PAPERS

SET IN THE

MATHEMATICAL TRIPOS, PART I

IN THE

UNIVERSITY OF CAMBRIDGE

1913–1917

CAMBRIDGE
AT THE UNIVERSITY PRESS
1919

MATHEMATICAL TRIPOS

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CONTENTS

MATHEMATICAL TRIPOS

PART I

1913

THURSDAY, 29 *May*. 9—12.

1. Prove that the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

The sides AB, DC of a quadrilateral $ABCD$ meet at E and the sides AD, BC meet at F. Prove that the circles described about the four triangles ABF, BCE, CDF and DAE intersect in a point.

Prove further that, if $ABCD$ is cyclic, the point of intersection of the circles lies on EF.

2. Prove that the tangent to an ellipse makes equal angles with the focal distances of the point of contact.

SP, $S'P'$ are focal radii of an ellipse drawn in the same direction and the tangents at P and P' meet $S'P'$, SP in Q' and Q respectively. Prove that QQ' is parallel to PP'.

3. (i) Solve the equations

$$x(1-y) = y(1-z) = z(1-x) = c$$

in the case $c = -2$; and prove that the solution is indeterminate when $c = 1$.

(ii) The equation

$$A(x-p)^2 + B(x-q)^2 = ax^2 + 2bx + c$$

is to be an identity for all values of x; determine A, B, q in terms of a, b, c, p, verifying that $A = (ac - b^2)/(ap^2 + 2bp + c)$.

4. Write down the series, in powers of x and y respectively, for

$$(1)\ \sin x;\quad (2)\ \log(1+y).$$

Deduce that, when x is small,

$$\log(\sin x) = \log x - x^2/6 - x^4/180$$

approximately.

Obtain the tabular value for the logarithmic sine of 5°, using the approximations

$$\log_{10} e = \cdot 4343,\quad \pi = 3\cdot 1416.$$

5. Prove that if $A+B+C=2\pi$

(1) $\cos^2 A+\cos^2 B+\cos^2 C-2\cos A\cos B\cos C=1$;

(2) $\sin A+\sin B+\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$;

(3) $\sin^3 A+\sin^3 B+\sin^3 C$

$$=3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}.$$

6. From a certain station A the angular elevation of a mountain peak P to the north is α. A hill B of height h above A is ascended. From B the angular elevation of P is β, the bearing of A is δ west of south, and the bearing of P is γ north of A. Shew that the height of P above A is

$$\frac{h\tan\alpha\sin\gamma}{\tan\alpha\sin\gamma-\tan\beta\sin\delta}.$$

7. Apply the method of integration by parts to evaluate

$$\int\log x\,dx,\quad \int x\log x\,dx,\quad \int x\,e^{ax}dx.$$

Shew that

$$\int_0^1 x\log(1+\tfrac{1}{2}x)\,dx=\tfrac{3}{4}(1-2\log\tfrac{3}{2}),$$

and prove in any way that this is less than

$$\int_0^1 \tfrac{1}{2}x^2dx.$$

8. If y is of the form ax^3+bx^2+cx+d, and y_1, y_2, y_3 are the values of y corresponding to values x_1, x_2, x_3 such that

$$x_2-x_1=x_3-x_2=h,$$

prove that $$\int_{x_1}^{x_3} y\,dx=\tfrac{1}{3}h(y_1+4y_2+y_3).$$

Deduce Simpson's rule for approximating to an integral

$$\int_a^b y\,dx=\tfrac{1}{3}h\{(y_1+y_{2n+1})+4(y_2+y_4+\dots+y_{2n})$$
$$+2(y_3+y_5+\dots+y_{2n-1})\},$$

where $x_1=a,\ x_2=a+h,\ x_3=a+2h,\ \dots,\ x_{2n+1}=b=a+2nh.$

The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in miles per hour.

2 min.	10 m./h.	12 min.	20 m./h.
4	18	14	11
6	25	16	5
8	29	18	2
10	32	20	at rest

Estimate approximately the total distance run in the 20 minutes.

9. Find the equation of a plane through three points whose coordinates are given.

Find the equations of the two planes through the points $(0, 4, -3)$, $(6, -4, 3)$, other than the plane through the origin, which cut off from the axes intercepts whose sum is zero.

10. Find the condition that two diameters of an ellipsoid should be conjugate.

Shew that any set of three equal conjugate diameters of the spheroid whose equation is $x^2/a^2 + (y^2 + z^2)/b^2 = 1$ lie on a right circular cone and that the cosine of the angle between any two is $(a^2 - b^2)/(a^2 + 2b^2)$.

THURSDAY, 29 *May*. 2—5.

1. The sides of a triangle are cut by a transversal; state and prove the relation between the segments into which the sides of the triangle are divided.

A circle concentric with the circumcircle of the triangle ABC cuts AC in E, E', and cuts AB in F, F'. EF, $E'F'$ cut BC in D and D'. Prove that D and D' are equidistant from the centre of the circle.

2. The section through the axis of a right circular cone whose vertex is A is a triangle BAC in which $AB = AC = 9$ inches and $BC = 12$ inches. P is a point on AB distant 6 inches from A. By considering the sector of a circle which can be rolled into the form of the given cone, calculate the length of the shortest path on the surface from P to C and find how near this path approaches A.

3. Obtain an expression of the fourth degree in x, which shall have the values a, b, c, d, e respectively corresponding to the values $-2, -1, 0, +1, +2$ of x.

Reduce your expression to the form

$$y = c + px + \tfrac{1}{2}qx^2 + \tfrac{1}{6}rx(x^2-1) + \tfrac{1}{24}sx^2(x^2-1),$$

where
$$p = \tfrac{1}{2}(d-b),\quad q = d - 2c + b,$$
$$r = \tfrac{1}{2}(e - 2d + 2b - a),$$
$$s = e - 4d + 6c - 4b + a.$$

4. Resolve the expression $\dfrac{9}{(x-1)(x+2)^2}$ into partial fractions and find the general term of this expression when expanded in ascending powers of x.

5. Shew that the equations of any two circles can be put in the form

$$x^2 + y^2 + 2gx + c = 0,\quad x^2 + y^2 + 2g'x + c = 0.$$

A variable circle is one of a definite co-axal system, and a perpendicular is drawn from a fixed point on its polar with respect to the variable circle. Shew that the locus of the foot of the perpendicular is a circle whose centre is on the common radical axis of the system of circles.

6. Obtain a quadratic equation whose roots are the lengths of the radii PQ, PR, drawn from the point P (x_0, y_0) in the direction making an angle θ with the axis of x, to meet the parabola $y^2 = 4ax$ in the points Q, R; and deduce an expression for the product of the lengths PQ, PR.

If P is a point on the parabola, prove that the length of the chord through P in the given direction is equal to

$$4a \sin(\alpha - \theta) \operatorname{cosec}^2\theta \operatorname{cosec}\alpha,$$

where α is the inclination of the tangent at P to the axis of x.

7. Find the coordinates of the pole of the line $lx + my = n$ with respect to the conic $\alpha x^2 + \beta y^2 = \gamma$, and deduce the condition that the line should touch the conic.

The points A, A' are the ends of the major axis of a conic and PAQ, $P'A'Q'$ are tangents to the conic there; if PP', QQ' are two other tangents to the conic, prove that

$$AP \,.\, A'P' = AQ \,.\, A'Q'$$

and that the lines PQ', $P'Q$ intersect on AA'.

8. Prove that the nth differential coefficient of $e^{ax}\cos bx$ is

$$(a^2+b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + n\tan^{-1}\frac{b}{a}\right).$$

Prove that, if $y = (x + (1+x^2)^{\frac{1}{2}})^m$ and y_n denotes $\frac{d^n y}{dx^n}$,

(1) $(1+x^2)\,y_2 + xy_1 - m^2 y = 0$;

(2) $(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0$.

9. Differentiate $\tan^{-1}\frac{y}{x}$ partially with respect to x and with respect to y.

Explain the meanings of $\frac{\partial x}{\partial r}$ and $\frac{\partial r}{\partial x}$, where x, y are the rectangular coordinates of a point, r, θ its polar coordinates, and illustrate them geometrically.

Prove that $$\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2.$$

10. A circle of radius a rolls on a fixed circle of radius $2a$. Shew that the equation of the curve described by a point on the circumference of the rolling circle can be expressed in the form $p = 4a\sin\frac{\psi}{2}$, where p is the perpendicular from the origin on the tangent and ψ is the angle the perpendicular makes with a fixed line.

Find the radius of curvature at any point of this curve, and shew that the (p, r) equation of the locus of the centres of curvature is $4(r^2 - a^2) = 3p^2$.

FRIDAY, 30 *May*. 9—12.

1. Fig. 1 is a perspective view of a light bent rod in which ACB and ADB are vertical and horizontal planes.

C is distant 12 inches from AB. The vertical through C is 12 inches from A and 18 inches from B. D is distant 8 inches from AB. The horizontal through D perpendicular to AB is 22 inches from A and 8 inches from B. A load of 100 lbs. is suspended from D and the equilibrium of the rod is maintained by a horizontal force P at C and by vertical and horizontal reactions V_A, H_A, V_B, H_B at A and B.

Determine the magnitudes of these reactions and the value of P, all the forces being perpendicular to AB.

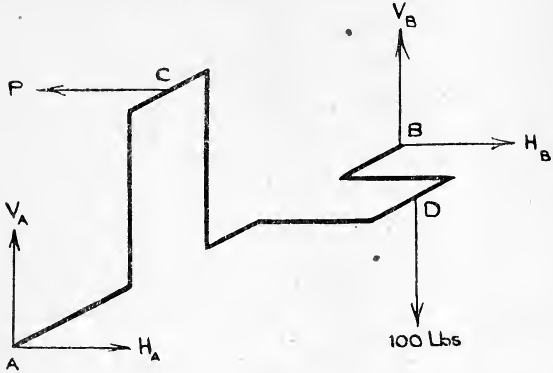


Fig. 1.

2. State the principle of Virtual Work, and deduce from it the conditions of equilibrium of a rigid body under the action of a system of co-planar forces.

In the framework of light rods illustrated in Fig. 2

$$AB = AC = a,\ DF = DG = EF = EG = b,\ DA = DH = DK.$$

The bars AB, AC are hinged to fixed points B and C, and are fitted with frictionless sliding pieces, H and K. Apply the

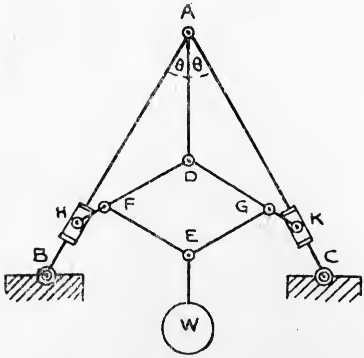


Fig. 2.

principle of work to prove that the horizontal component of the reactions at B and C is $W(a\sin\theta - 4b\sin 2\theta)/(2a\cos\theta)$.

3. Fig. 3 shews a pair of pincers for lifting a plate. By forcing the end of the screw FG against the block carrying the hinges C and D, the lower ends P and Q of the levers ACP, BDQ are made to grip the plate. P and Q are 4 inches below and E is 8 inches above CD. AE and BE make angles of 75° with the vertical, C and D are 6 inches apart. Calculate the pressure exerted on the plate at P and Q when the screw is made to exert a thrust of 500 lbs. at G.

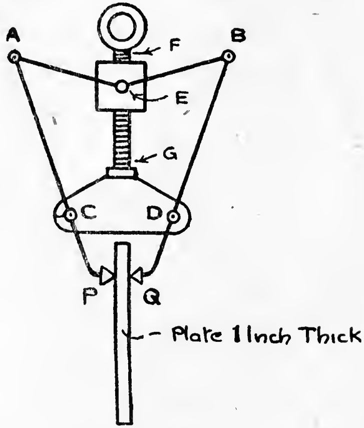


Fig. 3.

The plate which weighs 2000 lbs. is then lifted by the ring and owing to the "give" in the rods and levers the screw ceases to exert a thrust at G. Calculate the horizontal thrusts exerted on the plate at P and Q.

4. Prove that the kinetic energy of a body of mass M moving in two dimensions with angular velocity ω is $\frac{1}{2}MV^2 + \frac{1}{2}I\omega^2$, where V is the velocity of the centre of inertia and I is the moment of inertia about an axis through the centre of inertia, perpendicular to the plane of motion.

A four-wheeled railway truck has a total mass M, the mass and radius of gyration of each pair of wheels and axle are m and k respectively, and the radius of each wheel is r. Prove that, if the truck is propelled along a level track by a force P, the acceleration is $P/(M + 2mk^2/r^2)$; and find the horizontal force exerted on each axle by the truck.

[Axle friction and wind resistance are to be neglected.]

5. A light vertical spiral spring is suspended by one end and a weight A of 10 lbs. is attached to the other. Another weight B of 20 lbs. is hung from A by an inextensible string. The weight B is initially supported, and under the influence of A alone, the spring elongates ·01 feet. B is then let go and acquires a velocity of 16 feet per second just before the string becomes taut. Calculate the maximum elongation produced in the spring.

6. A and C are two concentric wheels. Four light rods freely hinged to A carry concentrated masses M fixed to their outer ends B. When A rotates, centrifugal action swings these heavy ends into contact with C. Shew that (neglecting the effect of gravity) the turning couple which A can exert on C in this manner is

$$\frac{4Mr^2\omega^2 \sin\beta \sin\phi}{\sin(\beta+\phi)},$$

where ω is the counter-clockwise angular velocity of A,
r is the radius of the inside of C,
ϕ is the angle of friction,
β is the obliquity of the rods.

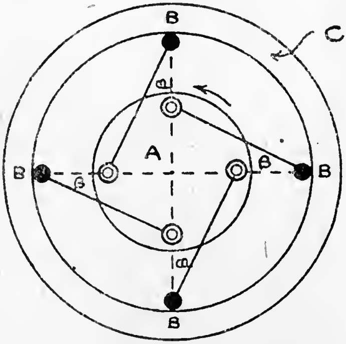


Fig. 4.

7. Fig. 5 illustrates a siphon for drawing water from a cask. The lengths of pipe DB, BA are 40 and 160 inches

respectively. The height of the section B above D is 36 inches. The end A is initially closed and the pipe from A to C filled with water, the air enclosed being at atmospheric pressure. The end A is then opened and as the water level in CA decreases, the level in DB rises. If a balance is obtained just before the water has risen to B, what length of water will remain in the arm CA?

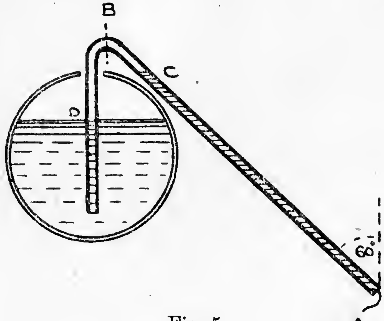


Fig. 5.

Shew that the siphon cannot be started by the process described above if the initial length of water in the arm AC is less than 120 inches.

[Assume the height of the water barometer to be 33 feet and neglect the inertia of the water.]

8. Prove that, if a ray of light passes through a refracting prism in a principal plane, the deviation is a minimum when the ray passes through symmetrically.

The angle of a glass prism, refractive index $\frac{3}{2}$, is 60° and the angle of incidence is 45°; find the difference between the deviation in this case and the minimum deviation.

9. Shew that if two sheets of a condenser are at a potential difference V, a circular area of radius a in the centre of either sheet is attracted towards the opposite sheet by a force equal to

$$\tfrac{1}{8}(Va/t)^2$$

where t is the thickness of the condenser.

The trap-door of an electrometer is a circle of 6 cm. diameter, and is at a distance of 2 mm. from the lower plate; it is found that a weight of $\frac{1}{2}$ gramme will balance the attraction between the plates. Calculate the potential difference in volts.

[The C.G.S. electrostatic unit of potential is 300 volts.]

10. To measure the insulation resistance of a given length of electric cable, the cable is placed in a tub of water and one end is carefully dried and insulated as shewn in Fig. 6 (*a*). A battery consisting of a large number of cells has one pole attached to the other end of the cable and the other pole is connected to the water surrounding the cable. The small current forced through the insulation produces a deflection δ in a very sensitive galvanometer.

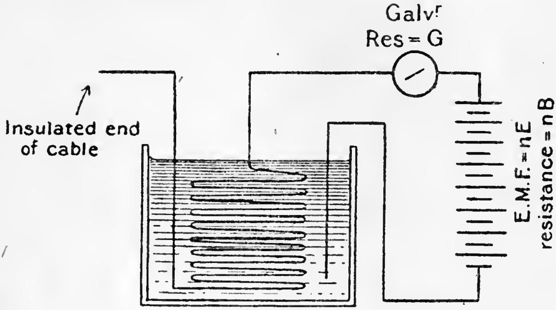


Fig. 6 *a*.

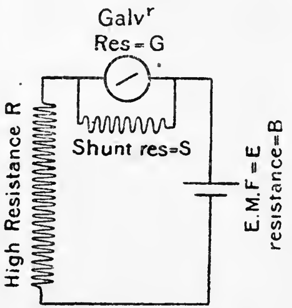


Fig. 6 *b*.

To interpret the reading of the galvanometer, it is now shunted with a resistance S and joined up with one cell and a known high resistance R, in the manner shewn in Fig. 6 (*b*). The deflection in this case is δ'.

Assuming that the currents through the galvanometer are proportional to the deflections, prove that the insulation resistance of the cable is

$$\frac{n\delta'}{\delta}\left[G+(B+R)\frac{G+S}{S}\right]-nB-G.$$

FRIDAY, 30 *May*. 2—5.

1. A chord QQ' is drawn in a circle whose centre is C; the tangents at Q, Q' meet in T, and QQ' is bisected in V. Prove that CVT is a straight line, and if it cuts the circle in P, shew that

$$CV \, . \, CT = CP^2.$$

Deduce that the line joining the mid-points of TQ, TQ' does not meet the circle.

Apply the method of orthogonal projection to obtain corresponding theorems for an ellipse.

2. A given pencil of four rays is cut by a transversal. Prove that the cross-ratio of the intercepts on the transversal is constant.

The angular points of a triangle move on fixed straight lines which meet in a point; prove that, if two of the sides pass through fixed points, the third side also passes through a fixed point, which is collinear with the first two fixed points.

3. If $$y = ax + bx^2 + cx^3 + \dots$$
shew that if x is expanded in a series of ascending powers of y the first three terms of the expansion are

$$x = \frac{y}{a} - \frac{by^2}{a^3} - \frac{(2b^2 - ac)y^3}{a^5} + \dots.$$

4. A chain of three rods AB, BC, CD, of lengths 5, 10, 5, feet, respectively, has its ends A and D attached to fixed points at the same level 18 feet apart. The chain initially hangs with BC horizontal and 3 feet below AD. The rod AB is then rotated about A in a vertical plane until B is 2·5 feet above AD. Calculate the angular displacement produced in CD.

5. Find the radius of the inscribed circle of a triangle in terms of the sides. Find also the segments into which the sides are divided by the points of contact with the circle.

The inscribed circle of the triangle ABC touches BC at P, AQ is the perpendicular from A on BC, AR the bisector of the angle A, and D is the middle point of BC. Prove that

$$DQ \, . \, DR = DP^2.$$

6. Find the angle between the two straight lines whose equations are

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0.$$

The equation of the bisector of the angle between two straight lines is $7x - 4y + 1 = 0$. The equation of one of the lines is $3x + 4y = 11$; find the equation of the other.

7. Find an expression for the length of the tangent to a sphere from a point, having given the coordinates of the point and the equation of the sphere.

Prove that, if a sphere cuts orthogonally two spheres whose equations are $S = 0$ and $S' = 0$, it will cut orthogonally the sphere whose equation is $S + \lambda S' = 0$.

8. Shew how to find the asymptotes of an algebraic curve when the terms of highest degree can be resolved into unrepeated linear factors.

Find the asymptotes of the curve whose equation is

$$x(x^2 - y^2) + x^2 + y^2 + x + y = 0.$$

9. Shew how to find the maximum and minimum values of a function of one variable.

P is a point on the circle whose equation is

$$(x - h)^2 + (y - h)^2 = a^2, \ (h > a),$$

and PM, PN are the perpendiculars on the axes. Find the positions of P when the triangle PMN is a maximum or a minimum, and shew that there are two maximum positions or one, according as h is less or greater than $a\sqrt{2}$.

10. Apply the transformation $t = \tan \frac{1}{2}x$ to the integrals

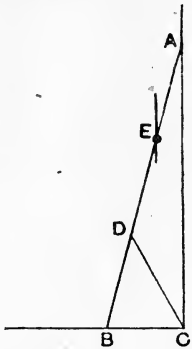
$$\int \frac{4dx}{5 + 3\cos x}, \quad \int \frac{4dx}{3 + 5\cos x}.$$

Hence or otherwise evaluate these integrals, to the nearest hundredth, when the limits are $x = 0$ and $\frac{1}{2}\pi$. Prove in any way that the second is the greater of the two integrals, when taken between 0 and $\frac{1}{2}\pi$.

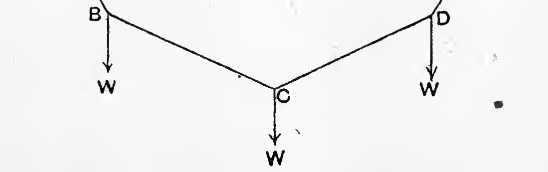
SATURDAY, 31 *May*. 9—12.

1. A uniform ladder AB weighing 100 lb. is 30 ft. long and rests against a smooth vertical wall at A making an angle of 15° with the vertical. The foot of the ladder rests on a smooth horizontal surface, but is kept from slipping by a cord

CD, where *BD* is 10 ft. A man weighing 150 lb. climbs up the ladder to *E*, where *BE* is 20 ft.: find graphically or otherwise the tension in the cord and the reaction at *A*, and verify that the reaction at *B* is 400 lb.



2. Four equal smoothly jointed rods *AB*, *BC*, *CD*, *DE* hang in equilibrium from two pins *A*, *E* on the same level: the rods are light, but carry weights *W* as indicated. Draw a force-diagram taking the horizontal component of the reactions at *A*, *E* to be *W*. Calculate the length of each rod when *AE* is 40 feet.



If the weights *W* are removed, shew that four equal heavy rods of the same sizes will hang in a figure of the same shape: and construct from the original force-diagram the reactions at the pins *A*, *B*, *C*, in this case.

3. A particle moving in a straight line is subject to a resistance which produces the retardation kv^3, where v is the velocity and k is a constant. Shew that v and t (the time) are given in terms of s (the distance) by the equations

$$v = \frac{u}{1 + ksu}, \qquad t = \frac{s}{u} + \tfrac{1}{2}ks^2,$$

where u is the initial velocity.

As a result of certain experiments with a rifle, it was estimated that the bullet left the muzzle with a velocity of 2400 ft. per sec. and that the velocity was reduced to 2350 ft. per sec. when 100 yds. had been traversed. Assuming that the air-resistance varied as v^3, and neglecting gravity, calculate the time of traversing a range of 1000 yds.

4. A ship is rolling with a period of 10 secs. A man at the masthead 100 feet above the deck is swung to and fro 25 feet on either side of the vertical with a motion which is approximately horizontal and simple harmonic. The man weighs 200 lbs. and his horizontal hold failing at 50 lbs. he is thrown off the mast. The width of the deck being 80 feet, prove that he falls clear of the ship.

[Assume $\pi^2 = 10$ and $g = 32$ f.s. units.]

5. A particle moves in an orbit under a central acceleration μ/r^2 along the radius vector r; obtain the equations of energy and angular momentum.

If the particle is projected with velocity u at right angles to the radius, at distance c from the origin, prove that

$$\left(\frac{dr}{dt}\right)^2 = \left\{\frac{2\mu}{c} - u^2\left(1 + \frac{c}{r}\right)\right\}\left(\frac{c}{r} - 1\right).$$

6. A body swings about a smooth horizontal axis, shew that the length of the simple equivalent pendulum (S.E.P.) is

$$l = h + k^2/h,$$

where h is the distance of the centre of gravity G from the axis, and k is the radius of gyration about a parallel axis through G.

An irregular bar hangs with the points A, B, in a vertical line, and the length of AB is c. When it swings about a knife edge at A, the length of the S.E.P. is $c - a$; and when it swings about B, in an inverted position, the length of the S.E.P. is $c-b$. Prove that AG is $bc/(a+b)$ and find the value of k^2 for the bar.

7. An eyepiece is formed of two thin convergent lenses of focal lengths 4 cm. and 10 cm. at a distance of 7 cm. apart. Calculate the focal length of the eyepiece, and draw a diagram to indicate the position with respect to the lenses of the focal and unit points of the eyepiece.

Draw in the diagram the course of a ray which strikes the first lens at right angles.

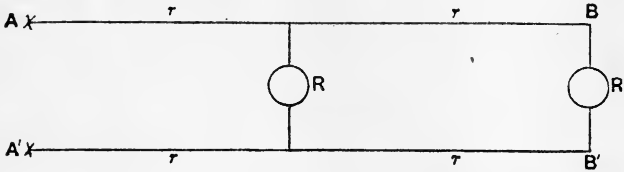
8. A plane mirror is placed behind a convergent lens; prove that if a needle is placed at the first point of the lens, its image, as seen by rays which have traversed the lens twice, will coincide with the needle; and that the image is equal to the object, but is inverted.

A scale is placed at the first focal point of a lens A, and is viewed through A and a second lens B; prove that the image is formed at the second focal point of B, and that the magnification of the scale is equal to the ratio of the focal lengths of the lenses. Is the image erect or inverted?

9. Two electric lamps, each of resistance R, are connected to supply mains by leads of resistance r as indicated in the

diagram. Shew that if P is the potential difference at A, A', while Q is that at B, B', then

$$\frac{P}{Q} = 1 + 6\frac{r}{R} + 4\left(\frac{r}{R}\right)^2.$$



If the lamps are listed as consuming 35 watts at a potential difference of 50 volts, calculate the resistance R; and if the difference $P-Q$ is not to exceed 2 per cent. of P, shew that r must be less than $\frac{1}{4}$ of an ohm.

[The watt is the rate of consumption of energy when a current flows along a conductor of resistance one ohm, the terminals being at a potential difference of one volt.]

10. Recent experiments lead to the inferences that in C.G.S. electrostatic units the charge on an electron is $3{\cdot}4 \times 10^{-10}$, and that the mass of an electron is $6{\cdot}1 \times 10^{-28}$.

If electrons pass between two metal plates at a potential difference of 2000 volts, and the whole of their lost potential energy is converted into kinetic energy, calculate the velocity with which they strike the second plate, on the above data.

[The C.G.S. electrostatic unit of potential is 300 volts.]

SATURDAY, 31 *May*. 2—5.

1. Define similar triangles, and state the various independent conditions that two triangles should be similar.

Two circles touch internally at A. A chord BC of the larger circle touches the smaller at D. Prove that BD and DC subtend equal angles at A.

2. Prove that, if two planes are each perpendicular to a third plane, their line of intersection is perpendicular to that plane.

Prove that, if the perpendiculars from two vertices of a tetrahedron on the opposite faces intersect, then the perpendiculars from the other two vertices on the opposite faces also intersect.

3. Shew how to find the condition that a rational integral algebraic equation may have equal roots.

Find the values of a for which the equation

$$ax^3 - 9x^2 + 12x - 5 = 0$$

has equal roots, and solve the equation in one case.

4. Prove that if m and n are unequal integers

$$\int_0^\pi \sin mx \sin nx \, dx = 0, \qquad \int_0^\pi \sin^2 mx \, dx = \tfrac{1}{2}\pi.$$

Find the area A between the curve

$$y = a\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x\right)$$

and the axis of x between the limits 0 and π; and the volume V obtained by rotating this area about the axis of x. Prove that $4V = \pi^2 aA$.

5. Investigate the types of motion given by the equation

$$\frac{d^2x}{dt^2} + 2\mu\frac{dx}{dt} + n^2x = A\cos nt,$$

according as $\mu < n$ or $\mu > n$, μ and n being positive.

Find the amplitude of the oscillations when t is large.

Solve completely the equation

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = Ar^3.$$

6. In starting a tram of mass 3200 lbs. the pull exerted by a horse is initially 200 lbs. and this pull decreases uniformly with the time until at the end of 10 secs. it has fallen to 40 lbs., an amount just sufficient to overcome the frictional resistance of the tram. Shew by means of a curve the variation in the velocity and find the distance run during this period of 10 secs.

Shew that the horse power is a maximum at the end of five seconds, and find its maximum value. (Assume $g = 32$ f.s. units.)

7. A wheel 30 inches in diameter, which can rotate in a vertical plane about a horizontal axis through its centre O, carries a mass of $\frac{1}{4}$ lb. concentrated at a point P on its rim. The wheel is held with OP inclined at 30° above the horizontal and then released. Owing to a friction couple of constant magnitude L at the bearing, the first swing carries OP to a

position only 45° beyond the vertical. Determine the value of L; and prove that, in the next swing, OP will come to rest before reaching the vertical.

8. A spherical condenser is formed of two concentric conducting metal spheres, of radii a, b where a is less than b; find the potential at any point when charges E_1, E_2 are communicated to the spheres.

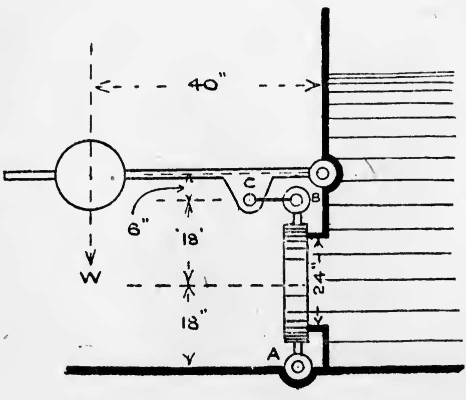
If the sphere a is raised to potential A, and the sphere b to potential B, shew that the charges communicated are given by

$$E_1 = \frac{ab}{b-a}(A-B), \quad E_2 = \frac{b}{b-a}(Bb - Aa).$$

If the two spheres are afterwards joined by a fine wire, express the loss of electrical energy in terms of A, B, a, b.

9. A circular plate of radius a has its plane vertical and its centre at a depth h below the surface of a liquid. Prove that its centre of pressure is at a depth $\frac{a^2}{4h}$ below the centre of the plate.

The figure illustrates a circular opening 24 inches in diameter in a water-tight partition. The opening is closed by a plate turning about a hinge at A, and this plate is forced into position by the horizontal link CB operated by a loaded lever. Take dimensions from the figure and calculate the value of W necessary to maintain a water-level 3 feet above the centre of the opening.



10. A small pencil of light is refracted directly at a spherical surface, find the relation between the distances of conjugate foci from the centre of curvature, explaining carefully the conventions as to the signs of the symbols used.

A concave mirror is made of glass $\frac{1}{4}$ inch thick (refractive index $\frac{3}{2}$), silvered at the back, and the radius of curvature of the silvered surface is 2 feet. Find the position of the image of a small object placed six inches from the silvered surface and the magnification produced. Shew that the image is $\frac{3}{8}$ in. nearer the object than it would be if the glass were indefinitely thin, the silvered surface being in the same position as before.

1914

THURSDAY, 4 *June*. 9—12.

1. If two triangles are equiangular to one another, prove that they are also similar.

Two circles cut orthogonally in A and B. A diameter of one of the circles is drawn cutting the other circle in C and D. Shew that $BC \cdot AD = AC \cdot BD$.

2. Two conjugate diameters CP, CD of an ellipse are drawn, and Y is the foot of the perpendicular from C on the tangent at P. Prove that

$$CP^2 + CD^2 = a^2 + b^2, \quad CY \cdot CD = ab.$$

If the length PY is given and equal to q, prove that there are in general two corresponding points P_1, P_2 in each quadrant of the ellipse; and that when these two points coincide

$$q = a - b, \text{ and } CY = \sqrt{(ab)}.$$

3. If x, y, z satisfy the equations

$$\frac{1}{x} + \frac{1}{y+z} = \frac{1}{a},$$
$$\frac{1}{y} + \frac{1}{z+x} = \frac{1}{b},$$
$$\frac{1}{z} + \frac{1}{x+y} = \frac{1}{c},$$

shew that $x(b+c-a) = y(c+a-b) = z(a+b-c)$

and hence solve the equations.

4. If the expression

$$S = \frac{1}{1-x} - \frac{3}{(1-2x)^{\frac{1}{2}}} + \frac{3}{(1-3x)^{\frac{1}{3}}} - \frac{1}{(1-4x)^{\frac{1}{4}}}$$

is expanded in the form

$$S = a_0 + a_1 x + a_2 x^2 + \ldots,$$

find the coefficients a_0, a_1, a_2, a_3 and a_4.

5. Write down the series for $\sin\theta$ and for $\cos\theta$ in powers of θ.

Determine a, b, c so that, as θ tends to zero, the function

$$\frac{\theta(a + b\cos\theta) - c\sin\theta}{\theta^5}$$

shall tend to the limit unity.

6. If the sides of a triangle ABC are a, b, c and if $a + b + c = 2s$, shew that

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

and that S, the area of the triangle, is given by

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

Find, to the nearest minute, the angles of the triangle in which

$$a = 7, \quad b = 6, \quad c = 5.$$

7. Evaluate the integrals

$$\int \frac{x\,dx}{(x+1)^2(x-1)}, \qquad \int \cos^2 x\,dx.$$

If $$s_n = \int \frac{\sin(2n-1)x}{\sin x}\,dx, \qquad v_n = \int \frac{\sin^2 nx}{\sin^2 x}\,dx,$$

prove the reduction formulae

$$n(s_{n+1} - s_n) = \sin 2nx, \qquad v_{n+1} - v_n = s_{n+1};$$

and shew that if v_n is taken between the limits 0 and $\frac{1}{2}\pi$, its value is $\frac{1}{2}n\pi$, when n is an integer.

8. A uniform solid is bounded by the surface obtained by revolving the curve

$$y^2 = ax^2 + 2bx + c$$

about the axis of x. A slice is cut from the solid by two plane sections perpendicular to the axis, at a distance h apart; prove that the volume of the slice is V, where

$$V = \tfrac{1}{2}(A+B)h - \tfrac{1}{6}\pi a h^3,$$

A and B being the areas of the two plane faces of the slice.

Shew also that the distance of the centroid of V from the face A is equal to

$$\frac{h}{2} + \frac{(B-A)h^2}{12V}.$$

9. Find the cosine of the angle between two straight lines whose direction cosines are given.

Prove that the straight lines which cut two given non-intersecting straight lines, such that the length intercepted is constant, are parallel to the generators of a right circular cone.

10. Find the equation of the polar plane of the point (f, g, h) with respect to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Prove that the plane section whose centre is at the point $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$ passes through three of the extremities of the principal axes of the ellipsoid.

THURSDAY, 4 *June*. 2—5.

1. If a straight line touch a circle, and from the point of contact a chord be drawn, prove that the angles which this chord makes with the tangent are equal to the angles in the alternate segments of the circle.

In any triangle ABC the straight line AD is drawn perpendicular to BC, meeting it in D. On AD as diameter a circle is described cutting AC and AB in P and Q respectively. Prove that the tangents at P and Q meet on the straight line joining A to the middle point of the side BC.

2. Prove that the volume of the tetrahedron $ABCD$ is $\frac{1}{6}AB \,.\, CD \,.\, EF \sin\theta$ where EF is the shortest distance and θ is the angle between the edges AB and CD.

Prove that the product of the lengths of a pair of opposite edges of a tetrahedron, self polar with respect to a sphere, is inversely proportional to the shortest distance between that pair.

3. The variables x and y are connected by the relation

$$\left(\frac{1}{x}+\frac{1}{a}\right)\left(\frac{1}{y}+\frac{1}{b}\right)=\frac{1}{c^2},$$

where c^2 is not equal to ab. Determine (i) h, k, l and (ii) p, q, f so that the relation may be written in the forms

$$\text{(i)} \quad (x+h)(y+k)=l^2,$$

$$\text{(ii)} \quad \frac{1}{x+p}+\frac{1}{y+q}=\frac{1}{f}.$$

4. Prove that, if x is not less than 100,

$$x^{\frac{1}{3}}-\tfrac{1}{2}\{(x+1)^{\frac{1}{3}}+(x-1)^{\frac{1}{3}}\}=\tfrac{1}{9}x^{-\frac{5}{3}},$$

the approximation being correct to at least eight decimal places.

In Barlow's Tables the following cube-roots are given:

$$(157)^{\frac{1}{3}}=5{\cdot}3946907,$$

$$(159)^{\frac{1}{3}}=5{\cdot}4175015.$$

Calculate to seven decimal places from these data (and the four-figure tables) the cube-root of 158.

5. In a triangle ABC, shew that the orthocentre P, the centroid G and the circumcentre O are collinear.

Shew also that if OP is equally inclined to AB and AC, then A is 60°.

6. Find the general equation of a circle, and shew that, in general, a circle can be found to satisfy three conditions. State three exceptions.

Find the equation of the circle which is inscribed in the triangle formed by the straight lines $x+9=0$; $y-2=0$; $3x+4y=5$.

7. Prove that, in general, three normals can be drawn to a parabola from any point P.

If P is on the parabola, shew that the pole of the chord through the extremities of the other normals, which pass through P, lies on a fixed straight line.

8. If $y=\cot x$, prove from first principles that

$$\frac{dy}{dx}=-(1+y^2).$$

Assuming that for small values of x, the function $\cot x$ can be expanded in the form

$$\cot x = \frac{1}{x} + A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots,$$

obtain the values of A_0, A_1, A_2, A_3, either by substituting in the above equation, or in any other way.

Prove that as x tends to zero, $(\cot^2 x - 1/x^2)$ tends to $-2/3$.

9. The perpendiculars from the origin on the tangent and normal at a point (x, y) of a plane curve are denoted by p and q respectively; shew that, with a convention as to sign, $p = x \sin\phi - y\cos\phi$, where ϕ is the angle between the tangent and the axis of x. Deduce that

$$dp = q\,d\phi, \quad dq = ds - p\,d\phi, \quad r\,dr = q\,ds,$$

where r is the radius from the origin and s is the arc measured from some fixed point; and prove that the radius of curvature is

$$\rho = p + \frac{d^2p}{d\phi^2} = \frac{r\,dr}{dp}.$$

The radius of curvature of a plane curve always subtends a right angle at the origin: shew that the curve cuts the radius r at a constant angle.

10. Obtain the complete solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-2t}$$

and determine the constants so that $x = 0$ and $dx/dt = 0$ when $t = 0$.

From the complete solution, or in any other way, show that, when x is expressed as a power series in t,

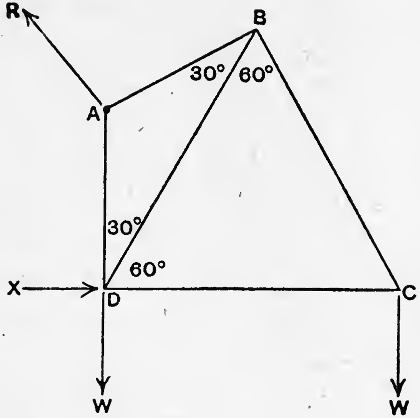
$$x = \tfrac{1}{2}t^2 - \tfrac{2}{3}t^3 + \tfrac{1}{2}\tfrac{1}{4}t^4 - \ldots.$$

FRIDAY, 5 *June*. 9—12.

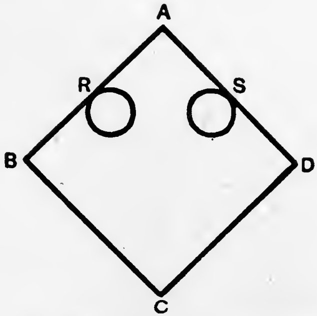
1. The accompanying diagram represents a smoothly jointed frame-work of light rods; it is hung from a smooth pin at A, it carries two equal loads W at C, D, and it is kept in place by a horizontal thrust X applied at D.

Draw a stress diagram for the frame-work, marking the rods so as to distinguish between tensions and thrusts.

Shew that the reaction at the pin A makes an angle of rather more than 49° with the horizontal.



2. State the principle of virtual work; and prove that in a small angular rotation $\delta\theta$ of a rigid body, acted on by a system of coplanar forces, the work done is equal to $L \,.\, \delta\theta$, where L is the moment of the forces about the instantaneous centre of rotation.



The diagram represents a rhombus formed of four equal heavy uniform rods, smoothly jointed at A, B, C, D; the rods AB, AD rest symmetrically on two equal smooth rails of circular section. The rails are on the same level and are both perpendicular to the plane of the rhombus, which is vertical. Prove that in equilibrium the horizontal through A, the vertical through B and the perpendicular to AB at the point of contact R, meet in a point.

3. In starting a train the pull of the engine on the rails is at first constant, and equal to P; and after the speed attains a certain value u, the engine works at a constant rate $R = Pu$. Prove that when the engine has attained a speed v greater than u, the time t and the distance x from the start are given by

$$t = \frac{1}{2}\frac{M}{R}(v^2 + u^2), \quad x = \frac{1}{3}\frac{M}{R}\left(v^3 + \frac{1}{2}u^3\right),$$

where M is the mass of the engine and train together.

Calculate the time occupied in attaining a speed of 45 miles an hour, when the total mass is 300 tons, the engine has 420 H.P. and can exert a pull equal to 12 tons weight.

[The acceleration of gravity may be taken as 32 in foot-second units, and 1 H.P. can do 550 foot-pounds of work in one second.]

4. Two particles of masses m, m' are moving in the plane of xy, under an attraction R along the radius r joining the particles. Shew that the centre of gravity moves with constant velocity in a straight line; and that if x, y are the rectangular coordinates of either particle with respect to the other, then

$$\frac{mm'}{m+m'}\frac{d^2x}{dt^2} = -R\frac{x}{r}, \quad \frac{mm'}{m+m'}\frac{d^2y}{dt^2} = -R\frac{y}{r}.$$

If the relative orbit is a circle of radius r, described in a period T, prove that

$$R = \frac{mm'}{m+m'}\frac{4\pi^2 r}{T^2}.$$

Assuming Newton's law of attraction, and that the moon describes a circle of radius r, relative to the earth, establish the equation

$$1 + \frac{M}{E} = \frac{4r^3}{a^2 l N^2},$$

where a is the earth's radius, l is the length of the seconds pendulum, M and E are the masses of the moon and earth respectively, and N is the number of seconds in the moon's period.

5. A solid uniform circular cylinder of mass m and of radius r rolls (under the action of gravity) inside a fixed hollow cylinder of radius R, the axes of the cylinders being parallel to each other and also horizontal. At any time t during the motion the plane containing the axes of the cylinders makes an angle θ with the vertical. Shew that the potential energy of the moving cylinder is

$$mg(R-r)(1-\cos\theta),$$

and that its kinetic energy is

$$\frac{3m}{4}(R-r)^2\left(\frac{d\theta}{dt}\right)^2.$$

Hence, or otherwise, shew that the time, T, of a small oscillation is

$$T=2\pi\sqrt{\frac{3(R-r)}{2g}}.$$

6. State the conditions of equilibrium of a body floating in a liquid.

A uniform log, whose cross section is a square, floats horizontally with one edge in the surface and one edge above the surface of the liquid. Shew that the ratio of their specific gravities is 3 : 4 and find the depth of the centre of pressure on the face, of which all the edges are below the surface of the liquid.

7. A photographic camera has a thin lens of 15 cm. focal length and of 3 cm. diameter. The camera is adjusted so that an object on the axis of the lens 240 cm. from the lens is in perfect focus on the plate. When an object point P lies on the axis at any distance greater or less than 240 cm. from the lens, the cone of rays emerging from the lens is cut by the plate in a circle. Shew that the diameter (c cm.) of this circle and the distance (p cm.) of P from the lens are connected by the relation

$$5cp=|240-p|,$$

where $|x|$ denotes the magnitude of x without regard to sign.

Find the two values of p corresponding to $c=0{\cdot}01$.

8. Prove that the electrostatic capacity of a condenser formed by two parallel plates of area A, at a small distance t apart, is approximately $A/4\pi t$.

Apply this to calculate the capacity of a condenser formed by two circular plates of tinfoil mounted on glass, taking the diameter of each plate to be 40 cm. and the thickness of the glass to be equivalent to an air-thickness of 1·5 mm.

Express your result in terms of a microfarad.

[The microfarad is the capacity of a sphere, whose radius is 9 kilometres.]

9. A dome of conducting material is built in the form of a hemisphere on the ground; a small charged conductor is situated midway between the dome and the ground, on the vertical through the centre of the dome. Obtain the system of images; prove that the mechanical force on the small conductor is upwards, and equal to $\frac{47}{225}E^2/a^2$, where E is the charge on the conductor and a is the radius of the dome.

10. Three wires APB, AQB, ARB are arranged "in parallel" between the points A, B; their resistances are p, q, r ohms respectively. Batteries of negligible resistances and of electromotive forces E, F volts are now inserted in the branches APB, AQB; the negative pole of each battery is connected to A. If the rate at which electrical energy is expended by the two batteries together is W watts, shew that

$$W = \frac{r(E-F)^2 + pF^2 + qE^2}{pq + rp + rq}.$$

FRIDAY, 5 *June*. 2—5.

1. A circle of radius a is inverted with respect to a second circle of radius k whose centre is O: shew that the inverse curve is a circle, and that its radius is ak^2/t^2, where t is the length of the tangent from the origin of inversion O to the first circle.

Let T denote the length of the tangent from a point P to the first circle, and T' the tangent from the inverse point P' to the inverse circle: prove that

$$T'/T = OP'/t.$$

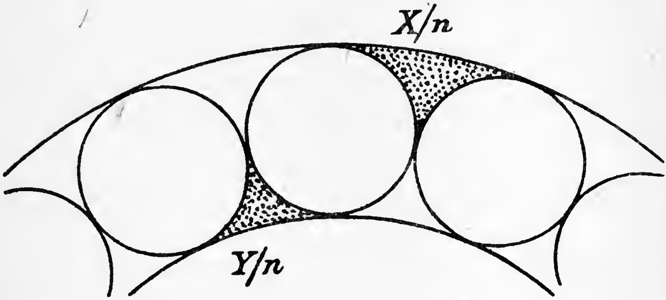
2. Write down the series for $\log_e(1+x)$ in powers of x. Shew that

$$\log_2 e - \log_4 e + \log_8 e - \log_{16} e + \dots = 1.$$

Employ the tables to obtain to 4 significant figures the value of x less than 10 which satisfies the equation

$$x^{10} = 10^x.$$

3. In the annulus between two concentric circles of radii $R-r$ and $R+r$ and centre O are described n circles each of radius r. Find the relation between R and r when each of the n circles touches its two neighbours.



If C be the sum of the areas of the n circles and A be the area of the annulus, shew that

$$\frac{C}{A}=\frac{n}{4}\sin\frac{\pi}{n}.$$

If X, Y be the sums of the areas of those parts of the annulus which lie without and within the circle of centre O and radius $\sqrt{R^2-r^2}$ and are not included within any of the n circles of radius r, find X and Y and shew that, as n is increased, $(X-Y)/\pi r^2$ approaches the limit $\frac{4}{3}$.

4. From a station A a balloon is seen in a direction 16° 20′ east of north at an elevation of 24° 30′, and from another station B it appears to be 15° 40′ west of north. If the distance between A and B is 1000 feet and the direction of the line AB is 27° 30′ north of east, find the height of the balloon.

5. Find the length of the perpendicular of the point (x', y') from the straight line $ax+by+c=0$.

Two sides of a parallelogram are formed by the straight lines (i) $3x-4y=4$; (ii) $y=mx$ and the other two sides are formed by the two straight lines through the point $(5,-1)$ which are parallel to the lines (i) and (ii). Find the two values of m for which the area of the parallelogram is 12.

6. Find the condition that the straight line $lx + my = n$ should touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and shew that the locus of the point of intersection of perpendicular tangents to the ellipse is a circle.

If P is any point on the director circle, shew that the locus of the middle point of the chord, in which the polar of P cuts the ellipse, is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2 + y^2}{a^2 + b^2}.$$

7. Shew that the equations of any two spheres can be put in the form

$$x^2 + y^2 + z^2 + 2\lambda_1 x + d = 0, \quad x^2 + y^2 + z^2 + 2\lambda_2 x + d = 0.$$

Shew that all the spheres, that can be drawn through the origin and each set of points where planes parallel to the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$$

cut the coordinate axes, form a system of spheres which are cut orthogonally by the sphere $x^2 + y^2 + z^2 + 2fx + 2gy + 2hz = 0$, if $af + bg + ch = 0$.

8. The rectangular coordinates x, y of points on a plane curve are expressed as functions of a variable t: prove that at an inflexion on the curve

$$x'y'' - y'x'' = 0,$$

where accents indicate differentiation with respect to t.

Apply the formula to determine the points of inflexion on the curve

$$x = a\cos t + \tfrac{1}{2}b\cos 2t, \quad y = a\sin t + \tfrac{1}{2}b\sin 2t,$$

where a, b are positive; and shew that the inflexions are real if a lies between b and $2b$.

9. The angles of a triangle are calculated from the sides a, b, c: if small changes δa, δb, δc are made in the sides, shew that approximately

$$\delta A = \frac{a}{2\Delta}(\delta a - \delta b \,.\, \cos C - \delta c \,.\, \cos B),$$

where Δ is the area of the triangle; and verify that

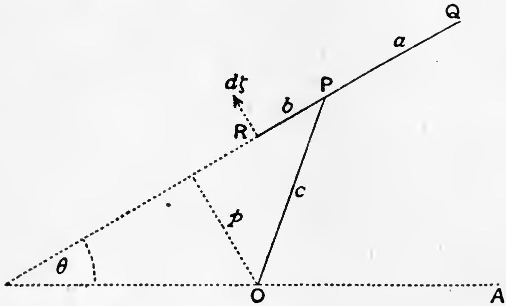
$$\delta A + \delta B + \delta C = 0.$$

Apply the formula to calculate in degrees and minutes the angles of a triangle in which $a = 99{\cdot}5$, $b = 100{\cdot}2$, $c = 100{\cdot}7$.

10. Prove that for a plane curve

$$x\,dy - y\,dx = 2dS,$$

where dS is the polar element of area.



Two bars OP, RPQ of lengths c, $a+b$ respectively turn round a fixed pin at O, the point P representing a hinge; dS_1, dS_2 denote the polar elements of area (about O) of the curves traced by P, Q respectively; prove that

$$dS_2 - dS_1 = a\,d\zeta + a\left(\tfrac{1}{2}a + b\right)d\theta - \tfrac{1}{2}a\,dp,$$

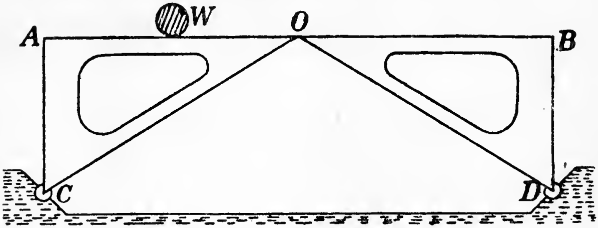
where $a = PQ$, $b = RP$, p is the perpendicular from O on RPQ, $d\zeta$ is the displacement of R perpendicular to RPQ, and θ is the inclination of RPQ to a fixed line OA.

SATURDAY, 6 *June*. 9—12.

1. Two equal frames AOC, BOD, are supported by hinges at C and D and are connected by a smooth joint at O. The line AOB is horizontal and AC, BD are vertical. The distance AB is 100 feet and O is 30 feet above CD. The weight of each frame is 50 tons and the distance between their centres of gravity is 80 feet.

When a load, W, of 20 tons rests on AO at a distance of p feet from A, the horizontal and vertical forces exerted by the frame BOD on AOC at O are (in tons weight) X towards A and Y upwards respectively.

Find X and Y and shew that $3X - 5Y = 50$.



2. A light rod $PABQ$ is suspended by two vertical springs PR, QS, and is horizontal when unloaded. When a load of X grammes is hung from A and a load of Y grammes from B, the rod takes the position $P'A'B'Q'$, where $AA' = x$, $BB' = y$. The loads are supposed so small that P', A', B', Q' may be considered as being vertically below P, A, B, Q. Each spring is such that when a load of one gramme is suspended from it the elongation is λ centimetres. The length of the rod is $a + b + c$ and $PA = a$, $AB = c$, $BQ = b$.

Find x and y in terms of X and Y and shew that, if the only load is one of M grammes hung from either of the two points A, B, the depression of the other point is

$$\frac{M\lambda(a^2 + b^2 + ac + bc)}{(a + b + c)^2} \text{ centimetres.}$$

3. Two rough planes intersect in a horizontal line and make equal angles α with a horizontal plane. A rough circular cylinder of radius r is placed, with its axis horizontal, in the trough formed by the two planes. The distance of the centre of gravity of the cylinder from its axis is h.

Shew that, when α is greater than ϕ, the angle of friction, the cylinder cannot be in equilibrium unless it is in contact with both planes and that, when $\alpha > \phi$, it will rest in any position in the trough, if

$$h < \frac{r \sin 2\phi}{2 \cos \alpha}.$$

4. A steel ball is released from rest and falls upon a fixed steel anvil and rebounds, the coefficient of restitution being

0·9. The lowest point of the ball is initially at a distance of one foot above the anvil, and the gravitational acceleration is 32 feet sec.$^{-2}$. Find the position and velocity of the ball half a second after its release.

Shew that the ball finally comes to rest on the anvil 4·75 seconds after its release and that the total distance travelled by the ball is $\frac{181}{19}$ feet.

5. A rigid pendulum OG swings about a horizontal axis through O, its centre of gravity being at G. The pendulum is released from rest when OG is horizontal. When OG becomes vertical the pendulum is brought to rest by an inelastic buffer B which is such that the line of the reaction between B and the pendulum is horizontal and at a distance l below O. The mass of the pendulum is m, its moment of inertia about a horizontal axis through G is mk^2 and $OG = h$.

Shew that, if the impulse of the force exerted by B upon the pendulum during the impact is P,

$$P = \frac{m}{l}\sqrt{2gh(h^2+k^2)}.$$

Deduce the impulse Q of the horizontal force exerted on the pendulum during the impact by the axis O and shew that it vanishes when l is equal to the length of the simple equivalent pendulum.

6. A particle moves in a straight line under a retardation kv^{m+1}, where v is the velocity at time t. Shew that, if u is the velocity at $t = 0$,

$$kt = \frac{1}{m}\left(\frac{1}{v^m} - \frac{1}{u^m}\right),$$

and obtain a corresponding formula for the space in terms of v.

A bullet fired with a horizontal velocity of 2500 feet per second is travelling with a velocity of 1600 feet per second at the end of one second. Assuming that $m = \frac{1}{2}$, calculate k and the space traversed in the first second, neglecting the effect of gravity.

7. State the laws of refraction of light and find to the nearest minute the deviation experienced by a ray which passes symmetrically in a principal plane through a prism of 60° and refractive index 1·5.

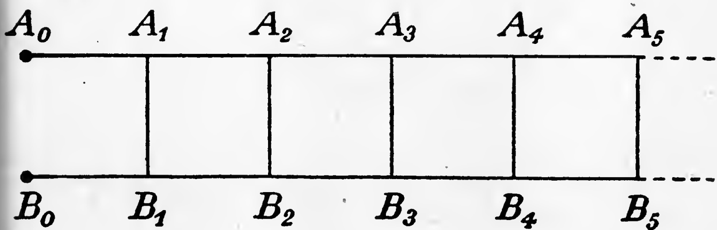
A prism ABC of refracting angle $ABC = \frac{1}{4}\pi$ and index μ_1 is cemented to a prism BCD of refracting angle $BCD = \theta$ and index μ_2; the principal planes are parallel, and the refracting edges are turned in opposite directions. Shew that a ray which falls normally on the face AB of the first prism will emerge from the face CD of the other prism parallel to its original direction if

$$\tan\theta = \frac{\mu_1 - 1}{(2\mu_2^2 - \mu_1^2)^{\frac{1}{2}} - 1}.$$

8. Two thin convergent lenses A, C of focal lengths F, G are placed at a distance $F + G$ apart. A third thin convergent lens B of focal length f is placed between them so that $AB = F$, $BC = G$, and the three lenses are coaxial. Shew that the magnification of any small object is independent of the position of the object and of the value of f and that the image is inverted.

Shew also that the image of any object point on the axis will be at a constant distance from the object if $F = G$ and that this distance will vanish if $F = G = 4f$.

9. Two uniform cables $A_0A_1 \ldots$, $B_0B_1 \ldots$ are divided into parts A_0A_1, A_1A_2, ..., B_0B_1, B_1B_2, ... each of resistance r, and wires, each of resistance s, are used to join A_1 to B_1, A_2 to B_2, ... When the cables are joined by the n wires A_1B_1, A_2B_2, ... A_nB_n, the resistance of the system from A_0 to B_0 is R_n.



Shew that $$R_n = 2r + \frac{sR_{n-1}}{s + R_{n-1}},$$

and that $$R_2 - R_3 = \frac{s^4}{(2r+s)(2r+2s)(2r+3s)}.$$

If R_∞ be the resistance of the system between A_0 and B_0 when the number of connecting wires is infinite, shew that

$$R_\infty = r + \sqrt{r^2 + 2rs}.$$

10. At the points A and B on the straight line AOB are placed electrical charges e and $-e$, and $AO = OB = a$. Find the electric force at any point on the symmetrical plane through O at right angles to AB. Deduce the surface density induced on a plane at zero potential by a charge e at a distance a from the plane.

If Q be any point on a line of electric force, shew that the line is given by

$$\cos\theta_1 + \cos\theta_2 = \text{constant},$$

where $\theta_1 = QAB$, $\theta_2 = QBA$.

A line of electric force meets AOB at right angles at A and B, and cuts the symmetrical plane in P. Shew that

$$OP = a\sqrt{3}.$$

SATURDAY, 6 *June*. 2—5.

1. Four points A, B, C, D are taken in a plane; AK is drawn parallel to DC, to meet BD in K; and KL is drawn parallel to BC, to meet AC in L. Shew that DL is parallel to AB.

Deduce or prove in any way the following result: Four other points P, Q, R, S are constructed so that PQ is parallel to CD, QR to AD, RP to DB, PS to BC and QS to AC; then RS will be parallel to AB.

2. Two circles, in different planes, meet in the two points A, B; shew that a sphere S can be drawn to pass through both circles.

If the circles intersect at right angles (so that their tangents at A and at B are perpendicular), prove that the plane of either circle passes through the pole of the other with respect to the sphere S.

3. On the summit of a hemispherical hill stands a flagstaff; from a position on the horizontal plane through the centre of the hill the angle of elevation of the top of the flagstaff is observed to be α, and that of the hill β. At a distance c

nearer the hill the top of the flagstaff is just seen above the hill, and its angle of elevation is γ. Shew that $\sin\beta = \cos\gamma \tan\alpha$; and that the height of the top of the flagstaff above the hill is

$$2c\,\frac{\sin\alpha\sin\gamma\sin^2\tfrac{1}{2}\gamma}{\sin(\gamma-\alpha)}.$$

4. Shew that y, a function of x, is a maximum or minimum at any value of x for which dy/dx changes sign, and explain how to distinguish between the two cases.

Find the maximum and minimum values of $y = x + \dfrac{4}{x+2}$; illustrate your results by drawing a rough graph of the function.

5. If $y^2 = a^2x^2 + c$, differentiate with respect to x the functions

$$\log(ax+y),\quad xy,$$

and express the results in terms of y.

Hence or otherwise evaluate the integrals

$$\int\frac{dx}{y},\quad \int y\,dx.$$

Prove that $$\int_2^{\frac{5}{2}} \sqrt{(x^2-4)}\,dx = \tfrac{15}{8} - 2\log 2.$$

6. A car whose mass is 2000 lb. starts from rest; the resistance to the motion is equal to 50 lb. weight. When it has travelled S feet the force exerted by the engine is P lb. weight where

S	0	10	20	30	40	50	60	70	80	90	100
P	644	634	622	607	587	565	537	509	475	440	404

Find, approximately, the velocity after the car has travelled 100 feet, assuming that the acceleration due to gravity is 32 in foot-second units.

7. Shew that, if a particle moving in a circle of radius r has at any instant a velocity v, the normal component of its acceleration is v^2/r.

A horizontal bar AB of length a is made to rotate with a constant angular velocity ω about a vertical axis through the end B. If a particle is attached to A by a string of length l, the string makes an angle θ with the vertical when the motion is steady. Prove that

$$l \cos \theta + a \cot \theta = g/\omega^2.$$

8. Shew that, if a body describes an ellipse about a centre of force at the focus, the law of force is that of the inverse square of the distance, and find an expression for the velocity of the body at any point in terms of its distance from the focus.

9. Describe an experimental method for verifying the laws of reflexion of light.

If a lighted candle is placed in front of a thick parallel plate glass mirror silvered at the back, shew that the distance between consecutive visible images of the flame is constant.

10. A cylindrical diving bell, whose inner and outer cross sections are 2 and $2\frac{1}{5}$ square metres respectively, and whose inner and outer heights are 4 and $4\frac{1}{11}$ metres respectively, is forced down into deep water. If it weighs 6000 kilograms, find the depth at which no force is required to keep it in equilibrium.

Find the force that would have to be applied to keep the bell in this position if 1·9 cubic metres of air at atmospheric pressure were pumped in.

[1 cubic metre of water weighs 1000 kilograms. The height of the water barometer is 10 metres.]

1915

THURSDAY, 3 *June*. 9—12.

1. Prove that the lines which bisect internally and externally the angle A of a triangle ABC divide BC in the ratio $BA : AC$.

Shew that, if the bisectors meet BC in X, X', the circle AXX' cuts the circle ABC at right angles; and that, if the other angles of the triangle ABC are bisected, and the two circles analogous to AXX' drawn, these three circles have two common points.

2. Prove that perpendicular tangents to a parabola meet on the directrix.

Two perpendicular focal chords of a parabola meet the directrix in T and T' respectively; shew that the tangents to the parabola, which are parallel to these chords, intersect in the middle point of TT'.

3. Prove that, if x and y are the values of s which satisfy the equation

$$s^2t^2 + 1 + a(s^2 + t^2) + bst = 0,$$

for any given value of t, then will

$$x^2y^2 + 1 + A(x^2 + y^2) + Bxy = 0,$$

where $A = \dfrac{(1-a^2)^2}{ab^2}$, $B = \dfrac{2(1-a^2)^2 - b^2(1+a^2)}{ab^2}$.

4. The successive coefficients $u_0, u_1, u_2, u_3 \ldots$ of the infinite series

$$1 + x + u_2x^2 + u_3x^3 + \ldots,$$

are connected by the relation

$$u_r - 2u_{r-1} - 8u_{r-2} = 0 \qquad (r = 2, 3 \ldots);$$

find the sum of the series, stating the necessary condition for its convergence.

Shew that the sum of the finite series

$$1^2 + 3^2x + 5^2x^2 + \ldots\ldots + (2n-1)^2 x^{n-1},$$

is

$$(1 + 6x + x^2)\frac{1 - x^n}{(1-x)^3} - 4x^n \frac{n^2(1-x) + n(1+x)}{(1-x)^2}.$$

5. Prove the relation

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},$$

for all values of the angles A and B.

Shew that, if A, B, C, D are the angles of a plane quadrilateral,

$$\cos 2A + \cos 2B + \cos 2C + \cos 2D$$
$$= 4\cos(A+B)\cos(A+C)\cos(A+D).$$

6. In the infinite series

$$S = \sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2^2}\right) + \ldots + \sin\left(\frac{\theta}{2^n}\right) + \ldots,$$

expand each term in a series of powers of θ and hence obtain an expression for S in the form

$$S = \frac{\theta}{a} + \frac{\theta^3}{b} + \frac{\theta^5}{c} + \frac{\theta^7}{d} + \dots.$$

Give the numerical values of a, b, c, d, and shew that when $\theta = 1$, $S = 0{\cdot}97645$ correct to 5 decimal places.

7. Evaluate the integrals

$$P = \int \frac{dx}{a^2 + x^2}, \quad Q = \int \sqrt{a^2 - x^2}\, dx, \quad R = \int \tan x\, dx.$$

If $$T_n = \int_0^x \tan^n x\, dx,$$

shew that $$T_n = \frac{\tan^{n-1} x}{n-1} - T_{n-2}.$$

Given that $\pi = 3{\cdot}141592\dots$, $\log_e 2 = 0{\cdot}693147\dots$, shew that

$$\int_0^{\pi/4} \tan^5 x\, dx = 0{\cdot}09657\dots, \quad \int_0^{\pi/4} \tan^4 x\, dx = 0{\cdot}11873\dots.$$

8. Indicate by a rough drawing the shape of the curve given by the equation $y^2(a + x) = x^2(3a - x)$.

Shew that the coordinates of any point on the curve may be taken as

$$x = \frac{a \sin 3\theta}{\sin \theta}, \quad y = \frac{a \sin 3\theta}{\cos \theta};$$

and prove that the area of the loop of the curve and the area between the curve and its asymptote are both equal to $3\sqrt{3}a^2$.

9. Find the lengths of the perpendiculars drawn from any point (f, g, h) to the lines

$$\text{(i)}\ \frac{x - \alpha}{\lambda} = \frac{y - \beta}{\mu} = \frac{z - \gamma}{\nu}: \quad \text{(ii)}\ y = x \tan\theta;\ z = c.$$

The sum of the squares of the perpendiculars from a point to the lines

$$y = x \tan\theta;\ z = c: \text{ and } y = -x\tan\theta;\ z = -c:$$

is $2k^2$. Prove that the locus of the point is an ellipsoid, and state the lengths of its principal axes.

10. Define conjugate diameters of an ellipsoid; and prove that, if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are the coordinates of the extremities of three conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \text{.....................(i)}$$

then $\quad x_1^2+x_2^2+x_3^2=a^2$, and $y_1z_1+y_2z_2+y_3z_3=0$.

Prove that, if the chord which joins two points of the ellipsoid (i) touches the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{2},$$

the two points must lie at the extremities of conjugate diameters of (i), and the point of contact must bisect the chord.

THURSDAY, 3 *June*. 2—5.

1. Shew how to draw a perpendicular to a plane from any external point.

Through the orthocentre P of a triangle ABC a line PQ is drawn at right angles to the plane of the triangle. Prove that, in any tetrahedron $ABCD$ having the triangle ABC as base, the perpendiculars let fall from A, B, C upon the opposite faces BCD, CAD, ABD all intersect PQ.

2. Defining a conic by the property of the focus and directrix, prove that the tangents drawn from any point subtend equal (or supplementary) angles at the focus.

The foci of an ellipse are S, S', and P, P' are two points of the curve on opposite sides of the major axis. The tangents at P, P' meet in T; SP meets $S'P'$ in Q; SP' meets $S'P$ in Q'. Prove (i) that a circle with centre T touches the four lines SP, SP', $S'P$, $S'P'$; (ii) that Q, Q' lie on another ellipse with foci S, S', and that QT, $Q'T$ are the tangents to this ellipse at Q and Q'.

3. Give the series for $\log(1+x)$ and shew that, if x is positive and less than 1,

$$\log\frac{1}{1-x} > x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^{n-1}}{n-1}+\frac{x^n}{n},$$

and that $$\log\frac{1}{1-x} < x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^{n-1}}{n-1}+\frac{x^n}{n(1-x)}.$$

Shew that, if $3\tan^2 x < 1$,

$$\log\left(\frac{\tan 3x}{3\tan x}\right) = \frac{8}{3}\tan^2 x\left\{1 + \frac{5}{3}\tan^2 x + \frac{91}{27}\tan^4 x + \dots\right\}.$$

4. If $f(x)$ be a polynomial in x, shew that between consecutive real roots of the derived equation $f'(x) = 0$ there occurs either one root or no root of $f(x) = 0$.

Find the range of values of k for which the equation

$$x^4 - 14x^2 + 24x - k = 0$$

has all its roots real.

5. Prove that, if D, E, F are the feet of the perpendiculars from the vertices A, B, C of a triangle on the opposite sides, the triangles AEF, DBF, DEC are similar to the triangle ABC; and state the ratio of corresponding sides in each case.

Prove that, if P_1, P_2, P_3 are the orthocentres of these three triangles, the sides of the triangle $P_1P_2P_3$ are equal and parallel to those of the triangle DEF, and that AP_1, BP_2, CP_3 are concurrent in the centre of the circle ABC.

6. Find a formula for the perpendicular distance of a point (x', y') from a line $Ax + By + C = 0$; and, assuming C to be positive, explain how the ambiguity of sign is settled.

Sketch roughly the lines

$$x + y = 4;\quad x - y = 2;\quad 17x + 7y = 28.$$

Shew that the point $(-4\frac{1}{2}, 1)$ is the centre of a circle which touches the three lines, and find the coordinates of the centre of the circle inscribed in the triangle formed.

7. Prove that the equation of the tangent at a point (x', y') of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is

$$b^2x'x + a^2y'y = a^2b^2,$$

and discuss the meaning of this equation when (x', y') is a point outside or inside the ellipse.

A conic $Ax^2 + By^2 = 1$ and a point $P(h, k)$ being given, prove that the locus of a point Q, whose polar makes a constant angle with QP, is a conic passing through P and through the origin. What is the nature of the locus when the constant angle is (i) zero, (ii) a right angle?

8. State Taylor's theorem, and apply it to express $\sin(\frac{1}{4}\pi + \theta)$ in a series of powers of θ.

Find by Maclaurin's theorem the expansion of

$$\frac{e^x}{e^x+1}$$

as far as the term in x^3; and shew that no even powers of x can occur in the expansion to any number of terms.

9. Why is $\frac{d\psi}{ds}$ regarded as the measure of the curvature at a point of a plane curve, in which s is the length of the arc, and ψ the inclination of the tangent to a fixed line?

Prove that in rectangular Cartesian coordinates

$$\frac{d\psi}{ds} = \frac{d^2y}{dx^2}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{3}{2}};$$

and shew that at any point (x, y) of the curve

$$(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1,$$

the measure of curvature is $\frac{1}{2}ab(ax+by)^{-\frac{3}{2}}$.

10. Solve the differential equations

$$\text{(i)} \quad \sin x \frac{dy}{dx} - 2y = \tan^4 \frac{x}{2};$$

$$\text{(ii)} \quad \frac{d^2x}{dt^2} + n^2 x = k \sin pt,$$

considering in (ii) the cases when p and n are unequal and when they are equal.

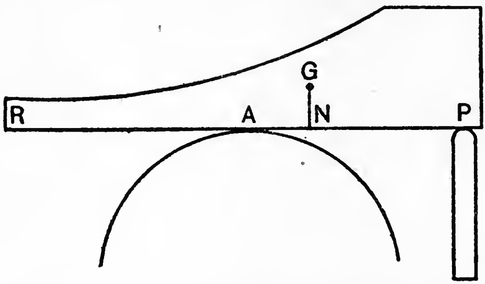
FRIDAY, 4 *June*. 9—12.

1. A string $OABCD$ is attached to a fixed point O. A weight W is hung from each of the points A, B, C, and the position of D is such that CD is horizontal and the tension in it is W. Explain how to draw a vector diagram shewing the tensions in OA, AB, BC, CD. Calculate to the nearest minute the angles which OA, AB, BC make with the horizontal, and shew that the product of the tensions in OA, AB, BC is $10W^3$.

2. Two equal uniform rods AC, CB are freely jointed at C, and rest in a vertical plane with the ends A and B in contact with a rough horizontal plane. Shew that in limiting equilibrium the angle ACB is $2\tan^{-1}(2\mu)$.

When the rods are in this position a weight is hung from one rod at a distance from C equal to xa and an equal weight from the other at a distance from C equal to ya, a being the length of either rod. Shew that, if equilibrium is preserved, the inequalities $3x+y \geqslant 2$, $3y+x \geqslant 2$ must be satisfied; and illustrate graphically the limitations on the possible values of x and y.

3. A circular cylinder of radius r is fixed with its axis horizontal, and a body, whose centre of gravity is G and whose weight is W, having a plane base PR, is supported with the base horizontal (see figure) resting upon the cylinder at A and



upon a prop at P. The prop is now lowered until the base of the body is inclined to the horizontal at an angle θ, the cylinder and body being so rough that no slipping takes place between them. Prove that the loss of potential energy of the body is

$$W\{(r+k)(1-\cos\theta)+(h-r\theta)\sin\theta\},$$

where h and k denote the lengths AN and NG respectively in the figure.

Deduce or prove otherwise that the body will be in equilibrium in this new position without support from the prop if

$$r\theta = h + k\tan\theta;$$

and that the equilibrium will be stable if

$$k < r\cos^2\theta.$$

4. Shew that a projectile moving under the action of gravity describes a parabola, and that its velocity at any point is the same as if it had fallen from rest from the directrix of the parabola.

A projectile is fired from a point O with a velocity due to a fall of 100 feet from rest, and hits a mark at a depth of 50 feet below O and at a distance of 100 feet from the vertical line through O. Shew that the two possible directions of projection are at right angles, and find to the nearest minute their inclinations to the horizontal.

5. A particle of mass m describes a parabola under an attraction to the focus; prove that the attraction at distance r is $m\mu/r^2$, and that the velocity is $(2\mu/r)^{\frac{1}{2}}$.

Two particles describe in equal times the arc of a parabola bounded by the latus rectum, one under an attraction to the focus, and the other with constant acceleration g parallel to the axis. Shew that the acceleration of the first particle at the vertex of the parabola is $\frac{16}{9}g$.

6. Enunciate the laws of the Conservation of Linear Momentum and of the Conservation of Energy.

A bead of mass M can slide on a smooth straight horizontal wire, and a particle of mass m is attached to the bead by a light string of length l. The particle is held in contact with the wire (the string being just taut), and is then let fall. Prove that, when the string is inclined to the wire at an angle θ, the bead will have slipped a distance $ml(1-\cos\theta)/(M+m)$ along the wire, and that the angular velocity (ω) of the string will be given by the equation

$$(M + m\cos^2\theta)\,l\omega^2 = 2\,(M+m)\,g\sin\theta.$$

7. Shew that a prism of *small* angle i produces a deviation $(\mu-1)\,i$ in any ray which passes nearly symmetrically through it in a principal plane, μ being the refractive index.

Hence, or otherwise, find the relation between the distances from the lens of two conjugate points on the axis of a thin lens in terms of μ and the radii of the faces of the lens.

8. A solid uniform hemisphere of radius a and of density σ can turn about O, the centre of its base. The point O is fixed in the surface of a liquid of density ρ. Shew that for a certain value of ρ/σ, the hemisphere will rest in any position.

If the hemisphere is completely immersed with its centre fixed and is kept at rest with its base vertical by means of a couple, determine, for any values of ρ and σ, the reaction at the centre and the moment of the couple. [The centroid of a solid hemisphere of radius a is at a distance $3a/8$ from the centre of the base.]

9. A battery of electromotive force E and internal resistance A, when connected to a tangent galvanometer of resistance R, gives a deflexion α. A second battery of the same E.M.F. but of internal resistance B, when connected to the galvanometer, gives a deflexion β. When the two batteries are connected in series to the galvanometer, the deflexion is γ. Shew that

$$A = R\,\frac{2\cot\gamma - \cot\beta}{\cot\alpha + \cot\beta - 2\cot\gamma},$$

and that the current through the galvanometer when the batteries are connected to it in series is

$$\frac{E}{R}\cdot\frac{\cot\alpha + \cot\beta - 2\cot\gamma}{\cot\gamma}.$$

10. A point-charge is placed outside an insulated spherical conductor having a given charge. Explain how the potential at any point of the field can be determined by help of two image point-charges.

A point-charge e is placed at a distance $6a$ from the centre of an insulated sphere of radius a, which has a charge E. Shew that if there is a point of equilibrium in the field, whose distance from the centre is $4a$ taken towards e, then $E = 4\frac{47}{3174}e$.

Give a rough drawing to indicate the nature of the equipotential surfaces in the field.

FRIDAY, 4 *June*. 2—5.

1. Prove the theorem of Menelaus that, if the sides BC, CA, AB of a triangle ABC (produced if necessary) are met by any straight line in L, M, N respectively,

$$\frac{BL}{CL}\cdot\frac{CM}{AM}\cdot\frac{AN}{BN} = 1;$$

and in the same figure apply the theorem (without proof) to the triangle AMN, the sides of which are met by a straight line in L, B, C.

A second straight line meets the sides BC, CA, AB of ABC in L', M', N', in such a way that

$$BN : BN' = CM : CM'.$$

Prove that $$L'M' : L'N' = LM : LN.$$

2. Define inversion; and, if P', Q' are the inverses of two points P, Q with respect to an origin of inversion O, find the ratio of $P'Q'$ to PQ.

Shew, by inversion with respect to any one of the points, that if A, B, C, D are any four non-coplanar points, the sum of any two of the three products $BC.AD$, $CA.BD$, $AB.CD$ is greater than the third.

3. One of two motor cars, travelling over the same route in the same direction, passed two places on the route at 2.15 and 3.57 respectively; the other car passed the same two places at 2.28 and 3.40. (i) Supposing the speed of each car to have been uniform, find at what time the former car was passed by the latter. (ii) Supposing the speed of each car to have been uniform, except that owing to a breakdown the former car was stationary for six minutes, shew that, if the breakdown occurred between 2.43 and 3.7, the former car was stationary when it was passed by the latter.

Indicate how a graphical method can be applied to solve these problems.

4. Observations are made at places A and B, whose distance apart is 328 yards, upon two objects P and Q. The angles PAB, QAB, PBA are respectively $64^\circ\ 30'$, $53^\circ\ 10'$, $72^\circ\ 35'$, and Q lies directly between B and P. Shew that the distance from P to Q is about 111 yards.

5. Prove the formulae

$$r = S/s = a \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \sec \tfrac{1}{2}A,$$

for the radius of the inscribed circle of a triangle ABC.

A line IF drawn parallel to AC from I the centre of the circle inscribed in ABC meets AB in F, and BC is trisected at P so that $3.BP = BC$. Shew that if $A = 60^\circ$, the angle BFP is $\tfrac{1}{2}B$.

6. Find the equation of the lines bisecting the angles between the two straight lines given by the equation

$$ax^2 + 2hxy + by^2 = 0.$$

Shew that the equation of the lines bisecting the angles between the bisectors is

$$(a-b)(x^2-y^2) + 4hxy = 0;$$

and that, if this pair of straight lines be taken as new axes of reference OX, OY, the equation of the original pair of lines may be written as

$$(a+b)(X^2+Y^2) + 2\mu XY = 0,$$

where $$\mu^2 = (a-b)^2 + 4h^2.$$

7. Find the equation of the line joining two points $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ on the rectangular hyperbola $xy = c^2$.

Shew that the orthocentre of the triangle formed by three points P, Q, R on the curve also lies on the curve; and that the fourth point, in which the circle PQR cuts the curve, is the other end of the diameter through the orthocentre.

8. If s be the length of the arc AQ, measured from a fixed point A on a plane curve, p be the perpendicular from the origin upon the tangent at Q, and ρ be the radius of curvature of the curve at Q, shew that, using polar coordinates r, θ,

$$\frac{ds}{d\theta} = \frac{r^2}{p}, \quad \rho = r\frac{dr}{dp}.$$

Shew that in the cardioid $r = a(1+\cos\theta)$, if s be measured from the point $r = 2a$, $\theta = 0$, and if θ lie between 0 and π,

$$9\rho^2 + s^2 = 16a^2.$$

9. Prove that

(i) $$\int_0^1 \frac{(x^2+4x+1)\,dx}{(x^2+1)(x+1)} = \frac{\pi}{2},$$

(ii) $$\int_0^1 \frac{x^3+4x^2+x-1}{(x^2+1)(x+1)^2}\,dx = \frac{1}{4}(1-\log 2).$$

Shew that the integral of

$$\frac{x^4+2x^3+4x^2-1}{x^2(x^2-1)^2}$$

is a rational algebraic function of x; and determine it.

10. Obtain a formula of reduction for $\int \sin^n x\,dx$; and find the values of $\int_0^{\frac{1}{2}\pi} \sin^{2n} x\,dx$ and $\int_0^{\frac{1}{2}\pi} \sin^{2n-1} x\,dx$, where n is integral.

By multiplying the inequality $1 \geqslant 2\sin x - \sin^2 x$ by $\sin^{2n-1} x$ and by $\sin^{2n} x$, and integrating between 0 and $\frac{1}{2}\pi$, shew that

$$\left\{\frac{(4n+3)(2n+1)}{(4n+4)}\frac{\pi}{2}\right\}^{\frac{1}{2}} > \frac{2.4\ldots 2n}{1.3\ldots 2n-1} > \left\{\frac{2n(2n+1)\pi}{4n+1}\right\}^{\frac{1}{2}}.$$

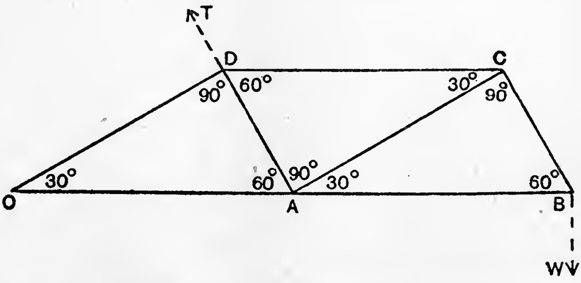
SATURDAY, 5 *June*. 9—12.

1. State the principle of Virtual Work.

$ABCD$ is a parallelogram of freely jointed rods; a point P on AB is joined to a point Q on CD by a string, and R on AD to S on BC by another string. If the tensions in these strings be T and T', shew that for equilibrium

$$\frac{T}{PQ} \cdot \frac{AP-DQ}{AB} = \frac{T'}{RS} \cdot \frac{BS-AR}{AD}.$$

2. The diagram represents a framework of seven light, smoothly jointed rods; OA and AB are horizontal and the various angles are either 30°, 60° or 90°. The frame carries a load W at B and is kept in equilibrium by a force T applied at D in the direction AD and by a force R applied at O.



Draw a stress diagram for the framework, and state which of the rods act as struts and which as ties. Shew that the stress in OA is five times that in AB.

3. State the principles employed in determining the velocities of two spheres after a direct impact with each other, when their masses, their initial velocities and the coefficient of restitution are given.

The masses of three spheres A, B, C are $7m$, $7m$, m; their coefficient of restitution is unity. Their centres are in a straight line and C lies between A and B. Initially A and B are at rest and C is given a velocity along the line of centres in the direction of A. Shew that it strikes A twice and B once, and that the final velocities of A, B, C are proportional to 21, 12, 1.

4. Explain what is meant by simple harmonic motion; and give a graphical method of representing the time between any two points of the path.

A heavy particle of mass m is attached to one end of an elastic string of natural length a, whose other end is fixed at O. The particle is let fall from rest at O. Shew that part of the motion is simple harmonic, and that, if the greatest depth of the particle below O is $a \cot^2 \frac{1}{2}\theta$, the modulus of elasticity of the string is $\frac{1}{2} mg \tan^2 \theta$, and that the particle attains this depth in time

$$\sqrt{\frac{2a}{g}} \{1 + (\pi - \theta) \cot \theta\},$$

where θ is a positive acute angle.

5. Prove that, in the orbit described by a small body P under the action of a force which is always directed towards a fixed point S, the velocity varies inversely as the perpendicular from S upon the tangent at P. Hence shew that, if the tangent at P meet any line through S in T, the component of the velocity of P in a direction perpendicular to ST varies inversely as ST.

In an elliptic orbit described under the action of a force to a focus, find the two points of the orbit at which the component of the velocity in any direction LM and the component in the opposite direction ML have maximum values; and shew that the sum of the two maximum values is the same for all directions of LM.

6. A rigid body moves under gravity and is free to turn about a fixed horizontal axis. Shew that the equation of motion is the same as that of a simple pendulum.

A thin uniform rod of mass m and length $2a$ can turn freely about one end which is fixed, and a circular disc of mass $12m$ and radius $a/3$ can be clamped to the rod so that its centre is on the rod. Shew that, for oscillations in which the plane of the disc remains vertical, the length of the simple equivalent pendulum lies between $2a$ and $2a/3$.

7. State the laws of refraction of light, and explain what is meant by the critical angle.

Light is incident normally on the plane base of a refracting glass hemisphere of radius a and refractive index μ. Shew that the emergent light crosses the axis of the hemisphere produced at a point whose distance from the centre is not less than $\dfrac{\mu a}{\sqrt{\mu^2-1}}$ and not more than $\dfrac{\mu a}{\mu-1}$.

8. From a point P on the axis of a thin lens of focal length f, a perpendicular PP' of small height h is erected. Find graphically the position and magnitude of the image of PP'.

If the point P lies between two thin lenses A, B of focal lengths f, g, the image of PP' formed by A is of height a and that formed by B is of height b. Shew that

$$\frac{1}{fb}+\frac{1}{ga}$$

has a constant value independent of the position of P, and that this value is $1/Fh$, where F is the focal length of the system of lenses.

9. Prove that the electrostatic energy of a system of charged conductors is half the sum of the products of the charges and the corresponding potentials.

Three concentric spherical conductors, radii a, b, c $(a < b < c)$, have charges E_1, E_2, E_3 respectively, and potentials V_1, V_2, V_3. Write down the potentials in terms of the charges; and shew that the electrostatic energy is

$$\frac{1}{2}\left\{\left(\frac{1}{a}-\frac{1}{b}\right)E_1^2+bV_2^2+\left(\frac{1}{c}-\frac{b}{c^2}\right)E_3^2\right\}.$$

10. An electrical point-charge Q is placed at a distance a from an infinite conducting plane at zero potential. Find the surface density σ induced at any point of the plane, and shew

that the mechanical force experienced by Q is equal to the integral

$$2\pi \int \sigma^2 dS,$$

where dS is an element of area, the integration extending over the whole plane.

Two equal charges are placed each at a distance a from the plane and at a distance $3a/2$ from each other. Shew that, if the charges have the same sign, the resultant mechanical force experienced by either charge makes an angle $\tan^{-1} \frac{224}{243}$ with the normal to the plane.

SATURDAY, 5 *June*. 2—5.

1. Prove that, when four concurrent lines are cut by two transversals, the two sets of four collinear points have equal anharmonic ratios.

Four given collinear points A, O, Q, Q' are such that

$$\mu'/AQ' - \mu/AQ = (\mu' - \mu)/AO;$$

prove that their anharmonic ratio depends only on the value of μ'/μ; hence, taking any point P upon the line $AOQQ'$, and using purely geometrical methods involving only straight lines, find the point P' such that

$$\mu'/AP' - \mu/AP = (\mu' - \mu)/AO.$$

2. Reckoning y in inches, and the angles $(x+30)^\circ$ and $(3x-30^\circ)$ in degrees with the scale of 30 degrees to one inch, draw in one figure the graphs of the functions

(i) $y = 3\sin(x+30)^\circ$, (ii) $y = \sin(3x-30)^\circ$,

for values of x ranging from 0 to 180.

Prove that within this range the function

$$y = 3\sin(x+30)^\circ - \sin(3x-30)^\circ$$

has turning values when $x = 0$, 30, 90, 180; and draw in a second figure the graph of this function, with the same scale and for the same range of values of x.

3. Prove Leibnitz' theorem for the nth differential coefficient of the product of two functions of x.

Shew that, if

$$x(1-x)\frac{d^2y}{dx^2}-(4-12x)\frac{dy}{dx}-36y=0,$$

then
$$x(1-x)\frac{d^{n+2}y}{dx^{n+2}}-\{4-n-(12-2n)x\}\frac{d^{n+1}y}{dx^{n+1}}$$
$$-(4-n)(9-n)\frac{d^ny}{dx^n}=0.$$

Hence prove by Maclaurin's theorem that the value of y, which vanishes when $x=0$, and is such that its fifth differential coefficient is unity when $x=0$, is

$$\frac{4\,!}{9\,!}\{126x^5-84x^6+36x^7-9x^8+x^9\}.$$

4. Explain what is meant by a partial differential coefficient of a function of two or more variables.

If
$$u=3(ax+by+cz)^2-(x^2+y^2+z^2),$$
and
$$a^2+b^2+c^2=1,$$
find the value of
$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}.$$

If
$$\theta=t^n e^{-r^2/4t},$$
find what value of n will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}.$$

5. Find the equation of the sphere through the origin O and three points A, B, C, whose coordinates are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

Shew that, if O' is the centre of this sphere, the sphere on OO' as diameter passes through the middle points of the six edges of the tetrahedron $OABC$, and through the feet of the perpendiculars from O on the sides of the triangle ABC.

6. Prove the formula $s=ut+\frac{1}{2}ft^2$ for the space travelled by a particle moving in a straight line with uniform acceleration.

A body moving in a straight line travels distances AB, BC, CD, of 153 feet, 320 feet, 135 feet respectively in three successive intervals of 3 secs., 8 secs., and 5 secs. Shew that these facts are consistent with the hypothesis that the body is subjected to uniform retardation. On this hypothesis find the distance from D to the point where the velocity vanishes, and the time occupied in describing this distance.

7. A flywheel has a horizontal shaft of radius r; the moment of inertia of the system about the axis of revolution is K. A string of negligible thickness is wound round the shaft and supports a mass M hanging vertically. Find the angular acceleration of the wheel when its motion is opposed by a constant frictional couple G.

If the string is released from the shaft after the wheel has turned through an angle θ from rest, and if the wheel then turns through a further angle ϕ before it is brought to rest by the frictional couple, shew that

$$G = \frac{KMgr\theta}{K\theta + (K + Mr^2)\phi}.$$

8. Obtain the formulae connecting the positions of conjugate foci for a concave spherical mirror when distances are measured from (i) the centre, (ii) the vertex, (iii) the focus.

The curved surface of a hemisphere of radius a of glass of refractive index μ is silvered. Shew that, for rays nearly coinciding with the axis and falling first on the plane surface of the hemisphere, the system acts as a mirror of radius a/μ, and find the positions of the centre and vertex of this mirror.

9. Shew that in an atmosphere of uniform temperature the pressure, p, at a height z above the ground, is given by the equation

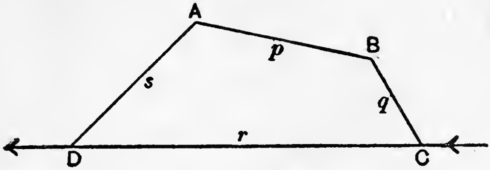
$$p = p_0 e^{\mu - z},$$

where $\mu = g\rho_0/p_0$, and ρ_0 and p_0 are the density and pressure of the air at $z = 0$.

A hollow gas-tight sphere containing hydrogen requires a force mg to prevent it from rising when the lowest point touches the ground; the total mass of sphere and hydrogen is M. Shew that the sphere can float in equilibrium with its lowest point at a height h above the ground, where

$$h = \frac{1}{\mu} \log \frac{M + m}{M}.$$

10. A quadrilateral is formed of wire and A, B, C, D are its corners taken in order. The resistances of the wires AB, BC, CD, DA are p, q, r, s respectively. The excess of the potential of A over that of B when unit current enters the quadrilateral at C and leaves it at D by wires applied at those points, as in the diagram, is denoted by $[AB.CD]$. The resistance of the quadrilateral when A and B are the electrodes



is denoted by $[AB]$. The other symbols have corresponding meanings. Calculate $[AB.CD]$ in terms of p, q, r, s, and shew that

$$[AD]+[BC]-[AC]-[BD]=2[AB.CD],$$
$$[AB.CD]+[BC.AD]+[CA.BD]=0.$$

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WEDNESDAY, 31 *May*. 9—12.

1. Shew that angles in the same segment of a circle are equal.

A rod PQ slides with its ends P, Q on the two straight arms of a bent rod. At each position of P and Q lines PR, QR are drawn perpendicular respectively to the arms on which P and Q move. Shew that, when the bent rod is fixed and PQ moves, the locus of R is a circle, and that, when PQ is fixed and the bent rod is moved, the locus of R is again a circle, of radius half that of the former circle and touching it at R.

2. Prove that the feet of the perpendiculars from the foci of an ellipse on any tangent lie on the auxiliary circle, and that their product is equal to the square on the semi-axis minor.

The feet of the perpendiculars from the foci S and S' on the tangent to the ellipse at P are Y and Y', and YQ, $Y'Q'$

are drawn to touch the ellipse in Q and Q'. Shew that QS and $Q'S'$ meet the ellipse again at the end of the diameter through P.

3. Express $\dfrac{20}{(x^2+x+4)(x-2)^2}$ in real partial fractions; and expand this expression in ascending powers of x as far as the term in x^4.

4. Find (i) the number of permutations, (ii) the number of combinations of n different things taken r at a time.

Shew that a sum of n pence can be made up of pennies, halfpennies and farthings in $(n+1)^2$ different ways.

5. Obtain expressions for all angles, θ, satisfying the equations

(i) $\cos\theta = \cos\alpha$, (ii) $\tan\theta = \tan\alpha$.

Shew that the roots of the equation

$$\cos\theta\cos(\theta-\alpha)\cos(\theta-\beta)\cos(\theta-\gamma) + \sin\theta\sin(\theta-\alpha)\sin(\theta-\beta)\sin(\theta-\gamma) = \cos\alpha\cos\beta\cos\gamma$$

are $$\theta = n\pi/2 \text{ and } (n\pi+\alpha+\beta+\gamma)/2,$$

where n is any integer.

6. An arm OP of length a makes complete revolutions about its end O, and by means of a straight link PQ of length b $(>a)$, attached to its other end P, draws a block Q to and fro in a straight line, whose direction passes through O. Shew that, when OP makes an angle θ with OQ, the distance of Q from its position when θ is zero is $a + \dfrac{a^2}{4b} - a\cos\theta - \dfrac{a^2}{4b}\cos 2\theta$, where terms in a^4/b^3 and smaller terms are neglected.

If $b=4a$, shew that, when $\theta = 60^\circ$, this expression gives a value within 0·2 per cent. of the true value.

7. If θ be the angle between two straight lines, whose direction cosines are l, m, n, and l', m', n', prove that

$$\cos\theta = ll' + mm' + nn'.$$

The feet ABC of a tripod, having legs of equal length, are at the points (0, 0), (3, 9), (7, 1), referred to rectangular axes in a horizontal plane. The apex P of the tripod is at height 12 above the plane. Determine the cosines of the angles between (i) the lines AP and BC, (ii) the planes PAB and PAC.

8. Shew that the condition that the plane $lx + my + nz = p$ should touch the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \text{ is } p^2 = a^2l^2 + b^2m^2 + c^2n^2.$$

Perpendiculars are drawn on any tangent plane from the ends of the principal axes of the ellipsoid; shew that the sum of the products of the perpendiculars from the ends of each axis is twice the square of the perpendicular from the centre.

Prove that the same property holds if the perpendiculars are drawn from the ends of any three conjugate diameters.

9. Prove the rule for integration by parts; and apply it to evaluate

$$\text{(i)} \int e^{ax} \cos bx\, dx, \qquad \text{(ii)} \int x^3 (\log x)^2\, dx.$$

Shew that

$$\int_0^1 x^{2p-1} \log(1+x)\, dx = \frac{1}{2p}\left(\frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} + \dots + \frac{1}{(2p-1)\,2p}\right).$$

10. Give a rough drawing of the curve, whose polar equation is

$$r = a(2\cos\theta + \cos 3\theta).$$

Shew that the radius vector has maximum values $3a$ and $a/3\sqrt{3}$, and that the area of the larger loop of the curve is $\dfrac{10\pi + 9\sqrt{3}}{12}a^2$, and the area of a smaller loop is $\dfrac{5\pi - 9\sqrt{3}}{24}a^2$.

WEDNESDAY, 31 *May*. 2—5.

1. Define a system of coaxal circles, distinguishing the three types of such systems.

Two circles are given, each being external to the other. Determine by geometrical constructions their radical axis and limiting points, and the circle coaxal with them, which passes through any given point. (Each step of the construction must be stated, but proofs need not be given.)

2. Prove that, if two straight lines in space are not parallel, one other straight line meets them both at right angles.

Two given straight lines AC, BD are both at right angles to AB, and are not parallel. From any point P of AC a perpendicular PM is let fall upon BD. Prove that the ratio $PM^2 - AB^2 : AP^2$ is the same for all positions of P; and shew that there is a certain fixed point S upon BA produced such that $SP : PM = SA : AB$.

3. Find the sum of a series of terms in arithmetical progression.

The first term of an arithmetical progression being a and the common difference b, and S_n being the sum of the first n terms, a new series is constructed whose nth term is $(S_1 + S_2 + \ldots + S_n)/n(n+1)$. Shew that this series is also an arithmetical progression, and that, if a new series be derived from this by the same process and so on, the nth term of the rth series so obtained from the original progression is

$$a/2^r + (n-1)\,b/6^r.$$

4. Assuming the exponential theorem, find the expansion of $e^{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4}$ correct to x^5.

Prove that, if $x > 1$,

$$\frac{\left(1+\frac{1}{x}\right)^{\frac{1}{2}x+\frac{1}{3}}}{\left(1-\frac{1}{x}\right)^{\frac{1}{2}x-\frac{1}{3}}} > e > \frac{\left(1+\frac{1}{x}\right)^{\frac{1}{2}x+\frac{1}{2}}}{\left(1-\frac{1}{x}\right)^{\frac{1}{2}x-\frac{1}{2}}}.$$

5. TP, TQ are tangents to a circle at P and Q. Shew that when the arc PQ bisects the area of the triangle PTQ, the angle, 2θ, subtended by PQ at the centre of the circle is given by the equation $2\theta = \tan\theta + \sin\theta\cos\theta$.

Prove that this equation has a root slightly less than 60°, and determine it graphically or otherwise to the nearest degree.

6. Interpret the equation $u - kv = 0$, where $u = 0$ and $v = 0$ are the equations of two given straight lines and k is any constant.

A triangle is formed by the three lines

$$u \equiv 8x + y - 7 = 0, \quad v \equiv 4x - 7y = 0, \quad w \equiv 5x - 3y - 8 = 0.$$

Prove that $u - kv = 0$ (i) passes through its orthocentre if $41k = 37$, (ii) bisects one of its angles if $k = 1$, (iii) passes through its circumcentre if $37k = 41$.

7. Shew that the equation of the chord joining two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, whose eccentric angles are ϕ_1 and ϕ_2, is

$$\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2} + \frac{y}{b}\sin\frac{\phi_1+\phi_2}{2} = \cos\frac{\phi_1-\phi_2}{2}.$$

Chords of an ellipse are drawn parallel to the tangent at a point of eccentric angle ϕ; shew that the locus of the point of intersection of the normals at the ends of a chord is the hyperbola

$$\left(\frac{ax}{\cos\phi} + \frac{by}{\sin\phi}\right)(ax\cos\phi + by\sin\phi) = (a^2 - b^2)^2\cos^2 2\phi.$$

8. Two variables x, y are given functions of a parameter t. Express

$$\frac{dy}{dx},\quad \frac{dx}{dy},\quad \frac{d^2y}{dx^2},\quad \frac{d^2x}{dy^2}$$

in terms of their differential coefficients with regard to the parameter.

Shew that

$$\left(\frac{dx}{dy}\right)^2 \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 \frac{d^3x}{dy^3} + 3\,\frac{d^2y}{dx^2}\,\frac{d^2x}{dy^2} = 0.$$

9. Give a geometrical construction for points on the curve $\cos\theta - \cos\theta' = c$, $(c<1)$, where θ and θ' are the angles POX and $PO'X$, O and O' being fixed points at a given distance a apart on the axis of x.

Shew that the normal to the curve at P meets the lines drawn through O and O' perpendicular to OP and $O'P$ respectively in points p and p' such that

$$Pp : Pp' = PO' : PO.$$

10. Solve the differential equation

$$(x+1)(x+2)\frac{dy}{dx} + 2y = (x+2)(x+3).$$

Find the solution of

$$\frac{d^2x}{dt^2} + 2n\cos\alpha\,\frac{dx}{dt} + n^2x = a\cos nt,$$

subject to the conditions $x = 0$, $\dfrac{dx}{dt} = 0$, when $t = 0$.

FRIDAY, 2 *June*. 9—12.

1. Find the position of the centre of gravity (i) of a circular arc, (ii) of a sector of a circle.

Any point P is taken upon the diameter AB of a semicircular area, and the semicircles upon AP, BP as diameters are removed. Find the centre of gravity of the area that is left; and shew that, for different positions of P on AB, the centre of gravity always lies midway between P and a certain fixed point.

2. A suspension chain carries a load uniformly distributed on a horizontal platform. Neglecting the weight of the chain, draw a force diagram for a portion of the chain extending from the lowest point of the chain, and hence or otherwise shew that the chain hangs in a parabola.

The load on the platform is $\frac{1}{2}$ ton per foot length of the span of 600 feet, and the height of the points of support above the lowest point of the chain is 50 feet. Find the greatest and least tensions in the chain, neglecting its weight.

3. Define *friction*, *limiting friction*, *coefficient of friction*.

A uniform plank AB, 20 ft. in length, lies horizontally upon two fixed rods at P and Q, AP being 7 ft. and AQ 15 ft. The rods are at the same level and parallel, and are equally rough; the plank lies across them at right angles to them. Prove that, if a constantly increasing horizontal force is applied to the plank at A in a direction perpendicular to its length, equilibrium will be broken by the plank slipping at P.

4. Two particles of masses M and m are connected by a fine inextensible string passing over a fixed smooth pulley, and the motion of the heavier particle, M, is limited by a fixed horizontal inelastic plane, on which it can impinge. The system starts from rest with M at a given height above the plane; shew that the successive heights of M at which it comes to instantaneous rest form a geometrical progression of ratio $\{m/(M+m)\}^2$, and that the whole time during which the system is in motion is three times the interval from the beginning of the motion to the first impact on the plane.

5. Prove that, if a particle move under a central attraction varying as the inverse square of the distance, the orbit described is a conic, having the centre of force as a focus.

Shew that, in elliptic motion about a focus under attraction μr^{-2}, the radial velocity is given by the equation

$$r^2\left(\frac{dr}{dt}\right)^2 = \frac{\mu}{a}\{a(1+e)-r\}\{r-a(1-e)\}.$$

6. Shew that, for a body moving in one plane, the moment of momentum about a fixed axis perpendicular to the plane is constant, if the external forces acting on the body have no moment about that axis.

A horizontal wheel with buckets on its circumference revolves about a frictionless vertical axis. Water falls into the buckets at a uniform rate of mass m per unit of time. Treating the buckets as small compared with the wheel, find the angular velocity of the wheel after time t, if Ω be its initial value; and shew that, if I be the moment of inertia of the wheel and buckets about the vertical axis and r the radius of the circumference on which the buckets are placed, the angle turned through by the wheel in time t is $\dfrac{I\Omega}{mr^2}\log_e\left(1+\dfrac{mr^2t}{I}\right)$.

7. Prove that the resultant thrust on a solid immersed in a fluid is equal to the weight of the fluid displaced.

A sphere, of radius a and specific gravity $\frac{1}{2}$, is held completely immersed at the bottom of a circular cylinder of radius b, which is filled with water to depth d. The sphere is set free and takes up its position of equilibrium; shew that the potential energy lost is $W\left(d-\dfrac{11}{8}a-\dfrac{1}{3}\dfrac{a^3}{b^2}\right)$, where W is the weight of the sphere.

[The centroid of a solid hemisphere of radius a is at distance $\frac{3}{8}a$ from its centre.]

8. Shew that when a ray of light passes through a prism in a principal plane, the deviation is a minimum when the ray passes symmetrically.

Shew that the same property holds, if the ray, after being reflected at the two faces of the prism, emerges at the second face.

9. Explain Faraday's method of indicating the intensity at any point of an electric field by the distribution of tubes of force.

Two simple closed conductors are external to one another; one has a given charge, the other is at zero potential. Indicate generally the nature of the equipotential surfaces and of the lines of force in the field, shewing that in each case there are two families.

10. State Ohm's law; and shew how to find the equivalent resistance of a battery of mn cells, of which sets of m cells are connected in series and the n sets are connected in parallel.

A battery of 8 cells, placed 2 in series in 4 sets in parallel, is connected with a galvanometer of 100 ohms resistance, which is shunted with a resistance of 100/9 ohms. Each cell has E.M.F. 2 volts and internal resistance 12 ohms. Find the deflection of the galvanometer, if a current ·0005 ampere gives a deflection of one division.

FRIDAY, 2 *June*. 2—5.

1. Prove that the orthogonal projection of a circle is an ellipse, and that conjugate diameters of the ellipse are the projections of perpendicular diameters of the circle.

Prove that in an ellipse any chord which joins the ends of two conjugate diameters divides the ellipse into two areas, one of which is very approximately ten times as great as the other.

2. Prove that, if A, B, C, D are four fixed points on a conic, and P is any point of the curve, the cross-ratio of the pencil $P\{ABCD\}$ is the same for all positions of P.

Deduce or prove otherwise that the intersection of the tangents at A and B to any conic, which passes through A, B, C and D, lies on a fixed straight line.

3. Shew that the squares of the roots of the equation

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} - a_3x^{n-3} + \ldots + (-1)^n a_n = 0,$$

are the roots of the equation

$$b_0x^n - b_1x^{n-1} + b_2x^{n-2} - b_3x^{n-3} + \ldots + (-1)^n b_n = 0,$$

where $b_0 = a_0^2,\ b_1 = a_1^2 - 2a_0a_2,\ b_2 = a_2^2 - 2a_1a_3 + 2a_0a_4, \ldots$

$$b_r = a_r^2 - 2a_{r-1}a_{r+1} + 2a_{r-2}a_{r+2} - 2a_{r-3}a_{r+3} + \ldots.$$

The equation, whose roots are the squares of the roots of the cubic,

$$x^3 - ax^2 + bx - 1 = 0,$$

is found to be identical with this cubic. Prove that either (i) $a = b = 0$, (ii) $a = b = 3$, or (iii) a and b are the roots of $z^2 + z + 2 = 0$.

4. Explain a method of solving a triangle, whose sides are of known lengths, proving the formulae required.

Apply the method to determine the angles of the triangle, whose sides are proportional to 679, 607 and 486 (using the four figure tables supplied).

5. The centres of the circumcircle and the inscribed circle of a triangle are O and I, the radii are R and r; prove that

$$OI^2 = R^2 - 2Rr.$$

Triangles are inscribed in a circle, centre O, and circumscribed to a circle, centre I; shew that the centres of their escribed circles lie on a circle of radius $2R$, whose centre I' is such that O bisects II'.

6. Find the equation of the tangent at any point of the parabola $y^2 = 4ax$, and the equation of the polar of any point with respect to the curve.

Shew that, if parabolas are drawn corresponding to different values of a, the feet of the perpendiculars from a fixed point on its polar lines all lie on a circle passing through the point.

7. Find the equation of the pair of tangents from the origin to the circle

$$x^2 + y^2 + 2gx + 2fy + k^2 = 0;$$

and shew that their intercept on the line $y = h$ is $\dfrac{2hk}{k^2 - g^2}$ times the radius of the circle.

A sphere of 10 inches diameter lies upon a table, and a spot of light is held 16 inches above the table and 2 inches horizontally from the centre of the sphere. Find the length of the major and minor axes of the shadow cast on the table by the sphere.

8. Explain what is meant by partial differentiation.

Verify that, if $$z = 3xy - y^3 + (y^2 - 2x)^{\frac{3}{2}},$$

then $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, \text{ and } \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2.$$

9. Integrate

$$\int \frac{x\,dx}{(3x^2 + 2x + 1)^{\frac{1}{2}}} \text{ and } \int \sec x\,dx.$$

Prove that $$\int_0^1 \frac{dx}{(x+1)(x+2)} = 0{\cdot}288.$$

10. By means of a formula of reduction or otherwise evaluate the integral $\int_0^{\pi/2} \sin^{2m}\theta\,d\theta$, where m is a positive integer.

The expression

$$\frac{1-a}{(1 - a\sin^2\theta)^{\frac{3}{2}}} - (1 - a\sin^2\theta)^{\frac{1}{2}},$$

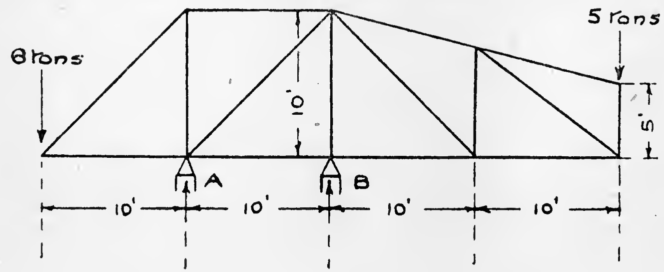
where $1 > a > 0$, is expanded in ascending powers of a, and the coefficient of a^n is denoted by u_n. Prove that $\int_0^{\pi/2} u_n\,d\theta = 0$.

SATURDAY, 3 *June*. 9—12.

1. Find in magnitude and position the resultant of two given like parallel forces P and Q.

If two equal and opposite forces S in any two parallel lines at distance b apart in the plane of P and Q are combined with them, shew that the resultant is displaced a distance $bS/(P+Q)$.

2. Determine the stresses in each member of the given frame, supported at A and B, and loaded as shewn. The



members are freely jointed at their extremities, and their weights may be neglected. Distinguish between the members in thrust and in tension.

3. Two particles of masses m_1 and m_2 are connected by a fine elastic string of modulus of elasticity λ and natural length l. They are placed on a smooth table at distance l apart, and equal impulses I in opposite directions in the line of the string act simultaneously upon them, so that the string extends. Prove that in the ensuing motion the greatest extension of the string is

$$I\{(m_1+m_2)\,l/m_1m_2\lambda\}^{\frac{1}{2}},$$

and that this value is attained in time

$$\tfrac{1}{2}\pi\,\{m_1m_2l/(m_1+m_2)\lambda\}^{\frac{1}{2}}.$$

4. A particle moves under gravity upon a smooth curve in a vertical plane. The velocity of the particle at one point of the curve being given, find its velocity and the pressure between it and the curve at any point.

A heavy particle, hanging from a fixed point by a light inextensible string of length a, is projected horizontally with velocity V. Shew that during the circular motion the tension of the string at any time is proportional to the depth of the particle at that moment below a certain horizontal line; and find the values between which V must lie that the string may become slack.

5. Establish the theorem connecting the moments of inertia of a rigid body about a number of parallel lines.

An arc of a circle is formed of thin wire (whose density may or may not be uniform) and hangs from a point P of the arc. Shew that, if in the position of equilibrium Q is the point of the circle vertically below P, then PQ is the length of the simple equivalent pendulum when the wire oscillates about P in its own plane.

6. The horse-power required to propel a steamer of 10,000 tons displacement at a steady speed of 20 knots is 15,000. If the resistance is proportional to the square of the speed, and the engines exert a constant propeller thrust at all speeds, find the acceleration when the speed is 15 knots.

Shew that the time taken from rest to acquire a speed of 15 knots is about $1\frac{1}{2}$ minutes, given $\log_e 7 = 1{\cdot}946$, one knot = 100 ft. per min.

7. A small pencil of rays issuing from a point is incident normally upon a concave spherical surface and is refracted. Find the position of the geometrical focus of the refracted pencil.

Water is contained in a spherical bowl, centre O, of internal radius a, made of glass of thickness t. A small object Q in the water is viewed by an eye E so placed that E, O, Q are in a straight line. Prove that the apparent position and magnitude of Q will be the same as if there were no glass and Q were in a concentric sphere of water of radius $a + 4at/(3a - t)$. (The index of refraction from air to water is 4/3 and from air to glass 3/2.)

8. A person, whose distance of distinct vision extends from 6 inches to 36 inches, uses a field-glass, composed of a convergent object-glass of focal length 6 inches and a divergent eye-glass of focal length 1 inch. The eye is placed close to the eye-glass. Prove that he can adjust the instrument once for all so as to see clearly, by the power of accommodation of the eye, all objects, which are more than 18 feet in front of the object-glass.

9. Explain what is meant by the capacity of a condenser.

A condenser is formed of a sphere, radius a, and a concentric spherical shell, whose inner and outer radii are b and c respectively ($a<b<c$). Shew that, when the shell is earth-connected, the capacity is $ab/(b-a)$; and that, when the sphere is earth-connected, the capacity is $c + ab/(b-a)$.

10. Find the electrical image of a point-charge in (i) an infinite plane, (ii) an uninsulated spherical conductor.

An infinite plane has a hemispherical boss, and a unit point-charge is placed in front of the boss on the common normal to the plane and the hemisphere. Shew that the charge induced on the part of the plane external to the boss is $-\cos 2\theta \,.\, \sec\theta$, where θ is the angle subtended at the charge by any radius of the base of the hemisphere.

SATURDAY, 3 *June*. 2—5.

1. Shew that, if G is the mean centre of n points $A_1, A_2 \dots A_n$, and P any point in space, ΣAP^2 exceeds $n \,.\, GP^2$ by a constant quantity.

A tetrahedron has its opposite edges equal in pairs; shew that its centroid, the centre of the circumscribing sphere, and the centre of the inscribed sphere coincide. Prove also that the orthocentres of the faces and the feet of the perpendiculars from each vertex on the opposite face lie on a sphere, having the centroid as centre.

2. Shew that $a\cos\theta + b\sin\theta$ may be written in the form $A\sin(\theta + \alpha)$, and determine A and α.

A long plank of uniform rectangular section of breadth b and thickness t is being lowered with the side b horizontal through a plane circular hole in the deck of a ship. The hole is of diameter d, and the thickness of the deck is h. Shew that the least inclination to the horizontal at which the plank will pass through the hole is

$$\sin^{-1}\frac{t}{(d^2+h^2-b^2)^{\frac{1}{2}}} + \sin^{-1}\frac{h}{(d^2+h^2-b^2)^{\frac{1}{2}}}.$$

3. Prove that the equation of any real sphere is of the form

$$x^2 + y^2 + z^2 + 2Ax + 2By + 2Cz + D = 0,$$

where

$$D < A^2 + B^2 + C^2.$$

Find separately the equations of the two planes, which pass through the line $x = y = z$, and touch the sphere

$$x^2 + y^2 + z^2 + 2x + 4y - 6z + 7 = 0.$$

Shew that they are at right angles to each other.

4. Shew how to calculate the change in a function of several variables due to small changes in the variables.

The height of a tower is determined by observing its elevations θ and ϕ at two points in a direct line with the foot of the tower and at distance a apart. Shew that, if small errors $d\theta, d\phi$ are made in the angles, the error in the deduced height of the tower is approximately

$$a(\sin^2\theta\, d\phi - \sin^2\phi\, d\theta)\operatorname{cosec}^2(\theta - \phi).$$

Prove that if $a = 300$ ft., $\theta = 58^\circ\, 15'$, $\phi = 32^\circ\, 25'$, and the angles may be $1'$ in error, the error in the calculated height is certainly less than 6 inches.

5. Prove that on the curve

$$y^2 = (x+2)(2x^2 - 3x + 2),$$

the values $x = -1$ and $x = \frac{2}{3}$ give turning values of y, and the value $x = 0$ gives points of inflexion.

Taking the unit of length both for x and y as one inch, plot the points of the curve, whose abscissae are $x = 0, \pm 1, \pm 2$; and sketch roughly the part of the curve upon which they lie.

6. A particle is projected with given velocity from a point P to pass under the action of gravity through a point Q; shew that there are, in general, two possible directions of projection.

Prove that the product of the times of flight in the two paths from P to Q is $2PQ/g$.

7. The motion of a rigid body is uniplanar; state, without proof, the equations of motion in their simplest forms.

The speed of a railway truck, weighing 5 tons, is reduced uniformly from 25 to 20 miles an hour on the level in a distance of $695\frac{3}{4}$ ft. by the brakes. Shew that, if no slipping takes place between the wheels and the rails, the normal pressure between the rails and each of the front wheels is 50 lb. wt. greater than the corresponding pressures on the back wheels, given that the distance between the axles is 12 ft. and that the centre of gravity of the truck is $4\frac{1}{2}$ ft. above the ground and equidistant from the axles, while the diameter of each wheel is 3 ft. and the moment of inertia of each pair of wheels and axle about its axis is 3600 lb. ft^2. units.

8. Determine the magnitudes and lines of action of the vertical and horizontal water pressures on the given dam (see fig. on p. 67); and find the point where the resultant water pressure meets the face of the dam.

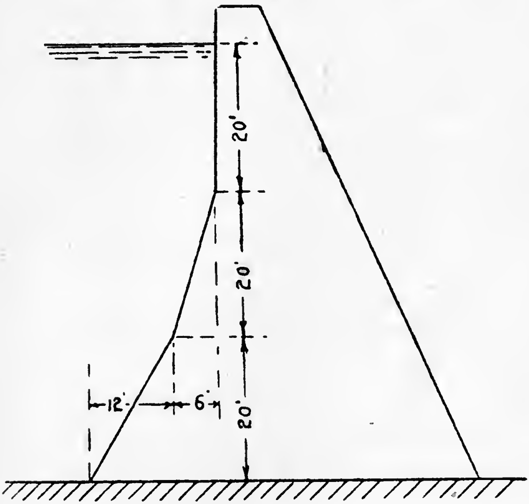
9. Prove that a transparent sphere of radius a and refractive index μ is exactly equivalent for rays passing near the centre to a thin lens at the centre, of power $-2(\mu - 1)/\mu a$.

Two such equal spheres, for which $\mu = \frac{3}{2}$, are placed in contact; determine the positions of the principal foci and the unit planes of the combination.

10. Two conducting circuits OPQ, $O'P'Q'$ are connected from P to P' and Q to Q' by wires of resistances r and r' respectively. A current enters the circuit at O and leaves at O'. Shew that, if the resistances of the lengths OP, PQ, QO are

A, B, C, and of the lengths $O'P'$, $P'Q'$, $Q'O'$ are a, b, c respectively, the currents in PP' and QQ' are in the ratio

$$\frac{BC}{A+B+C}+\frac{bc}{a+b+c}+r' : \frac{AB}{A+B+C}+\frac{ab}{a+b+c}+r.$$



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THURSDAY, 31 *May*. 9—12.

1. Shew that the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

Two straight lines AB, DC intersect in O, and E is any point on AD. Shew that the circles BAE, CDE intersect on the circle OBC.

2. Prove that two tangents to a parabola intersect on the diameter bisecting the chord of contact.

The axis of a parabola meets the directrix in X and the tangents from X meet any other tangent in P, Q. Prove that the projection of PQ on the tangent at the vertex is constant and that this tangent bisects PQ.

Prove also that $XP+XQ$ is constant.

3. Shew that there is a value of k for which $x^6-15x^3-8x^2+2$ is divisible by x^2+kx+1, and determine this value.

4. Shew that the equation

$$axy+bx+cy+d=0, \qquad (a \neq 0)$$

may be written in the form $\dfrac{x-\alpha}{x-\beta}=\lambda\dfrac{y-\alpha}{y-\beta}$, where λ is a root of the quadratic

$$(\lambda^2+1)(bc-ad)+\lambda(b^2+c^2-2ad)=0.$$

Find also the quadratic whose roots are α, β.

5. Prove the formulae for $\cos 2\theta$, $\cos 4\theta$ in terms of $\cos\theta$. Prove that

$$\tan^2(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta)=(\sec\theta+\tan\theta)/(\sec\theta-\tan\theta).$$

Also if $\qquad (1+e\cos\theta)(1-e\cos\phi)=1-e^2,$

shew that $\qquad \tan\tfrac{1}{2}\theta/\tan\tfrac{1}{2}\phi=(1+e)^{\frac{1}{2}}/(1-e)^{\frac{1}{2}}.$

6. A belt is stretched round two pulleys of diameters D and d whose centres are distant a apart. Shew that the length of the belt when tight is

$$\pi D-(D-d)\cos^{-1}\frac{D-d}{2a}+\sqrt{\{4a^2-(D-d)^2\}}.$$

If the pulleys are of diameters 6 and 3 feet, find the distance between the centres of the pulleys when the belt is of length 30·75 feet.

7. Indicate how to determine the maxima and minima of a function of one variable.

A rectangular sheet of tin, the lengths of whose sides are a and b, has four equal square portions removed from the corners, and the sides are then turned up to form an open rectangular box. Find the area of the portion removed, when the volume of the box has its maximum value.

8. Integrate

$$\int\frac{dx}{1+e^x}, \qquad \int\frac{dx}{\sqrt{\{\sin^3 x\sin(x+\alpha)\}}}.$$

Prove that

$$\text{(i)} \qquad \int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)}=\frac{\pi}{4},$$

$$\text{(ii)} \qquad \int_1^\infty \frac{(x^2+3)\,dx}{x^6(x^2+1)}=\frac{1}{30}(58-15\pi).$$

9. Prove that the area between the axis of x and the curve

$$y = a + bx + cx^2 + dx^3$$

from $x = 0$ to $x = 3h$ is

$$\frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3),$$

where y_0, y_1, y_2, y_3 are the values of y when $x = 0, h, 2h, 3h$ respectively.

Hence find an approximation to the value of the integral

$$\int_0^{\frac{\pi}{6}} (1 + 6 \sin \theta)^{\frac{1}{2}} \, d\theta.$$

10. Find the length of the shortest distance between two lines whose equations are given.

Shew that the length of the shortest distance between the line $z = x \tan \alpha$, $y = 0$ and any tangent to the ellipse $x^2 \sin^2 \alpha + y^2 = a^2$, $z = 0$ is constant.

THURSDAY, 31 *May*. 2—5.

1. Prove the projective property of a cross-ratio.

Shew how to draw through a point P, by ruler only, a straight line passing through the non-accessible intersection of two (nearly parallel) given straight lines.

2. E, F are the middle points of the sides BC, CD of a square $ABCD$, and a perpendicular is drawn from B to AE meeting AC in O. Prove that OC is equal to one-third of a diagonal of the square. Prove also that a tetrahedron can be formed by holding the square round AE, EF, FA, and that the height of the tetrahedron is one-third of a side of the square.

3. Prove that the sum of the coefficients of the odd terms in a finite binomial expansion is equal to the sum of the coefficients of the even terms.

Also prove that

$$\sum_{r=1}^{r=n} (-1)^{r-1} \frac{{}_nC_r}{r} = \sum_1^n \frac{1}{r}.$$

4. A number of corresponding values of two related variables x and y are given. Explain how it may be graphically verified

whether either (i) $y=ax^2+b$, or (ii) $y=ax^n$ approximately expresses the relation between them.

The breaking weight of a cast-iron column 15 feet high varies with the diameter according to the following table

Diameter in inches	...	3	4	5	6
Breaking weight in tons	...	18·5	55·3	124	238

Shew that the relation between the weight w and the diameter d is approximately of the form $w=ad^n$, and find the values of a and n.

5. A patrol boat lies 18 miles south-west of a harbour, and a merchantman leaves the harbour in the direction $11\frac{1}{4}°$ south of east at the rate of 15 miles an hour. On what course, and at what speed, must the patrol boat travel in order to overtake the merchantman in an hour and a half?

Determine to one-tenth of a degree the errors in the courses of the two ships which would result, with the same rates of steaming, in their meeting one another in one hour and thirty-one minutes after starting.

6. The angular displacement of a pendulum is given by $\theta=\theta_0 e^{-kt}\sin nt$. Shew that the successive maxima of θ form a series in geometrical progression.

If the time of a complete oscillation is one second, and if the ratio of the first and fifth angular displacements on the same side is 4 : 1, shew that the time taken in swinging out from the position of equilibrium to an extreme displacement is 0·241 second.

7. Find the angle between the two straight lines

$$ax+by+c=0, \quad a'x+b'y+c'=0,$$

and write down the condition that they may be perpendicular.

A variable rectangle $PQRS$ has its sides parallel to fixed directions. Q and S lie respectively on the lines $x=a$, $x=-a$, and P lies on the line $y=0$. Prove that the locus of R is a straight line, and that for all directions of the sides of the rectangle, this straight line always touches a fixed ellipse.

8. Two chords of an ellipse are drawn parallel to the minor axis, and equidistant from it. Any tangent to the curve meets the chords in P, Q. Prove that SP^2+SQ^2 is always proportional to PQ^2, S being either of the foci.

9. Find the equation of the tangent plane at (x_1, y_1, z_1) to the ellipsoid

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1.$$

Shew that the normals at the points (x_1, y_1, z_1), (x_2, y_2, z_2) will intersect if

$$\frac{(b^2-c^2)\,x_1}{x_1-x_2} + \frac{(c^2-a^2)\,y_1}{y_1-y_2} + \frac{(a^2-b^2)\,z_1}{z_1-z_2} = 0,$$

and that if (X, Y, Z) is the point of intersection,

$$a^2X\left(\frac{1}{x_1}-\frac{1}{x_2}\right) = b^2Y\left(\frac{1}{y_1}-\frac{1}{y_2}\right) = c^2Z\left(\frac{1}{z_1}-\frac{1}{z_2}\right).$$

10. Find the linear differential equation satisfied by

$$(A/x)\cos(\mu x + \epsilon),$$

for all values of A and ϵ.

Solve the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0,$$

subject to the conditions that $y = 1$ when $x = 0$ and when $x = 1$.

FRIDAY, 1 *June*. 9—12.

1. Explain the terms 'shearing stress' and 'bending moment' at a point of a beam under the action of given forces.

A light horizontal rod 16 feet long is supported at points distant 2 feet from one end and 3 feet from the other, and is loaded with a weight of 5 lbs. at its middle point and weights of 3 lbs. at each end. Draw diagrams shewing the shearing stress and bending moment at all points of the rod.

2. A uniform plank with one end on the ground rests upon a fixed cylinder which is placed on the ground with its axis perpendicular to the length of the plank. Shew that the angle α of inclination of the plank to the horizontal satisfies the inequalities

$$\tan\tfrac{1}{2}\alpha \not> \mu, \qquad \frac{\tan\frac{1}{2}\alpha}{2\sec\alpha - 1} \not> \mu'$$

where μ and μ' are the coefficients of limiting friction of the plank with the cylinder and ground respectively.

3. Explain and prove the principle of virtual work as applied to a system of rigid bodies.

Thence or otherwise shew that the force P under the piston F necessary to hold up the crank CB, connecting rod AB, and piston and rod AF in the position indicated is

$$W_1 + p_2 W_2 \frac{CN}{AC} + (p_3 W_3 + W_2) \frac{AN}{AC},$$

where W_1, W_2, W_3 are the weights of the parts and act as shewn in the figure; while

$$BD/BA = p_2, \quad CE/CB = p_3.$$

If the pin joints at A, B, C are not frictionless but resist motion with couples C_a, C_b, C_c respectively, shew that the force required under the piston is diminished by

$$\frac{(C_a + C_b)\,CN + (C_b + C_c)\,AN}{AC\,.\,BN}.$$

4. Determine in their simplest forms the equations of uniplanar motion of a rigid body.

A reel of radius r with rims of radius R rests on a plane inclined at an angle α to the horizontal. A thread fixed to the reel passes round and under it and then upwards, parallel to the plane, and over a smooth pulley, a mass m hanging freely from this end. The thread lies in the vertical plane of symmetry of the reel, which is also perpendicular to the inclined plane. Calculate the tension on the thread and the acceleration of m (i) when the inclined plane is smooth, (ii) when the plane is rough, and there is no slipping of the reel on the plane.

5. A bullet of 0·1 lb. weight is fired with a speed of 2200 feet per second into the middle of a block of wood of 30 lbs. weight, which is at rest but free to move. Find the speed of the block and bullet afterwards, and the loss of kinetic energy in foot-lbs. What becomes of this energy?

6. A shell fired with velocity V at elevation θ hits an airship at height H, which is moving horizontally away from the gun with velocity v. Shew that if

$$(2V\cos\theta - v)(V^2\sin^2\theta - 2gH)^{\frac{1}{2}} = vV\sin\theta$$

the shell might also have hit the airship if the latter had remained stationary in the position it occupied when the gun was actually fired.

7. A pressure gauge consists of a U tube of uniform bore containing mercury, one arm of the tube being closed at the top and containing 15 cubic centimetres of air at atmospheric pressure, and the other arm being connected to the receiver of a condenser. Initially the pressure in the receiver is atmospheric, and by the working of the condenser the volume of air in the gauge is reduced to 3·5 cubic centimetres and the mercury is raised 15 cm. Shew that the pressure in the receiver is then about 4·68 atmospheres, assuming that the height of a mercury barometer is 760 mm. and that the temperature in the gauge is unaltered.

8. A ray of light enters a prism and emerges after being reflected at the second face. Shew that if the incident and emergent rays are perpendicular, the angle of incidence is given by

$$\cos 2i \sin 2\phi = 1 - \mu^2 \sin^2 2i,$$

where i is the angle of the prism.

Shew also that for a glass prism of refractive index 3/2 this is only possible if $i < 20^\circ 54'$ approximately.

9. Determine the internal and external electrostatic potentials of a uniformly conducting spherical shell possessing a total charge Q.

A spherical conductor of radius 10 centimetres has initially a charge of 6284 units. Electrical connection is made, for an instant, with a distant insulated non-charged spherical conductor of radius 15 centimetres. What are the final charges of the two spheres?

10. Shew how the loss of energy in a conductor carrying a current depends upon the current and the resistance of the conductor.

A battery of mn equal cells is arranged in m files, in parallel, each containing n cells in series. The electromotive force of each cell is E, and its internal resistance is r. The battery supplies current to a circuit of external resistance R. Find the relations between m and n which give (a) the maximum current through R, or (b) twice as much energy expenditure in the external circuit as that wasted within the battery.

FRIDAY, 1 *June*. 2—5.

1. State and prove the harmonic property of a quadrilateral.

Each of a system of conics touches the sides PQ, PS of a quadrilateral $PQRS$ at Q, S. If QR, RS cut one of these conics in A, B, prove that AB and QS intersect in a fixed point.

2. Prove that when x is sufficiently small

$$1/\log(1+x) = \frac{1}{x} + \frac{1}{2} - \frac{1}{12}x + \frac{1}{24}x^2 + \ldots.$$

Prove also that if $n > 1$

$$1\Big/\log\left(1 + x + \frac{x^2}{2\,!} + \ldots + \frac{x^n}{n\,!}\right) = \frac{1}{x} + \frac{x^{n-1}}{(n+1)\,!} - \frac{x^n}{(n+2)\,n\,!} + \ldots.$$

3. Shew that the radius of the circumcircle of a triangle ABC is

$$R = \frac{a}{2\sin A} = \frac{abc}{4\sqrt{\{s(s-a)(s-b)(s-c)\}}}.$$

The legs of a tripod are each 10 feet in length and their points of contact with the horizontal table upon which the tripod stands form a triangle ABC whose sides are 7, 8 and 9 feet in length. Shew that the apex lies vertically above the centre of the circle ABC, and determine the inclinations of the legs to the horizontal and the height of the apex.

4. A right circular cylinder 2 feet in diameter and standing upon its base has its upper part cut away by a plane making an angle of 30° with the ground and cutting the axis of the cylinder 10 feet from the ground. Shew that if the surface of the cylinder is unwrapped and laid out flat, its upper edge will form a sine curve.

If the cylinder is further divided by a vertical plane through its axis in such a way that its line of intersection with the previous cutting plane meets the line of greatest slope of that section at an angle of 15°, find the lengths of the two generators of the cylinder lying in this plane.

5. Differentiate $\log_{10} x$ from first principles.

Find the differential coefficients with respect to x of $\sin^2 \frac{a}{x}$ and $\log_e \frac{1+x+x^2}{1-x+x^2}$.

6. Prove that if the axis of x touches a curve at the origin, the radius of curvature at this point is $\underset{x \to 0}{\text{Lt}}\ x^2/2y$.

A circle rolls inside another circle of double its radius but slips uniformly, making n revolutions where without slipping it would have made m revolutions. Shew that the radius of curvature of the curve traced by a point on the circumference of the rolling circle at the points furthest from the circumference of the fixed circle is $\frac{n+m}{n-m}$ times the radius of the rolling circle.

7. Prove the rule for integration by parts.

Find $\int \sin^{-1} x\, dx$ and $\int x^2 \sec^2 x \tan x\, dx$.

Prove that
$$\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x\, dx}{(1-x^2)^{\frac{3}{2}}} = \frac{\pi}{4} - \frac{1}{2}\log_e 2.$$

8. Find the equations of two circles of radii a, b having the points $(\pm c, 0)$ for centres of similitude, and prove that their radical axis is the axis of y if
$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}.$$

9. From any point on the normal to a rectangular hyperbola at a given point A the other three normals are drawn to the curve. Shew that the centroid of the feet of these three normals always lies on the diameter of the hyperbola parallel to the normal at A.

10. Find the angle between the two planes

$$lx + my + nz = p, \quad l'x + m'y + n'z = p'.$$

A chimney whose right section is a regular hexagon rises symmetrically from a roof which slopes at an angle of 45° to the horizontal, and one diagonal of the section of the chimney lies along the ridge of the roof. Find (i) the slopes of the lines of intersection of the chimney and the roof, (ii) the angles between the roof and the faces of the chimney.

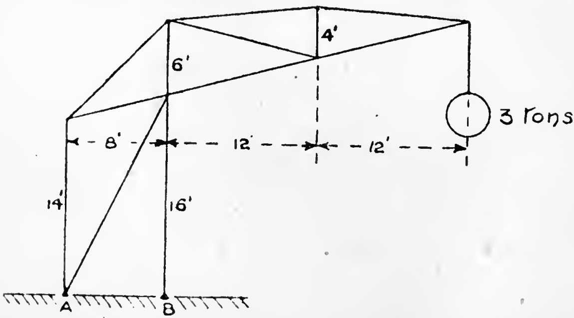
SATURDAY, 2 *June*. 9—12.

1. State the conditions of equilibrium of any number of coplanar forces.

Four forces acting along the sides of a quadrilateral $ABCD$ are in equilibrium. If one of them be completely represented by AE, where E is a given point in AB, and if the quadrilateral $AEFG$ be constructed having its sides parallel and proportional to the four forces, prove that

$$AE \,.\, FG \,.\, AD \,.\, BC = AB \,.\, CD \,.\, AG \,.\, EF.$$

2. Determine the stresses in each member of the given frame, hinged at A and B and loaded as shewn.



The members are freely jointed at their extremities and their weights may be neglected. Their lengths are given in the figure. Distinguish between the members of the frame in thrust and in tension.

3. What purpose is served by a flywheel?

A flywheel of a pressing machine has 150,000 foot-lbs. of kinetic energy stored in it when its speed is 250 revolutions per minute. What energy does it part with during a reduction in speed to 200 revolutions per minute? If 82 °/。 of this energy given out is imparted to the pressing rod during a stroke of 2 inches, what is the average force exerted by the rod?

4. A weight can slide along the spoke of a horizontal wheel but is connected to the centre of the wheel by means of a spring. When the wheel is fixed the period of a small oscillation of the weight is $2\pi/n$: shew that when the wheel is made to rotate with constant angular velocity ω the period of oscillation is

$$2\pi/\sqrt{(n^2 - \omega^2)}.$$

If the wheel is a light frame whose mass may be neglected, and is started to rotate freely with angular velocity Ω, shew that if $\Omega = 6n/5\sqrt{11}$ the greatest stretch of the spring is 20 °/。 of its original length.

5. A small sphere is suspended from a fixed point by an inextensible string. Another small sphere of equal mass moving in a direction making 60° with the downward vertical impinges directly on the first sphere with velocity V. Prove that the velocity of the first sphere after impact is $\frac{2\sqrt{3}}{7}(1+e)V$ where e is the coefficient of restitution.

6. The resistance to the motion of a car varies as the square of its speed and the effective horse-power exerted at the road wheels is constant and equal to 22. If the weight of the car is 1 ton and the maximum speed of the car against its resistance is 45 miles an hour, determine the distance in which the car can accelerate from 15 to 30 miles an hour.

7. Prove the formula $PF.QF = f^2$ for reflection in a spherical mirror.

A small object is placed between a plane mirror A and a convex spherical mirror B and upon their common normal. Images are formed successively by reflection in B, A, B. Shew that the third image is the same in all respects as if it had been produced by reflection in a single convex mirror placed at the image of A in B and find the focal length of this mirror.

8. An image of a small object is thrown by a convex lens upon a screen at a distance c from the object. The object and screen being fixed, shew that there are two positions of the lens for which this is possible, and that if d is the distance between them the focal length of the lens is $(c^2-d^2)/4c$, and find the magnification produced in each case.

9. Shew how to find the electric field due to a point charge in the presence of an uninsulated spherical conductor.

Within a hollow uninsulated conductor formed by a hemisphere of radius a together with its base, and at a distance $\frac{1}{2}a$ from the base, is placed a point charge e. Shew that the charge will be acted on by a force $\frac{47}{225}\frac{e^2}{a^2}$.

10. Explain the principle of the tangent galvanometer.

Find the electromotive force of a Daniell cell, in C.G.S. units, from the following data. Five Daniell cells were connected in series with a tangent galvanometer, the coil of which had 10 turns of 11 centimetres radius. The deflection produced by the current was 45°. The total resistance of the circuit was $16{\cdot}9 \,.\, 10^9$ C.G.S. units, and the horizontal component of the earth's magnetic field at the point was 0·180 C.G.S. units. (The axial magnetic field at the centre of a coil of N turns of radius r, due to a current i circulating through it, is $2\pi Ni/r$, all in C.G.S. units.)

SATURDAY, 2 *June*. 2—5.

1. Prove that inversion does not alter the angle of intersection of coplanar curves.

The three pairs of circles, PAB and PCD, PAD and PBC, PAC and PBD, are such that the intersections of the first two pairs are orthogonal. Prove that the third pair also intersect orthogonally.

2. Prove that conjugate diameters of an ellipse are the orthogonal projections of perpendicular diameters of the circle from which the ellipse is projected.

The tangents at the extremities A, B of any diameter of an ellipse are cut by a variable tangent in P and Q respectively. If C is the centre of the ellipse, shew that CP, CQ are conjugate to one another and that the rectangle $AP \cdot BQ$ = the square of the semi-diameter parallel to the tangents at A and B.

3. Prove, geometrically or by making use of differentiation, that if $0 < \theta < \frac{1}{2}\pi$,

$$1 > \cos\theta > 1 - \frac{\theta^2}{2}, \quad \theta > \sin\theta > \theta - \tfrac{1}{6}\theta^3,$$

θ being reckoned in circular measure.

Prove the following construction for the length of an arc AB of a circle less than a quadrant. Draw a circle with radius half that of the given circle touching it externally at A; produce BA to meet the smaller circle in C and with centre C and radius CB describe an arc to cut the tangent at A in D, then AD is approximately equal to the arc AB.

4. Given that $\frac{d}{dx}(e^x) = e^x$, deduce the differential coefficients of $\tanh x$ and $\operatorname{sech} x$.

Prove that the differential coefficient of $\tan^{-1}\tanh\frac{1}{2}x$ is $\frac{1}{2}\operatorname{sech} x$.

5. Find formulae of reduction for

$$\text{(i)} \int \sin^n x\, dx, \qquad \text{(ii)} \int \frac{x^n\, dx}{\sqrt{(ax^2 + 2bx + c)}}.$$

Prove that the perimeter of an ellipse of small eccentricity e and semi-axes a, b is equal to $\frac{1}{2}\pi\{3(a+b) - 2\sqrt{(ab)}\}$, neglecting e^7 and higher powers.

6. Define energy and distinguish between kinetic and potential energy.

The pull upon a train of 320 tons varies with the distance from rest according to the following table:

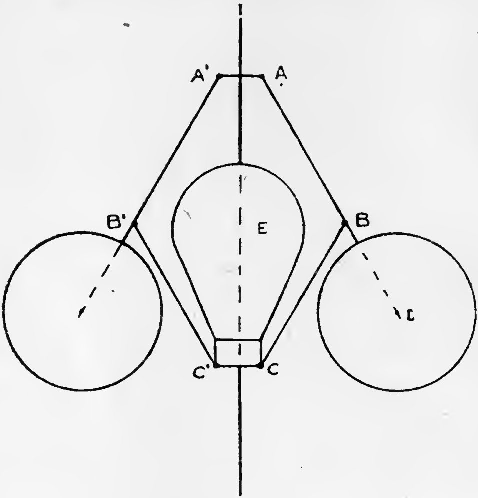
Distance in yards......	0	600	1000	2000	3000	4000
Pull in tons weight...	6·5	6·3	4·6	3·7	2·7	2·2

If the frictional resistance to motion of the train is 14 lbs. weight per ton, find the velocity of the train after travelling 2 miles on the level.

Explain how a curve shewing the distance and the time taken may be calculated.

7. A particle is moving in a circle of radius r with a velocity v; shew that the acceleration towards the centre is v^2/r.

In the figure the framework revolves about the vertical axis and the balls move outwards for an increase of speed, the weight E sliding up the axis. Shew that the angular velocity



with which the frame rotates in the position in which the arm ABD makes an angle α with the vertical is

$$\omega = \left\{ \frac{\left(1 + \frac{M'b}{Ml}\right) g \tan \alpha}{a + l \sin \alpha} \right\}^{\frac{1}{2}},$$

where $AB = BC = b$, $AD = l$, $AA' = CC' = 2a$. The balls are each of mass M and are fixed to the arms: the sliding weight E is of mass M' but the weight of the arms may be neglected.

8. Find the potential energy of a condenser with plane parallel plates, in terms of the difference of potential and the distance between the plates, and shew that the work done by the electric forces when the potentials are kept constant and the distance between the plates slightly decreased is equal to the gain in energy of the condenser.

9. The distance from the surface of a liquid to the centre of gravity of a plane area submerged in it is h, when measured in the plane of the area. Shew that the distance to the centre of pressure is $h + \frac{k^2}{h}$, where k is the radius of gyration of the area about a horizontal line drawn in its plane and through its centre of gravity.

A hole in the side of a ship is closed by a circular door 5 feet in diameter, hinged at the highest point and held inside against the water pressure at its lowest point by a fastening. If the highest and lowest points of the door are at a depth of 4 and 8 feet, shew that the least force exerted by the fastening must be 1·78 tons.

10. Shew that with the help of a plane mirror any coaxial system of lenses which gives a real image of an object on its axis can be made to produce an image coinciding in position with the object.

Shew that for a glass hemisphere of radius r and refractive index μ, when the object is at a distance u from the curved surface the mirror must be at a distance $r(u+r)/\mu\{(\mu-1)u-r\}$ from the plane face.

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