 a.m., from which the following results for the lunar tide have been deduced; the amplitude is expressed in mm.

10^h p.m.	$0.605 \; cos \; (2\,t - 253°\,52')$	118
11	$0.748 \; cos \; (2\,t - 261°\;\;9')$	134
12	$0.501 \; cos \; (2\,t - 266°\,21')$	146
1 a.m.	$0.500 \; cos \; (2\,t - 277°\,28')$	159
2	$0.451 \; cos \; (2\,t - 256°\,19')$	167
3	$0.625 \; cos \; (2\,t - 239°\,41')$	167
4	$0.650 \; cos \; (2\,t - 234°\,10')$	154
5	$0.732 \; cos \; (2\,t - 245°\,57')$	98

The numbers behind the formulae denote the corresponding number of observations; taking the weight of each result proportionate to these numbers in calculating the average value of amplitude and argument, we find

$$0.5913 \; cos \; (2t - 254°33').$$

To the amplitude a correction must be applied because the variation of the argument within 3 hours is not 90° but $3 \times 28.°98$; therefore the amplitude must be divided by

$$1 - cos\,(3 \times 28.°98) = 0.9465.$$

As the deflections have been measured out corresponding to the hour-signal, and this is given 5.5 minutes before Batavia mean time, the argument has to be diminished by 2°40′, so that the final expression becomes

$$0.6248 \; cos \; (2t - 251°53').$$

3. The sensitiveness of the seismograph has been determined after two different methods.

1. From the weight and the dimensions of the different parts of the instrument the total weight and the height of the centre of gravity above the pivot were deduced, and the deflections of the pointer, caused by small weights being put on the upper side of the bob, were measured out.

2. By determination of the period of vibration and direct measurement of the indicator-magnification.

The deflections were determined twice monthly; for this purpose a weight of 5 grammes was placed at a distance of 342.5 mm. from the vertical passing through the centre of gravity, first on the east side, then on the west side and then again on the east side. The apparatus does not assume immediately its equilibrium position but shows some sluggishness. Therefore in each experiment the weight was left on the bob for half an hour in the same position. In this way the following values were obtained for the double deflection given in mm.

2 July	21.5	17 Oct.	21.0
14 ,,	22.0	3 Nov.	19.7
29 ,,	20.2	16 ,,	19.7
14 Aug.	20.9	4 Dec.	19.1
3 Sept.	20.8	17 ,,	19.9
18 ,,	20.4	3 Jan.	19.4
8 Oct.	21.0		

or, on the average; 20.43 mm.

The height of the centre of gravity was found to be 895 mm., the total weight of the pendulum 985 K.G.; hence the change of inclination corresponding with 1 m.m. on the diagram is

$$\frac{2 \times 5 \times 342.5 \times 206265}{20.43 \times 985000 \times 895} = 0''.03923$$

For the period of vibration the following determinations were made

July	10.0,	10.0,	10.2,	10.0	mean	10.0^5 sec.
Aug.	10.0,	10.2,	10.2,	10.4	,,	10.1 ,,
Sept.	10.0,	10.0,	10.0		,,	10.0 ,,
Oct.	10.0,	9.8,	9.0		,,	9.6 ,,
Nov.	9.7,	9.7			,,	9.7 ,,
Dec.	9.7,	9.6			,,	9.6^5 ,,

from which the average value

9.85 sec.

And the equivalent length of penduluan

24.04 meters.

In order to determine the indicator-magnification each of the horizontal adjusting screws, by means of which the upper side of the pendulum can be clamped, was alternately screwed down so as to ensure good contact with the pendulum and then turned 90° to the right and left. The deflections on the diagram were:

60.3, 60.2, 60.1 and 60.2 mm.
mean value 60.2 mm.

corresponding with 0.25 of the screw's thread
$= 0.3517$ mm.

The adjusting screws are 122.5 c.m., the centre of gravity is 89.5 c.m., above the pivot; hence the indicator-magnification becomes

$$\frac{60.2 \times 122.5}{0.3517 \times 89\ 5} = 234.3$$

and the change of inclination corresponding with 1 mm.:

$$\frac{206265}{234\ 3 \times 24040} = 0''.03661$$

For the N—S component the two methods gave respectively
0''.04515 and 0''.04376

For either component the second method leads to somewhat smaller values than the first, on the whole the results are fairly congruent.

The systematic difference might be ascribed, on the one hand to a remaining influence of sluggishness in determining the deflections, on the other hand to a small increase of the period of vibration with the amplitude. Neither effect could be proved practically.

If the small weight is left for some time on the pendulum, disturbances of different kinds occur which are large in proportion to the small differences to be measured, on the other hand there is a limit to the accuracy to be obtained in determining the time of vibration owing to the pretty rapid decrease of the amplitude; the effect of the two influences is however in an opposite direction. It has further been proved that the deflections for the amplitudes as used are proportionate to the applied weights.

E. g. for 5 grammes the deflections were:

15.4, 15.5, 15.3, 15.2, 15.5, 15.6. mean 15.42 m.m.

for 2.5 gr.

7.7 7.7 7.6 7.6 7.8 7.8 mean $\frac{15.40}{2}$ m.m.

Probably of the two methods of determining the sensitiveness the second is to be preferred. The fact that for the E—W pendulum, more sluggish than the N—S pendulum, a difference is found larger than for the latter, seems to indicate that, in applying the first method, the effect of sluggishness was not quite eliminated.

Moreover, the determinations after the second method are the less complicate; therefore the value 0.''03661 has been taken as a base for the neat calculations.

The E-W component of the semi-diurnal lunar tide is then represented by the formula

$$0.''01144 \; cos \; (2t - 251°53').$$

4. The amplitude of the theoretical tide, on the assumption that the earth is perfectly rigid, is

$$\frac{3m}{2M}\left(\frac{a}{r}\right)^3 cos \; \Phi \; cos^4 \frac{I}{2}\left(1 - \frac{5}{2} e^2\right)$$

m and M denoting the mass of moon and earth, a and r the radii of the earth and the moon's orbit, Φ the latitude, I the obliquity of the moon's orbit to the equator and e the excentricity of the moon's orbit. The assumed values are:

$$\frac{m}{M} = \frac{1}{81.4}\,, \quad \frac{a}{r} = \frac{1}{60.27}\,. \quad \Phi = 6°11'\,, \quad I = 25°35'$$

and $e = 0.055$.

The lunar hour 0 corresponds with the time of the moon's upper transit.

Finally we find for the theoretical tide:

$$0.''0155 \; cos \; (2t - 270°)$$

and for the real tide:

$$0.''0114 \; cos \; (2t - 251°53').$$

Mathematics. — "*Infinitesimal iteration of reciprocal functions.*" By M. J. van Uven. (Communicated by Prof. Jan de Vries).

(Communicated in the meeting of April 29, 1910).

§ 1. A function $\varphi(x)$ will be called a *reciprocal function of order* n, when it satisfies the functional equation

$$\varphi_n(x) = \underbrace{\varphi\,[\varphi\,\{\ldots\;\varphi(x)\,.\,.\}]}_{n} = x.$$

The solution of this equation is known by the name of "the problem of Babbage" [1]).

In what follows we shall occupy ourselves exclusively with the reciprocal functions of order 2 which therefore satisfy

$$\varphi_2(x) = \varphi\,\{\varphi(x)\} = x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and which for short we shall call reciprocal functions.

The solution of the problem of Babbage shows us that the functional equation (1) must be satisfied by all the functions $y = \varphi(x)$

1) See inter alia Laurent: Traité d'analyse t. VI, Paris 1890, p. 243.

connected to x by a symmetrical equation

$$S(x,y) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We now make it our task to build up these functions by infinitesimal iteration.

Let us call the index of iteration n, we have then to find a function f in such a way that

$$f(y) = f(x) + 1 \quad , \quad f(y_n) = f(x) + n,$$

where y_n is put equal to $\varphi_n(x)$.

If we still put $f(x) = \mathbf{r}$, we find

$$x = f_{-1}(\mathbf{r}) = g(\mathbf{r}) \quad , \quad y = g(\mathbf{r} + 1) \quad , \quad y_n = g(\mathbf{r} + n).$$

From (1) and (2) follows that y_n and y_{n+1} are connected by the relation

$$S(y_n \, , \, y_{n+1}) = 0.$$

As $y_2 = g(\mathbf{r} + 2) = x = g(\mathbf{r})$, then $g(\mathbf{r})$ must depend exclusively on a periodical function with period 2 for which function we shall choose

$$\sigma = e^{i\pi\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3$$

The function $g(\mathbf{r})$ can therefore be written as a function of σ, in other words:

$$g(\mathbf{r}) = h(\sigma).$$

Consequently we have

$$g(\mathbf{r} + 1) = h(-\sigma),$$

so that the function h is determined by the equation

$$S\{h(\sigma) \, , \, h(-\sigma)\} = 0.$$

§ 2. A reciprocal function $y = \varphi(x)$ is evidently determined by the equation $S(x, y) = 0$. We have therefore to examine the various symmetrical equations $S(x, y) = 0$. We begin with the equation

$$S(x, y) \equiv x + y - 2k = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation passes on account of the substitutions

$$x = h(\sigma) \; , \; y = h(-\sigma)$$

into

$$h(\sigma) + h(-\sigma) = 2k$$

or

$$h(\sigma) - k = -\{h(-\sigma) - k\},$$

which is satisfied by choosing for $h(\sigma) - k$ an arbitrary odd function $\sigma \, . \, \omega(\sigma^2)$. So we put

$$h(\sigma) - k = \sigma \, . \, \omega(\sigma^2) \qquad (\omega \text{ arbitrary, but univalent}).$$

In this way we arrive at

$$\left.\begin{aligned} x &= h(\sigma) &&= k + \sigma \, . \, \omega(\sigma^2) = k + e^{i\pi\nu}\,\omega(e^{2i\pi\nu}), \\ y &= h(-\sigma) &&= k - \sigma \, . \, \omega(\sigma^2) = k - e^{i\pi\nu}\,\omega(e^{2i\pi\nu}), \\ y_n &= && = k + e^{i\pi(\nu+n)}\,\omega(e^{2i\pi(\nu+n)}). \end{aligned}\right\} \quad . \quad (5)$$

In order to build up the function $y = \varphi(x) = 2k - x$ by infinitesimal iteration we have only to let n increase gradually. It is as easy to interpolate between x and y a certain number of functions.

The indefinite elements in the solutions are 1 the quantity ν, 2 the function ω.

If we have once chosen a function ω, then by the choice of ν we can assign to the variable x a given value. If we start e. g. from an initial value x_0 then we find ν out of the first equation (5). It goes without saying that this initial value ν_0 of ν can turn out complex. If e. g. $\nu_0 = \lambda + i\mu$, then by iteration the real part will increase, the imaginary one will remain constant.

If to give an example, we wish to interpolate one function between x and y and if we choose for ω

$$\omega(\sigma^2) = 1,$$

we find

$$x = y_0 = k + e^{i\pi\nu}\, , y_{\frac{1}{2}} = k + ie^{i\pi\nu}\, , y_1 = y = k - e^{i\pi\nu}\, , y_{1\frac{1}{2}} = k - ie^{i\pi\nu}\, .$$

If x is to have the initial value x_0 then ν_0 is determined out of

$$x_0 = k + e^{i\pi\nu_0}$$

or

$$\nu_0 = \frac{1}{i\pi} \log (x_0 - k).$$

For the relation existing between $y_{\frac{1}{2}}$ and x we find

$$y_{\frac{1}{2}} = k + i\,(x - k) = (1 - i)\,k + ix,$$

and in general

$$y_{n+\frac{1}{2}} = (1 - i)\,k + iy_n$$

§ 3. It is easy to see that all symmetrical equations of the form

$$S(x\,,\,y) = \psi(x) + \psi(y) - 2k = 0 \quad \ldots \ldots \quad (6)$$

can be treated in the way followed in § 2.

We have but to put

$$\psi(x) = k + \sigma\,.\,\omega\,(\sigma^2) \quad , \quad \psi(y) = k - \sigma\,.\,\omega\,(\sigma^2),$$

hence

$$x = \psi_{-1}\,\{k + \sigma\,.\,\omega\,(\sigma^2)\} \quad , \quad y = \psi_{-1}\,\{k - \sigma\,.\,\omega\,(\sigma^2)\},$$

or

$$x = \psi_{-1}\,\{k + e^{i\pi\nu}\,\omega(e^{2i\pi\nu})\} \quad , \quad y = \psi_{-1}\,\{k - e^{i\pi\nu}\omega\,(e^{2i\pi\nu})\}.$$

$$y_n = \psi_{-1}\,\{k + e^{i\pi(\nu+n)}\,\omega\,(e^{2i\pi(\nu+n)})\} \quad \ldots \ldots \quad (7)$$

If we write the symmetrical equation in the form

$$S\,(x,y) = K\,,$$

then it is perhaps possible to regard $S(x,y)$ as a function of the expression $\psi(x)+\psi(y)$, so that

$$S(x,y)=F\{\psi(x)+\psi(y)\}=K, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

from which ensues

$$\psi(x)+\psi(y)=F_{-1}(K)=2k.$$

And with this we have returned to the preceding case.

If $S(x,y)$ is to be regarded as a function of the expression

$$\psi(x)+\psi(y)=T(x,y)$$

it must satisfy a certain differential equation. Let us now trace this equation.

It is clear that $T(x,y)$ satisfies

$$\frac{\partial^2 T}{\partial x \partial y}=0.$$

Let us put

$$\frac{\partial S}{\partial x}=S_x\,,\ \frac{\partial S}{\partial y}=S_y\,,\ \frac{\partial^2 S}{\partial x^2}=S_{xx}\,,\ \frac{\partial^2 S}{\partial x \partial y}=S_{xy}\,,\ \text{etc.},\quad \frac{dF}{dT}=F'\,,\ \frac{d^2F}{dT^2}=F'',$$

we then find in the first place

$$S=F(T),$$

$$S_x=F'T_x\quad,\quad S_y=F'T_y\quad,\quad S_{xy}=F''T_xT_y+F'T_{xy}=F''T_xT_y\,;$$

hence

$$\frac{S_{xy}}{S_xS_y}=\frac{F''}{F'^2}=H(T)=G(S),$$

or

$$S_{xy}=GS_xS_y,$$

and therefore also

$$S_{xxy}=G'S_x^2S_y+GS_{xx}S_y+GS_xS_{xy}\quad,\quad S_{xyy}=G'S_xS_y^2+GS_{xy}S_y+GS_xS_{yy},$$

from which ensues by elimination of G and G'

$$S_xS_y\,(S_yS_{xxy}-S_xS_{xyy})=S_{xy}(S_{xx}S_y^2-S_{yy}S_x^2). \quad . \quad . \quad . \quad (9)$$

Let us still put

$$S_x=p,\quad S_y=q,\quad S_{xx}=r,\quad S_{xy}=s,\quad S_{yy}=t,\quad S_{xxy}=u,\quad S_{xyy}=v,$$

we then find

$$pq\,(qu-pv)=s\,(q^2r-p^2t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

So each integral $S(x,y)$ of this differential equation can be regarded as a function of $T=\psi(x)+\psi(y)$.

The function F is determined as follows:

$$\frac{F''(T)}{F'^2(T)}=G(S)=G(F)$$

or

$$-\frac{d}{dT}\left(\frac{1}{F'}\right)=-\frac{dF}{dT}\cdot\frac{d}{dF}\left(\frac{dT}{dF}\right)=-\frac{\frac{d^2T}{dF^2}}{\frac{dT}{dF}}=G(F).$$

The solution of this is

$$\left.\begin{aligned} T &= C\int e^{-\int G(F)\,dF}\,dF + C' = \Phi(F) \\ F &= \Phi_{-1}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

so

As example we choose

$$S(x,y) = xy = K = k^2$$

or

$$S(x,y) = e^{\log x + \log y} = e^{2 \log k},$$

consequently

$$\log x + \log y = 2 \log k,$$

from which ensues

$$\log x = \log k + \sigma \,.\, \omega(\sigma^2), \qquad \log y = \log k - \sigma \,.\, \omega(\sigma^2)$$

or

$$x = k\, e^{\sigma \,.\, \omega(\sigma^2)}, \qquad y = k\, e^{-\sigma \,.\, \omega(\sigma^2)},$$

or

$$x = k\, e^{e^{i\pi\nu}\omega(e^{2i\pi\nu})}, \quad y = k\, e^{-e^{i\pi\nu}\omega(e^{2i\pi\nu})}, \quad y_n = k\, e^{e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})}.$$

This result we can express somewhat differently. We put

$$\sigma \,.\, \omega(\sigma^2) = \chi(\sigma) - \chi(-\sigma)$$

and we then arrive at

$$x = k\, e^{\chi(\sigma)-\chi(-\sigma)} = k\,\frac{e^{\chi(\sigma)}}{e^{\chi(-\sigma)}} = k\,\frac{\Omega(\sigma)}{\Omega(-\sigma)} = k\,\frac{\Omega(e^{i\pi\nu})}{\Omega(-e^{i\pi\nu})}$$

therefore

$$y_n = k\,\frac{\Omega(e^{i\pi(\nu+n)})}{\Omega(-e^{i\pi(\nu+n)})}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We now put $k = 1$ and $\Omega(\sigma) = 1 - \sigma$ and we find in that way

$$x = \frac{1-\sigma}{1+\sigma} = \frac{1-e^{i\pi\nu}}{1+e^{i\pi\nu}}, \qquad y_n = \frac{1-e^{i\pi(\nu+n)}}{1+e^{i\pi(\nu+n)}},$$

consequently

$$i\pi(\nu+n) = \log\frac{y_n-1}{y_n+1} + i\pi,$$

$$i\pi\nu = \log\frac{x-1}{x+1} + i\pi,$$

and

$$\log\frac{y_n-1}{y_n+1} = \log\frac{x-1}{x+1} + i\pi n \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If on the contrary we put $k=1$ and $\Omega(\sigma)=e^{\frac{\sigma}{2}}$, we find

$$x=\frac{e^{\frac{\sigma}{2}}}{e^{-\frac{\sigma}{2}}}=e^{\sigma}=e^{e^{i\pi\nu}}\ ,\quad y_n=e^{e^{i\pi(\nu+n)}}\ ,$$

therefore

$$i\pi\,(\nu+n)=\log\log y_n\ ,$$
$$i\pi\nu=\log\log x$$

and

$$\log\log y_n=\log\log x+i\pi n\quad\ldots\ldots\quad(13)$$

Now we have formerly [1]) shown that the equation (12) determines the iteration of $y=\dfrac{1}{x}$, when $\dfrac{1}{x}$ is taken as a linear-broken function of x, whilst (13) indicates how $y=x^{-1}$ is iterated when x^{-1} is regarded as exponential function. From the above-mentioned it is evident that these two solutions of the iteration problem of $y=\dfrac{1}{x}$ are but two of an infinite number.

§ 4. If a certain symmetrical relation is given between x and y, e·g.

$$S(x,y)=0,$$

it may happen that by a symmetrical transformation

$$x=\Psi(\xi,\eta)\quad,\quad y=\Psi(\eta,\xi)\quad\ldots\ldots\quad(14)$$

of the equation $S(x,y)=0$ we can arrive at a likewise symmetrical equation $\Sigma(\xi,\eta)=0$ of the form

$$\Sigma(\xi,\eta)=\psi(\xi)+\psi(\eta)-2k=0.$$

In this case we have

$$\xi=\psi_{-1}\{k+\sigma\,.\,\omega(\sigma^2)\}\quad,\quad\eta=\psi_{-1}\{k-\sigma\,.\,\omega(\sigma^2)\}\,,$$
$$x=\Psi[\{k+e^{i\pi\nu}\,\omega\,(e^{2i\pi\nu})\}\,,\{k-e^{i\pi\nu}\,\omega\,(e^{2i\pi\nu})\}],$$
$$y=\Psi[\{k-e^{i\pi\nu}\,\omega\,(e^{2i\pi\nu})\}\,,\{k+e^{i\pi\nu}\,\omega\,(e^{2i\pi\nu})\}],$$
$$y_n=\Psi[\{k+e^{i\pi(\nu+n)}\,\omega\,(e^{2i\pi(\nu+n)})\}\,,\{k-e^{i\pi(\nu+n)}\,\omega\,(e^{2i\pi(\nu+n)})\}].$$

We shall dwell particularly on the *projective transformation*

$$x=\frac{\alpha\xi+\beta\eta+\gamma}{\delta(\xi+\eta)+\varepsilon}\quad,\quad y=\frac{\beta\xi+\alpha\eta+\gamma}{\delta(\xi+\eta)+\varepsilon},\quad\ldots\quad(15)$$

where for abbreviation we shall put

$$\delta(\xi+\eta)+\varepsilon=\lambda.\quad\ldots\ldots\ldots\quad(16)$$

If $S(x,y)$ is a symmetrical *algebraical* function of order m, then

[1]) M. J. van Uven: "On the orbits of a function obtained by infinitesimal iteration n its complex plane. Proceedings of the Kon. Akad. Vol. XII, pages 503—512.

$S(x, y)$ will pass after the substitution (15) into an expression $S[\xi, \eta]$ of the form

$$S[\xi, \eta] = \frac{\Sigma(\xi, \eta)}{\lambda^m}.$$

The *equation* $S(x, y) = 0$ is then transformed into the equation $\Sigma(\xi, \eta) = 0$. The function $\Sigma(\xi, \eta)$ must now satisfy

$$\frac{\partial^2 \Sigma}{\partial \xi \partial \eta} = \Sigma_{\xi\eta} = 0.$$

So the differential condition becomes

$$\Sigma_{\xi\eta} = \frac{\partial^2}{\partial \xi \partial \eta} \{\lambda^m S[\xi, \eta]\} = 0,$$

or

$$\lambda^2 S_{\xi\eta} + m\delta\lambda\,(S_\xi + S_\eta) + m\,(m-1)\,\delta^2 S = 0 \quad . \quad . \quad . \qquad (17)$$

We now have

$$S_\xi = S_x x_\xi + S_y y_\xi \quad , \quad S_\eta = S_x x_\eta + S_y y_\eta \, ,$$

$$S_{\xi\eta} = S_{xx} x_\xi x_\eta + S_{xy}(x_\xi y_\eta + x_\eta y_\xi) + S_{yy} y_\xi y_\eta + S_x x_{\xi\eta} + S_y y_{\xi\eta} \, ;$$

$$x_\xi = \frac{(\alpha-\beta)\,\delta\eta + (\alpha\varepsilon - \delta\gamma)}{\lambda^2} \quad , \quad y_\xi = \frac{-(\alpha-\beta)\,\delta\eta + (\beta\varepsilon - \delta\gamma)}{\lambda^2}$$

$$x_\eta = \frac{-(\alpha-\beta)\,\delta\xi + (\beta\varepsilon - \delta\gamma)}{\lambda^2} \quad , \quad y_\eta = \frac{(\alpha-\beta)\,\delta\xi + (\alpha\varepsilon - \delta\gamma)}{\lambda^2} \, ,$$

$$x_{\xi\eta} = \delta\,\frac{(\alpha-\beta)\,\delta\,(\xi-\eta) - \{(\alpha+\beta)\,\varepsilon - 2\delta\gamma\}}{\lambda^3} \, ,$$

$$y_{\xi\eta} = \delta\,\frac{-(\alpha-\beta)\,\delta\,(\xi-\eta) - \{(\alpha+\beta)\,\varepsilon - 2\delta\gamma\}}{\lambda^3} \, .$$

From (15) ensues

$$\lambda = \delta\,(\xi+\eta) - \varepsilon = \frac{-\{(\alpha+\beta)\,\varepsilon - 2\delta\gamma\}}{\delta\,(x+y) - (\alpha+\beta)} \, ,$$

$$\xi = \frac{-(\alpha\varepsilon - \delta\gamma)\,x + (\beta\varepsilon - \delta\gamma)\,y + (\alpha-\beta)\,\gamma}{(\alpha-\beta)\,\{\delta\,(x+y) - (\alpha+\beta)\}} \, ,$$

$$\eta = \frac{(\beta\varepsilon - \delta\gamma)\,x - (\alpha\varepsilon - \delta\gamma)\,y + (\alpha-\beta)\,\gamma}{(\alpha-\beta)\,\{\delta\,(x+y) - (\alpha+\beta)\}} \, .$$

If we now put

$$\delta\,(x+y) - (\alpha+\beta) = l,$$

$$(\alpha+\beta)\,\varepsilon - 2\delta\gamma = c,$$

we finally find after reduction

$$x_\xi = \frac{l}{c}\,(\delta x - \alpha), \; y_\xi = \frac{l}{c}\,(\delta y - \beta), \; x_\eta = \frac{l}{c}\,(\delta x - \beta), \; y_\eta = \frac{l}{c}\,(\delta y - \alpha)$$

$$x_{\xi\eta} = \frac{l^2}{c^2}\,\delta\,\{2\delta x - (\alpha+\beta)\}, \; y_{\xi\eta} = \frac{l^2}{c^2}\,\delta\,\{2\delta y - (\alpha+\beta)\},$$

whilst at the same time holds

$$\lambda = -\frac{c}{l}.$$

The equation (17) now passes into

$$\begin{aligned} S_{xx}(\delta x-\alpha)(\delta x-\beta) &+ S_{xy}\{(\delta x-\alpha)(\delta y-\alpha)+(\delta x-\beta)(\delta y-\beta)\} + \\ + S_{yy}(\delta y-\alpha)(\delta y-\beta) &+ \delta S_x\{2\delta x-(\alpha+\beta)\} + \delta S_y\{2\delta y-(\alpha+\beta)\} - \\ - m\delta S_x\{2\delta x-(\alpha+\beta)\} &- m\delta S_y\{2\delta y-(\alpha+\beta)\} + m(m-1)\delta^2 S = 0, \end{aligned}$$

or

$$\begin{aligned} \delta^2\,[x^2S_{xx}+2xyS_{xy}+y^2S_{xy}-2(m-1)(xS_x+yS_y)+m(m-1)S] - \\ -(\alpha+\beta)\,\delta\,[xS_{xx}+(x+y)S_{xy}+yS_{yy}-(m-1)(S_x+S_y)] + \\ +[\alpha\beta S_{xx}+(\alpha^2+\beta^2)S_{xy}+\alpha\beta S_{yy}] = 0. \end{aligned}$$

In order to give to this equation a more concise form we shall make the equation S homogeneous by introduction of a third variable, z.

We then have

$$\begin{aligned} m(m-1)S &= x^2S_{xx}+2xyS_{xy}+y^2S_{yy}+2xzS_{xz}+2yxS_{yz}+z^2S_{zz}, \\ (m-1)S_x &= xS_{xx}+yS_{xy}+zS_{xz}, \\ (m-1)S_y &= xS_{xy}+yS_{yy}+zS_{yz}; \end{aligned}$$

so

$$\begin{aligned} x^2S_{xx}+2xyS_{xy}+y^2S_{yy}-2(m-1)(xS_x+yS_y)+m(m-1)S &= z^2S_{zz}, \\ xS_{xx}+(x+y)S_{xy}+yS_{yy}-(m-1)(S_x+S_y) &= -z(S_{xz}+S_{yz}). \end{aligned}$$

If we now put $z=1$ we find for the differential condition

$$\delta^2S_{zz}+(\alpha+\beta)\,\delta(S_{xz}+S_{yz})+[\alpha\beta S_{xx}+(\alpha^2+\beta^2)S_{xy}+\alpha\beta S_{yy}]=0. \quad (18)$$

If we exclude for the present the case $\delta=0$, corresponding to the *affine* transformation, we may put into the equation (18) without any objection $\delta=1$; by this (18) takes the form

$$S_{zz}+(\alpha+\beta)(S_{xz}+S_{yz})+[\alpha\beta S_{xx}+(\alpha^2+\beta^2)S_{xy}+\alpha\beta S_{yy}]=0. \quad (18a)$$

We can now dispose arbitrarily of the quantities α and β.

If $S(x,y)$ is of order *two*, then all second derivatives are constant, so that the equation (18a) forms a connection between the constants of the equation and the constants of the transformation. So we can say:

The general symmetrical quadratic equation can be brought by an *infinite number of projective transformations* into the form $\psi(x)+\psi(y)=2k$.

If e.g. is given

$$S(x,y) \equiv a_2(x+y)^2+2b_2xy+2a_1(x+y)+a_0=0,$$

then we have

$$S_{xx}=2a_2\,.\;S_{xy}=2(a_2+b_2)\,,\;S_{yy}=2a_2\,,\;S_{xz}=2a_1\,,\;S_{yz}=2a_1\,,\;S_{zz}=2a_0.$$

The condition ($18a$) now runs

$$a_0 + 2a_1(\alpha + \beta) + (a_2 + b_2)(\alpha + \beta)^2 - 2b_2\alpha\beta = 0 \quad . \quad . \quad (19)$$

Consequently if we choose α and β in such a way that (19) is satisfied, then S is brought to the form

$$(A\xi^2 + B\xi) + (A\eta^2 + B\eta) = 2C,$$

or

$$(\xi^2 - 2B'\xi + C') + (\eta^2 - 2B'\eta + C') = 2k,$$

or if we choose $C' = B'^2$

$$(\xi - B')^2 + (\eta - B')^2 = 2k,$$

so that

$$\psi(\xi) = (\xi - B')^2 = k + \sigma\,.\,\omega(\sigma^2) = k + e^{i\pi\nu}\omega(e^{2i\pi\nu}),$$

or

$$\xi = B' + \sqrt{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})},$$
$$\eta = B' + \sqrt{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})},$$

whilst

$$\left.\begin{aligned} x &= \frac{\alpha\{B' + \sqrt{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})}\} + \beta\{B' + \sqrt{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})}\} + \gamma}{\delta\{2B' + \sqrt{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})} + \sqrt{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})}\} + \varepsilon}, \\ y_n &= \frac{\alpha\{B' + \sqrt{k + e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})}\} + \beta\{B' + \sqrt{k - e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})}\} + \gamma}{\delta\{2B' + \sqrt{k + e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})} + \sqrt{k - e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})}\} + \varepsilon}. \end{aligned}\right\} \quad (20)$$

If $S(x,y)$ is of order *three*, then the two derivatives are of order one, therefore of the form $p_1(x+y) + p_0$. The equation ($18a$) becomes therefore likewise of order one, e.g.

$$P_1(x+y) + P_0 = 0.$$

As this relation must hold for all values of $x + y$, we have to satisfy

$$P_1 = 0 \quad , \quad P_0 = 0,$$

so that we have now obtained *two* relations between the constants of the equation and the *two* constants α and β of the transformation. So we conclude from this :

The general symmetrical cubic equation can be brought by a *finite number of projective transformations* into the form $\psi(x) + \psi(y) = 2k$.

If we put e.g.

$$S(x,y) = a_3(x+y)^3 + 3b_3(x+y)xy + 3a_2(x+y)^2 + 6b_2xy + 3a_1(x+y) + a_0 = 0,$$

we have

$$S_{xx} = 6\{a_3(x+y) + b_3y + a_2\}, \quad S_{xy} = 6\{(a_3+b_3)(x+y) + a_2 + b_2\},$$
$$S_{yy} = 6\{a_3(x+y) + b_3x + a_2\}, \quad S_{xz} = 6\{a_2(x+y) + b_2y + a_1\},$$
$$S_{yz} = 6\{a_2(x+y) + b_2x + a_1\}, \quad S_{zz} = 6\{a_1(x+y) + a_0\}.$$

So equation ($18a$) now becomes

$$[a_1 + (2a_2 + b_2)(\alpha + \beta) + (a_3 + b_3)(\alpha + \beta)^2 - b_3\alpha\beta](x + y) +$$
$$+ [a_0 + 2a_1(\alpha + \beta) + (a_2 + b_2)(\alpha + \beta)^2 - 2b_2\alpha\beta] = 0,$$

so that α and β are determined by

$$a_1 + (2a_2 + b_2)(\alpha + \beta) + (a_3 + b_3)(\alpha + \beta)^2 - b_3\,\alpha\beta = 0 \quad , \quad . \quad (21)$$
$$a_0 + 2a_1(\alpha + \beta) + (a_2 + b_2)(\alpha + \beta)^2 - 2b_2\,\alpha\beta = 0 \, . \quad . \quad . \quad . \quad (19)$$

Out of these equations we find two values for $\alpha+\beta$ and two corresponding values of $\alpha\beta$, thus two sets (α,β) or (β,α). So in general *two* projective transformations are possible transferring the symmetrical cubic equation into the standardform desired by us. This is

$$(A\xi^3 + B\xi^2 + C\xi) + (A\eta^3 + B\eta^2 + C\eta) = 2D.$$

We can modify the constants in such a way that we find

$$\{(\xi - B')^3 + 3\mu(\xi - B')\} + \{(\eta - B')^3 + 3\mu(\eta - B')\} = 2k,$$

so that

$$\psi(\xi - B') = (\xi - B')^3 + 3\mu(\xi - B') = k + \sigma \, . \, \omega(\sigma^2) = k + e^{i\pi\nu}\,\omega(e^{2i\pi\nu}) \, ,$$
$$\psi(\eta - B') = (\eta - B')^3 + 3\mu(\eta - B') = k - \sigma \, . \, \omega(\sigma^2) = k - e^{i\pi\nu}\,\omega(e^{2i\pi\nu}) \, ,$$

hence

$$\xi = B' + \psi_{-1}\{k + e^{i\pi\nu}\,\omega(e^{2i\pi\nu})\} \, , \; \eta = B' + \psi_{-1}\{k - e^{i\pi\nu}\,\omega(e^{2i\pi\nu})\} \, ,$$

$$\left.\begin{aligned} x &= \frac{\alpha[B' + \psi_{-1}\{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})\}] + \beta[B' + \psi_{-1}\{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})\}] + \gamma}{\delta[2B' + \psi_{-1}\{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})\} + \psi_{-1}\{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})\}] + \varepsilon} \, , \\ y_n &= \frac{\alpha[B' + \psi_{-1}\{k + e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\}] + \beta[B' + \psi_{-1}\{k - e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\}] + \gamma}{\delta[2B' + \psi_{-1}\{k + e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\} + \psi_{-1}\{k - e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\}] + \varepsilon} \, . \end{aligned}\right\} (23)$$

If we now regard the *affine* transformation, we have but to put in equation (18) $\delta = 0$; we then find

$$\alpha\beta(S_{xx} + S_{yy}) + (\alpha^2 + \beta^2)S_{xy} = 0$$

or

$$\frac{S_{xy}}{S_{xx} + S_{yy}} = -\frac{\alpha\beta}{\alpha^2 + \beta^2} = const. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For the quadratic equation this can always be satisfied and that by two values of the ratio $\alpha : \beta$; hence:

the *general symmetrical quadratic equation* can be brought by *two affine transformations into* the form $\psi(x) + \psi(y) = 2k$.

For the cubic equation, the equation (24) demands

$$\frac{(a_3 + b_3)(x + y) + a_2 + b_2}{(2a_3 + b_3)(x + y) + 2a_2} = const.,$$

therefore

$$\frac{a_3 + b_3}{2a_3 + b_3} = \frac{a_2 + b_2}{2a_2}$$

or

$$2a_3 b_2 + b_2 b_3 - a_2 b_3 = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The *general symmetrical cubic equation* can be brought by an *affine transformation* into the form $\psi(x) + \psi(y) = 2k$ *only* when condition (25) is satisfied.

This condition expresses that the three asymptotes of the cubic curve represented by the given equation pass through *one* point.

In connection with this we might have obtained equation (25) also in a geometrical way. Of a cubic curve which has as equation

$$A\xi^3 + B\xi^2 + C\xi + A\eta^3 + B\eta^2 + C\eta = 2D,$$

the three asymptotes pass namely through *one* point, a property which can stand an affine transformation.

Chemistry. — "*On the appearance of a maximum and minimum pressure with heterogeneous equilibria at a constant temperature*". By Dr. F. E. C. Scheffer. (Communicated by Prof. A. F. Holleman.)

(Communicated in the meeting of April 29, 1910).

In the spacial figure of a binary system in which occurs a complete miscibility in the liquid condition, a complete separation in the solid condition and where the vapour pressures of the liquid fall continuously from $x = 0$ to $x = 1$, two three-phase lines appear at the place where one of the two components in the solid condition coexists with liquid and vapour. Whereas the pressure values on the three-phase line of the first component increase continuously with the temperature, this is not the case with the line of the second component; Roozeboom suspected that the latter in its P-T-projection always possessed a maximum [1]) Later, Kohnstamm [2]) showed that this maximum need not appear always; from the equation of the three-phase line deduced in 1897 by van der Waals [3]), the condition could be deduced when a maximum appeared and when not, because in the former case the value of $(\eta_v - \eta_s) - \frac{x_v}{x_l}(\eta_l - \eta_s)$ must be 0. This condition, however, may point to the appearance of a minimum as well as that of a maximum.

The appearance of a minimum pressure on the three-phase line of the second component becomes even very probable when a minimum occurs in the P-x-lines of the liquid-vapour plane. For this case the

[1]) Bakhuis Roozeboom, Heterogene Gleichgewichte. II. 331.
[2]) Kohnstamm, Proc. 1907, Febr. 23.
[3]) Van der Waals, Proc. 1897, April 21.

P-T-projection of the spacial figure has been indicated in fig. 1 and the T-x-projection of the two three-phase lines in fig. 2. The three-phase line O_AE is always situated above the sublimation line of A; the line O_BE always above that of B; in point R the minimum line meets the three-phase line; liquid and vapour, under three-phase pressure, get here the same composition which, in fig. 2, amounts to an intersection of liquid- and vapour branches of the three-phase line of the second component. If we arrange the three phases according to their B-content, the order in which they follow at temperatures above that of R is LGS_B, below R, GLS_B. We now see that at temperatures below R the above condition may be fulfilled ($x_v > x_l$ where these values relate to the first component), so that the value of $\frac{dp}{dt}$ becomes 0, which, therefore, in our case causes the appearance of a minimum.

The condition for the appearance of a minimum is, however, not dependent on the presence of a minimum pressure line on the liquid-vapour plane; of this it is totally independent, so that there exists the possibility that a minimum pressure may appear in the P-T-projection of the three-phase line of the second component while the liquid-vapour plane is still falling continuously from $x = 0$ to $x = 1$.

In the case of a line of maximum pressure occurring on the liquid-vapour plane, this line will, in quite an analogous manner, come in contact with the three-phase line of the first component (fig. 3 and 4). In point R the value of $\frac{dp}{dt}$ is positive $\left(= \frac{Q_{lv}}{V_v - V_l}\right)$; between R and O_A there is now a possibility of the appearance of a maximum which has been mentioned previously by KOHNSTAMM. In this case two maxima may therefore occur, although neither of them need necessarily appear at all. It appears from the T-x-projection that at temperatures between R and O_A (fig. 4) the condition $x_v > x_l$ is complied with and that the value of $\frac{dp}{dt}$ may become nought.

Now whereas in the spacial figure of the above mentioned form only maximum and minimum pressure lines can appear on the liquid-vapour plane, because the concentration of the solid matter never differs from 1 or 0, it becomes possible that maximum and minimum pressure lines for the coexistence of solid and liquid and of solid and gas also appear when the concentration of the solid matter is not limited to 1 and 0.

In the first place this may happen when one or more compounds occur in the spacial figure; as to the presence of minimum pressure lines on the SG and SL planes, which come in contact with the maximum sublimation point and minimum melting point on the three-phase line, several communications have appeared in these Proceedings from VAN DER WAALS, BAKHUIS ROOZEBOOM, SMITS and others.

In the second place similar lines may occur in systems where the solid matter is a mixture of the components in varying proportions (mixed crystals). If we suppose that complete mixing takes place in the solid condition we can readily observe the possible cases from fig. 5. In this figure is drawn a circular curve of which the three-phase line forms a part; this gives us a simple means for observing all the possibilities. We conclude from this figure that six types are possible in which one line of maximum or minimum pressure comes in contact with the three-phase line (ab, bc, cd, de, ef, fa, where these letters indicate each time the triple points of the components). We can also distinguish six cases where two contacts occur ($a(b)c$, $b(c)d$, $c(d)e$, $d(e)f$, $e(f)a$, $f(a)b$) and six, where three of the said lines have a point in common with the three-phase line ($a(bc)d$, $b(cd)e$, $c(de)f$, $d(ef)a$, $e(fa)b$, $f(ab)c$) [1].

The other conclusions may, I think, be passed over, but it should be pointed out that it is shown, from fig. 5, in the plainest manner that when on the three-phase line occurs a point in which $x_s = x_l$ and another in which $x_l = x_v$ there must also occur on the three-phase line a point in which solid and gas have the same composition, a condition which we notice occurs in quite an analogous manner in the appearance of a dissociating compound.

Let us consider more particularly the case in which three lines of minimum pressure occur in the spacial figure (fig. 6). We then obtain a P,T projection which exhibits a great analogy with that of a dissociating compound which is less volatile than both components. The line where S and L have the same composition meets the three-phase line in R_1, the one where S and G have the same composition does so in R_2; in analogy with the corresponding remarkable points of a dissociating compound we may call these points the minimum melting point and maximum sublimation point of the mixtures behaving as a simple substance; both these points which in a dissociating compound exhibit one composition in their positions occur here with different x. This causes also the P,T line for the constant x to pass

[1]) In this communication it has been assumed that the solid matter, on melting, always increases in volume.

through R_1 (which therefore, possesses a cusp) and not through R_2 and reversedly, so that the two cusps of the P,T line of the compound occur in our case no longer on one but on two different lines [1]).

The T,x-projection of the three-phase line represented in Fig. 7 exhibits the intersections of the liquid and solid branch in R_1, of the solid and gas branch in R_2, of the liquid and gas branch in R_3; the three said points correspond with the homonymous ones in the P,T-projection (fig. 6).

Finally, it should be mentioned that the appearance of a maximum or a minimal pressure on the three-phase line may be readily ascertained from the equations given by VAN DER WAALS in 1897.

$$\frac{dp}{dt} = \frac{(\eta_v - \eta_s) - \dfrac{x_v - x_s}{x_l - x_s}(\eta_l - \eta_s)}{(v_v - v_s) - \dfrac{x_v - x_s}{x_l - x_s}(v_l - v_s)}$$

has the value 0 when the numerator becomes 0, for this we require $x_v - x_s > x_l - x_s$; this condition is fulfilled on the three-phase line between O_A and R_1 where, consequently, a maximum pressure may appear, whilst between O_B and R_3 a minimum pressure may occur.

It will be a simple matter to deduce also the possible forms for the other cases; the collection of facts at our disposal, is as yet too incomplete for us to be certain as to the occurrence of the above cases. However, the melting point lines of the system $HgI_2 - HgBr_2$,[2]) *d*- and *l*-carvoxime [3]), monochlore-monobromohydrochinondiacetate [4]), *p*-azoxyphenetol-*p*-methoxycinnamic acid [5]), *p*-azoxyphenetol-cholesteryl benzoate [5]) and of many others point to the appearance of maximum and minimum lines on the *SL*-plane and a good many examples are already known of maximum and minimum lines on the *LG*-plane, although in the latter cases there is generally no complete certainty as to the nature of the solid phase.

1) SMITS, Proc. 1906 April 27.

2) REINDERS, Dissertation, Amsterdam 1899.

3) ADRIANI, „ „ 1900.

4) KÜSTER, Zeitschr. f. Physik. Chemie **8**. 583.

5) A. PRINS, Dissertation, Amsterdam 1908.

Physics. — "*The magnetic separation of absorption lines in connection with Sun-spot spectra*". (Second Part)[1]). By Prof. P. ZEEMAN and Dr. B. WINAWER.

(Communicated in the meeting of April 29, 1910).

21. The outer components of a magnetically divided line, if observed in a direction inclined to the lines of force under an angle ϑ, are elliptically polarized.

In our experiments of §§ 12 -17 we frequently referred to this elliptical polarization. In § 12 were resumed the simple rules, which relate to the ellipses characterizing the state of polarization of the outer components, if *very narrow* spectral lines are observed in a *strong* field.

The linear vibrations of the central component of a triplet lie according to the elementary theory in the plane passing through the ray and a line of force, and the amplitude is proportional to *sin* ϑ.

RIGHI's theoretical considerations in his paper cited in § 1 also agree with this conclusion.

22. In VOIGT's[2]) theoretical investigation of the magnetic effect in a direction inclined to the lines of force, the remarkable conclusion is drawn that also the central component of a triplet may execute an elliptical vibration. This result is most closely connected with the taking into account of the mutual action between neighbouring molecules.

LORENTZ's considerations concerning our present subject (cf. § 1 above) give results which we may be permitted to summarize here briefly.

For arbitrarily chosen values of the angle ϑ between the ray and the magnetic force for every frequency two elliptical vibrations of opposite directions can be transmitted.

In the case of the *outer* components of a sharp triplet one of the two elliptic vibrations is absorbed.

If we have not to deal with a sharp triplet, i.e. three absorption bands that are completely separated, we can still say something about the vibration ellipses of the outer components.

Let axes OY and OX' be chosen, the one normal to the plane

1) Continuation of the paper published in these Proceedings. Vol. XII, p. 584. 1910.

2) W. VOIGT, Weiteres zur Theorie der magneto-optischen Wirkungen. Ann. d. Phys. I. (1900) p. 389.

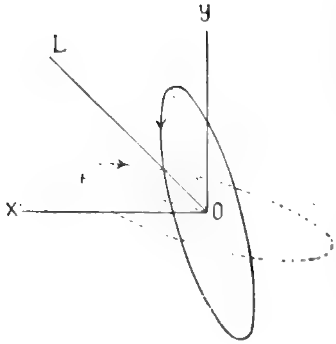


Fig. 1.

passing through the ray and the magnetic force, the other perpendicular to the ray and lying in the plane just mentioned. Then one of the characteristic vibration ellipses can be considered as the reflected image of the other with respect to a line bisecting the angle $X'OY$. This rule also applies to the direction of motion in the two ellipses.

The nature of the phenomena that will be observed for rays of a frequency corresponding to the *central* line of the triplet, depends upon the value of ϑ being greater or smaller than a certain angle ϑ_1. This latter is determined by the equation

$$tang\ \vartheta_1\ sin\ \vartheta_1 = \frac{g}{r}.$$

The quantity g may be regarded as a measure of the width of an absorption line and depends upon the constants of the vapour; r is determined by the change of the frequency of the free vibrations of the electrons and has a value proportional to the strength of the field.

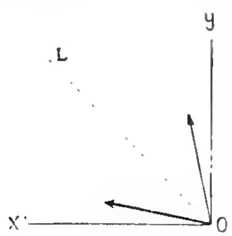


Fig. 2.

If $\vartheta > \vartheta_1$, then two linearly polarized beams with equal indices of refraction and different absorption indices can be propagated. The rectilinear vibrations make equal angles with the line OL bisecting the angle XOY'. The absorption is strongest for the beam whose vibrations make the smaller angle with the direction of the field. In the figure the most strongly absorbed vibration is indicated by a thicker arrow.

As ϑ decreases the vibrations of the two principal beams approach more and more to OL, so that for $\vartheta = \vartheta_1$ both directions coincide with the bisectrix. The two principal beams are now equally absorbed also.

When $\vartheta < \vartheta_1$, the state of things is wholly different.

In this case two elliptically polarized beams can be propagated; they are equally absorbed, but have different velocities of propagation. For both beams the characteristic ellipses are the same, but described in opposite directions. One of the axes of the ellipses coincides with the line OL in Fig. 2.

The ellipses become less and less eccentric as the wave becomes less inclined to the direction of the field. For $\vartheta = 0$ the ellipses become circles described in opposite directions.

A further approximation for $\vartheta = \vartheta_1$ shows, that in this case the two vibrations do not coincide exactly. As in the general case there are two distinct beams with different characteristic ellipses, both deviating somewhat from the line OL of Fig. 2.

The regions of the longitudinal and the transverse magnetic effect overlap to a certain extent and are not sharply separated from each other at the angle ϑ_1.

23. There are three results of LORENTZ's theory that probably admit of experimental verification.

Let us imagine the absorbing vapour placed in such circumstances that the elementary theory cannot be applied. The components of a divided line are now not neatly separated by practically transparent regions. The vapour density must be chosen relatively great and the magnetic intensity rather small. As always in the present paper we suppose the lines of force to be horizontal; we examine the propagation of the light also in a horizontal plane.

The three predictions referred to and which apply, if we exclude the cases of the true longitudinal and transverse effects, are:

1st. the major axes of the vibration ellipses of the outer components deviate from the vertical line.

2nd. the vibrations of the middle component (c.q. components) are, depending on circumstances, either linear and not horizontal or elliptic, the axes of the ellipse being inclined to the horizon.

3rd. there exists an angle ϑ_1 separating the regions of the longitudinal and the transverse magnetic effect.

Oblique position of the vibration ellipses of the outer components.

24. We succeeded establishing experimentally the oblique position of the vibration ellipses in the inverse magnetic effect of the D-lines; the amount of the slope of the axes we could measure.

The obliquity is far from striking.

When ϑ was already such that the ellipticity was very marked, we first only after some difficulty could make sure of the obliquity.

Some details of a definite case may be given. With $\vartheta = 69°$ and a field of about 18000 gauss the first observations were made.

Attention was given to D_2, the vapour density being regulated so that the outer components of the sextet could not be seen separately. Before the slit of the spectroscope a Nicol was placed with its plane of vibration under an azimut of say 35° with the horizon. The central part of the resolution figure is now very dark, the outer components of the pseudo-triplet however are only faintly visible. This has the advantage to increase the visibility of small changes of the intensity of the outer components.

The direction of the field we denote as field direction 1.

With the reversed field direction 2, the outer components became darker.

This experiment was repeated several times with the same result.

The Nicol then was placed in a position symmetrical to the one just mentioned. Now with field direction 1, the outer components were darker. From these experiments we must conclude that a vertical line is not an axis of symmetry of the vibration ellipses of the outer components, hence that the position of these ellipses is oblique.

25. The direction of the smaller axis of the vibration ellipse we measured for $\vartheta = 69°$, the vapour density being between the first and second phase (§ 13). Before the slit of the spectroscope a Nicol was introduced, mounted upon a divided circle, which gives the rotation of the Nicol in degrees. The vanishing or reappearing of the outer components gave a good criterion for the determination of the smaller and therefore of the major axis of the vibration ellipse.

The measurements gave the result that under the circumstances of the experiment the major axis made an angle of 5 degrees with the vertical. The obliquity was the same in amount and direction for the components towards the red and towards the violet. The diagram Fig. 3 illustrates the relation between the slope of the ellipses and the direction of the field.

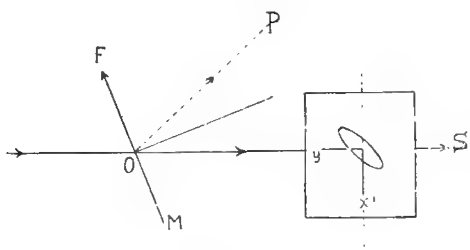


Fig 3.

Let OS be the beam, which traverses the source of light placed in O and OF the direction of the magnetic force. For an observer looking in the direction SO, the upper part of the vibration ellipse is inclined towards the right. The plane YX' containing the ellipse is normal to the ray and in the figure has been rotated round the dotted line until brought into coincidence with the plane SOM. That side of the plane which was visible from S, can now be seen. Both the ellipse towards the red and the ellipse, described in opposite direction, towards the violet have the same slope with a given direction of the magnetic field, as was remarked above.

26. *The same* sodium flame investigated as to the inverse effect in the direction OS, we studied in the direction OP (i.e, for an angle $FOP = MOS = 180° - \vartheta$) for the phenomenon of partial polarization, discovered by EGOROFF and GEORGIEWSKY. A small telescope focussed upon the flame was used and provided with a SAVART plate and a Nicol. This polariscope is mounted upon a divided circle graduated in degrees. The direction in which the fringes were most brilliant was determined in order to detect a possible deviation of the plane of maximum polarization from the vertical. It was easily seen that there was such a deviation. The fringes were most clear if for the observer in P their direction was from the upper left to the lower right quadrant, the direction of the field being always as indicated in the figure. After reversal of the magnetic field the fringes became indistinct. They became distinct again if the principal direction of the polariscope was from the upper right to the lower left quadrant. The result of these observations at least proves that the whole phenomenon is asymmetrical with respect to the vertical and hence proves the presence of oblique vibrations. In a conversation with one of the authors Prof. LORENTZ had kindly communicated that he observed phenomena of the kind described in this §.

27. In the experiment of the last § the axis of the telescope must be placed carefully in a horizontal plane passing through the poles of the electromagnet. If the observation is made in a plane which is not horizontal an apparent slope of the axes of the vibration ellipses becomes manifest, as is easily seen from a geometrical consideration.

28. The position of the plane of maximum polarization can be determined rather accurately. The obliquity of the major axis of the outer ellipses of sextet and quartet in one experiment was 5°; with the very same vapour density and the same strength of field the plane of partial polarization made an angle of 21° with the vertical.

At first sight it seems rather startling that the polariscope of SAVART is so sensitive to the obliquity of the ellipses.

The phenomenon of the partial polarization of the emitted light is very complicated and the complete theory still outstanding.

It seems not doubtful, however, in what direction we have to look for the explanation of the remarkable difference between the indications of the two instruments. They measure different quantities.

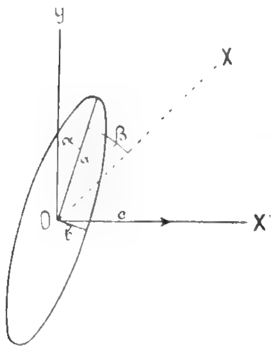


Fig. 4.

As long as the inclination of the vibration ellipses of the emitted light is zero, the total light also vibrates symmetrically relatively to the vertical.

If the inclination is not zero, however, but has the value α, the plane of maximum resultant luminous motion is inclined at an angle $\alpha + \beta$, which may be occasionally much greater.

The light emitted by the sodium flame contains:

1. horizontal vibrations of intensity c^2. (We neglect here a change mentioned in § 30 below).

2. elliptic vibrations, the major axes of which form an angle α with the vertical. Let the principal axes of these ellipses be a and b.

The intensity I_x in a direction OX becomes

$$I_x = c^2 \sin^2 (\alpha + \beta) + a^2 \cos^2 \beta + b^2 \sin^2 \beta \quad . \quad . \quad . \quad . \quad (1)$$

This expression becomes a maximum for a value of β satisfying

$$c^2 \sin 2(\alpha + \beta) + (b^2 - a^2) \sin 2\beta = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Hence it follows already that β cannot be zero, for otherwise α ought to be zero also.

From (2) we obtain

$$\frac{\sin 2(\alpha + \beta)}{\sin 2\beta} = \frac{a^2 - b^2}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence the value of β depends upon the intensities of the horizontal and vertical vibrations. Always $a > b$; in the emitted light the vertical vibrations generally preponderate hence also $a > c$. We conclude that β can only be positive.

If we take $\alpha = 5^\circ$ $\alpha + \beta = 21^\circ$ $b = 0{,}3\ a$ (cf. § 29) equation (3) gives

$$\frac{a^2}{c^2} = 1{,}4.$$

This is a plausible value. Hence there is no contradiction between the observations made with the polariscope and the results obtained with the Nicol alone.

29. We made, with the inverse effect, some measurements of the ellipticity of the outer components at different angles of incidence. We used for this investigation the well known method of the quarter wave plate and Nicol. The axes of the quarter wave

plate being placed parallel to the axes of the original ellipse, the resulting light is plane polarized. Let b and a be the horizontal and vertical or the nearly horizontal and the nearly vertical axes then

$$\frac{b}{a} = tang\ \alpha.$$

The mica quarter plate used proved to be very accurate for light of the refrangibility of the sodium lines, when tried by the method described on a former occasion [1]. Three determinations gave for the deviation from an exact quarter wave plate the values 1,8, 0,1, 1.0 %.

For our present determinations this accuracy of the plate is quite superfluous. The measurements are very difficult, relating as they do to the mean of the outer components of the sextet, hence to an extremely narrow part of the spectrum. Moreover the density of the vapour can be defined only approximately (10).

The following table embodies the results concerning the ellipticity of the outer components of the sextet obtained in a somewhat extended series of measurements.

ϑ	b/a		Remarks
$69^\circ\,^1/_2$	0.31 0.31 0.28	0.30	Vapour of intermediate density (§ 10)
47°	0.45 0,45	0.45	,,
	0.47 0.50	0.47	Vapour somewhat denser
39°	0.67 0.70 0.70 0.60 0.64 0.67 0.63 0.65 0.65	0.66	Very dilute vapour (§ 10)

The ratio of the axes at a certain angle undoubtedly somewhat depends upon the vapour density. Part of the oscillations of the results obtained at the same angle must be described to this cause.

At $\vartheta = 69^\circ\,^1/_2$ and with dense vapour the inclination of the major

[1]) ZEEMAN, These Proceedings October 30. 1909.

axis of the ellipse was 6°; with very dilute vapour the value zero was obtained.

At $\vartheta = 47°$ and with vapour of intermediate density the inclination was $4° \, ^1/_2$. The SAVART fringes then made an angle of 28° with the vertical.

Oblique position of the vibrations of the middle components.

30. Whereas the inclination of the vibration ellipses of the outer components could be demonstrated first for the sextet, it was for the quartet on the contrary we first succeeded in verifying the second of LORENTZ's above mentioned conclusions (23).

The deviation of the vibrations of the middle components of the quartet from the horizontal line can be shown in the same manner as the inclination of the ellipses (24).

The principal section of the Nicol before the slit was placed at an angle of about 30° with the horizon. The outer components of the quartet of D_1 are then hardly visible. The inner components are rather dark. The direction of the field be indicated as direction 1. Under the influence of the reverse field 2, the middle components become more black. If the Nicol be placed in the symmetrical position then it is with the field direction 1 that the middle components are most distinct.

The angle ϑ in this experiment was 47°.

Two different attempts to measure the angle between the vibration and the horizon gave the results $4° \, ^1/_2$, resp. $5° \, ^1/_2$. These measurements are very difficult, however and perhaps indicate only the order of magnitude of the inclination. The vicinity of the outer components largely interferes with the accuracy of the adjustment of the Nicol, for while it is moved about near the position of extinction and approaches to a vertical direction the greater intensity of the outer components distracts the eye.

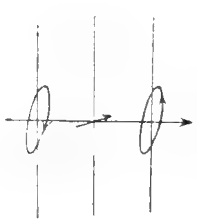


Fig. 5.

31. We have made yet another experiment which confirms the result of (30) for both the sodium lines and also exhibits the relation between the inclinations of the different components. This connection is for a triplet diagrammatically shown in Fig. 5. For the result obtained with the middle components of the quartet and the sextet certainly can be applied qualitatively to the triplet.

The experiment was the following: the principal section of the Nicol made an angle of $+40°$ with the vertical; the positive direction in Fig. 5 be anti-clockwise. Then the Nicol was placed at 320° (i.e. in the symmetrical position). The last position be indicated as position *B*, the first mentioned as position *A*.

The direction of the field remains unchanged.

In position *A* all lines were weaker than in position *B*.

Hence we conclude that the ellipses as well as the vibrations of the middle components are inclined, moreover that the relative position of the vibrations must be that shown in Fig. 5.

32. In the important paper already frequently mentioned RIGHI (Note p. 291 of the paper cited § 1 above) says that VOIGT's theoretical investigation of the general case of propagation of light in a direction inclined to the lines of force was published too late to guide him in his investigation. RIGHI expresses the opinion that it is rather improbable that in the course of his numerous observations particulars in the behaviour of the middle components as indicated by VOIGT could have escaped him and that LORENTZ's elementary theory is in accordance with all the observed phenomena.

This seems in contradiction with our experiments. This contradiction vanishes, however, if we assume that the vapour in RIGHI's experiments was very dilute, or the field so intense that the components were neatly separated. Under such circumstances also our observations are in complete accordance with the elementary theory, at least as to the polarization of the components and the direction of the vibrations.

Neither was it in RIGHI's experiments a matter of course to reverse the direction of the magnetic field, the procedure which most easily exhibits any obliquity of the vibrations.

Application of the results of §§ 24—31 to the interpretation of sunspot spectra.

33. The vibrations of the middle component of a triplet are parallel to the lines of force. The outer components vibrate linearly at right angles to the field. These rules also apply to dense vapours, if only the pure transverse magnetic effect be under consideration. If we assume that the direction of observation is oblique to the lines of force then only in the case of very dilute vapours the projection of the magnetic force on a plane normal to the line of vision can be found according to the rules of the elementary theory from the direction of the vibrations. If, however, the components of an inverse triplet are not neatly separated by practically transparent parts, — and the sun-spot lines seem to belong to this class of lines, — the particulars diagrammatically illustrated by Fig. 5 are to be taken into consideration.

In drawing charts of the magnetic fields in sun-spots, showing the intensity, the direction and the polarity of the magnetic force, the determination of the direction of the force will give some difficulties.

The value of the correction to the indications of the elementary theory necessary in some cases shall be given on another occasion.

The rule, which determines the direction of the deviation, may be indicated here.

The direction of rotation in the vibration ellipses of the outer components towards the red and towards the violet shows whether ϑ is acute or obtuse. If ϑ is obtuse (Fig. 3), then the relative position of the directions of the magnetic force, of the major axis of the vibration ellipses and of the vibration of the middle component is shown in Fig. 5.

From any point O draw a line OB parallel to the major axis of the vibration ellipses of the outer components and a line OM parallel to the vibration of the middle component, the angle BOM being always chosen acute. The projection OF of the magnetic force on a plane normal to the line of sight then makes a positive acute angle with OB, the angle BOF being greater than BOM, the positive direction being reckoned from OB to OM.

By ascertaining whether or not the major axes of the ellipses and the vibrations of the middle component are perpendicular to each other we can make sure whether the elementary theory may be applied or not.

Mathematics. — *"On linear polar groups belonging to a biquadratic plane curve"*. By Prof. JAN DE VRIES.

(Communicated in the meeting of April 29, 1910).

1. With respect to a biquadratic curve γ^4, with the symbolic equation $a_x^4 = 0$, the points X, Y, Z, W form a polarquadruple when the relation $a_x a_y a_z a_w = 0$ is satisfied. If we take X, Y, Z arbitrarily on a line r and if we take as fourth point W the point of intersection of r with the "triple polar line" p_{xyz} of X, Y, Z, we get a "linear" polarquadruple. The linear polarquadruples on r evidently form an involution I_3^4, its "*principal points*" (fourfold elements) are the points of intersection of γ^4 with r.

If we assume for the points on r such a parameter representation that two of its principal points are indicated by $\lambda = 0$ and $\lambda = \infty$ we find for the groups of the I_3^4 the relation

$$\sum_4 \lambda_1 \lambda_2 \lambda_3 + p \sum_6 \lambda_1 \lambda_2 + q \sum_4 \lambda_1 = 0, \quad \ldots \ldots \quad (1)$$

by which to each triplet $\lambda_1, \lambda_2, \lambda_3$ one value λ_4 belongs, unless at the same time

$$\lambda_1\lambda_2\lambda_3 + p(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + q(\lambda_1 + \lambda_2 + \lambda_3) = 0$$

and

$$(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + p(\lambda_1 + \lambda_2 + \lambda_3) + q = 0,$$

are satisfied.

So there is a "*neutral*" involution I_1^3, the groups of which are completed by each point to quadruples. The right line r is thus triple polar line for ∞^1 triplets of poles lying on it.

2. If we put $z_k = x_k + \lambda y_k$ and $w_k = x_k + \mu y_k$ we find for the linear polarquadruples on the line XY the relation

$$a_x a_y (a_x + \lambda a_y)(a_x + \mu a_y) = 0$$

or

$$a_x^3 a_y + (\lambda + \mu) a_x^2 a_y^2 + \lambda\mu a_x a_y^3 = 0.$$

If we choose X and Y in such a way, that at the same time the three relations

$$a_x^3 a_y = 0, \qquad a_x^2 a_y^2 = 0, \qquad a_x a_y^3 = 0$$

are satisfied, then X and Y form a "*neutral pair*" of the I_3^4; i. e. a pair forming with each two points of r a quadruple of the I_3^4 [1].

In this case the points of intersection of γ^4 with r are indicated by

$$a_x^4 + \lambda^4 a_y^4 = 0.$$

So for them the relation [2]

$$(\lambda_1\lambda_2\lambda_3\lambda_4) = -1,$$

holds, so that the principal points form a harmonic group.

The neutral points of such a "*harmonic*" I_3^4 we shall call "*associated*" with respect to γ^4. They form with each point of their connecting line r a triplet of poles, of which r is the triple polar line.

The line connecting two points U and V of γ^4 intersects it in two points more determined by

$$4a_u^3 a_v + 6\lambda a_u^2 a_v^2 + 4\lambda^2 a_u a_v^3 = 0.$$

They lie harmonically with respect to U and V when $a_u^2 a_v^2 = 0$ is satisfied.

For a given point U this equation represents a polar conic of U. As this conic touches the curve γ^4 in U and intersects it in six

[1] Since $a_x(\rho a_x + \sigma a_y)^3 = 0$ has the same scope as $\rho^3 = 0$, XY is the tangent in the point of inflection Y of the polar cubic of X. So XY forms a part of the polar conic of X and Y and it is therefore polar line of any point Z on XY.

[2] We have namely

$$\frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} : \frac{\lambda_1 - \lambda_4}{\lambda_2 - \lambda_4} = \frac{1-i}{-1-i} : \frac{1+i}{-1+i} = \frac{(1-i)^2}{(1+i)^2} = -1.$$

points Y, each point of γ^4 bears six right lines cut by γ^4 into harmonic groups. With this we have proved the known property according to which there is a curve of *class six* of which the tangents are cut by γ^4 into *harmonic* groups.

3. If we write the equation (1) in the form

$$(\lambda_1+\lambda_2)+p]\lambda_3\lambda_4+[\lambda_1\lambda_2+p(\lambda_1+\lambda_2)+q](\lambda_3+\lambda_4)+[p\lambda_1\lambda_2+q(\lambda_1+\lambda_2)]=0,$$

it is evident that I_3^4 will have a *neutral pair* when at the same time the three relations

$$(\lambda_1+\lambda_2)+p=0, \quad \lambda_1\lambda_2+p(\lambda_1+\lambda_2)+q=0, \quad p\lambda_1\lambda_2+q(\lambda_1+\lambda_2)=0$$

are satisfied, so when we have $2pq=p^3$.

If $p=0$, the neutral pair is determined by $\lambda^2=q$, if $q=\frac{1}{2}p^2$, we have for it $\lambda^2+p\lambda+\frac{1}{2}p^2=0$.

From this ensues that the neutral points coincide when we have $p=0$ and $q=0$. But then we find the principal points out of $\lambda^3=0$, the neutral points out of $\lambda^2=0$. Two associated points can therefore only coincide in a point of inflection of γ^4, and the 24 inflectional tangents are the only right lines on which the I_3^4 of the polarquadruples have a neutral double point.

The degree of the locus of the associated points must therefore be a multiple of six.

To determine that degree we make X to describe a line r. Out of the relations found above

$$a_x^3 a_y=0, \quad a_x^2 a_y^2=0, \quad a_x a_y^3=0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is evident, that the polar cubic p_x^3 describes a pencil, the polar conic p_x^2 a system with index 2, the polar line p_x a system with index 3. From this it is easy to deduce that the points of intersection of p_x^2 and p_x^3 describe a curve γ^8 which has double points in the 9 basepoints of the pencil (p_x^3), whilst the points of intersection of p_x^3 with p_x describe a curve γ^{10} possessing in the basepoints just mentioned threefold points.

In each point of intersection of r with γ^4 the curves p_x^3, p_x^2, and p_x touch each other; those four points are therefore at the same time points of contact of γ^8 and γ^{10}. As those two curves have 54 points of intersection in the 9 basepoints mentioned above there will be 18 points Y, where three corresponding polar lines p_x^3, p_x^2, and p_x concur.

The locus of the associated points is therefore a curve of order 18 which osculates γ^4 in its points [1]) of inflection.

[1]) According to a wellknown rule we find out of (2) by elimination of y_k an equation of order 26 in x_k. From the above follows that this can be broken up into an equation of order 18 and two times the equation of γ^4.

4. The principal points of a I_3^4 can in general not be united to a group of the involution. If again I_3^4 is represented by (1), then $4\lambda^3 + 6p\lambda^2 + 4q\lambda = 0$ furnishes its principal points; so these are determined by $\lambda_1 = 0$, $\lambda_2 = \infty$, $\lambda_3 + \lambda_4 = -\frac{3}{2}p$, $\lambda_3 \lambda_4 = q$.

If we put in (1) $\lambda_1 = 0$ and $\lambda_2 = \infty$, we find

$$\lambda_3 \lambda_4 + p(\lambda_3 + \lambda_4) + q = 0.$$

This is satisfied by the principal points λ_3 and λ_4, when we have $3p^2 = 4q$. But then the four principal points form an *equianharmonic* group, as is easily evident by substitution into the wellknown condition.

If U and V are two principal points of such a particular I_3^4, the other two are determined by

$$4a_u^3 a_v + 6\lambda a_u^2 a_v^2 + 4\lambda^2 a_u a_v^3 = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

with the condition

$$a_u a_v (a_u + \lambda_1 a_v)(a_u + \lambda_2 a_v) = 0$$

or

$$a_u^3 a_v + (\lambda_1 + \lambda_2) a_u^2 a_v^2 + \lambda_1 \lambda_2 a_u a_v^3 = 0 \quad . \quad . \quad . \quad . \quad (4)$$

Out of (3) follows however

$$\frac{1}{2 b_u b^3_v} = \frac{\lambda_1 + \lambda_2}{-3 b^2_u b^2_v} = \frac{\lambda_1 \lambda_2}{2 b^3_u b_v}.$$

By substitution in (4) we now find

$$4 a^3_u a_v b_u b^3_v = 3 a^2_u a^2_v b^2_u b^2_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This relation can be interpreted in a peculiar way. The points of intersection of the polar cubic of U, $a_u a_v^3 = 0$, with the line through U and the point V taken arbitrarily, are found out of

$$3 a^3_u a_v + 3 \lambda a^2_u a^2_v + \lambda^2 a_u a^3_v = 0.$$

If now (5) is satisfied, this equation has two equal roots λ and UV touches the polar cubic of U.

The tangents out of the point U lying on γ^4 to its polar cubic are therefore cut by γ^4 into equianharmonic groups.

With this we have likewise proved the wellknown property according to which the lines divided *equianharmonically* by γ^4 envelop a curve of *class four*. The tangents of this curve cut γ^4 into *linear polarquadruples*.

If besides (5) also $a^2_u a^2_v = 0$ is satisfied, we have either $a^3_u a_v = 0$ or $a_u a^3_v = 0$.

In both cases the equation $(a_u + \lambda a_v)^4 = 0$ has three equal roots ($\lambda^3 = 0$ or $\lambda^3 = \infty$). By this the well known property is confirmed according to which the 24 *inflectional tangents* of γ^4 are the common tangents of a curve of class 4 with a curve of class 6.

The polar conic of a point of inflection possessing the inflectional tangent as component part, this replaces according to § 2 two of the tangents out of the inflectional point to the envelope of the harmonic quadruples; this curve therefore touches the inflectional tangents of γ^4 in the inflectional points.

Indeed, this follows also from the fact, that no tangent of γ^4 can bear a harmonic group unless its point of contact is inflectional point.

5. If U and V are the points of contact of a double tangent of γ^4, then as $(a_u + \lambda a_v)^4 = 0$ shows, $a^3_u a_v = 0$ and $a_u a^3_v = 0$ are satisfied; each of those points is then the point of intersection of the polar line and the polar cubic of the other point

If we allow U to describe the curve γ^4 then p_u and p^3_u touch each other in U and *their point of intersection describes a curve of order* 32. For, p_u and p^3_u describe respectively systems with index 12 and 4, since the poles of the polar lines and of the polar cubics passing through a point V are generated on γ^4 by p^3_v and p_v. On a right line the two systems determine a (4,36) correspondence and as γ^4 belongs twice to the generated locus, the locus of the point of intersection of p_u and p^3_u is a curve γ^{32}.

In each point of contact of γ^4 the line p_u and curve p^3_u have three points in common; therefore γ^4 is osculated there by γ^{32}. The remaining 56 common points of the two curves are evidently the points of contact of the 28 double tangents of γ^4.

Physiology. — *"About exchange of gases in cold-blooded animals in connection with their size."* By F. J. J. Buytendijk. (Communicated by Prof. H. Zwaardemaker).

In a previous communication [1]) I have been able to prove that in fishes as well as in a number of invertebrate sea-animals the consumption of oxygen of the smaller individuals is considerably larger than that of the larger ones of the same kind.

Through the kindness of the Director of the Royal Zoological Society "Natura Artis Magistra" at Amsterdam I have been enabled to examine the gas-exchange of a great number of cold-blooded animals, in order to see whether the phenomenon stated in sea-animals occurs also in amphibia and reptilia.

The older investigations of Reignault and Reiset [2]), Moleschott,

[1]) These Proc. XII p. 48.

[2]) Reignault and Reiset, Annales de Chimie et de Phys 1849. Vol. 26. p. 299.

Pott and others [1]) concerning the exchange of gases in the lower vertebrata include only few kinds of animals and had not been made in connection with my question. A series of experiments made with animals of different weight, is given by Krehl and Soetbeer [2]), who examined calorimetrically the warmth given off by some animals.

They found at a temperature of 25,3° C.

Lacerta	weight	110	gr.	produces	per	hour	and	K.G.	0.8	calories.
Rana mugiens	,,	600	,,	,,	,,	,,	,,	,,	0.5	,,
Alligator	,,	1380	,,	,,	,,	,,	,,	,,	0.3	,,
Uromastix	,,	1250	,,	,,	,,	,,	,,	,,	0.26	,,

They thought they had to explain these contradictory values and especially the change of the numbers with the temperature from the differences in the kinds of the sample-animals and thought they ought to attribute the higher production of warmth in the first two animals not to their smaller dimensions but to their manner of living in the temperate zone, while they found that the protoplasma of the tropical animals works more economically, especially at a higher temperature.

With the smaller animals examined by me I determined the change in the composition of the air of a space in which the sample-animal had found itself for a certain time. This time was chosen so as to make the air at last contain 3—4 $^0/_0$ CO_2. In this case the CO_2 quantity did not disturb the intensity of the respiration of the animal and the error in the experiment, arising from the analysis of the air, had been diminished [3]). The analysis of the air was made according to Hempel [4]). The CO_2 absorption took place with concentrated $NaOH_2$, the O_2 binding in a gas-pipette filled with phosphor (the temperature of the room was always above 15° C.). Bigger animals I put in a space through which the current of air was led. The air that entered was made free from CO_2, of the air that was going out the CO_2 was kept back in bottles by means of baryte-water, and the quantity was determined by titrating the remaining $Ba(OH)_2$.

In the different series of experiments I have brought together animals agreeing as much as possible in their bodily shape, and most of which belong to very closely related species. By this it seems to me that a comparison of the results is possible and the differences that show themselves are to be attributed to the difference

[1]) See Zuntz in Hermann's Handbuch der Phys.

[2]) Krehl and Soetbeer, Pflügers Archiv. Bd. 77 p. 611—638.

[3]) For warmblooded animals, see Frédéricq in Diction. de Physiologie Vol. II. p. 449.

[4]) Hempel, Gasanalytische Methoden.

in size. That the origin of the animals from different hot climates is not of decisive influence, as KREHL and SOETBEER think, appears from the regular course of the numbers in the tables. Further, in connection with this, I draw attention to the fact that the Platyd. maur. (experiment II, Table I) of North Africa produces much more CO_2 p. hour and K.G. than the alligator species and the crocodile, which also originate from the torrid zone.

TABLE I

Temp. 18—19° C.	O_2 p. hour and K.G.	CO_2 p.hour and K.G.	Bodily weight
I Lacerta viridis var. maculosa (very movable)	—	551 cc	7.25 gr.
II Platydactylus mauritanica (in rest)	270 cc	171	8.055
III Molga torosa (in rest)	156	78	17.5
IV Salamandra maculosa (in rest)	128.7	99	28.32
V Alligator sclerops (in rest)	—	43	530.—
VI Alligator lucius (very movable)	—	90	797.—
VII Krokodillus porosus (moderately movable)	—	39	1467.—

It is seen that at an increase of the bodily weight the exchange of gas calculated per K.G. and hour, decreases. All the animals were full-grown and had been imprisoned for a long time at a temperature of ± 20° C., whilst the nutritive power was as good as possible. The diverging figures, which are found concerning the exchange of gas in frogs at an equal temperature and season, are for a great part to be attributed to the different nutritive power. Hence the importance of the following table, where all the animals had been fed as well as possible.

TABLE II

Temp. 18—19° C.	O_2 p. hour and K.G.	CO_2 p. hour and K.G.	Bodily Weight
I Pachytis Bombinator	325 cc	240 cc	5.88 gr.
II Hyla arborea	314	240	7.78
III Rana esculenta	210.9	152.7	30.—
IV " "	80.7	58.3	116.—
V " aspersa	—	45	563.—
VI Bubo marinus	—	22.4	1200.—

During the experiment all the animals moved as little as possible, so that the results are to be compared directly and once more show the same result as the experiments in Table I. In these animals I could not trace the influence of motion on the exchange of gases. Uneasiness, however, betrays itself in the toads by the so-called blowing up (air-swallowing). A Bubo marinus in this state produced 60.7 ccCO_2 instead of 22.1 cc. CO_2 p. hour and K.G.

TABLE IIIa

	CO_2 p. K.G. and hour	Bodily Weight	Temp.
I Angius fragilis (moving)	473 cc	22.21	21° C.
II Amphysbaena alba (very movable)	168	373.4	20°

TABLE IIIb

III Coronella austriaca	292 cc	24.8	19° C.
IV Tropidonotis natrix	432	46.2	20°
V „ „	324	74.3	19°
VI Boa constrictor	52.5	810.—	19°
VII „ „	32	2830.—	19°

All the snakes and lizards had had no food that day, the ring-snakes (IV and V) had not eaten anything for some months past, the two Boas had had no food for a fortnight.

The influence of alimentation on the metabolism in the Boa constrictor is perceptible, but not so great as would be expected, which appears from the following table:

TABLE IV

		CO_2 p. hour and K.G.	Temp.
Boa constrictor 2830 Gr.			
a without food (see Table IIIb pr VII)		32	19° C.
b devoured a pigeon	after 4 hours	52.9	20.5°
c	52	49.4	20°
d	124	55.5	—
Boa constrictor 3123 Gr.			
a devoured 2 rats	after 2 hours	41.5	19°
b	24	42.5	19°
c	48	52.9	19°

Thus far nearly all the experiments had been made upon full-grown animals; in the following experiments on the exchange of gases in tortoises a difference in bodily weight is accompanied by a difference in age.

TABLE Va

	CO_2 p.hour and K.G.	Bodily weight	Temp.
I Chelodina longicollis	53.3 cc	505 Gr.	19° C
II „ „	39	805	20°

TABLE Vb

III Chrysemus chineria (var. belli)	48.4	273	19°
IV „ „ „ (very movable)	50	1570	19°

TABLE Vc

V Testudo graeca	115	636	19°
VI „ tabulata	62	1099	20°
VII „ „	68	1650	23°

TABLE Vd

VIII Emys orbicularis	220	71	19°
IX „ „	74	135	19°
X „ „	33	589	19°
XI „ „	18	1190	19°

Besides the phenomenon already mentioned we may, from this table, also observe the influence of the growth upon the metabolism, the exchange of gases in the growing Emys orbicularis increasing more strongly with the fall of bodily weight than in full-grown animals.

In the Emys orbicularis an increase of bodily weight in proportion of	1 : 17
gives a decrease in the production of CO_2 p. KG. and hour of	12 : 1
In the snakes with an increase of	1 : 17.5
is found a decrease of CO_2 production p. KG. hour	8.3 : 1
in the amphibia of Table II an increase of	1 : 15
shows a decrease of CO_2 production per KG. hour	4 : 1
In the amphibia of Table I these figures are: increase	1 : 18.7
decrease of CO_2production	2.3 : 1

It follows, therefore, from the experiments and investigations described here, which, moreover, have formerly been made upon warm-blooded animals by other investigators, that of similar or closely related animals the smaller ones have a relatively much stronger metabolism than the larger animals.

It seems to me that the explanation of this general phenomenon is not to be found in the influence of the size of the body on metabolism, as neither in the experiments mentioned here, nor in those formerly made upon sea-animals, anything of a proportion between exchange of gases and the size of the body (equalled to $K\sqrt[3]{\text{weight}^2}$) has appeared to me. Therefore it also seems to me that the theory, projected by VON HOESSLIN [1]), should be put to the test by means of experiments. His supposition that metabolism of cold-blooded animals, just as that of warm-blooded, is proportional to their size, does not seem to be corroborated by experiments.

Further the writer thinks, that the alimentary matter with which the blood supplies the textures in the different animals, is proportional to the size of the animals; at which of the many moments at least a few have been left out of consideration.

Besides, in the animal organism the degree of the supply of food for the textures is determined by the degree of metabolism, and not the reverse: metabolism by the circulating quantity of blood. Moreover the influence of the movements of smaller and larger animals on their metabolism cannot be calculated without any investigation.

It may, however, be concluded that the law of dependence of consumption of oxygen on the size of warm-blooded animals, as has so irrefutably been proved by RUBNER [2]) for dogs, finds its origin in the general vital phenomenon mentioned here. According to this principle the regulation of the temperature in the smaller warm-blooded animals does not take place by a relatively decreased giving off of warmth (thicker skin- or skin-covering, narrower vessels, etc.), but by a relatively stronger production of warmth.

Utrecht, 25 April.

[1]) v. HOESSLIN, Archiv. f. (Anat. u.) Physiol 1888 p. 323—380.

[2]) RUBNER Zeitschr. f. Biolog. Bd. 19.

Physics. — *"Isotherms of monatomic gases and of their binary mixtures. IV. Remarks on the preparation of argon. V. Vapour pressures above* — 140° *C., critical temperature and critical pressure of argon."* By C. A. CROMMELIN. Comm. N°. 115 from the physical laboratory at Leiden. (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the Meeting of April 29, 1910).

IV. Remarks on the preparation of argon.

§ 1. *Purely chemical methods.* The most important problem encountered in the preparation of argon is the removal of nitrogen from a mixture of argon and nitrogen. The chief substances that have up to the present come into prominence as absorbents of nitrogen are calcium, a mixture of calcium carbide and calcium chloride and a mixture of quick lime, magnesium and sodium.

a. Calcium. According to MOISSAN and RIGAUT [1]) calcium at a dark red heat readily unites with nitrogen to form calcium nitride (Ca_3N_2), while according to SODDY [2]) calcium unites with nitrogen (and also with all other gases except the monatomic) only at temperatures far above those that can be reached with the ordinary chemical combustion furnace.

According to my own experience calcium at a light red heat in a chemical combustion furnace unites but sluggishly with nitrogen. For this experiment I used an apparatus built upon the lines of that used by RAYLEIGH and RAMSAY [3]) in which the gas was passed in a cycle through the furnace by means of a continuous action mercury circulating pump. That nitrogen really united with calcium was shewn not only by the diminution of the pressure in the apparatus, but also by the calcium nitride that came from the tube after the experiment, a yellow substance (not brown, as is stated in some chemical text-books) that in moisty air smelt strongly of ammonia. This method did not lead to satisfactory results.

b. Mixture of 90°/₀ *calcium carbide and* 10°/₀ *calcium chloride.* FISCHER and RINGE [4]) have interested themselves in this method. It has the great advantage that oxygen and nitrogen are absorbed at the same time. In collaboration with Mr. H. FILIPPO, Chem. Docts.,

1) H. MOISSAN and A. RIGAUT. Ann. de Chim. et de ph. (8). 2. 433. 1904.
2) F. SODDY, Proc. R. S. Nov. 15th 1906, excerpt in Nature Jan. 24th 1907.
3) LORD RAYLEIGH and W. RAMSAY, Phil. Trans. R. S. (A) 186. 187. 1895.
4) F. FISCHER and O. RINGE, Chem. Ber. 41. 2017. 1908.

to whom I take this opportunity of expressing my warmest thanks for his invaluable assistance, I constructed an apparatus which enabled us to prepare a large quantity of crude argon in a short time. A description of the peculiarities of this apparatus may well be omitted; it was arranged so that larger quantities of air could be treated than was possible with that of FISCHER and RINGE. With 1 K.G. of the mixture we treated about 216 L. of air, while FISCHER and RINGE only managed about 180 L. I can well recommend this method for the preparation of large quantities of crude argon.

c. A mixture of 20 parts by weight of quick-line, 4 of magnesium and 1 of sodium. This mixture recommended by HEMPEL[1]) as a modification of the MAQUENNE[2]) mixture I shall call the HEMPEL mixture; if prepared with great care it will be found to be of great service in the separation of nitrogen from crude argon. I used it for the final purification of the argon that was to be used for quantitative measurements.

The analysis of the gas obtained[3]) shewed the excellence of this mixture as an absorbent.

As far as I am aware, there are no determinations of the *dissociation-pressure* of magnesium- and calcium nitride at the temperature at which we worked. By heating the residue left from an experiment with HEMPEL mixture in a porcelain tube with ground joints, and attaching to one end a mercury-manometer for low pressures and to the other a TÖPPLER pump I was unable to find any evidence of a dissociation-pressure with a cathetometer on which $^1/_{20}$ mm. could be read with certainty. The dissociation-pressure is, therefore, certainly smaller than $^1/_{20}$ mm. Hence, under the most favourable circumstances there must not remain more than $^1/_{15000}$ nitrogen in the mixture.

§ 2. *Electrochemical methods.* The union of nitrogen with oxygen under the influence of electric discharge which was employed by RAYLEIGH and RAMSAY[4]), RAYLEIGH[5]), BECKER[6]), and others in the preparation of argon is especially useful for removing the last traces of nitrogen. The great difficulty inherent in this method — its slowness — is partially overcome by the use of a transformer for high tension alternating currents instead of an induction coil. Fol-

1) W. HEMPEL. Gasanalytische Methoden 3th Ed. p. 150.

2) L. MAQUENNE. C. R. 121. 1147. 1895.

3) See V, § 4. of this communication.

4) LORD RAYLEIGH and W. RAMSAY, Phil. Trans R. S. (A.) 186, 187, 1895.

5) LORD RAYLEIGH, Journ. Chem. Soc. (Trans) 71, 181, 1897.

6) A. BECKER, Ztschr. f. Electrochemie 9, 600. 1903.

lowing the principles of that used by Rayleigh and Ramsay I constructed an apparatus with transformer with great care for this purpose, and used it several times for a last purification. A flame arc between two fixed platinum electrodes was used as it was found in practice to be more suited to this purpose than the long narrow sparks [1]) which one would from a theoretical point of view probably choose [2]) in preference to a flame arc. To get an idea of the efficiency of the apparatus I made several measurements of the ratio between the watts consumed and the number of volt-amperes in the low-tension circuit. This ratio is practically the same in both circuits of a well constructed transformer.

I found:

25 volts,
44 amps,
480 watts,

when the flame burned quietly. The ratio is therefore

$$\frac{480}{25 \times 44} = 0.44.$$

Seeing that I had no oscillograph available I had to leave unanswered the question if this unfavourable ratio was due to a large phase difference between current strength and potential (high inductive resistance), or to other causes (deformation of the current and potential curves as indicated by Scheuer [3]) electromotive forces at the electrodes, etc.). Leaving this factor out of account, the efficiency of my apparatus gave an absorption of 11.8 L. per K. V. A. hour from a mixture of oxygen and nitrogen in the correct proportion to form nitric acid. Rayleigh [4]) absorbed 17.5 L. per K. V. A. hour.

§ 3. *Separation of argon and neon by fractional distillation.* The quantities of krypton, xenon and helium still present in the argon prepared [5]) according to one of the above methods could not influence the measurements (isotherms, vapour pressures) that were to be made with it. It was desirable, however, to get rid of the 0.1 or 0.2 °/₀ neon. This was done by fractionation at the temperature of liquid oxygen

[1]) C. S. Bradley and R. Lovejoy, Ztschr. f. Electrochemie, 9, 868. 1903.

[2]) F. Haber, Thermodynamik technischer Gasreaktionen (München and Berlin Oldenbourg, 1905).

Ph. A. Guye, Conf. Soc. Chim. de France May 24th 1909.

[3]) O. Scheuer, Zeitschr. f. Electroch. 11. 565. 1905. Thesis for the doctorate, Genève. 1905.

[4]) Rayleigh, Journ. Chem. Soc. (Trans) 71.181, 1897.

[5]) M. W. Travers, Experimental studies of gases.

boiling under greatly reduced pressure (— 217° C.). This temperature is above the critical temperature of neon while extrapolation of the vapour pressures of solid argon measured by RAMSAY and TRAVERS [1]) by means of an equation of the form

$$log\, p = A + \frac{B}{T}$$

gave me an estimate of 3 mm. as the value of the vapour pressure of solid argon at the temperature of — 217° C. This result which is in agreement with what the analogy of oxygen would lead us to expect was confirmed by the experiments themselves.

V. *Vapour pressures above — 140° C., critical temperature and critical pressure of argon.*

§ 1. The *object* with which these measurements were undertaken was two-fold: 1. I wished to control the vapour pressure determinations of RAMSAY and TRAVERS and also the critical quantities found by RAMSAY and TRAVERS [1]) and by OLSZEWSKI [2]); and 2. I wished, before proceeding to determine isotherms, to get an idea of the purity of the argon used by an application of KEESOM's [3]) theory of the condensation of substances with small quantities of admixture.

§ 2. The *apparatus* with which the measurements were made was designed and used by KAMERLINGH ONNES and HYNDMAN for some unpublished researches on oxygen, and it has already been noticed in a few words [4]).

To a great extent the apparatus corresponds with the piezometers of variable volume for low temperatures already described by KAMERLINGH ONNES and HYNDMAN [5]), with the exception that an alteration has been made in the small glass reservoir that was used for low temperatures and in the steel capillaries attached to it. This reservoir is 8 cm. long, has an internal diameter of 3.4 mm. and, particularly with a view to the observation of the critical phenomena, contains a glass stirrer. This stirrer can be moved up and down by means of a glass thread passing through the glass capillary and attached to a similar stirrer enclosing a piece of soft iron that can be moved by means of a small electromagnet up and down in a

1) W. RAMSAY and M. W. TRAVERS, Phil. Trans. R. S. (A) 197. 47, 1901.

2) K. OLSZEWSKI, Phil. Trans. R. S. (A), 186. 253. 1895.

3) Proc. R. A. Amsterdam, April 1902, Comm. Phys. Lab. Leiden. No. 79.

4) Proc. April 1902 Comm. No. 78 and Proc. March 1903 Comm. No. 83.

5) Proc. April 1901 Comm. No. 69.

small reservoir of the same dimensions above the capillary (where it projects above the cryostat). The lower reservoir is divided so that the position of the liquid meniscus may be read off and the motion of the meniscus may be followed.

As this piezometer was not intended for volume measurements and one is therefore not confined to one definite known quantity of gas (the normal volume) the steel capillary is provided with a steel T-piece to which a second capillary is soldered; in the middle of this capillary there is a steel high pressure valve; it connects the piezometer with a glass reservoir of argon, the pressure in which can be raised to ± 1.5 atm. by mercury. In this way the range of densities that can be investigated with a single piezometer by the addition of gas is greatly extended.

At the time the dimensions of the piezometer were calculated for oxygen. Seeing that this substance has about the same critical constants as oxygen the piezometer can be used for argon without alteration.

The cryostat used was that constructed by KAMERLINGH ONNES in 1902 for apparatus of small dimensions and has already been described [1])

The temperatures desired were reached by means of liquid ethylene boiling under reduced pressure, and the constancy of the temperature was regulated by a platinum resistance thermometer [2]).

The pressure was measured by a closed hydrogen manometer [3]) that had been calibrated with the KAMERLINGH ONNES [4]) open manometer. The accuracy of this pressure determination reached 1 in 3000.

The temperatures were measured by a platinum resistance thermometer constructed wholly of glass and platinum according to the directions of KAMERLINGH ONNES and CLAY [5]). This thermometer was compared with a standard resistance thermometer at a large number of temperatures between —103° C. and —140° C. The standard thermometer was very carefully compared with the hydrogen thermometer at various temperatures including —102° C. and —138° C. and a formula giving good agreement was calculated. Errors of a few hundredths of a degree may indeed occur in temperatures in the neighbourhood of —120° C. and —130° C. where there may be

1) Proc. Febr. and March 1903, Comm. No. 83, where a drawing of the cryostate is to be found on plate IV.

2) Proc. Febr. 1903, Comm. No. 83, § 5.

3) Proc. April 1902, Comm. N⁰. 78. § 17 and Proc. March 1907, Comm. N⁰. 97*a*. § 3.

4) Proc. Nov. 1898, Comm. N⁰. 44.

5) Proc. Sept. 1907, Comm. N⁰. 99*b*.

some doubt as to the magnitude of the deviations from the formula; but the temperatures are undoubtedly accurate to $^1/_{20}{}^\circ$. The temperatures calculated in this way on the scale of the hydrogen thermometer were reduced to the absolute scale by means of the corrections, given by KAMERLINGH ONNES and BRAAK [1]). The thermometer was calibrated and the temperatures were measured and calculated by Miss G. L. LORENTZ, phil. doct^a., to whom I take this opportunity of expressing my heartiest thanks for the valuable assistance she kindly gave me in my measurements. I am also indebted to Mr. W. J. DE HAAS phil. docts. for his help in my determinations.

§ 3. The *measurements* were made in the following fashion. After the formation of a small quantity of liquid the apparatus was left alone for some time to ensure temperature and pressure equilibrium, and then the position of the meniscus in the reservoir, the manometer, and the position of the meniscus in the glass stem of the piezometer were read off. The pressure was then raised sufficiently until the reservoir was almost filled with liquid when the measurements were repeated.

As is usual the critical temperature was approached gradually and then by steps of 0.1°. The fact that a temperature was below the critical was shewn by the occurrence of condensation with extremely slow compression (so as to make the adiabatic warming as small as possible and thus remain on the experimental isotherm). Being above the critical temperature was shewn by first raising the pressure above the critical and then lowering it by small expansions, of which the adiabatic cooling could be only a few hundredths of a degree; if while doing this no meniscus appeared then it was certain that the temperature was above the critical.

The pressure at which the meniscus disappeared at about the middle of the reservoir was taken to be the critical pressure.

For various reasons I gave up the idea of a searching investigation of the critical phenomena, and of a very exact determination of the critical constants. In the first place I suspected that my argon was not pure enough for that purpose, and this was confirmed later on by my measurements. In the second place since the description of the apparatus of KAMERLINGH ONNES and FABIUS [2]) the piezometer I used may be considered too antiquated for the purpose. The cryostat, too, left something to be desired in the constancy of the temperature.

1) Proc. Dec. 1907, Comm. N⁰. 101*b*.

2) Proc. Sept. 1907, Comm. N⁰. 98. More elaborate in G. H. FABIUS, Thesis for the doctorate, Leiden 1908 where drawings of the apparatus are to be found.

§ 4. *Calculations.* The pressures and temperatures were first calculated, and then by means of KEESOM's formula[1]) for the change of condensation pressure of a substance with small quantities of admixture an idea of the purity of the argon was obtained. I assumed, that the admixtures, which could produce a perceptible increase of pressure, must be nitrogen, for by far the greater part. The experience collected in the preparation justify this assumption.

KEESOM's formula gives

$$(v_2-v_1)\,(p_1-p_2) = M\,RT\,x\,(e^k + e^{-k} - 2) \quad . \quad . \quad . \quad (1).$$

in which

$$k = \left\{\alpha\frac{T}{p}\,\frac{dp}{dT} - \beta\right\}\frac{p\,(v_2-v_1)}{MRT}.$$

p is the vapour pressure of the pure substance at an absolute temperature T; p_1 and p_2 are the pressures at the beginning and the end of condensation respectively, v_2 and v_1 are the molecular volumes of the vapour and liquid respectively; x is the required molecular proportion of admixture, and finally α and β are the two constants introduced by KAMERLINGH ONNES[2]), viz:

$$\alpha = \frac{1}{T_k}\left(\frac{dT_{xk}}{dx}\right)_{x=0}$$

and

$$\beta = \frac{1}{p_k}\left(\frac{dp_{xk}}{dx}\right)_{x=0}$$

This formula, however, is valid for the beginning and the end of condensation, that is, for two corresponding points on the boundary curve (pv-diagram), while for experimental reasons I had arranged that my observations should be made just after the beginning and just before the end of condensation, that is, on two points on the experimental isotherm within the boundary curve.

To obtain a formula applicable to my measurements following a friendly hint given me by Dr. KEESOM, I derived from the two equations

$$\left.\begin{aligned}(v_2-v_1)\,(p_1-p) &= M\,RT\,.\,x\,.\,(e^{-k}-1)\\(v_2-v_1)\,(p_2-p) &= M\,RT\,.\,x\,.\,(1-e^{k})\end{aligned}\right\}$$ [3])

the difference between which gives equation (1), two other equations for points on the experimental isotherms in the neighbourhood of the boundary curve. These were

1) Proc. April 1902, Comm. N°. 79.

2) Proc. Dec. 1901, Comm. No. 75.

3) Proc. April 1902, Comm. N°. 79.

$$\left.\begin{aligned}(v_2-v_1)(p_b-p) &= \frac{MRT\,x\,(e^{-k}-1)}{1+y_b-y_b\,e^{-k}} \\ (v_2-v_1)(p_a-p) &= \frac{MRT\,x\,(1-e^{k})}{1-y_a+y_a\,e^{k}}\end{aligned}\right| \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which

$$y_a = \frac{v_{a_2}-v_a}{v_{a_2}-v_{a_1}}$$

$$y_b = \frac{v_{b_2}-v_b}{v_{b_2}-v_{b_1}}.$$

In the new notation introduced a refers to a state shortly after the commencement and b to one shortly before the end of condensation; 1 refers to the liquid, and 2 to the vapour, all expressed in molecular volumes. From equations (2) we get for the relation between x and the difference between the initial and final pressure in my experiments:

$$p_b-p_a = \frac{MRT\,x}{v_2-v_1}\left[\frac{e^{-k}-1}{1+y_b-y_b\,e^{-k}} - \frac{1-e^{k}}{1-y_a+y_a\,e^{k}}\right] \quad . \quad . \quad (3).$$

This equation can be directly applied to my measurements for the calculation of x. One difficulty, however, arises from the occurrence of various quantities which for the present cannot be calculated with great accuracy, so that the values of x given below must be regarded as rather rough approximations, which, however, undoubtedly give the order of magnitude of x.

The greatest uncertainty is in the estimate of the values of the constants α and β for the proper calculation of which measurements with mixtures of argon and nitrogen are necessary. Seeing that as yet such measurements are wanting I followed KEESOM's [1]) method and by making various simplifying assumptions came to the following estimates:

$$\alpha = -0.216 \qquad \beta = -0.484\ ^{2})$$

The values of $\dfrac{T\,dp}{p\,dT}$ and of the vapour pressure of the pure substance were taken from the determinations made by RAMSAY and TRAVERS [3]), v_2 and v_1 were calculated from KEESOM's formula [4]) for

[1]) Proc. April 1902, Comm. N°. 79.

[2]) For further values of α and β see J. E. VERSCHAFFELT, Proc. Dec. 1904, Suppl. N°. 10.

[3]) W. RAMSAY and M. W. TRAVERS, Phil. Trans. R. S. (A) 197, 47. 1901.

[4]) Proc. April 1902, Comm. N°. 79, p. 11.

coexisting liquid and vapour densities. To apply this formula I estimated the critical density from the liquid densities of argon given by BALY and DONNAN [1]) using the law of the diameter of CAILLETET and MATHIAS [2]) and the "loi du tiers" of MATHIAS [3]); the results given by these two methods were in satisfactory agreement.

In this way I calculated the following values of x from three observations of the vapour pressure:

x	$Vol\ ^0/_0$
0.00066	0.066
58	58
72	72

Considering how approximate values must be taken for various quantities occurring in equation (3) we may be well satisfied with this correspondence, and we may consider it as highly probable that the admixture is less than $0.1\ ^0/_0$.

Making use of equations (2) and of the mean of the values given above for x I was able to reduce the observed vapour pressures to those of the pure substance.

These corrections are extremely small, and are only very little outside the limits of experimental accuracy. The critical quantities observed (true plait-point quantities) were corrected by means of the following two equations deduced by KEESOM [4]):

$$\left.\begin{aligned} \frac{1}{T_k}\frac{dT_{pl}}{dx} &= \alpha - \frac{\left(\beta - \alpha\dfrac{\partial\pi}{\partial\tau}\right)^2}{C_4\dfrac{\partial^2\pi}{\partial\omega\partial\tau}} \\ \text{and} \qquad \frac{1}{p_k}\frac{dp_{pl}}{dx} &= \beta - \frac{\partial\pi}{\partial\tau}\frac{\left(\beta - \alpha\dfrac{\partial\pi}{\partial\tau}\right)^2}{C_4\dfrac{\partial^2\pi}{\partial\omega\partial\tau}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad , \qquad (4)$$

in which $C_4 = \dfrac{MRT_k}{p_k v_k}$ and the Greek letters as usual refer to reduced quantities.

Assuming now with VAN DER WAALS [1]) that for all substances

$$\frac{\partial\pi}{\partial\tau} = 6.7$$

1) E. C. C. BALY and F. G. DONNAN, Journ. Chem. Soc. 81, 911. 1902.
2) L. CAILLETET and E. MATHIAS, Journ. d. Phys. (2). 5, 549. 1886.
3) E. MATHIAS, Ann. d. l. Fac. d. Sc. Toulouse (6). 1892.
4) Proc. Dec. 1901, Comm. N⁰. 75, p. 6.
1) J. D. VAN DER WAALS, Proc. Apr. 1901.

and with KEESOM [1]) that

$$C_4 \frac{\partial^2 \pi}{\partial \omega \partial \tau} = -32.2,$$

and also that we may write

$$\frac{dT_{pl}}{dx} = \frac{T_{pl} - T_k}{x}$$

and

$$\frac{dp_{pl}}{dx} = \frac{p_{pl} - p_k}{x}$$

and, finally, writing for T_k and p_k the observed quantities (hence really T_{pl} and p_{pl}), which assumption will only very slightly alter the very small corrections $T_k - T_{pl}$ and $p_k - p_{pl}$, equations (4) take the form:

$$T_k - T_{pl} = -T_{pl}\, x \left[\alpha - \frac{\left(\beta - \alpha \frac{\partial \pi}{\partial \tau}\right)^2}{C_4 \frac{\partial^2 \pi}{\partial \omega\, \partial \tau}} \right]$$

and

$$p_k - p_{pl} = -p_{pl}\, x \left[\beta - \frac{\left(\beta - \alpha \frac{\partial \pi}{\partial \tau}\right)^2}{C_4 \frac{\partial^2 \pi}{\partial \omega \partial \tau}} \right]$$

From these equations I calculated the (very small) corrections for the critical quantities ($+ 0^\circ.02$ C. and 0.009 atm.).

§ 5. *Results.* After reducing the observed results to those for the pure substance by the method shown in § 4 I obtained the following results.

Date	Series	t	p (atm.)
1910 10 Febr.	VI	$-140^\circ.80$	22.185
10 „	V	-134.72	29.264
9 „	III and IV	-129.83	35.846
9 „	II	-125.49	42.457
12 „	VIII	-122.70	47.503
14 „	IX	-122.49	47.890

Critical quantities

14 Febr.	X	**$-122^\circ.44$**	**47.996**

[1]) Proc. Dec. 1901, Comm. No. 75.

These observations are shown graphically on the accompanying figure on which the pressures and critical point determined by RAMSAY and TRAVERS [1]) and the OLSZEWSKI [2]) critical point are shown for comparison.

The RAMSAY—TRAVERS critical point especially (— 117°.4 C. and 52.9 atm.) deviates largely from mine, and this may perhaps be explained by the primitive method in which these experimenters had arranged their temperature bath.

§ 6. *Representation of the vapour pressure curve by empirical formulae.* I attempted to represent the results given in the previous section by RANKINE's [3]) formulae.

A two constant formula, viz.:

$$\log p = A + \frac{B}{T},$$

identical with the well-known VAN DER WAALS [4]) vapour pressure formula, in which but one of the constants is determined by the vapour pressures in order that the formula may give the observed critical pressure for the observed critical temperature did not give very satisfactory results; this may be seen from the following table in which $p(O)$ represents the observed pressures and $p(C)$ the calculated:

Series	$p(O)$	$p(C)$	$p(O)-p(C)$
VI	22.185	21.642	+ 0.543
V	29.264	28.841	+ 0.423
III and IV	35.846	35.697	+ 0.149
II	42.457	42.628	— 0.171
VIII	47.503	47.521	— 0.018
IX	47.890	47.903	— 0.013

A much better correspondence with the observations was given

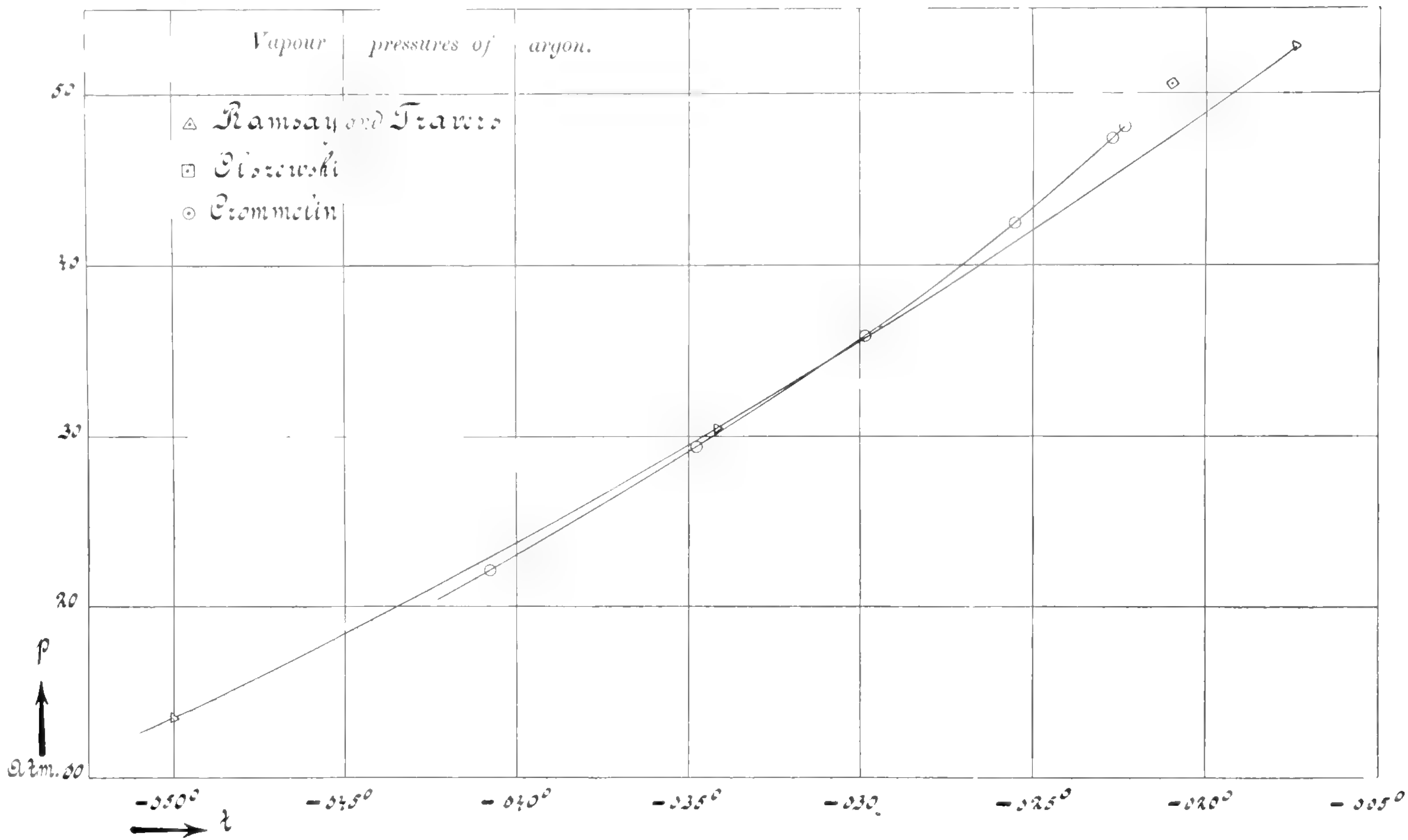
1) W. RAMSAY and M. W. TRAVERS, Phil. Trans. R. S. (A) 197, 147. 1901.

2) K. OLSZEWSKI, Phil. Trans. R. S. (A) 186. 253, 1895.

3) W. J. M. RANKINE, Edinb. New. Phil. Journ. 1849 and Phil. Mag. 1854, Misc Scient. Papers p. 1 and 410.

4) J. D. VAN DER WAALS Cont. I (2d edition) p. 158.

C. A. CROMMELIN. Isotherms of monatomic gases and of their binary mixtures. IV. Remarks on the preparation of argon. V. Vapour pressures above —140° C., critical temperature and critical pressure of argon."



by a three-constant formula:

$$log\, p = A + \frac{B}{T} + \frac{C}{T^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is identical with the vapour pressure formula given by KEESOM [1]) and has two constants to be determined by the vapour pressures. Calculating the constants by the method of the least squares I found

$$\left. \begin{aligned} A &= + 4.661764 \\ B &= - 524.3169 \\ C &= + 11343.28 \end{aligned} \right\} {}^{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which give the following correspondence with observation:

Series	$p\,(O)$	$p\,(C)$	$p(O)-p(C)$
VI	22.185	22.210	— 0.025
V	29.264	29.172	+ 0.092
III and IV	35.846	35.856	— 0.010
II	42.457	42.674	— 0.217
VIII	47.503	47.526	— 0.023
IX	47.890	47.905	— 0.015

Owing to the paucity of experimental data I did not calculate the constants for a four-constant formula (three dependent upon the vapour pressures).

Equation (5) with the constants (6) gives

$$\frac{T_k}{p_k}\left(\frac{dp}{dT}\right)_k = 5.712 \; {}^{3})$$

[1]) Proc. Jan. 1904, Comm. No. 88, p. 53.

[2]) These and the following figures differ somehow from those of the original Dutch publication. The differences are due to an improved calculation.

[3]) See H. HAPPEL, Ann. d. Phys. (4) 21, 342, 1906. and Phys. Ztschr. 10. 1026. 1909.

G. VAN RIJ, Thesis for the doctorate, Amsterdam, 1908 p. 13 sqq.

Physiology. — "*The influence of small amounts of Calcium on the motion of Phagocytes.*" By Prof. H. J. HAMBURGER.

Former investigations have shown that small amounts of calcium are able to promote phagocytosis considerably [1]). An addition for instance of 0.005 °/₀ $CaCl_2$ to the serum caused an increase of about 22 °/₀ in the phagocytarian power. This favourable effect of chloride of calcium becomes even more strongly manifest when, instead of being added to the serum, it is added to NaCl-solutions.

These investigations have been continued now in two directions.

In the first place we have asked ourselves whether the influence of Ca would also manifest itself *in the living body*. All experiments had hitherto been made outside the body. If — we argued — the phagocytarian power, heightened by Ca-Ions, is based upon an acceleration of the amoeboid motion then it may be expected that by Ca chemotaxis will be promoted likewise. And therefore we determined *the chemotaxis with and without the addition of chloride of calcium* in the manner described below.

At the same time this investigation would furnish an answer to a question raised in another quarter. In the Zeitschrift für Balneologie of August 15, 1909 we read that the Prussian Ministry of Public Worship, Education and Medical Affairs has addressed the following question to the Kaiserliche Gesundheitsamt: "Ist ein Mineralwasser, das eine isotonische Kochsalzlösung darstellt, durch einen Gehalt von 0.1 °/₀ Chlorcalcium gemäss den Untersuchungen des Prof. HAMBURGER in Groningen geeignet, dem Körper Stoffe zuzuführen, die in dem Serum die Aufgabe haben, den Verdauungsprocess der Bakterien vorzubereiten, die Phagocytose erheblich zu steigern? Sind einschlägige Untersuchungen in staatlichen Instituten mit einem Mineralwasser, das jene chemischen Vorbedingungen erfüllt, zu empfehlen?"

Prof. Dr. H. KIONKA (Jena) sent in a report on the matter, setting forth the great desirability of these investigations.

With a view to the interest from balneological quarters I have therefore at the same time made some experiments with a mineral-water containing much Ca; I took for it the water of the Virchow-Quelle at Kiedrich near Eltville (Wiesbaden).

Still in another direction we have carried on our investigations. We have namely made attempts *to penetrate further into the nature of the remarkable influence exercised by calcium.*

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1) HAMBURGER and HEKMA. These Proceedings Vol. X p. 144.
Biochemische Zeitschrift 9, 275, 1908.

A. Influence of calcium on the chemotaxis.

To test the influence of calcium upon the chemotaxis two methods were applied.

The first method consisted in placing under the skin of a rabbit small capillary tubes, closed at one end, and filled with a suspension of B. coli commune in NaCl solutions, containing or not containing $CaCl_2$. After some time the lengths of the leucocyte columns which had entered were measured.

The second method consisted in $CaCl_2$ being introduced into the intestinal canal of some rabbits and not in that of others; and investigating after some time to what extent the capillary tubes filled with the coli-suspension had attracted in the first rabbits a longer phagocyte-column than in the second.

Besides cultures as such, we also put into the capillary tubes instead of them, the liquid without bacteria, that is to say the products of the bacteria.

For the technical details we refer to the Biochemische Zeitschrift. Here we may mention that the capillary tubes were fastened into small flat pieces of cork, in which holes had been pricked beforehand, and further that for the experiments rabbits were used viz. the inside of the thigh. It is easy to make a pocket in the skin there, in which the piece of cork with the capillary tubes can find a place. This having been put in, the wound was closed.

We will now mention the result of some of the experiments.

First Method.

Experiments with bacteria suspensions with and without calcium.

For the following experiment three pieces of cork were each provided with two capillary tubes. In the two tubes of the first cork we put a suspension of B. Coli in NaCl 0.9 $^0/_0$. In the two tubes of the second cork the fluid was NaCl 0.9 pCt. + 0.01$^0/_0$ $CaCl_2$ and of the third cork NaCl 0.9 $^0/_0$ + 0.05 $^0/_0$ $CaCl_2$. The first and the third cork were placed under the skin of the right leg, the second (the one with 0.01 $^0/_0$ $CaCl_2$), under that of the left leg. They remained there for 24 hours. Then the lengths of the leucocyte columns which had entered, were measured.

The following table gives the results of one of the experiments.

It is seen that by the addition of $CaCl_2$ to the bacteria suspension, chemotaxis has increased.

TABLE I.

Influence of calcium upon the chemotaxis.

Suspension of B. Coli commune in a solution of NaCl 0.9 % in which has been dissolved:	Lengths of the phagocyte columns in two capillary tubes:
0 % $CaCl_2$	$\frac{1}{2}+2=3\frac{1}{2}$ m.M.
0.01 „ „	$2+2=4$ „
0.05 „ „	$2\frac{1}{2}+2\frac{1}{2}=5$ „

The same result was obtained when instead of the suspension as a whole we took the suspension freed from bacteria. We subjoin as an example:

TABLE II.

Influence of Calcium upon Chemotaxis.

	Right Leg	Left Leg
	Extract of Colibacteria in NaCl 0.9%	Extract of Colibacteria in NaCl 0.9% + $CaCl_2$ 0.01%
Total of 3 leucocyte-columns after 24 hours	4.25 mM.	4.75 mM.

Of the many other experiments made in the same way as those on which Table I and II are based, a detailed account will be found in the Biochemische Zeitschrift. The results were in all cases the same.

Second method.

Introduction into the intestinal canal of fluids containing calcium.

The difference between this method and the first lay in the fact that we added Ca not to the contents of the capillary tubes, but to the tissue fluid. We accomplished this by rectal injection of fluids containing Ca. As such we used in the first place NaCl sol 0.9 % in which $CaCl_2$ had been dissolved, and secondly a mineral water containing calcium.

a. NaCl-solutions containing Ca.

After the faeces in the rectum had been removed by soft pressure on the belly 60 cc.fluid were brought into the rectum of each of 4 rabbits, 15 cc. four times a day. The first rabbit got 60 cc. NaCl

0.9 %; the second 60 cc. NaCl 0.9 % in which 0.1 Gr. $CaCl_2$ had been dissolved; the third received in the same way 0.2 Gr. $CaCl_2$ and the fourth 0.5 Gr. $CaCl_2$. In each case a cork-slice with three capillary tubes containing a filtrated culture of B. coli commune had been placed under the skin. The result is found in the following table.

TABLE III.

Influence of calcium upon chemotaxis.

	Fluids introduced into the rectum:		Total length of the 3 leucocyte columns:
Rabbit 1	60 cc NaCl sol	of 0.9%	5 mM.
„ 2	60 „ „	„ +1 Gr. $CaCl_2$	9.5 „
„ 3	60 „ „	„ +0.2 „	9 „
„ 4	60 „ „	„ +0.5 „	9.30 „

It follows from these experiments:

1. that all solutions containing $CaCl_2$, have effected a more extensive chemotaxis than the pure NaCl-solution.

2. that an introduction of more than 0.1 Gr. $CaCl_2$ has caused no further increase of chemotaxis.

Further we wished to know whether the phenomenon would repeat itself the next day, if the experiment was continued, in other words if new capillary tubes were put in, and fresh injections were added. The following table may serve as an answer.

TABLE IV.

Influence of Calcium upon Chemotaxis.

	Fluid injected into the rectum:		Total length of the 3 lococyte columns
Rabbit 1	60cc NaCl sol of 0.9%		4.5 mM.
„ 2	60 „ „ „	+0.1 Gr. $CaCl_2$	8.7 „
„ 3	60 „ „ „	+0.2	10.5 „
„ 4	60 „ „ „	+0.5	8 „

It appears that when the experiment is continued a second day Ca has promoted chemotaxis likewise.

We may add that the same results were obtained when *unfiltrated cultures* were used. Comparative experiments, however, with filtrated and unfiltrated cultures upon one and the same animal showed that the leucocyte columns were longer in the capillary tubes with the filtrated culture. It is not difficult to explain this, seeing that the white blood corpuscles which have entered, are partly filled up with a considerable amount of coli-bacteria and are destroyed in consequence.

b. Mineral water rich in calcium (Virchow-Quelle).

The experiments were identical with those made on the influence of NaCl sol. containing $CaCl_2$. First, however, it had to be calculated, how much mineral water had to be injected into the rectum. According to the analysis of H. Fresenius it contains about 0.1°/₀ $CaCl_2$. To administer therefore 0.1 Gr. $CaCl_2$, as in the above experiments, it would be necessary to take 100 cc. of the water per day. In doing this there was no reason to expect that part of the water would be thrown out, but yet the volume could not be called small. Guided by the amount which is given to man, we come to a smaller volume for a rabbit than 100 cc. For in the case of men an average quantity of 1 L. of the water is prescribed per day. Calculating the weight of a man at 65 K.G., that of the rabbits at 3.5 KG., the rabbits would have to receive in proportion $\frac{3.5}{65} \times 100 = 54$ cc. Therefore we have given to the animals 60 cc. per day, distributed over four times, that is to say 15 cc.every time. *So they got 0.06 Gr. $CaCl_2$ per day.*

Let us now communicate some of the results. It need hardly be said that to control the experiments rabbits were also injected with pure NaCl-sol. 0.9°/₀.

Results: The total length of the 6 leucocyte-columns (2 legs) amounts to:

7.25 m.M.	in the NaCl rabbit.
9 m.M.	in the Virchow-Quelle rabbit.

Similar results are given by the following experiment:

Result: The total length of the 6 leucocyte columns (2 legs) amounts to:

8.4 m.M.	in the NaCl rabbit.
12 m.M.	in the Virchow-Quelle rabbit.

These experiments show that an introduction of only 0.06 Gr. $CaCl_2$ promotes chemotaxis considerably. Even without measuring the lengths of the phagocyte columns, one may convince oneself that this conclusion is the correct one. When opening the skin wound, it is immediately seen that *in the Virchow-rabbit a much thicker mass of phagocytes has gathered round the tubes than in the NaCl-rabbit.* The same thing we observed invariably in all experiments where NaCl-solutions containing $CaCl_2$ were injected.

After some reflection one is surprised at the great influence of this exceedingly small amount of Ca. The increase of Ca-percentage in the lymph must be very slight indeed. Let us assume for a moment that the 0.06 Gr. $Ca\,Cl_2$ have been distributed equally over the blood- and tissue-fluids of the animal, then the increase in Ca-percentage can only be very small. A rabbit of 3500 Gr. contains $\frac{3500}{100} \times 8$ Gr. $= 280$ Gr. of blood i. e about $280 \times \frac{2}{3} = 185$ cc. of serum. If further we assume that the animal contains 100 cc. tissue-fluid, then the Ca percentage of the tissue-fluid will have been raised by $\frac{100}{285} \times 0.06\,\% = 0.02\,\%$. As has been said above we take for granted, that the Ca has been distributed entirely and exclusively over the 285 cc. of fluid, in other words that nothing has penetrated into the tissue-cells or into the blood corpuscles [1]) or has left the kidneys. This calculation is very arbitrary, but still it gives some idea of the slight increase of calcium concentration, necessary to raise the chemotaxis from 7.25 to 9 or from 8.4 to 12 i. e by $\frac{12-8.4}{8.4} \times 100 = \pm 40\,\%$.

And this increase represents only a minimum value. For when a column of leucocytes has entered the capillary tube it impedes a further entrance of the movable cells, chiefly by the fact that now the liquid contents of the capillary tubes cannot diffuse freely into the surroundings. This furnishes an explanation why the influence of calcium is not so manifest when the capillary tubes are left under the skin for a longer time, for instance for 48 hours instead of 24. From this point of view it would have been advisable to leave the capillary tubes for a shorter time than 24 hours. Then the difference in percentage between the leucocyte-columns in normal and in calcium animals would undoubtedly have been greater. A technical drawback would have been, however, that the absolute lengths of all columns would have been smaller, and not so easy to measure accurately.

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[1]) Which is indeed the case with the red blood corpuscles. Compare our treatise on the permeability of blood cells to ions of Ca. These Proc. XI p. 718.

It is obvious that this remark also holds good as regards the results with Na Cl-solutions, containing or not containing calcium.

All these experiments show plainly that chemotaxis is considerably promoted even by slight quantities of calcium.

We now come to the second question: how to account for this promotion of chemotaxis and for the increased phagocytosis observed before?

B. Why are phagocytarian power and chemotaxis heightened by calcium?

As regards chemotaxis the answer is pretty obvious. The entering of a large number of phagocytes inte the capillary tubes can scarcely be explained in any other way than by a greater mobility of the cells. Is the increase of phagocytosis to be explained in the same way? Or are we to think of a greater development of force, manifesting itself by the presence of calcium in the phagocytes, and enabling cells which under normal circumstances would be too weak to take up particles, to do so now. There are grounds for thinking of this possibility, if we remember the way in which the phagocytarian capacity was determined by us. To a suspension of leucocytes, carbon-particles were added, and now it was investigated which percentage of leucocytes, both with and without calcium, had absorbed carbon.

It seemed not difficult to establish which of the two factors must be held responsible for the favourable effect of calcium: the acceleration of the amoeboid motion or the increase in force of this motion.

All we had to do to investigate this, was to take two equal suspensions of leucocytes, add calcium to one and not to the other, add carbon particles to both and to examine if the suspension without calcium after sufficient lapse of time would develop as great a phagocytarian capacity as the one with calcium. If this was really the case, then the favourable action of calcium had only to be attributed to an increased velocity of the amoeboid motion.

The following experiment may serve as an answer to the question. The method we adopted is shown in the following table.

To form a correct idea of it, it must be observed that before the addition of carbon the suspension had been heated to 37° and further that when the time of action was finished, the leucocyte-carbon suspensions were immediately placed in cold water to cut short the phagocytarian process as soon as possible.

TABLE V.

Influence of the Time on the extent of Phagocytosis [1]).

Time during which the phagocytes could take up carbon:	Percentage of Leucocytes having absorbed carbon:		Increase of phagocytosis by Calcium in %
	The leucocytes are in: NaCl 0.9 %	The leucocytes are in: NaCl 0.9 % + $CaCl_2$ 0.05 %	
10 minutes	$\frac{156}{500} \times 100 = 31.2\%$	$\frac{155}{506} \times 100 = 30.6\%$	2 %
20 „	$\frac{146}{460} \times 100 = 31.7$ „	$\frac{171}{428} \times 100 = 39.9$ „	26 „
30 „	$\frac{219}{529} \times 100 = 41.5$ „	$\frac{246}{420} \times 100 = 58.5$ „	40.1 „
1 hour	$\frac{270}{560} \times 100 = 48.2$ „	$\frac{339}{514} \times 100 = 65.9$ „	37 „
1½ „	$\frac{322}{503} \times 100 = 64$ „	$\frac{359}{602} \times 100 = 59.6$ „ (?)	
2 „	$\frac{302}{478} \times 100 = 65.1$ „	$\frac{309}{467} \times 100 = 66.1$ „	1.5 „

This table shows, that already after 10 minutes a considerable number of phagocytes have taken up coal. The influence of calcium cannot yet be observed here. This is the case, however, where the leucocytes have been in contact with carbon particles for 20 minutes; still more when the time was 30 minutés, whilst the greatest difference is to be observed after they have been together for one hour. It is seen that in the suspension without calcium 48.2 % of the leucocytes have taken up carbon, whilst in the suspension containing calcium this figure was already 65.9 %. This is evidently the maximum. This same maximum, however, is very nearly reached in the fluid without Ca, but about *half an hour after*.

This experiment, made with blood-corpuscles of another animal gives similar results as the preceding one: after one hour the phagocytosis in the suspension containing Ca is still considerably greater than in the suspension without Ca. After two hours they are about equal (59.8 and 59.4).

[1]) The experiments on phagocytosis mentioned in this treatise have been made in collaboration with Mr. J. DE HAAN, Med. Cand., assistant at the Physiologica Laboratory.

Repetition of the Experiment.

TABLE VI.

Effect of the time on the extent of Phagocytosis.

Time during which the phagocytes could take up carbon-particles:	Percentage of Leucocytes having taken up carbon:		Increase of Phagocytosis by Ca in %
	The leucocytes are in: NaCl 0.9%	The leucocytes are in: NaCl 0.9% + $CaCl_2$ 0.05%	
10 minutes	$\frac{104}{284} \times 100 = 36.6\%$	$\frac{214}{574} \times 100 = 37.2\%$	1.7%
30 „	$\frac{127}{327} \times 100 = 38.8$ „	$\frac{146}{330} \times 100 = 44.2$ „	14 „
1 hour	$\frac{171}{404} \times 100 = 42.3$ „	$\frac{235}{458} \times 100 = 51.3$ „	21.3 „
2 „	$\frac{304}{520} \times 100 = 58.4$ „	$\frac{200}{334} \times 100 = 59.8$ „	2.4 „
4 „	$\frac{165}{302} \times 100 = 57.9$ „	$\frac{153}{253} \times 100 = 60.4$ „	4.3 „

The fact of its remaining a little greater in the solution containing Ca than in the liquid without Ca must probably be attributed to the circumstance that with the first the phagocyte-contents are better balanced, in other words sustain less change than in the NaCl-solution without Ca.

Finally a third experiment may be mentioned. It gave the same results as the 2 first.

TABLE VII.

Influence of the time on the extent of phagocytosis.

Time during which the phagocytes could take up carbon-particles:	Percentage of leucocytes, having taken up carbon:		Increase of phagocytosis by Ca in %
	The leucocytes are in: NaCl 0.9%	The leucocytes are in NaCl 0.9% + $CaCl_2$ 0.05%	
10 minutes	$\frac{183}{493} \times 100 = 37.1\%$	$\frac{185}{405} \times 100 = 45.9\%$	23.7%
1 hour	$\frac{142}{294} \times 100 = 48.2$ „	$\frac{241}{369} \times 100 = 65.3$ „	35.4 „
2 „	$\frac{214}{333} \times 100 = 64.2$ „	$\frac{243}{384} \times 100 = 64.2$ „	0 „

The results obtained in the *experiments on chemotaxis* and in *those on phagocytosis show uniformly that the influence of Ca is based upon an acceleration of the amoeboid motion.*

It may further be asked *why* calcium accelerates the amoeboid motion of the phagocytes. We might suppose a modification in the agglomeration of the colloid protoplasm-particles. This might be occasioned by a modification in the electric charge owing to the entering of the bi-valent kation. If this were the case then it would not be improbable that the other bi-valent metal-ions such as barium, strontium or magnesium would promote phagocytosis likewise. The experiment has taught, however, that this is by no means the case. I shall mention here only one of the many experiments made on the subject.

To three solutions, viz. NaCl 0.9%, NaCl 0.9% + 0.11% $BaCl_2$ + 2 Aq and NaCl 0.9% + 0.05% $CaCl_2$ equal quantities of a leucocyte suspension were added. After the fluids had acted upon it for 2 hours, carbon was added and half an hour after, it was investigated what percentage of the leucocytes had taken up carbon. The following table gives the results of the experiments.

TABLE VIII.
Influence of Barium and Calcium.

Solution:	Percentage of leucocytes having taken up carbon
NaCl 0.9%	$\frac{58}{234} \times 100 = 24.8\%$
	$\frac{54}{229} \times 100 = 23.4$ „
	$\frac{68}{270} \times 100 = 25.1$ „
NaCl 0.9% + 0.11% $BaCl_2$ 2 Aq	$\frac{69}{279} \times 100 = 24.7$ „
NaCl 0.9% + 0.05% $CaCl_2$	$\frac{192}{377} \times 100 = \mathbf{50.9}$ „

This experiment shows that barium has exercised no determinable influence upon phagocytosis, calcium on the other hand in a very high degree.

This result is confirmed in the case of the same leucocytes, after they have been left to themselves *for 24 hours in a 0.9°/₀ NaCl-solution*. After that time a fixed amount of the leucocytes is added to a fresh solution of NaCl 0.9,°/₀, of NaCl 0.9°/₀ + 0.11°/₀ $BaCl_2$ and of NaCl 0.9°/₀ + 0.05°/₀ $CaCl_2$.

TABLE IX.
Influence of Barium and Calcium.

Solution:	Percentage of leucocytes having taken up carbon:
NaCl 0.9°/₀	$\frac{15}{473} \times 100 = 3.2$°/₀
„ + 0.11°/₀ $BaCl_2$	$\frac{16}{453} \times 100 = 3.5$ „
„ + 0.05°/₀ $CaCl_2$	$\frac{113}{216} \times 100 = 52.3$ „

These experiments show that when the phagocytes, by being exposed a long time to NaCl 0.9°/₀, have almost entirely lost their power, *they cannot be revived by barium*. An isosmotic quantity of *calcium however, produces this effect in a very marked degree.*

The action of strontium was identical with that of barium.

Finally we may add an experiment with magnesium.

TABLE X.

Solution:	Percentage of leucocytes containing carbon:
NaCl 0.9°/₀	$\frac{1}{529} \times 100 = 0.2$°/₀
„ + 0.05°/₀ MgCl	$\frac{8}{807} \times 100 = 2.2$ „
„ + 0.05°/₀ CaCl	$\frac{261}{532} \times 100 = 49$ „

Here it appears again that the disabled phagocytes are somewhat revived by the addition of some magnesium, but that the effect is incomparably much greater, an isosmotic amount of $CaCl_2$ being added.

From these experiments a greater number of which, with more detailed descriptions will be found in the Biochemische Zeitschrift, it may be concluded that the considerable increase of phagocytarian capacity effected by calcium, cannot be explained by the electric charge inherent in Ca as a bi-valent ion, but that we have to deal here with *a specific, biochemical property of this element.*

To throw more light upon the special significance of calcium we may incidentally mention that it is especially this element which represents the favourable effect of RINGER's fluid on phagocytosis. The following experiment may serve as an illustration.

Equal amounts (0.15 cc.) of the same leucocyte-suspension were mixed with equal amounts (2 cc.) of a sol. of NaCl 0.9 %, of RINGER's fluid without Ca (viz NaCl 8, $NaHCO_3$, KCl 0.075, 1000 aq.) and of RINGER's fluid containing different quantities of $CaCl_2$.

The leucocyte-suspension having been exposed to carbon for 30 minutes it was determined in the usual way what was the percentage of leucocytes containing carbon. The following table gives the results of an experiment.

TABLE XI.

Importance of Calcium in RINGER's fluid.

Solutions:	Percentage of leucocytes containing carbon:	Average:
NaCl 0.9%	38.6%—42 % [1]	40.3%
RINGER's fluid without $CaCl_2$	37.1 „ —41 „	39 „
„ „ with 0.005% $CaCl_2$	41.5 „ —43.1 „	42.3 „
„ „ „ 0.01 „ „	42.8 „ —45.5 „	44.1 „
„ „ „ 0.05 „ „	49.3 „ —51.7 „	50.5 „

It is seen that RINGER's fluid without calcium is not more favourable to phagocytosis than NaCl 0.9 % alone; it would rather seem to be a little impeded by it. *Addition of calcium, however, even of mere traces, promotes phagocytosis considerably.*

What influence this element exercises here can only be guessed

[1]) The two values are in all cases given by two observers. It is remarkable that the one always gets a higher figure for phagocytosis than the other, although the preparations were taken from the same suspension. Evidently the one sees coal in a cell sometimes, where the other does not.

at the present moment. About the real cause of the amoeboid motion of living protoplasm we know at present nothing with any amount of certainty. We have only suppositions; so we have thought for instance of the possibility that calcium would bring about a decrease in the surface tension of the phagocytes. Now the surface tension or rather the molecular constant of two contiguous layers is, as we know, expressed by the formula $K_{1\cdot 2} = K_1 + K_2 - A_{1\cdot 2}$, K_1 representing here the molecular constant of the extreme layer of phagocytes, K_2 the molecular constant of the surrounding fluid, and $A_{1\cdot 2}$ representing the energy resulting from the contact of the two surfaces. It would be of importance to be able to demonstrate that under the influence of calcium $K_{1\cdot 2}$ decreases. Hitherto, however we have failed to determine this value, even approximately. The only thing we could do was to establish whether the surface tension of the surrounding medium (fluid) viz. K_2 underwent any change under the influence of calcium. But we have not been able to discover any such change, neither in a positive nor in a negative sense.

We are still occupied with a further investigation concerning the nature of the effect produced by calcium. Perhaps in its turn it lays open a road to penetrate into a more general and more important problem viz. the cause of the motion of living protoplasm.

SUMMARY.

The following are the principal conclusions derived from the above described experiments.

1. *Chemotaxis is considerably promoted by slight amounts of calcium.*

This was demonstrated in two ways:

a. by placing under the skin capillary tubes containing bacteria cultures (B. Coli) with and without calcium and comparing the lengths of the columns of leucocytes which had entered into the tubes.

b. by injecting NaCl solutions with and without Ca into the intestinal canal and measuring subsequently in both cases the lengths of the columns of leucocytes which had been attracted into the capillary tubes filled with the bacteria suspension.

The experiments sub *a*, and sub *b* were carried out with bacteria suspensions as such, and with filtrates obtained by means of CHAMBERLAND's filters.

The results were similar in both cases.

The fluids containing calcium which were brought into the intestinal canal were:

1. NaCl-solution containing $CaCl_2$.

2. The water of the Virchow-spring (Kiedrich near Eltville, Wiesbaden) which contains a great amount of Ca. The influence of both fluids turned out to be very considerable:

If only 60 cc. of the above mineral water was injected daily into the intestinal canal of rabbits, a quantity corresponding with 0.06 Gr. $CaCl_2$, the chemotaxis increased by about 40 %. It must be observed that this increase represents only a minimum value.

2. *These chemotactic investigations have proved that calcium increases the activity of phagocytes to a very considerable extent, not only in vitro but also in the living organism.*

During 48 hours this influence remained undiminished. Very probably it extends over a much longer period. The way in which the experiments were conducted, however, did not admit their being continued for a longer period, with the same animal.

3. *This increased activity of the phagocytes cannot be accounted for by an increased intensity of the cell contractions, but finds its cause in an acceleration of the amoeboid motion.*

As regards chemotaxis this needs no further proof; as regards phagocytosis this could be demonstrated by the following experiment: when suspensions of leucocytes without calcium are left only sufficient time to take up carbon particles, the percentage of leucocytes having taken up carbon becomes equal to that which is observed, in a shorter time indeed, in suspensions with Ca.

4. If we ask ourselves what may be *the cause* of calcium accelerating the amoeboid motion of phagocytes, we might be inclined to think of a modification in the agglomeration of the colloid protoplasm-particles as a consequence of the electric charge, caused by the entering of a number of bi-valent calcium ions. This explanation however can hardly be the correct one. For the experiment teaches that other bi-valent kations namely barium, strontium, magnesium do not cause an acceleration of the amoeboid motion.

It must be assumed then, that the action of calcium in this case, is based upon a specific, hitherto unknown, biochemical property. As another example of the great influence of calcium we may also mention the fact that the favourable effect of RINGER's fluid, on phagocytosis, must be exclusively attributed to this metal.

(June 23, 1910).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING of Saturday June 25, 1910.

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CONTENTS.

Mathematics. — "*A quadruple involution in the plane and a triple involution connected with it.*" By Prof. JAN DE VRIES.

(Communicated in the meeting of May 28, 1910).

1. In a paper entitled "*An arrangement of the pointfield in involutory groups*" (Versl. van de Kon. Akad. v. Wet., series 3 vol. VI, p. 92—102, 1888; Archives Néerlandaises, vol. XXIII, p. 355—366) I have considered the involutions, the groups of which consist of basepoints of pencils comprised in a net of plane curves of degree n with $\frac{1}{2} n(n+3) - 2$ fixed basepoints. Lately Dr. W. VAN DER WOUDE (These Proceedings of March 26th 1910) has investigated a special cubic involution of the first rank in the plane. In the following paper I shall treat the involution, each group of which consists of the points of intersection of two conics α^2 and β^2 belonging to two pencils (α) and (β) with the basepoints A_k and B_k $(k=1, 2, 3, 4)$. Let this quadruple involution be indicated by $(P)^4$.

2. The eight basepoints are evidently *singular points of* $(P)^4$. For on the conic β^2_k which can be brought through A_k the conics α^2 describe a cubic involution, of which each triplet forms with A_k a quadruple of $(P)^4$; I call β^2_k and α^2_k the *singular conics*.

On an arbitrary right line (α) and (β) determine two involutions; their common pair belongs to a quadruple (P). The lines $a_{kl} \equiv A_k A_l$ and $b_{kl} \equiv B_k B_l$ contain an infinite number of pairs; on a_{12} we find that (β) determines an involution of which the pairs are completed to quadruples by the pairs which (β) describes on a_{34}. Lateron (§ 4) it will be evident that these 12 lines are not the only singular lines.

Each α^2 contains 6 quadruples with a double point (coincidence $P_1 \equiv P_2$) belonging to the biquadratic involution which is described by (β); the points P_3, P_4 which still appear in such a quadruple I call branchpoints of $(P)^4$. In each singular point we find such a coincidence, where the corresponding singular conic is touched by a conic of the second system. The locus of the coincidences has therefore with α^2 ten points in common; the *curve of coincidence* is therefore a curve of *order five*, γ^5, passing through the *eight singular points*.

The cubic involution on the singular conic β^2_k has four groups with a double point; so A_k is branchpoint in four quadruples, so the locus of the branchpoints passes four times through each singular point. As an arbitrary α^2 contains in six quadruples twelve branchpoints, the *branchcurve* is a curve of *order fourteen*, φ^{14}, with eight fourfold points in A_k and B_k.

3. If we regard A_2, A_3, A_4 as principal points of a quadratic transformation, then (α) passes into a pencil of lines (A'_1), (β) into a pencil (β') of biquadratic curves through the four points B'_k, with double points in A_2, A_3, A_4. Evidently the curve of coincidence γ^5 is transformed into the polar curve π^7 of A'_1 with relation to (B'), i. e. the locus of the points of contact of the curves B'^4 with lines through A'_1.

In A_2 the polar curve π^7 has a threefold point, of which a tangent passes through A'_1, because the curve β'^4 touching A'_1A_2 in A_2 is touched there at the same time by the polar cubic of A'_1 [1]).

In B'_k the polar curve π^7 touches the line $A'_1B'_k$. It is easy to see that in A'_1 15 tangents of π^7 concur; these right lines, inflectional tangents of curves β'^4, are changed by the quadratic transformation into conics α^2, each osculated by a β^2.

The quadruple involution $(P)^4$ possesses consequently fifteen threefold points $P_1 \equiv P_2 \equiv P_3$.

In each of these points γ^5 and φ^{14} will have to touch each other. So besides their 32 sections lying in the singular points, they have 8 points more in common. These must form four pairs R'_k, R''_k, each consisting of *two coincidences*; i. o. w. $(P)^4$ contains four quadruples, where $P_1 \equiv P_2$ and $P_3 \equiv P_4$. Evidently R'_k and R''_k are the points of contact of two conics α^2, β^2 touching each other twice; the contact chord $R'_k R''_k$ is indicated by c_k.

4. If we make a line l to rotate around a point T_1, the pair P_1, P_2 of $(P)^4$ lying on it describes a curve τ^5 with threefold point T_1, the tangents of which are directed to the points forming a quadruple with T_1. This curve passes through the singular points A_k, B_k and through the points which T_1A_k and T_1B_k have in common with the singular conics β^2_k and α^2_k. Each of the eight tangents t which τ^5 sends through T_1 bears a coincidence $P_1 \equiv P_2$. The lines t, containing two coincided points of P^4, envelop therefore a *curve of the eighth class*, τ_8.

As each t is conjugated to a definite point of γ^5 we find that τ_8 is of genus six just as the former; so it possesses 15 double tangents. To this belong the 12 singular lines a_{kl} and b_{kl} indicated above containing each an involution (P_1, P_2), hence two coincidences.

1) The other two tangents of the threefold point are the tangents to both β'^4, containing a cusp in A_2.

If d is one of the remaining three double tangents the quadratic involutions determined on them by (α) and (β) have the double points in common; so they are identical. Therefore the *three* lines d are singular too and $(P)^4$ possesses *fifteen singular lines.*

If we conjugate to each other any two conics α^2 and β^2 cutting d in the same pair of points P_1, P_2, the pencils (α) and (β) are projective and generate a cubic curve δ^3 on which they describe a selfsame central involution (P_3, P_4). The lines $P_3\,P_4$ concur in the opposite point D of the groups A_k and B_k.

The locus τ^5 of the pairs lying collinear with D evidently breaks up into δ^3 and a conic with double point D not passing through A_k, B_k, consisting therefore of the other two singular right lines d. The three points D are therefore the vertices of a triangle having the lines d as sides; this is lateron confirmed in a different way (§ 7).

5. We find that (α) and (β) determine on a_{12} and a_{34} two projective involutions in half-perspective position; for the point of intersection $A_{12,34} \equiv S$ of a_{12} and a_{34} appears in two pairs belonging to one and the same quadruple.

From this ensues that the lines connecting the points of a pair P_1, P_2 with the points P_3, P_4 of the corresponding pair envelop a *curve of class three* having a_{12} and a_{34} as tangents. If $Q_{1,2}$ and $R_{1,2}$ are the coincidences of the involution described on a_{12}, Q_3, Q_4 and R_3, R_4 the points of a_{34} forming with them two quadruples, then $Q_3Q_{1,2}$ is the tangent in Q_3, etc. As the indicated curve is cut by a_{34} in Q_3, Q_4, R_3, R_4 and is touched in the point S_4 belonging to $S_{1,3}$, it is a curve of *order six.*

The lines P_1P_3 and P_2P_4 are conjugated tangents; P_2P_3 and P_1P_4 form a pair of the same system. From this ensues that the locus of the diagonal points $N' \equiv (P_1P_3, P_2P_4)$ and $N'' \equiv (P_1P_4, P_2P_3)$ of quadrangle (P) is a *cubic curve*, $\alpha^3{}_{12,34}$; its points of intersection with a_{13} are $Q_{1,2}$, $R_{1,2}$ and $S_{1,3}$.

The line $n \equiv N'N''$ describes a pencil; for, the lines n are the polar lines of $S_{1,3}$ with respect to the pencil (β). The pairs (N', N'') form thus on the curve $\alpha^3{}_{12,34}$ a central involution. Where in future we speak of one of the three points $A_{kl,mn}$ and the cubic curve $\alpha^3{}_{kl,mn}$ conjugated to it, these will be indicated by A^* and α^*; an analogous signification have B^* and β^*.

6. When P_1 describes the line l, the points P_2, P_3, P_4 describe a *curve of order seven*, λ^7; with l it has in common the pair lying on that line besides the points in which l cuts the curve of coin-

cidence γ^5. As l has two points in common with each of the singular conics $\beta^2{}_k$, $\alpha^2{}_k$, A_k and B_k are double points of λ^7. With the curve λ^7 belonging to l' it has 32 sections in the singular points; of the remaining seventeen three form a quadruple with the point (ll'), 14 belong to 7 quadruples, each having a point on l, a second on l'.

To find the class of the curve enveloped by the lines P_2P_3, P_3P_4, P_4P_2, we determine the number of the lines passing through A_1. In the first place belong to these the lines through P_2 and P_3, which form triplets with the two points P_1 of $\beta^2{}_1$ lying on l. As A_1A_k contains a pair of $(P)^4$ lying with the point of intersection of l and $A_m A_n$ in a quadruple, A_1A_k is also one of the indicated tangents. The lines $p \equiv P_kP_l$ envelop therefore *a curve of class seven*, π_7, having the 12 singular lines a_{kl}, b_{kl} as tangents, the three singular lines d as *threefold* tangents; for, l has with δ^3 three points P_1 in common. The curves π_7 and π'_7 belonging to l and l' have thus in the singular lines 39 common tangents; of the remaining ten three belong to the point of intersection of l and l', 7 to as many quadruples, of which one point lies on l, an other point on l'.

If l passes through a singular point λ^7 breaks up into the corresponding singular conic ($\alpha^2{}_k$ or $\beta^2{}_k$) and a λ^5. For $l \equiv a_{12}$ we find that λ^7 consists of the conics $\beta^2{}_1$ and $\beta^2{}_2$, the line a_{12} and twice the line a_{34}. For $l \equiv A_1B_1$ we find the conics $\beta^2{}_1$ and $\alpha^2{}_1$ with a cubic curve through the remaining six singular points. For $l \equiv d$ we find that λ^7 consists of d and twice δ^3.

The system consisting of l and λ^7 is invariant with respect to the transformation which makes the points of a quadruple to correspond to each other. In general we shall have an invariant curve by assuming a correspondence (m, n) betweén the pencils (α) and (β). With projective conjugation we find a general biquadratic curve.

7. The conics which can be laid through the quadruples (P) form a linear system of order three (∞^3) which can be represented by an equation

$$\alpha a^2{}_x + \beta b^2{}_x + \gamma c^2{}_x + \delta d^2{}_x = 0.$$

A pair of lines with double point in O_3 belonging to it has as equation

$$(\sum_4 \alpha a_{11})\, x^2{}_1 + 2\,(\sum_4 \alpha a_{12})\, x_1x_2 + (\sum_4 \alpha a_{22})\, x^2{}_2 = 0,$$

where the parameters α, β, γ, δ are connected by the equations

$$a_{13}\alpha + b_{13}\beta + c_{13}\gamma + d_{13}\delta = 0,$$
$$a_{23}\alpha + b_{23}\beta + c_{23}\gamma + d_{23}\delta = 0,$$
$$a_{33}\alpha + b_{33}\beta + c_{33}\gamma + d_{33}\delta = 0,$$

which furnish in general but *one* solution.

We may conclude from this that an arbitrary point bears but *one* pair of opposite sides of a quadrangle (P) so that the diagonal points N, N', N'', of the quadrangles (P) can be arranged in the groups of a triple involution which will be indicated by $(N)^3$.

The lines $s = P_k P_l$ and $s' = P_m P_n$ are consequently conjugated to each other in an *involutory correspondence*, of which we can easily show that it is *quadratic*. For, if we make s to rotate round a point O, then s' will pass in two of its positions through O, namely when it coincides with one of the lines of the pair of lines (s, s') having O as double point; i. e. s' envelops a conic when s describes a pencil.

According to § 4, the *quadratic involution* (s, s') has the lines d as *principal lines*; for d forms a pair with each line through D. From this ensues again that the three points D are the vertices of the triangle formed by the lines d.

The *coincidences* (double lines) of the involution are the *chords of contact* c_k indicated above (§ 3). According to a wellknown property of the quadratic involution the principal lines d are the diagonals of the quadrangle formed by the double lines. In connection with this we put $c_k c_l = C_{kl}$ and $C_{kl} C_{mn} = d_{klmn}$. Apparently C_{kl} is the centre of an involution of rays, having c_k and c_l as double rays.

8. The triple involution $(N)^3$ has 6 *singular points* in C_{kl}; for, C_{kl} bears ∞^1 pairs of rays (s, s'), so it is a diagonal point N of ∞^1 quadrangles (P). Lateron it will be evident, that the locus of the corresponding pairs N', N'' is a *biquadratic curve* (§ 10).

Also the poles C_k of the four lines c_k with respect to the pairs of conics α^2, β^2 of which they form the chords of contact, are *singular points* of $(N)^3$. Each point C_k is as diagonal point N conjugated to the pairs N', N'' of an involution placed on c_k having R'_k and R''_k (§ 3) as double points.

Finally A^* and B^* are also *singular points*. As was evident in § 5, the corresponding *singular curves* α^* and β^* are of *order three*.

The involution $(N)^3$ has thus *sixteen singular points*.

Singular lines of $(N)^3$ are evidently the three right lines d and the four right lines c.

The triplets (N) determined by the quadruples (P) of a conic α^2, lie on a cubic curve ν^3, which cuts α^2 in the six coincidences of the biquadratic involution $(P)^4$. For the singular conic α^2_k (passing through B_k) ν^3 has a double point in B_k, of which the tangents are directed to the branchpoints of the coincidence lying in B_k.

9. When $N \equiv ss'$ describes the line l, then s and s' envelop a *curve of class three* touched by l in the point N which it has in common with the line l' conjugated to it. This curve λ_3 intersects l on the four right lines c, is thus a curve of *order six*.

We now determine the order of the locus of the quadruples (P) lying on the pairs of tangents of λ^6_3. The curve r^3 belonging to a definite conic α^2 determines on l three points N, so it contains three quadruples of the locus. This passes three times through each point A_k, because the rational r_3 belonging to β^2_k determines on l three diagonal points N of quadruples, in which A_k appears. It passes through the eight points R'_k, R''_k lying on the lines c_k and touches there the lines $C_kR'_k$, $C_kR''_k$.

The curves π_1^{12} and π_2^{12} belonging to l_1 and l_2 have $8 \times 9 = 72$ sections in A_k, B_k, $4 \times 4 = 16$ in the points R'_k, R''_k and 4 in the quadruple (P) for which $(l_1 l_2)$ is one of the diagonal points. The remaining 52 sections form 13 quadruples (P), of which one diagonal point lies on l_1, and a second on l_2.

From this ensues that to the points N of a line l correspond the pairs N', N'' of a *curve of order thirteen*, λ^{13}. With l the curve λ^{13} has five points of the curve of coincidence γ^5 in common, which is at the same time curve of coincidence of the involution $(N)^3$; the remaining eight form four pairs (N, N'). Each line bears thus four pairs of $(N)^3$.

10. The curve λ^{13} passes three times through each of the six singular points A^*, B^*, because l has three points in common with the corresponding singular curve α^*, respect. β^*. It also passes through the four singular points C_k and with a number of branches to be determined more closely through each of the six singular points C_{kl}.

The curves λ_1^{13} and λ_2^{13}, belonging to l_1 and l_2 have $6 \times 9 = 54$ sections in A^* and B^*, 4 in the points C_k; fartheron they have in common the pair of points conjugated to $N \equiv l_1 l_2$ besides the 13 points N forming each with a point of l_1 and a point of l_2 a triplet of $(N)^3$. As the remaining 96 sections must lie in the 6 points C_{kl}, we find that λ^{13} passes four times through each point C_{kl}. To the *singular point* C_{kl} belongs therefore a *singular biquadratic curve* γ^4_{kl}.

When l coincides with c_1, we find that λ^{13} breaks up into the line c_1 and the three singular curves γ^4_{12}, γ^4_{13}, γ^4_{14}. These pass all through the singular points A^*, B^*, because these points are threefold on λ^{13}. As the three curves γ^4_{1k} pass together four times through the points C_{23}, C_{24} and C_{34}, we find that γ^4_{1k} has a double point in C_{mn}. That γ^4_{kl} must have at least *one* double point, was deducible from the fact, that on a general biquadratic curve no involutions of pairs appear. On the uninodal γ^4_{kl} exists but *one* involution of pairs; the pairs (N', N'') belonging to $N = C_{kl}$ lie thus collinear with the double point C_{mn} of γ^4_{kl}.

As γ^4_{12} passes through C_{12} we find that C_{12} is a coincidence of $(N)^3$ and at the same time of $(P)^4$. The third point N'' of the corresponding triplet must lie on $C_{12}C_{34} = d_{12,34}$. From this ensues that C_{12} is one of the double points of the quadratic involution determined by (α) and (β) on $d_{12,34}$; then the second double point is C_{34}. The curve γ^5 cuts $d_{12,34}$ in C_{12}, C_{34} and in the three points which $d_{12,34}$ has in common with $\delta^3_{12,34}$.

The curves γ^4_{12} and γ^4_{34} have six points of intersection in A^*, B^*, 4 points of intersection in C_{13}, C_{14}, C_{23}, C_{24} and 4 points of intersection in C_{12} and C_{34} (for, C_{12} is double point of γ^4). The remaining two points of intersection are diagonal points of two quadruples having each a diagonal point in C_{12} and in C_{34}; the lines connecting these two points N with C_{34}, are apparently the tangents in the double point C_{34} of γ^4_{12}.

The curves γ^4_{12} and γ^4_{13} have 8 sections in the points C_{kl}, 6 in the points A^*, B^* and both of them pass through C_1; their 16th point of intersection is a point N forming a triplet with C_{12} and C_{13}. We see that N as point of γ^4_{12} must lie on the line $C_{13}C_{24}$, so this point must coincide with C_{23}. So the two curves must touch each other in C_{23}.

11. As γ^4_{kl} passes through the singular points A^* and B^* the singular points C_{kl} lie on the singular curves α^* and β^*. The curves $\alpha^3_{12,34}$ and $\alpha^3_{13,24}$ intersect each other in the 6 points C_{kl} and in the 3 points B^*; the last follows from the consideration of the quadruple that is determined by the right lines b_{kl}, b_{mn} on the right lines a_{pq}, a_{rs}.

The curves α^* and β^* have therefore in common the 6 points C_{kl}, the two points A^*, B^* and finally the point N forming a triplet with the last two points.

For the *singular points* and *lines* we have therefore the following orientation:

c_k contains the three points C_{kl};

$\alpha^3_{kl,mn}$ contains $A_{kl,mn}$, the three points B^* and the six points C_{kl};

$\beta^3_{kl,mn}$ contains $B_{kl,mn}$, the three points A^* and the six points C_{kl};

γ^4_{kl} has a double point in C_{mn} and passes through the remaining points C_{pq}, through the points C_k and C_l and through the six points A^*, B^*;

$d_{kl,mn}$ contains the points C_{kl} and C_{mn}.

For the singular line $d_{12,34}$ the curve λ^{13} consists of $\gamma^4_{12}, \gamma^4_{34}$ and a curve $\delta^5_{12,34}$ passing through the points A^*, B^*, C_{12} and C_{34}, and having double points in the remaining four points C_{kl}. It is a curve of *genus two*, so it contains only one involution of pairs; the pairs (N', N'') are determined by the conics containing the four double points, and the lines $n \equiv N'N''$ envelop a conic δ^2, touching δ^5 in five points [1]).

We find that $d_{12,34}$ is cut by $\delta^5_{12,34}$ in C_{12}, C_{34} and in the three points which $d_{12,34}$ has in common with $\delta^3_{12,34}$; these five points lie also on the curve of coincidence γ^5 (§ 10).

12. If N describes the line $d_{12,34}$, the line n envelops a figure of class four composed of the points C_{12}, C_{34} and a conic $\delta^2_{12,34}$. From this we may conclude that n will envelop *a curve of class four*, λ_4; when N describes the line l.

Between N and n there is no birational correspondence; N does determine in general *one* right line n, but on a non-singular n lie (§ 9, four pairs (N', N''), so that to n belong four points N.

The lines $s \equiv P_1P_2$ bearing the coincidences of $(P)^4$ envelop (§ 4) a curve of class eight, τ_8, having a_{kl}, b_{kl} and the principal lines d of the quadratic involution (s, s') as double tangents. Therefore the line $s' \equiv P_3P_4$ envelops a curve of class ten, τ'_{10}, possessing three fourfold tangents d and six double tangents a_{kl} and b_{kl} (a_{kl} corresponds in the involution to a_{mn}). The point of intersection of s' with s is the branchpoint N'' belonging to the points N and N' coinciding with P_1 and P_2. As none of the lines s coincides with the s' conjugated to them, the locus of the point (s, s') is a curve of order 18. The *branchcurve* of the involution $(N)^3$ is therefore of *order eighteen*, has double points in C_k, fourfold points in A^*, B^* and of course passes through the fifteen threefold points of $(N)^3$.

13. If the basepoints A_4 and B_4 coincide in the point E, then the quadruple involution $(P)^4$ passes into a *triple involution* with the *singular points* A_k, B_k ($k = 1, 2, 3$) and E. If to each conic α^2 the conic β^2 is conjugated which it touches in E then the biquadratic curve ε^4 containing the points of intersection of corresponding curves has with an arbitrarily chosen α^2 three points, A_k and two points P_1, P_2, in common; so it passes three times through E. So to the singular point E belongs a singular biquadratic curve with threefold

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[1]) See my paper "*Ueber Curven fünfter Ordnung mit vier Doppelpunkten*". (Sitz Ber. Akad. Wien, CIV, 46).

point in E bearing the pairs of points forming triplets with E; it of course passes through the six singular points A_k, B_k.

The curve of coincidence γ^5 possesses now a *threefold point* in E; for, with an α^2 it has in common the three points A_k and the four coincidences of the cubic involution lying on α^2. The curves γ^5 and ε^4 have in E the same three tangents.

On an arbitrarily chosen α^2 lie four branchpoints; as E is branchpoint for triplets, in which the two coincidences lying on ε^4 appear, and also the points A_k and B_k are each branchpoints for two groups, the branchcurve is of order six, φ^6, and it has double points in the *seven* singular points.

The curves γ^5 and φ^6 have in the singular points 18 sections; as fartheron they can only touch each other $(P)^3$ has *six threefold points*.

14. The pairs (P_1, P_2) lying collinear with a point T_1 form a curve τ^4 with double point T_1 where six tangents t concur. The bearers of the coincidences of $(P)^3$ envelop therefore a *curve of class six*, τ_6. As τ^4 has still 5 points in common with γ^5 besides the 6 points of contact of the tangents concurring in T_1 and the 7 singular points, the lines which connect each a point of coincidence with the corresponding branchpoint envelop a *curve of class five*.

The curve τ_6 is, like γ^5, of genus three, so it has seven double tangents. To this belong the six singular lines a_{kl}, b_{kl}; also the *seventh* indicated by d, *is singular* because (α) and (β) determine the same involution on it; the third movable point of intersection of two curves α^2 and β^2 conjugated in this way describes a cubic curve δ^3 with double point in E.

If P_1 describes the line l, then P_2 and P_3 describe a curve λ^7, passing four times through E, twice through A_k, B_k and cutting l into a pair and into five coincidences of $(P)^3$.

The line $p_1 \equiv P_2P_3$ envelops a curve of *class four*, π_4, for the positions of p_1 passing through E are furnished by the lines to the points P_2, which form triplets with E and the points of intersection P_1 of l and ε^4. This π^4 has a_{kl} and b_{kl} as tangents; for, on a_{kl} lies e. g. a pair P_2, P_3 belonging to the point of intersection P_1 of l with A_mE. The singular line d is *threefold tangent* of π_4; the three pairs P_2, P_3 lying on it correspond to the points of intersection P_1 of l and δ^3.

The curves π_4 and π'_4 belonging to l and l' have therefore in the seven singular lines 15 tangents in common; the 16[th] common tangent p is conjugated to the point of intersection of l and l'. By the birational transformation (P, p) a pencil is therefore transformed into a curve *of class four*.

When p rotates around T_1 the pair P_2, P_3 lying on it describes the above mentioned curve τ^4, possessing with λ^7 sixteen sections in the singular points; four points of intersection form each a pair with a point of l; the remaining ones belong to four pairs P_2, P_3, for which P_1 lies on l. To a pencil described by p corresponds therefore a *biquadratic curve* π^4 described by P.

As τ^4 has with ε^4 in common besides the singular points a point lying on ET_1 and three pairs P_2, P_3 placed on lines through T_1 for which P_1 falls in E, this E is a threefold point on π^4. In an analogous way is evident that A_k, B_k are points of π^4. Two curves π^4 have thus 15 sections in the singular points; the 16th common point corresponds to the common ray of the two pencils.

15. Finally we note the case, in which (α) and (β) have in common the basepoints E_1 and E_2, thus determine an involution of pairs (P_1, P_2).

To the *singular points* A_1, A_2; B_1, B_2 conics β^2_1, β^2_2; α^2_1, α^2_2 are conjugated, of which the points form a pair with the corresponding singular point.

If we conjugate again each α^2 to the β^2 touching it in E_1 its movable point of intersection describes a figure of order four, passing three times through E_1 and twice through E_2, thus composed of the line $e \equiv E_1 E_2$ and a cubic curve ε^3_1, having E_1 as double point and passing through E_2, A_1, A_2, B_1, B_2; it contains moreover the point of intersection C of $A_1 A_2$ and $B_1 B_2$.

As evidently C belongs also to the *singular curve* ε^3_2, of which the points form pairs with E_2, therefore C is also a *singular point*; it corresponds to each point of e.

The *curve of coincidence* has double points in E_1 and E_2; it is biquadratic and passes through the four points A_k, B_k.

If P_1 describes the line l, then P_2 describes a λ^6 through C with four double points A_k, B_k and two threefold points E_1, E_2., So we have here a *birational involutory* transformation of order six and class one (a pair on an arbitrary right line), with 7 principal points of which 2 are threefold, 4 twofold and 1 single.

The pairs on rays through T form a cubic curve τ^3 through the 7 principal points; two curves τ^3 have besides the principal points in common the pair on the line connecting the corresponding points T. As four tangents of τ^3 pass through T, the bearers of the coincidences envelop a *curve of class four*.

Physics. — *"On the scattering of light by molecules".* By Prof. H. A. LORENTZ.

(Communicated in the meeting of January 29, 1910)

§ 1. It was pointed out many years ago by Lord RAYLEIGH[1]) that a beam of light can be scattered to all sides not only by particles of dust, but also by the molecules of the medium in which the propagation takes place. According to his theory the coefficient of extinction due to this cause in the case of a body of small density, a gas for instance, is determined by the formula

$$h = \frac{32\pi^3(\mu-1)^2}{3N\lambda^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which μ is the index of refraction, λ the wave-length and N the number of molecules per unit of volume, the meaning of the coefficient h itself being that the intensity is diminished in the ratio of 1 to e^{-hl} when a distance l is travelled over.

RAYLEIGH has deduced his equation by calculating the energy radiating from the molecules whose particles are put in motion by the incident rays, and by taking into account that the quantities of energy traversing two successive sections of the beam must differ from each other by an amount equal to the energy that is emitted by the molecules lying between those sections.

The problem may, however, also be treated in a different manner. In many theories the ordinary absorption of light is explained by a resistance opposing the motion of the vibrating particles and giving rise to a development of heat. Similarly, the extinction which we are now considering may be ascribed to a certain resistance which, however, is not accompanied with a heating effect, but is intimately connected with the radiation from the molecules. According to the theory of electrons[2]) a force of this kind acts on an electron whenever its velocity $\mathfrak{v}$[3]) is variable; it is represented by the expression

$$\frac{e^2}{6\pi c^3}\frac{d^2\mathfrak{v}}{dt^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which e is the charge of the electron, and c the velocity of light in the ether.

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[1]) RAYLEIGH, On the transmission of light through an atmosphere containing small particles in suspension, and on the origin of the blue of the sky, Phil. Mag. (5) **47** (1899), p 375 (Scientific Papers **4**, p. 397).

[2]) See, for instance, LORENTZ, Math. Encyklopädie, V, 14, § 20.

[3]) German letters represent vector quantities.

In the case of a simple harmonic motion the sign of the second differential coefficient of $\mathfrak{v}$ is opposite to that of $\mathfrak{v}$ itself, so that, like the resistance assumed in the theory of absorption, the force (2) is opposite to the velocity. As to the connexion between this force and the radiation from the vibrating electron, it becomes apparent if we remark that during a full period the work of the force which is required for maintaining a constant amplitude, and which must be equal and opposite to (2), is exactly equal to the amount of the radiated energy.

In a recent paper NATANSON [1]) has shown that RAYLEIGH's formula can be obtained by introducing the force (2) into the equation of motion of each vibrating electron.

§ 2. This result is very satisfactory, but still there are some points which require further consideration.

In RAYLEIGH's theory it is necessary to take into account the interference between the vibrations which are produced, at some definite point of space, by all the molecules in the beam, and, on the other hand, a consideration of the resistances will be incomplete if one does not keep in view the mutual action between the molecules. Whether we prefer one course or the other, it may be shown that a scattering can only take place when the molecules are irregularly distributed, as they are in gases and liquids; in a body whose molecules have a regular geometrical arrangement, a beam of light is propagated without any diminution of its intensity.

Let us begin with the second method, and let us observe in the first place that, according to (2), the resistance per unit of charge is given by

$$\frac{e}{6\pi c^3}\frac{d^2\mathfrak{v}}{dt^2}.$$

If $\mathfrak{r}$ is the displacement of an electron from the position of equilibrium which it has in a molecule, this expression may be replaced by

$$\frac{e}{6\pi c^3}\frac{d^3\mathfrak{r}}{dt^3},$$

for which we may also write

$$\frac{1}{6\pi c^3}\frac{d^3\mathfrak{p}}{dt^3}, \quad \ldots \ldots \ldots \ldots \quad (3)$$

if we put

1) L. NATANSON, On the theory of extinction in gaseous bodies, Bulletin de l'Acad. des Sciences de Cracovie, déc. 1909, p. 915.

$$e\mathfrak{r} = \mathfrak{p}\,.$$

This latter quantity is the electric moment of the molecule, if e is the only movable electron contained in it.

The above expression contains the *third* differential coefficient of $\mathfrak{r}$ or $\mathfrak{p}$ with respect to the time, and it is easily seen that terms of this kind, or, in general, terms of odd order, are the only ones in the equations determining the propagation of light which can give rise to an extinction of the beam. This circumstance will enable us to distinguish the terms with which we shall be principally concerned, from others which determine, not the extinction but the velocity of propagation, and which it will not be necessary to consider in detail.

§ 3. It is important to remark that the field belonging to a molecule with an alternating moment $\mathfrak{p}$ acts with a force like (3), not only on the electron e in the molecule itself, but also on electrons lying outside the particle, at distances that are very small in comparison with the wave-length.

At a point (x, y, z), at a distance r from the molecule, the scalar potential φ and the vector potential $\mathfrak{a}$ are determined by the equations

$$\varphi = -\frac{1}{4\pi}\, div\, \frac{[\mathfrak{p}]}{r}\,, \quad \ldots \ldots \ldots \quad (4)$$

$$\mathfrak{a} = \frac{1}{4\pi c r}\left[\frac{d\mathfrak{p}}{dt}\right], \quad \ldots \ldots \ldots \quad (5)$$

in which the square brackets serve to indicate that, if we want to know the potentials for the time t, we must use the values of the enclosed quantities corresponding to the time $t - \frac{r}{c}$. Hence, $[\mathfrak{p}]$ is a function of x, y, z, t, and we may write for the vector potential

$$\mathfrak{a} = \frac{1}{4\pi c r}\,\frac{\partial[\mathfrak{p}]}{\partial t}\,.$$

Now, if r is very small with respect to the wave-length, we have

$$[\mathfrak{p}] = \mathfrak{p} - \frac{r}{c}\frac{d\mathfrak{p}}{dt} + \frac{r^2}{2c^2}\frac{d^2\mathfrak{p}}{dt^2} - \frac{r^3}{6c^3}\frac{d^3\mathfrak{p}}{dt^3} + \ldots .$$

For our purpose it will suffice to consider the part of φ corresponding to the fourth term of this series, and the part of $\mathfrak{a}$ corresponding to the second term. In equation (4) the quantity $\frac{[\mathfrak{p}]}{r}$ may therefore be replaced by

$$-\frac{r^2}{6c^3}\frac{d^3\mathfrak{p}}{dt^3},$$

a vector whose components are

$$-\frac{r^2}{6c^3}\frac{d^3\mathfrak{p}_x}{dt^3}, \quad -\frac{r^2}{6c^3}\frac{d^3\mathfrak{p}_y}{dt^3}, \quad -\frac{r^2}{6c^3}\frac{d^3\mathfrak{p}_z}{dt^3},$$

and whose divergence is

$$-\frac{1}{3c^3}\left(x\frac{d^3\mathfrak{p}_x}{dt^3}+y\frac{d^3\mathfrak{p}_y}{dt^3}+z\frac{d^3\mathfrak{p}_z}{dt^3}\right),$$

if the point from which r is reckoned, is taken as origin of coordinates.

We have therefore

$$\varphi\ (=)\ \frac{1}{12\pi c^3}\left(x\frac{d^3\mathfrak{p}_x}{dt^3}+y\frac{d^3\mathfrak{p}_y}{dt^3}+z\frac{d^3\mathfrak{p}_z}{dt^3}\right),$$

denoting by the symbol $(=)$ that terms irrelevant to our purpose have been omitted.

The differential coefficients of the quantity within the brackets with respect to x, y, z are

$$\frac{d^3\mathfrak{p}_x}{dt^3}, \quad \frac{d^3\mathfrak{p}_y}{dt^3}, \quad \frac{d^3\mathfrak{p}_z}{dt^3},$$

so that we find

$$grad\ \varphi\ (=)\ \frac{1}{12\pi c^3}\frac{d^3\mathfrak{p}}{dt^3}.$$

Combining this with

$$\mathfrak{a}\ (=)\ -\frac{1}{4\pi c^2}\frac{d^2\mathfrak{p}}{dt^2},$$

we are led to the expression

$$\frac{1}{6\pi c^3}\frac{d^3\mathfrak{p}}{dt^3},$$

which has already been mentioned, for the force acting on unit charge (which is given in general by $-\frac{1}{c}\dot{\mathfrak{a}} - grad\ \varphi$).

Simple examples may serve to show that this result agrees with the law of energy. Suppose, for instance, that two molecules placed very near each other contain equal electrons vibrating with equal amplitudes and phases along parallel straight lines. Then the flow of energy across a closed surface surrounding the molecules will be equal to four times the flow that would belong to one of the particles taken by itself. Hence, for each molecule, the work necessary for maintaining its vibrations must be doubled by the influence of the other particle. This is really the case because the resistance is doubled, each molecule contributing an equal part to it.

Again, if the two vibrations have opposite phases, the amplitudes still being equal, the two forces acting on one of the electrons according to our formulae — one produced by the field of the electron itself and the other by the field of the other molecule — will annul each other. But in this case the system of the two molecules does not lose any energy by radiation.

§ 4. The preceding considerations show that a correct explanation of the extinction of light, by means of the forces acting on the vibrating electrons, can only be obtained by examining the mutual actions between the molecules. In order to take these into account I shall follow the same method which I have used on previous occasions.

We shall start from the fundamental equations by means of which the electromagnetic field between the electrons and even inside these small particles can be described in all its details. Let $\mathfrak{d}$ and $\mathfrak{h}$ be the electric and the magnetic force, ϱ the density of the electric charge, and $\mathfrak{v}$ its velocity. Then

$$div\ \mathfrak{d} = \varrho\ ,$$

$$div\ \mathfrak{h} = 0\ ,$$

$$rot\ \mathfrak{h} = \frac{1}{c}\,(\dot{\mathfrak{d}} + \varrho\,\mathfrak{v})\ .$$

$$rot\ \mathfrak{d} = -\frac{1}{c}\,\dot{\mathfrak{h}}\ .$$

Any electromagnetic state which satisfies these conditions may be represented by means of a scalar potential φ and a vector potential $\mathfrak{a}$. These are determined by the equations

$$\varphi = \frac{1}{4\pi}\int\frac{[\varrho]}{r}\,d\,S\,, \quad \ldots\ldots\ldots\ldots \quad (6)$$

$$\mathfrak{a} = \frac{1}{4\pi c}\int\frac{[\varrho\mathfrak{v}]}{r}\,d\,S\,, \quad \ldots\ldots\ldots\ldots \quad (7)$$

in which the integrations are to be extended over all space, and we have

$$\mathfrak{d} = -\frac{1}{c}\frac{\partial\mathfrak{a}}{\partial t} - grad\ \varphi\ .$$

We may now pass on to the equations that may be used for a description of the phenomena in which the details depending on the molecular structure and inaccessible to our means of observation are omitted. We obtain these by simply replacing each term in the above formulae by its mean value over a space S surrounding the point considered, whose dimensions are so small that, in so far as it can be observed, the state of the medium may be regarded as the same at all points of S, and at the same time so great that S contains a large number of molecules. A space of this kind may be called "infinitely small in a physical sense" and the mean value of any scalar or vector quantity A is defined by the equation

$$\bar{A} = \frac{1}{S} \int A \, d\,S ,$$

in which the integration extends over the small space S.

We shall suppose the medium to contain neither conduction- nor magnetization-electrons, but only polarization-electrons, i. e. charged particles whose displacement from their positions of equilibrium produces the electric moments of the molecules. Let $\mathfrak{P}$ be the electric polarization (the electric moment per unit of volume). Then [1])

$$\bar{\varrho} = - \, div \, \mathfrak{P},$$

$$\overline{\varrho \mathfrak{v}} = \dot{\mathfrak{P}} ,$$

and, if we put $\bar{\mathfrak{d}} = \mathfrak{E}$ (electric force), $\mathfrak{E} + \mathfrak{P} = \mathfrak{D}$ (dielectric displacement), $\bar{\mathfrak{h}} = \mathfrak{H}$, $\bar{\varphi} = \Phi$, $\bar{\mathfrak{a}} = \mathfrak{A}$,

$$\left.\begin{array}{l} div \, \mathfrak{D} = 0, \\ div \, \mathfrak{H} = 0, \\ rot \, \mathfrak{H} = \dfrac{1}{c} \, \dot{\mathfrak{D}}, \\ rot \, \mathfrak{E} = - \dfrac{1}{c} \, \dot{\mathfrak{H}}, \end{array}\right\} \quad \ldots \ldots \ldots \quad (8)$$

$$\mathfrak{E} = - \frac{1}{c} \frac{\partial \mathfrak{A}}{\partial t} - grad \, \Phi \, . \quad \ldots \ldots \ldots \quad (9)$$

In those cases in which the field is produced by polarization-electrons only, we have by (6) and (7)

$$\Phi = - \frac{1}{4\pi} \int \frac{[div \, \mathfrak{P}]}{r} \, dS,$$

$$\mathfrak{A} = \frac{1}{4\pi c} \int \frac{1}{r} \left[\frac{\partial \mathfrak{P}}{\partial t} \right] dS. \quad \ldots \ldots \ldots \quad (10)$$

In the first of these two equations it has been tacitly assumed that there is nowhere a discontinuity in the polarization $\mathfrak{P}$. Whenever such a discontinuity exists at some surface σ, the equation must be replaced by

$$\Phi = - \frac{1}{4\pi} \int \frac{[div \, \mathfrak{P}]}{r} \, dS - \frac{1}{4\pi} \int \frac{1}{r} \left\{ [\mathfrak{P}_{n_2}] - [\mathfrak{P}_{n_1}] \right\} d\sigma, \quad . \quad (11)$$

[1]) Math. Encyklopädie V 14, § 30.

where n means the normal to the surface σ, drawn from the side 1 towards the side 2.

§ 5. The fundamental equations show that the field may be considered as produced by the electrons contained in the source of light and in the media traversed by the rays. Let σ be a closed surface in the medium with which we are concerned and let the value of $\mathfrak{E}$ at some point on the inside of σ be decomposed into two parts, the first of which ($\mathfrak{E}_1$) is due to all the electrons lying outside the surface, whereas the second part ($\mathfrak{E}_2$) has its origin in the state of the medium within σ. This latter part can be determined by the equations (9), (10) and (11), if, for a moment, we confine ourselves to the matter enclosed by σ, with the values of $\mathfrak{P}$ existing in it. Then, drawing the normal to σ towards the outside, we have $\mathfrak{P}_{n_2} = 0$ and we may write

$$\Phi = \frac{1}{4\pi}\int \frac{[\mathfrak{P}_n]}{r}\, d\sigma, \quad \ldots \ldots \ldots \quad (12)$$

if we omit the index 1 in $\mathfrak{P}_{n_1}$ and if we take for granted that the vibrations are transverse, so that $div\ \mathfrak{P} = 0$.

Confining the integration in (10) to the space within σ, we find for the second part of $\mathfrak{E}$

$$\mathfrak{E}_2 = -\frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t} - grad\ \Phi. \quad \ldots \ldots \ldots \quad (13)$$

As to the first part

$$\mathfrak{E}_1 = \mathfrak{E} - \mathfrak{E}_2$$

it represents the value which $\mathfrak{d}$ would have at a point within the surface, if we removed all the particles contained in it, without changing anything in the state of the matter on the outside.

In what follows we shall conceive the cavity made in this way to be infinitely small in a physical sense. But, nevertheless, we shall suppose its dimensions to be very great in comparison with those of the space S that has been mentioned in the definition of the mean values. Under these circumstances and if we except those points of the cavity which are very near the walls, there will be no difference between the mean value of $\mathfrak{d}$ and this vector itself. Hence, $\mathfrak{E}_1$ may be considered as the real value of $\mathfrak{d}$ within the cavity.

§ 6. In order to find the laws of the propagation of light, we have to combine the equations (8) with the relation between $\mathfrak{D}$ (or $\mathfrak{P}$) and $\mathfrak{E}$, which can be deduced from the equation of motion of the electrons vibrating in the molecules.

We shall simplify by assuming that each molecule contains no more than one vibrating electron. Let us fix our attention on a single molecule M and let us denote by $\mathfrak{r}$ the displacement of its movable electron from the position of equilibrium, by $\mathfrak{p} = e\,\mathfrak{r}$ the moment of the molecule, and by m the mass of the electron. The forces acting on the electron are: 1. the quasi-elastic force, for which we shall write $-f\mathfrak{r}$, 2. the resistance (3), and 3. the force $e\,\mathfrak{d}$, if $\mathfrak{d}$ is the electric force produced at the place of M by all the surrounding electrons. Now, after having described around M an infinitely small surface σ, such as has been considered in § 5, we may conceive $\mathfrak{d}$ to be made up of two parts, the vector $\mathfrak{E}_1$ that has already been mentioned, and the part that is due to the molecules Q surrounding M and lying within the surface σ. Let $\mathfrak{d}_q$ be the part contributed by one of these molecules, and let the symbol Σ refer to all the molecules Q. Then, the equation of motion becomes

$$m\frac{d^2\mathfrak{r}}{dt^2} = -f\mathfrak{r} + \frac{e}{6\pi c^3}\frac{d^3\mathfrak{p}}{dt^3} + e(\mathfrak{E} - \mathfrak{E}_2) + e\,\Sigma\,\mathfrak{d}_q \quad . \quad . \quad . \quad (14)$$

and here, on account of what has been said in § 3, we may put

$$\Sigma\,\mathfrak{d}_q \,(=)\, \frac{1}{6\pi c^3}\,\Sigma\,\frac{d^3\mathfrak{p}_q}{dt^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

if we confine ourselves to the resistances.

The determination of the sum occurring on the right-hand side would be a very simple matter, if the molecules were arranged in some regular way, if, for example, they occupied the points of a parallelepipedic net. In such a case, the moment $\mathfrak{p}_q$ of any one of the molecules Q may be considered as equal to that of the particle M itself, for which we want to write down the equation of motion (because the dimensions of σ are very small with respect to the wave-length). On the contrary, in a system of particles having an irregular distribution, unequalities may arise from the mutual electromagnetic actions; this is easily seen if one considers that the distance to the nearest particle is not the same for the different molecules. On account of this circumstance, it would be very difficult accurately to calculate the sum for a liquid body.

In the case of a gas the problem becomes more simple. Indeed, it can be safely assumed that in such a body the influence of the molecules on the propagation of light is rather feeble. It is only in a small measure that the state in a definite molecule depends on that of the surrounding ones; it is chiefly determined by the state of the ether, and this may be taken to be nearly the same that could exist if the beam were propagated in a vacuum. Consequently, in the equation of motion of the electron belonging to a definite molecule, the terms expressing the action of the other molecules are small in comparison with the remaining terms, and we shall neglect only quantities that may be said to be of the second order, if, in calculating the terms in question, we reason as if the moments of the molecules Q and that of M itself were wholly independent of the mutual action between these particles. But in this case all these moments would be equal to each other. Therefore, in calculating the sum in (15), we shall take each $\mathfrak{p}_q$ to be equal to the mean value of $\mathfrak{p}$ for all the molecules M contained in an infinitely small space. Distinguishing mean values of this kind by a double bar above the letter, and writing ν for the number of the molecules Q, i. e. for the number of particles, with the exception of M, lying within the closed surface σ, we may replace (15) by

$$\Sigma \mathfrak{d}_q (=) \frac{\nu}{6\pi c^3} \frac{d^3 \overline{\overline{\mathfrak{p}}}}{dt^3}.$$

§ 7. It remains to consider the electric force $\mathfrak{E}_2$ determined by (10), (12) and (13). Let us put for this purpose

$$[\mathfrak{P}] = \mathfrak{P} + \mathfrak{Q},$$

and let each of the three quantities ϕ, $\mathfrak{A}$ and $\mathfrak{E}_2$ be decomposed into two parts in a way corresponding to this formula. The first part of $\mathfrak{E}_2$ depends only on the values of $\mathfrak{P}$ which are found, at the definite moment t, on the surface σ and inside it, and even if account had to be taken of the changes of $\mathfrak{P}$ from one point to another — which can be represented by means of the differential coefficients of $\mathfrak{P}$ with respect to the coordinates, it could be shown that the part in question contains differential coefficients of even order only, at least if the form of σ is symmetrical with respect to three planes passing through M and parallel to the planes of coordinates. It will therefore suffice for our purpose to consider the second part of $\mathfrak{E}_2$, and to substitute in (13) the values

$$\phi = \frac{1}{4\pi} \int \frac{\mathfrak{Q}_n}{r} \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$\mathfrak{A} = \frac{1}{4\pi c} \int \frac{1}{r} \frac{\partial \mathfrak{Q}}{\partial t} \, dS. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In the following transformations, whose object is the determination of $\mathfrak{E}_2$, the coordinates of the point M for which we want to know ϕ, $\mathfrak{A}$ and $\mathfrak{E}_2$ are denoted by x', y', z', and those of a point on the surface σ or within it, by x, y, z.

It may be remarked in the first place that (16) may be written in the form

$$\phi = \frac{1}{4\pi} \int \left(\frac{\partial}{\partial x} \frac{\mathfrak{Q}_x}{r} + \frac{\partial}{\partial y} \frac{\mathfrak{Q}_y}{r} + \frac{\partial}{\partial z} \frac{\mathfrak{Q}_z}{r} \right) d\,S \quad . \quad . \quad . \quad (18)$$

and that here the differential coefficients with respect to x, y, z, may be replaced by those with respect to x', y', z' with the signs inverted. In order to show this, put

$$\mathfrak{P}_x = f_1\,(x, y, z, t),\ \mathfrak{P}_y = f_2\,(x, y, z, t),\ \mathfrak{P}_z = f_3\,(x, y\ z, t)$$

and write $f'_{1x}\,(x, y, z, t)$ etc. for the partial derivatives, taken for a constant t, of these expressions with respect to x, y, z. The vibrations being transverse, we have

$$f'_{1x}\,(x, y, z, t) + f'_{2y}\,(x, y, z, t) + f'_{3z}\,(x, y, z, t) = 0, \quad . \quad (19)$$

and also

$$f'_{1x}\left(x, y, z, t - \frac{r}{c}\right) + f'_{2y}\left(x, y, z, t - \frac{r}{c}\right) +$$
$$+ f'_{3z}\left(x, y, z, t - \frac{r}{c}\right) = 0, \quad . \quad . \quad . \quad (20)$$

because (19) is true for any value of t.

Now,

$$\frac{\mathfrak{Q}_x}{r} = \frac{1}{r}\left\{ f_1\left(x, y, z, t - \frac{r}{c}\right) - f_1\,(x, y, z, t)\right\},$$
$$\frac{\mathfrak{Q}_y}{r} = \frac{1}{r}\left\{ f_2\left(x, y, z, t - \frac{r}{c}\right) - f_2\,(x, y, z, t)\right\},$$
$$\frac{\mathfrak{Q}_z}{r} = \frac{1}{r}\left\{ f_3\left(x, y, z, t - \frac{r}{c}\right) - f_3\,(x, y, z, t)\right\},$$

and, if this is substituted in (18), we get two groups of terms, some depending on the explicit occurrence in $\frac{Q_x}{r}$ etc. of x, y, z and the remaining ones arising from the variability of r. Equations (19) and (20) show that the terms of the first group annul each other, and we may replace (18) by

$$\phi = -\frac{1}{4\pi}\int \left(\frac{\partial}{\partial x'} \frac{\mathfrak{Q}_x}{r} + \frac{\partial}{\partial y'} \frac{\mathfrak{Q}_y}{r} + \frac{\partial}{\partial z'} \frac{\mathfrak{Q}_z}{r} \right) d\,S \quad . \quad . \quad . \quad . \quad (21)$$

because

$$\frac{\partial r}{\partial x'} = -\frac{\partial r}{\partial x}, \text{ etc.}$$

Let us next substitute in (21) (cf. § 3)

$$\mathfrak{Q} = [\mathfrak{P}] - \mathfrak{P} = -\frac{r}{c}\frac{\partial \mathfrak{P}}{\partial t} + \frac{r^2}{2c^2}\frac{\partial^2 \mathfrak{P}}{\partial t^2} - \frac{r^3}{6c^3}\frac{\partial^3 \mathfrak{P}}{\partial t^3} + \ldots, \quad . \quad (22)$$

where the differential coefficients $\frac{\partial \mathfrak{p}}{\partial t}$ etc. are independent of x', y', z'.

After this expansion none of the terms in $\frac{\mathfrak{Q}}{r}$ contains a negative power of r, and in differentiating (21) with respect to x', y', z', as is necessary for the determination of *grad* Φ, we may effect the operation under the sign of integration. Thus

$$-\frac{\partial \Phi}{\partial x'} = \frac{1}{4\pi}\int\left(\frac{\partial^2}{\partial x'^2}\frac{\mathfrak{Q}_x}{r} + \frac{\partial^2}{\partial x' \partial y'}\frac{\mathfrak{Q}_y}{r} + \frac{\partial^2}{\partial x' \partial z'}\frac{\mathfrak{Q}_z}{r}\right) d\,S, \text{ etc.}$$

or, confining ourselves to the part of this expression corresponding to the last term in (22),

$$-\frac{\partial \Phi}{\partial x'} (=) -\frac{1}{12\pi c^3}\int \frac{\partial^3 \mathfrak{p}_x}{\partial t^3}\, d\,S, \text{ etc.},$$

i. e.

$$-\,grad\ \Phi\,(=) -\frac{1}{12\,\pi\, c^3}\int \frac{\partial^3 \mathfrak{p}}{\partial t^3}\, d\,S.$$

As to the term $-\frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t}$, it will suffice to substitute in (17) the first term of (22), so that

$$-\frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t} (=) \frac{1}{4\pi c^3}\int \frac{\partial^3 \mathfrak{p}}{\partial t^3}\, d\,S.$$

The result of our calculation is

$$\mathfrak{E}_2\,(=) \frac{1}{6\pi c^3}\int \frac{\partial^3 \mathfrak{p}}{dt^3}\, d\,S,$$

or, since $\frac{\partial^3 \mathfrak{p}}{\partial t^3}$ may be considered as constant throughout the small space enclosed by σ, if the magnitude of that space is denoted by U,

$$\mathfrak{E}_2\,(=) \frac{1}{6\pi c^3}\, U\, \frac{\partial^3 \mathfrak{p}}{\partial t^3}.$$

Finally, the equation of motion (14) takes the form

$$m\frac{d^2 \mathfrak{r}}{dt^2} = -f\,\mathfrak{r} + e\,\mathfrak{E} + \frac{e}{6\pi c^3}\left(\frac{d^3 \mathfrak{p}}{dt^3} + \mathfrak{r}\,\frac{d^3 \overline{\overline{\mathfrak{p}}}}{dt^3} - U\,\frac{\partial^3 \mathfrak{p}}{\partial t^3}\right) + \mathfrak{F}, \quad . \quad . \quad (23)$$

where several actions of which we have not spoken and which are not to be reckoned among the resistances, are taken together in the term $\mathfrak{F}$.

§ 8. We have now to distinguish two cases.

a. Let the molecules have a regular arrangement in such a manner that each occupies the centre of one of a system of equal parallelepipeds which are formed by three groups of planes. In this case there is no difference between $\mathfrak{p}$ and $\overline{\overline{\mathfrak{p}}}$. Further, if $\frac{1}{N}$ is the volume of one of the elementary parallelepipeds, and if we take for the space U a parallelepiped consisting of k elementary ones,

$$\mathfrak{P} = N\mathfrak{p},$$

$$U = \frac{k}{N}.$$

By this the expression enclosed in brackets in (23) becomes

$$(1 + \mathfrak{r} - k)\frac{\partial^3 \mathfrak{p}}{\partial t^3}.$$

But, $\mathfrak{r} + 1$ being the total number of particles in the space U, we have

$$\mathfrak{r} + 1 = k,$$

so that, after all, there is no resistance, and there can be no extinction of the rays of light.

b. The case of an irregular distribution of the molecules is best treated by applying equation (23) to each of the molecules within an infinitely small space and taking the mean value of each term. Since

$$\mathfrak{P} = N\overline{\overline{\mathfrak{p}}},$$

N being the number of molecules per unit of volume, we get

$$m\frac{d^2\bar{\mathfrak{r}}}{dt^2} = -f\bar{\mathfrak{r}} + e\bar{\mathfrak{E}} + \frac{e}{6\pi c^3}(1 + \bar{\mathfrak{r}} - NU)\frac{\partial^3 \mathfrak{p}}{\partial t^3} + \bar{\delta}. \quad . \quad . \quad (24)$$

Now, the number of particles in the space U considered in § 7 was $1 + \mathfrak{r}$, and therefore it would almost seem at first sight as if the mean value $1 + \bar{\mathfrak{r}}$ were equal to NU. In fact, however, we have, in the case of an irregular distribution

$$\bar{\mathfrak{r}} = NU. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

In order to see this, we must remember that $1 + \mathfrak{r}$ represented the total number of particles lying in a space *U that had been chosen around a molecule M on which we had previously fixed our attention.* Let us imagine in the gas a volume V very great in comparison with the infinitely small space U, and let us conceive the NV molecules which this volume is to contain, to be placed in it at random, no difference being made between one part of space and another. After having assigned its position to the first molecule, we choose around it the small space U and we ask how many of the remaining $NV - 1$ particles will, in the mean, come to lie in that space, if the experiment of placing the $NV - 1$ molecules in the volume V is repeated many times. Obviously, this mean number, which we may take for $\bar{\bar{r}}$, is

$$\frac{U}{V}(NV-1) = NU - \frac{U}{V}$$

and this may be replaced by (25), because $\frac{U}{V}$ is a very small fraction.

Our conclusion must therefore be that the coefficient $1+\bar{\bar{r}}-NU$ in (24) has the value 1, and we may express this by saying that among the terms in (23) which represent resistances, one only remains, namely the term that is due to the field belonging to the molecule itself which we are considering.

Finally, in order to give a more convenient form to the equation of motion (24) we shall multiply it by $\frac{eN}{m}$, replacing at the same time the vector $Ne\bar{\bar{r}} = N\bar{\bar{p}}$ by $\mathfrak{P}$. We shall also put

$$\frac{eN}{m}\bar{\bar{\mathfrak{F}}} = \gamma\,\mathfrak{P}.$$

where, with sufficient approximation, γ may be considered as a constant coefficient, and

$$\frac{f}{m} - \gamma = n_0^2.$$

In this way we are led to the formula

$$\frac{\partial^2 \mathfrak{P}}{\partial t^2} = -n_0^2\,\mathfrak{P} + \frac{e^2 N}{m}\mathfrak{E} + \frac{e^2}{6\pi c^3 m}\frac{\partial^3 \mathfrak{P}}{\partial t^3},$$

from which, if it is combined with (8), RAYLEIGH's extinction coefficient can be deduced.

§ 9. We shall conclude by briefly showing that, like the method which we have now followed, that of RAYLEIGH, namely the direct calculation of the energy emitted by the molecules, leads to a scattering of the light, only for a system whose molecules are irregularly distributed.

Let us consider a bundle of parallel homogeneous rays, and let L be a line or a very narrow cylinder having the direction of the rays, AB a part of L very long in comparison with the wave-length, AP a line making a certain angle with AB, and P a point of that line whose distance from A is many times greater than AB. We shall take the axis of x along AB and we shall simplify by assuming that, for each molecule situated on the line L or in the narrow cylinder, the electric moment may be represented by an expression of the form

$$a \cos (n\,t + p),$$

in which p is a linear function of x. The amplitude a may be regarded as constant, if we neglect the unequalities that may arise from the mutual action between the molecules of a gas or a liquid (comp. § 6), and if we suppose the extinction along the length of $A\,B$ to be very feeble.

For one of the components of the light vector at P, so far as it depends on one molecule, we may now put

$$b \cos (n\,t + q),$$

where b is a constant, and q a linear function of x, and we have to calculate the sum

$$s = \Sigma\, b \cos (n\,t + q), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

extended to all the molecules.

Suppose in the first place that k molecules occupy equidistant positions on the line AB. Then the values of q form an arithmetical series $q_1, q_1 + \Delta, q_1 + 2\,\Delta$, etc. and we have

$$s = \frac{b}{2 \sin \frac{1}{2}\Delta} [\sin \{nt + q_1 + (k - \tfrac{1}{2})\,\Delta\} - \sin \{nt + q_1 - \tfrac{1}{2}\Delta\}] =$$

$$= b \frac{\sin \frac{1}{2}k\Delta}{\sin \frac{1}{2}\Delta} \cos \{nt + q_1 + \tfrac{1}{2}\,(k-1)\,\Delta\}.$$

It appears from the first form that the resulting disturbance of equilibrium can be conceived as consisting of two vibrations emitted by points near the extremities of the row of molecules, and the second form shows that, when the length of the row is increased constantly, the amplitude of s remains comprised between $+\dfrac{b}{\sin \frac{1}{2}\Delta}$ and $-\dfrac{b}{\sin \frac{1}{2}\Delta}$. Though there is a certain residual vibration, its intensity cannot be said to increase with the length of AB.

§ 10. This conclusion also holds when the molecules of a gas are distributed in such a manner over the cylinder L that equal parts of it, separated from each other by normal sections, contain exactly equal numbers of particles. Then, for an element dx, the number will be $f dx$, with a constant f, and we have instead of (26)

$$s = bf \int \cos (nt + q)\, dx = bf \frac{l}{q'' - q'} \{\sin (nt + q'') - \sin (nt + q')\},$$

l being the length of AB, and q', q'' the extreme values of q. While

l increases, the ratio $\frac{l}{q''-q'}$ remains constant, and, like in the former case, the resulting vibration may be considered as made up of two components emitted by the extremities of AB.

In order not to encumber our formulae with this small residual vibration, I shall suppose the difference $q'' - q'$ to be a multiple of 2π.

When the distribution of the molecules is an irregular one, equal parts of the cylinder L will *not* contain exactly the same number of particles, and we shall now show that these differences must cause a real scattering of the rays. For this purpose we begin by dividing the cylinder AB into a number of parts AA', $A'A''$ etc., such that along each of them q changes by 2π. Next, always using normal sections, we divide each of these parts into a great number, say k, of smaller ones, all of equal length dx. Having done this, we take together the first part of AA', the first of $A'A''$, etc., considering their sum as *one* part of the cylinder AB; in the same manner we combine into a *second* part of it the second part of AA', the second of $A'A''$, and so on, so that after all the whole cylinder is divided into k parts of equal volume. For all the molecules lying in one of these parts the phases of the vibrations which they produce at the point P, may be taken to be equal. Let the k phases be determined by the quantities $q_1, q_2, \ldots q_k$, which form an arithmetical series.

Now, if $g_1, g_2, \ldots g_k$ are the numbers of molecules contained in the k parts of the cylinder, we have

$$s = b\,[g_1 \cos (nt + q_1) + g_2 \cos (nt + q_2) + \ldots + g_k \cos (nt + q_k)] \quad . \quad (27)$$

According to what has been said, this would be zero if all the numbers $g_1, g_2, \ldots g_k$ were equal. Consequently we may also write

$$s = b\,[h_1 \cos (nt + q_1) + h_2 \cos (nt + q_2) + \ldots + h_k \cos (nt + q_k)],$$

if we understand by $h_1, h_2, \ldots h_k$ the deviations of the numbers $g_1, g_2, \ldots g_k$ from their mean value. We shall denote this mean value itself by g.

The radiation across an element of surface lying at the point P is determined by the square of s, and our problem may therefore be put as follows: What will be the mean value of s^2 in a large number of experiments in which, all other things remaining the same, the distribution of the particles is different, a number kg of molecules being each time distributed at random over the k parts of the cylinder?

In considering this we must keep in mind that, among the numbers $h_1, h_2 \ldots h_k$ there must always be negative as well as positive ones;

since $h_1 + h_2 + \ldots + h_k = 0$, neither the positive nor the negative values will predominate.

Now it is clear that the mean value of any product of two different h's, relating to any two definite among the k parts, must of necessity be zero, in as much as there is no reason for a different probability of equal or unequal signs of those two deviations.

Hence, the mean value in question becomes

$$b^2 \left[\overline{h_1^2} \cos^2 (n\,t + q_1) + \overline{h_2^2} \cos^2 (n\,t + q_2) + \ldots + \overline{h_k^2} \cos^2 (n\,t + q_k)\right],$$

and on an average, for a full period,

$$\tfrac{1}{2}\, b^2 \left(\overline{h_1^2} + \overline{h_2^2} + \ldots + \overline{h_k^2}\right).$$

But, by a well known theorem in the theory of probabilities,

$$\overline{h_1^2} = \overline{h_2^2} = \ldots = \overline{h_k^2} = q,$$

so that our result becomes

$$\tfrac{1}{2}\, k q\, b^2,$$

showing that, in order to find the intensity of the radiation issuing from the cylinder L, we must multiply the intensity $\frac{1}{2} b^2$ that is produced by one molecule, by the number kq of particles in the cylinder. This conclusion can easily be extended to a part of the beam of any size. Indeed, the k vibrations occurring in (27) mutually destroy each other for the greater part by interference, and the vibration of which we have calculated the intensity is no more than a small residual disturbance of equilibrium. It may have any phase whatever according as the molecules happen to be disseminated in one way or another. Now, if a part of the beam of any magnitude is divided into a number of cylinders L such as we have considered in the last paragraphs, there will be no connexion between the distribution of the molecules in these several cylinders. The phases of the residual vibrations due to each of them will be wholly independent of each other, and it will be allowable, simply to take the sum of their intensities.

Physics. — *"Quasi-association or molecule-complexes."* By Prof. J. D. van der Waals.

(Communicated in the Meeting of May 28, 1910).

In the Meeting of this Academy of January 1906 I delivered an address on what I then called "Quasi-association". I demonstrated that the phenomena, particularly in the liquid state, led to the conclusion that the equation of state $p = \dfrac{RT}{v-b} - \dfrac{a}{v^2}$ was not in harmony with what experience teaches, even if the quantity b is assumed to be variable with the volume, but that with the great density of liquids, and at temperatures below the critical temperature, a circumstance must be taken into account which hardly makes its influence felt with slight density of the substance and at higher temperatures, but the influence of which can become so considerable with very great density that it quite decides about the course of the phenomena. In the above form the equation of state has been derived by taking the cohesion of the substance and the space occupied by the molecules themselves into account, but for the rest it is assumed that the molecules are distributed homogeneously throughout the space occupied by the substance. Already repeatedly, among others by REINGANUM, the thought has been expressed that there is reason to suppose that the distribution in the occupied space is *heterogeneous*, that molecule-complexes are to be expected, the number of which will certainly depend on the temperature and the degree of density. If this is the case this circumstance will make its influence felt on the value of p if T and v are given, and something is therefore to be added to the equation of state.

For myself I had formed a fairly clear general idea in what way the heterogeneous distribution of the molecules in the given space would make its influence felt on the form of the equation of state — but I was still doubtful about different particulars. This was the cause that I did not publish the contents of my address, and occupied with other investigations, I did not take up this subject again for a long time. Since then Dr. G. VAN RIJ discussed the contents of my address fully in his Thesis for the Doctorate "Schijn-associatie etc." He was enabled to do so by Dr. HALLO's kindness, who had given a shorthand account of my address in the Academy.

As regards what I said in my address of 1906 I might refer to the above-mentioned Thesis, and confine myself here to communicating what renewed investigation has taught as probable. But as I cannot take for granted that VAN RIJ's Thesis is sufficiently known, I shall have to enter into some repetitions for so far as this is necessary for the true insight into my reasoning. That I have resumed this investigation was due, among other causes, to the circumstance that in the investigation of the BROWN motion and allied phenomena it has again appeared of late that large molecule-complexes behave kinetically as a single molecule. I had already assumed this a priori, but the mentioned new investigations have made me the more confident in assuming that the number of molecules which can combine to a complex is great.

I. *The equation:* $p = \frac{RT}{v-b} - \frac{a}{v^2}$ *is not sufficient in the liquid state with* a *and* b *independent of* T, *and* b *a function of volume.*

From the thermodynamic equation $T\left(\frac{dp}{dT}\right) - p = \frac{d\varepsilon}{dv}_T$ follows for coexisting liquid and vapour phases:

$$T\frac{dp}{dT} - p = \frac{\frac{a}{v_1} - \frac{a}{v_2}}{v_2 - v_1}$$

or

$$\left(\frac{T}{p}\frac{dp}{dT} - 1\right) p\, v_1\, v_2 = a$$

and for the critical state:

$$\left\{\left(\frac{T}{p}\frac{dp}{dT}\right)_{kr} - 1\right\} p_k\, v_k^2 = a.$$

If these two equations are divided one by the other, and if $\left(\frac{T}{p}\frac{dp}{dT}\right)_{kr} = f$, $\frac{T}{p}\frac{dp}{dT}$ at lower temperatures being approximately $= \frac{f}{m}$, $\left(m = \frac{T}{T_k}\right)$, we find:

$$\frac{\left(\frac{f}{m} - 1\right) p\, v_1\, v_2}{(f-1)\, p_k\, v_k^2} = 1.$$

If we write $\frac{p}{p_k} = \pi$, $\frac{v_1}{v_k} = \mathfrak{r}_1$ and $\frac{v_2}{v_k} = \mathfrak{r}_2$, this equation becomes

$$\frac{\left(\frac{f}{m} - 1\right)}{(f-1)}\, \pi\, \mathfrak{r}_1\, \mathfrak{r}_2 = 1.$$

If we test this equation by SYDNEY YOUNG's well-known observations (Proc. Phys. Society of London 1892; Phil. Trans. **178** etc.), we find for the second member not 1, but the following series of values; e.g. for ether:

m	*second member*
1	1
0,9920	1,101
0,9825	1,156
0,89	1,342
0,81	1,414
0,68	1,425

$\frac{T}{p}\frac{dp}{dT}$ can be directly derived from these observations, and so the approximative character of $\frac{f}{m}$ may be avoided. The value f, namely, increases slowly with falling temperature.

Near T_k the value above unity can be represented by $\sqrt{1-m}$ with a high degree of approximation, and perhaps still better by $\sqrt{1-m} - \frac{1-m}{2}$ for lower T. It is to be regretted that for much lower temperatures the observation becomes impossible by the appearance of the solid state. But judging from the given values the value above unity seems to verge towards a limiting value. I shall represent this value above unity by the symbol φ.

II. *Is the assumption of a and b as function of the temperature sufficient to explain this deviation?*

On account of the existence of this value above unity which increases with decreasing temperature we might think that the assumption of a as temperature function or also of b as temperature function would be sufficient as an explanation. We have then (See among others Lehrbuch der Thermodynamik p. 76 etc. or VAN RIJ's Thesis):

$$\frac{\left(\frac{f}{m}-1+\frac{T}{b_g}\frac{db_g}{dT}\right)\pi\,\mathfrak{r}_1\,\mathfrak{r}_2}{f-1+\left(\frac{T}{b_g}\frac{db_g}{dT}\right)_{kr}} = \frac{1+\frac{T}{b_g}\frac{db_g}{dT}-\frac{T}{a}\frac{da}{dT}}{1+\left(\frac{T}{b_g}\frac{db_g}{dT}\right)_{kr}-\left(\frac{T}{a}\frac{da}{dT}\right)_{kr}}\,\frac{a}{a_k}.$$

In this equation b_g represents the limiting value of b for $v=\infty$.

That the assumption of CLAUSIUS' value for a, viz. $a\frac{T_k}{T}$ is altogether insufficient, appears immediately. If $\frac{T}{b_g}\frac{db_g}{dT}=0$, we find with this value of a for $\frac{T}{a_T}\frac{da_T}{dT}$ the value of -1 and

$$\frac{\left(\frac{f}{m}-1\right)\pi\,\mathfrak{r}_1\mathfrak{r}_2}{f-1}=\frac{2}{2}\frac{1}{m}=\frac{1}{m}.$$

The value above unity, which near T_k is equal to $1-m$, and not to $\sqrt{1-m}$, and so too small, is much too great for low T, and would even rise to ∞. But we need not examine other functions for a, which change continuously with the temperature for T_k. From

$$\frac{a_m - m\dfrac{da_m}{dm}}{\left(a_k - \dfrac{da_k}{dm}\right)} = 1 + \sqrt{1 - m}$$

follows by differentiation with respect to m:

$$\frac{-m\dfrac{d^2a_m}{dm^2}}{\left(a_k - \dfrac{da_k}{dm}\right)} = -\frac{1}{2}\frac{1}{\sqrt{(1-m)}},$$

an expression, which is equal to ∞ for $m = 1$, and shows that only a value of a which is dependent on the temperature and would show a sudden discontinuity at T_k, would be able to account for this value above unity. To accept this as possible would be tantamount to rejecting everything on which the equation of state rests. Then the critical temperature would not be that at which $\frac{dp}{dv}_T$ and $\frac{d^2p}{dv^2}_T$ can be equal to 0, and which has nothing remarkable for the rest. Then e.g. the characteristics of CO_2 also in the gaseous state would change discontinually when we pass from 30° to 31°.

A similar investigation reveals that the assumption that b would be a function of the temperature, cannot account for the value above unity either, unless we assume a sudden discontinuity in this function at T_k, and also that other assumptions about the value of the molecular pressure are unable to account for this value above unity, for these too would always require a sudden change at T_k.

III. *Two empiric formulae for the determination of liquid and vapour density for coexisting phases.*

Before proceeding to the explanation of what in my opinion causes the extent of this value above unity, I will just discuss the result obtained in one of its significations.

According to the empiric rule, known under the name of MATHIAS' rectiliniar diameter we have, calling the density in the liquid state ϱ_1, and the density in the vapour state ϱ_2:

$$\frac{\varrho_1 + \varrho_2}{2\varrho_k} = 1 + \gamma(1 - m).$$

The quantity γ is slightly different for different substances, but in the following discussion I shall assume $\gamma = 0{,}8$. According to the above formula:

$$\frac{\frac{f}{m}-1}{f-1}\,\frac{p\;v_1\,v_2}{p_k\;v_k{}^2}=1+\varphi=1+\sqrt{1-m}-\frac{1-m}{2}$$

or

$$\frac{\varrho_1\varrho_2}{\varrho_k{}^2}=\frac{\left(\frac{f}{m}-1\right)\pi}{(f-1)(1+\varphi)}$$

This may be considered as a second empiric formula which gives the product of the reduced densities. As now both the sum and the product of the reduced densities is known, they are given by the quadratic equation:

$$\left(\frac{\varrho}{\varrho_k}\right)^2-2\left[1+\gamma(1-m)\right]\left(\frac{\varrho}{\varrho_k}\right)+\frac{\left(\frac{f}{m}-1\right)\pi}{(f-1)(1+\varphi)}=0.$$

At T_k the factor of $\frac{\varrho}{\varrho_k}=2$ and the known term $=1$, and so we find both $\frac{\varrho_1}{\varrho_k}=1$ and $\frac{\varrho_2}{\varrho_k}=1$. But at lower temperatures the factor of $\frac{\varrho}{\varrho_k}$ becomes greater, the factor of the known term, on the other hand, much smaller, chiefly on account of the quantity π, which may be represented approximatively by $e^{-f\frac{1-m}{m}}$.

The relation which has served to form the known term, viz.

$$\frac{\left(\frac{Tdp}{pdT}-1\right)pv_1v_2}{(f-1)\,p_kv_k{}^2}=1+\varphi$$

gives occasion for the following remarks for lower temperatures. If we think m decreased to within the neighbourhood of $^1/_2$, in which case the vapour phase follows almost quite the laws of BOYLE and GAY-LUSSAC, $pv_2=RT=mRT_k$. If we substitute this value of pv_2 in the above equation, we find:

$$\frac{\left(\frac{Tdp}{pdT}-1\right)mRT_kv_1}{(f-1)\,p_xv_x{}^2}=1+\varphi.$$

The quotient $\frac{RT_k}{p_kv_k}$ has been determined for a number of substances by SYDNEY YOUNG and others. Henceforth we shall denote this relation by s. By introduction of this quantity the given relation becomes:

$$\frac{\left(\frac{T}{p}\frac{dp}{dT}-1\right)}{f-1}\, sm\, \frac{v_1}{v_k} = 1+\varphi$$

According to the rule of the rectilinear diameter the value of $\frac{v_1}{v_k}$ is then equal to $\frac{1}{2\,[1+\gamma(1-m)]}$ with a high degree of approximation, because the vapour-density may be neglected by the side of the liquid density; and writing for $\left(\frac{T}{p}\frac{dp}{dT}-1\right)m$ the value $f'-m$, choosing the symbol f' to denote that at lower temperatures the value of f has risen somewhat above that which this quantity has at T_k, we get:

$$\frac{f'-m}{f-1}\,\frac{s}{2\,[1+\gamma(1-m)]} = 1+\varphi$$

So we have here a relation which must exist between the 4 quantities, viz. f', s, γ, and φ at lower temperatures when the rule of the rectilinear diameter holds. If we use this relation with f' little above f $(f=7)$, $s=3.7$ and $\gamma=0.8$, we find the value $\frac{13}{12}\frac{3.7}{2.8}=1.43$ for $1+\varphi_{\frac{1}{2}}$, m being $\frac{1}{2}$. The calculation of φ from the value $\varphi=\sqrt{1-m}-\frac{1-m}{2}$ yields $1+\varphi_{\frac{1}{2}}=1.447$. This equation for the calculation of φ however, cannot be used to predict the course of φ with certainty for still lower temperatures. To do this we should have to know among others f'. If for $m=0$ the value of φ should still correspond to the given equation, and be equal to $\frac{1}{2}$, f' would have to have risen to nearly 9 for $m=0$.

In passing I draw attention to the equality or almost perfect equality of s and $2(1+\gamma)$. So the rule holds either with perfect validity or with a high degree of approximation that as many times as in the critical state the density is greater than would be the case according to the laws of the perfect gases, the limiting density of the substance would be greater than the critical density.

The relation:

$$\frac{\frac{f}{m}-1}{f-1}\,\pi v_1 v_2 = 1+\varphi$$

does not only draw our attention to some properties of the coexisting phases of a substance which we should else hardly notice, but enables us also to calculate beforehand the temperature at which they occur. As an example I point to what follows. For T_k of course $\pi r_1 r_2 = 1$. If we put the question what this product is for other temperatures it appears that with decreasing temperature this product rises at first, reaches a maximum value at a certain value of T_1, has descended again to 1 at a certain other temperature T_2, and falls below, with further decrease of the temperature.

Let us first calculate T_2. For the determination of the value of m belonging to this temperature we have then:

$$\frac{f}{m} - 1 = (f-1)(1+\varphi) = (f-1) + \varphi(f-1)$$

or

$$f\frac{1-m}{m} = (f-1)\left\{\sqrt{1-m} - \frac{1-m}{2}\right\}$$

or

$$\frac{f}{f-1}\frac{\sqrt{(1-m)}}{m} + \frac{\sqrt{(1-m)}}{2} = 1$$

or

$$\left(\frac{f}{f-1} + \frac{m}{2}\right)\frac{\sqrt{1-m}}{m} = 1$$

For $m = 0{,}75$ the first member is $(1{,}541)\frac{1}{1{,}5}$, and so somewhat greater than the second member, and for $m = 0{,}7975$, for which $1-m = 0{,}2025$ and $\sqrt{1-m} = 0{,}45$, the first member is appreciably smaller. So for m somewhat greater than 0,75 $\pi r_1 r_2$ will have to be equal to 1.

Now for $m = 0{,}7371$ we find for ether $\pi = 0{,}088465$, $r_1 = 0{,}4033$, and $r_2 = 28{,}3$.

and for $m = 0{,}7798$ we find $\pi = 0{,}14744$, $r_1 = 0{,}4209$ and $r_2 = 17{,}1$.

The value of $\pi r_1 r_1$ is equal to 1,02 for $m = 0{,}7371$ with these data, and equal to 1,06 for $m = 0{,}7798$. So according to the observations the temperature, at which $\pi r_1 r_2 = 1$, would lie somewhat below 0,7371. But according to the observations for $m = 0{,}6866$ the product $\pi r_1 r_2$ has already far decreased below 1, viz. to 0,93.

As to the determination of T_1, the temperature at which the product $\pi r_1 r_2$ has a maximum value, this lies at about $m = 0{,}9$. The observations for ether give for the value of this product:

m	0,9728	0,9573	0,9277	0,8923	0,8472
$\pi r_1 r_2$	1,14	1,16	1,165	1,173	1,13

It is easy to see that it can be calculated with a sufficient degree of approximation by determining at what value of m the product $m(1+\varphi)$ has a maximum value. That a maximum value for $m(1+\varphi)$ exists, is to be seen a priori. With decrease of m below 1, $1+\varphi$ increases comparatively rapidly, and though m decreases, the product becomes greater. Afterwards $1+\varphi$ remains almost constant, and m continuing to decrease at the same rate, the product decreases. If we now calculate $m(1+\varphi)$, we find for:

$1-m$	0,01	0,04	0,09	0,16
$m(1+\varphi)$	1,08	1,13	1,14	1,09

So sufficient agreement.

Now if we reverse the roots in the quadratic equation, which serves for the calculation of $\frac{\varrho}{\varrho k}$, the new equation serves for the calculation of r_1 and r_2. We find then e.g.

$$\pi \frac{r_1+r_2}{2} = [1+\gamma(-m)] \frac{f-1}{f-m} m(1+\varphi)$$

In the πr diagram $\frac{r_1+r_2}{2}$ is the abscissa of the point which lies halfway on the straight line of evaporation, and π the ordinate of that point. For the critical point both members of this equation are equal to 1, but there exists a value of m, for which also the second member is equal to unity, and so the point belonging to this temperature lies on the same hyperbola as the critical point. For $m=0{,}64$ the second member is equal to 1,09, and for $m=0{,}51$ equal to $\frac{1}{1{,}08}$. Between $m=1$ and the value of m, for which $m\frac{r_1+r_2}{2}$ is again equal to 1, the point lying halfway the nodal line lies inside the hyperbola — at lower temperatures on the other hand outside it. More such particulars might be observed. But the particulars mentioned are certainly sufficient to set forth the significance of the discussed equation.

If we know $\frac{\varrho_1+\varrho_2}{2\varrho k}$ and $\frac{\varrho_1\varrho_2}{\varrho k^2}$, of course $\frac{\varrho_1-\varrho_2}{2\varrho k}$ is also known, a quantity for which it is often necessary to know how it depends on m.

Then we find:

$$\left(\frac{\varrho_1-\varrho_2}{2\varrho k}\right)^2 = [1+\gamma(1-m)]^2 - \frac{(f-m)}{(f-1)} \frac{\pi}{m(1+\varphi)}$$

I have calculated the value of $\frac{\varrho_1-\varrho_2}{\varrho k}$ from this formula with $\gamma=0{,}8$

and $f = 7$ for a few values of m, and compared them with the observations.

Thus we find for $1-m = 0,09$ the value 1,73 from the formula for $\frac{\varrho_1-\varrho_2}{\varrho k}$, whereas $\frac{\varrho_1-\varrho_2}{\varrho k} = 1.83$ would follow from the observations at $m = 0.89$, so at slightly lower value of m.

For $1-m = 0,16$ we find the value 2,1 by calculation, while observation yields the value of somewhat more than 2,0 for $m = 0,8472$. Of course at very low temperatures the difference between $\frac{\varrho_1+\varrho_2}{\varrho k}$ and $\frac{\varrho_1-\varrho_2}{\varrho k}$ is not worth mentioning, and entirely neglecting $\frac{\varrho_2}{\varrho k}$ for $m = {}^1/_2$, we should find 2,8.

For values of m very near 1 we might draw up the following approximative formula:

$$\left(\frac{\varrho_1-\varrho_2}{2\varrho k}\right)^2 = \sqrt{1-m} + f(1-m)$$

For $1-m = 0,01$ this formula yields:

$$\frac{\varrho_1-\varrho_2}{2\varrho k} = \sqrt{0,17}$$

or

$$\frac{\varrho_1-\varrho_2}{\varrho k} = 0,82$$

the observations yielding the value of 0,775 for $m = 0,9915$, so for somewhat higher value of m. For $1-m = 0,04$ this approximative formula gives $\frac{\varrho_1-\varrho_2}{\varrho k} = 1,384$. The observations gave the value 1,345 for $m = 0,9573$. So there is sufficient agreement.

In my thermodynamic theory of capillarity I came to the conclusion that the capillary constant is proportional to $\left(\frac{\varrho_1-\varrho_2}{\varrho k}\right)^3$ in the neighbourhood of the critical temperature, and not knowing the quantity φ at the time, I ascribed the form $\sqrt{1-m}$ to $\frac{\varrho_1-\varrho_2}{\varrho k}$. Now we have found a more complicated form for $\frac{\varrho_1-\varrho_2}{\varrho k}$, which however passes into $\sqrt{1-m}$ if $\varphi = 0$. For exceedingly small value of $1-m$ $\frac{\varrho_1-\varrho_2}{\varrho k}$ appears to be proportional to $(1-m)^{3/4}$, but then the amount of the capillary constant is so small that it cannot be measured accurately.

For somewhat higher value of $1-m$ also the term $f(1-m)$ begins to make its influence felt in the expression:

$$\frac{\varrho_1-\varrho_2}{2\varrho_k}=[\sqrt{1-m}+f(1-m)]^{1/2},$$

and then the capillary constant is neither proportional to $(\sqrt{1-m})^{3/2}$, nor to $(1-m)^{3/2}$, as would be the case for $\varphi=0$. Over a great range of temperature there is found proportionality with

$$(1-m)^{1.23}$$

so with an exponent which is greater than $^3/_4$ and smaller than $^3/_2$. So that what was inexplicable about the exponent 1,23 has been solved for the greater part. But really this whole calculation would now have to be revised, which, however, I shall have to put off.

IV. *The critical quantities.*

If in the equation of state a and b are put constant, and if we do not admit any other influence on the behaviour of a substance, we find:

$$v_k=3b \quad RT_k=\frac{8}{27}\frac{a}{b},\quad p_k=\frac{1}{27}\frac{a}{b^2},\quad \frac{p_k v_k}{RT_k}=\frac{3}{8},\quad \frac{RT_k}{p_k}=8b,\quad \left(\frac{T}{p}\frac{dp}{RT}\right)_{kr}=4$$

$$\frac{a}{v_k\,RT_k}=\frac{9}{8},\quad \frac{v_k}{v_k-b}=\frac{3}{2}.$$

A change takes place in this if we put b variable with the volume. Then there is question of b_g, by which I shall denote the value of b for infinitely large volume. The value of $\frac{b}{b_g}$ is derived by means of theoretical considerations, and found to have the form:

$$\frac{b}{b_g}=F\left(\frac{b_g}{v}\right)$$

but this function is not entirely known. If we develop $F\left(\frac{b_g}{v}\right)$ in a series of ascending powers of $\frac{b_g}{v}$, there are at most a few coefficients of these therms known. Enough however, to show that the factor of $v_k=rb_g$ is not 3, but lies much nearer to 2 than to 3. But then also the other coefficients occurring in the value of the critical quantities, must change. Thus we find $\frac{R}{(v-b)_{kr}}$ from $\left(\frac{dp}{dT}\right)_{kr}$ in the critical point; in this expression v is much smaller than 3, and b only little smaller than b_g; and if in the equation of state we substitute $\left(\frac{dp}{dT}\right)_{v_k}$, for which $\frac{dp}{dT}$ can be put as the value of tension of the saturate vapour in the critical point, we find $p_k = \frac{a}{b_g^2}\frac{1}{r^2(f-1)}$, if f represents the value of $\left(\frac{T}{p}\frac{dp}{dT}\right)_{kr}$. It has already been remarked that f will have to be much larger than 4; but it is remarkable that $r^2(f-1)$ will be found exactly or almost equal to 27. As f is known through the observations, and may be put equal to 7, we find then $\sqrt{4.5}$ or slightly more than 2.124 for r; and this hardly changes if f is made to descend to 6.7 — then we find $r = 2.17$. It was, indeed, to be expected a priori that the factors occurring in the two quantities which are generally used for the calculation of a and b_g, viz. p_k and RT_k, will have the same value, or almost the same value as was originally given by me. For the values of a and b_g thus calculated appeared to be satisfactory. Using the factors r and f, and the above-mentioned factor s, we have:

$$\alpha \ldots\ldots v_k = rb_g$$

$$\beta \ldots\ldots \frac{p_k v_k}{RT_k} = \frac{1}{s}$$

$$\gamma \ldots\ldots \frac{RT_k}{p_k} = rs\, b_g$$

$$\delta \ldots\ldots \left(\frac{T}{p}\frac{dp}{dT}\right) = f$$

$$\alpha' \ldots\ldots p_k = \frac{a}{bg^2}\frac{1}{r^2(f-1)}$$

$$\beta' \ldots\ldots RT_k = \frac{a}{bg}\frac{s}{r(f-1)}$$

$$\gamma' \ldots\ldots \frac{a}{v_k RT_k} = \frac{f-1}{s}$$

$$\delta' \ldots\ldots \left(\frac{v}{v-b}\right)_{kr} = \frac{f}{s}\,.$$

A similar remark as I have made for p_k, viz. that the factor $\frac{1}{27}$ holds now too, may also be made for RT_k. Now too $RT_k = \frac{8}{27}\frac{a}{bg}$ holds. To show this we must calculate $\frac{s}{r(f-1)} = \frac{sr}{r^2(f-1)} = \frac{sr}{27}$. So we must find $sr = 8$. If we put r again equal to 2.124, we find $s = 3.77$, a value lying quite in the series of little diverging values, which SYDNEY YOUNG found for s for normal substances. I do not mean to assert that the factors $\frac{1}{27}$ and $\frac{8}{27}$ will always hold with perfect validity; this would require a further investigation, also an experimental one. But yet I found in the above remarks an indication pointing to the fact that if I would explain the whole behaviour of liquids, and wanted to assume the existence of molecule complexes for this purpose, I ought to examine whether the number of molecules which combines to a complex, might not be such that, even if there is quasi-association in the critical state, this quasi-association is without influence on the critical quantities. For the above list of quantities has been drawn up by assuming the equation of state as perfectly valid in the critical state without any further addition.

According to the above the whole cause of the deviation of the critical constants is to be found in the existence of the variability of b with the volume. This law of variability may perhaps be different for the different shapes of molecules, and then this might give rise to the deviations from the law of the corresponding states too. If according to a somewhat different law of variability r becomes smaller, f and s must become greater. Above we concluded to the constancy of $r^2(f-1)$ and rs — without my feeling justified in considering this perfect constancy as convincingly proved. Else we might add $\frac{s^2}{f-1}=\frac{64}{27}$, which of course would yield $s=3.77$ again with $f=7$.

V. *Quasi association.*

I have tried to account for the existence of the quantity φ by assuming that molecule complexes may form in the substance, which behave as a simple molecule from a kinetical point of view without their being simple molecules from a chemical standpoint. The quantity of lost energy in the complex is then entirely due to the ordinary molecular attraction, and is equal to the limiting value of the internal latent heat. So much smaller than when real chemical combination to double or multiple molecules took place. By way of distinction I speak of *quasi*-association, though from a physical point of view there would be little reason for this distinction.

If we put the quantity of substance we deal with $=1$, and assume the fraction $1-x$ to be present as simple molecules, and the fraction x as molecule complexes, formed by combination of n simple molecules to a complex molecule, the number of molecules has decreased from 1 to $1-\frac{n-1}{n}x$ from a kinetical point of view. So a first change must be made in the equation of state by multiplication of RT by the factor $1-\frac{n-1}{n}x$. Further if n is large, the quantity a will have to be subjected to a change. It is true that I preserved the quantity a unchanged in (Théorie Moleculaire § 29 Cont. 2, p. 29), where I treated a similar problem for real association to double molecules.

There I started from the consideration that if in part of the space occupied by the substance the quantity of lost energy is $=-\frac{a}{b}$, this lost energy is to be considered as a middle value. In this space the substance moves. One moment a point of the considered space is empty, the next moment it is filled. If the molecules suddenly all passed into double molecules, the time during which a point may be considered as filled would be twice as small, but filled with double the quantity of substance. Accordingly the forces which have come into play to form the double molecules, are new forces, and are not derived from part of the molecular forces. But this consideration holds no longer for the case considered now, where n will have to be taken as large, and where the forces forming the complex, are the molecular forces of attraction themselves. I shall come to the conclusion that n approaches somewhat to the number for which one molecule is surrounded in all directions by another, — so the number of spheres that can touch a given sphere at the same time. And then part of the molecular forces of the molecules of the covering layer is directed inward, and so this serves to keep the complex together, only the forces of the outer layer working outward remaining active to serve as internal pressure, to keep in conjunction with the external pressure p the moving substance together. So I shall multiply a by the factor $\{1-(1-k)x\}^2$ — and put later on $1-k=\frac{1}{2}$ as probable.

In this way we get the form $a[1-(1-k)x]^2$. The contribution to the cohesion constant of $(1-x)$ simple molecules is $a(1-x)^2$. The attraction between the complex molecules and the substance in simple-molecule form is equal to $akx(1-x)$, in which k is smaller than 1. And the inverse attraction is just as great, while the complex molecules attract each other with an amount equal to ak^2x^2. Together this gives the indicated amount. [1])

Is b also to be modified? Already the consideration that the com-

[1]) Now that we always deal with the same substance, and so the specific qualities have disappeared, we may assume $a_1a_2=a_{12}^2$.

plex molecules may be considered as the same substance but in a more condensed state, and that in a more condensed state b is smaller, shows that there will be question of a value $\left(\frac{db}{dx}\right)_v$, and that this quantity will be negative. But the difficulty to find the exact form of b if the complex molecules should not exist, and which is greatly exhanced if they do exist, have led me to neglect $\left(\frac{db}{dx}\right)_v$ at least for the present. So this is one of the causes why this investigation cannot be considered as quite completed, but I think that this will only exert influence on some details. Hence the equation of state used will have the form:

$$p = \frac{RT\left(1 - \frac{n-1}{n}x\right)}{v-b} - \frac{a\{1-(1-k)x\}^2}{v^2}$$

in which b is an unknown function of v and x. But the dependence on x will be neglected in the applications.

VI. *Determination of the value of x for given v and T.*

If we determine the value of ψ (Théorie Moléculaire § 14 Cont. 2, p. 28 et. seq.) for constant values of x chosen arbitrarily, we find the value of x, which corresponds in the state of equilibrium to given value of v and T, by putting $\left(\frac{d\psi}{dx}\right)_{vT} = 0$.

We have successively:

$$\psi = -RT\left[1 - \left(\frac{n-1}{n}\right)x\right]\int\frac{dv}{v-b} - \frac{a\,[1-(1-k)\,x]^2}{v} +$$

$$+ RT\left\{(1-x)\,l\,(1-x) + \frac{x}{n}\,lx\right\} + E_1\,(1-x) + E_2\,x - T\,[H_1\,(1-x) + H_2]$$

and

$$0 = RT\,\frac{n-1}{n}\int\frac{dv}{v-b} - 2\,(1-k)\,a\,\frac{1-(1-k)\,x}{v} +$$

$$+ RT\left\{\frac{lx}{n} - l\,(1-x) - \frac{n-1}{n}\right\} + E_2 - E_1 - T\,(H_2 - H_1).$$

The quantity $E_2 - E_1$ is equal to the limiting value of the internal latent heat, but we shall represent it by $-E$. The quantity $H_2 - H_1$ must be considered as unknown. If it were known, $\left(\frac{d\psi}{dx}\right)_{vT} = 0$

might serve to determine the value of x for any value of v and T. So also the value of x_{kr} by putting $T = T_k$ and $v = v_k$. Reversely this constant can be calculated if x is known in any volume for given T.

Let us on the other hand determine the values of $T\frac{dp}{dT} - p$ for coexisting phases. Then this value being equal to $\frac{\varepsilon_2 - \varepsilon_1}{v_2 - v_1}$ we must be able to express the value of the energy ε in the quantities defining this phase. If there was no quasi-association, the energy would be equal to $-\frac{a}{v}$, to which a function of T would have to be added, which, however, would disappear again in the difference $\varepsilon_2 - \varepsilon_1$. Now that there is quasi-association, the value of a for this phase must be used, viz. $a\,[1-(1-k)x]^2$, to which $-Ex$ is then to be added.

We get then:

$$T\frac{dp}{dT} - p = \frac{E(x_1 - x_2) + \dfrac{a\,[1-(1-k)\,x_1]^2}{v_1} - \dfrac{a\,[1-(1-k)\,x_2]^2}{v_2}}{v_2 - v_1}$$

or

$$\left(\frac{Tdp}{pdT} - 1\right)\frac{p\,v_1v_2}{a} = \frac{\dfrac{E}{a}(x_1 - x_2)\,v_1v_2}{v_2 - v_1} + \frac{v_2[1-(1-k)x_1]^2 - v_1[1-(1-k)x_2]^2}{v_2 - v_1}$$

or

$$\left(\frac{Tdp}{pdT} - 1\right)\frac{pv_1v_2}{a} = \frac{Ev_k}{a}\frac{x_1 - x_2}{\dfrac{\varrho_1 - \varrho_2}{\varrho_k}} + \frac{\varrho_1 + \varrho_2}{2(\varrho_1 - \varrho_2)}\{[1-(1-k)x_1]^2 - [1-(1-k)x_2]^2\} +$$

$$+ \frac{1}{2}\{[1-(1-k)x_1]^2 + [1-(1-k)x_2]^2\}$$

or

$$\left(\frac{Tdp}{pdT} - 1\right)\frac{pv_1v_2}{a} = \frac{x_1 - x_2}{\varrho_1 - \varrho_2}\varrho_k\left\{\frac{Ev_k}{a} - (1-k)\frac{\varrho_1 + \varrho_2}{\varrho_k}\left[1 - \frac{1-k}{2}(x_1 + x_2)\right]\right\} +$$

$$+ 1 - (1-k)(x_1 + x_2) + \frac{(1-k)^2}{2}(x_1^2 + x_2^2).$$

In the absence of association the second member is simplified to 1, and so the quantity φ disappears.

Let us take the extreme case. For T_k $x_1 = x_2 = x_k$ and $v_1 = v_2 = v_k$ or $\varrho_1 = \varrho_2 = \varrho_k$. For $\frac{x_1 - x_2}{\varrho_1 - \varrho_2}$ we must then take $\left(\frac{dx}{d\varrho}\right)_{k}$.

We have then:

$$\left(\frac{T\,dp}{p\,dT}-1\right)_{kr}\frac{p_k v_k^2}{a}=\left(\frac{dx}{d\varrho}\right)_{kr}\varrho_k\left\{\frac{Ev_k}{a}-2(1-k)[1-(1-k)x_k]\right\}+$$
$$+\,1-2(1-k)x_k+(1-k)^2\,x_k^2.$$

Then too the second member is equal to 1 in the absence of association.

This would also be the case if in the critical state x_k and $\left(\frac{dx}{d\varrho}\right)_k$ should be equal to zero. But though it is probable that x_k is small, $\left(\frac{d\psi}{dx}\right)_{vT}=0$ argues against the assumption $x_k=0$. In this case $Nep.\,log\,x_k$ is equal to $-\infty$, and this equation cannot be satisfied. I then asked myself whether it could be possible that all the terms which occur in the second member besides 1, could be together $=0$. Then $\frac{d\varepsilon}{dv_T}$ is equal to $\frac{a}{v_k^2}$ in the critical point, both when there is association and when there is not, and in the same way $T\left(\frac{dp}{dT}\right)$ $-p$ is equal to $\frac{a}{v_k^2}$, or $\frac{T\,dp}{p\,dT}-1=\frac{a}{pv_k^2}$ and so $(f-1)=\frac{a}{p_k r^2 b_g^2}$ or $p=\frac{a}{b_g^2}\frac{1}{(f-1)\,r^2}$. Though the quantities r, s, and f should be slightly changed in consequence of the existing association, the before discussed relations between them, and their relations with the critical quantities would continue to exist. Only in the quantity $\left(\frac{v}{v-b}\right)_k$, which is little known as yet, a change comes in case of association, which change I will discuss later on, after I shall have shown what follows with regard to n from the mentioned assumption.

VII. *Possible value of n.*

Let us examine the obtained formula for $\left(\frac{d\psi}{dx}\right)_{kr}=0$ more closely. It can be written in the form:

$$-\frac{\int\frac{dp}{dx_{vT}}dv}{RT}+\left\{\frac{lx}{n}-l(1-x)\right\}-\frac{E}{RT}=constant$$

and leads to the differential equation:

$$-\frac{\left(\dfrac{dp}{dx}\right)_{vT}}{RT}\,dv + dx\left\{\frac{1}{nx}+\frac{1}{1-x}-\frac{\displaystyle\int\frac{d^2p}{dx^2{}_{vT}}\,dv}{RT}\right\}+\frac{dT}{T}\left\{\frac{E+\dfrac{1}{v}\dfrac{da_x}{dx}}{RT}\right\}=0$$

or

$$\frac{dv}{v}\left\{\frac{n-1}{n}\frac{v}{v-b}-\frac{2a(1-k)[1-(1-k)x]}{vRT}\right\}+dx\left\{\frac{1}{nx}+\frac{1}{1-x}-\frac{2a\,(1-k)^2}{vRT}\right\}+$$

$$+\frac{dT}{T}\,\frac{E-\dfrac{2a\,(1-k)\,[1-(1-k)\,x]}{v}}{RT}=0$$

To find $\left(\dfrac{dx}{d\varrho}\varrho\right)_{kr}$ or $-\left(\dfrac{dx}{dv}v\right)_{kr}$, we must put $dT=0$ in this differential equation, and take the critical values for the other quantities.

We find then the following intricate equation:

$$\frac{\dfrac{\dfrac{Ev_k}{a}-2(1-k)[1-(1-k)x]\left\{\dfrac{n-1}{n}\,\dfrac{1+(f-1)[1-(1-k)x]^2}{1-\dfrac{n-1}{n}x}-2(1-k)n(f-1)[1-(1-k)x]\right\}}{s}}{1+\dfrac{nx}{1-x}-nx2\,(1-k)^2\,\dfrac{f-1}{s}}=$$

$$=2\,(1-k)+x\,(1-k)^2.$$

To get an idea about the value of n from this equation, as x is no doubt small in the critical state, we may put this quantity $=0$. In the denominator nx does occur, and if we put $x=0$, it may seem as if nx was neglected. But $\dfrac{nx}{1-x}-\dfrac{nx2(1-x)^2}{s}(f-1)$ may be probably also neglected if nx should not be small. Then this equation becomes:

$$\left\{\frac{Ev_k}{a}-2(1-k)\right\}\left\{(n-1)f-2\,(1-k)\,n\,(f-1)\right\}=2\,(1-k)\,s$$

or

$$n\,\{f-2\,(1-k)\,(f-1)\}=s\,\frac{2\,(1-k)}{\dfrac{Ev_k}{a}-2(1-k)}+f.$$

As for E the limiting value of the internal latent heat is to be expected, so $\dfrac{a}{v_{minimum}}$, and the smallest value of v is $2(1+\gamma)$ times contained in v_k according to the rule of the rectilinear diameter, $\dfrac{Ev_k}{a}=2(1+\gamma)$. Hence $\dfrac{Ev_k}{a}$ is always greater than $2(1-k)$. Now in this last equation all the quantities are known except n and $1-k$.

But at any rate a relation has been found between these two quantities, which will have to be fulfilled approximately. The value of $2(1-k)$ cannot be larger than $\frac{f}{f-1}$, for then n would be negative: for $2(1-k)=\frac{f}{f-1}$ we should have $n=\infty$. For $2(1-k)=1$ we find: $n=f+\frac{s}{\frac{Ev_k}{a}-1}$ or $n=7+\frac{3,77}{2,6}$. We had concluded to $1-k=\frac{1}{2}$ for such a value of n, and therefore this value of n seemed probable to me. At any rate I concluded to $n>f$. This conclusion approaches to certainty when we examine the value of $\frac{T}{x}\frac{dx}{dT}$, which is found from the same differential equation at low temperatures for the vapour phase. If v is put very large, the equation is simplified to:

$$\frac{T}{v}\frac{dv}{dT}\frac{n-1}{n}+\frac{T}{x}\frac{dx}{dT}\left\{\frac{1}{n}+\frac{x}{1-x}\right\}+\frac{E}{RT}=0.$$

For the third term $\frac{E}{mRT_k}=\frac{\frac{Ev_k}{a}a}{mRT_kv_k}=\frac{f-1}{s}\frac{Ev_k}{a}\frac{T_k}{T}$ may be written. Supposing the value of x to be very small, because we examine the behaviour of normal substances, and so assuming the equation $pv=RT$ or $\frac{T}{p}\frac{dp}{dT}+\frac{T}{v}\frac{dv}{dT}=1$ for the vapour phase, we get for the above formula

$$\frac{n-1}{n}\left\{1-\frac{T}{p}\frac{dp}{dT}\right\}+\frac{T}{x}\frac{dx}{dT}\frac{1}{n}+\frac{f-1}{s}\frac{Ev_k}{a}\frac{T_k}{T}=0$$

or

$$\frac{T}{x}\frac{dx}{dT}=(n-1)f\frac{T_k}{T}-n(f-1)\frac{\frac{Ev_k}{a}}{s}\frac{T_k}{T}-(n-1).$$

Already before I pointed out the equality or almost equality of $2(1+\gamma)$ and s, from which follows $\frac{\frac{Ev_k}{a}}{s}=1$.

From this follows:

$$\frac{T}{x}\frac{dx}{dT}=[(n-1)f-n(f-1)]\frac{T_k}{T}-(n-1)$$

And as we must conclude to a course of x with T as drawn in fig. 58 (Lehrbuch der Thermodynamik) $\frac{T}{x}\frac{dx}{dT}$ will be positive for the vapour phase at least for low temperatures, or $(n-1) f > n (f-1)$ or $n > f$.

The formula shows the possibility of reversal of sign at higher temperatures, and so a course for which the x of the coexisting vapour phases shows a maximum value and a minimum value, for $\frac{dx}{dT}$ will at any rate have become positive again before T_k is reached. So if we wish to make an accurate determination of the molecular weight at temperatures below T_k by means of the density of the vapour phase, not only a correction is to be applied for the existence of a and b, but it must also be investigated whether also x has an appreciable value — which we can do by heating in constant volume; then the a and the b retain the same value, but the value of x decreases rapidly with rising temperature. As I pointed out before only in this way the too great density of saturated vapour of 100° can be accounted for.

It appears with certainty from the foregoing remarks that if we wish to ascribe the existence of the quantity φ to molecule complexes, which behave from a kinetical point of view as simple molecules, we shall have to put the degree of multiplicity n greater than f, and that we shall have to make a decrease at the same time in such a way that $1-k$ is not very far from $^1/_2$; whether we might also explain the existence of φ by another kind of complexes, in which the number of molecules, present from a kinetical point of view, remains unchanged, or changes in another way, remains an open question. But the difficulty to find an accurate form for the equation of state in this case has made me relinquish such an investigation. If it had only been our purpose to give values for $\frac{Ev_k}{a}$ and x_1 and x_2 satisfying the equation:

$$\left(\frac{Tdp}{pdT}-1\right) p\,\frac{v_1 v_2}{a} = 1+\varphi = \frac{\frac{Ev_k}{a}(n_1 - x_2)}{(\varrho_1-\varrho_2)}\,\varrho_k + \frac{v_2 f(x_1) - v_2 f(x_2)}{v_2 - v_1}$$

we might e.g. choose $f(x_1) = f(x_1) = 1$, so unchanged values of a, and further:

$$\varphi = \frac{Ev_k}{a}\,\frac{x_1 - x_2}{\varrho_1 - \varrho_2}\,\varrho_k$$

If the value of φ were $= \sqrt{(1-m)} - \frac{1-m}{2}$ up to $m=0$, then follows:

$$\frac{1}{2} = \frac{\frac{Ev_k}{a}}{2(1+\gamma)}$$

or

$$\frac{Ev_k}{a} = (1+\gamma)$$

and $x_1 - x_2$ can then be calculated at any temperature by the aid of this value of $\frac{Ev_k}{a}$ and the knowledge of $\frac{\varrho_1 - \varrho_2}{\varrho_k}$. Then for T_k however $\frac{dx}{d\varrho} = 0$ follows, a value which is highly improbable. But there must exist a fixed rule, which indicates in what way x depends on v and T. And this law must be satisfied, if the reduction of the explanation of the value of φ to the existence of the quantity x is to mean anything. This it is what I have attempted to do, though numerous questions remain. We will just give a single instance of such questions. It follows from our assumption on the cause of the existence of the quantity φ, as v_2 is infinite, that:

$$1 + (\varphi)_{m=0} = \frac{\frac{Ev_k}{a}}{2(1+\gamma)} + \{1 - (1-k)\}^2$$

$$1 + (\varphi)_{m=0} = 1 + k^2$$

or

$$(\varphi)_{m=0} = k^2.$$

As according to our assumption $\varphi = \sqrt{1-m} - \frac{1-m}{2}$, and so $\varphi_{m=0}$ would be $= \frac{1}{2}$, and $k^2 = \frac{1}{4}$, we have here a contradiction.

But that this contradiction should be of sufficient weight to reject our explanation, does not seem true to me. Of the course of the quantity φ we do not know anything more than between $m=1$ and m about 0.6. This is only almost perfectly certain that for very small value of $1-m$, the value of φ is chiefly given by $\sqrt{1-m}$. But the theoretical, the real value will no doubt have quite another form, and whether it has been reduced to $\frac{1}{4}$ for $m=0$, cannot be denied with any certainty. Indeed, the value of $1 - m$ cannot rise much above $\frac{1}{2}$ for liquids. Then the solid state sets in.

To the derivation of $n > f$ from the sign of $\frac{T}{x}\frac{dx}{dT}$ in the gas-phase the objection might be made, that in this proof $\frac{T}{p}\frac{dp}{dT}$ was put equal to $\frac{f}{m}$, and not to $\frac{f'}{m}$, as we ourselves put before. In spite of this I have retained this proof, because it seemed suggestive to me in many respects. But to exclude doubt about the great value of n I shall indicate a strict course of reasoning.

According to the definition of φ: $\frac{f'-m}{f-1}\frac{s}{2\,[1+\gamma\,(1-m)]} = 1 + \varphi_m$ at temperatures for which the vapour phase does not appreciably deviate from $pv_2 = RT$, as we already indicated before.

The condition that $\frac{Tdx}{xdT}$ be positive, is:

$$\frac{n-1}{n}\left(\frac{T}{p}\frac{dp}{dT} - 1\right) > \frac{f-1}{s} E \frac{v_k}{a} \frac{1}{m}$$

or

$$\frac{n-1}{n}(f'-m) > \frac{f-1}{s} E \frac{v_k}{a}.$$

And so:

$$\frac{n-1}{n}(1+\varphi_m) > \frac{E\frac{v_k}{a}}{2\,[1+\gamma\,(1-m)]}$$

As the value of $\frac{T}{x}\frac{dx}{dT}$ will certainly have to be positive for $m = 0$

$$\frac{n-1}{n}(1+\varphi_0) > \frac{E\frac{v_k}{a}}{2\,(1+\gamma)}.$$

Now we found above for the value of the second member:

$$\frac{E\frac{v_k}{a}}{2\,(1+\gamma)} = 1 + \varphi_0 - k^2.$$

Hence

$$\frac{n-1}{n}(1+\varphi_0) > 1 + \varphi_0 - k^2$$

or

$$\frac{n-1}{n} > 1 - \frac{k^2}{1+\varphi_0}$$

or

$$\frac{k^2}{1+\varphi_0} > \frac{1}{n}$$

or

$$n > \frac{1+\varphi_0}{k^2}$$

must hold. With $\varphi_0 = \frac{1}{2}$ and $k^2 = \frac{1}{4}$ this relation becomes:

$$n > 6$$

Neglect of $\left(\frac{db}{dx}\right)_{vT}$ has no influence on the derivation of the amount of n by means of properties of the gas phase. On the other hand the existence or non-existence of $\left(\frac{db}{dx}\right)_{vT}$ *has* influence on the derivation of the value of n from the critical phase. Negative value of $\left(\frac{db}{dx}\right)_{vT}$ makes n decrease in this derivation. The fact that these two results agree so closely gives rise to the supposition that this quantity, if it exists, will be small.

I will add here a single remark about the value of the quantity $\frac{Ev_k}{a}$. I have called E the limiting value of the internal latent heat, and derived from it that $\frac{Ev_k}{a}$ is equal to $2(1+\gamma)$. From the equation:

$$1+\varphi_{m=0} = \frac{\frac{Ev_k}{a}}{2(1+\gamma)} + k^2$$

follows, if we continue to assume $\varphi = \sqrt{1-m} - \frac{1-m}{2}$,

$$\frac{\frac{Ev_k}{a}}{2(1+\gamma)} = \frac{5}{4}$$

For a moment I have assumed the appearance as if I agreed that this implied a contradiction. If we, however, consider that the internal latent heat is greater for a mass of spherical shape than for one bounded by a plane, and that this can and will become noticeable when the sphere has such a small radius as is the case for our complex molecules, it appears that we might have expected beforehand that $\frac{Ev_k}{a}$ would be $> 2(1+\gamma)$. In the estimations about molecular dimensions (Chapter X Continuity) I have arrived at the conclusion that the ratio $\frac{H}{K}$ is of the same value as the diameter of a molecule. According to the usual formula for the molecular pressure

$$K + \frac{H}{R} = K \left\{ 1 + \frac{\frac{H}{K}}{R} \right\},$$ if we should also preserve it unchanged for the case that R has decreased to so small an amount, but now that these complex molecules are surrounded by matter of normal density, we need not wonder that $1 + \frac{\frac{H}{K}}{R}$ has increased to $^5/_4$.

VIII. *Formulae indicating the relation between x_1 and x_2 at given temperature.*

In the case that there is no quasi-association, there are only two unknown quantities, viz. v_1 and v_2 at given temperature, and we have two equations to determine these unknown quantities, viz.:

$$p = p'$$

and

$$\int_{v_1}^{v_2} p dv = p(v_2 - v_1)$$

This latter equation may also be written:

$$\mu = \mu'$$

In case of quasi-association there are 4 unknown quantities, viz. v_1, v_2, x_1, and x_2. If we thought it feasible to solve the value of x from $\left(\frac{d\psi}{dx}\right)_T = 0$, and express it in v and T, that value of x could be substituted in p and then p could be integrated with respect to v, and then there were again only two unknown quantities, and two equations for the determination. Since this elimination of x is not possible, we must retain the 4 unknown quantities, and so we also require 4 equations to determine them. Indeed, there is now question of the thermodynamic potential of the simple molecules, and of that of the complex molecules. Let us call the former μ_1 for the liquid, and $(\mu_1)'$ for the vapour, — in the same way the latter μ_2 and $(\mu^2)'$. Then we have for the determination of the 4 unknown quantities:

$$p = p'$$
$$\mu_1 = (\mu_1)'$$
$$\mu_2 = (\mu_2)'$$

to which we add as the 4th equation $\left(\frac{d\psi}{dx}\right) = 0$ or $\mu_1 = \mu_2$ or $(\mu_1) = (\mu_2)'$.

Then there still remains an obstacle to the accurate determination in the unknown constant which occurs in $\left(\frac{d\psi}{dx}\right)_{vT}$, but this would also be the case if we could have solved x from $\left(\frac{d\psi}{dx}\right)_{vT} = 0$. The matter, however, is simplified in so far that we may consider the two quantities v_1 and v_2 as known by the two given empirical formulae, and so there are only x_1 and x_2 left to be determined.

The functions which then can be used for the determination of x_1 and x_2 are:

1. $\left(\frac{d\psi}{dx}\right) = 0$ or $\mu_2 - \mu_1 = 0$. When we put

$$(\mu_2 - \mu_1) = (\mu_2 - \mu_1)'$$

the unknown constant disappears, and we get:

$$RT\left(\frac{n-1}{n}\right)\int^{v_1}\frac{dv}{v-b} - \left(\frac{da}{dx}\frac{1}{v}\right)_1 + RT\left\{\frac{lx}{n} - l(1-x)\right\}_1 =$$

$$= RT\frac{n-1}{n}\int^{v_2}\frac{dv}{v-b} - \left(\frac{da}{dx}\frac{1}{v}\right)_2 + RT\left\{\frac{lx}{n} - l(1-x)\right\}_2$$

2.
$$\mu_1 = (\mu_1)'$$

As $\mu_1 = \psi - x\left(\frac{d\psi}{dx}\right)_{vT} + pv$

$$\mu_1 = pv - RT\int\frac{dv}{v-b} - \frac{a_x - x\frac{da_x}{dx}}{v} + RTl(1-x) + xRT\frac{n-1}{n}$$

3.
$$\mu_2 = (\mu_2)'$$

As $\mu_2 = \psi + (1-x)\left(\frac{d\psi}{dx}\right)_{vT} + pv$

$$\mu_2 = pv - \frac{RT}{n}\int\frac{dv}{v-b} - \frac{a_x - x\frac{da_x}{dx} + \frac{da_x}{dx}}{v} + RT\frac{lx}{n} - (1-x)RT\frac{n-1}{n}$$

4. $$'n\mu_2 = (n\mu_2)'$$

As both in μ_1 and in $n\mu_2$ the quantity $\int \frac{dv}{v-b}$ occurs, which cannot be replaced by $l(v-b)$, and which therefore prevents an accurate application, we can make this quantity disappear by calculating $n\mu_2 - \mu_1$; so we use:

5. $$(n\mu_2 - \mu_1) = (n\mu_2 - \mu_1)'$$

We find for the value of $n\mu_2 - \mu_2$, or $(n-1)\mu_1$:

$$(n-1)pv - (n-1)\frac{a_x - x\frac{da_x}{dx}}{v} - n\frac{\frac{da_x}{dx}}{v} + RTl\frac{x}{1-x} - RT(n-1)\left[1 - \frac{n-1}{x}x\right].$$

Each of these forms can of course, also be used to calculate $\left(\frac{dx}{d\varrho}\right)_{kr}$, e.g. the last form by differentiating with respect to x and v, and keeping T and p constant. Of course we then find back the value obtained before.

The given relations may particularly serve to calculate the ratio of x and x_2 in coexisting phases. We shall use the last equation:

$$(n\mu_2 - \mu_1) = (n\mu_2 - \mu_1)'$$

to demonstrate that for temperatures for which the vapour phase may be considered as a perfect gas the value of x_2 has decreased to an exceedingly small amount. For the gas phase $pv = RT$ may then be put. We find then:

$$(n-1) + l\frac{x_2}{1-x_2} + \frac{(n-1)^2}{n}x_2 = l\frac{x_1}{1-n_1} + \frac{(n-1)^2}{n}x_1 - \frac{(n-1)\left(a_x - x\frac{da_x}{dx}\right) + x\frac{da_x}{dx}}{vRT}.$$

The value of:

$$(n-1)\left(a_x - x\frac{da_x}{dx}\right) + n\frac{da_x}{dx} = a\,[1-(1-k)\,x]\,[n-1-2n\,(1-k) + (n-1)\,(1-k)\,x],$$

or with $1 - k = \frac{1}{2}$ equal to $a\left[1 - \frac{x}{2}\right]\left[-1 + \frac{n-1}{2}x\right]$.

With this the relation between x_1 and x_2 becomes:

$$l\frac{x_1}{1-x_1}\,\frac{1-x_2}{x_2} + \frac{(n-1)^2}{n}(n_1 - x_2) = n-1 + \frac{(f-1)\,2\,[1+\gamma\,(1-m)]}{sm}\left(1 - \frac{x_1}{2}\right)\left(-1 + \frac{n-1}{2}x_1\right).$$

A first conclusion to which this equation leads is that for such values of m $\left(\text{e.g. } m = \frac{1}{2}\right)$ the quantity $-1 + \frac{n-1}{2}x_1$ must be positive, or $x_1 > \dfrac{2}{n-1}$. We must come back to this later on, and will only state here that for $m = \dfrac{1}{2}$, and lower, $x_1 > \dfrac{1}{2}$.

With $x_1 = 0{,}6$ and $m = \dfrac{1}{2}$ we find for x_2 a value of the order of 10^{-7}. So when for water-vapour a much higher degree of abnormality of the density is found, this must be ascribed to still other causes. Accordingly all that has been observed here refers only to the so called normal substances.

IX. *Approximate relations between x_1 and x_2 at given temperatures.*

If we integrate the differential equation:

$$-\frac{d\varrho}{\varrho}\left\{\frac{n-1}{n} - \frac{v}{v-b} - \frac{2x(1-k)[1-(1-k)x]}{vRT}\right\} + \frac{dx}{nx}\left\{1 + \frac{nx}{1-x} - nx\frac{2a(1-k)^2}{vRT}\right\} + \frac{dT}{T}\left\{\frac{E - \dfrac{2a(1-k)[1-(1-k)x]}{v}}{RT}\right\} = 0$$

which at the critical point can be simplified to:

$$-\frac{d\varrho}{\varrho_k}\left\{\frac{n-f}{s}\right\} + \frac{dx}{x_k} + \frac{dT}{T}n\left\{\frac{Ev_k}{a} - 1\right\}\frac{f-1}{s} = 0$$

we get the approximate equations:

$$\frac{x_1 - x_k}{x_k} = \frac{n-f}{s}\frac{\varrho_1 - \varrho_k}{\varrho_k} + (1-m)\,n\left(\frac{Ev_k}{a} - 1\right)\frac{f-1}{s}$$

and

$$\frac{x_2 - x_k}{x_k} = \frac{n-f}{s}\frac{\varrho_2 - \varrho_k}{\varrho_k} + (1-m)\,n\left(\frac{Ev_k}{a} - 1\right)\frac{f-1}{s}.$$

By adding these two equations, we get:

$$\frac{x_1 + x_2 - 2x_k}{2x_k} = \frac{n-f}{s}\frac{\varrho_1 + \varrho_2 - 2\varrho_k}{2\varrho_k} + (1-m)\,n\left(\frac{Ev_k}{a} - 1\right)\frac{f-1}{s}$$

or

$$\frac{x_1 + x_2 - 2x_k}{2x_k} = (1-m)\left\{\gamma\frac{n-f}{s} + n\left(\frac{Ev_k}{a} - 1\right)\frac{f-1}{s}\right\}$$

If this equation should hold not only in the neighbourhood of the critical point, but at all temperatures, there wonld be question for the value of x of a rectilinear diameter and we should have a form from which x_k can be calculated. Then we should find about 0.01 for x_k. This equation might then also be written as follows:

$$x_1 + x_2 - 2x_k = (1 - 2x_k)(1 - m)$$

and for low value of m when x_2 may be quite neglected,

$$x_1 = 2x_k + (1 - 2x_k)(1 - m)$$

or

$$x_1 = 1 - m + 2x_k\, m.$$

That this value of x_1 is the perfectly accurate one I should not dare to assert, but that it holds by approximation for temperatures $m < \frac{1}{2}$, can be shown; but this does not prove its approximative accuracy over the whole range of temperature. To calculate x at lower temperatures, we may make use of the equation which holds as definition of the quantity φ, viz.:

$$\left(\frac{T}{p}\frac{dp}{dT} - 1\right) p \frac{v_1 v_2}{a} = 1 + \varphi = \frac{\frac{Ev_k}{a}(x_1 - x_2)}{\frac{\varrho_1 - \varrho_2}{\varrho_k}} + \left(1 - \frac{1}{2}x_1\right)^3.$$

If we take v_2 very large, and neglect $\frac{v_1}{v_2}$, this is the form which holds for lower temperatures. If we also neglect ϱ_2 or $\frac{1}{v_2}$ and x_2 in this form, we get:

$$\frac{Ev_k}{a}\frac{x_1}{\frac{\varrho_1}{\varrho_k}} - x_1 + \frac{x_1^2}{4} = \sqrt{1-m} - \frac{1-m}{2} = \varphi$$

or

$$\frac{\frac{5}{4}\, 2(1+\gamma)\, x_1}{2\{1+\lambda(1-m)\}} - x_1 + \frac{x_1^2}{4} = \sqrt{1-m} - \frac{1-m}{2} = \varphi.$$

$$\begin{array}{llll} 1 - m = 0{,}49 & \text{for} & \varphi = 0{,}455 \\ 1 - m = 0{,}64 & ,, & \varphi = 0{,}48 \\ 1 - m = 0{,}81 & ,, & \varphi = 0.495 \\ 1 - m = 1 & ,, & \varphi = 0{,}5. \end{array}$$

We find for the value of x_1 successively:

$$x_1 = 0{,}6 \qquad x_1 = 0{,}71 \qquad x_1 = 0{,}85 \qquad x_1 = 1.$$

These values, however, point to a greater value of x_k than was assumed above, and so too to a non-rectilinear diameter for $\frac{x_1 + x_2}{2}$, so that the value of x_k still remains doubtful.

If we assume a form for $\frac{x_1 + x_2 - 2x_k}{2x_k}$ of the following shape:

$$\frac{x_1 + x_2 - 2x_k}{2x_k} = A(1 - m) - B(1 - m)^2$$

the values found are satisfied with $A = \frac{1.4}{2x_k} - 3$ and $B = \frac{0.4}{2x_k} - 1$. And then x_k would be equal to 0.015. *(To be continued).*

Physics. — "*Determination of the pressure of a gas by means of* GIBBS' *statistical mechanics.*" By Dr. L. S. ORNSTEIN. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of February 27, 1909).

Dr. O. POSTMA has made some remarks on the way in which I have calculated the pressure of a gas by means of GIBBS' statistical mechanics.

The first objection relates to formula (5) of my dissertation

$$\bar{A} = -\frac{\partial \Psi}{\partial a},$$

where $\bar{A}$ is the average in the ensemble of the force corresponding to the parameter a which is exerted by the system, and where Ψ is given by the relation

$$e^{\frac{\Psi}{\Theta}} = \int e^{-\frac{\varepsilon}{\Theta}} d\lambda_p\, d\lambda_q\,;$$

in this equation ε in the energy, $d\lambda_p$ an element of the extension in velocity and $d\lambda_q$ an element of the extension in configuration. The energy ε depends on the momenta, the coordinates and also on the parameter a. The force exerted by a single system is given by the relation

$$A = -\frac{\partial \varepsilon}{\partial a},$$

the coordinates and the momenta being kept constant in the differentiation. Assuming that the kinetic energy does not depend on the coordinates and integrating with respect to the velocities, we obtain:

$$e^{\frac{\Psi}{\Theta}} = C \int e^{-\frac{\varepsilon_q}{\Theta}} d\lambda_q.$$

The magnitude of the part S of the extension in configuration where systems can be represented and over which therefore the integration has to be extended, depends on the parameter a; this is easily seen in those cases in which a determines the position of walls within which the system is confined. We may say that in the parts of the extension outside S the density of the distribution is zero because ε_q is infinite; and this is also true at the boundary of S.

Let us now consider the increment of $e^{-\frac{\Psi}{\Theta}}$ when a increases by an infinitely small amount da. The integral on the right-hand side changes for two reasons: 1 because ε_q changes in the part S of the extension over which the integral has to be taken when the parameter has the value a; 2 because there must be added an integral taken over a part of the extension S' which surrounds S as an infinitely thin layer. We obtain therefore

$$\delta \int e^{-\frac{\varepsilon q}{\Theta}} d\lambda_q = \int_S \delta e^{-\frac{\varepsilon q}{\Theta}} \cdot d\lambda_q + \int_{S'} e^{-\frac{\varepsilon q}{\Theta}} d\lambda_q .$$

The second integral however is zero because ε_q is infinite everywhere in the layer S'. We obtain finally

$$\frac{\partial \Psi}{\partial a} = C \int_S \left(\frac{\partial \varepsilon_q}{\partial a}\right)_q e^{\frac{\Psi - \varepsilon q}{\Theta}} d\lambda_q = -\bar{A} .$$

Treating the integral, which we found for $e^{-\frac{\Psi}{\Theta}}$ as an ordinary multiple one whose limits depend on the parameter a, and differentiating this integral in the usual way with respect to a, we obtain the same result. There are no objections against this differentiation, as the integrated function has no singularities within the limits of integration. The theorem in question is now proved quite generally and we may immediately apply it to the particular case in which a is the volume v of a gas.

The first objection of Dr. Postma being removed in this way, his second objection amounts to this that I should have calculated Ψ by a wrong method. I cannot however, admit the truth of this remark. Dr. Postma says that I have limited myself to the most frequent system. It may be that page 62 of my dissertation makes this impression; but in the more detailed calculation which I have given page 111 I have not at all confined myself to the most frequent system. On the contrary, I have considered systems differing greatly from it. It is true that not all the systems of the ensemble have been taken into consideration, but the systems that were neglected fill only a very small part of the extension in configuration and the density of their distribution in the ensemble is very small. The value found for Ψ by formula (131) differs by a factor from that found by (43), but I have shown on page 127 that we may replace this factor by unity. The objection that ε_1 and ε_2 have been supposed to be discontinuous can easily be removed. Indeed, we may begin by considering ε_1 and ε_2 as continuous functions of the coordinates q and the parameter a; the case of discontinuity may then be treated as a limiting case. The validity of this method is shown on page 91 of my dissertation, where I have determined the virial of repulsive forces in this way. The forces on the walls admit of a similar treatment.[1])

It is also possible to calculate ε_1, ε_2, and ε_3 for the most frequent system either directly or by means of the virial, and to determine $\frac{\partial \varepsilon_0}{\partial v}$ $\left(\text{which differs from } \overline{\frac{\partial \varepsilon}{\partial v}}\right)$. The result is A_0 and we find that $A_0 = \overline{A}$. A direct proof of this latter theorem is obtained if after having determined A for each group of equivalent systems (of the number ζ) we calculate the mean value $\overline{A} = \frac{\Sigma_e (A\zeta)}{N}$. The sum must be taken for all possible groups of equivalent systems contained in the ensemble.

I hope to have shown by the above considerations that my determination of the pressure is free from the objections raised by Dr. Postma.

Geology. — *"Diluvial boulders from the island of Borkum."* By J. H. Bonnema. (Communicated by Prof. G. A. F. Molengraaff).

Some years ago four boulders which had been found by Dr. Lorié on the beach of the island of Borkum were sent me with the request to ascertain their age. They excited my interest in a degree that I resolved to go to the island myself with the result that I found 21 pieces more. These 25 boulders are the subject of this short communication.

Both Dr. Lorié's boulders and those I have collected are from the northern beach of the middle part of the above-mentioned island, which part is called Tüsskendoor. In accordance with their locality the surface often shows the peculiar gloss produced by flying sand.

They are all of sedimentary origin, and consist of limestone, dolomite, or sandstone.

So far as they consist of one of the two first-mentioned substances they usually show long and narrow holes at the surface somewhat tighter in the middle, so that they resemble the shape of an 8. These holes are mostly about 2 mm. long. Their length, however, varies from 1 to 6 mm.

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1) An equation of state may also be established if, without passing to the limiting case of discontinuity, we regard ε_1 and ε_2 as certain given functions of the coordinates and the parameters.

Not very deep channels correspond to these holes, which channels are generally slightly curved and more or less at right angles to the surface. These channels are frequently covered on the inside with a thin skin of lime.

These channels, I felt sure, had been made by some boring animal. Since I only knew of the boring of sponges, sea-urchins and lamellibranchiata into stone, I suspected that on account of their small size they had to be ascribed to boring sponges. I sent some pieces to Prof. VOSMAER to obtain certainty, and was informed that my hypothesis was wrong.

Meanwhile Dr. GRÖNWALL of Copenhagen, whom I had sent a photo of one of these boulders wrote me that similar boulders are frequently found on the Danish coasts and that the bore-holes are attributed by Danish geologists to a worm viz. *Polydora ciliata.*

With the assistance of Prof. VOSMAER, whom I herewith tender my cordial thanks I found some treatises on the boring of worms into stone [1]. By means of these publications I could ascertain that the bores had really been made by *Polydora* (*Leucodore*) *ciliata* JOHNST.

In several pieces the mouths of the channels are found at the bottom of shallow funnel-shaped holes as also appears distinctly from the picture added. In my opinion these holes have been caused by the circumstance that when *Polydora ciliata* had bored slight holes into the boulders they were rubbed to and fro at the bottom of the sea for a longer or shorter period. Thus the sand could develop its erosive force most effectually at the mouths of the channels.

The boulders that show the finest bore-holes have been left intact, so that their age could not be ascertained. The others may be arranged as follows.

1. **Scolithus-sandstone** [2].

Two pieces of this rock are among the collection. The one is light-gray with yellow spots and has thin tubes, 1.5 mm. in diameter. The other is yellowish-brown, sometimes slightly reddish, and the tubes occurring in it are 3 mm. in diameter.

As is sufficiently known this boulder is of the lower cambrian

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[1]) E. RAY LANKESTER, On Lithodomous Annelids. Ann. and Mag. of Nat. Hist., 1868, Ser. 4, Vol. I, p. 233—238.

W. C. M' INTOSH, On the Boring of certain Annelids. Ann. and Mag. of Nat. Hist., 1868, Ser. 4, Vol. II, p. 276—296.

[2]) H. G. JONKER, Beiträge zur Kenntnis der Sedimentärgeschiebe in Niederland. Mitteil. a. d. Mineral.-Geol. Institut zu Groningen, 1905, Bd. I, Heft 1, S. 91.

age, has been proved to occur in the diluvium of the Netherlands and Germany, and is found as firm rock in the neighbourhood of Kalmar.

According to Prof. DEECKE [1]) the tubes have originated in the rising of air-bubbles through the sand of which this rock consists. According to this hypothesis the air through the strong dashing of the waves got under the sand where it remained, till the sand at the surface was dry. I have not yet had an opportunity to observe the phenomenon described by DEECKE, although I have seen at Scheveningen how after low tide air-bubbles rose to the surface every time a wave flowed over a dry part of the beach, leaving behind vertical tubes in the wet sand.

If DEECKE's explanation is correct I think it highly remarkable that sandstone with similar tubes occurs only in lower cambrian strata both in America and Europe.

2. Sandstone with brownish-violet strata crossing each other.

JONKER [2]) wrongly calls this boulder "Sandstein mit discordanter Parallelstruktur". This would be correct if the two systems of strata only touched, but here they cross each other.

This rock of which I found one piece corresponds in age with the former and is also frequently found in the diluvium. As firm rock it is unknown, but presumably it is present as such at the bottom of the northern part of the Kalmar Sund, as it is often found as boulders on the west coast of Oeland.

3. "Backsteinkalk." [3])

Of this species too I found one piece. The internal unweathered part consists of compact, splintery, bluish-gray, silicious limestone, which is rich in spicules of siliceous sponges. This boulder also contains rests of a *Cyclocrinus*-species. The external part has become brownish by weathering. Further weathering would doubtless make it assume the structure of "Backsteinkalk", as I think I

1) W. DEECKE, Einige Beobachtungen am Sandstrande (mit 6 Textfiguren). Centralblatt f. Mineral., Geol. und Paläontol., 1906, S. 726.

2) H. G. JONKER, Beiträge zur Kenntnis der Sedimentärgeschiebe in Niederland. Mitt. a. d. Mineral.-Geol. Institut zu Groningen, 1905, Bd. I, Heft 1, S. 94.

3) H. G. JONKER, Beiträge zur Kenntnis der Sedimentärgeschiebe in Niederland. Mitteil. a. d. Mineral.-Geol. Institut zu Groningen, 1905, Bd. I, Heft 1, S. 132.

am entitled to conclude from observations made in the island of Oeland. Boulders of this rock weathered more or less into typical "Backsteinkalk" I found there in great numbers on the east coast near Segerstad. This lower silurian rock therefore is probably present as firm rock at the bottom of the Baltic to the east of Oeland. It is most likely that the age of this rock comes nearest that of the Ifter stratum (C_3) of the Russian Baltic provinces.

4. Phaseolus-limestone. [1])

The piece of this rock that I found consists of yellowish-gray, nearly compact limestone with dark crystalline parts, which on examination proved to be stromatopora and corals petrified into calcite. Because it also contains rests of *Proetus conspersus* Ang., *Beyrichia protuberans* Boll, *Primitia mundula* Jones en *Spirifer elevatus* Dalm. sp., it appears sufficiently that this stone corresponds in age with the upper Oesel stratum (K). In various parts of our diluvium, however, I found boulders of the same rock, which besides the above-mentioned fossils also contained such as are peculiar to the lowest (yellow) zone of the above stratum, e.g. *Meristina didyma* Dalm. sp. For this reason I do not hesitate to consider the age of this stone to be the same as that of this zone (K_1).

5. Chonetes-limestone (Beyrichia-limestone).

To this species I consider to belong in the first place a piece of bluish-gray limestone, which consists almost entirely of valves of *Orthis canaliculata* Lindström (*Orthis orbicularis* v. Schmidt). Besides this piece also contains fish-scales, some specimens of *Ptilodictya lanceolata* Lonsdale and a glabella of a *Calymene*-species. This limestone was already classed by Roemer [2]) under *d* as one of the species of Beyrichia-limestone.

Further a piece of light-gray, sometimes yellowish, fine-grained crystalline limestone which in the process of disintegration appears to contain many rests of little organisms, must be assigned to this group. Smooth shells and valves of Ostracoda especially belong here. By the occurrence of *Kloedenia Wilckensiana* Jones, *Phclidops antiqua* Schloth. sp., and fish-scales (e. g. of *Thelodus parvidens*

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[1]) H. G. Jonker, Bijdragen tot de kennis der Sedimentaire zwerfsteenen in Nederland. Verhand. Kon. Akad. v. Wetensch. te Amsterdam, (2e sectie), 1906, Dl. XII, No. 3, pag. 25.

[2]) F. Roemer, Lethaea erratica, 1885, p. 93.

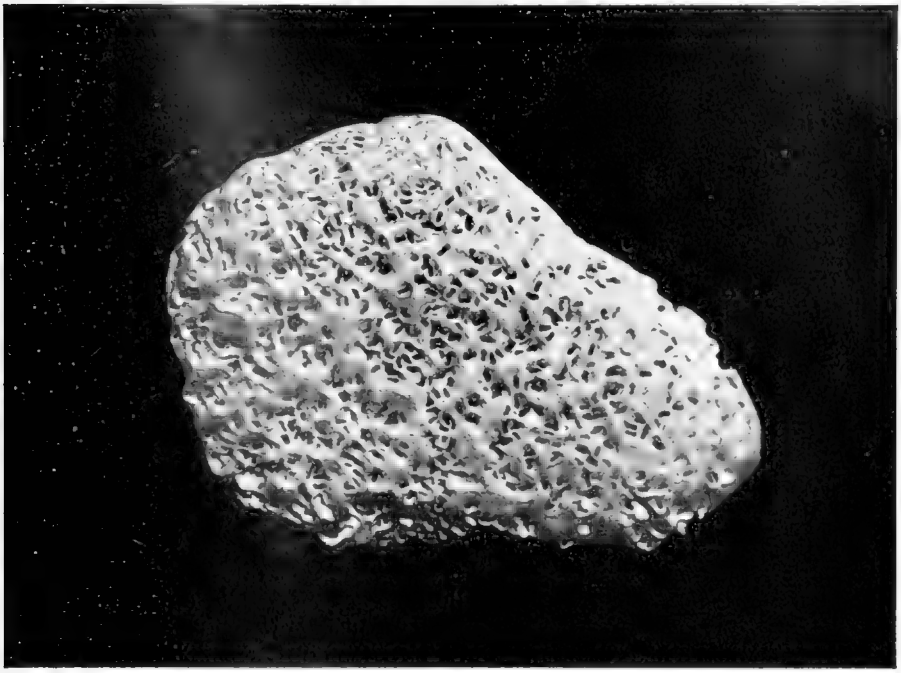

Fig. 1.

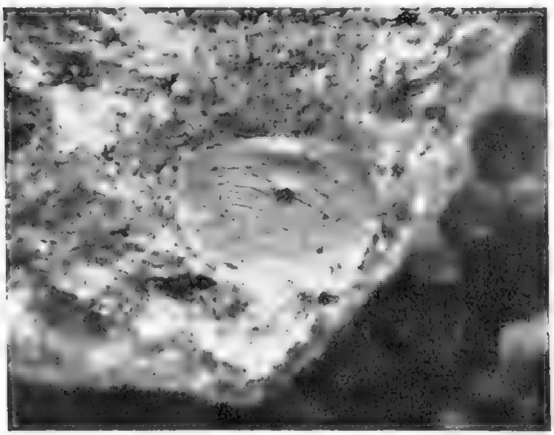

Fig. 2.

Fig. 1. Boulder from Borkum, bored by *Polydora (Leucodore) ciliata* JOHNST. Natural size.

Fig. 2. Right valve of *Kirkbya (?) Loričі* BONNEMA, × 30.

AGASSIZ) it is sufficiently proved that this rock as well as the preceding one belongs to the gray zone of the Upper Oesel stratum (K_2).

In the boulder described last I also found a valve of an Ostracod still unknown, which has received the name of ***Kirkbya (?) Loriéi*** and is represented as figure 2 on the annexed plate. The valve found I consider to be a right one. It is 0,9 mm. in length and 0,6 mm. in height. Its shape is nearly oval. Along the hinge line there is a narrow dorsal plane. Along the rounded edge which separates the dorsal plane from the lateral plane this has a flat part. Otherwise the lateral plane is equally and slightly convex except that it has a little round pit about the middle which certainly indicates where the adductor muscle was attached inside. Very characteristic are narrow ridges which are separated from each other by nearly as broad furrows, and are absent at both ends of the valve. Continuations of the furrows sometimes penetrate a short way into the ridges, which run nearly from the posterior part of the hinge line to the anterior part of the ventral margin. This valve of which too little can be seen to ascertain to what genus the animal belonged to which it pertained reminds one of those which JONES [1]) assigns to *Kirkbya* (?) *Walcotti* and comes from the Devonian beds of Canada. Still stronger does it remind one of the lower-carboniferous *Kirkbya* (? *Barychilina*) *costata* MC COY [2]).

6. Dolomite with fish-rests.

Of this I possess two pieces. One is a yellowish-brown, in some parts dark gray, fine-grained dolomite with many cavities, the sides of which are set with dolomite crystals. The other is different, its colour being more reddish while there are hardly any holes.

As regards the age of this species of dolomite it must be assigned to the youngest silurian strata, certainly not younger on account of the presence of fish-rests. It is even possible that they are not older than devonian.

7. Eocene lime-sandstone.

It is a boulder of soft gray, fine-grained lime-sandstone partly

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[1]) T. RUPERT JONES, On some Ostracoda from the Cambro-Silurian, Silurian and Devonian rocks. Contributions to Canadian Micro-palaeontology, Part III, 1891, p. 96, pl. XI, figs. 12a, 12b.

[2]) E. O. ULRICH, New and little known American palaeozoic Ostracoda. Journal of the Cincinnati Society of Natural History, 1890, p. 208, pl. XVIII, figs. 2a,b.

thin-laminated by weathering, and containing grains of glauconite and little laminae of mica.

Besides indeterminable rests of molluscs, many zoaria of *Vincularia* and shells of foraminifera (presumably *Rotalia)* occur. Not having found a similar boulder before, and taking it to be tertiary, I sent a piece to Dr. GRÖNWALL, asking him whether a similar boulder was known to him. He was kind enough to inform me that such boulders are of frequent occurrence near Ystad in the south of Sweden and assigned to the Eocene period.

From the occurrence of boulders on the northern beach of Borkum I think I may conclude that boulder-clay is present in its neighbourhood at the bottom of the sea. As regards the nature of the boulders found there this boulder-clay bears a close resemblance to that which I [1]) have become acquainted with at Kloosterholt (Heiligerlee) in the province of Groningen and at Hemelum in Friesland, and not with that of Groningen (at least not with that of the upper part of the Hondsrug). We have therefore here another instance of the West-Baltic type, which is met with almost everywhere in the western part of the North German plain.

Boulders with fish-rests, it is true, do not occur as firm rock in Sweden, but are found in Russia. Boulders of dolomite with fish-rests, however, I also found in the boulder-clay of Hemelum which has a western Baltic character. I believe therefore that I am right in supposing that this boulder formerly occurred also farther to the west as firm rock.

Beyrichia-lime with *Orthis canaliculata* LINDSTRÖM, *Ptilodictya lanceolata* LONSDALE [2]) and fish-rests is not known to me from that locality. At Groningen, where the boulder-clay (the upper part at least) has an Eastern Baltic character, I have often found this rock. It is possible that this boulder has been taken from older boulder-clay.

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[1]) J. H. BONNEMA, Cambrische zwerfblokken van Hemelum in 't Zuidwesten van Friesland. Proceedings Acad. Amsterdam, Vol. V, p. 140.

[2]) H. G. JONKER, De oorsprong van het glaciaal diluvium in Nederland. Delft, 1907, pag. 16.

Anatomy. — "*Nucleus facialis dorsalis, nucleus trigemini posterior, nucleus trochlearis posterior*". By Dr. C. T. van Valkenburg. (Communicated by prof. L. Bolk.)

Studying some fetal human brainstems, my attention was drawn to the constant presence of a few groups of cells — nuclei — not yet described, which I could demonstrate too in the neonatus, and in the fullgrown man. They are the following:

1. *Nucleus facialis dorsalis*: By Kölliker [1]) and Ziehen [2]) the existence of a dorsal facial nucleus in monotremata (ornithorhynchus, echidna) has been demonstrated (normal brains). In other mammals, inclusive man, it has not yet been found. Experimentally Kohnstamm [3]) and Yagita and Hayama [4]) have shown a number of more or less scattered cells dorsally to nucl. VII, degenerated after resection of the submaxillary nerve resp. chorda tympani.

The japanese authors localise these cells closely to the medial part of the substantia gelatinosa Rolando of the descending trigeminusroot. According to the periferal lesion, the cells are supposed to represent a nucleus salivatorius nervi facialis.

The brainstem of a *fetus of* 27 *c.m. length*, cut transversally, was first examined.

Near the frontal end of the nucleus facialis, 1.4 mm. in front of the caudal ending, a cell-group appears dorsomedially from this nucleus, lateroventrally from the nucl. abducentis. This cell-group has a length of 200 μ (fig. 1).

The number of the cells is about 70; they are of the motor type, show distinct dendrites in the direction of the common facial nucleus (fig. 2). The staining with haematoxylin does not permit us to distinguish the cylinderaxis. 200 μ frontally to the rostral end of the cellgroup, the ventral nucl. VII disappears.

The brainstem of a *fetus of* 14 *cm.* was examined in the same

1) Kölliker: Oblongata u. Vierhügelgegend von Ornithorhynchus und Echidna. Leipzig Engelmann 1901.

2) Ziehen: Das Centralnervensystem der Monotremen und Marsupialier 2er Theil, 2er Abschnitt. Semon's Forschungsreisen 1908.

3) Kohnstamm: Vom Centrum der Speichelsecretion etc. XX Congress für innere Medicin.

Ders. und Wolfstein: Versuch einer physiolog. Anatomie der Vagusursprünge etc. Journ. f. Psychol. u. Neurol. 1907. S. 190 flgg.

4) Yagita und Hayama: Ueb. das Speichelsecretionscentrum Neurol. Ctrbl. 1909. N⁰. 14.

Yagita: Weitere Untersuchungen üb. das Speichelzentrum. Anat. Anz. 1909. N⁰. 2 u. 3.

way. 950 μ before the caudal beginning of the nucleus facialis a cell-group appears at the same place as mentioned above; the longitudinal dimension is 150 μ, the number of cells about 66. The protoplasmatic dendrites extend ventrolaterally in the path, also taken by the ascending axones of the ventral nucleus. Frontally the cellgroup ends in the same transversal plane with the ventral nucleus VII.

A *fetus of* 7.4 *cm.* showed principally the same features. The sections being made in a more horizontal direction, it was impossible to make quite exact and comparable measurings without modelling. About the relation of this cellgroup to the nucl. VII, it must be mentioned that the former was situated somewhat more laterally. The frontal endings of both were visible in one and the same plane; the horizontal length of the little nucleus was about 20 μ; it was impossible to determine exactly the number of cells in it.

I think it to be in a high degree probable that the indicated little nucleus is a dorsally situated cellgroup belonging to the facialis. The quite constant situation above the most frontal part of the nucleus VII, amongst the ascending rootfibres of this nerve, which after KAPPERS' [1]) description indicate the way phylogenetically and ontogenetically taken by the facial nucleus, is a strong support to this interpretation. Moreover the cells are of a purely motor type, and I am not able to find a relation with any other bulbar nucleus or root, particularly not with the nucleus abducentis, whose ascending way from the base to the fourth ventricle lies far more medially.

Of course it is desirable to indicate exactly by means of appropriated methods (silverimpregnation) cylinderaxes passing from the described cells into the nervus facialis.

To complete my observation I looked for this nucleus in the brainstem of a *child of* 14 *days,* and in that of a fullgrown man. Both showed the cell-group in absolutely the same situation. The former was stained with the PAL-method and it was due to the accidental colouring of the nerve cells by the chrom-haematoxylin that I could identify the elements in the ascending facialisroot (fig. 5). By the incompleteness of the series it was impossible to make trustworthy measurements.

The series of the *fullgrown,* stained after the method of VON GIESON, was strictly without gaps; the characteristic local connexions above mentioned were easily to be seen (fig. 6). In 14 preparations I counted 59

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[1]) See e. g.: ARIËNS KAPPERS u. H. VOGT: Die Verlagerung der motorischen Oblongatakerne in phylogenetischer und teratologischer Beziehung. Neur. Ctrbl. 1908.

cells, in very satisfactory accord with the numbers obtained in fetal brains. Summarizing I believe I am qualified to conclude that a dorsal nucleus facialis occurs in man with great probability. The situation of this nucleus, during ontogenesis, changes no more, or at best very little, with respect to the neighbouring elements, particularly to the ventral nucleus facialis, as soon as the latter has reached its definite ventral place (in every case before the 3^{d} month). It seems to be very probable, that the described cellgroup may be considered as a nucleus salivatorius giving origine to the secreto-motor fibres in the chorda tympani; my investigations cannot decide this question. The fact of the dorsal situation gives — in reference to Ariëns-Kappers' neurobiotaxis-doctrine — a hint to a close connexion with the dorsal part of the substantia gelatinosa Rolando nervi trigemini [1]). After the examination of Wallenberg [2]), Eisenlohr [3]), which I could confirm in a pathological case, it is highly probable that this part receives mandibular rootfibres. Therefore it is obvious to localise in this nucleus a function, closely connected to the sensibility of the oral mucosa. It is clear, that the action of the glandulae submaxillaris and sublingualis must be considered in the first place.

Security will only be obtained by cases of degeneration, which at the same time will indicate or explain the apparent difference from the experimental results in the dog.

The nucleus described here is most likely the same as van Gehuchten believed to have found in a hen-embryo, and which afterwards was brought to degeneration in the rabbit by this author. He considered it as a nucleus abducentis ventralis (See: Les nerfs moteurs oculaires Journ. de Neurol. 1898). Lugaro confirmed the existence of this nucleus (Sull' origine di alc. nerv. encef. Archi di Ottalmol. 1894), and also Pacetti (Sull' origine dell' abducente. Ric. fatte nel Labor. d'Anat. norm. di Roma 1896). Siemerling and Boedeker contradicted it (Chron. fortschr. Augenmuskellähm. etc. A. f. Psych. Bd 29).

My preparations are contrary to v. Gehuchten's interpretation. Moveover I am able to confirm Siemerling and Boedeker's observation, who did not find — in a case of abducens-paralysis — any cell-degeneration out of the region of the nucleus VI. In a case of atrophy of the eye muscle-nuclei the little cellgroup in question was undamaged.

The results of Kaplan and Finkelnburg (Beitr. z. Kenntn. des

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1) And with the frontal continuation of the nucleus of the fasc-solitarius?

2) Deutsche Ztschr. f. Nervenheilk. 1897, p. 400.

3) Arch. f. Psych. 1892, p. 314.

sogen. ventr. Abducenskernes. A. f. Psych. Bd 33) are somewhat uncertain.

2. *Nucleus trigemini posterior.*

Fetus 27 *c.m.* 200 μ frontally to the rostral border of the described nucleus, a cellgroup appears in a somewhat more lateral situation, but in the same horizontal plane. Proximally it continues directly into the nucleus trigemini, which has a much larger dorsoventral dimension (fig. 7). It would be unnecessary to take this cellgroup as a separate nucleus if in younger stadia clear limits did not exist.

Fetus 14 *c.m.* shows the little nucleus in the middle between nucl. fac. dors. and nucl. trigemini; the number of its cells is about 96. As in the former fetus, it lies more laterally than the dorsal facial, dorsomedially to the trigeminus nucleus (fig. 8). The dendrites run partly in a lateral direction, but most of them are not visible: this is due to the plane of section, which cuts the cells vertically on their longitudinal diameter. The common nucleus motorius V lies, as said above, more ventrolaterally, increasing in frontal direction especially to the ventrolateral side, approaching the end-nucleus of the sensory trigeminus [1]).

Fetus 7.4 *c.m.* In consequence of the more horizontal sections, the connexions are not distinct enough to permit any conclusion. Therefore I examined an embryo of 6.5 c.m., where the plane of section through the stem was more strictly transverse. Here the existence of the same cellgroup in quite homologous connexions was esaily to be seen. I was not able to make a sufficient photogram.

In the neonatus, nor in the full-grown man the described nucleus was to be found. It is true that a remarkable diminution in the number of cells is obvious in the distal parts of the nucl. V motor, which is followed behind and before by a rather sudden increase. But it is difficult to value exactly such a grouping, which occurs in other parts of the same nucleus too, and in a similar way in several other nuclei.

In every case we have to do with an ontogenetical confluence of two parts of the trigeminal nucleus between the 3[d] en the 5[th] month. It cannot be stated with absolute certainty whether the nucleus posterior shifts forward, or the nucleus principalis goes backward. The connexions with other nuclei which could serve as points de repère are not reliable by the fact that the latter have not yet —

[1]) In teleosts and some reptiles a distinct cellgroup belonging to the motor V nucleus, behind this latter, occurs; it remains in this place during the whole life (Ariens-Kappers).

at least in all details — obtained their definite situation and form. The sagittal direction of the greater part of cell dendrites gives a strong support to the presumption that nucl. posterior is "drawn" to the nucl. principalis. The cells of the latter send their protoplasmatic projectures mainly ventrolaterally, in the direction of the sensory V nucleus, where the stimuli arrive, by the influence of which the motor nucleus descends.

I am not able to give any interpretation of the initial separation and the later confluence of the two nuclei. Nor do I know anything of the functional nature of the nucleus posterior.

3. *Nucleus trochlearis posterior.*

Fetus 27 *cm.* 1.5 mm. behind the caudal pole of the nucleus trochlearis a cellgroup appears in a quite analogous part of the fasciculus longitudinalis posterior. It consists of large, motor cells, about 26 together, and measures sagitally 200 μ (left side). On the right side the distance between the caudal poles of nu. IV and nu. IV post. is 850 μ. De cell-corpuscle with its dendrites, is generally stretched from medioventral to laterodorsal, sometimes sagittally.

Fetus 14 *cm.* On the right side the same little nucleus as in fetus 27 cm. is present. It continues, with a few cells directly in the nucl. IV propr.; its length is about 250 μ. In the left half it appears already 600 μ behind the nucl. IV and measures 200 μ. The horizontal dimension of the nucl. IV is no more than 250 μ, whereas the right one measures 500 μ.

The left nuclei together are 550 μ, the right ones 750 μ. (fig. 9 and 10, 13, and 14).

Fetus 7.4 *cm.* could not give any information because it was cut in a too horizontal plane.

The same occurred in fetus 6.5 cm.

In the brainstem of a neonatus and of a normal fullgrown no nucl. trochl. post. was to be found.

In a case of ophthalmoplegia completa dextra I found behind the atrofied trochlearis nucleus another smaller nucleus equally atrofied On the other side this nucleus was unimpaired (fig. 11). The rootfibres go off in exactly the same way as from the nucl. IV principalis, which lies 1.680 mm. frontally to the posterior one. The sagittal length of the latter is 1.260 mm.

The nucl. trochlearis changes phylogenetically [1]) and ontogenetically its place in a frontal direction; a result of the frontal migration of the nucleus is the caudal situation of the point, where the root leaves the brain.

[1]) TRETJAKOFF: Das Nervensystem von Ammocoetes Arch. f. mikrosk. Anat. Bd. 74.

It is evident that certain, still unknown influences cause in some cases an incomplete migration, in consequence of which a group of cells remains on its way at a smaller or greater distance from the spot, that should be reached. Highly characteristic with regard to the inconstancy of this phenomenon is the above mentioned asymmetry [1]).

The cause can only be spoken of in general terms. The principal ground for the migration of the nucl. IV lies in the fact, that it acts under the influence of stimuli by which the nucl. oculomot. is stimulated at the same time. Regarding the very predominant role of the oculomotorius it is obvious, that the common stimuli of both nuclei, and the coordinated action of musculi innervated by them, draw the trochlearis nucleus forward more than the oculomotor one backward. Of course the stimuli received by nu. IV together with nu. VI for a common action, must be regarded too, though these may be of much less importance than the above mentioned ones. Now it seems possible to see in the caudal staying of a part of the nucl. IV the anatomical expression of the influence of common stimulation and common action with nucl. VI.

Only in a minority of cases this influence, which of course always exists, can be demonstrated in such a striking way.

EXPLANATION OF THE FIGURES.

Fig. 1. Transverse section through the oblongata of a human embryo 27 cm.
Fig. 2. The same, stronger magnif.
Fig. 3. Section on the same level, embryo 14 cm.
Fig. 4. „ „ „ „ „ „ 7.4 cm.
Fig. 5. „ „ „ „ „ child 14 days.
Fig. 6. „ „ „ „ „ fullgrown man
Fig. 7. Transverse section through the pons of human embryo, 27 cm.
Fig. 8. „ „ „ „ „ „ „ „ 14 cm.
Fig. 9. Transverse section through the mesencephalon of a human embryo 14 cm.
Fig. 10. „ „ „ „ „ „ „ „ „ 27 cm.
Fig. 11. „ „ „ „ „ „ „ fullgrown man.
Fig. 12. „ „ „ „ „ „ the same 1.300 m m. frontally to fig. 11.

Fig. 13. Projection on a sagittal plane of nucleus VI, VII ventralis, VII dorsalis, V posterior, V principalis, IV posterior, IV principalis; embryo 27 cm. Magnif. 20 : 1.

Fig. 14. The same projection, embryo 14 cm. In figs 13 and 14 only the sagittal distances are exact.

v = ventralis. *d* = dorsalis. *p* = posterior. *pr* = principalis. *de* = dexter. *si* = sinister. *o.s.* = oliva superior. *ne* = nervus. *atr.* = atrophicus.

1) In a rabbit I saw the same nucleus occurring only in the right brainstem half. Westphal described (A. f. Psych. Bd. 18, p. 846) a group of little cells, lying dorsocaudally to the trochlearis nucleus. The cells of the nu. trochl. post. mihi are neither little nor do they lie more dorsally than the nu. trochl. principalis.

Meteorology. — *"Preliminary report upon the investigation of the upper air-layers begun at Batavia in* 1909". By Dr. W. VAN BEMMELEN and Dr. C. BRAAK.

In the beginning of 1909 the necessary funds for the purchase of apparatus for an aërological investigation at the Batavia observatory were placed at the disposal of the first named of the authors by the Colonial Secretary.

As it seemed desirable not to proceed to liberating registering balloons before having acquired some more knowledge about the wind in different atmospheric layers above the island of Java by means of pilot-balloons, a number of pilots with instruments were sent out.

With pilot-balloons of the Continental Caoutchouc & Guttapercha Company of Hannover no heights greater than 5.5 K.M. could be reached and a great number burst already during inflation.

In September 1909 pilot-balloons were received (45 gr. weight) from the firm Paturel of Paris, which gave much better results.

In 124 experiments 11 balloons burst during the process of inflation but, almost without exception, considerable heights were attained; e. g. 25 times a height of more than 10 K.M., 14 times of more than 12 K.M. and once a height of 15 K.M. was reached.

These very satisfactory results are due to the circumstance that in the early morning hours at Batavia the sky is mostly clear, and that the velocity of the wind above Java is small, so that it was possible to give the balloon a buyoancy of only half that of the amount usual in Europe.

Up to April 1st the last named of the authors conducted the experiments; afterwards the first named, on his return from Europe, resumed the management.

A most valuable assistance was offered by the naval lieutenant A. E. RAMBALDO who, in the beginning of September 1909, arrived at Batavia with an equipment for kite and balloon work, and was detached at the Observatory.

Ascents of kites and captive balloons were organised and on November 22 the first experiments took place on the "Koningsplein" at Batavia.

The balloon had a capacity of 30 $M.^3$ and reached a height of 1800 M., the kites a height of 2200 M.

Three new balloons of a capacity of 36 $M.^3$ and two registering apparatus with ventilation, sent out by Prof. Dr. R. ASSMANN, the

Director of the Aëronautical Observatory at Lindenberg, have just been received.

We are under great obligations to Prof. Assmann for his assistance, as well as to Prof. Dr. H. Hergesell who, on the occasion of our visits to Strassburg, gave us his valuable advice.

In January 1910 the last named of the authors made a voyage with Mr. Rambaldo to the Natuna Isles in the South China-Sea, and then found an opportunity to conduct nine successful kite-ascents; the greatest height, attained with a team of four kites, was 3075 M.

Twice a kite was lost in a squall.

Not till December 1909 some registering balloons (of the Continental Company at Hannover) were sent out; six registering apparatus, of the firm Bosch at Strassburg, were received some weeks previously.

The diameter and weight of the balloons were 1.5 M. and 1.5 K.G.

As we were afraid that the balloons, when liberated so near the seaside as Batavia, would fall into the sea, the first were sent up, tandem-fashion, at Depok, half way between Batavia and Buitenzorg.

The first tandem attained a height of 12 K.M. and was soon brought back; the registration was in good order. Two tandems subsequently liberated were soon lost in the clouds and have as yet not been recovered.

Twice during the month of May a balloon provided with a parachute was sent up at the Observatory.

Both balloons have been recovered, but of the first the diagram was lost, having been wiped out by an inquisitive native and, owing to a cloudy sky, the trigonometrical measurements failed.

Measures have now been taken against the spoiling of instruments and records.

The second balloon, sent up on May 19 during the passage of the earth through the tail of Halley's comet, was immediately recovered. Its position was measured from a basis line of 1.5 K.M. and the diagram is in perfect order; the balloon however burst at a height of 7 K.M.

Once a 1.5 M. balloon was let up without recording instrument and as a pilot balloon; this balloon could be followed from two points situated at a distance of 4.5 K.M. and attained a height of more than 18 K.M.

The considerable amount of data obtained up to the present time has been nearly all worked out.

We found very variable circumstances, which make it difficult even to draw preliminary conclusions, but nevertheless we will try to give some of our results which, to a considerable extent, may be considered fairly accurate.

Temperature-Gradient. The temperature-gradient of the lower 2 K.M. of the atmosphere has been determined in three ways, namely:

1. above the land with the captive balloon and light wind.
2. above the land with a moderate westerly wind and with kites.

TEMPERATURE-GRADIENT

(Decrease per 100 M.)

Height in M.	Captive balloon observations		Number of cases	Kite-observations above the land		Number of cases	Kite-observations above the sea		Number of cases	Coast and adjacent sea, East-Africa after BERSON
0— 100	0°.81		(24)	0°.85		(13)	1°.17		(15)	
100— 200	78		(24)	94		(13)	1.00		(16)	
200— 300	89	0°.77	(24)	81	0°.87	(13)	80	0°.91	(15)	1°.08
300— 400	78		(24)	87		(13)	80		(15)	
400— 500	60		(24)	86		(13)	78		(14)	
500— 600	57		(24)	87		(13)	66		(12)	
600— 700	73		(24)	67		(13)	79		(12)	
700— 800	52	0°.57	(23)	60	0°.72	(13)	58	0°.59	(12)	0°.50
800— 900	53		(22)	61		(14)	62		(12)	
900—1000	52		(19)	87		(11)	30		(10)	
1000—1100	30		(14)	46		(11)	64		(10)	
1100—1200	47		(14)	44		(11)	52		(10)	
1200—1300	45		(11)	38	0°.44	(10)	69	0°.71	(10)	0°.57
1300—1400				49		(10)	44		(8)	
1400—1500							29		(6)	
1500—2000								0°.34	(3)	0°.49
2000—2500								0°.50	(3)	0°.46
2500—3000								0°.46	(2)	0°.50

3. above the sea (14—20 Jan.), the sky being cloudy and the weather rainy, with kites.

These three series of gradients are not immediately comparable as, above the land, the observations with the captive balloon took place in earlier morning hours than the kite observations, whilst the kite experiments at sea were made in the morning as well as in the afternoon.

For the sake of comparison the values of the gradient, as found by Prof. Berson during his aëronautical expedition to East-Africa above the coast and littoral, is given in the last column. The gradient for the first 500 M., as stated by Berson, is still larger than that found here.

Temperature-inversion. At a height of about 1 K.M. the gradient found by means of kite- as well as by means of balloon-ascents shows a sudden decrease. The reason of this is to be found in the inversion often found at this height, occurring in the so-called fine weather cumuli.

Owing to the rising air currents the formation of these clouds is seen to commence about 10—11^h a.m.; sometimes they pile up to high cumuli, but often they remain floating as small white clouds.

In the latter case an inversion of temperature and humidity has repeatedly been observed, commencing at the cloud base.

Thus a confirmation is here afforded of what has been found and communicated as a still unknown phenomenon by Prof. Rotch (Nature Oct. 14, 1909, p. 473). Aitken (Nature Nov. 18, p. 67) ascribes this increase of temperature to diffuse sun's radiation within the cloud. The radiation on the rather thin cumulus layer is very important at Batavia, owing to the high position of the sun. In the interior of larger cumuli, where an active air motion exists and the influence of the sun's radiation is unimportant, this inversion does not occur; on the contrary, during the passage of these clouds a decrease of temperature was always observed and the humidity approximated to saturation as the apparatus entered the cloud.

During kite ascents only once, on January 19, inversions have been found to occur, as shown by the following data.

Height	Temperature	Humidity
374 M.	23°.2	100%
567	23 .1	
741	23 .2	
1235	19 .5	73
1300	19 .4	
1381	19 .5	
2040	14 .7	78
2198	14 .7	

Southerly Winds. As to the direction of the wind, it can be noticed that, besides the south wind which can be regarded as a land breeze, often another south wind is found, the origin of which is probably to be sought in a pushing forward of air layers from the Indian Ocean; the height of these layers is 1.0 to 1.5 K.M.

They were not observed on January and February; perhaps the west-monsoon was then too strong to be pushed aside. A synoptic summary of the wind's direction from kilometer to kilometer for the period September to May clearly shows that the general air-current has easterly components up to the greatest heights attained (10—15 K.M.) and how during shorter or longer periods the west-monsoon thrusts itself below it. Nothing is to be seen of an anti-trade wind; the east-monsoon consists of one mighty air-current.

The rapid increase of the wind's velocity with increasing height in the lower layers is remarkable.

Whilst at the earth's surface during the night a perfect calm always reigns and during the day the motion of the air (at least at land) is slight, the wind's velocity increases to about 3 M. at a height of 100 M.

The small gradients which at this height are capable of causing motion in the air are evidenty too small to overcome friction at the earth's surface.

At the division between easterly currrents above and westerly currents in the lower parts alto-cumuli may often be observed which explains their general occurrence in the west-monsoon and their absence during the east-monsoon.

The average height of the Alto-cumuli above Batavia, determined at 5.4 K.M., during the international cloud year 1896—97, from numerous measurements by VAN DER STOK and FIGEE is in perfect agreement with the average height of the division of west- and east-monsoon as found by the authors.

In the following 19 experiments the division could be determined with considerable accuracy.

Date 1909	Height of Westmonsoon	Date 1910	Height of Westmonsoon
2 Sept.	5.0 K.M.	6 Jan.	2.0 K.M.
8 Nov.	5.7	21 Febr.	2.5
9 ,,	5.5	16 Mrch.	5.8
15 ,,	4.2	15 Apr.	6.0
8 Dec.	7.4	24 ,,	9.9
9 ,,	7.0	14 May	4.0
11 ,,	6.0		
13 ,,	3.7		
15 ,,	4.5		
16 ,,	4.2	Average	
24 ,,	8.0	Sept.-May	5.4
27 ,,	8.5		
31 ,,	2.0		

The higher easterly, as well as the lower westerly winds are sometimes affected by strong northerly or southerly components.

So e. g. on September 15 a south wind of a velocity from 2—8 K.M. was observed; on the contrary on September 22 a north wind of from 5—8 K.M.

On May 27 we found the following directions:

0— 1 K.M.	W.
1— 4	S.
4— 5	N.
5— 7	W.
7— 9	S.
9—10	Calm
10—14	N. E.

Influence of the earth's rotation. Often the direction of the wind shows a well marked veering to the left, mostly in the lower layers, which may be ascribed to the influence of the earth's rotation. The deviating force at a latitude of 6°11′ is certainly a small quantity, the sine being not more than 0.11, but, on the other hand, the force determined by the pressure gradient is also very small. Therefore the influence on the direction can become important; it must be noticed however that veering to the right occurs as well, although much more rarely.

Inversions. Sudden turnings of the wind within small intervals of height, mostly accompanied by a notable reduction of velocity, have been observed in many cases up to a height of 10 K.M.

Probably these are also accompanied by inversions of the temperature.

W. wind at 17 K.M. During his expedition to East-Africa, Prof. Berson found the unexpected occurrence of strong westerly winds at heights from 10—20 K.M., between or above the general easterly current, a phenomenon which is still waiting for an explanation.

It is very remarkable that, on the first occasion upon which a balloon attained 18 K.M., at Batavia a wind of the same description was encountered.

As the balloon was followed by means of theodolites from two points favourably situated at a distance of 4340 M. apart, (the angles of inclination being still 54° and 53° at the moment of the bursting of the balloon), the following data are certainly quite trustworthy.

Height	Direction of the wind	Velocity of the wind, m. p. sec.
16.6—16.9 K.M.	N.E.	9.0
16.9—17.3	W.	0.8
17.3—17.6	W.S.W.	5.6
17.6—17.9	W.S.W.	5.6
17.9—18.3	W.S.W.	5.5

Velocity of rising. This balloon ascent may be cited as an example showing the possible errors made when a calculated velocity of rising is assumed; whereas the calculated velocity was 200 M. per minute, the actual velocity was

0.0— 2.0 K.M.	198 M. p. min.	=	3.3 M. p. sec.
2.0— 4.8	280		4.7
4.8— 7.3	256		4.3
7.3—10.2	288		4.8
10.2—13.3	312		5.2
13.3—16.6	330		5.5
16.6—18.3	324		5.4

The considerable increase of velocity up to a height of 5 K.M. may be explained by assuming that the gas of the large balloon (161 c.m. diameter) gave off its heat but slowly so that the difference of its temperature and that of the surrounding layers continually increased. With pilot-balloons, where the proportion between capacity and area is more than three times less, these differences will be much smaller; in fact they appear to have a constant velocity up to a height of 12 K.M.

When using balloons of 80 cm. diameter, sent up tandem-fashion, Berson also found an increase of the same order as that found by the authors in the three experiments cited here:

Aug. 6, 1908.		Aug. 30, 1908.		Sept. 5, 1908.	
	m. p. s.		m. p. s.		m. p. s.
2.8 — 3.6 K.M.	3.4	9.3 — 11.1	3.7	5.9 — 8.3	3.2
3.6 — 5.6 ,,	4.7	11.1 — 13.0	4.0	8.3 — 9.6	3.6
		13.0 — 15.0	4.4	9.6 — 11.3	3.7
		15.0 — 18.0	5.1	11.3 — 14.1	4.6
				14.1 — 16.9	5.2

Registering-balloons. In the following table the results are given obtained by means of the registering-balloon, let up at Depok on February 16, 1909.

Depok is situated 95 m. above the sea-level.

The balloons of the tandem-system had a diameter of 150 cm. They were inflated until a buoyancy of resp. 3.5 and 2.2 KG. was attained. The free buoyancy of the whole system was 2.75 KG.

Local time.	Height in meters.	Temp.	Temp. gradient per 100 m.	Relat. humidity.
8 h. 50 m. a. m.	95	27°.0 C		79 %
			0°.67	
53	869	21 .8		83
			0 .54	
9 0	3072	9 .8		66
			0 .47	
8	5415	— 1 .2		67
			0 .63	
14 [1])	6740	— 9 .5		79
			0 .60	
20 [2])	8150	—18 .0		73
			0 .44	
26	9370	—23 .4		59
			0 .74	
33 [3])	10711	—33 .5		52
59	11543	—42 .6		45
			0.68	
10 8	9287	—27 .2		53
			0.75	
15 [4])	7002	—10 .0		75
			0.54	
22	5284	— 0 .8		92
			0.52	
30	3265	9 .6		76
			0.50	
38	1635	17 .8		99
			0.63	
45	95	27 .5		76

Remarks. [1]) From 9 h 14 m to 21 m the wind veers from WSW to ENE; [2]) Afterwards the temperature decreases somewhat more quickly (above 8000 m.); [3]) Here one balloon becomes leaky and the system is floating, showing influence of radiation. Finally the balloon bursts and a quick descent sets in. In descending the ventilation is sufficient; at a height of 2000 m. during the ascent there seems to be some influence of radiation. [4]). The increase of the humidity can be explained by the occurrence of alto-cumuli which in the mean time are being formed at a height of 5 K.M., and of cumuli at 1.5 K.M.

The diagram of the registering-balloon on May 19, during the passage of the earth through the tail of Halley's comet has as yet not been worked out.

It shows no other remarkable feature than an inversion of temperature between 6 and 7 K.M.; as stated above, the balloon burst at a height of somewhat more than 7 K.M.

Chemistry. — *"On the continuous connection between the three-phase lines which indicate the equilibria between the two components in the solid condition with liquid and vapour respectively, in a binary system."* By Dr. F. E. C. SCHEFFER. (Communicated by Prof. J. D. VAN DER WAALS).

In consequence of the theoretical essay of Prof. VAN DER WAALS on the equilibrium between a solid substance and a fluid phase, particularly in the neighbourhood of the critical condition [1]) SMITS studied in 1905 the hidden equilibria in the P-x-sections of BAKHUIS ROOZEBOOM's spacial figure below the eutectic point [2]). The two lines for the fluid phases which coexist with solid A and solid B, respectively, intersect each other in the isothermic sections below the eutectic point at two three-phase pressures: $S_A\,S_B\,L$ and $S_A\,S_B\,G$ which indicate stable conditions for the case that $\frac{dp}{dt}$ of the solid-liquid line of the components is negative, which was supposed to be so in the above mentioned paper.

As a further study as to the course of the lines S_A-Fluid and S_B-Fluid can enlighten us as to the continuous connection of the two said three-phase lines in the spacial figure I will trace its position for the most simple case: complete miscibility in the liquid condition, separation in the solid condition and gradual fall of the coexistence pressure $L-G$ from the one component to the other.

If for this purpose we consider the condition at a temperature situated a little above the quadruple point, the constant pressure lines will show in the V-x figure a course as indicated in Fig. 1, where the pressure continuously increases along an isometric line from $x=0$ to $x=1$. These isopiests have a vertical tangent on the lines ehf and ilk, the geometrical place where $\frac{d^2\psi}{dv^2}$ on the ψvx plane of the fluid phases is zero.

The points H and G, S and R situated on the binodal line ab and cd indicate the liquid and vapour which under three-phase pressure can coexist with S_A and S_B, respectively.

Let us now imagine that all the ψ values belonging to the $v-x$ points are deposited perpendicularly on the plane of drawing. The tangent plane turning through $\psi_{S_A}(V)$ over the surface of the fluid phases will describe a line $LGEDBAK$ indicating the fluid

[1]) Proc. 1903 Oct. 31.
[2]) Proc. 1905 Dec. 30.

phases which may coexist with the first component in the solid condition. In complete analogy herewith, by the moving of the tangent plane through $\psi_{S_B}(W)$, a line $QRPONMSK$ is formed indicating the fluid phases coexisting with S_B; on LG and QR solid A and solid B, respectively, exists with a gasphase; on HK and SK with a liquid phase [1]).

The two lines have each a point of maximum and a point of minimum pressure at the place where they intersect the spinodal line (egf and imk). The fact that in the intersecting points with the spinodal line the isopiest really touches the coexistence line solid-fluid is obvious from the equation deduced by van VAN DER WAALS [2]):

$$\frac{d^2\psi}{dv^2_f} v_{sf} \frac{dp}{dx_f} = (x_s - x_f) \left\{ \frac{d^2\psi}{dx_f^2} \frac{d^2\psi}{dv_f^2} - \left(\frac{d^2\psi}{dx_f dv_f} \right)^2 \right\}.$$

On the spinodal line, the factor of $x_s - x_f$ is zero; as the other quantities in this equation have generally all a finite value $\frac{dp}{dx_f}$ will be 0, which consequently points to the appearance of a maximum or a minimum pressure and which, therefore takes place in the points B and M (minimum) and P and D (maximum).

In the points A, E, N, and O where the nodal line Solid-Fluid touches the isopiest, $v_{sf} = 0$. From the above equation it, therefore, follows that in the said points the value of $\frac{dp}{dx_f}$ is infinite.

If, now, we observe the progressive change of the pressure values on the two coexistence lines S_A-Fluid and S_B-Fluid, and by the aid of this construct the corresponding P-x figure (Fig. 2), the P-x lines will show a vertical tangent in A, N, E, and O ($V_{sf} = 0$), in D, P, B, and M a horizontal tangent (points of the spinodal line) and in the points K, I, and F an intersection (three-phase pressures). From a joint examination of the V-x and the P-x figure it will then appear that the other numerous intersections are only incidental and do not indicate a coexistence of S_A, S_B and one fluid phase, because the intersecting points do not represent *one* but *two* different fluid phases with a different volume.

At the temperature to which Figs. 1 and 2 refer five three-phase pressures may occur: S_ALG, S_BLG and S_AS_BL (stable), S_AS_BG (metastable) and S_AS_BFl labile [3]).

[1]) It is assumed here that thesolid substances increase in volume on being melted.

[2]) Cont. II. 13 and l. c.

[3]) For clearness' sake the fluid phases between the spinodal line have been indicated here by "Fl".

If, now, we observe what changes occur at various temperatures, it will be noticed that ψ_S, which value is situated under the ψ-plane for the fluid phases, falls with elevation of temperature $\left(\frac{d\psi}{dt} = -\eta, \text{ in which } \eta \text{ is taken as positive}\right)$ and as η_G and $\eta_L > \eta_S$, shows a *smaller* fall than the ψ-plane for the same increase of temperature. The points of the solid substance, therefore approach the ψ-plane of the fluid phases; consequently the line described by the tangent plane turning over the surface will shift more towards the side of the solid phase; the line *LGEDBAK* therefore, shifts towards the left; *QRPONMSK* towards the right.

The consequence of this shifting will, therefore, be that the points I and F approach each other finally coinciding in a point where the two curves of the fluid phases, coexisting with S_A and S_B, come in contact with each other.

In quite an analogous manner, it will be obvious, that on lowering the temperature the points I and K approach each other and finally coincide also in one point. Before, however, I and K can meet, K will have to arrive within the binodal line and therefore in the metastable region; just at the moment that K passes the binodal line, F will pass from the metastable region into the stable one: this transition takes place in the quadruple point, where the stable coexistence of four phases is possible.

We, therefore, conclude that there exists a temperature traject where three three-phase pressures may occur which is limited at the higher temperature by the coincidence of I and F, at the lower temperature by the coincidence of K and I. The consequence of this will be, that the two three-phase lines $S_A\,S_B\,L$ and $S_A\,S_B\,G$ are continuously connected in their P-T- projection. In order to be able to judge about the shape of this connecting line we will observe a little more closely the transformation at temperatures where I and F approach each other.

When we consider that in D and P (points of the spinodal line) the isopiest touches the coexistence lines Solid-Fluid and that therefore when D and P coincide the two branches must necessarily have a common tangent, it is evident that the two intersecting points must necessarily coincide on the spinodal line. If this is to happen I will have to move through O, and F through E, a necessity which we read at once from the V-x-figure and which accounts for the fact that in the P-x-figure, just before the contact takes place, a situation occurs as indicated in Fig. 3^b. (In the V-x and P-x projection of

Fig. 3^a and 3^b the corresponding points are indicated by the same letters). The point T is here a point of incidental intersection; this will be clearly seen by looking at the line $\frac{d^2\psi}{dv^2}=0$, which is indicated in Fig. 3^a. The point where $\frac{d^2\psi}{dv^2}=0$ intersects the line IDF has a higher pressure than the points of equal x of the line IPF, and in the intersecting point of $\frac{d^2\psi}{dv^2}=0$ with the latter, the pressure is higher than in the point of equal x on the line IDF. It will be obvious, that somewhere on IDF there will be found a point where the pressure is equal to that at a point of equal x on IPF; this point is the intersecting point T.

At an elevation of temperature I shifts more and more towards the line $\frac{d^2\psi}{\partial v^2}=0$. When it has reached this line, I and T will so there is a contact in this point have coincided; as indicated in Fig. 4^a and 4^b.

If now, on a further elevation of temperature, the point I arrives between the line $\frac{d^2\psi}{dv^2}=0$ and the spinodal line, the points I and T will have exchanged their places (See Figs. 5^a and 5^b). It will be obvious that the point of incidental intersection T again corresponds here with a point in the figure 5^a of the line OV and UW of equal x and also lying on the same isopiest.

If the temperature is still further increased the points I and will finally coincide in D and P. As now p_I and $p_F < p_P$ and $\frac{dp_P}{dT}$ is always positive (because P moves along the spinodal line towards the side of higher pressure and the pressure in each point rises with an elevation of temperature) [1]) the coincidence of I and F with P will necessarily cause $\frac{dp_I}{dT}$ and $\frac{dp_F}{dT}$ to be positive also.

On the coincidence of I and F, $\frac{dp_I}{dT}$ and $\frac{dp_F}{dT}$ will moreover become equal, because in $\frac{dp}{dT_x}=\frac{W_{sf}}{TV_{sf}}$ the quantities W_{sf} and V_{sf} then relate

[1]) This is also shown from $\frac{dp}{dT_x}=\frac{W_{sf}}{TV_{sf}}$, because V_{sf} in P is negative and W_{sf} also.

to one and the same fluid phase. On continued elevation of temperature only incidental intersections in the P-x projection remain.

In the P-T-projection the three-phase line S_A-S_B-Fluid will, therefore, have always positive values for $\frac{dp}{dT}$ [1]); there exists a temperature traject where the pressure at constant temperature is trivalent; the stable-metastable branch is connected with the labile branch by means of a cusp. The connection is indicated in the P-T-projection of fig. 6.

Physics. — *"The magnetic separation of absorption lines in connection with sun-spot spectra."* (Third Part) [2]). By Prof. P. ZEEMAN and Dr. B. WINAWER.

Demonstration of oblique position of vibrations by means of half wave-length plate.

34. The observations published in our two preceding communications relate to the region between $\vartheta = 90^\circ$ and $\vartheta = 39^\circ$, the two principal directions inclusive. We now intend to describe in this third, conclusive, part of our paper experiments relative to the remaining region between $\vartheta = 39^\circ$ and 0°.

This region seemed very interesting because under suitably chosen circumstances it probably would contain the angle ϑ_1 of LORENTZ, separating the regions of the longitudinal and the transverse magnetic effect. The principal object we had in view in undertaking this third part of our investigation was to prove experimentally the existence of an angle of the kind mentioned. We think we attained our purpose.

Before proceeding to describe these experiments, we shall mention a method for verifying the results (24—32) relating to the oblique position of the vibration ellipses of the outer components and that of the vibrations of the inner components, but without commutation of the current in the electromagnet.

Whereas in our former experiments the *difference* of the intensity of the components by *commutation* of the current gives the proof for the obliquity of the components, the half wave-length plate demonstrates it at once.

A half wave-length plate with one of its principal directions situated horizontally and limited by a horizontal line is placed near the source

1) A quite analogous view may be applied to the coincidence of I and K.

2) Continued from these Proceedings Vol. XIII p. 35, 1910.

of light. Vibrations from the source, making a definite angle with the edge of the plate, after traversing it are rotated through twice that angle. The plate covers only half of the field of view. The directions of the emergent vibrations make the same angles with the horizontal edge as at first, but upon the further side.

An image of the edge is focussed upon the slit of the spectroscope; before the slit a Nicol is placed.

In one of our experiments, ϑ being 39°, the plane of vibration of the Nicol was under an angle of 35° with the horizon. The magnetic components are now seen unequally dark in the two halves of the field of view. It appeared possible to photograph the phenomenon; small variations of vapour density, which may possibly introduce errors with other methods of observation, are now without influence.

Reversion of the direction of the current, changes the sign of the difference of intensity of the two halves of the field of view.

Connection between the inclination of the ellipses in particular cases.

35. The direction of the magnetic field, and that of propagation of the beam, traversing the magnetized source of light, determine the sense of the inclination of the vibration ellipses (25). If the direction of the field be reversed, the sign of the inclination of the vibration ellipses also changes. In fig. 3 (§ 25) the connection established by our experiments, between the three mentioned directions is given.

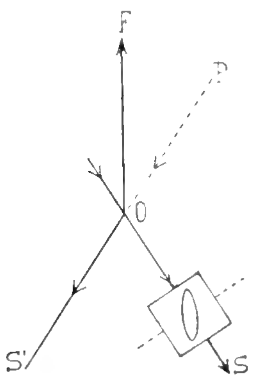

Fig. 6.

Let OF be the magnetic force, and let the beam, traversing the magnetized flame O, be propagated in the direction from O to S. The inclination of the ellipses in this case is indicated in fig. 6. The plane normal to the ray and containing the ellipse has been rotated round the dotted line until brought into coincidence with the plane of the paper.

What is the inclination, if the source of light be traversed by the beam in the direction OS'?

This question is easily answered by applying the well-known method of reflected images.

The geometrical outlines of all things composing a given system, together with the physical processes in the system, which we suppose may be all represented by geometrical figures, we imagine reflected at every instant in a plane V. The new system obtained by reflection and which we call the image of the original system is a possible one, as soon as the last mentioned one has an objective existence.

Applying this to our experiment (fig. 7) and placing the plane V parallel to OF and perpendicular to the plane of the paper we obtain from system I, the system II.

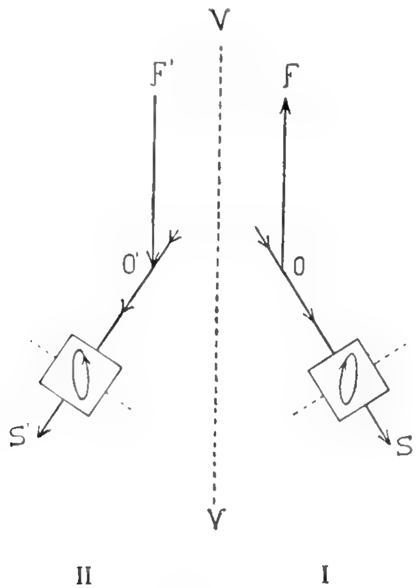

Fig. 7.

The magnetic field in the second system is the inverted image of the field in the first one; indeed, before taking the image of the field we have to substitute it by the equivalent Ampère currents.

Hence in II the arrow $F'O'$ is drawn from F' to O'.

Reversing afterwards the field in system II, the inclination of the ellipse changes its sign.

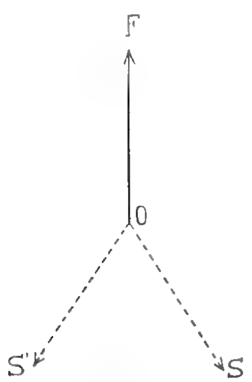

Fig. 8.

Hence we conclude that (fig. 8), if OF be the direction of the magnetic field, the inclination of the major axes of the ellipses, as observed from S as well as from S', is always from the lower left to the upper right quadrant.

By means of SAVART's polariscope all this could be experimentally verified.

We come to the same conclusion by using the experimental result of § 26, concerning the inclination of the ellipses in the beam emitted in the direction OP (see fig. 6).

The close connection, existing between emission and absorption, enables us to predict the phenomena to be seen if light traverses the source in the direction OS'. (cf. § 44).

Investigation concerning the existence of an angle ϑ_1 (§ 36—§ 46).

36. It seems possible to give by different ways experimental proof of the existence of an angle ϑ_1, separating the regions of the longitudinal and of the transverse effect.

The most direct proof would be given, if, with a chosen magnetic force, the vapour density could be changed in such a degree, that at last the direction of the vibrations in the issuing beam were inclined at an angle of 45° with the vertical. Then one would observe at the angle ϑ_1 itself, the values of density (width) and magnetic intensity corresponding. The following up of this plan gives rise however to serious difficulties.

The significance and the particularity of the angle ϑ_1 become however manifest also, if it be possible to establish the existence of the characteristic phenomena only observable for a direction of observation which forms an angle with the lines of force lying between 0° and ϑ_1. We have experimentally verified the theoretical inference.

We made many experiments belonging to each of the two classes of experiments mentioned and intend to give a few examples of each.

37. *Observations at* $\vartheta = 32°$. Soft iron cones with a vertex semiangle of 32° were made and adapted to a DU BOIS-electromagnet. The intensity of the magnetic field proved sufficient to establish the character of the resolution in the first order spectrum of the large ROWLAND grating.

The middle components were especially watched. It is easily established that the vibrations of these components deviate from the horizon. In order to demonstrate an inclination of 45°, a quartz plate, cut perpendicularly to the axis, and exactly 2 mm. thick, was introduced in the beam. This plate rotates the plane of polarization for sodium light $2 \times 21.7 = 43.4°$. Vibrations under azimut 45°, after traversing the plate, become either horizontal or vertical.

Between the plate and the spectroscope slit a calcspar rhomb was inserted and a horizontal slit placed near the source; two contiguous horizontal images of the slit are now formed on the slit. The one contains the vertical, the others the horizontal constituents of the beam.

The middle components, which at the angle ϑ under consideration are rather weak, are, dependent upon the direction of the current, visible either only in the upper or only in the lower of the two stripes, if the vapour density be properly chosen.

This experiment does not prove however definitely that the middle components may vibrate under an angle of 45° with the vertical. The rather limited sensitiveness of the method must be taken into account.

The experiment certainly proves that the vibrations are inclined relatively to the horizon, at an angle of perhaps 20° or 30°.

It is shown by an observation with the calcspar rhomb alone, after removal of the quartz plate, that the vibrations are not performed under 45°. A difference between the upper and the lower image is now manifest. This would be impossible, if the inclination of the vibrations were 45°. The difference of intensity in the two stripes decreases with increased density of the vapour.

All experiments undertaken in order to measure more accurately the inclination gave no decisive results. The weak intensity of the middle components, the feeble separation (just wanted according to theory for the observations in view), the perturbation by the vicinity of the outer components, and also the fact that the vibrations become probably slightly elliptic, account for the difficulty of the measurements.

We also investigated the emitted light without the aid of the spectroscope, with a SAVART polariscope alone; the emitted light appeared to be nearly unpolarized. The fringes in the polariscope were very weak. This is clearly due to the light containing equal portions of right-handed and left-handed nearly circularly polarized light; the intensity of the light of the middle components is relatively very small and therefore scarcely perceptible in the resulting total intensity.

The indistinctness of the fringes made only inaccurate determinations of the position of the plane of polarization possible. An inclination of 42° relatively to the vertical was found.

38. The method of the non-uniform field[1]) seemed to open the possibility of a direct reading of the field intensity corresponding to ϑ_1, the vapor density (i.e. the width of the spectralline) being given. At $\vartheta = 39°$, a diminished image of the cones of the electromagnet was focussed upon the slit plate of the spectroscope. The magnetic separation is different at different heights and in the spectroscope the spindle-shaped resolution figure, a photograph of which was given on a former occasion, is seen; but now, as the inverse effect is under consideration, rather dark lines on a luminous background are seen. A Nicol with its plane of vibration under 45°

[1]) ZEEMAN. These Proceedings April 1906, November 1907.

with the horizon is placed before the slit. If the vibrations occur under 45° somewhere in the divided lines, the components must become black at such a place. Width and field intensity, belonging to the mentioned part of the components, correspond to a value of ϑ_1 equal to 39°.

No clear result was obtained however by means of this method, which was tried with several vapour densities.

The change of the state of polarization in the resolution figure apparently is too gradual to prove the existence of ϑ_1 by direct observation.

Our following experiments (39—46), indeed, seem to leave no doubt as to the real existence of such an angle.

39. In order to extend observations to still smaller angles ϑ, the second order spectrum of the great ROWLAND grating was employed for all following observations. The brightness is still largely sufficient and more details are seen. Even with cones with a vertex semi-angle of 26° the characteristic phenomena may now be advantageously observed. With vapour of intermediate density (10) now only the outer components of the quartet and sextet are visible, the phenomenon closely resembling the pure longitudinal one. Middle components only make their appearance after the density is largely increased. The nature of these components appears (40) however to have changed, as is proved by an examination of their state of polarization.

The latter is more easily ascertained, if the components are more widely separated. This is the case in the experiments described in the next paragraphs and therefore we prefer to give some details of the observations made with the more efficient arrangement.

40. A still smaller angle between the directions of the beam and of the field may be employed and moreover wider separation obtained than in § 39, by looking through axial holes and deviating the beam in the field by means of two small prisms. A remark of Prof. WERTHEIM SALOMONSON induced us to give prisms a trial.

The arrangement for $\vartheta = 16°$ is shown in the next figure.

The prisms are fixed to copper tubes, which are put into the bored cones of a DU BOIS electromagnet and may be turned about their axes. It is therefore possible to adjust the parallelism of the planes of prisms and to arrange vertically the edges.

A drawback inherent to this method is, that after some time the interior surfaces of the prisms become covered with some white

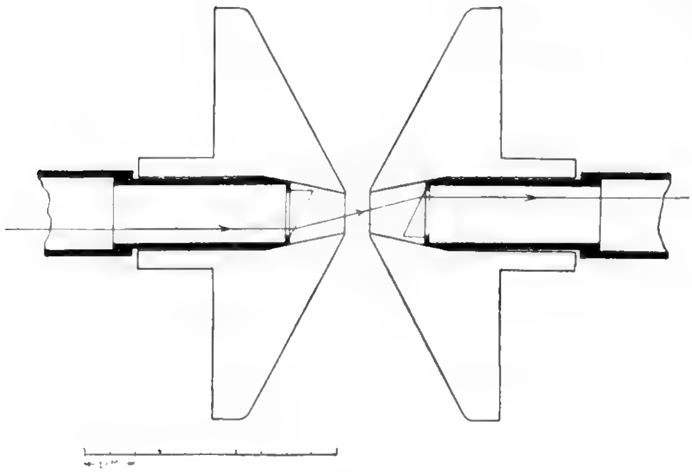

Fig. 9.

precipitate. With very dense vapours this inconvenience is rather troublesome.

Immediately after introduction of the flame into the interferrum aqueous vapour condenses upon the prism faces, soon disappearing, however, when the temperature of the prisms has increased. In order to avoid the danger of cracking, the prisms have been disposed at some distance from terminal planes of the cones.

Even with very dense vapour (third phase of § 10), the field being of the order of 20.000 Gauss, the phenomenon closely resembles the pure longitudinal one. No trace of middle components is visible.

After an increase, however, of the vapour density to the limit obtainable by the introduction of a glass rod, charged with melted salt, into the gas-oxygen flame, two new black lines appeared in the vicinity of D_1; they were clearly visible against the rather dark background formed by the broadened outer components.

These new lines, which have the same period as the middle components, are unpolarized (see 41—44).

41. We have come to this conclusion after trying in vain to detect any trace of polarization phenomena of the new components.

In the first place rotation of a Nicol, placed before the slit of the spectroscope, gave no change of intensity of the lines; only the background formed by the nearly, but not accurately, circularly polarized outer components was slightly changed.

42. After removal of the Nicol a quarter wave plate with its

principal direction under 45° was inserted in the beam and a broad horizontal slit placed near the field. By means of a calcsparrhomb two stripes are obtained, separating the oppositely polarized circular vibrations.

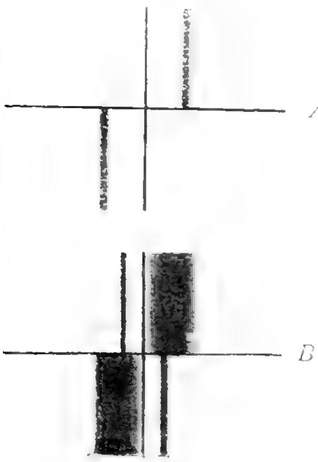

Fig. 10.

With vapour of intermediate density fig. 10^A gives the appearance for D_1. The vertical line represents the reversed line due to the arc light.

With very dense vapour, we get the phenomenon represented in fig. 10^B. New components appear in the initially bright parts of the field of view.

The positions of the new components correspond to those of the inner components of the quartet, at least as far as can be judged by eye observation. This observation is confirmed by measurements made on a photograph of, it must be said, only moderate quality.

As to the polarization of the new lines a few remarks may be made. From an inspection of fig. 10^B alone, one might conclude to a circular polarization of the inner components, of a sign opposite to that of the outer ones.

One might be tempted to infer that, under the circumstances of the experiments, the inner components are due to the motion of positive charges.

There is no need discussing the degree of probability of such a conclusion, as it is refuted by the next observation.

42. If the quarter wave plate be rotated in its own plane so that the principal direction more and more approaches to the horizontal position, the intensity of the outer components decreases. The inner components, which at first are invisible in two of the quadrants, being entirely hidden by the black, broad, outer components, are seen, already soon, as continuous bands crossing at right angles the horizontal separation line.

Finally, when the principal direction of the quarter wave plate has become horizontal, there is, as far as concerns the inner components, no difference at all between the upper and lower fields, and only a slight one as far as concerns the outer components.

43. From the observations recorded in §§ 41 and 42 we cannot

but conclude that under the circumstances of the experiment the inner components of the *new* quartet are unpolarized.

This result seems paradoxical, because one now has become accustomed to expect polarization of all magnetically separated and displaced lines.

The result, however, seems to be in perfect accordance with theory, at least if it be permitted to apply to the middle components of the quartet, the theoretical inference drawn for the central component of the triplet.

Lorentz has proved that in the case of a triplet for a frequency $n = n_0$ and $\vartheta < \vartheta_1$, two oppositely elliptically polarized beams may be transmitted, having the same index of absorption, but unequal velocities of propagation. The characteristic vibration ellipses for the two beams are the same, but described in opposite directions. (see also § 22 above).

Since the indices of absorption of the two beams are equal, we may expect that, under the circumstances mentioned, a magnetized vapour can produce in a continuous, unpolarized, spectrum only unpolarized absorption lines.

44. The consideration in § 35 of the reflected image of a system, was made in order to show that the inclination of the ellipses remains unaffected by a change of sign of the angle between the line of force and the ray.

45. *Quartet for* $\vartheta = 0$. By increasing still further the vapour density necessary for the § 43 experiment, we were able to observe even in the direction $\vartheta = 0$, the two unpolarized lines, corresponding to the inner components of the quartet. The outer components, however, have become then extremely diffuse.

It is certainly remarkable, that the two new components are still relatively narrow. The theoretical reason for this feature of the phenomenon has still to be worked out.

It is, however, in accordance with theory (always on the supposition that it does apply directly to the quartet) that for $\vartheta = 0^\circ$ the density of the vapour must exceed that for $\vartheta = 16^\circ$, in order to render visible the new lines. Indeed according to the formulae (42) and (26) of Lorentz's paper (cited in § 1 above) the absorption index decreases with decreasing ϑ.

The experiments (39—43) seem to give conclusive evidence that an angle ϑ_1 really exists.

Indeed, phenomena of the kind described in the last §§ are to be expected in a region only between ϑ_1 and 0°.

The experimental verification of LORENTZ's deductions, formulated in § 23 above, gives a new proof of the rational connexion established by VOIGT's theory of the inverse magnetic effect between diverse phenomena.

A more accurate measurement of ϑ_1, the vapour density and the field being chosen, must be postponed.

46. The new type of magnetic separation, with some components polarized, the other ones unpolarized, which returns to the ordinary separation by decrease of vapour density, we were able to observe also with D_2. Since the density of the vapour must be great in the present experiment, the effects observed with D_2, which splits up into a pseudo-triplet, are less clear and characteristic than with D_1. We, therefore, restricted the detailed description of our observations to the case of D_1.

Mathematics. — *"On continuous vector distributions on surfaces"* (3rd communication)[1]). By Dr. L. E. J. BROUWER. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of May 28, 1910).

§ 1.

The irrigating field on the sphere.

In order to get an insight into the structure of an arbitrary finite continuous vector field with a finite number of singular points on the sphere over its entire extent, we begin by investigating a particular case characterized by the *absence of simple closed tangent curves.*

In a field which possesses this property, and which we shall call an *irrigating field,* no spirals can appear as tangent curves and no rotation points as singular points. As farthermore a singular point can neither possess elliptic sectors or leaves, it is either a source point without leaves, or a vanishing point without leaves, or it possesses exclusively hyperbolic and parabolic sectors without leaves, in which case we shall speak of a *stroking point.*

The singular points of an irrigating field cannot all be stroking points. This follows from theorem 8 of the second communication [2]) in

1) For the first and second communication see these Proceedings Vol. XI 2, p. 850 and Vol. XII 2, p. 716.

2) l. c. p. 734.

connection with the observation, that the reduction of stroking points can lead only to reflexion points.

So there are certainly source points or vanishing points; to fix our thoughts we shall start from the existence of source points $B_1, B_2, \ldots\ldots B_m$.

In B_1 we start an arbitrary tangent curve which when pursued indefinitely can neither close itself, nor become a spiral. So it must stop at a singular point, which can be nothing but a vanishing point V_1.

If possible we then start in B_1 a second tangent curve, not crossing the first and stopping at an *other* vanishing point V_2.

If possible then in each of the two sectors generated in B_1 a tangent curve not crossing the two already existing ones and stopping either at a third vanishing point V_3, differing from V_1 and V_2, or, if that is excluded, stopping e.g. at V_1, but then in such a way that in B_1 a sector is determined limited by two tangent curves stopping at V_1, inside which we can draw a tangent curve not crossing the existing ones, starting from B_1 and stopping at V_2.

We continue this process of insertion as often as possible, whereby every time in each sector is inserted a tangent curve not crossing the existing ones which either stops at an other vanishing point as the two tangent curves limiting the sector, or, if that is excluded, determines a new sector, in which such an insertion is possible.

In this way it is impossible that at some moment a sector should appear limited by two tangent curves stopping at the same vanishing point, and within which no other vanishing point should lie.

So the number of tangent curves stopping at one and the same vanishing point, and appearing in this process of insertion, must remain smaller than the total number of vanishing points and from this ensues that the process of insertion ends after a finite number of insertions.

Of the then constructed finite system of tangent curves starting from B_1, which we shall call a *system of skeleton curves of* B_1, no two consecutive ones have the same vanishing point as their end-point.

Let for a certain sense of circuit those skeleton curves be consecutively $r_1, r_2, \ldots\ldots r_n$, stopping respectively at the vanishing points $V_1, V_2, \ldots\ldots V_n$, which of course need not be all different.

We then if possible introduce between every r_p and r_{p+1} a tangent curve starting in B_1 and stopping at a certain vanishing point, not crossing the already existing ones and reaching a distance as great as possible from r_p and r_{p+1}. In each of the sectors thereby generated at B_1 we repeat such an insertion, in each of the sectors thereby generated again and so on; finally after having repeated this process of insertion ω times, we add the limiting curves, which are likewise tangent curves starting from B_1 and stopping at certain vanishing points. After that, as ensues from the reasoning followed in § 2 of the second communication [1]), no new tangent curves starting from B_1 can be inserted, whilst the constructed tangent curves cover on the sphere a closed coherent set of points, to which belong all possible tangent curves starting from B_1, and which we shall call *the irrigation territory of* B_1.

The method according to which the skeleton curves have been constructed implies furthermore that between every r_p and r_{p+1} two tangent curves r''_p and r'_{p+1} appear, between which no further tangent curves starting from B_1 can be constructed, whilst all tangent curves, which have been constructed between r_p and r''_p, end in V_p, and all tangent curves, which have been constructed between r'_{p+1} and r_{p+1}, end in V_{p+1}.

From this ensues that these curves r''_p and r'_{p+1} coincide from B_1 up to a certain stroking point S_p, beyond which they diverge for good.

For, when diverging either in a non-singular point or immediately in B_1, insertion of new tangent curves starting from B_1 would be possible.

And also when rejoining after having previously diverged, an insertion of a new tangent curve starting from B_1 would be possible, namely of such a one that had with r''_p as well as with r'_{p+1} an arc in common.

So the irrigation territory of B_1, consisting of n sectors Σ_p, each limited by a tangent curve r'_p and a tangent curve r''_p, possesses an outer circumference $V_1 S_1 V_2 \ldots V_n S_n V_1$, consisting of $2n$ tangent arcs, which we shall call its "sides". It may happen here, that an even side $S_p V_{p+1}$, and an odd side $S_q V_q$ (p and q different) *touch each other outwardly* along an arc PV_{p+1} resp. PV_q (which can expand to an entire side $S_p V_{p+1}$ or $S_q V_q$, or reduce itself to a point V_{p+1} resp. V_q) but not in an other way.

For, when two such sides $S_p V_{p+1}$ and $S_q V_q$ have collided somewhere outwardly, they cannot leave each other any more before V_{p+1} resp. V_q has been reached. Otherwise a tangent curve coinciding partially with $S_p V_{p+1}$ and partially with $S_q V_q$ might be inserted, which would separate $S_p V_{p+1}$ and $S_q V_q$, so that these could not have collided with each other, but only with the newly inserted tangent curve. The sectors Σ_p connecting in this way B_1

[1]) l. c. p. 723.

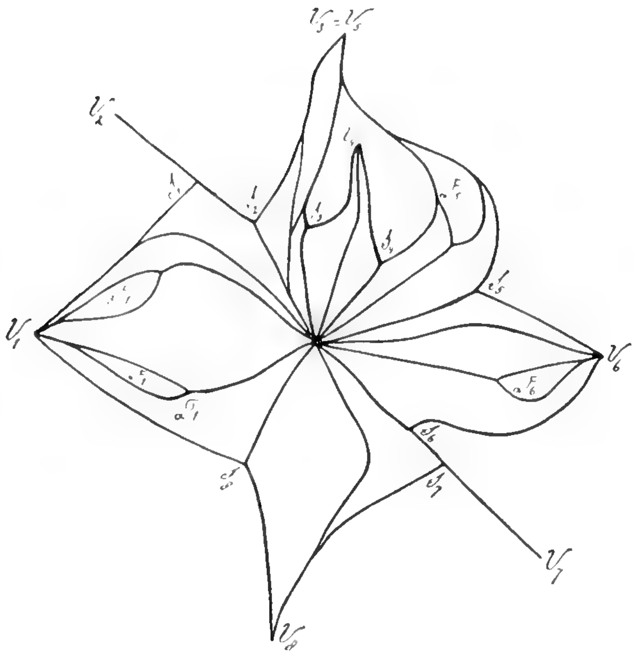

Fig. 1. Irrigation territory.

with one and the same vanishing point possess round about that vanishing point the same cyclic order as about B_1.

Let us consider a sector Σ_p. The limiting tangent curves r'_p and r''_p can collide inwardly in an arbitrary closed set of points (which in particular can entirely cover those curves). Farthermore it is not necessary that the entire inner domain determined by r'_p and r''_p belongs to Σ_p. However for each region ${}_\alpha\Gamma_p$ between r'_p and r''_p not belonging to Σ_p the property holds that it is limited by two tangent curves ${}_\alpha\varrho_p$ and ${}_\alpha\varrho'_p$ running from B_1 to V_p (between which no further tangent curves starting from B_1 can be constructed), which coincide from B_1 up to a certain stroking point ${}_\alpha\sigma_p$, then diverge, and finally after rejoining in a point ${}_\alpha H_p$ (which can also coincide with V_p) remain united to their end in V_p. If namely the latter property were lacking, then a new tangent curve starting from B_1 could be inserted. As finally the stroking point ${}_\alpha\sigma_p$ must give inside the region ${}_\alpha\Gamma_p$ two (and not more than two) hyperbolic sectors, only a finite number of points ${}_\alpha\sigma_p$ can coincide in one and the same stroking point, and from this ensues that there is only a finite number of regions ${}_\alpha\Gamma_p$.

The preceding shows that the residual regions determined on the sphere by the irrigation territory of B_1, are each bounded by a single inner circumference $V_{\alpha_1} S_{\alpha_1} V_{\alpha_2} S_{\alpha_2} \dots V_{\alpha_n} S_{\alpha_n} V_{\alpha_1}$, whose sides each join a stroking point and a vanishing point, in such a way that two successive sides concurring in a vanishing point V_{α_p} can touch each other inwardly from a certain point P up to V_{α_p}, but other inner contacts are excluded, and farthermore that each stroking point S_{α_p} possesses in the considered residual region *two* hyperbolic sectors.

The irrigation territory s_1 of B_1 possesses a finite distance from all the remaining source points.

If we construct for B_2 the irrigation territory analogously as for B_1, these two irrigation territories can partially penetrate into each other. This can however, when constructing the irrigation territory of B_2, be prevented by enforcing on its tangent curves starting from B_2 the condition that they may neither cross each other nor any tangent curve starting from B_1, whilst for the rest we act in the same way as before.

In that manner we have the *irrigation territory s_2 of B_2, independent of B_1*, containing all those tangent curves starting from B_2 which do not cross any tangent curve starting from B_1. The structure of s_2 is entirely the same as of s_1. Between s_1 and s_2 outward contact may take place on account of the coincidence of an even (resp. odd) side $S_\alpha V_\gamma$ of s_1 and an odd (resp. even) side $S_\beta V_\gamma$ of s_2 along an arc $P V_\gamma$, which can expand to an entire side $S_\alpha V_\gamma$ or $S_\beta V_\gamma$ or can reduce itself to the point V_γ. Farthermore s_2 lies entirely in *one* of the residual regions determined by s_1, however in such a way, that between two successive sides of this region which are inwardly pressed together, s_2 can very well penetrate to the vanishing point in which those sides concur. Together s_1 and s_2 contain all tangent curves starting from B_1 or B_2. For the residual regions which are determined on the sphere by s_1 and s_2 together the same properties hold as for the residual regions of s_1 alone.

In one of those residual regions lies B_3 at a finite distance from s_1 and s_2, and in that region we construct the *irrigation territory s_3 of B_3, independent of B_1 and B_2*, containing all those tangent curves starting from B_3 which do not cross any tangent curve starting from B_1 or B_2. Together s_1, s_2, and s_3 contain then all the tangent curves starting from B_1, B_2 or B_3. Outward contact between s_3 and s_1 or s_2 can take place in the same way as between s_1 and s_2.

In a quite analogous way we construct s_4 in one of the residual regions determined by s_1, s_2, and s_3. And in this way we go on. When we have constructed $s_1, s_2, \dots s_{m-1}$, then the sphere is not yet quite covered. For, the system of the tangent curves starting from

B_1, B_2, ... B_{m-1} cannot approach B_m within a certain finite distance. But after insertion of s_m the sphere is completely covered, for the set of the tangent curves starting from B_1, B_2, ... B_{m-1}, B_m is identical with the set of *all* tangent curves, so must cover the sphere entirely, and we have proved:

THEOREM 1. *An irrigating field divides the sphere into a finite number of irrigation territories each of which contains in its interior one of the source points.*

A clear example of an irrigating field is the force field of a finite number of positive and negative divergency points [1]).

The notion of irrigating field can be extended in the following manner:

Let be given on the sphere a multiply connected region γ, bounded by a finite number of coherent boundaries, and in γ a finite, continuous vector distribution, which continuity is uniform with the exception of a finite number of points. We then can construct of the region γ exclusive of its boundaries a continuous one-one representation on a sphere β in such a way that to the boundaries of γ correspond on β single points. The tangent curves of γ are thereby represented on a set of simple curves ϱ described in a certain sense. If among these curves ϱ no simple closed curves appear, they determine on β the structure of an irrigating field. In that case we shall call the given vector field in γ likewise an irrigating field.

This more general irrigating field differs thereby from the particular kind first considered that a boundary can play the part of a singular point. We accordingly distinguish *source boundaries, vanishing boundaries,* and *stroking boundaries.* From this ensues that in the more general irrigating field also spirals can appear as tangent curves, namely such whose windings converge uniformly to a source boundary or to a vanishing boundary.

§ 2.

The most general field with a finite number of singular points.

Let there be given an arbitrary finite continuous vector field on the sphere with a finite number of singular points. Let N be one of the singular points, then we shall say that a closed tangent curve *flows round about N*, if it does not contain a singular point, and encloses a region in which lies N but no other singular point. Fartheron we shall say that a closed tangent curve *flows round*

[1]) Compare my paper: "*The force field of the non-Euclidean spaces with positive curvature*", these Proceedings Vol. IX 1, p. 250.

against N, if it contains N but no other singular point, and encloses a region in which no singular point lies.

If there is neither a tangent curve flowing round about N nor a tangent curve flowing round against N, then we shall call N a *naked singular point*, otherwise a *wrapped singular point.*

We shall assume that N is a wrapped singular point and we shall distinguish two cases:

First case. There is no tangent curve flowing round about N. Let then ϱ be a tangent curve flowing round against N and let us agree about an arbitrary tangent curve r inside ϱ, that, when it reaches ϱ, we shall pursue resp. recur it along ϱ, until it reaches N; in this way r also becomes a tangent curve flowing round against N. We can thus fill the inner domain of ϱ with tangent curves flowing round against N and not crossing each other in the same way as in the second communication p. 727 was executed for an elliptic sector.

If we now construct a well-ordered series continued as far as possible of tangent curves flowing round against N, enclosing ϱ and bounding outside ϱ an ever increasing area, then it converges either to a tangent curve flowing round against N, or to a *circumference* consisting of simple closed tangent curves which can contain besides N still other singular points and which possesses all the properties deduced in the second communication p. 720 and 721 for the limiting

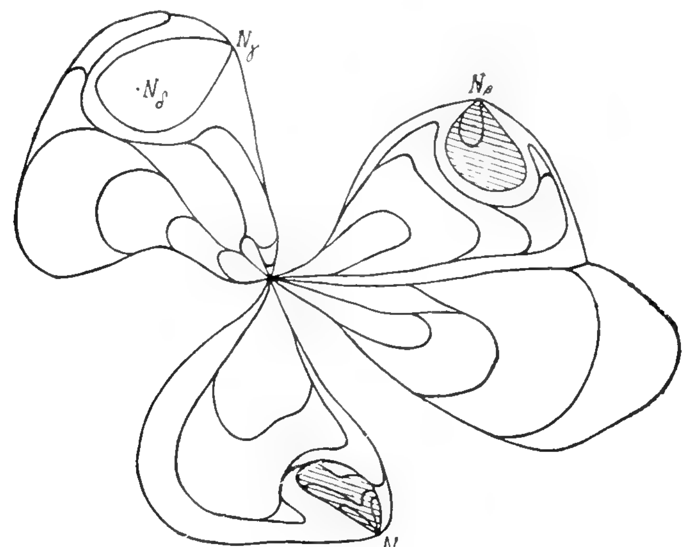

Fig 2. Circumfluence territory with (shaded) additional territories.
First case.

circumference of a spiral tangent curve. The inner region of that circumference, which can be entirely filled with tangent curves flowing round against N and not crossing each other, we shall call a *circumfluence sector of* N.

The singular point N can possess an infinite number of circumfluence sectors lying outside each other, but amongst these there are only a finite number, which reach an arbitrarily assumed finite distance from N.

The set of regions covered by the different circumfluence sectors of N we shall call the *circumfluence territory* of N.

We shall now regard of this circumfluence territory those residual regions which are bounded by a tangent curve flowing round against an other singular point N_x, and we shall fill them with tangent curves flowing round against N_x and not crossing each other. The set of regions filled in this way with tangent curves possesses at each of the points N_x entirely the structure of a circumfluence territory, and we shall call it an *additional circumfluence territory of* N. The point N possesses then only a finite number of additional circumfluence territories.

The circumfluence territory of N determines with its additional territories together a finite number of residual regions on the sphere.

Second case. There exists a tangent curve flowing round about N. Let ϱ be that curve, we then construct from ϱ outwards a well-ordered series continued as far as possible of tangent curves flowing round about N, enclosing ϱ and bounding outside ϱ an ever increasing area. The limit τ_1 to which this series converges is either a tangent curve flowing round about N, or a *circumference* containing singular points, consisting of simple closed tangent curves, and possessing all the properties deduced in the second communication p. 720, 721 for the limiting circumference of a spiral tangent curve.

Let us construct likewise from ϱ inwards a well-ordered series continued as far as possible of tangent curves flowing round about N, enclosed by ϱ, and limiting around N an ever decreasing area, then the limit τ_0 to which this series converges is either the point N, or a tangent curve flowing round about N, or a circumference consisting of a finite or countable set of tangent curves flowing round against N.

If τ_0 is a circumference containing N, we can fill up its inner regions with tangent curves flowing round against N and not crossing each other.

If τ_0 is a tangent curve flowing round about N, there can exist no tangent curve flowing round against N and having with a τ_0

point in common. For then in the terminology of § 3 of the second communication we should possess between N and τ_0 a positive as well as a negative curve of the third kind, from which we could start to fill the inner region of τ_0 with tangent curves not crossing each other. We should then have to find there the number of elliptic sectors equal to the number of hyperbolic sectors; so there would have to be at least *one* hyperbolic sector inside τ_0; this would however give rise to tangent curves flowing round about N and lying inside τ_0, which is excluded.

So if τ_0 is a tangent curve flowing round about N, then there exists inside τ_0 at a finite distance from τ_0 a circumference τ'_0 containing N, consisting of a finite or countable set of tangent curves flowing round against N, and inside which lie all existing tangent curves flowing round against N. If τ'_0 does not reduce itself to the single point N, its inner regions can be filled with tangent curves flowing round against N and not crossing each other.

The tangent curves not crossing each other with which the annular region between τ_0 and τ'_0 can be filled, must on one side either all enter into τ_0 or all converge spirally to τ_0, and on the other side either all enter into τ'_0 or all converge spirally to τ'_0.

In order to fill up the annular region between τ_0 and τ_1 with tangent curves not crossing each other, we construct in it a tangent curve $r_{\frac{1}{2}}$ flowing round about N and reaching from τ_0 and τ_1 a distance as great as possible. Between τ_0 and $r_{\frac{1}{2}}$ we then if possible insert a tangent curve $r_{\frac{1}{4}}$ flowing round about N and reaching from τ_0 and $r_{\frac{1}{2}}$ a distance as great as possible; likewise between $r_{\frac{1}{2}}$ and τ_1 if possible a tangent curve $r_{\frac{3}{4}}$ flowing round about N and reaching from $r_{\frac{1}{2}}$ and τ_1 a distance as great as possible. This inserting process we repeat as often as possible, eventually ω times, and finally we add the limiting curves. We are then sure that no more tangent curves flowing round about N can be inserted, so that eventually the regions between τ_0 and τ_1 remained empty of tangent curves must be annular regions.

Let α be such an annular region bounded by the tangent curves r_p and r_q flowing round about N, then α can be filled with tangent curves not crossing each other, which on one side either all enter into r_p or all converge spirally to r_p, and on the other side either all enter into r_q or all converge spirally to r_q.

The inner region of τ_1, in this manner entirely filled with tangent curves not crossing each other, we shall call the *circumfluence territory* of N.

We shall farther of this circumfluence territory fill each residual

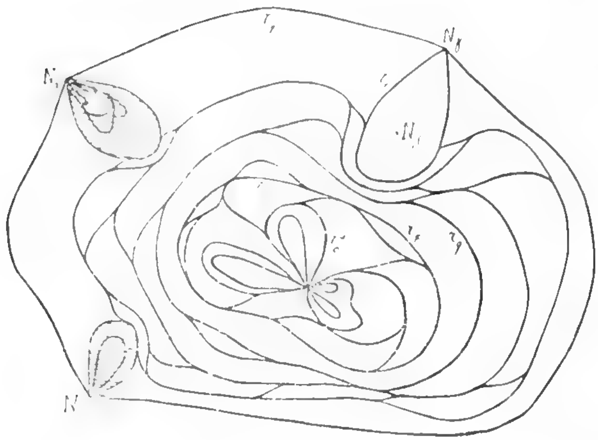

Fig. 3. Circumfluence territory with (shaded) additional territories. Second case.

region, bounded by a tangent curve flowing round against a singular point N_α, with tangent curves flowing round against N_α and not crossing each other. In this manner we add to the circumfluence territory of N a finite number of *additional circumfluence territories*, after which there remain on the sphere only a finite number of residual regions.

Let us now consider on the sphere a finite and, with the exception of a finite number of points, uniformly continuous vector distribution in a multiply connected region γ with a finite number of coherent boundaries. By a *closed tangent curve* we shall understand here, besides each tangent curve to which we have formerly given this name, each system of n simple tangent arcs not meeting each other and n cyclically ordered boundaries or singular points not contained in a boundary $N_1, N_2, N_3 \ldots N_n$, between which those tangent arcs run consecutively from N_1 to N_2, from N_2 to $N_3, \ldots$ and from N_n to N_1. In particular thus a simple tangent arc whose endpoints lie on one and the same boundary forms together with that boundary a closed tangent curve. Fartheron we shall understand by the *boundaries* of such a field for shortness' sake also the singular points which are not contained in a boundary. Finally we shall call a closed tangent curve not containing a boundary, and enclosing in γ a region in which lies N but no other boundary, *a tangent curve flowing round about* N, and we shall call a closed tangent curve containing N but no other boundary, and enclosing in γ a region in which lies no boundary, *a tangent curve flowing round against* N. *Naked* and *wrapped boundaries* we then define analogously as before naked and wrapped singular points.

For a wrapped boundary the circumfluence territory can be constructed in the same way as was done above for a wrapped singular point; the whole of its structure undergoes in this more general case no change, we only have to replace closed tangent curves in the narrower sense by closed tangent curves in the wider sense. The filling with tangent curves not crossing each other and the completion of the territories by means of its additional territories needs no modification either.

We shall understand by the *order* of the field twice the number of its naked boundaries plus three times the number of its wrapped boundaries.

We shall now start from a finite and, with the exception of only a finite number of points, uniformly continuous vector field in a region of the sphere with a finite number of coherent boundaries, each of which either reduces itself to a single point, or consists of tangent arcs turning one of their sides to the field, whilst in the latter case we assume that each fundamental series of consecutive points in a segment of a boundary determines only *one* limiting point, which property we express by calling the boundary *simple.* So the appearance of spirals in the boundaries is excluded.

We shall indicate two operations, both of which reduce this field to a finite number of fields of the same kind but of a lower order:

First reducing operation: We construct in the given field such a closed tangent curve which together with each of the two partial fields determined by it contains at least two of the boundaries of the given field.

Then namely each of the two partial fields is of a lower order than the original field.

Second reducing operation: we construct to a wrapped boundary the circumfluence territory with its eventual additional territories.

Then namely each of the residual fields is of a lower order than the original field.

It is clear that after a finite number of applications of these reducing operations either nothing of the original field is left or there remain only such fields to which neither of the two operations can any more be applied.

Then however in these residual fields there exists no closed tangent curve, so that they are irrigating fields.

If these last remaining residual fields are lacking, then the original field can be divided by *simple* boundaries consisting of tangent arcs into a finite number of circumfluence territories with additional territories which property we shall express by calling it a *circumfluent field.*

The circumfluent field can be regarded as the counterpiece to the irrigating field analysed in § 1.

A clear example of a circumfluent field is the force field without divergences of a finite number of positive and negative rotation points. [1])

We now have proved:

THEOREM 2. *A finite continuous vector field on the sphere with a finite number of singular points can be divided by simple boundaries consisting of tangent arcs into a finite number of irrigating fields and a finite number of circumfluence territories.*

At the same time we notice that among the tangent curves not crossing each other, with which in the preceding pages we have filled the field, spirals *cannot* appear in the *boundaries* of the irrigating fields or circumfluence territories meant in theorem 2, and in their *interior* exclusively in the following two ways:

1st. A circumfluence territory of the second kind can contain annular regions filled with spirals.

2nd. An irrigating field can possess source boundaries or vanishing boundaries round about which all tangent curves arrive resp. depart spirally.

§ 3.

The theorem of the invariant point on the sphere.

In the first communication on this subject (these Proceedings Vol. XI 2) we have on page 857 brought an arbitrary continuous one-one transformation of the sphere in itself into relation with the vector distribution for which in each point the vector direction is determined by the shortest arc of principal circle joining that point with its image point, for which distribution appear as singular points: 1st. the points invariant for the transformation. 2nd. the points having their antipodic points as their image points. The singular points of the latter kind form for transformations with inversion of the indicatrix as well as for transformations with invariant indicatrix a closed set of points of the most general kind which makes it pretty well

[1]) Compare my paper quoted above: "*The force field of the non-Euclidean spaces with positive curvature*".

impossible to deduce out of the properties of the vector distribution, either by means of theorem 2 of the first communication, or by means of theorem 8 of the second communication, the existence of at least *one* invariant point for transformations with invariant indicatrix.

The difficulty caused by this inconvenient set of points disappears however for an other vector distribution deduced from the transformation.

To construct this distribution we bring through each point P a circle containing its image point P' and a fixed point O, and we determine the vector direction in P by the arc of circle PP' *not* containing O. Let Q be the point having O as its image point, then as singular points of this vector distribution appear 1st. the point O. 2nd. the point Q. 3rd. the points invariant for the transformation.

If this vector distribution has an infinite number of singular points, then there are certainly points invariant for the transformation; so we assume in the following that the number of singular points is finite, and we investigate first the nature of the singularity in O.

For a point P in sufficient proximity of O the vector direction differs indefinitely little from the direction of the geodetic arc of circle OP. So by a circuit of a small circle about O the total angle which the vector turns with respect to the tangent to the small circle is zero, *so that when reduced the singularity gives rise to a radiating point.*

To investigate the nature of the singularity in Q, we represent the sphere stereographically on a Euclidean plane in such a way that O represents the infinite of the plane. Then in this plane the vector distribution is determined in each point by the straight line segment joining the point with its image point.

In the Euclidean plane the image of an infinitesimal circle about Q is an infinitely large circle; the infinitesimal circle and the infinitely large circle possess for transformations with invariant indicatrix *opposite senses of circuit*; for transformations with inversion of the indicatrix *equal senses of circuit.*

In the former case the vector describes in a circuit of the infinitesimal circle an angle 2π *in a sense opposite to the circuit*; in the latter case an angle 2π *in the same sense as that of the circuit.*

So when reduced the singularity in Q gives rise for transformations with invariant indicatrix to a reflexion point, for transformations with inversion of the indicatrix to a radiating point.

Thus the two radiating points, which according to theorem 8 of the second communication (p. 734) must be present in the reduced distribution, appear for a transformation with inversion of the indicatrix in the points O and Q; for a transformation with invariant indicatrix however the second radiating point can be furnished only by a point invariant for the transformation, *which therefore must necessarily exist.*

§ 4.

The index relation on the sphere for a finite number of singular points.

We shall now discuss the questions whether the number of singular points of a finite continuous vector distribution on the sphere, which according to theorem 2 of the first communication cannot be zero, is arbitrary for the rest, and farther whether the structure of the singular points, which according to theorem 8 of the second communication is not entirely free, is liable to still other restrictions than those expressed in that theorem.

These questions can be fully answered by means of the following reasoning, which is analogous to the proof of EULER's law, and which was indicated to me by Prof. HADAMARD.

The total angle which for a finite stereographic representation of the inner region of a simple closed curve enveloping only *one* singular point on a Euclidean plane the vector describes by a circuit in the sense of that circuit, and which according to theorem 5 of the second communication (page 731) is equal to $\pi\,(2+n_1-n_2)$, where n_1 represents the number of elliptic sectors, n_2 the number of hyperbolic sectors of the singular point, can be written in the form $2k\pi$, where k is an integer, which we call the *index* [1]) of the singular point.

For a simple closed curve, enveloping n singular points with indices $k_1, k_2, k_3, \ldots\ldots k_n$, the total angle which, for a finite stereographic representation of the inner domain of that curve on a Euclidean plane, the vector describes by a circuit in the sense of that circuit, is equal to $2\pi\,(k_1+k_2+\ldots\ldots k_n)$, as is immediately evident when we divide the inner domain under observation by means of arcs of simple curve into n inner domains of such simple closed curves, which each envelop only *one* of the singular points.

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[1]) This expression is used (not for the singular point itself but for a curve by which it is enclosed) by POINCARÉ: "*Sur les courbes définies par une équation différentielle*", 1[er] mémoire, Journ. de Math. (3) 7, p. 400. The univalent continuous vector distributions treated there are of a particular algebraic kind, so that only indices $+1$ and -1 appear for the singular points.

We now make on the sphere a circuit along a certain principal circle on which lies no singular point; the total angle, which in the sense of a certain indicatrix on the sphere the vector direction describes by that circuit with respect to the tangent direction, is equal to $2h\pi$, where h is an integer.

The sense of that circuit is with respect to one of the hemispheres, into which the sphere is divided by that principal circle, the same as the sense of the indicatrix, with respect to the other opposite to the sense of the indicatrix; so for a circuit of the first hemisphere the vector describes with respect to the tangent direction an angle $2h\pi$ in the sense of the circuit, for a circuit of the second hemisphere an angle $2h\pi$ opposite to the sense of the circuit.

The total angle which, for finite stereographic representation of the first resp. the second hemisphere on a Euclidean plane, the vector describes by a circuit in the sense of that circuit, is thus equal to $2(1+h)\pi$ resp. $2(1-h)\pi$.

If in the first hemisphere lie m singular points with indices $k_1, k_2, \ldots k_m$, in the second hemisphere $n-m$ singular points with indices $k_{m+1}, k_{m+2}, \ldots k_n$, we have

$$\begin{array}{l} k_1 + k_2 + \ldots k_m = 1 + h, \\ k_{m+1} + k_{m+2} + \ldots k_n = 1 - h, \\ \hline k_1 + k_2 + \ldots + k_{n-1} + k_n = 2, \end{array}$$

so that the sum of the indices of the singular points is equal to 2, a generalisation of the relation deduced by POINCARÉ for the particular case treated by him [1]), whilst the structure of the singular points is submitted to the following restrictive property:

THEOREM 3. *Twice the number of singular points plus the number of elliptic sectors is equal to the number of hyperbolic sectors plus four.*

The necessary existence of at least one singular point before reduction as well as of at least two radiating points after reduction lies included in this theorem and finds there its simplest proof.

We shall finally show that the set of singular points (supposed finite) is submitted to no other restriction than the one expressed in theorem 3.

Let us namely assume an arbitrary finite set of points as singular points, let us enclose them each by a suchlike simple closed curve that these curves do not intersect each other, and let us give inside and on these curves to the vector field a structure satisfying theorem 3 but for the rest arbitrary. We must then show that the outer domain

[1]) l.c. p. 405.

of these curves can be filled up with a finite continuous vector distribution *without* singular points and passing into the already existing ones.

To that end we take for the closed curves a certain cyclic order and join each of them with the succeeding one by such an arc of simple curve that these arcs do not intersect each other, so that on the sphere two free regions γ_1 and γ_2, bounded by simple closed curves, are determined. We then construct along the inserted arcs of curve suchlike finite continuous vector distributions without singular points and passing into the existing ones that the total angle, which for finite stereographic representation of γ_1 on a Euclidean plane the vector describes in a circuit, is zero. Then γ_1 can be filled, in the manner indicated in the second communication p. 732, 733, with a finite continuous vector distribution *without* singular points and passing into the existing ones.

As now however the singularities have been chosen in such a way that they satisfy theorem 3, the vector describes in a circuit of the complementary domain of γ_2, stereographically represented on a finite region, a total angle 4π in the sense of the circuit; thus by a circuit of the region γ_2 itself, when stereographically represented on a finite region, a total angle zero. Therefore γ_2 also can be filled with a finite continuous vector distribution *without* singular points and passing on its boundary into the existing ones, with which the lack of other restrictions than those expressed in theorem 3, has been proved.

As for the singular points (supposed to form a finite set) of a finite continuous vector distribution in the Euclidean plane, neither their number, nor their structure is submitted to any restriction.

E R R A T U M.

In the first communication on this subject, these Proceedings Vol. XI 2, p. 856, l. 3 and 7 from top

for: recure it, meets   read: recur it, it meets

Zoology. — *"The saccus vasculosus of fishes a receptive nervous organ and not a gland"*. By Prof. J. Boeke and K. W. Dammerman. (Communicated by Prof. A. A. W. Hubrecht).

(Communicated in the meeting of May 28, 1910).

In 1901 one of us came to the conclusion, based on the study of the development and of the histological structure of the saccus vasculosus in embryos and larvae of Teleosteans [1]), that in the saccus vasculosus of fishes we must not see a gland, as it is done generally ("Infundibulardrüse" of RABL-RÜCKHARD), but a sense organ, a nervous structure, that is stimulated by a distinct stimulus and reacts upon this stimulus in a certain way ("Infundibular organ"). In the following year (1902) in these same proceedings were published further observations on the saccus vasculosus of larvae and full grown specimina of Teleosteans, which seemed to point to the same conclusion [2]), and a similar organ was discovered in the ventral wall of the brain of *Branchiostoma lanceolatum* [3]). In 1902 JOHNSTON came to a similar conclusion for the saccus of *Acipenser rubicundus* [4]) after studying the nervous fibers and the form of the cells in the saccus. According to this author the saccus vasculosus is stimulated by variations of the pressure of the blood or of the ventricle-fluid inside the brain-ventricle. In response to this stimulus however the saccus may secrete some specific constituents of the ventricular fluid (comp. JOHNSTON 1906). Two years ago (1908) the development and the structure of the infundibular sense organ of amphioxus were described more fully by one of us [5]), and its homology with the saccus vasculosus of fishes was more clearly defined.

But although several authors [6]) acknowledged the value of the hypothesis of this homology, yet it is always (with the exception of JOHNSTON in the nervous system of vertebrates 1906 and KAPPERS in 1904 and 1906) taken for granted, that the saccus vasculosus is only a glandular structure [7]), secreting the whole or a part of the fluid, filling the ventricles of the brain.

Renewed investigations in this direction showed us, that this is not right, that the peculiar elements, which form the epithelial lining of the wall of the saccus of all the forms which possess a saccus vasculosus in its developed state, are not glandular cells (STUDNICKA), but that their interpretation of sensory cells, receptive

1) J. BOEKE. Die Bedeutung des Infundibulums in der Entwickelung der Knochenfische. Anat. Anzeiger Bd. 20, 1901.

2) J. BOEKE. in: Proceedings K. A. W. Math. phys. cl. Meeting of January 25, 1902.

3) J. BOEKE. in: Proceeding K. A. W. Math. phys. cl. Meeting of April 19, 1902.

4) J. B. JOHNSTON. The Brain of acipenser, in: Zool. Jahrbücher Bd. XV 1902.

5) J. BOEKE. Das Infundibularorgan im Gehirn von Amphioxus. Anat. Anzeiger Bd. 32. 1902 and Proceedings K A. W. Meeting of May 1908.

6) Among others by WIEDERSHEIM, in his "Vergl. Anatomie" 1909, JOHNSTON 1906, EDINGER 1909.

7) WIEDERSHEIM, B. HALLER, EDINGER, etc. Even JOHNSTON attributes a secretory function to the saccus epithelium, but does not say to which part of it.

nervous elements, set forth in the publications mentioned above, is right, and *that they are to be looked upon as sense cells, able to be stimulated by a distinct stimulus, and able to carry that stimulus along distinct nerve-tracts to sharply defined parts of the brain, cells which do not end with a rounded base, as gland cells would do, but are there prolonged each into a fine nerve fiber, all these fibers finding their way into and constituting a nerve tract, in the same way as the neurites of the olfactory cells form the fila olfactoria.*]

Typical gland cells are not to be found in the saccus vasculosus. As far as we could gather, the saccus vasculosus has no secretory function.

In this communication we intend to publish the general results of the renewed investigations we undertook in connection with the previous investigations mentioned above. A more detailed description will be found in the inaugural dissertation of the last-named of us two.

These general results may be laid down and summarised in the following points:

1. As to the infundibular organ in the brain of Amphioxus *(Branchiostoma lanceolatum)*, we cannot add much to the description of 1908, mentioned above [1]), and we will here only call attention to the interesting fact, that the cells of the infundibular organ are prolonged into fine nerve fibers, and that the two bundles of nerve fibers, formed by these cell prolongations run caudad at each side of the median plane and then show a rather typical decussation of the fibers in the median line, after which they are not to be followed any farther. In connection with the fact, that in many fishes the fibers of the nerve tracts, formed by the neurites of the cells of the saccus vasculosus (the *tracti sacci vasculosi*), cross each other at a certain point of the mid-brain in the median line, it is of a high importance, that this decussation obviously is already to be found in Amphioxus, not only so for the proposed homology of the saccus vasculosus of fishes and the infundibular organ in the brain of Amphioxus, but also for the interpretation of the amphioxus-brain not as an archencephalon (KUPFFER), but as a degenerated brain with a number of rudimentary fiber-tracts, to be

[1]) [It gives me great pleasure to note, that EDINGER (in a letter to the author) has taken back entirely his statement of 1907, that in front of the infundibular organ there should be a real infundibular cavity in Amphioxus so that now he agrees in all points with the description given by us in 1902 and 1908. See also EDINGER and WALLENBERG Bericht über die Leistungen auf dem Gebiete der Anatomie des Centralnervensystems. Vierter Bericht 1909. Seite 299. (BOEKE)].

compared in many points with the ichthyopsidan brain, defended some years ago by one of us [1]).

2. In all fishes, which are studied in this direction, the saccus vasculosus is found at least in "Anlage", as a hollow outgrowth of the brainwall in the bottom of the diencephalon, directed ventrally and growing out caudad. Still found in a developed state in Acipenser, Amia, Lepidosteus, it disappears in amphibians and is looked for in vain in terrestrial animals.

In a number of bony fishes however the "Anlage" of the saccus vasculosus is not developed any farther or it becomes rudimentary afterwards, so that in the adult animals only scanty traces of it are found (or none at all, cf. GENTÈS). These forms, in nearly all the cases (with only a few exceptions), appear to be freshwater fishes.

3. There where the saccus vasculosus attains its full development, it appears in Elasmobranchii as well as in Teleosts as a sort of bladder or sack with a thin wall, which is more or less folded and wrinkled up and often much branched; this wall is composed of the epithelial lining, the prolongation of the primary brain-wall, a layer of nerve fibers, glious fibers, a layer of flat endothelial cells which line the bloodvessels and lastly the bloodvessels themselves, chiefly a very highly developed system of bloodsinus, giving to the saccus in the living animal a red or purple colour (s. vasculosus).

During the whole life of the animal there remains a communication of the cavity of the saccus with the ventricles of the brain.

The structure of the wall is identical at all points of the saccus.

4. The description of the elements of the epithelial lining of the saccus, given by one of us (1901 and 1902), after examination of a great many forms appeared to be perfectly true.

The epithelial lining of the saccus wall shows two kinds of cells, in the papers mentioned above distinguished as "*sense cells*" and "*supporting cells*".

The *sense cells*, which we may call "crown cells" after the form of the fully differentiated elements, are large, bulky and more or less bottle-shaped. The broader basal part contains the large round nucleus, upwards the body of the cell gets more slender to end in a pear-shaped head, which is crowned with a large number (20—25) of stiff hairs ending in small knobs or vesicles. This part of the cell is protruding beyond the line of the supporting cells into the cavity of the saccus. It reminds one strongly of the receptaculum of

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[1]) J. BOEKE. Proceedings K. A. W. Matth. Phys. Cl. Meeting of May 31, 1908. Page 6.

a dandelion (taraxacum) with the seeds on it. The hairs themselves may be best compared in form to a cherry on its stalk. These knobs are described by LUNDBORG and especially by STUDNICKA as drops of fluid secreted by the cell. This however is not the case. In the first place they develop out of common hairs, which first take the shape of a club and then by further differentiation grow out to the stalked structures described above (c f. BOEKE 1901). Of this differentiation we could study all the intermediate stages, so as to leave no doubt whatever as to the course of the process of development. In the second place they are implanted in the cell-protoplasm on small basal bodies ("Basalkörperchen"). These are connected with a fine system of very thin fibers running through the cell towards the base. In the third place they may be seen on the living cells in transparent larvae (so for example in the exceedingly transparent larvae of the Muraenoids, in which they could be studied for hours at a stretch in the same larva [1]), and in small pieces cut out of the wall of the saccus of large adult fishes. In no case one of the small knobs was seen to fall off, to alter its form, to grow larger or form itself anew, even when the living saccus is studied with a high magnifying power for hours in the same larva.

They are nothing else but hairs swollen at their ends into small knobs. With a secretory process they have nothing to do.

And above all, it was possible to show by means of the m thods of APATHY, RAMON Y CAJAL and BIELSCHOWSKY, that these cells contain a very fine and regular neurofibrillar structure, the fibrillae being connected, as far as could be made out, with the basal bodies of the tuft of hairs, and running through the protoplasm of the cell as a bundle of very fine wavy threads, which passes the nucleus and on arriving at the basal end of the cell condensing itself into a bundle of finest neurofibrillae, leave the cell as a nerve fiber, a neurite, which neurite could be followed in an uninterrupted course to the bundle of nerve fibers running along the bases of the epithelial cells and from there into the tractus sacci vasculosi.

The cells take the Golgi-stain with difficultly, but nevertheless we succeeded in impregnating them in a number of preparations, and in several sections cut in the right direction (we imbedded the pieces in celloidine and cut them into sections of 100 μ) we were able to follow the nerve fibers from the impregnated cell-body as a fine non-varicose thread through the saccus as far as in the bundle of

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[1]) c.f. BOEKE. Proceedings K. A. W. Meeting of January 25, 1902, page 8.

nerve fibers connecting the saccus with the diencephalon, the tractus sacci vasculosi (cf. JOHNSTON 1902).

By this fact, which could be established several times with great clearness, the nature of these cells as nervous elements, as sense cells, seems to us to be proved beyond doubt.

The *supporting cells*, lying between the crown cells, are small cells, which fill up entirely the little room left between the bottle-shaped sense cells, and contain a curiously shaped threecornered or pyramidal nucleus, which often seems to fill up the whole cell, leaving room only for a very thin layer of protoplasm. At their basal end these cells seem to be prolonged into slender feet, which (as it is the case with the ependymal cells and glious cells) envelop the bundles of nerve fibers, springing from the crown cells (c. f. JOHNSTON 1906).

5. These nerve fibers, being therefore (for by far the greater part, see under 6) nothing else but the axones of the sense cells of the saccus, all run through the saccus towards the point where the walls of it are connected with the diencephalon, and there they condense into two bundles of nerve fibers, running in the diencephalon in the same direction at each side of the median plane, the *tracti sacci vasculosi*. This points to an *originally bilateral origin of the saccus vasculosus*, just as it could be established for the infundibular organ of amphioxus.

These tracti sacci vasculosi, seen and described more or less clearly already by a number of investigators (GORONOWITSCH, EDINGER, KUPFFER, BICKFORD, JOHNSTON, ARIENS KAPPERS, GOLDSTEIN) run at both sides of the recessus inferior, from the point of entrance up through the walls of the diencephalon obliquely and forward, through the corpora mamillaria to dorsally of the recessus inferior. Here they may decussate in the median line (trout), or they remain independent, running each at a side of the median plane (Anguilla, Zoarces) to end, in Elasmobranchii as well as in Teleosts, in two nuclei, lying at the end of the tuberculum posterius close to the median plane just over the aquaeductus. ARIENS KAPPERS has first described these nuclei in *Galeus*. GOLDSTEIN saw them in Teleosts. Here we found them in a great many forms. These nuclei contain large nerve cells from which fibers grow out mostly caudad. In BIELSCHOWSKY-preparations the connection of the terminal branches of the neurites of the sense cells from the saccus with these cells was clearly to be seen.

6. Eferent nerve fibers are also to be found in the saccus, coming

from the diencephalon, from the lobi inferiores; these fibers end probably all or most of them in the walls of the bloodvessels of the saccus.

It would take us too far to describe here the details of these different tracts and fibers, the secondary connections of the nuclei with each other and with the other parts of the brain, and to enter into a discussion of the results of our investigations in connection with the facts, found by previous writers. It would be impossible to enter into these things without the aid of a great many figures and in the few pages these proceedings allow us. All these things will be discussed at proper length in the publication of the last-named of us two, mentioned above. [1]) Here we only wanted to show, that renewed investigations convinced us of the truth of the hypothesis, put forth several years ago, that the saccus vasculosus of the lower aquatic vertebrates is not a gland but a receptive nervous structure, bilateral in origin, finding its homologon in the infundibular organ of amphioxus.

Leiden. Anatomical Cabinet. April 1910.

Palaeontology. — "*A further investigation of the pliocene flora of Tegelen.*" By CLEMENT REID, F.R.S. and Mrs. ELEANOR M. REID, B.Sc. (Communicated by Prof. G. A. F. MOLENGRAAFF).

The results obtained from our first examination of the Pliocene deposits of Tegelen [2]) pointed to so rich a flora, that we considered it advisable to make further researches. Accordingly in the summer of 1908 we asked Messrs. CANOY, HERFKENS and SMULDERS to send us a further quantity of the fossiliferous brick-earth from the bottom of their pit. They most kindly carried out our request, employing the same men who had assisted us to collect the samples in 1905. The amount of loam sent was nearly 300 Kilog., and we must thank Messrs. CANOY and Co. and their workmen for the great care taken in its collection. Recent seeds were quite absent, except for a few grass seeds, which fly everywhere and are almost impossible to exclude. This large quantity of material has taken us a long time to examine, and we have been interrupted by other work which could not wait; hence the delay in publishing our results.

The new material was not quite so prolific as our former gathering,

[1]) Appearing in the "Zeitschrift für wissensch. Zoologie."

[2]) Verhand. Kon. Akad. v. Wetensch. (Tweede Sectie). Deel XIII, No. 6 (1907); Proceedings Acad. Amsterdam 1908. Vol. X, p. 860.

for only a botanist can select on the spot the thin seams which contain most of the land-plants; but this new gathering yielded in profusion the aquatic species. The results are of great interest, both as confirming our previous conclusions, and as extending our knowledge of the Tegelen flora. The additions to the list number about 40, thus bringing the complete list to about 135. Most of the seeds found belong, as one would expect, to species we have already recorded; but in many cases we obtained much better specimens, enabling us to make more definite determinations.

Among the novelties less than half are now living in the Netherlands. Some we are unable to identify either specifically or generically. In a few cases we feel confident that our species are extinct, but the specimens we have are too few to permit us to make the necessary dissections before describing and naming them. Others belong to very large orders or genera, the seeds of which are not well represented in herbaria, e.g. *Labiateae* and *Hypericum*. Others again belong to genera which have their chief or only development at the present day in Eastern Asia. This is the case with such genera as *Staphylea*, *Prunus*, many genera of *Araliaceae* and *Cornaceae*, *Viburnum*, *Carpinus*, etc. In view of the fact that the fruits of many of these eastern species are quite unknown we again hesitate to describe our species belonging to these genera as new.

The newly discovered exotic forms are mainly related to species of Eastern Asia, a few are European. Thus we have, besides the Eastern species mentioned above, *Crataegus cuneata* now living in China and Japan and a species of *Hippomarathrum*, a genus now distributed round the shores of the Mediterranean and in Western and Central Asia. We have the Central and Southern European species *Valeriana tripteris*, *Physalis Alkekengi*, and *Equisetum ramosissimum*. It may be noted in passing that all point to somewhat warmer conditions than at present prevail in the Netherlands.

Our present investigation of the Tegelen flora brings out one fact very strikingly. We have already mentioned, both in this paper and our former, that the living species agreeing with, or most closely related to many Tegelen species are now living only in Eastern Asia. It would seem therefore that there is a close affinity between the Pliocene flora of Tegelen, and the existing flora of parts of Eastern Asia; and that the more we learn about the Tegelen flora, the more marked does this affinity become. It is at present too early to consider what this means; whether it implies that the flora of the Far East is a survival of one which originated in Western Europe but was driven eastward; or whether it may rather point to a wide-spread Palaearctic flora, now exterminated in the West, but surviving in the East. It will require much laborious research both in East and West to settle this interesting point.

In the Tegelen brick-earth we have discovered, mixed with the seeds, various remains of small vertebrates; these Mr. E. T. NEWTON, who has so carefully studied the similar remains of the Cromer Forest-bed, has now determined for us. It may be remarked that the specimens from Tegelen, like those from Cromer, are usually very fragmentary, but in neither case is there any reason to doubt that these small mammals and fishes were contemporaneous with the plants. Mr. NEWTON's determinations have already been published [1]; and it will be seen that we have obtained since 1907 several novelties. The complete list is as follows:

Abramis brama LINN.
Anguilla vulgaris LINN.
Carpinus rutilus LINN.
Esox lucius LINN.
Gasterosteus aculeatus LINN.
Leuciscus cephalus LINN.?
,, erythrophthalmus LINN.
Perca fluviatilis LINN.
Tinca vulgaris CUV.
Cypridoid teeth (not determined).
Minute curved spines (unknown).
Rana sp.
Talpa europaea LINN.?
Microtus (Mimomys) pliocaenicus F. MAJOR.
,, ,, intermedius NEWTON.

Gasterosteus and *Anguilla* have not yet been found in the Cromer Forest-bed; *Microtus intermedius* is abundant in that deposit; *Microtus pliocaenicus* is found in the slightly older Weybourn Crag, in the Norwich Crag, and in the Pliocene deposits of the Val d'Arno.

Notes on the Plants.

Clematis Vitalba LINN. (Fig. 1).

Three well-preserved carpels undoubtedly belonging to this species.

Ranunculus repens LINN. (Fig. 2)

Two well-preserved achenes.

[1]) Bull. Soc. belge de Géologie. Mémoires XXI, 591 (1907); and Procès-Verbaux XXIV, 231 (1910).

Magnolia Kobus DC. (Figs. 3, 4).

The beautifully-preserved seed shown in fig. 3 has been photographed side by side with a recent seed (fig. 4) grown at Kew. For other specimens, grown near Tokyo, we have to thank Dr. TOKUTARO ITO. There seems no doubt as to the determination of this Japanese plant.

Euryale limburgensis C. and E. M. REID (Fig. 6).

This specimen shows the embryotega in place and is figured to show this curious organ, and its exact resemblance to the recent species (Fig. 5).

Stellaria aquatica SCOP. (Fig. 7).

Numerous seeds of this species were found in 1908, though none occurred in our former collection.

Stellaria nemorum LINN. (Fig. 8).

A single broken seed evidently belongs to this species. It shows the attachment, and the characteristic tubercles with their digitate sutures.

Lychnis Flos-cuculi LINN.

Three more small seeds agreeing exactly with this species, but rather small, were found in 1908. They are slightly stipitate; but on comparing them with good series of recent seeds this stalk is occasionally found; it would probably become more prominent in the fossil through shrinkage of the testa.

Scleranthus annuus LINN. (Fig. 127 of former paper).

We can now identify this fruit with *S. annuus*, which occurs also not uncommonly in the Cromer Forest-bed.

Hypericum sp. 6 (Fig. 9).

Two more species have been found. Fig. 9 shows a minute seed resembling *H. montanum* in size and shape, but the sculpture seems finer and more irregular.

Hypericum sp. 7 (Fig. 10).

A short thick seed evidently belongs to a seventh species; but the genus is so large, and our collection of recent seeds is still so incomplete, that we cannot match it.

Staphylea sp. (Figs. 11, 12, 13).

We have now found a few fragments of seeds of this genus. The piece figured shows the very characteristic base and attachment;

but it belongs to no living species with which we have been able to compare it. It is a larger seed than either *S. pinnata* or *S. colchica* and differs also in various details. The attachment is round or ovate with longer axis parallel to the flattened sides. In the recent *S. pinnata* the attachment is transversely ovate. It may be extinct; but several new species of *Staphylea* have been collected by Dr. AUGUSTINE HENRY in China, and of some of them no fruits are in the Kew herbarium.

Prunus spinosa LINN. (Fig. 14).

We have found only a single worn stone of this species, and it looks as if it might have travelled some distance.

Prunus cf. lusitanica LINN. (Figs. 15, 16).

Two fragments of thin-walled plum-stones, showing sharp apex, somewhat flattened. They seem to be close to *Prunus lusitanica*, but the endocarp is much smaller.

Prunus Maximoviczii RUPRECHT (Figs. 17, 18).

We have now several more specimens of these small globose plum-stones; but most of them seem to have been digested by birds, or have travelled far. One (fig. 18) shows the highly characteristic loops or folds belonging to *Prunus Maximoviczii*, and has been photographed side by side with a recent specimen (fig. 17, from a specimen sent to us from Yokohama), to show the exact correspondence. None of our European species show loops of this character.

Rubus sp. (Figs. 19, 20, 21).

Two or three broken and much decayed endocarps and a few imperfect prickles are all the remains of *Rubus* we have yet found. They are quite indeterminable; but the endocarps are unusually thin and of a different shape from any recent species we have seen. Unfortunately ripe fruits of *Rubus* are seldom to be found in herbaria, and there are many species fruits of which we have not yet examined.

Crataegus cuneata SIEB & ZUCC. (Figs. 22, 23, 24, 25, 26).

A single bony carpel, the inner faces meeting at less than a right angle, agrees exactly with this Japanese 5-carpelled *Crataegus*. We have photographed the fossil and recent specimens side by side in two aspects.

Hippomarathrum n. sp.? (Fig. 32).

Two well-preserved but somewhat flattened inflated mericarps, showing 5 broad strong ribs and numerous small rugosities. They are greatly rounded dorsally and hollowed ventrally, bringing the curious triangular base into close proximity to the large triangular beak. We find this same very curious form of mericarp accompanied by the same large triangular beak in two species of *Hippomarathrum*, *H. vaginatum*, and an unnamed species from Turkestan. We have no hesitation therefore in referring our species to this genus, though it does not agree with any living species. It has the shape and size of *H. vaginatum*, but this species is almost without ribs, whereas ours is strongly and broadly ribbed. Other species have ribs such as ours, but do not agree with it in shape or size or in having the large beak. We do not describe our species as new as we cannot make a section to show the vitteae; we require more specimens.

The genus *Hippomarathrum* inhabits the Mediterranean region. It extends in Asia as far east as Soungaria.

Cryptotaenia? (Fig. 33).

One specimen, somewhat crushed and torn, but otherwise in a fair state of preservation. Ribs 5, very strong, merging into a beak $^1/_6$ the length of the fruit; vittae 3 between each pair of dorsal ribs, those nearest the ribs being in close contact with them; ventral face somewhat damaged. Length 2.5 mm., breadth 1.5 mm., length of beak 0.4 mm.

The only genus showing these characters appears to be the Japanese *Cryptotaenia*; but our fossil fruit does not agree with the living species, being much smaller.

Genus? (Fig. 34).

Possibly a large umbelliferous fruit, but not sufficiently well-preserved for determination.

Cornaceae or Araliaceae. (Figs. 27, 28, 29).

We have several minute fruits irregularly 5 or 6-angled and indehiscent. One cut transversely showed 2 complete cells and apparently 2 or 3 with abortive seeds. Indehiscent fruits of this character are found in the Cornaceae (cf *Toricellia*) and in the Araliaceae (cf *Heptapleurum*)

Viburnum sp. (Figs. 30, 31).

Two endocarps of *Viburnum* belong to no recent species we have been able to examine, but correspond exactly with an unknown *Viburnum* found by us in the Cromer Forest-bed (*Linnean Journal-Botany*, XXXVIII, pl. 13, figs. 75, 76, 77). Three species of *Viburnum* now live in Europe; but the fossil is quite unlike any of these. There are various species living in Asia, and of most of them we can find no fruits in the herbaria; we therefore do not describe this fossil as an extinct form.

Valeriana tripteris *Linn.* (Fig. 35).

A single specimen seems to correspond exactly with this Mid-European species; but as the fruit is much flattened it is difficult to photograph.

Bidens tripartita LINN. (Fig. 36).

Several fragments evidently belong to this species; but we have found no good specimen.

Carduus palustris WILLD. (Fig. 37).

We have found three fruits, about two-thirds the length of our recent specimens, but otherwise agreeing. Fossil composite-fruits have always shrunk considerably, and it is therefore useless to lay much stress on difference of size.

Physalis Alkekengi LINN. (Fig. 38).

Three seeds show the very peculiar sculpture of this species. They are somewhat small and have probably shrunk in carbonising, as happens with all fossil seeds belonging to this order.

Prunella vulgaris LINN. (Fig. 41).

One well-preserved nutlet has been found.

Stachys longiflora BOISS. & BAL.? (Fig. 43).

Two small nutlets, oval, somewhat truncate, narrowed below with a small terminal attachment, surface rugose and pustulate. Strikingly like *S. longiflora*, but somewhat small.

Genus? (Fig. 42).

A finely-tubercled nutlet of a labiate, which we are unable to identify.

Teucrium Botrys LINN. (Figs. 39, 40).

We have only a single nutlet, which we have photographed from above and below. The irregular rugosity and large basal aperture are very characteristic of this species.

Polygonum minus HUDSON. (Fig. 44).

A few small nuts seem to represent this species.

Polygonum Convolvulus LINN. (Fig. 45).

A single somewhat damaged nut corresponds exactly and shows the very characteristic granulation of this species.

Polygonum Bellardi ALL.

Fig. 68 of our previous paper by mistake was only magnified $^{6}/_{1}$, not $^{12}/_{1}$ as stated, like other species of the genus. The nut is large.

Rumex n. sp.? (Fig. 46).

Several nuts, often with remains of fruiting sepals, and one good fruit. This exceedingly minute *Rumex* shows short triangular fruiting sepals, one of which bears a longish-oval tubercle; sepals somewhat waved at margin with strong looped reticulation; nut small, broadly triangular, with rounded base, so that the greatest width is at one-third of the height, surface rugose. We can find no recent species at all like this; but we hesitate to give it a name, as we have only one good fruit.

Carpinus n. sp.? (Fig. 47).

Three minute nuts represent a second species of this genus; but they are much too small to be referred to the European forms. There are several small-fruited species of *Carpinus* in Japan and China; the one which corresponds most closely with our fossil is *C. laxiflora* Bl.; but the ribs in our fossil are more prominent.

Potamogeton crispus LINN. (Fig. 48).

Numerous fruits have now been found, though we saw none in our former gathering.

Cyperus sp. 1 (Fig. 49).

These minute nuts seem to belong to a species of *Cyperus*; they are very abundant.

Cyperus Sp. 2 (Fig. 50).

This form of Cyperus is almost as abundant as the last.

Dulichium vespiforme C. and E. M. REID.

This species has been found abundantly in our new collection from Tegelen, and one specimen distinctly shows 9 setae, though the usual number appears to be 7 or 8. In other respects the fruits show but little variation. A closely allied form was figured by Dr. J. STOLLER in 1909, from Friedrichshagen am Müggelsee and Lauenburg a. Elbe [1]).

Scirpus 3 sp. (Figs. 51, 52, 53).

The three species of Scirpus we have not been able to identify. The setae are not preserved.

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[1]) Jahrb. Königl. Preuss. Geol. Landesanstalt fur 1909, bd. XXX, t. 1. heft 1.

Scirpus Tabernaemontani GMEL. (Fig. 54).

We have only one nut of this species.

Incerta sedes.

Fig. 55 appears to be a minute hard seed or endocarp.

Figs. 56, 57 represent the inside of the split carpel, of which the outside was figured in 1907 (fig. 125). We are still unable to identify it.

Fig. 58 is a minute globular seed sculptured like *Papaver*; but we have been unable to refer it to any genus.

Fig. 59 is a hard endocarp with remains of succulent pericarp. It is probably unripe, but seems unlike anything else we have found.

More specimens have been found of the unknown seed fig. 124 of our former paper. The same species has also been sent to us by Baron GREINDL from Raevels; but we are still unable to suggest its botanical position. The seeds are small and hard, with thick bony testa, often facetted by mutual pressure, and an external curved "germination groove" along which the seed burst. The embryo was pendent, but has left no impression on the smooth interior of the testa.

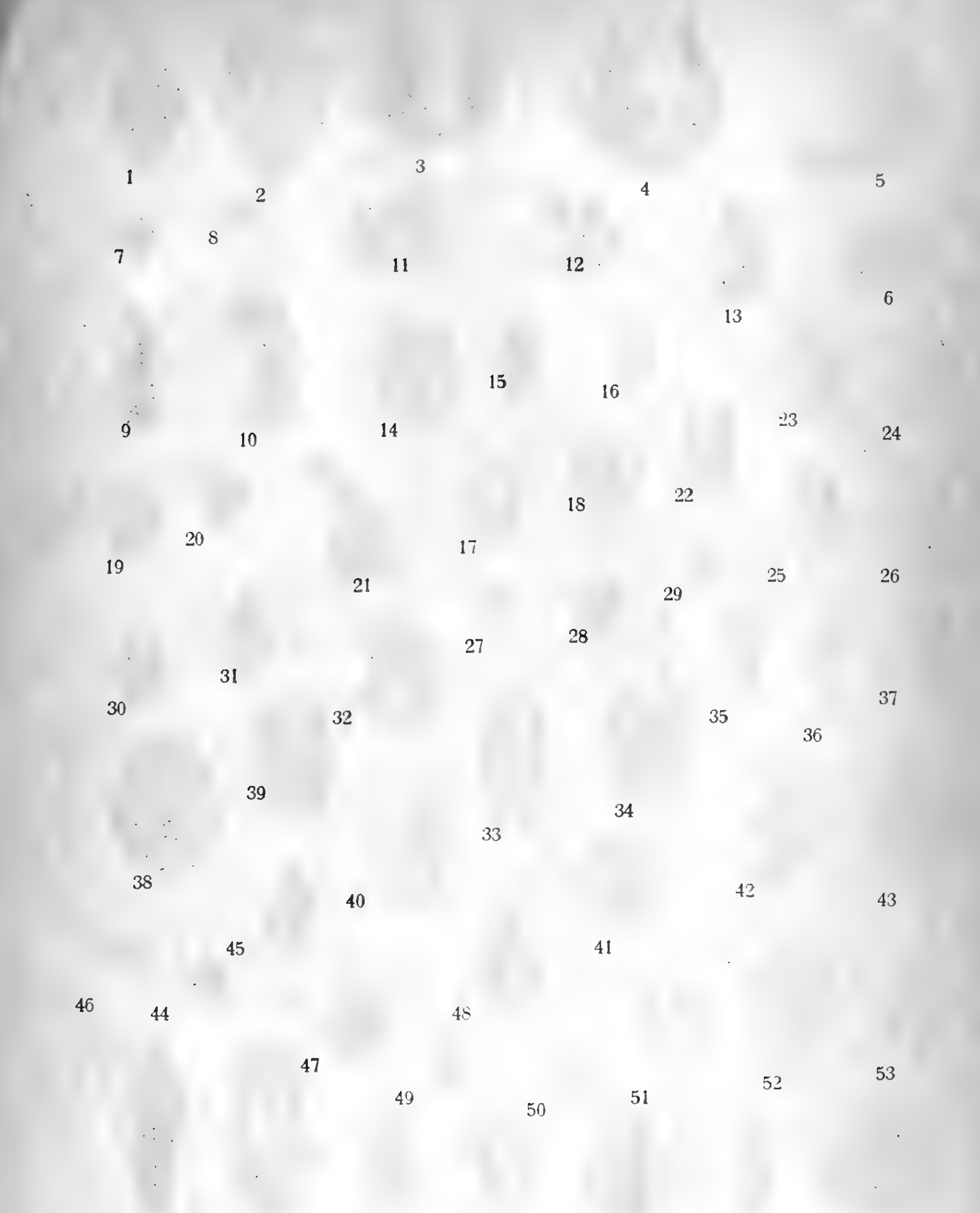
Equisetum ramosissimum DESF. (Fig. 60).

A fragment of *Equisetum* showing a hexagonal stem with irregular transverse rugosity, a six-toothed sheath, each tooth with a central rib, and two opposite buds. These characters are found combined in *E. ramosissimum* and we may refer our specimen to this species. It is probably a portion of a branch close to the rhizome. We have found such a branch on a specimen from the Canton Vaud, in Switzerland, which agrees in every detail with our fossil. The species is widely distributed throughout Southern Europe, Asia, and Africa, but does not now extend so far north as Tegelen.

DESCRIPTION OF THE PLATE.

Fig. 1. Clematis Vitalba LINN. Achene. $^6/_1$.
" 2. Ranunculus repens LINN. Achene. $^{12}/_1$.
" 3. Magnolia Kobus DC. Seed (fossil). $^3/_1$.
" 4. " " Seed (recent). $^3/_1$.
" 5. Euryale ferox SALISB. Seed (recent). $^3/_1$.
" 6. " limburgensis C. and E. M. REID. Seed. $^3/_1$.
" 7. Stellaria aquatica SCOP. Seed. $^{12}/_1$.

Clement Reid and Mrs. E. M. Reid. A further investigation of the pliocene flora of Tegelen.



Photo, C. & E. M. Reid.

... hard seed or endocarp.

... side of the split carpels of which the ... (fig. 123). We are still unable to ...

... angular seed sculptured like *Papaver*, but ... refer it to any genus.

... with remains of succulent pericarp. ... we have found ...

... found of the unknown seed fig. 124 ... which has also been sent to us by ... from Reuver, but we are still unable to suggest its ... The seeds are small and hard, with thick ... by unequal pressure, and an external curved groove ... which the seed burst. The embryo was ... left an impression on the smooth interior of the testa.

Equisetum ramosissimum Desf. (fig. 60).

... of *Equisetum* ... jointed stem with irregular ... teeth with a central ... found ...

... probably ... found such a ... near ... Vaud, ... which agrees in every detail with our fossil. The ... distributed throughout Southern Europe, Asia, and ... far north as Tegelen.

DESCRIPTION OF THE PLATE.

... Achene. 4/1.

... Kobe ... Seed (fossil). 4/1.

... Seed (recent) ...

... Seed (recent). 4/1.

... C. and E. M. Reid. Seed. 4/1.

... Stellaria ...

Clement Reid and Mrs. E. M. Reid. »A further investigation of the pliocene flora of Tegelen.«

Heliotypie. Van Leer Amsterdam

Fig. 8. Stellaria nemorum LINN. Broken seed. $^{12}/_{1}$.
„ 9. Hypericum sp. 6. Seed. $^{24}/_{1}$.
„ 10. „ „ 7. Seed. $^{24}/_{1}$.
„ 11. Staphylea sp. Interior of broken seed. $^{3}/_{1}$.
„ 12. „ „ Side view of attachment. $^{3}/_{1}$.
„ 13. „ „ Base of seed. $^{3}/_{1}$.
„ 14. Prunus spinosa LINN. Decayed stone. $^{3}/_{1}$.
„ 15, 16. Prunus cf. lusitanica LINN. Interior and exterior of broken stones. $^{3}/_{1}$.
„ 17. Prunus Maximoviczii RUPRECHT. Recent stone. $^{3}/_{1}$.
„ 18. „ „ „ Fossil stone. $^{3}/_{1}$.
„ 19. Rubus sp. Exterior of endocarp. $^{6}/_{1}$.
„ 20. „ „ Interior of endocarp (another specimen). $^{6}/_{1}$.
„ 21. „ „ Prickle. $^{6}/_{1}$.
„ 22. Crataegus cuneata SIEB. & ZUCC. (fossil). Inner face of carpel. $^{3}/_{1}$.
„ 23. „ „ (fossil). Outer „ „ „ $^{3}/_{1}$.
„ 24. „ „ (recent). „ „ „ „ $^{3}/_{1}$.
„ 25. „ „ (fossil). Side view „ „ $^{3}/_{1}$.
„ 26. „ „ (recent). „ „ „ „ $^{3}/_{1}$.
„ 27, 28, 29. Cornacae or Araliaceae. Three specimens, showing interior, side view, and apex. $^{6}/_{1}$.
„ 30. Viburnum sp. Dorsal view of endocarp. $^{6}/_{1}$.
„ 31. „ „ Ventral „ „ „ (another specimen). $^{6}/_{1}$.
„ 32. Hippomarathrum n. sp.? Mericarp. $^{6}/_{1}$.
„ 33. Cryptotaenia? Mericarp. $^{12}/_{1}$.
„ 34. Umbelliferous Mericarp.? $^{3}/_{1}$.
„ 35. Valeriana tripteris LINN. Fruit. $^{6}/_{1}$.
„ 36. Bideas tripartita LINN. Damaged fruit. $^{6}/_{1}$.
„ 37. Carduus palustris WILLD. Fruit. $^{6}/_{1}$.
„ 38. Physalis Alkekengi LINN. Seed. $^{12}/_{1}$.
„ 39, 40. Teucrium Botrys LINN. Top and base of nutlet. $^{12}/_{1}$.
„ 41. Prunella vulgaris LINN. Ventral face of nutlet. $^{12}/_{1}$.
„ 42. Labiate, genus unknown. $^{12}/_{1}$.
„ 43. Stachys longiflora BOISS & BAL.? Nutlet. $^{12}/_{1}$.
„ 44. Polygonum minus HUDS. Nut. $^{6}/_{1}$.
„ 45. „ convolvulus LINN. Nut. $^{6}/_{1}$.
„ 46. Rumex n. sp.? Fruit. $^{6}/_{1}$.
„ 47. Carpinus n. sp.? Nut. $^{6}/_{1}$.
„ 48. Potamogeton crispus LINN. Fruit. $^{6}/_{1}$.
„ 49. Cyperus? sp. 1. Nut. $^{12}/_{1}$.
„ 50. Cyperus? sp. 2. Nut. $^{12}/_{1}$.
„ 51, 52, 53. Scirpus 3 sp. Nuts. $^{12}/_{1}$.
„ 54. Scirpus Tabernaemontani GMEL. Nut. $^{12}/_{1}$.
„ 55. Unknown. $^{24}/_{1}$.
„ 56, 57. Inside of unknown carpel. $^{12}/_{1}$.
„ 58. Papaveraceae? Seed. $^{24}/_{1}$.
„ 59. Unknown endocarp with remains of pericarp. $^{6}/_{1}$.
„ 60. Equisetum ramosissimum DESF. $^{3}/_{1}$.

Mathematics. — *"Infinitesimal iteration of reciprocal functions"* (2nd Part.). By Dr. M. J. VAN UVEN. (Communicated by Prof. W. KAPTEYN.

(Communicated in the meeting of May 28, 1910).

§ 5. In § 4 of my first paper (Proceedings of the Kon. Akad. v. Wetenschappen, May 28, 1910) we have tried to let the symmetric equation

$$S(x,y)=0$$

pass by means of the symmetric projective transformation

$$x=\frac{\alpha\xi+\beta\eta+\gamma}{\delta(\xi+\eta)+\varepsilon}\quad,\quad y=\frac{\beta\xi+\alpha\eta+\gamma}{\delta(\xi+\eta)+\varepsilon}\quad . \quad . \quad . \quad (15)$$

into the equation

$$\psi(\xi)+\psi(\eta)=2k,$$

and we have found, that the function $S(x,y)$ had to satisfy the differential condition:

$$\delta^2 S_{zz}+(\alpha+\beta)\,\delta(S_{xz}+S_{yz})+[\alpha\beta S_{xx}+(\alpha^2+\beta^2)\,S_{xy}+\alpha\beta S_{yy}]=0, \quad (18)$$

which equation was obtained after by introduction of the factor of homogeneity z we had made the expression $S(x,y)$ homogeneous.

When excluding the affine transformation we might put $\delta=1$, by which (18) would have passed into

$$S_{zz}+(\alpha+\beta)(S_{xz}+S_{yz})+[\alpha\beta S_{xx}+(\alpha^2+\beta^2)\,S_{xy}+\alpha\beta S_{yy}]=0. \quad (18a)$$

As (18*a*) had to be satisfied independent of the value of the coordinates, this equation represented a series of relations between the constants α and β of the transformation (15) (with $\delta=1$) and the constants of $S(x,y)$. If we chose for $S(x,y)=0$ the symmetric quadratic equation

$$S_2=a_2(x+y)^2+2b_2xy+2a_1(x+y)+a_0=0,$$

then there proved to be *one* condition, namely

$$a_0+2a_1(\alpha+\beta)+(a_2+b_2)\,(\alpha+\beta)^2-2b_2\alpha\beta=0, \quad . \quad . \quad (19)$$

whilst when for $S(x,y)=0$ we chose the symmetric cubic equation

$$S_3\quad a_3(x+y)^3+3b_3(x+y)xy+3a_2(x+y)^2+6b_2xy+3a_1(x+y)+a_0=0$$

we arrived at *two* conditions, namely

$$a_1+(2a_2+b_2)(\alpha+\beta)+(a_3+b_3)(\alpha+\beta)^2-b_3\alpha\beta=0 \quad . \quad (21)$$

$$a_0+2a_1(\alpha+\beta)+(a_2+b_2)(\alpha+\beta)^2-2b_2\alpha\beta=0, \quad . \quad . \quad (19)$$

of which the last was the same as that of the quadratic equation.

We shall now point out that with an efficient notation of the equations the conditions which must hold for the symmetric equation of order $m-1$ are all contained in those belonging to the symmetric equation of order m, so that the reappearance of the condition **(19)** ensues for the cubic equation from a general principle.

Beforehand we shall make the symmetrical function of order m. $S_m(x,y)$ homogeneous with the aid of the factor z and then we shall arrange it according to ascending powers of z. We then put

$$S_m = s_m + m s_{m-1} z + \frac{m(m-1)}{2} s_{m-2} z^2 + \ldots + m s_1 z^{m-1} + s_0 z^m, \quad (26)$$

where s_k represents a homogeneous function of x and y of degree k.

The notations chosen by us for the quadratic and cubic equation are evidently in accordance with this precept.

We introduce the following notations:

$$\left.\begin{array}{l} \dfrac{\partial S}{\partial x} = D_x S, \ \dfrac{\partial^2 S}{\partial x \partial z} = D_{xz} S, \text{ etc.} \\ \displaystyle\int_0^z S dz = D_z^{-1} S, \end{array}\right\} \quad . \quad . \quad . \quad (27)$$

so that when integrating $D_z^{-1} S$ no term appears independent of z, therefore no term exclusively dependent of x and y.

From the equation (26) now follows immediately

$$D_z S_m = m S_{m-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and

$$S_m = s_m + m D_z^{-1} S_{m-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Fartheron we find, as a result of (29),

$$D_x S_m = D_x s_m + m D_z^{-1} D_x S_{m-1},$$
$$D_y S_m = D_y s_m + m D_z^{-1} D_y S_{m-1},$$
$$D_{xx} S_m = D_{xx} s_m + m D_z^{-1} D_{xx} S_{m-1},$$
$$D_{xy} S_m = D_{xy} s_m + m D_z^{-1} D_{xy} S_{m-1},$$
$$D_{yy} S_m = D_{yy} s_m + m D_z^{-1} D_{yy} S_{m-1}.$$

From (28) follows moreover

$$D_{xz} S_m = m D_x S_{m-1} = m D_x \{s_{m-1} + (m-1) D_z^{-1} S_{m-2}\} =$$
$$= m D_x s_{m-1} + m(m-1) D_x D_z^{-1} S_{m-2}.$$

Now holds

$$D_z S_{m-1} = (m-1) S_{m-2},$$

therefore

$$D_{xz} S_{m-1} = (m-1) D_x S_{m-2}$$

and

$$D_z^{-1} D_{xz} S_{m-1} = (m-1) D_z^{-1} D_x S_{m-2} = (m-1) D_x D_z^{-1} S_{m-2},$$

so that we find

$$D_{xz} S_m = m D_x s_{m-1} + m D_z^{-1} D_{xz} S_{m-1},$$

and likewise

$$D_{yx}\, S_m = m D_y\, s_{m-1} + m\, D_z^{-1}\, D_{yz}\, S_{m-1}\,.$$

Fartheron we have

$$D_{zz} S_m = m D_z S_{m-1} = m(m-1) S_{m-2} = m(m-1)\{s_{m-2} + (m-2) D_z^{-1} S_{m-3}\} =$$
$$= m\,(m-1)\, s_{m-2} + m\,(m-1)\,(m-2)\, D_z^{-1}\, S_{m-3}\,,$$

whilst

$$D_{zz}\, S_{m-1} = (m-1)\, D_z\, S_{m-2} = (m-1)\,(m-2)\, S_{m-3}\,,$$

and therefore also

$$D_z^{-1}\, D_{zz}\, S_{m-1} = (m-1)\,(m-2)\, D_z^{-1}\, S_{m-3}\,;$$

consequently we find

$$D_{zz}\, S_m = m\,(m-1)\, s_{m-2} + m D_z^{-1}\, D_{zz}\, S_{m-1}\,.$$

When reducing $D_{xz} S_m$ and $D_{zz} S_m$ we have had to make a round-about way, because the symbols D_z and D_z^{-1} do *not* annul each other.

The differential condition ($18a$) we can write as follows symbolically:

$$\{D_{zz} + (\alpha+\beta)\,(D_{xz}+D_{yz}) + [\alpha\beta D_{xx} + (\alpha^2+\beta^2)\, D_{xy} + \alpha\beta D_{yy}]\}\, S = 0, \quad (18a)$$

or if we represent the differential operator by Δ

$$\Delta S = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18b)$$

From the reductions found just now is evident that

$$\Delta S_m = m\,(m-1)\, s_{m-2} + m\,(\alpha+\beta)\,(D_x + D_y)\, s_{m-1} +$$
$$+ [\alpha\beta D_{xx} + (\alpha^2+\beta^2) D_{xy} + \alpha\beta D_{yy}] s_m + m D_z^{-1} \Delta S_{m-1} = t_{m-2} + m D_z^{-1} \Delta S_{m-1}.$$

The expression ΔS_m is of order $m-2$, ΔS_{m-1} is of order $m-3$, $D_z^{-1}\, \Delta S_{m-1}$ is *in x and y likewise of order $m-3$*. The term t_{m-2} contains therefore *all* expressions, which are in x and y of order $m-2$.

The condition $\Delta S_m = 0$, which must be satisfied independent of x and y, now demands that the coefficients of *all* terms with x and y are *zero*. From this ensues that the coefficients must vanish from all terms out of which t_{m-2} is built up, as well as *from all terms out of which ΔS_{m-1} is composed.*

This last condition, however, is also expressed by

$$\Delta S_{m-1} = 0,$$

so that the conditions ensuing from $\Delta S_{m-1} = 0$ are included in the conditions following out of $\Delta S_m = 0$.

Let us consider the *affine* transformation; then we must put in (18) $d = 0$, in consequence the differential condition runs:

$$[\alpha\beta\, D_{xx} + (\alpha^2+\beta^2)\, D_{xy} + \alpha\beta\, D_{yy}]\, S_m = 0 \quad . \quad . \quad . \quad (24a)$$

or

$$\Delta' S_m = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24b)$$

for which we can also write

$$\Delta' s_m + m\, D_z^{-1}\, \Delta' S_{m-1} = 0.$$

Also for the affine transformation the conditions ensuing from

$\Delta' S_{m-1} = 0$ *are included in the conditions which are consequences of* $\Delta' S_m = 0$..

We shall now find the number of relations between the constants α and β of the projective transformation (15) ($\sigma = 1$) and the constants of $S_m(x,y)$.

With respect to the terms t_0, t_1, etc. the following holds:

$$\begin{aligned}
t_0 &= p_0,\\
t_1 &= p_1\,(x+y),\\
t_2 &= p_2\,(x+y)^2 + 2q_2 xy,\\
t_3 &= p_3\,(x+y)^3 + 3q_3(x+y)\,xy,\\
&\vdots
\end{aligned}$$

$$t_{2h} = p_{2h}\,(x+y)^{2h} + 2h\,q_{2h}\,(x+y)^{2(h-1)}\,xy + \ldots + (\ldots)\,(xy)^h,$$

$$t_{2h+1} = p_{2h+1}(x+y)^{2h+1} + (2h+1)\,q_{2h+1}\,(x+y)^{2h-1}xy + \ldots + (\ldots)\,(x+y)\,(xy)^h,$$

so that

$$\begin{array}{lll} t_{2h} & \text{contains} & h+1 \text{ terms},\\ t_{2h+1} & \quad ,, & h+1 \quad ,, \;. \end{array}$$

If m is even, hence $m = 2k$, the number of terms of

$$\Delta S_m = \Delta S_{2k} = \sum_{i=0}^{i=m-2} t_i = \sum_{h=0}^{h=k-1} t_{2h} + \sum_{h=0}^{h=k-2} t_{2h+1}$$

is given by

$$\sum_{h=0}^{h=k-1} (h+1) + \sum_{h=0}^{h=k-2} (h+1) = 2\sum_{h=0}^{h=k-2} (h+1) + k = k^2 = \frac{m^2}{4}.$$

If on the other hand m is odd, hence $m = 2k+1$, then the number of terms of

$$\Delta S_m = \Delta S_{2k+1} = \sum_{i=0}^{i=2k-1} t_i = \sum_{h=0}^{h=k-1} t_{2h} + \sum_{h=0}^{h=k-1} t_{2h+1}$$

is determined by

$$\sum_{h=0}^{h=k-1} (h+1) + \sum_{h=0}^{h=k-1} (h+1) = 2\sum_{h=0}^{h=k-1} (h+1) = k\,(k+1) = \frac{m^2-1}{4}.$$

For *even* values of m the condition $\Delta S_m = 0$ as well as the condition $\Delta' S_m = 0$ represents $\frac{m^2}{4}$ relations between α and β and the constants of S_m.

For *odd* values of m this number of relations amounts to $\frac{m^2-1}{4}$.

As the expression ΔS_m is heterogeneous in α and β there must exist between the coefficients of the equation $S_m = 0$ resp. $\frac{m^2}{4} - 2$ and $\frac{m^2-1}{4} - 2$ relations for that equation to be brought by *projective* transformation into the form $\psi(\xi) + \psi(\eta) = 2k$.

As the expression $\Delta' S_m$ is homogeneous in α and β, there must exist between the coefficients of $S_m = 0$ resp. $\frac{m^2}{4} - 1$ and $\frac{m^2-1}{4} - 1$ relations, if this equation is to pass by means of *affine* transformation into the form $\psi(\xi) + \psi(\eta) = 2k$.

§ 6. If we consider the conic represented by $S_2(x,y) = 0$ and the cubic curve represented by $S_3(x,y) = 0$, it is evident that these are symmetric with respect to the line $y = x$.

Whilst in § 5 we have found for the coordinates x and y of a point of a curve $S_2(x,y) = 0$, resp. $S_3(x,y) = 0$ expressions, which were irrational with respect to the odd function $\tau = \sigma . \omega(\sigma^2)$, we can also express the coordinates of the conic and of the *unicursal* cubic curve in *rational* functions of τ.

For the conic we have only to put

$$x = \frac{p_2\tau^2 + p_1\tau + p_0}{q_2\tau^2 + q_0} \quad , \quad y = \frac{p_2\tau^2 - p_1\tau + p_0}{q_2\tau^2 + q_0} \quad . \quad . \quad . \quad (30)$$

and for the rational cubic curve

$$x = \frac{p_3\tau^3 + p_2\tau^2 + p_1\tau + p_0}{q_2\tau^2 + q_0} \quad , \quad y = \frac{-p_3\tau^3 + p_2\tau^2 - p_1\tau + p_0}{q_2\tau^2 + q_0} . \quad (31$$

Elimination of τ out of the equations (30) furnishes

$$(p_0q_2 - p_2q_0)^2 (x-y)^2 + p_1^2[q_0(x+y) - 2p_0][q_2(x+y) - 2p_2] = 0, (32)$$

whilst after elimination of τ out of the expressions (31) we arrive at

$$(p_0q_2 - p_2q_0)^2[q_2(x+y) - 2p_2](x-y)^2 +$$
$$+ [q_0(x+y) - 2p_0][(p_1q_2 - p_3q_0)(x+y) - 2(p_1p_2 - p_0p_3)]^2 = 0. \ (33)$$

The equations (32) and (33) can now very easily be identified with the standard forms $S_2 = 0$ and $S_3 = 0$.

Chemistry. — *"The equilibrium solid-liquid-gas in binary systems which present mixed crystals."* (2nd Communication). By Dr. H. R. KRUYT. (Communicated by Prof. P. VAN ROMBURGH).

(Communicated in the meeting of May 28, 1910).

In a previous communication [1]) I showed what forms the line of the monovariant three-phase equilibrium solid-liquid-gas can assume in systems in which a continuous series of mixed crystals forms the solid phase. The appearance of the three possible forms (maximum,

1) Proc. Vol. XII, p. 537.

minimum or without max. and min.) proved to be dependent on the difference of the triple point pressures of the components and of the form of the melting diagram.

I have started the experimental investigation of this kind of equilibria with the system *paradichloro-* and *paradibromobenzene*, a system in which a complete series of solid-gas equilibria and the boiling point lines have already been determined by KÜSTER [1]). That system had the great advantage that both its components could be determined analytically in a mixture.

The investigations have been carried out by means of an apparatus the principle of which is the same as that of KÜSTER's, but in which a number of technical improvements have been made in consultation with our mechanician Mr. DE GROOT, thus removing many difficulties. Full details of the experiments will not be given here, but later in the "Zeitschrift für physikalische Chemie". The Figures 1 and 2 will be understood sufficiently after a slight explanation.

In Fig. 1 is shown the section of a diving-bell which plunges into the thermostat A (also indicated in Fig. 2). Both the air and the partially fused substance contained in the small basin I are stirred by means of the stirrers (G and F) which are connected with the bell by metallic merceryseals (C and D). If after a number of hours the saturated vapour has formed in the bell, some litres of this are drawn off through the tube L. In Fig. 2 it will then be seen that this gas passes through a tube placed in the furnace m and filled with calcium oxide which decomposes the p $C_6H_4Cl_2$ and p $C_6H_4Br_2$ and retains the halogens as $CaCl_2$ and $Ca\,Br_2$, which may be readily determined quantitatively. The amount of gas which has passed out of the bell is ascertained by measuring the water which has run from the aspirator flask s.

defg is a constant level arrangement and l a steam jacket to prevent condensation of the saturated vapour between the thermostat and the combustion tube; e is the appertaining boiler.

The extra pressure in the bell is finally read off on the water-filled manometer O (Fig. 1).

The water drawn from the aspirator gives us the volume of the gas drawn from the bell in the following manner:

Let us call V_1 the volume wanted, V_2 that of the water passed out; T_1 and P_1 the temperature and pressure in the bell, T_2 and P_2 that in the aspirator; π_1 the saturated water vapour pressure at T_1°, π_2 that at T_2°, then we have:

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[1]) Zeitschr. f. physik. Chem. **50**, 65 and **51**, 222 (1905).

$$V_1 = V_2 \frac{P_2 - \pi_2}{P_1 - \pi_1} \frac{T_1}{T_2}$$

But this V_1 does not yet represent the exact volume. For during the passage through the tube the molecules of the substituted benzenes have been decomposed and have used the requisite amount of oxygen for their combustion [1]). Therefore, V_1 will have to be increased with

$$7\tfrac{1}{2} \times \frac{T_1}{273} \times \frac{760}{P_1} \times 22\ 43 \text{ c.c.m.}$$

for each millimol. which, according to the analysis, has been destroyed by the calcium oxide.

Now, if we know the volume drawn out and, from the analysis, the number of molecules of the compounds present therein we can calculate the pressure exercised by the saturated vapour in the bell. This indirect determination of small vapour pressures suffers, of course, from the defect that no notice is taken of any association of molecules. But this is of very little consequence in these experiments.

I refrain from giving, in this communication, full details as to the purification of the various materials, the preliminary experiments made to see whether my experiments were in accord with those of KÜSTER and the experiments made to find a simple analytical method for the determination of Br and Cl in presence of each other; also the results of blank experiments. I will only state that the melting points are: p-$C_6H_4Cl_2$, 53°,0 and p-$C_6H_4Br_2$, 87°,2. My preparations had, therefore, a higher m. p. than those of KÜSTER and were, therefore, presumably purer and in fact, on repeating one of KÜSTER's experiments I found a somewhat higher vapour pressure. The analytical method employed consisted in dissolving the calcium oxide in dilute HNO_3 and adding an excess of N/10 $AgNO_3$, the precipitate consisting of silver chloride and bromide was weighed in a GOOCH crucible and the excess of silver in the filtrate was determined by VOLHARD's method. From these *data* the two halogens may be readily calculated.

The branches of the melting point line were determined in the usual manner. It should be observed here that the branch of the composition of the liquid may be determined very sharply (initial solidifying points) but, on the other hand the determination of the end solidifying points and the initial melting points is beset with

1) The CO_2 formed is retained in a washbottle containing strong KOH (Fig. 2) and also by the liquid from the aspirator which has been rendered alkaline with KOH.

. R. KRUYT. "The equilibrium solid-liquid-gas in binary systems which present mixed crystals."
2nd Communication.

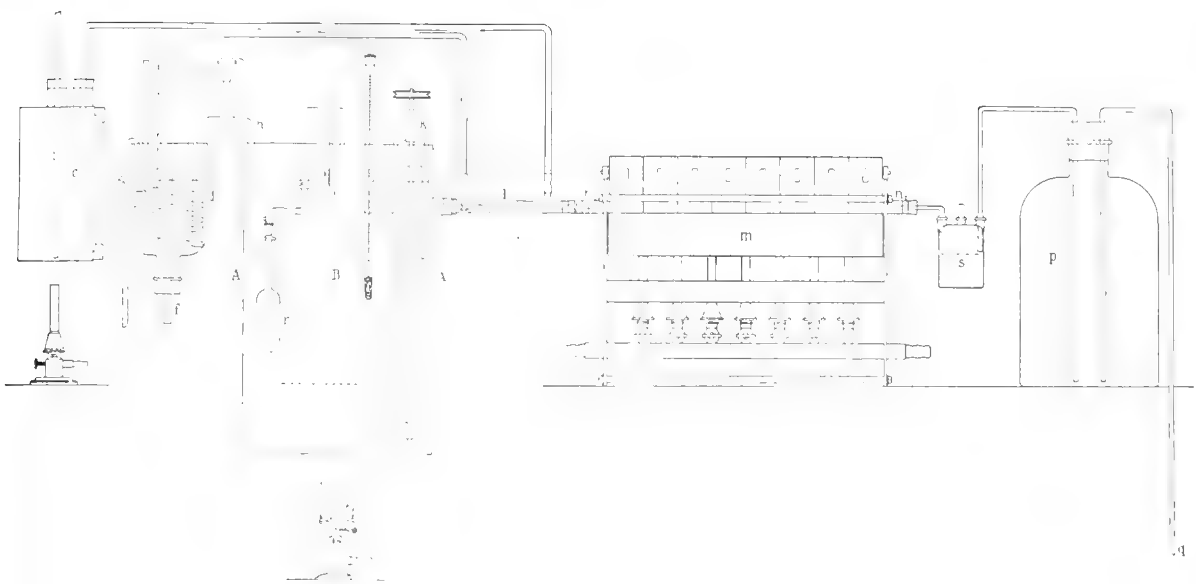

Fig. 2.

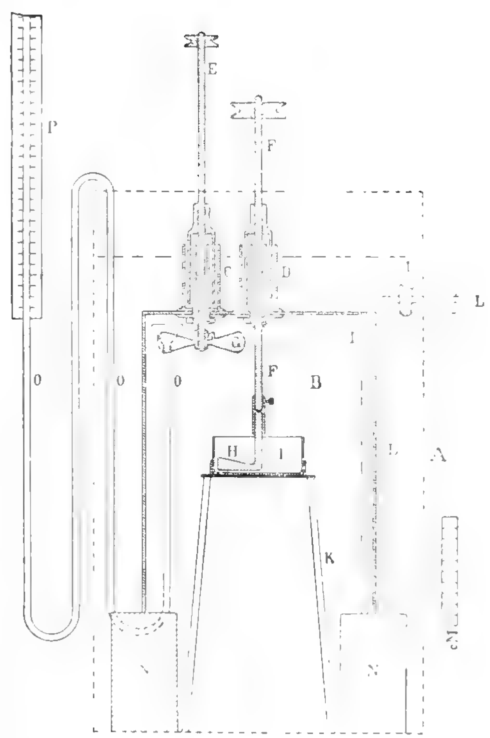

Fig. 1.

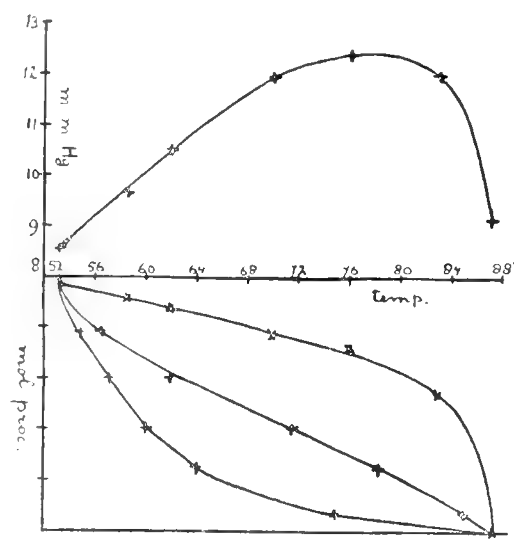

Fig. 3

great difficulties so that the accuracy of the figures for those branches is not so great. In table I are found the results of the respective determinations.

In table II are given the results of the determination of the two triple point pressures. From this we notice that they differ but very little and that one of the conditions for a course with a maximum or minimum in the three-phase line has, therefore, been complied with.

Finally, we find in table III the results of the three-phase tension and gas-composition for mixtures. In Fig. 3, a combined *pt* and *tx* projection, all the results have been united.

From this graphic representation it is shown that we are dealing here with a case which, in my previous paper, I have called case II*a*.

In addition to the demand for about equal triple point pressures, it is also necessary to comply with the demand that, at the side of p-$C_6H_4Cl_2$, the branches of the melting point line shall diverge widely. Now, this is not exactly what takes place here and at the beginning of this research, after finding that the triple point pressures were nearly the same, I fully expected to find case II*b* (minimum), because from the line of the initial solidifying points, as found by KÜSTER a closed figure at the side of the dichloro-component was to be expected. Nevertheless, the remark on pg. 544 of my first communication explains these results; because in the expression

$$(1-x_L)\,P_{T_1}\,e^{f\frac{RT_1}{q}(x_S-x_L)} + p_B < P_{T_1},$$

the term x_S-x_L occurs as an exponential, even a very small value of that term will cause an increase of the pressure proceeding from the triple point along the three-phase line.

The three-phase line with a maximum is therefore remarkably enough, the normal case in the systems with mixed crystals, as well as in those with pure crystallised components. I may add (and indeed this will be readily noticed) that such is also the case for systems with a limited miscibility in the solid phase; the study of the spacial

TABLE I.
Melting point line branches.

gr. p$C_6H_4Cl_2$	gr. p$C_6H_4Br_2$	mol.% $C_6H_4Br_2$	Initial solidifying point	Initial melting point
33.493	3.054	4.8	53°.2	53°.0 à 53°.1
23.633	10.502	21.7	56°.3	55°
15.392	15.973	39.3	62°.6	57°
9.041	21.360	59.5	71°.7	60°
5.679	27.669	75.[illegible]	78°.3	64°
1.502	31.826	93.0	84°.8	75°

diagrams of those systems, also of the peculiar properties of the three-phase line, has engaged my attention for a considerable time. I hope to refer to this later.

TABLE II.
Triple point pressures of the components.

Component.	Temp.	Duration of the experiment	Mols in 100.000 L found	Mols in 100.000 L mean	Pressure in mm. of Hg.
$pC_6H_4Cl_2$	53°.0	3 hours	41.7	41.9	8.53
	53°.0	6 „	42.1		
$pC_6H_4Br_2$	87°.2	4 hours	39.5	40.5	9.40
	87.°2	4 „	41.5		

TABLE III.
Tension on the three-phase line and composition of the gas.

No.	Experiment temp.	Duration of the experiment in hours	mol. % of $pC_6H_4Br_2$ in the mixture used	mols in 100.000 L found	mols in 100.000 L mean	Pressure in mm. of mercury	Composition of gas-phase in mol. % $pC_6H_4Br_2$ found	Composition of gas-phase in mol. % $pC_6H_4Br_2$ mean
1	58.5°	10½	26.1	46.9	46.6	9.64	7.8	8.5
2	58.5°	12½		46.3			9.2	
3	62.0°	7½	42.0	50.5	50.3	10.52	12.6	12.2
4	62.0°	7	41.9	50.1			11.8	
5	69.8°	10	59.5	55.5	55.9	11.96	23.6	23.7
6	69.8°	12		56.4			23.8	
7	76.0°	10	75.2	57.1	57.0	12.39	26.7	26.8
8	76.0°	25		56.9			26.8	
9	83.0°	21	93.0	53.6	54.0	12.00	45.8	46.0
10	83.0°	23		54.3			46.2	

Utrecht, VAN 'T HOFF-laboratory. May 1910.

Chemistry. — *"On the alkaloid content in the leaves of the Cinchonas."* By P. VAN LEERSUM.

(Communicated in the meeting of May 28, 1910).

Historical review.

In his report as to the alkaloid content of the bark and the leaves of the cinchona trees cultivated in Java (Jan. 1, 1864), JUNGHUHN states that according to the Calcutta Gazette Supplement (Aug. 15, 1863) Dr. TH. ANDERSON had prescribed successfully decoctions of the fallen leaves of *C. Succirubra* for fever in the hospital at Darjeeling.

On treating an acid decoction of these leaves with sodium carbonate he obtained small crystals which he thought might be quinine sulphate

JUNGHUHN proceeds: "Although Dr. J. E. DE VRIJ has already analysed cinchona leaves from Tji-Bodas and stated in his report that he has not found a trace of alkaloids therein still I have thought it necessary to test once more carefully the leaves of all our local cinchona species as to their possible alkaloid content, and this was carried out in exactly the same manner as the assay of the barks."

JUNGHUHN's process was as follows:

The leaves with the stalks adhering were cut up into small pieces and dried at 100° until no further loss of weight took place. 40 grams of the dry sample were now boiled gently for an hour with 10 times the weight of acid water (1 part of sulphuric acid to 300 parts of water), the water lost on evaporation being constantly replenished.

The decoction was filtered through flanel and the mass boiled once more with acid water and then twice with plain water. The united filtrates contained in a cylindrical glass were neutralised with ammonia and treated with a solution of tannic acid and the precipitate was collected on a filter.

JUNGHUHN was of opinion that the precipitate consisted of a bitannate of quinine and chinchonine and treated it with calcium hydroxide and alcohol. The alcoholic filtrate was evaporated in a little basin and the residue dissolved in water containing sulphuric acid and filtered. The acid liquid occupying not more than 7 or 10 cc. was collected in a beaker and rendered alkaline with ammonium carbonate which caused the alkaloids to be precipitated as white flakes. The precipitate was collected on a weighed filter, dried and weighed.

The alkaloid content found by JUNGHUHN in this manner amounted in the leaves of *C. Pahudiana* to 0.420 %, of *C. lancifolia* to 0,220 %, of *C. Calisaya* to 0.587 and of *C. succirubra* (fallen, partly green, partly reddish-brown and withered) to 0.520 %.

After JUNGHUHN, it was DE VRIJ who was engaged in the investigation of the leaves of the cinchona tree.

DE VRIJ first attempted to extract the powdered leaves with dilute hydrochloric acid but the results obtained were very unfavourable. He then operated as follows: [1])

The powdered leaves were mixed in a spaceous porcelain dish with one-fourth part by weight of calcium hydroxide and then made with water into a thin paste which was left for some days with

1) Kinologicol studies: *Ned. Tijdschr. Pharm. Chem.* and *Toxic. 1896.*

occasional stirring until the whole mixture had assumed a dark-red colour.

The object of this tedious repeated stirring was to convert the large quantity of kinotannic acid present in the leaves, which was probably the cause of the failure of the hydrochloric acid extraction, into cinchona red by the action of the air and the excess of calcium hydroxide and this object was fully obtained. The mixture was now dried and extracted with alcohol which was afterwards recovered by distillation. The residue was then warmed with dilute acetic acid and the calcium precipitated with ammonium oxalate. The liquid was filtered, a large quantity of chlorophyl being left on the filter. The perfectly clear filtrate had a very pale yellow colour and yielded with ammonia an abundant, voluminous, but very light precipitate, the weight of which, after washing and drying amounted to only 0.162 %. It was a deep yellow powder which did not melt on the waterbath, but dissolved in alcohol to a brown solution. This was again evaporated and the residue converted into acid sulphate; a comparatively large quantity of a reddish-brown substance remained insoluble and was removed by filtration. To the almost colourless filtrate was now added a solution of iodine in potassium iodide, which yielded a fairly abundant precipitate. This was collected on a filter, washed, dried and dissolved in a little warm alcohol.

In this liquid Prof. BEHRENS could not observe microchemically, a trace of any kind of crystalline herapathite [1]) from which it follows that the said precipitate is a compound of amorphous alkoloid with JHJ and H_2SO_4.

The conclusion arrived at by DE VRIJ from this investigation of the cinchona leaves is that they contain one (or more) amorphous alkaloids which are afterwards converted in the living plant into crystalline alkaloids such as occur in admixture with more or less amorphous alkaloid in the cinchona barks.

According to MOENS[2]) the leaves contain very little or no alkaloid and J. C. HOWARD[3]) found once a little in *Succirubra* leaves, but afterwards none in 20 pounds of the same.

From fresh *Succirubra* leaves BROUGHTON [4]) also obtained only 0.0041 % of alkaloid of which 0.0016 % was quinine; from the dry leaves 0.019 % of alkaloid of which 0.008 % was quinine; from *Ledgeriana* leaves MOENS obtained only traces.

1) Kinological Studies. Ned. Tijdsch. Pharm. Chem. and Toxicol. 1899 p. 104

2) The Cinchona culture in Asia 1854—1882.

3) Ph J. 1878 p. 541.

4) Blue book 1870 p. 278.

In fresh *C. officinalis* leaves, Broughton found 0.0035 % of alkaloid of which 0.0015 % was quinine.

Owing to the very divergent results obtained in the analysis of the cinchona leaves, Lotsy[1]) decided to investigate the leaves of the cinchona tree once more and to ascertain whether these organs play also a role in the formation of the alkaloid.

The *modus operandi*: employed by Lotsy is described as follows:

The parts of the leaf (to the left and the right of the midrib) were cut up into very small squares and boiled for half an hour in alcohol containing $^1/_2$ % of HCl (20cc. of strong hydrochloric acid per litre). This took place on the waterbath in small Erlenmeyer flasks closed with a cork fitted with a long tube serving as a reflux condenser. The alcohol was then poured into small porcelain dishes placed on the waterbath and evaporated nearly to dryness. Water was poured into the dishes and the solution again evaporated nearly to dryness in order to be certain that all the alcohol had been expelled.

More water was again added, the solution was filtered and the filtrate collected in a separatory funnel. After being rendered alkaline with KHO, the liquid was shaken with chloroform, which was then evaporated in a watchglass on the waterbath.

The residue was then taken up with water containing $^1/_2$ % of HCl and thoroughly rubbed with a glass rod to detach the resinous matters from the watch-glass. The solution was passed through a miniature filter and the filtrate then used for the alkaloid reactions. Lotsy (l.c.) now arrives at the following conclusions.

1. The amount of alkaloid present in the leaves of a *Cinchona succirubra* and in those of a *Chinchona Ledgeriana* is many times more than sufficient, when transported to the bark regularly, to form the amount of alkaloid present therein pg. 8.

2. *Cinchona succirubra* leaves can part with the whole of their alkaloid supply in 12 hours p. 99.

3. The extent of the formation and migration of the alkoloid is influenced by the weather.

4. The alkaloid disappearing from the *Succirubra* leaf is transported to the stem pg. 18.

5. The alkaloid which is found afterwards in the same leaf has been generated by the leaf itself.

And on pg. 19 it is further stated.

"We may, therefore, come to the final conclusion, without being unduly speculative, that in the cinchona trees the alkaloid is formed

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[1]) J. P. Lotsy. Physiological experiments carried out with *Cinchona Succirubra*. Communication from the Government Botanical Gardens 36 1899.

in the leaves and from thence migrates to the stem, where it is retained either in its original form or in that of a new compound (thus forming an alkaloid different from that derived from the leaves)" and further:

"It stands to reason that these experiments do not yet exclude the possibility of a formation of alkaloids in the bark itself but — looking at the experiments and the arguments held — we may safely assume that it is insignificant, in comparison to what is formed in the leaves and from thence transported to the stem.

Experimental.

In my own investigation the above mentioned process of LOTSY was tried first but this did not prove satisfactory, for if, according to LOTSY's directions, potassium hydroxide is added to an acid solution of cinchona alkaloid still containing impurities the first drops throw down no alkaloid, but all kinds of impurities and, considering the large number of substances occurring in the leaves, which pass together with the alkaloid into the different solvents, this separation of impurities is not trifling. It would have been better (and the results obtained would have looked quite different) not to have used an acid solution for testing of alkaloids, but to have neutralised the liquid (or rendered the same faintly alkaline) in order to get rid of the impurities.

Then, there would have been no risk of failing to obtain a precipitate by adding an insufficiency of alkali to a too strongly acid solution, and all danger of a coprecipitation of foreign matters would have been avoided.

The following process was therefore employed.

24 grams (or less) of the leaf powder (sieve B 40) were mixed with 12 grams of calcium hydroxide and then made into a coarse mass with 8 grams of 15°/₀ sodium hydroxide and 12 grams of ammonia. This mass was shaken for 3—4 hours with 600 cc. of ether and from the clear, green solution 500 cc. (= 20 grams of leaf) were taken. Before proceeding to distillation 10 cc. of 1°/₀ sulphuric acid and 20 cc. of water were added and the mixture was thoroughly shaken.

The ether was now distilled off very slowly.

If the ether is evaporated before addition of the acid water, the large quantity of vegetable fat prevents a thorough contact between the acid and the alkaloid; and a loss occurs.

When manipulating like this in the analysis of leaf rib and leaf stalk all the impurities were separated in a pulverous form, and the washing caused no trouble whatever.

When testing mesophyll particularly that of the *Ledgeriana* leaf, in which occurs more vegetable fat, this did not go so readily, and traces of alkaloids were retained.

The acid yellow coloured liquid was thoroughly shaken in the flask with a few pyropes and filtered.

The filtrate was collected in the separatory funnel and the flask containing the insoluble impurities washed repeatedly with water, until the washings were no longer acid.

After being rendered alkaline, the liquid was shaken four times in succession with 50 cc. of ether, and each time, the flask, in which the extracted liquid was collected before being returned to the funnel, was rinsed with 50 cc. of ether.

The ethereal liquids were collected in another separatory funnel and left at rest for some time to allow any alkali and suspended impurities to deposit.

The deposit formed was then removed with water and the washing continued until the water was no longer alkaline.

The pale yellow coloured ether, containing the alkaloid in solution, was first shaken with 10 cc. of $N/_{10}$ hydrochloric acid and a little water.

After the two layers had separated, the acid aqueous solution containing the alkaloid was collected in a beaker and the ether was again shaken four times in succession with pure water.

After the ether dissolved in the acid water had evaporated spontaneously the excess of acid in the liquid (measuring about 250 cc.) was titrated with $N/_{10}$ alkali, using haematoxylin as indicator.

The observation of the change from yellow into green requires some practice, but still the end reaction is plainly perceptible.

The above method, though tedious, gave good results and the following analyses of bark show that the alkaloid is completely extracted. Assay *A* is made by the method described and *B* by a totally different method of bark analysis.

Sample 1.

A. 7.70 °/₀ of quinine sulphate.
B. 7.60 °/₀ „ „ „

Sample 2.

A. 5.00 °/₀ of quinine sulphate.
B. 4.98 °/₀ „ „ „

Sample 3.

A. 5.40 % of quinine sulphate.

B. 5.40 % ,, ,, ,,

Sample 4.

A. 6.85 % of quinine sulphate.

B. 6.84 % ,, ,, ,,

The subjoined figures were obtained, by the method described, in the assay of *C. succirubra* leaves without mid-rib.

1st	analysis	0.739 %	of total alkaloid.
2nd	,,	0.721 %	,, ,, ,,
3rd	,,	0.739 %	,, ,, ,,
4th	,,	0.750 %	,, ,, ,,

Now, in order to ascertain whether the results obtained by LOTSY[1]) were correct, the process described by him was followed, that is to say, that two halves of the same leaf were always used for the research. These halves were always longitudinal ones.

They were obtained by cutting exactly along the mid-rib of the leaf. In this manner the leaf was divided into two unequal parts one with and one without mid-rib.

The piece without mid-rib was tested at once, that with the mid-rib remained attached to the tree. At the end of the experiment the mid-rib was removed and the remaining half of the leaf was then tested.

The pieces of leaf to the left and the right of the same mid-rib were in this manner compared with each other and LOTSY obtained the following results (l.c. pg. 9).

6 p.m. 18 Sept. '99	6 a.m. 19 Sept. '99.
N°. 284 full	empty
,, 285 ,,	,,
,, 286 ,,	,,
,, 287 ,,	,,
,, 288 ,,	,,
,, 289 ,,	,,
,, 291 ,,	,,
,, 292 ,,	,,
6 a.m. 21 Sept. '99	**6 p.m. 21 Sept. '99,**
N°. 305 full	empty
,, 308 ,,	,,
,, 310 ,,	,,

1) l.c. p. 4.

In order to ascertain the possible influence of light and darkness, I placed on 19/8 '08 a *Ledgeriana* tree about five years old entirely under a box lined with lead foil but first of all, from a large portion of the well developed tree one half of each leaf was removed, leaving the mid-rib attached to the other half.

After removing the box on 3/9 '08 the second half of the leaves was examined.

Result:

I. *A.* Investigation of the part without mid-rib viz. that removed from the leaf before covering with the box.

1st half. Total alkaloid 0.410 %.

B. Investigation of the other half of the leaf viz. that which had been excluded from the light for 16 days with its rib.

2nd half. Total alkaloid 0.430 %.

II. *A.* Investigation of the part without mid-rib, viz. the part removed from the leaf before covering with the box.

1st half. 0.412 % of total alkaloid.

B. Investigation of the other half of the leaf, viz. the part excluded from the light for 16 days with the rib.

2nd half. 0.410 % of total alkaloid.

Leaf rib and leaf stalk 0.695 % of total alkaloid.

In addition to the above experiments the following comparative experiments were made.

A cultivating bed planted with Ledgeriana seedlings was divided into two plots *A* and *B*.

The plants in plot *A* were on 19/8 '03 excluded from the light by means of a box lined with lead foil, but beforehand one-half of the leaf was removed and investigated.

Plot *B* remained uncovered and, therefore, kept growing under normal conditions but one-half of the leaf was also removed and tested.

On 4/9 '08, or 16 days afterwards, the box was removed and the other half of the leaf was tested, also the 2nd half of the leaf from plot *B*.

Plot *A* (leaves, the first half tested 19/8, the 2nd half after having been in darkness for 16 days).

Plot *A*. 1st half of the leaf. Total alkaloid 0.508 %.
2nd ,, ,, ,, ,, (darkness) Total alkaloid 0.530 %.

Plat *B*. 1st half of the leaf. Total alkaloid 0.447 %.
2nd ,, ,, ,, ,, (light) Total alkaloid 0.460 %.

If now, Lotsy's theory were correct that the alkaloid in the Cinchonas is a product of assimilation, therefore a substance formed like amylum in the leaf and ready to be conveyed to the stem, the leaves which have been excluded from the light for a considerable time ought no longer to contain alkaloids or, in any case, much less than the leaf under normal conditions.

The above results, however, show the reverse. Moreover, according to the said theory, none or little alkaloid ought to be present in the fallen leaves and this should have been transported previously to the fall either entirely or for the greater part.

The subjoined analyses, however show the reverse.

Mesophyll of plucked, still living, green Succirubra-leaves. I.	Mesophyll of fallen Succirubra leaves of the same tree. II.	Leaf stalk and mesophyll of the still living green leaf. III.	Leaf stalk and mesophyll of the fallen Succirubra leaf. IV.
Total alkaloid in proc.	Total alkaloid in proc.	Total alkaloid in proc.	Total alkaloid in proc.
a. 0,728	*a.* 0,739	*a.* 1,01	*a.* 1,23
b. 0,739	*b.* 0,721	*b.* 0,997	*b.* 1,15
	c. 0,739		
	d. 0,750		
Id. of Ledgeriana.	Id. of Ledgeriana. (same trees)	Id. of Ledgeriana.	Id. of Ledgeriana.
a 0,410	*a.* 0,420	*a.* 0,500	*a* 0,647
b. 0,440	*b.* 0,440	*b.* 0,580	*b.* 0,615
c. 0,390			
d. 0,408			

As, however, the possibility is not excluded that the tree when covered for 14 days or a month with a box exists under abnormal conditions, which was moreover indicated by the dropping of many leaves, the experiment was repeated and conducted in a different manner.

Of 50 leaves, the one-half along the mid-rib was removed and investigated.

The other half with the mesophyll and leaf-stalk was carefully wrapped in tin foil, thus absolutely excluding access of light.

After having been wrapped up for 12 hours or longer, the second half of the leaf was tested with the following result:

I.

C. succirubra leaves.

a. 1st half removed at 6 p. m. Total alkaloid: 0.197 gram in 50 half leaves.

b. 2nd half removed at 6 o'clock next morning. Total alkaloid: 0.212 gram in 50 half leaves.

II.

C. Succirubra leaves.

a. 1st half removed at 6. p. m. Total alkaloid: 0.248 gram in 50 half leaves.

b. 2nd half removed at 6 o'clock next morning. Total alkaloid: 0.254 gram in 50 half leaves.

III.

C. Succirubra leaves.

a. 1st half removed at 6 p. m. Total alkaloid: 0.233 gram in 50 half leaves.

b. 2nd half removed at 6 o'clock next morning. Total alkaloid: 0.207 gram in 50 half leaves.

This experiment was also made in a reverse sense; the entire leaf was first wrapped in tin foil for 14 days and then the first half of the leaf was removed along the mid-rib and tested; the 2nd half was then exposed to the light for 14 days and also tested without mid-rib.

IV.

C. succirubra leaves.

a. 1st half, after the entire leaf had been wrapped in tin foil for 14 days. Total alkaloid 0.213 gram in 50 half leaves.

b. 2nd half after the same had been again exposed to the light for 14 days. Total alkaloid: 0.198 gram in 50 half leaves.

Now, if alkaloids had formed in the leaf by the assimilation process, these values should have been reversed.

From these investigations it is now evident that when the plant is excluded from the light for ten days (or even a month) this has *no influence* on the alkaloid content of the leaf, whereas Lotsy thought he could even notice a change *after* 12 *hours.*

The conclusion arrived at by Lotsy that the alkaloid in the Cinchonas is a product of *assimilation* is also incorrect.

If this theory were correct, none, or but very little alkaloid should occur in the leaves which have been excluded from the light for a considerable time, or in fallen leaves; in the latter case the alkaloid ought to have been transported entirely, or for the greater part, before the falling of the leaves. The investigations show, however, that the mesophyll of plucked, still living, green leaves contains as much amorphous alkaloid as the mesophyll of fallen, brown, no longer living leaves.

It is, therefore, obvious that the alkaloids are products of metabolism which are formed in the leaf or in other organs and remain accumulated there without being of importance, in ordinary circumstances, for the metabolic change.

In order to test the correctness of LOTSY's thesis (l. c. p. 18): "The alkaloid disappearing from the *succirubra* leaf is transported to the stem", the following additional experiments were made.

Twenty well-developed branches of *C. Ledgeriana* were decorticated, that is to say, a strip of bark 4 c.m. in width was removed from the branches and submitted to analysis. The part laid bare was well cleaned to remove all the cambium so that a fresh formation of bark was impossible on that spot.

Eighteen days after the decortication the branch was sawed off at the stem and another 3 c.m. wide strip of bark was removed from both below and above the decorticated piece.

As a new tissue (callus) was beginning to form at the injured surface a strip of a few m.m. wide was left between the injured surface and the new sample, so as to avoid the influence of fresh tissue.

For, it might be possible that the alkaloid content in this new abnormal tissue was also not normal.

1st Experiment.

a. Analysis of the first circular strips of bark tested at once after removal.

7.20 °/₀ of quinine, or in 10 pieces of absolutely dry bark 1.69 gram of quinine (alkaloid).

b. Analysis of the sample of bark situated below the circular strip and tested about 14 days after the decortication.

7.35 °/₀ of quinine, or in 10 pieces of absolutely dry bark 1,69 gram of quinine (alkaloid).

c. Analysis of the sample of bark situated above the circular strip and tested 14 days after the decortication during which a supply of alkaloid from the leaves might have taken place.

6.60 °/₀ of quinine, or in 10 pieces of absolutely dry bark 2.89 grams of quinine (alkaloid),

2nd Experiment,

a. Analysis of the first circular strips of bark tested at once.

8.66 °/₀ of quinine, or in 10 pieces of absolutely dry bark 2.89 gram of quinine (alkaloid).

b. Analysis of the sample of bark situated below the circular strip and tested about a month after the decortication.

9.01 °/₀ quinine, or in 10 pieces of absolutely dry bark 3.17 grams of quinine (alkaloid).

c. Analysis of the sample of bark situated above the circular strip and tested about a month after the decortication, during which time a migration of alkaloid from the leaves might take place.

7,53 °/₀ of quinine, or in 10 pieces of absolutely dry bark 2.79 grams of quinine (alkaloid).

The difference in quinine content, in both experiments, of the samples of bark above and below the ring must be attributed to age, for the bark of the pieces below the decorticated piece is older than that of the piece situated above it, nearer the top of the branch.

The ring test was also applied in a different manner as follows.

A tree about 20 years old with perfectly sound bark was decorticated by removing a strip of bark $13^1/_2$ c. m. in width at the height of a man's chest.

The wood deprived of bark was well scraped thus removing not only the candium but also a part of the young wood, which totally excluded the formation of renewed bark.

At the same time a strip of bark 47 c. m. in length and 2 c. m. in width was removed from just above the decorticated part and also a similar strip from below the same.

First 14 days, then 6 weeks, and finally 3 months after the decortication a second strip next to the first was removed from above and from below the decorticated part and all the four strips were examined.

The result obtained was as follows:

a. Weight of ring bark 236 grams (moist) = 89 grams (air—dry).

Content 7.53 °/₀ of quinine (alkaloid) or 2.30 grams of quinine in a strip of bark weighing 34 grams.

b^1. Weight of strip of bark from above the ring 88 grams (moist) = 34 grams (air—dry).

Content 8.10 /₀ of quinine (alkaloid) in the absolutely dry bark or 2.40 grams of quinine in a strip of bark weighing 34 grams.

b^2. Weight of strip bark from above the ring taken 14 days after the first sample.

90 grams moist = 34 grams air—dry.

Content 8.20 °/₀ of quinine (alkaloid) in absolutely dry or 2.55 grams of quinine in a strip of bark weighing 34 grams.

b^2. Weight of strip of bark from above the ring taken 6 weeks after the first sample.

106 grams moist = 34 grams air-dry.

Content 7.85 °/₀ quinine (alkaloid) in absolutely dry bark or 142 gram of quinine in a strip of bark weighing 34 grams.

b^4. Weight of strip of bark from above the ring taken 3 months after the first sample 104 grams moist = $41^1/_7$ grams air-dry.

Content 6.67 °/₀ quinine (alkaloid) in absolutely dry bark or 1.98 grams of quinine in a strip of bark weighing 34 grams.

c_1. Weight of strip of bark from above the ring tested at once.

90 grams moist = $34^1/_2$ grams air-dry.

Content 8.40 °/₀ quinine (alkaloid) in absolutely dry bark or 2.61 grams of quinine in a strip of bark weighing 34 grams.

c_2. Weight of the strip of bark taken 14 days after the first sample.

90 grams moist = 34 grams air-dry.

Content 8.23 °/₀ of quinine (alkaloid) in absolutely dry bark or 2.60 grams of quinine in a strip of bark weighing 34 grams.

c_3. Weight of the strip of bark from below the ring taken 6 weeks after the first sample.

105 grams moist = 34 grams air-dry.

Content 8.54 °/₀ quinine (alkaloid) in absolutely dry bark or 2.60 grams in a strip of bark weighing 34 grams.

c_4. Weight of strip of bark from below the ring taken 3 months after the first sample.

92 grams moist = 30 grams air-dry.

Content 9.09 °/₀ of quinine (alkaloid) in absolutely dry bark or 2.70 grams of quinine in a strip of bark weighing 34 grams.

From these investigations it is obvious that there can be no question of a migration of alkaloid from the leaves towards the stem; in fact we notice that in the strips of bark from above the ring (see b_2 and b_4) a decrease in quinine has taken place from 7.85 °/₀, six weeks after the decortication to 6.67 °/₀ after 3 months of the same or, from 2.42 grams of quinine to 1.98 grams in a strip of bark weighing 34 grams, respectively.

No explanation has been found as yet why the decrease of quinine takes place just in the strips of bark from above the ring while the strips from below the ring show hardly any differences.

One would feel inclined to attribute this to the influence of the irritation caused by the inflicted wound, although the tree kept to the last fairly healthy as regards the leaves, but in every case it appears that this is out of the question.

In N°. 10 of J. E. DE VRIJ's Kinological studies it is stated that BEHRENS did not succeed in obtaining a trace of a crystalline herapathite when adding a solution of iodine in potassium iodide to the colourless filtrate obtained in due course from the leaf powder.

In order to ascertain whether any crystalline alkaloids are present in leaf mesophyll, the following *modus operandi* was employed.

100 grams of *C. succirubra* leaf deprived of leaf stalk and midrib were dried and treated in the manner described. The acid solution finally obtained was again purified by adding alkali and shaking with ether; the ether was then evaporated and the residue dissolved in water, sulphuric acid being added to acid reaction.

The filtrate was evaporated on the waterbath to a syrupy consistency and then dissolved in alcohol.

The alcoholic filtrate was evaporated and the residue dissolved in water and again filtered.

The aqueous fairly colourless solution was rendered alkaline and shaken with ether; the ether was evaporated and the residue subjected to sublimation.

The very small sublimate was taken up with a trace of HCl-containing water, evaporated to dryness in a desiccator and the residue dissolved in a drop of water and filtered.

On heating this filtrate with a trace of a strong solution of sodium hydrogen carbonate, a crystal of chinchonine was obtained.

The presence of a crystalline alkaloid in the mesophyll could also be shown in another, indirect manner.

Attacks of cinchonas by Atlas caterpillars are not rare and as they principally feed on the mesophyll and the leaf stalk and leave the mid-rib untouched both the contents of the stomach and the excreta[1]) of the Atlas caterpillars were examined chemically and in each case cinchonine could be detected but no other crystalline alkaloid.

In the analysis of leaf stalk and mid-rib cinchonidine could be detected as well as chinchonine and judging by the fluorescence, quinine was also present.

In order to compare the results obtained in the analysis of Cinchona, I think I ought to mention briefly the results of the investi-

[1]) When starting from 200 grams of excreta, the other crystalline alkaloids could also be detected.

gation of *Datura strammonium* [1]) by JULIUS FELDHAUS, and of tea by DU PASQUIER [2]) and TH. WEEVERS [3]).

On p. 88, FELDHAUS arrives at the following result in his investigation of *Datura strammonium*.

Blätter: Die Zeit der Einsammlung ist ohne Einflusz auf den Alkaloidgehalt, denn in einem Falle enthielten die Ende Juli und die Ende August und in einem anderen Falle die Anfang September und Anfang Oktober gesammelten Blätter derselben Pflanzen eine nicht wesentlich verschiedene Menge Alkaloid. 0,46°/₀ respektive 0,46°/₀ Alkaloid und 0,30°/₀ respektive 0,39°/₀ Alkaloid.

Der Gehalt junger, an der Basis noch gelbgefärbter, etwa 5—10 cm. langer Blättchen mit 0,48°/₀ Alkaloid war nicht wesentlich verschieden von dem vollentwickelter, zu gleicher Zeit von denselben Pflanzen gesammelter Blätter, der 0,49°/₀ Alkaloid betrug. Damit ist die Ansicht SÜM-JENSENS, zu der er auf Grund seiner mikrochemischen Betrachtungen bei Hyoscyamus gelangt war, nicht bestätigt, dass in jungen Blättern der Alkaloidgehalt relativ grösser zu sein scheine

Die weiteren mikrochemischen Untersuchungen von SÜM-JENSEN sowohl die von PH. MOLLE zeigten, dass die grösste Alkaloidmenge in den Gefässbündeln, wenig oder gar nicht im Mesophyll der Blätter zu finden sei.

Ich fand im Assimilationsgewebe 0,48°/₀, in Mittel- und Sekundärnerven 1,39°/₀ und in den Blattstielen derselben Blätter 0,69°/₀ Alkaloid.

Bei Hyoscyamusblättern hatte E. SCHMIDT eine Trennung in Blattflächen und Blattstiele vorgenommen und fand in den Blattflächen 1) 0,2726°/₀ und 2) 0,2861°/₀ Alkaloid, in den Blattstielen 1) 0,36°/₀ und 2) 0,365°/₀ Alkaloid. Also auch bei Hyoscyamus ein höherer Gehalt an Alkaloid in den Blattstielen als in den Blattflächen.

Eine ergiebige Chilisalpeterdüngung ist ohne Einfluss auf den Alkaloidgehalt. Blätter von Pflanzen, die auf ungedüngtem Beete gewachsen waren, hatten 0,49°/₀ Alkaloid, von Pflanzen, die auf dem Salpeterbeete gewachsen waren, 0,50°/₀ Alkaloid. Samen von Pflanzen der ersten Sorte hatten 0,34°/₀ Alkaloid, von der zweiten Sorte 0,34₀/₀ Alkaloid.

Assimilation. Verdunkelung ist auch von keinem Einflusse auf den

[1]) Quantitative Untersuchung der Verteilung des Alkaloides in den Organen von Datura stramonium. Inaugural Dissertation von JULIUS FELDHAUS, Marburg 1903.

[2]) Beiträge zur Kenntnis des Thees. Inaugural Dissertation von PAUL A. DU PASQUIER, Zürich 1908.

[3]) Die Physiologische Bedeutung des Koffeins und des Theobromins von TH. WEEVERS. Annales du Jardin Botanique de Buitenzorg (Volume XXI) 2e Serie. (Volume VI) 1e Partie.

Alkaloidgehalt. Im Dunkeln aufgewachsene Keimpflanzen hatten, 0,66 °/₀ Alkaloid, normal aufgewachsene derselben Samen 0,67 °/₀ Alkaloid.

Ebenso konnte CLAUTRIAU keinen Unterschied im Alkaloidgehalte hell und dunkel erwachsener Keimpflänzchen von Coffea- und Thea-Arten beobachten.

Blatthälften am Abend gesammelt hatten 0,48 °/₀ Alkaloid, die zugehörigen Blatthälften am folgenden Morgen gesammelt hatten 0,40 °/₀ Alkaloid. Blatthälften abends gesammelt hatten 0,51 °/₀ Alkaloid, die zugehörigen Blatthälften, nach dreitägiger Verdunkelung gesammelt, hatten 0,51 °/₀ Alkaloid. Es findet also während der Nacht oder künstlicher Verdunkelung keine Ableitung des Alkaloides statt.

Es tritt aber auch bei Tage keine wesentliche Vermehrung des Alkaloidgehaltes in ausgewachsenen Blättern ein, ich müsste sonst, da ja keine Ableitung stattfindet, in den an verschiedenen Tagen gesammelten Blatthälften derselben Blätter einen wesentlich höheren Alkaloidgehalt in den später gesammelten Hälften gefunden haben. Ich fand in Blatthälften 0,33 °/₀ Alkaloid, in den zugehörigen, nach drei Tagen ohne künstliche Verdunkelung gesammelten Blatthälften fand ich 0,33 °/₀ Alkaloid.

Die Verletzung des Blattes veranlasste also auch nicht eine stärkere Alkaloidproduktion.

Aus allen Versuchen geht hervor, dass das Alkaloid kein direktes Produkt der Wirkung des Lichtes auf die Blätter ist, also auch kein Assimilationsprodukt.

On pg. 36, DU PASQUIER arrives at the following result in his investigation of tea.

"Alle drei Wege führten mithin zum selben Resultate: Koffein spielt in der Theeflanze die Rolle eines Abfallproduktes."

DU PASQUIER found in 50 fallen tea leaves (weighing when dry 11.000 grams) a total weight of 0.1001 gram of caffeine = 0.91 °/₀.

On pg. 12 he further states:

Vergleicht man diese Zahlen mit meiner früheren Reihe (Seite 21, Tabelle VIII), so sieht man, dass sie sich aufs schönste an jene Zahlen angliedern würden, sodass also ein Rückgang oder sogar ein Verschwinden im Koffeingehalt bei den abgefallenen Blättern nicht zu erkennen ist.

This result, however, does not agree with that of **WEEVERS**, who does not find any caffeine in the fallen tea leaves.

In regard to this DU PASQUIER offers the following explanation:

Es war mir denn auch nicht schwierig, die Erklärung für das Nichtauffinden von Koffein durch WEEVERS—DE GRAAF zu geben.

Dieselben verwendeten zur Koffeinbestimmung eine Methode, in der die Blätter mit ungelöschtem Kalk behandelt werden. Nun hat aber A. Beitter 1901 in seiner Arbeit "Neuere Erfahrungen über Koffeinbestimmung" nachgewiesen, dass beim Behandeln mit Kalk die Hälfte des ganzen Koffeins zersetzt wird. Bedenkt man ferner, dass Beitter seine Beobachtungen an mehrere Prozent Koffein enthaltenden Thees machte und dass der Prozentgehalt der abgefallenen Blätter an Koffein nicht einmal 1 % beträgt, so wird man leicht einsehen können, dass die geringe Menge Koffein leicht übersehen wurde, und Weevers Begründung mithin nicht stichhaltig ist. Dazu kommt, dass sie keine quantitativen Bestimmungen machten, sondern nur den qualitativen Nachweis zu führen suchten.

By way of comparison the leaves of a tea shrub were also investigated. The results obtained agree with those of du Pasquier in so far that caffeine could be plainly detected in the leaves of a tea shrub which had been excluded from the light for 14 days.

Owing to the want of material no investigation could be made as to the presence of caffeine in the fallen leaves.

I wish to give my best thanks to Dr. A. Rant, botanist at the Government Cinchona exploitation for his suggestions and advice in this investigation.

CONCLUSIONS.

The conclusions arrived at in this research are as follows:

1. The contention of J. P. Lotsy that an exposure of the leaf to light or darkness affects the alkaloid content is incorrect.

2. His view that the formation and migration of the alkaloid is affected by the weather is also incorrect.

3. The alkaloid is not an assimilation but a metabolic product.

4. The mesophyll and the veins of both *C. Ledgeriana* and *C. succirubra* leaves contain most decidedly crystalline alkaloids and also quinine.

5. The leaf stalk and the mid-rib of *C. Ledgeriana* and *C. succirubra* contain besides cinchonine also quinine.

EXPLANATION OF THE ILLUSTRATIONS.

Fig. I. Cinchonine from leaves of C. succirubra.

Fig. II. Cinchonine crystals obtained with sodium hydrogen carbonate from the excreta of the Atlas-caterpillar.

Fig. III. Cinchonine crystals obtained with sodium hydrogen carbonate from the contents of the Atlas-caterpillar.

Fig. IV. Cinchonine crystals obtained with sodium hydrogen carbonate from the chrysalis of the Atlas-caterpillar.
Fig. V. Sublimate of theïne recrystallised from water (dark).
Fig. VI. Sublimate of theïne recrystallised from water (light).
Fig. VII. Theïne (dark) with sodium acetate.
Fig VIII. Cinchonine from mid-rib and leaf stalk.
Fig. IX. Cinchonine from mesophyll and veins.
Fig. X. Cinchonidine from mid-rib and leaf stalk.

Physiology. — *"The temperature optimum of physiological processes"*. By Miss J. VAN AMSTEL and Prof. G. VAN ITERSON Jr. (Communicated by Prof. M. W. BEIJERINCK).

(Communicated in the meeting of May 28, 1910).

The destruction at high temperature of the active principle, so conspicuous in physiological processes, has been subjected by TAMMANN [1]) for a few enzyme processes to a nearer research, which led that investigator to the result, that destruction of the enzyme by heating takes place after the equation of monomolecular chemical reactions. In accordance with this view the relation between the quantity of enzyme y, which after heating at a constant temperature during a time t, is still active, and that time, would be represented by the formula: $k = \frac{1}{t} \log \frac{a}{y}$, where a represents the originally present quantity of enzyme and k a constant.

DUCLAUX [2]) suggested a relation between the destruction of the active agency by high temperature and the optimum of enzyme action, and explained the occurrence of this cardinal point by admitting that the velocity of the reaction continually increases with the rising of the temperature, whilst the bending of the curve, which represents the relation between velocity and temperature, should exclusively be ascribed to a steadily increasing destruction of the enzyme by the heating. The views of DUCLAUX were absolutely theoretical and he made no experiments to test them.

The idea which forms the base of DUCLAUX' theory we find back in a treatise of BLACKMAN [3]), but here the views put forward à priori are tested by observations and in particular by the results of studies made conjointly by this investigator and Miss MATTHAEI [4]) on the relation of the carbonic acid assimilation with the temperature.

[1]) Zur Wirkung ungeförmter Fermente, Ztf. f. Physikal. Chem., Bd. 18, 1895, S. 429.
[2]) Traité de Microbiologie, T. II, 1899, p. 193.
[3]) Optima and Limiting Factors, Annals of Botany, Vol. XIX, 1905, p. 281
[4]) Phil. Trans. Roy. Soc., Vol 197 B, 1904, p. 85.

BLACKMAN also takes into account the duration of the heating, a factor which, as TAMMANN's researches show, is of much weight and which DUCLAUX had quite neglected.

From the quantities of carbonic acid, assimilated during four consecutive hours at different temperatures (after $1^1/_2$ hours' heating before the beginning of the first observation), were approximated by extrapolation the velocities of the carbonic acid assimilation which would have been observed if it had been possible instantly to bring the leaf to the desired temperatures and then immediately to measure the initial velocity. The thus estimated velocities would then, after BLACKMAN, quite like those at low temperatures, follow the rule of VAN 'T HOFF after which for every rise of 10° C. the rate of reaction velocity is about doubled or trebled, a rule applicable to a great number of chemical reactions.

In the year 1909 the theory of BLACKMAN was tested by KUIPER[1]) for the process of respiration of the higher plants. For different low and high temperatures he determined the quantities of carbonic acid absorbed during each of 6 consecutive hours. The main impression which KUIPER obtained from these observations was that BLACKMAN's theory holds also good for the respiration function, yet he noted that a somewhat improbable course ought to be given to the extrapolation curves in order to find values for the velocities in absence of destruction, which would be in accordance with VAN 'T HOFF's rule.

When now, as here, there is little known about the nature of the function, or when it actually represents a logarithmic curve, extrapolation is a rather dangerous means to approximate the sought for quantity. This will especially be the case when the times during which the observations of the reaction velocities must be made, are so ong hat already considerable modifications may have appeared.

Moreover, in such long periods as were required in the described experiments, adaption to the high temperature may occur. Lastly, cells of quite different nature were subjected to the heating and it is a well known fact that the resistance to heating for different tissues differs also very much.

Hence we thought it desirable to submit the above views to a renewed research for physiological processes which may be studied on a large number of equal cells at a time, and which go on

[1]) Influence of temperature on the respiration of higher plants. These proceedings Vol. XVIII, 1909/1910, p. 210. See also the dissertation with the same title, Utrecht, 1909.

so rapidly that the periods between the successive observations need but be very short. As such we considered various physiological processes proper to alcohol yeast and a short survey will be given of the results obtained by the investigation of alcohol fermentation and the inversion of cane sugar by that yeast. The detailed description of the experiments and of observations concerning other functions will be given at another place.

Although it now proved possible for these processes to study with fairly great precision the course of the reaction at heating, here also some incertainty continued to exist when applying the extrapolation method. In order to remove it, a somewhat modified method of observation was followed. A determined quantity of yeast was heated during different times at ultra-optimal temperature and after quickly cooling the velocity of the process for this yeast was observed. Thus could be calculated to what portion of the original quantity of yeast at a noxious temperature the observed velocity was to be ascribed, and with a simple and certainly permitted allowance could then be estimated what velocity would have been read if all the yeast at that temperature had still been in possession of the function. Hence, by this way the sought for relation between the velocity of the reaction and the temperature, in absence of any destruction, could be estimated.

Once this relation established it was possible to ascertain whether the thus found values might also have been expected with sufficient probability after the extrapolation method.

I. Alcohol fermentation [1]).

§ 1. *Arrangement of the experiments.* As a criterion for the velocity of the alcohol fermentation was taken the quantity of CO_2 in cM^3. which is evolved per second in the fermentation of a glucose solution by pressed yeast, which was regularly put at our disposal by the "Dutch Yeast- and Spirit Works" at Delft. The yeast was rubbed up with a known quantity of water and fore-heated by the side of the sugar solution. Not before both solutions were at the temperature of the bath they were mixed together [2]). To obtain a homogeneous medium the suspension was moved during the whole of the experiment by a shaking apparatus. The temperature was kept constant within limits only differing 0.05° C.

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1) Some eight years ago this process was studied by us in the Laboratory of Prof. Beijerinck; by making use of more exact methods of observation we have now obtained definite results.

2) See for precautions to take at noxious temperatures: § 4.

Only after 20 to 25 cM³ of gas have been formed the evolving of gas (at least at harmless temperatures) becomes sufficiently regular; therefore the observations were not begun before this quantity had escaped. Then a volume of at most 100 cM³ was still caught and the periods determined which elapsed by the evolution of the consecutive 2-, 5-, 10-, or 25 folds of cM³, according to the more or less rapid gas evolution.

It should besides be noted that the concentration of the sugar solution was constantly made so strong that the quantity of glucose wanted for the development of 125 cM³ caused no actual modification in the concentration of the sugar so that we always determined the *initial velocity*.

§ 2. *The relation between the initial velocity of the fermentation and the concentration of the yeast at constant sugar concentration.* The obtaining of a constant sugar concentration with varying quantities of yeast gave some trouble by the circumstance that the pressed yeast, such as it was supplied to us, always contains water between the cells, which will take part in the dilution of the sugar solution. By determining the fall of the concentration of a known glucose solution after mixing with a certain quantity of yeast (which took so short a time and was effected at so low a temperature that no sugar was fermented), this quantity was found to be about 35 % of the weight of the yeast.

For each experiment was now used 10 cM³ of a 30 % glucose solution; the yeast was mixed with so much water that it was finally suspended in 47 cM³ of liquid which quantity must be calculated for each experiment taking into account the above mentioned 35 % of water. By this arrangement the sugar concentration for all the experiments was constant and amounted to $\frac{300}{47} = 6.4\ \%$. All the experiments were carried out at 45° C., at which temperature during the periods used, no destruction of the function is observable.

Fig. 1, Plate I, represents graphically the relation of the fermentation velocity to the sugar concentration and shows that at low concentration there is almost proportionality, whilst at a higher rate rather considerable deviations arise.

§ 3. *The relation between the velocity of the fermentation and the concentration of sugar at constant yeast concentration.* Although the knowledge of this relation is not strictly necessary for our further research, still it seemed suitable to elucidate this point too. This relation (6 gr. of yeast with 35 cM^3 of water and 10 cM^3 of various glucose solutions, temp. 45° C.) is represented in fig. 2 of Plate I by a parabolic curve so that there is no question of proportionality of velocity to sugar concentration. Indeed, as we intend to demonstrate elsewhere, this relation may be represented by the well-known adsorption formula.

§ 4. *The relation between the initial velocity and the temperature at constant yeast and sugar concentrations.* For the experiments to determine the optimum curves were always taken 16 G. of yeast and 31.5 cM^3 of water with the addition of 10 cM^3 of a 30 % glucose solution. The yeast suspension and the sugar solution were, beforehand, brought to the desired temperature.

It resulted now that the observed fermentation velocities up to 45° C. are independent of the time of fore-heating if, at least, as in our experiments, it be taken maximal 20′, whilst moreover these velocities were constant during the fermentation experiments.

From this it follows that at a temperature of 45° C. and lower during these heating-periods there can be no question of destruction of the function. The observations hereabout are to be found in the last column of Table I and led to the construction of the curve in fig. 1 of Plate II for temperatures below 45° C.

Above 45° C. the velocity of the fermentation depends certainly on the duration of the foreheating and besides, it also falls during the fermentation experiment. We have for this reason fore-heated the yeast suspension at various noxious temperatures, respectively for 5, 10 and 20 minutes, then mixed it with the sugar solution (of the same temperature) and determined the fermentation velocity.

Now, various points should thereby taken into consideration.

Firstly, the yeast suspension, when introduced into a thermostat of high temperature, will not immediately have adopted this temperature. If no account were kept herewith the fore-heating at the deleterious temperature would be shorter than desired. To prevent this the suspension was first heated in a separate thermostat to the highest temperature that was not yet noxious, and this reached, instantly placed in the definitive bath, where, owing to the shaking apparatus and the thin glass walls, it attained the noxious temperature within $^1/_4$ minute. [1])

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[1]) The little error still existing is also made at the determination of the "destruction-curve" and thereby removed.

The second difficulty is that always 20 to 25 cM.^{3} of gas must escape before the reading can begin. During the time therefor required the destruction of the fermentation function goes on, and if this circumstance were left out of account the heating before the first observation would be longer than was meant. This difficulty was removed by determining by a preliminary experiment the time wanted for the evolution of these 25 cM.^{3} and by mixing the yeast suspension and the glucose solution so much earlier, the reading being commenced only after the 5, 10, 15 or 20 minutes had elapsed [1]).

Thirdly it should be observed that where the velocity of the fermentation during the experiment at a noxious temperature is seen to fall, even the first observation made of the fermentation velocity does not give the value which must have existed immediately after the fore-heating. Now, this latter velocity, for this it is we want to know, could with some precision be estimated by extrapolation from the different observations made at one and the same fermentation experiment, owing to the circumstance that the velocity thereby regularly falls. In order to demonstrate this in an example the observations for 52° C. for the four various fermentation experiments (for 5, 10, 15 and 20′ of fore-heating) are graphically represented by fig. 6 Plate I, where on the ordinate axis the fermentation velocities are marked, calculated from the times wanted for the evolution of the 1st, 2nd etc. tenfolds of cM^3. We now find that the points of our figure for each of the four series of observations lie on rather straight lines so that the fermentation velocity, which will set in immediately after the fore-heating, can easily be read.

Still it should be noted that the course of the four lines in the different graphic representations for the different temperatures gives an interesting insight into the process of dying of the function. At 45° C. they fall together and form one horizontal line; at 46° C. they lie very close together and still have a rather horizontal course; at higher temperatures they diverge more and more, get lower and have a steadily increasing inclination.

Be it moreover stated here that these graphics showed us that the destruction after the mixing with the sugar solution proceeded with less rapidity than before but that the difference was not very great. We shall later return to this point.

The initial velocities found in the above way by extrapolation of

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[1]) As it will be shown, the destruction in the sugar solution occurs less rapidly than in the aqueous suspension, but the hereby caused error falls within the limits of observation.

the fermentation after 5, 10, 15 and 20 minutes of fore-heating are represented in Table I and were used for the construction of the four

TABLE I.

Temperature in degr. C.	Velocity after 5 min. fore-heating	Velocity after 10 min. fore-heating	Velocity after 15 min. fore-heating	Velocity after 20 min. fore-heating
20				0.087
25				0.139
30				0.199
33				0.239
36				0.298
39				0.360
42				0.385
45				0.405
46	0.41^{5}	0.39^{5}	0.37	0.34
47	0.42^{5}	0.38	0.33^{5}	0.30^{5}
48	0.40	0.35	0.30^{5}	0.27
50	0.40	0.23^{5}	0.20^{5}	0.17^{5}
52	0.29^{5}	0.13^{5}	0.10	0.06^{5}
53	0.16^{5}	0.10^{5}	0.05^{5}	0.02
54	0.12	0.04	0.01^{5}	
55	0.06			
56	0.02			
57	0.00			

curves, which are given fig. 1 Plate I for temperatures above 45° C. and we clearly see now that the place of the optimum is influenced by the time of the fore-heating.

Let us now make a short return to the curve below 45° C. If we test this part by the rule of VAN 'T HOFF we find:

$$\frac{v_{30}}{v_{20}} = 2{,}3 \qquad \frac{v_{35}}{v_{25}} = 2{,}9 \qquad \frac{v_{40}}{v_{30}} = 1{,}8 \qquad \frac{v_{45}}{v_{35}} = 1{,}5.$$

So it is evident that the value of the proportion at high temperatures is considerably smaller than for low ones, only for the latter it attains the value which was to be expected in accordance with the rule.

But also from the graphic figure the said fact can at once be read. For low temperatures, we know, the optimum curve has a convex course with regard to the axis of temperature and after VAN 'T HOFF's rule a similar course might be expected for higher temperatures too. Instead of this, however, we see the curve become concave already at ± 35° C. According to the theory of DUCLAUX and BLACKMAN this concavity, respectively the said deviation from the rule, below the optimum, should be explained from a deleterious influence on the agent caused by the heating. Of such an influence, however, at temperatures below 45° C. during heating periods as used by us, nothing is to be seen. This follows already from the fact that below that temperature the four optimum curves fall together, but moreover from direct observation.

Hence it should be emphatically stated that also in virtue of the course of the optimum curve below injurious temperatures the theory of DUCLAUX and BLACKMAN must be refuted.

§ 5. *Relation between the degree of dying and the temperature.* For the study of this relation 16 Gr. of yeast were mixed with 31.5 cM³. of water and then during 5, 15, or 20 minutes heated to a noxious temperature. By fore-heating to the highest possible temperature the required degree could very quickly be reached. The cooling was now effected with the greatest possible rapidity and the fermentation velocity determined at 45° C. after mixing with 10 cM³. of a 30°/₀ glucose solution. As it was known for the used quantity of yeast what would have been its fermentation velocity in the here obtained concentration in case nothing had been destroyed, and as furthermore by fig. 1, Plate I, the relation between fermentation velocity and concentration of yeast is given, it could be calculated what percentage of the yeast had lost the function by the heating. The result of these calculations are graphically shown in fig. 3, Plate I.

We must by the bye observe here that these curves, which we shall call destruction-curves, exhibit an interesting resemblance to the so-called distribution-scheme of GALTON, a resemblance which, as will be demonstrated later, must probably not be considered as an accidental one.

§ 6. *Calculation of the optimum curve for 0 minutes fore-heating.*

From fig. 3, Plate I may now at once be read to what percentage of the original yeast an observed fermentation velocity at a certain noxious temperature and with a determined period of fore-heating is due. Fig. 1, Plate I, shows furthermore what velocity that yeast would have at 45° C. as well in its original state as after having been destroyed for the found percentage; the relation of the former to the latter velocity can thus at once be calculated. If it is now admitted that the relation between the fermentation velocity and the concentration for temperatures above the optimum is represented by the same curve as we gave in fig. 1, Plate I for 45° C. [1]), an admission based on great probability, then the velocity, such as we read it for a noxious temperature from fig. 1 Plate II, multiplied by the just now mentioned proportion, will produce the value which the fermentation velocity would have had if nothing had died at that noxious temperature, i.e. at 0′ fore-heating at that temperature.

It is now possible to make the here meant calculation for the different points of the 4 optimum curves and then by four ways to find a curve which represents the relation between the initial velocity of the fermentation at 0′ fore-heating and the temperature. If our view is right the four curves must fall together. This is actually not the case but the corresponding points of the 4 curves come so closely together that the aberrations must be ascribed to observation errors. In our graphic representation fig. 1, Plate II, these points are respectively marked with 1, 2, 3, and 4 and the dotted line more precisely represents the probable course of the optimum curve for 0′.

§ 7. *The relation between the fermentation velocity and the time of heating at constant noxious temperature.* In the method followed by Blackman for the approximation of the said curve, the relation between the reaction velocity and the time of heating for constant temperatures was graphically represented and by extrapolation the velocity for 0′ heating was found.

For the reasons mentioned before we have restricted ourselves to demonstrating that this relation, when using the values found in the preceding paragraph, displays a very regular course. The hereby concerned curves for different temperatures are those of fig. 4 Plate I.

If we assume that the initial velocity is directly proportional to

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[1]) Here is also the explanation why we chose for our observations for fig. 1 a temperature so near the noxious one.

the yeast concentration — an assumption only true for the lower concentrations, — these lines, according to TAMMANN, represent logarithmic curves and they may indeed be fairly well rendered by such a formula.

Although the graphic extrapolation was thus not used here as a means to determine the optimum curve for 0′ fore-heating, yet the fact that for the relation between velocity and time of heating a very probable curve is found may be considered as a strong argument for the exactness of our views.

§ 8. *Relation between time of fore-heating and temperature at constant fermentation velocity.* Also along another way than that followed by BLACKMAN the optimum curve for 0′ fore-heating might have been found by extrapolation, namely by graphically representing the relation between the time of heating and the temperature for different constant fermentation velocities. We shall again not effect that extrapolation but demonstrate that the values calculated for our optimum curve for 0′ give a quite natural direction to the curves in question. These curves are seen fig. 5 Plate I. This circumstance also argues for the accuracy of our calculated curve.

II. THE INVERSION OF CANE SUGAR.

Our experiments on the inversion of cane sugar have been carried out with a preparation made by precipitating pressed yeast with alcohol and quickly drying that precipitate. This powder, after prolonged keeping in a well-closed bottle, preserved its full activity.

The velocity of the inversion was determined by colorimetrically establishing the quantity of invert sugar in a similar way as KJELDAHL followed for his experiments on the action of diastase. About the various precautions to be taken in these researches we cannot be long and will only observe that for the inversion the same determinations have been made as for the alcohol fermentation. Fig. 2, Plate II, shows the relation between velocity and temperature for constant enzyme- and sugar concentration and we see that the observed and the calculated curves show a course similar to that of the alcohol fermentation.

Of the other results we shall only give a few particulars. The inversion velocity of the invertase concentration used was in perfect proportion to the quantity of invertase, a circumstance which makes the matter much simpler than for the alcohol fermentation where this relation is rendered by the curve fig. 1, Plate I.

The relation of the inversion velocity to the sugar concentration was again represented by a curve showing resemblance to a parabola. The destruction-curves showed as to their form the greatest possible likeness to those we found for the alcohol fermentation, but deviated in so far as they were displaced to the higher temperatures and (on the same scale) also displayed a less steep slope, so that the destruction region of the invertase (at the same duration of fore-heating) extends over a much larger and higher temperature interval than that of the alcohol fermentation.

The curves which indicate the relation between the velocity of the inversion and the time at constant noxious temperature, when adopting the values of our calculated optimum curves for 0′ fore-heating, proved to have a very probable course and the same holds good for those which represent the relation between the time of fore heating and the temperature at constant inversion velocity.

Lastly it should be remarked that the optimum curve below the noxious temperatures, now again shows a perceptible deviation from VAN 'T HOFF's rule:

$$\frac{v_{30}}{v_{20}} = 1{,}75 \quad \frac{v_{35}}{v_{25}} = 1{,}63 \quad \frac{v_{40}}{v_{30}} = 1{,}50 \quad \frac{v_{45}}{v_{35}} = 1{,}46.$$

Yet the divergency is less pronounced here and the curve does not even become concave before the noxious temperatures are reached.

III. FINAL CONSIDERATIONS.

The chief result to which our researches have led is, no doubt, that the curve related to a fore-heating period of 0 minutes is a distinctly pronounced optimum curve, as much in the case of the alcohol fermentation as in that of the action of the invertase. To this may be added that our investigations on the reduction of methyleneblue by pressed yeast have convinced us that the same is the case for this function although here complications arose which cannot now be discussed. Hence it may be considered as highly probable that the same conclusion will also hold good for other physiological processes. The reverse results at which BLACKMAN arrived, and with some reserve KUIPER also, should, in our opinion, be ascribed to the circumstance that the velocity of the physiological reactions, measured by these investigators, was small when compared to that of the destruction.

Two important points still deserve attention. In the first place it might be that the still living portion of the yeast which has been heated to noxious temperatures and then cooled had experienced a pernicious influence, so that in reality a smaller portion had been destroyed than we supposed. Nothing can be said against this consideration, but, accepting it, our curve for 0′ must lie lower than we found it and consequently the optimum would still more strongly be pronounced. Secondly, it might be observed that we have here studied the influence of the temperature on the agent suspended in an aqueous solution and that this influence might be quite different if the agent had always been active in a sugar solution. We found indeed that the alcohol function in the glucose solution is more resistent to heating than in an aqueous suspension and the same holds good for the action of invertase. Yet we have not made our studies with such solutions because the rate of resistibility depends on the degree of concentration of the sugar, and there is no degree of concentration, excepting 0°/₀, which deserves the preference. Moreover, the theory of DUCLAUX-BLACKMAN is independent of this consideration and ought also to be found true after our method, which, as we saw, is not the case.

That the theory of DUCLAUX and BLACKMAN must be refuted follows already from the fact of the stated deviation from the rule of VAN 'T HOFF below the noxious temperatures, which sometimes becomes so great that the curve grows concave, which is inconsistent with the hypothesis.

It is interesting to observe how also EULER [1]) has come to the conclusion that, at least for the invertase action, VAN 'T HOFF's rule cannot unreservedly be applied even at temperatures that do not destroy the invertase. He states that the quotient $\frac{v_{t+10}}{v_t}$ for the invertase action is always found smaller than for the splitting of cane sugar by acids. So TAMMANN found for the enzyme action: $\frac{v_{30}}{v_{20}} = 1.4$, KJELDAHL: $\frac{v_{40}}{v_{30}} = 1.5$, O'SULLIVAN and TOMPSON: $\frac{v_{50}}{v_{40}} = 1.4$, VISSER: 2, EULER and AF UGGLAS for the temperature interval 0 – 20° : 2.0 (compare to this our results). For the splitting of cane sugar by acids SPOHR, on the other hand, found this quotient to be 3.6 for the temperature interval 25—50°. Now EULER remarks that this considerable difference might be explained by accepting that the invertase at rise of temperature is not only irreversibly decomposed but, besides, rendered reversibly inactive.

[1]) Allgemeine Chemie der Enzymen, Wiesbaden, 1910, S. 175.

EULER accepts the views of DUCLAUX and BLACKMANN and, besides, the said hypothesis. According to our results the said theory must now be rejected: also without destruction the function shows an optimum curve. We have thus to admit that if heating above this optimum were possible without destruction a reversible inactivation of the enzyme would be observed, but there is no cause to speak with EULER of such an inactivation already below that temperature and therewith to explain the deviation from VAN 'T HOFF's rule.

The obtained result also throws light upon other observations: the occurrence of the "Wärmestarre" to explain which the hypothesis of DUCLAUX-BLACKMAN meets with great difficulties, is now easily explained by considering that in cases where this phenomenon takes place, the destruction proceeds extremely slowly so that the optimum curves for different (not too long) times of fore-heating actually fall together with those of 0′ fore-heating.

Moreover, the influence of the temperature proves now to follow the same law as was observed for other influences on physiological processes, by which the process is accelerated to a certain degree, then slackened and finally stopped.

Lastly we bring our best thanks to the Direction of the Dutch Yeast- and Spirit Works here for the readiness with which the yeast for the described experiments was supplied us.

Delft, May 1910.

Astronomy. — "*Researches into the structure of the galaxy.*" By Dr. A. PANNEKOEK. (Communicated by Prof. E. F. VAN DE SANDE BAKHUYZEN).

§ 1. The researches into the structure of the universe propose to ascertain the star-density (quantity per unit of volume) as a function of their place in space, i.e. for any direction the star-density as a function of the distance to the sun. If all the stars had the same absolute luminosity, the apparent brightness m expressed in magnitudes would be a direct measure for the distance r according to the relation $0{,}2\, m = \log r$. The number of stars of a given magnitude $A(m)dm$ would then indicate immediately the density Δ for the corresponding distance r, according to $A(m)dm = \Delta\,(r)\,.\,r^3dr$.

The stars, however, are of different luminosity and therefore the number of stars $A(m)$ is in a more intricate way dependent on the density Δ. If the function expressing the number of stars of different luminosity is known, inversely Δ can be concluded from the values of A. For the present we must assume that the function expressing the distribution of stars over different luminosities, is the same everywhere. This function has been determined by KAPTEYN. The values of the logarithm of the number of stars per unit of volume $log\, \psi$ as a function of the logarithm of the luminosity $log\, L$ [1]) can be represented by a parabola with a maximum for $log\, L = 8.2$, if for the sun we put $L = 1$. If the luminosity is expressed by the apparent brightness H in magnitudes which the star would show at a distance $\pi = 0''.1$, then we find $H = 5.5 - 2.5\, log\, L$ (the sun having the magnitude 5.5 at this distance), so the maximum lies at $H = 10$ and the function ψ can be expressed thus:

$$log\, \psi = c - 0.025\, (H - 10)^2 \text{ or } \psi = C 10^{-0.025\, (H - 10)^2}.$$

All researches into the structure of the universe must start from the knowledge of the function $A(m)$, the number of stars of a given magnitude. In practice we do not as a rule use A, but $N(m)$, the total number of stars down to a given limit of brightness, and so related to the former that $dN(m) = A(m) dm$.

Our want of knowledge of $N(m)$ was mainly due until a short time ago to the deficiency of good photometric measurings of the fainter stars on account of which the value of the limiting brightness m for a given numbering was unknown. Only these last few years this want has been to some extent removed. In N. 18 of the *Groningen Publications* the measurings extant have been collected and discussed; the results thus attained form at present the only reliable and firm basis for researches into the structure of the universe.

For the whole of the sky $N(m)$ is at present known to the 14th or 15th magnitude. From the brightest stars down to about the 11th magnitude the function is almost linear and we may put $N(m) = C + 0.50 m$; after this the increase is less rapid. We know that a linear function $N(m) = c + \lambda m$, independently of the function $\psi(H)$ determines the distribution of densities; this distribution is then expressed by $\Delta(r) = r^{-5(0.60 - \lambda)}$. If the linear function mentioned with the coefficient 0.50 held good for the whole range of magnitudes, then we might conclude that $\Delta\, r = r^{-\frac{1}{2}}$, i.e. our star-system is densest in the middle and decreases in density towards the outsides, about inversely proportional with $\sqrt{r}$. That the fainter stars increase less in number than indicated by the formula, shows

[1]) On the luminosity of the fixed stars. Public. Groningen N. 11. p. 16, 19.

that on the outside of our system the density decreases in a degree even greater than $1 : \sqrt{r}$.

But as it is the universe has not to be regarded as one whole nor the stellar system as a globular mass. The Milky Way forms a girdle across the sky where the star-density is greatest and from where it decreases to both sides. In second approximation N is no longer a function of m only but of b as well, the galactic latitude. KAPTEYN has given this function in N. 18 of the *Groningen Publ.* in a tabular form: $N(m, b)$, generally as well as $N(m)$ for three different parts of the sky, for the galactic zone, for the vicinity of the galactic poles and for an intermediate zone between 20° to 40° galactic latitude. The analytical functions deducted there do not, however, give an easy insight into their numerical values and these may be represented about as accurately by simpler and more easily manageable functions. For the galactic zone a linear function suffices:

$$log\ N_0^{20} = \overline{9}.70 + 0.49\ (m - 7)$$

For both the other zones the addition of a quadratic term is required and we may put:

$$log\ N_{20}^{40} = \overline{9}.48 + 0.49\ (m - 7) - 0.007\ (m - 7)^2$$
$$log\ N_{40}^{90} = \overline{9}.40 + 0.47\ (m - 7) - 0.009\ (m - 7)^2$$

By these formulae the structure of the universe is determined as a figure of revolution, a flat disk, its axis at right angles with the Milky Way. The star-density depends on two co-ordinates, the distance to the central plane z and the distance to the axis $\sqrt{x + y^2}$, or in polar co-ordinates: on the galactic latitude b and the distance to the sun r. These formulae show that the density decreases from the centre to all sides, fastest in the direction z at right angles with the plane of the galaxy, slower to all sides of the same. In the galaxy itself there clearly exists a rapid and regular decrease of density with the distance, according to the law $r^{-0.55}$.

This conclusion, however, is *in direct opposition to the appearance of the galaxy*. We see the galaxy as a belt of more or less circular masses, patches and drifts designating a totally different structure. Progressing in the direction of such a star-cloud we first expect an increase on this side of the cloud and then a decrease on the other side, which differs absolutely from what KAPTEYN's result for the galactic zone leads to. The appearance of the galaxy shows too that the zone between +20° and —20° galactic latitude should by no means be treated as one whole. In that way parts of the universe of really great diversity of structure would be mixed up; the galactic zone consists partly of star-clouds, forming the real phenomenon of the Milky Way, partly of intermediate and adjacent celestial regions possibly agreeing in formation with the galactic poles. It may be necessary to take all these different parts together for arriving at an average representation of the distribution of the stars in space, but this is *obscuring* the especially striking *character* of this distribution, which shows in the aggregation of stars into clouds and drifts; and it is giving a false impression of the real Milky Way if the star-density is represented as a simple function of r and b.

In order to obtain a true representation we must go on to a 3rd approximation; treat the special parts of the zone, the great patches and drifts in the Milky Way individually, determine for them separately N as a function of m and derive from this the value of Δ as a function of r.

The investigation communicated here is a first attempt to determine these functions for some parts of the galaxy, particularly to see what conclusions may be drawn to this purpose from the available material. Three regions were chosen to this end:

1. part of the large, bright Cygnus-patch, reaching from β to γ Cygni and forming the largest and brightest patch on the northern sky; this was chosen partly on account of its peculiar position[1]), partly because Herschel's gauges are most numerous here. As these do not as a rule reach farther than 36° declination, only the region below this was taken. For limits were chosen parts of parallels at 1°, and parts of declination-circles at 4^m distance from each other, following as well as possible a boundary line on Easton's map[2]). The sketch in fig. 2 designates these limits (region A).

2. part of the other branch of the galaxy in Aquila and Sagitta between 10° and 20° declination; the boundary line was taken from my own maps of the galaxy. This region was taken on account of its being a characteristic part of the main-branch in 18^h—19^h RA. as well as because Herschel and Epstein[3]) found here their richest fields. For this region too a broken line was assumed for boundary, as sketched on fig. 3 (region A).

3. For comparison a region on the other side of the sky at 6^h RA. was investigated, though less minutely. There is less contrast here between well defined smaller patches and a background of faint diffused light, and owing to this phenomenon larger regions

[1]) See Easton, La distribution de la lumière galactique, p. 45. (Verh. K. A. v. W. VIII, N. 3. 1903).

[2]) loc. cit. Carte isophotique.

[3]) Mitth. der V. A. P. Vol. III, p. 118.

were used corresponding to the trapezia counted by SEELIGER, viz. the areas $0°-5°$, $6^h40^m-7^h20^m$; $5°-10°$, $6^h0^m-6^h40^m$; $10°-25°$, $5^h20^m-6^h40^m$.

§ 2. Let us first consider how a structure such as is to be expected from the aspect of the patches of the Milky Way, must present itself in the distribution of the stars $N(m)$. The luminosity-function is $log\, \psi = c - \alpha H^2$, assuming $H = 0$ for a star of magnitude 10.0 at a distance corresponding with $\pi = 0''.1$. For the distance r we introduce a new variable x, so that $x = 5\; log\, r$ and $x = 0$ for $\pi = 0''.1$; so the scale of x corresponds to the scale of magnitudes. We assume an agglomeration of stars at the distance r_0, the density of which decreases to both sides of r_0 according to the law

$$\Delta = 10^{-\mu(x-x_0)^2}.$$

Then the number of stars of brightness m is represented by

$$A_m = \int_{-\infty}^{+\infty} \Delta(x)\, 10^{0,6x - \alpha(m-x)^2}\, dx.$$

From this we conclude

$$log\, A_m = (0.6 + 2\mu x_0)\frac{\alpha}{\alpha+\mu}\, m - \frac{\mu\alpha}{\alpha+\mu}\, m^2$$

$$= C - \frac{\mu\alpha}{\alpha+\mu}\left[m - \left(x_0 + \frac{0,3}{\mu}\right)\right]^2.$$

We put $\dfrac{1}{\mu} = \sigma^2$, $\dfrac{1}{\alpha} = \tau^2$, then

$$log\, A_m = C - \frac{1}{\sigma^2 + \tau^2}\,[m - (x_0 + 0,3\sigma^2)]^2$$

So A is just like Δ and ψ an exponential function, having the form of the law of errors. If we express the function Δ by $log\, \Delta = -\left(\dfrac{x-x_0}{\sigma}\right)^2$, then we may call σ the dispersion of this function. Because, for $x - x_0 = \pm\sigma$ the value of the function becomes $^1/_{10}$; so practically that is the limit. For the agglomeration of stars σ, the dispersion toward both sides gives us an idea of its size. In the same way when in the luminosity-function we substitute $\dfrac{1}{\alpha} = \tau^2$, τ is the dispersion of that function; for $H = \pm\tau$, the number of stars becomes $^1/_{10}$ of the maximum. As $\alpha = 0.025$ we have $\tau^2 = 40$ and $\tau = \sqrt{40} = 6.3$ magnitudes. Now the formula shows that the maximum of the function A lies in the cluster of stars, i. e. in that magnitude which is most numerous for the distance of the cluster; not in the centre, however, but $0.3\,\sigma \times$ size of the cluster farther away; this is owing to the increase of the sector-volume with the distance. The dispersion of the A is the square sum total of the dispersions σ and τ of the curves of density and luminosity; the accumulation of Δ is considerably lessened in the A.

Now the dimensions of the galactic clouds in the radius vector cannot be very great, the largest patches, which seem to be round, stretch across the sky from 15°—20°, and if they have just as much depth in the radius vector as breadth in the perpendicular plane then their depth must be about $^1/_3$ of their distance; for $r = {}^2/_3$ and $^1/_3$ x becomes -0.9 and $+0.6$. So σ^2 is not much greater than 1, whereas $\tau^2 = 40$. The dispersion of the luminosity-function is therefore of considerably more influence in A, than that of the density. The great diversity in the luminosity of the stars causes each aggregation of them to be reflected only very faintly and diffusedly in the distribution of their numbers over different magnitudes.

The function N has then the form $\int_{-\infty}^{m} 10^{-\beta^2(m-m_0)^2}\, dm$ and can be calculated numerically. The function $\log N$ first goes straight upwards and then approaches asymptotically the logarithm of the total number of stars of the cluster; the maximum m_0 lies where the curve is at a distance of 0.3 below this maximum (curve 1 in fig. 1). Now another

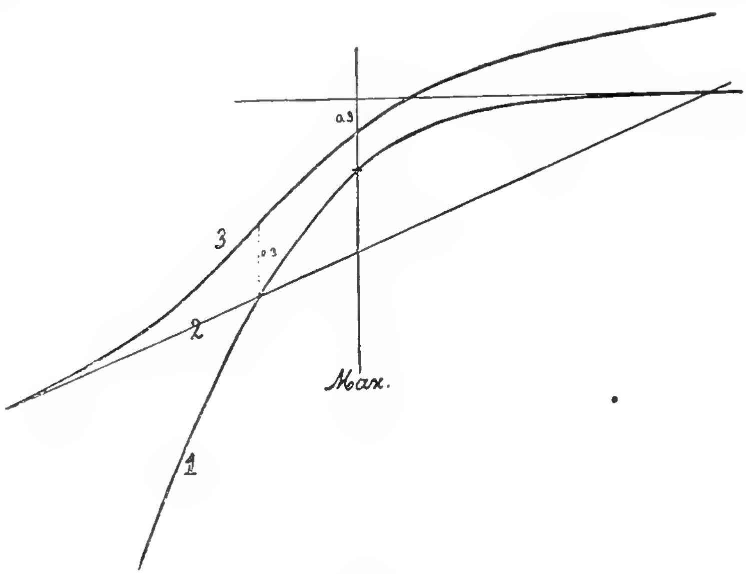

Fig. 1.

mass of stars, lying between us and the cluster, perhaps reaching even farther is added to the latter. The values of N for these stars, the same as for regions outside the Milky Way, may be represented by curve 2. So the total number of stars will progress like curve 3; first it will coincide about with curve 2, then it will continually rise above it, get a larger gradient and finally, past the maximum of the cluster, it will again go down to curve 2 and show a smaller gradient than this one.

If the density is the same along the whole of the radius vector the gradient of function N will be 0.60. If the density decreases regularly the gradient is < 0.60, if the density increases it is > 0.60. If this happens only in parts of the radius vector, the different gradients will be mixed up and strongly levelled, but still the general rule will be that a gradient below 0.60 indicates decreasing density and a gradient above 0.60 increasing density.

§ 3. The following material of star-countings could be used for our investigation:

1. The Bonner Durchmusterung. For magnitudes 6.55, 8.05 and 9.05, which we used as limiting ones, the photometric magnitude is known exactly. We could partly use the numbers of SEELIGER and STRATONOFF, partly they had to be counted anew. The total number 9.5 incl. was of no use, as its limiting brightness could not be sufficiently determined.

2. The gauges by W. HERSCHEL as published by HOLDEN in the 2nd vol. of the Washburn Observations. As nothing has been done during the whole of the 19th century to correct or complete HERSCHEL's gauges, they still form by the low limit to which they reach the most valuable, indeed an inestimable and indispensable material for researches into the structure of the universe. But owing to this, each want of homogeneity in these countings becomes an impediment to accomplishing such an investigation. Some parts of the Milky Way — especially the Cygnus-patch at β Cygni — are very rich in counted fields, while not a single gauge occurs in the most northern parts of Cygnus towards Cassiopeia and Auriga. The gauges by J. HERSCHEL at the Cape are of no use for our purpose, as the results of the separate fields have not been published. The limiting magnitude has been computed by KAPTEYN on the photometric scale and found to be 13.9; through this determination only HERSCHEL's star-gauges have realized their full value.

3. The gap between the B.D. und HERSCHEL being considerable

it is of the utmost importance to have star-countings like Herschel's for intermediate limiting magnitudes as well. These have been executed by Th. Epstein at Frankfurt on the Main. From 1877 till 1888 he has gauged about 2700 fields distributed over the whole sky with a 6in. telescope. It is much to be regretted that the outcome of this interesting work has not been published and it is certain that the great value of such an investigation for the knowledge of the distribution of the fainter stars in the sky can only be realized by a detailed publication of its results. Mr. Epstein has kindly communicated to me the results needed for my researches and it will appear below how valuable they were.

4. The photographic *Carte du Ciel* as far as it has been published. This furnishes two sorts of data: for a brighter limit in the catalogue-plates, for a fainter one in the chart-plates. The limit not being the same for all parts of a plate owing to the curvature of the field — which has been found out to exist particularly in the plates of Oxford and Potsdam — these plates have always been taken *as one whole*; only the total numbers of stars on a plate or chart has been used as data.

In how far these data for the celestial regions examined are available, may be seen in the following sketches, fig. 2 and fig. 3, where the limits of the areas have been represented by broken lines. Herschel's gauges have been indicated by dots, those of Epstein by crosses.

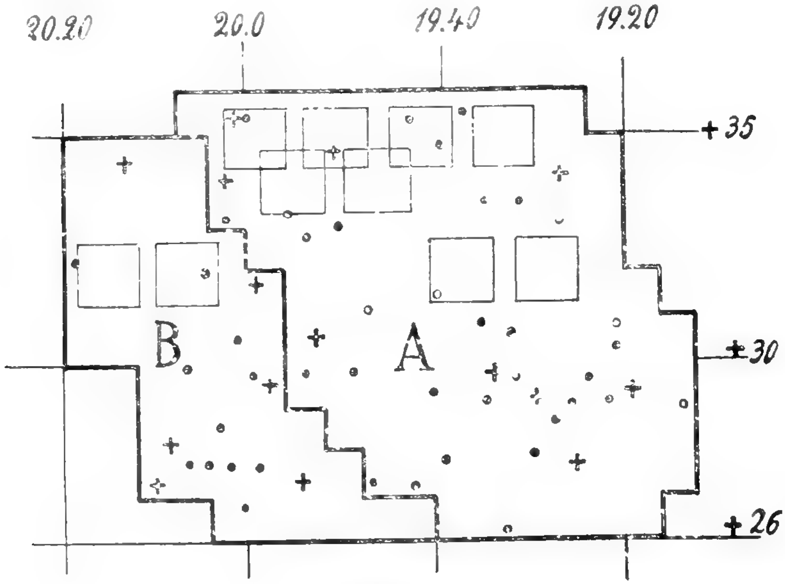

Fig. 2.

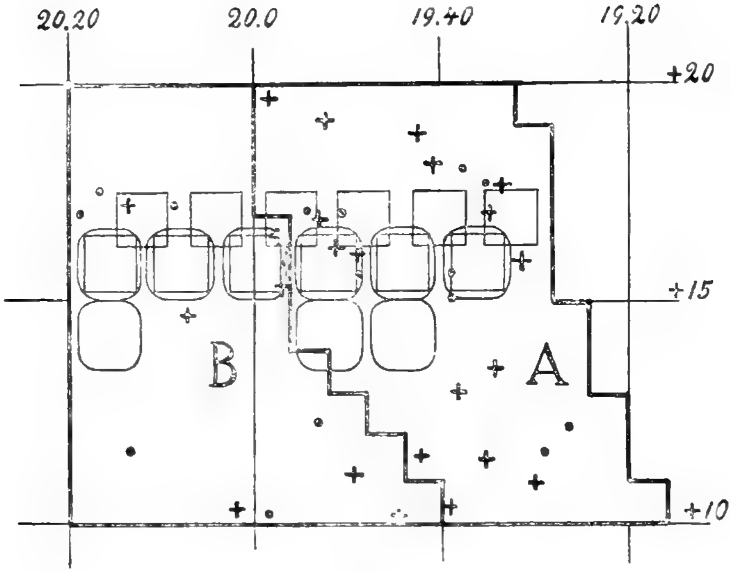

Fig. 3.

The fields of 2° length that are covered by catalogue-plates have been designated by small squares; those of which chart-plates are extant by bigger squares with rounded corners. For the Cygnus-region only catalogue-plates of Potsdam can be used, since by the strict equality of kind of plates and of time of exposure these form a homogeneous whole. The Oxford plates below 32° I dared not use; although the total number of stars is given for those plates that have not been completely measured, the diversity of kind of plates and of time of exposure made me fear a want of homogeneity that might be absolutely fatal here. For the Aquila-region I could only find catalogue-plates of the zones at 16° and 17° declination and chart-plates with centra at 14° and 16° declination, all of them from Bordeaux.

This is the weak point in our method of investigation: for with this irregular distribution of data the accidental irregularities in the distribution of the stars may not be sufficiently neutralized. The results for the number of stars N cannot be absolutely comparable unless they stand for the same region. If Herschel's or Epstein's gauges have been spread in a satisfactory number over the examined area then one may expect that from their average the irregularities have sufficiently disappeared. This is more doubtful for the plates of the *Carte du Ciel.* The outcome will have to show whether this want of identity of the regions examined has a more or less obnoxious influence.

§ 4. Now there remains to be determined the limiting magnitude for the numbers of stars per square degree found from each of these sources. For the *BD* it may be computed from SEELIGER's correction-formulae of the *BD*-magnitudes as a function of the star-density. For HERSCHEL it has already been determined by KAPTEYN to be 13.90. For the other sources it has been computed in the same way by using KAPTEYN's Table 1 in the *Gron. Publ.* N. 18 where *log N* is given as a function of *b* and *m*. For each of the sources used *N* was deduced for as many different parts of the sky as possible; for the numbers found and the *b* of each of these parts the table then furnished the corresponding *m* = the limiting magnitude. So these magnitudes and with them the whole of our research is based on the photometric measures and on the countings on charts and star-plates that have been collected in Summary 11 of the work mentioned.

Before communicating the numbers thus found we must first face some difficulties. The irregularities of density are greater inside the Milky Way than outside; this is of the more importance as each division of the *Carte du Ciel* comprises only a narrow zone, and the average of both places where this zone crosses the Milky Way may differ considerably from the average of the whole of the galaxy. From this point of view it may seem desirable to use only the regions outside the Milky Way for the determination of the limiting magnitude.

On the other hand it is possible that there exists a systematic difference between the regions inside and those outside the Milky Way. The photometric scale on which KAPTEYN's tables are founded is a visual one, and his numbers though counted on photographs still indicate the numbers for visual magnitudes, supposing that the stars taken as standards have everywhere the same average colour as the great mass of the other stars. If the average colour of the fainter stars inside and outside the Milky Way is different — when the stars of small luminosity are on an average yellower than those of great luminosity, the average colour of the fainter stars in the Milky Way must be more blue than that of those outside — then $N(m, b)$ must be different for visual and photographic *m*, and the limiting magnitude deduced from the photographic numbers with the help of KAPTEYN's tables will be found different for the inside of the galaxy and for the outside. In the hypothesis mentioned above *m* must be found greatest in the Milky Way. Such an error will be even more striking in the brighter patches than in the average galactic zone. By using all data, including those of the Milky Way itself, the error will be somewhat less than by using only the extra-galactic regions.

The visual countings by HERSCHEL and EPSTEIN are free from such an error, and this fact together with their homogeneity determines their great value even now that we have the photographic *Carte du Ciel.* But here another difficulty arises. Can it not be that the limiting brightness lies deeper in poor regions because in richer fields not all the fainter stars are counted? By the great influence of this error the numbers of the B.D. are, as we know, of no use for investigations such as ours. In so far as this error is occasioned by the fact that in rich regions many stars are neglected, it is a priori probable that it does not occur in simple *countings*. In the B.D. the stars were not counted but measured and that was the chief reason why very faint stars were taken in poor fields and omitted in richer ones. KAPTEYN's research has already shown that in HERSCHEL such a divergency from homogeneity cannot be indicated for certain.

Such a difference may also be due to the use of too low a power, while the emerging pencil of light is larger than the pupil; each increase of light in the field of view owing to a greater amount of stars then causes a contraction of the pupil and therefore a diminution of the actual aperture, hence a brighter limiting magnitude. In the introduction to vol. 8 of the Bonner Beobachtungen SCHÖNFELD communicates the constants of the instrument used at Bonn for the northern Durchmusterung: Aperture 78 mm., magnifying power 9 times, diameter of the emerging beam of light $8^2/_3$ mm., diameter of the diaphragma often put before it 8 mm. "Dieser letztere ist immer noch grösser als die Pupillenöffnung der Beobachter unter mittleren, vielleicht selbst grösser als unter den günstigsten Umständen." So it is very likely that this circumstance has played some part in the B.D. The aperture of the instrument used by EPSTEIN was according to the communications of the observer 16 cm., the magnifying power used was 80 times, so the diameter of the emerging beam of light was 2 mm., which certainly is always smaller than the pupil. Here of course there can be no question of such a systematic error.

The question may be put if an error like this can occur in the photographic *Carte du Ciel* owing to stars having been neglected in the richer fields. According to Prof. SCHEINER's communications in the introduction to the 1st vol. of the Potsdamer *Catalog der photographischen Himmelskarte* p. xxx, this is quite impossible; the reverse should rather be feared, because in poor plates where there are many squares devoid of stars faint traces are more likely to escape observation than in rich plates where each square keeps the eye much longer. This cannot be, however, of great significance.

For the determination of the limiting brightness for EPSTEIN 48 fields were used which were taken at random from the mass and are distributed over the whole sky. Each field gave a value for $N(m, b)$, from which the m was found for the given b. Divided according to the zones the averages are

b 40°—90°	12.56	(21 fields)
,, 20 —40	12.62	(10 ,,)
,, 0 —20	12.37	(17 ,,)

If from the latter one strongly diverging value is excluded (an excessively poor field between the branches of the Milky Way) this average rises to 12.49 (16 fields). So no systematic difference between the zones is indicated. As a general average we find the limiting brightness of all EPSTEIN's countings to be **12.51.**

The question may arise whether it would be better to leave that one diverging value out or perhaps to use only the regions outside the Milky Way, but that alters the value only less than 0.1 magnitude. From all EPSTEIN's material of course a much more accurate result might be found, but we may expect the number found here to be accurate to 0.1 magnitude.

The published plates of Potsdam are not distributed very regularly over the whole zone. After the b had been computed for the centre of each plate they were taken together in the order of RA. in greater or smaller groups covering about 10° galactic latitude, a little more in regions of which fewer plates exist. The plates being more numerous in 19^h RA. than elsewhere, many more plates were grouped together there not to give them too great a weight. The following table gives first the average values of b, then the average number of stars per plate, i. e. per 4 square degrees, next the number of plates of each group and finally the deduced limiting brightness m.

b	$4N$	n	m	b	$4N$	n	m
— 30°	194	9	11.79	+ 75	97	7	11.71
— 25	147	11	11.35	+ 83	86	10	11.59
— 14	209	8	11.27	+ 70	73	7	11.33
— 3	455	12	11.61	+ 55	100	5	11.53
+ 4	652	16	11.94	+ 41	137	7	11.69
+ 14	319	6	11.67	+ 24	316	5	12.07
+ 25	225	6	11.77	+ 14	423	5	11.94
+ 34	203	10	11.95	+ 4	723	29	12.04
+ 44	146	5	11.80	— 5	718	29	12.05
+ 54	132	7	11.84	— 12	442	10	11.88
+ 64	114	7	11.80	— 18	223	5	11.49

The 9 results belonging to the galactic zone give an average of 11.77, the 8 for galactic latitudes above 50° give 11.68; so there does not appear a systematic difference of any importance. The great abundance of stars in the Cygnus-regions happens to be so counterbalanced by the poverty of the Auriga-Perseus-region, that their average agrees with the general average. The average of all the plates together gives as

Limiting brightness of the Potsdam catalogue-plates **11.73**.

The catalogue-plates of Bordeaux have been measured belt-wise and in the first two volumes of the catalogue all plates with centres at 17° declination and all those with centres at 16° declination appear complete. In each belt we have grouped the plates together by fives in the order of RA. then computed for the average the b, the $4N$ and from this the m. Taking the results together again in 3 zones in the order of the b, we find:

b	Zone 17° $m =$	Zone 16° $m =$
40°—90°	12.13 (14)	11.74 (15)
20—40	11.92 (14)	11.71 (12)
0—20	11.76 (8)	11.72 (9)
	11.97 (36)	11.73 (36)

In the first belt there seems to be a considerable difference between the pole and the Milky Way, which does not occur at all in the second one. More striking still is the evident difference between the two adjacent belts (partly covering each other) although the instrument and the observers at the measuring-apparatus were the same. That a systematic difference does exist here, which does not find its origin in the sky, appears from the fact that the smaller fluctuations of richer and poorer fields in both belts run parallel with nearly always the same systematic difference. The explanation must be found in the observers at the measuring-apparatus having been unpractised in the beginning, so that they took the utmost care to discover each almost imperceptible star-spot and measure it; while after more practice they regularly left out the faintest traces as uncertain and took only those that were more definitely visible. The tendency to take into consideration even the faintest spots being likely to have been greater in poor regions than in rich ones, this explains at the same time the systematic difference between the pole and the Milky Way in the first belt. If this explanation is right then the later parts will show about the same results as zone 16° and we may assume as

Limiting brightness for the Bordeaux catalogue-plates **11.73**.

The plates of zone 17° might be reduced to the later method of measuring by subtracting an average of $^2/_{11}$ from the number of stars. But as we have only to do with plates of the Milky Way where the number counted is not too great it seems best to reduce also these plates of zone 17° with the same limiting brightness.

On the Bordeaux charts that have been published the total number of stars appearing on the cliché has been printed at the bottom; we assume these to stand for an area of 130′ length and breadth. These charts have not been distributed so regularly over the whole sky. We have arranged them, just as mentioned above for the Potsdam plates, in the order of the RA., taken them in groups of 2—7 plates, and computed the m for these groups. Thus we found for

9 groups	with b	40°—90°	$m = 13.03$
6 ,,	,,	20 —40	13.30
6 ,,	,,	0 —20	13.35

As a general average computed from 83 plates we find:

Limiting brightness of the Bordeaux chart-plates **13.20.**

An increase of the m towards the galaxy seems indicated.

§ 5. In the following list the results of the countings in the regions examined have been collected:

	Cygnus-region (area 85.3 □°)		Aquila-Sagitta-region (area 68.4 □°)		Monoc.-Taurus-region (area 385 □°)	
	Total	Per □°	Total	Per □°	Total	Per □°
BD 0—6.55	25	0.29	19	0.28	82	0.21
0—8.05	192	2.25	126	1.84	520	1.35
0—9.05	916	10.74	541	7.91	2800	7.27
All Stars	2958	34.68	1998	29.21	10833	28.14
	Per field	Per □°	Per field	Per □°	Per field	Per □°
Herschel	269	5461 (32)	437	8871 (9)	97.6	1981 (37)
Epstein		426 (9)		361 (16)		262 (18)
	Per plate	Per □°	Per plate	Per □°	Per plate	Per □°
Catal. pl.	729	182 (8)	710	177.5 (7)	549	137 (19)
Chart pl.	—	—	7528	1603 (5)	—	—

Now there remains the limiting brightness for the B.D. to be reduced to the photometric scale. From the total number per square degree we find:

	Cygnus	Aquila-Sagitta	Monoc-Taurus
D (Seeliger)	1.39	1.17	1.12
6.5 DM =	6.50	6.51	6.51
8.0 „ =	8.02	8.03	8.04
9.0 „ =	9.03	9.08	9.10

Putting together the numbers found as functions of the deduced limiting magnitudes and computing the *log N* and their gradients we find the following results to which Kapteyn's numbers for the average galactic zone have been added.

The values of the gradients $\frac{d}{dm} \log N$ have each time been formed from the value of *log N* on the next higher and the next lower line.

Cygnus-region

m	N	$\log N$	$\frac{d}{dm} \log N$
6.55	0.29	9.46	
			0.59
8.07	2.25	0.352	
			0.67
9.08	10.74	1.031	
			0.46
11.73	182	2.260	0.46
12.51	426	2.629	0.68
			0.80
13.90	5461	3.737	

Aquila-Sagitta-region

m	N	$\log N$	$\frac{d}{dm} \log N$
6.56	0.28	9.45	
			0.54
8.08	1.84	0.265	
			0.60
9.13	7.91	0.898	
			0.52
11.73	177.5	2.249	0.49
12.51	361	2.557	0.65
13.20	1603	3.205	1.00
			1.06
13.90	8871	3.948	

Monoceros-Taurus-region

m	N	$\log N$	$\frac{d}{dm} \log N$
6.56	0.21	9.33	
			0.52
8.09	1.35	0.131	
			0.69
9.15	7.27	0.862	
			0.49
11.73	137	2.137	0.46
12.51	262	2.418	0.54
			0.63
13.90	1981	3.297	

Average galactic zone

m	$\log N$	$\frac{d}{dm} \log N$
6.55	9.467	
		0.51
8.08	0.239	
		0.50
9.10	0.745	
		0.48
11.73	2.013	
		0.46
12.51	2.375	
		0.45
13.90	3.003	

At first sight the gradients show an irregular up and down movement; past the 8^h magnitude they rise, then they go down from the 9^{th} to the 12^{th} magnitude to a much lower value only to rise again rapidly after the 12^{th} magnitude. This course appearing in all three regions the supposition seems obvious that it does not correspond to a real phenomenon in the sky, nor that it is the consequence of accidental error, but that it is caused by systematic errors in the *m*. It might be explained if the magnitudes in the vicinity of the 9^{th} magnitude were all taken too low and in the vicinity of the 12^{th} magnitude too high.

Now the limiting magnitudes for the B.D. have been found in a different way from those for the countings of the fainter stars; and KAPTEYN has already observed that the magnitudes accepted for the B.D. do not correspond with his tables; for these magnitudes he found the numbers diverge from the tables in the same sense as here, viz. they are 12°/₀ too great. With regard to this he says: "That the irregularity must be looked for not in the sky but either in the photometric determinations or in the countings, seems probable from the fact that for the most strongly diverging results the deviations for the zones 40—90, 20—40, 0—20 have the same sign and, speaking roughly, the same amount" [1]). If inversely one computes the limiting magnitude from the numbers reduced to 9.25 with the help of the tables, one will find not 9.25 but 9.36. Without looking into the reasons for this difference it is plain that for the sake of greater homogeneity it will be better to base the limiting brightness for the 9^{th} magnitude also on KAPTEYN's tables. We shall therefore add 0.11 to all magnitudes of our table, standing for 9.0 B.D.

The question whether the magnitudes in the vicinity of the 12^{th} are too high, is more difficult to answer. Taking into account that for the Potsdam catalogue-plates 11.0 on ARGELANDER's scale is meant for limit and that therefore the time of exposure, 5 min., was chosen in such a way as to first determine empirically the times of exposure for 7.0 and 9.0 B.D., after which the latter was once more enlarged in the same proportion, then one can expect at most 11.5 on the photometric scale. On the other hand KAPTEYN's tables correspond so well to the photometric measurings of the fainter stars that no error of great significance can be assumed here. Of course it would be of the utmost importance to control the accuracy of the deduced limiting brightness independently of KAPTEYN's countings. This might be done by finding among all the series of fainter stars that have been

[1]) Groningen Publications N. 18 p. 39.

measured at Harvard or by Parkhurst as comparison-stars for variables, those that occur in the Potsdam zone and by simply finding out which do and which do not occur in the Potsdam catalogue. This control I could not execute because it appeared that in the four volumes of Potsdam published till now there happens to be only one of the series of comparison-stars. As soon, however, as more volumes will be ready this course may be taken. For the present we have not a single definite indication that the magnitudes in the vicinity of the 12th are systematically too high. To remove the whole difference between the lower and the higher magnitudes an error of half a magnitude had to be assumed and this seems improbable. So for the 12th magnitude we keep to the magnitudes given above. After correction of the magnitudes in the vicinity of the 9th the gradients become

Cygnus		Aquila-Sagitta		Monoc-Taurus	
6.55		6.56		6.56	
	0.59		0.54		0.52
8.07		8.08		8.09	
	0.61		0.55		0.62
9.19		9.24		9.26	
	0.48		0.54		0.52
11.73	0.48	11.73	0.51	11.73	0.48
12.51	0.68	12.51	0.68	12.51	0.56
	0.80				0.63
13.90		13.20	1.00	13.90	
			1.06		
		13.90			

Here it must be observed that the course that is shown by these gradients and that completely depends on the values for the 12th magnitude, is found in the same way, with only a slight difference in the numerical value, from the photographic catalogue-plates and from Epstein's countings. From this it appears that the accidental errors in our numbers, as consequences of the deficient identity of the fields for which they stand and of the irregularities in the distribution of the stars, are not so great as to obscure the result of our research. For what might seem doubtful before: whether the accidental irregularities would be sufficiently removed in our result, appears to be indeed the case as proved by two absolutely different sources corroborating each other.

The same holds good as well for the systematic differences between the photographic and the visual numbers of stars; these too cannot give rise to greater errors than the slight differences we find between the gradients computed from the magnitudes 11.73 and 12.51. Here it appears how valuable EPSTEIN's countings are; without this material it would have been impossible to state if the results from the photographic *Carte du Ciel* did not lead to an absolutely wrong outcome and false conclusions.

Now what are the conclusions that may be drawn from these numbers?

In the Cygnus-region as well as in that of Aquila-Sagitta the number of stars down to the 9th magnitude increases more rapidly than in the average galactic zone (gradient 0.50). Granting the accidental uncertainty in these numbers to be great on account of the smallness of the areas used still the difference will seem real; by further investigation it has to be decided whether the same holds good for all bright parts of the galaxy. Past the 9th magnitude the gradient for Cygnus goes very low down, just as low as for the average zone, while the decrease goes less far for Aquila-Sagitta. Past the 12th magnitude the gradient increases rapidly to far over 0.60. From this it appears *that in the direction of the bright galactic patches the star-density decreases at first and then increases again at a greater distance so that there occurs a real star-clustering, the influence of which is not felt before the numbers after the 12th magnitude.* This cluster is separated from the dense mass of stars surrounding us by an intermediate poor region that is especially perceptible in Cygnus. The increase is still greater in the Aquila-drift than in Cygnus; evidently the cluster is denser there. On the Monoceros-Taurus-side of the galaxy there occurs an increase in the gradient past the 12th magnitude but it does not or hardly reach over 0.60; so here there seems to be after a poor region a very slight, hardly perceptible aggregation.

In the same way we have also treated some regions of the sky on the eastern boundary of the regions in Cygnus and Aquila, so that the first is situated between the two branches, the other outside the Milky Way. Their limits have also been indicated in fig. 2 and fig. 3, where they form the regions *B*. The last named region has been examined because in HERSCHEL as well as in EPSTEIN and in the Bordeaux plates the richest fields are not found in the middle of the galaxy, but towards the eastside and because they even continue till outside the bright light of the Milky Way. The results of these countings and computations have been put down in the following tables:

Cygnus between the branches

m	N	$log\ N$	$\frac{d}{dm}$
6.56	0.25	9.40	
			0.62
8.08	2.22	0.346	
			0.59
9.22 [1]	10.48	1.020	
			0.46
11.73 [4]	150.	2.176	0.41
12.51 [2]	326.	2.513	0.46
			0.48
13.90 [2]	1522.	3.182	

Aquila-Delphinus

m	N	$log\ N$	$\frac{d}{dm}$
6.56	0.32	9.51	
			0.49
8.10	1.87	0.272	
			0.52
9.28 [1]	7.59	0.880	
			0.50
11.73 [5]	124.	2.093	0.51
12.51 [3]	326.	2.513	0.61
13.20 [6]	978.	2.990	0.90
			1.11
13.90 [3]	5826.	3.765	

So the Aquila-Delphinus-region to the east of the galaxy has the same structure as this galaxy itself. The gradients are the same; after a continually decreasing density in the beginning they increase rapidly after the 12th magnitude. The density is everywhere somewhat less than in the central parts; so this region must be regarded as part of the body of the Milky Way, an outside part where the stars are less densely aggregated. A quite different picture gives the Cygnus-region between the branches of the Milky Way. It is hardly less rich in bright stars than the bright patch itself — compare with this ARGELANDERS's remark that in the B.D.-stars the bifurcation of the Milky Way is hardly perceptible —; on the other hand it lacks the increase of density past the 12th magnitude. This proves still more plainly that behind a region of stars, getting thinner with the distance and stretching over the whole breadth of the Milky Way, there occurs in one direction a dense star-cluster, which forms the bright Cygnus-patch, while we do not see the galactic light in the adjacent region where no such cluster occurs; this is the dark stroke between the two branches.

It does not seem advisable to draw still farther-reaching conclusions from this first material. It appears that with the stars as far as the 13.9 magnitude we only just reach into the greater star-clusters

1) Already corrected with 0.11.
2) EPSTEIN 6, HERSCHEL 11 fields.
3) EPSTEIN 7, HERSCHEL 7 fields.
4) 2 plates.
5) 5 plates.
6) 4 plates.

forming the Milky Way, and in order to ascertain more about their structures and distances we have to go on to still lower magnitudes. That is why we do not venture here a comparison between our numbers and the light of the Milky Way. We only want to observe that the views based on former investigations have been rather contradicted than corroborated by this research. What has been found here indicates *that no organic relation exists between the great mass of stars of the 9th magnitude and perhaps as far as the 11th, and the star-clusters forming the Milky Way.* Before putting this down as a certainty, however, it is desirable that we should wait till we have more material available.

The completion and the publication of the photographic *Carte du Ciel* promises important results; it will be some time, however, before the charts fully cover the regions that are to be examined. But however much may be expected from a systematic treatment of the thus completed material, through combining the B.D., the catalogue-plates and the chart-plates of the *Carte du Ciel* and the star countings by Herschel and Epstein for different parts of the sky, still there remains the lack of homogeneity and of exact identity of the celestial regions for which these numbers stand. Another time I hope to describe a method free from these drawbacks.

Physiology. — "*The permeability of red blood-corpuscles in physiological conditions, especially to alkali- and earth alkali metals* [1]). By Prof. H. J. Hamburger and Dr. F. Bubanović (Croatia).

Introduction.

In a former communication one of us [2]) has demonstrated by means of quantitative chemical determinations that red blood corpuscles are in both directions permeable to Ca. At the same time the conditions were investigated under which this permeation took place. We have now extended our investigations to other Kations viz. magnesium, potassium, and sodium, and have finally connected with it the question whether, under the same physiological conditions under which the permeation of calcium, magnesium, sodium and potassium, was inves-

1) More explicit communications on this subject will appear in the *Archives Internationales de Physiologie* publ. par Léon Fredericq.

2) On the Permeability of blood cells to Calcium. These Proceedings of March 27 1909. See also a more detailed account in the Zeitschrift für Physikalische Chemie. Bd. 69, S. 663, 1909. (Festband f. Arrhenius).

tigated, it would also be possible to demonstrate a permeation of anions, such as chlorine and alkali. In order to promote the permeation of the above-mentioned ions, the same modifications were made in the composition of the serum, as had been made in the former experiments on Ca. They simply consisted in the blood being made hyperisotonic or hypisotonic, and that, to an extent corresponding to the fluctuations which may daily occur in normal life. These investigations were also made with a view to collecting data which afterwards might serve perhaps to explain a phenomenon, noticed by HEDIN [1]) several years ago and which has hitherto been left unexplained. We mean the fact that the volume of red blood-corpuscles is equal in isosmotic-isotonic solutions of different salts, but unequal in isosmotic-anisotonic solutions. Also on former occasions our attention was drawn by this phenomenon [2])

Method of Investigation.

As has been said the permeability of the blood-cells was investigated by modifying the composition of the serum within physiological limits. The modification consisted in the blood-serum being made anisotonic, that is to say hyperisotonic by an addition of 0.2 °/₀ NaCl, hypisotonic by an addition of 10 °/₀ water. To accomplish this in an efficient manner a certain quantity of blood was centrifugalized, the serum was partly removed and mixed with the necessary amount of NaCl, or water; then it was added to the rest of the blood and well mixed with it. The suspension thus acquired was left to itself for an hour, to enable the blood corpuscles to get balanced with their new surroundings. After that time the suspension was centrifugalized, and the serum which was thus removed, could be examined as to its percentage of magnesium, potassium, sodium, etc.; this percentage could then be compared with the relative amount of these substances in the original serum. To control the result of the experiment we have in most cases made a quantitative determination of these substances as found in the red blood-corpuscles, and thus we could easily verify whether a decrease in the amount of certain serum substances was accompanied by a corresponding increase of them in the blood corpuscles or vice versa. For an exact determi-

[1]) HEDIN, Skandinavisches Archiv. f. Physiol. 1895 S. 377.

[2]) HAMBURGER and HEKMA, Zur Biologie der Phagocyten III, Biochem. Zeitschr. **9**, 281, 1908.

HAMBURGER and DE HAAN, Zur Biologie der Phagocyten V, Biochem. Zeitschr. **24**, 317, 1910.

nation of the increase or decrease it was necessary to know in all cases the volumes of the serum and of the blood corpuscles, as they were modified by anisotony, for it is obviously impossible to obtain and analyse *all* the serum of a certain amount of blood. For, however strongly we may centrifugalize, yet it remains impossible to remove all the serum from the sediment; a thin layer is always left behind. Whenever it was necessary, we have therefore determined the relative volumes of blood corpuscles and serum by centrifugalizing e.g. 0.06 cc blood in our funnel shaped tubes, until the volume of the sediment remained constant.

Finally it must be observed that all the blood, used for these experiments, had been shaken with 5 volume percent carbonic acid. This was done to render eventually a more extensive interchange of substances possible, thus causing permeability when present, to manifest itself in a more marked degree.

Permeability to Potassium and Sodium.

The permeability to potassium and sodium was investigated in the following way. As has been said above, a great volume of blood (3 litres) was shaken with 5 vol. perc. carbonic acid and left to itself for three hours to enable the carbonic acid to act; then 3×12 tubes were each filled with 75 cc. of this blood and closed at the top with india rubber covers; at the same time 0.06 cc. of the same blood were put in 3 funnel shaped tubes provided with well-fitting stoppers in order to determine the relative volumes of blood corpuscles and serum. Then in 12 tubes, part of the clear serum was removed, and Na Cl was dissolved in it. To the serum of the 12 other tubes water was added, then these sera were replaced in their original tubes and well mixed with the rest of the serum and the blood corpuscles. The amount of salt added was just sufficient to cause an increase of 0.2 $^0/_0$ NaCl in the serum. The amount required was calculated by means of the comparative volumes of blood corpuscles and serum, as they appeared from volumetrical determinations in the funnel shaped tubes. In the same way the amount of water was determined which had to be added to the serum in the 12 other tubes.

Now the blood corpuscles were left for an hour to get into a state of equilibrium with the surrounding fluid. This fluid was centrifugated and the clear serum was removed as much as possible. The volume of it having been measured, it was evaporated in a platinum basin at 110° and the residuum was exposed to a soft glowing heat.

Then distilled water and HCl were added to turn the metals into chlorides. The solution was filtrated and washed, and $BaCl_2$ and BaH_2O_2 were added to the filtrates to remove sulphuric acid and magnesium. Then the filtrate was mixed with $(NH_4)_2CO_3$ to remove the surplus of barium and also the calcium. These having been removed by filtration, the filtrate could be evaporated in a platinum basin and glowed to remove the superfluous $(NH_4)_2 CO_3$. What was left behind now was only KCl and NaCl. The total amount of it was weighed.

In this mixture the potassium could be determined. HCl and aq. dist. were added; then an excess of H_2PtCl_6, and the whole was concentrated on a water bath into a semi fluid mass, $80\,^0/_0$ alcohol being added. In this way sodium chloroplatinate (Na_2PtCl_6) remained in solution and yellow crystals were formed of K_2PtCl_6. These were placed on a filter, which had previously been dried and weighed; the crystals were washed with alcohol, dried at 120° and weighed. To determine the amount of sodium we had only to subtract the amount of KCl from the total amount of KCl and NaCl.

In the blood corpuscles the determination of potassium and sodium was carried out in the same way as in the serum. As a matter of course we had to take into account that there was still serum left among the red blood corpuscles. The amount of it was established in the usual way by centrifugalizing the thick suspension in funnel shaped tubes.

The results which we obtained may be summarized in the following table. (p. 262).

From this table it appears:

1. *that when serum is made hyper-isotonic by the addition of NaCl* to an amount which is also observed in normal life, *sodium enters the blood corpuscles and potassium leaves them* (Comp. the first two numbers of column 4*a*, of column 4*b*, 5*a* and 5*b*.

2. *when water is added to the serum* to an amount which is also observed in normal life, *sodium likewise enters the blood corpuscles whilst potassium leaves them*. (Comp. in all cases the first and third numbers of the above mentioned columns).

The entrance of sodium into the blood corpuscles appears not only from a decreased percentage of this metal in the serum but also from an increase in the blood corpuscles, whilst the fact that *K* leaves the blood corpuscles not only appears from a decreased percentage of this metal in the cells, but also from an increased percentage of this metal in the serum.

TABLE I.

Permeability to Potassium and Sodium.

1	2*a*	2*b*	3*a*	3*b*	4*a*	4*b*	5*a*	5*b*
900 cc blood	In the 900 cc blood are:		Total amount of K and Na determined as KCl and NaCl, in:		Total amount of K determined as KCl, in:		Total amount of Na determined as NaCl, in:	
	serum	blood corp.	the serum	the blood corp.	the serum	the blood corp.	the serum	the blood corp.
normal	594 cc	306 cc	4.9805 Gr.	1.6577 Gr.	0.3479 Gr.	1.2379 Gr.	4.6326 Gr.	0.4198 Gr.
to the serum 0.2% *NaCl* has been added	618 „	282 „	4.9323 „	1.7666 „	0.4438 „	1.0761 „	4.4885 „	0.6905 „
to the serum has been added *10% water*	570 „	330 „	4.9170 „	1.7755 „	0.4006 „	1.2132 „	4.5164 „	0.5623 „

Permeability to Magnesium.

With a view to the slight amount of magnesium found in blood, and likewise with a view to its lightness, a great quantity of blood was operated upon viz. 3 × 1200 cc. The experiments were identical with those made on the determination of K and Na. The ash was treated in a different way, of course. After it had been moistened with some HCl and dissolved in hot water, it was diluted to 100 cc. in a beaker and neutralised with $(NH_4)OH$ till a precipitate was formed. Then it was acidified with acetic acid and a concentrated solution of ammonium acetate was added. After being boiled the precipitate was filtrated and washed.

In the acetic acid filtrate the Ca was precipitated by ammonium oxalate at boiling-heat and the whole was left to itself for 4 hours. The filtrate was then mixed with ammonia and some Na_2HPO_4 and the mixture was left to itself for 24 hours at a low temperature. The resulting $Mg(NH_4)PO_4$ could now be washed, dried, and glowed in a platinum basin.

The following table contains the results obtained.

From this table it appears that *if the serum is made hyperisotonic by an addition of 0.2°/₀ NaCl, the amount of magnesium decreases* (from 0.1445 to 0.1385) and that accordingly the magnesium increases in the blood corpuscles (from 0.0166 to 0.0221 gr.) *whilst if the serum is made hypisotonic by an addition of water, magnesium acts just the other way, that means to say that it leaves the blood corpuscles.* (Comp. the first and third numbers of column 2*a* and of column 2*b*).

TABLE II.

Permeability to Magnesium.

	1*a*	1*b*	2*a*	2*b*
	In the 1200 cc blood are		Amount $Mg_2P_2O_7$ indicating the amount of Mg. in	
	serum	blood corpuscles	serum	blood corpuscles
a. 200 normal blood	768 cc	432 cc	0.1445 gr.	0.0166 gr.
b. 200 cc blood (a), to the serum of which was added *0.2°/₀ NaCl.*	804 „	396 „	0.1385 „	0.0221 „
c. 1200 cc blood (a), to the serum of which was added 10°/₀ water.	736 „	464 „	0.1319 „	0.0159 „

Permeability to Calcium.

As we said above, former experiments[1]) had shown that calcium can both enter and leave the blood corpuscles. This motion of calcium was discovered to find its cause in a disturbance of the equilibrium between blood corpuscles and serum. This disturbance was, amongst other causes, brought about by an addition of some NaCl or of a slight amount of water to the serum. *We now wished to investigate to what extent a disturbance of the equilibrium caused by 5 vol. perc. carbonic acid would likewise cause a transfer of Ca.*

As we know an addition of 5 % CO_2 to arterial blood falls within physiological limits.

After what has been said, it may be esteemed superfluous to enter into technical details as to the method of investigation. Let us only state that the Ca was determined in the ash by ammonium oxalate, that after being heated the oxalate was weighed as CaO, and further that in this case only the serum was examined as to its amount of Ca. This seemed sufficient after the detailed investigations formerly made on Ca[1]).

The result of the experiments now made was that under the influence of 5 vol. perc. CO_2, calcium had entered the blood corpuscles and that owing to a further disturbance in the equilibrium, caused by the addition of NaCl to the blood after it had been shaken with carbonic acid, a new amount of Ca had entered the blood corpuscles

TABLE III.

Permeability to Calcium.

	In the 900 cc blood are	Amount of CaO indicating the amount of Ca in the serum
a. 900 cc blood	600 cc serum and 300 cc bl. corp.	0.1592 gr. CaO
b. 900 cc blood shaken with 5 vol. perc. CO_2	588 cc „ „ 312 cc „	0.1526 „ „
c. 900 cc blood shaken 5% CO_2, to the serum of which 0.2% NaCl has been added	606 cc „ „ 294 cc „	0.1383 „ „

[1]) These Proceedings of March 27 1909; Zeitschr. f. physik. Chemie, Festband, Arrhenius l. c.

If we compare the 3 numbers of the third column, *it becomes manifest that under the influence of CO_2 calcium enters the blood corpuscles, and that this is the case to a much greater extent if to this blood a physiological amount of NaCl has been added.*

We may add to this that in connection with these experiments we also investigated the effect of an addition of NaCl to blood which had *not* been treated with CO_2. The amount of CaO now found in the serum was 0.1444 gr. whilst the volumes of serum and blood cells had become 618 cc. and 282 cc. respectively. This may serve as a confirmation of what had formerly been observed in the above-mentioned investigation concerning the permeability of red blood corpuscles to Ca-Ions.

Permeability to Chlorine.

Though at the present moment there is probably no one who doubts the fact that red blood corpuscles are permeable to chlorine, yet we have thought it expedient, in connection with the above-mentioned experiments, to investigate whether a motion of chlorine could be demonstrated under the same conditions under which the kations K, Na, Mg, and Ca passed through the blood corpuscles. Hitherto, indeed, we examined the permeability of red blood corpuscles to this anion almost exclusively by allowing physiological amounts of CO_2, H_2SO_4 and KOH to act upon the blood [1]. Would it be possible to establish likewise a permeation of chlorine if the normal equilibrium between red blood corpuscles and serum was broken by adding to the latter 0.2 % NaCl or 10 % water?

For this purpose we have experimented in exactly the same way as above, that is to say a known volume of serum as well as a known volume of blood corpuscles were dried, made into ash and in the ash the chlorine was determined. This was done after the method of VOLHARD.

It need hardly be said that to obtain an estimate as to the absolute amount of chlorine in blood corpuscles and serum the volumes of both had to be established.

1 Litre of blood was again shaken with 5 vol. perc. CO_2. Of this quantity we took 3 times 300 cc.

To one of these 3 quantities Na Cl 0,2 % was added, to another

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[1]) HAMBURGER, Zeitschr. f. Biologie 1891, S. 405; Archiv f. (Anat. u.) Physiologie 1892, S. 513; 1893, S. 153; 1893, S. 157; Zeitschr. f. Biologie 1897, S. 352; Archiv. f. (Anat. u.) Physiol. 1898.

HAMBURGER und VAN LIER, Archiv f. (Anat. u.) Physiol. 1902, S. 492.

water, whilst the third as such was examined as to its amount of chlorine.

TABLE IV.
Permeability to Chlorine.

	2a	2b	3a	3b
	In the 300 cc blood are		Amount $^1/_{10}$ n. $AgNO_3$ indicating the amount of Cl in	
	Serum	blood corp.	serum	blood corp.
a. 300 cc. normal blood	189 cc.	111 cc.	110 06 cc.	33.34 cc.
b. 300 cc. blood *a*, to the serum of which 0.2 % NaCl had been added	197 „	103 „	110 34 „	34 16 „
c. 300 cc. blood *a*, to the serum of which 1 % water had been added	181 „	119 „	112.20 „	31.18 „

From these experiments it appears that by the addition of 0.2 % NaCl to the serum a certain amount of chlorine enters the blood corpuscles, and that conversely by the addition of water to the serum chlorine leaves the blood corpuscles. (Comp. of the same columns the first and third numbers).

So it appears that under the same circumstances or in other words by the same equilibrium disturbances which cause blood corpuscles to be permeated by kations, also a permeation of chlorine takes place.

Finally we shall examine whether the same holds good for alkali.

Permeability to Alkali.

Already on the occasion of former investigations as to the permeability of red blood corpuscles and other cells to chlorine and other anions, the permeability to alkali has been set forth. In this paper we have investigated to what extent an addition of slight quantities of NaCl or water to the serum caused the blood corpuscles to yield or to take in alkali. The amount of alkali in the serum was determined by means of lacmoid paper. As we know, in this way the total amount of diffusible and non diffusible alkali is titrated.

We summarise the results in the following table, allowing for the modification of the volumes, mentioned before.

TABLE V.

Permeability to Alkali.

	In the 100 cc blood are	Amount of $^1/_{25}$ norm. tartaric acid corresponding with the alk. of the serum
a. 100 cc. normal blood (with 5% CO_2)	61 cc serum + 36 cc bl. corp.	28.8 cc
b. 100 cc. blood (*a*), to the serum of which 0.2% NaCl had been added	67 cc „ + 33 cc „	27 5 cc
c. 100 cc. blood (*a*), to the serum of which 10% water had been added	61.3 cc „ + 38.7 cc „	27 1 cc

These experiments show that by making the serum hyperisotonic by the addition of NaCl, alkali enters the blood corpuscles, whilst by the addition of water to the serum the same thing takes place, but in a higher degree.

Summary.

The investigations described above have chiefly led to the following results:

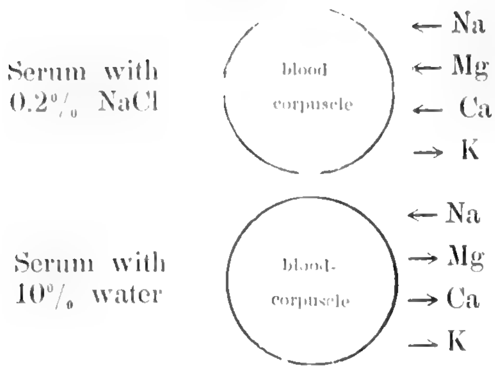
1. If in the composition of blood we bring about a disturbance in the equilibrium between blood corpuscles and serum, falling within physiological limits, a redistribution of the anorganic components takes place over blood corpuscles and serum.

2. This redistribution relates to kations as well as anions.

a. As regards the kations it has been seen that by the addition of 0.2% *NaCl to the serum, Na, Mg, and Ca enter the blood corpuscles, whilst K leaves them.*

When the serum is diluted with 10% *water Na enters, whereas K, Mg, and Ca leave the blood corpuscles.*

A survey of these movements may be given in the following way:



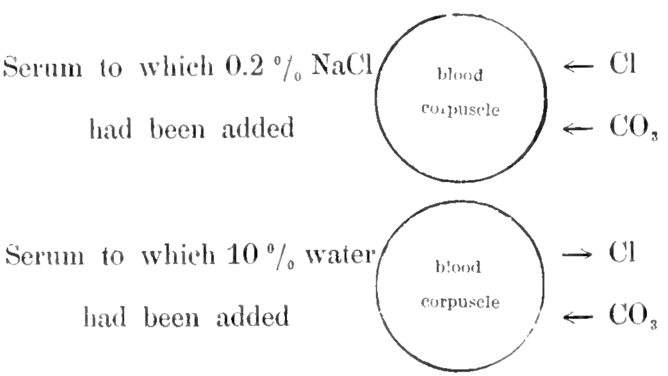
As regards Ca, this result is a confirmation of what had previously been found as an outcome of detailed investigations.

In this paper it has moreover been demonstrated that Ca enters the blood corpuscles too when the blood is shaken with a slight quantity (5 vol. perc.) of CO_2 and that this transfer becomes more considerable still, if to this blood containing CO_2 some NaCl is added.

b. The addition of Na Cl or water to serum as mentioned sub. *a* caused not only a movement of kations, but also of *anions*. *By the addition of some NaCl to the serum, chlorine was found to enter the blood corpuscles; by the addition of water to the serum chlorine left them.*

The alkali (CO_3) *likewise participated in this movement.* An addition of NaCl to the serum caused alkali to enter the blood corpuscles, whilst an addition of water had the same effect, but in a somewhat higher degree.

A survey of these movements may be given again in the following way:



These observations have proved again, and that by methods entirely different from those formerly employed by us and by others, that red blood corpuscles in physiological conditions, are permeable to anions.

3. The conclusion concerning the permeation of Na, K, Mg, Ca and Cl is based upon the results of quantitative-chemical analyses of these substances in the serum; these results were in all cases confirmed by the quantitative determinations of these substances in the corresponding blood corpuscles.

4. Briefly **the results described above justify us in asserting that blood corpuscles under physiological conditions are permeable to kations** as well as **to anions**, or if we do not wish to view the matter in the light of the ion-theory, **to metals and acid anhydrids.**

As to the kations (metals) this result is opposed to the current view.

The latter is based upon an investigation of GÜRBER according to which the serum retained its amount of sodium and potassium when blood was saturated with carbonic acid; and tacitly the impermeability to potassium- and sodium ions has been extended to calcium and magnesium. If, however, we examine GÜRBER's experiment more closely, it is seen that the blood used by this investigator for his analyses, amounted to only 100 cc., a quantity much too small to arrive at definite conclusions as to the permeability to K and Na. The amount of potassium indeed, contained in the serum of 100 cc. blood (0.018 gr. K_2O), is so small that it is impossible to demonstrate with certainty an increase or a decrease of 5 %. In order to get results which are at all reliable, it is necessary to experiment upon a much greater quantity of blood; besides it is necessary to analyse as a further test not only the serum, but also the corresponding blood corpuscles; this was neglected by GÜRBER.

More explicit critical remarks are found in our treatise in the Archives Internationales de Physiologie. In the same article the grounds may be found which induce us to look upon the permeation of substances as an interchange of *ions*.

5. As to the conditions under which the permeation of kations occurs, we assume also on account of former investigations on Ca, that it takes place only where an exchange is possible with equivalent kations on the other side. And this possibility occurs whenever the equilibrium is disturbed in the normal chemical composition of serum and blood corpuscles. If further we investigate by what

causes again this disturbance may be brought about, then it appears that in the first place this becomes possible by a change in the osmotic pressure of the blood, by which the dissociation is modified. This dissociation occurs, in the blood corpuscles in another way than in the serum. Likewise the equilibrium is disturbed when substances are added to the serum such as CO_2 and different salts. Of these two causes for the disturbance in the equilibrium the change of osmotic pressure was found, at least as regards calcium, to have a paramount influence (l.c.).

6. That the movement of kations and anions through cells is of importance to life has been plainly set forth for instance by comparative investigations on the influence of KCl, NaCl, NaBr, KI, NaI, NaFl, and also of Ca on phagocytosis.

Groningen, June 1910.

Physiology. — "*Experimental researches on the segmental innervation of the skin in dogs.*" By Prof. C. WINKLER from researches made in collaboration with Prof. G. A. VAN RIJNBERK. (VIth Communication).[1])

On form and situation of the dermatomata of the posterior extremity.

It has been for some years now that we have been occupied by attempts to obtain some insight into the manner in which the dermatomata are ranged on the posterior extremity in dogs.

These researches have been made partly at the Laboratory for Neurology in Amsterdam, partly at the Laboratory for Physiology of Prof. LUCIANI in Rome.

Albeit we were acquainted with the work of TÜRCK, SHERRINGTON, BOLK and others, and though we took from their researches the startingpoint for our experiments, still it has taken a long time before we obtained any reliable result, because we were not prepared for so great a variability in the innervation of the skin as we found.

The first difficulty that presents itself, is of course the definition of the boundary between the posterior extremity and the trunk.

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[1]) The 5 preceding notes are printed in *Proc. Kon. Akad. v. Wetensch. te Amsterdam* 1901, vol: IV. p 266, p. 308, p. 508, and 1903 vol. VI p. 347, p. 392.

Ventrally the groin marks a natural boundary. There, if from the highest point of the inguinal fold, which advances far cranially on the abdomen, a line is drawn, joining the mid-ventral line at the cranial border of the symphysis, this line will follow almost precisely the groin.

Dorsally there is no natural boundary. There a line may be drawn from the highest point of the inguinal fold, over the top of the crista ilei, joining the mid-dorsal line at the upper border of the sacrum.

The extremity being thus confined from the trunk by two boundaries, it becomes necessary to mark on it for topographical purposes a few fixed points.

Besides the salient vertebrae, the crista ilei and the symphysis pubis, there may be taken as such in the first place salient bones, which can be felt through the skin. Tuber ischii, trochanter femoris, the lateral and the medial epicondylus femoris, the patella, the tuberositas tibiae, the capitulum fibulae, the malleolus lateralis (fibulae), the malleolus medialis (tibiae), the calcaneum. Also sinews or muscles forming a distinct relievo, were made use of to this purpose.

Our dogs were operated with rigorous aseptical precautions, in deep morphium-chloroform-narcosis.

The vertebral canal having been laid open, after the method of SHERRINGTON, the nerve-roots situated cranially and caudally from those chosen for examination, are cut through externally from the dura mater. As the normal dog possesses seven lumbar nerve-roots, three sacral roots and from four to seven coccygeal roots, orienteering is ordinarily very difficult.

This difficulty is connected with the relatively great variability of the sacro-lumbar vertebrae and nerve-roots.

A normal dog possesses seven cervical vertebrae, thirteen thoracic vertebrae (with their costae) and seven lumbar vertebrae cranial from the sacrum. It is however not at all uncommon to find dogs, provided with only twelve thoracic vertebrae at which ribs are surely inserted. Such dogs still possess seven lumbar vertebrae, but the most cranial of these often bears a rudiment of a rib and then is signally a thoracic vertebra. In such cases therefore the most caudal or seventh lumbar vertebra, has been absorbed wholly or partly in the sacrum.

As the nerve-root leaving the vertebral canal between the 21st and the 22nd vertebra (reckoned from the foramen occipitalis) is always reckoned to be the first lumbar root, it is evident that in these cases the 7th lumbar nerve-root possesses in reality through the most cranial sacral hole. Such dogs present always difficulties in the orienteering. For localisation depends for a great part on the accuracy of the estimation of the 7th lumbar nerve-root, on its size and its relation to the first sacral root, which is always thinner, but often only slightly so, and which is often closely allied to the 7th lumbra root.

After the operation the dogs generally thrive very well. During the first days following it, the intensely hyperalgetic areas bounding the analgetic ones, are easily defined by pinching the skin with nippers. In this manner is found on the skin the hyperalgetic area corresponding with the uninjured nerve-root, and at the meantime the caudal border is defined of a dermatoma situated cranially, and also the cranial border of a dermatoma situated caudally.

These boundaries are marked on the skin. As soon as they have become constant, they are photographed, their course relative to the fixed points mentioned above is carefully described and lastly a design of them is taken on a model in plaster. After this the animal is sacrificed. The skin of the posterior part is cut open along beforehand indicated ventral or dorsal lines, then it is tanned, varnished and preserved, the hyperalgetic areas having been made distinct by some striking colour.

At the autopsy it is settled how many cervical, costal thoracic and lumbar vertebrae the animal possessed. Afterwards preparations are made of the sacro-lumbar medulla with all the nerve-roots. The connection of the intact nerve-roots with their inter-vertebral ganglia is not severed, this naturally being impossible if they are cut through during the operation (even if only posterior roots were cut through). The entire preparation is then fixed on cardboard by means of pins and hardened in formaline. After a few days the pins are taken away. The preparation then preserves its form and it may be distinguished at first sight, which nerve-roots have been left intact.

In this way we have at our disposal for each nerve-root: 1. photograms showing the area on the skin from that root; 2. a description of the boundaries of that area; 3. a model in plaster on which this area is designed; 4. a tanned hide on which it is coloured and 5. the preparation of the sacro-lumbar medulla corresponding to it.

Besides a certain number of combination-experiments, we defined, after the method of SHERRINGTON the following radicular areas:

The skinfield of	L_1	was defined	once	(dog Nr. 3)
,, ,, ,,	L_2	,, ,,	twice	(,, Nrs. 4 and 5)
,, ,, ,,	L_3	,, ,,	three times	(,, Nrs. 8, 15, 16)
,, ,, ,,	L_4	,, ,,	nine ,,	(,, Nrs. 6, 13, 14, 19, 24, 26*R*, 31*L* and *R*)
,, ,, ,,	L_5	,, ,,	six ,,	(,, Nrs. 2, 10, 17, 25, 28 and 30)
,, ,, ,,	L_6	,, ,,	four ,,	(,, Nrs. 9, 18, 27*R*, 32)
,, ,, ,,	L_7	,, ,,	five ,,	(,, Nrs. 12, 20, 27*L*, 29*L*, 33*R*)
,, ,, ,,	S_1	,, ,,	five ,,	(,, Nrs. 11, 21, 28, 29*R*, 33*L*)
,, ,, ,,	S_2	,, ,,	three ,,	,, Nrs. 13, 25, 30)
,, ,, ,,	S_3	,, ,,	twice	(,, Nrs. 23, 18)
,, ,, ,,	$Cocc_1$	,, ,,	once	(,, Nr. 34)

Moreover we got at our disposal still several caudal or cranial borders of cranial or caudal dermatomata, which were found in defining the boundaries of one isolated dermatoma.

Here follow examples of each of these radicular areas.

The skinfield of *L*I.

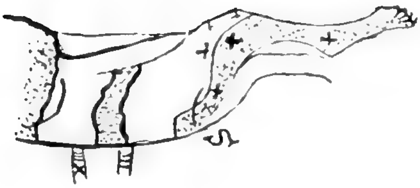
In dog Nr. 3 (see fig. I). The sensible area, which was left between two analgetic ones is bounded in the following manner:

Its cranial boundary leaves the middorsal line at the upper border of the XIIIth thoracic vertebra, goes somewhat caudally, at a distance of 3 cm. from the costal arch, towards the highest point of the inguinal fold, then straight towards the mid-ventral line, coming to one cm. distance of it a little above the top of the prepuce, takes a cranial turn parallel to the m. v. line and is caught within the caudal boundary of Th. IX.

Its caudal boundary leaves the m. d. line near the IVth lumbar vertebra, about 3.5 cm. lower than the cranial boundary, runs parallel to this latter, straight between the crista ilei and the costal arch, crosses the inguinal fold between its cranial and its central third, bends with a very small caudal flap into the groin, and goes converging with the upper boundary towards the mid-ventral line coming to one cm. distance of it at the cranial third of the prepuce. Then it takes a turn caudalward, runs parallel to the mid-ventral line on the prepuce and passes into the cranial boundary of S_1.

For dog Nr. 17, where *L*v is isolated by dividing the roots of *L*II, *L*III and *L*IV cranially and *L*VI, *L*VII, *S*I and *S*II caudally, we find the description of the caudal boundary almost conform, as follows (see fig. 10). The boundary leaves the mid-dorsal line at the 4th lumbar vertebra, continues vertically on this vertebra, between the crista ilei and the costal arch, 3.5 cm. from the former, 4 cm. from the latter towards the cranial third of the inguinal fold, crosses this at 11 cm. distance from the tuberositas tibiae, takes a sharp turn caudalward, forms a small

Plate I.
*L*ı. Dog Nr. 3.



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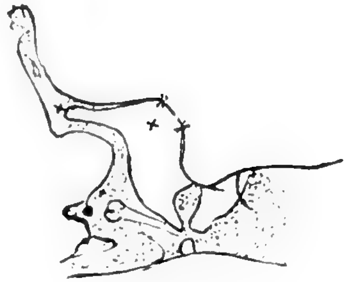

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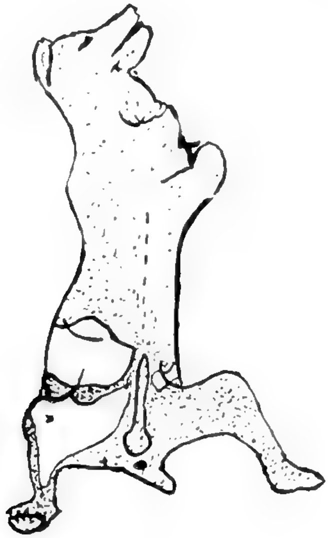

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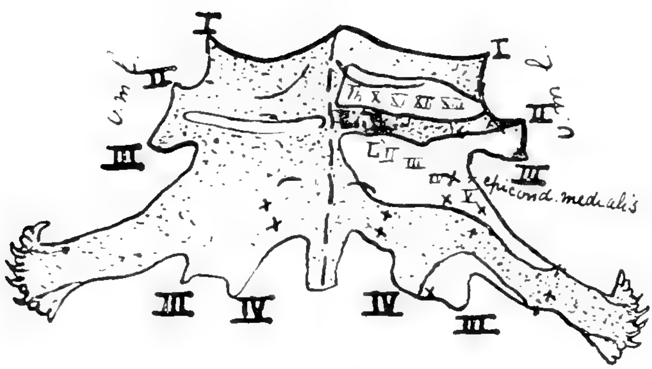


4

Isolation of the skin-area of *L*ı. To the right are cut through: *Th*x. *Th*xı. *Th*xıı. *Th*xııı. *L*ıı. *L*ııı. *L*ıv and *L*v. To the left *L*ı has been cut through.

Thus the caudal boundary of *Th*ıx, the cranial boundaries of *L*vı, *L*vıı and *S*ı have been partly defined on the right side, together with the isolated skinfield.

1 lateral portion of the extremity. 2 medial portion of the extremity. 3 ventral part of the dog. 4 the skin cut open along the mid-ventral line, i. e. the mid-ventral line is followed unto the symphysis; from thence the section passes over epicondylus medialis femoris, malleolus medialis, the palmar surface of the foot, divides the great trigonal planta pedis, and the planta of the second medial toe. Along this line the skin is cut open and stretched.

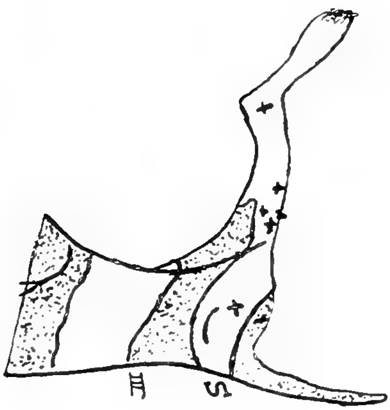
tongue into the groin, and bending again cranially, it comes to within $1/2$ cm. distance of the mid-ventral line at the top of the prepuce. Then it turns caudalward, runs parallel to the mid-ventral line on the prepuce, and is caught within the cranial boundary of *S*ɪɪ.

Evidently the influence of the extremity is already felt on the caudal border of *L*ɪ. This skin-area advances with a narrow tongue towards the groin. But this influence is as yet only slight. The first nerveroot that partakes distinctly in the innervation of the skin of the groin and the medial crural region is *L*ɪɪ.

The skinarea of *L*ɪɪ.

Plate II.

*L*ɪɪ. dog 4.



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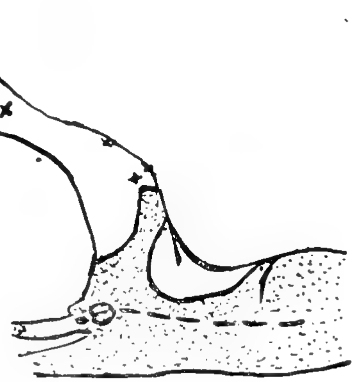

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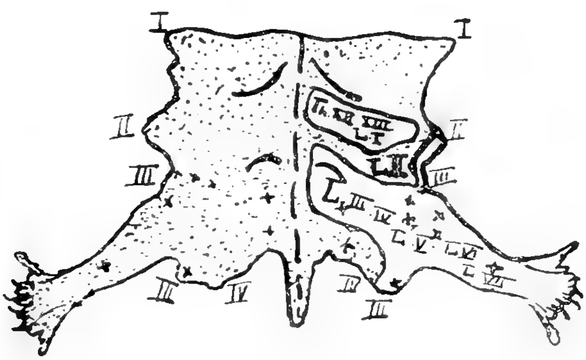


3

Isolation of the skinfield of *L*ɪɪ. To the right are cut through: *Th*xɪɪ, *Th*xɪɪɪ and *L*ɪ, and *L*ɪɪɪ, *L*ɪv, *L*v, *L*vɪ and *L*vɪɪ.

On the right are defined the caudal boundary of *Th*xɪ, the cranial boundary of *S*ɪ and the skinfield of *L*ɪɪ. 1 lateral portion of the extremity, which however is turned in part medially, 2 medial portion of the extremity, 3 the skin, cut open along the **mid-ventral** line, and stretched

The skinarea of *L*II that remains between the two analgetic areas is bounded in the following way: (see fig. II).

The cranial boundary leaves the mid-dorsal line at the second lumbar vertebra, passes in the middle between costal arch and crista ilei towards the cranial third of the inguinal fold, crosses and then takes a turn straight towards the mid-ventral line. It does not reach this latter, but comes to within 2.5 cm. distance of it at about 5 cm. above the symphysis, then it bends cranially and passes parallel to the mid-ventral line in the caudal boundary of Th. XI.

The caudal boundary leaves the mid-ventral line at the fourth lumbar vertebra, and goes 2.5 cm. cranially from the crista ilei towards the anterior crural surface. It crosses in caudal direction the inguinal fold in its posterior third 3 cm. above the patella, reaches the medial crural surface, and takes a sharp cranial bend 3 cm. cranially from the epicondylus medialis femoris. After making this tongue it passes through the groin without reaching however the mid-ventral line, as it meets within the cranial boundary of *S*I at 5 cm. laterally from the vulva.

The skinarea of *L*II which is left here (incompletely) in the analgetic area is bounded in the following manner (see fig. III): The cranial boundary leaves the middorsal line at the second lumbar vertebra, and turns in caudal direction towards the upper third of the inguinal fold, crosses it, and goes straight towards the mid-ventral line, without reaching it however. Opposite the middle of the skin-fold of the prepuce, it turns in caudal direction at somewhat more than 3.5 c.m. distance from the m.v.l. running parallel to this latter, and so passes into the caudal boundary of the area of *L*II.

This caudal boundary leaves the m.d.l. at the 4th lumbar vertebra, running over the lateral crural surface. 4 cM. above the crista ilei, towards the caudal end of the inguinal fold, crossing this latter 3 cM. above the patella and goes, taking a cranial bend 4 cm. above the epicondylus medialis femoris, through the groin to the m.v.l. It does not reach this, but passes into the cranial boundary opposite the caudal third of the prepuce, at 3.5 cm. distance from the m.v.l.

The connexion with the m.v.l. therefore no longer exists in this case. The radicular area is incomplete. It is a caricature that has lost its ventral portion, such as we have found the "ventral area" in trunk dermatomata in our former communications. If the boundaries are continued along the dotted lines, the area of *L*II is equal to the one described before.

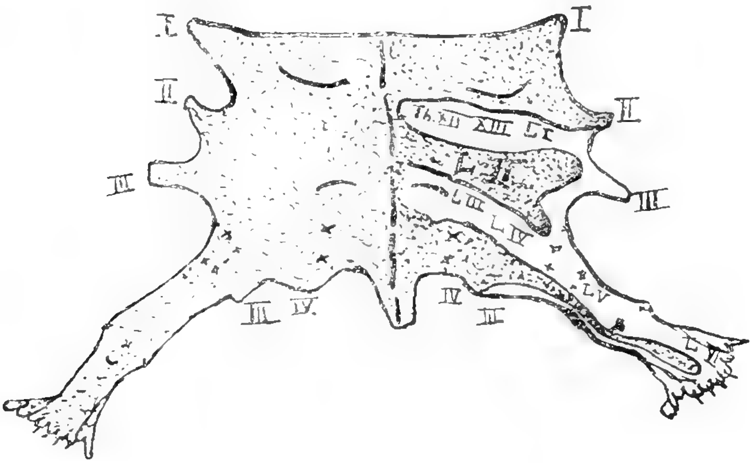
Consequently *L*II commands a skinfield situated cranially and basally on the extremity. The medial third of the inguinal fold is innervated by it. In males it contributes to the innervation of the medial third of the skin-fold of the prepuce, in females to that of the cranial half of the mons veneris. The ventral portion overlaps that of *L*III, of *L*IV, in such a manner that it is at the m.v.l. even connected with that of *S*I. It will be shown by other isolations that the "crossed overlapses" of *L*II, *L*III and *L*IV on the other side are not large enough to bring about the complete innervation of the crossed prepuce or of a band, broad 2.5 à 3 c.m. above the crossed mons Veneris.

Consequently the area of *L*II is at the m.v.l. in connection to the

Plate III.
*L*ii. dog V.

1

2



3

Imperfect isolation of the skinfield of *L*ii. On the right are cut through: *Th.* xii, *Th.* xiii, *L*i and *L*iii, *L*iv, *L*v, *L*vi.

Consequently to the right are defined the caudal boundary of *Th.* xi, the cranial boundary of *L*vii and of *S*i and the skinfield of *L*ii. The isolation is not complete, for the sensible area does not reach the mid-ventral line. 1. lateral portion of the extremity, 2. medial portion of the extremity, 3. the skin, cut open along mid-ventral lines and stretched.

field of S_I. At the m.d.l. there is no connection between the field of L_{II} and that of S_I. The lateral portion of the area of L_{II} sends out a neck branch through the groin on the anterior surface of the extremity, passing at the middle third of the skinfold of the groin.

The skinfield of L_{III}.

This root has been isolated three times.

1. on dog 8. To the right, between Th. XIII, L_I, L_{II} and L_{IV}, L_V, L_{VI}, L_{VII}. The sensible area is bounded in the following manner:

The cranial boundary leaves the m.d.l. to 1 c.m. distance, about the 4th lumbar vertebra, going in caudal direction 1 c.m. above the crista ilei. towards the medial part of the inguinal fold, and crossing this at 7 c.m. above the patella. Having reached the interior crural surface, it continues at first in caudal direction, and turns cranially through the most medial portion of the groin, towards the inferior mammilla. There, 2.5 c.m. from the m.v.l. it bends cranially parallel to it and passes into the caudal boundary of Th. XII.

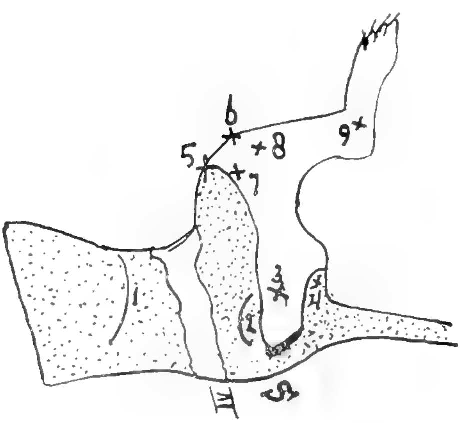
The caudal boundary leaves the m.v.l. to 1 c.m. from it at the cranial end of the sacrum, and goes between crista ilei and trochanter towards the exterior crural surface in the lateral furrow next the m. quadriceps to the patella; having crossed this, it takes a turn towards the symphysis on the medial crural surface, cranially from the epicondylus medialis.

When arrived at the middle of the groin, without reaching this, it goes somewhat more cranially, and at 4.5 c.m. distance from the symphysis, it takes a caudal bend, parallel to the m.v.l., into the s. bicip. int. and so passes into the cranial boundary of S_1.

2. on dog 15. To the right, between ThXIII, L_I, L_{II}, and L_{IV}, L_V, L_{VI}, L_{VII} and S_1. (see fig. IV).

Plate IV.

L_{III}. dog 15.



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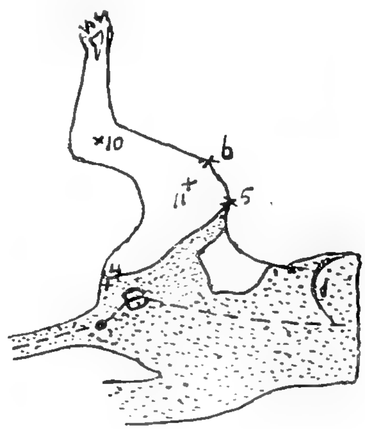

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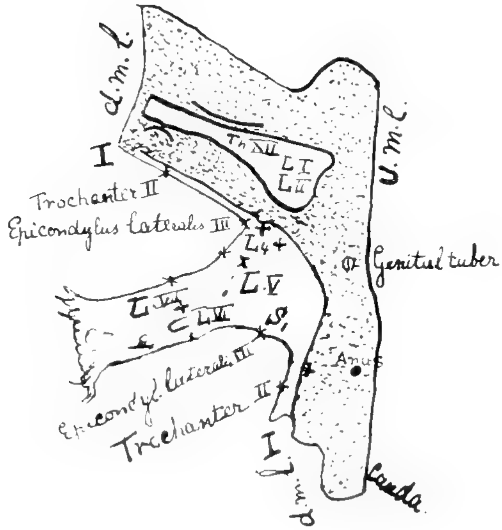


3

Isolation of the area on the skin from *L*III. To the right are cut through: *Th*XIII, *L*I, *L*II, and *L*IV, LV, *L*VI, *L*VII and *S*I. Consequeutly on the right side are isolated the caudal boundary of *Th*XII, the cranial boundary of S_2, and the skin-area of *L*III. 1 the lateral portion of the extremity, 2 the medial portion of the extremity, 3 the skin cut open along **dorsal** lines. This means that the mid-dorsal line has been followed on to the sacrum; the section then passes over trochanter, epicondylus lateralis femoris, malleolus lateralis and the dorsum of foot towards the dorsum of the 2nd medial toe (the 3rd lateral one). The toes are cut open in the same manner as was done on the fore-leg in former experiments. *L*III passes into *S*II both at m.d.l. and at the m.v.l. Fixed points are: 1 costal arch, 2. crista ilei. 3. trochanter. 4. tuber ischii. 5. patella. 6. tuberositas tibiae. 7. epicondylus lateralis, 8. capitulum fibulae. 9. malleolus fibulae. 10. malleolus medialis. 11. epicondylus medialis.

The sensible area of *L*III is here bounded in the following manner: The cranial boundary leaves the m.d.l. near the 4th lumbar vertebra, and goes cranially from the crista ilei in caudal direction on the exterior crural surface towards the inguinal fold, crossing it 4 c.m. above the patella. Arrived on the interior crural surface, it runs on to the middle of the groin, and there takes a cranial turn to the 4th mammilla (beginning to count from above). At somewhat more than 3c.m. distance from the m.v.l., it passes parallel to it, into the caudal boundary of Th. XII.

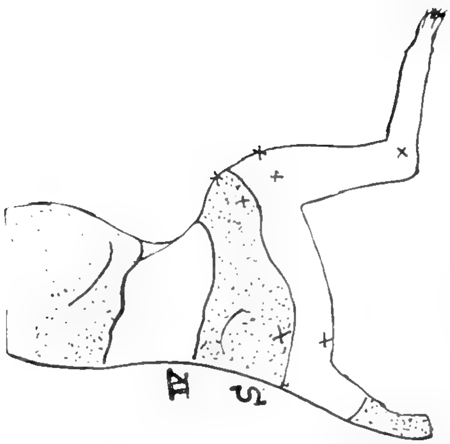
The caudal boundary has its origin 2.5 c.m. lower, near the sacrum, and passes between crista ilei and trochanter, 1.5 c.m. cranial from the latter, to the patella, crossing it. Arrived on the medial crural surface, it goes straight towards the conus genitalis, approaching this to 3 c.m. distance and returns caudally parallel to the m.v.l., encircling the anus, as the cranial boundary of *S*II, between tuber ischii and trochanter to the m.d.l.

3⁰. On dog 16. To the right, between *Th. X*III, *L*I, *L*II and *L*IV — Cocc. *II*. (See plate V).

The sensible area of *L*III is here bounded in the following manner: The cranial boundary leaves the m.d.l. near the 5th lumbar vertebra, crosses in caudal direction

Plate V.

*L*III. dog 16.



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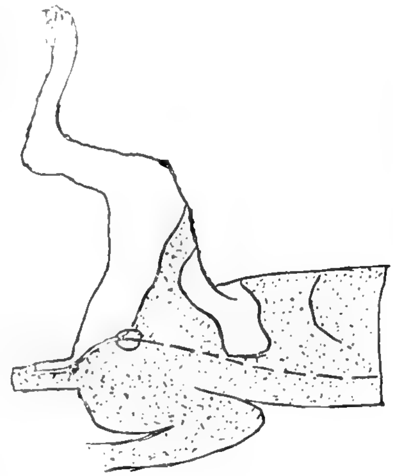

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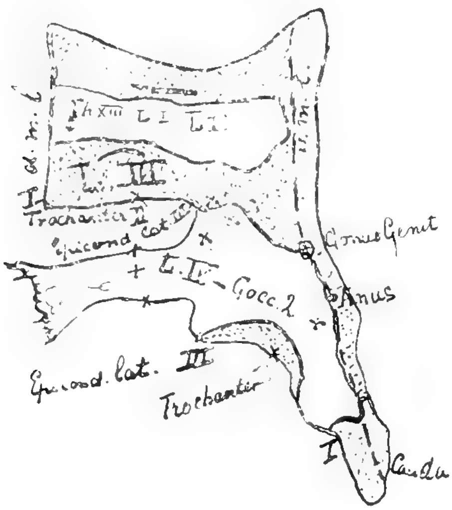


3

Isolation of the skinfield of *L*III. To the right are cut through: *Th*XIII, *L*I, *L*II, and *L*IV, *L*V, *L*VI, *L*VII, S_I S_{II}, S_{III}. On the right side are defined the caudal boundary of *Th*XII, the cranial boundary of *Cocc.*II and the area on the skin from *L*III. 1 the lateral portion of the leg, 2 the medial portion of the leg, 3 the skin cut open along **dorsal** lines and stretched.

the crista ilei, goes to the middle of the inguinal fold, and having crossed it, takes a cranial turn towards the middle of the groin. From thence it turns, describing an angle of 45 degrees, towards the m. v. l., where it is joined by the caudal boundary of Th. XII. The caudal boundary leaves the m.d.l. 4 c.m. beneath the cranial one, 3 c.m. above the root of the tail, at the middle of the sacrum, and passes between trochanter and tuber ischi, on the lateral crural surface parallel to the cranial boundary. From the centre of this surface it converges with the cranial boundary, goes caudally from the epicond. lat. femoris to the inferior border of the patella and crosses it; then it goes strongly convergent with the cranial boundary (the sensible area is here narrowed to 2.5 cm.) towards the middle of the vulva. There it joins the m. d. l. continues along with it over the anus, and joins finally, as cranial boundary of *C*II, 3 c.m. under the root of the tail, the m. d. l.

The skinfield of *L*III varies as to situation and extension. Dorsally it may be found from the 4th lumbar vertebra to the sacrum, and likewise from the 5th lumbar vertebra to the middle of the sacrum. The cranial boundary may cross the inguinal fold either above or below the middle of it. The caudal boundary may pass either cranially or caudally from the trochanter, and also either above the patella, over it or just below it.

This area is connected with the area of *S*I at the mid-dorsal and at the mid-ventral line, more cranially it overlaps *L*II and *L*IV and only when all the sacral nerve-roots are cut through, it may be isolated completely. The area is extremely narrow in the medial third of the groin (as was likewise the case with *L*II), and in cases of incomplete isolation it is frequently discontinued by analgetic zones in this region. The area on the skin from *L*III may be counted, like that of *L*II, to belong to the cranial marginal dermatomata, and among these it is the largest area.

The skinfield of *L*IV.

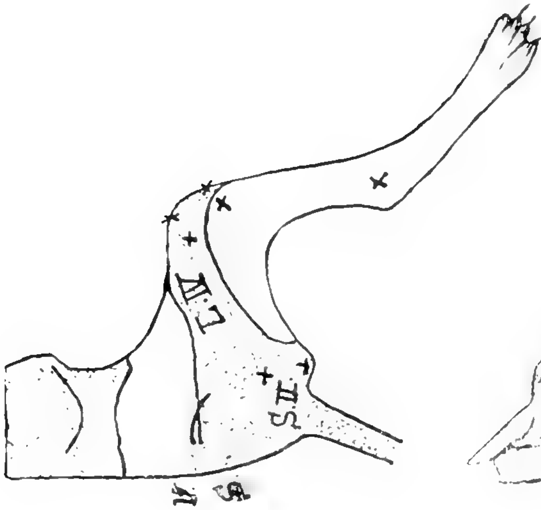
This area has been isolated nine times. The description of some of our results may follow here.

1. On dog Nr. 13. To the right are cut through *L*I, *L*II, *L*III, further *L*V, *L*VI, *L*VII, *S*I, also *S*III and the upper rad. coccygei, consequently *L*IV together with *S*II have been isolated.

The sensible area of *L*IV is bounded as follows:

It does not reach the m. d. l., but remains 2 cm. distant from it. The line connecting its cranial and caudal boundaries has a length of 2.5 cm. and runs parallel to the m d. l. 2 cm. distant from this latter. The cranial boundary goes right under the crista ilei transversally over the exterior crural surface towards the inferior border of the patella, it crosses there the ligamentum interarticulare and goes straight towards the symphysis over the medial crural surface, distally from the epicond. med. fem. It reaches the m. d. l. 3 cm. above the symphysis opposite the root of the prepuce, and passes cranially along the m. d. l. into the caudal margin of *L*III. The caudal boundary originates 2 cm. distant from the m. d. l., opposite the middle of the sacrum, caudally of the trochanter and goes between this and the tuber ischii (4 cm. above this latter) right over the cap. fib., distally from the tuberositas tibiae towards the crista tibiae, crossing it. On the medial crural surface it converges with the cranial boundary (the narrowest region of the

Plate VI.
*L*IV. dog 14.



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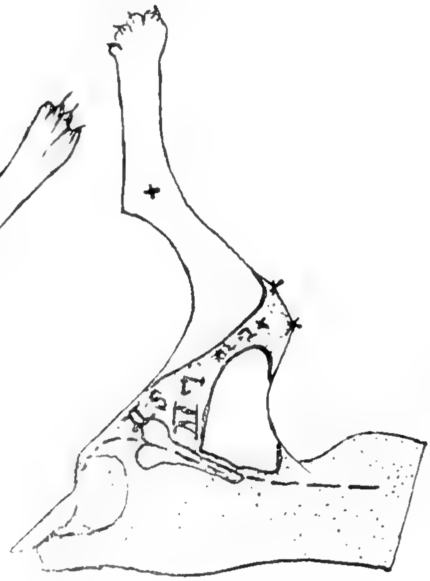

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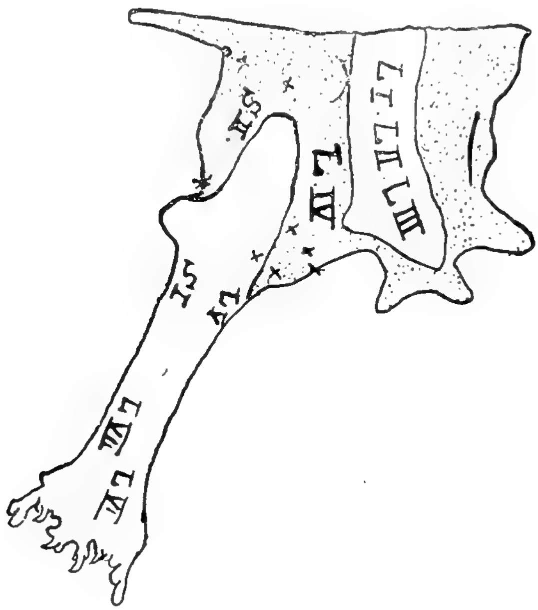


3

Isolation of the skinfield of *L*IV. To the right are cut through: *L*I, *L*II, *L*III and *L*V, *L*VI, *L*VII and *S*I. Consequently to the right are defined the caudal boundary of *Th*.XIII, the cranial boundary of *S*II and the area on the skin from, *L*IV. 1, the lateral portion of the extremity, 2, the medial portion of the extremity 3, the skin cut open along **ventral** lines and stretched. *L*IV passes into *S*II both at the mid-dorsal and at the mid-ventral line.

area) and turns towards the symphysis. Just below this it joins the m. v. l.. leaving it again immediately, as cranial boundary of *S*II.

This area is incomplete. The dorsal portion fails entirely and the ventral is too small. More complete is:

2. On dog 14. To the right are cut through: *L*I, *L*II, *L*III and *L*V, *L*VI, *L*VII and *S*I (see plate VI).

The sensible area of *L*IV is here bounded in the following manner: The cranial boundary leaves the d.m.l. at the 5th lumbar vertebra, goes over the crista ilei to the inferior third of the inguinal fold and crossing this, continues in caudal direction into 1 cm. proximal from the epic. med. fem. There it takes a cranial turn and goes through the inguinal fold to the middle of the prepuce. Going upwards along this latter, parallel to the m v. l. $^1/_2$ cm. distant from it, it passes into the caudal boundary of Th. XIII. The caudal boundary has its origin not at the m. d. l., but at a point situated somewhat lateral from the line connecting trochanter and tuber ischii (the dorsal connection between *L*IV and *S*II), it passes over the lateral crural surface as an arch opened distalward, between epicond. fem. lat. and capit. fibulae, crossing the crista tibiae 2 cm. below the tuberositas tibiae. On the medial crural surface it returns to its origin, running distally (at $1^1/_2$ cm.) from the epic. fem. med., almost parallel to a line connecting this with the tuber ischii. Consequently *L*IV and *S*II are connected both at the mid-dorsal and at the mid-ventral line.

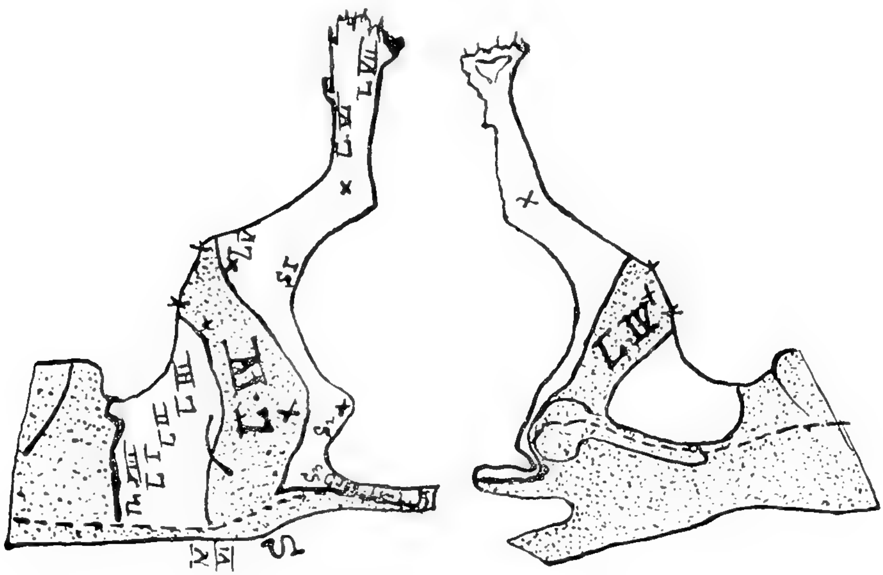
Still more complete is the isolation of the radicular area of *L*IV, in the dogs n⁰. 26 and n⁰. 24.

3. On dog 26, where to the right are cut through: Th XIII *L*I, *L*II, *L*III and *L*V, VI, *L*VII, *S*I, *S*II, *S*III and the upper Coccygei. (see Plate VII 1 and 2).

The area is bounded in the following manner: The cranial boundary originates at the m. d. l. at the 6th lumbar vertebra opposite the crista ilei, it goes over the

Plate VII.

*L*IV. dog 26.



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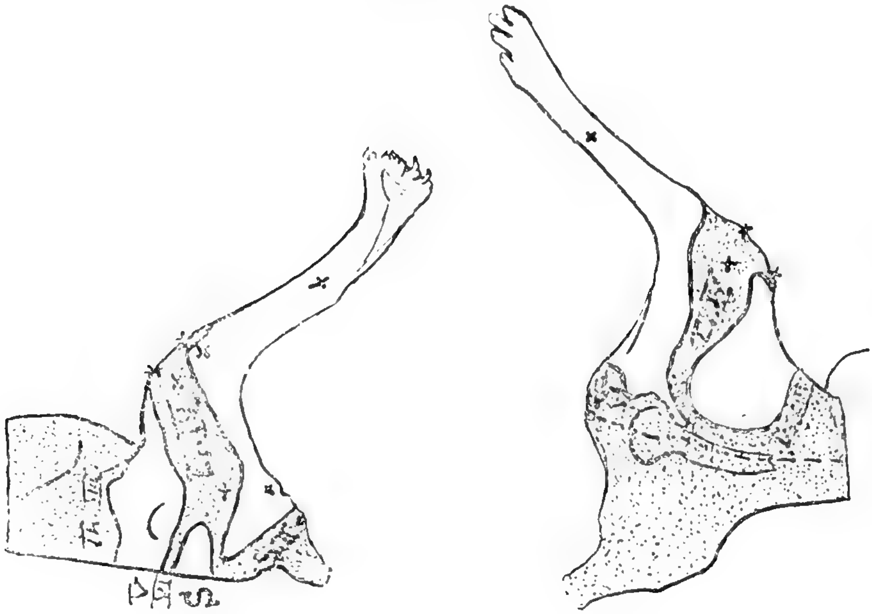
*L*IV. dog 24.



3 4

Isolation of the skinfield of *L*IV (dog 26 and dog 24). On dog 26 to the right are cut through: *Th*XIII to *L*III including this and *L*V to *Cocc*II, including this.

1 and 3 the lateral and dorsal regions of the extremity. 2 and 4 the medial and ventral regions of the extremity.

On dog 26 the isolation of *L*IV is complete. On dog 24 *L*IV is still connected with *S*II, but the area is a caricature, the dorsal portion fails almost entirely, the ventral one is very narrow.

lateral crural surface towards the anterior surface crossing this latter $2^1/_2$ cm. above the patella, continues in a straight line over the medial surface to the symphysis, approaching there the m. v. l. $^1/_2$ cm distance at the root of the prepuce, and turning cranialward parallel to the m. v. l. it passes into the caudal boundary of Th XIII.

The caudal boundary has its origin 4 cm. lower at the m.d.l. on the middle of the sacrum, at an angle of 45°; it passes between trochanter (1 cm. caudally from it) and tuber ischii (3 cm. cranially from it) and goes in a straight line over the lateral crural surface, between epic. fem. lat. and cap. fib. It crosses the crista tibiae one cm. below the tub. tibiae. Then it goes over the medial crural surface, 4 cm. below the epic. fem. med. straight towards the conus genitalis, and passing over it, reaches the m. v. l. and along with this, over perineum, anus and tail, in caudal direction the cranial boundary of Cocc. III.

The radicular area of *L*IV is feeblest dorsally. Contrary to what we found for the dermatomata of the trunk, its caricatures are most easily found at the m. dors. l. Sometimes disappearance (dog 13), sometimes also the narrowing (dog 24) of the dorsal portion is found for instance.

4. On dog 24 the radicular area of *L*IV has been isolated. (See plate VII fig. 3 and 4). To the right *L*I—*L*III inclusive, also *L*V—S_1 inclusive, have been cut through. The radicular area is bounded in the following manner: The cranial

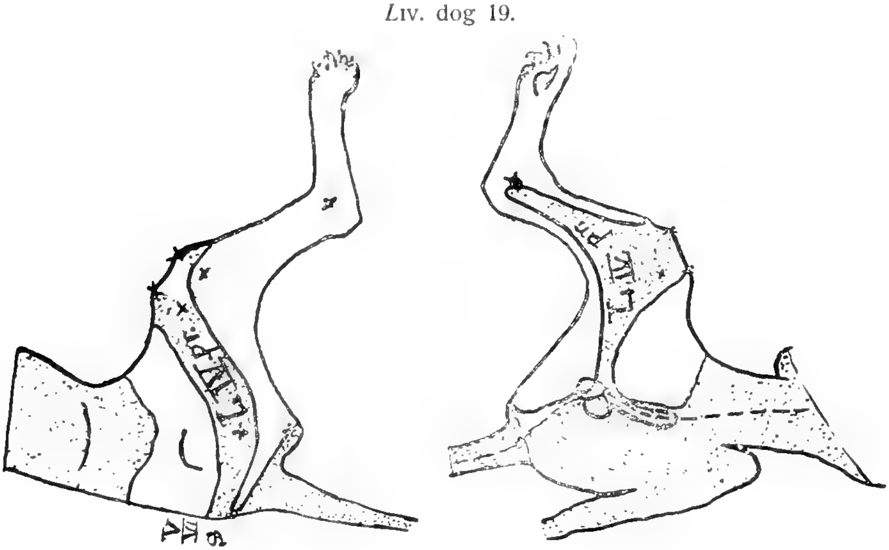
boundary originates at the 6^{th} lumbar vertebra at the m.d.l. goes 1.5 c.m. caudally from the crista ilei over the lateral crural surface towards the upper border of the patella, then taking a cranial turn, it is directed to the symphysis. In the inguinal fold it goes towards the root of the prepuce and comes there to within 3 c.m. of the m.v.l., goes upwards parallel to this line and passes into the caudal boundary of *L*I.

The caudal boundary leaves the m.d.l. immediately below the cranial boundary, and apparently an insensible quadrangular area lies between two sensible fields. (It ought to have been a great sensible area of the connected fields of *L*IV and *S*I, and it has become such a field after a few days). It goes 1.5 c.m. caudally from the trochanter over the lateral crural surface, between epic. fem. med. and capit. fib. to the crista tibiae, crosses this and turns on the medial surface, converging with the cranial boundary towards the basis of the scrotum. There the area of *L*IV has become very narrow ($^1/_2$ cm.). The boundary then turns caudalward // to the m.v.l., 3 cm. from the perineum, encircles the anus and passes into the cranial boundary of *S*II, which in its turn passes on the back into the caudal boundary of *L*IV.

Still more remarkable becomes the radicular area of *L*IV as soon as the dog possesses only twelve thoracic vertebrae or when the 7^{th} lumbar vertebra has become a part of the sacrum, as was the case for dog 19. Then there is, what has been called by SHERRINGTON: prefixion of the extremity. In such cases the radicular area of *L*IV has its utmost variety.

5. On dog 19 to the right are cut through: *L*I—*L*III inclusive and *L*V—*S*III inclusive. The radicular area of *L*IV is bounded in the following manner (see plate VIII): The cranial boundary leaves the m.d.l. $^1/_2$ cm. below the cranial border of the sacrum, goes 2.5 c.m. below the crist. ilei over the lateral crural surface towards the cranial border of the patella, and crossing this bends in a cranial-concave arch to the root of the prepuce where it nearly reaches the m.d.l. and runs parallel to this latter upward, passing into the caudal boundary of Th XIII.

Plate VIII.
*L*IV. dog 19.



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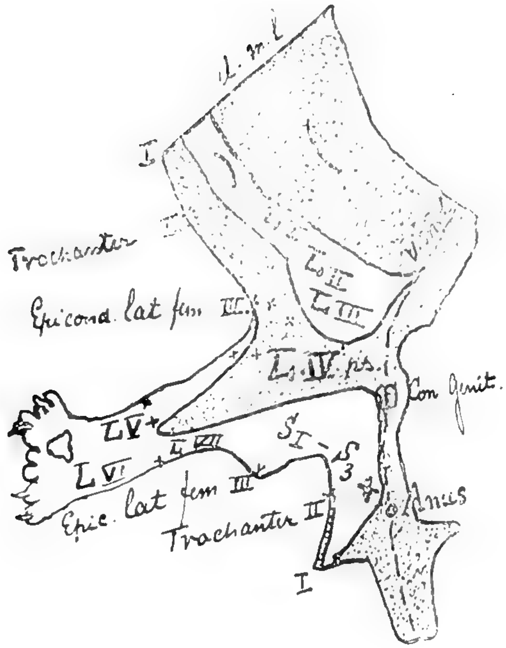


3

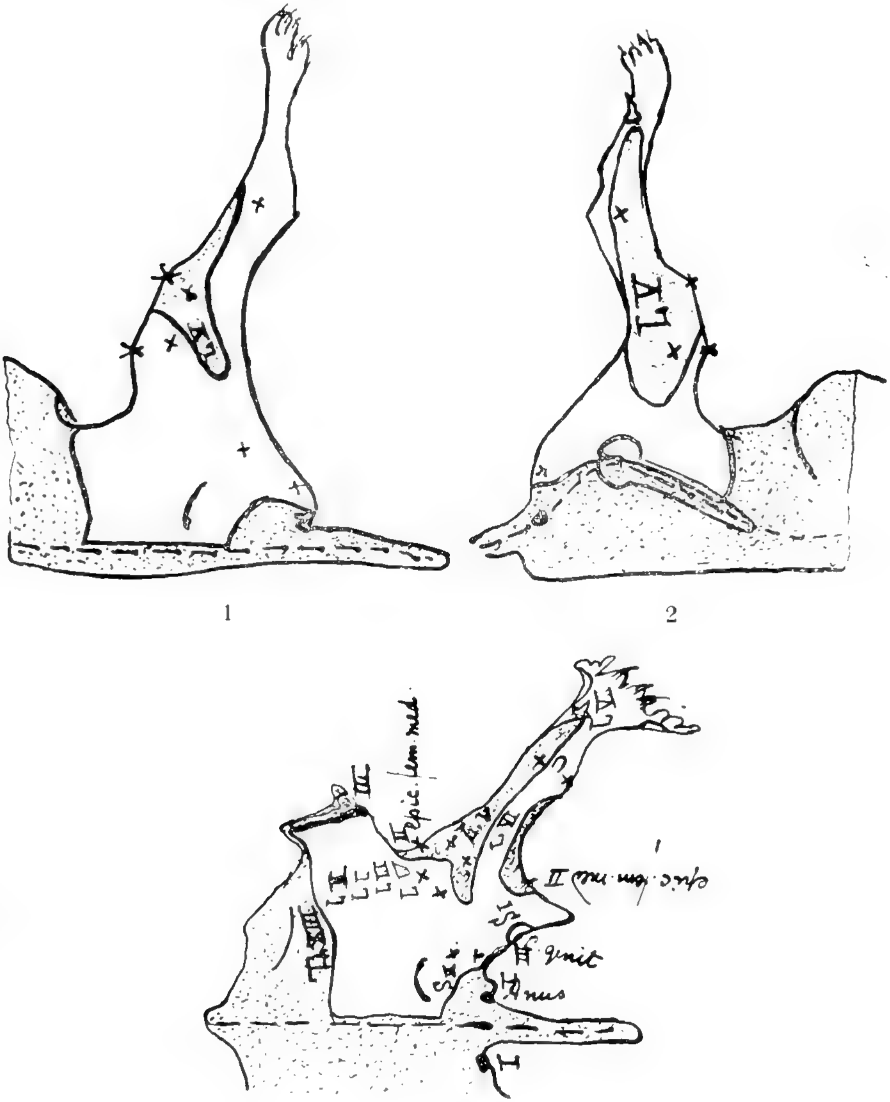
Isolation of the skinfield of *L*IV, made on a dog with 7 cervical vertebrae, 12 thoracic vertebrae bearing ribs, 1 thoracic vertebra without rib, 6 lumbar vertebrae whilst the 7th of these is entirely caught into the sacrum. According to Prof. Bolk, by whom the plexus was examined, the animal has a double N. furcalis. (Ihering). All this is on the right side. To the left it is less complete. To the right are cut through: *L*I—*L*III inclusive and *L*V—S^3 inclusive. 1. latero-dorsal portion of the extremity. 2. medio-ventral portion of the extremity, with the linguiform prolongation of the area on the medial surface of the fore-part of the extremity. 3. the skin cut open along **dorsal** lines.

The caudal boundary originates 1.5 c.m. lower, and goes just below the trochanter, along the longitudinal axis of the lateral crural surface, between epicond. fem. lat. and cap. fib. to the crista tibiae below the tuberositas. On the medial surface of the fore-part of the leg it continues caudally along the crista tibiae to the malleolus tibiae med. Between tibia and tendo Achillis it returns cranial-wards to the upper leg, and crossing the flexors of the knee it bends to the m.v.l. reaching it on the middle of the scrotum, and then takes a caudal turn parallel to the m.v.l.

Like the areas of *L*II and *L*III, the area on the skin from *L*IV presents a connexion both with the mid-dorsal and with the mid-ventral line. Like these therefore it may be reckoned to belong to the basal or marginal dermatomata of the posterior extremity. These three are the cranial of the marginal dermatomata. Together with those of *L*II and *L*III the dorsal and ventral portions of this radicular area are situated one upon another, at the sacrum and at the symphysis. The connection with the mid-dorsal line has become less solid for *L*IV than it was for *L*II and *L*III. This connection is the first to fail whenever the isolation was defective. The area is very variable in extension, i.e. its lateral portion is subject to very great changes in form, and the most remarkable manner of extension is shown, when the area sends out a long tongue on the medial surface of the forepart of the leg. An instance of this is presented by dog 19; where with a so-called prefixion of the extremity, the area of the

Plate IX.

*L*v. dog 10.



3

Isolation of the skinfield of *L*v. On dog 12 to the right are cut through L_{I}—L_{IV} inclusive and L_{VI}—S_{II} inclusive. 1 dorso-lateral portion of the extremity. 2 ventro-medial portion of the extremity. 3. skin cut open along **ventral** lines and stretched.

skin innervated by *L*IV assumes partly the place which generally is taken by that of *L*V.

Both these last facts support our opinion, that the dermatomata send their lateral portions on the extremity, but do not pass on it as a whole.

The skinfield of *L*V.

To this area a very particular place must be assigned. It has been isolated by us six times, a few instances among these follow here.

1st. On dog 10. To the right are cut through: *L*I—*L*IV inclusive and *L*VI—*S*I inclusive. (See plate IX).

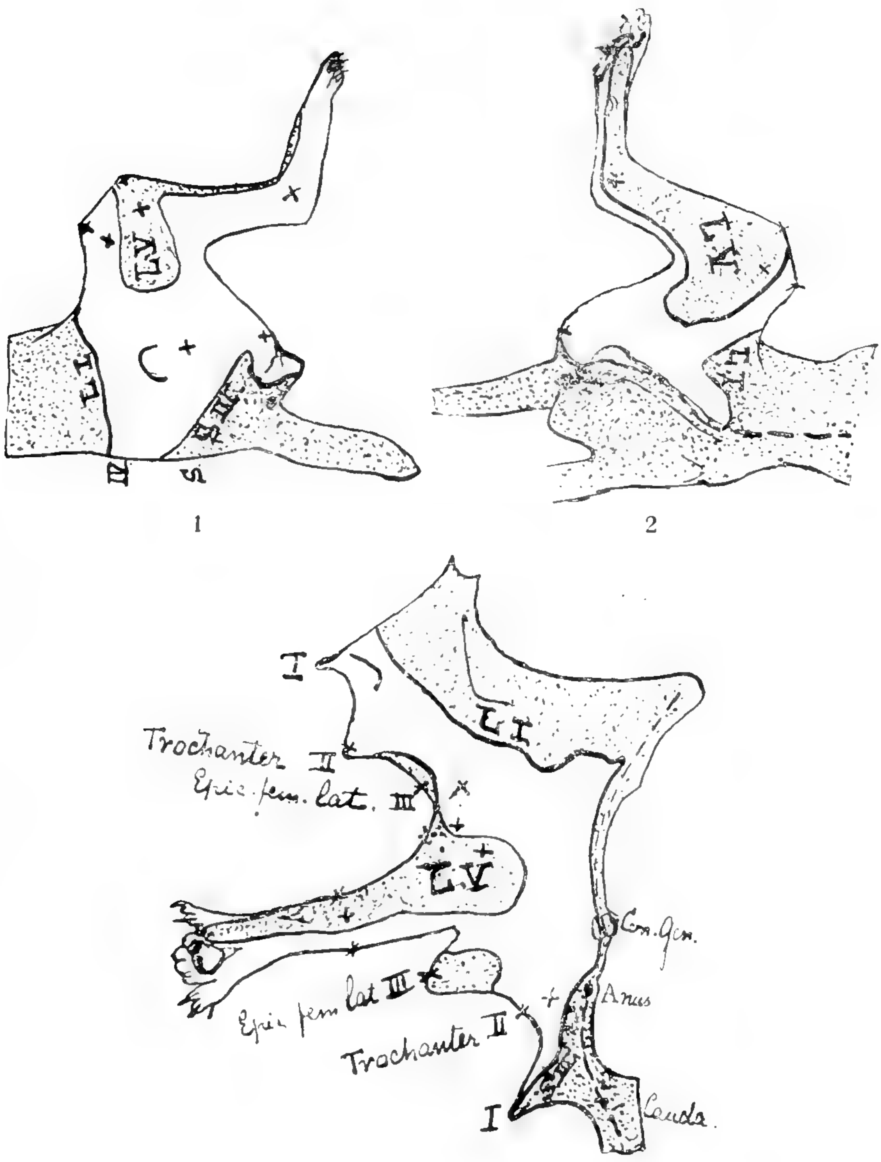
There is an insensible zone bounded in the following manner: laterally by the m.d.l. from the 5th lumbar vertebra unto halfway the sacrum; medially by the m.v.l. from the orefic. urethrae over the scrotum and the perineum unto halfway the anus; cranially by the caudal margin of *L*I; caudally by the cranial margin of *S*II; within these boundaries the sensible area of *L*V, detached both from the m.v.l. and from the m.d.l., is situated, whilst the rest of the extremity is insensible. At a point, situated 4 cm. lateral from the trochanter on the line uniting this with the epic. fem. lat. originates a line which, passing between patella and tuberositas tibiae over the lateral crural surface in an arch opening cranialward, crosses the ligamentum interarticulare. Arrived on the medial crural surface it takes a turn towards the symphysis, approaching this to 4 cm. (this part of the line may be called the cranial boundary). Then it takes a vertical bend distalward, and follows the m. biceps, along the popliteal space, unto the medial surface of the forepart of the extremity. Between tendo Achillis and mall. medialis it passes on the lateral surface of the foot unto the solitary (1st) medial nail. Right above this it returns over the middle of the dorsum offoot in proximal direction, goes upward between both malleoli along the crista tibiae unto the anterior surface of the calf opposite the tuberositas tibiae and then turns laterally, distally from the cap. fibulae, // to its first part and reaches its point of departing again by taking a vertical bend. The here isolated area on the skin from *L*V is excessively small, if compared with others.

2nd. On dog 17. To the right are cut through: *L*II, *L*III and *L*IV and *L*VI—*S*II inclusive. (See plate X).

The boundary originates at a point somewhat above the lateral third of the line uniting the trochanter to the epic. lat. femoris, goes in distal direction on the lateral crural surface towards the tuberositas tibiae, crosses this and passes on the medial crural surface. Then it goes cranially over the epic. fem. med., parallel to the groin, directed towards the symphysis. At 5 cm. distance from the m. v. l. it takes a vertical turn caudalward unto near the poplideal spaces. Going along the medial surface of the forepart of the extremity, it passes between malleolus med. and calcaneum on the medial border of the foot, and along the media border of the great planta pedis it reaches the plant of the most medial (2nd) toe. Returning then cranialward on the dorsum of the foot (the solitary nail, of the (1st) medial toe, situated more proximally, is found within the sensible area) it passes between the malleoli, slowly crosses the anterior surface of the forepart of the extremity, the crista tibiae and goes towards the cap. fibulae. 1 cm. distally

Plate X.

*L*v. dog 17.



Isolation of the skinfield of *L*v. On dog 17. To the right are cut through: *L*II—*L*IV, inclusive and *L*VI—*S*II inclusive. 1. dorso-lateral portion of the extremity. 2. ventro-medial portion of the extremity. 4. the skin cut open along **dorsal** lines and stretched.

from this latter, it diverges suddenly to the m.d.l. and runs parallel to its beginning, but remaining 3 c.m. distant from it nearly along the line uniting the cap. fibulae with the tub. ischii. Opposite the point of departure, it takes a sudden cranial bend and returns into its origin after a course of 3 c.m.

In order to facilitate the survey of the situation occupied by *L*v on the medial and anterior portion of the foot we still add the following experiment.

3rd. the skinfield of *L*v on dog 25 (See plate XI, fig. 1 and 2).

Here to the left were cut through: *L*i—*L*iv inclusive, also *L*vi—*S*i inclusive and lastly S_3—*Cocc*iii inclusive.

Its boundary goes from a (dorsally and proximally) point, situated on the lateral crural surface near the epic. fem. let., along the cranial margin of the cap. fibulae over the lig. interartic. patellae towards the medial crural surface It passes just above the epic. fem. med. cranialward towards the symphysis, in the longitudinal axis of the upper-leg, right between the central and the medial third of the line connecting the ep. fem. med. with the symphysis and attains its highest point (ventrally and proximally) on the medial crural surface. Then it takes a caudal turn over the flexores of the knee describing an arch open on the posterior side along the flexores and the popliteal space on the medial surface of the forepart of the extremity, between malleolus med. and calcaneum it reaches the medial surface of the foot, passes on the sole, divides in two the large sole, and reaches the dorsum of the foot over the sole of the most medial (2nd) toe along the nail. Turning now cranialward, it crosses the dorsum and reaches medially along the malleolus the lateral portion of the extremity, following its longitudinal axis, but continually diverging laterally. Having reached the sulcus bicip. lateralis, it continues to below the dorsal and proximal point of departure, and then takes a sudden rectangular bend towards that point, rejoining it after 3 cm.

But the radicular area of *L*v has not always completely lost its connection with the mid ventral line of the trunk. Sometimes it reaches the m. v. l.; a connection with the m. d. l. however we found in no case.

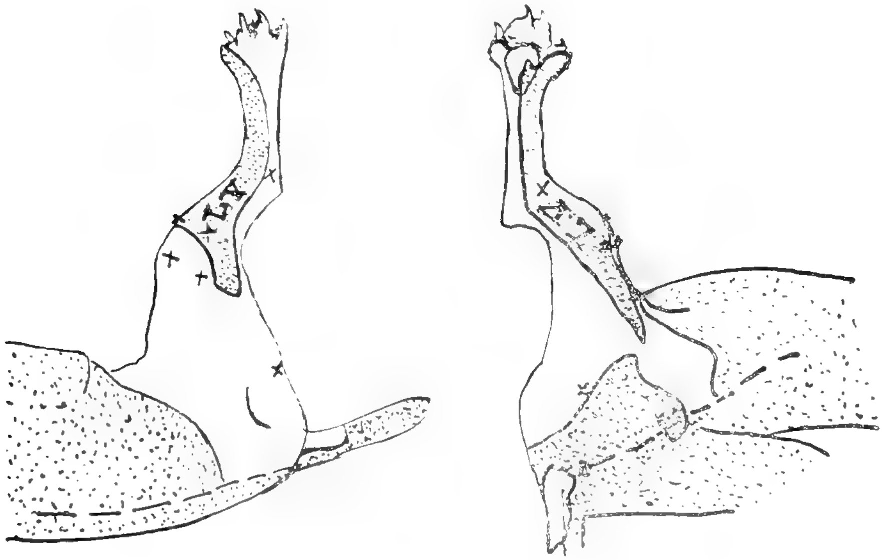
As an instance we give the following:

4. On dog 30 are cut through to the right: L_1—*L*iv inclusive, *L*vi—S_1 inclusive and *S*iii, whilst to the left are cut through S_2, S_3 and Co_1 (see Plate XI, fig. 3 and 4).

Within an insensible area on the extremity, bounded by the caudal margin of L_1 and the cranial margin of *S*ii, there is a sensible area bounded in the following manner: The boundary originates at a point situated on the lateral crural surface nearly at the midle of the line uniting ep. fem. lat. and trochanter. It bends cranialward to the superior margin of the patella, and at 5 cm cranially from the epic. medialis it goes linea recta to the superior margin of the vulva, where it meets the m. v. l. (cranial boundary).

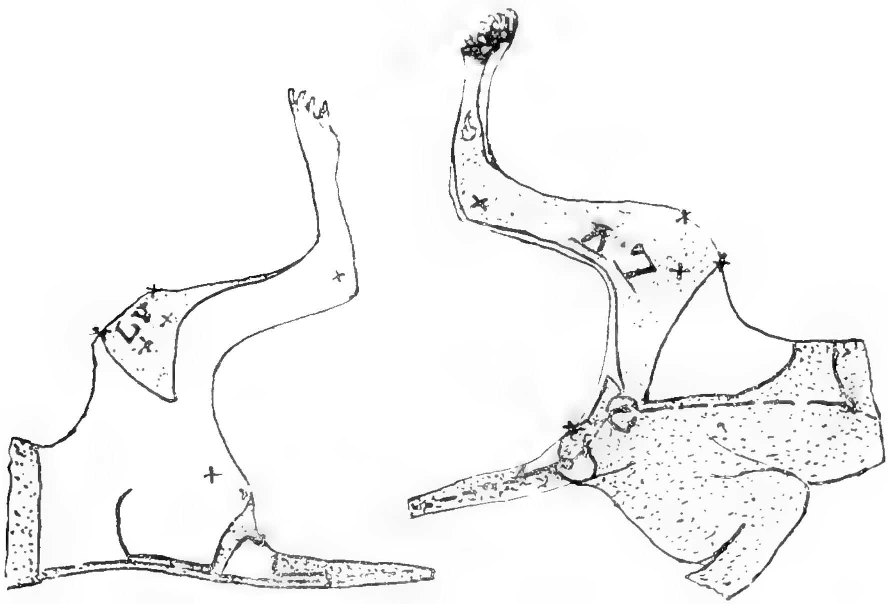
From the same (dorso-proximal) point originates the caudal boundary. It goes caudalward on the forepart of the extremity, lateral from epic. fem. lat. and capit. fib., and descends along the crista tibiae to the ankle-joint. Having crossed this between the malleoli, it continues on the limit between medial margin and dorsum of the foot towards the sole of the most medial toe, passes between nail and sole of this toe and running over the middle of the large sole it takes a bend proximalward and goes along the limit of plantar and medial surface of the foot, towards the fore part of the extremity. Having reached this between calcaneum and malleolus medialis, it continues along the medial margin of the tendo Achillis and over the medial surface of the peroneal muscles, along the popliteal space and crossing the medial crural surface it goes straight to the inferior border of the vulva, where it meets the cranial boundary of *S*ii.

Plate XI.
Lv. dog 25.



1 2

Lv dog 30



3 4

Isolation of the skinfield of *Lv* on dog 25, where to the right are cut through: L_I—L_{IV} inclusive and L_{VI}—S_I inclusive. 2 On dog 30 to the right are cut through: L_I—L_{IV} inclusive, L_{VI}—S_I inclusive and S_{II}, whilst to the left are cut through S_{II}, S_{VIII} and S_I.

Dog 25: 1 dorso-anterior portion of the foot. 2 ventro-posterior portion of the foot (plantar surface)

Dog 30: 3 lateral portion of the foot, 4 ventro-medial portion of the foot. On this dog the radicular area of *Lv* reaches the mid ventral line.

The skinfield of *L*v is the most difficult to understand, but at the same time the most interesting of the skin-areas of the posterior extremity.

It may be represented as a large triangle, whose base is stretched between two points. One of these points (the proximo-dorsal one) is situated on the lateral crural surface in the midst between trochanter and epic. fem. lat. The other (the proximo-ventral one) is found on the medial crural surface in the midle between symphysis and epicondylus femoris medialis. Its apex on the contrary is fixed against the sole of the most medial (2^{nd}) toe. The proximally and medially situated solitary nail (really the first medial toe) is constantly to be found within this area on the skin.

In this way the area is situated "à cheval" on the ligamentum interarticulare patellae. Its extension may vary considerably. The proximo-dorsal point sometimes may be in the centre of the lateral crural surface, and it may approach very closely the epicond. fem. lateralis. The proximo-ventral point may be found in the centre of the medial crural surface and it may appear removed unto the symphysis. In the latter cases the area on the skin from *L*v is connected with the m. v. l. A similar connection with the m. d. l. however was never found. Consequently this area cannot be conceived as one of the cranial marginal dermatomata like *L*II, *L*III and *L*IV, it rather shows a greater affinity with the apical or top-dermatomata, examples of which we shall learn to know *L*VI and *L*VII. Though a top-dermatoma, it nevertheless has also affinity with the basal or marginal dermatomata, because it still shows a propensity to retain its connection with the m. v. l. As it was already observed in *L*IV -- (viz. its feeble contact with the mid-dorsal line of the trunk, causing the dorsal portion to become insensible before the ventral portion and in this way inverting the rule, that we found in trunk-dermatomata), here the contact of the dorsal portion of *L*v with the mid-dorsal line of the trunk now is definitely lost. There is no longer any connection with the mid-dorsal line. Incidentally however the connection with the m. v. l. returns, though this likewise failed in most cases. This behaviour of the skinfield of *L*v indicates that, whenever there is spoken of axial lines or differential-lines, separating the cranial maginal dermatomata from the caudal marginal dermatomata, they must be very short at the ventral i. e. medial crural surface, much shorter than the dorsal ones, situated on the lateral crural surface.

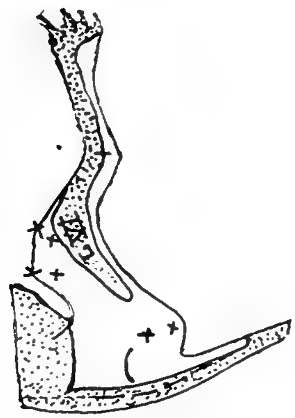
The skinfield of *L*vi.

This skinfield has been isolated 4 times by us, and we give again a few instances of isolation.

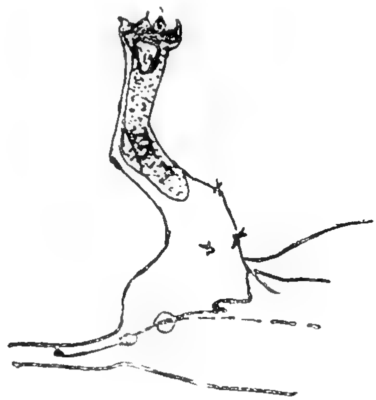
1. On dog 9 are cut through to the right: *L*ii to *L*v inclusive and *L*vii to *S*iii inclusive (see Plate XII).

The sensible area of *L*vi, enclosed within the analgetic area on the extremity, which is bounded by the caudal boundary of *L*i and the cranial boundary of Cocci,

Plate XII.
*L*vi dog 9.



1



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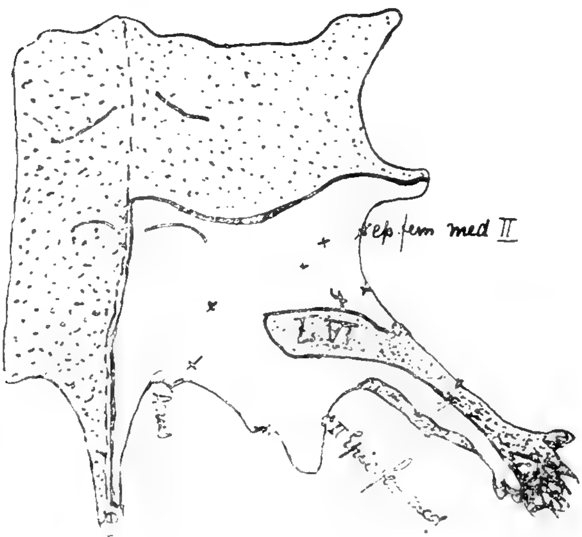


3

Isolation of the skinfield of *L*vi. On dog 9 are cut through to the right: *L*ii to *L*v inclusive and *L*vii to *S*iii inclusive. 1, latero-dorsal portion of the extremity. 2, ventro-medial portion of the extremity. 3, skin cut open along dorsal lines and stretched.

presents the following aspect: On the lateral crural surface a line may be drawn from the crista ilei to the patella. About 1 cm. distally from the middle of this line originates the area that is stretched into a pointed form. From this point of departure the boundary goes // to the line just mentioned towards the crista tibiae, reaching this distally from the tuberositas and descending along with it, it goes on the medial surface of the forepart of the extremity at the downmost third then following the medial side of the tendo Achillis, it goes between maleolus med. and calcaneum on the foot, passes on the medial margin of the palmar surface of the little (most lateral) toe and arrives on the dorsal surface of this toe.

Then it takes a cranial bent, returning over the dorsum of the foot along the malleolus lateralis, and along the lateral side of the tendo Achillis, over the peroneal muscles and the m. biceps lateralis, and converges towards its point of departure on the lateral crural surface This area is smaller than the area on the skin from *L*VI is found in most cases.

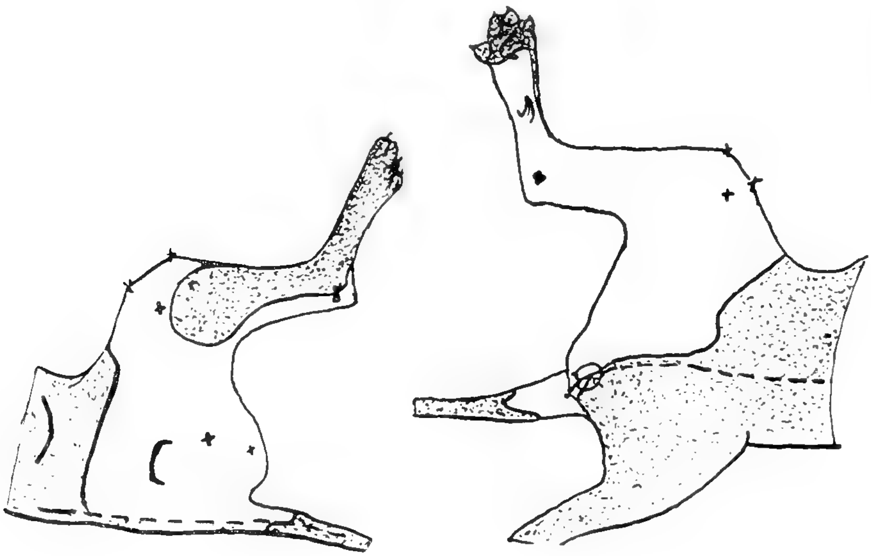
2. On dog 18 the area is already more extensive.

Here were cut through to the right: *L*III to *L*V inclusive and *L*VII to *S*III inclusive (see Plate XIII).

Within the analgetic surface of the extremity, bounded by the caudal boundary of *L*II and the cranial boundary of CocCI, a sensible area is found.

From a point on the lateral crural surface, 1 cm. caudal from the epicondylus femoris lateralis, originates a boundary which runs parallel to the line connecting the crista ilei with the patella It passes between capitulum fibulae and tub. tibiae and follows in distal direction the longitudinal axis of the forepart of the extremity, continually diverging medialward, until it arrives at $1^1/_2$ cm. distance from the malleolus medialis on the dorsum of the foot. It passes the isolate medial nail (of the hallux) at 1 cm. distance and takes a turn towards the medial surface of the foot on the first phalanx of the medial toe, crosses the large sole of the 4th and

Plate XIII.
*L*VI. dog 18.



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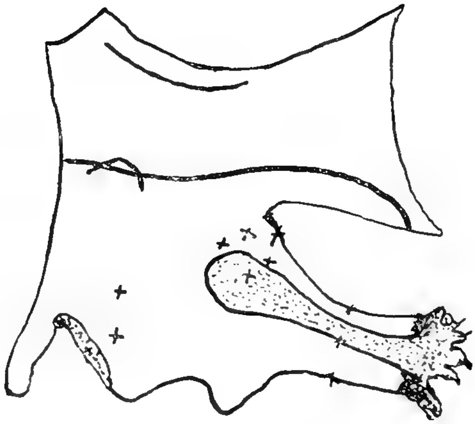


3

Isolation of the skinfield of *L*VI. To obtain this, on dog 18 to the right are cut through: *L*III—*L*V inclusive and *L*VII—*S*III inclusive Consequently we defined the caudal boundary of *L*II, the cranial boundary of *Cocc*I and the sensible area of *L*VI. 1 latero-dorsal portion of the extremity. 2 medio-ventral portion of the extremity 3 skin cut open along **ventral** lines and stretched.

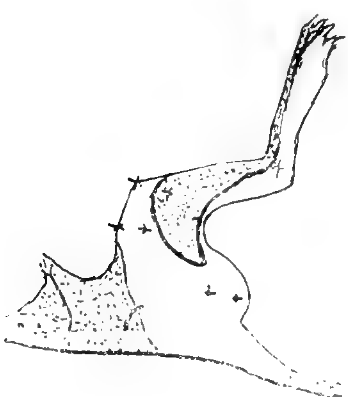
the 5th (lateral) toe, and then follows its course on the dorsal surface of this latter. It then takes a sudden proximal turn, goes on the limit between the dorsum of the foot and the lateral margin, over the malleolus lateralis, follows the longitudinal axis of the forepart of the extremity, along the tendo Achillis, unto the popliteal space, and returns converging with the cranial boundary to the point of departure

Still larger and extending further medialward on the foot than usually is *L*VI in the following case.

3. on dog 27, where to the right are cut through: *L*III—*L*V inclusive and *L*VII—*S*III inclusive. (see Plate XIV fig. 1 and 2).

On the analgetic surface of the extremity of dog 27 the area is bounded in the following manner: Originating at a point situated just distally from the epicond. med. fem., it takes between this point and the cap. fibulae a turn towards the

Plate XIV.
*L*VI. dog 27.



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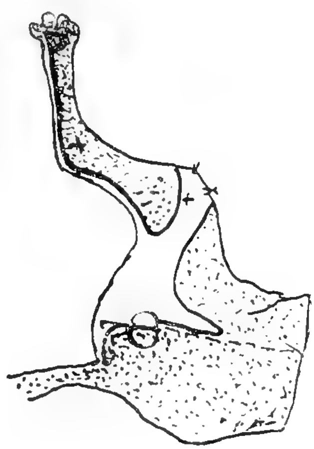
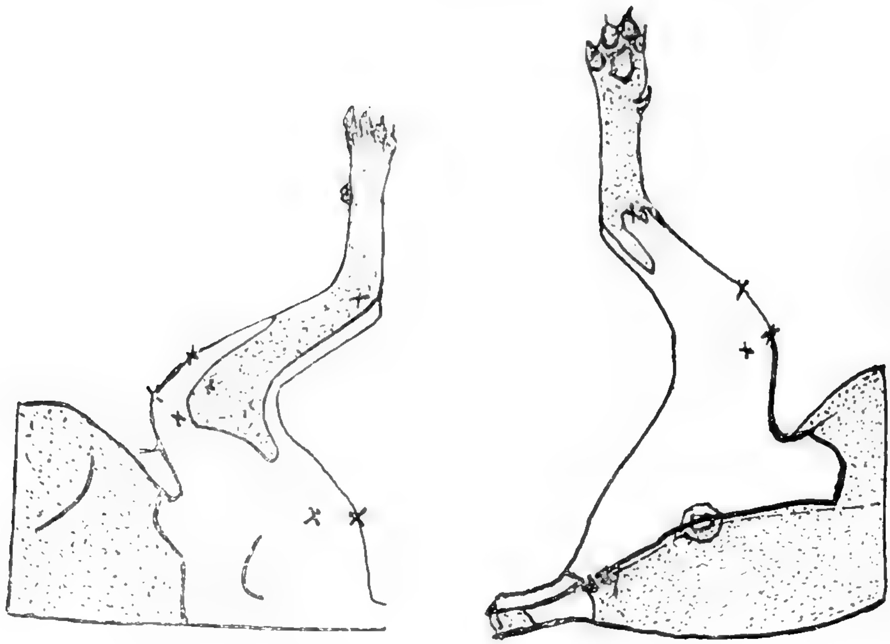


2

*L*vi. dog 32.



3 4

Isolation of the skinfield of *L*vi. On dog 27 *L*vi has been isolated by cutting through to the right *L*iii—*L*v inclusive and *L*vii—*S*iii inclusive. On dog 32 *L*vi has been isolated by cutting through to the right *L*i—*L*v inclusive and *L*vii—*S*iii inclusive. On dog 27 the area of *L*vi covers besides the dorsum of the foot a great part of the medial portion. On dog 32 the area of *L*vi covers nearly the whole foot. 1 and 3 latero-dorsal portions of these extremities. 2 and 4 their ventro-medial portions.

anterior surface of the forepart of the extremity, crosses this 1 c.m. distally from the tub. tibae, and goes 1 cm. distally from the epic. fem. medialis towards the middle of the medial crural surface.

It then takes a rectangular bend distalward to the popliteal space and goes over the tendo Achillis and the calcaneum on the palmar surface of the foot towards the middle of the large sole, reaches along this the second lateral (really the 4th) toe (whose lateral portion is analgetic) and passes on the dorsum of the foot. Continuing proximally over the malleolus lateralis on the forepart of the extremity, it diverges towards the popliteal region, goes to the middle of the lateral crural surface, ascends 2 cm. capitalward and returns to its point of departure along the line that was drawn from the trochanter to the epicondylus lateralis.

Still more extensive and encompassing the whole of the foot, is *L*vi in the following case.

4. On dog 32. Here to the right are cut through: *L*i—*L*v inclusive, *L*vii—*S*iii inclusive, Cocc II and to the left *S*i, *Cocc* i and *Cocc* ii. (see Plate XIV fig. 3 and 4).

Within the analgetic surface of the foot there is a sensible area bounded in the following manner: From a point, situated on the line connecting tuber ischii and epic. lat. femoris, in the midle between these, 6 cm. distally from the trochanter, the boundary goes // to the popliteal space, on the lateral surface of the forepart of the extremity, along the lateral part of the tendo Achillis, between malleolus ext. and calcaneum on the lateral surface of the foot, takes a rectangular bend on the palmar surface of the foot, then another rectangular bent, continuing for 3 cm. proximalward over the medial side of the calcaneum and along the medial side of the tendo Achillis. Then it returns // to itself distalward to the malleol. med., crosses this and goes now proximalward on the anterior surface of the forepart of the extremity, crosses the tibia, goes on the lateral surface of the forepart of the extremity between cap. fib. and tub. tibiae and passes at 1 cm. from the epic. fem. lat. into the line connecting this point with the tuber ischii and so returns to its point of departure.

The skinfield of *L*VI is a true top-dermatoma, and the most cranial one of these. It has no longer any connection neither with the mid-dorsal nor with the mid-ventral line. It covers the anterior surface of the forepart of the extremity, the dorso-medial, but also the medio-palmar surface of the foot. Thus it appears wound spirally around the extremity. Moreover its extension and likewise its situation are very variable. These apparently enormous alternations however are less capricious than they appear at first sight. They leave on the proximo-medial portion of the foot a space for the skinfield of *L*V that supplies the medial covering of the foot. The dorso-lateral and the latero-palmar portion of the foot remain to be covered by the area of *L*VII. Even in cases, when this radicular area, beginning from the dorso-medial side (as e.g. on dog 32) encompasses the whole foot, we still may find the extrance of *L*V on the medial surface of the foot and of *L*VII on its lateral and palmar surface, expressed by the peculiar bayonet-shaped lines of the boundaries. The radicular area of *L*VI is a very variable one. It swings as it were from one side to the other perpendicular on the longitudinal axis of the forepart of the leg.

It retains a relative fixation at the lateral crural side, and swings over the anterior part of the foreleg and the medio-dorsal part of the foot. In most cases it covers the foot far to the medio-palmar side, sometimes passing thence on the lateral side, and in a few cases on both sides, thus encompassing the whole foot. But in its swinging across the foreleg and foot, it is inseparably united with its neighbours *L*V and *L*VII. Its variations are limited by the boundaries set by *L*V and *L*VII, even as the variable areas of them both, which swing from one side to the other in the same way, are limited by the variations of *L*VI.

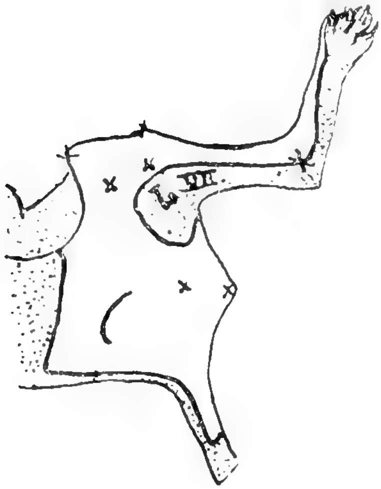
The skinfield of *L*VII.

This area has been completely isolated by us 5 times, and we will again in the first place give a few instances of its isolation.

1st. On dog 12. Here to the right were cut through: *L*II—*L*VI inclusive, S_1—$Cocc_2$ inclusive. (See plate XV).

On the analgetic surface of the extremity we find a sensible area bounded in the following way: On the line drawn between tuber ischii and epic. fem. lat.,

Plate XV.
*L*VII dog 12.



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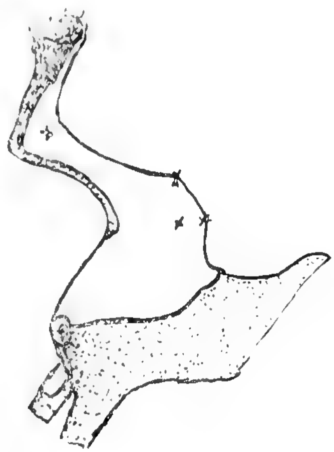

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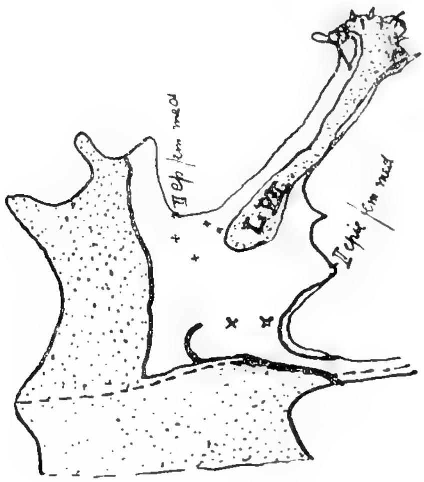


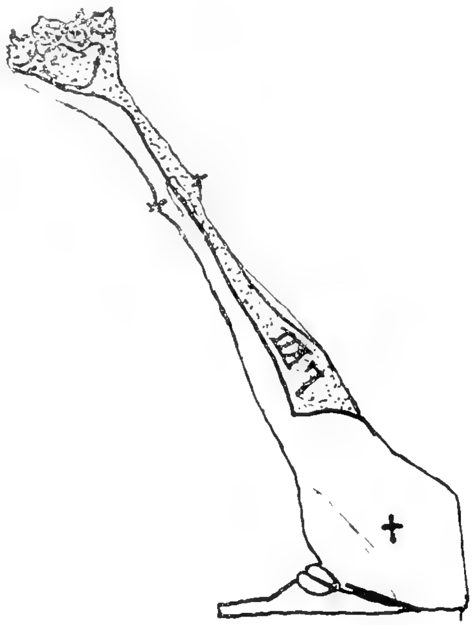
3

Isolation of the skinfield of *L*VII. On dog 12 to the right are cut through: *L*II—*L*VI inclusive, S_1, II, III and Cocc. I and II. 1. dorso-lateral portion of the upper-leg and the dorsum of the foot. 2. ventro-medial portion of the upper-leg and palmar surface of the foot. 3. skin cut open along **ventral** lines.

there is a point situated at the limit between its central and lateral third 1.5 c.m. dorsally from the femur. The boundary begins at this point, goes towards the epic. fem. lat. and at 1.5 c.m. distally from it, takes a turn to the forepart of the extremity, 1 c.m. distally from the capitulum it reaches the fibula, follows this and after crossing the mall. lateralis reaches the dorsum of the foot, crosses this, directed towards the 4th (most lateral) toe, passing over the lateral border of this

Plate XVI.
Lvii. dog 20.

1



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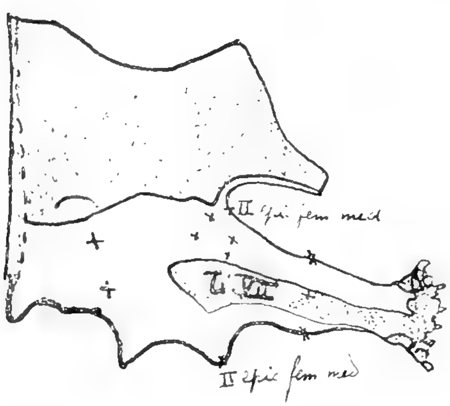


3

Isolation of the skinfield of *L*vii. On dog 20 are cut through *L*iv—*L*vi inclusive, and *S*i – Cocc.ii inclusive. Defined were in this way the area on the skin from *L*vii. 1. lateral and dorsal portion of forepart of the extremity and of the foot. 2. ventral portion of both. 3. skin cut open along ventral lines and stretched.

toe, it reaches the plantar border of the foot, and goes over the large sole straight over the plantar surface of the foot to the calcaneum. It then goes along the tendo Achillis towards the popliteal space and returns over the lateral crural surface to its starting-point, coming to 3 c.m. distance of the trochanter.

2nd. On dog 20 to the right are cut through: *L*IV—*L*VI inclusive and *S*I—Cocc. I inclusive. (See plate XVI).

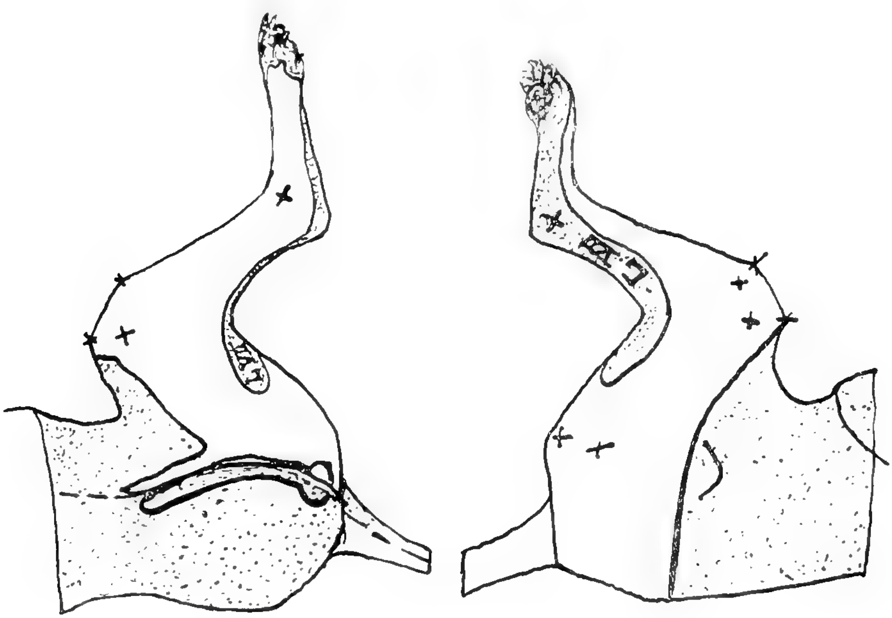
On the analgetic surface of the extremity there is a sensible area, bounded in the following manner: The starting-point of the boundary is in the middle of the line connecting tuber ischii and epicond. fem. lat., 5 c.m. distant from either. From this point it goes on the lateral crural surface to within 2 c.m. distance of the epic. fem. lat. and going 2 c.m. laterally from the cap. fib. it takes a turn distalward between fibula and tendo Achillis, crosses the mall. lateralis, and when it has reached the boundary between the dorsum and the lateral portion of the foot, it follows this unto the 4th (most lateral) toe; then crossing the dorsum of the foot, it diverges medialward (all the webs are sensible) to the medial portion of the foot on the plantar side; and goes medial from the large sole over the plantar surface to the calcaneum, along the tendo Achillis through the popliteal space to the lateral crural surface. About 3 c.m. below its origin it takes a sudden bend and returns to it.

3rd. On dog 27 to the left are cut through *L*IV—*L*VI inclusive and *S*I—Cocc. I inclusive. (See plate XVII fig. 3 and 4).

On the analgetic surface of the extremity, which is bounded cranially by the caudal margin of *L*IV we find a sensible area bounded in the following manner: From a point, situated on the dorsal surface of the extremity, near the capitulum fibulae, the boundary goes medially towards the anterior surface of the forepart of the extremity, descends over the calf of the m. tib. antic., passes the ankle-

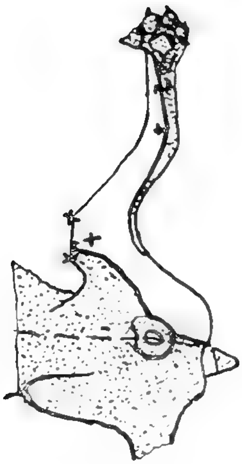
Plate XVII.

*L*VII. dog 33 to the left.



1 2

*L*vii. dog 27 to the left.



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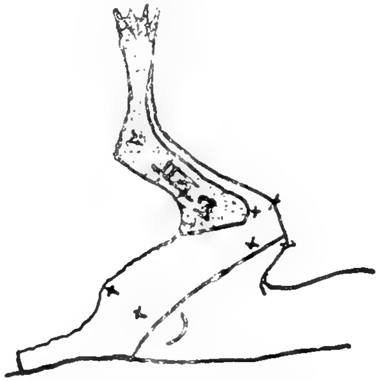


4

Isolation of the skinfields of *L*vii on dog 33 and dog 27.
On dog 27 to the left are cut through *L*iv—*L*vi inclusive and *S*i—*Cocc*i inclusive.
On dog 33 to the left are cut through *L*iii - *L*vi inclusive and *S*i—*Cocc*i inclusive.
Fig. 1 and fig. 3 the ventral portion of leg and foot.
Fig. 2 and fig. 4 the lateral portion of the extremity with the plantar portion of the foot turned towards the reader.

joint between the two malleoli, crosses the dorsum of the foot medialward and near the basal ossicle of the 2nd medial toe, it takes a definite turn to the medial surface of the foot, medially from malleolus medialis along the medial side of the tendo Achillis, follows the popliteal region unto the inferior third of the medial crural surface, then it takes a rectangular bend and returns to its origin on the femur.

4. On dog 33 to the left are cut through *L*iii—*L*vi inclusive and S_2—Cocc 3 inclusive. (See Plate XVII, fig. 1 and 2).

On the analgetic surface of the extremity, bounded by the caudal margin of *L*ii, there is a sensible area bounded in the following manner: The boundary originates from a point 1 cm. distally from the middle of the line, connecting trochanter and epic. fem. lat., 5 cm. distant from either. It turns distalward on the lateral surface of the forepart of the extremity, 3 cm. from the cap. fib., between fibula and tendo Achillis. Arrived on the anterior surface it passes the ankle joint medially from the mall. lat. It crosses transversally the dorsum of the foot, goes along the most medial toe on the palmar surface, takes a proximal turn medially from the large sole and follows the medial margin of the foot, the medial side of the calcaneum and the tendo Achillis to the knee. Arrived on the medial crural surface it takes a turn, goes in the direction of the anus towards the middle of the femur, passes the posterior crural surface, takes again a bent and returns to its point of egression along the line connecting tuber ischii and cap. fib.

Thus the skinfield of *L*vii, like that of *L*vi is a top-dermatoma. Whilst *L*vi covers the anterior surface of the forepart of the extremity, and the dorso-medio-plantar surface of the foot, the skinfield of *L*vii covers the posterior surface of the forepart of the extremity, and the dorso-latero-plantar surface of the foot. It is the caudal top-dermatoma. Even as in covering the medial portion of the foot *L*vi is supported by *L*v, which in this regard, may be called the *cranial* marginal dermatoma that advances farthest, so *L*vii is supported by the nearest, most advancing *caudal* marginal dermatoma *S*i and this latter aids in covering completely the lateral portion of the foot.

Only after having treated the skinfield of *S*i too, it will be possible to give a survey of the different ways, by which *L*v, *L*vi, *L*vii and *S*i support each other in covering the foot.

What distinguishes however the top-dermatomata from the basal or marginal ones is the manner in which they swing from one side of the foot to the other perpendicularly on the longitudinal axis of the extremity. From *L*v and *L*iv, *L*iii and *L*ii we see longer or shorter tongues or flaps projected in distal direction. When in *L*v appears its significance as a marginal dermatoma, we observe the shorter or longer tongue issuing towards the 1st medial toe (the solitary nail), but we also see *L*v displacing itself transversally over the forepart of the leg ("à cheval" on the knee). In this way it has also a significance, and a much greater one as a top dermatoma.

The skinfield of *S*i.

This skinfield has likewise been isolated several times, and we will first give a few instances.

1. On dog 33 to the right were cut through: *L*iii—*L*vii inclusive, *S*ii—Cocc. III inclusive. (See Plate XIX, fig. 1 and 2).

On the buttock of the extremity, that is analgetic, as is likewise the tail, we find a sensible area, connected both with the mid-dorsal and with the mid-ventral line. This area is bounded in the following way:

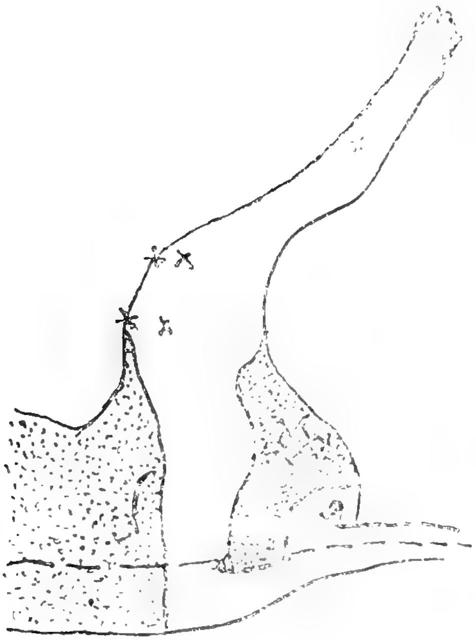
The cranial boundary begins near the sacrum forming with the m. dors. line an angle of 60° opening caudalward, passes over the trochanter, follows on the lateral crural surface the line connecting trochanter and epic. fem. lat. unto the middle of it, takes a turn towards the popliteal space, passes likewise in the middle the line connecting tuber ischii and cap. fib., reaches the popliteal fold at its highest point, goes linea recta through the groin towards the symphysis, 2 cm. above this latter it comes to within 2 cm of the m. v. l. and ascending // with this it reaches the caudal boundary of *L*ii.

The caudal boundary leaves the mid-dorsal line at the root of the tail forming with it an angle of 45° opening caudalward, encircles the anus at a distance of 3 cm., passes in the middle between anus and tuber ischii, reaching the perineum 1 cm. above the anus. In the mid-ventral line (to the left all nerve-roots, *L*vii excepted, have been cut through) it goes straight over the scrotum, the skin-fold of the praeputium and passes there into the left caudal boundary of *L*ii.

2. On dog 21 are cut through *L*iii to *L*vii inclusive, *S*ii to Cocci inclusive (see Plate XIX fig. 3 and 4).

Plate XIX.

S_{I}. dog 33 to the right.



1

2

S_1. dog 21 *R*.

3

4

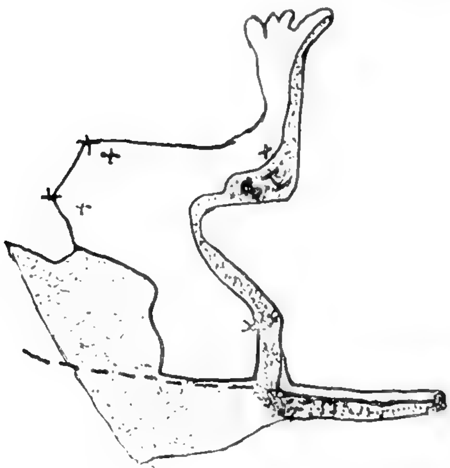
Isolation of the skinfield of S_{I}. On dog 33 to the right are cut through L_{III}—L_{VII} inclusive and S_{II}—$Cocc_{\text{III}}$ inclusive. To the left are cut through L_{III}—L_{VI} inclusive, and S_1 - $Cocc_{\text{III}}$ inclusive. So to the right the caudal margin of L_{II} and the radicular area of S_1 have been defined. This area is very small but the caudal margin of L_{II} also descends very low. The area of S_{I} is connected ventrally with that of L_{II}, dorsally it is not the case. On dog 21 to the right are cut through: L_{III}—L_{VII} inclusive, S_{II}—$Cocc_{\text{I}}$ inclusive. Here the area of S_{I} is larger, at the same time the caudal margin of L_{II} remains higher. Fig. 1—2 both sides of the extremity on dog 33. Fig. 3—4 both sides of the extremity on dog 21.

On the posterior side of the analgetic extremity is a sensible area, bounded in the following way.

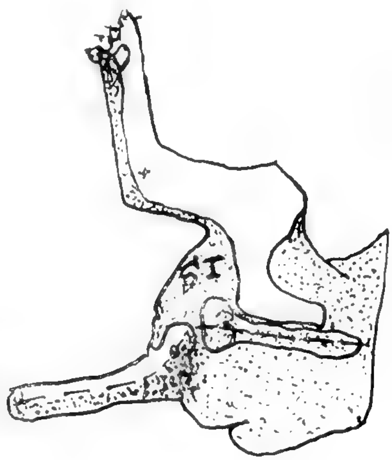
The cranial boundary starts from the m. d. l. forming with it an angle of nearly 45°, passes over the trochanter, thence to the middle of the line connecting tuber ischii and cap. fib., into the lateral sulcus of the biceps, along the lateral margin of the popliteal space, goes on the lateral border of the tendo Achillis, along the lateral border of the calcaneum, on the plantar surface of the foot, stretched to a fine point advancing on the middle of the foot. Then it takes a sudden proximal turn, diverges towards the medial border of the calcaneum, and

Plate XVIII.

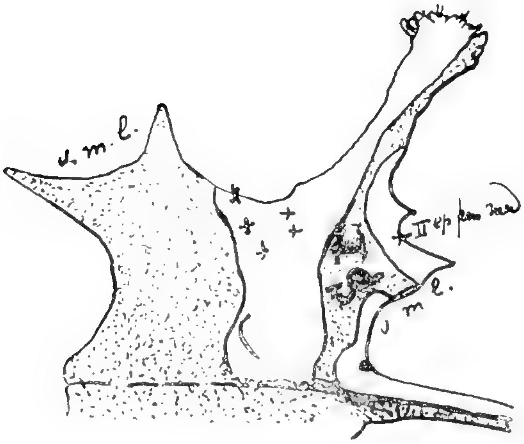
*S*I. dog 11.



1



2



3

Isolation of the skinfield of *S*I. On dog 11 to the right are cut through *L*III—*L*VII inclusive and all the Coccygei. The skinfield of *S*I covers the lateral plantar portion of the foot. Fig. 1 dorso-lateral portion of foot and extremity. Fig. 2 ventral portion of both. Fig. 3 skin cut open along ventral lines and stretched.

goes along the medial margin of the tendo Achillis, along the popliteal space towards the medial sulcus of the biceps, passing transversally over the medial portion of the upperpart of the extremity, over the adductores and the groin, towards a point 2 cm. above the symphysis, it comes to within 1 cm. from the m. v. l. and ascending // to it, passes into the sensible caudal border of *L*II.

2 cm. below this line the caudal boundary departs from the m. d. l., a little above the root of the tail, forms with the m. dors. l. an angle of 40° opening caudalward, passes just below the tuber ischii and reaches the perineum $^1/_2$ cm. caudally from the vulva, bends along the m. v. line, along the anus, and goes straight over this latter and the m. v. l. on the tail unto the 3rd Cocc. vertebra where it takes a rectangular bent towards the m. d. l.

In the following case the radicular area from *S*I appears still larger.

3. On dog 11 to the right are cut through *L*III to *L*VII inclusive, *S*II and the Cocc, to the left *S*I (see Plate XVIII).

The cranial boundary departing from the m.d.l. departs at an acute angle opening caudalward, and goes over the trochanter and along the femur unto the middle of the lateral crural surface. In the s. bic. lateralis it crosses this muscle and the peroneal muscles, goes laterally from the tendo Achillis near to the mall. lat. on the dorsum of the foot unto the lateral toe, and takes a turn on the plantar side of the foot, along the linea interdigitalis, between fourth and third toe, and even over this. Then it returns proximalward, diverging laterally over the large foot-sole, over the plantar surface of the foot, and reaches the medial surface of the underleg between calcaneum and malleolus med. Having reached along tendo Achillis, peroneal muscle and popliteal space the medial crural surface, it goes straight towards the middle of the scrotum, encircles this and ascends 1.5 c.m. from the mid-ventral line and from the praep., passing towards the orefic. urethrae into the caudal boundary of *L*II. The cranial boundary leaves the m.d.l. under an acute angle opening caudalward, near the root of the tail, passes over the tuber ischii to the perineum, and having reached this right in the middle between scrotum and anus, it takes a turn along the m.d.l., straight over the anus, continues along the tail and returns at its end (9th coccyx-vertebra) to the m. d. line.

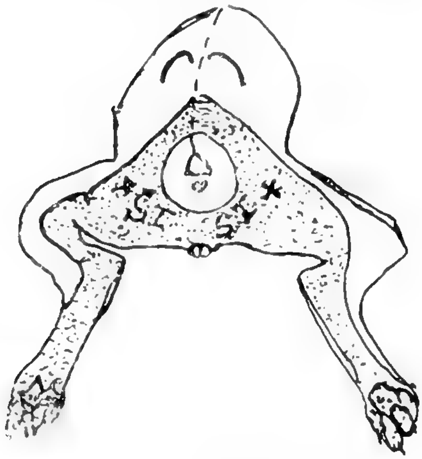
The skinfield of *S*I is the most cranial of the *caudal* marginal dermatomata, the one advancing farthest on the extremity. It is connected both with the mid-dorsal and the mid-ventral line. At the mid-ventral line the ventral portions of the dermatomata are lying one upon another and overlap very much. Therefore at the m. v. l. the ventral cranial margin of *S*I is still connected with the caudal margin of *L*II. At the mid-dorsal line such is not the case: There is a gap between *L*II and *S*I.

The area of *S*I provides in the innervation of the external genital skinfolds, but it does not reach the anus.

This becomes most evident when all the nerve-roots caudal from *S*I are cut through on both sides.

On dog 29 all roots below S_1 have been cut through, in order to isolate *S*I + *L*VII (see plate XX).

Plate XX.
On both sides *S*I + *L*VII. dog 29.



On dog 29 to the left and to the right the skinfield of *L*VII + *S*I have been isolated by cutting through *L*IV—*L*VI inclusive, *S*II. *S*III and a few of the Coccygei. Around the anus there is an analgetic area.

There is a caudal analgetic area, encircling the tail that is very short, and the anus. This area touches the perinaeum at 1 cm. from the anus, passes in the middle between anus and tuber ischii 2 cm. distant from both, and ends 4 c.m. above the anus, above the root of the tail.

The extension of the radicular area of *S*I varies very much. Situated on the buttock it is linguiform, and the stretching of its tongue may be very different. It may be found extending unto the popliteal region, or unto the posterior surface of the forepart of the extremity, or unto the palmar and lateral portion of the foot or even unto the sole of the lateral toes. Evidently *S*I aids *L*VII in covering the latero-plantar portion of the foot, just as *L*VI is aided by *L*V in covering the medio-plantar portion of the foot.

It can never however cover together with *L*V or with *L*VII the whole of the foot, because the dorsal portion of the dorsum of the foot needs *L*VI to cover it.

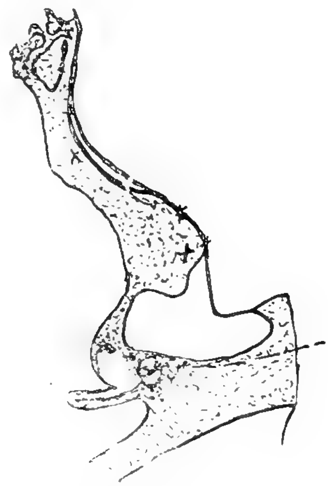
As an instance of this will serve the joint isolation of *L*V and *S*I.

On dog 28 *L*I—*L*IV inclusive, *L*VI, *L*VII, *S*II, *S*III and a few of the Coccygei were cut through (see plate XXI).

On the extremity there is a sensible area covered in the following manner. The cranial boundary leaves the m. d. l. $4^1/_2$ c.m. below the crista ilei describing an angle opening caudalward, it goes over the trochanter on the lateral crural surface towards the popliteal space, passes in the middle between cap. fib. and popliteal fold on the lateral surface of the forepart of the extremity, then medially from the mall. lat. first on the lateral and then on the medial surface of the foot, in the direction of the most medial toe. Along the webs it then continues proximally

Plate XXI.
Si + *Lv*. dog 28.

1



2

Joint isolation of the skinfield of *Lv* and *Si*. To obtain this *Li*—*Liv* inclusive, *Lvi*, *Lvii*, *Sii*, *Siii* and a few of the coccygei were cut through on dog 28, fig. 1 dorsal portion fig. 2 ventral portion of the extremity. *Si* + *Lv* cover completely the palmar portion of the extremity.

over the medial side of the foot and passes medially from the malleolus medialis on the medial surface of the under-leg, then to its anterior surface and having crossed this, it returns between cap. fib. en epic. fem. lat. to the medial crural surface and goes through the groin to the mid-ventral line. Opposite the vulva there is a dubious zone, through which it becomes difficult to follow the line when, ascending // to the m. v. l. it passes into the caudal boundary of Th XIII.

The caudal boundary originates at the d. m l. near the root of the coccyx, describing an extremely acute angle towards the tub. ischii, passes above this latter, encircles the anus at a distance of 4 c.m. going to the perineum where it remains separated from the genital tuberosity by another dubious zone.

On the contrary *Lv* + *Lvi* is capable of covering the whole of the foot, and even the single skinfield of *Lvi* may do this.

Lv + *Lvii* may under circumstances be capable of doing it, but not so *Lvii* + *Si* (see dog 29).

It follows that the covering of the foot is provided for by the two top-dermatomata, *Lvi* + *Lvii* aided by *Lv* and by *Si*; it may oftentimes occur that *Si* does not take any part in it.

For the covering of the foot *Lvi* is of the greatest importance. It covers the foot dorso-medially. Next to it follows *Lvii*, which covers the foot dorso-laterally. The manner in which both partake in the covering of the plantar surface, may vary. In most cases the part taken by *Lvi* in this, is of more importance than that of *Lvii*.

Generally *L*v aids more or less to the medio-ventral covering. The latero-ventral covering is often supported by *S*i, but not in all cases.

*The skinfield of S*ii *and S*iii.

The area skinfield of *S*ii has been alluded to already a few times, and in Plate XI on dog 25 and dog 30 a design has been given of it. It is presented now, photographed under a somewhat different aspect on dog 25.

On dog 25 *L*i—*L*iv inclusive, *L*vi, *L*vii, *S*i, *S*iii and some of the Coccygei have been cut through. (See plate XXII fig. 1 and 2).

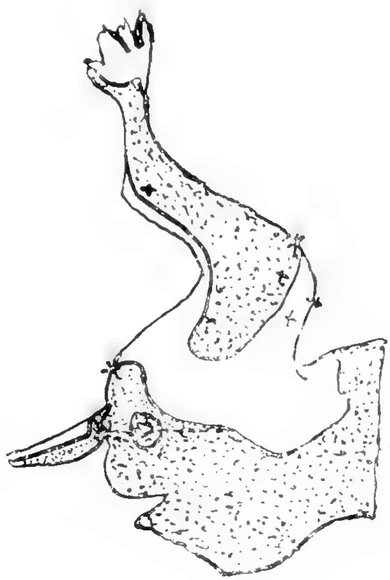
On the analgetic extremity we find, besides the sensible area of *L*v, a small sensible area near the nates, bounded in the following manner:

The cranial boundary originates near the root of the tail and forms with the m. dors. line an exceedingly acute angle that opens caudalward, it goes to the tuber ischi which remains 1¼ c.m. dorsal from it, 5 c.m. from the anus it sends a small tongue on the nates and then taking a turn it goes straight towards the

Plate XXII.

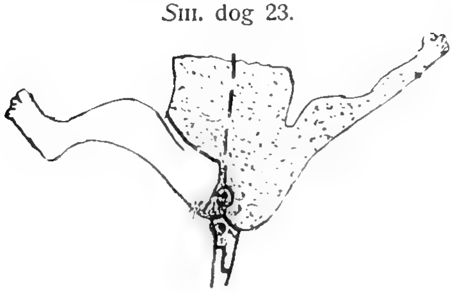
*S*ii. dog 25.

1



2

*S*iii. dog 23.



3

4

5

Isolation of the skinfield of *S*II and of *S*III. On dog 25 to the right are cut through: *L*I–*L*IV inclusive, *L*VI—*S*I inclusive, *S*III and a few of the Coccygei. In this way *S*II has been isolated. Fig. 1 and 2 dorsal and ventral portion of the extremity. On dog 23 to the right are cut through: *L*I—*S*II inclusive, and a few of the coccygei. In this way *S*III has been isolated. Fig 4 and 5 dorsal and ventral portion of the extremity; fig. 3 aspect ot anus and vulva.

symphysis, thus passing into the mid-ventral line (the whole of the vulva is sensible). The caudal boundary originates on the dorsum of the tail near the first cocc. vertebra and turns rectangularly on the m d. l. towards the mid. ventral l., reaching it 1 c.m. distalward from the anus.

2nd. On dog 30 to the right have been cut through: *L*I—*L*IV inclusive, *L*VI—*S*I inclusive, *S*III and a few of the Coccygei (on both sides).

On the analgetic extremity there is, besides the sensible area of *L*V, (see dog 30 Plate XI fig. 3 and 4) a caudal sensible area of *S*II on the nates.

The cranial boundary leaves the m. d. l. near the root of the tail under an angle of 30° opening caudalward, it goes in caudal direction 1 cm. below the tuber ischii, between this and the anus, remaining 4 cm. distant from it. It passes between tuber ischii and vagina about 5 cm. on the posterior surface of the upper leg, takes a rectangular bend, and 1 cm. farther another, then going in the direction of the symphysis it approaches the genital tuberosity to within 3 cm. distance and a little above the symphysis it passes under an angle of 45° into the m. v. l.

The caudal boundary of the zone of *S*II originates 1 cm. distally from the cranial one at the m. d. l. describing an extremely acute angle opening caudalward, it first goes somewhat caudally, and then takes a turn straight towards the anus, where it passes into the m. v. l. The cranial border of the anus is sensible, the caudal border analgetic. The mucous membrane is sensible everywhere.

The skinfield of *S*III has likewise been designed already a few times. A few more instances follow here.

1. On dog 23 to the right are cut through: *L*I—*S*II inclusive and the two nearest coccygei. (See Plate XVII, fig. 3, 4 and 5).

On, the analgetic surface of the extremity and the tail there is a sensible area bounded in the following manner:

The cranial boundary originates at the inferior border of the sacrum, leaving the m. d. l. under an angle of ± 25°, it goes caudally from tuber ischii and anus, and takes a bend towards the mid-ventral line halfway the perineum between anus and vulva, then it ascends again and reaches near the vulva the m. v. l. (the genital tuberosity).

The caudal boundary originates at the 2^{nd} coccygeus $1/_2$ c.m. from the mid-dorsal line and takes a rectangular bend towards the m. v. l., which it reaches 2 cm. distally from the anus.

2. On dog 22 to the right are cut through: *L*VI, *L*VII *S*I and *S*II, Cocc. I, II and III.

There is a sensible area on the buttock.

The cranial boundary originates at the m. d. l. near the root of the tail, leaving it under an extremely acute angle opening caudalward, it passes between tuber ischii and anus, 1 cm. distant from the former, and then in a straight line towards the scrotum, where it meets the m. v. l. in the inferior third. The caudal boundary leaves the m. d. l. between the 2^{nd} and the 3^{rd} cocc. vertebra at $1/_2$ cm. distance, describing a right angle, encircles the tail and reaches the m. v. l. 3 cm. distally from the anus.

*S*II and *S*III are the two last, most caudal nerve-roots that participate in the innervation of the skin on the extremity. Their areas on the skin, together with that of *S*I form the three caudal specimina of basal or marginal dermatomata. Following the indications, furnished by the preceding researches, we intend erelong to publish some conclusions about the manner in which these different areas are ranged on the skin and their reciprocal variations.

ERRATUM.

In the Proceedings of the meeting of May 1910:

p. 63, l. 15 from the bottom: the fraction in the second member of the equation has to be multiplied by $\frac{\partial \pi}{\partial r}$ (cf. p. 62 equation (4)).

(September 1, 1910).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING of Saturday September 24, 1910.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 24 September 1910, Dl. XIX).

CONTENTS.

Physiology. — "*Physiological sclerometry*". By Dr. A. K. M. Noyons. (Communicated by Prof. H. Zwaardemaker).

(Communicated in the meeting of March 26, 1910).

Already in a former communication [1]) I found an opportunity to draw the attention to the quality of the physiological hardness so little studied, in casu the hardness of the muscles. Then it was pointed out how hardness is a collective idea, comprising and typifying in mineralogy an amount of qualities such as cohesion, elasticity, plasticity, gliding, splitting, and fracture. For physiological purposes it is only the three qualities: elasticity, plasticity and cohesion which naturally come under discussion. Hence the definition of physiological hardness and the method of determining it must be based upon the three above mentioned qualities. Therefore it appears on second thoughts that Auerbach's [2]) definition of mineralogical hardness cannot unaltered be transferred to the physiological hardness, when he says: "Härte is eine Art von Festigkeit nämlich der Widerstand gegen die Bildung von Unstetigkeiten oder dauernden Deformationen beim Drucke zweier sphärischer Oberflächen gegen einander und kann Eindringungsfestigkeit genannt werden . . . Sie ist quantitativ durch den Grenzeinheitsdruck im Mittelpunkte der Druckfläche bestimmt."

Definition and method of determination start from the principle, — which is hardly ever desirable for physiological cases — that the moment at which a permanent deformation appears, is used as a criterion. This permanent deformation is in mineralogy required for the determination of *absolute* hardness. For physiological objects, therefore, we have to look out for another principle.

In the above communication I described a suitable apparatus, by means of which differences of hardness in muscles under different circumstances could be pointed out. The oscillations of a little falling hammer beating a muscle, is under certain premises a measure for the hardness. In judging about the hardness of the object that is to be examined, various data in the photographs may be taken into account:

1. the total number of reverberations;
2. form and height of each reverberation separately;
3. amount of the heights of all reverberations;
4. the process of penetration of the hammer into the object, which is derived from the situation of the lowest points of the sclerometric figure.

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[1]) Noyons, A. K. M. About the determination of hardness in muscles. Kon. Akad. v. Wetensch. Amsterdam. Proceedings of the meeting of June 12, 1908.

[2]) Auerbach, Fel. Kanon der Physik. p. 119. Leipzig 1899.

With the aid of the above-mentioned apparatus, for which I choose the name of "ballistic sclerometer" to discriminate it from another apparatus that I am going to describe hereafter, changes of hardness were proved among other in muscles through which passed a galvanic current, in muscles moistened with different isotonic salt-solutions, in m. gastrocnemii excited to tetanus without being capable to contract, lastly with muscles exposed to different temperatures. Also objects of quite different nature and hardness present, at an examination with the ballistic sclerometer, fine differences of hardness. The subjoined figures may throw some light upon this. The objects were examined with a hammer of the same weight, falling constantly from the same height, whilst in the perpendicular position the convex surface of the hammer at rest just touched the surface of the object. Fig. 1 gives the ballistic sclerogram of an Hirudo medicinalis, killed by being kept for some hours in ethervapour. Fig. 2 gives an image of the hardness of the m. gastrocnemius of Rana esculenta an hour after its death cut out. Fig. 3 is the ballistic sclerogram of the eye of the hog some hours after death.

Gelatin-plates of the same size and the same thickness but each time of a different concentration, and which at digital touching show quite certain sensoric differences of hardness, may be tapped with the ballistic sclerometer. In accordance with the concentrations we then find differences in the ballistic sclerograms of those plates. The disk-shaped gelatin-plates have an area of 47.3 cM^2 and a thickness of 2.11 cM.

GELATIN-PLATES DETERMINED WITH BALLISTIC SCLEROMETER.

Gelatin-concentration	Transparency [1])	Number of Reverberations	Amount of the heights of all reverberations	Average height of a reverberation
2%	195	25	37.3 cM.	1.49 cM.
3	185	29	44.6	1.53
4	181	31	49	1.60
5	176	34	59.2	1.74
6	168^5	36	72.4	2.01

[1]) Transparency was determined by examination whether a letter-type $D = 1$ of SNELLEN's optotypes was still to be recognised when viewed through the gelatin-plate and two smoke-glasses.

The figures are nothing but the average product of the numbers of the two smoke-glasses, which every time were wanted by twos from always the same series of smoke-glasses, in order to reach the limit of recognition of the letters.

Though, as appears among others from the above table, the sclerometric curves denote the mutual differences of hardness very accurately, yet the ballistic method cannot be used to determine the absolute hardness, as it occurs that objects of different structure, which seem to be equally hard by digital touching, make a deviating impression with the ballistic sclerometer. This is because in the ballistic sclerometer it is especially the elasticity of the object that comes to the front. If this is taken into consideration, the method is suitable to the circumstances of the case.

Yet this fact made me look out for another method which was perhaps to show proportionate data at digital touching on the one side and at the sclerometrical examination of the same object on the other.

Mineralogy has at its disposal numbers of methods of a static nature which are not to be used for physiological purposes, so long as one sticks to the permanent deformation as a criterion for hardness. However, another criterion may be used and as a measure for the hardness may be taken the depth of the penetration of a certain object into the object that is to be examined, whilst this penetrating object is charged in a definite way.

This principle somewhat reminds of the principles applied at the determination of hardness in mineralogy according to Brinell and Ludwik [1]). In this method we may also speak of absolute hardness, provided the data are every time reduced to the corresponding results arrived at in a material which is considered as unity of hardness. In the static sclerometer the principle is applied as follows. A cone of ebonite hangs by means of a little bar which can move without any incorrect movement, on one arm of a little lever. This same arm of the lever bears a hook in order to hang up different weights, and further a weak iron plate, which by means of an electro-magnet, fed with 4 à 6 volt., can be held fast, so that the cone is prevented from indenting the object. When the current is broken, the cone sinks into the object that is to be examined and the extent of this indenture is indicated by the other arm of the lever magnified 30 times, either by simple reading of the position of the lever along a measuring-lath or by registration on a kymographion or by fixation by means of photography. The registering part of the apparatus with the cone can be moved up and down by means of a metal ring, so that at the determination we first bring about and start from, the condition in which the cone just touches the surface of the object to be examined without any

[1]) See Dr. Vikt. Pöschl, Die Härte der festen Körper. Dresden 1909.

pressure at all. The object itself lies on a firm substratum. In order to be certain that always the same point of the surface is examined this may be marked with colouring matter.

In order to trace whether the form of the indenting object exercises influence upon the degree of indenting I have caused the identure to be brought about by balls of different radii, every time with the same series of weight. When a curve is projected of the size of the indenture, got at different loads, this curve shows, especially at the harder objects, a peculiar course. At the outset the curve has an irregular form, which afterwards passes into a linear course. Now according as the indenture is brought about with balls of greater radii, this linear course shows itself earlier, but at the same time the absolute amount of the indenture is smaller, because the acting force has to spread over a larger surface.

By the analogy of this we should a priori expect that a small plate which might be considered as a ball with an infinitely large radius as penetrating object, must yield a curve, on the whole with a well nigh linear course. Indeed the experiment proves this, provided the condition that the plate has a sufficient size, be satisfied. But as soon as, on the other hand, this size becomes somewhat considerable, so small an indenture is got that the method consequently becomes practically less fit. The subjoined curves obtained from static sclerograms by plotting the depth of the indenture and the corresponding weight, clearly show the influence of the length of the radius of ball or disk. From this it appears that the conus remains the fittest penetrating object, at least in this case, where as an object to be examined was chosen a ± 5 % gelatin plate, which, as for its hardness, borders upon the hardness of the M. gastrocnemius of Rana.

The conus may be considered as a little ball with a very small radius. If e. g. the conuspoint is measured under the microscope, it is seen that this point has a certain roundness, for which a radius may be fixed. The indenture got with the conus at a definite weight, but introduced into the system of coordinates as being caused by a ball with 0.4 mM. radius at the same weight, gets a place in accordance with the theoretical plan, on the curve of the conus, which was determined experimentally.

By means of the static sclerometer, with the conus as penetrating object, I have as to their hardness examined gelatin plates of different concentration with the intention, to take as unity of hardness a definite gelatin plate. Gelatin namely is a pretty constant material, easily obtainable for every one (fig. 5). On the whole, if we work with not too large weights, we get regular curves; only the 2 % gelatin proves not to be able to carry a conus loaded with 100 mgr. Evidently the conus at a given moment destroys the coherence of the gelatin, which manifests itself by the sudden steepness in the curve. At this moment we have reached what in mineralogy is called the 'Grenzeinheitsdruck'.

From the curves may be derived:

1. With how much weight the conus must be loaded in order to make a gelatin plate of a definite concentration undergo an indenture resp. 1 mm., 2 mm., 3 mm. deep.

2. How deep a gelatine plate of a definite concentration is indented at a weight on the conus of resp. 100, 200, 300 and 400 mgr.

In the subjoined tables these amounts have been given.

WEIGHT OF THE CONUS FOR A DEFINITE INDENTURE OF THE GELATIN PLATES.

Concentration of the gelatin plate	Weight necessary for 1 mm. indenture	Weight necessary for 2 mm. indenture	Weight necessary for 3 mm. indenture
2%	49 mgrm.	88 mgrm.	101 mgrm.
3	66	131	195
4	105	229	337
5	141	300	487
6	216	439	735
8	337	735	1125

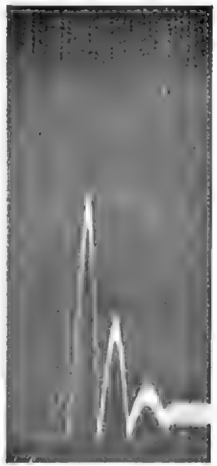
INDENTURE BY A DEFINITE WEIGHT ON THE CONUS.

Concentration of the gelatin plate	Indenture at a weight of 100 gr.	Indenture at a weight of 200 gr.	Indenture at a weight of 300 gr.	Indenture at a weight of 400 gr.	Indenture at a weight of 500 gr.
2%	∞	∞	∞	∞	∞
3	1.53	3.06	4.16	∞	∞
4	0.93	1.78	2.53	3.20	3.83
5	0.72	1.40	2.00	2.56	3.06
6	0.46	0.95	1.40	1.83	2.23
8	0.31	0.63	0.90	1.17	1.41

Dr. A. K. M. NOYONS. "Physiological Sclerometry."

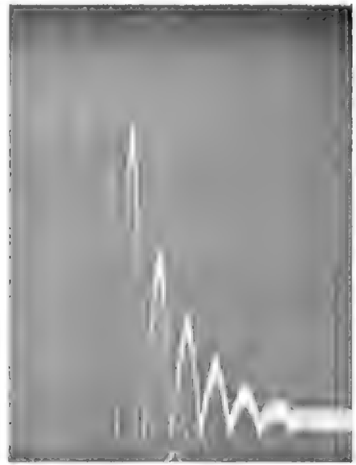
Photographs of the ballistic sclerometer

Fig. 1.



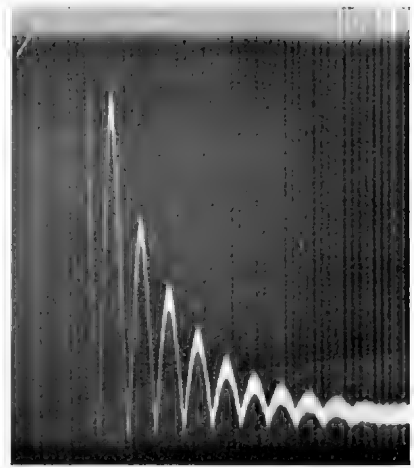
Hirudo medicinalis killed by aether vapour.

Fig. 2.



M. gastrocnemius cut from Rana esculenta.

Fig. 3.



Pig's eye some hours after death.

Lastly we may communicate here the results got by means of the static sclerometer with the three objects, of which the ballistic sclerograms have already been represented.

The Hirudo medicinalis, at a weight on the conus of 94 mgr., showed an indenture of 3.3 mm. and consequently possesses a hardness smaller than a gelatin plate of 2 % gelatin.

The results for the m. gastrocnemius of Rana are found in the following table:

DETERMINATION OF HARDNESS
M. GASTROCNEMIUS OF RANA.

Weight on conus	Dept of indenture	As to its hardness agreeing with gelatin plate of:
75 mgr.	0.78 mM.	3—4 %
94	1.00	3—4
188	1.60	4
282	2.00	4—5
376	2.40	5

From this it may be concluded that the muscle at deeper indentures of the conus, comparatively speaking, gets harder, which may perhaps be attributed to the presence of the many membranes in the muscles. That these differences are, at deeper indentures, no mistakes in the method, probable is made by the fact that at superficial and stronger digital touching also differences in hardness may be perceived.

DETERMINATION OF HARDNESS
OF A PIG'S EYE.

Weight on the conus	Depth of the indenture	In hardness agreeing with a gelatin plate of:
94 mgr.	0.43 mM.	6 %
188	0.80	6—7
376	1.06	6—7
752	2.50	6—7

The determination of hardness happened by making the conus press upon the cupola of the cornea in the pig's eye. At this indenture of the cornea very peculiar optic changes show themselves microscopically under low magnification.

From what precedes it appears that in the hardness of physiological objects we should distinguish well between relative and absolute determination of hardness. This is unmistakably connected with the fact that one of the three qualities which are implied in hardness, viz. elasticity, plasticity, and cohesion, comes to the front. Which part each of these qualities has in definite cases and how they are perhaps to be separated in sclerometry, I hope to show later on.

Chemistry. — *"On the unary tri-molecular pseudo-ternary system acet-, par-, and met-aldehyde".* By Prof. A. Smits and Dr. H. L. de Leeuw. (Communicated by Prof. A. F. Holleman).

(Communicated in the meeting of June 25, 1910).

During the investigation of the system acetaldehyde-alcohol a great quantity of metaldehyde, which deposited in the shape of needles, was formed in one of the mixtures during the cooling without our being able at the first moment to indicate the reason.

This phenomenon, which recalled to our memory the many contradictory accounts which are to be found in the literature about the behaviour of metaldehyde, induced us to undertake the following investigation on the connection between acet-, par-, and met-aldehyde, in which we were fortunate enough to find a solution, which brings unity in the work of many and makes apparent contradictions conform to a perfect harmony.

In 1872 Kekulé and Zincke[1]) found that the formation of metaldehyde from acetaldehyde, just as that of paraldehyde from the same substance, takes place in the presence of certain substances, but that while the formation of paraldehyde takes place at the usual and higher temperatures that of metaldehyde is generally to be observed at lower temperatures. The paper by Kekulé and Zincke cited here is distinguished by the great accuracy of the description of the observed phenomena, and contains a passage, whose meaning has been evidently overlooked by others, as it with great clearness points out the direction in which the solution of the problem is to be found.

The passage in question runs as follows:

"Fügt man zu reinem Aldehyd kleine Mengen von Salzsäure-gas, Chlorkohlenoxyd, Schwefliger Säure oder verdünter Schwefelsäure und kühlt dann sofort, am besten mit einer Kältemischung ab, so

[1]) Ann. d. Chemie u. Pharm. **162**, 125, (1872).

scheiden sich feine lange Nadeln von Metaldehyd ab, die bisweilen die ganze Flüssigkeit wie ein feines Netzwerk durchziehen. Stets wird nur ein kleiner Theil des Aldehyds in Metaldehyd verwandelt und die Menge des letzteren nimmt bei längerem stehen *nicht zu; sie kann sich vielmehr vermindern und der Metaldehyd kann ganz verschwinden, namentlich wenn ein energischer wirkendes Ferment in einigermassen beträchtlicher Menge zugegen ist und wenn die Temperatur nicht niedrig genug gehalten wird.*

Neben Metaldehyd entsteht immer Paraldehyd in mehr oder minder grosser Menge." [1])

That metaldehyde originates from cooled acetaldehyde in the presence of a katalyser, in which paraldehyde is also always formed, and the statement that this metaldehyde can disappear again when there is enough present of an energetically working katalyser, this result, which has been left unused up to now, joined to the fact stated by TRÖGER [2]), FRIEDEL [3]), ORNDORFF and WHITE [4]), that gradually metaldehyde is converted into paraldehyde and a little acetaldehyde at the usual temperature, and at 120° almost exclusively into acetaldehyde, as KEKULÉ and ZINCKE state, all this leads us to suppose that we have to deal here with a pseudo-ternary system, which passes into a unary-trimolecular system in case of internal equilibrium.

This supposition suggested itself, for it was found that metaldehyde can be formed from acetaldehyde at lower temperature, whereas the reversed reaction takes place at higher temperature.

So this points to the following reversible conversion:

$$\text{acetaldehyde} \rightleftarrows \text{metaldehyde.}$$

In the second place it was found that metaldehyde is converted into paraldehyde. If we assume reversibility also here, we get as second reversible reaction:

$$\text{metaldehyde} \rightleftarrows \text{paraldehyde.}$$

It was further known that acetaldehyde is easily convertible into paraldehyde, and also reversely paraldehyde into acetaldehyde.

Now it might naturally be supposed that in this latter reversible conversion metaldehyde always appears as middle-product, but as we have not the slightest reason to suppose this, and the supposition that also the reversible reaction

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1) The italics are ours. In a still earlier paper by FEHLING [Ann. **27**, 319 (1838)] it is also mentioned that metaldehyde disappears again after some time.

2) Ber. **25**, 3316 (1892).

3) Bull. **9**, 384 (1893).

4) Americ. Chem. Journ. **16** 43 (1894).

acetaldehyde $\rightleftarrows$ paraldehyde

exists, leads us to an exceedingly plausible explanation of the different phenomena, we do not hesitate for a moment to insert this latter assumption. In this way we arrive at the existence of a great equilibrium [1]), which is composed of three others, viz.:

$$\begin{array}{ccccc} & & 1 & & \\ & \text{acetaldehyde} & \rightleftarrows & \text{paraldehyde} & \\ 2 & \searrow\!\!\nwarrow & & \swarrow\!\!\nearrow & 3 \\ & & \text{metaldehyde.} & & \end{array}$$

Now it is clear that these three reactions of equilibrium will in general be influenced in a different way by one and the same katalyser which may convey the impression to us as if exclusively or mainly only one reaction takes place.

This is illustrated by an experiment, which we arranged in the same way as KÉKULÉ and ZINCKE's experiments.

If acetaldehyde is cooled in ice, and a very small amount of strong H_2SO_4 is added, e. g. 1 mgr., solid metaldehyde is formed when the solution has been shaken for some time, which metaldehyde does not seem to change at the temperature of the room, but which in reality disappears very slowly.

This slow disappearance may be accelerated by the addition of more sulphuric acid.

This points to the fact that for the equilibrium reaction acetaldehyde $\leftrightarrows$ metaldehyde H_2SO_4 is a much stronger katalyser than for the other two conversions, so that a very small quantity H_2SO_4 mainly causes metaldehyde to be formed at 0°.

From the fact that metaldehyde deposits here in solid state follows that the internal equilibrium in the system acetaldehyde—metaldehyde lies in the supersaturate region of metaldehyde.

So if the other two reactions were not at all influenced by this small quantity of H_2SO_4, the whole amount would of course be converted to solid metaldehyde, but this is not the case. Though much less quickly a small quantity of H_2SO_4 makes also the other two equilibrium reactions proceed towards their state of equilibrium, this explains the fact that by the side of metaldehyde always paraldehyde is formed, as KÉKULÉ and ZINCKE observed.

It is clear that also with an exceedingly small quantity of sulphuric acid, but then only after a very long time, internal equilibrium will be established; if however, we add more sulphuric acid, every equi-

1) BANCROFT, Journ. phys. chem. 5, 182 (1910) arrives at the conclusion, which is erroneous in our opinion, that metaldehyde is always metastable.

librium will set in with greater rapidity, and if at last the quantity of sulphuric acid has become so large, that each of the three equilibria sets in momentaneously, the three aldehydes will be in equilibrium in all possible circumstances, so internal equilibrium will prevail, and the trimolecular system will behave as a unary system.

Now we observe that the solid metaldehyde, which has originated from acetaldehyde with a little sulphuric acid at a lower temperature disappears again when more sulphuric acid is added with a rapidity which is the greater as the temperature is chosen higher.

Thus it appears among others that at the ordinary temperature the unary liquid phase cannot be in equilibrium with *solid* metaldehyde, or in other words, that this liquid is unsaturate with respect to metaldehyde.

Now it would be natural to suppose that the internal equilibrium in the pseudo-binary system acetaldehyde-metaldehyde already lies in the region unsaturate with metaldehyde at the usual temperature, but this is certainly not the case, as it appears from an experiment by FEHLING [1]), which was repeated by us, that when acetaldehyde is brought into contact with some pieces of $CaCl_2$, at the temperature of the room crystals of metaldehyde deposit on the $CaCl_2$ after some hours, the quantity of paraldehyde formed being very slight.

This is an experiment which does *not* succeed with a trace of sulphuric acid, because H_2SO_4 at the ordinary temperature already too greatly accelerates the conversion of metaldehyde to paraldehyde, whereas the influence of $CaCl_2$ on this conversion is exceedingly small at the ordinary temperature, as appears from the very slight formation of paraldehyde.

Now it may be clearly shown in the following way that it is

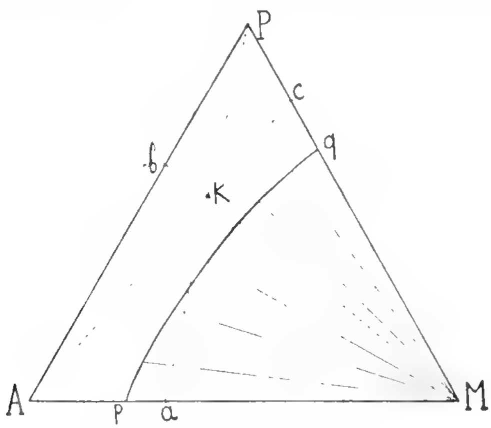


Fig. 1.

1) Ann. **27** 319, 1838.

very well possible that whereas the inner equilibrium in the pseudo-binary system acetaldehyde-metaldehyde lies in the region that is supersaturate with respect to metaldehyde, the great trimolecular inner equilibrium is unsaturate with respect to metaldehyde.

The adjoined figure 1 holds for constant temperature and pressure, and not to complicate the question needlessly it has been further assumed, that no mixed crystals are formed. If we choose the ordinary temperature, metaldehyde is the only substance which appears in the solid state, and pq represents the melting-point isotherm of this substance, the points a, b, and c denoting the inner equilibria in the three pseudo-binary systems.

The point b lies as follows from TURBABA's[1]) investigation at about 16 mol. $^0/_0$ of acetaldehyde, and 84 mol. $^0/_0$ of paraldehyde.

In accordance with the observations the point a lies in the region which is supersaturate with metaldehyde, and if we take only these two points as quite certain, the following remarks may be made.

Let us suppose that we start from the point a, and that we add paraldehyde to this supersaturate solution, which is in inner equilibrium. Then if we assume that the inner equilibrium continues to exist, and the paraldehyde is not converted, the concentration would proceed along the straight line $a\,P$, at least if the law of mass-action continued to hold perfectly.

Now we know, that the law of mass-action will certainly not continue to hold perfectly, and that consequently the line aP will in reality be curved, but this does not affect the essential part of our reasoning, and therefore the line aP has been taken straight for the take of simplicity.

If we then start from b, the concentration will proceed along bM with addition of metaldehyde on the above conditions.

Now we see that the two lines $a\,P$ and $b\,M$ intersect in the point K, and it is now easy to see what this intersection means. In K we have a liquid in which inner equilibrium prevails, not only between *acet*- and *met*-aldehyd, but also between *acet*- and *par*-aldehyde, from which also follows that there is also equilibrium between *met*- and *par*-aldehyde, and that the line $c\,A$ must also pass through this same point K.

So we see from this derivation that it is possible that while a lies in the supersaturate region, the great inner equilibrium K lies in the *unsaturate* region.

Thus KEKULÉ and ZINCKE's observations have been explained in a rational way.

[1]) Tomsk. Verlag. d. Techn. Hochschule 1901. Aus dem Gebiet der Katalyse

As we may say now with certainty that the great inner equilibrium K contains dissolved metaldehyde at the ordinary temperature, we may predict with a high degree of probability, that this will still be the case in an appreciable degree at the unary boiling-point, more than 20° higher, which lies at 41.6° according to HOLLMANN [1]).

In order to investigate this the boiling liquid was suddenly poured into water of 18° in which the metaldehyde, if it was found in the boiling liquid, would certainly deposit in solid state in consequence of its slight solubility in mixtures rich in water.

In this way a slight but very distinct deposition of metaldehyde takes really place, which proves that HOLLMANN did not deal with a unary bimolecular system, as he thought, but with a unary trimolecular one.

If we ask what will be the change in situation of the point K when the temperature is raised, the answer is easy to give.

In gaseous state and dissolved in phenol paraldehyde consists of mol. $(CH_3COH)_3$, and metaldehyde dissolved in the same solvent consists of mol. $(CH_3COH)_4$ [2]), so that metaldehyde is the most complicated substance of the three aldehydes.

If we now assume for simplicity that the size of the molecules of the different aldehydes dissolved in each other does not differ from that in the phenol solutions we have this inner equilibrium:

$$
\begin{array}{c}
(1) \\
12\ CH_3\ COH \rightleftarrows 4\ (CH_3\ COH)_3 \\
(2) \searrow\!\!\nwarrow\ \ 3\ (CH_3\ COH)_4\ \ \nearrow\!\!\swarrow\ (3)
\end{array}
$$

With rise of temperature each of the three unary bimolecular equilibria will shift in the endothermic direction, or in other words the dissociation will increase for these three inner equilibria.

The equilibrium (2) will move more considerably with the temperature than the equilibrium (1), whereas (3) is probably not very susceptible to variations of temperature, at least in comparison with the other equilibria (2) and (1).

So if the temperature rises, the point a will move to the left, the point b downward, and the point c will be slightly moved to the paraldehyde side.

If we now consider that the melting-point isotherm retracts with rise of temperature, it is clear that a will soon lie in the unsaturate region, and that therefore no katalyser will be able any more to make

1) Zeitschr. f. phys. Chem. **43**, 157 (1903)..

2) W. BURSTIJN. Sitzungsberichte *Wien* (1902), 511. HANTZSCH. Ber. **40**, 434 (1907).

solid metaldehyde deposit from acetaldehyde, which is in agreement with the fact ascertained by us that $CaCl_2$ does not do so any

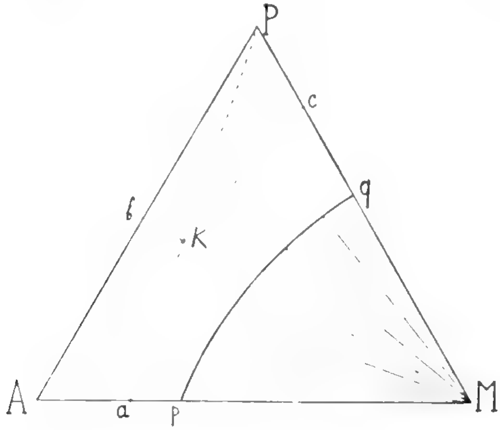


Fig. 2.

more above 40°.

It further follows from all this that the mixture in the state of internal equilibrium becomes richer in acetaldehyde as the temperature rises, as was also found by HOLLMANN.

To give a survey of the whole system with all its peculiarities a spacial figure has been constructed in an equilateral prism, on each of the side-planes of which the T-X-projection of the three-phase regions of one of the pseudo-binary systems has been given (fig. 3). A represents acetaldehyde, P paraldehyde, and M metaldehyde.

In agreement with what was predicted already before by one of us it was found that none of these pseudo-binary systems possesses a eutectic point, which has been taken into account in the diagram.

On the plane APT the T,X-projection of the three-phase regions in the system acetaldehyde-paraldehyde has been indicated.

a_0 is the triple-point of acetaldehyde situated at —123°3.

The temperature of the four-phase equilibrium vapour + liquid + solid acetaldehyde + solid paraldehyde, which is denoted by the letters h, c, d, e, in the figure was found to lie at about —123°. As is known, 4 three-phase regions intersect on this line $hcde$.

If we denote the mixed crystals which chiefly consist of acetaldehyde by S_A, those which chiefly contain paraldehyde by S_P, and the mixed crystals of metaldehyde by S_M, we can easily indicate the intersecting three-phase regions.

Thus we have in the first place the three-phase region of $S_A + L + G$ indicated by the lines ad, ac, and ah. In the second place the three-phase region of $S_P + S_A + G$ indicated by ge, fd, and $h'h$. In the third place the three-phase region for $S_P + L + G$, which is indicated here by eb_0, cb_0, and hb_0, in which b_0 is the triple-point of paraldehyde, lying at 12,55°. In the fourth place the three-phase region for $S_A + S_P + L$, but this has been omitted in the figure.

On each of the three side-planes of the trilateral prism such a figure is found, of which in each case the three most important points have been determined, viz.: the triple points of the components, and the four-phase temperatures. Thus the temperature of the equilibrium between $S_M + S_A + L + G$, indicated in the figure by the letters e_1, d_1, c_1, h_1, was found to be —122.8°. The temperature for the four-phase equilibrium between $S_M + S_P + L + G$, which in the figure is to be found on the line e_2, d_2, c_2, h_2 was found to be 12°.9.

So it appears from this that the four-phase temperatures in the three pseudo-binary systems lie only little higher than the temperature of the triple-point of the component with the lowest melting-point, from which follows that the TX,-projections will show a very one-sided situation.

In the figure this one-sidedness has not been made too pronounced, because this would have impaired the clearness of the figure.

The attempts to determine the triple point of metaldehyde had failed up to now in consequence of the conversion of metaldehyde into paraldehyde and acetaldehyde. Now this point has been determined by us by the method of Soch[1]), which consists in this that a substance is placed in a thin-walled capillary melted together at the bottom, after which it is examined at what temperature of a bath the contents of the capillary when immerged in this bath, show melting after a few seconds (here 2).

As the triple-point-pressure of metaldehyde lies above 1 atmosphere, the capillaries had, of course, to be fused together, in which the vapour volume was always chosen as small as possible.

The result was that under its vapour pressure metaldehyde melts at 246°.2, whereas Bakhuis Roozeboom[2]) gives 184° and Hollman 167°.

The spacial figure has been made clearer by sections.

Above the triple point of paraldehyde $b_0 = 12{,}55°$ this section is exceedingly simple.

If we take the section U_1, V_1, W_1 as an example, we see there a continuous melting-point isotherm L_4L_5, a continuous vapour line G_4G_5, and a continuous mixed crystal line S_4S_5.

If, however, we get below the triple point of paraldehyde, the

[1]) Journ. Phys chem. 2, 364 (1898).

[2]) Heterogene Gleichgewichte.

sections get more intricate, specially on account of the discontinuity in the mixed crystal series of paraldehyde-metaldehyde. In consequence of this we get two melting-point isotherms, three vapour-lines, and four mixed crystal lines in the section U, V, W.

Before proceeding, however, to the discussion of this section, we will point out, that of the four coexisting phases S_M, S_P, L, G, which are denoted by the points e_2, d_2, c_2, h_2, four lines proceed into space. The first two are mixed crystal lines, the third is a melting-point line, or rather a eutectic line under the vapour pressure, and the fourth is a vapour line.

Four such lines also start from the points e, d, c, h and e_1, d_1, c_1, h_1.

At the ternary eutectic temperature, to which the base corresponds, the solid lines, the eutectic lines, and the vapour lines which belong together meet, and so we get the coexistence of five phases, three solid ones $S_A + S_P + S_M$, a liquid one L_e, and a vapour one G_e.

If we now return to the section UVW we may remark that the line $S_{m_2}S_m$ indicates the metaldehyde mixed crystals which coexist on one side with the paraldehyde mixed crystals $S_{p_2}S_p$, and on the other side with the gases G'_2G_2.

The line $S_{p_1}S_P$ indicates the paraldehyde mixed crystals, which coexist with the liquids LR, and the gases GG_2, and the line $S_{m_1}S_m$ indicates the metaldehyde mixed crystals which coexist with the liquids L_3R and the gases G_3G_2.

It follows from this that at this temperature four phases can coexist, viz.: $S_m + S_P + R + G_2$, i. e. mixed crystals which consist chiefly of metaldehyde, mixed crystals which contain chiefly paraldehyde, a saturated solution, and a gas coexisting with these phases.

These four phases lie in the angles of a quadrangle, which is composed of *four* three phase triangles.

If we now assume that the temperature of this section agrees with the unary melting-point, we are certain that the liquid in which internal equilibrium prevails, will have to lie on one of the liquid lines LR and L_3R.

Hollmann, who thought he had to deal with a pseudo-binary system, found that the liquid which behaved as a unary one deposited solid paraldehyde at 6°,75. If we now correct this result by means of the newly-acquired knowledge, we arrive at the following result.

If the temperature of a liquid, which is in inner equilibrium, decreases, and we assume that this liquid is in equilibrium with its vapour, which is then, of course, also in inner equilibrium, the liquid will move with decrease of temperature along the line $\mathbf{L}_0, \mathbf{L}'_0, \mathbf{L}''_0$, and the coexisting vapour along the line $\mathbf{G}_0, \mathbf{G}'_0, \mathbf{G}''_0$.

If we have now descended to the unary melting temperature, the liquid line has reached the melting plane of the paraldehyde mixed crystals in $\mathbf{L}_1$, and the vapour the vapour plane coexisting with it in $\mathbf{G}_1$, at which moment solid substance deposits, which is a mixed crystal phase $\mathbf{S}_1$, which is also in inner equilibrium, as one of us showed already before.

This unary three-phase equilibrium can only exist at one temperature and pressure, so that below this temperature we get two-phase equilibria between solid phases and vapour phases both in internal equilibrium.

The solid phases move along the line $\mathbf{S}_1\mathbf{S}_2$ and the vapour phases along $\mathbf{G}_1\mathbf{G}_2$.

A representation which is in closer agreement with the investigations which have been made up to now, is obtained when the T_1X-sections for constant pressure, e g. for one atmosphere, are indicated on the sides of the prism.

If we do this, we get the T,X-section for the pseudo-binary system acetaldehyde-paraldehyde on the APT-plane, as it was determined by HOLLMANN, only with this difference, that the eutectic point has disappeared (fig. 4).

This T, X-section is indicated by the melting-point figure a, c, b, e, g, d, f, and by the boiling-point lines kl. The temperatures of the most important points are indicated in the figure, so that this section does not require any further elucidation.

We get a more complicated figure on the plane for paraldehyde-metaldehyde, the PMT-plane, because metaldehyde sublimates under the pressure of 1 atm. In consequence of the conversion of metaldehyde into acetaldehyde and paraldehyde it is impossible to determine how high this sublimation point is, for the method followed for the determination of the triple-point cannot be applied here.

The only thing that can be said about it at present is this that this sublimation point probably lies little under the triple-point temperature, as the triple-point pressure probably does not lie much above 1 atmosphere.

The T,X-section of this system is indicated by b, c_2, n_2 l, m_2, o, p_2, e_2, g_2, d_2, f_2. We determined the temperature of the three-phase equilibrium $S_M + L + G$, indicated by the line p_2, n_2, m_2 at about 124°, when we worked very quickly, from which it therefore follows that the boiling solution which coexists with aldehyde mixed crystals, contains comparatively little metaldehyde.

On the front plane, the plane for acetaldehyde-metaldehyde, we have the T,X-section for this system, indicated by a, c_1, n_1, k, m_1,

o, p_1, e_1, g_1, d_1, f_1, which section, in its nature closely agrees with that for paraldehyde-metaldehyde.

We found the three-phase equilibrium $S_M + L + G$, or in other words the boiling-point of the solution which is in equilibrium with metaldehyde-mixed crystals, at 21°.0, so only 0°.2 higher than the boiling-point of acetaldehyde, from which follows that the solubility of this solid phase in acetaldehyde is exceedingly slight at 21°.

With regard to the most important parts of the spacial figure, they are elucidated by sections here too, which will not require any further explanation in connection with what precedes.

The only thing that calls for explanation is the situation of the unary system in the trimolecular, pseudo-ternary system.

Hollmann found 41°.6 for the boiling-point of the unary system, and now we shall assume that the section $U_1 V_1 W_1$ holds exactly for this temperature.

This section is perfectly analogous to the section U, V, W, which holds for a higher temperature.

In the section $U_1 V_1 W_1$, however, the points of the three-phase triangle have shifted so considerably towards the front-plane, that this triangle could no longer be indicated clearly, which, however, is of minor importance here.

What we want to draw attention to in this section is the coexisting phases which are in inner equilibrium, so the boiling liquid phase $\mathbf{L}_1$, and the vapour phase $\mathbf{G}_1$.

Hollmann thought that this liquid consists of 53.4 mol. $^0/_0$ of paraldehyde, and 46.6 mol. $^0/_0$ of acetaldehyde, but our investigations have taught that this liquid also contains metaldehyde, though this quantity is very small; the same remark holds of course for the vapour phase $\mathbf{G}_1$.

If we now examine what we find at temperatures above and below the boiling-temperature in the unary system, we see at once that only vapour can exist above this temperature, and so that only vapour phases in inner equilibrium are possible; these vapour phases are indicated in the spacial figure by the line $\mathbf{G}_1\mathbf{G}_0$, the direction of which shows that these phases get richer in acetaldehyde with rise of temperature. Below the boiling-temperature of the unary system only liquids in inner equilibrium are possible, so that a liquid line runs from $\mathbf{L}_1$ towards lower temperature, till the melting point plane of paraldehyde is reached in $\mathbf{L}_2$ at the temperature of 6°.75.

At this temperature the mixed crystal phase $\mathbf{S}_2$ deposits, and from there another line runs again towards lower temperatures, viz. $\mathbf{S}_2\mathbf{S}_1$, which curve denotes the inner equilibria in the solid phases.

So it has appeared from what precedes that the observed phenomena are in perfect harmony with the conception that the three considered aldehydes really form a unary, trimolecular, pseudoternary system, the connections of which have been examined theoretically here for the first time.

Anorg. Chem. Laboratory of the University.
Amsterdam, June 17, 1910.

Chemistry. — *"On the system acetaldehyde-alcohol"*. By Prof. A. SMITS and Dr. H. L. DE LEEUW. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated in the meeting of June 25, 1910).

PERKIN [1]) was the first who observed that there exists a close analogy between the behaviour of the system aldehyde-water and aldehyde-alcohol.

He found for both systems that when the two liquids are mixed, first an absorption of heat appears, and then generation of heat.

PERKIN naturally ascribed the absorption of heat to the physical process, the mixing, and the subsequent generation of heat to a purely chemical reaction, the formation of a compound.

Also Miss HOMFRAY and COLLES [2]) found indications by different ways for the existence of different compounds in the system aldehyde-water, so that a systematical investigation of the system *aldehyde-alcohol* promised to yield a positive result.

An important question, which had first to be answered was this: when compounds are formed, are these compounds addition products or are they bodies which form from these substances with separation of water.

It is clear that this is an important question, because the system can be considered as binary in the case of the existence of addition products, whereas in the other case the system is much more complicated viz. quaternary.

[1]) Journ. chem. soc. **51**, 826 (1887).
[2]) Journ. chem. soc. **87**, 1434 (1905).
„ „ „ **89**, 1249 (1906).

To decide this mixtures of aldehyde and alcohol were placed in glass tubes, which were then fused together.

After the lapse of different times these tubes were opened, and anhydrous copper sulphate was added.

The result was that mixtures which had been preserved at the ordinary temperature for *a year*, do not assume a blue colour with $CuSO_4$ even after a quarter of an hour's contact, the same result being obtained with mixtures which had not been kept for a year at the ordinary temperature, but had been heated to 100° for some hours.

In this way it was proved that no water had split off, and that when a compound is formed when aldehyde and alcohol are mixed, this must be no acetale, but an addition product, an aldehyde-alcoholate.

In the above-mentioned experiments another remarkable phenomenon was found, which is worth mentioning here.

It appeared namely that when aldehyde-alcohol mixtures are left in contact with $CuSO_4$ for a *long* time, a blue colour does really appear, and that this is to be ascribed to the fact that $CuSO_4$ is a katalyser for the formation of acetale according to the equation:

$$CH_3\,C\begin{cases}O\\H\end{cases} + 2\,C_2\,H_5\,OH \rightarrow CH_3\,C\begin{cases}OC_2H_5\\OC_2\,H_5\\H\end{cases} + H_2O$$

To show this clearly a mixture of 1 mol. of aldehyde and 2 mol. of alcohol was placed with $CuSO_4$ in a glass tube, which was then fused together.

After a few days the tube was opened, the liquid filtered off from the copper sulphate, which had become *blue*, and then the liquid was distilled by fractions during which process a great quantity of *acetale* could be isolated. In this way a very convenient method of preparing acetale was found at the same time.

As the strong contraction which occurs when aldehyde and alcohol are mixed, led us to suspect, that the determination of the specific weight might give some indication about the existence of a compound, the systematical investigation was opened with these determinations.

The difficulty we met with here, was this, that just as PERKIN had found for aldehyde and water, the spec. weight was *not* constant for some time after the mixing, as the contraction continues for a long time.

To ascertain how long after the mixing the spec. weight yields reliable results, it was examined by means of a very sensible dilatometer, when the volume of a mixture had become constant, for which we found at 18° one day after the mixing.

So before the specific weight could be determined, every mixture was placed in a glass tube which was fused at the end, in a thermostat of 18° for more than a day.

The result of the determinations carried out in this way, was as follows:

mol. °/₀ of aldehyde	spec. weight 18°/4°
100	0.7834
84.28	0.8277
75.08	0.8474
66.86	0.8601
53.86	0.8715
50.32	0.8719
44.56	0.8709
36.50	0.8627
29.70	0.8501
18.48	0.8296
13.02	0.8200
0	0.7907

If we represent this result graphically, we get the following diagram (p. 332).

So this curve of the specific weights exhibits a very distinct maximum at 50 mol. °/₀, which makes it probable that the compound

$CH_3C\!\begin{array}{l}\diagup\!\!\diagup O\\ \diagdown H\end{array}\! . C_2H_5OH$ is formed in the liquid.

We see further from this curve that a determination of the spec. weight can make the concentration of the mixture known to us with fairly great accuracy, if we namely know what component is present in excess. Only in the neighbourhood of 50°/₀ this method becomes too inaccurate, on account of the level shape of the curve at that place.

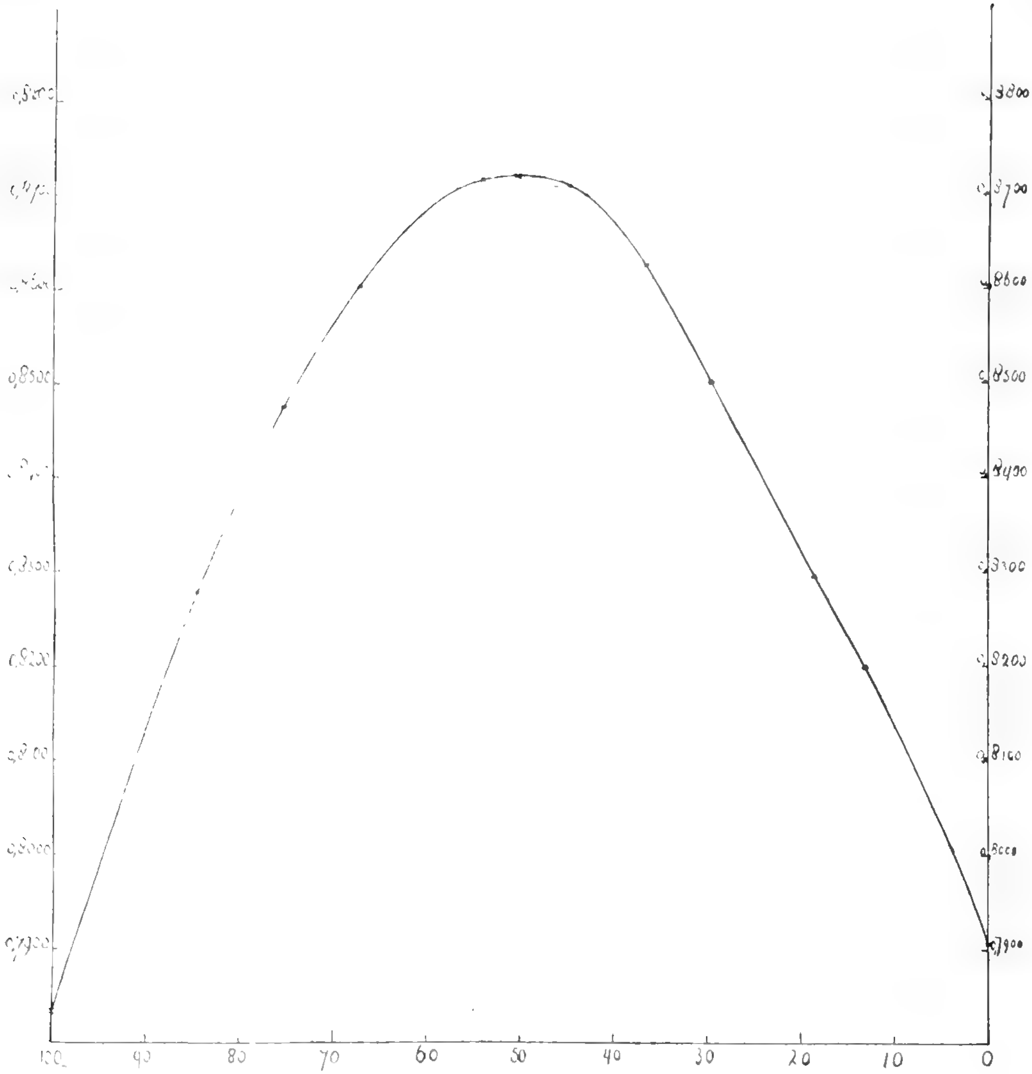


Fig. 1.

After having obtained some indications in this way, we proceeded to the determination of the boiling-points of different mixtures, and the concentration of the coexisting phases at different pressures.

The following results were obtained:

Pressure **699** m.m. Hg.

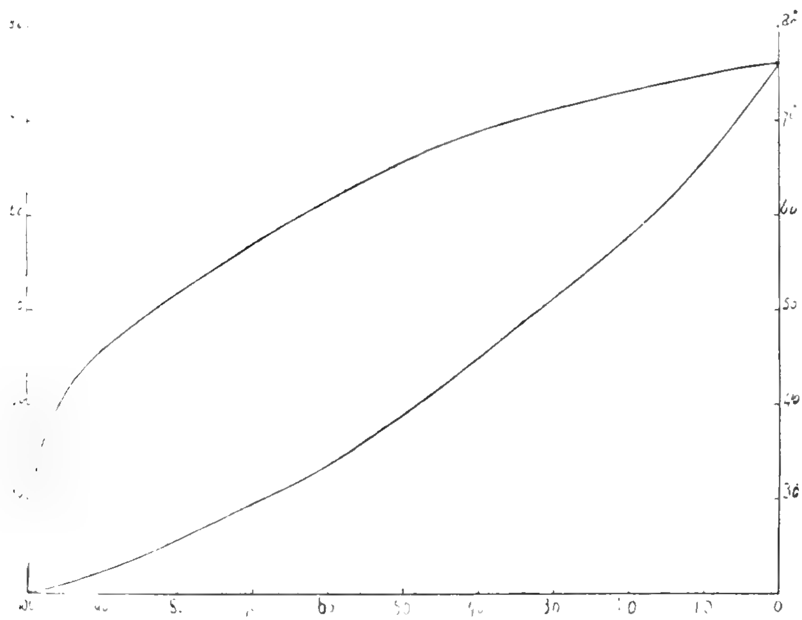
temp.	concentration liquid	concentration vapour
20.°1	100 mol.%aldehyde	
25. 3	81.2 " " "	—
29. 6	69.7 " " "	—
34. 7	57.8 " " "	—
40. 1	48.2 " " "	95.8 mol.%aldehyde
48. 8	34.6 " " "	85.2 " " "
57. 7	20.5 " " "	69.2 " " "
65. 3	10.8 " " "	51.1 " " "
76. 1	0 " " "	—

Pressure **398** m.m. Hg.

temp.	concentration liquid	concentration vapour.
5.°8	100 mol.% aldehyde	—
11. 0	81.2 " " "	—
15. 9	69.7 " " "	—
21. 9	57.8 " " "	—
29. 2	51.5 " " "	95.5 mol.% aldehyde
30. 4	46.5 " " "	—
32. 3	42.3 " " "	—
37. 2	34.6 " " "	91.7 " " "
44. 2	24.7 " " "	
45. 4	22.6 " " "	67.1 " " "
51. 8	13.6 " " "	—
53. 2	11.4 " " "	44.5 " " "
62. 8	0	—

Pressure 97 m.m. Hg.

temp.	concentration liquid	concentration vapour.
— 23. 9	100 mol. % aldehyde	—
— 15. 7	66.4 „ „ „	—
— 7	33.8 „ „ „	—
— 1. 8	51.6 „ „ „	—
+ 3. 6	46.5 „ „ „	—
5. 4	—	94.8 mol. % aldehyde
7. 8	40.9 „ „ „	—
8. 5	—	91.0 „ „ „
11. 1	36.9 „ „ „	—
15. 9	31.6 „ „ „	79.5 „ „ „
21. 2	24.8 „ „ „	—
22. 3	—	67.8 „ „ „
23. 3	20.3 „ „ „	65.3 „ „ „
25. 1	15.5 „ „ „	—
27. 8	10.2 „ „ „	—
29. 7	—	38 8 „ „ „
29. 9	—	34.3 „ „ „
30. 1	6.9 „ „ „	24.1 „ „ „
31. 4	—	20.9 „ „ „
34. 3	0. „ „ „	—



Aldehyde mol. % aldehyde Alcohol

Fig. 2.

If we represent these results in diagrams, we get figs. 2, 3, and 4.

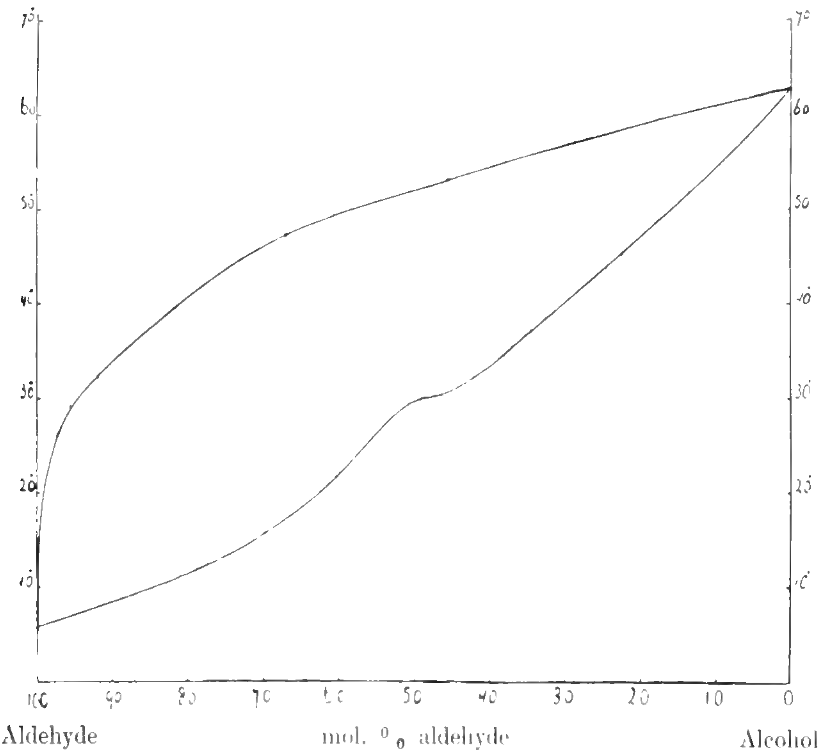


Fig. 3.

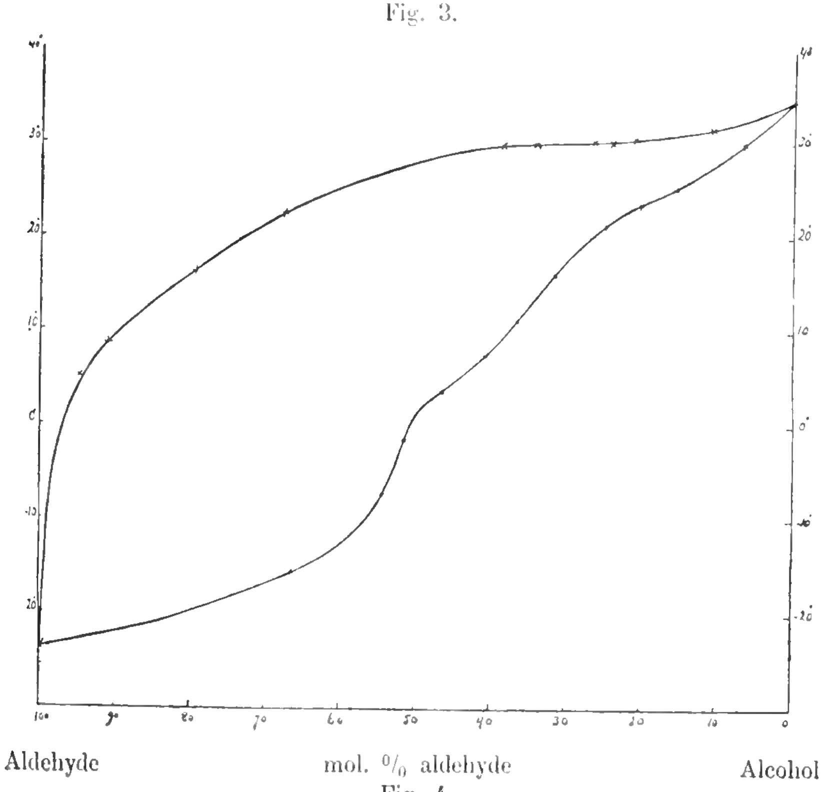


Fig. 4.

Fig. 2, which represents the T, X-section of the liquid-vapour surface corresponding to the pressure of 699 mm. Hg., does not present any particularity; the liquid and the vapour curve lie very far apart, which is a consequence of the pretty large difference in vapour tension between aldehyde and alcohol.

Fig. 3, the T, X-section at the pressure of 398 mm. Hg. shows a constriction at about 50 mol. $^0/_0$, which as is known, points to a compound, whose existence was already suspected on the ground of the density determinations. The vapour which coexists with the liquid of 50 mol. $^0/_0$ is much richer in aldehyde, contains, namely, about 95 mol. $^0/_0$ of aldehyde, as a proof that the compound is already greatly dissociated at 29°.2.

This is in perfect accordance with the fact that the boiling-point line at the pressure of 699 mm. Hg. does not show anything of a compound.

The most interesting is fig. 4, in which the T, X-section has been drawn which corresponds to the pressure of 97 mm. Hg., for this figure not only points to the existence of a compound of 1 mol. of aldehyde to 1 mol. of alcohol, but also to a second compound of 1 mol. of aldehyde and 2 or 3 mol. of alcohol, because in its neighbourhood the liquid line also shows a constriction, which is, indeed faint, but without doubt essential.

This second compound, whose existence is made probable in this way, must be still more greatly dissociated than the first, which is in accordance with the fact that the boiling-point line of 398 mm. Hg. does not exhibit anything that would point to its existence.

The third method which was followed to get to know the character of the system aldehyde-alcohol was the *calorimetric* one.

As was said before when aldehyde and alcohol are mixed, first absorption of heat occurs, and then generation of heat.

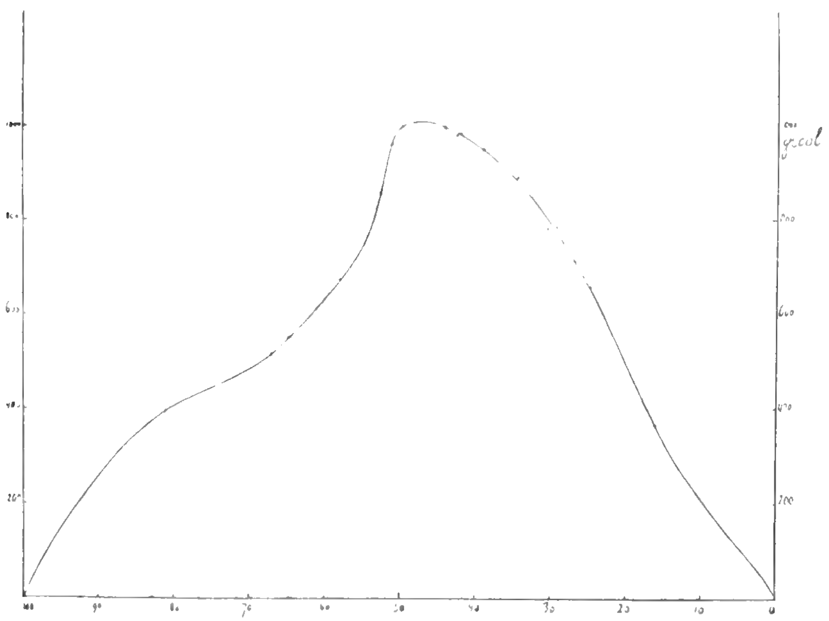
Now it is clear that the heat-effect of the first period is not to be accurately determined from the fall of the temperature, as the exothermic reaction also goes on during this time, the endothermic process, however, at first predominating.

Besides, this fall of the temperature is very greatly dependent on the completeness of the mixing, which is certainly not reached with equal rapidity in the different experiments. So this is the reason that oscillating values are obtained for the negative heat of mixing, and that calculations could be based only on the rise of temperature.

The heats of reaction calculated from this rise of temperature are represented in the following table.

Composition of the mixture, obtained by mixing, in mol. % of aldehyde.	Mol. heat of reaction in gr. Cal.
81.10	— 396.2
67.17	— 516.2
64.78	— 554.5
58.42	— 675.3
54.70	— 737.1
52.85	— 859.9
51.14	— 962.8
49.73	— 1000.8
44.22	— 998.2
42.44	— 980.6
42.11	— 987.2
39.06	— 949.5
34.22	— 897.4
30.59	— 781.2
29.10	— 788.5
27.09	— 714.8
24.75	— 657.0
16.21	— 385.5

If we reproduce this result graphically (Fig. 5), we get a curve which shows a maximum for the concentration of 50 mol. %, and



Aldehyde mol. % aldehyde Alcohol

Fig. 5.

further presents this peculiarity that the course on the lefthand side of the maximum is very different from that on the right.

The curve lies higher on the alcohol side than on the aldehyde side, which proves that special heat-effects are active on the alcohol side, which may find an explanation in the formation of a second compound which is richer in alcohol.

To obtain perfect certainty we proceeded to the determination of the melting-point line, which investigation required much liquid air, for which the arrangement was not yet ready at the beginning of our experiments.

Though we really met with the anticipated obstacles, the difficult crystallisation of the liquids, yet we succeeded in determining the most important parts of the melting-point line, producing in this way the most striking proof for the existence of *two* compounds in the discussed system.

Concentration in mol. % aldehyde.	Final melting-point.
100	— 123°.3
90.31	— 125.4
83.90	— 127.6
80.19	not to be determined very viscous mass probably below −132°
77.34	— 126.0
74.45	— 126.05
66.01	— 124.3
59.70	— 123.5
50.73	— 122.3
44.53	— 125.3
39.50	— 128.05
34.33	— 123.2
29.25	— 126.8
25.06	— 132.2
17.32	— 130.6
9.78	— 120.6
0	— 114.

The investigation, which was carried out with a very sensible resistance thermometer, made by Messrs. DE LEEUW and ZERNIKE, yielded the results given in the table on p. 338.

These results enable us to draw also the T,X-section of the P,T,X-spacial figure corresponding to the pressure of 1 atmosphere in so far as the equilibria with solid phases are concerned or in other words the melting-point lines.

From the course of these melting-point lines (fig. 6), in which

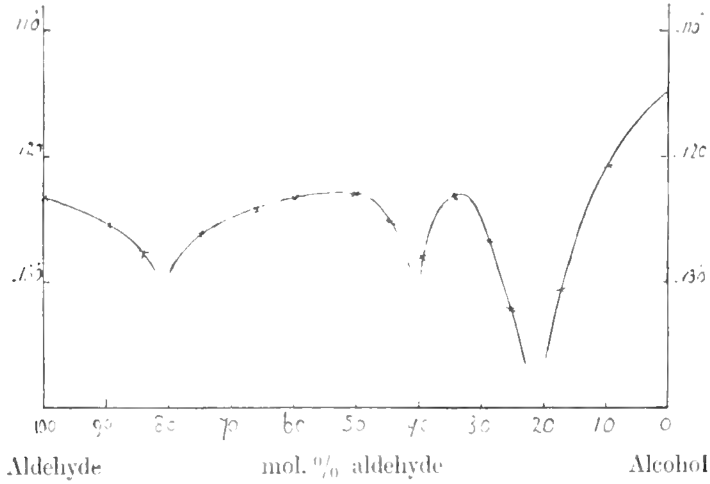


Fig. 6.

two maxima occur, one at 50 % and one at 33 % aldehyde, follows with sufficient certainty the existence of the two compounds $CH_3COH.C_2H_5OH$ and $CH_3COH\,2C_2H_5OH$, which are considerably dissociated even at the low melting-point temperatures —122° and — 123°.

Amsterdam, June 18. *Anorg. Chem. Lab. of the University.*

Chemistry. — "*On the system chlorine-sulphurdioxyde*". By Prof. A. SMITS and W. J. DE MOOY. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated in the meeting of June 25, 1910).

In our search for a clear example of the influence of light on heterogeneous equilibria we have fixed upon the above-mentioned system, of which it was already known that the conversions $SO_2 + Cl_2 \rightleftarrows SO_2Cl_2$ do *not* take place in the dark and in the absence of a katalyser, while light or a katalyser as camphor, animal carbon, etc. causes the equilibrium between sulphuryl-chloride and its dissociation products to set in comparatively quickly.

The purpose was first to study the T,X-figure corresponding to the pressure of 1 atm. in the dark and in the absence of a katalyser, and then take the same experiments in the light.

The investigation in the dark and in the absence of a katalyser is over now, and has yielded the result we anticipated, as appears from the subjoined figure, which only represents what has been found at and below the melting-point temperatures of the components, the determination of the boiling-point curves being postponed till afterwards.

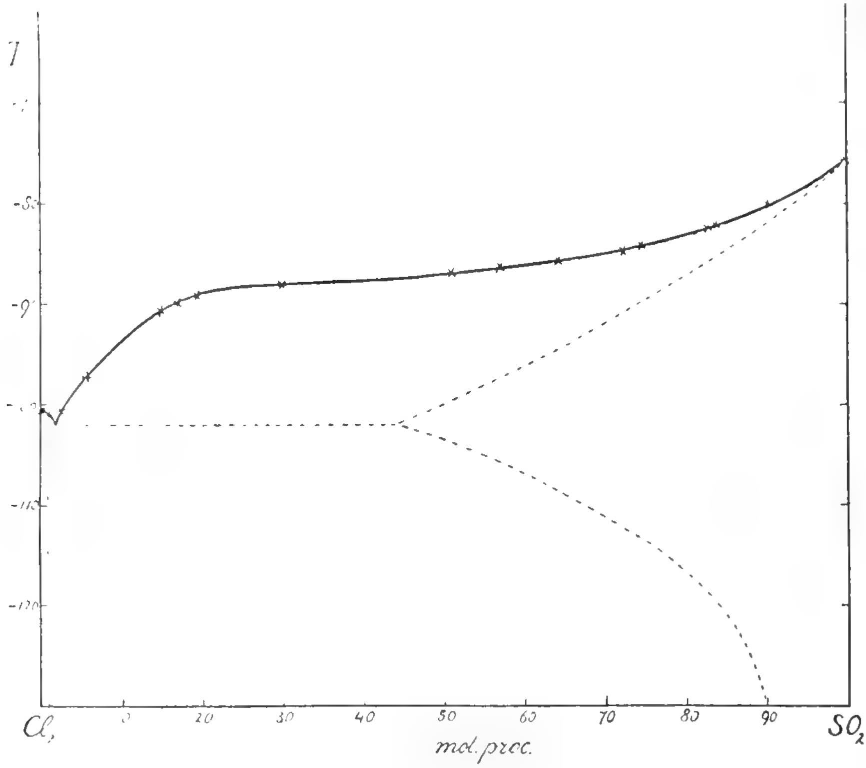


Fig. 1

The determinations, which were made with a very sensible resistance-thermometer, made after a device of Messrs DE LEEUW and ZERNIKE gave for the meltingpoint of Cl_2 the temperature of —100°.45, and —75°.6 for that of SO_2.

On addition of very small quantities of SO_2 to Cl_2 a lowering of the end-melting-point was found, so this proved that the melting-point lines form a eutectic point.

It appeared that this eutectic point lay at — 102°.3 and at a concentration of about 1.5 mol. $^0/_0$ SO_2.

As this eutectic point lies very near the Cl_2-side, and so not very far below the melting-point of SO_2, the shape of the melting-point line of SO_2, which presents a distinct point of inflection is rather level, from which it appears at the same time that if the experiment is made in the dark, there is no question of the formation of a compound, which, however, is easy to verify after the experiment is finished. As is known, SO_2Cl_2 hardly dissolves in water, so that its presence is at once betrayed when the mixture is poured into water.

Further the diagram shows that mixed crystals very clearly appear. As it, however, appeared, that probably on account of the low temperature, the conversions may be easily retarded, the mixed crystal curves are dotted, because it is possible that when the experiments are repeated with baths of low temperature specially arranged for the purpose, small deviations will be found, though, of course, the type is fixed already now.

The experiment in the light, for which sunlight and also a quartz-mercury lamp was used, has already proved that rapid formation of sulfuryl-chloride takes place under these circumstances, in consequence of which the melting-point figure undergoes a considerable modification.

For the present we shall confine ourselves here to the statement that a mixture of 47.1 mol. $^0/_0$ SO_2, in which equilibrium had been established in sunlight, presented an initial point of solidification of about — 80°, which point, therefore, lies considerably above the melting-point curve, as a proof that we have to deal here with another system.

What the shape will be of the whole T,X-figure of the system in equilibrium in the light, will be communicated on another occasion. We will investigate also the heterogeneous equilibria in the dark and in the presence of a katalyser, in which probably results will be obtained, deviating in some regards from those obtained in the light.

Amsterdam, June 22, 1910.

Anorg. Chem. Laboratory
of the University.

Chemistry. — *"On critical end-points in ternary systems"*. By Prof. A. Smits. (Communicated by Prof. J. D. van der Waals).

(Communicated in the meeting of June 25, 1910).

In a previous paper [1]) some remarks were made about the ternary system which is obtained when a third substance is added to ether and anthraquinone, which does not yield critical end-points p and q either with ether or anthraquinone, and is miscible with the other two components in all proportions in the liquid state.

It was then pointed out that with addition of this third substance the two critical end-points p and q at first continue to exist, but that at last with greater quantities of the third substance they disappear in consequence of the fact that the points p and q mentioned approach each other more and more, and at last coincide.

It was demonstrated on the same occasion that interesting phenomena must precede this coinciding, which was another incitement to investigate experimentally the case under consideration.

This investigation though not quite completed is far enough advanced to be fit for publication and in order to set forth clearly what has been found, some theoretical considerations must be premised.

At the beginning of the investigation the surprising phenomenon occurred that with increase of volume the three phase equilibrium $S + L + G$ could form from a mixture which was at a temperature a few degrees *above* the critical temperature.

This phenomenon seemed so surprising to me that I thought at first that it was to be ascribed to impurities, but it soon appeared to be essential. It is very significant that the V,x-diagram, which also solved so many questions in the system ether-anthraquinone, showed the true connection of the equilibria in the clearest way here too, and indicated the necessity of the above surprising phenomenon with great clearness.

This has again proved the advantage of this way of representation, and this is the reason why the figures discussed here will be derived from the V,x-diagram for the ternary system.

The adjoined figure holds for the system alcohol-ether anthraquinone, and for a temperature lying between that of the two critical end-points p and q of the system ether-anthraquinone, so between 203° and 247°, e. g. 230°.

In the front plane of the trilateral prism the V,X-figure of alcohol-anthraquinone has been drawn. The critical point of alcohol lies at 243°.1; so pure alcohol is still below its critical temperature in this

[1]) These Proc. Sept. 1909, p. 182

figure, and this is the reason why the liquid point d and the vapour point a still lie comparatively far apart.

The field $abde$ is the region for the coexistence of unsaturate liquid and vapour, and the points e and b indicate the liquid and the vapour, which are in equilibrium with solid anthraquinone, lying in f. So the triangle ebf is the three-phase triangle, which is bounded on the right by the region for solid anthraquinone + fluid, which latter phases lie on the line bc, which is one of the stable branches of the continuous solubility isotherm $cbeh$, of which the second branch eh indicates the liquids coexisting with solid anthraquinone.

In the plane for alcohol-ether the drawing is exceedingly simple for, as 230° lies far above the critical temperature of ether, liquid curve and vapour curve have continuously flowed together, and so we have got a continuous binodal curve with a plaitpoint in K.

On the plane for ether-anthraquinone the V,X-figure is equally simple; there we have the continuous solubility isotherm ch_2, which is stable over its full length. Further we see in this plane the line gK_2P, which indicates the metastable continuous binodal curve, which may be realised if the solid substance did not appear, so if the critical phenomenon, as has been found already, is to be realised for a supersaturate solution.

If we now start from the three-phase equilibrium $S + L + G$ in the system alcohol-anthraquinone, and gradually add more ether, the quantity of anthraquinone always exceeding that of ether, the points b and e will move in the space, because the liquids and vapours which now coexist with solid anthraquinone, will contain also ether, and the vapour of course more than the liquid.

Hence the three coexisting phases no longer lie in the same plane for a constant proportion alcohol-ether; such a section can contain only two of the three coexisting phases, $S + L$ or $S + G$, or in other words on each section lie two pair of coexisting phases, but to $S + L$ belongs a vapour, which contains more ether than L, and to $S + G$ belongs a liquid, which contains more alcohol than G.

Thus the three points $f\,l_1\,g_1$ form a three-phase triangle, and it is evident that g_1 lies farther back in the figure, consequently it contains more ether than l_1, and the same thing is to be observed for the succeeding three-phase triangles.

Now it is clear that this ternary liquid and vapour line cannot proceed to the plane for ether-anthraquinone, for in this plane no stable liquids can exist at the temperature under consideration.

So we see that before this time the lines mentioned will have to merge continuously into each other, and so that the critical phenomenon will appear for a saturate solution at the moment of this continuous transition, just as this can be the case in the system ether-anthraquinone.

So if K is this ternary critical end-point, the liquid point l and the vapour point g coincide there and the spacial solubility isotherm touches the three-phase coexistence curve bK_1l exactly in K_1.

Before we proceed it is necessary to mention what are the principal modifications which the figure undergoes, when the temperature is varied.

These modifications are obvious; for it is clear that at the temperatures of the critical end-points p and q the three-phase coexistence curve bK_1l will just touch the plane for ether-anthraquinone namely in the critical end-points p and q.

Between these temperatures no contact with the plane for ether-anthraquinone can occur, because then a stable solution cannot occur in this system.

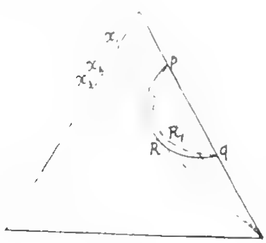


Fig. 2.

So it is to be expected in the simplest case that the ternary plaitpoint curve pRq in projection on the concentration triangle has a course as is indicated in the adjoined drawing, and from which it is to be seen at the same time that the concentration R is the last at which a critical end point still occurs.

So it is clear from what precedes that the three-phase coexistence curve bK_1l between the two critical end-temperatures beginning at p will first recede into the space, and approach the ether-anthraquinone plane again afterwards, and finally touch it again at q for the second time.

It follows from this that it is easy to derive from the v-x-spacial representation what will have to be observed when a mixture of alcohol and ether x_1 with an excess of anthraquinone is studied at different temperatures, for the phenomena must on the whole agree with those which would be met with if at constant temperature we first made the alcohol-ether mixture richer in alcohol, and then poorer in alcohol, till the original concentration was reached again. In this case, however, we get exactly to the same point, whereas this is not the case with change of temperature.

So we shall begin with projecting a plane through the axis of anthraquinone and the liquid point l_1; we then get the following section, it being noteworthy, however, that now l_1, f, and g are no coexisting phases now.

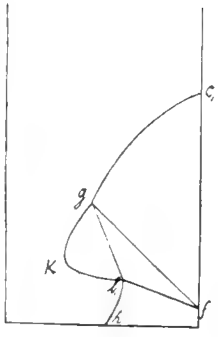


Fig. 3.

l_1 without vapour, can indeed be in equilibrium with f, and also g, when there is no liquid present, but the three phases cannot be in equilibrium together, because there belongs a vapour phase to the liquid l, which contains more ether than g.

Nor do the coexisting liquid and vapour phases lie on the line l_1Kg, for they contain different quantities of the three components, and can therefore, never lie in the same section.

The lines gc_1 and l_1h represent vapour and liquid phases which can coexist with solid anthraquinone.

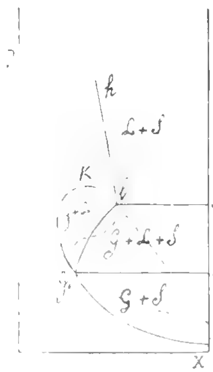


Fig. 4.

If in accordance with this v-x-section we project the corresponding p-x-figure, we get this.

The point g corresponding with a liquid which contains less ether than l_1, the three-phase pressure in g is smaller than in l_1, and this is the reason that we now get a p-x-figure with a three-phase *region* $l_1S_1g_1S$, and the boundary between this region and that for $G+L$ is formed by the line g_1l_1.

As to the continuous curve l_1Kg_1, no more than in the v-x-figure do coexisting liquid and vapour phases lie on this line here, so that we must regard it as a mere line of demarcation.

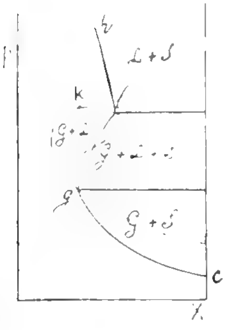


Fig. 5.

The lines hl_1 and g_1c_1 are also boundary lines, but there lie phases on these lines which can coexist with solid anthaquinone [1]).

If we now project a plane through the axis for anthraquinone and the liquid point l_3, then the p-x-figure corresponding with this v-x-section is as indicated in fig. 5.

The region for $G+L$ has become smaller and the points l_1 and g have risen.

If we now think a plane projected through the axis of anthraquinone and the critical endpoint K_1, the p, x figure corresponding with this v-x-section has the following shape (fig. 6).

We see that the plaitpoint K has coincided with the point l

[1]) The continuity is indicated schematically in all the figures.

and so that the critical phenomenon is observed for a solution saturate with solid anthraquinone.

The particularities which present themselves here for a ternary system, are now very evident, for we see that at the temperature of this critical end-point no region has disappeared as yet, and so that no continuity exists as yet between the region for $L+S$ and $G+S$, which is the case in a binary system at the corresponding critical end-temperature.

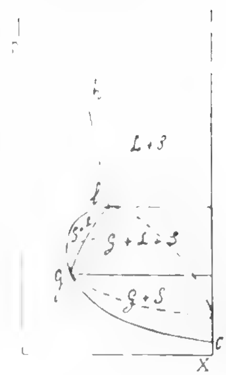


Fig. 6.

If we now proceed to still greater concentrations of ether, the point l moves downwards along the vapour branch, g upwards, and if we now project a plane which, passing through the axis of anthraquinone just touches the three-phase coexistence line bK_1e, the points l and g, which past K_1 may be most appropriately called two different fluid phases, and at first differ very much in density, have coincided. For this case we get then the p, x diagram, which has been drawn in fig. 7.

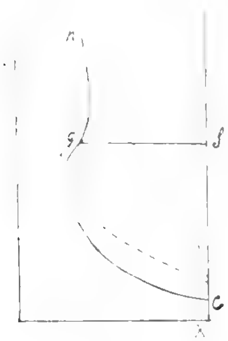


Fig. 7.

In the point g, where the curve for solid-fluid just touches the p-x-loop, a three-phase equilibrium is possible for the last time.

If we now take a section, which corresponds with still more ether, equilibrium in stable state is possible only between fluid phases and solid anthraquinone, as the p-x-loop for liquid-vapour has no longer any point in common with the curve for solid-fluid, as fig. 8 shows.

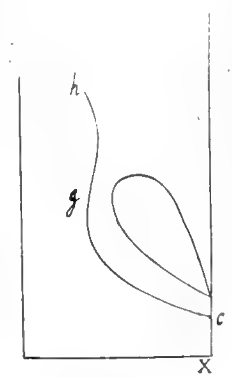


Fig. 8.

If we now pass to greater contents of alcohol, in which the just-discussed sections, but in reversed order, are obtained, this succession gives us an *idea* of what we get when a liquid mixture x (see Fig. 2) with an excess of anthraquinone is studied for a series of temperatures, if we also take into account what will generally be the influence of the temperature on the concentration and the pressure.

If we now indicate these sections in a perspective spacial representation, we get figure 9, from which follows that the curve which connects the liquid points or the *maximum three-phase points* l and

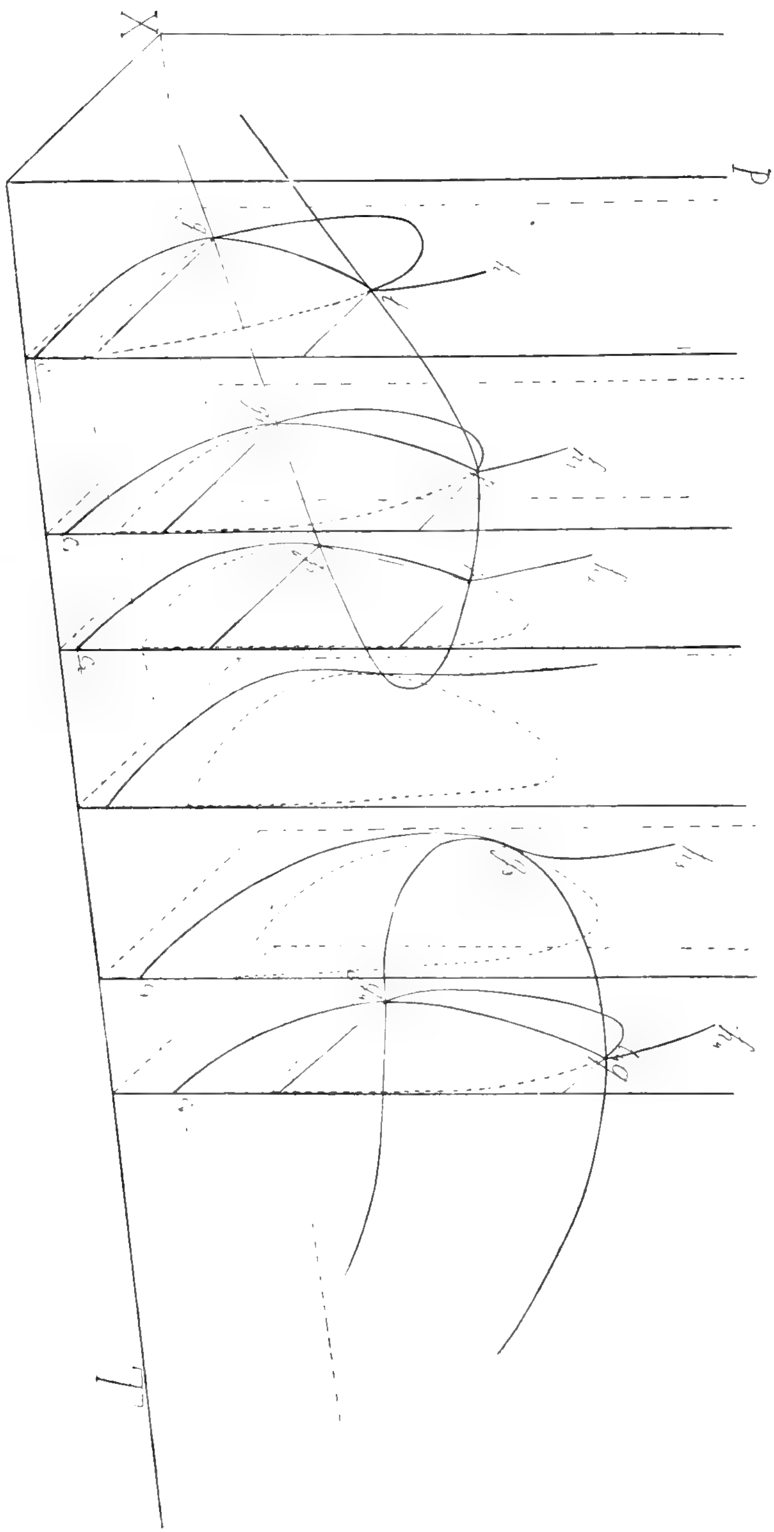


Fig. 9

the line on which the vapour points g or the *minimum three-phase points* lie, are two curves which continuously merge into each other at the maximum three-phase temperature, so past the first ternary critical end-point p.

Above the maximum three-phase temperature we find for a series of temperatures only fluid phases or equilibrium between fluid phases and solid anthraquinone, till at a certain temperature, which we may now call a minimum three-phase temperature, the just-mentioned phenomena repeat themselves, but now in reversed order.

It is now easy to derive from this spacial figure what will be observed when mixtures of ether and alcohol of the concentration x_1 (Fig. 2) are examined with varying quantities of anthraquinone at different temperatures.

If we now assume that the anthraquinone-concentration is not sufficient to reach the ternary first critical end-point, the $(PT)_x$-section which we study, will lie beyond p, and have the following shape (fig. 10).

If the concentration of the anthraquinone just suffices to realize the critical end-point, the $(PT)_x$-section is that of fig. 11 from which follows that the plaitpoint K and the minimum-three-phase point g

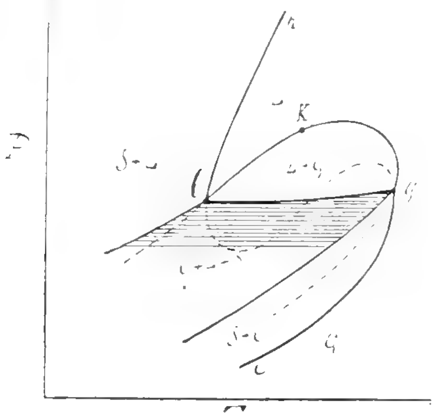


Fig. 10.

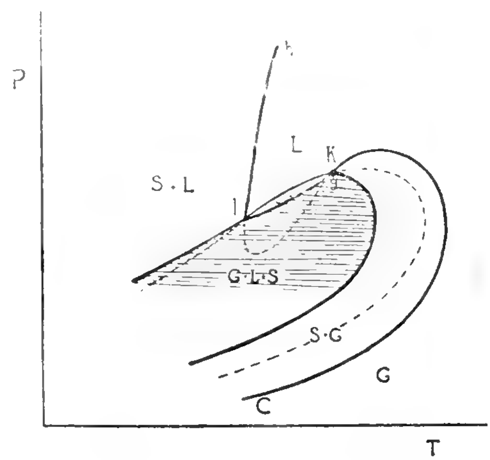


Fig. 11.

have coincided, and that a liquid saturate with solid anthraquinone shows the critical phenomenon.

In the mean time we see that the three-phase equilibrium $S + L + G$ can occur with increase of volume above the critical temperature.

If we now take a mixture with still more anthraquinone, the points l and g, the latter of which is now also a maximum three-phase point draw nearer and nearer to each other, and they finally coincide, the maximum and minimum three-phase-curves merging continuously into each other, and giving rise in this way to the following $(PT)_x$-section (fig. 12).

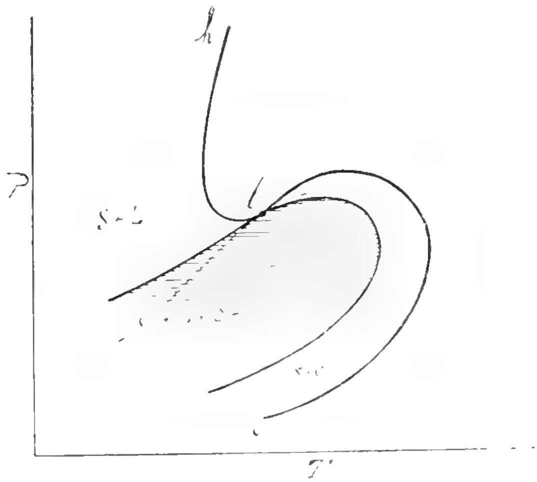


Fig. 12.

For a still greater quantity of anthraquinone the continuous three phase line of demarcation and the line for solid fluid get detached.

For a certain content of anthraquinone the second three phase region will now be reached. At this moment, i. e. with this content of anthraquinone a $(PT)_x$-diagram will be found as is shown in Fig. 13, consisting only of a continuous curve of solid-fluid touching the metastable loop liquid-vapour.

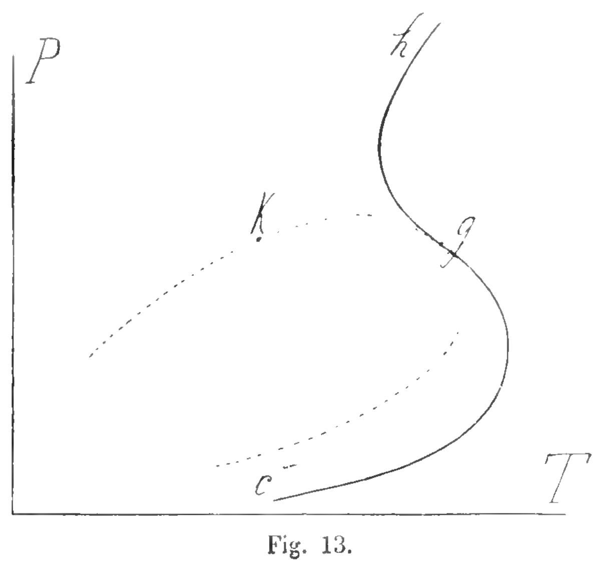


Fig. 13.

If we then choose a concentration which still contains too little anthraquinone to realise the second critical end-point, we get **Fig. 14**, of which it is noteworthy that the plaitpoint K still lies in the metastable region.

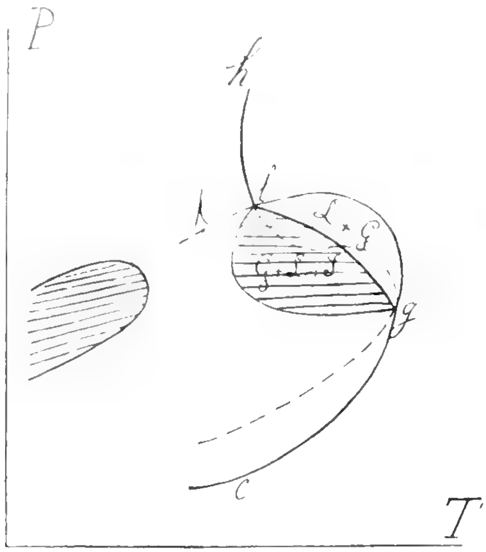


Fig. 14.

For a slightly greater content of anthraquinone the critical end-point can just be reached, and the $(PT)_x$-section has the following shape, l and K having coincided.

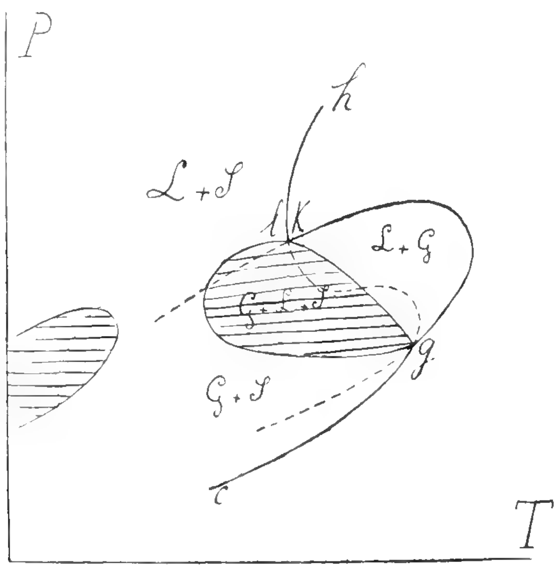
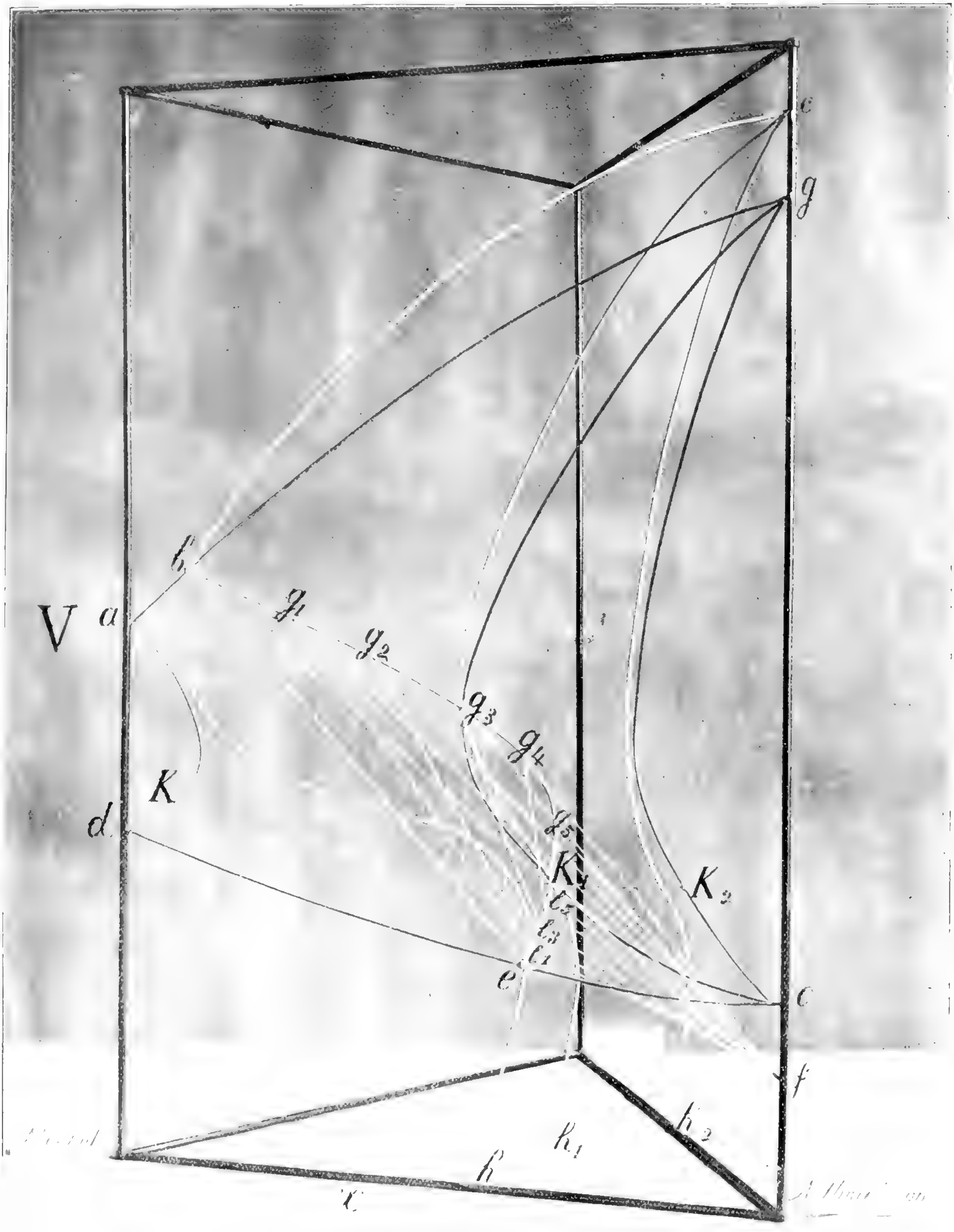


Fig. 15.


V
a
b
c
d
e
f
g
K
K_1
K_2
g_1
g_2
g_3
g_4
g_5
t_1
t_2
t_3
h
h_1
h_2
x

If we finally take a concentration of anthraquinone, which is slightly greater than that which corresponds with the second critical end-point, the corresponding $(PT)_x$-section is represented in Fig. 16.

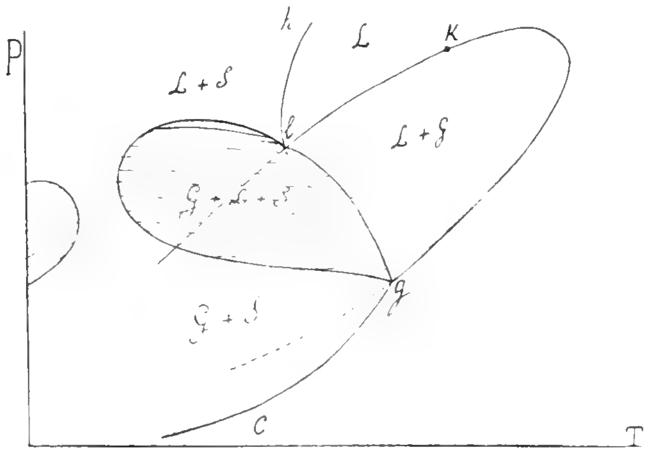


Fig. 16

For a mixture of alcohol-ether with more alcohol, we shall be able to realise two critical endpoints, as appears from fig. 2, till the alcohol-ether concentration has become x_2, for if we join this point x_2 with the point which represents anthraquinone, this joining line just touches the line pR_1q, and this is the line on which the points lie of the maximum three-phase temperature, so that we should get a contact of the three-phase regions for this alcohol-ether concentration, which, however, changes into intersection, as fig. 17 shows.

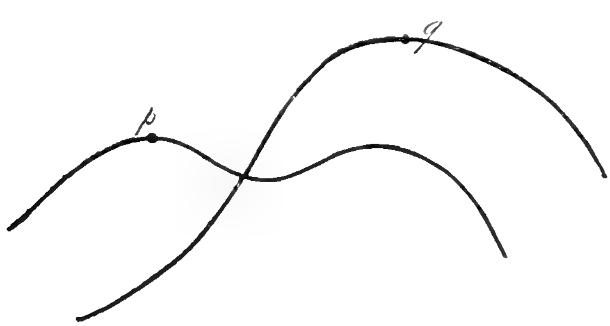


Fig. 17.

If we have a mixture alcohol-ether, lying between x_2 and x_3 communication has been brought about between the two three-phase regions, as Fig. 18 shows.

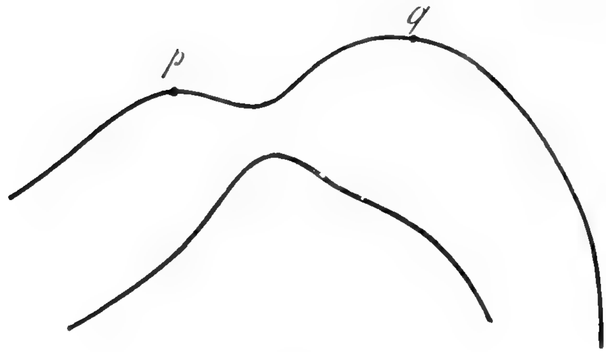


Fig. 18

And p_1 and p_2 coincide for the alcohol-ether concentration x_3, and all the peculiarities have disappeared except this one that there still exists one saturate solution which shows the critical phenomenon (Fig 19), but this too disappears, when we take a mixture with still more alcohol.

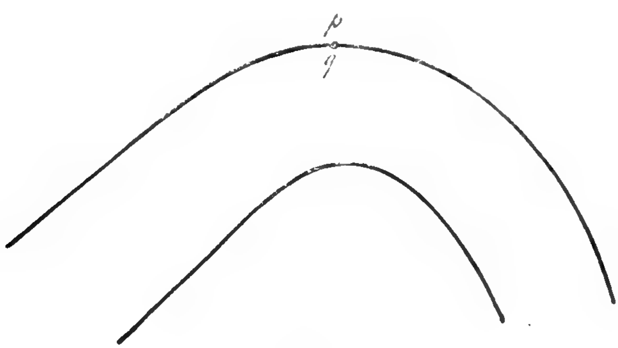


Fig. 19.

In how far this theory has already been corroborated by the experiment, will appear from the communication of Dr. ADA PRINS, who has not studied the system *alcohol-ether-anthraquinone*, but the system *naphthalene-ether-anthraquinone*.

Amsterdam, June 24.

Anorg. Chem. Laboratory of the University.

Chemistry. — "*Critical phenomena of the ternary system ether-anthraquinone-naphthalene.*" By Dr. ADA PRINS. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated in the meeting of June 25, 1910).

In a communication to this Academy [1]) Prof. SMITS has pointed out what changes may be theoretically expected to occur in the system ether-anthraquinone on addition of a third component.

As an experimental study of such a ternary system was still wanting, Prof. SMITS suggested to me the reseach, an account of which will now be given.

As a third component we chose naphthalene, because the melting point of this substance (79,3°) lies considerably lower than the critical point of ether (193°) and its solubility in ether is pretty great; therefore the binary system naphthalene-ether will show no critical phenomena for saturate solutions; nor is this the case for the binary system naphthalene-anthraquinone, and so we may expect a similar behaviour as Prof. SMITS l.c. gave for ether-anthraquinone.

The experimental investigation has confirmed the theory perfectly. On addition of a small amount of naphthalene to the system ether-anthraquinone the two critical end-points p and q continue to exist, draw near to each other on addition of a greater percentage of naphthalene, and so have already disappeared for a quantity of $4^1/_2$ °/₀ of naphthalene. So a spacial interval is formed which does not extend very far, and a projection of which in a concentration triangle has just the shape of fig. 2 in SMITS' paper. As, however,

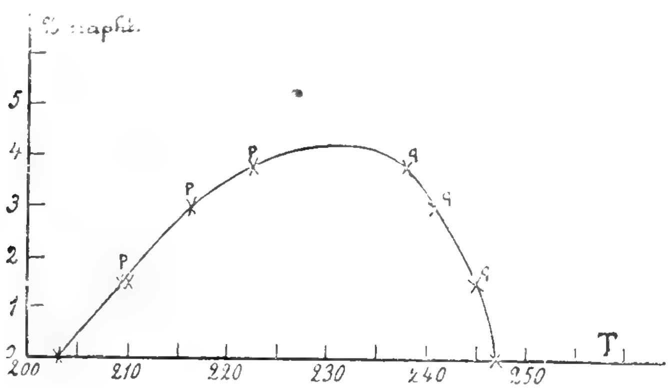


Fig. 1.

1) See the preceding paper.

I have not determined the anthraquinone concentrations of the fluid phases, the exact shape of the curve cannot be given; we can only say that the line anthraquinone — x_3 takes such a course that x_3 is smaller than 0.045 of naphthalene.

If, however, we think the triangle as basis of a trilateral prism, the erect side of which serves as an axis of temperature, and if we then project p and q on the side plane naphthalene-ether, we get fig. 1, from which we can also see, how the point p and q approach each other with an increasing proportion of naphthalene, and finally coincide.

Besides the temperature I also determined the pressure of the critical end-points; from this we can draw up a p-t-projection of the ternary plaitpoint curve (see fig. 2, and the table below):

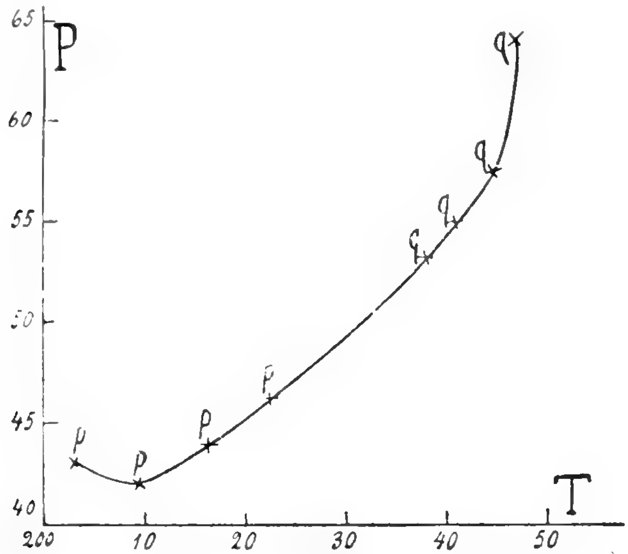


Fig. 2.

Percentage of naphth. with regard to ether	T	P in atm.
$1^1/_2$	p 210.0	42.0
$1^1/_2$	p 209.4	41.8
	q 244.7	57.4
3	p 216.3	43.8
	q 240.8	54.9
3.8	p 222.5	46.2
	q 238.0	53.3

Then this curve indicates the temperatures and pressures at which ternary liquids and vapours become critical in the presence of solid anthraquinone. This line too connects the two critical points p and q of the binary system ether-anthraquinone continuously. The pressure of the point p does not rise at once; on the addition of very little naphthalene (1 %) it falls somewhat, so that the curve shows a minimum lying near p.

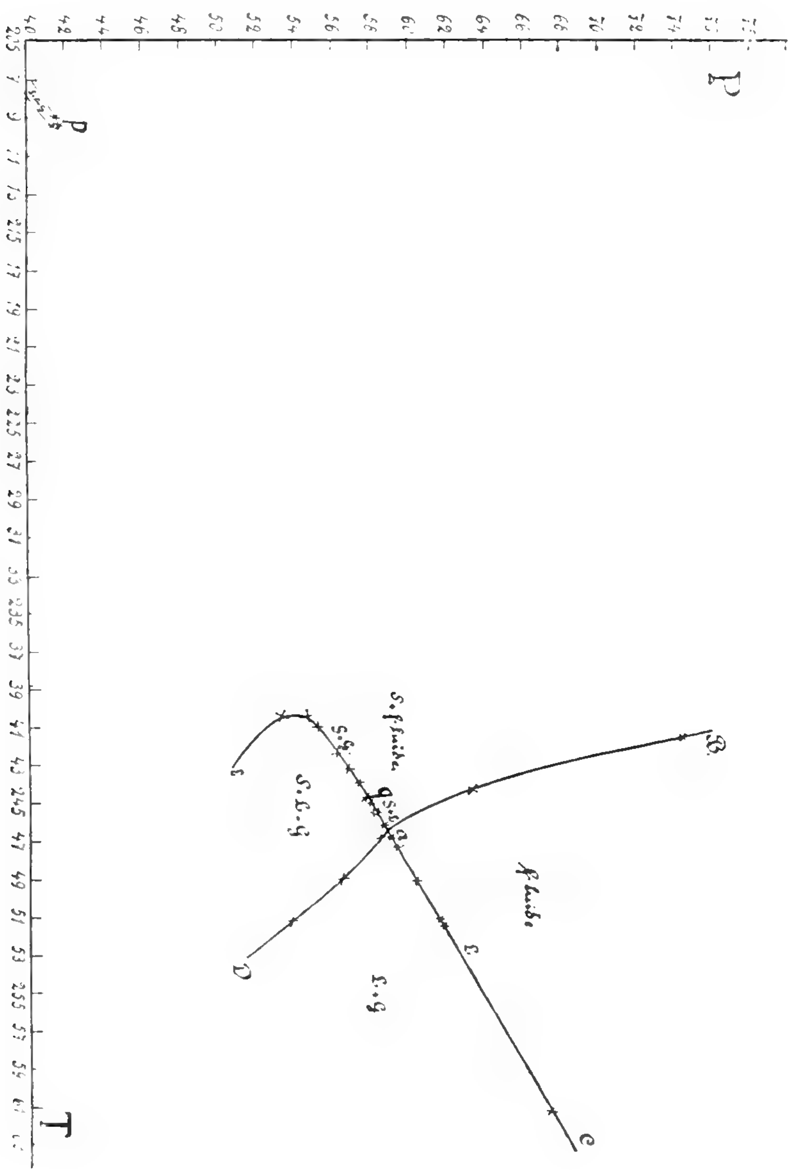


Fig. 3.

The different points of these two curves were found by determining the P-T-sections of mixtures with different quantities of naphthalene. These P-T-sections will in general present a shape as drawn in fig. 16 in Prof. SMITS' paper. It is noteworthy that K, i. e. the critical point $L+G \rightarrow$ fluid moves more to the right on increase of the quantity of anthraquinone. On the left of K the meniscus disappears in the top of the tube, i. e. the gas-phase becomes smaller and smaller, whereas on the right of K the meniscus disappears at the bottom, because the quantity of liquid phase decreases here. On the other hand on the right of the critical endpoint q at the transition $S+L+G \rightarrow S+$ fluid we shall see the meniscus disappear at the top, on its left at the bottom in the tube.

The mixtures with a ratio of $1\frac{1}{2}$, 3, 3,8, and 5 % of naphthalene with regard to the quantity of ether were studied, and gave rise to the graphical representations drawn in the figs. 3, 4, 5, and 6.

None of them have been completed, as the trouble required for this would be too great in comparison with the increase of knowledge it would yield us; for the essential part of the problem is perfectly represented by the determined curves.

Fig. 3 gives the P-T-projection of a section with $1\frac{1}{2}$ % of naphthalene. The quantity of anthraquinone is so great that the point q can be realized, viz. 25 % of the total quantity of the mixture taken.

The region of coexistence $S+L+G$ is bounded by the regions $S+$ fluid, and $L+G$. On the upper loop-line $S+L+G \rightarrow S+$ fluid lies the point q. So at this temperature and pressure the meniscus disappears just in the middle of the tube, solid substance being present. The line AB indicates the transition $S+$ fluid $\rightarrow$ fluid; the line AC the transition $L+G \rightarrow$ fluid. So when this figure is completed the line AC would have to meet the continuation of AD, and the loop AE would have to terminate at the same point. Fig. 4 represents the behaviour of a mixture of the same ratio of ether and naphthalene, but with a smaller content of anthraquinone. This content, viz. 20 % is too small to reach the point q. I succeeded for this mixture to demonstrate the point K in a supersaturated solution.

Figs. 5 and 6 give the confirmation of the possibility mentioned by SMITS that there exist still two critical end-points without the P-T-figure showing a hiatus (see fig. 18 in the cited paper).

For the mixture with 3.8 % of naphthalene (fig. 6) p and q approach each other more than in the 3 % mixture, though their distance apart remains comparatively great; the $S+L+G$ region for the first-mentioned mixture is wider than for the last-mentioned, and in both the minimum and the two maxima are to be observed,

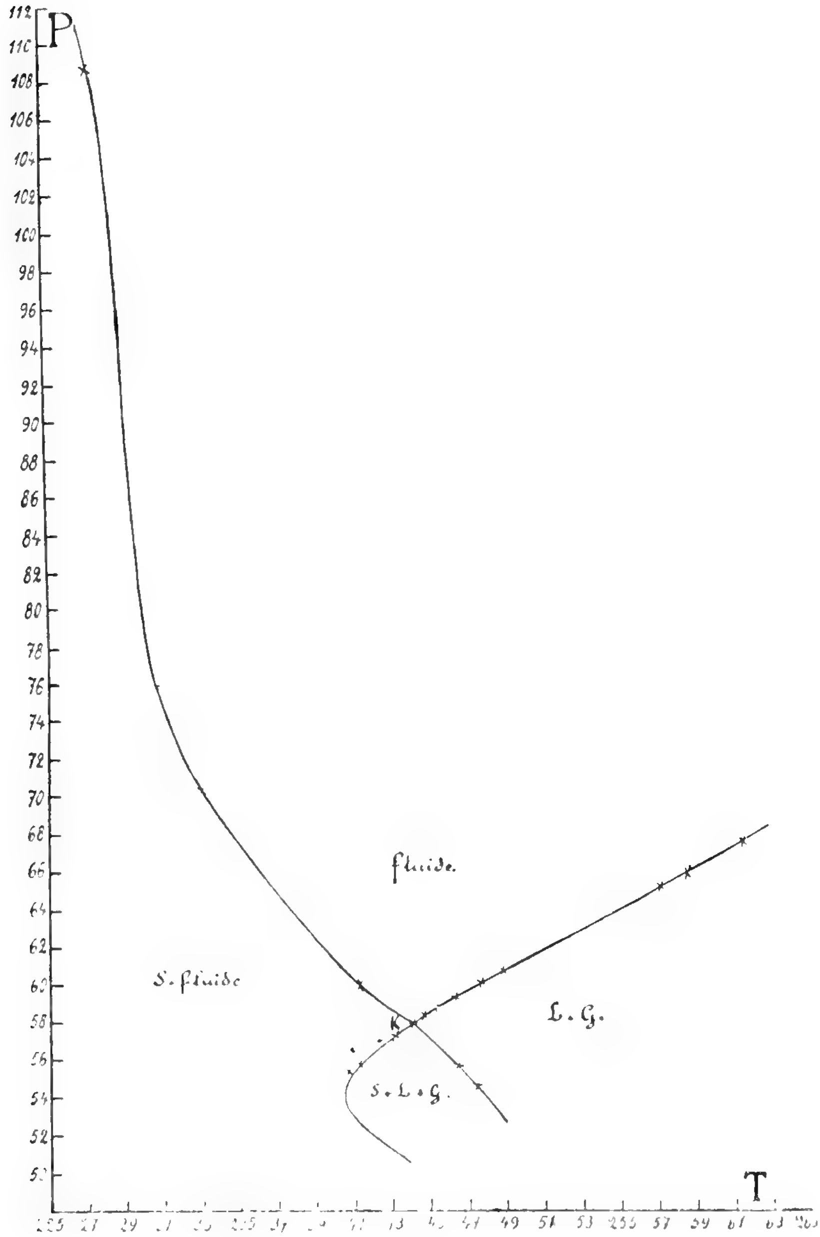


Fig. 4.

though faintly, in the continuous $S + L + G \rightarrow S +$ fluid curve between the points p and q.

It appeared convincingly from the observations of the $5\,^0/_0$ mixture, that the critical end-points have already disappeared here, so that a graphical representation would present nothing particular.

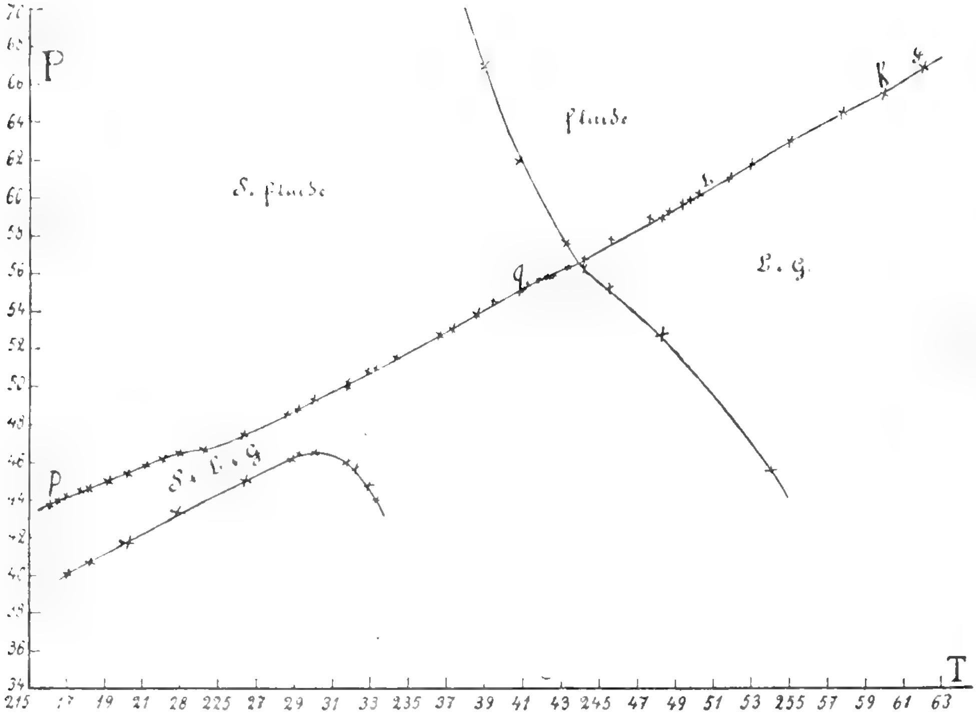

P
T
S. fluide
fluide
L. G.
S. L. G.
p
q
L
K
g

Fig. 5.

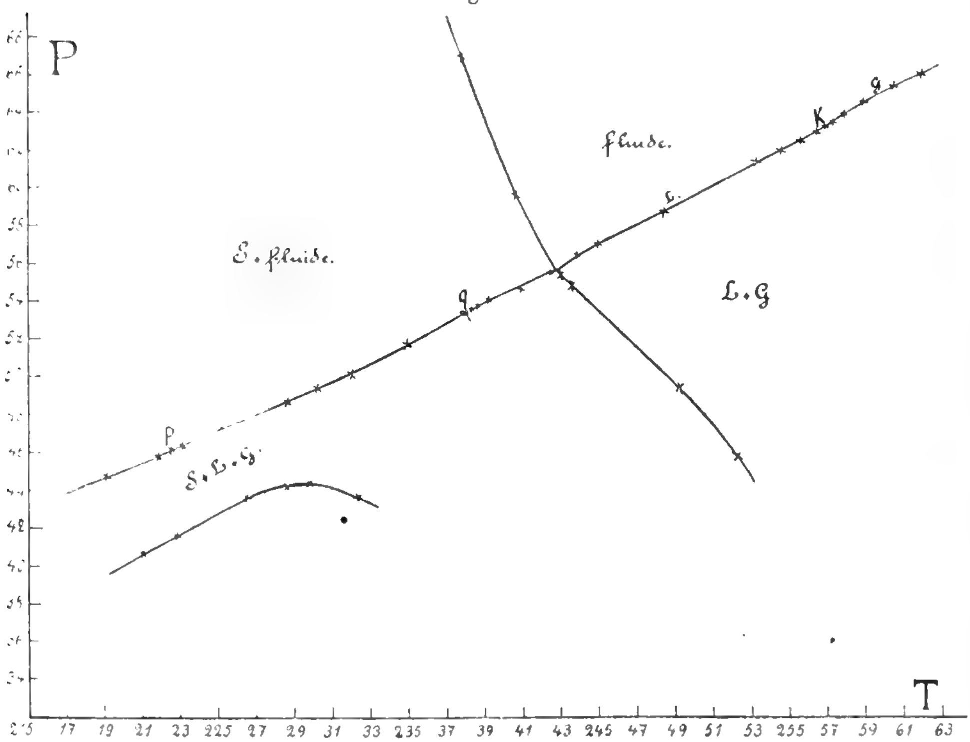

P
T
S. fluide.
fluide.
L. G
S. L. G.
p
q
L
K
g

Fig. 6.

Mixture 5% of naphthalene

	T	P		
	241°	45.6		$S+L+G$
	237°	43.7		$S+L+G$
cooled down to	200°		remains	$S+L+G$

The experiments were arranged in the well-known way [1]). The pressures were read on a manometer of SCHÄFFER and BUDENBERG, and provided with the required correction. The heating was brought about by means of α-monobromonaphthalene boiling under low pressure.

Anorg. Chem. Labor. of the University.

Amsterdam June 23, 1910.

Chemistry. — *"Investigations on the radium content of rocks."* I. By Dr. E. H. BÜCHNER. (Communicated by Prof. A. F. HOLLEMAN.)

(Communicated in the meeting of June 25, 1910)

Introduction. The fact that everywhere in the atmosphere radio-active emanations are found, in connection with the observation — at different places of the surface of the earth — of a very penetrating radiation, suggest very clearly that radio-active substances are found everywhere in the earth's crust. As far as radium is concerned this conclusion was experimentally confirmed for the first time by STRUTT [2]), who by means of the new methods of radium-determination could ascertain not only that a number of typical rocks contain radium, but even succeeded in measuring how great the content of radium was. The quantitative character of his results gave a still greater significance to his investigation in another respect, namely with reference to the question whether radio-active processes can be the cause of the internal heat of the earth. It was already known — from calculations by RUTHERFORD — that the presence of a quantity of radium of 4.6×10^{-14} gr. per gramme of the earth would be sufficient to keep the surface of the earth at a constant temperature, in other words to maintain the thermal equilibrium of the earth. And now STRUTT arrived at the surprising result, that on an average about 1.5×10^{-12} gr. of radium is present per gramme of rock, considerably more than the quantity calculated by RUTHERFORD. It appears

1) See SMITS, Z. phys. Chem. LII, p. 587.

2) Proc. Roy. Soc. A 77, 472 (1906) and 78, 150 (1906).

immediately from this value that calculations of the age of the earth or of the time during which life on earth has been possible, as they have been given among others by Lord KELVIN, must be thoroughly revised; on account of the presence of radium, the earth can have had its present temperature for a very long time already. But leaving this on one side, the too great amount found by STRUTT brings us in a great difficulty, a solution of which may be looked for in different directions; a perfectly satisfactory explanation, though has not yet been given. That the earth should get hotter, as has been asserted, is of course out of the question, if it were only on ground of the consideration that the cooling, in consequence of radiation of heat, can never have gone beyond the point at which the radiation was in equilibrium with the heat which was generated in the interior of the earth and flowed to the surface. So we shall have to take recourse to other suppositions, e. g. that radium is of cosmic origin, or that the desintegration proceeds more slowly under the conditions of the interior of the earth — high temperature and pressure — and accordingly generates less heat, or that the radium accumulates in the earth's crust, in other words that the different rocks at the surface of the earth contain more of it than the interior.

I will not enter as yet into what is to be said in favour or against these hypotheses, as first of all the fact itself requires confirmation. Also some English investigators have seen this; thus EVE[1]) has investigated some ten rock varieties from the neighbourhood of Montreal, and shortly ago COLERIDGE FARR and FLORANCE[2]) rocks from New-Zealand. It is further particularly JOLY[3]) who has occupied himself with these questions; among others he examined the radium content of the different rocks through which the St. Gotthard and Simplon tunnels have been bored. Though the values found by JOLY are on the whole much higher than those of the other investigators mentioned, yet they all arrive at the same result in so far that really the radium content is of the order of magnitude of 10^{-12} gr. per gramme of rock as was found by STRUTT. So we have investigations of rocks from England, Canada, British India, and New-Zealand; the continent of Europe is, however, hardly represented. As however extension of experimental material is very desirable in view of the far-reaching conclusions which may be attached to the results of these investigations, I have taken up the investigation of a number of European and Dutch-Indian rocks.

1) Phil. Mag. [6] 14, 231 (1907).
2) Phil. Mag. [6] 18, 812 (1909).
3) Phil. Mag. [6] 18, 140 (1909); also Radioactivity and Geology, Londen 1909.

The results may contribute at the same time to the solution of the question if the radium content, which may differ pretty considerably for different rocks, is connected with other properties, e.g. chemical composition or age.

As a first series the results of the investigation of ten rocks of the West-coast of Sumatra are given in this communication. I gladly avail myself of the opportunity to express my cordial thanks also here to professor MOLENGRAAFF at Delft for the kindness with which he placed the required material at my disposal.

Method. The methods to determine such slight quantities of radium quantitatively, have been given by STRUTT and BOLTWOOD and others.

The principle on which they rest is this: the solution containing radium is stored till the equilibrium quantity of emanation has formed; then it is expelled by boiling, collected, and conveyed to a so-called emanation-electroscope, in which the measurement takes place in the well-known way. As is known, the quantity of emanation is proportional to the radium which is present in the solution, and so we can calculate from the accelerated movement of the gold leaf in the electroscope, how much radium is present. The easiest way to do this is by subjecting a solution with a known quantity of radium to the same process: expelling the generated emanation by boiling, conveying it to the electroscope and measuring it.

Starting from these principles, I arranged the experiments as follows: 25 grams of the rock which had been ground to a fine powder beforehand was fused in a platinum dish together with 80 à 100 grams of potassiumsodiumcarbonate in a furnace during four or six hours. Then the melted substance was chilled, after which it easily separated from the dish; it was then reduced to powder in a high mortar, and digested on the waterbath in a beaker for some hours, the mass being continually stirred by means of a hot-air-engine. Then it was filtered at the pump, and the filtrate was poured into a flask, which was kept firmly closed. The remaining carbonate mixture, which still contained silicates [1]) was then evaporated to dryness with hydrochloric acid; after having been moistened again with hydrochloric acid, and after having stood for twenty minutes, hot water was poured over it, and it was filtered again. Thus an acid solution was obtained, which was also preserved. A remaining residue of silicic acid was dissolved in boiling sodiumhydroxyde, and

[1]) Washing till all the sodium silicate has disappeared from the precipitate, takes very long, and often gives a turbid filtrate; therefore the method indicated in the text was chosen, as a much shorter one.

the solution was added to the before mentioned alkaline filtrate. So the total quantity of the rock is found back in two solutions, an alkaline and an acid one, which were separately preserved and boiled to prevent precipitation of a voluminous silicic acid precipitate.

After the solutions had been set aside for at least a month, the emanation was expelled in the way as will be clear from the sub-

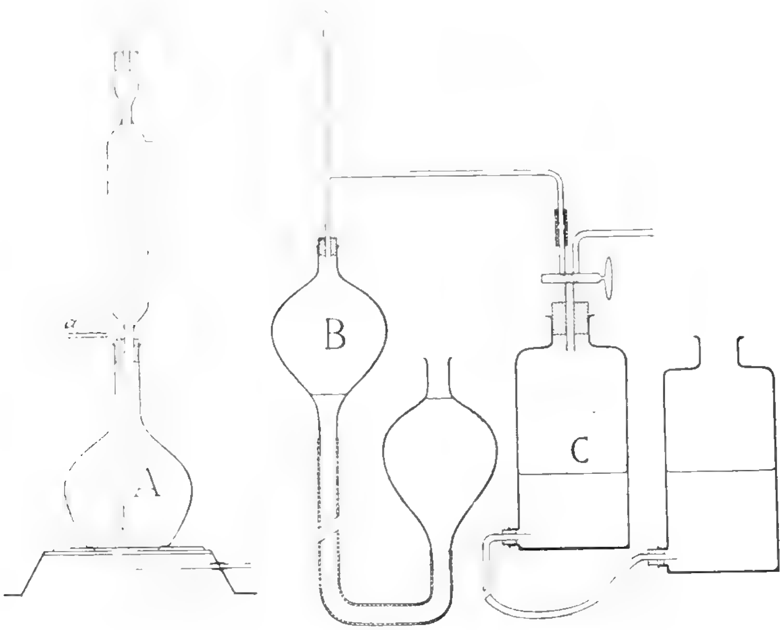


Fig. 1.

joined figure. In A the solution is boiled; the water-vapour condenses in the cooler; the generated gases with the emanation collect in the flask B over a saturate common salt solution [1]); after 25 à 30 minutes the boiling is stopped; the gas is sucked from the flask B into the bottle C, which had been beforehand exhausted, and a clip is opened at d; air flowing in from outside drives all the emanation which might still be found above the solution in the flask or in the cooler in this way into the bottle C, again over a salt solution. Finally the emanation is now transferred from C into the electroscope, for which purpose the latter had been first exhausted. Then gas and emanation flows through a tube with lime, a tube with phosphorpentoxyde and at last a tube with cotton wool into the electroscope; when the contents of the bottle C have been quite

[1]) At 20° the absorption coefficient of emanation in water amounts to 0,28 in saturate salt solution to 0.04.

transported, the electroscope is further filled with air, so that what still remains in the drying tubes, also flows in.

The electroscope was of the WILSON type; it is represented schematically in fig. 2. It consists of a copper cylinder, 16 cm.

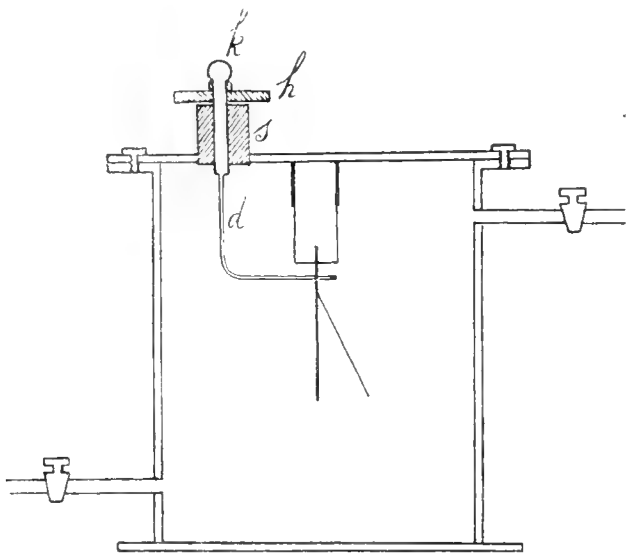


Fig. 2.

high, and with a diameter of 12 c.m., so that the capacity is $\pm$ 1700 cm³. By means of two cocks the air can be sucked out, and the gas charged with emanation admitted; further two glass windows are adjusted diametrically, on a level with the leaf system. This consisted of a fixed copper strip, and a movable aluminium leaf, and was attached to a piece of amber, which tightly fitted in a copper tube, soldered to the lid; it was charged by means of the copper wire *d*, which turned airtight in an ebonite stopper *s*. For this purpose the knob *k* was connected with the negative pole of a storage battery of 160 cells[1]), the wire *d* being in contact with the leaf system. By turning the handle *h*, which was also made of ebonite the connection between *d* and the leaf system was broken; then that of *k* with the battery was interrupted, and finally *d* was turned so far till it was stopped by the wall of the electroscope, which is connected with the earth. The reading takes place by means of a telescope, the eye-piece of which is supplied with a scale; the time is noted that the movable leaf requires to pass a definite number of scale divisions. These were the same in all the measure-

[1]) This battery was placed at my disposal by the "Amsterdamsche Universiteits-vereeniging". I will once more express my sincere thanks to the directors of this institution here.

ments, so that the inequivalence of different points of the scale has no influence. By means of the γ-rays of 1 mg. of radium bromide, which was placed at a fixed place and level above the electroscope, it was ascertained whether the capacity varied. Slight variations actually now and then occurred; therefore the measurements were all corrected to one and the same capacity. The normal leak was regularly determined and subtracted; the measurements did not take place until $2^1/_2$ a 3 hours after the introduction of the emanation, because as is known, in consequence of the formation of the active precipitate RaA, B, and C, constant values for the velocity of discharge are not obtained until then.

In conclusion a word on the gauging of the electroscope. This was generally done by dissolving a uranium mineral, and expelling the emanation by boiling, and conveying it into the electroscope; if then by chemical analysis it is determined how much uranium the solution contains, the content of radium may be calculated by the aid of the ratio of uranium and radium, which is known by BOLTWOOD's investigations. I preferred a direct method to this, viz. a comparison with a solution of a known quantity of radium bromide. For this purpose Professor E. RUTHERFORD of Manchester kindly sent me a solution, which according to his statements contained $0{,}157 \times 10^{-9}$ grammes of Ra per $cm.^3$ [1]). 4 $cm.^3$ of this solution were used for the comparison; the emanation which had generated after three weeks, was conveyed into the electroscope. It was found in this way that a velocity of the leaf of 10 scale divisions an hour corresponded to $1{,}08 \times 10^{-11}$ gr. of Ra.

Results. The values obtained by the described method have been compiled in the subjoined table, which indicates the quantity of radium per gramme of rock.

Quartz porphyrite,	river Malakoetan	$1{,}3 \times 10^{-12}$ gr.
Granite,	Siboemboen	2,5
Basalt,	volcano of Asar	13,0
Andesite,	Padang	5,1
Augite andesite,	Soengei Landei	1,3
Augite andesite,	Ajer Kolbing	0,56
Granitite,	river Pasier	1,5
Granitite,	Soengei Lumani	3,1
Diorite,	Ahoer Tampoeroengo	0,30
Diabase,	Siboemkang	0,34

1) I gladly express my hearty thanks to Professor RUTHERFORD for the readiness with which he complied with my request.

It may further be mentioned here that all the chemicals used were examined separately in the same way to ascertain whether they contained radium; this appeared not to be the case. It is further noteworthy that every solution, both the acid and the alkaline ones, were boiled two or three times, and that the values inserted in the table are the average ones of the results obtained in the different experiments. By far the greater part of the radium is found in the acid solution; it was even often, — particularly for the rocks poor in radium — not to be demonstrated at all in the alkaline liquid.

It is seen that this investigation yields a similar result as the preceding ones: the rocks from Sumatra have a same relatively high content of radium of the order of magnitude 10^{-12} gr. per gramme.

We shall not yet draw any conclusions concerning the problems mentioned in the introduction, but postpone them till a number of rocks from *Borneo* have been discussed in a following communication.

Anorg. Chem. Laboratory University of Amsterdam.

Botany. — *"On the structure of the nucleus and karyokinesis in Closterium Ehrenbergii Men."* By Prof. C. VAN WISSELINGH. (Communicated by Prof. J. W. MOLL).

(Communicated in the meeting of June 25, 1910).

While the structure of the nucleus and the karyokinesis of *Spirogyra* have been repeatedly examined, the nuclei of the genus *Closterium* have rarely been the subject of investigation. This is the more remarkable, because the nuclei attain a considerable size. The few statements made in the literature about the structure of the resting nucleus of *Closterium* chiefly amount to this that the nucleus agrees with that of other algae, especially *Spirogyra*; thus for instance DE BARY [1]) states: Ein Zellkern von der für *Spirogyra*, *Zygnema* beschriebenen Structur nimmt die Mitte der Desmidieenzelle ein. DE WILDEMAN [2]) says: Le noyau des *Closterium* est du même type que celui des *Cosmarium* et des *Spirogyra*. The latter also gives some particulars of the nucleus. According to DE WILDEMAN the nucleus is formed by a rounded or rectangular mass, containing a large nucleolus at its centre. The nucleus contains hardly any

[1]) A. DE BARY, Untersuchungen über die Familie der Conjugaten, 1858, p. 40.

[2]) E. DE WILDEMAN, Recherches au sujet de l'influence de la température sur la marche, la durée et la fréquence de la caryokinèse dans le règne végétal, Extrait des Annales de la Société belge de microsc., t. XV, 1891, p. 47 and following.

chromatin, while the nucleolus stains very deeply, no matter what reagent is used. It is remarkable, that DE WILDEMAN in his investigations of living material, came to results somewhat different from those obtained with fixed material. In the living material he found considerable variation. In some cases the nucleolus was rounded, as in the fixed material, in other cases, however, the central mass was of a granular substance, and missed definite shape. DE WILDEMAN could often distinguish small globules, which were separate or united. The number of these small bodies decreased by fusion.

The accounts in the literature of the nuclear division of *Closterium* are as scarce as those of the structure of the nucleus. Some investigators, especially FISCHER [1]), HAUPTFLEISCH [2]) and LÜTKEMÜLLER [3]) have examined the division of *Closterium* in detail, but their investigations refer almost exclusively to the cell-wall. Several investigators, including FISCHER [4]) and DE WILDEMAN [5]) have directed attention to the division of the chromatophores, which begins before or during the nuclear- and cell-division, and to the movement of the daughter-nuclei along the cell-wall to the places where the chromatophores are divided into two. FISCHER further mentions that several nuclear bodies occur in the daughter-nuclei.

The most important data concerning the nuclear division have certainly been furnished by KLEBAHN [6]). His investigations refer to the germinating zygotes, in which he observed the union of both nuclei to one, the mitotic division of this nucleus into two equal daughter-nuclei and the subsequent mitosis of these daughter-nuclei, which by that process each produce two unequal nuclei. The figures of KLEBAHN clearly show that the nuclei divide by mitosis, and that in this division spindle formation takes place. The nuclear- and cell-division of the vegetative cells was not examined by KLEBAHN.

It is evident from the above summary of the results of various authors, that our knowledge about the nuclear structure of *Closteterium* is still very incomplete. The accounts of some investigators of the resemblance of the nuclei of *Closterium* to those of

1) A. FISCHER, Ueber die Zellteilung der Closterien, Bot. Zeitung, 1883, N⁰. 14, p. 225.

2) P. HAUPTFLEISCH, Zellmembran und Hüllgallerte der Desmidiaceen, Inaugural-Dissertation, 1888.

3) J. LÜTKEMÜLLER, Die Zellmembran der Desmidiaceen, Beiträge zur Biologie der Pflanzen (Cohn), VIII. Bd., 1902, p. 347.

4) l. c. p. 226, 232 and 233.

5) l. c. p. 50, 51 and 52.

6) H. KLEBAHN, Studien über Zygoten, I, Die Keimung von Closterium und Cosmarium, Pringsheim's Jahrb. für wiss. Botanik, XXII. Bd., p. 420 et seq.

Spirogyra and other *Conjugatae* have especially little value, because there is not even a definite consensus of opinion among botanists as to the structure of the *Spirogyra* nucleus, which has been so frequently examined. For instance, how different are the views about the nucleolus of *Spirogyra*. Some take it as identical with the nucleoli which occur in the vegetable kingdom generally, while others regard it as a small nucleus lying in a larger one. Hence the mere statement that the nucleus of *Closterium* agrees with that of *Spirogyra* means but little. Further investigations will have to show whether the nucleolus of *Closterium* indeed agrees with that of *Spirogyra*; i. e. whether it is an ordinary nucleolus, or something corresponding to a small nucleus, or something else. The variations in the nucleus of *Closterium*, mentioned by DE WILDEMAN certainly heighten the interest in this point of investigation.

The investigations of the vegetative nuclear division in *Closterium* have brought but little to light. Thus it is not even certain, whether the nuclear division is a mitosis. This may only be thought probable in connection with the results which KLEBAHN obtained with germinating zygotes and because the nuclei also divide by mitosis into other *Conjugatae*. Whether chromosomes arise, whether a spindle is formed what changes the nucleolus undergoes, of all these cardinal points in the investigation of the karyokinesis of *Closterium* nothing is known as yet.

Many years ago I intended to examine the nuclear and cell division of *Closterium*. More than once I had to give up my attempts for lack of sufficient material, until in March and April 1910 I was able to cultivate *Closterium Ehrenbergii Men.* for some time successfully so that at last I had at my disposal a very abundant and healthy material with numerous stages of division, which enabled me to examine repeatedly all occurring stages of division.

Fixed material had to be used for the investigation because not much is visible of the nuclear division in living material, even less than in *Spirogyra*; FLEMMING's mixture was used for fixation (1 g. chromic acid, 6 g. glacial acetic acid, 0.5 g. osmic acid, 120 c.c. distilled water). To bring the nuclear figures into prominence a solution of chromic acid was used. With the help of this the cytoplasm with the chromatophores and the starch was dissolved. When this has happened the flat nuclei turn over, which is of great advantage, because it enables one to examine microscopically the same nucleus in a horizontal and in a vertical position; this applies to the resting nucleus as well as to the various stages of division. After more prolonged action the chromic acid also dissolves the nuclei, but the different parts are dissolved to an unequal extent. This circumstance may therefore also contribute to a wider knowledge of the nuclear structure. Sometimes the material was examined after it had been acted on by chromic acid for some time, washed and stained with "Brillantblau extra grünlich". I will omit the details of this method because I have already described it in an earlier publication [1]. I only wish to point out that the material has to be very carefully treated with FLEMMING's solution. By this treatment the nuclei must obtain a great resistance to chromic acid; on the other hand the cytoplasm with the chromatophores and the starch must slowly dissolve in the chromic acid solution, without contracting or losing their definite outline. For this purpose the material was fixed with a small quantity of FLEMMING's mixture and was daily examined to see whether the action had been sufficient and if necessary more of the FLEMMING's mixture was added.

In this paper the results of my investigation will be mentioned, as far as they concern the structure and the division of the nucleus. Before doing so I must briefly indicate my stand-point with respect to the different views of the nuclear structure and the karyokinesis in *Spirogyra*, for otherwise it would not be clear what I mean by such expressions as corresponding to or different from *Spirogyra*. After my last publications on the karyokinesis of *Spirogyra* I have more than once returned to the subject, not only with species about which I had written before, but also with others. In no case did these investigations raise doubt as to the earlier results. On the whole the newly examined species differed very little from those that had been examined before. The investigation of a species received from England alone led to new results of which I hope to give an account later.

Nevertheless all the species examined agree in this, that the nucleolus or the nucleoli must be regarded as small nuclei inside a

1) Ueber den Nucleolus von Spirogyra, Ein Beitrag zur Kenntnis der Karyokinese, Bot. Zeitung, 56. Jahrg. 1898, 1. Abt. p. 199.

Ueber das Kerngerüst, Zweiter Beitrag zur Kenntnis der Karyokinese, Bot. Zeitung, 57. Jahrg., 1899, 1. Abt., p. 155.

Ueber Kernteilung bei Spirogyra, Dritter Beitrag zur Kenntnis der Karyokinese, Flora oder Allgem. Bot. Zeitung, 1900, 87. Bd. 4. Heft, p. 356.

Ueber abnormale Kernteilung, Fünfter Beitrag zur Kenntnis der Karyokinese, Bot. Zeitung, 61. Jahrg., 1903, 1. Abt. p. 210.

Ueber die Karyokinese bei Oedogonium, Sechster Beitrag zur Kenntnis der Karyokinese, Beihefte zum Botan. Centralblatt, Bd. XXIII, 1908, Abt. 1, p. 138, 139, 148 et seq.

large one, a view which agrees with the opinion of CARNOY[1]), who first drew attention to the interesting structure of the nucleolus of *Spirogyra*. As I have found,[2]) all the details which can be distinguished in a nucleus can also, by suitable means, be demonstrated in the nucleolus of *Spirogyra*, namely: a wall and contents, containing one or two threads, or a network, such as nuclei usually have, in addition to a substance which may be compared to that of ordinary nucleoli. As a rule these elements of the contents do not entirely fill up the space inside the nucleolus and cavities containing fluid may further be distinguished inside it. Also in their division the nucleoli of *Spirogyra* show very important points of agreement with nuclei, for instance the dissolving of the wall and of the substance agreeing with that of ordinary nucleoli and the longitudinal splitting of bodies which are comparable to chromosomes.

The nuclei of *Spirogyra* are, as far as the research extends, distinguished from all vegetable nuclei by their remarkable nucleoli. It is self-evident therefore that in examining the nuclei of *Closterium*, which are still so little known, I paid special attention to the nucleoli, the peculiar appearance of which had already attracted the attention of investigators. The answer to the question, whether *Closterium* possesses as remarkable nucleoli as those of *Spirogyra* and whether therefore both these Conjugatae agree in this respect, was thus an important point of investigation for me. In other respects also I have, however, endeavoured to bring to light as much as possible concerning the nuclear structure and karyokinesis.

Resting nucleus. The unicellular plants possess a single nucleus. As a rule it is found near the centre of the cell, i.e. it is about equidistant from both apices of the cell and everywhere about equidistant from the cell-wall, which with regard to the nucleus is concave on one side, convex on the opposite side. Often, however, the nucleus is somewhat nearer to one end than to the other and it occasionally lies considerably nearer to that part of the wall, which turns its concave side towards the nucleus.

As far as its shape is concerned the nucleus of *Closterium* agrees with that of *Spirogyra*, for it is flattened, appearing oval when viewed from above and circular when viewed sideways. The position of the nucleus in the cell also agrees with that seen in *Spirogyra*. The flattened poles are turned towards the apices of the

[1]) J. B. CARNOY, Biologie cellulaire, fasc. 1, p. 236.

[2]) Ueber den Nucleolus von Spirogyra, l. c. p. 220 et seq. Ueber Kernteilung bei Spirogyra l. c. p. 374 et seq. p. 359 and 360. Ueber abnormale Kernteilung, l. c. p. 215 et seq. and 241.

cell. The size of the nuclei surpasses that of the nuclei of *Spirogyra*. Observations on the diameter of nuclei of *Closterium Ehrenbergii* and of some thick species of the genus *Spirogyra* are given below.

Closterium Ehrenbergii Men. from 37 to 66 μ, average 53 μ, found near Groningen.

Spirogyra crassa Ktz. from 40 to 44 μ, average 42 μ, found near Utrecht and determined by MOLL [1]).

Spirogyra maxima (Hass.) Wittz. from 31 to 40 μ, average 36 μ, found near Groningen.

Spirogyra triformis n. sp. (with 6 chromosomes in the equatorial plate [2]) from 27 to 31 μ, average 28,5 μ, found near Steenwijk.

Spirogyra setiformis (Roth.) Kg. from 27 to 31 μ, average 27 μ, found near Steenwijk.

In the nucleus of *Closterium Ehrenbergii* there may be distinguished the same component parts, as generally occur in nuclei namely the nuclear wall, the network, the nucleolus or the nucleoli and the nuclear fluid. The wall of the nucleus is thin; it seems to be thinner than that of *Spirogyra*. It cannot long resist the influence of chromic acid. The network has a delicate, regular, reticulate structure. The nucleolus has a peculiar appearance. It consists of a collection of more or less rounded polyhedral bodies, which are mostly attached to one another, but still may quite well be distinguished separately. When the network has dissolved in chromic acid, it may easily be observed that many of the small bodies are joined. Each body may be regarded as a separate small nucleolus and the whole as a collection of small nucleoli. Amongst these occasionally one or a few occur which are considerably larger, and also more or less spherical. It seems to me that the small nucleoli lie in the meshes of the network which probably prevents their fusion to one great nucleolus. In dealing with the karyokinesis we will show that there are good reasons for this view, as well as for the conclusion that they consist of a fluid substance. The small nucleoli agree with those which are generally found in the vegetable kingdom and not with those normally occurring in *Spirogyra*. They are not at all to be compared with small nuclei. They have no wall, neither is the collection of nucleoli surrounded by a wall, nor can threads be distinguished in it as integral elements or be liberated from it by means of chromic acid, as is the case with the nucleoli in the nuclei of *Spirogyra*.

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1) J. W. MOLL, Observations on Karyokinesis in Spirogyra, Verhandelingen der Koninkl. Akad. van Wetensch. te Amsterdam, 2e sect. D. 1. N⁰. 9, p. 16.

2) C. VAN WISSELINGH, Ueber Kernteilung bei Spirogyra, l. c. p. 356 and 362.

In *Closterium Ehrenbergii* I have not found important variations of the nucleoli in different nuclei, such as should occur according to DE WILDEMAN. It is true that the small nucleoli in the nucleus seem to form a more compact mass in the one than in the other which probably has led DE WILDEMAN to distinguish two different types. I have not however found essential points of difference.

Karyokinesis. When in *Closterium Ehrenbergii* division is about to take place, modifications occur in the cytoplasm as well as in the nucleus. At some distance from the nucleus both chromatophores show a constriction as the beginning of a division into two. Cytoplasm collects near the nucleus and the latter also shows considerable modifications. The nucleoli become distributed in the nucleus. The nuclear wall is dissolved and the network forms visible threads. The most striking of these three processes is the distribution of the numerous nucleoli in the nucleus. The threads arising from the network are at first rosary-like. They slowly contract to form a great number of short thick threads or chromosomes. Meanwhile many nucleoli unite so that often large globules arise. The modifications which the network undergoes, seem to go hand in hand with the union of the nucleoli. By the contraction of the network to broad threads the nucleoli seem to have more opportunity for fusion. The numerous nucleoli sometimes prevent the distinct observation of the chromosomes. When the action of the chromic acid somewhat disintegrates the nuclear figure the chromosomes become distinctly visible. The fusion of the nucleoli to globules shows that they consist of a fluid substance. A large proportion of the nucleoli get outside the nucleus; in consequence of this a great number of globules of different sizes are seen on either side of the nucleus. Gradually these dissolve in the cytoplasm.

When the nucleus has undergone the above mentioned modifications, the formation of the equatorial plate begins. The chromosomes move to the plane passing through the equator of the nucleus and finally they all lie in that plane. Thus the equatorial plate has been formed. The latter has the following peculiarities. It is flat, seen sideways it is almost round. It is of a considerable size. The diameter is from 26 to 40 μ on an average 35 μ. Although the structure of the equatorial plate is rather favourable for the determination of the number of chromosomes, the latter are so numerous that I have not succeeded in counting them exactly. There are more than sixty. Just as in other cases (*Spirogyra* [1]), embryosac [2]) of

[1]) Ueber den Nucleolus von Spirogyra, l.c. p. 209.
[2]) Ueber das Kerngerüst, l.c. p. 168.

Fritillaria and *Leucojum*, *Oedogonium*[1]) the chromosomes are not quite free in the cytoplasm but form a connected whole by means of delicate fibres. The cytoplasm may be dissolved and the entire equatorial plate isolated by treating material, fixed with FLEMMING's mixture, with chromic acid solution. The overturned equatorial plate is found floating and at first the chromosomes keep their original position with respect to each other. Only after prolonged action they become loose and separate. The chromosomes differ in length. In general they are short, most of them very short; the longer ones protrude from one of the sides of the equatorial plate. Their shape varies; some, especially the smaller ones are straight or slightly bent. Others are bent in different ways and form J-, S-, L-, U-, V-shaped and other figures. Longitudinally they show a line which indicates the place where they will split into two.

As follows from what has been said above, *Closterium* again supplies an example of a nucleus with chromosomes of varying length. Formerly this phenomenon attracted little attention in the vegetable kingdom. In 1898 I[2]) pointed out that in *Spirogyra* two of the twelve chromosomes differ from the others in having a thinner end from which a small thread-like body could be isolated which was rather resistant to the action of chromic acid, when the rest of the chromosomes had already been dissolved. The two aberrant chromosomes were often a little longer than the others. Later on I noticed two corresponding chromosomes in a *Spirogyra*-species in which the whole number of chromosomes was six[3]). In *Closterium* the chromosomes only differ in length. In 1905 this phenomenon was noticed in the vegetable kingdom by ROSENBERG[4]) namely in *Listera* and in 1908 by myself[5]) in *Oedogonium*; later it has also been noticed in other plants.

The division of the equatorial plate into two halves and the separation of these halves takes place in the ordinary way. The chromosomes split longitudinally. When the halves of the plate separate the ends of the chromosomes remain together longest. Consequently both halves often form rhomboidal figures and later on when separated V-formed ones with the arms pointing to each other. This especially occurs with the smaller chromosomes. With the longer chromosomes first the parts that are

[1]) Ueber die Karyokinese bei Oedogonium, l.c. p. 140.

[2]) Ueber den Nucleolus von Spirogyra, l.c. p. 205 et seq.

[3]) Ueber Kernteilung bei Spirogyra, l.c. p. 147.

[4]) Zur Kenntnis der Reduktionsteilung in Pflanzen, Botan. Notiser, 1905, Separatabdr. p. 9.

[5]) Ueber die Karyokinese bei Oedogonium, l.c. p. 141,

united with other chromosomes split and afterwards the free protruding portions. When the halves of the equatorial plate separate the halves of these chromosomes remain connected for a longer time at their free ends than at the ends which are not free. Finally these halves of the chromosomes also separate completely. During the action of the chromic acid the above mentioned particulars are readily observable. The rhomboidal, V-shaped and other figures which arise from the splitting chromosomes become entirely free.

The halves of the equatorial plate become smaller while they separate. At the same time the free parts of the longer chromosomes turn more or less outwards; thus there is also developed in *Closterium* the typical aspect peculiar to the phase known as diaster.

As I have remarked above, the nucleoli get into the cytoplasm, where they form on both sides of the nuclear-figure a number of greater and smaller globules. It not infrequently happens that at first a portion of the nucleoli remains behind attached to the equatorial plate between the chromosomes and sometimes considerable globules and masses are found between and on the separating halves of the equatorial plate.

When the equatorial plate has been formed the formation of the transverse wall also begins. The transverse wall develops in a way corresponding to that of *Spirogyra*. The process begins at the cell-wall and continues inwards until the cell is divided into two daughter-cells by a flat diaphragm. In *Closterium* this process is followed by another one, namely by the splitting of the cell-wall. There where the transverse wall arises, the cell acquires a constriction, which becomes deeper and deeper and is accompanied by a fission of the wall of the mother cell and with a splitting of the transverse wall, the halves of which develop strongly after splitting.

The equatorial plate is surrounded by the spindle. This arises from the cytoplasm surrounding the nucleus. In accordance with the size of the nucleus and of the equatorial plate the spindle is wide but it is not strongly developed, far less than in *Spirogyra*. The spindle fibres are delicate and thin I have pointed out before [1]) that the spindle of the nucleus in *Spirogyra* most probably contributes to the regulation and acceleration of the separation of the daughter-nuclei; that when no spindle or an imperfect spindle is formed, this separation goes much more slowly and that through the development of the spindle the daughter nuclei are also driven apart, when the nucleus has been forced from its place by centrifuging, and with

[1]) Zur Physiologie der Spirogyrazelle, Beihefte zum Botan. Centralblatt, Bd. XXIV (1908), Abt. 1, p. 147.

chromatophores and protoplasm has been pressed against the wall. On account of what has been said above I am inclined to connect the inferior development of the spindle in *Closterium*, with the way in which the daughternuclei go to their appointed places in the daughtercells.

In *Spirogyra* the daughter nuclei are widely separated by the development of the spindle, so that each almost immediately takes up its appointed place, while in *Closterium*, in which the spindle does not develop as strongly as in *Spirogyra* and dissolves more quickly in the cytoplasm, the daughter nuclei at first do not separate so far. After the transverse wall has been formed, the daughter-nuclei in *Closterium* are at a small distance from this on either side. Next they move along the cell wall to the places, where the chromatophores divide into two. First they move along the transverse wall and next along the wall of the mother-cell, on that side which is bent most. In the daughter-nuclei they take a place between the two chromatophores, which have arisen by division from a single chromatophore of the mother-cell.

As a rule the nuclear spindle in *Closterium* is developed regularly. If, however, the nucleus is not situated in the middle of the transverse plane of the cell, but more on that side which is most strongly curved, the spindle-fibres extend more on the opposite side.

The halves of the equatorialplate develop to daughter-nuclei in the following way. They become surrounded by a wall and the chromosomes gradually develop to a fine network. At first these get a looser structure and more and more resemble threads of beads; soon they can no longer be distinguished from one another; they seem to form a tissue of fine threads of beads and in the end they constitute a delicate network. It is difficult to indicate the exact moment at which the young daughter-nuclei become surrounded by a wall. Very soon numerous little nucleoli appear in the daughter-nuclei between the threadwork. They come nearer to each other, form some small masses and finally one great central whole or a collection of nucleoli. When the young daughter-nuclei are still found near the transverse wall, the nucleoli are still spread in great quantities all over the nucleus. When the nucleus arrives between the two chromatophores of the daughter cell, they form a few masses. This has led to the conclusion that at first the daughter-nuclei have not one, but several nucleoli. The daughter-nuclei are flat, like the resting nucleus. On their way along the cellwall their shape more or less adapts itself to the circumstances; in making curves the nuclei are bent.

Above I have briefly described the structure of the nucleus and the details of the karyokinesis in *Closterium Ehrenbergii.* As appears from what has been said the nucleus, as far as the nucleoli are concerned, does not agree with the nuclei of *Spirogyra,* as earlier investigators have supposed. In this respect the nucleus of *Closterium* differs on an important point from those of *Spirogyra,* namely, it does not possess a nucleolus which may be identified as or compared with a nucleus. The nuclei agree with nuclei, which are generally found among plants, especially the higher plants. Nevertheless they show one peculiarity: the nucleoli which are indeed present in great numbers, form in the middle of the nucleus a conglomeration.

The nucleus divides by karyokinesis or mitosis. All phenomena which generally occur, also take place in *Closterium.* In *Closterium* the nuclear division presents the following particulars: the distribution of the nucleoli in the nucleus and their extrusion into the cytoplasm, the great flat equatorial plate, the great number of chromosomes which is more than 60, the different length of the chromosomes, which in general are short and of which the longer ones only have free ends, protruding sidewards, the wide, feebly developed spindle and the translocation of the daughter-nuclei along the cellwall.

Later I hope to give a more detailed account of the karyokinesis in *Closterium* and to illustrate with figures the above mentioned results. In this paper hardly anything has been said about the cell division and the growth of the cellwall. To this I also hope to refer later.

Mathematics. — *"On the relation between the vertices of a definite sixdimensional polytope and the lines of a cubic surface".* By Prof. P. H. Schoute.

1. In his investigation about semiregular polytopes and polytopes possessing a higher degree of regularity Mr. E. L. Elte, whose dissertation is to appear shortly has met with a sixdimensional polytope of degree of regularity $\frac{3}{4}$ with 27 vertices. Our aim here is to point out the complete correspondence in relations of position between the 27 vertices of this polytope and the 27 lines of a cubic surface.

The symbol of the characteristic numbers of this polytope is

$$(27,\ 216,\ 720,\ 1080,\ 432 + 216,\ 72 + 27),$$

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i. e. the polytope has 27 vertices, 216 edges, 720 faces, 1080 limiting bodies, 648 fourdimensional limiting polytopes and 99 fivedimensional ones. Here the numbers 27, 216, 720, 1080 between the brackets are left undivided, as the corresponding elements are of the same kind: all the edges have the same length and — with respect to the whole figure — the same position, all the faces are equal equilateral triangles in the same position, all the limiting bodies are equal regular tetrahedra in the same position. On the other hand the 648 equal regular fivecells forming the fourdimensional limiting polytopes split up according to their position into two groups, while the 99 fivedimensional limiting polytopes consist of 72 regular simplexes $S(6)$ with six vertices and 27 regular cross polytopes $Cr(10)$ with ten vertices; of the 648 fivecells 432 are common to an $S(6)$ and a $Cr(10)$, the remaining 216 to two $Cr(10)$.

2. In order to be able to enter into our subject immediately we start from the 27 points with the coordinates

$$\begin{array}{cccccc l}
0 & 0 & 0 & 0 & 0 & -\frac{4}{3}\sqrt{3} & \ldots\ a_0, \\
(1 & -1 & -1 & -1 & -1) & -\frac{1}{3}\sqrt{3} & \ldots\ 5a_i \\
1 & 1 & 1 & 1 & 1 & -\frac{1}{3}\sqrt{3} & \ldots\ b_0 \\
(-1 & -1 & 1 & 1 & 1) & -\frac{1}{3}\sqrt{3} & \ldots\ 10c_{ik} \\
(2 & 0 & 0 & 0 & 0) & \frac{2}{3}\sqrt{3} & \ldots\ 5b_i \\
(-2 & 0 & 0 & 0 & 0) & \frac{2}{3}\sqrt{3} & \ldots\ 5c_{0i}
\end{array}$$

In this scheme the symbols $a_0, a_1, .. a_5, b_0, c_{12}, .. c_{45}, b_1, .. b_5, c_{01}, .. c_{05}$ of the last column represent the points in a transparent manner; moreover this notation is entirely the same as that generally used for the 27 lines of the cubic surface. Indeed, if — by means of the well known formula for the distance of two points with given coordinates — it has been shown, that any of the 27 points is at distance $2\sqrt{2}$ from 16 and at distance 4 from 10 other points and it has been found for each of the 27 vertices which are the 16 adjacent ones and which the 10 remote ones, it is immediately evident that in using the same symbols a, b, c for the 27 vertices of the polytope and the 27 lines of the cubic surface two *adjacent* vertices (edge distance $= 2\sqrt{2}$) always correspond to two *crossing* lines, two *remote* vertices (diagonal distance $= 4$) always correspond to two intersecting lines. We will show that this correspondence leads to simple geometrical results; but to this end we have to know the projections of the new polytope on different axes of symmetry.

3. All the 27 vertices are at the same distance $\frac{4}{3}\sqrt{3}$ of the origin. So the origin is the centre of the polytope and all its axes of symmetry pass through this point.

The projection of the polytope on the axis OX_6 passing through the vertex a_0 can be deduced immediately from the coordinates. It has been given in the known manner in fig. 1. Moreover the List I gives the names of all the edges, faces, etc.

From this projection (1, 16, 10) it is evident that a limiting cross polytope $Cr(10)$ is oppositely placed to the vertex a_0. We say that these elements are *rightly* opposite to each other, as the line from the vertex to the centre of the polytope passes if produced through the centre of the opposite cross polytope $Cr(10)$.

4. We repeat in fig. 2^a the position of the 27 vertices in the projection (1, 16, 10) and indicate now how the other projections (2, 10, 10, 5), etc. given there have been obtained. We thereby enter into detail with respect to the first new case (2, 10, 10, 5) of 2^b, where the axis passing through the midpoint of the edge $a_1 a_2$ is the axis of projection.

The coordinates of the midpoint of the edge $a_1 a_2$ are

$$0, \quad 0, -1, -1, -1, -\frac{1}{3}\sqrt{3}.$$

So

$$\frac{(x_3 + x_4 + x_5)\sqrt{3} + x_6}{\sqrt{10}} = const.$$

is the standard equation of any space S_5 perpendicular to the axis under consideration. The constant of the second member takes for the groups of vertices $(a_1 a_2)$, $(a_0 a_3 a_4 a_5\ c_{03} c_{04} c_{05} c_{34} c_{35} c_{45})$, $(b_1 b_2\ c_{01} c_{02} c_{13} c_{14} c_{15} c_{23} c_{24} c_{25})$, $(b_0 b_3 b_4 b_5\ c_{12})$ indicated in fig. 2^b successively the values $-\frac{10}{\sqrt{30}}$, $-\frac{4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{8}{\sqrt{30}}$; by means of these values the position of the points of the axis where the 2, 10, 10, 5 vertices project themselves, with respect to the origin indicated by the dotted vertical line, is easily found.

The centre of gravity of the regular fivecell $(b_0 b_3 b_4 b_5 c_{12})$, lying opposite to the edge $(a_1 a_2)$, i.e. the point with the coordinates

$$0,\ 0,\ \frac{4}{5},\ \frac{4}{5},\ \frac{4}{5},\ \frac{4}{15}\sqrt{3},$$

is situated on the axis of projection. So the edge $(a_1 a_2)$ and the fivecell $(b_0 b_3 b_4 b_5 c_{12})$ are *rightly* opposite to each other. From the number 216 of the edges it follows, that each of the opposite fivecells must be common to two cross polytopes $Cr\,(10)$; really the fivecell opposite to the edge $(a_1 a_2)$ is common to the two $Cr(10)$ opposite to the vertices a_1, a_2.

5. In an analogous manner the other projections are found.

So fig. 2^c deals with the case of the axis passing through the centre of the face $a_1 a_2 a_3$. The standard equation

$$\frac{x_1 + x_2 + x_3 + 3\,(x_4 + x_5) + x_6\sqrt{3}}{2\sqrt{6}} = const.$$

corresponding to this case gives for the groups of vertices

$$(a_1 a_2 a_3),\ (a_0 a_4 a_5 c_{04} c_{05} c_{45}),\ (c_{01} c_{02} c_{03} c_{14} c_{24} c_{34} c_{15} c_{25} c_{35}),\ (b_1 b_2 b_3 c_{12} c_{13} c_{23}),\ (b_0 b_4 b_5)$$

successively the values $-\frac{4}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{4}{\sqrt{6}}$ of the constant. So we find the projection (3, 6, 9, 6, 3), showing that the faces of the polytope are placed in pairs rightly opposite to each other. For the centre of gravity of the triangle $b_0 b_4 b_5$ lies on the axis of projection.

So fig. 2^d treats the case of the axis through the centre of the tetrahedron $a_1 a_2 a_3 a_4$, by means of the standard equation $\frac{(x_1 + x_2 + x_3 + x_4 + 2x_5)\sqrt{3} + 2x_6}{2\sqrt{7}} = const.$ and of the values $-\frac{7}{\sqrt{21}}, -\frac{4}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{5}{\sqrt{21}}, \frac{8}{\sqrt{21}}$ of the constant, the projection (4, 3, 8, 6, 4, 2). Here the edge $b_0 b_5$ corresponding to the value $\frac{8}{\sqrt{21}}$ is placed *obliquely* opposite to the tethrahedron $a_1 a_2 a_3 a_4$, for the midpoint of the edge does not lie on the axis of projection. A closer examination shows that in this manner each edge is placed obliquely opposite to *five* limiting tetrahedra, i.e. to the five limiting tetrahedra of the fivecell placed rightly opposite to the edge. In accordance to this the number 1080 of the limiting tetrahedra is five times that of the edges.

Farthermore fig. 2^e gives the projection on the axis passing through the centre of the fivecell $a_1a_2a_3a_4a_5$ common to the simplex $a_0a_1a_2a_3a_4a_5$ and a cross polytope $Cr(10)$. The standard equation is

$$\frac{3(x_1+x_2+x_3+x_4+x_5)\sqrt{3}+5x_6}{4\sqrt{10}} = const.,$$

the values of that constant are $\frac{8}{\sqrt{30}}, -\frac{5}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{7}{\sqrt{30}}, \frac{10}{\sqrt{30}}$; the fact that in this arithmetical series the term $\frac{4}{\sqrt{30}}$ is lacking will be accounted for in a natural way later on. The opposite point b_0 lies obliquely opposite to the fivecell from which we started. A closer investigation shows the following. There are — we have already stated this — 216 fivecells, each of which is common to two $Cr(10)$; i.e. of the 27×32 limiting fivecells of the cross polytopes 432 cover each other by pairs, while the 432 remaining ones are covered by the 72×6 limiting fivecells of the $S(6)$. So the 32 limiting fivecells of each $Cr(10)$ are coloured alternately white and black, if we call a fivecell in contact with an $S(6)$ white, a fivecell in contact with a $Cr(10)$ black; now each vertex is obliquely opposite to the 16 white limiting fivecells of the $Cr(10)$ rightly opposite to it. Indeed the number 432 of the fivecells common to two five-dimensional polytopes of different kind is 16 times the number of vertices.

Finally 2^f represents the case of the axis through the centre of gravity of the simplex $a_0a_1a_2a_3a_4a_5$. To this corresponds the standard equation $\frac{x_1+x_2+x_3+x_4+x_5+x_6\sqrt{3}}{2\sqrt{2}} = const.$ with the values $-\sqrt{2}, 0, \sqrt{2}$ of the constant and the simple projection (6, 15, 6) of the points a, c, b given in fig. 3, while the List II gives the names of all the limiting elements [1]). This projection shows us that the 72 limiting $S(6)$ are placed by pairs rightly opposite to each other.

6. Before we consider the obtained projections in connexion with the 27 lines of a cubic surface it will be well to extend our terminology by putting side by side the simpler corresponding properties of the two systems of 27 objects. In this comparison "the polytope"

[1]) Here the subscript 0 has been replaced by 6.

stands for the figure with the 27 vertices, "the configuration" for the 27 lines of the cubic surface.

The polytope has	The configuration has
216 edges and 135 diagonals, lying three by three in 45 planes;	216 twocrossers and 135 two-intersecters, the points of intersection of which form 45 triangles;
720 triangular faces, forming 360 pairs of rightly opposite triangles;	720 threecrossers, forming 360 pairs of threecrossers lying on the same quadratic surface;
1080 limiting tetrahedra;	1080 fourcrossers;
648 limiting fivecells;	648 fivecrossers;
72 limiting simplexes $S(6)$, forming 36 pairs of rightly opposite $S(6)$;	72 sixcrossers, forming 36 double-sixers;
27 limiting polytopes $Cr(10)$, placed rightly opposite to the vertices;	27 tenlines admitting one of the other lines as common transversal;
27 fivedimensional sections with sixteen vertices;	27 sixteenlines admitting one of the other lines as common crossing line;
of the 648 fivecells 432 belong to the limits of an $S(6)$.	of the 648 fivecrossers 432 belong to half a double-sixer.

7. We now consider the obtained projections in connexion with the lines of a cubic surface and distinguish the element placed in the diagram at the lefthand side as the "starting element", the element placed at the righthand side as the "end element". In this comparison we immediately find this particularity that the property — following in fig. 2^a from the assumed relation between the vertices of the polytope and the lines of the cubic surface — i.e. that the 10 lines of the end element intersect the line forming the starting element, maintains itself up to fig. 2^e in this form that all the lines of the end element are common transversals of the lines of the starting element, while in fig. 2^f each line of the end element cuts only five of the six lines of the starting element. It is easy to express this by a rule without exception indicating the relations of position much more accurately, if we measure as in fig. 4 on a horizontal line OX from the origin O equal segments, mark the points of division by the row of numbers $0, 1, 2, 3 \ldots$, place under O the lines of the different starting elements [1]) and write under $0, 1, 2, 3 \ldots$

[1]) As to this point, according to the last sentence of this article, the process has to undergo a small amplification, which will be perfecly clear to the reader if he has gone through the whole article.

the groups of lines, any line of which cuts respectively $0, 1, 2, 3 \ldots$ of the lines forming the starting element. We then really fall back on the projections $(1, 16, 10), (2, 10, 10, 5)$, etc.

The cases in which the starting element contains one, two, three or four lines, give at most rise to the remark, that we find back in fig. 4 the old projections represented on a different scale; for the *mutually* equal segments of each projection have really *different* length for the *different* projections of fig. 2, while *all* the segments have been taken equal to each other in fig. 4.

For the four remaining projections of fig. 4 the starting element is in the language of the configuration successively:

a fivecrosser not belonging to half a double-sixer;

a fivecrosser belonging to half a double-sixer;

a sixcrosser;

a tenline.

Of these four cases still to be discussed the first is that of fig. 2^b, the last that of fig. 2^a, both taken reversely, i.e. with interchange of starting and end element, while the second and the third correspond to fig. 2^e and fig. 2^f.

We treat of the second of the four cases, that of the projection $(5, 1, 5, 10, 0, 5, 1)$ in the first place, in order to fix the attention on the point bearing no projection indicated by the nought. Wherefore has this empty place (fig. 2^e) to present itself? Because the number of lines cutting respectively $0, 1, 2, 3, 4, 5$ of the five lines a_1, a_2, a_3, a_4, a_5 forming the starting element is $1, 5, 10, 0, 5, 1$; in other words any of the 27 lines cutting three of the five lines a_i at least cuts four of them. By this rational explanation of the mentioned hiatus the three other projections are also explained. If we take the projection of fig. 2^b in the reverse sense, we find that each of the ten lines $(b_1\, b_2\, c_{01}\, c_{02} \ldots c_{25})$ cuts three, each of the two lines $(a_1\, a_2)$ cuts five of the lines of the starting element $(b_6\, b_3\, b_4\, b_5\, c_{12})$. The third of the four cases, that of the projection $(6, 15, 6)$ of fig. 2^f, can be explained in the same way. Finally we have still to remark that in the last case the displacement of the starting element, the tenline, over *one* segment to the right of the origin, is no mistake; it corresponds to this that the ten lines do not cross each other altogether, but that each of them intersects *one* of the nine others.

8. So the subject proper of this communication is exhausted. However we will finally move the question if it may be possible that considerations analogous to those mentioned above lead from

other known configurations of lines to unknown polydimensional polytopes with a certain degree of regularity and reversely newly discovered polytopes of this character to unknown configurations of lines. According to our opinion there can be no doubt about the answer to this question.

But instead of entering into this new subject just now we will only point out the configuration of the 16 lines crossing one of the 27 lines, to which corresponds the fivedimensional polytope with the 16 vertices $(5a_i, b_0, 10c_{ik})$. As is known this configuration presents itself on the quartic surfaces with double conic. On the other hand the 16 points $(5a_i, b_0, 10c_{ik})$ are the vertices of the half measure polytope $\frac{1}{2}[1\,1\,1\,1\,1]$ of the fivedimensional space $x_6 = -\frac{1}{3}\sqrt{3}$. So we have here before us a second example of a correspondence as the one treated above. In order to enable the reader to study this correspondence we have repeated in the fig. 5 and 6 the part of the fig. 2 and 4 which relates to these systems of 16 objects.

LIST 1.

Vertices.

$a_0 - 5a_1, b_0, 10c_{12} - 5b_1, 5c_{01}$.

Edges.

$5a_0a_1, a_0b_0, 10a_0c_{12} - 10a_1a_2, 30a_1c_{23}, 10b_0c_{12}, 30c_{12}c_{13} -$
$5a_1b_1, 20a_1c_{02}, 5b_0b_1, 30b_1c_{23}, 20c_{01}c_{12} - 10b_1b_2, 20b_1c_{02}, 10c_{01}c_{02}$.

Faces.

$10a_0a_1a_2, 30a_0a_1a_{23}, 10a_0b_0c_{12}, 30a_0c_{12}c_{13} -$
$10a_1a_2a_3, 30a_1a_2c_{34}, 60a_1c_{23}c_{24}, 30b_0c_{12}c_{13}, 20c_{12}c_{13}c_{14}, 10c_{12}c_{13}c_{23} -$
$30a_1a_2c_{03}, 30a_1b_1c_{23}, 60a_1c_{02}c_{23}, 30b_0b_1c_{23}, 60b_1c_{23}c_{24}, 30c_{01}c_{12}c_{13} -$
$20a_1b_1c_{02}, 30a_1c_{02}c_{03}, 10b_0b_1b_2, 3\ b_1b_2c_{34}, 60b_1c_{02}c_{23}, 10c_{01}c_{02}c_{12} -$
$10b_1b_2b_3, 30b_1b_2c_{03}, 30b_1c_{02}c_{03}, 10c_{01}c_{02}c_{03}$.

Tetrahedra.

$10a_0a_1a_2a_3, 30a_0a_1a_2c_{34}, 60a_0a_1c_{23}c_{24}, 30a_0b_0c_{12}c_{13}, 20a_0c_{12}c_{13}c_{14},$
$10a_0c_{12}c_{13}c_{23} - 5a_1a_2a_3a_4, 10a_1a_2a_3c_{45}, 30a_1a_2c_{34}c_{35}, 20a_1c_{23}c_{24}c_{25},$
$20a_1c_{23}c_{24}c_{34}, 20b_0c_{12}c_{13}c_{14}, 10b_0c_{12}c_{13}c_{23}, 5c_{12}c_{13}c_{14}c_{15} -$
$20a_1a_2a_3c_{04}, 60a_1a_2c_{03}c_{34}, 60a_1b_1c_{23}c_{24}, 60a_1c_{02}c_{23}c_{24},$
$60b_0b_1c_{23}c_{24}, 20b_1c_{23}c_{24}c_{25}, 20b_1c_{23}c_{24}c_{34}, 20c_{01}c_{12}c_{13}c_{14} -$
$30a_1a_2c_{03}c_{04}, 60a_1b_1c_{02}c_{23}, 31a_1c_{02}c_{03}c_{23}, 30b_0b_1b_2c_{34}, 30b_1b_2c_{34}c_{35},$
$60b_1c_{02}c_{23}c_{24} - 30a_1b_1c_{02}c_{03}, 20a_1c_{02}c_{03}c_{04}, 10b_0b_1b_2b_3, 10b_1b_2b_3c_{45},$
$60b_1b_2c_{03}c_{34}, 30b_1c_{02}c_{03}c_{23} - 5b_1b_2b_3b_4, 20b_1b_2b_3c_{04}, 30b_1b_2c_{03}c_{04},$
$20b_1c_{02}c_{03}c_{04}, 5c_{01}c_{02}c_{03}c_{04}$.

Fivecells common to $S(6)$ and $Cr(10)$.

$5a_0a_1a_2a_3a_4, 30a_0a_1a_2c_{34}c_{35}, 20a_0a_1c_{23}c_{24}c_{31}, 20a_0b_0c_{12}c_{13}c_{14},$
$5a_0c_{12}c_{13}c_{14}c_{15} - a_1a_2a_3a_4a_5, 10a_1a_2c_{34}c_{35}c_{45}, 5b_0c_{12}c_{13}c_{14}c_{15} -$

$20a_1a_2a_3c_{04}c_{15}$, $20a_1b_1c_{23}c_{24}c_{25}$, $20a_1c_{02}c_{23}c_{24}c_{25}$, $20b_0b_1c_{23}c_{24}c_{34}$ -
$10a_1a_2a_3c_{04}c_{05}$, $30a_1a_2c_{03}c_{04}c_{34}$, $60a_1b_1c_{02}c_{23}c_{24}$, $30b_0b_1b_2c_{34}c_{35}$,
$10b_1b_2c_{34}c_{35}c_{15}$, $20b_1c_{02}c_{23}c_{24}c_{25}$ — $20a_1b_1c_{02}c_{03}c_{04}$, $5a_1c_{02}c_{03}c_{04}c_{05}$,
$5b_0b_1b_2b_3b_4$, $20b_1b_2b_3c_{04}c_{45}$, $30b_1b_2c_{03}c_{04}c_{34}$, — $b_1b_2b_3b_4b_5$,
$10b_1b_2b_3c_{04}c_{05}$, $5b_1c_{02}c_{03}c_{04}c_{05}$.

Fivecells common to two $Cr(10)$.

$10a_0a_1a_2a_3c_{45}$, $20a_0a_1c_{23}c_{24}c_{25}$, $10a_0b_0c_{12}c_{13}c_{23}$ $5a_1a_2a_3a_4c_{05}$,
$30a_1a_2c_{03}c_{34}c_{35}$ $20a_1b_1c_{23}c_{24}c_{34}$, $20a_1c_{02}c_{23}c_{24}c_{25}$, $5c_{01}c_{12}c_{13}c_{14}c_{15}$ -
$10b_0b_1b_2b_3c_{45}$, $30b_1b_2c_{02}c_{34}c_{35}$, $10a_1a_2c_{03}c_{04}c_{55}$, $30a_1b_1c_{02}c_{03}c_{23}$ —
$5b_1b_2b_3b_4c_{05}$, $10b_1b_2c_{03}c_{04}c_{05}$, $c_{01}c_{02}c_{03}c_{04}c_{05}$.

Simplexes $S(6)$.

$a_0a_1a_2a_3a_4a_5$, $10a_0a_1a_2c_{34}c_{35}c_{45}$, $5a_0b_0c_{12}c_{13}c_{14}c_{15}$ — $10a_1a_2a_3c_{04}c_{05}c_{45}$,
$20a_1b_1c_{02}c_{23}c_{24}c_{25}$, $10b_0b_1b_2c_{34}c_{35}c_{45}$ — $5a_1b_1c_{02}c_{03}c_{04}c_{05}$.
$b_0b_1b_2b_3b_4b_5$, $10b_1b_2b_3c_{04}c_{05}c_{45}$.

Cross polytopes $Cr(10)$.

$5a_0a_1b_0b_1c_{23}c_{24}c_{25}c_{34}c_{35}c_{45}$, $5a_0a_1a_2a_3a_4c_{05}c_{15}c_{25}c_{35}c_{45}$ —
$a_1a_2a_3a_4a_5c_{01}c_{02}c_{03}c_{04}c_{05}$, $10a_1a_2b_1b_2c_{03}c_{04}c_{05}c_{34}c_{35}c_{45}$,
$5b_0b_1b_2b_3b_4c_{05}c_{15}c_{25}c_{35}c_{45}$ — $b_1b_2b_3b_4b_5c_{01}c_{02}c_{03}c_{04}c_{05}$.

LIST II.

Vertices.

$6a_1$, $6b_1$, $15c_{12}$.

Edges.

$$\begin{array}{l|l|l} 15a_1a_2 & 6a_1b_1 \quad 60a_1c_{23} & 60c_{12}c_{13} \\ 15b_1b_2 & \quad\quad\quad\;\; 60b_1c_{23} & \end{array}$$

Faces.

$$\begin{array}{l|l|l} 20a_1a_2a_3 & 90a_1a_2c_{34} & 180a_1c_{23}c_{24} \\ 20b_1b_2b_3 & 90b_1b_2c_{34} & 180b_1c_{23}c_{24} \end{array} \; 60c_{12}c_{13}c_{14} \; 20c_{12}c_{13}c_{23}$$

Tetrahedra.

$$\begin{array}{l|l|l|l} 15a_1a_2a_3a_4 & 60a_1a_2a_3c_{45} & 180a_1a_2c_{34}c_{35} & 120a_1c_{23}c_{24}c_{25} \\ 15b_1b_2b_3b_4 & 60b_1b_2b_3c_{45} & 189b_1b_2c_{34}c_{35} & 120b_1c_{23}c_{24}c_{25} \end{array}$$

$$\begin{array}{l|l|l} 60a_1c_{23}c_{24}c_{34} & 180a_1b_1c_{23}c_{24} & 30c_{12}c_{13}c_{14}c_{15} \\ 60b_1c_{23}c_{24}c_{34} & & \end{array}$$

Fivecells common to $S(6)$ and $Cr(10)$.

$$\begin{array}{l|l|l|l} 6a_1a_2a_3a_4a_5 & 60a_1a_2a_3c_{45}c_{46} & 60a_1a_2c_{34}c_{35}c_{45} & 30a_1c_{23}c_{24}c_{25}c_{26} \\ 6b_1b_2b_3b_4b_5 & 60b_1b_2b_3c_{45}c_{46} & 60b_1b_2c_{34}c_{35}c_{45} & 30b_1c_{23}c_{24}c_{25}c_{26} \end{array} \; 120a_1b_2c_{23}c_{24}c_{25}.$$

Fivecells common to two $Cr(10)$.

$$\begin{array}{l|l} 15a_1a_2a_3a_4c_{56} & 60a_1a_2c_{34}c_{35}c_{36} \\ 15b_1b_2b_3b_4c_{56} & 60b_1b_2c_{34}c_{35}c_{36} \end{array} \; 60a_1b_1c_{23}c_{24}c_{34} \; 6c_{12}c_{13}c_{14}c_{15}c_{16}.$$

Simplexes $S(6)$.

$$\begin{array}{l|l} a_1a_2a_3a_4a_5a_6 & 20a_1a_2a_3c_{45}c_{46}c_{56} \\ b_1b_2b_3b_4b_5b_6 & 20b_1b_2b_3c_{45}c_{46}c_{56} \end{array} \; 30a_1b_1c_{23}c_{24}c_{25}c_{26}.$$

Cross polytopes $Cr(10)$.

$$\begin{array}{l|l} 6a_1a_2a_3a_4a_5c_{16}c_{26}c_{36}c_{46}c_{56} & 15a_1a_2b_1b_2c_{34}c_{35}c_{36}c_{45}c_{46}c_{56}. \\ 6b_1b_2b_3b_4b_5c_{16}c_{26}c_{36}c_{46}c_{56} & \end{array}$$

Mathematics. — *"Reciprocity in connexion with semiregular polytopes and nets."* By Mrs A. BOOLE STOTT and Prof. P. H. SCHOUTE.

1. The tables added to the memoir "Geometrical deduction of semiregular from regular polytopes and space fillings", recently published by this Academy (*Verhandelingen*, eerste sectie, deel XI, N°. 1) [1]), show that the same semiregular polytope or net may sometimes be derived from different regular polytopes or nets by different operations. It was stated there (p. 13) that this is due to the "reciprocity of the figures". We propose to examine here the influence of this reciprocity on the mutual relationship between the results of the different operations of expansion explained there. Before doing so it will be well to give a definition of what is understood here by reciprocity of two polytopes in space S_n, where these polytopes have either a finite or an infinite number of limiting elements l_{n-1}, the first case referring to two polytopes in S_n and the second to two nets of polytopes in S_{n-1} considered as two polytopes in S_n.

2. *Definition of reciprocity.* Two regular polytopes in S_n are reciprocal to one another if the number of limiting elements l_p passing through a limiting element l_q of the one is equal to the number of limiting elements l_{n-p-1} lying in a limiting element l_{n-q-1} of the other, where $p > q$.

We illustrate this by some examples which we divide into two groups, the first *a*) dealing with *pairs* of polytopes of different forms, the second *b*) with *self reciprocal* polytopes.

a. In S_3 we have two pairs of reciprocal regular polyhedra, C and O, I and D, in S_4 we have once more two pairs of regular polytopes, C_8 and C_{16}, C_{120} and C_{600}, and one pair of regular nets, NC_{16} and NC_{24}. So for $n = 4$ the number 3 of faces passing through an edge in C_8 (see the "Table of incidences" in the memoir quoted) is equal to the number of edges lying in a face in C_{16}. So for $n = 5$ the number 8 of faces passing through an edge in NC_{16} (see the same table) is equal to the number of faces lying in a limiting body in NC_{24}, while the number 12 of limiting bodies passing through an edge in NC_{16} is equal to the number of edges lying in a limiting body in NC_{24}, etc.

b. In S_3 we have only one self reciprocal regular body, T, and one self reciprocal regular net, the net of cubes, in S_4 we have the

[1]) The figures alluded to in the following pages will be found in the memoir quoted.

two self reciprocal regular polytopes C_5, C_{24} and one self reciprocal regular net, the net NC_8. In passing we may remark that in space S_n the net NM_n of measure polytopes M_n is self reciprocal.

3. By the application of the operation of expansion e_{n-1} to a regular polytope A in S_n each vertex, each edge, each face, etc. is replaced by a limiting polytope of $n-1$ dimensions, filling up the *gaps* caused by the expansion; these polytopes will be indicated respectively by the symbols g_0, g_1, g_2, etc., the last one g_{n-1} being the *original* limiting $n-1$-dimensional polytope itself in an other position. The subscripts $0, 1, \ldots, n-1$ of these symbols $g_0, g_1 \ldots, g_{n-1}$ represent the *import* of the limiting polytopes Now, if we apply the operation e_{n-1} to two polarly related polytopes A and A' of S_n, the gaps $g_0, g_1, \ldots g_{n-1}$ of $e_{n-1}A$ are respectively equal, in form — and in number as long as this remains finite —, to the gaps g_{n-1}, $g_{n-2}, \ldots g_0$ of $e_{n-1}A'$, in other words the polytopes $e_{n-1}A$ and $e_{n-1}A'$ have their gaps of *reciprocal import* equal (p. 9 of the memoir quoted). We will try to make this clear by a few examples [1]).

In the simple case of C and O in S_3 the e_2 expansion applied to both gives an RCO (fig. 3^a and 3^b), where the gaps g_0, g_1, g_2 of the one are equal to the gaps g_2, g_1, g_0 of the other. In the case of the cells C_8 and C_{16} in S_4 the e_3 expansion leads up to the same form (fig. 6^a and 6^b); here we have $g_0\, e_3\, C_8 = g_3\, e_3\, C_{16} = T$, $g_1\, e_3\, C_8 = g_2\, e_3\, C_{16} = P_3$, $g_2\, e_3\, C_8 = g_1\, e_3\, C_{16} = P_4$, $g_3\, e_3\, C_8 = g_0\, e_3\, C_{16} = C$ (see the numbers indicated in the diagrams). In the case of the nets NC_{16} and NC_{24} (fivedimensional reciprocal polytopes) the two polytopes $e_4 NC_{16}$ and $e_4 NC_{24}$ (fig. 26) show the relations $g_0 e_4 NC_{16} = g_4 e_4 NC_{24} = C_{24}$, $g_1\, e_4\, NC_{16} = g_3\, e_4\, NC_{24} = PO$, $g_2\, e_4\, NC_{16} = g_2\, e_4\, NC_{24} = (3\ ;\ 3)$, $g_3\, e_4\, NC_{16} = g_1\, e_4\, NC_{24} = PT$, $g_4\, e_4\, NC_{16} = g_0\, e_4\, NC_{24} = C_{16}$.

4. We have shown above that the application of the operation e_{n-1} (with the highest subscript) to two reciprocal polytopes A and A' in S_n produces the same form with reciprocal imports. If any second operation e_k be applied to $e_{n-1}A$, will it be possible to find an operation $e_{k'}$ by which $e_{n-1}A'$ may be transformed so as to make $e_{k'}\, e_{n-1}\, A' = e_k\, e_{n-1}\, A$?

The answer to this question is very simple: in order to obtain the same result in both cases we have only to take care that the two operations e_k and $e_{k'}$ act upon the *same subject*. Now the limiting

[1]) An analytical proof of this theorem and the following one will be published later on.

polytopes of k import in the first are the same as those of $n-k-1$ import in the second; so k' has to be equal to $n-k-1$, i.e. we have $k+k'=n-1$. So we get

$$e_k\, e_{n-1}\, A = e_{n-k-1}\, e_{n-1}\, A',$$

i.e.: If we apply respectively to $e_{n-1}\, A$ and $e_{n-1}\, A'$ any two *reciprocal* operations e_k and e_{n-k-1} the result is the same but the imports are reciprocal.

This simple general theorem accounts for the equality of all the pairs of polytopes (and nets) indicated in the tables added to the memoir quoted.

Hever, Kent, England. September, 1910.

Physics. "*An improved semicircular electromagnet.*" II. By Prof. H. E. J. G. du Bois. (Communication from the Bosscha-Laboratory.)

Recently I described a new type of semicircular electromagnet together with some results obtained with it.[1]) In the present paper I beg to communicate a few more measurements; and also its adaptation to special purposes, which lately have come to prominent notice.

Influence of polar windings. The reproduction given previously exhibited the windings as split into two divisions by a rectangular flange: *a.* polar windings, which are in the neighbourhood of the pole-pieces, the efficiency of which can be increased by supplementary loose polar coils; *b.* circuital windings round the other parts of the magnetic circuit. A second instrument was wound and connected in a somewhat different way; the field was determined again under different circumstances by means of a ballistic moving-coil galvanometer; this was standardised by means of a normal solenoid, and the proportionality of the readings ascertained. A small test-coil was made with a diameter of 3 m.m. and a thickness of 0,3 m.m.; the thickness of the bare copper wire used was 0,025 m.m., silk-covered 0,07 m.m.; it was wound in collodium. The equivalent area of the 45 windings was 1,544 cm²., determined by comparison with a slightly smaller normal coil of 1,530 cm²., measured geometrically. The results are given in the subjoined table:

[1]) H. du Bois. These Proc. **18** p. 189, 1909.

End planes / Polar distance	3,6 mm.		6 mm.	
	Field (Kilogauss) with:		Field (Kilogauss) with:	
	a) 135 K. A. T.	*a*, *b*) 255 K.A. T.	*a*) 135 K. A. T.	*a*, *b*) 255 K. A.T.
0 mm.	—	—	53,3	53,9
0,5 »	54,9	55,4	51,3	51,9
1,0 »	51,3	51,9	49,3	50
1,5 »	48,9	49,7	47,3	48,1
2,0 »	46,5	47,4	45,8	46,6

These values are a little higher still than those previously given; the first limiting values have again been linearly extrapolated, and they are all considerably greater than is calculable from a saturation-value of about 1710 C. G. S.[1]) according to the usual formulae. This fact, of advantage from an empirical point of view, is difficult to explain as yet. Sub *a*) we find the fields measured with polar windings only (135 kiloampère-turns); sub *a*, *b*) those obtained with polar and circuital windings (255 K. A. T.). Hence it is convincingly shown, as might be foreseen, that the influence of the former greatly preponderates; under certain circumstances the share of the latter amounts to only one per cent of the whole field; in other cases, however, it is greater. Yet for various reasons it does not seem desirable to omit these inferior windings altogether, as has been the general custom with designers of the Rühmkorff type.

For the investigation of this problem — also of importance with a view to the economical construction of field-magnets in general — under better defined circumstances, a complete ring-electro-magnet was used, provided with 12 separate coils according to Fig. 1. They were connected in various ways, but always so that two coils, numbered alike, symmetrical with respect to the air-slit, were excited at the same time. It appears from a great number of field-curves — as a function of the kiloampère-turns — that up to $^1/_2$ or $^2/_3$ saturation the azimuth of the coils remains indifferent, as has been generally supposed. The more, however, the iron becomes saturated, the more the action of the coils near the air-slit begins to preponderate, so that their influence is determined by the order of the

[1]) E. Gumlich. Elektrotechn. Zeitschr. **30**, p 1096, 1909. P. Weiss, Journ. de Phys. (4) **9**. p. 373, 1910. Comp. also B. Beattie and H. Gerrard, the Electrician, **64** p.p. 750, 811, 1910.

numbers in Fig. 1. This is in accordance with KIRCHHOFF's saturation-law; P. WEISS also drew attention to this fact. On the other hand J. HOPKINSON did not postulate such a difference in his well-known theory of the magnetic circuit; in spite of this such views have also been gradually introduced into electrical practice. As dynamos and motors were more and more saturated and the air-slits became narrower — the value of the induction sometimes reaches 20000 C. G. S. — the field-windings were moved as near as possible to the armature-space; this tendency is finally checked by the fact that a given number of accumulated windings has a higher resistance than when they are uniformly distributed, so that their periphery on an average is evidently smaller.

Gradient-pole-pieces are used for investigations in a non-uniform field; this is the opportunity to describe the arrangement, alluded to in a previous communication [1]). The test-piece takes up a position in the equatorial plane such that both $\mathfrak{H}_x$ as well as the transverse gradient $\partial \mathfrak{H}_x/\partial y$, and also the product $\mathfrak{H}_x \partial \mathfrak{H}_x/\partial y$ retain values as great as possible; this product determines the attraction or repulsion exerted. Besides this chief condition, some practical requirements concerning the necessary space etc. must be fulfilled. The calculation of an optimum would be exceedingly difficult, and even if feasible, might prove more laborious than the empirical method, by which the configuration represented in Fig. 2 was developed after much experience. The axes of the polar pieces form an angle of 25°; the pole-tops are provided with conic cores slightly rounded and just protruding. The field was determined by means of a standardised spherical test-coil of diameter 3 mm. Inside the smaller angle (direction $+y$) the maximum of $\mathfrak{H}_x \partial \mathfrak{H}_x/\partial y$ in general lies further away and is flatter than in the opposed direction $(-y)$; as the axial angle increases the maximum moves away from the origin A towards $+y$; the distance between the pole-tops and the strength of the current have less influence on its position.

In Fig. 3 some curves have been traced referring to this, and corresponding to the configuration of Fig. 2; the abscissae $\pm y$ represent the distances from the origin on a ten-fold scale. The ordinates of I represent the field $\mathfrak{H}_x$ in kilogauss (right-hand scale) those of II the value of $\mathfrak{H}_x \partial \mathfrak{H}_x/\partial y$ in millions of C.G.S. units (left-hand scale). This experiment was made with a distance between the pole-tops of 0,3 cm. and 50 kiloampère-turns. So it appears possible to

[1]) H. DU BOIS and KŌTARŌ HONDA, These Proc. **18**, p. 596, 1910. Cf. P. CURIE, Ann. Chim. & Phys. (7) **5** p 295, 1895; Oeuvres p. 237, Paris 1908.

make use of a non-uniform field of more than 25 kilogauss with a perfectly sufficient gradient. Now the pole-tops may be insulated from the shoes e.g. by means of horn-discs, or by surrounding them with a somewhat pliable leather case, so that e.g. immersion in liquid air can take place; thus we can easily work within a temperature range of -200° to $+200^\circ$. In addition a similar arrangement was made for pyromagnetic investigations at high temperatures up to 1300°, in which case the available field amounted to only 15 kilogauss on account of the larger space required.

Oblique-vision pole-pieces. By EGOROFF and GEORGIEWSKY, and afterwards by RIGHI the ZEEMAN-effect was investigated in directions forming an arbitrary angle ϑ with the direction of the field [1]. The last-mentioned physicist already pointed out the necessity of special electromagnets for this purpose, and could observe within a range $42^\circ < \vartheta < 90^\circ$ with the aid of pointed conical polar-pieces and coils. When recently this problem again came under consideration in connection with the spectrum of the solar spots, it was treated theoretically by LORENTZ, experimentally by ZEEMAN and WINAWER [2]. They extended the interval from 90° to 26°; with such pointed polar-pieces, however, the field is very much weakened; with the aid of glass prisms inserted within the polar-pieces [3]) it was also possible to observe with one single smaller angle $\vartheta = 16^\circ$. In consequence of a conversation with Prof. ZEEMAN I have lately tried to design an arrangement which allows of gradually varying the angle of observation ϑ from 0° to 90°.

Within the range $0^\circ < \vartheta < 45^\circ$ the rays must pass inside the iron; these small angles are of the greatest importance because the critical angle ϑ_1 of LORENTZ will probably always lie within this interval for a strong field. The pole-tops Q_1 and Q_2 (Fig. 4) as usual have a half vertex angle of 55° gradually increasing to 57°; they are kept separate by a strong, unmagnetic mounting V, which is provided with openings. At the back they are spherical and ground into the hollow cups of the pole-shoes P_1 and P_2, whose half vertex-angle amounts to 59°. The bore B had the shape of an excentrical rect-

1) N. EGOROFF & N. GEORGIEWSKY. Compt. Rend. **124** p. 949, 1897. A. RIGHI, Mem. accad. Bologna (5) **8** p. 277 (Fig. 3.) 1899. Cf. A. COTTON, le Phénom. ZEEMAN, "Scientia" No. 5. p. 48, 74, Paris 1899.

2) H. A. LORENTZ, These Proc. **12** p. 321, 1909. P. ZEEMAN & B. WINAWER, These Proc. **12** p. 584; **13** p. 35, 1910.

3) For this artifice proposed by WERTHEIM SALOMONSON the use of a magneto-optically inactive ceriteborosilicate crown-glass may prove efficient; cf. H. DU BOIS & G. J. ELIAS, Verh. D. phys. Ges. **11** p. 710., 1909.

angular pyramid, which fits round the conical beam as determined by the usual conic prolongation of the bores B' in pole-shoe and further core. Now it appears sufficiently clear from Fig. 4, how we can arbitrarily change the angle between the field-axis $x'x'$ and the direction of the light xx and read it on the divided circle C. For a definite polar distance the conical faces as well as the segments of the two pole-tops are concentric — and so also those of the pole-shoes; for other distances these spherical surfaces can always be made to coincide by a slight lateral shifting of the cores carrying the cup-shaped pole-shoes.

This arrangement proved satisfactory; the subjoined table gives some measurements of the field (in kilogauss), firstly before boring and

Polar distance	2 mm.		4 mm.		6 mm.	
ϑ	without bore	with bore	without bore	with bore	without bore	with bore
0°	41,2	29,2	35,3	28,2	31,3	26,4
25°	—	26,2	—	25,5	—	23,9
45°	33 9	21,6	28,2	21,2	24,0	19,5

secondly after the boring had been made; the diameter of the end-planes was 6 mm.; the measurements were always made with 122 kiloampère-turns. So when ϑ increases from 0° to 25°, the field decreases by about 3, between 0° and 45° by about 7 kilogauss. For unbored tops no deviation of the field was found from the direction normal to the end-planes, which, indeed, agrees with a known property of lines of force. After the boring, however, the field is somewhat strained, so that ϑ is a few degrees greater than the angle between $x'x'$ and xx; the difference diminished as ϑ became greater, and disappeared at 45°; then the well-known counteracting influence of the polar windings just balanced the deviation caused by the borings.

Lately CORBINO [1]) described an optical method of photographing the distribution of the iso-dynamics in such cases by means of the birefringency in BRAVAIS-iron. However, it is not necessary to use the round bores assumed by him on account of the straining of the field; slit apertures should, no doubt, always be used. This is also possible in the case in question: for so far as one observes at a definite

[1]) O. M. CORBINO. Phys. Zeitschr. **11**, p. 521, 1910.

angle ϑ it will be advisable to fill up the superfluous parts of the bores B with a set of loose wedge-shaped cores; then the field can be only slightly weakened and strained.

Angles $45° < \vartheta < 90°$ are easier to realise, because the course of the light remains outside of the pole-pieces, the half vertex-angle of which must, however, be smaller than ϑ. If this were the case, we might e. g. already go to 45° in Fig. 4, the line of vision LL going alongside the truncated pole-flange F_2. For this work the ball-frame with the divided circle, on which the whole electromagne rests, proves convenient; it was, in fact first used by RIGHI (loc. cit.) for similar purposes.

With the arrangement described Dr. ELIAS investigated the oblique emission-effect for some spark-spectra [1]). It may also prove serviceable in other cases, e. g. for the KERR-effect.

If we now consider Fig. 4 as normal section of pole-tops, bounded by truncated bi-planes and cylinders, we have a configuration that may be useful e. g. for string-galvanometers, for the observation of the transversal bi-refringency and for similar cases. We can, in addition, prove that the normal optimum-value of the bi-plane angle amounts to $2 \times 45°$ in this case, instead of to $2 \times 54°44'$ for the cone vertex. I am greatly indebted to Mr. MORRIS OWEN for the measurement of the magnetic fields.

Physics. — "*On the* ZEEMAN-*effect for emission-lines in a direction oblique with regard to the lines of force.*" By Dr. G. J. ELIAS. (Communicated by Prof. H. E. J. G. DU BOIS.)

The modifications to which emission and absorption lines are subjected in a magnetic field, have been studied up to now chiefly in two special cases, namely those for which the direction of the magnetic field coincides with the direction of the rays of light, and those for which it is normal to it.

The theory of the phenomenon for the case that the rays of light form an arbitrary angle with the direction of the magnetic field, was developed by LORENTZ [2]).

Experiments of ZEEMAN and WINAWER [3]) refer to the modifications to which the absorption lines of natrium vapour are subjected in a magnetic field with oblique passage of the light.

1) Comp. the subjoined communication.

2) H. A. LORENTZ, These Proc. 12 p. 321; 1909.

3) P. ZEEMAN and B. WINAWER, These Proc. 12 p. 584. 1909; 13 p. 35. 1910

In this paper I will communicate the results of some preliminary experiments on emission lines in a magnetic field.

As source of light I used a condensed electric spark of an inductorium capable of giving a 30 c.m. spark maximum with WEHNELT-interruptor.

The primary current amounted about to 10 ampères, the length of the spark was 2 m.m., four Leyden jars being connected parallel to the spark. I further used DU BOIS' first improved semicircular electromagnet [1]) with the pole-pieces described above [2]), by means of which the angle ϑ between field-direction and ray-direction can be varied from 0° to 45° at pleasure. For greater angles the magnet was rotated round a vertical axis, and the light did not issue through the borings, but on the outside of the pole-pieces. The polar distance amounted to 4 m.m., with which for $\vartheta = 0°$ a field was reached of 27 Kgs., for $\vartheta = 45°$ one of 20 Kgs. As spectroscope the large ROWLAND grating was used, belonging to the University laboratory at Berlin, and kindly put at our disposal by Geheimrat Prof. RUBENS. The observation was made in the spectrum of the third order; for so far as the polarisation by the grating was troublesome, it was eliminated by bringing the light always back to the same plane of polarisation by means of a $\frac{\lambda}{2}$mica plate. This was, however, only the case in the first experiments, when the incident and the issuing light made a large angle with each other; in later experiments this angle was made much smaller by rotation of the grating, and the grating appeared to have no influence on the plane of polarisation.

The state of polarisation of the emitted light was investigated with a nicol, either in connection with a $\frac{\lambda}{4}$ mica plate or not. For the investigation of elliptically polarised light the plate was placed with its optical axis normal resp. at an angle of 45°, and in both cases the position of the nicol was determined, in which the intensity of the emitted light was a minimum. If the two components of the elliptically polarized light, which are normal to each other, are resp. $u = F \cos (\nu t - f)$ and $v = G \cos (\nu t - g)$, a simple calculation teaches that the position of the nicol, at which the intensity of the radiated light is a minimum, is given by $tg\, 2\varphi = \frac{2F\, G}{F^2 - G^2} \sin (f - g)$ for the case that the optical axis of the mica plate is vertical, and by

1) H. DU BOIS. These Proc. **12** p. 189. 1909.

2) H. DU BOIS. These Proc. **13** p. 388. 1910.

$tg\, 2\varphi = -\cot(f-g)$, when the axis makes an angle of 45° degrees with the vertical line. So $f-g$ and $\frac{F}{G}$ can always be calculated from this, and the state of polarisation of the light is determined.

First I made some experiments with electrodes of copper, and chose the line 510.575. As, however, the splitting in the magnetic field is comparatively small for copper-lines, I made most experiments with the line 520.620 of the chrome-spectrum; for this line MILLER[1]) made some measurements about the transversal ZEEMAN-effect.

For the copper-line 510.575 the middle line of the triplet was exceedingly dimly visible for $\vartheta = 10°$, too dimly to determine its state of polarisation. For $\vartheta = 20°$ it was more clearly (though still faintly) visible, and the direction of the vibrations appeared to be pretty nearly horizontal.

In the same way the middle line for the chrome-line 520.620 was very faintly visible for $\vartheta = 12°$, the direction of vibration made an angle of about 30° with the horizontal direction. Even for $\vartheta = 10°$ the middle line was still exceedingly dimly visible; the angle between the direction of the vibrations and the horizontal line seemed to amount to about 40°. For angles ϑ smaller than 10° the middle component could not be observed on account of its slight intensity. For $\vartheta = 17°$ the angle between the horizontal and the vibration direction amounted only to 15°, and for the greater values of ϑ the direction of vibration was found to be pretty well horizontal; it is true that there were still some deviations, but most likely they are due to the unreliability of the observation.

Hence the angle ϑ_1 must amount to about 10°.

For the outer components values were always found for φ of about 90° resp. 180° when the axis of the mica-plate stood under 45°, from which accordingly appears that the axes of the ellipses of vibration are situated horizontally and vertically. For values of ϑ up to 30° the light was found to be almost circularly polarized.

For $\vartheta = 40°$ the ratio of the length of the axes was 0.80; for $\vartheta = 67.5°$ 0.39; when we consider that $\cos 40° = 0.77$ and $\cos 67.5° = 0.39$, we see that the agreement with the theory is pretty close.

The ratio of the intensity of the components inter se does not quite conform to the theory. The intensities of the outer and middle components are proportional to resp. $\frac{1+\cos^2\vartheta}{2g}$ and

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[1]) W. MILLER, Ann. d. Phys. **24** p. 105; 1907.

$\dfrac{g \pm 2\,(q + \sqrt{q^2 - 1})\, r \cos\vartheta}{4\, r^2 + g^2}$ [1]), if ϑ is not too small, so that h_{II} may be neglected by the side of h_I. In this case $\dfrac{q \cos\vartheta}{r} = \dfrac{\sin^2\vartheta}{g}$ may be put for the second expression, as r is pretty large compared with g. The angle ϑ, for which the intensity of the outer and middle components is the same, is given by

$$2 \sin^2\vartheta = 1 + \cos^2\vartheta, \text{ which gives } \vartheta = 54^\circ,$$

whereas the equality of the components was observed at about 67°.5; at 45° the middle component was clearly fainter than the outer ones.

If the observation is made with a nicol which transmits the horizontal vibrations, the outer components will only have the intensity

$$\frac{\cos^2\vartheta}{1 + \cos^2\vartheta} \times \frac{1 + \cos^2\vartheta}{2g} = \frac{\cos^2\vartheta}{2g} .$$

The angle ϑ for equal intensity is then given by $\cos^2\vartheta = 2 \sin^2\vartheta$, which yields $\vartheta = 35^\circ$, whereas in this case equal intensity was observed at 27°; at 35° the middle component was already stronger than the outer ones. The observed differences are undoubtedly in connection with the fact that with purely transversal observation ($\vartheta = 90^\circ$) the ratio of the intensities of the middle and the outer components amounts only to about 1.32, instead of 2; with observation by means of a nicol the components were namely seen with equal intensity when the direction of the transmitted vibrations made an angle of 49° with the horizontal direction. In another respect, too, the examined chrome line is not perfectly normal: the middle component is distinctly broader than the original line, so that it is possibly, double.

Physics. — *"Diffraction of a single pulse wave through a slit according to* Kirchhoff's *theory."* By Prof. C. H. Wind.

1. Some years ago Prof. Haga in conjunction with the author of this paper studied experimentally the image which is formed of a slit lighted by Röntgen-rays on a photographic plate placed behind it. [2]) By comparison [3]) of the obtained photographs with the known diffraction images which we get when lighting the slit with homo-

[1]) H. A. Lorentz loc. cit.

[2]) H. Haga and C. H. Wind, These Proc. I, p. 420, 1899, and V, p. 247, 1902.

[3]) Id. Ibid., I, p. 423, 1899; cf. also Physik. Zschr. 2, p. 265, 1900.

geneous light of different wave-lengths, it was possible, by approximation to find the region of wave-lengths, inside which the R-rays considered as a mixture of rays of different wave-lengths, but homogeneous in themselves, possess their greatest energy.

Such a conception of the nature of the R-rays was not inconsistent with the supposition advanced then already on different sides, that these rays would owe their origin to impulsive disturbances of the equilibrium of the ether, following each other irregularly. For also a radiation arising from such "single pulse waves" may be conceived as a mixture of homogeneous rays, though it be of an infinite number of wave-lengths [1]). This was elucidated by me at the "Deutsche Naturforscherversammlung" at Aix-la-Chapelle (1900) and afterwards more fully in the Physik. Zeitschr. [2]). It was then also pointed out, what connection there exists between what on this latter supposition may be called the "length" [3]) of the single pulse waves, viz. the distance between the first and the last wave-front in one of them and what we had found, starting from the experiments, as "*wavelength of maximum energy*" in the mixture of homogeneous radiations.

The latter conception renders it possible to reduce the problem of diffraction of R-rays through a slit to the problem already fully worked out by KIRCHHOFF of diffraction of homogeneous light through a slit. [4]) But it is, also possible to derive the diffraction of a single pulse wave through a slit *directly* from HUYGENS-KIRCHHOFF's principle. [5]) This will be done in the following pages. Where the problem of the diffraction of the R-rays is still actual even now, [6]) this new treatment cannot be considered as superfluous. It may serve at the same time as an introduction to a reply to the objections brought forward by WALTER and POHL [7]) to HAGA and WIND's conclusions from their experiments. It is true that about simultaneously with these investigations SOMMERFELD has devoted an extensive study to

[1]) G. JOHNSTONE STONEY, Phil. Mag. (5) 45, p. 532, and 46, p. 253, 1898.

[2]) C. H. WIND, Physik. Zschr. l.c. and 2, p. 189, 1900, p. 292, 1901.

[3]) It seems more in harmony with the denomination "wavelength" for periodic disturbances, to call the distance between front and rear wave-front "*length* of the single pulse wave", than to speak of its "breadth" ("Breite des Impulses") with SOMMERFELD) (Physik. Zschr. 1, p. 105, 1899, and 2, p. 55, 1900).

[4]) G. KIRCHHOFF, Vorlesungen üb. math. Physik, II, 7te Vorl., p. 129, 1891.

[5]) G. KIRCHHOFF, Ibid., 2te Vorl., p. 22, 1891.

[6]) Comp. i. a. E. MARX, "Zweite Durchführung der Geschwindigkeitsmessung der R.-strahlen" (Abh. math. phys. Kl. k. sächs. Ges. d. Wiss. 32, N°. 2, p. 156, 1910).

[7]) B. WALTER u. R. POHL, Ann. d. Physik 25, p. 715, 1908, and 29, p. 331, 1909.

the same subject [1]). But he took a different course, by the side of which it may once more be shewn that the original way indicated by KIRCHHOFF leads just as well to the purpose. Further I have been enabled, thanks to the collaboration of our fellow-member W. KAPTEYN [2]), for which I am greatly indebted to him, to carry out the numerical calculation of the intensity of radiation for every point of our special diffraction image, and so to get to know the distribution of this intensity over the image in all the details required. This is of importance on account of the doubt which has again risen [1]) [3]) with respect to the interpretation of our experiments of diffraction.

2. According to KIRCHHOFF [4]) in any point O behind an opaque screen, which is provided with one or more apertures, but which for the rest extends into infinity (fig. 2), and receives on its front side a radiation determined by a function φ, which outside the sources of radiation satisfies the equation:

$$\Delta \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the value of this function at any time t may be expressed by:

$$\varphi_{O,t} = \frac{1}{4\pi}\int_S dS \left\{ \frac{\partial r_0}{\partial N}\frac{\partial}{\partial r_0}\frac{\overline{\varphi}_{t-\frac{r_0}{c}}}{r_0} - \frac{\left(\frac{\partial \overline{\varphi}}{\partial N}\right)_{t-\frac{r_0}{c}}}{r_0} \right\} \quad . \quad . \quad . \quad (2)$$

In this expression the integral is to be taken over a surface S consisting of as many parts as there are apertures in the screen and bounded by the edges of these apertures, while in every point of this "slit plane" N denotes the normal to it directed backward, r_0 the distance to the point O, and $\overline{\varphi}$ the value which the function φ would have in the point in the case of presence of the same sources of radiation, but absence of the screen.

In the case of a single pulse wave emitted by an electrical point charge during a change in its state of movement, we are free to take for φ either the electromagnetic potential ϕ or the electromagnetic vector potential $\mathfrak{A}$ [5]), or e.g. the electric or magnetic force.

[1]) A. SOMMERFELD, Zschr. f. Math. u. Physik. 46, p. 11, 1901.
[2]) See the following communication in these proceedings p. 405.
[3]) See note 6 and 7 p. 395.
[4]) G. KIRCHHOFF, l.c. 2e Vorl..
[5]) M. ABRAHAM, Elektromagn. Th. d. Strahlung, § 6, Leipzig, 1905.

3. If the disturbance in the source is accomplished from the moment t' to the moment t'', then $\bar{\varphi}$ in the point P of the slit-plane, at a distance r_1 from the point L, differs from zero only from $t = t' + \frac{r_1}{c}$ to $t = t'' + \frac{r_1}{c}$. If the greatest absolute value which this function reaches there during this period be $\frac{K}{r_1}$ and the flux of energy may be put proportional to the square of the function φ, K may

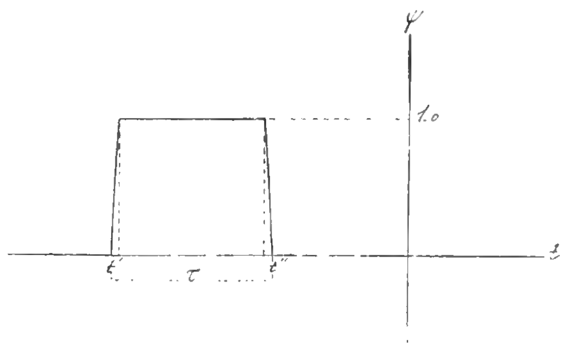


Fig 1.

evidently be considered as dependent on the source but not on the place of the point P in the slit plane.

Then if we put:

$$\bar{\varphi}(t) = \frac{K}{r_1}\,\psi\left(t - \frac{r_1}{c}\right), \quad \ldots \ldots \ldots \quad (3)$$

ψ is a function, of which for all the points of the slit-plane (cf. fig. 1, in which for a reason which will become clear later on t' has been taken negative and $t'' - t'$ is indicated by τ) we know already, that

$$\left.\begin{array}{lrr} \text{for} & t \leqq t' & \psi(t) = 0, \\ \text{,,} & t' < t < t'' & 0 < |\psi(t)| < 1, \\ \text{,,} & t \leqq t'' & \psi(t) = 0. \end{array}\right\} \quad \ldots \quad (4)$$

Further we have:

$$\left(\frac{\partial \bar{\varphi}}{dN}\right)_t = -K \cos\theta_1 \left[\frac{\psi\left(t - \frac{r_1}{c}\right)}{r_1^{\,2}} + \frac{\psi'\left(t - \frac{r_1}{c}\right)}{r_1 c}\right],$$

$$\frac{\partial r_0}{\partial N}\frac{\partial}{\partial r_0}\frac{\bar{\varphi}_{t-\frac{r_0}{c}}}{r_0} = -K \cos\theta_0 \frac{\partial}{\partial r_0}\frac{\psi\left(t - \frac{r_0 + r_1}{c}\right)}{r_0 r_1},$$

if θ_0 and θ_1 denote the angles which resp. the radius vector r_0 from P to O and the continuation of the radius vector r_1 from L to P form with the normal N (fig. 2),

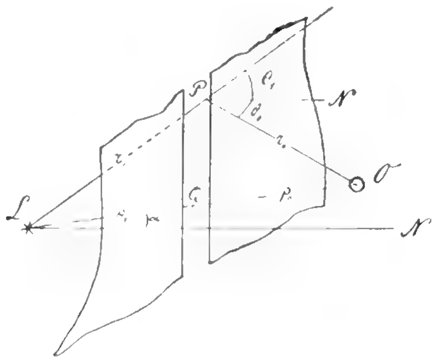


Fig. 2.

and equation (1) is changed into:

$$\varphi_{(t)} = \frac{K}{4\pi} \int_S dS \left\{ \left(\frac{\cos\theta_0}{r_0} + \frac{\cos\theta_1}{r_1} \right) \frac{\psi\left(t - \frac{r_0 + r_1}{c}\right)}{r_0 r_1} + \right.$$

$$\left. + (\cos\theta_0 + \cos\theta_1) \frac{\psi'\left(t - \frac{r_0 + r_1}{c}\right)}{c\, r_0 r_1} \right\} \quad (5)$$

4. For every point of the slit-plane the length of path $r_0 + r_1$ from L to T over P has a definite length. It is shortest for the pole Q of the point O, the point of intersection of the slit-plane with the line LO; for this point it is $\varrho_0 + \varrho_1$ (c.f. notation indicated in fig. 2). We put:

$$\varrho_0 + \varrho_1 = R$$

and term

$$\zeta = r_0 + r_1 - R$$

the difference of path for the point P of the slit.

We now consider in the slit-plane those lines which are loci of points with definite values of this difference of path ζ, and term these lines ζ-curves. Two such curves (fig. 3), belonging to values of ζ which differ from each other an infinitely small amount $d\zeta$, inclose an infinitely small region of the slit-plane, the whole of which, bounded as it is by the limits of the slit-plane, and often consisting of as many separate pieces as there are apertures in the screen, we call a $d\zeta$-zone. The area of such a $d\zeta$-zone we may denote by $l d\zeta$, in which l is a function of ζ.

We now introduce such a $d\zeta$-zone as surface-element dS into the integral of (5), noticing that under the integral sign in the denominators

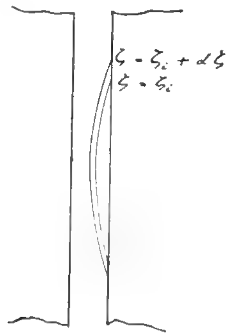


Fig. 3.

we may substitute the constant distances ϱ_0 and ϱ_1 for the distances r_0 and r_1, varying from point to point in the slit plane, and likewise the constant $\cos\alpha$ for $\cos\theta_0$ and $\cos\theta_1$, behaving in the same way, provided we consider the radiation only in such points O, for which the values r_0 and r_1 are very large compared with the dimensions of the slit plane, in other words at great distances from the slit.

By these substitutions and simplifications (5) becomes:

$$\varphi_{O,t} = \frac{K}{2\pi}\,\frac{\cos\alpha}{\varrho\, R}\int_0^\infty \left\{ \frac{2}{\varrho}\,\psi\left(t - \frac{R+\zeta}{c}\right) + \frac{2}{c}\,\psi'\left(t - \frac{R+\zeta}{c}\right)\right\} l\,d\zeta, \quad . \quad . \quad (6)$$

if be put:

$$\frac{1}{\varrho_0} + \frac{1}{\varrho_1} = \frac{2}{\varrho},$$

and therefore

$$\varrho_0\varrho_1 = \tfrac{1}{2}\,\varrho R$$

5. To get a better understanding of the quantity l we imagine the confocal revolution ellipsoids, which may be described with the ascending values of $R+\zeta$ as lengths of the major axis, round L and O as foci, and which meet the slit plane in the ζ-curves.

Excluding cases in which the angle α comes very near 90° we may for those values of ζ which are of importance for us consider the projection of the ζ-curve on the plane, brought $\perp LO$ through Q, to be an arc of the circle along which this plane is intersected by the same ellipsoid which meets the slit plane in the ζ-curve. If we call the radius of this circle ξ and the extent of that arc, in radians, β, then, whith a sufficient degree of approximation,

$$\zeta = \frac{\xi^2}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Considering finally the area of the projection of the $d\zeta$-zone on the plane mentioned we have

$$\cos\alpha \, . \, l \, d\zeta = \beta\xi \, . \, d\xi = \tfrac{1}{2}\, \beta d\xi^2 = \tfrac{1}{2}\, \beta\varrho d\zeta,$$

from which follows

$$l = \frac{\varrho}{2\cos\alpha}\, \beta.$$

By substitution of this value (6) passes into:

$$\varphi_{0,t} = \frac{K}{2\pi R_{\cdot}} \int_0^{\infty} \frac{1}{\varrho}\, \psi \left(t - \frac{R+\zeta}{c}\right) \beta d\zeta + \frac{K}{2\pi R_{\cdot}} \int_0^{\infty} \frac{1}{c}\, \psi' \left(t - \frac{R+\zeta}{c}\right) \beta d\zeta. \quad (8)$$

6. As $\psi\left(t - \dfrac{R+\zeta}{c}\right)$ at every definite moment differs from zero only between two values of ζ lying $c\,(t''-t')$ from each other, and as it is then $=1$ at the utmost, whereas β cannot exceed 2π, it is clear that

$$\frac{K}{R} \frac{c\,(t''-t')}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

is an upmost limit which connot be exceeded in any case by the value of the first term of (6). Hence so long as

$$\frac{c\,(t''-t')}{\varrho}$$

is a very small value [1]), that term may be neglected by the side of a term of the order of magnitude of $\dfrac{K}{R}$, and it is allowed to write for (8):

$$\varphi_{0,t} = \frac{K}{2\pi R} \int_{\zeta=\infty}^{\zeta=0} \beta \, d\psi \left(t - \frac{R+\zeta}{c} \right). \quad . \quad . \quad . \quad . \quad (10)^{2)}$$

---

1) For Röntgen rays $c\,(t'-t'')$ is of the order of magnitude of 10^{-10}, and in the diffraction experiments ϱ is of the order of 10^2 cm.

2) That it appears to be allowed to neglect the first integral in the second member of (8) is of much interest in connection with the application of HUYGENS' principle in its more primitive form to the problem of diffraction: it proves that if one forms a conception of the propagation of radiation aggreeing with this principle, one must bear in mind that the (secondary) emission of elements of disturbance in ether which one then imagines to issue from every element of the slit plane, does not depend on the amplitude which the disturbance itself possesses in the considered element, but only on the *rapidity of change of this amplitude* at that place.

We assume — to take a special case — that $\psi(t)$ increases within a very short time from zero to its value 1 (fig. 1), then preserves this value unchanged, and afterwards decreases again as rapidly to zero, and that in such a way that it is allowed for the values of ζ corresponding to the periods of change of ψ, to take no value for β but that corresponding to the beginning or end of these periods. Then (10) becomes simply:

$$\varphi_{0,t} = \frac{K}{2\pi R}(\beta' - \beta''), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

when we put for:

$$\left.\begin{aligned} \zeta = \zeta' &= c\,(t-t') - R \qquad \beta = \beta' \\ \text{and for} \qquad \zeta = \zeta'' &= c\,(t-t'') - R \qquad \beta = \beta''. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

7. Now if the slit is bounded by parallel edges, if the distance from the point Q to the nearest edge be denoted by n, to the furthest by m, n being taken as negative when Q falls outside the slit, then

$$\left.\begin{aligned} \beta &= 2\left(arcsin\frac{m}{\sqrt{\zeta\varrho}} + arcsin\frac{n}{\sqrt{\zeta\varrho}}\right), && \text{for} \quad \zeta\varrho > m^2, \\ \beta &= 2\left(\frac{\pi}{2} + arcsin\frac{n}{\sqrt{\zeta\varrho}}\right), && \text{,,} \quad m^2 > \zeta\varrho > n^2, \\ \beta &= 2\left(\frac{\pi}{2} + \frac{\pi}{2}\right), && \text{,,} \quad n^2 > \zeta\varrho > 0,\ n > 0, \\ \beta &= 2\left(\frac{\pi}{2} - \frac{\pi}{2}\right), && \text{,, ,, ,, ,, ,, ,,} \quad n < 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (13)$$

All these expressions for β, holding in the different cases, may be united to a single one, if we consider that in general

for $a^2 > 1$ and $a > 0$, $arc\ sin\ a = \frac{1}{2}\pi\, i\, l_{nat}\,(a + \sqrt{a^2-1})$,

,, ,, ,, ,, ,, $a < 0$, $arc\ sin\ a = -\frac{1}{2}\pi - i\, l_{nat}\,(-a - \sqrt{a^2-1})$

and ,, $a^2 < 1$, $arc\ sin\ a =$ real.

Then we may write in each of the 4 cases considered:

$$\beta = 2\,\Re\left(arc\ sin\frac{m}{\sqrt{\zeta\varrho}} + arc\ sin\frac{n}{\sqrt{\zeta\varrho}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Even for moments for which (1) yields a negative value of ζ, this

Therefore, if a single pulse wave of the type represented in fig. 1, traverses the element, *two* periods of secondary emission of radiation are to be ascribed to the element, one during the "*immersion*" of the element in the single pulse wave, the other during the "*emersion*" (cf. footnote 1 p. 403).

expression accurately indicates the value of β, which then of course is zero. Indeed ζ being negative, ξ (according to (7)) and also $\frac{m}{\xi}$ and $\frac{n}{\xi}$ will be purely imaginary, while, generally, for imaginary values of a

$$arc\ sin\ a = -i\, l_{nat}\, (ia + \sqrt{1-a^2})$$

and hence

$$\Re\ arc\ sin\ a = 0\,.$$

If we now take (14) as the general expression for β and substitute $\sqrt{\varrho\zeta}$ for ξ (cf. (7)), we get:

$$\beta = 2\,\Re\left(arcsin\frac{m}{\sqrt{\varrho\zeta}} + arcsin\frac{n}{\sqrt{\varrho\zeta}}\right). \quad . \quad . \quad . \quad . \quad (15)$$

and so, according to (11) and (12)

$$\varphi_{O,t} = \frac{K}{2\pi R}\,.\,2\,\Re\left(arcsin\frac{m}{\sqrt{\varrho\zeta'}} - arcsin\frac{m}{\sqrt{\varrho\zeta''}} + \right.$$

$$\left. + arcsin\frac{n}{\sqrt{\varrho\zeta'}} - arcsin\frac{n}{\sqrt{\varrho\zeta''}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

or, introducing the values of ζ' and ζ'' indicated in (12):

$$\varphi_{O,t} = \frac{K}{\pi R}\,\Re\left(arc\,sin\frac{m}{\sqrt{\varrho c(t-t')-\varrho R}} - arc\,sin\frac{m}{\sqrt{\varrho c(t-t'')-\varrho R}} + \right.$$

$$\left. + arc\,sin\frac{n}{\sqrt{\varrho c(t-t')-\varrho R}} - arc\,sin\frac{n}{\sqrt{\varrho c(t-t'')-\varrho R}}\right) \quad . \quad . \quad (17)$$

If in this formula we put

$$t' = -\frac{R}{c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which means that time is reckonned from the moment at which, in the absence of the diffracting screen, the beginning of the disturbance would reach the point O; if further we put

$$t'' - t' = \tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

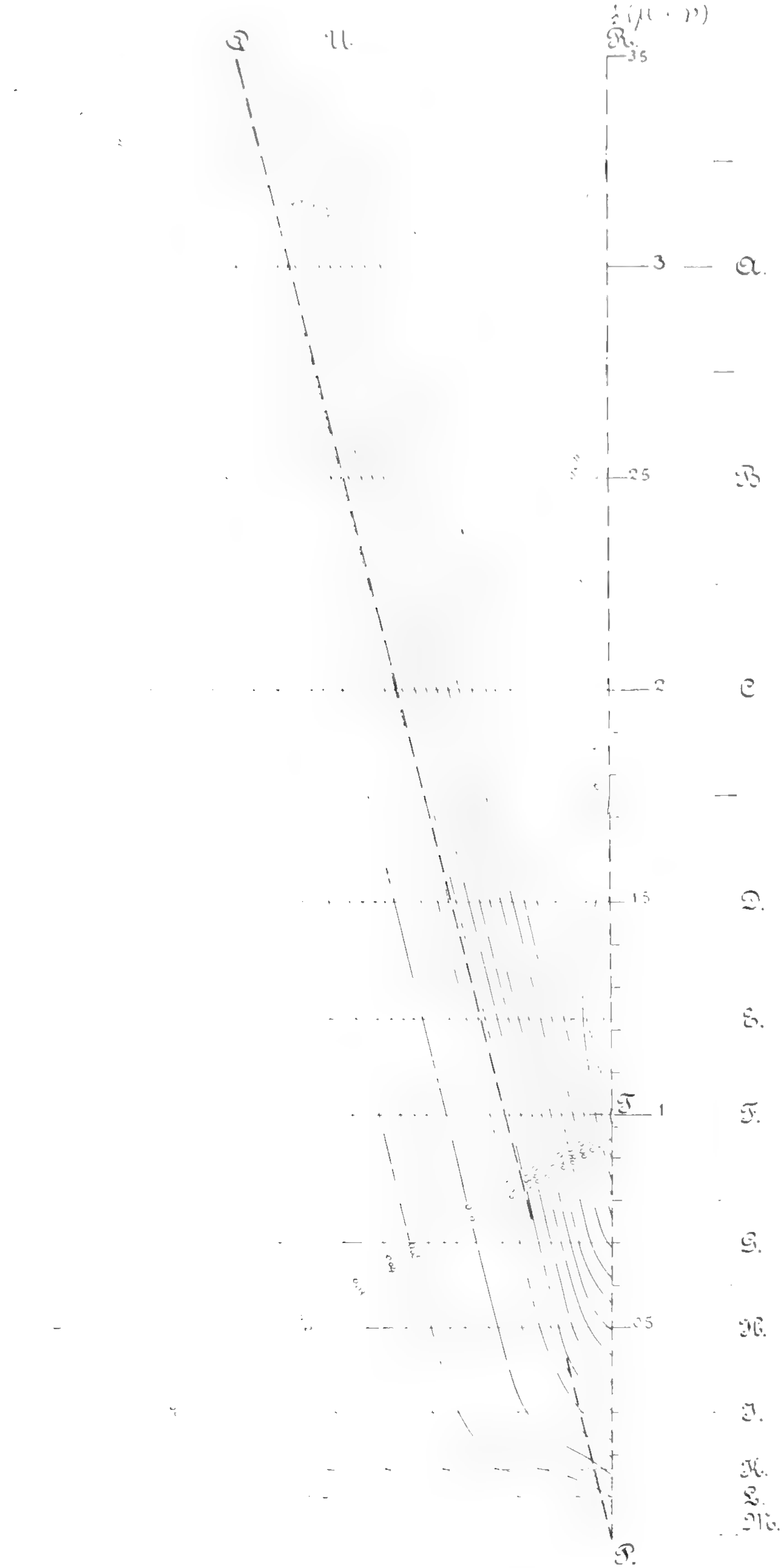
$$c\tau = \lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

indicating by τ and λ the duration and the "l e n g t h" of the single pulse wave, and finally:

$$\frac{m}{\sqrt{\varrho\lambda}} = \mu\,, \quad \frac{n}{\sqrt{\varrho\lambda}} = \nu, \quad . \quad . \quad . \quad . \quad (21),\ (22)$$

we get:

C. H. WIND. "Diffraction of a single pulse wave through a slit according to KIRCHHOFF's theory." Fig. 4.



$$\varphi_{O,t} = \frac{K}{\pi R} \Re\left(arc\,sin \frac{\mu}{\sqrt{t/\tau}} - arc\,sin \frac{\mu}{\sqrt{t/\tau - 1}} + \right.$$

$$\left. + arc\,sin \frac{\nu}{\sqrt{t/\tau}} - arc\,sin \frac{\nu}{\sqrt{t/\tau - 1}} \right). \quad . \quad . \quad . \quad (23)$$ [1])

The phenomenon observable in O, by which we may judge of the intensity of the radiation produced by the single pulse wave, may be of different kinds, e. g. photographic action on a sensitive plate, or generation of heat in case of absorption by matter, or ionisation of a gas and the resulting discharge of a charged body. With phenomena of this kind it is usual, and to a certain extent justifiable, to put the intensity of the action produced during a definite lapse of time proportional to the quantity of energy which, with the wave, and per unit of area, traverses a surface element placed in the point of observation normal to the direction of radiation. Per unit of time this quantity may be put proportional to $\varphi^2_{O,t}$, whence for the whole duration of the disturbance in O it will be adequately represented by:

$$J = \int_{-\infty}^{+\infty} \varphi^2_{O,t}\, dt.$$

This becomes, by introducing (23) and putting

$$\frac{t}{\tau} = x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$J = \left(\frac{K\tau}{\pi R}\right)^2 I, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in which:

$$I = \int_{-\infty}^{+\infty} \left[\Re\left(arc\,sin \frac{\mu}{\sqrt{x}} - arc\,sin \frac{\mu}{\sqrt{x-1}} + arc\,sin \frac{\nu}{\sqrt{x}} - arc\,sin \frac{\nu}{\sqrt{x-1}} \right) \right]^2 dx.$$

This expression essentially agrees with that which SOMMERFELD arrived at in his lastly mentioned paper and which has also been there numerically evaluated. These calculations, however, have been confined — at least as far as is seen from the paper — to values in points situated along certain special curves, and, moreover, have

[1]) In connection with footnote 2 p. 400 we may point out that in the expression (23) the first and the third term between parentheses jointly correspond to the "immersion" of elements of the slit plane into the single pulse wave, the second and the fourth term jointly to the "emersion" from it.

not been pushed to a very high degree of accuracy. For the purpose, pursued by Mr. SOMMERFELD at the time, of course they were quite sufficient.

The present author in the contrary, having grown anxious to learn further particulars on the distribution of intensity, found himself in the necessity of approximating the expression with some accuracy, and that not only along curves, along which the integration of the expression can be obtained in finite form, but also in a great number of points arbitrarily chosen. By the series expansions deduced by Prof. W. KAPTEYN [1]) this was rendered possible, though it remained a laborious business.

The result, being a nearly complete survey of the way in which the energy of radiation of the single pulse wave in a definite point depends on u and v, or any other couple of parameters equivalent to these, is graphically represented in fig. 4. In this diagram every section normal to the axis PR indicates the distribution of the energy in the horizontal section of the diffraction image of a parallel edged slit, for a definite value of $\frac{1}{2}(u+v)$. For some of the sections the value of this quantity, viz.

$$\tfrac{1}{2}(u+v) = \tfrac{1}{2}\frac{m+n}{\sqrt{\varrho\lambda}} = \tfrac{1}{2}\frac{\textit{breadth of the slit}}{\sqrt{\varrho\lambda}}$$

is indicated by the numbers put along the axis PR.

The points of equal intensity in the successive sections are connected by curves, and this enables us at once to form an idea also of the distribution of intensity which may be expected in the diffractionimage of a slit with edges not parallel, but slowly converging towards the base [2]). When using the diagram in this way for single pulse waves of very small length, we must, bear in mind that in the direction normal to PR it is represented at a much larger scale than in the direction of PR itself.

What the diagram, looked upon from this point of view, represents is the distribution of intensity over one half of the slit image, the broken line PQ indicating in it the projection of one edge of the slit from the point L on the plane of observation.

This distribution of intensity presents a number of noteworthy particulars. To some of these I hope to draw your attention on a following occasion.

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1) See the following communication in these proceedings.

2) In the slits, used by HAGA and WIND in their experiments, the angle formed by the edges never amounted to more than 0°.03.

Mathematics. — *"On the final Integral occurring in* Prof. WIND's *paper: "Diffraction of a single pulse wave by a slit, according to* KIRCHOFF's *theory."* By Prof. W. KAPTEYN.

1. In Prof. WIND's paper the problem is reduced to the integral

$$I = \int_0^\infty \left[R_e \, as \frac{\mu}{\sqrt{x}} - R_e \, as \frac{\mu}{\sqrt{x-1}} - R_e \, as \frac{b}{\sqrt{x}} + R_e \, as \frac{b}{\sqrt{x-1}} \right]^2 dx$$

wherein $-b$ is written instead of $\mathbf{r}$. R_e means the real part of the function which follows and as represents the function sin^{-1}. The object of this paper is to reduce the preceding integral so that it is ready for numerical computation.

Let

$$R_e \, as \frac{\mu}{\sqrt{x}} = A, \quad R_e \, as \frac{\mu}{\sqrt{x-1}} = B \quad R_e \, as \frac{b}{\sqrt{x}} = A' \quad R_e \, as \frac{b}{\sqrt{x-1}} = B'$$

we have

$$I = I(\mu, b) = \int_0^\infty (A-B)^2 \, dx + \int_0^\infty (A'-B')^2 \, dx -$$

$$- 2\int_0^\infty (AA' - AB' - A'B + BB') \, dx \quad . \quad . \quad . \quad . \quad (1)$$

For $b = 0$ and $\mu = 0$ this reduces to

$$I(\mu, 0) = I(\mu) = \int_0^\infty (A - B)^2 \, dx$$

$$I(0, b) = I(0, -b) = I(b) = \int_0^\infty (A' - B')^2 \, dx$$

thus, if we put

$$K = AA' - AB' - A'B + BB'$$

the equation (1) may be written

$$I(\mu, b) = I(\mu) + I(b) - 2\int_0^\infty K \, dx \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we call μ, b and $\mu, -b$ corresponding points, it is evident that the values of the integral in corresponding points may be deduced from one another by the relation

$$I(\mu, -b) = 2\,I(\mu) + 2\,I(b) - I(\mu, b) \quad . \quad . \quad . \quad . \quad (3)$$

for, $I(b)$ being the same as $I(-b)$, we have

$$I(\mu, -b) = I(\mu) + I(b) + 2\int_0^\infty K\,dx$$

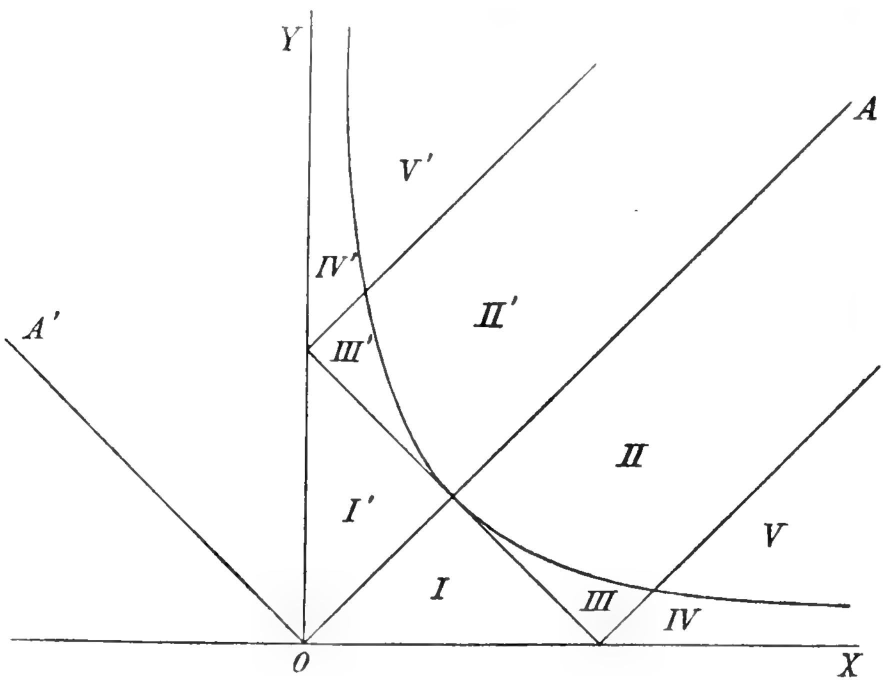
and this equation together with (2) gives the equation (3).

We may therefore limit our investigation to positive values of b only.

2. We must now distinguish the following cases

I.	$\mu^2 < 1$	$b^2 < 1$	$\mu^2 < b^2 + 1$
II.	$\mu^2 > 1$	$b^2 < 1$	$\mu^2 > b^2 + 1$
III.	$\mu^2 > 1$	$b^2 < 1$	$\mu^2 < b^2 + 1$
IV.	$\mu^2 > 1$	$b^2 > 1$	$\mu^2 < b^2 + 1$
V.	$\mu^2 > 1$	$b^2 > 1$	$\mu^2 > b^2 + 1.$

To represent these cases by a figure, we draw from the origin of a rectangular system of axes XOY the lines OA and OA' so that $\angle AOY = \angle A'OY = \varphi$. Considering these lines as the limits of the slit and remembering the signification of μ and b, we get for the coordinates of a point P of the plane



$$x = \frac{1}{4}(\mu + b) \qquad\qquad y = \frac{1}{4}(\mu - b)\, ctg\, \varphi .$$

The limits

$$\mu^2 = 1 \qquad\qquad b^2 = 1 \qquad\qquad \mu^2 = b^2 + 1$$

may therefore be represented by the lines

$$4\,(x + y\, tg\, \varphi) = 1, \qquad\qquad 4\,(x - x\, tg\, \varphi) = 1$$

and the hyperbola

$$16\, xy\, tg\, \varphi = 1.$$

These limits divide the plane in different regions which are represented in our figure by the numbers corresponding with the preceding cases. The accentuated numbers are inscribed in the regions where b is negative.

3. In the first place we shall consider the integral $I(\mu)$.
If $z > 1$, we have

$$as\, z = \frac{\pi}{2} - il\,(z + \sqrt{z^2 - 1})$$

so

$$R_e\; as\, z = \frac{\pi}{2}.$$

Therefore, according to the values of μ

$\mu < 1$				$\mu > 1$			
x		A	B	x		A	B
0 until	μ^2	$\frac{\pi}{2}$	0	0 until	1	$\frac{\pi}{2}$	0
μ^2 ,,	1	$as \frac{\mu}{\sqrt{x}}$	0	1 ,,	μ^2	0	0
1 ,,	$1+\mu^2$	$as \frac{\mu}{\sqrt{x}}$	$\frac{\pi}{2}$	μ^2 ,,	$1+\mu^2$	$as \frac{\mu}{\sqrt{x}}$	$\frac{\pi}{2}$
$1+\mu^2$,,	∞	$as \frac{\mu}{\sqrt{x}}$	$as \frac{\mu}{\sqrt{x-1}}$	$1+\mu^2$,,	∞	$as \frac{\mu}{\sqrt{x}}$	$as \frac{\mu}{\sqrt{x-1}}$

So if $\mu < 1$ the value of the integral reduces to

$$I(\mu) = \frac{\pi^2}{2}\mu^2 - \pi\int_1^{1+\mu^2} as \frac{\mu}{\sqrt{x}}\, dx + 2\int_{\mu^2}^{\infty}\left(as \frac{\mu}{\sqrt{x}}\right)^2 dx - 2\int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}}\; as \frac{\mu}{\sqrt{x-1}}\, dx \quad (4)$$

and if $\mu > 1$, to

$$I(\mu) = \frac{\pi^2}{2} - \pi\int_{\mu^2}^{1+\mu^2} as \frac{\mu}{\sqrt{x}}\, dx + 2\int_{\mu^2}^{\infty}\left(as \frac{\mu}{\sqrt{x}}\right)^2 dx - 2\int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}}\; as \frac{\mu}{\sqrt{x-1}}\, dx \quad (5)$$

In both these equations the second and third integrals of the second members are infinite, their sum however is finite. We may escape

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this difficulty by considering the infinite limits of those integrals as $Lim \frac{1}{\varepsilon^2}$ for ε approaching to zero.

Differentiating both members of (4) we find

$$\frac{\partial}{\partial\mu}\int_1^{1+\mu^2} as \frac{\mu}{\sqrt{x}}\, dx = 2\mu\, as \frac{\mu}{\sqrt{1+\mu^2}} + 2 - 2\sqrt{1-\mu^2}$$

$$\frac{\partial}{\partial\mu}\int_{\mu^2}^{\infty}\left(as \frac{\mu}{\sqrt{x}}\right)^2 dx = -\frac{\pi^2}{2}\mu + 4\mu - 4\mu\, l\, \mu\varepsilon$$

$$\frac{\partial}{\partial\mu}\int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\, dx = -\pi\mu\, as \frac{\mu}{\sqrt{1+\mu^2}} +$$

$$+\int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x-1}} \frac{dx}{\sqrt{x-\mu^2}} + \int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}} \frac{dx}{\sqrt{x-1-\mu^2}}$$

We shall now transform the two last integrals. Integrating by parts we obtain

$$\int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x-1}} \frac{dx}{\sqrt{x-\mu^2}} = 2\mu - \pi + \mu\int_{1+\mu^2}^{\infty} \frac{\sqrt{x-\mu^2}}{(x-1)\sqrt{x-1-\mu^2}}\, dx$$

$$\int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}} \frac{dx}{\sqrt{x-1-\mu^2}} = 2\mu + \mu\int_{1+\mu^2}^{\infty} \frac{\sqrt{x-1-\mu^2}}{x\sqrt{x-\mu^2}}\, dx$$

where

$$\int_{1+\mu^2}^{\infty} \frac{\sqrt{x-\mu^2}}{(x-1)\sqrt{x-1-\mu^2}}\, dx = \int_{1+\mu^2}^{\infty} \frac{dx}{\sqrt{(x-\mu^2)(x-1-\mu^2)}} +$$

$$+ (1-\mu^2)\int_{1+\mu^2}^{\infty} \frac{dx}{(x-1)\sqrt{(x-\mu^2)(x-1-\mu^2)}}$$

$$\int_{1+\mu^2}^{\infty} \frac{\sqrt{x-1-\mu^2}}{x\sqrt{x-\mu^2}}\, dx = \int_{1+\mu^2}^{\infty} \frac{dx}{\sqrt{(x-\mu^2)(x-1-\mu^2)}} -$$

$$-(1+\mu^2)\int_{1+\mu^2}^{\infty} \frac{dx}{x\sqrt{(x-\mu^2)(x-1-\mu^2)}} \quad .$$

and

$$\int\limits_{1+\varepsilon^2}^{\infty} \frac{dx}{\sqrt{(x-\mu^2)(x-1-\mu^2)}} = l\,\frac{4}{\varepsilon^2}$$

$$\int\limits_{1+\varepsilon^2}^{\infty} \frac{dx}{(x-1)\sqrt{(x-\mu^2)(x-1-\mu^2)}} = \frac{2}{\mu\sqrt{1-\mu^2}}\, as\, \sqrt{1-\mu^2}$$

$$\int\limits_{1+\varepsilon^2}^{\infty} \frac{dx}{x\sqrt{(x-\mu^2)(x-1-\mu^2)}} = \frac{2}{\mu\sqrt{1+\mu^2}}\, l\,(\sqrt{1+\mu^2}+\mu)$$

Hence finally

$$\frac{dI(\mu)}{d\mu} = -\,8\mu\, l\, 2\mu + 2\mu\sqrt{1-\mu^2} - 4\sqrt{1-\mu^2}\, as\, \sqrt{1-\mu^2} +$$
$$+\, 4\sqrt{1+\mu^2}\, l\,(\sqrt{1+\mu^2}+\mu)$$

or

$$\frac{dI(\mu)}{d\mu} = -\,8\mu\, l\, 2\mu + 4\sqrt{1-\mu^2}\, as\, \mu + 4\sqrt{1+\mu^2}\, l\,(\sqrt{1+\mu^2}+\mu).$$

If now we integrate this equation again we shall obtain $I(\mu)$ in the required form. For

$$\int \mu\, l\, 2\mu\, d\mu = \frac{\mu^2}{2}\, l\,(2\mu) - \frac{\mu^2}{4}$$

$$\int \sqrt{1-\mu^2}\, as\, \mu\, d\mu = \int\left(\frac{1}{2\sqrt{1-\mu^2}} + \frac{1-2\mu^2}{2\sqrt{1-\mu^2}}\right) as\, \mu\, d\mu =$$

$$= \frac{1}{4}\,(as\,\mu)^2 + \frac{1}{2}\,\mu\sqrt{1-\mu^2}\, as\, \mu - \frac{1}{4}\,\mu^2$$

$$\int \sqrt{1+\mu^2}\, l\,(\sqrt{1+\mu^2}+\mu)\, d\mu =$$

$$\int\left(\frac{1}{2\sqrt{1+\mu^2}} + \frac{1+2\mu^2}{2\sqrt{1+\mu^2}}\right) l\,(\sqrt{1+\mu^2}+\mu)\, d\mu =$$

$$= \frac{1}{4}\, l^2\,(\sqrt{1+\mu^2}+\mu) + \frac{1}{2}\,\mu\sqrt{1+\mu^2}\, l\,(\sqrt{1+\mu^2}+\mu) - \frac{1}{4}\,\mu^2$$

So because $I\,(0) = 0$

$$I\,(\mu) = -\,4\mu^2\, l\, 2\mu + 2\mu\sqrt{1-\mu^2}\, as\, \mu + (as\,\mu)^2 +$$
$$+\, 2\mu\sqrt{1+\mu^2}\, l\,(\mu+\sqrt{\mu^2+1}) + l^2\,(\mu+\sqrt{\mu^2+1}) \qquad (\mu < 1) \qquad (6)$$

In the same way differentiating (5) we get

$$\frac{dI(\mu)}{d\mu} = -\,8\mu\, l\, 2\mu + 4\sqrt{\mu^2-1}\, l\,(\mu+\sqrt{\mu^2-1}) + 4\sqrt{\mu^2+1}\, l\,(\mu+\sqrt{\mu^2+1}),$$

which gives by integration

$$I(\mu) = \frac{\pi^2}{4} - 4\mu^2\, l\, 2\mu + 2\mu \sqrt{\mu^2-1}\; l\,(\mu+\sqrt{\mu^2-1}) - l^2\,(\mu+\sqrt{\mu^2-1}) +$$

$$+ 2\mu \sqrt{1+\mu^2}\; l\,(\mu+\sqrt{\mu^2+1}) + l^2\,(\mu+\sqrt{\mu^2+1}). \qquad (\mu > 1) \qquad (7)$$

4. In order to simplify the computation of the numereal, values, it seems important to expand $I(\mu)$ in a convergent series.

Writing therefore in equation (6)

$$lg\,(\mu - \sqrt{1+\mu^2}) = i\, as\, \frac{\mu}{i}$$

we may introduce the known expansions of $as\, z$ and $(as\, z)^2$. In this way we easily obtain, if $(\mu < 1)$

$$I(\mu) = -4\mu^2\, l\, 2\mu + 6\mu^2 - \frac{8}{15}\left[\frac{\mu^6}{3} + \frac{4.6}{7.9}\,\frac{\mu^{10}}{5} + \frac{4.6.\ 8.10}{7.9.11.13}\cdot\frac{\mu^{14}}{7} + \ldots\right] (8)$$

a series which is sufficiently convergent for all values from $\mu = 0$ to $\mu = 0.7$.

If $\mu > 1$, we have, putting: $\mu = \frac{1}{m}$

$$I(\mu) = \frac{\pi^2}{4} - \frac{4}{m^2}\, l\, \frac{2}{m} + \frac{2}{m^2}\sqrt{1+m^2}\; l\, \frac{1+\sqrt{1+m^2}}{m} +$$

$$+ \frac{2}{m^2}\sqrt{1-m^2}\; l\, \frac{1+\sqrt{1-m^2}}{m} - l^2\, \frac{1+\sqrt{1-m^2}}{m} + l^2\, \frac{1+\sqrt{1+m^2}}{m}$$

and differentiating this equation

$$\frac{dI(\mu)}{dm} = \frac{8}{m^3}\, l\, \frac{2}{m} - \frac{4\sqrt{1+m^2}}{m^3}\, l\, \frac{1+\sqrt{1+m^2}}{m} - \frac{4\sqrt{1-m^2}}{m^3}\, l\, \frac{1+\sqrt{1-m^2}}{m}$$

or

$$\frac{dI(\mu)}{dm} = \frac{4}{m^3}\, l\, \frac{2}{m}\,[2 - \sqrt{1+m^2} - \sqrt{1-m^2}] -$$

$$- \frac{4}{m^3}\left[\sqrt{1+m^2}\; l\, \frac{1+\sqrt{1+m^2}}{2} + \sqrt{1-m^2}\; l\, \frac{1+\sqrt{1-m^2}}{2}\right].$$

Writing

$$\varphi(m) = l\, \frac{1+\sqrt{1+m^2}}{2} \qquad \psi(m) = l\, \frac{1+\sqrt{1-m^2}}{2}$$

we get by differentation

$$\varphi'(m) = \frac{1}{m}\left(1 - \frac{1}{\sqrt{1+m^2}}\right) = \frac{1}{2}\, m - \frac{1.3}{2.4}\, m^3 + \frac{1.3.5}{2.4.6}\, m^5 - \frac{1.3.5.7}{2.4.6.8}\, m^7 + .$$

$$\psi'(m) = \frac{1}{m}\left(1 - \frac{1}{\sqrt{1-m^2}}\right) = -\frac{1}{2}\, m - \frac{1.3}{2.4}\, m^3 - \frac{1.3.5}{2.4.6}\, m^5 - \frac{1.3.5.7}{2.4.6.8}\, m^7 - \ldots$$

and by integrating, the constants being zero

$$\varphi(m) = \frac{1}{2}\cdot\frac{m^2}{2} - \frac{1.3}{2.4}\cdot\frac{m^4}{4} + \frac{1.3.5}{2\ 4.7}\cdot\frac{m^6}{6} - \frac{1.3.5.7}{2.4.6.8}\cdot\frac{m^8}{8} + \dots$$

$$\psi(m) = -\frac{1}{2}\cdot\frac{m^2}{2} - \frac{1.3}{2.4}\cdot\frac{m^4}{4} - \frac{1.3.5}{2.4.6}\cdot\frac{m^6}{6} - \frac{1.3.5.7}{2.4.6.8}\cdot\frac{m^8}{8} - \dots$$

Thus

$$\frac{dI(\mu)}{dm} - 8\left[\frac{1.1}{2.4}m + \frac{1.1.3.5}{2.4.6.8}m^5 + \frac{1.1.3.5.\ 7.\ 9}{2.4\ 6.8.10.12}m^9 + \dots\right] l\frac{2}{m} -$$
$$- 8\left[\frac{1}{32}m + \frac{59}{3072}m^5 + \frac{1417}{122880}m^9 + \dots\right]$$

$$I(\mu) = \frac{\pi^2}{4} + \left(\frac{m^2}{2} + \frac{5}{96}m^6 + \frac{21}{1280}m^{10} + \dots\right) lg\frac{2}{m} +$$
$$+ \frac{1}{8}m^2 - \frac{13}{768}m^6 - \frac{233}{30720}m^{10} + \dots$$

and finally

$$I(\mu) = \frac{\pi^2}{4} + \left(\frac{1}{2\mu^2} + \frac{5}{96\mu^6} + \frac{21}{1280\mu^{10}} + \dots\right) lg\, 2\mu +$$
$$+ \frac{1}{8\mu^2} - \frac{13}{768\mu^6} - \frac{233}{30720\mu^{10}} \dots \dots \qquad (9)$$

This series is sufficiently convergent for all values of $\mu \geq 2$.

5. Proceeding now to the integral $\int_0^x K\,dx$ in the different cases we immediately get the following results.

Case I. $b^2 < 1 \quad \mu^2 < 1 \quad \mu^2 < b^2 + 1$.

x	A	A'	B	B'
0 *until* b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
b^2 ,, μ^2	$\frac{\pi}{2}$	*as* $\frac{b}{\sqrt{x}}$	0	0
μ^2 ,, 1	*as* $\frac{\mu}{\sqrt{x}}$	*as* $\frac{b}{\sqrt{x}}$	0	0
1 ,, $1 + b^2$	*as* $\frac{\mu}{\sqrt{x}}$	*as* $\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$1 + b^2$,, $1 + \mu^2$	*as* $\frac{\mu}{\sqrt{x}}$	*as* $\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	*as* $\frac{b}{\sqrt{x-1}}$
$1 + \mu^2$,, ∞	*as* $\frac{\mu}{\sqrt{x}}$	*as* $\frac{b}{\sqrt{x}}$	*as* $\frac{\mu}{\sqrt{x-1}}$	*as* $\frac{b}{\sqrt{x-1}}$

$$\int_0^{\infty} K\,dx = \frac{\pi^2}{2} b^2 + \pi \int_{b^2}^{\mu^2} as \frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2} \int_1^{1+\mu^2} as \frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2} \int_1^{1+b^2} as \frac{\mu}{\sqrt{x}}\,dx +$$

$$+ 2 \int_{\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}}\, as \frac{b}{\sqrt{x}}\,dx - \int_{1+\mu^2}^{\infty} as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\,dx - \int_{1+b^2}^{\infty} as \frac{\mu}{\sqrt{x}}\, as \frac{b}{\sqrt{x-1}}\,dx.$$

Case II. $b^2 < 1 \quad \mu^2 > 1 \quad \mu^2 > b^2 + 1.$

x	A	A'	B	B'
0 *until* b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
b^2 „ 1	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	0	0
1 „ $1+b^2$	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$1+b^2$ „ μ^2	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x-1}}$
μ^2 „ $1+\mu^2$	$as \frac{\mu}{\sqrt{x}}$	$as \frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x-1}}$
$1+\mu^2$ „ ∞	$as \frac{\mu}{\sqrt{x}}$	$as \frac{b}{\sqrt{x}}$	$as \frac{\mu}{\sqrt{x-1}}$	$as \frac{b}{\sqrt{x-1}}$

$$\int_0^{\infty} K\,dx = \frac{\pi^2}{4} b^2 + \frac{\pi}{2} \int_{b^2}^{1} as \frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2} \int_{\mu^2}^{1+\mu^2} as \frac{b}{\sqrt{x}}\,dx + \frac{\pi}{2} \int_{\mu^2}^{1+\mu^2} as \frac{b}{\sqrt{x-1}}\,dx +$$

$$+ 2 \int_{\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}}\, as \frac{b}{\sqrt{x}}\,dx - \int_{1+\mu^2}^{\infty} as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\,dx - \int_{\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}}\, as \frac{b}{\sqrt{x-1}}\,dx.$$

Case III $b^2 < 1 \qquad \mu^2 > 1 \qquad \mu^2 < 1 + b^2.$

x	A	A'	B	B'
0 *until* b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
b^2 „ 1	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	0	0
1 „ μ^2	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$

μ^2 *until* $1+b^2$	$\text{as}\,\frac{\mu}{\sqrt{x}}$	$\text{as}\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$1+b^2$,, $1+\mu^2$	$\text{as}\,\frac{\mu}{\sqrt{x}}$	$\text{as}\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\text{as}\,\frac{b}{\sqrt{x-1}}$
$1+\mu^2$,, ∞	$\text{as}\,\frac{\mu}{\sqrt{x}}$	$\text{as}\,\frac{b}{\sqrt{x}}$	$\text{as}\,\frac{\mu}{\sqrt{x-1}}$	$\text{as}\,\frac{b}{\sqrt{x-1}}$

$$\int\limits_0^\infty K dx = \frac{\pi^2}{4}(1+2b^2-\mu^2) + \frac{\pi}{2}\int\limits_{b^2}^{1} \text{as}\,\frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2}\int\limits_{\mu^2}^{1+\mu^2} \text{as}\,\frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2}\int\limits_{\mu^2}^{1+b^2} \text{as}\,\frac{\mu}{\sqrt{x}}\,dx +$$

$$+\frac{\pi}{2}\int\limits_{1+b^2}^{1+\mu^2} \text{as}\,\frac{b}{\sqrt{x-1}}\,dx + 2\int\limits_{\mu^2}^{\infty} \text{as}\,\frac{\mu}{\sqrt{x}}\,\text{as}\,\frac{b}{\sqrt{x}}\,dx - \int\limits_{1+\mu^2}^{\infty} \text{as}\,\frac{b}{\sqrt{x}}\,\text{as}\,\frac{\mu}{\sqrt{x-1}}\,dx - \int\limits_{1+b^2}^{\infty} \text{as}\,\frac{\mu}{\sqrt{x}}\,\text{as}\,\frac{b}{\sqrt{x-1}}\,dx.$$

Case IV $b^2 > 1 \quad \mu^2 > 1 \quad \mu^2 < b^2+1.$

x	A	A'	B	B'
0 *until* 1	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
1 ,, b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
b^2 ,, μ^2	$\frac{\pi}{2}$	$\text{as}\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
μ^2 ,, b^2+1	$\text{as}\,\frac{\mu}{\sqrt{x}}$	$\text{as}\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
b^2+1 ,, μ^2+1	$\text{as}\,\frac{\mu}{\sqrt{x}}$	$\text{as}\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\text{as}\,\frac{b}{\sqrt{x-1}}$
μ^2+1 ,, ∞	$\text{as}\,\frac{\mu}{\sqrt{x}}$	$\text{as}\,\frac{b}{\sqrt{x}}$	$\text{as}\,\frac{\mu}{\sqrt{x-1}}$	$\text{as}\,\frac{b}{\sqrt{x-1}}$

$$\int^\infty K dx = \frac{\pi^2}{4}(2+b^2-\mu^2) - \frac{\pi}{2}\int\limits_{\mu^2}^{1+\mu^2} \text{as}\,\frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2}\int\limits_{\mu^2}^{b^2+1} \text{as}\,\frac{\mu}{\sqrt{x}}\,dx + \frac{\pi}{2}\int\limits_{b^2+1}^{\mu^2+1} \text{as}\,\frac{b}{\sqrt{x-1}}\,dx +$$

$$+2\int\limits_{\mu^2}^{\infty} \text{as}\,\frac{\mu}{\sqrt{x}}\,\text{as}\,\frac{b}{\sqrt{x}}\,dx - \int\limits_{\mu^2+1}^{\infty} \text{as}\,\frac{b}{\sqrt{x}}\,\text{as}\,\frac{\mu}{\sqrt{x-1}}\,dx - \int\limits_{b^2+1}^{\infty} \text{as}\,\frac{\mu}{\sqrt{x}}\,\text{as}\,\frac{b}{\sqrt{x-1}}\,dx.$$

Case V. $b^2 > 1$ $\mu^2 > 1$ $\mu^2 > b^2 + 1$.

x	A	A'	B	B'
0 *until* 1	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
1 „ b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
b^2 „ $1+b^2$	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$1+b^2$ „ μ^2	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x-1}}$
μ^2 „ $1+\mu^2$	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x-1}}$
$1+\mu^2$ „ ∞	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$as\,\frac{\mu}{\sqrt{x-1}}$	$as\,\frac{b}{\sqrt{x-1}}$

$$\int_0^\infty K dx = \frac{\pi^2}{4} - \frac{\pi}{2}\int_{\mu^2}^{1+\mu^2} as\,\frac{b}{\sqrt{x}}\,dx + \frac{\pi}{2}\int_{\mu^2}^{1+\mu^2} as\,\frac{b}{\sqrt{x-1}}\,dx +$$

$$+ 2\int_{\mu^2}^{\infty} as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x}}\,dx - \int_{1+\mu^2}^{\infty} as\,\frac{b}{\sqrt{x}}\,as\,\frac{\mu}{\sqrt{x-1}}\,dx - \int_{\mu^2}^{\infty} as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x-1}}\,.$$

6. The integrals containing only one function *as* may be easily determined. We shall therefore consider only those integrals which contain the product of two functions *as*.

In the first place we have

$$\int as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x}}\,dx = x\,as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x}} + b\sqrt{x-b^2}\,as\,\frac{\mu}{\sqrt{x}} +$$

$$+ \mu\sqrt{x-\mu^2}\,as\,\frac{b}{\sqrt{x}} + 2b\mu l\,(\sqrt{x-\mu^2} + \sqrt{x-b^2}) -$$

$$- (b^2+\mu^2)\,l\,\frac{\mu\sqrt{x-b^2} + b\sqrt{x-\mu^2}}{\sqrt{x}}$$

so in each case

$$\int_{\mu^2}^{\infty} as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x}}\,dx = 3b\mu + 2b\mu\,lg\,\frac{2}{\varepsilon} - \frac{\pi}{2}\,\mu^2\,as\,\frac{b}{\mu} - \frac{\pi b}{2}\sqrt{\mu^2-b^2} +$$

$$+ \frac{(\mu-b)^2}{2}\,l\,(\mu-b) - \frac{(\mu+b)^2}{2}\,l\,(\mu+b).$$

The second integral containing the product of two functions *as* is different according to the value of b.

If $b < 1$, as in the three cases I, II, III.

$$\int as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\, dx = x\, as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}} + \mu\, as \frac{b}{\sqrt{x}} \cdot \sqrt{x-1-\mu^2}$$

$$+ b\, as \frac{\mu}{\sqrt{x-1}} \cdot \sqrt{x-b^2} + 2b\mu\, l\,(\sqrt{x-b^2} + \sqrt{x-1-\mu^2})$$

$$+ b\sqrt{1-b^2}\, as \frac{\sqrt{1-b^2}\sqrt{x-1-\mu^2}}{\sqrt{1+\mu^2-b^2}\sqrt{x-1}}$$

$$- \frac{\mu\sqrt{1+\mu^2}}{2}\, l\, \frac{\sqrt{1+\mu^2}\sqrt{x-b^2} + b\sqrt{x-1-\mu^2}}{\sqrt{1+\mu^2}\sqrt{x-b^2} - b\sqrt{x-1-\mu^2}}$$

$$+ \frac{\mu}{2}\int as \frac{b}{\sqrt{x}}\, \frac{dx}{(x-1)\sqrt{x-1-\mu^2}}.$$

Putting

$$x = 1 + \frac{\mu^2}{\sin^2 v}$$

we have

$$\frac{\mu}{2}\int_{1+\nu^2}^{x} as \frac{b}{\sqrt{x}}\, \frac{dx}{(x-1)\sqrt{x-1-\mu^2}} = \int_0^{\frac{\pi}{2}} as \frac{b \sin v}{\sqrt{\mu^2+\sin^2 v}}\, dv - T'$$

so introducing the limits

$$\int_{1+\nu^2}^{\infty} as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\, dx = 3b\mu + 2b\mu\, l\, \frac{2}{\varepsilon} - \frac{\pi}{2} b\sqrt{1+\mu^2-b^2}$$

$$- \frac{\pi}{2}(1+\mu^2)\, as \frac{b}{\sqrt{1+\mu^2}} + b\sqrt{1-b^2}\, as \frac{\sqrt{1-b^2}}{\sqrt{1+\mu^2-b^2}}$$

$$- \frac{\mu\sqrt{1+\mu^2}}{2}\, l\, \frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} - b\mu\, l\,(1+\mu^2-b^2) + T'.$$

If $b > 1$, as in the cases IV and V

$$\int as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\, dx = x\, as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}} + \mu\, as \frac{b}{\sqrt{x}} \cdot \sqrt{x-1-\mu^2}$$

$$+ b\, as \frac{\mu}{\sqrt{x-1}} \cdot \sqrt{x-b^2} + 2b\mu\, l\,(\sqrt{x-b^2} + \sqrt{x-1-\mu^2})$$

$$- b\sqrt{b^2-1}\, l\, \frac{\mu\sqrt{x-b^2} + \sqrt{b^2-1}\sqrt{x-1-\mu^2}}{\sqrt{x-1}}$$

$$-\frac{\mu\sqrt{1+\mu^2}}{2}\, l\, \frac{\sqrt{1+\mu^2}\sqrt{x-b^2}+b\sqrt{x-1-\mu^2}}{\sqrt{1+\mu^2}\sqrt{x-b^2}-b\sqrt{x-1-\mu^2}}$$

$$+\frac{\mu}{2}\int as\,\frac{b}{\sqrt{x}}\,\frac{dx}{(x-1)\sqrt{x-1-\mu^2}}$$

and

$$\int_{1+\mu^2}^{\infty} as\,\frac{b}{\sqrt{x}}\,as\,\frac{\mu}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\, l\,\frac{2}{\varepsilon} - \frac{\pi}{2}\, b\sqrt{1+\mu^2-b^2}$$

$$-\frac{\pi}{2}(1+\mu^2)\,as\,\frac{b}{\sqrt{1+\mu^2}} - \frac{b\sqrt{b^2-1}}{2}\, l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}$$

$$-\frac{\mu\sqrt{1+\mu^2}}{2}\, l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} - b\mu\, l\,(1+\mu^2-b^2) + T'.$$

The third integral containing two functions *as*, which has different limits in the two cases II and V, may be easily deduced from the preceding integrals.

When $\mu < 1$ we find

$$\int_{1+b^2}^{\infty} as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\, l\,\frac{2}{\varepsilon} - \frac{\pi}{2}\mu\sqrt{1+b^2-\mu^2}$$

$$-\frac{\pi}{2}(1+b^2)\,as\,\frac{\mu}{\sqrt{1+b^2}} + \mu\sqrt{1-\mu^2}\,as\,\frac{\sqrt{1-\mu^2}}{\sqrt{1+b^2-\mu^2}}$$

$$-\frac{b\sqrt{1+b^2}}{2}\, l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}+\mu} - b\mu\, l\,(1+b^2-\mu^2) + T$$

and when $\mu > 1$

$$\int_{1+b^2}^{\infty} as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\, l\,\frac{2}{\varepsilon} - \frac{\pi}{2}\,\mu\sqrt{1+b^2-\mu^2}$$

$$-\frac{\pi}{2}(1+b^2)\,as\,\frac{\mu}{\sqrt{1+b^2}} - \frac{\mu\sqrt{\mu^2-1}}{2}\, l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}$$

$$-\frac{b\sqrt{1+b^2}}{2}\, l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu} - b\mu\, l\,(1+b^2-\mu^2) + T$$

wherein

$$T = \int_0^{\frac{\pi}{2}} as\,\frac{\mu\, sin\, v}{\sqrt{b^2+sin^2 v}}\,dv.$$

Similarly we get

$$\int_{a^2}^{\infty} as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\, l\,\frac{2}{\varepsilon} - \frac{\pi}{2}\,b\sqrt{\mu^2-1-b^2}$$

$$-\mu\sqrt{\mu^2-1}\,l(b+\sqrt{\mu^2-1}) + \frac{\mu\sqrt{\mu^2-1}}{2} - 2b\mu\, l(\mu^2-1-b^2) -$$

$$-\frac{b\sqrt{1+b^2}}{2}\,l\,\frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}} - \frac{\pi}{2}\,\mu^2\,as\,\frac{b}{\sqrt{\mu^2-1}} + T'''$$

where

$$T''' = \frac{b}{2}\int_{a^2}^{\infty} as\,\frac{\mu}{\sqrt{x}}\,\frac{dx}{(x-1)\sqrt{x-1-b^2}}.$$

This integral may be transformed as follows:
Putting

$$x = \frac{\mu^2}{\sin^2 v}$$

we obtain

$$T''' = b\,\mu^2\int_0^{\frac{\pi}{2}} \frac{v\cos v\,dv}{(\mu^2-\sin^2 v)\sqrt{\mu^2-(1+b^2)\sin^2 v}} = \int_0^{\frac{\pi}{2}} v\,d\,.\,as\,\frac{b\sin v}{\sqrt{\mu^2-\sin^2 v}}$$

$$T''' = \frac{\pi}{2}\,as\,\frac{b}{\sqrt{\mu^2-1}} - \int_0^{\frac{\pi}{2}} as\,\frac{b\sin v}{\sqrt{\mu^2-\sin^2 v}}\,dv$$

or

$$T''' = \frac{\pi}{2}\,as\,\frac{b}{\sqrt{\mu^2-1}} - U$$

where

$$U = \int_0^{\frac{\pi}{2}} as\,\frac{b\sin v}{\sqrt{\mu^2-\sin^2 v}}\,dv.$$

7. It is evident from the preceding article that in all cases $\int_0^{\infty} K\,dx$ may be reduced to the three definite integrals T, T'' and U. Introducing these we obtain, after some slight reductions, the following results

Case I. $\mu^2 < 1 \qquad b^2 < 1 \qquad \mu^2 < b^2 + 1$

$$\int_0^\infty K\,dx = -T - T' + \frac{\pi}{2}\operatorname{as} b + \frac{\pi}{2}\operatorname{as}\mu$$

$$+ b\sqrt{1-b^2}\operatorname{as}\frac{\mu}{\sqrt{1+\mu^2-b^2}} + \mu\sqrt{1-\mu^2}\operatorname{as}\frac{b}{\sqrt{1+b^2-\mu^2}}$$

$$+ (\mu-b)^2\, l\,(\mu-b) - (\mu+b)^2\, l\,(\mu+b)$$

$$+ b\mu\, l\,(1+\mu^2-b^2) + b\mu\, l\,(1+b^2-\mu^2)$$

$$+ \frac{b\sqrt{1+b^2}}{2}\, l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu} + \frac{\mu\sqrt{1+\mu^2}}{2}\, l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$

Case II $\mu^2 > 1 \quad b^2 < 1 \quad \mu^2 > b^2 + 1.$

$$\int_0^\infty K\,dx = U - T' + \frac{\pi}{2}\operatorname{as} b + b\sqrt{1-b^2}\operatorname{as}\frac{\mu}{\sqrt{1+\mu^2-b^2}}$$

$$+ (\mu-b)^2\, l\,(\mu-b) - (\mu+b)^2\, l\,(\mu+b)$$

$$+ b\mu\, l\,(\mu^2+1-b^2) + b\mu\, l\,(\mu^2-1-b^2)$$

$$+ \frac{b\sqrt{1+b^2}}{2}\, l\,\frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}} + \frac{\mu\sqrt{1+\mu^2}}{2}\, l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$

$$+ \frac{\mu\sqrt{\mu^2-1}}{2}\, l\,\frac{\sqrt{\mu^2-1}+b}{\sqrt{\mu^2-1}-b}.$$

Case III $\mu^2 > 1 \quad b^2 < 1 \quad \mu^2 < b^2 + 1.$

$$\int_0^\infty K\,dx = \frac{\pi^2}{4} - T - T' + \frac{\pi}{2}\operatorname{as} b + b\sqrt{1-b^2}\operatorname{as}\frac{\mu}{\sqrt{1+\mu^2-b^2}}$$

$$+ (\mu-b)^2\, l(\mu-b) - (\mu+b)^2\, l(\mu+b)$$

$$+ b\mu\, l(1+\mu^2-b^2) + b\mu\, l(1+b^2-\mu^2)$$

$$+ \frac{b\sqrt{1+b^2}}{2}\, l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu} + \frac{\mu\sqrt{1+\mu^2}}{2}\, l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$

$$+ \frac{\mu\sqrt{\mu^2-1}}{2}\, l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}.$$

Case IV $\mu^2 > 1 \quad b^2 > 1 \quad \mu^2 < b^2 + 1.$

$$\int_0^\infty K\,dx = \frac{\pi^2}{2} - T - T'$$

$$+ (\mu-b)^2\, l\,(\mu-b) - (\mu+b)^2\, l\,(\mu+b)$$

$$+ b\mu\, l\,(1+\mu^2-b^2) - b\mu\, l\,(1+b^2-\mu^2)$$

$$+\frac{b\sqrt{1+b^2}}{2}\, l\, \frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu}+\frac{\mu\sqrt{1+\mu^2}}{2}\, l\, \frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$

$$+\frac{b\sqrt{b^2-1}}{2}\, l\, \frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}+\frac{\mu\sqrt{\mu^2-1}}{2}\, l\, \frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}.$$

Case V. $\mu^2>1 \quad b^2>1 \quad \mu^2>b^2+1.$

$$\int_0^\infty K dx=\frac{\pi^2}{4}+U-T''$$

$$+(\mu-b)^2\, l\,(\mu-b)-(\mu+b)^2\, l\,(\mu+b)$$
$$+b\mu\, l\,(1+\mu^2-b^2)+b\mu\, l\,(\mu^2-1-b^2)$$
$$+\frac{b\sqrt{1+b^2}}{2}\, l\, \frac{\mu+\sqrt{b^2+1}}{\mu-\sqrt{b^2+1}}+\frac{\mu\sqrt{1+\mu^2}}{2}\, l\, \frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$
$$-\frac{b\sqrt{b^2-1}}{2}\, l\, \frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}+\frac{\mu\sqrt{\mu^2-1}}{2}\, l\, \frac{\sqrt{\mu^2-1}+b}{\sqrt{\mu^2-1}-b}.$$

8. After these reductions it seems to be necessary to expand the three integrals T', T'', and U in convergent series [1]). It is however preferable to get an expansion for the general integrals $\int_0^\infty K\,dx$ and $I(\mu, b)$. To this we shall now proceed, beginning with the two cases II and V which are different from the rest.

Case II.

Differentiating U and T'' with respect to b, we have

$$\frac{\partial U}{\partial b}=\int_0^{\frac{\pi}{2}}\frac{\sin v\, dv}{\sqrt{\mu^2-(1+b^2)\sin^2 v}}=\frac{1}{2\sqrt{1+b^2}}\, l\, \frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}}$$

$$\frac{\partial T''}{\partial b}=\int_0^{\frac{\pi}{2}}\frac{\sin v\, dv}{\sqrt{\mu^2+(1-b^2)\sin^2 v}}=\frac{1}{\sqrt{1-b^2}}\, as\, \frac{\sqrt{1-b^2}}{\sqrt{1+\mu^2-b^2}}$$

thus generally

$$\frac{\partial}{\partial b}\int_0^\infty K\,dx=\frac{\partial L}{\partial b}=\sqrt{1+b^2}\, l\, \frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}}+2\sqrt{1-b^2}\, as\, \frac{\mu}{\sqrt{1+\mu^2-b^2}}$$
$$-2(\mu-b)\, l\,(\mu-b)-2(\mu+b)\, l\,(\mu+b)$$
$$+\mu\, l\,(\mu^2+1-b^2)+\mu\, l\,(\mu^2-1-b^2).$$

1) The expansions for T and T'' are to be found: Nieuw Archief voor Wiskunde (2) Vol. IX,

Now the different terms of this equation may be expanded as follows

$$2\sqrt{1-b^2}\,\mathrm{as}\,\frac{\mu}{\sqrt{1+\mu^2-b^2}} = 2\sqrt{1-b^2}\left(\frac{\pi}{2} - \mathrm{as}\,\frac{\sqrt{1-b^2}}{\sqrt{1+\mu^2-b^2}}\right)$$

$$= \pi\sqrt{1-b^2} - \frac{\sqrt{1-b^2}}{i}\, l\,\frac{\mu + i\sqrt{1-b^2}}{\mu - i\sqrt{1-b^2}}$$

$$-\frac{\sqrt{1-b^2}}{i}\, l\,\frac{\mu + i\sqrt{1-b^2}}{\mu - i\sqrt{1-b^2}} =$$

$$= -2\left\{\frac{1-b^2}{\mu} - \frac{1}{3}\frac{(1-b^2)^2}{\mu^3} + \frac{1}{5}\frac{(1-b^2)^3}{\mu^5} - \frac{1}{7}\frac{(1-b^3)^4}{\mu^7} + \dots\right\}$$

$$\sqrt{1+b^2}\; l\,\frac{\mu + \sqrt{1+b^2}}{\mu - \sqrt{1+b^2}} =$$

$$= 2\left\{\frac{1+b^2}{\mu} + \frac{1}{3}\frac{(1+b^2)^2}{\mu^3} + \frac{1}{5}\frac{(1+b^2)^3}{\mu^5} + \frac{1}{7}\frac{(1+b^2)^4}{\mu^7} + \dots\right\}$$

$$\mu\, l\,(\mu^2+1-b^2) =$$

$$= 2\mu\, l\mu + \frac{1-b^2}{\mu} - \frac{1}{2}\frac{(1-{}^2)^2}{\mu^3} + \frac{1}{3}\frac{(1-b^2)^3}{\mu^5} - \frac{1}{4}\frac{(1-b^2)^4}{\mu^7} + .$$

$$\mu\, l(\mu^2-1-b^2) =$$

$$= 2\mu\, l\mu - \frac{1+b^2}{\mu} - \frac{1}{2}\frac{(1+b^2)^2}{\mu^3} - \frac{1}{3}\frac{(1+b^2)^3}{\mu^5} - \frac{1}{4}\frac{(1+b^2)^4}{\mu^7} - \dots$$

$$-2(\mu-b)\, l(\mu-b) - 2(\mu+b)\, l(\mu+b) =$$

$$= -4\mu\, l\mu - 2\frac{b^2}{\mu} - \frac{2\;b^4}{2.3\;\mu^3} - \frac{2\;b^6}{3.5\;\mu^5} - \frac{2\;b^8}{4.7\;\mu^7} - \dots$$

thus by addition

$$\frac{\partial L}{\partial b} = \pi\sqrt{1-b^2} + \frac{2}{3\mu^3} + 2\,\frac{3b^2}{3.5\,\mu^5} + 2\,\frac{1+6b^4}{4.7\,\mu^7} +$$

$$+ 2\,.\,\frac{5b^2+10b^6}{5.9\,\mu^9} + 2\,.\,\frac{1+15b^4+15b^8}{6.11\,\mu^{11}} + \dots$$

or by arranging this series according to ascending powers of b

$$\frac{\partial L}{\partial b} = \pi\sqrt{1-b^2} + 2\sum_1^\infty \frac{1}{(2n)\,(4n-1)}\;\frac{1}{\mu^{4n-1}}$$

$$+ 2b^2\sum_1^\infty \frac{1}{4n+1}\;\frac{1}{\mu^{4n+1}}$$

$$+ \frac{2b^4}{2\,!}\sum_1^\infty \frac{2n+1}{4n+3}\;\frac{1}{\mu^{4n+3}}$$

$$+ \frac{2b^6}{3\,!}\sum_1^\infty \frac{(2n+2)\,(2n+1)}{4n+5}\;\frac{1}{4^{4n+5}}$$

$$+ \dots\dots\dots\dots\dots\dots$$

Finally integrating and remarking that the constant is zero, we have this result

$$L = \int_0^{\xi} K\,dx = \frac{\pi}{2}\left(as\, b + b\sqrt{1-b^2}\right)$$

$$+ 2b \sum_1^{\infty} \frac{1}{(2n)(4n-1)} \frac{1}{\mu^{4n-1}}$$

$$+ \frac{2b^3}{3} \sum_1^{\infty} \frac{1}{4n+1} \frac{1}{\mu^{4n+1}}$$

$$+ \frac{2b^5}{2!5} \sum_1^{\infty} \frac{2n+1}{4n+3} \frac{1}{\mu^{4n+3}}$$

$$+ \frac{2b^7}{3!7} \sum_1^{\infty} \frac{(2n+2)(2n+1)}{4n+5} \frac{1}{\mu^{4n+5}}$$

$$+ \frac{2b^9}{3!9} \sum_1^{\infty} \frac{(2n+3)(2n+2)(2n+1)}{4n+7} \frac{1}{\mu^{4n+7}}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

This series is sufficiently convergent for values of b between 0 and 0.4, and for values of μ from 1.2 upwards.

Case V.

Here again we have the same value for $\frac{\partial U}{\partial b}$ as in the preceding and further

$$\frac{\partial T''}{\partial b} = \frac{1}{2\sqrt{b^2-1}}\, l\, \frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}$$

thus

$$\frac{\partial L}{\partial b} = \sqrt{1+b^2}\, l\, \frac{\mu+\sqrt{b^2+1}}{\mu-\sqrt{b^2+1}} + \sqrt{b^2-1}\, l\, \frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}$$

$$- 2(\mu-b)\, l\,(\mu-b) - 2(\mu+b)\, l\,(\mu+b)$$

$$+ \mu\, l\,(\mu^2+1-b^2) + \mu\, l\,(\mu^2-1-b^2).$$

Expanding now $\sqrt{b^2-1}\, l\, \frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}$ we get the same series as that for $-\frac{\sqrt{1-b^2}}{i}\, l\, \frac{\mu+i\sqrt{1-b^2}}{\mu-i\sqrt{1-b^2}}$ in the former case; therefore we may write

$$L = 2b \sum_1^{\infty} \frac{1}{(2n)(4n-1)} \frac{1}{\mu^{4n-1}}$$
$$+ \frac{2b^3}{3} \sum_1^{\infty} \frac{1}{4n+1} \frac{1}{\mu^{4n+1}}$$
$$+ \frac{2b^5}{2!5} \sum_1^{\infty} \frac{2n+1}{4n+3} \frac{1}{\mu^{4n+3}}$$
$$+ \frac{2b^7}{3!7} \sum_1^{\infty} \frac{(2n+2)(2n+1)}{4n+5} \frac{1}{\mu^{4n+5}}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ const.$$

It is evident that this series and the series for L in case II must agree for $b=1$. Therefore the value of the constant must be $\frac{\pi^2}{4}$ so that finally

$$L = \int_0^{\infty} K\,dx = \frac{\pi^2}{4} + 2b \sum_1^{\infty} \frac{1}{(2n)(4n-1)} \frac{1}{\mu^{4n-1}}$$
$$+ \frac{2b^3}{3} \sum_1^{\infty} \frac{1}{4n+1} \frac{1}{\mu^{4n+1}}$$
$$+ \frac{2b^5}{2!5} \sum_1^{\infty} \frac{2n+1}{4n+3} \frac{1}{\mu^{4n+3}}$$
$$+ \frac{2b^7}{3!7} \sum_1^{\infty} \frac{(2n+2)(2n+1)}{4n+5} \frac{1}{\mu^{4n+5}}$$
$$+ \frac{2b^9}{4!9} \sum_1^{\infty} \frac{(2n+3)(2n+2)(2n+1)}{4n+7} \frac{1}{\mu^{4n+7}}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The values of this series have been computed for $b=1.2$ $\mu \geq 1.8$; $b=2.4$ $\mu > 3.4$; $b=3.6$ $\mu \geq 3.8$; it has been found to be sufficiently convergent for these values.

9. In the three cases I, III, IV we will try to expand at once $I(\mu,b)$ according to ascending powers of $\mu - b = a$.

Let

$$I(\mu,b) + 4b^2\, l\, 2b + 4\mu^2\, l\, 2\mu + 2(\mu-b)^2\, l(\mu-b) - 2(\mu+b)^2\, l(\mu+b) = f(\mu)$$

and

$$f(\mu) = f(b) + a f'(b) + \frac{a^2}{2!} f''(b) + \frac{a^3}{3!} f'''(b) + \ldots.$$

then it is evident that $f(b)=0$ for $I(b,b)=0$ and $(\mu-b)\, l(\mu-b)=0$ if $\mu=b$.

Dividing $f(\mu)$ in two parts $\varphi(\mu)$ and $\psi(\mu)$ we may write:

Case I.

$$\varphi(\mu) = 2\mu\sqrt{1-\mu^2}\,as\,\mu + (as\,\mu)^2 + 2\mu\sqrt{1+\mu^2}\;l\,(\mu+\sqrt{\mu^2+1}) +$$
$$+ l^2\,(\mu+\sqrt{\mu^2+1}) + 2b\sqrt{1-b^2}\,as\,b + (as\,b)^2 +$$
$$+ 2b\sqrt{1+b^2}\,l\,(b+\sqrt{b^2+1}) + l^2\,(b+\sqrt{b^2+1})$$

$$\psi(\mu) = 2T + 2T' - \pi\,as\,\mu - \pi\,as\,b - 2\mu\sqrt{1-\mu^2}\,as\,\frac{b}{\sqrt{1+b^2-\mu^2}} -$$
$$- 2b\sqrt{1-b^2}\,as\,\frac{\mu}{\sqrt{1+\mu^2-b^2}} - 2b\,\mu\,l\,(1+\mu^2-b^2) -$$
$$- 2b\,\mu\,l\,(1+b^2-\mu^2) - \mu\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} -$$
$$- b\sqrt{1+b^2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu}$$

$$\frac{\partial\varphi}{\partial\mu} = 4\sqrt{1+\mu^2}\,l\,(\mu+\sqrt{\mu^2+1}) + 4\sqrt{1-\mu^2}\,as\,\mu + 4\mu$$

$$\frac{\partial\psi}{\partial\mu} = -2\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} - 4\sqrt{1-\mu^2}\,as\,\frac{b}{\sqrt{1+b^2-\mu^2}} -$$
$$- 2b\,l\,(1+\mu^2-b^2) - 2\,b\,l\,(1+b^2-\mu^2) - 4b$$

and if $\dfrac{\partial\varphi}{\partial\mu} = \dfrac{\partial\varphi_1}{\partial\mu} + \dfrac{\partial\varphi_2}{\partial\mu}$, $\dfrac{\partial\psi}{\partial\mu} = \dfrac{\partial\psi_1}{\partial\mu} + \dfrac{\partial\psi_2}{\partial\mu}$

$$\frac{\partial\varphi_1}{\partial\mu} = 4\sqrt{1+\mu^2}\,l\,(\mu+\sqrt{\mu^2+1}) + 2\mu\ ; \quad \frac{\partial\varphi_2}{\partial\mu} = 4\sqrt{1-\mu^2}\,as\,\mu + 2\mu$$

$$\frac{\partial\psi_1}{\partial\mu} = -2\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} - 2b\,l\,(1+\mu^2-b^2) - 2b;$$

$$\frac{\partial\psi_2}{\partial\mu} = -4\sqrt{1-\mu^2}\,as\,\frac{b}{\sqrt{1+b^2-\mu^2}} - 2b\,l\,(1+b^2-\mu^2) - 2b$$

Hence

$$(1+\mu^2)\frac{\partial^2\varphi_1}{\partial\mu^2} - \mu\frac{\partial\varphi_1}{\partial\mu} - (4\mu^2+6) = 0,\ (\mu^2-1)\frac{\partial^2\varphi_2}{\partial\mu^2} - \mu\frac{\partial\varphi_2}{\partial\mu} - (4\mu^2-6) = 0$$

$$\frac{\partial^2\psi_1}{\partial\mu^2} = -\frac{2\mu}{\sqrt{1+\mu^2}}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b},\quad \frac{\partial^2\psi_2}{\partial\mu^2} = \frac{4\mu}{\sqrt{1-\mu^2}}\,as\,\frac{b}{\sqrt{1+b^2-\mu^2}}$$

$$\mu\,(1+\mu^2)\,(1+\mu^2-b^2)\frac{\partial^3\psi_1}{\partial\mu^3} - (1+\mu^2-b^2)\frac{\partial^2\psi_1}{\partial\mu^2} - 4b\mu^3 = 0\,,$$

$$\mu\,(\mu^2-1)\,(1+b^2-\mu^2)\frac{\partial^3\psi_2}{\partial\mu^3} + (1+b^2-\mu^2)\frac{\partial^2\psi_2}{\partial\mu^2} + 4b\mu^3 = 0$$

so that

$$\left(\frac{\partial\varphi_1}{\partial\mu}\right)_b+\left(\frac{\partial\psi_1}{\partial\mu}\right)_b=0\ ,\quad \left(\frac{\partial\varphi_2}{\partial\mu}\right)_b+\left(\frac{\partial\psi_2}{\partial\mu}\right)_b=0$$

$$\left(\frac{\partial^2\varphi_1}{\partial\mu_2}\right)_b+\left(\frac{\partial^2\psi_1}{\partial\mu^2}\right)_b=6\ ,\quad \left(\frac{\partial^2\varphi_2}{\partial\mu^2}\right)_b+\left(\frac{\partial^2\psi_2}{\partial\mu^2}\right)_b=6$$

$$\left(\frac{\partial^3\varphi_1}{\partial\mu^3}\right)_b+\left(\frac{\partial^3\psi_1}{\partial\mu^3}\right)_b=4b\ ,\quad \left(\frac{\partial^3\varphi_2}{\partial\mu^3}\right)_b+\left(\frac{\partial^3\psi_2}{\partial\mu^3}\right)_b=-4b.$$

Thus in the required expansion the coefficients of a and a^3 will be wanting and the coefficient of $\frac{a^2}{2!}$ will be 12. We will defer the determination of the further coefficients until we have considered also the three first coefficients in the two cases III and IV.

Case III.

Here we have

$$\varphi(\mu)=2\mu\sqrt{\mu^2-1}\,l\,(\mu+\sqrt{\mu^2-1})-l^2\,(\mu+\sqrt{\mu^2-1})+$$
$$+\,2\mu\sqrt{1+\mu^2}\,l\,(\mu+\sqrt{\mu^2+1})+l^2\,(\mu+\sqrt{\mu^2+1})$$

$$\psi(\mu)=-\frac{\pi^2}{4}+2T+2T'-\pi\ as\ b-2b\sqrt{1-b^2}\ as\frac{\mu}{\sqrt{1+\mu^2-b^2}}$$
$$-\,2b\mu\,l\,(1+\mu^2-b^2)-2b\mu\,l\,(1+b^2-\mu^2)$$
$$-\,b\sqrt{1+b^2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu}-\mu\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}+b}-$$
$$-\,\mu\sqrt{\mu^2-1}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}$$

$$\frac{\partial\varphi}{\partial\mu}=4\sqrt{\mu^2-1}\,l\,(\mu+\sqrt{\mu^2-1})+4\sqrt{1+\mu^2}\,l\,(\mu+\sqrt{\mu^2+1})+4\mu$$

$$\frac{\partial\psi}{\partial\mu}=-\,2\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}-2\sqrt{\mu^2-1}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}$$
$$-\,2b\,l\,(1+\mu^2-b^2)-2b\,l\,(1+b^2-\mu^2)-4b.$$

Putting now

$$\frac{\partial\varphi_1}{\partial\mu}=4\sqrt{1+\mu^2}\,l\,(\mu+\sqrt{\mu^2+1})+2\mu\ ,$$

$$\frac{\partial\varphi_2}{\partial\mu}=4\sqrt{\mu^2-1}\,l\,(\mu+\sqrt{\mu^2-1})+2\mu$$

$$\frac{\partial\psi_1}{\partial\mu}=-\,2\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}-2b\,l\,(1+\mu^2-b^2)-2b\ ,$$

$$\frac{\partial\psi_2}{\partial\mu}=-\,2\sqrt{\mu^2-1}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}-2b\,l\,(1+b^2-\mu^2)-2b$$

we find

$$(1+\mu^2)\frac{\partial^2\varphi_1}{\partial\mu^2}-\mu\frac{\partial\varphi_1}{\partial\mu}-(4\mu^2+6)=0\,,$$

$$(\mu^2-1)\frac{\partial^2\varphi_2}{\partial\mu^2}-\mu\frac{\partial\varphi_2}{\partial\mu}-(4\mu^2-6)=0$$

$$\frac{\partial^2\psi_1}{\partial\mu^2}=-\frac{2\mu}{\sqrt{1+\mu^2}}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}\qquad\frac{\partial^2\psi_2}{\partial\mu^2}=-\frac{2\mu}{\sqrt{\mu^2-1}}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}$$

$$\mu(1+\mu^2)(1+\mu^2-b^2)\frac{\partial^3\psi_1}{\partial\mu^3}-(1+\mu^2-b^2)\frac{\partial^2\psi_1}{\partial\mu^2}-4b\mu^3=0\,,$$

$$\mu(\mu^2-1)(1+b^2-\mu^2)\frac{\partial^3\psi_2}{\partial\mu^3}+(1+b^2-\mu^2)\frac{\partial^2\psi_2}{\partial\mu^2}+4b\mu^3=0$$

thus

$$\left(\frac{\partial\varphi_1}{\partial\mu}\right)_b+\left(\frac{\partial\psi_1}{\partial\mu}\right)_b=0\qquad\left(\frac{\partial\varphi_2}{\partial\mu}\right)_b+\left(\frac{\partial\psi_2}{\partial\mu}\right)_b=0$$

$$\left(\frac{\partial^2\varphi_1}{\partial\mu^2}\right)_b+\left(\frac{\partial^2\psi_1}{\partial\mu^2}\right)_b=6\qquad\left(\frac{\partial^2\varphi_2}{\partial\mu^2}\right)_b+\left(\frac{\partial^2\psi_2}{\partial\mu^2}\right)_b=6$$

$$\left(\frac{\partial^3\varphi_1}{\partial\mu^3}\right)_b+\left(\frac{\partial^3\psi_1}{\partial\mu^3}\right)_b=4b\qquad\left(\frac{\partial^3\varphi_2}{\partial\mu^3}\right)_b+\left(\frac{\partial^3\psi_2}{\partial\mu^3}\right)_b=-4b$$

and evidently the expansion in this case will agree with that in the preceding case.

Case IV.

In this case we may write

$$\begin{aligned}\varphi(\mu)=&\;2\sqrt{\mu^2-1}\,l\,(\mu+\sqrt{\mu^2-1})-l^2\,(\mu+\sqrt{\mu^2-1})\\&+2\,\mu\sqrt{1+\mu^2}\,l\,(\mu+\sqrt{\mu^2+1})+l^2\,(\mu+\sqrt{\mu^2+1})\\&-2\,b\sqrt{1+b^2}\,l\,(b+\sqrt{b^2+1})+l^2\,(b+\sqrt{b^2+1})\\&+2\,b\sqrt{b^2-1}\,l\,(b+\sqrt{b^2-1})-l^2\,(b+\sqrt{b^2-1})\end{aligned}$$

$$\begin{aligned}\psi(\mu)=&-\frac{\pi^2}{2}+2T+2T'-2b\mu\,l\,(1+\mu^2-b^2)-2b\mu\,l\,(1+b^2-\mu^2)\\&-b\sqrt{1+b^2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu}-\mu\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}\\&-b\sqrt{b^2-1}\,l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}\quad\mu\sqrt{\mu^2-1}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}\end{aligned}$$

$$\frac{\partial\varphi}{\partial\mu}=4\sqrt{\mu^2-1}\;l\;(\mu+\sqrt{\mu^2-1})+4\sqrt{\mu^2+1}\;l\;(\mu+\sqrt{\mu^2+1})+4\mu$$

$$\begin{aligned}\frac{\partial\psi}{d\mu}=&-2\sqrt{1+\mu^2}\;l\;\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}-2\sqrt{\mu^2-1}\;l\;\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}\\&-2\,b\,l\,(1+\mu^2-b^2)-2\,b\,l\,(1+b^2-\mu^2)-4\,b\end{aligned}$$

what is in perfect accordance with the preceding case.

Thus we have the same expansion for all the three cases.

10. To determine the coefficients of this expansion we must differentiate repeatedly the differential equations obtained.

From

$$(1+\mu^2)\frac{\partial^2\varphi_1}{\partial\mu^2}-\mu\frac{\partial\varphi_1}{\partial\mu}-(4\mu^2+6)=0$$

we derive successively

$$(1+\mu^2)\frac{\partial^3\varphi_1}{\partial\mu^3}+\mu\frac{\partial^2\varphi_1}{\partial\mu^2}-\frac{\partial\varphi_1}{\partial\mu}-8\mu=0$$

$$(1+\mu^2)\frac{\partial^4\varphi_1}{\partial\mu^4}+3\mu\frac{\partial^3\varphi_1}{\partial\mu^3}-8=0$$

$$(1+\mu^2)\frac{\partial^5\varphi_1}{\partial\mu^5}+5\mu\frac{\partial^4\varphi_1}{\partial\mu^3}+3\frac{\partial^3\varphi_1}{\partial\mu^3}=0$$

$$(1+\mu^2)\frac{\partial^6\varphi_1}{\partial\mu^6}+7\mu\frac{\partial^5\varphi_1}{\partial\mu^5}+8\frac{\partial^4\varphi_1}{\partial\mu^4}=0 \text{ etc.}$$

and from

$$\mu(\mu^2+1)(\mu^2+1-b^2)\frac{\partial^3\psi_1}{\partial\mu^3}-(\mu^2+1-b^2)\frac{\partial^2\psi_1}{\partial\mu^2}-4b\mu^3=0$$

the following

$$(\mu^2+1)(\mu^2+1-b^2)\frac{\partial^4\psi_1}{\partial\mu^4}+[5\mu^3+(5-3b^2)\mu]\frac{\partial^3\psi_1}{\partial\mu^3}-2\frac{\partial^2\psi_1}{\partial\mu^2}-12b\mu=0$$

$$(\mu^2+1)(\mu^2+1-b^2)\frac{\partial^5\psi_1}{\partial\mu^5}+[9\mu^3+(9-5b^2)\mu]\frac{\partial^4\psi_1}{\partial\mu^4}+$$
$$+[15\mu^2+(3-3b^2)]\frac{\partial^3\psi_1}{\partial\mu^3}-12b=0$$

$$(\mu^2+1)(\mu^3-1-b^2)\frac{\partial^6\psi_1}{\partial\mu^6}+[13\mu^3+(13-7b^2)\mu]\frac{\partial^5\psi_1}{\partial\mu^5}+$$
$$+[42\mu^2+(12-8b^2)]\frac{\partial^4\psi_1}{\partial\mu^4}+30\mu\frac{\partial^3\psi_1}{\partial\mu^3}=0 \text{ etc.}$$

If now in the latter equations we put $\mu=b$ and write

$$D^n\psi_1=\left(\frac{\partial^n\psi_1}{\partial\mu^n}\right)_b$$

we obtain

(p) $b(b^2+1)D^3\psi_1-D^2\psi_1-4b^4=0$

(q) $(b^2+1)D^4\psi_1+(2b^3+5b)D^3\psi_1-2D^2\psi_1-12b^2=0$

(r) $(b^2+1)D^5\psi_1+(4b^3+9b)D^4\psi_1+(12b^2+3)D^3\psi_1-12b=0$

(s) $(b^2+1)D^6\psi_1+(6b^3+13b)D^5\psi_1+(34b^2+12)D^4\psi_1+30b\,D^3\psi_1=0$

(t) $(b^2+1)D^7\psi_1+(8b^3+17b)D^6\psi_1+(66b^2+25)D^5\psi_1+114b\,D^4\psi_1+30D^3\psi_1=0$

etc.

Multiplying the equation (p) by 2 and subtracting this product from the equation (q) we arrive at

$$(b^2+1)\,D^4\psi_1+3bD^3\psi_1=12b^2-8b^4.$$

Now multiplying this equation by $4b$ and subtracting it from the equation (r) we obtain

$$(b^2+1)\,D^5\psi_1+5bD^4\psi_1+3bD^3\psi_1=12b-48b^3+32b^5.$$

Multiplying again this equation by $6b$, the preceding one by 4 and subtracting these from the equation (s) we find

$$(b^2+1)\,D^6\psi_1+7bD^5\psi_1+8D^4\psi_1=-120b^2+320b^4-192b^6.$$

In the same way we may deduce the following equations

$$(b^2+1)D^7\psi_1+9bD^6\psi_1+15D^5\psi_1=-120b+1440b^3-2880b^5+1536b^7$$
$$(b^2+1)D^8\psi_1+11bD^7\psi_1+24D^6\psi_1=3360b^2-20160b^4+32256b^6-15360b^8$$
$$(b^2+1)\,D^9\psi_1+13bD^8\psi_1+35D^7\psi_1=3360b-80640b^3+$$
$$+322560b^5-430080b^7+184320b^9$$
$$(b^2+1)D^{10}\psi_1+15bD^9\psi_1+48D^8\psi_1=-60480b^2+$$
$$+1935360b^4-5806080b^6+6635520b^8-2580480b^{10}$$

etc.

Adding to these

$$\begin{aligned}
(1+b^2)\,D^4\varphi_1&+3bD^3\varphi_1-8=0\\
(1+b^2)\,D^5\varphi_1&+5bD^4\varphi_1+3D^3\varphi_1=0\\
(1+b^2)\,D^6\varphi_1&+7bD^5\varphi_1+8D^4\varphi_1=0\\
(1+b^2)\,D^7\varphi_1&+9bD^6\varphi_1+15D^5\varphi_1=0\\
(1+b^2)\,D^8\varphi_1&+11bD^7\varphi_1+24D^6\varphi_1=0\\
(1+b^2)\,D^9\varphi_1&+13bD^8\varphi_1+35D^7\varphi_1=0\\
(1+b^2)\,D^{10}\varphi_1&+15bD^9\varphi_1+48D^8\varphi_1=0
\end{aligned}$$

etc.,

we get

$$(1+b^2)\,D^4(\varphi_1+\psi_1)+3bD^3(\varphi_1+\psi_1)=8+12b^2-8b^4$$
$$(1+b^2)\,D^5(\varphi_1+\psi_1)+5bD^4(\varphi_1+\psi_1)+3D^3(\varphi_1+\psi_1)=$$
$$=12b-48b^3+32b^5$$
$$(1+b^2)\,D^6(\varphi_1+\psi_1)+7bD^5(\varphi_1+\psi_1)+8D^4(\varphi_1+\psi_1)=$$
$$=-120b^2+320b^4-192\,b^6$$
$$(1+b^2)\,D^7(\varphi_1+\psi_1)+9bD^6(\varphi_1+\psi_1)+15D^5(\varphi_1+\psi_1)=$$
$$=-120b+1440b^3-2880b^5+1536b^7$$
$$(1+b^2)\,D^8(\varphi_1+\psi_1)+11bD^7(\varphi_1+\psi_1)+24D^6(\varphi_1+\psi_1)=$$
$$=3360b^2-20160b^4+32256b^6-15360b^8$$
$$(1+b^2)\,D^9(\varphi_1+\psi_1)+13bD^8(\varphi_1+\psi_1)+35D^7(\varphi_1+\psi_1)=$$
$$=3360b-80640b^3+322560b^5-430080b^7+184320\,b^9$$
$$(1+b^2)\,D^{10}(\varphi_1+\psi_1)+15bD^9(\varphi_1+\psi_1)+48D^8(\varphi_1+\psi_1)=$$
$$=-181440b^2+1935360b^4-5806080b^6+6635520b^8-2580480b^{10}$$

etc.

Between the second members of these equations $P_4 P_5 \ldots P_{10}$ there exists a recurrent relation which from $n = 7$ upwards is given by

$$P_n = -2\,(n-3)\,b P_{n-1} - (n-2)\,(n-5)\,P_{n-2}.$$

From the preceding relations the following values are easily found

$D^3\,(\varphi_1 + \psi_1) = 4b$
$D^4\,(\varphi_1 + \psi_1) = 8(1 - b^2)$
$D^5\,(\varphi_1 + \psi_1) = -8b\,(5 - 4b^2)$
$D^6\,(\varphi_1 + \psi_1) = -32\,(2 - 9b^2 + 6b^4)$
$D^7\,(\varphi_1 + \psi_1) = 96b\,(11 - 28b^2 + 16b^4)$
$D^8\,(\varphi_1 + \psi_1) = 192\,(8 - 87\,b^2 + 160\,b^4 - 80\,b^6)$
$D^9\,(\varphi_1 + \psi_1) = -192\,b\,(279 - 1480\,b^2 + 2160\,b^4 - 960\,b^6)$
$D^{10}\,(\varphi_1 + \psi_1) = -4608\,(16 - 325\,b^2 + 1150\,b^4 - 1400\,b^6 + 560\,b^8)$
etc.

In the same way the functions φ_2 and ψ_2 give

$D^3\,(\varphi_2 + \psi_2) = -4b$
$D^4\,(\varphi_2 + \psi_2) = -8\,(1 + b^2)$
$D^5\,(\varphi_2 + \psi_2) = -8b\,(5 + 4\,b^2)$
$D^6\,(\varphi_2 + \psi_2) = -32\,(2 + 9\,b^2 + 6\,b^4)$
$D^7\,(\varphi_2 + \psi_2) = -96\,b\,(11 + 28\,b^2 + 16\,b^4)$
$D^8\,(\varphi_2 + \psi_2) = -192\,(8 + 87\,b^2 + 160\,b^4 + 80\,b^6)$
$D^9\,(\varphi_2 + \psi_2) = -192\,b\,(279 + 1480\,b^2 + 2160\,b^4 + 960\,b^6)$
$D^{10}\,(\varphi_2 + \psi_2) = -4608\,(16 + 325\,b^2 + 1150\,b^4 + 1400\,b^6 + 560\,b^8)$
etc.

the resulting expansion is therefore

$$\begin{aligned} I\,(a,b) = & -2a^2\,l\,a - 4a^2\,l\,2a - 4b^2\,l\,2b + 2\,(a+b)^2\,l\,(a+b) \\ & + 6a^2 - \frac{2}{3}\,b^2\,a^4 - \frac{2}{3}\,b\,a^5 - \frac{8}{45}\,(1 + 3b^4)\,a^6 - \frac{16}{15}\,b^3\,a^7 \\ & - \frac{b^2}{105}\,(87 + 80\,b^4)\,a^8 - \frac{b}{105}\,(31 + 240\,b^4)\,a^9 \\ & - \frac{8}{1575}\,(8 + 575\,b^4 + 280\,b^8)\,a^{10} \ldots \end{aligned} \qquad (10)$$

This series has been computed for the following values

$a = 0.04$	$b = 0.2$	to	3.6
$a = 0{,}08$	$b = 0.2$	,,	2.4
$a = 0.14$	$b = 0.2$	,,	2.4
$a = 0.2$	$b = 0.2$	,,	2.4
$a = 0.4$	$b = 0.3$	,,	0.8
$a = 0.6$	$b = 0.2$	,,	0.4

The convergency for all these values proved to be tolerably well.

Chemistry. — *"On a method for the quantitative analysis of ternary mixtures."* By Prof. A. F. HOLLEMAN, T. VAN DER LINDEN and J. J. P. VALETON.

(Communicated in the meeting of February 26, 1910).

Chemistry. — *"The melting diagram of the system of the three isomeric nitranilines."* By J. J. P. VALETON (Communicated by Prof. A. F. HOLLEMAN).

(Communicated in the meeting of February 26, 1910).

Chemistry. — *"On the introduction of halogen atoms into the core of phenols."* By Prof. A. F. HOLLEMAN and I. J. RINKES.

(Communicated in the meeting of May 28, 1910).

Chemistry. — *"On the introduction of halogen atoms into the core of the monohalogenbenzols."* By Prof. A. F. HOLLEMAN and T. VAN DER LINDEN.

(Communicated in the meeting of June 25, 1910).

These papers will not appear in these Proceedings.

(October 27, 1910).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING of Saturday October 29, 1910.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 29 October 1910, Dl. XIX).

CONTENTS.

Physiology. — *"Experimental researches on the segmental innervation of the skin in dogs."* By Prof. C. WINKLER from researches made in collaboration with Prof. G. A. VAN RIJNBERK. (VII[th] Communication).

(Communicated in the meeting of September 24, 1910).

On the manner in which the dermatomata of the posterior extremity are ranged, and on their variability.

In our VI[th] communication we gave an elaborate description of form and situation of the skin-areas of the seven lumbar and the three sacral posterior nerve-roots. These researches led to the following results:

1. The 6 inferior lumbar roots and the 3 sacral posterior roots contribute to the innervation of the skin of the posterior extremity.

2. The skin-fields of *L*VI and of *L*VII have always lost all connection with the mid-dorsal and the mid-ventral lines of the trunk,

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that of *L*v has lost this connection in most cases. These areas cover the apex of the extremity, this latter being considered as a cone. They were therefore called *apical* or *top*-dermatomata.

3. All other skin-fields on the contrary retain a connection both with the mid-dorsal and the mid-ventral line. They are situated on the basal margin of the cone, to which the extremity was compared and were therefore called *marginal* dermatomata. The skin-fields situated cranially from the top-dermatomata — *L*iv, *L*iii and *L*ii we called *cranial marginal dermatomata*, as opposed to those situated caudally from them, which were designated as *caudal marginal dermatomata.*

4. The skinfield of *L*v (and in some cases even that of *L*iv) must be ranged between the marginal and the apical dermatomata, generally it behaves like an apical dermatoma, occasionally however it retains a connection with the mid-ventral line of the trunk, and whenever such is the case, it presents in this respect the characteristics of a marginal dermatoma.

What now remains to be elucidated is the reciprocal relation between these areas.

The three *cranial* marginal dermatomata cover together a smaller zone of the skin on the *dorso-lateral* surface of trunk and extremity than on the *medio-ventral* surface. With the *caudal* marginal dermatomata exactly the opposite is the case. At the same time however the *caudal* marginal dermatomata cover a *larger* field on the *dorso-lateral* surface of trunk and extremity than that covered by the *cranial* marginal dermatomata. On the ventro-medial surface these conditions are reversed.

If *L*vii, *L*vi, and *L*v are cut through, the sensibility of the apical dermatomata being consequently destroyed, sensibility is retained along the mid-dorsal and mid-ventral lines of the trunk.

The sensible zone along the mid-dorsal line of the trunk extends far on the dorso-lateral surface of the extremity, nearly unto the epicondylus lateralis femoris. The remaining sensible zone along the mid-ventral line is far less extended on the ventro-medial surface of the extremity, it continues only unto a few centimeters laterally from the symphysis pubis.

After carefully comparing the *caudal boundaries* of the *cranial* marginal dermatomata with the *cranial boundaries* of the *caudal* marginal dermatomata we find at the mid-dorsal line of the trunk the following conditions:

The caudal boundary of the skin-field of *L*ii goes from the V[th] vertebra along the upper margin of the crista ilei; the cranial boundary of the skinfield of *S*II goes from the inferior margin of the sacrum between tuber ischii and anus towards the perineum. These boundaries therefore do not touch one another anywhere.

The caudal boundary of the skinfield of *L*III originates near the sacrum, and passes between crista ilei and trochanter just above this latter in the direction of the epicondylus lateralis femoris; the cranial boundary of the field of *S*II originates on the sacrum and goes from thence towards the tuber ischii. Between sacrum and trochanter these two boundaries run together for some extent.

The caudal boundary of the skinfield of *L*IV originates at the sacrum and goes towards the capitulum fibulae, either just above or just below the trochanter; the cranial boundary of *S*I has likewise its origin on the sacrum, and goes either just above or just below the trochanter in the direction of the epicondylus lateralis femoris, diverging to the popliteal space.

In this manner the joint caudal boundaries of the cranial marginal dermatomata *L*III and *L*IV form a line from the sacrum to the epicondylus lateralis femoris passing over the trochanter, as is likewise done by the joint cranial boundaries of the caudal marginal dermatomata *S*II and *S*I.

The skin-fields of *L*III and *L*IV, of *S*II and *S*I only slightly overlap one another here, they only are bounded by one another, whilst *L*II and *S*III are absolutely unconnected together.

As far as it is bounded by marginal areas, the line leaving the mid-dorsal line of the trunk at the sacrum and passing over the trochanter in the direction of the epicondylus lateralis femoris, may be considered as SHERRINGTON's mid-dorsal-line of the extremity or as BOLK's dorsal boundary of differentiation.

At the mid-ventral line of the trunk the relation between the boundaries of the marginal areas is different. There, the caudal boundaries of the cranial areas *L*II, *L*III and *L*IV all converge towards the short line, measuring only a few centimeters, which may be traced from the symphysis to a point situated in females alongside of the genital tuberosity or even cranially from it, in males alongside of the root of the prepuce.

The cranial boundaries of the caudal marginal areas *S*III, *S*II, and *S*I likewise converge to this line.

At the mid-ventral line of the trunk, the line along which the cranial and caudal marginal areas are bounded by one another is accordingly very short. This short line going from the symphysis towards the point alongside of the genital tuberosity and cranial from it, may be considered as SHERRINGTON's mid-ventral line of the extremity or as BOLK's ventral boundary of differentiation, (as far as bounded by marginal dermatomata).

The dorsal axis-line of the extremity, or the dorsal boundary of differentiation, corresponding to marginal dermatomata is therefore much longer than the equivalent ventral axis-line.

The subsequent course of these axis-lines however can only be demonstrated, when some insight is obtained into the manner in which the apical dermatomata are ranged and when their reciprocal variability and that of the marginal dermatomata have been treated at some length.

In order to give an instance of the way in which the skin-fields are ranged on the posterior extremity, the same image may serve that we made use of already when treating of the anterior extremity. There we compared the ranging of the skin-fields with that of floral leaves covering a flowerbud. The marginal dermatomata, like the basal floral leaves, perform this covering by pairs.

In applying this comparison likewise to the posterior extremity it must be kept in mind however:

a. that the hindleg is larger and longer than the foreleg;

b. that the posterior extremity, growing in caudo-cranial direction, suffers a greater rotation than the anterior one, the most cranial toe (the solitary nail) becoming situated thereby far medialward.

The greater length of the extremity finds an expression in the fact that 9 posterior nerve-roots contribute to its innervation (against 8 for the fore-leg), and still more in this, that three apical dermatomata — *L*V, *L*VI and, *L*VII — cover the nether-leg and the foot (against 2 covering the top of the fore-leg).

That this extremity suffers a greater rotation finds an expression in the fact that at the mid-ventral line of the trunk the cranial and caudal marginal dermatomata overlap one another so much with their ventral portions, that in this region the areas *L*II, *L*III, *L*IV, *S*I, *S*II, and *S*III overlap one another. Such is in no wise the case at the mid-dorsal line of the trunk, where the caudal margin of *L*II in that region does not join the cranial margin of *S*III.

Moreover the remarkable behaviour of the area *L*IV, is in accordance with this fact. It sometimes occurs that this skinfield does not join the mid-dorsal line of the trunk. In all cases however, where we observed this variation, it had to be ascribed to so-called formation of caricatures [1]), as it never persisted. Nothwithstanding that, the

[1]) In former communications we described how skinfields, isolated under unfavourable auspices (profuse hemorrhagy, prolongated narcosis, shock etc.) suffer alterations in form, appearing in a regular way, but not persisting. We called this "formation of caricatures", and made it an object of research.

skin-field of *L*IV behaved otherwise than the dermatomata of the trunk, where the first signs of a caricature was never found in the *dorsal*, but always in the *ventral* region. It is likewise in accordance with this line of thought, that the skinfield of *L*V never presents a connexion with the mid-dorsal line, whilst it not rarely presents a connexion with the mid-ventral line. Apparently both skinfields are dislocated farther from the mid-dorsal than from the mid-ventral line.

Still other facts may also be considered as an expression of the intense rotation of the extremity, in which the skin takes part. Firstly the situation of the cranial marginal dermatomata, which occupy much more territory on the ventro-medial surface of the extremity than on its latero-dorsal surface; then the behaviour of the skinfield of *L*V which, being fixed to the first cranial toe (solitary medial nail), has turned almost completely on the medial surface of the extremity and presents only a small top on the dorso-lateral surface; further the manner in which are situated (the topdermatoma *L*VII and) the caudal marginal dermatomata, whose extension, contrary to that of the cranial areas, is much larger on the latero-dorsal surface of the extremity; and lastly the shape of the topdermatomata, among which *L*VI occupies a middle position. They are wound spirally around the forepart of the extremity, in such a manner that the longitudinal axis of each of these skinfields is placed in latero-medial direction.

Taking all this into account, the following scheme may be designed for the ranging of the dermatomata on the posterior extremity. The areas of *L*II and *S*III represent the most basal floral leaves, leaves, joined together at the mid-ventral line of the trunk, but not rejoining one another at the mid-dorsal line.

The second pair of floral leaves is formed by the skin-fields of *L*III and *S*II, joined together both at the mid-ventral and at the mid-dorsal line, but covering a larger portion of the basis of the cone at the latter.

Still more clearly expressed are these relations in the third pair of floral leaves, the skin-fields of *L*IV and *S*I, which are still connected both at the mid-ventral and mid-dorsal lines, but whilst they encompass the dorso-lateral crural surface unto past the middle of it, this encompassment on the ventro-medial basis of the extremity, continues at the utmost to the medial fourth of the inguinal fold.

This deficiency of the marginal dermatomata in covering the ventro-medial surface of the extremity is filled up by the skin-field of *L*V, which has been removed almost entirely to the ventral (medial) surface.

The apex (in casu the foot) is covered then by the skinfields of

*L*VI and *L*VII. The skinfield of *L*VI is situated in the middle and for the covering of the foot it is supported on the medial side by that of *L*V, on the latero-plantar side by that of *L*VII.

These differences in the covering of the dorso-lateral and of the medio-ventral basis of the cone, the medial overlapping of the area of *L*V and the spiral winding of the areas of *L*V, *L*VI, and *L*VII are alike expressed by the image of a bud wrung sideways (see the scheme).

On such a bud it may easily be demonstrated, how the skinfields of the nerve-roots have maintained their segmental successive order, if a line is drawn through it, crossing one after another all radicular areas in their successive order.

This line begins at the origin of the inguinal fold, and continues in proximo-distal direction over the patella, between tuberositas tibiae and epicondylus femoris medialis, over the solitary nail to the medial second toe, crosses the foot along the dorso-plantar limit-line of the toes, and goes in disto-proximal direction between calcaneus and malleolus lateralis over the popliteal space to the tuber ischii. In following this line in the direction here indicated we find that it passes successively the tips of the different tongues, by which the areas of *L*II, *L*III, *L*IV, and *L*V are characterized, crossing transversally the top-dermatomata *L*VI and *L*VII, and returning over the tongues of the skinfields of *S*I, *S*II, en *S*III. (Red line on the scheme). The skinfields are ranged along this line.

The skinfields, covering in this manner the extremity, though bound to fixed rules as regards their situation and the way in which they are ranged, are nevertheless within certain limits subject to important variations concerning their situation, extension, and shape.

The cranial marginal areas *L*II and *L*III still behave partly as trunk-dermatomata. In their variations they are dislocated more or less far in cranial or caudal direction, and are thus lying opposite a higher or a lower vertebra. As far as they partake in covering the extremity, their dislocation is combined with their linguiform prolongation being drawn in or protruded distalward.

The variations of the caudal marginal areas *S*III and *S*II are different. Their dorsal cranial boundaries are lying very near to one another, between sacrum and root of the coccyx, but the angle opening caudalward, described by these boundaries in leaving the mid-dorsal line is less acute for *S*II ($\pm$ 45°) than for *S*III ($\pm$ 30°). Their variations present no dislocation in caudal or cranial direction but the angle opening caudalward at their origin is reduced or enlarged, and at the same time the linguiform prolongation is displaced either a little or further distalward. In this way it may occur that the tongue of the skinfield of *S*II advances on the surface of the nates between anus and tuber ischii, unto some centimeters past this latter.

The marginal dermatomata *L*IV and *S*I, which are already of far greater importance in covering the extremity, suffer hardly any dislocation, but follow in cranial or caudal direction lines which are called SHERRINGTON's axial lines or BOLK's boundaries of differentiation. On the other hand, in their variations, variability of length and the advancing distalward of the linguiform protrusion are an essential moment. The tongue of the skin-field of *L*IV may extend unto the tuberositas tibiae and in extreme cases it may even reach the medial surface of the foot. Still more forcibly this is expressed in the skin-field of *S*I, where the tongue sometimes extends to the popliteal space and in extreme cases, along the posterior surface of the under-leg, even to the lateral portion of the foot and the lateral toes.

The skinfield of *L*V varies again in a different way, along the medial surface of the under-leg it sends out on the medial margin of the foot a tongue of variable length, now reaching not farther than the solitary nail (1st medial toe), then again extending unto the medial sole of the foot and the 2nd medial toe. So far this area behaves in the same manner as that of *L*IV.

But at the same time this skinfield, fixed at the medial surface of the foot, extends transversally over the knee and the underleg. In medial direction this may continue so far, that the mid-ventral line of the trunk is attained. In lateral direction it varies likewise. The dorso-proximal head of this skin-field may extend from the epicondylus lateralis unto far on the latero-dorsal crural surface, whilst its lateral boundary may reach the front margin of the underleg, sometimes crossing this and even extending unto the malleolus lateralis. Here therefore we observe two differently directed variations. The marginal areas advanced tongues along the direction of the line above described by us, they varied accordingly in the direction of this line, whilst the skin-field of *L*V varies (likewise in the direction of this line) partly as *L*IV, partly perpendicularly on the direction in which succeeds the variation of the marginal areas. The variation perpendicular to the variation of the marginal areas is characteristic for the true top-dermatomata; corresponding to this fact the described line takes upon the paw a course rectangular to itself.

The skin-field of *L*VI is situated laterally on the thigh, on the underleg on its front-surface, on the foot dorso-medially, and from thence it may encompass the plantar surface of the foot and all the toes. Accordingly this skinfield lies around the leg like a spiral wound around it, and it is subject to the following variations: 1. the spiral area around the foot may be dislocated in toto, sometimes more medially, in other cases more laterally, and 2. there may be some territory added to it, either medially or laterally or on both sides, and this may go so far that the whole of the foot is encompassed by it.

The dorso-proximal head of this area is relatively fixed in place, though it may vary greatly in breadth. The ventro-proximal head suffers more important displacements, and it may even fail altogether. Accordingly the variations of this skin-field consist in a dislocation (especially on the foot) in the direction of the described line, which takes here a rectangular bend, i. e. rectangular to the direction of the variation of the marginal areas

The skin-field of *L*VII is situated dorsally on the thigh, dorso-laterally on the under-leg, ventro-laterally on the foot, it encompasses the dorsal side of the toes, and a larger or smaller portion of the soles of the toes and of the large plantar sole. Accordingly this skin-field, laterally from that of *L*VI and parallel to it, is wound as a spiral zone around the under-leg and the foot.

Here again it is the distal part that varies most. The lateral surface of the foot indeed always belongs to *L*VII, but the toes and the plantar surface are covered by this dermatoma farther or less in different cases. This skin-field varies likewise rectangularly to the direction of the variation of the marginal dermatomata.

Remarkable is the manner in which behave the proximal territories of the topdermatomata, which we called their heads. In this regard the topdermatomata differ between them, for the area of *L*V, placed "à cheval" on the knee, has a large ventro-proximal head and a small dorso-proximal one, that of *L*VI a large dorso-proximal one and a small ventro-proximal one, whilst in that of *L*VII the ventro-proximal one is failing, The proximo-dorsal heads of *L*V and *L*VI overlap one another, that of *L*VII approach nearer to the mid-dorsal line of the trunk, whilst of all three, the ventro-proximal head of *L*V, if it does not quite reach the mid-ventral line of the trunk, approaches nearer to it than that of *L*VI.

Accordingly it may be said, that the three top-dermatomata originate at the same point of a sufficiently well-defined region on the lateral surface of the thigh. Subsequently they extend next to one another, like the divisions of a fan, wound spirally round the under-leg and the foot, and in cases where there are variations, they swing to and fro in this region together.

We will now turn our attention again to the axial lines (SHERRINGTON)[1]) and the boundaries of differentiation (BOLK) on the posterior extremity.

As we observed, the origin of the three topdermatomata on the lateral thigh-surface is limited to a pretty well defined region, situated nearly at the point unto which the line was drawn, where the caudal boundaries of the cranial marginal areas reached dorsally the cranial boundaries of the caudal marginal areas.

It would not be justifiable to continue the dorsal axial line of the extremity farther than this point. Three successive areas *L*v, *L*vi and *L*vii, overlapping one another in a large measure, extend from this place, like divisions of a fan.

The dorsal axial line therefore does not continue farther than the middle of the lateral thigh-surface.

On the medio-ventral surface of the extremity the case is different. We found here that the region where cranial marginal areas (*L*ii, *L*iii, and *L*iv) and the caudal ones (*S*iii, *S*ii and *S*i) rejoin one another, is a very short one. Here however it is possible to draw between the marginal dermatoma *S*i and the topdermatomata *L*v and *L*vi, a line answering to the definition given by SHERRINGTON for the axial line or by BOLK for the boundary of differentiation.

For our different isolations of the skinfields of *L*v and *L*vi all showed us medial boundaries, situated on the medial surface of the thigh and of the underleg nearly at the same place as is occupied there by the medial boundaries of *S*i and *L*vii. Furthermore we have seen that in the important variations, characteristic for the skinfields of *L*iv and *L*v, the medial margins of these are dislocated in different directions along that line. It is along this line that the skinfield of *L*v reaches the mid-ventral line, and along it too that the skinfield of *L*iv extends a distal tongue on the medial surface of the under-leg. The same conditions prevail in the exceedingly great variations of *S*i. Whether the tongue of this skinfield extends to the popliteal space or to the toes, it is always this line that forms its medial boundary.

Accordingly, the ventral axial line must be drawn further from the middle of the genital tuberosity over the medial surface of thigh and under-leg unto the malleolus medialis.

1) SHERRINGTON's axial lines of the extremity are characterized by the very slight degree in which the areas bounded by it overlap one another. In this it reminds the "crossed overlapse", at the mid-dorsal and mid-ventral lines on the trunk, of the homonymous skin-fields of both halves of the body.

BOLK's boundary of differentiation is a boundary between dermatomata, which originally do not follow one another in the serial order, but are placed next to one another by the development of the extremity.

As the skin-fields vary in the direction of the line we have described, the variations of the marginal fields are found to move more or less in the direction of the longitudinal axis of the extremity. The topdermatomata swing to and fro rectangular to it, whilst *L*v varies in both directions.

Still another circumstance must be added. As soon as a skinfield changes as to place and extension, its form changes likewise, and in such a manner that it shows an inclination to take the form of the field, whose place it usurps. This fact is illustrated by the fields of *L*v and *L*vi.

That of *L*v varies in two directions. The tongue extends more or less far distalward along the medial border of the foot (from the solitary nail to the sole of the second medial toe), the heads are dislocated in dorso-ventral direction, and the ventro-proximal one may even reach the mid-ventral line. But that of *L*iv too presents somewhat similar conditions. If the tongue, which in most cases does not pass the tuberositas tibiae, extends farther, it may go along the medial underleg unto the solitary nail. In such cases the skinfield of *L*iv may be on the point of leaving the mid-dorsal line (formation of caricatures), and *L*iv has then completely taken the form of *L*v, just as *L*v whenever it extends to the mid-ventral line, resembled *L*iv in its form.

However great the variability of the single skin-fields may be, — still in their variations they are collectively bound to certain rules. In every individual there is a constant correlation between the topographical characteristics of all the skinfields situated on the extremity.

This correlation is expressed especially in the behaviour of the cranial and caudal marginal dermatomata and in some cases it can be demonstrated how a *reciprocal correlation* prevails in their variation. For an instance of this we refer to the isolation of the skinfield of *S*i, which was found on dog 11 [1]) extending a very long tongue unto the toes, whilst on dog 33 [1]) the tongue was very short, only extending to the popliteal space. In both cases the caudal margin of *L*ii is known. In the first case (tongue of *S*i very long, protruded far cranialward), the caudal margin of *L*ii is likewise shifted cranialward, passes above the crista ilei, and extends a tongue that goes unto 2 centimeters' distance from the patella. The further *S*i advanced on the extremity, the farther *L*ii drew back. These two skinfields, situated far from one another, show therefore a reciprocal correlation, or what may be called also a dislocation in the same direction, both cranialward or caudalward.

1) Cf. VIth Communication p. 304 and p. 303.

It may only be demonstrated in an indirect way, that the same holds good for top-dermatomata, situated next to one another. Still a comparison between several isolations sometimes tends to prove this. On dog 27 for instance the skinfield of *L*VI is isolated to the left and that of *L*VII to the right[1]), whilst the behaviour of the caudal boundaries of *L*III to the left and *L*IV to the right indicate symmetrical relations between the two halves of the body. In this case the skinfield of *L*VI (to the left) leaves uncovered a large portion of the dorso-lateral border of the foot, it has accordingly been displaced somewhat cranialward, and that of *L*VII (to the right) fits completely in this gap and has accordingly been displaced in the same direction.

Without pretending to include all possible variations, we still have found a few fixed rules to which they are submitted.

1. The variations of the single skinfields take place in determined directions. The marginal dermatomata vary nearly in the direction of the longitudinal axis of the extremity (linguiform protrusion). The top-dermatomata vary in a direction rectangular to it. Both of them however along the "ranging line", above described by us.

2. Each single skinfield in an individual may occupy, wholly or partly, in several gradations, the place that is occupied in other individuals by the adjacent skinfield. Whenever a skinfield does this, it changes form at the same time and it bears the greater resemblance to this adjacent area, the more completely it usurps its place. We never met with a case, where a nerve-root undertook the innervation of the whole area, which in individual one or another got its innervation from the third nerve-root in cranial or caudal direction.

3. However great may be the separate variabilities, these variations ordinarily do not concern only one separate skinfield. Apparently in most cases they vary collectively, and then in such a manner that the whole series of them is displaced along the ranging line and the direction of the collective displacement is in most cases cranialward.

In 1856 TÜRCK[2]) proved, that in dogs the posterior nerve-roots following on one another, supply the innervation of skin-fields,

[1]) Cf. Ibidem p. 295 and p. 301.

[2]) L. TÜRCK, Vorläufige Ergebnisse von Experimentaluntersuchungen zur Ermittelung der Hautsensibilätsbezirke der einzelnen Rückenmarksnervenpaare. Sitzber. der Math. Nat. Cl. der K. Ak. der Wiss. Wien 1856.

ranged in successive order, and that the serial ranging of the nerve-roots may be discovered also on the extremities. With the aid of the material collected by TÜRCK in his researches, WEDL designed in 1869 schemata for the areas of innervation, also for those of the posterior extremity. Yet TÜRCK had not given any special opinion about variability, and we are inclined to seek even in his silence hereupon the reason why these schemata came into existence. For if TÜRCK could have rightly valued the variability, as it was made possible for us to value it by the experiences of later anatomists, the difficulty of compressing the varying configurations of the dermatomata within the bounds of a scheme, would have become still more evident to WEDL [1]) too, than he found it already.

Since then however FÜRBRINGER and HERRINGHAM have first put forth the principle, which was afterwards amply elaborated by the researches of LANGLEY, SHERRINGTON and BOLK, and which has led to the modern conception of the segmental variations as being serial variations.

The meaning of this is, that the composition of the peripherical nerve-plexus of the extremity, and of the innervation of muscles and skin of the extremity is dependent on the different segmental level on which the extremity was developed. It may be displaced cranialward or caudalward for a segment.

If the origin of the extremity has been displaced one segment cranialward, it gets the material for its innervation from a region of the medulla situated one segment more cranialward, and the innervation of the periphery corresponds to more craniai elements of root-fibres. In such cases a segment is added cranially, caudally a segment falls off, but the successive order of the innervation areas for skin and muscles is maintained (cf. LANGLEY [2]), SHERRINGTON'S [3])

1) WEDL, weiland Prof. L. TÜRCK, Ueber die Hautsensibilätsbezirke etc. Abhandlungen der Math. Nat. Cl. der K. Ak. der Wiss. zu Wien, 1869, writes in his preface to the publication of researches left by TÜRCK: "Die Varianten der Sensibilitätsbezirke hat der Verfasser .... auf die Schablone transponirt. Es ist hierbei allerdings der missliche Umstand eingetreten, dasz einzelne Figuren an Klarheit eingebüsst, und es selbst mir trotz Zuhilfenahme der einzelnen Experimente nicht immer möglich war zu entscheiden, ob so manche Variante zu dem Bezirke des einen oder anderen Nerven gehöre.

2) J. N. LANGLEY, On the course and connection of the secretory fibres supplying the sweat glands of the foot of the cat. Journal of Physiology. London 1891. Vol. XII. N⁰. 4, p. 347.

3) C. S. SHERRINGTON, An experimental investigation of the nerve-roots which enter into the formation of the sacro-lumbal plexus of Macacus Rhesus. Abstr. of the Proceedings of the Royal Soc. of London 1893, Vol. 53, p. 459, Vol. 54, p. 213.

prefixed type). Mutatis mutandis the same holds good for cases, where the extremity originates one segment more caudally (SHERRINGTON's postfixed type). Independently of SHERRINGTON, BOLK[1]) arrived at the same conclusion, when he represented the series of dermatomata on the extremity as the links of a chain, which may be drawn hither and thither round a fixed bar.

Our results again are for the greater part in accordance with the rules set down by SHERRINGTON and BOLK. A great many of the variations we found, fit perfectly in the frame of the serial displacement of the dermatomata.

The apparent contradiction between the direction of the variations of marginal and of apical dermatomata, the former nearly following the longitudinal axis of the extremity, the latter rectangular to it, means only a variation along the course of the "ranging line" described by us (see page 436), and accordingly fits in the frame of the serial displacement.

BOLK's image of the linked chain involves likewise, that the direction of the variations of the marginal dermatomata must be placed rectangular to that of the apical ones.

Less obvious appeared the fact, that with serial displacement the form of the skin-fields should change so completely. Still this was found to be the case, but the remarkable behaviour of *S*I, where the extension of the tongue may alternate from the popliteal space to the lateral border of the foot, or that of *L*IV, where this extension may alternate from the tub. tibiae to the malleolus medialis, fits nevertheless perfectly into the frame-work of a serial displacement.

If the more distal region of the origin of the extremity is represented by an exceedingly steep cone, then a very slight displacement of this steep cone (the nerves growing into it, being imagined as a series of parallel fascicles) may be the cause that the extremes of this series may penetrate either very little or very far distalward into the cone. When the displacement is caudalward, *S*I advances far distally on the caudal border of the cone, when it is cranialward, *S*I advances less far, but instead of it *L*IV, situated at the cranial border, goes farther. And as the basis of the cone is broad and short, this relation becomes less evident than it is for *L*IV and *S*I, for *L*II, *L*III, *S*III and *S*II, which supply the innervation of the broad and low superior part of the conic circumference.

Finally the behaviour of the cranial and caudal marginal dermatomata, showing a reciprocal correlation (as described by us

[1]) L. BOLK, Een en ander uit de segmentaal-anatomie van het menschelijk lichaam, Weekbl. van het N. T. v. Geneesk. 1897, Deel I 24, p. 982, Deel II, p. 366.

for *L*III and *S*I, *L*VI and *L*VII) tends likewise to prove a serial variation, dependent on the different segmental level of the origin of the extremity.

Most important of all in demonstrating this is perhaps the tendency of the skin-fields to take also one another's form whenever they happen to have taken one another's place.

We have described this for *L*IV and *L*V. The skinfield of *L*IV may show an almost perfect or even perfect likeness to that of *L*V and the reverse.

But in cases where *L*IV took the place, and together with it the form of *L*V, still other characteristic alterations were found in the segmental relations of other organs (vertebral column and plexus).

Whenever *L*IV sends its long tongue medialward (we found this in three cases), we always found only 12 thoracic vertebrae with their costae. The 13th without costa then appears as 1st lumbar vertebra and the seventh lumbar vertebra has been absorbed entirely or partly in the sacrum. Accordingly the vertebral column presented a reason for assuming that it had been shortened by one vertebra, a fact corresponding to a cranial displacement of the origin of the extremity [1]).

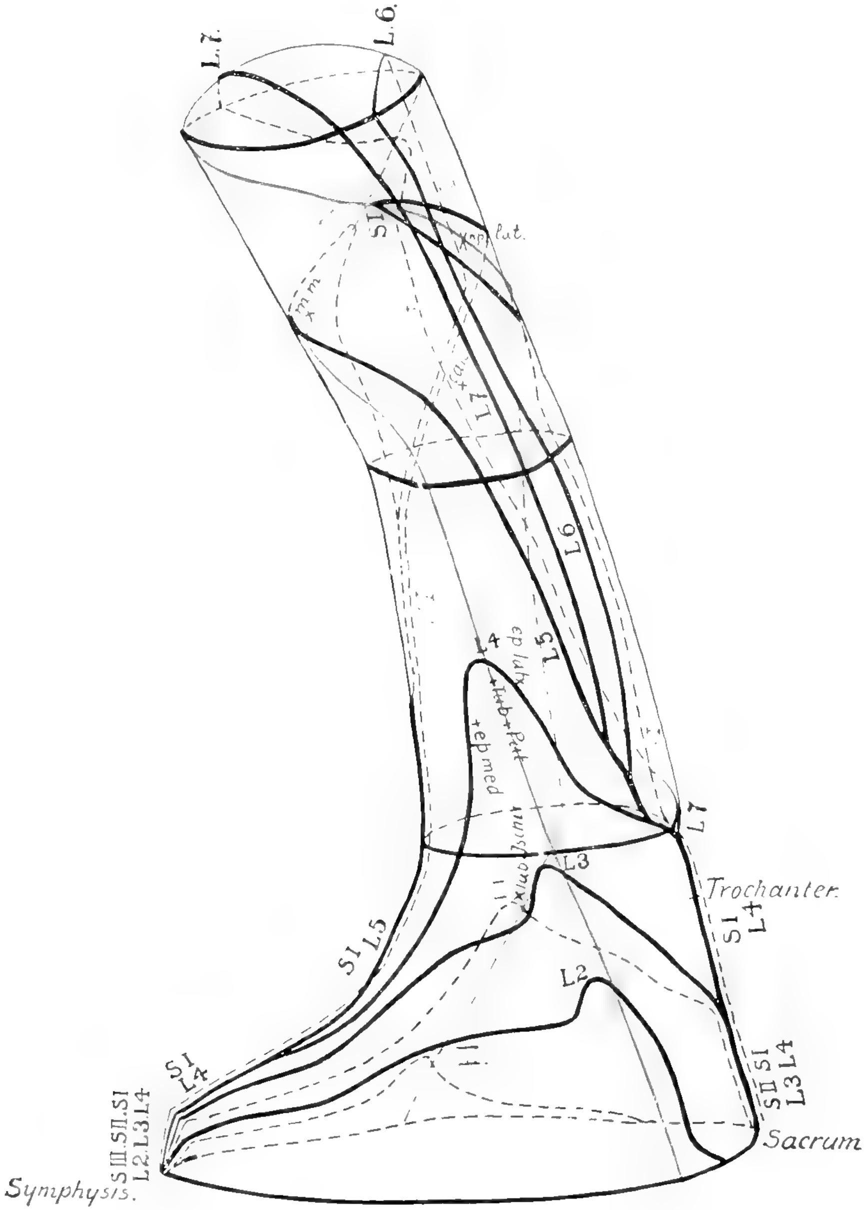
Dr. FRETS, who has made researches on the sacro-lumbar plexus of our and other cases, obtained accordant results. They will be published by him separately.

But the here mentioned observations, that apparently it is decided by the periphery what form the skin-field will assume; that it seems to be quite indifferent for its form, whether the material for its innervation has been supplied by a more cranial or a more caudal level of the medulla; that an identical form of the skinfield may be realized, independent of its innervation being derived either from *L*IV or from *L*V, if only the periphery presents an attitude favourable to such a form, — seem to us of the greatest importance.

Though a great many of the above described instances are in accordance with the theory of the serial displacement, still there

1) We are inclined to think that the expression in a scheme of these dermatomata by WEDL and TÜRCK was based upon their reckoning without these variations. There is sufficient reason for this. If TÜRCK thought to cut through on a praefixed type with e. g. six true lumbar vertebrae (the 7th being absorbed in the sacrum) the 5th lumbar nerve-root, it is evident that he cut through in reality the 4th, but found none the less a skinfield, corresponding to that of *L*V in not prefixed individuals. TÜRCK did not valuate rightly the extremes of the variations, because in such cases the 13th thoracic vertebra frequently bears no costa. In such cases however, the root that seems to be *L*VII is *L*VI etc

C. WINKLER and G. A. VAN RIJNBERK. "Experimental researches an the segmental innervation of the skin in dogs."
(VII[th] communication).



Scheme of the skin-areas of the posterior extremity which is represented as a cone. The red line is the ranging-line described in our paper. This line runs between patella (+ pat), tuberositas tibiae (+ tub) and epicondylus medialis femoris (+ ep. med), towards the back-side of the malleolus medialis (+ m m.). It crosses the toes, returns on the fore-side of the foot and passing the maleolus lateralis (+ m. lat) it regains the fossa poplitaea and goes to the tuber ischii (+ tub. ischii).

remains a certain number of cases, that will not fit into this theory. We will indicate two groups of them.

It may happen, that *L*IV advances very far on the extremity, assuming the place and likewise the form of *L*V, a cranial displacement of the cranial margin of the extremity being thereby apparently indicated (and affirmed moreover by the vertebral column), whilst yet there is no reciprocal correlation with the caudal marginal dermatomata, and the skin-field of *Cocc.*I is partly situated before the anus. This indicates a caudal displacement of the caudal margin of the extremity whose cranial margin was displaced cranialward. In other words, in these cases the origin of the extremity appears to be not displaced but enlarged.

We found also, in examining the different medullae operated upon, that the dorsal nerve-roots may vary very much in thickness.

For instance the relative thickness of *L*VII compared with *S*I is subject to much variation. *L*VII indeed is always thicker than *S*I, sometimes however they are almost equal in thickness, whilst in other cases *S*I appears only as a thin thread compared with *L*VII. Frequently the two nerve-roots have a common course, and it even happens that their intervertebral ganglia are partly grown together.

A similar, but never so strongly expressed reciprocal relation exists between *L*IV and *L*V. *L*IV too is always thinner than *L*V. *L*VI is the thickest of the lumbar nerve-roots. We did not always find. however, that a large skin-field corresponded to a thick isolated nerve-root or vice versa. We found indeed *S*I sometimes exceedingly thin, *L*VII exceedingly thick, and at the same time *L*IV very thick compared with *L*V,, a relation which, (if a larger skinfield corresponds to a thicker nerve-root) indicates a reciprocal correlation and consequently a serial displacement.

But we did observe still very different relations in the thickness of single nerve-roots, relations we will not insist upon at present.

Whether besides the variations that may be considered as based upon serial displacement (and that may be then observed in numerous gradations, even extending to one segment), other variations may occur, dependent on enlarged origin of the extremity or based on mutual interchange of root-fibres, or lastly based on the possibility of more or less oblique out-growth of the origin of the extremity, — a possibility we did not treat of in this paper — these are questions for the elucidation of which the material we dispose of at present, is not sufficient. It is our purpose to resume them later on.

Physiology. — *"Lens measurements and Emmetropisation"*. By Dr. W. P. C. ZEEMAN. (Communicated by Prof. T. PLACE).

(Communicated in the meeting of September 24, 1910).

Since DONDERS' pioneer writings on the refraction of the human eye, a great amount of research has been devoted to the study of the refractive anomalies, but conspicuously little attention has been given to emmetropia.

Nevertheless the question of the origin of emmetropia is of the greatest importance, not only to physiology but also for a right understanding of the refractive anomalies.

STRAUB has repeatedly pointed this out, and has endeavoured to give an explanation of the origin of emmetropia. From a point on the illuminated retina, a bundle of rays of light goes out with a certain divergence. The divergence of this bundle is modified by the optic system of the eye, which has a certain converging power. The peculiarity of emmetropia now lies in the fact that the converging power of the optic system is just equal to the divergence of the retina bundle. An explanation of how this equality comes about has been given by STRAUB in his theory of emmetropia.

According to this theory the tone of the ciliary muscle gives to the lens the exact form to attain this equality. The lens form, or tone of the ciliary muscle is, therefore, the factor whereby in every eye emmetropia can be reached and maintained.

The ophthalmometric measurements, which I made and the results of which I shall give here, do indeed show that the production and maintenance of emmetropia is the work of the lens. Measurements of the eyes of hypermetropes and myopes prove that in these eyes there is also a tendency towards emmetropia, that in them the lens has such a curvature as to lessen the degree of refractive anomaly.

My researches extended over 75 persons (25 emmetropes, 25 hypermetropes, 25 myopes) of about the same age.

The refraction was determined by means of the shadow test, spectacle glasses or by the direct method of ophthalmoscopy; the radius of the cornea was measured with JAVAL's ophthalmometer ("Kagenaar" model), the accuracy of which was tested by means of a quartz ball with a radius of 15.4 mm.; the angle α, the position and curvature of the anterior and posterior surfaces of the lens were determined by TSCHERNING's ophthalmophakometer, the method of using which instrument is described minutely in the Encyclopédie franc. d'Ophthalmologie.

We introduced only a few slight modifications in the method; a cross-shaped fixation mark, illuminated from behind, for which a lens was set up, forced the subject to relax his accommodation.

The depth of the anterior chamber was measured by means of TSCHERNING's ophthalmophakometer, but according to the method of VON HELMHOLTZ, by fixing the point of convergence by two lines intersecting each other in the centre of the pupil.

With the exception of finding the depth of the anterior chamber of the eye, the measurements were taken after the pupil had been dilated by a mydriatic.

From the results we calculated the position of the principal planes, principal foci and nodal points of the optic system, and finally the length of the axis of the bulb.

After fixing angle α, it appeared that a good centering is a great rarity. In accordance with EHRNROOTH the centre of the cornea appeared to lie at the temporal side of the axis of the lens.

Properly, therefore, we cannot speak of an principal axis. We shall therefore give the name of principal axis to the connecting line of the centres of cornea and anterior surface of the lens. Further we found that angle α was smaller in the case of the myope than in that of the emmetrope, and in the latter smaller than in that of the hypermetrope. Considering that the size of angle α is dependent on 3 factors, viz. the position of the nodal point, the position of the retina, and the distance of the fovea centralis retinae, from the principal axis, it was of importance to investigate the influence of these factors further. For this purpose I calculated the position of the nodal point in respect to cornea and retina, and the distance of the fovea from the principal axis.

From the curves formed with these results the following conclusions could be drawn.

1. The differences in angle α in refractive anomalies are dependent upon the differences in the length of the axis.

2. The differences in angle α in persons of one and the same refraction are exactly proportional to the distance of the fovea from the principal axis.

3. The distance of the fovea from the principal axis has no relation whatever to the refraction.

The radius of the cornea was found with JAVAL's ophthalmometer. The myopes proved to have a shorter, and the hypermetropes a longer cornea-radius than the emmetropes, which is in accordance with the results of earlier investigators (SCHIÖTZ, PLANTENGA etc.).

The peculiar fact that the investigators who took their measurements with JAVAL's apparatus as a rule obtained figures larger than those taken with HELMHOLTZ' ophthalmometer (no good reason for which could be found) was evident in my case also, as was seen in comparing my measurements with a series which I had formerly made with the ophthalmometer of HELMHOLTZ. I obtained then as averages in emmetropes, hypermetropes and myopes respectively: 7.8, 7.66 and 7.66 mm. against 8.07, 7.92 and 7.87 mm. in the present case.

By means of a quartz ball with a radius[1]) of 7.7 mm. both instruments were subjected to a new test, which revealed that our HELMHOLTZ ophthalmometer had indicated too low values. The averages of our first series were after correction, therefore, 8.1, 7.96 and 7,96 mm., and thus practically agree with the averages of our new cases examined with JAVAL's apparatus.

I therefore venture to express the supposition that the low values found by some early investigators for the cornea radius are to be attributed likewise to similar inaccuracies of the instruments employed.

The *depth of the anterior chamber* was originally determined by TSCHERNING's method. This method requires the greatest care if reliable results are to be arrived at. In my opinion it is of the utmost importance to repeat the examination after the interchange of lamp and glass, a point also mentioned by GULLSTRAND in the 3rd edition of HELMHOLTZ' "Physiologische Optik". The depth of the anterior chamber was generally determined by fixing the point of intersection of the two lines running through the centre of the pupil (after the example of the method indicated by HELMHOLTZ).

The depth of the anterior chamber proved, in accordance with the results of former investigators as well as with my own, to be smaller in the hypermetrope than in the emmetrope, and in the latter smaller than in the myope.

The differences in the depth of the anterior chamber are undoubtedly for a part the direct result of the differences in curvature of the anterior surface of the lens.

The radii of the *anterior and posterior surfaces*, and the *thickness of the lens* were determined by the method indicated by TSCHERNING with his ophthalmophakometer.

The examination of the curvature of the lens surfaces, especially that of the anterior surface of the lens, showed considerable differences in persons with dissimilar refraction, in the hypermetrope a

[1]) The diameter of the quartz ball was found with a pair of adjustable compasses to be 15.4 mm. and the regular concavity at different points was controlled with the ophthalmometer. Finally, by very careful weighing, Prof. ZEEMAN fixed the diameter at 15.42—15.43 mm.

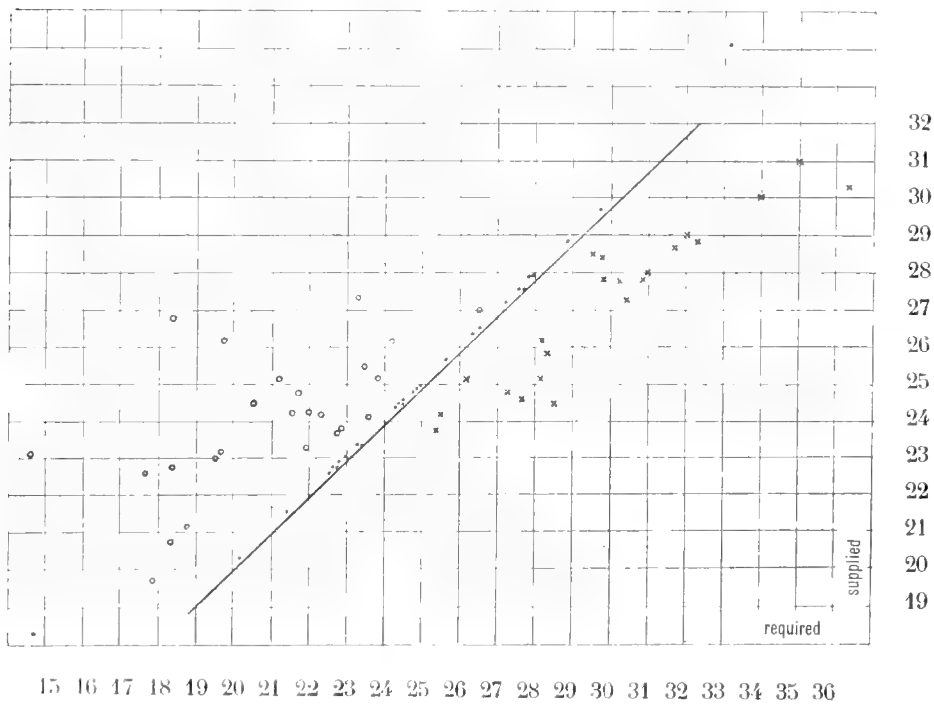
more decided, in the myope a less pronounced curvature of the lens surface.

The *thickness of the lens* did not vary in persons with different refraction. The errors, however, of measurement, are rather considerable.

The result of our measurements is, therefore:

Emmetropes, hypermetropes and myopes differ in respect to the curvature of the cornea, the size of angle α, the depth of the anterior chamber and the radii of the surfaces of the lens. The differences in the curvature of the cornea are such as to increase the refractive anomaly. The differences in the other measures are of such a nature as to lessen the refractive anomaly. In how far this is the case will be clear from the following computations.

From the data we possess at present we are able to estimate the strength of the lens as a whole. This proved to be greater in the hypertrope than in the emmetrope, and in the latter greater than in the myope. The importance of these figures is at once seen in



Refractive power of the lens in dioptres.

××× Hypermetropes
.... Emmetropes
oo Myopes

Fig. 1.

looking at fig. 1. We have computed the hypermetropia which each of the eyes examined should have from the curvature of their corneae and length of axis in the absence of the lens, and have arranged the eyes according to this hypertropia, after which we indicated, in dioptres, on the ordinates the refractive power of the lens.

Assuming that it is the work of the lens to correct the hypermetropia occasioned by the curvature of the cornea and the length of the axis, the eyes are classed according to the work required of the lens, while the ordinates indicate in how far the lens has answered to these requirements. Where these two figures are the same, there is emmetropia, where the lens has supplied more dioptres than are desirable there is myopia, where it had a relatively weaker refractive power there is hypermetropia.

Thus, in a hypermetropical eye x the refractive power of the lens amounted to 31 dioptres, the eye was 4 D. hypermetropic, so that the lens would have had to supply 35 D. in order to reach emmetropia.

Now in this figure we see a regular ascension from left to right, that is to say the more there is required of the lens the stronger is its refractive power. The lens has thus apparently the tendency to reduce the refractive anomaly. It goes without saying that the emmetropes lie on one line, as the refractive power here invariably answers to the demand put upon it.

The hypermetropical lens supplies more, and the myopical lens fewer dioptres. This clearly points to a tendency towards emmetropia.

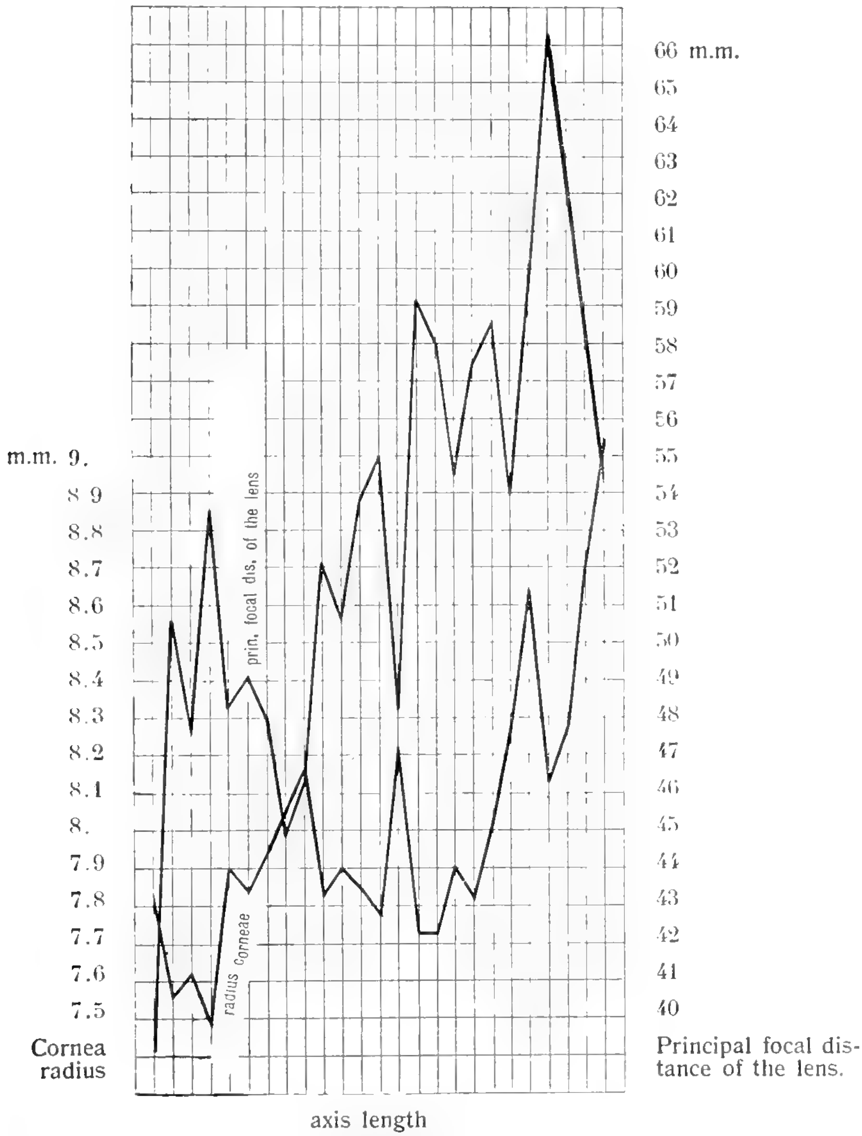
Without such a tendency, without an emmetropisation, we might expect to find in hypermetropes and myopes a lens of equal refractive power, and in our figure all these would have to be arranged on a horizontal level.

If we know the position and curvature of the refracting surfaces and the refraction of every eye, we are able to calculate the position of the retina.

In doing this it is assumed that the refractive indices of aqueous humour and lens are the same in the various eyes.

In the emmetrope the distance from the posterior principal focus to the cornea is equal to the length of the axis, in the ametropė we can determine the axial length approximately by placing the retina for each 3 dioptres 1 mm. before respectively behind the posterior principal focus.

In the emmetrope it has been seen that the length of the axis fluctuated between wide limits. If, nevertheless, emmetropia is present in these eyes the cornea or the lens must possess an accordingly diminished or increased refractive power, i. e. a greater or less curvature. This is seen most clearly in fig. 2. The cornea radii and principal focal distances of the lens increase regularly as the length of the axis increases. In the case of a greater length of axis we find, therefore, a greater focal distance of the lens. The greater fluctuations of these two lines are invariably in contrast. This contrast is a characteristic of emmetropisation. The eyes with



Emmetropes arranged according to *length of axis.*
Mutual relation of the cornea radius to the principal focal distance of the lens.
Fig. 2.

a relatively greater cornea radius have been kept emmetropic by a weaker lens refraction.

The length of the axis is closely related to the refraction. To demonstrate this unambiguously we must endeavour to exclude the influence of the differences in size. For this purpose the different measures of each eye must be reduced to one and the same cornea.

From the actual axis length of each of the eyes examined, I have determined the axis length which each should have if the cornea radius measured 8 mm. If now these axis lengths are brought together in curves, it will be found that the axis length of the emmetrope can vary very greatly, that at the most, however, it measures 23.5 mm. while in hypermetropes and myopes an axis length of 22 mm., and 24.5 mm. respectively are most frequently met with. It seems to me that we may consider the 9 emmetropes, in whom the reduced axis length amounts to 23.5 mm., the 10 myopes in whom it is 24.5 mm., and the 11 hypermetropes in whom it is 22 mm., as types of emmetropia, myopia, and hypermetropia.

We have, therefore, reduced the other measures also of these types to a cornea radius of 8 mm. and then found:

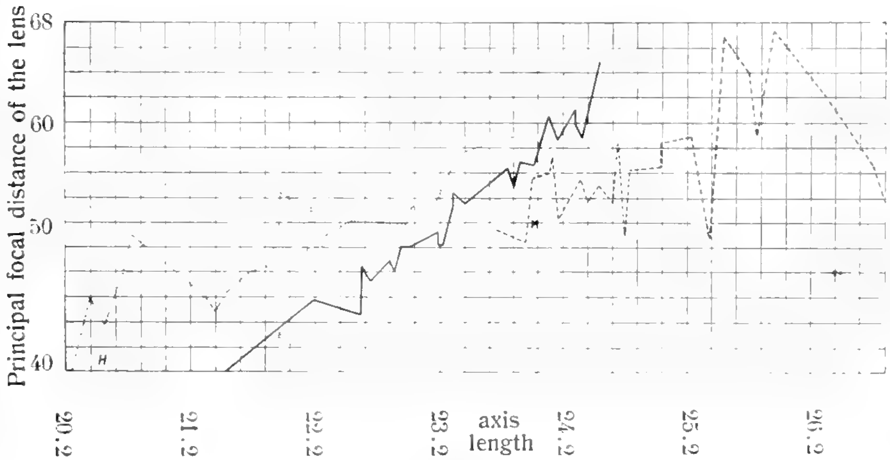
	Cornea radius	Length of axis	Depth of anterior chamber	Radius of anterior surface	Radius of posterior surface	Thickness of the lens	Focal distance of the lens	Dist. cornea and posterior focus	Refractive power of the lens	Refractive power of whole system
1st group (Emmetropes)	8 mm.	23.54	3.81	11.5	6.17	3.7	54.5	23.54	24 D.	62.3
2nd group (Myopes)	8 mm.	24.4	4.03	12.7	6.1	3.5	56.	23.7	23 85 D.	61.5
3rd group (Hypermetropes)	8 mm.	22.	3.65	10.03	5.95	3.67	50.8	22.95	26.3 D.	64.3

Finally we have tried to demonstrate the connection between length of axis and principal focal distance of the lens by arranging the eyes, after reducing all the measures to a cornea radius of 8 mm. according to axis length, and indicating the principal focal distance of the lens on the ordinates (fig. 3).

We have now to do with eyes of the same dimensions; we might make the corneae coincide, and then we could best study the relationship between axis length and principal focal distance.

As was to be expected the emmetropes lie again on one line; greater axis lengths are of course compensated by a greater principal focal distance; possible deviations from this line must be attributed to errors of measurement.

The end points of this line give us the limits between which in



Principal focal distance of the lens and axis length reduced to a cornea radius of 8 mm. with different refractions.

Fig 3.

emmetropes the axis length and the focal distance of the lens can oscillate. On the one side of this line the hypermetropes must lie and on the other the myopes, as the focal distance of the former is relatively greater, and of the latter relatively less, than with emmetropes.

Hypermetropes and myopes are thus separated by the slanting line of the emmetropes; this is simply a result of the definition.

It is interesting, however, to note that they can also be divided by a vertical and a horizontal line, which was not to be expected a priori.

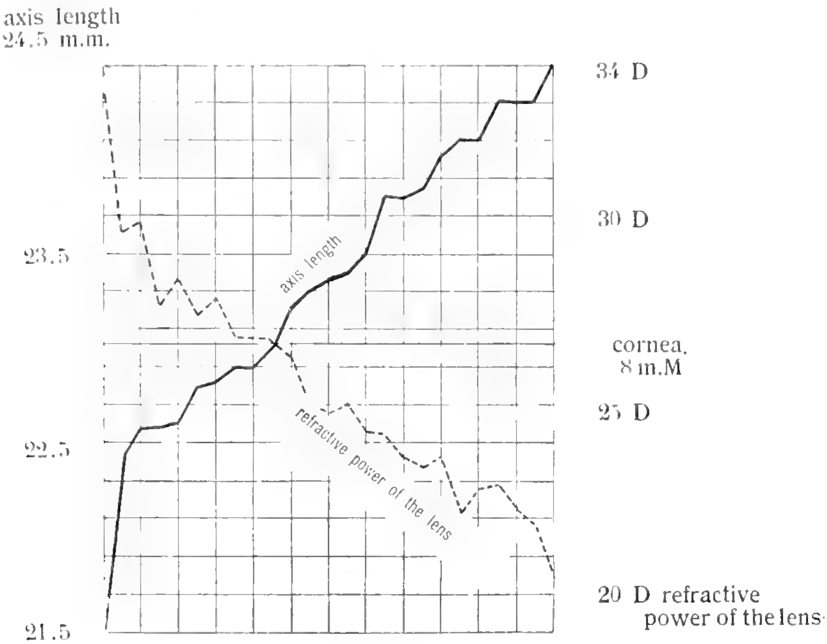
These dividing lines show that the hypermetropes and myopes can not only be divided by a difference in the ratio between axis length and focal distance, but moreover by differences in the absolute size of axis length and of focal distance of their lenses separately.

The vertical dividing line falls on 23.7, which bears out the well-known fact that the hypermetropes possess a shorter, and the myopes a longer axis length. Below 23.7 mm. there is no myopia, and above 23.7 mm. no hypermetropia.

The horizontal dividing line, which would have to be drawn at 53.5, shows that the focal distance of the lens in the hypermetrope (in 84 % of the cases) is less, and in the myope (in 80 % of the cases examined) is more than 53.5 mm.; hypermetropes and myopes are thus separated by their axis length and by the focal distance of their lens. The first factor causes the refractive anomaly, the second factor tends to diminish it.

This arrangement of our figures also shows in the clearest possible way that the lens tends to diminish the refractive anomaly, and that it is undoubtedly the lens which, by adapting itself to the axis length, reduces so many eyes to emmetropia, so that STRAUB's theory of Emmetropisation by the lens is confirmed by our measurements.

The nature of emmetropia is best seen in fig. 4, in the varying course of the lines representing the *axis length* and the *refractive power of the lens.*



Emmetropes, arranged according to the axis length reduced to a cornea radius of 8 m.m.
Fig. 4.

Physics. — "*On the solid state.*" V. By J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of Sept 24, 1910).

17. More than a year ago I published the fourth part of my Treatise on the solid state. (These Proceedings June, 1909); the continuation announced there, had, however, to be postponed to the present day in consequence of all kinds of interruptions.

Before proceeding with the further examination of the coexistence-curve liquid-solid, the equation of which was derived by me in IV (formula (16) on p. 134), I will first reduce this equation to a somewhat simpler form. According to (a) and (b) on p. 133, is namely in case of equilibrium:

$$\left(\frac{\partial \Omega'}{\partial n_1} - RT \log c_1\right)_{liq.} = (id.)_{solid} \quad . \quad . \quad . \quad . \quad . \quad (a)$$

In this (see p. 134 above):

$$\frac{\partial \Omega'}{\partial n_1} = RT \log \frac{RT}{p + a/v^2} - RT - b_1 (p + a/v^2) + \frac{2a}{v},$$

when namely $v-b$ is replaced by $\frac{\Sigma n_1 . RT}{p + a/v^2}$, and a_1 by a (see I p. 769).

Now c_1 being $= \frac{1-\beta}{1+\beta}$, ($a$) passes into

$$\log \frac{p + a/v^2}{p + a/v'^2} + \frac{b_1}{RT}\left(\frac{a}{v^2} - \frac{a}{v'^2}\right) - \frac{2a}{RT}\left(\frac{1}{v} - \frac{1}{v'}\right) + \log \left(\frac{1-\beta}{1+\beta}\,\frac{1+\beta'}{1-\beta'}\right) = 0,$$

after division by RT, when the quantities which refer to the solid state are accentuated. So we find finally:

$$\log \left(\frac{p + a/v^2}{p + a/v'^2}\,\frac{1-\beta}{1+\beta}\,\frac{1+\beta'}{1-\beta'}\right) = \frac{a}{RT}\left[2\left(\frac{1}{v} - \frac{1}{v'}\right) - b_1\left(\frac{1}{v^2} - \frac{1}{v'^2}\right)\right], \quad (19)$$

which form is simpler than (16).

The relation derived just now, however, has the drawback that when β or β' are very near 1 (almost complete dissociation of double molecules), it becomes practically useless. So the equation (19) can be successfully applied, when β and β' are both near 0 (slight dissociation).

If β and β' are both in the neighbourhood of 1, (19) may be easily reduced by the aid of the equation of dissociation (loc. cit. p. 136)

$$\frac{\beta^2}{1-\beta^2} = \frac{cT^{\gamma+1}}{p + a/v^2}\, e^{\frac{(p + a/v^2)(-\Delta b) - q_0}{RT}}.$$

From this follows namely:

$$\log (1-\beta) = \log \left[(p + a/v^2)\frac{\beta^2}{1+\beta}\right] - \log \theta + \frac{(p + a/v^2)\,\Delta b}{RT},$$

when for brevity θ is substituted for $cT^{\gamma+1}\, e^{-\frac{q_0}{RT}}$. So we have also:

$$\log \left[(p + a/v^2)\frac{1-\beta}{1+\beta}\right] = 2 \log \left[(p + a/v^2)\frac{\beta}{1+\beta}\right] - \log \theta + \frac{(p + a/v^2)\Delta b}{RT}, \quad (\beta)$$

so that (19) passes into (a perfectly identical relation holds namely for the accentuated quantities):

$$2\,log\left(\frac{p+{}^a/_{v^2}}{p+{}^a/_{v'^2}}\,\frac{\beta}{1+\beta}\,\frac{1+\beta'}{\beta'}\right)=\frac{a}{RT}\left[2\left(\frac{1}{v}-\frac{1}{v'}\right)-b_1\left(\frac{1}{v^2}-\frac{1}{v'^2}\right)-\Delta b\left(\frac{1}{v^2}-\frac{1}{v'^2}\right)\right].$$

Now $b_1+\Delta b=b_1+(-b_1+2\,b_2)=2b_2$, hence we get finally

$$2\,log\left(\frac{p+{}^a/_{v^2}}{p+{}^a/_{v'^2}}\,\frac{\beta}{1+\beta}\,\frac{1+\beta'}{\beta'}\right)=\frac{a}{RT}\left[2\left(\frac{1}{v}-\frac{1}{v'}\right)-2\,b_2\left(\frac{1}{v^2}-\frac{1}{v'^2}\right)\right]\ldots\ (19^a)$$

If in the derivation of the equation (19) we had directly started from simple molecules, and not from double ones, we should at once have found (19^a), which equation is only distinguished from (19) in this that b_1 has been replaced by $2b_2$, $c_1=\frac{1-\beta}{1+\beta}$ by $c_2=\frac{2\beta}{1+\beta}$, and c'_1 by c'_2. Further the whole term under the sign *log* is in the second power, in accordance with the dissociation relation $c_2^2 : c_1 =$ etc.

Now in reality the case that β is near 1 (the molecules of the liquid almost entirely dissociated), and β' near 0 (the molecules of the solid phase almost undissociated) occurs most frequently, and so we shall have to transform the equation (19) with a view to the latter case.

If we, namely, only substitute in (19) the second member of (β) for $log\left[(p+{}^a/_{v^2})\frac{1-\beta}{1+\beta}\right]$, and leave the accentuated quantities unchanged, we get:

$$log\left[\frac{(p+{}^a/_{v^2})^2}{p+{}^a/_{v'^2}}\,\frac{\beta^2}{(1+\beta)^2}\,\frac{1+\beta'}{1-\beta'}\,\frac{1}{cT^{\gamma+1}}\right]=$$

$$=\frac{a}{RT}\left[2\left(\frac{1}{v}-\frac{1}{v'}\right)-b_1\left(\frac{1}{v^2}-\frac{1}{v'^2}\right)\right]+\frac{(p+{}^a/_{v^2})\,(-\Delta b)-q_9}{RT}.\ (19^b)$$

So far this equation is perfectly accurate. When however β is near enough 1 and β' near enough 0, so that also v is in the neighbourhood of $2b_2$ and v' in that of b_1, it is possible to simplify (19^b) very considerably, by putting

$$\beta=1,\qquad \beta'=0,\qquad v=2b_2,\qquad v'=b_1.$$

That this is really allowed in many cases, appears from the plate and the tables of III. Even at 100°, where — for the hypothetic substance supposed by us — the pressure of coexistence is about $=0$, and hence the p-T line solid-liquid intersects the axis $p=0$ in the neighbourhood of the triple-point solid-liquid-vapour, β is $=1$ and β' is about $=0{,}06$ (between 0,093 and 0,041) for $p=0$, according to the table on p. 122. So at all temperatures between 0° (absolute) and 100° $\beta=1$ and $\beta'=0$ may be put a fortiori in our case [for $T=0$ it is exactly so (see the plate of III)].

For $2\left(\frac{1}{v}-\frac{1}{v'}\right)-b_1\left(\frac{1}{v^2}-\frac{1}{v'^2}\right)$ we may then write $-\frac{1}{b_1}\left(\frac{-\Delta b}{2b_2}\right)^2$.

If we add to this $\frac{-\Delta b}{v^2}=\frac{-\Delta b}{(2b_2)^2}$, the sum is $\frac{-\Delta b}{b_1 \cdot 2b_2}$, as $b_1+\Delta b=2b_2$.

So the equation (19^b) passes into

$$log\left[\frac{\left(p+\frac{a}{(2b_2)^2}\right)^2}{p+{}^a/_{b_1^2}}\,\frac{1}{4cT^{\gamma+1}}\right]=\frac{a}{RT}\,\frac{-\Delta b}{b_1 \cdot 2b_2}+\frac{p(-\Delta b)-q_0}{RT}.$$

Now (see IV, p. 136) the pressure of coexistence solid-liquid for $T=0$ is:

$$p_0=\frac{q_0}{-\Delta b}-\frac{a}{vv'}=\frac{q_0}{-\Delta b}-\frac{a}{b_1 \cdot 2b_2},$$

so that we may write for the second member of the above equation:

$$\frac{-\Delta b}{RT}(p-p_0).$$

Hence we get finally:

$$p-p_0=\frac{RT}{-\Delta b}\,log\left[\frac{\left(p+\frac{a}{4b_2^{\,2}}\right)^2}{p+\frac{a}{b_1^{\,2}}}\,\frac{1}{4cT^{\gamma+1}}\right], \quad . \quad . \quad . \qquad (20)$$

which equation will only hold, when really $\beta=1$ and $\beta'=0$ may be put, i. e. if we do not approach the critical point solid-liquid (see farther on) too much. If this is no longer exactly the case, (20) will hold in any case as an approximation. In the neighbourhood of $T=0$ the equation found may be considered as quite accurate.

If we write

$$log\left[\frac{\left(p+\frac{a}{4b_2^{\,2}}\right)^2}{p+\frac{a}{b_1^{\,2}}}\,\frac{1}{4c}\right]=C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\gamma)$$

C is a variable quantity on account of p. In practice, however, C will not greatly vary when p decreases from p_0 to 0, because p occurs under the sign *log*. In many cases we may, accordingly, consider C as practically constant all over the line $f(p,T)=0$, and we shall thus have obtained an exceedingly simple form for the line of coexistence solid-liquid. In our example, where $a=2700$, $b_1=1$,

$2b_2 = {}^1/_2$, $c=2$, we have e.g. for $p=0$ the quantity $C = log\, \frac{(10800)^2}{2700.8} =$ $= log\, 5400 = 8{,}594$, whereas for $p = 1100$ (the maximum pressure close to $T=0$) this quantity $= log\, \frac{(11900)^2}{3800.8} = log\, 4658 = 8{,}446$, so not quite $2\,{}^0/_0$ smaller.

Equation (20) now reduces to the very simple form:

$$p - p_0 = \frac{RT}{-\Delta b}\left[C - (\gamma + 1)\, log\, T\right], \quad . \quad . \quad . \quad . \quad (21)$$

and it is this which we shall subject to a closer examination.

18. So the expression found for $p - p_0$ is of the order $aT - \omega T\, log\, T$, which agrees with what was found in IV p. 137—138. For $\frac{dp}{dt}$ we find:

$$\frac{dp}{dt} = (a - \omega\, log\, T) - \omega = \frac{p - p_0}{T} - \omega \quad . \quad . \quad . \quad . \quad (22)$$

For $T=0$ this approaches to $+\infty$. The maximum of pressure in the neighbourhood of $T=0$ is found from $\frac{dp}{dt} = 0$, from which follows:

$$log\, T_m = \frac{a}{\omega} - 1 = \frac{C}{\gamma + 1} - 1\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

hence

$$p_m - p_0 = \omega\, T_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

on account of $a - \omega\, log\, T_m = \omega$. Thus the temperature of the maximum of pressure will be about independent of the quantity $-\Delta b$. With $C = 8{,}446$, $\gamma + 1 = {}^5/_2$ we find $log\, T_m = 2{,}378$, from which $T_m = 10^\circ{,}8$.

As $\omega = \frac{(\gamma+1)R}{-\Delta b}$, ω will be $= 10$, when $-\Delta b = {}^1/_2$, and thus $p_m - p_0 = 10\, T_m = 108$. So with $p_0 = 1000$ we find $p_m = 1108$ [1]).

Let us also calculate the temperature at which $p = p_0$ for the second time (point L in fig. 14 of the plate). Then not $T=0$, but $a - \omega\, log\, T = 0$ in the expression $p - p_0 = T(a - \omega\, log\, T)$, hence:

[1]) With regard to the units in which all these quantities are expressed, compare II p. 27.

$$log\, T' = \frac{\alpha}{\omega} = \frac{C}{\gamma + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

So this temperature too is almost independent of the value of $-\Delta b$. As $C = 8{,}456$ for $p = 1000$, we find in our case for $log\, T'$ the value 3,382, from which $T' = 29°{,}4$.

We may point out here that also from the general expression for $\frac{dp}{dt}$, viz. (see IV, p. 137; $\angle V$ is $= v - v'$)

$$\frac{dp}{dt} = \frac{\angle E}{T \angle V} = \frac{1}{T}\left[\left(p + \frac{a}{vv'}\right) + \frac{\beta - \beta'}{v - v'}(q + \gamma RT)\right],$$

the equation (22) for $\frac{dp}{dt}$ derived above follows. Now the expression $\left(p + \frac{a}{vv'}\right)(v - v')$ has been derived from $\left(-\frac{a}{v} + pv\right) - \left(-\frac{a}{v'} + pv'\right)$. In this, however, $-\frac{a}{b} + pb + \Sigma n_1 . RT$ may be written for $-\frac{a}{v} + pv$ [see inter alia p. 219 of my paper on the melting-point lines in the Z. f. physik. Ch. **63** (1908)], so that the expression mentioned becomes:

$$\left(p + \frac{a}{bb'}\right)(b - b') + (\beta - \beta')\, RT,$$

because the same holds for $-\frac{a}{v'} + pv'$, and $\Sigma n_1 = 1 + \beta$, $\Sigma n'_1 = 1 + \beta'$.

Hence:

$$\frac{dp}{dt} = \frac{1}{T}\left[\left(p + \frac{a}{bb'}\right)\frac{b - b'}{v - v'} + \frac{\beta - \beta'}{v - v'}\left(q_0 + (\gamma + 1)\, RT\right)\right].$$

By approximation $\frac{b - b'}{v - v'} = 1$ in this. Further $v - v' = b - b' =$ $= (b_1 + \beta \Delta b) - (b_1 + \beta' \Delta b) = (\beta - \beta') \angle b$, hence $\frac{\beta - \beta'}{v - v'} = \frac{1}{\Delta b}$, so that we get:

$$\frac{dp}{dt} = \frac{1}{T}\left[\left(p + \frac{a}{b_1 . 2b_2}\right) + \frac{1}{\Delta b}\left(q_0 + (\gamma + 1)\, RT\right)\right],$$

as $b = 2b_2$ and $b' = b_1$, by approximation. And as

$$\frac{a}{b_1 . 2b_2} + \frac{q_0}{\Delta b} = -p_0$$

(see above), it follows that

$$\frac{dp}{dt} = \frac{1}{T}\left(p - p_0 - \omega T\right) = \frac{p - p_0}{T} - \omega,$$

because $\omega = \frac{(\gamma+1)R}{-\Delta b}$ (see above). And so (22) has been found back.

Equation (21) may be successfully used to determine the temperature T_0 of the point where $p=0$, i. e. the *melting-point* at small pressure (to be identified in most cases with the triple-point). If we namely write:

$$p - p_0 = T\,(a - \omega \log T)\,,$$

it follows from this, when $p=0$, that:

$$-p_0 = T_0\,(a - \omega \log T_0)\,, \quad \ldots \ldots \quad (26)$$

from which T_0 can be found. As $a = \frac{RC}{-\Delta b} = 4 \times 8{,}594$ and $\omega = 10$ $T_0 = 92°{,}1$ with our data. The second member is then

$$92{,}1\,(34{,}38 - 45{,}23) = 92{,}1 \times -\ 10{,}85 = -999{,}6,$$

and the first member $= -1000$.

As the critical temperature $= 400°$, in our case

$$T_0 = 0{,}23\ T_c.$$

In general the value of $T_0 : T_c$ will depend, besides on that of Δb, also on the values of γ and c (the constant of the dissociation equation), so that the above-mentioned ratio can assume very different values. That for this ratio a value is so often found in the neighbourhood of $^1/_2$, is certainly to be attributed to an accidental concurrence of circumstances. We intend to return to this very important subject later on, viz. when we shall discuss the influence on our formulae of the association, not to double, but to multiple molecules.

It moreover follows from (22) that for T_0 the value of $\frac{dp}{dt}$ is given by the expression $-\left(\frac{p_0}{T_0} + \omega\right)$. So this duly gives a negative value in our case. That for *negative* values of Δb $\frac{dp}{dt}$ can never be positive for $T = T_0$, the possibility of which was erroneously assumed in fig. 7 of the plate of IV, follows from the fact that then necessarily a vertical tangent would have to be present somewhere in A. This is impossible, for then the denominator would have to be $\Delta V = v - v' = 0$ in the general expression for $\frac{dp}{dt}$. But if $v = v'$, it follows from the equation of dissociation that also $\beta = \beta'$ (for equal p and T). For in the equation mentioned (see p. 455), β occurs only in the first member, in the form $\frac{\beta^2}{1-\beta^2}$, so that only one definite value of

β belongs to every value of v. If, however, $v=v'$, $\beta=\beta'$, also all other quantities (energy etc.) are the same in A, and we should have to deal there with a critical end-point. Such a critical point solid-liquid can, indeed, occur, as we shall see presently; but first of all $\frac{dp}{dt}$ need not necessarily be ∞ then, and secondly the coexistence curve *terminates* then in the point A.

In order to find an expression, from which T_0 can be derived by approximation, we may also start from (19^a). With $p=0$, $v=2b_2$, $v'=b_1$, $\beta=1$, $\beta'=0$, this equation becomes:

$$2\,log\left(\frac{b_1^2}{4b_2^2}\,\frac{1}{2}\,\frac{1}{\beta'}\right)=\frac{a}{RT_0}\left(\frac{1}{2b_2}-\frac{1}{b_1}\right)\left[2-2b_2\left(\frac{1}{2b_2}+\frac{1}{b_1}\right)\right],$$

i. e.

$$RT_0=\frac{a}{2b_2}\left(\frac{-\Delta b}{b_1}\right)^2\,:\,2\,log\left(\frac{b_1^2}{4b_2^2}\,\frac{1}{2\beta'}\right). \quad . \quad . \quad (27)$$

Now as $a_2=\frac{1}{4}a_1=\frac{1}{4}a$, $RT_c=\frac{8}{27}\frac{a_2}{b_2}=\frac{8}{27}\times\frac{1}{2}\frac{a}{2b_2}$ (a_2 and b_2 refer namely to a simple molecular quantity, a, a_1, and $2b_2$ to a double molecular quantity). Hence $\frac{a}{2b_2}=\frac{27}{4}RT_c$, and so we may also write for (27):

$$\frac{T_0}{T_c}=\frac{27}{8}\left(\frac{-\Delta b}{b_1}\right)^2:\,log\left(\frac{b_1^2}{4b_2^2}\cdot\frac{1}{2\beta'}\right) \quad . \quad . \quad . \quad . \quad (27^a)$$

In this it is supposed that at T_c (here about 400°) all the molecules are entirely dissociated (according to the table in III, p. 131 this is really the case, viz. $\beta=0{,}9975$ for K), so that we may write $RT_c=\frac{8}{27}\frac{a_2}{b_2}$.

Now we can determine from (27^a) the value of β', so that the value of $T_0:T_c$ may become e.g. $=\frac{1}{2}$ for given values of $-\Delta b:b_1$. Thus e.g. with $b_1=1$, $-\Delta b=0{,}5$, $2b_2=0{,}5$ we should find the value 0,733 for $log^{10}\frac{2}{\beta'}$, from which $\beta'=0{,}37$.

From this we see that — as β' must be near 0, — only very great values of $\frac{-\Delta b}{b_1}$, greater than 0,5 [1]), can give a value in the

[1]) As we shall presently see, there is no longer a coexistence curve solid-liquid which runs on to $p=0$ for values of $-\Delta b<0{,}45$, but it terminates in a critical end-point for a higher or lower positive value of p.

neighbourhood of $^1/_2$ for $T_0 : T_c$. But as we already said the *number* of molecules associating to one will be of influence on this — which we shall investigate more closely in a following paper.

If, however, $T_0 : T_c$ is smaller than $^1/_2$, e.g. $^1/_4$, as in our example, the value of log^{10} becomes larger for $-\Delta b = 0{,}5$, viz. 1,466, and we find a much smaller value for β', viz. 0,07, agreeing with what we found before. Even for $T_0 : T_c = {}^1/_3$, for which log^{10} becomes $= 1{,}100$, we still find a rather low value for β', viz. 0,15 with $-\Delta b = 0{,}5$.

The calculation of formula (21) for different values of p now gives the following corresponding values of T. In this $-\Delta b = {}^1/_2$, $\gamma = {}^3/_2$ and $p_0 = 1000$.

$p =$ 0	$C = 8.594$	$T = 92.4$
100	8.576	87.0
200	8.560	81.7
300	8.544	76.3
400	8.529	70.8
500	8.515	65.0
600	8.502	59.0
700	8.489	52.8
800	8.478	45.9
900	8.467	38.4
1000	8.456	29.4 en 0
1050	8.451	23.8 „ 1.8
1108	8.445	10.8 „ 10.8

The course of this curve is represented in fig. 14 of the plate As we already observed in IV, p. 139—140, the point N will. approach more and more to 0 with smaller values of q_0.

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19. We now proceed to the examination of the course of the p,T-curve solid-liquid for smaller values of $-\Delta b$.

$$-\Delta b = 0{,}1 \ (b_1 = 1 \ , \ 2\, b_2 = 0{,}9).$$

The values of β are found from (see I, p. 773):

$$\frac{\beta^2}{1-\beta^2}=\lambda\,\theta^{3/2}\,e^{-1/\theta}\,\frac{e^{\varphi}}{\varphi},$$

or

$$log^{10}\frac{\beta^2}{1-\beta^2}=log^{10}\,\lambda-\left({}^3/_2\,log\frac{1}{\theta}+0,4343\,.\frac{1}{\theta}+(0,4343\,\varphi-log^{10}\,\varphi\right),\;(a)$$

in which $\lambda=\dfrac{cq_0^{\gamma}}{R^{\gamma+1}}(-\Delta b)$, $\theta=\dfrac{RT}{q_0}$, different values being successively assumed for $\varphi=\dfrac{p+{}^a/_{v^2}}{RT}(-\Delta b)$. Then the value of v is determined by (see p. 773 loc. cit.).

$$v=b_1-\left(\beta-\frac{1+\beta}{\varphi}\right)(-\Delta b)\,,\quad\ldots\ldots\quad(b)$$

the corresponding value of p being found from:

$$p=\frac{RT}{-\Delta b}\varphi-\frac{a}{v^2}\quad\ldots\ldots\ldots\ldots\quad(c)$$

On p. 774 loc. cit. we find for λ the value $64000\times{}^1/_2$. Now with $-\Delta b=0{,}1$ this value is only 6400.

For $\underline{T=0}$ (see Fig. 15) we find $p_E=\dfrac{q_0}{-\Delta b}-\dfrac{a}{(2b_2)^2}=\dfrac{3200}{0.1}-$ $-\dfrac{2700}{(0{,}9)^2}=32000-3333=28667$. Further $p_D=\dfrac{q_0}{-\Delta b}-\dfrac{a}{b_1^{\,2}}=$ $=\dfrac{3200}{0{,}1}-\dfrac{2700}{1^2}=32000-2700=29300$. The pressure of coexistence p_0 is $=\dfrac{q_0}{-\Delta b}-\dfrac{a}{b_1\,.\,2b_2}=\dfrac{3200}{0{,}1}-\dfrac{2700}{1\,.\,0{,}9}=32000-3000=\underline{29000.}$

Further $p_C=-\dfrac{a}{b_1^{\,2}}=-\dfrac{2700}{1^2}=-2700.$

As $\theta=\dfrac{1}{800}$, the equations:

$$\left.\begin{array}{l} log^{10}\dfrac{\beta^2}{1-\beta^2}=-347{,}984+0{,}4343\,\varphi-log^{10}\,\varphi \\ v=1-0{,}1\left(\beta-\dfrac{1+\beta}{\varphi}\right);\qquad p=40\,\varphi-\dfrac{a}{v^2}\end{array}\right\}$$

hold for $\underline{T=2.}$

If we examine only the course from E to D, so if we confine ourselves to such values of φ that $log^{10}\dfrac{\beta^2}{1-\beta^2}$ varies between about $+2$ and about -2, we get the following table.:

$T = 2$

φ	log^{10}	β	v	$a/_{v^2}$	p
812	1.754	0.992	0.901	3326	29154
810	0.886	0.941	0.906	3289	29111 (E)
808	0.018	0.716	0.929	3131	29189
806	—0.849	0.352	0.965	2899	29341
804	—1.717	0.137	0.986	2775	29385 (D) ($\beta = 0.145$)
802	—2.584	0.051	0.995	2727	29353

So there is still a distinct minimum at E and a maximum at D, though the distance between the two extreme pressure values, which still amounted to 633 for $T = 0$, has now already decreased to 274.

For $T = 5$, where $\theta = \frac{1}{320}$, we have the equations:

$$log^{10} \frac{\beta^2}{1 - \beta^2} = - 138,926 + 0,1343\ \varphi - log^{10}\ \varphi\ ;\quad p = 100\ \varphi - \frac{a}{v^2}.$$

This gives the following values:

$T = 5$

φ	log^{10}	β	v	$a/_{v^2}$	p
328	1.006	0.954	0.905	3295	29505
326	0.141	0.762	0.924	3166	29434 (E) ($\beta = 0.80$)
324	—0.726	0.398	0.961	2926	29474 (D) ($\beta = 0.39$)
322	—1.591	0.158	0.985	2785	29415

The distance is only 40 units, and it is to be expected that at somewhat higher temperature the points D and E will coincide to a horizontal point of inflection. Above this temperature we shall have a gradual decrease from $p = \infty$ to $p = p_C$ on the isotherm, so that the coexistence of solid-liquid has become impossible.

This *critical point* solid-liquid lies at 6°,2, so still below the temperature of the maximum in the p, T-line, which is 10°,8 (see § 18).

Below this point there is, therefore, a *continuous* transition from the liquid to the solid state, when the temperature is lowered. The liquid will gradually become more viscous, finally assume the glassy amorphous state — and in this it will entirely depend on the mutual situation of the particles, whether eventually *crystallisation* sets in, i.e. whether the irregular situation of the molecule-complexes passes into a regular crystalline structure. But however it be: there is no abrupt change in the solidification, it takes place quite continuously.

According to the above this behaviour may be expected for all substances where $-\Delta b$ has a *low* value. No *distinct melting-point* is then found for the ordinary pressures with abrupt changes of the properties.

For higher value of $-\Delta b$, e.g. 0,2, 0,3 etc. this critical point solid-liquid moves to ever lower pressures, but yet there remains a considerable region of pressure, where *continuity* of the solid and liquid state exists, as we shall see in what follows.

The temperature of the critical point can be calculated by approximation in the following way. According to I p. 778, we have approximately in the points D and E, when viz. φ is great enough to allow us to write $^1/_2\,\beta(1-\beta)\,\varphi^2$ for $1+{}^1/_2\,\beta(1-\beta)(1-\varphi)^2$:

$$\frac{2a}{v^3}=\frac{RT}{{}^1/_2\,\beta(1-\beta^2)(-\Delta b)^2}.$$

So the two values of β may be calculated from:

$$\frac{\beta(1-\beta^2)}{(b_1+\beta\Delta b)^3}=\frac{RT}{a(-\Delta b)^2},$$

when namely $v=b$ may be put, while $b=b_1+\beta\Delta b$. With $R=2$, $a=2700$, $b_1=1$, $-\Delta b=0{,}1$ this becomes:

$$\frac{\beta(1-\beta^2)}{(1-0{,}1\beta)^3}=\frac{2}{27}\,T.$$

Now the two values of β, which can be solved from this for different values of T, coincide when the first member has reached its maximum value, i.e. when $\beta=0{,}612$. The fraction then becomes $=0{,}463$, and we have $T_c=\frac{27}{2}\times 0{,}463=6°{,}25$. If we take the neglected influence of φ into account, this value becomes only slightly less, viz. 6°,2.

For $-\Delta b=0{,}1$ we now find ($p=$ pressure of coexistence solid-liquid):

$-\Delta b = 0{,}1$

T	p	β	β'	v	v'
0	29000	1	0	0.90	1
2	29270	0.1	0.02	0.90	1
5	29460	0.81	0.26	0.91	0.97^5
$T_c = 6.2$	29500	0.61	0.61	0.94	0.94

The values of β and v have been found by interpolation; those of p by taking the average of the pressures in the maximum and the minimum.

20. Let us now proceed to the calculation of the case

$$\underline{-\Delta b = 0{,}2} \; (b_1 = 1, \; 2b_2 = 0{,}8).$$

For λ we find now $64000 \times 0{,}2 = 12800$.

For $\underline{T=0}$ (see Fig. 16) $p_0 = \frac{3200}{0{,}2} - \frac{2700}{1 \times 0{,}8} = 16000 - 3375 =$ $= \underline{12625}$; $p_E = 16000 - \frac{2700}{(0{,}8)^2} = 11781$; $p_D = 16000 - \frac{2700}{1^2} = 13300$.

For $\underline{T=10}$ we have with $\theta = \frac{1}{160}$ for β, v, and p the equations

$$\left.\begin{array}{l} \log^{10} \dfrac{\beta^2}{1-\beta^2} = -63{,}685 + 0{,}4343\,\varphi - \log^{10}\varphi \\ v = 1 - 0{,}2\left(\beta - \dfrac{1+\beta}{\varphi}\right); \quad p = 100\varphi - \dfrac{a}{v^2} \end{array}\right\},$$

from which we calculate:

$T = 10$

φ	$\log^{10}$	β	v	a/v^2	p
168	2.050	0.995	0.803	4183	12612
166	1.186	0.969	0.809	4130	12470 *(E)*
164	0.323	0 823	0.838	3848	12552
162	−0.541	0.473	0.907	3281	12919
160	−1.403	0.195	0.962	2915	13085 *(D)*
158	−2.267	0.073	0.987	2773	13027

The distance between maximum and minimum pressure has decreased from 1519 units (at $T = 0$) to 615 units.

For $\underline{T=20}$ we have with $\theta = \frac{1}{80}$ the equations:

$$log^{10} \frac{\beta^2}{1-\beta^2} = -33,491 + 0,4343\, \varphi - log^{10}\, \varphi \; ; \; p = 200\, \varphi - \frac{a}{v^2} \, .$$

From this follows:

$T = 20$

φ	log^{10}	β	v	a/v^2	p
84	1.066	0.960	0.813	4085	12715
82	0.207	0.785	0.847	3766	12634 *(E)*
80	—0.650	0.428	0.918	3202	12798 *(D)*
78	—1.508	0.174	0.968	2881	12719

The distance between E and D amounts only to 154 units. We can again calculate by approximation that the two coinciding values of β occur at $\beta = 0,648$, the maximum value of the fraction $\frac{\beta(1-\beta^2)}{(1-0,2\,\beta)^3}$ now being 0,570. Then 30,8 is further found for the critical temperature, which value is however too high, and falls to about 29° in consequence of the influence of φ (see above).

So we have the following survey for $-\Delta b = 0,2$.

$-\Delta b = 0,2$

T	p	β	β'	v	v'
0	12625	1	0	0.80	1
10	12800	1	0.02	0.80	1
20	12720	0.97	0.19	0.81	0.965
$T_c = 29$	12640	0.65	0.65	0.88	0.88

This table has been calculated in the same way as the corresponding one in § 19.

21. Let us now consider the case:

$$\underline{-\Delta b = 0,3}\; (b_1 = 1\,,\; 2b_2 = 0,7)$$

The value of λ is $64000 \times 0,3 = 19200$.

For $\underline{T=0}$ (see Fig. 17) we find $p_0 = \frac{3200}{0,3} - \frac{2700}{1 \times 0,7} =$

$= 10667 - 3857 = \underline{6810}$. Further $p_E = 10667 - \dfrac{2700}{(0,7)^2} = 5157$;
$p_D = 10667 - 2700 = 7967$.

For $\underline{T = 50}$ we have with $\theta = \dfrac{1}{32}$:

$$\left. \begin{aligned} log^{10} \frac{\beta^2}{1-\beta^2} &= -11,872 + 0,4343\,\varphi - log^{10}\,\varphi \\ v = 1 - 0{,}3\left(\beta - \frac{1+\beta}{\varphi}\right); &\quad p = \frac{1000}{3}\varphi - \frac{a}{v^2} \end{aligned} \right\},$$

from which we calculate:

$T = 50$

φ	log^{10}	β	v	a/v^2	p
35	1.784	0.992	0.720	5215	6452
33	0.941	0.947	0.734	5017	5983 *(E)*
31	0.100	0.747	0.793	4294	6039
29	—0.739	0.393	0.897	3360	6307 *(D)*
27	—1.577	0.161	0.965	2902	6098

The difference of pressure between maximum and minimum has decreased from 2810 units at $T = 0$ to 324 units.

For $\underline{T = 75}$ we have with $\theta = \dfrac{3}{64}$:

$$log^{10} \frac{\beta^2}{1-\beta^2} = -6,975 + 0,4343\,\varphi - log^{10}\,\varphi\,; \quad p = 500\,\varphi - \frac{a}{v^2}.$$

From this we calculate:

$T = 75$

φ	log^{10}	β	v	a/v^2	p
20	0.410	0.849	0.773	4520	5480
19.5	0.203	0.784	0.792	4300	5450
19	—0.003	0.706	0.815	4063	5437
18	—0.413	0.528	0.867	3951	5409

So the critical point lies somewhat below 75°.
Hence we have for $-\Delta b = 0,3$:

$$-\Delta b = 0{,}3$$

T	p	β	β'	v	v'
0	6810	1	0	0.70	1
50	6150	0.96	0.22	0.73	0 95
$T_c = 75$	5440	0.70	0.70	0.82	0.82

The values for 75° have only been given by approximation.

22. Now we calculate the case:

$$\underline{-\Delta b = 0{,}4} \quad (b_1 = 1,\ 2b_2 = 0{,}6).$$

For λ we have to put the value $64000 \times 0{,}4 = 25600$.

For $\underline{T = 0}$ (see Fig. 18) $p_0 = \dfrac{3200}{0{,}4} - \dfrac{2700}{1 \times 0{,}6} = 8000 - 4500 = \underline{3500.}$

Further $p_E = 8000 - \dfrac{2700}{(0{,}6)^2} = 500$; $p_D = 8000 - 2700 = 5300$. As from this moment the minimum near C will lie in the neighbourhood of the critical point solid-liquid (E, D), we may point out that for $T = 0$ $p_C = -\dfrac{a}{b_1{}^2} = -2700$.

For $\underline{T = 50}$ we find with $\theta = \dfrac{1}{32}$:

$$\left.\begin{aligned} log^{10} \frac{\beta^2}{1-\beta^2} &= -11{,}747 + 0{,}4343\,\varphi - log^{10}\,\varphi \\ v = 1 - 0{,}4\left(\beta - \frac{1+\beta}{\varphi}\right); &\quad p = 250\,\varphi - \frac{a}{v^2} \end{aligned}\right\},$$

and from this the following table is calculated.

$$T = 50$$

φ	log^{10}	β	v	a/v^2	p
35	1.909	0.994	0.625	6907	1843
33	1.066	0.960	0.640	6609	1650 *(E)*
31	0.225	0.792	0.706	5411	2339
29	−0.614	0.442	0.843	3797	3453
27	−1.452	0.185	0.944	3032	3718 *(D)*
25	−2.288	0.071	0.989	2763	3487
5	—	0	1.080	2315	−1065
3	—	0	1.133	2103	−1353 *(C)*
1	—	0	1.400	1378	−1128

The distance between E and D has decreased from 4800 units at $T=0$ to 2068 units.

For $\underline{T=100}$ we find with $\theta=\frac{1}{16}$:

$$log^{10}\frac{\beta^2}{1-\beta^2}=-4{,}347+0{,}4343\,\varphi-log^{10}\,\varphi\;;\quad p=500\,\varphi-\frac{a}{v^2},$$

from which we calculate:

$T=100$

φ	log^{10}	β	v	$^a/_{v^2}$	p
16	1.398	0.981	0.657	6251	1749
14	0.587	0.891	0.698	5549	1451 *(E)*
12	—0.214	0.616	0.808	4140	1860
10	—1.004	0.300	0.932	3108	1892 *(D)*
8	—1.775	0.128	1.005	2672	1328
4	—3.212	0.025	1.093	2262	—262
2	—3.779	0.013	1.197	1883	—883 *(C)*
1	—3.912	0.011 (min.)	1.400	1378	—878

The difference in pressure between E and D is now only 451 units. As we already observed in II, p. 28, the minimum value of β is always found at $\varphi=1$.

For $\underline{T=128}$ we find with $\theta=\frac{2}{25}$:

$$log^{10}\frac{\beta^2}{1-\beta^2}=-2{,}666+0{,}4343\,\varphi-log^{10}\,\varphi\;;\quad p=640\,\varphi-\frac{a}{v^2},$$

and from this we calculate:

$T=128$

φ	log^{10}	β	v	$^a/_{v^2}$	p
10	0.677	0.909	0.723	5168	1232
9	0.289	0.812	0.756	4730	1030 *(E)*
8	—0.094	0.668	0.816	4051	1069 *(D)*
6	—0.838	0.356	0.948	3004	836
2	—2.098	0.089	1.182	1932	—652
1	—2.232	0.076 (min.)	1.400	1378	—738 *(C)*
0.5	—2.146	0.084	1.834	803	—483

So the critical point solid-liquid is found only very little above 128°. Now we have the following survey for $-\Delta b = 0{,}4$.

$-\Delta b = 0{,}4$

T	p	β	β'	v	v'
0	3500	1	0	0.60	1
50	2900	1	0.02	0.61	1
100	1700	0.96	0.23	0.66[5]	0.97
$T_c = 128$	1050	0.74	0.74	0.79	0.79

For 128° the minimum at C lies at $p = -740$.

23. Now the question rises what takes place between $-\Delta b = 0{,}4$ and $-\Delta b = 0{,}5$. For with $-\Delta b = 0{,}4$ we have still concurrence of the solid and the liquid phase at a certain temperature in consequence of the coincidence of the minimum E in a horizontal point of inflection with the maximum D (See also fig. 20). But for $-\Delta b = 0{,}5$ we have, as we saw in our previous papers, coincidence of the maximum D with the minimum C, whereas now at higher temperatures the minimum E continues to exist, just as at smaller values of $-\Delta b$ the minimum C continues to exist after the coincidence of E with D. So there must exist a value of $-\Delta b$ somewhere between $-\Delta b = 0{,}4$ and $0{,}5$, for which *at the same time* the points E, D, and C coincide in one contact of higher order. For lower values of $-\Delta b$ we have the case that only E and D coincide in a critical point solid-liquid Cr, above or below the curve of coexistence vapour-liquid, and that C continues to exist [1]); on the other hand for higher values of $-\Delta b$ the case will present itself that only D and C coincide, and E continues to exist. In the latter case we have a point of inflection but evidently no *critical* point *solid-liquid*. In the

[1]) As at the point N of the curve $NMCr$ (Fig. 20) the two phases solid and liquid diverge as much as possible in their properties ($\beta = 1$, $v = 2b_2$; $\beta' = 0$, $v' = b_1$), whereas in the critical point the phases become identical, we see clearly, that when from the triple point S we proceed to *higher* pressures, the phases *diverge*, and do not gradually assimilate, as this is the case for Δb *positive*, as we shall demonstrate in the following paper. So even at the highest pressures there is no vertical tangent possible in the melting-point line NMS (see also § 18), for there necessarily v would have to be $= v'$, so also $\beta = \beta'$.

latter case the curve NMS (see fig. 14 and 20), however, will not terminate in the point where this concurrence D, C takes place, but already before (at negative pressure) in a point P, where the coexistence curve $V_l—V_r$ would still just touch the branch DC in the point C. [Something similar takes also place with the prolongations of the curves OS and KS through the point O].

This transition takes place very near $-\Delta b = 0{,}455$, as we shall show in what follows.

$-\Delta b = 0{,}46 \quad (b_1 = 1,\ 2\, b_2 = 0{,}54).$

The value of λ is here $64000 \times 0{,}46 = 29440$. Let us determine the values of p of the isotherm of 160°, i. e. those values which lie in the neighbourhood of the points E, D, and C. With $\theta = \frac{1}{10}$, the following formulae hold:

$$\left.\begin{aligned} \log^{10} \frac{\beta^2}{1-\beta^2} &= -1{,}374 + 0{,}4343\,\varphi - \log^{10}\varphi \\ v = 1 - 0{,}46 \left(\beta - \frac{1+\beta^2}{\varphi}\right); &\quad p = 695{,}7\,\varphi - \frac{a}{v^2} \end{aligned}\right\},$$

from which we calculate (see fig. 19):

$T = 160$

φ	$\log^{10}$	β	v	a/v^2	p
8	1.197	0.970	0.667	6069	—504
7	0.821	0.932	0.698	5539	—669 (E)
6	0.454	0.860	0.747	4840	—666
5	0.098	0.746	0.817	4041	—563
4	—0.239	0.605	0 906	3288	—505 (D)
3	—0.548	0.470	1.009	2651	—564
2	—0.806	0 368	1.145	2054	—663 (C)
1	—0.940	0.321 (min.)	1.460	1267	—571

So it is not yet clear at this temperature whether E and D, or D and C will coincide. Let us now calculate the isotherm of 170°. With $\theta = \frac{17}{160}$ we have:

$$\log^{10} \frac{\beta^2}{1-\beta^2} = -1{,}079 + 0{,}4343\,\varphi - \log^{10}\varphi; \quad p = 739{,}1\,\varphi - \frac{a}{v^2},$$

and from this follows the subjoined table.

$T = 170$

φ	log^{10}	β	v	$^{a}/_{v^2}$	p
6	0.749	0.921	0.724	5158	—723
5	0.393	0.844	0.782	4421	—725 (*E*)
4	0.056	0.730	0.863	3626	—669
3	—0.253	0.5[illegible]9	0.970	2872	—655 (*D*)
2	—0.511	0.485	1.119	2157	—679 (*C*)
1	—0.645	0.430 (min.)	1.460	1267	—528

Now it is clear that D and C will coincide, and that $-\Delta b = 0{,}46$ is, therefore, past the transition value. The coincidence of D and C takes place at 174°. Then $\theta = \frac{174}{160}$ and we have:

$$log^{10}\frac{\beta^2}{1-\beta^2} = -\,0{,}970 + 0{,}4343\,\varphi - log^{10}\,\varphi \;; \quad p = 756{,}5\,\varphi - \frac{a}{v^2},$$

from which we calculate:

$T = 174$

φ	log^{10}	β	v	$^{a}/_{v^2}$	p
6	0.858	0.937	0.718	5243	—704
5	0.502	0.872	0.771	4543	—760 (*E*)
4	0.165	0.771	0.849	3744	—718
3	—0.144	0.646	0.955	2958	—688 } *D,C*
2	—0.402	0.533	1.107	2203	—690 } *D,C*
1	— 0.536	0.475 (min)	1.460	1267	—510

And from this the coincidence of D and C appears immediately, while the minimum E continues to exist.

It is now very probable that the transition in question takes place at $-\Delta b = 0{,}455$, for the pressure at $E\,(= -760)$ is no longer very far from that of the coinciding points D and C $(= -690)$.

$$-\Delta b = 0{,}455\ (b_1 = 1,\ 2b_2 = 0{,}545).$$

For λ we have $64000 \times 0{,}455 = 29120$. Let us first calculate the isotherm of 175°. With $\theta = \frac{175}{1600}$ we have:

$$\left.\begin{array}{l} \log^{10} \dfrac{\beta^2}{1-\beta^2} = -0{,}948 + 0{,}4343\,\varphi - \log^{10}\varphi \\ v = 1 - 0{,}455\left(\beta - \dfrac{1+\beta}{\varphi}\right); \quad p = 769{,}2\,\varphi - \dfrac{a}{v^2} \end{array}\right\},$$

and this yields the following table.

$T = 175$

φ	$\log^{10}$	β	v	a/v^2	p
6	0.880	0.940	0.719	5220	—605
5	0.524	0.877	0.772	4535	—689 (*E*)
4	0.187	0.778	0.848	3755	—678
3	—0.122	0.656	0.953	2975	—667 (*D*)
2	—0.389	0.542	1.104	2215	—677 (*C*)
1	—0.514	0.484 (min.)	1.455	1275	—506

So $-\Delta b = -0{,}455$ is still above the transition value, for it is already clear from the above table, that at slightly higher temperature D and C will coincide, and not E and D. Really we find at 177° $\left(\theta = \dfrac{177}{1600}\right)$ with:

$$\log^{10} \frac{\beta^2}{1-\beta^2} = -0{,}896 + 0{,}4343\,\varphi - \log^{10}\varphi\,; \quad p = 778{,}0\,\varphi - \frac{a}{v^2}:$$

$T = 177$

φ	$\log^{10}$	β	v	a/v^2	p
5	0.576	0.889	0.768	4583	—693
4	0.239	0.797	0.842	3808	—696 (*E*)
3	—0.070	0.678	0.946	3018	—684
2	—0.328	0.566	1.099	2236	—680
1	—0.464	0.506 (min.)	1.455	1275	—497

Now we find easily by interpolation, that the temperature of transition amounts to 176°,4, for which $p_E = -694$ and $p_{D,C} = -679$.

Now for $-\Delta b = 0{,}46$ the distance between p_E and $p_{D,C}$ still amounted to as much as 70 units; this distance, however, is no more than 15 units for $-\Delta b = 0{,}455$, and we find by interpolation that the transition value of $-\Delta b$ will amount to

$$-\Delta b = 0{,}454,$$

for which at the critical temperature 177°,0 the points E, D, and C will coincide at a pressure of —676 units.

Finally summarizing what has been found with regard to the critical points, we have the following survey:

$-\Delta b$	$T_c = T_{E,D}$	p	
0.1	6.2	$p_{E,D} = 29500$	
0.2	29	12630	
0.3	75	5440	
0.4	128	1050	$p_C = -740$
0.454	177.0	$p_{E,D,C} = -676$	
	$T_{D,C}$		
0.455	176.4	$p_E = -694$	$p_{D,C} = -679$
0.46	174	-760	-690
0.5	160	-1540	-755

So when $-\Delta b$, i.e. the difference between the volume of a quantity of double molecules b_1 and the volume of the equivalent quantity of simple molecules $2b_2$, has reached a sufficient value, there is a chance of a continuous melting-point line as in Fig. 14. But for slight values of $-\Delta b$, in our example from 0,1 to 0,4, this line necessarily ends in a *critical point* solid-liquid [1]); hence at the usual pressures a *continuous* transition will take place from the liquid state to the solid state, when the temperature is lowered. Then a melting point in the strict sense does not exist, unless at very high pressures — and so we have got to know a new cause why a great number of substances become solid without a clearly defined point of transition where the properties undergo an abrupt change. So this behaviour is easily explained from what was said in IV, p. 140—141, and in what was discussed now.

In a following Paper the case Δb *positive* will be treated, and moreover the influence will be discussed of the coincidence of more than one molecule to one complex molecule.

[1]) It is self-evident that for $-\Delta b = 0$ this critical point will lie at $T = 0$, and that then coexistence solid-liquid is no longer possible at all, not even at the highest pressures.

Botany. — *"The influence of temperature on the presentation-time in geotropism."* By Dr. A. A. L. Rutgers. (Communicated by Prof. F. A. F. C. Went).

(Communicated in the meeting of Sept. 24, 1910).

§ 1. *Introductory.*

In 1905 and 1908 there appeared two papers [1]) by Blackman in which he dealt with the influence of temperature on physiological processes in general, while in addition in the first of these papers he tested on a special case his new views on this subject and showed that the results arrived at by Miss Matthaei on assimilation as a function of temperature [2]) confirmed his theory.

One of the chief points in Blackman's argument is the proposition that van 't Hoff's law of reaction velocity as a function of temperature must also hold good in the field of physiology. According to this law the reaction, for certain chemical transformations increases two- to three-fold for every 10° C. rise of temperature. The connection between temperature and a physiological process is in general represented by a curve with an inversion-point, the so-called optimum curve. Blackman maintains that the inversion-point owes its origin to secondary influences, that in consequence this optimum does not express a primary relation which universally holds good between temperature and a physiological process.

With the aid of the figures available for this purpose, Blackman shows further that in general the law of van 't Hoff applies in the field of botany for temperatures roughly between 10° C. and 27° C. Above 27° C. a quick falling off takes place, so that at higher temperatures the values obtained do not nearly reach those which might be expected, if calculated by van 't Hoff's law.

Blackman in his explanation of this phenomenon lays stress on a new point of view, calling attention to the time-factor which here comes into play. With higher temperatures too low a value is found in consequence of the harmful influence of such temperatures. The longer the plant remains exposed to these harmful temperatures, the greater is the damage. So also conversely the shorter the time they remain at this temperature, the less is the

1) F. F. Blackman, Optima and Limiting Factors, Annals of Botany, Vol. XIX, 1905.
F. F. Blackman, Opening Address of the Botanical section of the British Association, Nature, Vol. 78. 1908.

2) G. L. C. Matthaei, Experimental Researches on Vegetable Assimilation. Phil. Trans. Series B, Vol. 197, 1905.

harm done. Blackman holds that according to van 't Hoff's law the theoretical value would be found, if only an observation could be made after an exposure of 0 minutes to the higher temperature. This value after time 0 cannot however be experimentally determined and so Blackman has recourse to extrapolation from the curve which can be drawn through the points representing the values obtained after an exposure to the higher temperature of shorter and shorter duration. In this way by extrapolating the time curves obtained by Miss Matthaei for assimilation at high temperatures, Blackman indeed finds values which fairly well agree with those calculated according to van 't Hoff's law.

From these considerations it also follows that the optimum must vary with the time of observation. If the subject of the experiment is warmed for a short time only before the observation, the optimum will be found at a higher temperature than after longer warming.

Although the author is evidently convinced that his theories will have to apply over the whole field of plant physiology, there are nevertheless processes to which he has not yet been able to extend his conclusions, at least at the end of his second paper he says: "Finally superposed upon all this comes the first category of phenomena that we are content still to regard as stimulatory." "From our present point of view vision does not extend to the misty conceptions of stimulation upon our horizon".

In the investigation of which a preliminary account is here given, an attempt is also made to apply the ideas developed by Blackman to the field of pure physiology of stimulus and to test their general validity experimentally.

§ 2. *Methods.*

In order to determine the influence of temperature in connection with the time-factor, the experimental objects (coleoptiles of *Avena sativa*) were kept before and during the experiments for a definite time at that temperature of which the influence had to be determined. After having been warmed for a certain time the oat-seedlings were stimulated for some minutes by means of gravity at an angle of 90° and were afterwards placed vertically at a temperature at 20° C. In this way the presentation-time for temperatures between 0° C. and 40° C. was determined after various periods of warming. The warming took place in a thermostat specially constructed for this purpose which was electrically warmed and kept at a constant temperature by means of an electrical regulator, so that there was no need to use gas for the experiments as it considerably impairs the power of geotropic curvature.

All the experiments took place in the excellently fitted dark room of the Botanical Laboratory at Utrecht under the direction of Prof. WENT. The most important source of error was in the difficulty of keeping the air in the laboratory quite pure and in the individual variations of the objects of experiment. Great care was bestowed on the elimination of these sources of error, in the first place by keeping the atmosphere as pure as possible and further by using for every experiment as great a number of plants as possible.

Determinations were made at temperature-intervals of 5° C. and at each temperature after warming for 1, 2, 4, 6, 12, and 24 hours, unless it was evident from the experiments that the time-factor was absent, when two determinations sufficed.

§ 3. *Results.*

The results of this investigation are summarised in the following table. The horizontal rows give the values of the presentation-times

SUMMARY OF PRESENTATION-TIMES.

Temperature.	Warming for 1 hour	Warming for 2 hours	Warming for 4 hours	Warming for 6 hours	Warming for 12 hours	Warming for 24 hours
0°	72′	—	72′	—	—	—
5°	16′	—	16	—	—	—
10°	10′40″	—	10′40″	—	—	—
15°	6′	—	6′	—	—	—
20°	4′20″	—	—	[1]) 4′10″	—	—
25°	2′20″	2′20″	2′20″	2′20″	—	—
30°	3′30″	3′10″	2′10″	1′50″	1′40″	1′40″
35°	2′30′	3′30″	4′	4′	[2]) 5′	5′
37°	9′20″	16′	—	21′40″	—	21′40″
38°	11′30″	19′10″	38′	53′	75′	347′
39°	23′	40′	—	—	—	—
40°	260′	—	—	—	—	—

1) After warming for 8 hours. 2) After warming for 18 hours.

corresponding to the temperature at the beginning of the row. The length of warming is given at the top of the vertical columns.

According to this table the presentation-time shows a clear dependence on temperature, while at higher temperatures, the length of warming is evidently of great importance.

If we now ask how far VAN 'T HOFF's law holds good, we cannot simply take the ratio of the presentation-time for the determination of the temperature-coefficient. The presentation-time is not itself a chemical process, but can serve as the measure for the perception process. If the rate of this process is greater, then the presentation time will be less, and conversely.

For the determination of the temperature-coefficient, we shall therefore be obliged to take the reciprocal values of the presentation-time, or, which comes to the same thing, instead of $\frac{K_{20}}{K_{10}}$, the reciprocal $\frac{K_{10}}{K_{20}}$.

We then find:

$$\frac{K_0}{K_{10}} = \frac{72'}{10'40''} = 6.8\,, \quad \frac{K_5}{K_{15}} = \frac{16'}{6'} = 2.6, \quad \frac{K_{10}}{K_{20}} = \frac{10'40''}{4'20''} = 2.5\,,$$

$$\frac{K_{15}}{K_{25}} = \frac{6'}{2'20''} = 2.6\,, \quad \frac{K_{20}}{K_{30}} = \frac{4'20''}{1'40''} = 2.6\,, \quad \frac{K_{25}}{K_{35}} = \frac{2'20''}{2'30''} = 0.93\,,$$

$$\frac{K_{30}}{K_{40}} = \frac{1'40''}{260'} = 0.0064\,.$$

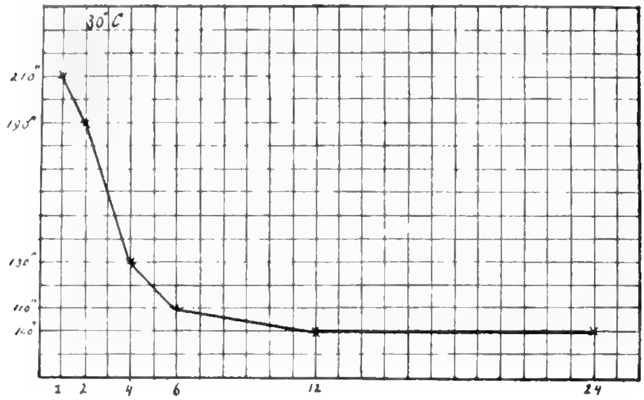
As is evident from these coefficients VAN 'T HOFF's law for the presentation-time holds good in geotropism from 5° C. to 30° C. At 0° C. we notice a sudden increase of the presentation-time, through which the temperature coefficient between 0° C. and 10° C. reaches the unusually high value 6.8. Possibly this is connected with the cessation of growth at 0° C.

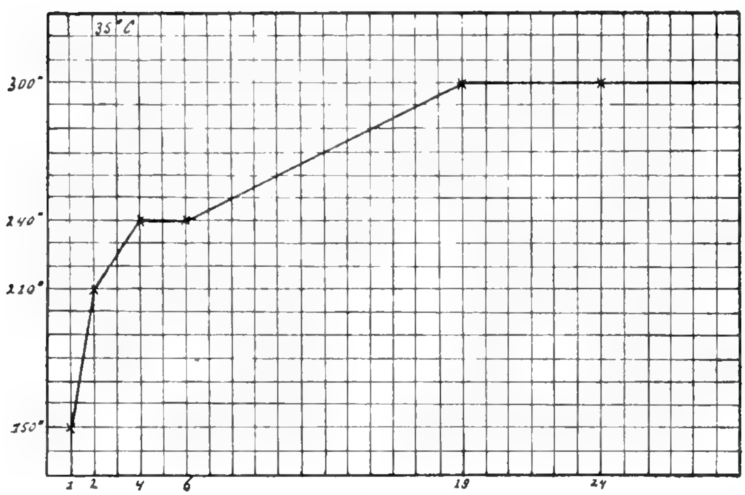
The above table also gives a very good idea of the significance of the length of the previous warming. From 0° C. to 25° C. no influence of the length of warming can be traced, at 30° C. and higher the time-factor, in BLACKMAN's sense, plays an important part.

The accompanying figures represent graphically the change in the presentation-time at 30° C. and 35° C., as a function of the time f warming. The most remarkable thing about these more or less ogarithmic curves, is the fact that the presentation-time at 30° C. ecreases and at 35° C. increases. This therefore means that at

30° C. the presentation-time decreases under the influence of the longer warming, at 35° C., however, it increases.

The temperature of 35° C. has, therefore, a distinctly injurious influence, while the favourable influence of the temperature of 30° C. appears to be a function of time.





Nevertheless there also seems to be a harmful influence acting at 30° C. This is evident when we compare the values obtained after warming for 1 hour at 30° C. with the corresponding values at 25° C. The latter is 2′20″, the former 3′30″, which means that in the first hour at a temperature of 25° C., the presentation-time fell from 4′20″ at 20° C. to 2′20″ at 25° C., and that in the first hour at a temperature of 30° C. there was only a decline from

4′20″ at 20° C. to 3′30″ at 30° C. Thus there is here clearly a harmful influence at work which was only gradually overcome by the favourable influence of this temperature.

We might represent this influence in the following way, that for the greater rate of transformation at 30° C., a greater quantity of an enzyme is required. The first effect of this temperature is the destruction of a quantity of enzyme and only gradually a sufficient quantity of the enzyme is formed, in order to accelerate the process and so to obtain a smaller presentation-time. Whether we are indeed concerned with the action of an enzyme, cannot, however, be made out.

One may readily assume that not only at 30° C., but also at higher temperatures, the favourable influence of these temperatures which expresses itself by a shortening of the presentation-time, acts only gradually. Only at 35° C. and higher temperatures this phenomenon can no longer show itself in consequence of the much stronger opposite influence of these temperatures. In one way alone can the fact still be traced that the shortening of the presentation-time at increased temperatures does not immediately occur, namely, that after one or more hours previous warming, those values which we should expect according to BLACKMAN's theory, are not found. This is indeed the case. Extrapolation from the time-curves does not here give the values for time 0, which, according to VAN 'T HOFF's law, we could calculate for it from the values found at lower temperatures. Nor can this be, if the favourable influence of the higher temperature is a function of time, for then this theoretical value after a time 0 does not exist, but the starting-point of the time-curve for a time 0 lies at a higher value of the presentation-time.

The same circumstance explains also the fact that the optimum here is only in very slight degree variable with the time of observation. After 1 hour's previous warming, we find a not very distinct optimum at 22° C., after 12 hours' previous warming it is shifted to 30° C.

The whole course of the presentation-time as a function of temperature and of time of previous warming, is represented in Plate I. The thick continuous line is the presentation-time at varying temperatures after 1 hours' previous warming.

From 30° C. upwards this line is continued by an interrupted line which connects the points calculated by VAN 'T HOFF's law, starting from the values found at lower temperatures. The above plate also gives the time-curves which, for temperatures of 30° C.

upwards, show the connection between the presentation-time and the period of previous warming. For this purpose the abscissae axis has been taken as time axis and for each temperature the ordinate of that temperature as starting-point. The dotted lines with which these time-curves begin connect the values, found after 1 hour's previous warming, with the values calculated according to VAN 'T HOFF's law.

§ 4. *Comparison with the results of previous investigators.*

In two directions the results of this investigation lead to a comparison with previous work. In the first place we must consider to what extent earlier papers on the influence of temperature on the presentation-time in geotropism are confirmed by this investigation and in the second place the results of this inquiry must be used to ascertain the correctness of BLACKMAN's theory.

CZAPEK [1]) and BACH [2]), the former with germinating roots of Lupinus albus, the latter with seedlings of Vicia Faba,, have examined the influence of temperature on presentation-time. CZAPEK found in this way a falling of the presentation-time from 0° C. to 15° C., from 15° C. to 30° C. it was constant, after which up to 40° C. there was a rise. BACH found from 14° C. to 30° C. a continuous decrease above 30° C. a rise in the presentation-time. Thus both found in the main the same curve, because the stationary character of the presentation-time between 10° C. and 30° C. in CZAPEK's experiments must no doubt be attributed to secondary influences.

In CZAPEK's and BACH's work there are also a few indications that VAN 'T HOFF's law applies, although their observations are not complete enough to attach great value to their figures from this point of view. From CZAPEK's figures we can calculate: $\frac{K_5}{K_{15}} = \frac{45'}{20'} = 2{,}25$, and from BACH's figures: $\frac{K_{20}}{K_{30}} = \frac{7'.5}{2'} = 3.75$, values which make it appear not improbable that also with the objects of experiment used by them, if the investigation were more complete, VAN 'T HOFF's law would be found operative.

There have been only a few investigations since the appearance of BLACKMAN's first published paper, in which his above-mentioned views have been taken into account. In 1907 SMITH [3]) mentioned in a few lines that in Hydrilla verticillata the intensity of respiration

[1]) F. CZAPEK. Weitere Beiträge zur Kenntniss der geotropischen Reizbewegungen. Jahrb. f. wiss. Botan., Bd. XXXII, 1898.

[2]) H. BACH. Ueber die Abhängigkeit der geotropischen Präsentations- und Reaktionszeit von verschiedenen Aussenbedingungen. Jahrb. f. wiss. Botan., Bd. XLIV, 1907.

[3]) A. M. SMITH. Respiration of Hydrilla verticillata. Proceedings of the Cambridge Phil. Soc. Vol. XIV, 1907.

rose from 7° C. to 50° C. according to VAN 'T HOFF's law with a coefficient 2.2 for every 10° C. rise of temperature.

In 1908 BALLS [1]) published figures on the growth of fungus hyphae, from which he concluded that between 15° C. and 30° C. the growth in this case followed VAN 'T HOFF's law.

In 1909 KUYPER [2]) published a detailed account of the influence of temperature on respiration and came to the conclusion that BLACKMAN's theory is only partly applicable to respiration.

Up to 10° C. the same quantity of CO_2 is expired in successive hours and from 10° C. to 20° C. there is a slight increase during successive hours, then there follows a period in which the production of CO_2 oscillates, while above 40° C. a regular decrease takes place which graphically represented gives an almost logarithmic curve. VAN 'T HOFF's law holds good for Pisum and Triticum at 0°—20° C., for Lupinus up to 25° C.; the coefficient for a rise of 10° C. of temperature lies between 2 and 3. The optimum is variable with the time of observation. Extrapolation from the time curves in order to obtain the values after 0 time, did not give the values which should be obtained if BLACKMAN's theory applied fully.

The results of the above-mentioned investigations were all more or less a confirmation of BLACKMAN's theory; there is therefore no need to discuss them in further detail. This is, however, not the case with a paper which appeared in 1910 by VAN ITERSON and Miss VAN AMSTEL [3]), in which the writers come to the conclusion that BLACKMAN's theory must be rejected. Since, on the ground of my own investigation, I have come to the opposite conclusion, I will briefly explain to what extent, in my opinion, VAN ITERSON's figures can be employed against BLACKMAN's theory.

In the determination of the influence of temperature on alcoholic fermentation the writers find the following values for the temperature coefficient at temperatures below the optimum·

$$\frac{V_{30}}{V_{20}} = 2{,}3, \quad \frac{V_{35}}{V_{25}} = 2{,}0, \quad \frac{V_{40}}{V_{30}} = 1{,}8, \quad \frac{V_{45}}{V_{35}} = 1{,}5.$$

On account of this decrease of the temperature coefficient with rise of temperature, the writers conclude: "it should thus be pointed

1) W. L. BALLS. Temperature and Growth. Annals of Botany. Vol. XXII. 1908.

2) J. KUYPER. De invloed der temperatuur op de ademhaling der hoogere planten. Diss. Utrecht. 1909. Also published in Recueil des Trav. Botan. Néerl. Vol. VII, 1910.

3) G. VAM ITERSON JR. and Miss J. VAN AMSTEL. On the temperature optimum of physiological processes. Proc. Royal Academy Sciences. Amsterdam. 1910.

out very emphatically that already on account of the course of the optimum curve below harmful temperatures the theory of DUCLAUX and BLACKMAN must be rejected."

This conclusion is not justified, because there is also the same decrease of the temperature-coefficient to be observed *in vitro*. Thus PLOTNIKOW [1]) found the temperature-coefficient 6.2 for the reaction between ethylene and bromine at — 78° C. TRAUTZ and VOLKMAN [2]), for the saponification of ethyl-acetate by baryta, give the following values for the temperature-coefficient :

$$\frac{10^\circ}{0^\circ} = 1{,}96 \quad \frac{20^\circ}{10^\circ} = 2{,}04 \frac{30^\circ}{20^\circ} = 1{,}90 \frac{40^\circ}{30^\circ} = 1{,}75 \frac{50^\circ}{40^\circ} = 1{,}60 \frac{60^\circ}{50^\circ} = 1{,}45.$$

For the saponification of propylacetate the corresponding values: 1.63, 2.00, 1.81, 1.70, 1.55, 1.43.

COHEN [3]) also points out that the temperature-coefficient in a chemical reaction is in general liable to vary with change of temperature. At high temperatures the temperature-coefficient decreases, at low ones it rises. The other ground on which VAN ITERSON believes BLACKMAN's theory to be untenable, is that the curve which represents the connection between the alcoholic fermentation and the temperature, is also a pronounced optimum-curve for a previous-warming time of 0 minutes. In my opinion, the authors have attached too great weight to this objection also. Various points can be brought forward to explain this phenomenon.

In the first place, there is the fact already mentioned, that the temperature-coefficient decreases with a rise of temperature.

Further it must be pointed out that VAN 'T HOFF's law applies less strictly in the field of botany than in that of chemistry, for the living organism may not be regarded simply as a homogeneous system.

Moreover even in this case a special factor comes into account, through which an important deviation at higher temperatures is *a priori* probable. The reaction takes place here between the zymase which is enclosed within the cell-wall and the sugar solution outside it.

Thus the transformation only takes place when the sugar diffuses inwards and the reaction products diffuse out in the opposite direction. Now, since after 10° C. rise of temperature the velocity of

[1]) J PLOTNIKOW, Reaktionsgeschwindigkeiten bei tiefen Temperaturen. Zeitsch. f. phys. Chemie LIII, 1905.

[2]) M. TRAUTZ, und K. TH. VOLKMANN, Der Temperaturkoëfficient chemischer Reaktionsgeschwindigkeiten. Zeitschr. f. phys. Chemie LXIV, 1908.

[3]) E. COHEN, Vorträge für Aerzte über Physikalische Chemie. 2e Aufl. Leipzig, Engelmann. 1907.

diffusion only rises about 20%[1]) and the velocity of fermentation 150—200%, it is to be expected that at higher temperatures the velocity of fermentation will remain considerably under the theoretical values, in consequence of the diffusion not proceeding quickly enough.

Finally, the possibility must be considered that also in alcoholic fermentation the favourable influence of higher temperatures first makes itself felt as a function of time, in the same way as was the case in this inquiry at 30° C., and that hence also there the theoretical values according to VAN 'T HOFF's law have no real existence. For if it takes a certain time for the reaction velocity to reach the value belonging to that temperature then this value will never be reached, because, before that happens the harmful influence of the high temperature will already have made its action felt.

The values obtained for the reaction velocity at high temperatures will then, especially after a short time of previous warming, be lower than ought to be the case according to BLACKMAN's theory. The values when extrapolated for time 0 will also be found too low.

Summarising our results, we can therefore say that BLACKMAN's theory in the investigation of the influence of temperature on the presentation-time in geotropism is in the main confirmed, while the investigations which have hitherto taken this theory into account, give no reason to reject it.

On the contrary, in this investigation it is clear, that also in the field of the pure physiology of stimulus the laws of physical chemistry hold.

For the perception of the stimulus of gravitation it follows from this investigation that, with reference to temperature, perception behaves as a chemical process.

Geophysica. — *"On the volcanic eruption in the island of Téon (Tijau) in 1659".* By Prof. ARTH. WICHMANN.

(Communicated in the meeting of Sept. 24, 1910).

In his criticism of RUMPHIUS and VALENTIJN as historiographers of Ambon F. DE HAAN says: "We do not intend to express by this the "desirability of publishing before long the History of Ambon (by "RUMPHIUS). VALENTIJN has plundered it in such a degree, that only "a scanty gleaning of details of little importance is left for a later

[1]) E. COHEN, Vorträge für Aerzte über Physikalische Chemie. 2e Aufl. Leipzig. ENGELMANN. 1907, p. 126.

"investigator" [1]). Now when a few weeks ago the manuscript in question was published by the Koninklijk Instituut voor de Taal-, Land- en Volkenkunde, it was immediately evident that on a great number of details it affords information which had hitherto been looked for in vain. What follows may serve as an instance, how it is the details that are of the greatest consequence.

Only a comparatively short time ago the attention was directed to the fact that VALENTIJN describes a volcanic eruption in the island of Téor (Tior or Tjor) [2]) in the following terms: "A little S. E. "of Koerekofe lies the little island of Tewer, situated 35 or 36 miles "to the East of Banda [3]). It is very mountainous 2 miles in circum-"ference and full of cocoa-trees yielding much oil. Here is likewise "a high burning mountain which burst asunder with great violence "in the year 1659" [4]). Though VALENTIJN distinctly indicates the situation, and the map added to his work (vide fig. 2) neither admits of the least doubt that the island of Téor (Tior or Tjor) was meant P. A. LEUPE started, as early as 1871, the question, whether the mentioned report could not have related to the island of Téon, belonging likewise to the South-Western Islands, but situated at a quite different place [5]). Attention was moreover attracted by the fact that, whilst the range of islands, constituting the inner girdle by which the Banda Sea is bounded on the East, consists entirely of islands of volcanic origin, we find in the following range, running parallel with the former, only two to which a similar origin is ascribed viz. Moa and Téor [6]). This pretended fact even suggested to E. SUESS [7]) the idea of a Northern continuation of the volcanic range of the South-Western Islands (Roma to Nusa Manuk) towards New-Guinea between which consequently Téor was to serve as a link (vide fig. 3).

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1) RUMPHIUS-Gedenkboek. Amsterdam, 1902, blz. 25.

2) This report was known to JUNGHUHN but, by mistake, he has set down this eruption to the island of Koerkaf (Java. III. Leipzig 1854, page 834). Though 5 eruptions are known of Téon, as will appear hereafter, this island is not mentioned in any of the volcanic catalogues.

3) The distance is in reality 27 geographical miles (200 km.).

4) Oud en Nieuw Oost Indiën. III. 2. Dordrecht—Amsterdam 1726, page 38. (ed. S. KEIJZER. III. 's-Gravenhage 1858, page 32).

5) Téor (highest mountain) is situated 4°45′ S. Lat., 131°44′ E. Longt. Téon 7°1 S. Lat. 129°9′ E. L.

6) Gesteine von Timor und einiger angrenzender Inseln. Beiträge zur Geologie Ostasiens. II. Leiden 1882—87, page 200. — Der Wawani auf Amboina und seine angeblichen Ausbrüche. Tijdschr. K. Nederl. Aardr. Gen. (2) XVI. 1899, page 136.

7) Das Antlitz der Erde. II. Wien 1888, blz. 208.

R. D. M. VERBEEK [1]) succeeded however in 1899 in demonstrating in the most convincing manner, that in the island of Téor there is

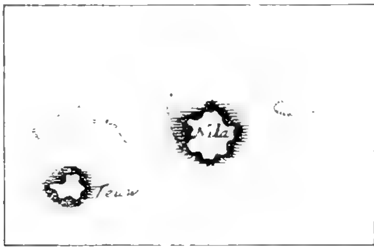

Fig. 1

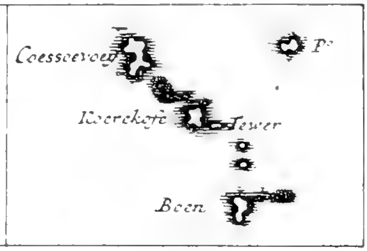

Fig. 2.

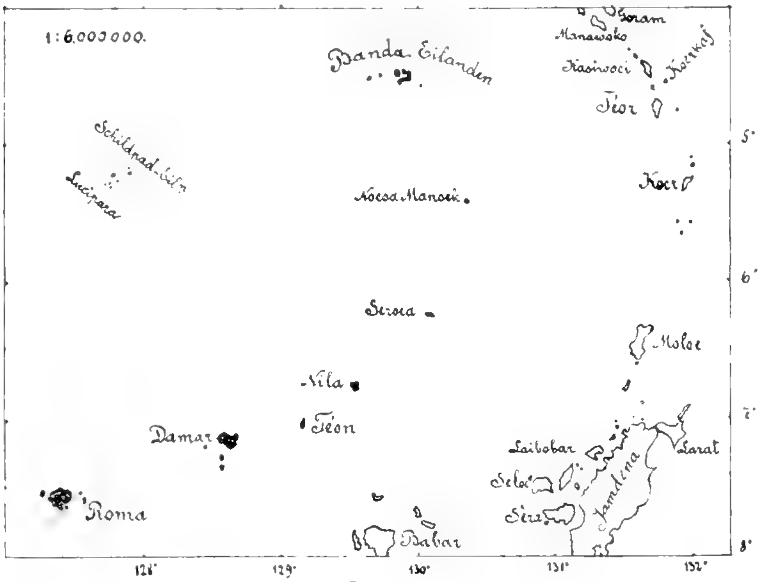

Fig 3.

found no vestige of any volcanic rock, but serpentine, phyllite, sandstone, and limestone are met with [2]). As similar observations were made in the island of Moa, not a single so-called volcano exists in the second girdle. What regards the above-mentioned report of VALENTIJN, which is doubtless incorrect: VERBEEK thinks it an open question, whether this report refers to the eruption of 1660

[1]) Voorloopig verslag over eene geologische reis naar het oostelijk gedeelte van den Indischen Archipel in 1899. Batavia 1900, pag 5,28,29— Molukken-Verslag. Jaarboek van het Mijnwezen Ned. O. Indië. XXXVII. Wet. Ged. 1908, pag. 531—532, 8.

[2]) MAX WEBER communicated that part of the 359 m. high mountain consists of coral-limestone (Siboga-Expeditie. Introduction et description de l'expédition. I. Leiden 1902, page 117).

in the island of Téon, or to the appearance of a new island between Tajando and Kaimeer in 1649 [1]).

RUMPHIUS gives an answer to this question. "On the 11th of November "[1659] in Banda a noise was heard like the detonation of cannon "and muskets which were continually and regularly fired, this made "all the people crowd on the walls, supposing that some ships were "fighting at sea, as they usually heard such a noise on Amboina "[sic!]; on the same day the water began to rise and to fall so "rapidly that it was like a miracle, and people could hardly escape, "afterwards they understood that at the same time the island of "Teeuw consisting chiefly of a high mountain had sprung up amidst "great noise and a dreadful creaking which had been the cause of "those false cannon shots, the islanders (having been warned two "days previously by subterranean rumbling and trembling of the "earth) having fled to the nearest islands Nila and Damme" [2]).

VALENTIJN gathered all his information concerning earth-quakes and volcanic phenomena in the Great-East from RUMPHIUS' manuscript in so far as they took place before his arrival in Amboina. Consequently it remains unexplicable how he could write instead of Teeuw (read Téon) Tewer (read Téor) for, as appears from the map (see fig. 1 and 2) the difference between the two islands was also known to him. The idea of a clerical error is likewise rather inadmissible, because RUMPHIUS expressly mentions the vicinity of Damar and Nila. He excerpted perhaps incompletely during his residence in Amboina, and committed the mistake afterwards, when working out his notes. At all events we may admit as irrefutable that the eruption of 1659 took place in the island of Téon. On the other hand it is decidedly not the same as that of 1660, which was considered possible by LEUPE and VERBEEK.

Regarding the eruption of 1660 the Governor of Banda wrote, dated 4 May 1660, as follows: "In February last the burning mountain in "the island of Teeuw burst and exploded entirely, so that the little "villages lying in the vicinity and at the foot of the mountain were "entirely overwhelmed, nay all men, with the exception of 2 or 3 "and all the cattle were suffocated and killed under it. A most "deplorable spectacle to behold, the inhabitants of Nilla were in deep "affliction about it, as by their marriages, many people are related

1) J. S. WURFFBAIN. Vierzehenjährige Ost-Indianische Kriegs- und Oberkaufmanns-dienste. Nürnberg 1686, p. 62.

2) De Ambonsche Historie behelsende een kort verhaal der gedenkwaardigste geschiedenissen ..... door GEORGIUS EVERHARDUS RUMPHIUS. Tweede deel. Bijdr. t. de T. L. en Vk. (7) X. 1910, p. 131—132.

"to them" [1]). Whereas in November 1659 the inhabitants, warned by trembling of the ground, could fly in due time to Nila and Damar, they were in February 1660 unexpectedly overtaken by the eruption, so that nearly all of them lost their lives.

During the seventeenth century two more eruptions follow. About the first, that of 18th January 1660 the then governor of Banda JOHAN VAN DAM reported only briefly that Teeuw was "blown up" [2]).

The second in 1693 was likewise reported only incidentally by NICOLAAS WITSEN who, in addition to a report of an eruption in the island of Serua, says: "others have begun to open themselves and to cast out Fire, as in the Isle Chiaus" [3]).

It is true that LEOPOLD VON BUCH was of opinion that this report must refer to the island of Siau [Sijau] [4]) belonging to the Sangi islands. But it is evident that Tjau or Tijau (Téon) can be transcribed in English as Chau, but Siau cannot. Besides in the mentioned report there was only question of islands in the Banda Sea.

During the 18th and the 19th century nothing whatever is heard about volcanic activity of the island. Only on the 3rd of June 1904 a new eruption took place on which occasion, as the short report says, the gardens of the village of Mèsah, situated on the westside were destroyed [5]).

The name of the volcano of Téon is said to be Vunuweri (read Funuweri) [6]).

Physiology. — "*The permeability of red blood-corpuscles in physiological conditions, especially to alkali- and earth-alkali metals.*" By Dr. G. GRYNS.

(Communicated in the Meeting of September 24, 1910).

In the meeting of the Kon. Akademie van Wetenschappen of 25 June 1910 (proceedings p. 258) H. J. HAMBURGER, also in the name of F. BUBANOVIĆ, communicated about the above subject and came to the conclusion, that the red blood-corpuscles in physiological

[1]) P. A. LEUPE. Uitbarsting van den brandenden berg op het eiland Teeuw Bijdr. t. de T. L. en Vk. (3) VI. 1871, p. 231.

[2]) W. E. VAN DAM VAN ISSELT. Mr. JOHAN VAN DAM, Gouverneur van Banda 1661 en van Amboina 1665. De Indische Gids. XXX. 1. Amsterdam, 1908, p 137.

[3]) Account of the sad Misschief befallen the Inhabitants of the Isle of Sorea. Philosoph. Transact. XIX. London, 1695, p. 51.

[4]) Physicalische Beschreibung der Canarischen Inseln. Berlin, 1825, p. 376, also Gesammelte Schriften III. Berlin, 1877, p. 580.

[5]) Nieuwe Rotterdamsche Courant, Dinsdag 17 Januari 1905, Tweede Blad p. 3.

[6]) J. G. F. RIEDEL. De sluik- en kroesharige rassen tusschen Selebes en Papoea. 's-Gravenhage, 1886, p. 466. — Zeemansgids voor den Oost-Indischen Archipel V. 's Gravenhage, 1908, p. 14.

conditions are permeable to kations and anions, or if one does not wish to place oneself on the standpoint of the doctrine of ions, to metals and acidradicals.

This conclusion, as HAMBURGER remarks himself on page 269, being opposed to the current view, should be founded on very sound bases. Now HAMBURGER communicates, as usually in the "Proceedings", for every ion but one experiment, so that only for those experiments in which both in the serum and in the blood-corpuscles, the quantity of the investigated ion was ascertained, the accuracy of the analysis can be controlled.

If we do this for Table I and for Table IV, we come to peculiar results.

Table I Permeability to Kalium and Natrium.

In the first experiment (*b*) 0,2 °/₀ NaCl is added to the serum, and the latter is afterwards united again with the blood-corpuscles. The percentage of KCl in the mixture must consequently have remained constant.

In the second experiment *c* 10 °/₀ of water (according to the text; in the table stands 0,2 °/₀ which is evidently an error) is added to the serum. The serum amounted to 60 °/₀ of the blood, the quantity of water added was consequently 6 °/₀ of the blood; therefore in 900,00 of the diluted blood there must be found 1,5858 : 1,06 or 1,496 Gr. KCl.

H. found however:

	in the serum	in the bloodcorp.	total
normal	0,3479	1,2379	1,5858
with 0,2 °/₀ NaCl.	0,4438	1,0761	1,5199
with 10 °/₀ H_2O	0,4006	1,2132	1,6138

Still more peculiar are the results with natrium chloride.

H. found here:

	in the serum	in the bloodcorp.	total
normal	4,6323	0,4198	5,0524
with 0,2 °/₀ NaCl	4,4885	0,6905	5,1790
with 10 °/₀ H_2O	4,5164	0,5623	5,0787

The quantity of serum was 594,00; 0,2 °/₀ of it is 1.188 Gr. Of these 1,188 Gr. NaCl added only

$$5.1790 - 5.0524 = 0.127 \text{ Gr.}$$

was found back again.

In the second experiment (*c*), as we saw, 6 °/₀ of water was added to the blood. The total amount of common salt in 900 cc. of the mixture should consequently be 5,0524 : 1,06 or 4,766 Gr., i. e. 0,292 Gr. less than was found.

Table IV. Permeability to Chlorine.

Quantity of 1/10 n. $AgNO_3$ solution as measure for the percentage of chloride in:

	the serum	the bloodcorp.	total
normal	110,06	33,34	143,40
with 0,2°/₀ NaCl.	110,34	34,16	144,50
with 10°/₀ H_2O	112,20	31,18	143,38

In experiment *b* 0,2°/₀ of NaCl was added to the serum. The quantity of serum was 189 cc., consequently 0,378 Gr. NaCl was added. Out of this 63 cc. 1/10 normal salt solution could have been made, equivalent to as much 1/10 normal $AgNO_3$ solution. For the blood-corpuscles and the serum together consequently also 63 cc. solution more would have been required, H finds however only 1,1 cc. more.

In experiment *c* 10°/₀ of water was added to the serum or 18,9 cc. For 300 cc. of the diluted blood 143,40 cc. $AgNO_3$ solution were required. For 300 cc. of the diluted blood consequently 300 : 318,9 times 143,40 or 134,9 cc. are required i.e. 8,5 less than H found.

Consequently we see that in three of the four experiments that can be controlled the errors in the analysis are much greater than the differences on which the conclusions are based.

We do not doubt but both investigators have made more than one experiment with regard to each ion, but we may likewise admit that an investigator who in his publication communicates only one single experiment out of a series, will certainly choose such a one as he classes with those that have offered the best result. Consequently there is no reason to suppose a priori, that the experiments that are not mentioned, had more exact results.

Therefore, in my opinion, one will act wisely by not modifying one's views about the permeability of the red blood-corpuscles on the authority of the investigations discussed above.

E R R A T A.

In the Proceedings of the Meeting of April 29, 1910:

Vol. XII. p. 813 l. 2 and 9 from the top: for 61°.9 read 64°.9.
,, ,, ,, 832 l. 14 from the top: for 5 read 0.5.
l. 10 from the bottom: for 1.3 read amply 1.3.
,, XIII. ,, 382 plate: to interchange the subscripts Fig. 4 and Fig. 5.

(November 24, 1910).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING
of Saturday November 26, 1910.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 26 November 1910, Dl. XIX).

CONTENTS.

Physics. — *"Quasi-association or molecule-complexes." II.* By Prof. J. D. VAN DER WAALS.

(Communicated in the meeting of October 29, 1910).

IX. *A substance in quasi-association considered as a binary system.*

When a substance is in a state of quasi-association, it consists of molecules with different properties, viz. simple ones and complex ones. We have simplified in so far that we have only supposed two kinds of molecules, simple ones and n-fold ones, though it is probable, especially when n is great, that it may have a variable value. In this case n must be considered as a mean value. Of the complex molecules we have to consider the volume n times larger than that of the simple molecules, and we have come to the conclusion that the attraction which exists between the molecules has a twofold effect then. First of all it brings about the aggregation to molecule-complexes, and for another part, but then to a diminished amount, it remains present as molecular pressure. For that molecular pressure a is diminished to $a\left(1-\frac{x}{2}\right)^2$. That this aggregation is to be expected when the molecular attraction diminishes exceedingly rapidly, and only makes itself felt at distances which are comparable with the mean distance of the molecules, had already been foreseen by BOLTZMANN, as DEBYE remarks. (Ann. der Physik 1910). But then it should also be accepted, in my opinion, that this is accompanied by a decrease of the molecular pressure.

That such an aggregation, called by me quasi-association, exists, I derived in the preceding first part (These Proc. June 1910) from the differences which the experiment presents with every equation of state for which such an association is not assumed. For it was demonstrated there that the assumption of a as temperature function is not to be reconciled with the course of the existing differences. It has also been shown at length that the assumption of b as temperature function cannot account for the existing differences either, though I referred for the proof to VAN RIJ's thesis for the doctorate. That also other suppositions concerning the value of the molecular pressure are insufficient to do so, I have stated, though I have omitted the proof. And to increase the confidence in the existence of this quasi-association I will prove this here first of all. To a value of the molecular pressure of $\frac{a}{v^{\mu}}$ corresponds a value of the energy of $-\dfrac{1}{\mu-1}\dfrac{a}{v^{\mu-1}}$, and a value of

$$\left(\frac{T}{p}\frac{dp}{dT}-1\right)p=\frac{1}{\mu-1}\frac{\dfrac{a}{v_1^{\mu-1}}-\dfrac{a}{v_2^{\mu-1}}}{v_2-v_1}\text{, or putting } \frac{1}{v}=\varrho\text{:}$$

$$\left(\frac{T}{p}\frac{dp}{dT}-1\right)\frac{pv_1v_2}{a}=\frac{1}{\mu-1}\frac{\varrho_1^{\mu-1}-\varrho_2^{\mu-1}}{\varrho_1-\varrho_2},$$

while

$$\left(\frac{T}{p}\frac{dp}{dT}-1\right)_{kr}\frac{p_kv_k^2}{a}=\varrho_k^{\mu-2}.$$

By division of the two last equations by each other we find:

$$1+\varphi=1+\sqrt{1-m}-\frac{1-m}{2}=\frac{1}{\mu-1}\frac{\left(\dfrac{\varrho_1}{\varrho_k}\right)^{\mu-1}-\left(\dfrac{\varrho_2}{\varrho_k}\right)^{\mu-1}}{\left(\dfrac{\varrho_1}{\varrho_k}\right)-\left(\dfrac{\varrho_2}{\varrho_k}\right)}$$

The last term has a value of $=1$ for $\mu=2$, a value of $\frac{1}{2}\left\{\left(\dfrac{\varrho_1}{\varrho_k}+\dfrac{\varrho_2}{\varrho_k}\right)\right\}$ for $\mu=3$, or a value of $1+\gamma\,(1-m)$ as $\dfrac{\varrho_1+\varrho_2}{\varrho_k}=$ $=2\,\{1+\gamma\,(1-m)\}$.

So the rapid increase of the quantity φ, which already amounts to 0,1 for $1-m=0{,}01$ according to the observations, is not even explained by $\mu=3$, but would then amount to no more than 0,008. A value of $\mu=\dfrac{7}{3}$, which was put by KLEEMAN (Phil. Mag. Oct. 1910) would even be less adequate to do so.

For a value of μ between 2 and 3, φ also has a value between 0 and $\gamma\,(1-m)$. We can namely show that for equal value of $1-m$, the quantity φ increases with μ.

Putting $\dfrac{\varrho_1}{\varrho_k}=1+\Delta_1$ and $\dfrac{\varrho_2}{\varrho_k}=1-\Delta_2$, we have namely:

$$1+\varphi=\frac{1}{\mu-1}\frac{(1+\Delta_1)^{\mu-1}-(1-\Delta_2)^{\mu-1}}{\Delta_1+\Delta_2},$$

and from this follows:

$$\frac{d\varphi}{(1+\varphi)\,d\mu}=-\frac{1}{(\mu-1)^2}+\frac{(1+\Delta_1)^{\mu-1}\,Nep\,log\,(1+\Delta_1)-(1-\Delta_2)^{\mu-1}\,Nep\,log(1-\Delta_2)}{(1+\Delta_1)^{\mu-1}-(1-\Delta_2)^{\mu-1}}.$$

If we confine ourselves to small values of $1-m$, and so also to small values of Δ_1 and Δ_2, we find:

$$\frac{d\varphi}{(1+\varphi)\,d\mu} = -\frac{1}{(\mu-1)^2} + \frac{1}{\mu-1} = \frac{\mu-2}{(\mu-1)^2}.$$

Moreover, if we confine ourselves to small values of Δ_1 and Δ_2 we find for $1+\varphi$ the value:

$$\frac{(\mu-1)(\Delta_1+\Delta_2) + \frac{(\mu-1)(\mu-2)}{1\,.\,2}(\Delta_1^2-\Delta_2^2)}{(\mu-1)(\Delta_1+\Delta_2)}$$

or

$$1+\varphi = 1+\frac{\mu-2}{1\,.\,2}(\Delta_1-\Delta_2) = 1+(\mu-2)\,\gamma\,(1-m).$$

So φ would be $=\frac{1}{3}\gamma(1-m)$ for $\mu=\frac{7}{3}$, so much too small, at least for very small value of $1-m$.

So we come to the conclusion, that the observations in the neighbourhood of the critical point and a fortiori at lower temperatures and greater density lead to the assumption of quasi-association — and if we then take a single value for n by way of simplification, we have a binary system.

If we now determine the value of ψ at given temperature for all possible values of x, representing the fraction present as simple molecules by $1-x$, such a value of ψ as function of x and v represents a surface. It is true that on account of the possibility of the transition of the substance from the simple to the n-fold molecular state by no means all the points of such a surface represent states which can really occur. A second equation holds for the determination of those points of the ψ-surface which represent really occurring states. If the value of ψ has been determined for constant weight, this second equation is given by $\left(\frac{d\psi}{dx}\right)_{vT}=0$. If on the other hand we have determined the value of ψ for $1-y$ simple and y complex molecules, this second relation must be found by putting the molecular thermodynamic potential for a complex molecule n times that of a simple molecule. But, as immediately follows from the equation given in These Proc. October 1902, p. 306, this may immediately be reduced to the preceding form, if in the second case the value of ψ is first divided by the weight of $1-y$ simple molecules and y complex ones, so by $1+(n-1)\,y$.

But whatever form we may choose for ψ, we get a second equation — and it follows from this that only a single curve lying on the ψ-surface indicates the really occurring states. This curve may be considered as the intersection of the ψ-surface with another surface

$\left(\frac{d\psi}{dx}\right) = 0$; and so we find the points of this curve by seeking the smallest value of ψ in every section of the ψ-surface for $v =$ constant. Now too coexisting states will be given by points on the ψ-surface, for which the tangent planes coincide. If the double tangent plane is rolled when there is a spinodal and also a binodal line on the ψ-surface, only one single position will be of significance for really occurring states. The points of contact are then the points in which the curve under consideration intersects the binodal curve, the intersection with the spinodal curve giving the points between which unstable states are found. When the temperature has risen to above the critical temperature of the substance, and so when no coexisting states are possible any longer, the discussed curve must pass throughout its course, so between $v = \infty$ and $v = b$, through points of the ψ-surface representing stable phases and so neither the spinodal nor the binodal line can extend over the whole breadth of the ψ-surface. At the critical temperature, the two points of intersection of the curve with the binodal, and also with the spinodal line coincide, and so the critical point is a plaitpoint on the ψ-surface.

The conditions for stability of a phase on the ψ-surface of a binary system are:

$$\left(\frac{d^2\psi}{dv^2}\right)_{xT} > 0$$

$$\left(\frac{d^2\psi}{dx^2}\right)_{vT} > 0$$

and

$$\left(\frac{d^2\psi}{dv^2}\right)_{xT}\left(\frac{d^2\psi}{dx^2}\right)_{vT} > \left(\frac{d^2\psi}{dx\,dv}\right)^2$$

or

$$-\left(\frac{dp}{dv}\right)_{xT} > 0$$

$$\left(\frac{d^2\psi}{dx^2}\right)_{vT} > 0$$

and

$$-\left(\frac{dp}{dv}\right)_{xT} - \frac{\left(\frac{dp}{dx}\right)^2_{vT}}{\left(\frac{d^2\psi}{dx^2}\right)_{vT}} > 0$$

The last form may be written:

$$-\left(\frac{dp}{dv}\right)_T > 0$$

and so can assume the simple form which holds for a simple substance, either with or without association. For

$$\frac{dp}{dv_T} = \left(\frac{dp}{dv}\right)_{xT} + \left(\frac{dp}{dx}\right)_{vT} \frac{dx}{dv}.$$

From $\left(\frac{d\psi}{dx}\right)_{vT} = 0$ follows by differentiation:

$$\left(\frac{d^2\psi}{dx dv}\right)_T dv + \left(\frac{d^2\psi}{dx^2}\right)_{vT} dx + \left(\frac{d^2\psi}{dx\, dT}\right)_v dT = 0$$

or

$$-\frac{dp}{dx_{vT}} dv + \left(\frac{d^2\psi}{dx^2}\right)_{vT} dx - \left(\frac{d\eta}{dx}\right)_{vT} dT = 0$$

or

$$-\left(\frac{dp}{dx}\right)_{vT} dv + \left(\frac{d^2\psi}{dx^2}\right)_{vT} dx - \left(\frac{d\varepsilon}{dx}\right)_{vT} \frac{dT}{T} = 0$$

So for constant value of T we have $\frac{dx}{dv} = \frac{\left(\frac{dp}{dx}\right)_{vT}}{\left(\frac{d^2\psi}{dx^2}\right)_{xT}}$; if we substitute this value for $\frac{dx}{dv}$ in the equation for $\frac{dp}{dv_T}$, the third condition of stability becomes:

$$-\frac{dp}{dv} > -\left(\frac{dp}{dv}\right)_{xT} - \frac{\left(\frac{dp}{dx}\right)^2_{vT}}{\left(\frac{d^2\psi}{dx^2}\right)_{vT}}$$

as we had derived from the theory of a binary system.

So the limits within which unstable states are found, lie further apart than would follow from $\left(\frac{dp}{dv}\right)_{xT} = 0$. They are determined by

$$-\left(\frac{dp}{dv}\right)_{xT} = \frac{\left(\frac{dp}{dx}\right)^2_{vT}}{\frac{d^2\psi}{dx^2}}$$

Only for the case that also $\left(\frac{dp}{dx}\right)_{vT}$ is $= 0$, they coincide with those of $\left(\frac{dp}{dv}\right)_{xT} = 0$.

In harmony with all this is also the circumstance that the critical point of the binary mixture for constant value of x still lies in the unstable region. In the critical point for a mixture with constant x $\left(\frac{dp}{dv}\right)_{xT} = 0$, and so also $\frac{dp}{da}$ positive. Accordingly the critical point of the substance in association is a plaitpoint, as we saw above. More similar remarks might be made, but all following from and in agreement with the consideration of an associating substance as a binary system.

X. *Shape of the ψ-surface.*

For the ψ-surface for constant weight we must assume for p the form:

$$p = \frac{RT\left(1-x+\frac{x}{n}\right)}{(v-b)_x} - \frac{a\left(1-\frac{x}{2}\right)^2}{v_x^2}$$

if $1-x$ represents the fraction of the weight that is present in the form of simple molecules, and x the fraction which occurs as complex molecules.

For the ψ-surface for constant number of molecules we must put:

$$p = \frac{RT}{(v-b)_y} - \frac{a\left[1-y+\frac{n}{2}y\right]^2}{v_y^2}$$

if $1-y$ represents the fraction which is present as simple molecules and y the fraction which occurs as complex molecules.

Between the quantities x and y exists the relation:

$$\frac{ny}{1-y} = \frac{x}{1-x}.$$

For both the terms of this equation represent the ratio of the quantities by weight in the associated and unassociated form. Then we find:

$$x = \frac{ny}{1-y+ny}$$

$$1-x = \frac{1-y}{1-y+ny}$$

$$1-x+\frac{x}{n} = \frac{1}{1-y+ny}$$

$$1-\frac{x}{2} = \frac{1-y+\frac{n}{2}y}{1-y+ny}.$$

From the equality of p, whether this quantity is expressed in x or in y, follows:

$$\frac{1-x+\frac{x}{n}}{(v-b)x} = \frac{1}{(v-b)_y} \text{ of } (v-b)_y = (1-y+ny)(v-b)_x,$$

and

$$\frac{\left(1-\frac{x}{2}\right)^2}{v_x^{\,2}} = \frac{\left(1-y+\frac{n}{2}\right)^2}{v_y^{\,2}} \text{ or } v_y = (1-y+ny)v_x\,.$$

And so we find, what, indeed, might have been put at once:

$$b_y = b\,(1-y+ny)$$

and so:

$$p = \frac{RT}{v_y - b_x\,(1-y+ny)} - \frac{a\left(1-y+\frac{n}{2}\right)^2}{v_y^{\,2}}.$$

The quantity b_x is not dependent on x, and may be written without index. In the last form we can apply with the greatest certainty the criteria which have been found in the theory of a binary system to determine whether we have a mixture with decreasing or increasing value of T_k or perhaps with minimum value of T_k. The latter appears to be the case. The criterion for minimum T_k requires that a value of y can be determined between 0 and 1 which satisfies the relation:

$$\frac{1}{a_y}\frac{da_y}{dy} = \frac{1}{b_y}\frac{db_y}{dy}.$$

With $a_y = a\left(1-y+\frac{n}{2}y\right)^2$ and $b_y = b\,(1-y+ny)$ we find for the determination of y the equation:

$$\frac{n-2}{1-y+\frac{n}{2}y} = \frac{n-1}{1-y+ny}.$$

For $y=0$ the first term viz. $n-2$ is smaller than the second, which then becomes $n-1$. For $y=1$ the first term is equal to $2\frac{n-2}{n}$, and the second to $\frac{n-1}{n}$. So if $2n-4 > n-1$ minimum T_k is present, so if $n > 3$. For the value of $\frac{y}{1-y}$ we find $\frac{2}{n(n-3)}$, and so for $\frac{x}{1-x}$ the value $\frac{2}{n-3}$.

From the form of p as function of x we might also have arrived at the same result by investigating whether the quantity:

$$\frac{\left(1-\frac{x}{2}\right)^2}{1-x+\frac{x}{n}}$$

can assume a minimum value for values of x between 0 and 1. So we might determine x from:

$$-\frac{1}{1-\frac{x}{2}}=-\frac{1-\frac{1}{n}}{1-x+\frac{x}{n}}$$

or

$$\frac{1}{1-x+\frac{x}{2}}=\frac{n-1}{n(1-x)+x}$$

or

$$(n-1)(1-x)+\frac{x(n-1)}{2}=n(1-x)+x$$

or

$$\frac{x}{1-x}=\frac{2}{n-3}$$

or

$$x=\frac{2}{n-1}.$$

In harmony with this is the value

$$y=\frac{2}{(n-1)(n-2)}.$$

Especially on the ψ-surface for constant number of molecules the mixture with minimum critical temperature lies very near the side of the component which has the smallest volume of the molecules. And it is to be expected that a mixture for which the plaitpoint line (p,T-projection) touches the p,T-projection of the critical points, does not exist. Such a point, viz. lies still more shifted towards the side of the component with the smallest b. Originally I gave the formula:

$$\frac{1}{a_y}\frac{da_y}{dy}=\frac{2}{3}\frac{1}{b_y}\frac{db_y}{dy} \quad (Cont.\ II,\ p.\ 120)$$

for the determination of the concentration of such a mixture.

Later on (These Proc. March 1902, p. 548) I thought I had to conclude that the form:

$$\frac{1}{a_y}\frac{da_y}{dy} = \frac{f-2}{f-1}\frac{db_y}{b_y dy}$$

in which $f = \left(\frac{T}{p}\frac{dp}{dT}\right)_{kr}$, is more accurate.

Then we have for the determination of y the equation:

$$\frac{n-2}{1-y+\frac{n}{2}y} = \frac{f-2}{f-1}\frac{n-1}{1-y+ny}.$$

The value of y satisfying this equation is:

$$y = \frac{f-n}{f}\frac{2}{(n-1)(n-2)}$$

If $n > f$, to which I have thought I had to conclude, then y is negative. In other words, then the point of contact of the plaitpoint line and the curve of the critical points does not occur. But this does also away with the principal reason why in the drawing of the two p,T-curves, viz. that of the plaitpoints and that of the critical points, for mixtures with minimum critical temperature, the distance of the two curves has been chosen so small.

The p,T-projection of the critical points begins at a temperature given by $RT_{k_1} = \frac{s}{r(f-1)}\frac{a}{bg}$, ends at $RT_{k_2} = \frac{s}{r(f-1)}\frac{a}{bg}\frac{n}{4}$, and has a minimum temperature given by $RT_{km} = \frac{s}{r(f-1)}\frac{a}{bg}\frac{n(n-2)}{(n-1)^2}$. So whereas the final temperature is about or a little more than twice as high as the initial temperature, the temperature has first run back, and has fallen to the

$$\frac{n(n-2)}{(n-1)^2} = 1 - \frac{1}{(n-1)^2}$$

part of the initial temperature, so only a little lower than this, for $x = \frac{2}{n-1}$ or $y = \frac{2}{(n-1)(n-2)}$. The value of p_k has continually descended. In the initial point this value amounts to $\frac{a}{bg^2}\frac{1}{r^2(f-1)}$, at the minimum temperature it amounts to the $\left(\frac{n-2}{n-1}\right)^2$ part of $\frac{a}{bg^2}\left(\frac{n-2}{n-1}\right)^2$, so to somewhat more than $\frac{2}{3}$ of the original amount —

and the final value is the $\frac{1}{4}$ part of the initial value. The temperature for the curve of the plaitpoints of course coincides with that of the critical points for T_{k_1} and for T_{k_2}, but is higher for all intermediate values of x or y. Only if there should be a mixture for which the two curves are in contact, they will of course again coincide. If this point of contact exists, which if we start from the initial point, lies before the point where minimum temperature exists, it follows from this that also the plaitpoint curve must begin with retrograding to lower temperatures. But this cogent reason for the retrogression of the plaitpoint line is wanting here. And so the question may be raised, if when this point of contact is absent, the plaitpoint line may perhaps begin with running to higher temperatures. For the particular ψ-surface of an associating substance the answer to this question is of no or rather of very little importance. But for the theory of the binary systems in general it is of greater importance. If the question should have to be answered in the affirmative, the T,x-projection of the plaitpoint line need not present a minimum for T_{pl}, and the existence of double retrograde condensation, which I discussed (These Proc. March 1909) would not be necessary. Then we meet, however, with other difficulties, which I cannot discuss here.

For the ψ-surface of an associating substance the matter would be settled if it was possible to prove that the value of T_k for the substance when there is no association is just as high as or lower than the value of T_{pl} in the case of association.

If we seek T_k for the case when:

$$p=\frac{RT}{v-b}-\frac{a}{v^2}$$

we find for the determination of T_k:

$$\frac{RT_k}{(v-b)^2}\left(1-\frac{db}{dv}\right)=\frac{2a}{v^3}$$

and for the determination of v:

$$\frac{2}{(v-b)}\left(1-\frac{db}{dv}\right)+\frac{\frac{d^2b}{dv^2}}{1-\frac{db}{dv}}=\frac{3}{v}$$

or

$$\frac{v}{v-b}\left(1-\frac{db}{dv}\right)+\frac{v}{2}\frac{\frac{d^2b}{dv^2}}{\left(1-\frac{db}{dv}\right)}=\frac{3}{2}.$$

But even though we restrict ourselves to the simplified form:

$$\frac{b}{bg} = 1 - \alpha \frac{bg}{v}$$

the determination of v requires the knowledge of α, and also the determination of v becomes uncertain and so also the determination of:

$$RT_k = \frac{2a\,(v-b)^2}{v^3} \frac{1}{1 - \frac{db}{dv}}.$$

It appears that RT_k differs very little from $\frac{a}{bg} \frac{1}{r(f-1)}$ (p.118 These Proc. June 1910). But whether this difference is positive or negative is not to be stated with certainty.

For the present I shall have to leave this an open point.

In the equation

$$MRT\,v^3 - 2a\,(v-b)^2 = \frac{y(1-y)}{2}\,a_y \left(\frac{v}{a_y}\frac{da_y}{dy} - 2\frac{db_y}{dy}\right)^2$$

(of § 21 Cont. II) there is, indeed, a means to be found to get information about the different circumstances in the course of the spinodal line in the immediate neighbourhood of the axes $x=0$ and $x=1$, at least qualitatively, for mixtures for which the minimum value of T_k lies at very small value of y. The above simple equation namely holds, if $a_1 a_2 = a_{12}^2$, which will be the case for quasi-association.

Let us put the value of T equal to T_k for $y=0$, and so $MRT\,v_1^3 = 2a_1\,(v_1 - b_1)^2$, and further, what has always been assumed in the derivation of this equation $\frac{db_y}{dv} = 0$ and $MRT_k = \frac{8}{27}\frac{a_1}{b_1}$ and $v_1 = 3b_1$.

Let us now seek how many points of intersection a line $v = v_1$ possesses with the spinodal line. As $a_y = a_1\left(1 + \frac{n-2}{2}y\right)$ and $b_y = b_1\,[1 + (n-1)\,y]$, the above equation becomes:

$$MRT_k\,27b_1^3 - 2a_1\left(1 + \frac{n-2}{2}y\right)[3b_1 - b_1 - b_1\,(n-1)\,y]^2 =$$

$$= \frac{y(1-y)}{2}\,a_1\left(1 + \frac{n-2}{2}y\right)^2 \left\{\frac{3b_1(n-2)}{1 + \frac{n-2}{2}y} - 2\,(n-1)\,b_1\right\}^2$$

or if we divide by $8\,a_1 b_1^2$:

$$1-\left(1+\frac{n-2}{2}y\right)^2\left(1-\frac{n-1}{2}y\right)^2=\frac{y(1-y)}{16}\left\{3(n-2)-2(n-1)-(n-1)(n-2)y\right\}^2.$$

If for $v=0$ or $x=1$ we had just minimum critical temperature, then:

$$\left(\frac{1}{a_y}\frac{da_y}{dy}\right)_1=\left(\frac{1}{b_y}\frac{db_y}{dy}\right)_1$$

would be, or $n-2=n-1$, which might only be put for $n=\infty$; if the above mentioned point of contact of plaitpoint line and critical line was just there, then would be:

$$\left(\frac{1}{a_y}\frac{da_y}{dy}\right)_1=\frac{2}{3}\left(\frac{1}{b_y}\frac{db_y}{db}\right)_1$$

or

$$3\,(n-2)=2\,(n-1)$$

which would hold for $n=4$. Assuming intermediate properties for the initial point, we must put:

$$\left(\frac{1}{a_y}\frac{da_y}{dy}\right)_1=k\left(\frac{1}{b_y}\frac{db_y}{dy}\right)_1,$$

in which k lies between 1 and $\frac{2}{3}$. That I put here $\frac{2}{3}$, and not as above $\frac{f-2}{f-1}$ is in accordance with what I have said about the simplifications which have been applied in the derivation of the discussed equation.

So we have here:

$$(n-2)=k\,(n-1)$$

So k approaches unity in connection with the high value of n, and the minimum critical temperature lies only a very little distance from the axis $x=0$.

Now we have to examine the value of y in the equation:

$$1-\left(1-\frac{n-2}{2}y\right)^2\left(1-\frac{n-1}{2}y\right)^2=\frac{y(1-y)}{16}\{n-4-(n-1)(n-2)y\}^2$$

For $y=0$ this equation is satisfied, and so the critical point for $x=0$ is a point of the spinodal curve. It is self-evident that the line $v=v_1$ can only present points of intersection with the branch of the spinodal line which lies on the liquid side. If we divide both the members of the equation by y, we get a third-power equation apparently, which, however, simplifies to a 2nd power equation, because the quantity $-\frac{(n-1)^2(n-2)^2}{16}y^3$ occurs in both members, which quantity accordingly is cancelled. This quadratic equation can either have two positive roots, or one positive root and a negative one, which depends on the value of n.

But without ascertaining the significance of the different points of intersection in connection with the number of the roots of the equation we immediately find *for small values of y* an answer to the question which has been put here, if we replace the equality of the two members of the equation by, what is the criterion of stability,

$$1 \left(1 - \frac{n-2}{2}y\right)^2 \left(1 - \frac{n-1}{2}y\right)^2 > \frac{y(1-y)}{16}\left\{(n-4)-(n-1)(n-2)y\right\}^2$$

or for small value of y by :

$$y > \left(\frac{n-4}{4}\right)^2 y$$

So when $\left(\frac{n-4}{4}\right)^2 > 1$, the spinodal curve extends over the whole breadth, and the temperature must rise above T_k from the very beginning.

We might make this result more comprehensible by observing that for a mixture with minimum critical temperature splitting up of the line $\frac{d^2\psi}{dv^2}=0$ takes place at $(T_k)_{min}$; but that splitting up of the spinodal line does not take place until higher temperature, and then at another value of y, too. So the double point shifts, and according to our result, to the side from which $(T_k)_{min}$ is not far distant. For $\left(\frac{n-4}{4}\right)^2 = 1$ the double point reaches the edge. For smaller value of n it does not reach the edge yet. On account of the approximative character of the equation under discussion all this can of course, not be considered as a conclusive proof, and the result will not be numerically accurate. But the result is in such close agreement with what we could expect a priori, viz. that it must be possible that the plaitpoint line rises at first even for mixtures with $(T_k)_{min}$, that I think that we may safely assume this possibility. But all this can only occur if the value of y, for which T_k has minimum value, is very small. If y for $(T_k)_{min}=0$, this would be self-evident.

Physics. — "*The critical phenomena of dissolution of mixtures with normal components examined under variable pressure.*" By Dr. JEAN TIMMERMANS. "VAN DER WAALS-fonds" researches. I. (Communicated by Prof. J. D. VANDER WAALS).

(Communicated in the meeting of October 29, 1910).

1. *Purpose of the investigation. Choice of the examined mixtures.*
As a continuation of a preliminary work undertaken last year [1]), I have now taken in hand the detailed quantitative examination of the critical phenomena of dissolution under variable exterior pressure; in this first paper I shall examine the particularities which the study of mixtures of normal substances with an upper critical temperature of dissolution offers from this point of view; I think I shall have to enter into details concerning the mode of working and observation of these phenomena, which have not been studied as yet with so much care as the critical phenomena of evaporation.

I have fixed my choice on the three following mixtures: cyclohexane + aniline, whose critical temperature of dissolution (C.T.D. = 31°,05) rises on compression, and which therefore belongs to the type of the splitting up of the plaitpoint line; nitrobenzene+hexane (C.T.D.=21°,00) and + isopentane (C.T.D. = 32°,20), mixtures whose critical temperature of dissolution falls on compression, and which therefore belong to the type of the retrogression of the plaitpoint line; the similitude of composition of these two latter mixtures will enable us eventually to draw comparisons between them.

To obtain reliable results in the determination of the critical temperature of dissolution, great purity of the components is absolutely necessary, as former investigations have sufficiently shown [2]), (cf. e.g. KUENEN's experiments [6]).

The five components used in the course of this investigation have therefore been purified with much care; the details of the methods followed are found in a preceding publication [3]); it will suffice to say here that the specimens used were provided by KAHLBAUM, and that they had been subjected to fractional distillation till their boiling-point was absolutely constant. The isopentane and the hexane had previously been subjected to the chemical treatment recommended by S. YOUNG; the aniline was almost colourless after distillation, the nitrobenzene pale yellow; the cyclohexane had been prepared by reduction of pure benzene by the method of SABATIER and SENDERENS; then it was purified by repeated treatment with a mixture of sulphuric and nitric acid after which the last traces of benzene were extracted by repeated crystallisation, till the temperature of solidification was perfectly constant.

The pure substances thus obtained possessed the constants given in table N°. 1; in the successive columns are found: the name of the examined substance, its temperatures of boiling and of solidification and its density at 0° compared with water taken at its maximum density 4°.

TABLE N° 1.

Examined substance	Temperatures		Density at 0°/4°
	of boiling	of solidification	
Isopentane	27°.95	— 158°.55	0.63942
Normal hexane	68°.95	—	0.67713
Cyclohexane	80°.75	+ 6°.5	—
Aniline	184°.40	— 6°.17	1.03895
Nitrobenzene	210°.85	+ 8.70	1.22290

2. *Preparation of Cailletet tubes with great resistance to high pressures.*

In this investigation I wished to extend my experiments to the highestpr essures that a CAILLETET tube can bear [1]). Previous experiments by BRADLEY and BROWN, and afterwards by ONNES and BRAAK confirmed by a series of investigations made in the physical laboratory of the university of Amsterdam by Mr. MASSINK, have shown that tubes of borosilicate glass generally resist pressure better than those of Thuringen or Jena glass; and that the fragility of the tubes rapidly increased when the bore of the capillary was enlarged. In the experiments which I have carried out the use of KUENEN's electromagnetic stirrer is indispensable; the minimum diameter of the capillary which can be used, is reduced by this fact to about 2 mm.; even of this width Mr. MASSINK met with tubes capable of resisting pressures of more than 500 atmospheres; unfortunately when I began with the real experiments, I found the limit of resistance considerably lower for the CAILLETET tubes; this difference seems chiefly due to the following fact indicated by BRADLEY and BROWN: a tube of slight length, such as Mr. MASSINK used for his experiments would be protected to its end by the neighbourhood of the mounting of copper in which it is fixed; for tubes longer than 20 cms., like those I used, this protecting range would be exceeded, and the resistance of the tube would be greatly reduced.

Another cause of the fragility of the glass tubes of larger bore is found in the phenomena of tension, which are the inevitable consequence of the manipulations to which the tubes are subjected in the course of the experiments and during their preparation; it is of course the abrupt variations of temperature that are particularly injurious from this point of view, but the influence of tangential pressures makes itself also often felt, chiefly when the tube is no longer sufficiently rigidly fixed in its mounting of copper on account of a local loosening of the joint, either in consequence of a rise of temperature, or by the action of a solvent.

A last difficulty arises from the necessity of subjecting the whole of the mixture to the action of the stirrer; therefore it is impossible to use as usual tubes drawn out to capillaries, which are easy to fuse firmly together; part of the liquid would stay behind in the capillary, escape the action of the stirrer, and vitiate the apparent concentration; on the other hand, if the tube is fused after being filled, not at the capillary, but at its termination, it is impossible to make the cooling of the glass take place regularly; such a tube will always break under a weak pressure, often even at last the point will burst spontaneously at the ordinary pressure.

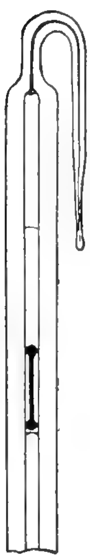

Fig. 1

To avoid the different difficulties which have just been indicated, I have finally adopted, after endless fruitless attempts, a mode of filling which is at the same time simple and rapid, and meets all the requirements mentioned before. The CAILLETET tube is drawn out at its upper part to a long very narrow capillary bent double in a semi-circle downward, parallel to the test tube (fig. 1); the bore of this capillary passes by an abrupt change into that of the test tube, to avoid the stirrer from sticking fast in the mouth of the capillary.

To fill the test tube with the liquids which are to be examined it is first completely filled with mercury, which is on a level with the free end of the capillary; then a small dish containing one of the liquids is placed in such a way that the extremity of the capillary is exactly on a level with the surface of the liquid, and then by slowly lowering the pressure, we siphon a fixed quantity of this liquid into the tube; slight alternative variations of pressure then enable us to drive out the last air-bubble that had adhered between the tube and the stirrer; then we preserve the constant pressure, wait till the level has become constant, after which we can take away the dish to determine the quantity of this first component brought into the test tube, from the difference in weight. This operation is repeated for the second component of the mixture; the latter, which is the lightest, is generally also the least viscous and takes along with it the traces of the first liquid, which had adhered to the capillary; it is easy to regulate at pleasure the quantity of the second component, which is thus brought into the tube. At last when the proportions required for the mixture have been reached, a small quantity of very pure mercury is siphoned through the capillary which washes its walls; when the capillary is full of mercury, we stop; freeze the mercury with a piece of solid carbonic acid soaked with alcohol, and fuse the capillary full of mercury together without any difficulty; the adherence to the glass suffices to keep the mercury further in its place. In this way we have obtained a mixture of known proportions, placed in a tube excluded from the air, which is kept between the meniscus of the mercury in the lower part of the test tube and the drop of mercury in the capillary; so it can be energetically stirred everywhere.

A tube filled in this way can resist a pressure of about 250 atmospheres over a length of 50 cm. with an inner diameter of 2 mm. and an outer diameter of 10 mm.; when the tube breaks under the action of high pressures, it is not in consequence of a local flaw, but because the limit of elasticity of the glass has really been exceeded; the rupture of the tube simultaneously all over its length and into very numerous fragments proves it convincingly.

3. *Methods of measurement.*

When the apparatus has been mounted, the temperature of mixing under different pressures must be exactly determined.

The regulation of the temperature was obtained by immerging the test tube into a heating bath of a capacity of 40 liters, which was rectangular, and two opposite sides of which were replaced by glass, allowing the observation of the tube, which was lighted from behind by means of an electric lamp. The CAILLETET tube reaches the bottom of the bath through a circular aperture left for this purpose, which is closed by means of a metal lid or a rubber stopper; escape of luke-warm water must be prevented here, for the mastic of the glass-copper joint might melt and the tube, subjected to tangential pressure, runs a risk of breaking; finally the bath is provided with a mechanical stirrer and a thermo-regulator of OSTWALD.

The temperature is indicated by a BECKMANN thermometer, which allows us to distinguish the thousandth part of a degree with a magnifying glass; the thermometer is incapable of following very rapid variations of temperature, but when we keep within a narrow range of almost constant temperatures oscillating between some hundredths of a degree round the critical temperature, its indications are exact on condition that it is subjected to some abrupt shakes every time before the reading to overcome the inertia of the column of mercury; what shows this clearly is that the critical temperatures read both with cooling and with heating of the mixture, coincide absolutely. The absolute value of the temperatures indicated by the BECKMANN thermometer, has been obtained by comparison with a normal thermometer which gives $\frac{1}{50}$ degree; for this purpose the two thermometers have been placed side by side in the heating bath, their columns completely immerged.

The pressure was measured on a metal manometer of SCHÄFFER and BUDENBERG, graduated in $\frac{1}{4}$ atmospheres allowing by means of a mirror in which the image of the needle is reflected, the observation of $\frac{1}{10}$ atmosphere up to 250 atms.; the indications on this manometer have been corrected by means of the SCHÄFFER and BUDENBERG "pressure-balance". Here too it is necessary to overcome the inertia of the manometer by repeated shakes, and we must avoid as much as possible variations of the pressure however slight in the course of the measurements for fear of causing errors in the critical temperature as a consequence of the VAN DER LEE effect (§ 6).

The constancy of the indications of the manometer is proved by the concordant values of the critical temperature observed several times for a same mixture and under a same pressure. We must remark here that the data of the manometer have not been corrected for the pressure of the mercury column in the CAILLETET tube, a correction which may be neglected, and is moreover, almost constant.

The stirring of the mixtures was brought about by KUENEN's electromagnetic stirrer; the circuit of which was immerged in the heating-bath itself, and was guided there by two vertical rods, lest its irregular movements should bring about shocks, and cause the rupture of the tube. The electric circuit is provided with an interruptor, without which the caloric radiation emitted by the magnet might locally vitiate the indications of the temperature; during the few moments necessary for stirring the mixture, the rise of temperature is too slight to influence the phenomena which take place in the tube, of which I have been able to convince myself by control experiments at constant temperature and closed electric circuit without stirring.

Another cause of error already pointed out by KUENEN in the use of the stirrer proceeds from the momentary elevation of the temperature, which takes place after the movement of the stirrer in consequence of its friction against the glass tube; if e. g. we keep the mixture at a temperature only very slightly below its critical temperature (some thousandths of a degree), it has the appearance of a white, opaque, and homogeneous cloud (see § 4), and if we then move the stirrer, we see very distinctly the liquid get clear for a moment after its passage, which accordingly is marked by a kind of transparent streak, which, however, disappears again almost at once. The stirring has, therefore, brought about a local elevation of the temperature to above the critical temperature. So to avoid this cause of error, it will be necessary to avoid the use of the stirrer during the measurements themselves; moreover it would be useless, for if we make the temperature oscillate round the critical temperature, the mixture always remains homogeneous, even below the critical temperature.

4. *General aspect of the critical phenomena.*

Let us now in detail examine the aspect of the critical phenomena observed in this case, in such a way as to be able to define very accurately what we understand by the critical temperature. When the temperature is sufficiently high, the mixture is homogeneous and offers no visible particularity; when the temperature is lowered gradually the critical opalescence is seen to appear at a temperature of about half a degree above the critical point; this opalescence increases gradually, it is very intense at 1 or 2 hundredths of a degree above the critical temperature, though the liquid remains entirely transparent; then all of a sudden the liquid is invaded by a white, opaque cloud, which prevents us from seeing the wire of the electric lamp placed at the back; this cloud does not break up or only very slowly (5 minutes or more) into two phases if we keep it at a temperature a few thousandths of a degree below the critical temperature; when we then raise the temperature again we see the liquid suddenly become transparent and homogeneous again without stirring being necessary; finally when the temperature falls still more, we see drops of the two phases clearly separate, beginning at a hundredth degree below the critical temperature; soon these drops re-unite, and the two phases appear separated by a meniscus, which is flat at first, but which curves more and more downward, when we get further from the critical point; the curvature is not appreciable before some degrees below the critical point. In this succession of phenomena I have chosen as critical temperature the temperature at which the white cloud appears in the transparent liquid; it is, indeed, this temperature that can be fixed most accurately in the succession of states observed.

The critical opalescence is very distinct in the examples which I have just studied; it is impossible to indicate the exact temperature at which it appears, but an experienced eye already recognizes traces at $\frac{1}{2}$ degree above the critical temperature; it is visible by reflection looking like a bluish cloudiness long before traces can be perceived by transparence; by transparence it can hardly be perceived before $\frac{1}{10}$ degree above the critical temperature, owing to the orange colour which the electric light assumes on its passage through the CAILLETET tube; then the intensity of the opalescence increases rapidly. At a hundredth degree above the critical temperature the aspect of the liquids is really beautiful: by reflexion the liquid emits a greyish blue very intense light, and appears quite opaque; by transparence on the other hand, it is still clear, but has taken an orange-red tint with green reflections due to the movements to which the liquid is subjected in its mass; when the liquid becomes opaque, the cloud which forms when it leaves the walls is on the other hand white; I have not been able to observe anything resembling a maximum of intensity of opalescence at a temperature somewhat above the critical point, as TRAVERS and USHER did for pure ether[5]). This difference may be owing to a difference of opinion about the exact definition of the critical temperature.

Here follows the description of an experiment: the liquid was first brought to a temperature above the critical region, and long shaken, till it was quite homogeneous, and till the equilibrium of the temperature had set in; then the temperature was slowly lowered $\left(\frac{1}{100}\text{ degree a minute}\right)$, without shaking, and once a minute the temperature of the bath and the aspect of the liquid was noted down; when the critical temperature had been reached, half a minute generally sufficed for the opaque cloud to pervade the liquid completely, and the critical temperature was taken as mean of the temperatures at which the cloud began to appear, and at which the liquid was perfectly opaque. From this moment the temperature was made to rise slowly, and the temperature was observed at which the liquid got clear; these variations of temperature were repeated several times keeping them between very narrow limits $\left(\frac{1}{100}\text{ degree}\right)$; then the liquid always remained homogeneous without the intervention of the stirrer, the temperatures of opaqueness and of clearness obtained in this way generally agreed down to some thousandths of degrees without it being possible to detect an effect of retardation of the thermometer in cooling or heating; this is also in harmony with the theory of VAN DER WAALS, according to which phenomena of supersaturation are impossible at the critical point.

So it follows from these experiments that it is easy to determine in these conditions the critical temperature to 2 or 3 thousandths of a degree; the repetition of the experiments really furnishes absolutely concordant results, and shows once more the extraordinary sensibility of the critical phenomena to the influence of the temperature.

5. *The critical concentration and the Cagniard de la Tour phenomenon.*

The phenomenon of CAGNIARD DE LA TOUR, known with regard to pure fluids, is here found back very clearly: the meniscus appears and disappears not always halfway up the tube of the mixture, but at the top or at the bottom of the tube; the critical opalescence is also often irregularly distributed in the liquid. The same causes which bring about the phenomena in the pure liquids: great variability of volume and influence of gravity, are found back here; their action must be greatly increased by the presence of traces of impurities (dissolved air and moisture), which it is particularly difficult to prevent during the long process necessary for the preparation of the mixtures. Besides the existence of the CAGNIARD DE LA TOUR phenomenon alone permits the exact determination of the critical temperature of dissolution under variable pressure; for the preparation of liquids of exactly the critical concentration would be exceedingly laborious if not impossible.

To bring mixtures of a determined concentration into a test tube is, indeed, exceedingly difficult on account of the small quantity of liquid with which we must work: the loss of 1 mgr. on 200 mgr. of substance already represents an error of $^1/_2{}^0/_0$. Indeed, if we try to prepare mixtures of the same concentration, we always observe variations round the critical concentration in spite of all precautions, as the various heights at which the meniscus disappears in the CAGNIARD DE LA TOUR phenomenon prove. It is, after all, at least as difficult to realize exactly the critical concentration of a mixture as to measure the critical volume of a pure liquid; so it would be better in the two cases to take an indirect course for fixing these constants and make use of the law of the rectilinear diameter. The feeble variations of the critical temperature (less than 0°.1) which I observed when I successively examined different quantities of the same mixtures are perhaps solely due to inevitable variations of the concentration.

On the other hand too only the phenomenon of CAGNIARD DE LA TOUR enables us to study the variations of the critical temperature of dissolution with the pressure for a single mixture, for strictly speaking there exists only one temperature corresponding to one pressure for every mixture of a definite concentration. Indeed if we consider VAN DER WAALS' *tvx*-diagram for the simple case of the retrogression of the plaitpoint line (fig. 2), and project the course of

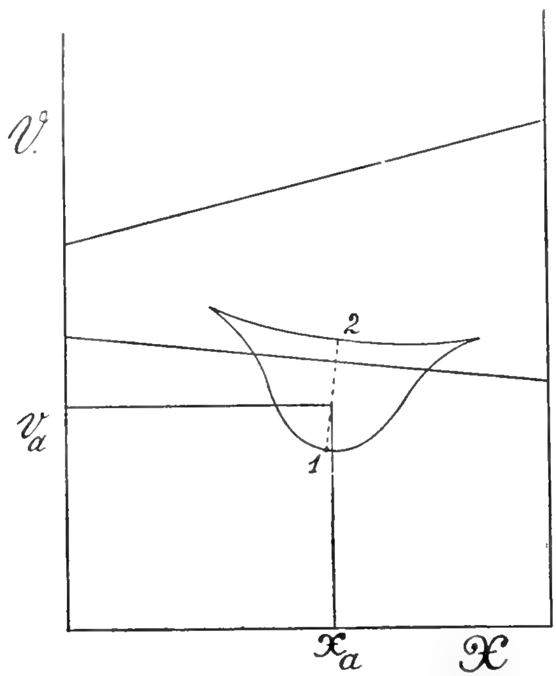

Fig. 2.

the plaitpoints at different temperatures on one of the horizontal sections *vx*, we get a line 1—2, oblique to the *x*-axis, and accordingly one single, well defined critical concentration x_a corresponds to every critical volume b_a; it is evident that this holds too for the critical temperature and pressure. So without the phenomenon of CAGNIARD DE LA TOUR it would only be possible to observe a single critical temperature for every mixture; fortunately the critical phenomena are measurable over a pretty large range of concentrations, and the inclination of the plaitpoint line to the axis of the concentrations being generally slight, we can determine a considerable portion of it with a single mixture. (This is not always the case when the critical phenomena of dissolution take place in the neighbourhood of the critical temperature of the components — compare on this subject KUENEN). [6])

So if the theory of VAN DER WAALS, which takes the critical line of dissolution as a part of the ordinary plaitpoint line, is correct, we must be able to observe variations of the relative volume of the two phases in equilibrium in the critical region, when we vary the concentration; it is this, indeed, that I have observed when comparing the volume of the two liquid layers of a same mixture under very different pressures; the following table 2 gives some examples where the volume of the layers is expressed as function of the number of cms. which they take up respectively in the CAILLETET tube.

TABLE N°. 2.

Isopentane + Nitrobenzene (3rd experiment).

		$P=2$ atm.	150 atm.	[illegible]00 atm.
Height of the upper layer	(pentane) in cms.	4	3,75	3,50
" " " lower "	(nitrobenzene)	3,25	3,25	3,25

System Cyclohexane + Aniline.

		$P=2$ atm.	250 atm.
Height of the upper layer	(cyclohexane)	2	$1\frac{1}{2}$
" " " lower "	(aniline)	3	3

In the first mixture we get nearer to the critical concentration, where the volume of the two phases is equal, by compression — in the second, on the other hand, we get further from it, it contained too little cyclohexane. In the two cases the inclination of the plait-point line takes place in the same sense: the critical concentration of the lightest and the most volatile component increases with the pressure.

This fact proves that a theory drawn up on this point by OSTWALD [7]) in opposition to VAN DER WAALS' views is erroneous, as BÜCHNER [8]) had already pointed out: OSTWALD distinguishes between critical points of the 1st order for pure liquids, which would not vary with the pressure — of the 2nd order for mixtures whose critical temperature of dissolution would on the other hand vary with the temperature. This conception implies a constant critical concentration under different pressures, which is in contradiction with the preceding data; so it is only improperly and by extension that we can speak of the influence of the pressure on the critical temperature of dissolution; properly speaking there is for every mixture, as is the case with the critical temperature of vaporisation only one critical temperature corresponding to a definite pressure and volume.

6. *The* VAN DER LEE *effect.*

I will now describe some new observations concerning a very curious phenomenon, which I propose to call "the VAN DER LEE effect", because when preparing his thesis for the doctorate [9]) in the physical laboratory of this university, VAN DER LEE was the first to notice this phenomenon: his observations referred to the system water-phenol, and have not been repeated as far as I know. The VAN DER LEE effect consists in this: when this mixture, after having been first compressed, is made to expand suddenly, being kept at a constant temperature very little above the critical temperature, we ought simply to observe that the liquid, already homogeneous under high pressure, remains so equally under low pressure, because this mixture belongs to the type of the splitting up of the plaitpoint line, and that its critical temperature *falls* with the *pressure*. The observation, however, yields an altogether different result: we see on the other hand the liquid, which was clear at first, get very turbid, a white cloud appearing as precursor of the separation into two phases; but this period of opaqueness is short, and soon the liquid becomes clear again, and remains so definitively.

To explain this strange phenomenon VAN DER LEE justly points out that the work of expansion must produce a passing cooling of the liquid; if the thermal changes with the thermostat are not instantaneous, this cooling may be sufficient to reduce the mixture to a temperature below its critical temperature; but soon the equilibrium will set in again, and the liquid will return to the region above the critical temperature. So the Van der Lee effect would be due to the cooling which accompanies the expansion of a compressed liquid near its critical point, the effect of which is used in the commercial production of liquid air.

Van der Lee has set forth this theory as a purely provisional one, and without supporting it by decisive proofs, which I think I can now furnish. At first when the opaque liquid reassumes its transparency, we see very clearly the region of transparency proceed gradually from the walls towards the centre of the tube, according as the wave of heat propagates. In the second place besides the preceding phenomenon, I have often been able to observe the reverse for the system cyclohexane + aniline. Starting from an opaque liquid kept at a temperature very little below the critical one, I abruptly raised the pressure; the liquid should remain opaque because here the critical temperature rises by compression; but quite on the contrary, it began by getting temporarily clear, and it is only afterwards that it got turbid again beginning at the walls; this phenomenon is, however, not so pronounced as the reversed one, but it is evidently due to the same cause: the heat of compression.

Finally, and this seems to me a decisive proof, in the course of my numerous experiments on more than seventy mixtures, I have often observed the van der Lee effect in the case of the splitting up of the plaitpoint-line (water + phenol, cyclohexane + aniline), hardly

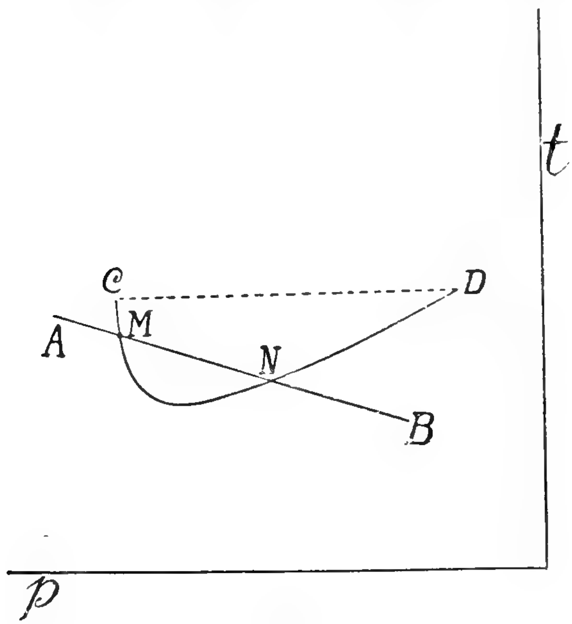

Fig. 3.

ever in the case of retrogression (nitrobenzene + hexane or isopentane). This opposition is easily explained when we compare the shapes of the two following lines in the *pt*-diagram (fig. 3—6): the plaitpoint line *AB* and the curve *CD*, which represents the succession of states

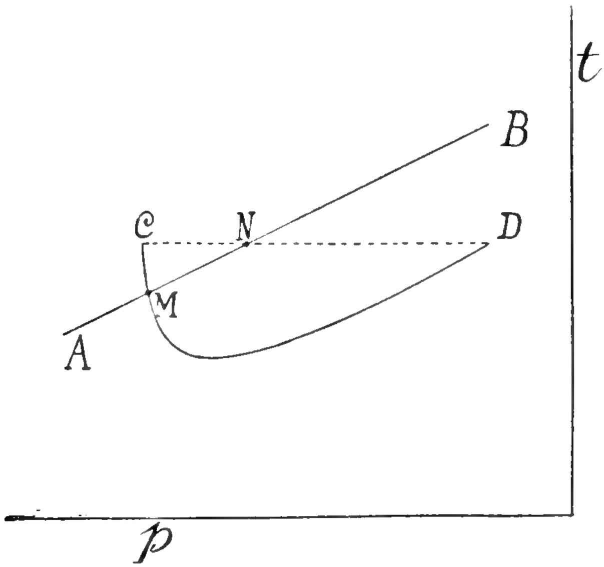

Fig. 4.

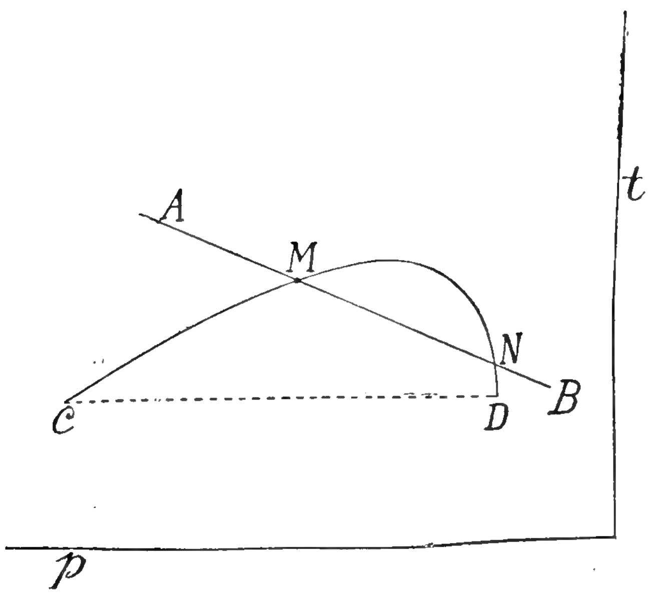

Fig. 5.

through which the mixture passes when subjected to an expansion (fig. 3 and 4) or to an abrupt compression (fig. 5 and 6).

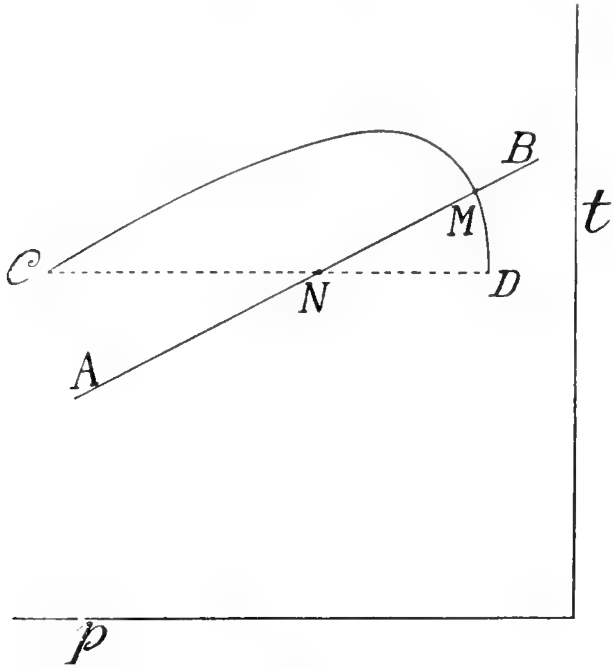

Fig. 6.

If there is splitting up of the plaitpoint line, the line AB descends slowly beginning with high pressures, the curve CD being greatly concave; so these two lines have every chance to intersect twice, once in M, when the liquid will get turbid, the second time in N, when it gets clear again, it is sufficient for this phenomenon to take place that the initial temperature is very little above the critical temperature of the mixture under this pressure, and that the expansion is sudden enough, and consequently the concavity of the curve CD pronounced enough; moreover in the case of splitting up the line AB descends only very slowly (less than 0°,03 per atmosphere), which promotes the effect. In the case of retrogression of the plaitpoint line, on the other hand, the abrupt expansion has the effect of making the liquid turbid starting from the point M, much quicker than when the expansion had been isothermal (cloud only in N), but when the liquid has once entered the infra-critical region, it will remain there, and the VAN DER LEE effect cannot take place. The consideration of the two following figures, which deal with the reversed phenomenon, will suffice to show that the facts are of the same order.

The VAN DER LEE effect is easy to obtain; reduction of the pressure by some atmospheres suffices to bring it about when the expansion is sufficiently sudden, and the initial temperature favourable. So it will be necessary to take this phenomenon into account in qualitative observations, when the pressure is made to vary rapidly to prevent our confounding cases of splitting up of the plaitpoint line, complicated by the VAN DER LEE effect, with cases of retrogression.

The remarks noted down in table 3 show how clearly the effect was observed.

Table 3. System cyclohexane + aniline. C. T. D. under 100 atms. of pressure = 4°,05 (BECKMANN thermometer).

Fall of pressure at 4°,06	Intensity of the VAN DER LEE effect.
100—90 atm.	very great.
100—95 ,,	,, ,,
100—97 ,,	great.
100—98 ,,	weak.
100—99 ,,	hardly visible.

7. *Course of the plaitpoint line under variable pressure.*

In the tables 4, 5, and 6 I have collected the experimental data obtained for the variation of the critical temperature with the pressure; these tables are composed as follows: in the first column the pressures are indicated in atmospheres — in the next the critical temperatures observed under these different pressures for the different examined mixtures: at the head of each of these columns the number of the examined mixture is indicated. The critical temperatures observed under higher pressures are expressed in thousandths of degrees, with the number of times that the measure has been repeated between parentheses; to facilitate the comparison we have reduced them to the same scale of temperatures, representing by 0°.000 the critical temperature examined under the feeblest pressure. In the last column but one the mean of the observed critical temperatures is given for every pressure, with in thousandths of degrees the maximum difference between the various observations — and the last column contains the value of $\frac{\partial t}{\partial p}$, the variation of the critical temperature for a change of pressure of 1 atm., measured separately between the different limits of examined pressures.

At the head of every table (and under it) some remarks are found on the particularities offered by every mixture (position of the meniscus, intensity of the critical opalescence, etc.), besides the total weight of the examined mixture, the concentration of the 2nd component, expressed in percents by weight and the absolute value of the critical temperature reduced by extrapolation to the pressure of one atmos-phere to facilitate comparison.

TABLE N° 4.

System Cyclohexane + Aniline.

N° of the mixture	Total weight in gr.	Concentration % in Aniline	C. T. D.
1	—	—	30°.932 (7)
(2)	0.390 gr.	49.72 %	32.367 (4) ± 3
3	1.113 »	49.64 »	31.032 (5) ± 2
4	0.549 »	49.99 »	31.011 (7) ± 3
5	0.455 »	49.27 »	31.053 (5) ± 2
6	—	—	31.095 (1)

P. in atm.	1	(2)	3	4	5	Mean	$\frac{dt}{dp}$ in degrees
1.80	0°.000	0°.000	0°.000	0°.000	0°.000	0°.000	
47.85	+0.285 (3)	+0.240 (7)	+0.277 (3)	+0.272 (4)	+0.278 (7)	+0.276 ± 4	+0°.0060
98.25	+0.584 (6)	+0.528 (2)	+0.596 (4)	+0.597 (3)	+0.598 (3)	+0.597 ± 1	+0.00635
148.50	+0.960 (2)	+0.826 (2)	—	—	+0.941 (4)	+0.941 —	+0.0068
199.25	—	+1.151 (3)	—	—	+1.303 (4)	+1.303 —	+0.00715
250.05	—	+1.484 (2)	—	—	<+1.70 —	<+1.70 —	<+0.0077

Remark. The very first experiment has furnished somewhat inconsistent results; the 2nd has been vitiated by the accidental presence of a trace of isopentane in the test-tube, which has been sufficient to raise the critical temperature considerably, greatly lowering the value of $\frac{dt}{dp}$ at the same time. All the mixtures contained a slight excess of aniline, for the meniscus disappeared at the top of the test-tube, but we were in the critical region with its fine opalescence.

TABLE N°. 5.

System Isopentane + Nitrobenzene.

N° of the mixture	Total weigth in gr.	Concentration 0/0 in nitrobenzene	C. T. D.
(1)	0.382 gr.	50.37 0/0	32°.617 (extrapolated)
2	0.517 „	49.54 „	32.219 (5) ± 3
3	0.740 „	49.66 „	31.817 (7) ± 2
4	0.548 „	49.71 „	32.179 (5) ± 4
5	0.471 „	50.14 „	32.116 (6) ± 3

P. in atm.	1	2	3α	3β	Mean	$\frac{dt}{dp}$
1.70	0°.000	0°.000	0°.000	0°.000	0°.000	
12.50	—0.446 (6)	—	—	—	—	—0°.0413
39.65	—	—1.577 (4)	—	—1.567 (4)	—1.572±5	
56.40	—2.257 (4)	—	—	—	—	—0.0344
81.35	—	—3.002 (6)	—2.996 (3)	—3.010 (2)	—3.003±7	
102.45	—3.781 (5)		—	—	—	—0.0294
123.55			—4.238 (4)	—4.248 (1)	—4.243±5	—0.0247
165.35				—5.323 (4)	—5.323 —	—0.0219
207.60				—6.251 (2)	—6.951 —	—0.0200
250.05				—7.100 (3)	—7.100 —	<—0.017
300.00				<—7.95 —	<—7.95 —	

The first experiment has furnished somewhat inconsistent results; in all of them there is a slight excess of isopentane, and the meniscus disappears at the bottom of the tube — the measurements of the 3rd mixture have been separately repeated by means of two different thermometers; its critical temperature is a little too low.

TABLE N°. 6.

System Hexane + Nitrobenzene.

N°. of the mixture	Total weight	Concentration % in nitrobenzene	C. T. D.
1	—	—	20°956 (3) ± 1
2	0.635 gr.	51.57 %	21.023 (5) ± 1
3	0.475 „	51.72 „	20.955 (3) ± 1

P. in atm.	1	$\frac{dt}{dp}$ in degrees
8.95	0°.000	
		— 0°.0186
47.85	— 0.727 (3) ± 3	
		— 0.0163
98.25	— 1.548 (2) ± 1	
		— 0.0140
148.50	— 2.253 (3) ± 1	
		— 0.0122
199.25	— 2.871 (3) ± 1	

In the first experiment the quantities used were not known, in the 1st and the 3rd there was a slight excess of nitrobenzene; in the second on the other hand the critical concentration has been realized, for the meniscus disappeared half way up the tube. The experience obtained in this series of experiments, allowed us to obtain a close agreement in all these measurements.

The examination of the obtained quantitative results shows that the mixtures of normal substances studied here belonged to two distinct types: that of the splitting up of the plaitpoint line $\left(\frac{\partial t}{\partial p}\text{ is positive}\right)$, and that of the retrogression $\left(\frac{\partial t}{\partial p}\text{ is negative}\right)$ — (compare on this subject the classification given in our previous paper). The case of splitting up (cyclohexane + aniline) is characterized by a very feeble value of $\frac{\partial t}{\partial p}$, as in all the other observed cases of splitting up $\left(\frac{\partial t}{\partial p}\right.$ positive then never exceeds $\left.+\,0°{,}03\right)$ — moreover this value gradually increases when the pressure rises; this proves that under the pressures of saturated vapour we are not yet at a distance from the point where the splitting up takes place, which is a minimum temperature (fig. 7 point *H*). Starting from this point the plaitpoint line gradually ascends. So we are probably very far from the point where the plaitpoint line passes through a maximum temperature (point I), before it finally descends, and it is even very well possible that this point is not yet reached, when the plaitpoint line cuts the line $v = b$. So in this case it would be impossible to make the components miscible even at the highest pressures.

The two examined cases of retrogression (nitrobenzene + hexane

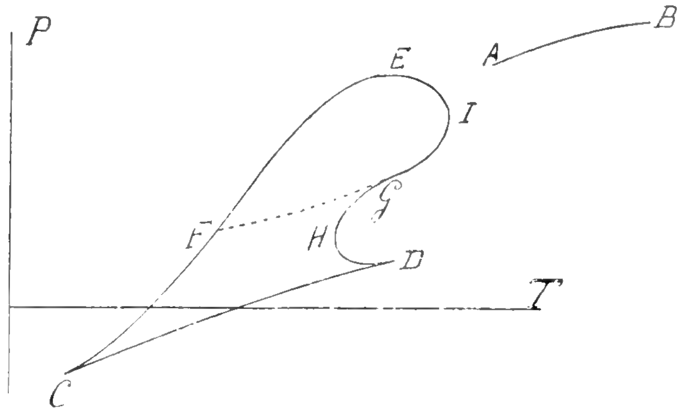

Fig. 7.

and isopentane) lead to a similar statement: the value of $\frac{dt}{dp}$ which is pretty high at feeble pressures descends very rapidly: for the system nitrobenzene + isopentane, it reduces from — 0°,0413 to — 0°,0200 (at 250 atm.); for the system nitrobenzene + hexane it falls from — 0°,0186 to — 0°,0122 (at 200 atm.). So it seems that the plaitpoint line tends to become parallel to the axis of the pressures and in this case the mixture could not be rendered homogeneous either at every temperature, even by the highest pressures. It would be interesting to verify this latter point for a mixture the $\frac{dt}{dp}$ of which is already very feeble under feeble pressures (the system aniline + decane has for its $\frac{dt}{dp}$ value : — 0°,001); this I hope to be able to do later on.

The critical temperature of the first mixture is reduced from 32°,2 under the tension of saturated vapour to 24°,25 under a pressure of 300 atms. — that of the second from 21°,0 to 18°,15 at 200 atms. — Hence both in consequence of the absolute value of the critical temperatures and in consequence of their variation with the pressure,

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these two curves are almost each other's continuation. Between 1 and 200 atms., the system nitrobenzene + hexane presents a plait-point line which is almost identical with that which the mixture isopentane + nitrobenzene would have from 300 atms. So we meet here again with the phenomena of gradual increase so often observed in a homologous series when we pass from one term to the next.

The whole of the results thus obtained is in accordance with the theoretical investigations of Prof. KOHNSTAMM, the first part of which will shortly be published in the Zeitschrift für phys. Chemie and which chiefly refer to the relative frequency of the different types of the saturation curve for the mixtures of normal substances. The systems of a completely closed saturation curve with two critical end-points, a higher one and a lower one, would be impossible (as yet indeed no mixture of normal substances is even known where the saturation curve presents a minimum of solubility at given temperature); on the other hand the systems with a lower critical end-point, the plaitpoint line of which enters the critical region proper would be possible; and finally the systems with a higher critical end-point would be very frequent, but the plait-point line would reach the $T=0$ axis both in case of retrogression and in case of splitting up.

Conclusion: In the course of this paper, I have described in detail the precautions necessary in an exact quantitative investigation of the critical phenomena of dissolution, and I have drawn attention to some interesting particulars which these phenomena present: the CAGNIARD DE LA TOUR phenomenon under variable pressure, the VAN DER LEE effect etc.; I have then made an exact determination of the position of the plait-point line of three mixtures of normal substances up to a pressure of 250 atmospheres.

Physical Laboratory of the University. *Amsterdam*, October 1910.

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Chemistry. — *"On nitrogen (or nitrilo)- trimethylnitraminomethylene".*
By Prof. A. P. N. FRANCHIMONT.

(Communicated in the meeting of October 29, 1910).

Many times I have pointed out the analogy, and also the difference in behaviour between hydrogen cyanide and methylnitramine. The analogy renders it probable that the nitramine reacts as $CH_3 . NH . NO_2$. The following is a striking instance.

ESCHWEILER has allowed hydrogen cyanide in aqueous solution to act on hexamethylenetetramine (urotropine) and on using six mols. of CNH for one of urotropine, he obtained iminodiacetonitrile. He advises the addition now and then of a little hydrochloric acid to neutralise the ammonia which is also formed and yields brown coloured products with the hydrogen cyanide. On using a larger quantity of hydrochloric acid he obtained nitrogen-triacetonitrile.

If methylnitramine and urotropine in aqueous solution are allowed to react on each other in the same proportion nothing happens apparently, but, on warming, formaldehyde is at once liberated. If the solution is exposed to the air (not in a desiccator over sulphuric acid) there are formed after some time splendid pillar-shaped crystals melting at 116°. If these are removed when they no longer increase but actually seem to be disappearing and are then pressed dry, or recrystallised from boiling chloroform, which is the best solvent, their analysis agrees with the formula $N\left(CH_2 . N\begin{matrix} CH_3 \\ NO_2 \end{matrix}\right)3$ which is nitrogen (or nitrilo)-trimethylnitraminomethylene.

This formula is confirmed by a molecular weight determination and by determining the products of decomposition by acids and alkalis which are formed according to the equation.

$$N\left(CH_2 . N\begin{matrix} CH_3 \\ NO_2 \end{matrix}\right)_3 + 3H_2O = NH_3 + 3CH_2O + 3\left(NH\begin{matrix} CH_3 \\ NO_2 \end{matrix}\right)$$

On boiling with solution of barium hydroxide barium methylnitramine could be obtained, but the two other decomposition products could not be isolated quantitatively, as a portion of the formaldehyde and the ammonia recombined to form urotropine, which is not decomposed by the alkali. On boiling with dilute sulphuric acid the formaldehyde could be distilled off and determined as such while the methylnitramine is decomposed into methylalcohol and nitrous oxide, which latter was measured; the ammonia is retained by the acid from which it is afterwards liberated by alkali and collected in standard acid.

The result of these three determinations agrees with the formula and the above equation of decomposition.

The yield, however, was very bad and, therefore, it was endeavoured in the first place not only to effect an improvement, but also to gain an insight into the progressive change of the reaction.

It appeared that the addition of hydrochloric acid, at least if care was taken to avoid an excess, caused each time the formation of the nitrilo-derivative, provided the product formed was removed each day before the addition of a little more acid; in this manner the yield finally amounted to about 70 %.

If one took less than six mols. of nitramine for one of urotropine, the latter crystallised from the solution. If eight mols. were taken a little more of the nitrilo-derivative was formed but only a trifling amount. If the solution is placed in a desiccator over sulphuric acid crystals of ammonium sulphate are noticed on the wall of the desiccator when this has been moistened with sulphuric acid; the liquid has an odour of ammonia, although it is acid to litmus, but yields no nitrilo-derivative.

When urotropine and methylnitramine are mixed in the dry state, the mixture, after a few moments, turns liquid and becomes very cold but nothing further takes place; on addition of water, however, a little separation of crystals may sometimes occur.

Obviously, urotropine in aqueous solution is decomposed by the feebly acid methylnitramine, but the compound of nitramine with ammonia is strongly hydrolysed in water and, therefore, the addition of a little hydrochloric acid to fix the ammonia promotes the formation of the nitrilo-derivative, which is decomposed by ammonia in aqueous solution, in which the latter acts as a base.

The nitrilo-derivative which is not soluble in water is, however, decomposed by water after some time, and the products of decomposition pass into solution which has a distinct odour of formaldehyde. If that solution is allowed to evaporate spontaneously, the nitrilo-derivative is reformed when the three components are present in the exact proportion.

As in the decomposition of urotropine, four mols. of ammonia are formed for six mols. of formaldehyde, two more than required, I therefore added to the urotropine solution another six mols. of formaldehyde and twelve mols. of nitramine. After half an hour, the nitrilo-derivative commenced to crystallise and after 24 hours 94 % of the theoretical yield was already obtained; further crystals were obtained from the mother liquor, so that the final yield was a quantitative one.

A still simpler procedure was to add to commercial formaldehyde solution as much ammonia as corresponded with one mol. to three mols. of formaldehyde, to cool the solution in water and then to add three mols. of nitramine which caused a strong cooling. A quarter of an hour after the liquid had been brought to the ordinary temperature by warming with the hand crystallisation set in. The yield, in this case, was also a quantitative one.

The reaction is, therefore, analogous to the one which I communicated in 1897 for piperidinomethyl alcohol with methylnitramine and to which I have recently added another viz. piperazine formaldehyde and methylnitramine. Like some other aldehydes, formaldehyde yields with ammonia and some amines amino-alcohols 1.1 which behave like bases and on which methylnitramine acts as an acid, so that a kind of salt (or ester) is produced with formation of water. These compounds are all decomposed by water, owing to hydrolysis.

The reaction of methylnitramine is quite analogous to that of hydrogen cyanide; the aminonitriles 1.1, however, are very permanent, because in them the carbon is linked to carbon, as in the case of the esters of hydrogen cyanide.

Usually, however, the aminonitriles are prepared from the oxynitriles 1.1. with ammonia or amines; this has not been found to apply to the nitramines for attempts to obtain compounds of aldehydes with nitramines have proved unsuccessful up to now.

It appears to me, however, that one goes too far when looking on the process of the formation of aminonitriles, — as often happens — as represented by the equation

$$\begin{array}{l}\mathrm{R} \\ \mathrm{HCOH} \\ | \\ \mathrm{CN}\end{array} + NH_3 = H_2O + \begin{array}{l}\mathrm{R} \\ \mathrm{HCNH_2} \\ \backslash \\ \mathrm{CN}\end{array}$$

as if a double decomposition between the oxynitrile and the ammonia, with formation of water, took place. For, if we wish to consider the oxynitriles 1.1. (cyanhydrins) as alcohols, that reaction with ammonia in aqueous solution is very strange and hardly probable. The acid character of the OH-group of the cyanhydrins must have been strengthened by the adjacency of the cyano-group, so that they may be rather looked upon as feeble acids. One might then expect the formation of an ammonium salt which will be strongly hydrolysed or dissociated in water, but not so much the elimination of water and the formation of an amide-like substance in the aqueous solution.

If we consider, however, that the cyanhydrins, like their analogous halogen compounds, are readily decomposed by bases with loss of hydrogen cyanide and formation of the aldehyde, and that ammonia, not only in aqueous solution reacts as a base, but even without the presence of water can, like amines, abstract the acid, the reaction of ammonia (and amines) on the cyanhydrins may be reduced to that of hydrogen cyanide on the amino alcohols 1.1.

The equation then certainly indicates the endproducts, but the progressive change of the reaction, considering the properties of the substances, would be that NH_3 removes CNH from the cyanhydrins to form ammonium cyanide, which is for the greater part hydrolysed, while with NH_3 the liberated aldehyde yields the amino alcohol, which then reacts with CNH.

Geophysics. — *"On the application of* DARWIN'S *method to some compound tides."* By M. H. VAN BERESTEYN. (Communicated by Dr. J. P. VAN DER STOK).

(Communicated in the meeting of October 29, 1910).

Sir G. H. DARWIN has given a method for reduction of tidal observations, which in the case of 24 hourly daily observations has been fully described in his "Scientific Papers" Vol. I, p.p. 216—257.

Briefly the method consists of evaluating a special hour corresponding to 12^h m. s. time of any day; taking the speed of the tide equal to $15^\circ p$ $(p = 1.2 \ldots)$ for the hours 12 . . . 0 and 12 . . . 23; summing the observations arranged under the same special hour; then by harmonic analysis from those sums (24) both components of the tide can be found.

It appears from the table on p. 241 l. c. that this method is also applied to the compound tides MS, $2SM$, and $2MS$. As no mention has been made of disturbing influences, which these tides may undergo from others, it is of some importance to show, that these tides, when calculated after DARWIN'S method, need a correction. Moreover as may appear from what follows, the two combining tides M_2, M_4 are in the same manner influenced resp. by MS $2SM$ and $2MS$.

Suppose the speed per m. s. hour of a tide to be $= p.\sigma$.

Then the speed of a compound tide (R_1, ζ_{r1}) consisting of one of the tides of this series and one of the S series. $(S_{2.4.6})$ is generally:

$$\sigma_1 = p\sigma + 15^\circ q$$
$$(p = \pm 1 \pm 2 \ldots$$
$$q = \pm 1 \pm 2 \ldots)$$

For a special hour $\tau = \tau' - t$ we have after DARWIN's assumption:
$$15^\circ (p+q)(\tau+\alpha) = (p\sigma + 15^\circ q)\,12 - 15^\circ (p+q)\,t + 24\,p\sigma i - n.2\pi\,, \quad (1)$$
if τ' be the special hour corresponding to mean solar time: 12^h, day i; $\alpha = -0.5 \ldots +0.5$ spec. hour and $n = 1.2 \ldots$

The observation entered in the column of this spec. hour τ is now that of m. s. time: $(12 - t)^h$, day i.

At this date the influence of another compound tide (R_2, ζ_{r2}) with speed per m. s. hour $\sigma_2 = k\,r\,.\sigma + ks\,.\,15^\circ$, where
$$k = \pm 1 \; r = \pm 1 \pm 2 \ldots s = \pm 1 \pm 2.$$
is
$$R_2 \cos\{(kr\,.\,\sigma + ks\,.\,15^\circ)(12 - t) + 24\,kri\sigma - k\zeta_{r_2}\}$$
In connection with (1) this can be written
$$\begin{aligned} R_2 \cos [15^\circ (p+q)(\tau+\alpha) &+ \{\sigma (kr - p) + 15^\circ (ks - q)\}\,12 \\ &+ \{15^\circ (p + q - ks) - k\,r\,.\,\sigma\}\,t \\ &+ 24\,\sigma\, i\,(kr - p) \\ &- k\,\zeta_{r_2} \qquad]. \end{aligned}$$

If $kr = p$ i. e. the two compound tides R_1, R_2 are composed of the same tide R_p and one of the tides $S_{2,4,6}$ this influence of R_2 at special hour τ becomes.
$$R_2 \cos[15^\circ (p+q)(\tau+\alpha) + (ks-q)\,\pi + \{15^\circ (p+q-ks) - p\sigma\}\,t - k\zeta_{r_2}].$$
The number of observations being great, α varies from $-0.5 \ldots +0.5$ and t will assume all 24 integral values between $-11 \ldots +12$. Therefore the influence of R_2 on the mean sum of the R_1 arrangement at τ hour special time is:
$$\frac{1}{24} \sum_{t_1=-11}^{t_2=+12} \int_{\alpha_1=-0.5}^{\alpha_2=+0.5} R_2 \cos[15^\circ (p+q)(\tau+\alpha) + \{15^\circ (p+q-ks) - p\sigma\}\,t + (ks-q)\,\pi - k\zeta_{r_2}]\,d\alpha$$
or
$$= \frac{1}{F_{p+q}} \frac{\sin\{15^\circ (p+q-ks) - p\sigma\}\,\frac{24}{2}}{24 \sin\{15^\circ (p+q-ks) - p\sigma\}\,\frac{1}{2}} R_2 \cos[15^\circ (p+q)\,\tau +$$
$$+ \tfrac{1}{2}\{15^\circ (p+q-ks) - p\sigma\} + (ks-q)\,\pi - k\zeta_{r_2}].$$
Where
$$F_{p+q} = \frac{(p+q)\,7^\circ.5}{\sin (p+q)\,7^\circ.5}.$$
If we put
$$a_{r_2} = \frac{1}{F_{p+q}} \frac{\sin\{15^\circ (p+q-ks) - p\sigma\}\,\frac{24}{2}}{24 \sin\{15^\circ (p+q-ks) - p\sigma\}\,\frac{1}{2}}$$
$$\Theta_{r_2} = \tfrac{1}{2}\{15^\circ (p+q-ks) - p\sigma\}$$

the influence of R_2 on the components of the tide R_1 arranged according to DARWIN's method, that is, on

$$\begin{matrix} A_{p+q} = \\ B_{p+q} = \end{matrix} \frac{1}{12}\left[h_\tau \begin{matrix} \cos \\ \sin \end{matrix} 15^\circ (p+q)\tau \right]$$

is then

$$\begin{matrix} \delta A_{p+q} = \\ \delta B_{p+q} = \end{matrix} \alpha_{r_2} R_2 \begin{matrix} \cos \\ \sin \end{matrix} \left\{ k\zeta_{r_2} - \Theta_{r_2} - \pi (ks - q) \right\}.$$

For the compound tides MS. $2\,SM$. $2\,MS$. we have

$$\sigma = 14^\circ.4920521$$

Now for the required tide:

$M_2, p=2\ q=0$; $MS, p=2\ q=2$; $2\,SM, p=-2\ q=4$;
disturbing tide:

$MS, k=1\ r=2\ s=2$; $M_2, k=1\ r=2\ s=0$; $M_2, k=-1\ r=2\ s=0$; $2SM = -1 = -2 = 4$; $2SM = -1 = -2 = 4$; $MS = -1 = 2 = 2$;
required tide: $2\,MS,\ p=4\ q=-2$; $M_4,\ p=4\ q=0$;
disturbing tide: M_4 ; $k=1\ r=4\ s=0$; $2MS$; $k=1\ r=4\ s=-2$;
whilst: $ks - q = \pm 2n$, $n = 1, 2$.

With these data the values of α_{r_2} and Θ_{r_2} for these tides are as given in the following table.

	Influence of:					
	M_2	MS	$2SM$		M_4	$2MS$
on:	$k=$ —	1	-1	on:	$k=$ —	1
M_2	$\alpha=$ —	-0.0348	0.0171	M_4	$\alpha=$ —	0.0595
	$\Theta=$ —	$-14^\circ.5$	$30^\circ.5$		$\Theta=$ —	$16^\circ.0$
	$k=1$	—	-1		$k=1$	—
MS	$\alpha=0.0314$	—	0.0118	$2MS$	$\alpha=-0.0704$	—
	$\Theta=15^\circ.5$	—	$45^\circ.5$		$\Theta=-14^\circ.0$	—
	$k=-1$	-1	—			
$2SM$	$\alpha=-0.0177$	-0.0124	—			
	$\Theta=29^\circ.5$	$44^\circ.5$	—			

It appears from the values of α_{r_2} that by applying DARWIN's method, the compound tides with the same absolute daily motion influence each other more or less. Equally, the combining tide is disturbed by them and inversely, independent of the hourly motion of the tide, provided the absolute daily motion be the same. When M_2 is great, compared with MS and $2SM$ the disturbance caused by it on the latter is important and a correction is necessary. These are for the components obtained from 24 hourly observations of

$$\left.\begin{aligned} MS: -\delta A_4 &= \\ -\delta B_4 &= \end{aligned}\right\} - 0.0314\, Rm_2 \left\{\begin{aligned} &cos . \\ &sin . \end{aligned}\right. (\zeta m_2 - 15°5)$$

$$\left.\begin{aligned} 2SM: -\delta A_2 &= + \\ -\delta B_2 &= - \end{aligned}\right. 0.0177\, R_{m_2} \left\{\begin{aligned} &cos. \\ &sin. \end{aligned}\right. (\zeta m_2 + 29°.5)$$

where R_{m_2} and ζm_2 represent the amplitude and phase at the beginning of the treated year, which can be calculated with sufficient accuracy from A_{m_2}, B_{m_2}. In the table the influence of the tide of long period MSf, on the other tides with the same daily motion has not been inserted. On account of the smallness of this tide the disturbance may be neglected by the side of M_2.

Again, the tides M_4 and $2MS$ can only be separated from each other by solving A_{m_4}, B_{m_4}, A_{2ms}, and B_{2ms} from the 4 equations:

$$A_4 = A_{m_4} + 0.0595\,(2MS)\, cos\, (\zeta_{2ms} - 16.0)$$
$$B_4 = B_{m_4} + 0.0595\,(2MS)\, sin\, (\zeta_{2ms} - 16.0)$$
$$A_2 = A_{2ms} - 0.0704\, M_4\, cos\, (\zeta_{m_4} + 14.0)$$
$$B_2 = B_{2ms} - 0.0704\, M_4\, sin\, (\zeta_{m_4} + 14.0).$$

where A_4 B_4 A_2 and B_2 are obtained from harmonic analysis of the arrangement of M and $2MS$.

It follows from what has been said above, that DARWIN's method of calculating these compound tides is not very suitable. A simpler and theoretically more accurate way for evaluating their components and those of the M series can however easily be computed from Dr. VAN DER STOK's arrangement of the observations at a same solar hour. In this way by a single arrangement all compound tides and principal tides are to be found.

Finally it may be noticed that only the mutual influence of the tides MS, $2SM$, $2MS$, M_2 and M_4 has been determined. For, though more compound tides of M and S may be proved to exist [1]), only those above mentioned have been frequently evaluated after DARWIN's method.

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1) Dr. VAN DER STOK's Etudes des Phénomènes de Marée sur les côtes néerlandaises. IV.

Chemistry. — *"The velocity of the Ring opening in connection with the composition of the unsaturated Ring systems."* By Prof. J. Böeseken and A. Schweizer. (Communicated by Prof. A. F. Holleman.)

(Communicated in the meeting of Oct. 29, 1910).

Although the investigations of this subject have only just commenced, the communication by Rivett and Sidgwick (Trans. Chem. Soc. 1910, 1677) forces us to make a preliminary communication as to the object of the research and of some experiments. As is known von Baeyer has pointed out that the stability of the five-ring systems is, according to the idea of van 't Hoff, a consequence of the distribution of the attraction centres or affinity directions on the quadrivalent carbon atom; it is, however, obvious that, even without accepting these attraction centres, but assuming that the atoms attached to the central atom distribute themselves as evenly as possible in space (Werner, Neuere Anschauungen), we must conclude that a ring composed of five quadrivalent atoms must possess a lesser shifting tension than a corresponding four or six-ring. This, however, only avails in so far as the ring-forming atoms are not attached by a multiple linking to other atoms.

If a quadrivalent atom is only attached to three others (to one of them, therefore, with a double bond) and if we assume that these arrange themselves again as evenly as possible in space around the quadrivalent atom they will place themselves in the same plane, while the connecting lines between them and this atom will form angles of 120°. This is the angle of the regular hexagon; if now we are dealing, as in the case of benzene and pyridine, with a similar six-ring system its stability must, probably, be attributed to the absence of the ring tension.

Reversely, the occurrence of a double bond in a saturated five-ring will cause an increase of the ring tension.

As a first orientating investigation we have compared the velocities of hydration of succinic and maleic anhydrides with each other; in the latter this is indeed much more considerable than in the former; the unsaturated five-ring is more quickly opened than the saturated one. To this single fact we must not, of course, attach an absolute value. In the first place the five-ring does not entirely consist of carbon atoms, the oxygen will no doubt cause a tension and, moreover, the double bonds of the carbonyl groups will probably exercise a loosening influence.

But as the oxygen-containing part of the two molecules is equal, we thought this difficulty had been greatly avoided.

There is, however, another matter which demands consideration: the double bond will cause, in addition to the change of equilibrium, also a modification in the affinity, which may also influence the velocity of hydration in a marked degree. This influence is, up to the present, quite unknown and will only be elucidated when a large variety of saturated and unsaturated systems have been investigated.

Provisionally, we mention, therefore the fact that the hydration constant of maleic anhydride at 25° was too large to admit of correct measurement, whereas we have found it at $0° = 2.3025 \times 0.125$, and that of succinic anhydride at $0° = 2.3025 \times 0.0088$. The constant is calculated for a unimolecular reaction with the minute as time unit.

VOERMAN (Rec. **23** 279 (1904)) found for the latter at $25° = 2.3025 \times 0.0736$.

RIVETT and SIDGWICK (l. c.) found at 25° for succinic anhydride 2.3025×0.0693 therefore considerably less than VOERMAN and for maleic anhydride $= 2.3025 \times 0.690$.

The method which we, like the English investigators have followed is the one employed by VOERMAN, namely, that of measuring the conductivity in given intervals of time and ascertaining in this way the concentration of the acid molecules formed and consequently the anhydride molecules which have disappeared; the conductivity for different concentrations of the acid had been first determined. The end value of the conductivity then also indicates the initial concentration of the anhydride. This concentration was very small so that we could feel sure of dealing with a unimolecular reaction. After every half minute a reading was taken. When calculating the reaction constant of succinic anhydride at 0° we had to disregard the first twelve observations, after dissolving the anhydride in the resistance cell. For a concentration 0.027 N, $K = 2.3025 \times 0.00868$; for $C = 0.0015$ N, $K = 2.3025 \times 0.0082$ average, therefore, 0.0088×2.3025.

The maleic anhydride was prepared from malic acid with PCl_5 and after a few distillations recrystallised from dry chloroform.

The opening of the ring was already completed at 25° after $2^1/_2$ minutes; notwithstanding violent shaking before taking the readings no constant could be obtained. RIVETT and SIDGWICK, who took an observation every 5 or 7 seconds have been able to calculate, therefrom, a constant; with them the opening of the ring was also completed after about $2^1/_2$ minutes.

At 0° we succeeded in getting very concordant values, namely, for the concentrations 0.037, 0.027, and 0.0356 N were calculated:

$$K = 2.3025 \times 0.124$$
$$\times 0.124$$
$$\times 0.126$$

Whereas the English investigators found that at 25° the hydration of maleic anhydride proceeds about ten times more rapidly than that of succinic anhydride, we found at 0° the relation of 14.2 : 1.

Delft. Org. Chem. Lab. Technical University.

Botany. — "*The cause determining the selection of food in some herbivorous insects*". By Prof. Ed. Verschaffelt. (Communicated by Prof. Moll).

(Communicated in the meeting of October 29, 1910).

I. *Pieris Brassicae* L. and *P. Rapae* L. In agriculture and horticulture, experience has shown that caterpillars of the above mentioned species of the genus *Pieris*, the greater and the lesser cabbage butterfly, are fairly strongly specialised in the choice of their host plants. Whilst various *Cruciferae*, especially cultivated ones, constitute the ordinary food of these caterpillars, plants of other orders are only exceptionally attacked by them, the most noteworthy of the latter being species of *Tropaeolum* and *Reseda* [1]). Yet it is exactly the nature of these two genera which furnishes the answer to the question, by what chemical constituents of its plant-food the choice of the *Pieris*-caterpillar is determined.

The glucosides, from which by hydrolysis the mustard-oils (alkylisosulphocyanates) are formed, are highly characteristic of *Cruciferae*, but are also found in *Tropaeolaceae* and *Resedaceae* [2]). This is shown by the pungent odour perceived after bruising these plants. Experimental investigation completely confirms, as will be shown, that these insects are to a great extent guided in their choice of plant-food by the presence of such glucosides.

So far as concerns the *Cruciferae* themselves, both species of *Pieris* appear to be able to attack all sorts of plants belonging to this order. At any rate leaves of the following species given to caterpillars in captivity were all eaten. Among these species are representatives of various indigenous subdivisions of the order.

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[1]) Compare for instance Sorauer. Pflanzenkrankheiten. 3e Aufl. Bd. 3. 1910 p. 398.

[2]) Literature on the point in Czapek. Biochemie der Pflanzen. Bd. 2. 1905. p. 232; and Kobert. Intoxikationen. 2e Aufl. Bd. 2. 1906. p. 539.

Cochlearia Armoracia L.
Sisymbrium officinale (L.) Scop.
,, *strictissimum* L.
Sinapis arvensis L.
Brassica oleracea L.
Crambe cordifolia Stev.
Barbarea vulgaris R.Br.
Cardamine hirsuta L.
Capsella Bursa Pastoris (L.) Mnch.
Aubrietia deltoidea (L.) DC.
Arabis alpina L.
Erysimum Perofskianum Fisch. et Mey.
Alyssum saxatile L.
Hesperis matronalis L.
Bunias orientalis L.

Nevertheless all species were not equally readily eaten. *Capsella Bursa Pastoris* even, whether offered alone or with another Crucifer, was only very slightly attacked by the caterpillars. Other plants of this order which are also less welcome food, though more of them was eaten than of shepherd's purse, are *Aubrietia deltoidea*, *Erysimum Perofskianum* and *Sinapis arvensis*. With regard to this phenomenon, I have observed no difference between *P. Brassicae* and *P. Rapae*. The reason for their distaste, which probably may be looked for in the presence of subsidiary constituents, is still to be explained.

Not only the leaves, but also the flowers and young fruits of the species mentioned are eaten; in the case of *Cochlearia Armoracia* even the root was not rejected; the caterpillars however refused to eat the pounded and moistened seeds of *Brassica nigra* L.

Glucosides of mustard oils are in general widely distributed in *Capparidaceae*. *Capparis spinosa* L., *Steriphoma paradoxum* Endl., *Cleome spinosa* L., species of this order which were investigated, appear very attractive to *Pieris*-caterpillars. Indeed, during this summer, the *Cleome's*, cultivated in the Amsterdam botanic garden, had suffered greatly from the cabbage-butterfly. *Resedaceae* and *Tropaeolaceae* have already been mentioned. Out of the former of these two orders, *Reseda lutea* L., *R. luteola* L., *R. alba* L., *R. virgata* Boiss. and Reut., were given to the insects, with the result that the plants were always gnawed, although not very eagerly. *Tropaeolum majus* L. and *T. peregrinum* L. belong on the other hand to the plants which are most quickly eaten up.

There are still three orders known to be characterised by the possession of mustard oils: the *Moringaceae*, the *Limnanthaceae*, and the *Caricaceae*. I had no representatives of the first two orders at my disposal; out of the last named *Carica Papaya* L. could be investigated: the leaves of this plant were indeed refused.

Afterwards the behaviour of both species of *Pieris* was observed in relation to a succession of plants, chosen at random from different subdivisions; there were species belonging to the following orders: *Liliaceae, Chenopodiaceae, Caryophyllaceae, Papaveraceae, Saxifragaceae, Rosaceae, Leguminosae, Geraniaceae, Malvaceae, Violaceae, Umbelliferae, Gentianaceae, Borraginaceae, Labiatae, Caprifoliaceae, Cucurbitaceae*, and *Compositae*. In general the parts of the aforementioned were avoided; in some instances they were gnawed to a small extent, though even then usually very slightly. This was the case with the root of *Daucus Carota* and with the leaves of some species of *Lathyrus*, namely *L. sylvestris* L. and *L latifolius* L., whilst *L. tuberosus* L., and other *Leguminosae* experimented upon remained unattacked. As there was nothing in the odour perceived after bruising such *Lathyrus*-leaves as were eaten, to lead one to suspect the presence of mustard-oils, and the existence of them in the carrot may be excluded, it follows from these observations that both *Pieris*-caterpillars occasionally, and to a small extent, attack plants in which these substances do not occur.

To what extent indeed these insects are attracted by mustard-oils is clear when the leaf of a species not otherwise eaten by them, namely, *Apios tuberosa* Mönch., is smeared with a paste or the juice obtained from the leaves of a Crucifer *(Bunias orientalis)* and is offered them as food. It was at once attacked and in a short time devoured. The same occurrence with the leaves of other plants needs no further explanation, though it will easily be understood that every species cannot be used for such experiments. Thus the leaves of *Salvia officinalis, Prunus Laurocerasus, Menyanthes trifoliata*, also rubbed with *Bunias*-juice, remain untouched, doubtless because they contain constituents which are distasteful to the caterpillars.

It is however unnecessary to place the Crucifer-juice on a living leaf. Wheat-flour or material of still simpler composition, maize-starch, for example, which is rejected by both *Pieris*-caterpillars when dry or moistened with water, I saw eaten with avidity when soaked with some drops of *Bunias*-juice. The excrements of the caterpillars were almost white in colour and consisted of a mass of starch grains not appreciably attacked. The insects behave in a similar manner towards filter-paper saturated with *Bunias*-juice. A microscopical examination of the excrements showed scarcely any other constituent in it than the matted paper fibres.

These experiments are also of interest because they prove that the *Pieris*-caterpillars are not only attracted by the unsplit glucosides as these occur in living plants, but that they also seek the fission-products themselves. For in bruising the fresh leaves the glucosides are hydrolysed, and the mustard-oils set free. There still remains the question whether the presence of a very small quantity of free mustard-oils in the leaves is not the reason why the caterpillars are attracted. In some Crucifers, for instance in *Bunias orientalis,* a faint yet distinct odour of mustard-oil can be perceived in the unbruised leaf. It is very possible that the larvae of *Pieris* perceive the same odour in Crucifers which are odourless to our organs; and since caterpillars do not as a rule taste other plants, it is fairly clear that they must be informed as to the nature of food offered by its odour. This is to a still greater extent necessary for the butterflies, which only lay their eggs on Crucifers or on plants chemically related to them, and will doubtless recognise these by the odour.

Now in all the foregoing experiments plants or juices are used in which all kinds of substances occur, and thus there is no guarantee that it is exactly those constituents which are more noticeable to us, that so specially attract *Pieris.* This is proved by wetting leaves of *Apios tuberosa* and *Rosa* with a fairly strong solution of pure sinigrin (potassium myronate), the glucoside from black mustard [1]). Such leaves are eagerly eaten by the larvae. Whether therefore the caterpillars find also the sinigrin as such agreeable, or whether the glucoside is split up in their mouth by their own saliva, it is proved by this that these insects are specially attracted by the presence of this or of like constituents in their food.

We can draw a further conclusion from the foregoing experiments. Cabbage-butterflies were found to eat widely different species of Crucifers. Chemical investigation has moreover shown that not all the plants of this order contain the same glucosides, but various ones differing according to the nature of the mustard-oil which can be set free from them. While out of sinigrin from black mustard the allyl-compound is obtained, other Crucifers yield an isosulphocyanate in which another alkyl-group is present. Also the mustard oil in *Tropaeolum* is peculiar to it (compare CZAPEK, l. c.). It is thus clear that the *Pieris*-caterpillars seek out various mustard oils, just like the various glucosides derived from them. They are clearly attracted by the whole group of substances.

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[1]) This substance was kindly placed at my disposal by Dr. N. H. COHEN of Amsterdam.

Finally this observation led to the question whether perhaps plants which possess more or less closely related constituents would also be eaten by *Pieris*. Special attention was given to the species of the genus *Allium*, in which no isosulphocyanates occur, although there are various alkylsulphides, whose pungent odour shows some likeness to that of the oils of the Crucifers. Experiments indeed showed that *P. Brassicae* and *P. Rapae* both eat fairly readily the organs of various species of *Allium*. I mention for instance the bulb-scales of *Allium Cepa* L., the leaves of *Allium Porrum* L., and those of *Allium azureum* Ledeb. The caterpillars therefore probably do not distinguish the odour of these plants from that of the Crucifers, although they refuse other strongly smelling plants such as *Salvia officinalis* and *Mentha piperita*.

II. *Priophorus Padi* L. The larvae of this leaf-wasp live at the expense of the leaves of various *Rosaceae*, especially species of *Prunus*, *Sorbus Aucuparia* L. and *americana* Pursh., *Crataegus Oxyacantha* L. They are not uncommon in gardens.

The following species were eaten by larvae in captivity:

Cotoneaster tomentosa Lindl.
Mespilus germanica L.
Amelanchier vulgaris Mönch.
Crataegus Oxyacantha L.
,, *Pyracantha* Pers.
Cydonia vulgaris Pers.
,, *japonica* Pers.
Sorbus Aucuparia L.
,, *americana* Pursh.
Prunus Persica Sieb et Zucc.
,, *avium* L.
,, *Cerasus* L.
,, *Laurocerasus* L.
,, *Padus* L.

Leaves of *Eriobotrya japonica* Lindl., *Photinia serrulata* Lindl., and species of *Pirus* (*P. Malus* L., *P. salicifolia* L.) were always refused. Once a little was gnawed from a leaf of *P. Ringo* Wenzig. No single species out of the subdivision *Rosoïdeae* could be utilised by the larvae, nor out of that of *Spiraeoïdeae*. All experiments with plants belonging to other orders yielded negative results [1]).

[1]) Some handbooks give still other plants for this insect, as for example the birch. (HENSCHEL. Die schädlichen Forst- und Obstbaum-Insekten, 1895, p. 244). My larvae, however, refused to eat from these.

In going over the composition of the above list, we see quite clearly that the plants which *Priophorus Padi* willingly uses for food, all belong to species in which a glucoside like amygdalin is present, nay, in which such a glucoside is so abundantly present that on bruising the leaves the odour of benzaldehyde and HCN can be perceived. Although other *Rosaceae*, including some of the species refused, possess a little prussic acid, as for instance species of *Spiraea*, *Kerria*, and *Pirus* [1]), they are very much less rich in this constituent than most forms of *Prunus*, *Crataegus*, etc. Also the numerous other plants, from which prussic acid has lately been obtained, usually only possess traces of it. I have not had the opportunity of placing before *Priophorus* the species of exotic plants which are rich in HCN. But the larvae did not eat species of *Thalictrum*, *Sambucus* and *Ribes*, all of which are known to contain a small quantity of the same substance.

In order to furnish proof that the glucosides, from which by fission benzaldehyde and prussic acid originate, represent indeed the constituent that determines the choice of food of the *Priophorus*-larva, various leaves of non-edible species were moistened with a solution of amygdalin from bitter almonds. The experiments were by no means so uniformly successful as those of *Pieris* with sinigrin, partly because the larvae were found to eat unwillingly leaves covered with fluid; when finally, however, the leaves were simply smeared with dry amygdalin, the insects were seen to attack *Apios*-leaves thus treated, albeit only after waiting for some days.

Leaves of *Apios*, *Fragaria*, *Rosa*, smeared with a paste of bruised leaves of *Prunus avium*, were similarly eaten, as soon as the paste had somewhat dried.

III. *Gastroidea viridula* Goerz. A little beetle of which both larva and imago feed on the leaves of species of *Rumex*. During the past summer this insect occurred in the Amsterdam Botanic Garden on *Oxyria digyna* Hill. and *Rumex scutatus* L. In previous years the damage in the Polygonaceous bed was often of greater extent; and in addition to *Rumex*, the leaves of cultivated species of *Rheum* were also occasionally eaten full of holes by this insect.

The well-known acid properties of the cell sap of the abovementioned *Polygonaceae* justified the question whether *Gastroidea* would not specially seek plants which were rich in oxalic acid or acid oxalates. Indeed, although the beetles show a special predilection

[1]) See, for instance, the list of cyanogenetic plants published by M. GRESHOFF in Bull. des Sciences pharmacolog. tome 13, 1906, p. 598 et seq.

for leaves of *Rumex*, *Oxyria* and *Rheum*, yet they also eat a little from the leaves and flowers of *Begonia* (*manicata* Cels., *tuberosa* Lam., *ricinifolia* A. Dietr. and others). Leaves of *Oxalis* (*floribunda* Lehm., *Deppei* Lodd.) were on the other hand always refused. It should however be stated that on account of the far advanced season the latter experiments could only be done with the imago, which also rejected species of *Portulaca*, *Celosia*, and *Sedum*, and only eat slightly from *Polygonum* (*cuspidatum* Sieb. et Zucc. for instance).

Also in the case of this insect direct proof can be adduced that the occurrence of a definite substance in the plants determines whether they shall or shall not serve as food. Leaves of *Lathyrus sylvestris* are never eaten fresh, but it was repeatedly observed that the beetles gnawed at them when for some time they had been immersed in a normal solution of oxalic acid. In the absence of material, through the disappearance of the last beetles, it was impossible to investigate further whether they are attracted by the acid reaction in general or whether they are exclusively dependent on oxalic acid. Next summer I hope to be able to decide this point and also to trace the behaviour of *Gastroidea* towards plants in which other organic acids (malic or citric) occur in considerable concentration.

Botany. — "*The mechanism of the absorption of water by the seeds of the Cucurbitaceae*". By Prof. ED. VERSCHAFFELT. (Communicated by Prof. J. W. MOLL.)

The seeds of the *Cucurbitaceae* are in general distinguished by the rapidity with which they take up water. This property is strongly developed in the ordinary cultivated varieties of the species *Cucurbita Pepo* L. and *Cucurbita maxima* Duch., our gourds and pumpkins. In the accompanying figure (fig. 1) one curve represents the increase in weight, during the first hours, of the air-dry seed of a variety of large yellow gourd, "Grosser gelber Zentner", when placed in water, and the other curve shows the same for a variety of *Vicia Faba* L., the Mazagan bean. The determination was made by taking the seeds out of water from time to time, drying them superficially with filterpaper, and weighing them. The estimation was made with a small number of seeds, so that, on account of the individual differences, a repetition of the experiment gives curves which do not agree completely with the one reproduced here. It is, however, easy to convince one's self that the general shape of the curv remains unaltered, and that the remarkable difference between the curves of pumpkin-seeds and beans always appears.

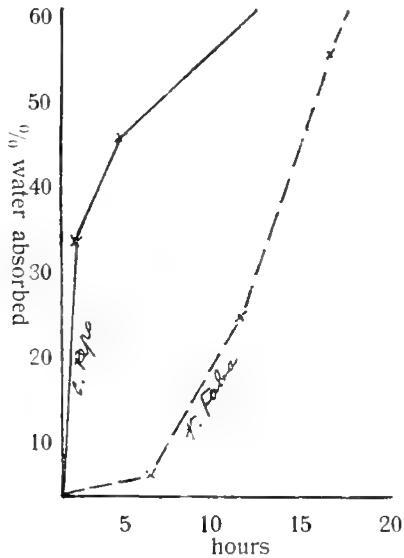

Fig. 1.

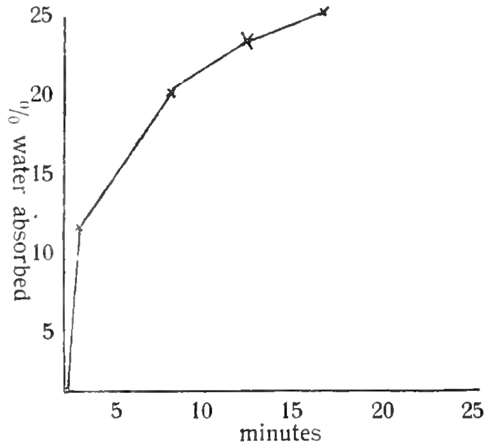

Fig. 2.

While therefore in *Vicia Faba* the increase of weight is at first a very gradual one, it begins in *Cucurbita Pepo* so rapidly, that already at the end of the first hour water to the extent of 34% of the dry weight was taken up, in the experiment of fig. 1; this is a third of the total quantity which the seed was found able to retain.

Even in the first few minutes the absorption is specially marked. It is represented graphically (fig. 2) for the same yellow variety of gourd ("gelber Zentner"). As will be seen, in twenty minutes water was taken up to the extent of more than 25% of the air-dry weight of the seed.

It is natural to hold the seed-coat responsible for such a rapid and considerable absorption of water, and we find indeed that the seed-coat of *Cucurbita*, in contradistinction to that of many other seeds, especially that of *Vicia*, is immediately wetted. If a drop of water be placed on the seed, one recognises by the darkening of the white surface round about the drop of water, that the liquid has been sucked up by the tissue, as if by blotting paper.

This is therefore the cause of the *rapid* imbibition. But that such a *large* quantity of water is taken up in a short time, is also a result of the properties of the seed coat, as is evident from a comparison between the absorption of water by a pumpkin-seed as a whole and that of the interior of the seed, freed from its hard testa. The former can easily be dissected out and then consists principally of the germ surrounded by a thin greyish green membrane, which is firmly connected with it.

Such seeds of the large yellow gourd, after being deprived of their testa, absorbed:

during	1^h	4^h	24^h	50^h
water	8 °/₀	20 °/₀	36 °/₀	42 °/₀

whereas in the same times, the entire seed took up:

	34 °/₀	48 °/₀	86 °/₀	96 °/₀

and afterwards this considerable difference was maintained. If we consider that the testa only constitutes $^1/_6$ to $^1/_5$ of the weight of the seed, it follows from this proportion that the testa must be very rich in water. Direct determinations indeed showed that a testa, separated off and placed in water, finally underwent an increase in weight of 194 °/₀. This was in the variety Courge gaufrée of *Cucurbita Pepo,* a variety which takes up less water than the large yellow gourd ("gelber Zentner") previously experimented on, where the increase in the testa was 228 °/₀. On comparison of the absolute quantity of water taken up by the seed freed from its testa, with that absorbed by the testa alone, the latter quantity is found to be the larger. In Courge gaufrée, where some seeds took up on the average 0.25 gram of water, the testa of a single seed contained on the average 0.16 gram of water.

We will now consider by what peculiarities of the anatomical structure the testa in Cucurbita is able to hold so much water. For this purpose the accurate description by F. v. Höhnel (*Sitzungsber. K. Akad. Wiss. Wien,* Math. Nat. Cl. Bd. 73 Abt. I. 1876. p. 297) can be used. As this paper and the plates attached to it can be referred to for the details, here only the anatomical arrangements which may have significance with regard to absorption, will be dealt with.

The thin transparent pellicule of which single pieces are frequently found still attached to the seed and which represents the inner layer of the fruit wall can here remain unnoticed, because it is so easily loosened and is without importance in the imbibition of the seed.

On the other hand the structure of the epidermis is in a high degree adapted to increase the absorption of water. This layer consists of elongated prismatic cells without cuticle or contents, whose walls swell very greatly in water, so that according to v. Höhnel's measurements the length of these cells in a dry condition is only 30 μ while in water it increases to 300 μ.

The presence of this epidermis explains how the seed so easily becomes moistened. Another question is moreover, whether all tissues which compose the testa in like measure take part in this storing up of water.

Apart from the innermost thin membrane, consisting of inner integument, perisperm and endosperm, which surrounds the germ and is firmly fixed to it, the testa of *C. Pepo* and *maxima* is divided into five layers. The two outer ones we are now acquainted with. Next within these there is a tissue, 4 to 6 cells thick, built up out of small elements with fairly thick walls (layer III). Layer IV is the schlerenchyma-layer, composed of a single row of hard, thick-walled and very irregularly formed cells; this gives hardness to the testa. Finally layer V is a strongly developed spongy tissue composed of cells filled with air, separated by numerous intercellular spaces.

This spongy tissue at once gives the impression of being specially fitted by capillarity to hold a large quantity of water. In how far this and the remaining tissues of the testa partake in this phenomenon, we shall endeavour to determine quantitatively.

By scraping with a knife, layers II and III are easily removed from the seed: thereupon the brownish schlerenchyma comes into view. If this is done partly with air-dry seeds partly with seeds soaked in water, then the change in weight will show how much water these two layers together take up. In the same way the spongy tissue (layer V) can be removed from a pealed testa and by a comparison of the loss of weight in the dry and soaked testa the water content can be deduced. Finally also layer IV, the schlerenchyma must be isolated and its absorption of water determined separately.

With seeds of the variety Courge gaufrée I obtained in this way the following results.

Layers.	Weight when dry.	Weight of water absorbed.
II—III.	0.02 gr.	0.02—0.03 gr
IV.	0.02—0.03 gr.	0.02 gr.
V.	0.01 gr.	0.08—0.10 gr.

The great importance of the spongy tissue as a water-reservoir is thus demonstrated.

The greater part of this water is absorbed by capillarity, while the air which fills the cells and the intercellular spaces is expelled.

Connected with this is the fact that the seed of the pumpkin, when thrown into water air-dry, floats, but after some time sinks to the bottom, when a sufficient quantity of air has been expelled from the spongy tissue. In Courge gaufrée this is the case as soon as the quantity of water absorbed amounts to 50—60 % of the air-dry seed.

Now its seems very likely *a priori* that the air finds an outlet through the hilum of the seed. There is indeed an opening by means of which the intercellular spaces communicate with the outside air; the schlerenchyma is interrupted at the level of the hilum (v. HÖHNEL l. c. p. 315). Bubbles of air are seen to escape from the hilum of a seed immersed in water, although slowly and not very profusely. Thus F. NOLL has also pointed out that this arrangement may assist the water-absorption in the seeds of *Cucurbita* (Landw. Jahrb. Bd. 30, Erg. Bd. III, 1902 p. 150; footnote). But in how far this actually takes place remained to be investigated experimentally.

For this purpose a comparison was made between the increase of weight in water of seeds with the hilum open or opened still further where necessary, and seeds in which the hilum had been closed by means of sealing wax or bees wax. In this way it was found that the significance of the hilum is not specially great. The water-absorption was but little retarded by the closing of the hilum; the differences between seeds treated in this way and those not so treated frequently falls within the limits of individual variations; and also in the case of carefully selected and apparently comparable seeds the difference is indeed generally in favour of those with open hilum, but it is always relatively slight.

A single example is quoted:

3 seeds of Courge gaufrée untreated, weigh air-dry	0.76 gr.
,, ,, whose hilum was closed with beeswax weigh air dry	0.77 ,,
after lying in water for 24 hours they weigh respectively	1.17 and 1.15 ,,
,, ,, ,, ,, ,, 48 ,, ,, ,, ,,	1.35 and 1.24 ,,
Thus there is water absorption after 48 hours of respectively	77.6 and 61.0 %

The difference which occurs here is among the most notable of those observed.

It is thus in the highest degree probable that the air, which is driven out of the spongy tissue, finds an outlet through the testa. This is also undoubtedly the reason why seeds with closed hilum do not in general remain floating longer than seeds which are not so treated. Moreover one can observe directly that water does not penetrate through the opening in the hilum to an appreciable extent. Even if this mouth is made somewhat wider, and it is immersed in a solution of a colouring matter, for instance 0.1 % methylene blue in water, so that the spongy tissue may come into contact with the fluid at the hilum, the fluid is only imbibed extremely slowly by capillarity. After 24 hours it has scarcely risen in the testa. On the other hand the coloured solution is quickly taken up by the surface of the seed, and it also diffuses very soon in the deeper layers.

It is different if the seed is immersed with the hilum in a fluid whose surface-tension is less than that of water, as for instance in alcohol. Such a fluid is soaked up much more quickly by capillarity (See inter alia L. Errera, Bull. Soc. belg. micr. t. 13, 1886, no. 3 and Rec. Inst. botan. Univ. Bruxelles, t. 2, 1906, p. 111). A solution of methylene blue in alcohol penetrates almost immediately into the hilum and in a very short time fills the whole spongy tissue. This can be seen very well if in a dry seed a piece of the testa is cut away from the extremity opposite to the hilum, and the latter is held in the solution. After a few seconds the blue fluid penetrates to the top, while the outer layer remains uncoloured and dry.

For the same reason a much quicker stream of air bubbles escapes from the hilum of a seed thrown into alcohol than when it is thrown into water. Also the weighings show important differences. A gourd-seed held with the hilum only in alcohol takes up in one minute 0.04 – 0.05 gr. of alcohol, whilst in the same time barely 0.01 gr of water is absorbed, and then moreover partly by imbibition of the outer layer. The outermost tissues of the testa only slightly imbibe alcohol; a seed immersed in alcohol so that only the hilum protrudes from the fluid, takes up in one minute 0.01—0.02 gr.; in water, under the same conditions about 0.05 gr. is taken up in the same time.

Along the narrow border and the whole way round on the outermost edge of the spongy tissue in the testa of *C. Pepo* and *C. maxima* there runs a canal filled with air that on its inner side contains the vascular bundle and opens out at the hilum, on the left and right (v. Höhnel l. c. p. 317).

In the experiment with coloured alcohol described above it is along this canal that the fluid first rises, in order to spread thence through the spongy tissue. It is therefore conceivable that the circumambient canal plays a dominant role in the capillary ascent. In order to decide this question a piece of the testa was removed half way up on both sides in such a way that the canal was interrupted on the left and right. As in the former experiment a piece of the testa was also removed from the round end of the seed and the hilum immersed in blue-coloured alcohol. Although now the fluid could penetrate to the top only through the spongy-tissue it did this about as quickly as in seeds in which the canal was uninterrupted: no important differences could be observed.

In order to give an idea of the rate at which water is imbibed by seeds of other *Cucurbitaceae,* the increase in weight is given in the following table for a number of species, expressed as a percentage of the dry weight, which the seeds undergo when they lie in water for one hour:

Acanthosicyos horrida Welw.	14.0 %
Momordica Charantia L.	9.6 „
Luffa cylindrica Roem.	4.1 „
Citrullus vulgaris Schrad. (with black seed)	25.6 „
„ „ „ (with red seed)	14.4 „
„ „ „ (American lemon)	16.4 „
Cucumis sativus L. (Excelsior)	13.9 „
„ „ „ (Indian giant net-cucumber)	24.4 „
Cucumis Melo L. ("cantaloup" from Algiers)	20.4 „
Benincasa cerifera Savi.	71.4 „
Lagenaria clavata	26.1 „
Trichosanthes Anguina L.	13.8 „
Cucurbita Pepo L. ("gelber Zentner")	33.7 „
„ „ „ ("Miracle")	18.2 „
„ *argyrosperma*	82.5 „
„ *melanosperma* A. Br.	16.3 „

It is seen from this table that the imbibition-velocity of different species varies considerably. The peculiarities of the structure of the testa, which explain these differences, will not be dealt with here for all the species mentioned. Attention will only be called to a few of the more notable cases. Only two species equal *Cucurbita Pepo* as regards the rate of absorption of water, viz. *Benincasa cerifera* and *Cucurbita argyrosperma*. Both these seeds finally take up very considerable quantities of water; after 4 or 5 days *Benincasa* increases in weight by 130%, and *C. argyrosperma* even by 150% of the original weight.

The seeds of *Benincasa* are found on microscopic investigation to possess an exceptionally thick layer of spongy tissue of the same structure as that observed in *C. Pepo* [1]). The spongy tissue in this case is most strongly developed on the *outer* side of the schlerenchyma instead of on the *inner* side; it is, in other words, layer III of

[1]) Compare H. A. Lothar. Anatomie comparée des Cucurbitacées Lille 1881 p. 215.

VON HÖHNEL which in respect of thickness as of power of imbibition, is the of most importance in this seed. A result of this is, that if the seeds are peeled down to the schlerenchyma and are then placed in water, they then take up much less water than seeds not so treated:

3 entire seeds weighing 0.18 gr. took up in 24 hours 0.13 gr. of water or 72%

3 peeled ,, ,, 0.15 ,, ,, ,, ,, ,, ,, 0.05 gr. of water or 33%

The great quantity of air present in the intercellular spaces is the reason why the seeds of *Benincasa* remain floating for a longer time on water than those of *Cucurbita Pepo*. Of ten *Benincasa*-seeds which were placed in water, two were still floating after 5 days, notwithstanding the increase in weight of the whole batch then already amounted to 132%.

The seeds of *Cucurbita argyrosperma* are very light and spongy to the touch and are specially characterised by the great development of the projecting edge of the seed, which, just as in *C. Pepo* and *maxima*, is divided by a groove from the body of the seed.

Especially this edge, which has a grey-blue colour while the rest of the seed is white, has a spongy structure, which appears very well adapted for taking up much water. Under the microscope it is found to consist of the same tissue with strongly developed air-spaces which constitutes layers III and V in *C. Pepo*. The schlerenchyma is not however present in the edge, which must therefore be considered as an excrescence of the testa which is formed by layer III and runs round the seed.

Its importance with regard to absorption is clear from the following observation:

3 entire seeds, weighing 1.22 gr. in 19 hours took up 1.19 gr. of water or 97%

3 seeds without edge, weighing 1.05 gr. in 19 hours took up 0.58 gr. of water or 55%.

The edge of the 3 seeds last mentioned was cut away beforehand. And this difference could still be observed three days later when the seeds which had been deprived of their edge had only taken up 78% of water, while on the other hand the entire seeds had taken up 131%.

When we now turn to the *Cucurbitaceae* whose seeds saturate themselves less thoroughly with water than those of *C. Pepo*, our attention is specially claimed by *Luffa cylindrica*, whose seeds, as will be seen from the above table, take up in the first hour eight times less water than those of the pumpkin and twenty times less than the seed of *C. argyrosperma*. Nevertheless the structure of the testa of *Luffa* is not so strikingly different from that of *Cucurbita* as might be expected from the above-mentioned fact. The testa of *Luffa* possesses, under a hard and thick schlerenchyma layer, a fairly well-developed spongy tissue. [1]). This contradiction is only apparent; the seed of *Luffa* indeed takes up water very slowly, but in the long run it absorbs a fairly large quantity of water, more than 80°/₀ of its own dry-weight.

The difference in the rate of imbibition between *Cucurbita* and *Luffa* is explained by a considerable difference in the structure of the outermost layer of the seed. This has a strongly developed cuticle, as appears on treatment with strong sulphuric acid, and undoubtedly on this account becomes moistened relatively slowly. The same reason does not account for the slighter absorption of *Cucumis sativus* and *C. Melo*. These seeds become quickly moistened, but they cannot store up much fluid, because the spongy parenchyma is more or less completely absent from the testa. [2]) Indeed the total quantity of water absorbed by the species of *Cucumis* mentioned does not exceed 60 °/₀ of the dry weight.

One word finally about the variety of *C. Pepo* included in the above list as "miracle", which possesses a low capacity for imbibition. The seeds of this variety are distinguished by their testa consisting only of a thin, soft, grey-green membrane that very easily becomes saturated with water, yet can only store up a small quantity of it. Spongy tissue and schlerenchyma are both entirely absent from this membrane, and the "miracle" seeds, when thrown into water, sink at once to the bottom. Finally their increase in weight amounts to no more than 50 °/₀; it is therefore comparable with that of the peeled seeds of *Curcurbita Pepo*.

Microbiology. — "*Individuality and heredity in a lower mould fungus, Trichophyton albiscicans*". By Prof. A. W. NIEUWENHUIS. (Communicated bij Prof. M. W. BEIJERINCK).

Introduction.

One of the most striking observations arising from a prolonged residence among the still primitive tribes of the East-Indian Archipelago, for example among the Dajaks of Borneo, is certainly that our dark fellow-men are endowed with so good mental dispositions, of the same nature as ours, that they may in general well be compared

[1]) Compare LOTHAR l. c. p. 219 and K. G. BARBER, Bot. Gaz. vol. 47, 1909, p. 305.
[2]) Description and figures in VON HÖHNEL, LOTHAR, and BARBER.

to those of the Europeans. The conditions of life of these tribes, however, unfavorable for their physical and psychical development, prevent in the first place the increase of their very small number of 1—3 per KM.[2] (on Java 230), and furthermore, in combination with the hence arising social relations, they largely contribute, in spite of the good intellectual powers of these tribes, to keep up among them a great want of knowledge, particularly respecting relations between natural phenomena. As to these their conceptions are so childish that to us, Europeans, it is very difficult to comprehend them. To a stranger in their surroundings and to one who does not positively reflect on them, the details of their society are the more deceiving as these primitive tribes know, of course, many particulars about their own milieu, so strange to a European, that the latter wants a long time to get acquainted with them. The connection and causes of those facts remain notwithstanding a closed book for the natives.

A proof for their power of observing visible phenomena which struck me particularly when I resided as a physician among the Dajaks of Middle Borneo, gave their diagnoses of parasitic skin diseases, so frequent among them, and about which their views differed from those common among the European physicians in the East Indies. They consider the decoloring of their hand-palms and foot-soles and contiguous parts of the skin, as the consequence of an independent cutaneous disease and separate it from other parasitic skin diseases, which, for the rest, they divide into groups, as is done in official physic.

A five years' practice among them, during which I was able to work out a treatment of that skin-disease, showed me that their diagnosis may be clinically well defended and that therapeutically much may be done against the disease by parasiticids as iodium and chrysarubine. After my return I found in 1901 in the Laboratory of the Department of Agriculture at Buitenzorg, Java, Trichophyton as probable cause of the disease and I then described this disease and the mould by the name of Tinea albigena in "Geneeskundig Tijdschrift voor Nederlandsch Indië", Dl. XLIV Afl. 6. Since this first treatise on this subject in 1902, Tinea albigena has been recognised as an independent skin disease, occurring in South East-India, as its appearance and clinical character has been established, besides by myself among the inhabitants of Java, Borneo, and Lombok, also on Java by KIEVIT DE JONGE, on Sumatra by SALM, on Ceylon by CASTELLANI, on New Guinea by VAN DER SANDE, and in the Bismarck Archipelago by SIEBERT.

On my return to Holland I succeeded at Leiden to cultivate this

Trichophyton and point out its parasitic properties by provoking Onychomycosis, occurring in Tinea albigena, by inoculating a nail with a culture of the mould-fungus from a diseased nail, containing numerous spores. Unter the title of "Tinea albigena und die Züchtung ihres Pilzes", the then obtained results were published in the "Archiv für Dermatologie und Syphilis" Bd. LXXXIX, Heft 1 and in "Geneesk. Tijdschrift voor Nederl. Indië" Vol. XLVIII, Afl. 1. The mould found and cultivated I named Trichophyton albiscicans as its growth and the strong pigmentatrophy in a dark coloured skin, attacked by Tinea albigena, justify this denomination.

Culture of Trichophyton albiscicans and its forms.

As I already showed more in detail in "Tinea albigena und die Züchtung ihres Pilzes", so many circumstances influence the macro- and microscopic form of this mould-fungus, that its description can only be given with a complete enumeration of its conditions of life, i. e. of the medium and its quantitative composition, the reaction of the latter, the shape of the culture vessel and the temperature of the growth, furthermore, the age of the culture, the illumination, the aeration etc. I have besides found that even the origin of the chemical ingredients, such as glucose, is not indifferent, and that pure white glucose of KAHLBAUM gives quite another result than a light yellow one of MERCK. That also peptones of different origin produce different results is a matter of course.

While working on solid media with various stocks of Trichophyton albiscicans, obtained from pathogenic material and otherwise, I have further found, that these distinct but pure stocks then only assume the same form, when growing on a favorable medium, and that their forms begin to vary very much as soon as modifications are introduced unfavorable to their functions, for instance, glycerine instead of sugars as source of carbon. The association of these changes will be shown below.

Though so large a variety of mycelia may be obtained by cultivation under varying conditions, it should be well kept in mind that this variety of forms does not exist when the mycelia have originated by inoculation of pieces from one and the same stock; also under unfavorable, but for the rest similar circumstances, they develop in quite the same way. This constancy in form of the same stock, which, as we shall see below, also shows itself in other biologic properties, is of great weight for the subsequent research and should therefore be kept in view.

As to its morphology this mould shows macroscopically what follows. A thallus of Trichophyton albiscicans, cultivated during 6 weeks on 4 KAHLBAUM glucose, 1 CHASSAING peptone, 2 agar, and 100 water, in a cotton-plugged ERLENMEIJER flask in ordinary daylight at 25—30° C., has the form of a flat round disk of $\pm 2^1/_2$ cm. diam., and the color of dirty-white velvet. The mycelium hardly penetrates into the medium, and develops so that the whole thallus may be removed from the still smooth agar surface. The spore-bearing hyphae appear later as a white, plushy substance, spread over the whole surface, especially when the nutrition grows worse by desiccation or other unfavorable conditions of assimilation. As said before, the forms vary by a change of medium, especially when this becomes worse. As transplantations show, the thallus is very coherent and a stiff iron needle should be used instead of a flexible platinum wire to cut off little bits.

Microscopically the hyphae are not very characteristic: they are mostly threads with walls and transverse septa of $1^1/_2\,\mu$ diameter, branching dichotomically. The length of the cells of the hyphae varies very much; there are round as well as oval ones, measuring from 15—20 μ, and more. Already at a very early stage, and later, there appear at the end of the hyphae spherical highly refracting corpuscles of 5 μ diam., with or without a short stalk. In cultures on a poor substratum, there appear, among the long hyphae cells short and nearly round ones, almost ten times larger, which are perhaps spores.

More characteristic than these parts of the thallus are the white, spore-bearing air-hyphae. The spores have the same oval or pear-shaped form as those found in the diseased nail substance in Tinea albigena. Like these they are seated immediately on the side of the air-hyphae, but they also occur terminal; they remain either single or they develop centrifugally to rows of from 2—8. Besides these rows of conidia, there appear on the air-hyphae grape-shaped groups, which consist of numerous rows of spores starting from one point or packed closely side by side. Other spores I did not find in this mould. The above has much in common with what is known of other corresponding lower moulds, in particular of those which were found in Europe as the cause of other trichophytic and allied skin diseases, cultivated afterwards.

Such was the state of my research in the publication "Tinea albigena und die Züchtung ihres PILZES" in 1908. At the "Congress Deutscher Naturforscher und Aertze" at Dresden in 1907, I could demonstrate these results with cultures in the section for "Tropenkrankheiten". I then also exposed my method for the culture of micro-organisms from one spore, which has rendered possible the following investigations, and to which I briefly referred in "Archiv für Dermatologie und Syphilis" Bd. LXXXIV p. 20, but which will more elaborately be described at the end of this research.

Investigation on Individuality.

While I was making the above observations, the cultures of Trichophyton proved to possess the following biological properties: 1 On a favorable nutrient medium, for example of 4 glucose, 1 peptone, 2 agar and 100 water (SABOURAUD) at 25° C. an acid is soon formed, which, by addition of litmus to the medium may already be pointed out after 10 days when the thallus has attained a diameter of 1 cm.; 2 such a medium without litmus becomes more and more gray after a few months, and finally, after 6 months, it gets a blackish color; 3 on a medium of 4 glucose, 1 peptone, 10 gelatin and 100 water already at $1^1/_2$—2 cm. diameter of the culture liquefying of the gelatine occurs, which later increases, whereby the gelatine, like the solid agar, assumes a brown color; 4 already after a month a white spore-bearing air-mycelium is formed, when the above agar is used at the places where, with an inclined surface, some part of the border of the thallus dries up; if the moisture is sufficient, however, it appears later; 5 the hyphae of the cultures show a strong catalase action.

H_2S is not formed, amylum not attacked; the fungus develops with much difficulty on an alkaline medium.

Though all the stocks possessed these various properties, they possessed them in different degrees; still, in a same stock certain forms and biological properties are fairly constant. The stocks with which I had operated had successively, from 1903—1904, originated from diseased nail tissue; moreover three stocks had each originated from one isolated spore. These three showed the same properties as the other stocks.

All this induced me to test the above mentioned differences of the properties in the mycelia, originated from a single spore, and thus representing one individual. The stocks obtained in the ordinary way by sowing of which it may be supposed that they consist of a mixture of mycelia, are unfit for investigations on individual properties. In order to be able to formulate the results with the greatest possible sharpness, the spores destined for the investigations, were taken from one individual, grown from a single spore.

Moreover to simplify as much as possible all the circumstances of the research, necessary rightly to appreciate the differences observed, the media were prepared of nutrient substances of quite the same quality, consisting only of glucose, agar, peptone and gelatine. The research ran as follows. Of a culture of Ag_{11} (11th generation of A) were successively isolated in September 1909 a number of 50 spores, which were put to develop in drops of nutrient liquid in moist chambers at $\pm$ 25° C.; from these 13 mould-fungi (numbered 1 to 13) developed. After a fortnight they had each grown sufficiently to be transferred from the liquid (4 glucose, 1 peptone, 100 water) to a 2 °/₀ agar medium of the same composition, where they grew out to vigorous mycelia. After plant N°. 2 had formed spore-bearing air-mycelium, it was selected to procure the spores for the individuals to be examined more particularly. From forty spores isolated in November 1909 were obtained 10 mycelia which shall be called I, II, III . . . X. Each of these, issuing from one and the same mould-fungus, was now examined with respect to the following biological properties: 1. characteristic form, 2. acid formation, 3. production of proteolytic ferment, 4. coloring of the medium and the thallus, 5. formation of spore-bearing mycelium.

For the preparation of the media, use was made of a kind of white glucose, peptone of CHASSAING in Paris, and always of the same agar or gelatine, sometimes colored with litmus or eosine and neutralised with KOH. For a same series one quantity of nutrient liquid was always boiled, which was filled into equal vessels, either PETRI dishes or ERLENMEIJER flasks, and the series of cultures put up side by side in the same cupboard, kept for those on agar at $\pm$ 25°, on gelatine at $\pm$ 20°. Perhaps it is not superfluous to add that I effected myself all the manipulations, also the cleaning of the vessels.

In each series only one mycelium of the different kinds was cultivated, so that the judged ones are not picked out from a series of the same kind.

In the beginning of 1910 the cultures I, II, III—X, had sufficiently grown to be transplanted, and accordingly, on 3 January 1910, in order to compare the forms, pieces of about 2 $\times$ 3 mm., of each of the mycelia were inoculated into media of the composition: 4 glucose, $^1/_{10}$ peptone, 2 agar and 100 water, in ERLENMEIJER flasks of 100 cm³. filled with a quantity of 60 grs.

By its small percentage of peptone this medium was unfavorable for the mould.

2. To compare the acid formation pieces of mycelium of equal size were at the same date transplanted to a medium of the favorable composition: 4 glucose, 1 peptone, 2 agar, 100 water, colored with

litmus and neutralised with KOH almost but not quite, in 40 grs. Petri dishes.

3. To compare the formation of proteolytic ferment, which liquefies gelatine, a medium was used of 4 glucose, 1 peptone, 10 gelatine, 100 water, into which were at the same time inoculated small bits of the different mycelia, in a 100 grs. Erlemeijer flask receiving 60 grs. of the medium.

The other characters, viz. the coloring of the medium and the origin of the white sporulating air-mycelium, could also be studied in these three series.

1. Regarding the form the following was observed during the growth: In the first three weeks not much can be perceived in the first series of cultures of mutual differences in quickness of growth and in form, and on all the substrata a white, slightly wrinkled, fleecy thallus of $1^1/_2$ cm. diameter has grown, with here and there a small, white, 1—2 mm. long erect "needle". On the 1st of March 1910, however, so after 3 months, the state was as follows:

I_{g_2} (2nd generation of I): The dirty-white, $\pm$ 3 mm. thick thallus has covered the whole surface of the substratum ($\pm$ 6 cm. in diam.), and consists of a central disk with a border of $\pm$ 1 cm. width. The disk is flat but here and there covered with small white, straight needles or bent threads from 1—3 mm. long. The border, however, is undulated with radiately directed ridges and grooves, without any needles, but the ridges have at their greatest height a radiately stretching fissure, enclosed by elevated margins (plate 2).

II_{g_2}: the thallus colored like I_{g_2} has a central disk of only 2 cm. with some single needles of 2 mm.; around it a smooth, only slightly undulated border.

III_{g_2}: the thallus like II_{g_2}, but the needles are distributed over the whole surface;

IV_{g_2}: the central disk of the thallus colored like the preceding, devoid of needles;

V_{g_2} became useless by infection;

VI_{g_2}: the whole central disk colored as before, without needles but the smooth surface is raised in folds as a five-rayed star;

VII_{g_2}: the thallus $= II_{g_2}$, only the border is fissured nearly as strongly as I_{g_2};

$VIII_{g_2} = VI_{g_2}$, but the border fissured on the ridges;

IX_{g_2}: the thallus, colored as before, has its central disk densely studded with needles, the border undulated and smooth;

$X_{g_2} = VIII_{g_2}$.

Consequently I_{g_2}, IV_{g_2} and IX_{g_2} differ most in form.

As is generally the case the differences in form increase in the course of time.

On 2/7'10 these differences had become stronger and were still multiplied by the more or less dark coloring of the substratum (see 4°) and the irregular formation of white air-mycelium (see 5°).

2. Regarding the acid production in stocks I_{g2}, II_{g2}—X_{g2}, it should be noted that not only the above described litmus media were used, but, on account of the often slight differences in this respect, also a substratum of the same composition colored with eosine. After addition of eosine tincture to the substratum the reaction was rather strongly acid, which, by partly decoloring with KOH, was reduced to a very feeble acid reaction.

On this medium the production of a minimal quantity of acid by the mycelium caused a distinct increase of the red color, and the degree of this red-coloring can often be better fixed thus than on a litmus medium. The coloring of the substratum with litmus has little influence on the growth of Trichophyton albiscicans, that with eosine retards the growth the more strongly as more eosine is added. Still these two substrata form a good mutual control.

Already on 19 Januari 1910, thus after 16 days, the acid formation and the mutual differences were very well to be seen. After the acid titre the following series could be established: $VIII_{g2}$, IV_{g2}, II_{g2}, X_{g2}, III_{g2}, VII_{g2}, VI_{g2} V_{g2}, IX_{g2} and I_{g2} in decreasing rate. A marked difference was thus observed between VIII and I, the transitions of the others were often very regular.

On 1/3'10 the state of the cultures had quite altered as the produced acid was then oxydised, which on the red litmus substratum caused a return of the deep blue color, on the eosine medium a return of the red to light-yellow. This occurs regularly through the whole mass, for although at first formed in the vicinity of the culture, the acid diffuses during the slow growth of the organism throughout the substratum and in the same way it disappears. So, at the said date, VIII and IV had turned again quite light-yellow orange, whilst IX and I continued deep red and only later passed into the more yellow state. Still all the cultures had grown into a vigorous thallus with of 4 to 5 cm. diameter. In the rate of acid formation and the disappearance of the acid the above order had continued to exist.

On a medium with lead carbonate for the detection of H_2S (after BEIJERINCK) the mycelium developed only slowly, but dissolved in its vicinity the carbonate so that a round transparent spot appeared; later, however, by the disappearance of the acid, the lead precipitated again as a white salt and the spot, transparent before, grew again opaque.

3. The formation of proteolytic ferment was examined also on 3/1'10 by transplanting the stocks I—X to 100 grs. ERLENMEYER flasks in 60 grs. of a substratum of the composition: 4 glucose, 1 peptone, 10 gelatine and 100 water, so a favorable medium. At ± 20° C. the culture soon develops to a white thallus beset with many needles, fairly alike in shape and size.

It was later found, on 1/3'10, that its diam. of $2^1/_2$—3 cm. had much increased, and that the liquefaction of the gelatine could already be well observed. The most obvious phenomenon was that the thallus, where the liquefaction had first set in, had most sunk down, by the evaporation of the water, so that the surface of the culture was concave. Most of the thalli were soaked with the liquid and had a moist appearance. None of them showed any trace of the formation of white air-mycelium.

Although mutual differences in the degree of liquefaction of the gelatine could be noted, it was difficult at that moment to fix their precise succession. This difficulty continued by the troublesome circumstance that some mycelia or portions of them were immersed in the liquid, and by deficient access of air began to grow more feebly than the floating parts.

It was found after some time that especially I_{g_2} and II_{g_2} began to form strong contrasts. At a for the rest equal development, the whole surface of the substratum in the ERLENMEYER-flasks being covered with the thallus, I found the gelatine in II_{g_2} to be liquefied and when kept in an inclined position, it flowed out from under the culture, whilst in I_{g_2} the liquefaction had only taken place in so far that the thallus had strongly sunk down in the centre. For the rest the gelatine of I_{g_2} was still solid. The other mycelia showed phenomena between these extremes, but a proportional succession in vigour could not be stated.

After 6 months, however, the gelatine of all the stocks had liquefied and all the mycelia sank down in the liquid, by which excepting the phenomenon mentioned under 4., they, all presented the same appearance of a white, gelatinous layer, immersed in an amber-yellow liquid. In none of them was as yet formed a white, spore-bearing air-mycelium, which in the course of time does take place in some cultures.

4. Already in the beginning of my researches on Trichophyton albiscicans I had observed as well liquid as solid substrata of the composition: glucose + peptone to color darker and darker in the

course of time. In this series of investigations the dark-coloring occurred first in II_{g_2} on the gelatine medium. After two months the top border at the wall of the flask began to darken, at first only from 2—3 mm. high, which later, however, extended over the whole surface and downwards. The hitherto light yellow gelatine grew more and more dark and finally brown, whilst in the beginning the other cultures preserved their colors. Only after three months this rather sudden change of color appeared in the gelatine culture of IX_{g_2}, and later and in different degrees in the others, least in I_{g_2}; I_{g_2} and II_{g_2} thus formed here the extremes of the series.

Also in the cultures of these stocks on agar media of 4 % glucose and $^1/_{10}$ % peptone, appeared this darkening of the color, which here passed into gray and here also it was strongest in stock II. Over the whole extent of the medium the dark-coloring was quite regular as likewise that of the thallus. As for the gelatine culture, IX followed II, then the others, I also here being the latest. For the first time after 2 months there appeared in II an obvious decoloring which after 5 months had given a lead color to the medium and the thallus. The other stocks had likewise darkened in different degrees, Ig_2 least.

5. Especially on a solid medium a white air-mycelium is formed on the thallus after a certain time, bearing the spores. The originating of this air mycelium highly depends on various circumstances, which act on the state of nutrition of the mould.

On a substratum of 4 glucose, 1 peptone, 2 agar and 100 water the hyphae are formed already when after a month by desiccation the assimilation becomes unfavorable. On any other part of the thallus, where the humidity is still sufficient, this spore-bearing mycelium is not yet present after 4 months. This appears, for example, in phials where the medium is solidified obliquely; the topside near the plug is then already after a month covered with white air-hyphae, whilst near the bottom in the moist part the thallus still remains spore-free for months. On the said favorable medium I could observe no individual differences in the cultures representing the different stocks.

Still these proved really to exist, which became clear in cultures on media of the composition: 4 glucose, $^1/_{10}$ peptone, 2 agar, 100 water, that is as in series 1. Here all the conditions of growth were the same and the several parts of one thallus were not exposed to different influences, and yet the spore-bearing air-mycelium of one culture appeared at quite another time than that of all the others. In this stock the IX^{th} generation was the earliest and, when it was kept at

± 25° C., the white air-mycelium was present after 5 months on the greater part of the culture. As the latter had then already assumed a dark color the appearing of this white air-mycelium was very obvious.

The second which then followed was stock II, where after 5 months a trace of the white mycelium began to develop on the central disk, then showing a strong contrast with the still darker color of the thallus. The others did not yet possess any spore-bearing air-hyphae. Later, however, they appeared in all.

Surveying all the results of this investigation on the individual properties of the descendants, obtained by fructification from a single mould-fungus, it appears that no property is found not possessed by any of the other stocks, but that each of them shows these properties in more or less degree.

The first question now arising in this connection is: are all these properties in each individual mould regularly fortified or enfeebled? and must those differences be considered as the utterances of the greater or smaller intensity of the functional activity in for the rest like individuals? Or do the examined properties appear in the various individuals irregularly fortified or enfeebled?

Basing on the above researches the second of these questions should be answered affirmatively. For if we arrange the results of the most characteristic individuals in such a way that the lowest degree of a property is represented by 1 and the highest by 5, we obtain the following survey.

	Form	Acid formation	Proteolyt. ferment	Dark color	Spore-formation
I	Particularly on unfavorable media all differing from one another	1	1	1	3
II		3	5	5	4
IV		4		3	1
VIII		5		2	2
IX		2		4	5

This table shows that for the lower mould-fungus Trichophyton albiscicans each mould springing from a single spore of one and the same plant possesses an individuality, which is observed as well in the degree of development of its chemical properties, particularly under unfavorable conditions of growth, as in the form.

It is a matter of course that the here found variations of properties give by no means an image of the full extent of these variations among the fructificative descendants of a single mould-fungus. Of the innumerable spores produced by one mould, the number of ten, used for the experiments, was much too small. Remarkable are notwithstanding the relatively great differences occurring among these ten descendants.

As pains were taken to render the conditions of growth as favorable as possible, and no strongly irritating means were applied in the research, which so often occur in biological isolations by using chemicals or by heating, we certainly have to do here with phenomena of normal individuality.

Among the variations proper to micro-organisms, noted by BEIJERINCK in the Meeting of the Academy of 27 Oct. 1900 they can hardly be ranged, as degeneration, transformation and common variation all appear in the course of some generations, which is not the case here. Here neither occurred, what at the same place is stated as characteristic for "ordinary variation": "Here the normal form continues unchanged but now and then throws off individuals, variants, which from the very first are likewise constant and remain so, but at times again throw off other variants, among which the normal form may occur as an atavist."

It is possible that in the course of time Trichophyton, too, is subject to such alterations. It is already established that it degenerates in form by prolonged culture and by repeated re-inoculation into unfavorable media and thereby slackens its growth.

The above individuality may be compared to the inconstancy of buds and seeds in the higher plants, in whose descendants, however, the differences in form are much more easily observed than those of their biochemical properties.

One weighty objection might be made against this individuality of the spores, namely: the spores which germinated did not originate from the same portion of thallus II; hence it might be that we had not to do here with a property possessed by the spores themselves, but with one springing from the dissimilarity of the various parts of the thallus. For it is accepted that different parts of a same mycelium can possess different properties.

As to Trichophyton albiscicans I have perceived nothing of this during the years I have observed this organism. But it is possible to prove the equality of all the parts of its mycelium by examining them in relation to form and properties. This was done in Sept. 1910 when culture II was 12 months old and hence had had every opportunity to deviate in any of its characters.

Parts of the mycelium taken at random were now examined on the various above differences in form, acid formation, proteolytic ferment etc., by transplantation on a medium as used before.

Although such old cultures are less active than much younger ones, all of them taken from II grew well and developed as usually. The result was that in the first six weeks nothing was to be perceived, neither in form nor in the production of acid or of proteolytic ferment, of the relatively great differences seen at multiplication by spores. This vegetative multiplication thus shows in this respect a strong contrast as compared with the fructificative.

Heredity.

The relatively great differences in individuality among the descendants of Trichophyton albiscicans, made the question arise whether these graduated properties of the species would be hereditary in vegetatively obtained descendants.

To answer this question, from the ten stocks, obtained from one spore, those were selected which in their different biological properties showed the strongest contrasts, namely I, II, IV, VIII, and IX. Of these was vegetatively cultivated from the 2nd generation a 3rd, from this a 4th, and from this again a 5th. The same precautions as to similarity of growth conditions were taken as before. Each generation was grown from the preceding by taking small bits from the mycelia on litmus substrata, as on this medium the cultures preserved their full vigour of growth. This was not the case with cultures on agar of the composition: 4$^0/_0$ glucose, $^1/_{10}{}^0/_0$ pepton, on account of their small protein percentage. As our object was the examination of normal descendants, the same way of re-inoculating was also to be followed for the cultures on an unfavorable medium. From the litmus culture of the related stock, bits of about the same size were cut off with a flat needle and transferred to the said media. All the generations were begun on the same day and exposed to the same conditions as to heating, illumination etc. So, after preparing the media, generation 2 was set up, as said above, on 3 January 1910, the generations from this on 7 March, generation 4 on 7 April and generation 5 on 18 May.

These experiments carried out with care, confirmed what I had already seen for years in Trichophyton albiscicans concerning the form, that vegetative multiplication of this mould produces equivalent descendants, also as to their biological properties.

1. Regarding the form the circle of fissures on the border of I is very much developed in opposition to II and IX, where these fissures do not or only sporadically occur.

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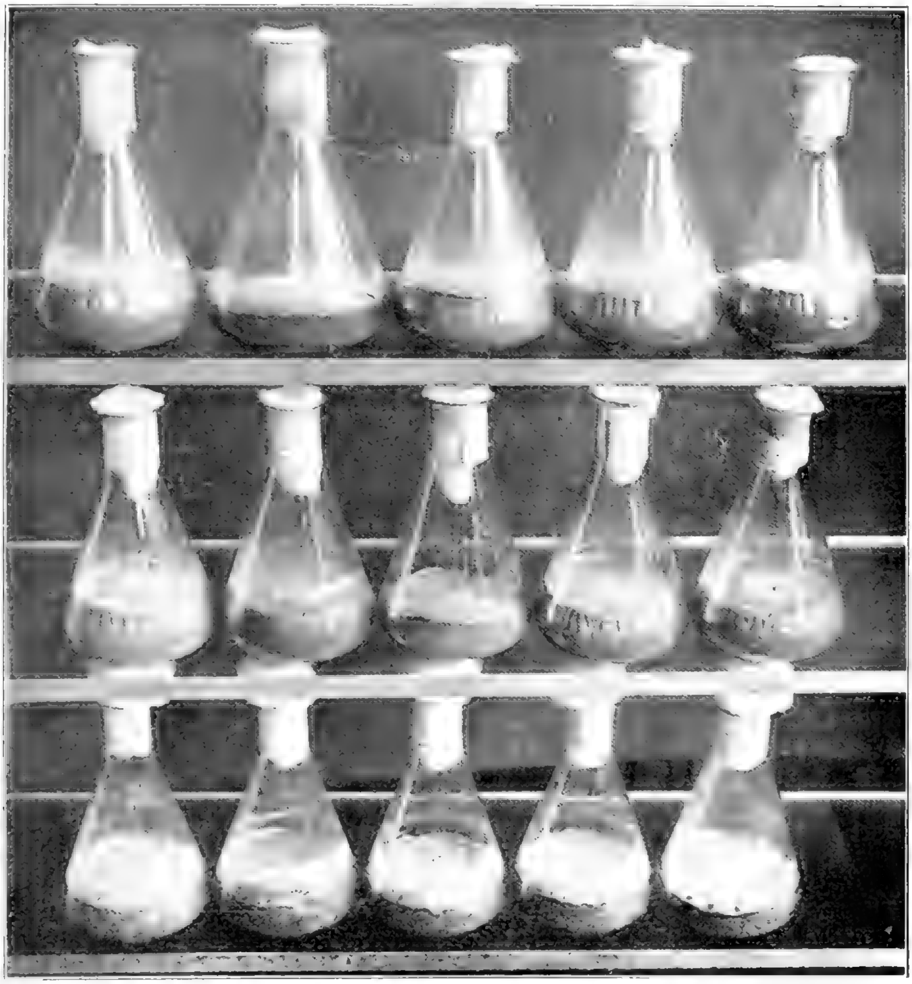
Plate 1.

Gen 2

Gen. 3

Gen. 4



I II IV VIII X

ERLENMEYER flasks with cultures of the five stocks I, II, IV, VIII and IX of Trichophyton albiscicans in 2d, 3d and 4th generations. The stocks are placed underneath each other, the generations side by side, to show the differences in de coloring of the media.

In all the cultivated generations of I this particularity was transmitted as well in I as in II and IX, as is shown by Plate 2, where the 2nd and 3rd generations are given, the latter in a stadium of the 5th month, the 2nd is one month older. After two months already some fissures at the border appear and then grow more and more numerous.

In the thallus of II, contrasting in so many respects with I, these fissures do not at all occur in the border of the 2nd generation, only in the course of the third month they appeared at a few spots sporadically and developed but to a small number, in the 4th generation they were absent. For IX the same may be observed on the plate.

Besides this, the distinct partition into central disk and outer border affords a useful distinguishing mark; likewise the presence or absence of threads or hairs which give the surface a velvet like appearance. That partition is most pronounced in IX and its descendants, least in II where of this character not much is to be seen, neither for the 3rd, 4th and 5th generations. There the central disk is entirely covered with little hairs as is also the case in the descendants. This is clearly seen in the 3rd generation of IX, plate 2, for the 2nd generation it has become less distinct by the prominence of the air-mycelium. The indistinct division into disk and border and the small number of hairs in the middle are best seen in the 3rd generation of II, as in the 2nd it is eliminated by the stronger growth of the air-mycelium.

Whereas already after a month the central disk of IX is closely beset with hairs, they appear in the centre of II only after 4 months and remain very rare.

2. The rate of acid formation for the vegetatively obtained descendants was traced in the same way as for the above described research on individuality on media of 4 glucose, 1 peptone, 2 agar and 100 water and then neutralised with KOH to a very feeble acidity. The greatest difference was here between I and VIII, the intervening II, IV and IX were, however, also cultivated. Each of the following generations was obtained vegetatively from the preceding one and transplanted to Petri-dishes on the same days as the series arranged for the observation of the differences in form.

For the generations Ig_2—$VIIIg_2$ to Ig_5—$VIIIg_5$ the same order continued to exist in the acid production. After six weeks it was at its highest, then passed slowly into an alkaline reaction, indicated by the deep blue color, which also took place on the eosine medium. Where the acid production was first strongest, as in VIII, this conversion of the acid, and in relation to this the change of color also set in first in all the cultures.

The rate of acid formation, too, proves thus as an individual property to be hereditary in the vegetatively obtained descendants.

3. The rate of production of proteolytic ferment also was observed in the vegetatively obtained descendants in a similar way as above; the medium then used of 4 glucose, 1 peptone, 10 gelatine and 100 water was also used for the successive generations, in quantities of 60 grs. in 100 grs. ERLENMEIJER flasks. They were inoculated on the same days as under 1 and 2, whereupon each series was continually exposed to the same conditions.

Respecting the difficulty to determine the degree of liquefaction, enough has been said already; at the estimation of the results for the successive generations the same trouble as before arose, but still, after 6 weeks, when the liquefaction was nearly complete, it could be stated with certainty, that the vegetative descendants of II always caused much stronger liquefaction on the gelatine media than those of I. Hence, the moment when the substratum of Ig_2 and its descendants was liquefied, was reached a few weeks later than for II.

So the heredity of the degree of vigour in the formation of proteolytic ferment is not doubtful, but can be fixed accurately.

4. The dark-coloring of the substrata and the mycelia, so irregular for the various individuals, occurs as well in the solid agar as in the liquefying gelatine media. For the estimation of the degree of coloring, agar media should, however, be preferred, as the liquefaction of the gelatine sometimes affects the growth and might render it very difficult rightly to judge only one culture of each species as was done here. The agar medium is moreover quite regularly dark-colored just as the liquefied gelatine.

Of the results obtained in the cultivation of three generations of the five stocks, plate 1 gives a good image. The stocks are placed side by side in this order: I, II, IV, VIII, IX, next each other and in the 2nd, 3d and 4th generations one under another; the first two killed after 6 und 5 months and preserved with formol, the third still alive and 4 months old.

In all the generations it could distinctly be stated in the flasks that in accordance with the darkness of the decoloring the order of succession was: II, IX, VIII, IV, I, of which II was the darkest. On the photography this can be seen with some difficulty by the colors of the media, but very distinctly we see there the continuous contrast of the series of I with light colored and that of II with a dark colored substratum.

This well proves (and it is still better seen in the cultures in the flasks) that by vegetative multiplication the dark-coloring, as yet so mysterious to us, is transmitted to the descendants, even with the same degree of intensity.

5. The formation of white, spore-bearing mycelium which, as said in the research on individuality, showed such great differences as to the time of its appearing for the various individuals on the medium: 4 glucose, $^1/_{10}$ peptone, 2 agar and 100 water, produced the same results at the investigation here described.

Only after 5 months the white hyphae began to form in $IXg_{,}$, first in the middle, and after 6 months they covered a great part of the culture, as plate 3 shows; its contrast with the state of II and I is less pronounced, as by the insolation of the light, also other parts have become whiter. Yet II has only produced spore-bearing mycelium in a slight degree, I hardly any. In the three figures of the second generation, only 5 months old, this inherited particularity is more marked. IXg_3 bears in the middle the disk covered with white mycelium, in opposition to the smooth border; IIg_3 has remained nearly quite smooth and only begins to produce spore-bearing organs, whilst $I_{/3}$ possesses hair-shaped, prominent parts but, as the culture shows, no spore-bearing hyphae.

Consequently as all the growth conditions were the same for these cultures, the results point out that also the spore-bearing air-mycelium of the vegetative descendants, which appears in the various individuals at different periods, comes in the same order as in the preceding generations, so that this difference in the individuality too proves hereditary.

This research on heredity shows that vegetatively obtained descendants from the examined individuals, had inherited the relative degree of the properties of the original cultures. In connection with the investigation on individuality follows from this, that by the cultivation of spores of a single mould of Trichophyton albiscicans, not only individuals are obtained with special properties, but that stocks with such properties also take rise by common propagation.

So for an inferior organism as this mould-fungus the same laws prevail as for the higher plants, namely, that individual properties are transmitted to the descendants especially by vegetative multiplication and that, by fructification, a variation in biological properties is observed, moving within the limits of the species and comprising also the form-properties.

It is not improbable that this will prove also to be the case for other moulds, albeit perhaps more troublesome to trace in the quickly growing species. The conception "species" would then comprise all the individuals possessed of all the variations in morphologic and biochemic properties, stated in the sporogeneous descendants of a single individual. Several moulds, hitherto considered as belonging to different species, will probably find their place between the limits of variation of other forms. It may furthermore be expected, that many cases of pleomorphy, and the doubt respecting the affinity of the Trichophyton group and the Favus moulds, will find their solution in that direction.

A particular attraction will be lent to such researches by the fact, that the chemical properties of the examined organisms can much more easily be observed than in the higher plants

Besides the fungi there are, however, many other low organisms which multiply partly vegetatively (by division), partly sporogenously (by fructification). To these the bacteria belong, and here much has been discovered about the variability of properties, which has given rise to a sometimes very doubtful distinction of stocks, varieties, related forms, etc, and moreover to a great uncertainty as to the affinity of these forms and the possibility of their mutual transition.

If a nearer research might prove that, just as for the here examined lower fungus, also for bacteria and other organisms multiplation by spores produces stocks with another combination of properties than by direct division, an extensive field of investigation might possibly be opened.

A practical, simple method of arranging one-cell cultures, whereby the organisms remain vigorous and possessed of their normal properties is therefore required.

For that reason I will describe the method which enabled me to make the above observations.

Microbiology. — "*Method to cultivate micro-organisms from one cell.*" By Prof. A. W. Nieuwenhuis. (Communicated by Prof. M. W. Beijerinck).

(Communicated in the meeting of October 29, 1910).

Among the characteristics of the present period of investigation of micro-organisms and the application of the thereby obtained results in behalf of the life of modern society, this is certainly a salient one, that by the more profound study of the properties of those smallest of beings a great uncertainty regarding the constancy of their life functions becomes prevalent, in connection with doubts about the independence and the more or less narrow affinity of certain, for instance of pathogenic and non-pathogenic species. To cite only a few examples, I mention, on account of their practical importance, the doubt about the constancy and affinity of many yeast species, of typhus and paratyphus bacilli and bacterium coli, and in mycology the contest as to the unity or plurality of the Favus- and Trichophyton moulds, especially in relation to their geographical distribution in the temperate zones.

Since the parasitic diseases in the tropics have been scientifically studied the existing uncertainty concerning organisms, such as typhus- and dysentery bacilli, still increases.

The important results obtained notwithstanding, which have become so valuable for industry, public health and physic, have somewhat broken the conviction that a solid, scientific insight into the morphological and biological properties of micro-organisms is necessary.

The great technical difficulties accompanying an investigation into the life conditions of some smaller microbes, have still more weakened that conviction.

For those, however, who seek the solution of the theoretical questions which are to form the firm base for important social measures, the study of the organisms themselves and of their properties continues to be a prominent demand.

Only recently Professor Dr. D.A. DE JONG, in his treatise "De Tuberkelbacillus", published as special number to the 4th year of "Tuberculose, het orgaan van de Centrale Vereeniging tot bestrijding der Tuberculose" pleaded eloquently for a conscientious investigation into the properties of the tubercle bacillus. How many important problems are still to be solved thereabout is shown even by some heads of this treatise as: 2. Properties of the tubercle bacillus in relation to the diagnosis; 3. Pseudo tubercle bacilli; 4. Culture of tubercle and pseudo tubercle bacilli; 5. Morphology of tubercle and pseudo tubercle bacilli; 6. Differences of tubercle bacilli mutually; 7 Tubercle bacilli of cold blooded animals. From the contents we further see, that also for these questions, touching the very heart of the matter, the differences of views are still very numerous.

For attaining our present standpoint, PASTEUR in the first place showed us the way by his evolutionary researches, and the great progress dates from the time when KOCH introduced his solid medium. According to many, however, this expedient does not afford the support, wanted for the following step to take on the long way before us of bacteriological and mycological research. Numerous are hence the other methods of investigation, whereby use is made of biological properties of the organisms to state their independent existence or their affinity, for instance of serum reactions (agglutination, precipitation, etc.) together with color methods. Hitherto these methods could not, however, solve very grave questions, and, by their number, they prove in some way how little efficacious they are in certain cases.

A long time already there have been hopes for the great support of a practical method to carry out a one-cell culture under most varied circumstances, not only to observe the properties of the individuals among the micro-organisms, or their affinity or similarity to others, but also to cultivate organisms, which are difficult to study, because they cannot grow on solid media and should therefore be transferred from liquid to liquid. How much the want of such a means is felt, is proved by a few citations, referring to methods for one-cell culture already published before. When describing the method of S. L. Schouten, E. Küster declares in his "Kultur der Microorganismen, Leipzig und Berlin 1907": "Eine ingeniöse Methode, welche die Koch'sche zu ergänzen berufen sein könnte ...." and when discussing a similar method of Marshall A. Barber, H. Pringsheim declares in the "Zentralblait für Bakteriologie, Parasitenkunde und Infektionskrankheiten", Bd. 23, N. 6/9, Abteilung II: "Vielleicht ist die Methode Barber's in noch höherem Grade, als der Autor annimmt, dazu bestimmt von den Einzelzellen ausgehend zu Reinkulturen solcher Microorganismen, wie z. B. mancher Flagellaten, Algen und Diatomeen zu kommen, die wegen ihrer Empfindlichkeit nur schwer in Anhäufungskulturen zu gewinnen sind und die sich deshalb von Bakterien und Schimmelpilzen nur schwer trennen lassen".

Such researches about the properties and affinities of certain organisms, cultivated strictly from one cell, were carried out by E. B. Hansen in the Carlsberg-Laboratory near Copenhagen, relatively to some yeast species. They have given much insight into the properties of these yeasts and exerted a great influence in the brewing industry. With his method of working, which reposed on the detecting and noting of separate yeast-cells in a liquid medium, the relatively large size of a yeast-cell enabled him to apply a microscopical magnification of $\pm$ 60. Also for the modification devised by Lindner such a large organism is required. Much smaller organisms, such as mould spores and bacteria cannot be found in this way and for these another method is necessary.

A practically good method to arrange one-cell cultures should answer the following requirements:

It must be fit to be applied with magnifications of 300 and higher; this needs no further explanation.

2. The organism to be isolated must be injured neither by chemical nor physical stimulants.

This is a demand which for other methods of research, too, is by no means generally recognised. It rests on the fact that as well higher plants as lower organisms react on even apparently insignificant stimuli, for example on a slight modification in nutrition, with a considerable deviation in their functions. Especially in biological isolation methods this principle is sometimes earnestly sinned against. As soon as temperatures to even 80° C., or chemical substances are used to kill other organisms present at the same time, it must be accepted that the remaining individuals are no more normal.

3. The greatest possible simplicity in the application is required, so that the method is within the reach of every experimenter, not exacting too much of personal dexterity, patience or time.

4. An easy maintaining of asepsis in the research. This, also, is sufficiently clear.

These demands are not satified by any of the hitherto published methods of research with one-cell culture. S. L. SCHOUTEN (1901), isolates the cells from suspensions in hanging drops at large magnification by means of two needles. MARSHALL A. BARBER (1907), does the same with fine glass capillaries and R. BURRI (1907) uses little drops of East-Indian ink.

The very ingenious method of S. L. SCHOUTEN is for a general application much too complex, as it requires too much from the dexterity and patience of the investigator; on account of the long time, too, wanted in the application it would be troublesome, even for a skilful experimenter, frequently to use it. Moreover, it has the drawback that the extremely fine, artfully made glass needles must be sterilised. How difficult this is with frequent use without stimulating and enfeebling the concerned micro-organisms by chemicals, the inventor himself proves on page 113 of his treatise.

MARSHALL A. BARBER substituted capillaries for SCHOUTEN's needles; with these he draws by suction some organism from a hanging drop. Especially when working with impure material the sterilisation of these capillaries must occasion still greater difficulties than SCHOUTEN's needles.

The "Tusche-method" of BURRI is again a culture method on solid substrata, in so far as it is used for the multiplication of an organism, whereas it does not properly effect the isolation of a single individual.

The above research on the "Individuality and Heredity of Trichophyton albiscicans" I carried out by means of a one-cell culture method, which possesses all the above requirements. It can be applied with large magnifications (I used those of 300 und 450), and requires a glass needle of easy make and every time a new one, which renders disinfection unnecessary. The management does not require much dexterity and time, nor very complex instruments, whilst asepsis can easily be maintained.

Likewise as by the methods of S. L. Schouten and Marshall A. Barber, I isolated the concerned organism from a drop of the suspended material hanging under the cover-glass, only with one needle, to subsequently transfer it to a drop of nutrient liquid in which the preliminary or the whole further development can take place. The two drops are hanging side by side under the cover-glass. The glass needle *a* is fixed on a stand *b* on which it can be moved mechanically in every direction by three micrometer screws (Plate 3).

As Plate 4 shows this stand can be placed beside every microscope *f*; the needle *a* should be so long that the point can be placed in the axe of the microscope. The end of the glass rod wherewith the isolation is effected, is a glass globule which may differ in size in accordance with that of the organism to be isolated.

At the previous research, whereby mould-spores of $2—2^1/_2\,\mu$ and mycelium cells of $1—1^1/_2\,\mu$ thickness, but of greater length, were transferred, I used globules of $20—30\,\mu$ diameter.

As to the execution of the isolation the following observations may be made.

a. The material from which the organism is to be isolated is distributed in a sterile liquid, in such a way, that it is suspended very finely divided, so that the spores in the drop under the microscope do not get too near one another, or too near strange ones. How this is to be contrived depends on the concerned material and may be arranged at will. I did it on a sterile object-slide in a flat glass box, likewise freed from germs. On the slide a drop of nutrient liquid was laid and with an iron needle a number of spores were distributed in it. To divide a bit of mycelium two iron needles were wanted.

b. The cover-glass on which a little of this material must be put, may be of the usual form, for example 22×26 mm.; it is, after a careful cleaning, very thinly smeared with pure vaseline on one side, and subsequently freed from germs by quickly passing that side a few times through a gas flame. The layer of vaseline should be extremely thin, as it only serves to prevent the drops from flowing over the glass surface.

It is advisable to prepare beforehand in a sterilised glass box a sufficient number of these cover-glasses, for example for one day.

c. As nutrient liquid I used that which had proved most favorable for the organism. In order to exert no weakening influence I used the same liquid for mixing; in the above research 4 parts of glucose, 1 peptone, 100 water.

d. A copper rim must serve as support for the cover-glass under the microscope, for as the experimenter works with the glass needle under the cover-glass with the hanging drops he wants a rather large free space. The most practical I found to be a copper rim 8 mm. high of the shape of three sides of a rectangle, having a side length of 18 mm. and from 1.5 to 2 mm. thick. The fourth open side serves for the introduction of the glass needle and for its movement.

To easily move the cover-glass I cement the copper rim with some vaseline on an ordinary object-slide lying on the stage. By greasing the flat top side of the copper square with a little vaseline, the coverglass adheres somewhat to it, which is desirable though not necessary.

e. The glass needle with which the isolation of the organism is effected is so simple of shape, and can so easily be made, that it implies the possibility of an extensive application of my method and much advantage over other needles. The part of the needle properly used is its terminal portion, which has the shape of a globule, in my investigation of 20—30 μ diameter, which dimension may, however, be varied according to the size of the organism. The foremost part was drawn out to $\pm$ 10 μ thickness and over a length of from 4—5 mm. bent upwards, with the globule at the top.

The making of such a needle is done as follows: two ordinary glass rods or tubes $\pm$ 4 mm. thick and $\pm$ 15 cm. long which touch each other with their ends, are melted together in a gas flame, then the still soft middle part is drawn out to 1 mm. thickness. After cooling the rod is divided into two by breaking it just in the middle.

To make the fine points a microburner is used and the flame lowered to a height of 2—3 mm. so that no yellow central part is seen. In this low flame the ends of the two glass rods are held in contact with each other. When half liquid they stick together and are drawn out to a thread of $\pm$ 10 μ thickness. After cooling and breaking in the middle, one has two of the desired needles but without terminal globule As the extremely delicate points must not lie in the axe of the rods but should be directed upwards, the drawing must be contrived so as to bring the needles in the same plane at angles of $\pm 120°$. The finest portions of the point then get also this direction.

The terminal globule is simply made by passing the broken end of the $10\,\mu$ thick glass thread so quickly through the microflame, that it just for a moment has a yellow blaze. If it is done too slowly the hair-thin glass thread melts into too large a globule.

When drawing out the glass to a thread of about $10\,\mu$ it often breaks or the ends get a somewhat irregular shape, particularly when the drawing goes too far. After the globule has been formed the point sometimes needs improvement on account of an abnormal bent. A very simple manner is then to keep the end of the needle for a moment over the flame, it grows soft and the rising current of hot air may place it in the right position.

After a few trials the making of the needles is quite easy for every experimenter.

Now that every one is able after a little trying to construct the desired needle within some ten minutes or less, without any other implements but a few glass rods and a microburner, it is in the first place possible to repair a needle that has become useless by refounding it, instead of by disinfection, further it may be replaced by others easily made in store.

f. There are diverse stands with which the thereon adjusted objects may very slowly and regularly be moved in three directions by means of micrometer screws; if they have only about the height of the stage of a microscope they may be rendered serviceable to this method by supplying them with a glassneedle. As Plate 4 shows, such a stand *b* is placed quite free beside a microscope *f*, only the needle *a* must reach to a certain height over the table that it may be placed with its point under the objective.

The stand used by me is shown on Plate 3 in all its parts, and consists of a foot *j*, on which a column *h* may be raised by screw *d*; on table *i*, adjusted on this column, is a sliding-piece destined for the fixation of the needle-holder and movable to the right and the left by screw *c*. The movement backwards and forwards is effected by screw *e*, which makes the whole upper part of the foot turn round on its base. All these movements are regulated by spiral-springs, which counteract the movements of the screws. The needle holders of which two, with their needles fixed by means of gypsum, lie beside the stand, are placed loose on the sliding-piece and are fastened by two pins in corresponding holes. The whole is made of copper. The up and downward movements, caused by screw *d*, were accompanied in this stand with a slight rotation; for the rest the movements of the point of the needle *a*, as seen under the microscope, were quite-regular, so that the apparatus proved very useful.

g. Any microscope, either with or without a nose piece may be adapted to this isolation method if only the room between the objective and the stage is large enough to place in it the glass slide, with the copper square and the cover-glass with the two drops. If the stand is not constructed for the microscope in use, the desired height can be obtained by placing disks of the required thickness under the base.

As may be seen on Plate 4, a movable stage of Zeiss was adjusted on the microscope *f*, by which the slide *l* and thus the copper square *m*, too, and the cover-glass *n* may be quite regularly moved in the horizontal plane. The movements of the Zeiss instruments I applied later instead of those of the stand as the screws ran more gently. With the stand only, without the movable stage *k* the method can still be very well carried out.

h. Finally, for moist chamber I made use of the following simple arrangement which can easily be sterilised. An objective is sterilised. in a gas flame and subsequently placed on a $\pm$ 7 m.m. high glass ring of 20 m.m. diameter inside and 1 m.m. thick. By holding such a ring with a forceps in the flame, it is soon freed from germs, and in that warm state greased with vaseline on one side it adheres well to the glass slide and prevents the air from entering. In a sterilised glass box a whole series of such moist chambers may be kept in store for some research. To cultivate in it the organism in the hanging drop, the top edge is also rubbed with some vaseline and the cover glass pressed so much that, also here all access of air is excluded.

At first, as is often advised, I put a drop of liquid on the bottom of the room to prevent evaporisation of the culture drop. This, however, gives some trouble by the condensation of vapour on the under surface of the cover-glass, which does not occur when the drop at the bottom is left out. When a hanging drop of for instance 4—5 m.m. diameter and $1^1/_2$—2 m.m. thick is used, the slight evaporisati on wanted to saturate the small room I never found injurious. Vapour at the bottom of the chamber is sufficiently prevented by placing it on a solid object in the warmed room by which the glass-slide becomes warmer than the coverglass.

The isolation and subsequent cultivation of an organism are managed as follows:

First the objective, with the copper square fixed on it with vaseline, is placed on the stage of the microscope so that its axis passes

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through the centre of the square and the open side is directed to one of the sides, as on Plate 4 to the left.

Now the stand *b* with the needle *a* is placed on the left side of the stage, in such a way, that the needle reaches over that side, its point being placed under the objective within the copper square (fig. 4). It is further desirable already now to bring the head of the needle in the axis of the microscope, which after some practice under control of magnifications of for instance 50, may be done with the hand or by means of the micrometer screws. When the head of the needle is in the desired position, the glass rod *a* is so far lowered by the corresponding screw, that its head is somewhat below the top edge of the copper square, whereby the head, of course, continues to move in the axis of the microscope. When now the cover-glass with the hanging drops is put in position the drops do not touch the head.

The cover-glass under which the organism is to be isolated, has been described before (see *b*). The hanging drops are transferred to it with the following precautions. With a sterilised glass rod a drop from a tube of germ-free nutrient liquid is laid on the fatted side, which drop is to serve for the cultivation of the organism and will therefore be called culture drop. Its size is best at 4 mm. diameter and $1^1/_2$—2 mm. thickness. With the same glass rod a small quantity of the suspended material can now be taken up and placed at little distance, for instance 1—$1^1/_2$ mm., as second or "material drop" beside the culture drop. It is best to give the material drop an oblong shape, so that a long side may be turned towards the culture drop. Micro-organisms situated near the edge are most easily isolated, and the small distance between the drops is desirable to make the way for the organism to be isolated as short as possible. This cover-glass supplied with two drops is now removed with a forceps from under the sterile cover glass under which they were formed, then quickly inverted, so that the drops do not coalesce which, however, does not easily occur on the fatted side; the cover-glass *n* with the two now hanging drops is placed on the copper square in such a way that the drops are in the middle and do not touch the edges.

By removing the glass slide *l*, eventually with the movable stage, it is possible to bring the space between the two drops in the axe of the microscope, which is easily controlled by fixing the edges of the drop.

After these preparatives the transference of an organism must be conducted as follows: The simplest way is to pick up the individual to be transferred at the side of the material drop which is turned towards the culture drop. Then the needle is screwed up by *e* so much that the rounded point is vaguely visible and consequently lies beneath the field.

The transferring itself reposes upon this, that, when the point of the needle is placed in the material drop, just against the greased under surface of the cover-glass, and the needle is then horizontally moved by screw *e* out of the drop, it carries along with it over the cover-glass a small quantity of the liquid, and at the same time the corpuscles present in it. The size of the droplet depends on that of the glass globule, the nature of the liquid, the degree of greasing, etc., but with the above given proportions of a globule of 20—30 μ, the quantity of liquid, when moving slowly, is large enough to carry along a suspended spore of $2^1/_2$ μ. By the greasing the liquid will, soon after its exit from the edge of the material drop, contract at the cover-glass into a droplet, in consequence of which it does not flatten out on its way. If now by means of screw *e* the globule is continually moved in the direction of the near culture drop, the latter and the droplet containing the spore will soon coalesce.

If thus the point of the needle is placed in the material drop in contact with the organism to be transferred (for easily working it is best in the centre of the field), it can in the way described be conveyed to the "culture drop" and there develop uninjured.

That the particles to be conveyed should be as far from each other as possible in the "material drop" is evident and explains the necessity of strong dilution.

The easiest way is to isolate an individual from a pure culture; however, from a mixture it is also very well possible, and as all can, to a certain extent, be controlled, at any magnification, contamination of the culture drop seldom occurs after some practice. Continued suspension of the organism during the transference is most desirable, but when the way becomes too long the quantity of the liquid diminishes by the lagging droplets, and a spore, for instance, will adhere to the under surface of the cover-glass. By placing the point of the needle behind it, then moving again, such a particle is carried along, but the less frequently this occurs the better on account of possible injury. This explains why the two drops are placed as close as possible to each other. Especially long, slender organisms are strongly apt to adhere.

The greasing with vaseline has also much influence in this respect, the thicker the layer the more trouble it causes. Contamination of the needle by the vaseline cannot be wholly avoided and cleaning is very troublesome. hence a simple refounding of the needle to obtain a new pure one is desirable.

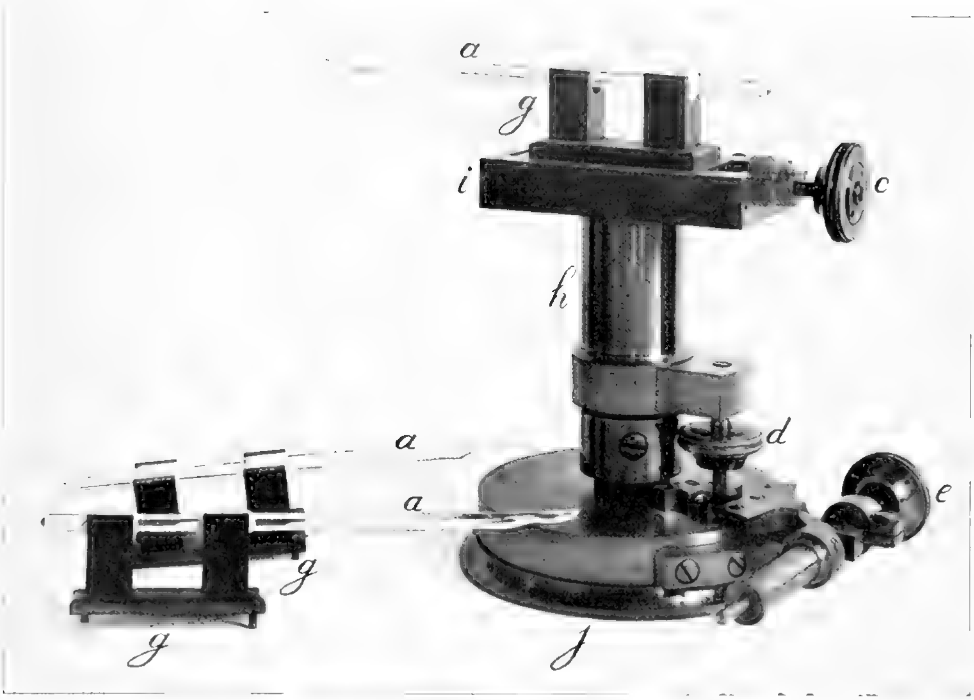
As soon as the organism is arrived in the culture drop the glass needle *a* is lowered by means of screw *e*, and the transported organism now lies free in the culture drop. Before the cover-glass can be placed on the moist room for further development of the germ, the material drop must be removed. This is done by means of a piece of filter paper, sterilised by being passed through the gas flame. The cover-glass is removed with a forceps from the copper square, and then a point of the filter paper is cautiously held in the bottom part of the material drop, which is quickly sucked up. Minimal rests of moist do no harm. By placing the cover-glass with the hanging drop in the moist chamber, as aforesaid, with addition of some vaseline, one has an opportunity to make the development proceed either with or without control of the microscope.

When working with strongly contaminated material, so that it is hardly possible to isolate the wished for organism from the material drop, it is convenient to place a "washing drop" closely beside the two others. With the greatest care, the spore is then first conveyed from the "material drop" to the sterile "washing drop" in the above manner. With the point of the needle the spore is conducted through the latter, whereby other accompanying organisms may be left behind and, with the same needle, or if this is supposed to be contaminated, with a new one, the spore is led into the culture drop. Material- and washingdrop are removed with the filter paper before the organism is placed in the moist chamber.

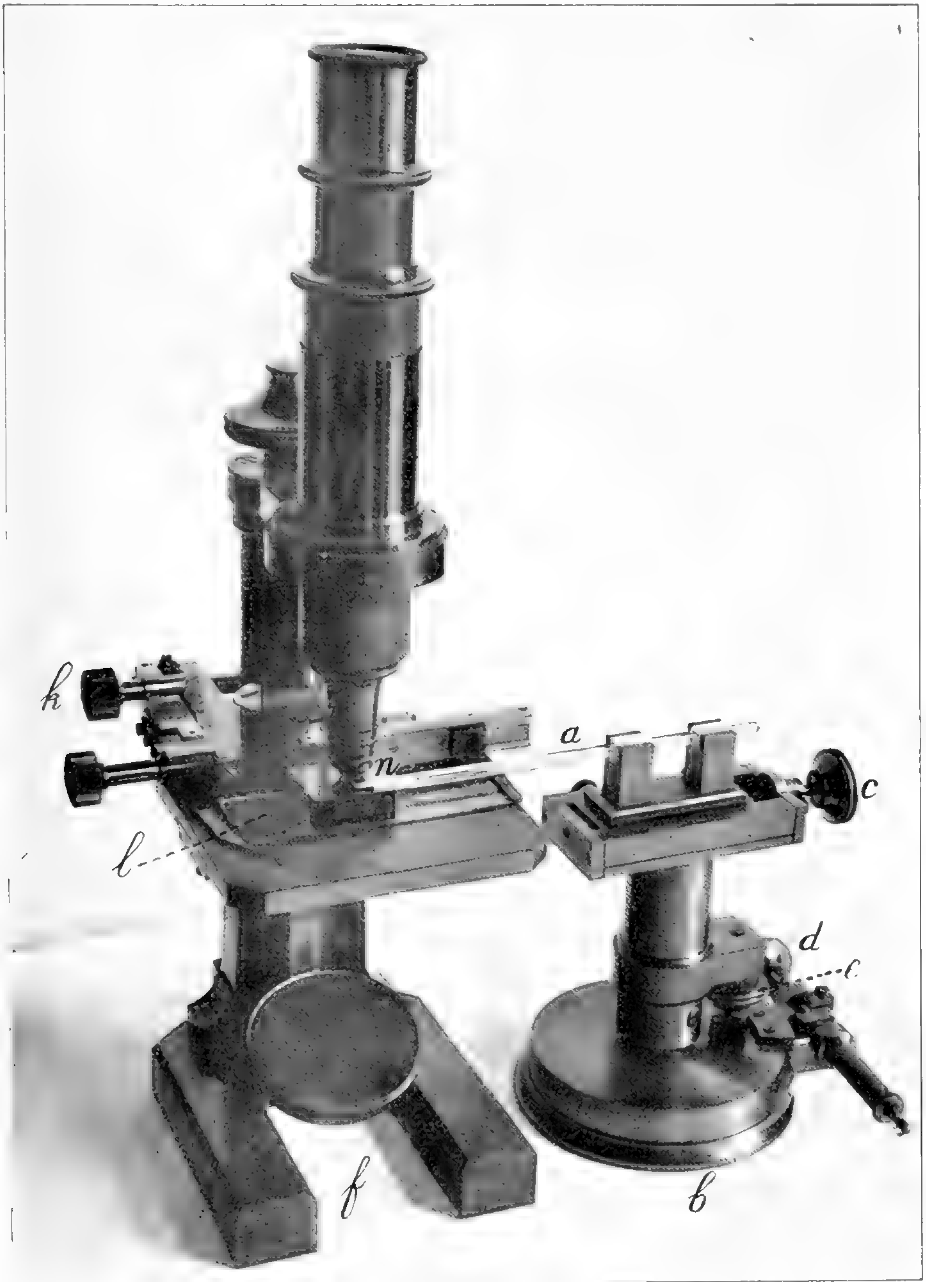
The relative simplicity of this method renders it possible to modify it according to circumstances, which will certainly further an extensive application.

Such a modification of the foregoing manipulation I have devised myself by using the movable stage of Zeiss *k*. With this it is easy to move the slide *e*, and hence the cover-glass *n* with the hanging drops, in the horizontal plane. For the isolation proper of the organism it is the same if either the needle or the cover-glass is moved. It depends chiefly on the fine workmanship of the used instruments which method is to be preferred. On Plate 4 we see the micoscope and stand with the movable stage of Zeiss.

I now wish to bring my kindest thanks to my colleague van Itallie for his receiving me in his Laboratory, and for his help and sympathy.



Copper stand for the movement of the glass needle a in three directions.



Microscope and stand for the separation of a microorganism and the culture from one cell.

Physiology. — "*On the consumption of oxygen by the nervous system.*" By F. J. J. BUIJTENDIJK. (Communicated by Prof. Dr. H. ZWAARDEMAKER).

(Communicated in the meeting of October 29, 1910).

By means of a method that I demonstrated at the 8th International Physiological Congress I succeeded in determining the quantity of O_2 absorbed from a fluid by a removed animal tissue. In the below mentioned experiments the brain, the spinal cord, or some other peripherical nerves were placed in a RINGER solution, of which after some time the percentage of O_2 was determined. By means of the method applied and the apparatus used, the experimental error could be reduced to 2—3 mm.^{3} O_2.

WINTERSTEIN [1]) determined the consumption of O_2 of the frog at 260—300 mm.^{3} per hour and gram, whilst at an equal temperature muscles of the same experimental animal consume 80—100 mm.^{3} O_2 per gram and hour (THUNBERG [2])).

I can also verify this strikingly high consumption of O_2 by the central nervous system. So I found that the spinal cord of a frog in a RINGER fluid consumed 180—250 mm.^{3} O_2 whilst for the muscles likewise $\pm$ 80 mm.^{3} was found. Fishes proved to be very fit experimental animals for the study of the assimilation of the brain. KULIABKO [3]) succeeded in transfusing such a fluid through the brain of fresh-water-fishes so that the respiration motions continued. Consequently this fluid is especially fit to make life continue in the removed brain of fishes.

I used for my experiments fresh-water-fishes as Esox lucius, Lucioperca sandra, Tinca vulgaris, Idus melanotus, Perca fluviatilis, for which I could always find a solution of salt which, during the transfusion through the brain, according to the method of KULIABKO, caused the motions of respiration to return. For the consumption of O_2 by the brain of salt-water-fishes I simply used a solution of NaCl with an osmotic tone like that of the experimental animal (according to the indications of BOTAZZI) [4]), I had at my disposal specimens of Gadus morrhua, Gadus merlangus, Trigla etc.

The figures for the brain originating from newly killed animals do not differ much. Some of the results are mentioned in the first

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1) WINTERSTEIN, Zeitschr. f. Allgem. Physiol. 1907 blz. 315—392.

2) THUNBERG, Scandin. Archiv. für Physiol. Bd. 17.

3) KULIABKO, Archiv. intern. de physiol. IV p. 437.

4) BOTAZZI, Ergelnisse der Physiol.

column (A) of Table I. In the second column (B) are mentioned the quantities of O_2 consumed by the brain of fishes that had died in the aquarium a few hours previous to the experiment.

TABLE I.

Quantity of O_2 consumed per gram and hour by the brain of fishes.

A.		*B.*	
1. Idus melanotus	124 mm^3	1. Trygon pastinaca	39 mm^3
2. „ „	103 „	2. Lucioperca sandra	36 „
3. „ „	117 „	3. Idus melanotus	23 „
4. Perca fluviatilis	127 „	4. Gadus morrhua	34 „
5. Tinca vulgaris	110 „		
6. Gadus merl.	84 „		
7. Trigla hirundo	94 „		

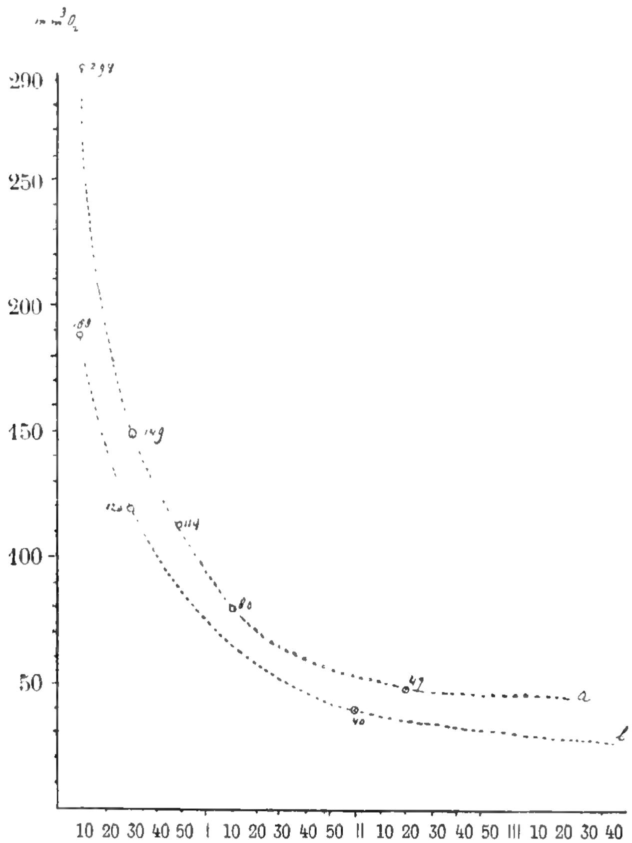
Besides the figures mentioned in the table I still obtained some results being much lower, and amounting e. g. to no more than 9 or 6 $mm.^3$ O_2 per gram and hour. An equal consumption of O_2 by the brain was likewise ascertained even 48 hours after the death of the individual, and is consequently only an accessory respiration according to BATELLI.

During the investigation it appeared to me that the percentage of O_2 of the fluid (at the beginning saturated with air) should not be less than 3 cc. O_2 per liter. If the latter was the case, the quantity of O_2 consumed by the tissue was considerably reduced.

Another fact that must be taken into account, is the mortification which takes place even when the fluid is of a good composition. So the brain of Perca fluviatilis consumed, during an experiment lasting 30 minutes, 184 mm^3 O_2 (calculated per gram and hour). After having been kept during an hour between two watch-glasses, the result was 162 mm^3 O^2 (for an experiment of 30 minutes). With another specimen these figures were respectively 173 mm^3 O_2 and 145 mm^3 O_2. One sees that the diminution of the consumption of O_2 by mortification in the first hours need not be of importance in the tissues of such cold-blooded animals. The quantity of O_2 consumed is however considerably reduced if the fluid is repeatedly renewed (e. g. after every 20 minutes).

In fig. 1 I have given a graphical representation of two of these experiments.

Fig. 1.



On the line of the abscisses the time is indicated that has elapsed since the beginning of the experiment, in the ordinates the quantity of O_2 per gram and hour. Curve *a* relates to the respiration of the brain of a Gadus merlangus, curve *b* to an experiment with an Esox lucius. One sees that the diminution of the consumption of O_2 is constantly going down by repeated renewing of the fluid. In how far this must be attributed to the extraction of vital (oxydative) ferments I wish for the present moment to leave undecided.

In a subsequent series of experiments I examined the quantity of O_2 that was consumed by the lobi optici, the lobi olfactori and the cerebellum conjointly, and likewise the quantity of O_2 consumed by the radix cerebri, and the medulla longata, originating from the same experimental animal. In all the experiments mentioned in table II, the consumption of O_2 by the radix ceribri was found to be considerably smaller than by the remaining part of the brain The duration of the experiments was always the same (20-25 minutes) the final percentage of O_2 3—4 cc. per liter.

TABLE II.

Quantity (per gram and hour) consumed by the radix cerebri (column *A*) and by the remaining part of the brain (column *B*).

A.		*B.*
1. Idus melanotus	133 mm³	202 mm³
2. Trigla hirundo	160 „	177 „
3. Gadus merlangus	64 „	84 „
4. Cyprinus carpeo	136 „	200 „
5. Tinca vulgaris	43 „	66 „

Had I tried hitherto to compose the fluid in which the brain respired in such a way as to preserve the normal qualities of the nervous system, I have likewise investigated some injurious influences, and these experiments give an impression of the part that the vital processes take in the consumption of O_2. The experiments of Table III show that ether, aqua distillata, and a little acidity of the solution of salt considerably reduce the consumption of gas whilst a little increase of alcalicity is only connected with a slight reduction.

TABLE III.

Quantity of O_2 (per gram and hour) consumed by the brain from a RINGER fluid (column *A*) and from another fluid (column *B*).

A.		*B.*	
1. Scardinius erythropht	133 mm³	Sol of ether-salt	37 mm³
2. Idus melanotus	202 „	„ „	84 „
3. Clupea harengus	44 „	„ „	4 „
4. Tinca vulgaris	31 „	„ „	3.5 „
5. Scardinius erythropht	124 „	Aqua dist.	43 „
6. „ „	70 „	„ „	34 „
7. Lucioperca sandra	72 „	$^1/_{100}$ N.HCl	19 „
8. Gadus merlangus	84 „	$^1/_{100}$ N.KOH	64 „

Some electrodes had been placed in the glass vessel in which the brain respired in the midst of the fluid, so that, during the experiment the tissue could be irritated by induction currents.

In the first place I can communicate that in the controlling experiments the solution of salt only or with the pieces of blottingpaper immerged in it did not show a reduction of the percentage of O_2 worth mentioning, if during 30—60 minutes induction currents (of the strength used for the irritation) were conducted through the fluid. It was however different, if the fluid contained brain. In table IV is shown that brain respiring strongly consumes considerably more when irritated. On the contrary hardly any increase is observed with brain respiring feebly (dead brain). In the first three experiments the consumption of O_2 after the irritation is stated.

TABEL IV

Quantity of O_2 consumed (per gram and hour) by the brain (column *A*) with irritation during 20—30 minutes (column *B*) and afterwards (column *C*).

	A	*B*	*C*
1. Lucio perca Sandra	208 mm.^{3}	275 mm.^{3}	229 mm.^{3}
2. Tinca vulgaris	110 „	150 „	98 „
3. Gadus morrhua	67 „	87 „	— „
4. „ „ [1]	23 „	28 „	— „

In connection with the current view about assimilation in the nervous system, the increase of consumption of O_2 demonstrated here, is not unexpected.

For the investigation of the consumption of O_2 by peripheric nerves, I used the head nerves of large specimens of Gadus morrhua, as with these experimental animals each individual gave a sufficient quantity of nervous tissue to consume such a quantity of O_2 from the Ringer solution as could easily be determined. Every experiment lasted 30—60 minutes.

I have given a graphical representation of the results in fig. II. The ordinate indicates the quantity of O_2 consumed (per gram and hour), the abscis the experimental numbers arranged in a special way. It is evident that with irritation (indicated in the figure by *) the

[1]) For Rana I find for the spinal cord a consumption of 150 mm.^{3} O_2 with irritation 178 mm.^{3} O_2, afterwards 141 mm.^{3} O_2 per gram.

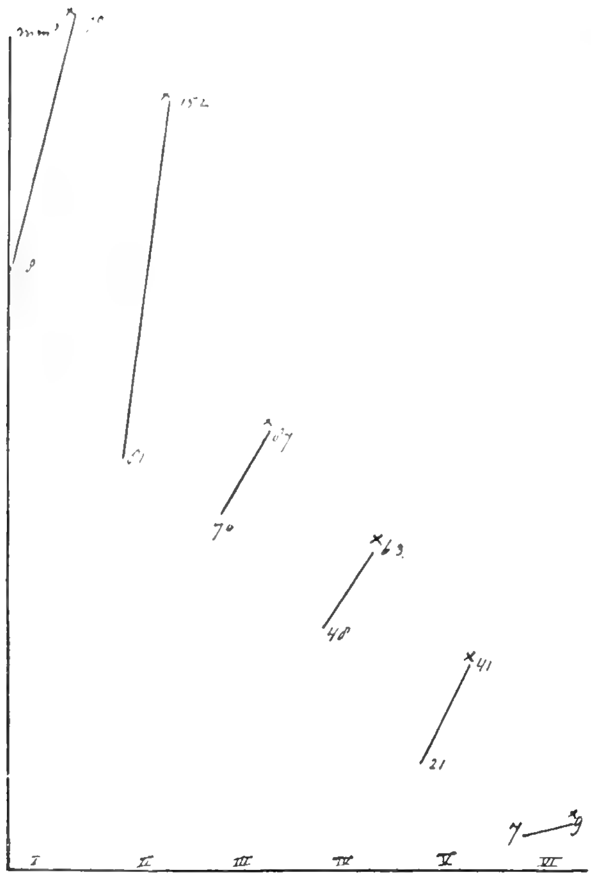


Fig. 2.

quantity of O_2 consumed is greater, and has increased in proportion as the quantity originally consumed is greater [1]).

By the kind coöperation of the Director of the Kon. Zoöl. Gen. Natura Artis Magistra, to whom I beg to pay here my sincere thanks, I was able to execute these investigations.

[1]) For the frog-nerve I could not possibly show an increased consumption of O_2 during irritation. In the following table stands under column *A* the quantity of O_2 consumed by Nervi ischiadici of Rana, under column *B* during irritation.

	A.	*B.*
1.	64	66
2.	48	52
3.	35	31
4.	23	25

The slight differences fall entirely within the experimental errors.

Zoology. — "*A new case of parental care among fishes.*" By Prof. MAX WEBER.

Cases of parental care are some of the remarkable phenomena among lower animals which have attracted very little attention — though amongst fishes they are so numerous that the well-known American naturalist TH. GILL has devoted an extensiv earticle to: "Parental care among fresh-water fishes" [1]). But this phenomenon is also found among sea fishes, although in these cases it more easily escapes observation.

The usual idea is that fishes are indifferent to the fate of their eggs and off-spring, that their care for both, if it does exist, generally does not extend further than a search for a suitable place for their development and that only in rare cases more care is bestowed, for instance by fastening the eggs on a suitable place, or by building a nest for them or by taking care of them in some other way.

Closer observation teaches us that such care can be shown not only in a more active but especially in a more lasting way.

It is known for instance that the male of the marine stickleback weaves a nest as large as a fist of vegetable parts and threads of slime, a secretion of its kidneys, which hardens when in water — in which nest the eggs and afterwards the fry are kept and bravely defended. In fact many cases are known of guarding and defending the brood and generally this is done by the male. But he can also concern himself in another way with the fate of the eggs and can give them what the Germans so aptly call: "Brutpflege". The male of the lumpsucker (*Cyclopterus lumpus*), whilst taking care of the large cluster of eggs, keeps the water round them refreshed and in circulation and also keeps poking his snout in the cluster of eggs so that the surface changes. In the same way the freshwater fish *Leucaspius delineatus* keeps the plants on which he has fastened the eggs, in constant motion by beating his tail against them so that the eggs may have enough oxygen.

Still more intimate is the connection between parents and eggs in a number of fishes (*Cichlidae, Siluridae, Apogonidae*). Sometimes the female, sometimes the male, more often the latter, hatches its eggs in its mouth cavity and when in danger even the young fishes can safely hide there.

No less peculiar is the phenomenon of the eggs being stuck to the skin of the abdomen of the male. In the male of the *Aspredidae* it grows in such a way round the eggs that these come to be pedunculated.

1) TH. GILL. SMITHSONIAN. Report 1905. p. 404.

Something like this is shown by the males of the sea-horses and needlefish (*Syngnathidae*). In the simplest cases the eggs are stuck by slime in two or more rows to the abdominal surface, in other cases the female brings the eggs into a brood-pouch which is developed during the time of propagation along the ventral surface of the male's tail.

In this short sketch the different methods of parental care (generally by the male) have not by a long way all been described.

The knowledge of one of the most remarkable cases is due to the two glorious expeditions to Dutch South New Guinea under leadership of Mr. H. A. LORENTZ. Both expeditions had their working basis in the Lorentz (Noord) river, which was navigated up very high and which had its fish fauna thoroughly investigated. It was then that the *Kurtus Gulliveri* de Castelnau which is remarkable as well for its shape as for its internal build, was discovered in that river; formerly it was only known from the Norman and the Strickland River, a side stream of the Fly River in British New Guinea. A second species *Kurtus indicus,* which is much smaller, (specimens of 430 mm. length were brought home of *Kurtus Gulliveri*) lives round the coasts of the Indian Archipelago and Britisch India. Together they form the small family of *Kurtidae* with the single genus *Kurtus,* one of whose characteristics is that the male, when fullgrown, has on its occiput a bony hook which is bent forward (Fig. 1). It comes from the supraoccipital and carries the remains of rudimentary dorsal spines. The females have no sign of this apparatus; in the male it developes gradually during the growth of the individual and appears only to reach its full size during propagation when the

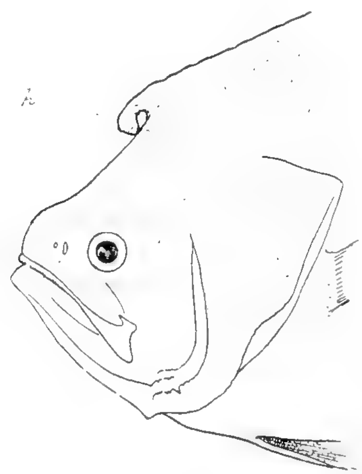


Fig. 1.

skin round the hook swells in a rind-like way. At any rate in *Kurtus Gulliveri*, the end of the hook becomes in this way so large that, as it bends downwards and forwards, it nearly touches the head and in this way forms an eye in which the eggs are carried. This is done by a round string which is held fast by the eye mentioned above and which branches off on either side, first in coarse branches, then in finer ones and finally in very fine fibres, at the ends of which the eggs are fastened, each in its strong but transparent membrane.

All the eggs together form a more or less round mass which rests on either side of the male's head. In this way the eggs develop till they are hatched. In one of the two clusters of eggs which I had at my disposal the eggs had nearly reached this stage — the young fishes had eyes, a well-developed tail and the yolk sack was on the point of disappearing.

There are more cases known of fishes' eggs being stuck together by some stuff formed by ovarium or oviduct and secreted in a more or less fluid or slimy condition with the eggs, but which hardens in water.

Probably in *Kurtus Gulliveri* the united ends of the two oviducts secretes the connecting string, and its collateral continuations with the ramificating branches that carry the eggs, are formed in the respective oviducts and ovaries. Then it must be supposed that at a certain moment this apparatus is discharged as a whole by the genital porus after which by the contact with water, those parts that carry the eggs harden.

Before trying to answer the difficult question of how this apparatus comes underneath and behind the male's hook I must point out that during the expedition of 1907 on October 6th the fishing net brought to the surface a specimen of *Kurtus Gulliveri* of 390 mm. length with its hook but slightly open whilst the net also contained an egg-apparatus. Mr. J. W. VAN NOUHUIJS, Mr. LORENTZ's companion then suggested that the cluster of eggs had possibly been carried behind the hook. This was confirmed by the capture on March 3rd 1910 of a specimen of 420 mm. length, that in fact still carried the eggs, as the accompanying illustration (Fig. 2) shows. On this specimen the hook and the head form together a nearly closed eye.

After removal of the thickened skin the hook is seen to consist of a sickle-shaped process of the supraoccipital which is composed in front of a very narrow, laterally compressed bone. Its appearance leads us to suppose that it had its origin in ossification of the subcutaneous connecting tissue and gradually gained in measurements during the growth of the fish. The first indication of the hook in

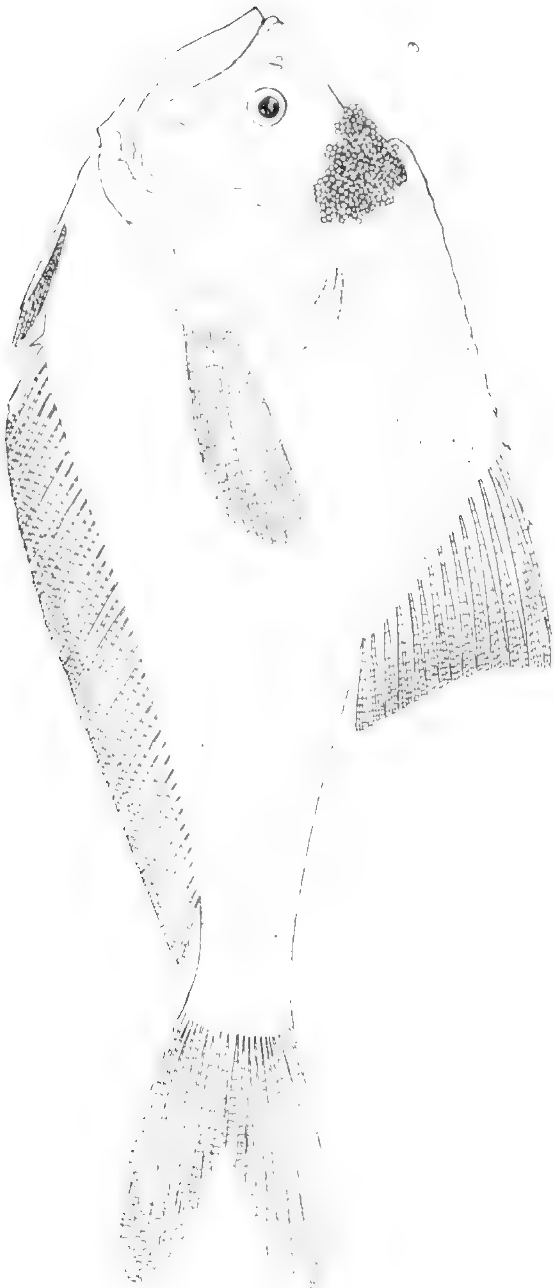


Fig. 2.

the young male is only visible as a slight protuberance, that in older specimens slowly takes the shape sketched above.

It will be superfluous to mention particularly that further investigaion confirmed that only the males possess the hook.

This curious apparatus first reminds us of the frontal clasper in males of different kinds of Chimaera. But this is movable and provided in front with dermal teeth and supported by a piece of cartilage that can be retracted into a dermal pouch. Though its function is not known, it certainly has nothing to do with the carrying of eggs. So functionally the two apparatuses are not to be compared either.

It appears to me one might sooner draw a comparison with the crestlike elevation of the occipital in *Selene* which gains in height with the growth of the individual; also with *Naseus* (*Acanthurus*) *nasicornis*, whose skull sends out a bony horn above the eye, which grows longer as the fish grows older.

A question not easy to find an answer to is how the hook in *Kurtus Gulliveri* is set to work, how in fact the short string with on either side its cluster of eggs, comes to lie under the hook in such a way that the two clusters hang symmetrically on the head of the male. The only line of action I can imagine the couple concerned in the laying of the eggs to take, is that the male should take up a vertical position under the female's genital porus. As soon as this ejaculates the string with its two clusters of eggs — the string now being still soft — it is caught by the male's head and pushed under the hook, possibly by a forward movement on the part of the male.

This is a purely hypothetical explanation — but I know of no better one for the intricate manoeuvres necessary to bring the eggs in the desired place. The advantage for the eggs, when once in that position, is apparent especially in a stream as the Lorentz River, which is a quick flowing stream and even more so as it floods its banks whenever there is a heavy fall of rain. When carried by the strong parent, there is small danger for the eggs of being swept against the banks or buried under mud and stones or of being harmed in some other way.

But this is not an answer to the question what the origin was of this strange line of action and many other questions in connection with this. It is not known if *Kurtus indicus* uses his hook in the same way.

Physics. — "*The electromotive force of the* WESTON *Normal Cell*".
By Prof. H. HAGA and J. BOEREMA.

At the international Conference on electrical Units and Standards held in London in Oct. 1908, some directions were given for the construction of the WESTON cell, as a standard of electromotive force. For its E. M. F. at 20° 1.0184 international volts was taken provisionally, till further measurements shall give a more accurate value.

The usual way of determining the E. M. F. of a cell consists in measuring the strength of the current, which gives rise to a difference of potential between the terminals of a known resistance equal to the E. M. F. of the cell.

The various determinations differ in the method of measuring the current. Restricting ourselves to the determinations of the five last years, in 1906 at the *Bureau of Standards* at Washington [1]) the current was measured by means of an electrodynamometer, consisting of two coils, whose axes were placed horizontally and at right angles to each other. The smaller of the two coils was suspended by a phosphor-bronze wire inside the other. From the dimensions of the coils, the modulus of torsion of the wire and the torsion required to keep the inner coil in the original position during the passage of the current, the strength of the current was calculated. In 1907 at the *National Physical Laboratory* at Teddington [2]) a current weigher was used, which had a coil with vertical axis suspended from each end of the beam inside of a fixed coil; the movable and the fixed coil hanging coaxially. The torque arising from the passage of the current was compensated by weights. From the amounts of these weights and the dimensions of the coils the strength of the current could be found.

In a similar way the E. M. F. was determined at the *Laboratoire central d'électricité* at Paris [3]) in 1908; the coils of the English current weigher, however, were long and wound with a single layer of bare wire round marble cylinders, those of the French instrument were much shorter and wound with several (12—18) layers, which, no doubt, rendered the accurate measurement of the effective area of the coils more difficult.

The following values of the E. M. F. of the Weston Normal Cell were found at 17° :

Bureau of Standards	1.01864	Volts
National Physical Laboratory	1.01830	,,
Laboratoire Central d'électricité	1.01869	,,

the current being expressed in C. G. S. ampères, the resistance in international Ohms.

It is in principle much simpler to measure the current by means of the tangent-galvanometer than by these methods, which require

[1]) Bulletin, Bureau of Standards vol. 2. Nr. 1. p. 33.

[2]) Phil. Trans. Roy. Soc. A. Vol. 207. p. 463.

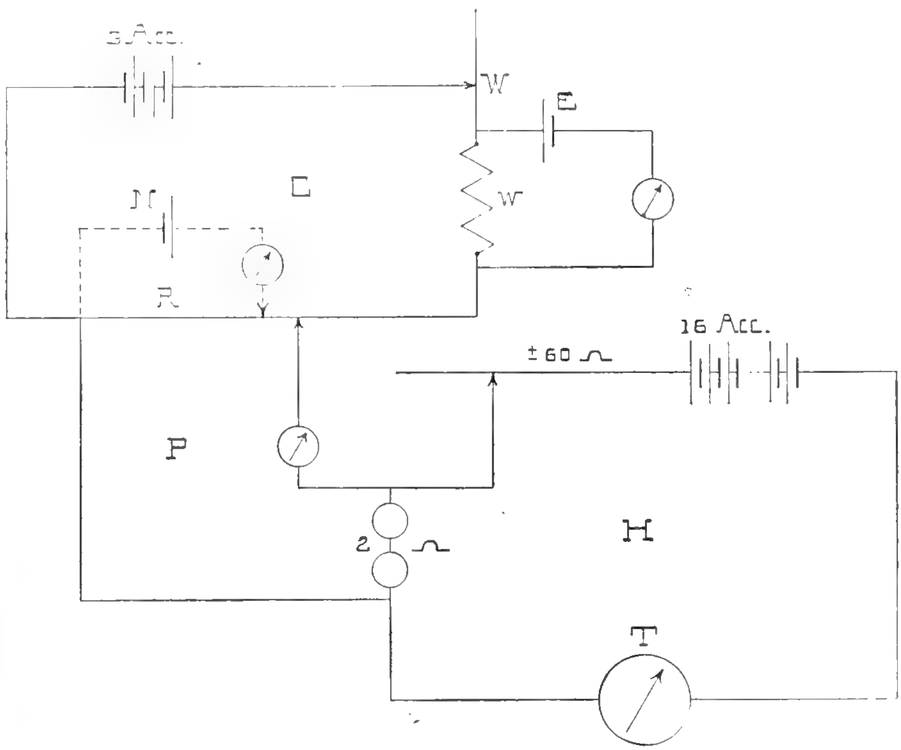
[3]) Bulletin de la Soc. internationale des Electriciens, 1908, 1910.

rather complicated expressions for the forces acting on the coils.

The *Physical Laboratory* of the *University* of Groningen being very suitable for these researches, and the importance of determining a quantity so essential for electrical measurements by different ways being great, a new measurement was undertaken by means of the tangent-galvanometer, though this method has the drawback that an error in the value of the horizontal component of the earth's magnetic field occurs in the value of the E. M. F. $5^1/_2$ times increased.

2. The adjoined figure represents the arrangement of the circuits diagrammatically.

In the main circuit a current of about $^1/_2$ ampère was produced by a battery of 16 accumulators; this current passed through a regulating resistance of manganine wire of about 60 ohms, and two resi-



stances of 1 ohm; its strength was measured by two tangent-galvanometers. These two resistances of 1 ohm had been constructed for this purpose by Otto Wolff, Berlin, from manganine wire of such a section that a current of 1 ampère would cause only a slight rise of temperature; so the current of $^1/_2$ ampère used in the experiment will only very slightly modify the temperature of the two resistances, which were immersed in a large tank with paraffin oil provided with a stirring-apparatus. Their resistance *in international ohms* was found by comparison with two standard resistances of 1 ohm, which had been tested with special care before and after the investigation at the *Physikalisch Technische Reichsanstalt*

The comparison was made by means of the WHEATSTONE bridge; the four branches being formed by a ratio coil (100, 0,05, 0,05, 100 ohms), the two resistances of 1 ohm, and the two standard resistances of 1 ohm. The galvanometer used was the JAEGER galvanometer, made by SIEMENS & HALSKE, with a movable coil with a resistance of 9,5 ohms; a deflection of 1 mm. with a distance of the scale of 1 meter was obtained by a current of 1.4×10^{-8} ampères; the resistances could be determined to one millionth of an ohm.

The two tangent-galvanometers were the same as had been used by Messrs. G. v. DIJK and J. KUNST in their determination of the electro-chemical equivalent of silver [1]); they were placed one to the north, the other to the south of a bifilarly-suspended magnet, so that the horizontal component of the earth's magnetic field could be determined immediately before and after the measurements of the current, the variometers for the declination and the horizontal intensity being read during the latter. The horizontal component of the earth's magnetic field was determined in the same way as has been at length explained in the above-mentioned paper. An improvement was only made in the method of the determination of the distance of the magnetometers from the bifilarly-suspended magnet, which consisted in this that in the frontside and in the backside of the glass tubes, containing the suspension threads of the magnetometers, holes are bored, 3 mm. wide and 1 cm. high, so that the place of the suspension threads could be accurately determined on a horizontal graduated glass scale placed behind it. Except in this determination the holes were closed by a paper tube. Moreover the wooden 3-meter scale, serving for the measurement of the scale-distances, was replaced by a brass one. The thick copper strips: the leads of the southern tangent-galvanometer, had been replaced by two thin copper wires, which were placed close to each other in a plane normal to the magnetic meridian.

By means of RAPS' potentiometer the difference of potential between the terminals of the resistance of 2 ohms was compared with that of the WESTON Normal Cell N, for which the element

[1]) Arch. Néerland. Série II, Tome IX, p. 442.

marked C_{20} was taken. By means of an auxiliary battery of 3 accumulators, a regulating resistance W, the fixed resistance W of 10190 ohms, the WESTON cell E — a current was obtained of about 0,0001 ampères in the well-known way, and the potentiometer resistance R was determined, required to make the current in the circuit of N zero.

Then the resistance in the main circuit H was adjusted so that with the same resistance R no current passed through the circuit P. When this was obtained the simultaneous reading of the two tangent-galvanometers gave the intensity of the current.

As, however, perfect equality of the potentiometer resistance in the two cases could not be obtained, part of the measurements were made with a somewhat smaller, part of the measurements with a somewhat greater potentiometer resistance in the circuit P than in the circuit of the normal cell, so that the accurate value of the current could be found by interpolation.

The galvanometer used with the potentiometer was an EDELMANN galvanometer with movable coil of 240 ohms; a deflection of 1 mm. for a distance of the scale of 1 meter was caused by a current of $3,6 \times 10^{-10}$ ampères; by means of this galvanometer it was possible to determine the potentiometer resistances occurring here with an accuracy of one tenthousandth percent.

On account of the field-magnets of the two galvanometers the circuits P and C were placed at a great distance from the tangent-galvanometers.

3. The WESTON Normal Cells were constructed by one of us (J. B.) according to the procedure at the *National Physical Laboratory*[1]): the mercury was distilled in a space of rarefied air, small air-bubbles being led through it, and then a few times in vacuo; the cadmium amalgam was prepared electrolytically, pure cadmium of KAHLBAUM being used as anode. A hundred parts by weight of the amalgam contained $12^1/_2$ parts by weight of cadmium. The cadmium sulphate furnished by KAHLBAUM, denoted in his catalogue "zur Arsenbestimmung", was recrystallised a few times. We obtained the mercurous sulphate by preparing an acid solution of mercurous nitrate from strong nitric acid and mercury, and by pouring this as a finely divided stream into hot dilute sulphuric acid, while shaking it vigorously. The precipitate was filtered, washed twice with diluted sulphuric acid and then several times with a neutral solution of cadmium sulphate.

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1) Phil. Trans Roy. Soc. A. 207, p. 393.

The cells were prepared at different times; they were sealed by the blow-pipe, and placed in a paraffin oil-bath. It appears from the subjoined table 3 that the E. M. F. of the cells differed little from each other; their E. M. F. was 38×10^{-6} volts higher than the E. M. F. of three cells which were kindly put at our disposal by the *National Physical Laboratory* in October 1908: S_5, S_6, and S_8, [1]) so that the experience obtained in other laboratories that the WESTON Normal Cell, if prepared with care, can be reproduced, is fully confirmed.

4. The dimensions of the instruments required for the determination of the current, the radii of the tangent-galvanometers, the length and the distance of the suspension-threads of the bifilarly-suspended magnet, etc. were determined by one of us (J. B.) by comparison with a standard invar-meter, whose errors of graduation were found by comparison with a double decimeter of invar, which had been examined at the *Bureau international des poids et mesures* at Breteuil.

5. The course of the measurements, which were made with the assistance of Messrs. E. OOSTERHUIS and R. PALSMA, was as follows:

a. Determination of the horizontal component of the earth's magnetic field by simultaneous reading of the positions of the bifilar magnetometer, the two magnetometers of the tangent-galvanometers, and the variometers for the horizontal intensity and the declination.

b. Determination of the deflections of the two tangent-galvanometers at the moment that the circuit P was without current, the potentiometer resistance R also being read. This measurement was made an odd number of times (generally 11), always after reversal of the current in the tangent-galvanometers. At the same time with the tangent-galvanometers the variometers were also read.

c. Determination as under *a*.

For the final determinations ten such series of measurements have been made; so the number of measurements of the current amounts to fifty for every tangent-galvanometer.

On account of the disturbing influence of the electric tram on the positions of the magnetometers and variometers, the measurements had to be made in the night; between half past eleven and two o'clock two series could be finished.

The results of these measurements are given in the tables 1 and 2.

In table 1 the 2nd and 3rd columns give the values of the three determinations of the horizontal component of the earth's magnetic field

[1]) These cells contained cadmium amalgam with 10% cadmium.

TABLE 1.

Date	H_z	H_n	$H(i_1)$	$H(i_2)$	i_z	i_n
26 Sept.	0.18154	0.18148	0.18150	0.18156	0.50915	0.50910
	0.18157	0.18154	0.18153	0.18159	0.50924	0.50915
	0.18161	0.18162	0.18156	0.18162		
27 Sept.	0.18172	0.18172	0.18175	0.18147	0.50918	0.50923
	0.18166	0.18168	0.18176	0.18148	0.50917	0.50916
	0.18147	0.18149	0.18170	0.18143		
28 Sept.	0.18181	0.18178	0.18164	0.18158	0.50925	0.50908
	0.18158	0.18161	0.18164	0.18157	0.50927	0.50922
	0.18164	0.18163	0.18162	0.18155		
29 Sept.	0.18124	0.18128	0.18136	0.18134	0.50919	0.50904
	0.18133	0.18133	0.18134	0.18132	0.50923	0.50907
	0.18131	0.18133	0.18140	0.18138		
30 Sept.	0.18153	0.18146	0.18149	0.18149	0.50919	0.50915
	0.18149	0.18144	0.18145	0.18145	0.50920	0.50924
	0.18152	0.18144	0.18149	0.18149		

TABLE 2.

Date	i	R_P	R_N	t_N	R	E. K. C_{20} at 17°
26 Sept.	0.50913	10190.2	10191.0	15°.6	2.00002_5	**1.01829**
	0.50920	10191.1	10191.0	15 .6	2.00002_0	**1.01834**
27 Sept.	0.50920	10191.7	10191.4	15 .7	2.00000_9	**1.01830**
	0.50916	10190.6	10191.0_5	15 .8	2.00001_5	**1.01833**
28 Sept.	0.50917	10190.7	10190.9	16 .0	2.00002_0	**1.01832**
	0.50924	10191.3	10190.9_5	16 .0	2.00002_5	**1.01841**
29 Sept.	0.50912	10190.6	10190.8	18 .2	2.00014_1	**1.01836**
	0.50915	10191.2_5	10190.8	18 .2	2.00014_1	**1.01838**
30 Sept.	0.50917	10190.7	10191.0	16 .5	2.00006_8	**1.01838**
	0.50922	10191.2_5	10191.0	16 .5	2.00006_2	**1.01842**
					Mean	**1.01835**

for every day, derived respectively from the deflections of the southern magnetometer: H_z, and from those of the northern magnetometer: H_n; the 4th and 5th columns give the value of H, corresponding with the mean position of the intensity-variometer, resp. during the 1st and 2nd measurement of the current, as they are derived from the three H-determinations. By application of the necessary corrections, the strength of the current in ampères was determined from the mean of the values of H obtained in this way, the known radii of the tangent-galvanometers and the angles of deflection; the 6th column gives the strength of the current of the 1st and the 2nd measurement of the current derived from the southern tangent-galvanometer for every day, the 7th column the same from the northern tangent-galvanometer.

TABLE 3.

Differences in the E. M. F. with those of C_{20} in microvolts (10^{-6} V),					
C_{13}	0	C_{24}	+ 1	C_{35}	− 2
C_{14}	0	C_{25}	+ 7	C_{36}	− 2
C_{15}	− 7	C_{26}	+ 3	C_{37}	− 6
C_{16}	+ 2	C_{27}	− 2	C_{38}	0
C_{17}	− 1	C_{28}	− 2	C_{39}	− 1
C_{18}	− 2	C_{29}	+ 4	C_{40}	+ 1
C_{19}	+ 8	C_{30}	+ 7	C_{41}	+ 8
C_{20}	—	C_{31}	+ 3	C_{42}	+ 6
C_{21}	+ 1	C_{32}	− 1	C_{43}	0
C_{22}	+ 5	C_{33}	+ 3		
C_{23}	+ 5	C_{34}	0		

In table 2 the second column gives for every day the value of the two intensities of the current as the mean value from the two last columns of table 1; the 3rd column the potentiometer resistance in the circuit P; the 4th column that of the circuit N; as is seen one value R_P is every day greater, the other smaller than R_N.

The 6the column contains the value of the 2 Ohm resistance for the temperature, which this resistance had during the passage of the current.

The E. M. F. of the cell C'_{20} at the temperature mentioned in the 5[th] column was now found as the product $i \cdot R \cdot \frac{R_X}{R_P}$. From the value found in this way the E. M. F. for 17° is calculated by the aid of the temperature formula. It is found in the last column.

As according to table 3 the mean E. M. F. of all the 31 cells is only 1 microvolt higher than the E. M. F. of C'_{20} the result of this investigation is that the E. M. F. of the **Weston Normal Cell at 17° is 1.0183$_5$ Volts** (intern. ohm; C.G.S.-ampère), which value may be considered as accurate down to the fourth decimal.

This value is in close agreement which that found in the *National Physical Laboratory*

On account of the remaining doubt as to the accurate value of the electro-chemical equivalent of silver the ratio between the C. G. S.-ampère and the international ampère is not yet accurately known, so that it is not yet possible to express the above result in international volts.

(December 22, 1910).

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