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THE

# GYROSCOPE.

BY MAJOR J. G. BARNARD, A. M.,

CORPS OF ENGINEERS, U. S. A.

FROM BARNARD'S AMERICAN JOURNAL OF EDUCATION.

NEW YORK: D. VAN NOSTRAND.
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1858.

## ERRATA.

In the first paper of this pamphlet the *references to pages* should, wherever met with, read instead of "52," "53," "54,"—"540," "541," "542."

In the second paper, for "spiral" "and "spiral motion," read "helix" and helical motion."

Gen: Blume
starb am 23.
Juni 1865.

Von dem Hauptmann
Fritz Muenther durch Ver-
mittelung seines Schwa-
gers, des General Blume
zum Geschenk erhalten,
Spandau. Frühjahr 1864.

Otto.

Erhalten vom Verfasser
Washington D.C. January 31st 1865
F. R. Muenther.
Capt. & A.D.C.

# THE PHENOMENA

OF

# THE GYROSCOPE,

## Analytically Examined

WITH TWO SUPPLEMENTS,

ON

THE EFFECTS OF INITIAL GYRATORY VELOCITIES, AND OF RETARDING FORCES ON THE MOTION OF THE

GYROSCOPE.

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# PREFATORY REMARKS.

No physical phenomenon has ever more highly excited the curiosity of the public generally than that exhibited by the simple instrument known as the "Gyroscope." None, based so directly upon the very *fundamental laws* of mechanics, (viz., those which refer to *inertia*, and to *gravitation*,) seems, at first sight, to exhibit so utter a violation of them.

It is not indeed the unskilled in mechanics alone who, seeing an apparent suspension, in this little instrument, of the very first law governing matter which addresses itself to the experience of childhood, is perplexed.

The scientific man too, the mathematician (unless his studies have happened to lead before in this very direction,) is startled, and is prone to ask himself if so paradoxical a phenomenon does not involve some new and hitherto unknown mechanical principle, or some modification of those already admitted.

Yet there can be perhaps no more beautiful illustration of those laws, no more convincing proof of their absolute truth, and adequacy to explain all purely mechanical phenomena, than is found in the solution of the problem of the Gyroscope.

To exhibit this *perfect harmony* of the phenomenon with laws universally known and understood (so far as the primal laws of matter can be understood) has been the governing idea in my mind in preparing these pages; and auxiliary to this, I desired to set at rest a vexed question, and, while correcting the numerous errors which had been circulated in popular and even scientific journals, to place *the analysis of the problem* in such a form that all who had so much knowledge of mechanics as may be derived from text-books, could follow it.

If I had addressed mathematicians alone, and sought *results* merely, it is proper to say that I could have arrived at them by much shorter methods.

To those who seek a *popular explanation* and do not find satisfactory that which I strive to give, independently of the analysis, in the latter part of my first paper, I can only say that all attempts at a purely popular explanation I have yet seen have been failures, and that the perplexity of terms, rather than the intrinsic difficulty of the subject, renders such explanations of little avail to those who cannot also comprehend the analysis.

The two supplementary papers of this pamphlet became necessary in order to apply the theory to the actual circumstances under which the Gyroscope is seen; the more so because at the first glance the *actual* motions of the instrument seem as paradoxically to violate the theory, as the theoretical motions seem to do the laws of nature.

The theory of the Gyroscope contained in these pages is not new, nor does it profess to be so: but the whole constitutes, so far as I know, the only thorough application of the laws of rotary motion to all the observed phenomena of the instrument, involving, as they do, the effects of friction, resistance of the air and of initial gyratory velocities.

J. G. B.

New York, April 21st, 1858.

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# XX. EDUCATIONAL MISCELLANY AND INTELLIGENCE.

## ROTARY MOTION AS APPLIED TO THE GYROSCOPE.

BY MAJOR J. G. BARNARD, A. M.

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AFTER reading most of the popular explanations of the above phenomenon given in our scientific and other publications, I have found none altogether satisfactory. While, with more or less success, they expose the more obvious features of the phenomenon and find in the force of gravity an efficient cause of horizontal motion, they usually end in destroying the foundation on which their theory is built, and leave an effect to exist *without a cause;* a horizontal motion of the revolving disk about the point of support is supposed to be accounted for, while the descending motion, which is the first and direct effect of gravity (and without which no horizontal motion can take place), is ignored or supposed to be entirely eliminated. Indeed it is gravely stated as a distinguishing peculiarity of rotary motion, that, while gravity acting upon a non-rotating body causes it to descend vertically, the same force acting upon a rotary body causes it to *move horizontally. A tendency to descend* is supposed to produce the effect of an *actual descent;* as if, in mechanics, a mere tendency to motion ever produced any effect whatever without that motion actually taking place.

Whatever 'mystification' there may be in analysis—however it may hide its results under symbols unintelligible save to the initiated, it is most certain that the greater portion of the physical phenomena of the universe are utterly beyond the grasp of the human mind without its aid. The mind can—indeed it *must*—search out the inducing causes, bring them together and adjust them to each other, each in its proper relation to the rest; but farther than that (at least in complicated phenomena) unaided, it cannot go. It cannot *follow* these causes in all their various actions and re-actions and at a given instant of time bring forth the results.

This, analysis alone can do. *After* it has accomplished this, it indeed usually furnishes a clue by which to trace how the workings of known mechanical laws have conspired to produce these results. This clue I now propose to find in the analysis of rotary motion as applied to the gyroscope.

The analysis I shall present, so far as determining the equations of motion is concerned, is mainly derived from the works of Poisson (vide "Journal de l'Ecole Polytech." vol. xvi—Traité de Mécanique, vol. ii, p. 162). Following his steps and arriving at his analytical results, I propose to develop fully their meaning, and to show that they are expressions not merely of a visible phenomenon, but that they contain within themselves the sole clue to its explanation: while they dispel all that is mysterious or paradoxical, and, in reducing it to merely a "particular case" of the laws of "rotary motion," throw much light upon the significance and working of those laws.

Although not unfamiliar to mathematicians, it may not be uninteresting to those who have not time to go through the long preliminary study necessary to enable them to take up with Poisson this special investigation; or whose studies in mechanics have led them no farther than to the general equations of "rotary motion" found in text books, to show how the particular equations of the gyroscopic motion may be deduced.

In so doing I shall closely follow him; making however some few modifications for the sake of brevity and of avoiding the use of numerous auxiliary quantities not necessary to the limited scope of this investigation.

The general equations of rotary motion are (see Prof. Bartlett's "Analytical Mechanics" Equations (228), p. 170):

$$\left.\begin{aligned} C\frac{dv_z}{dt}+v_x v_y(B-A)&=L_1 \\ B\frac{dv_y}{dt}+v_x v_z(A-C)&=M_1 \\ A\frac{dv_x}{dt}+v_y v_z(C-B)&=N_1 \end{aligned}\right\} \quad (1.)$$

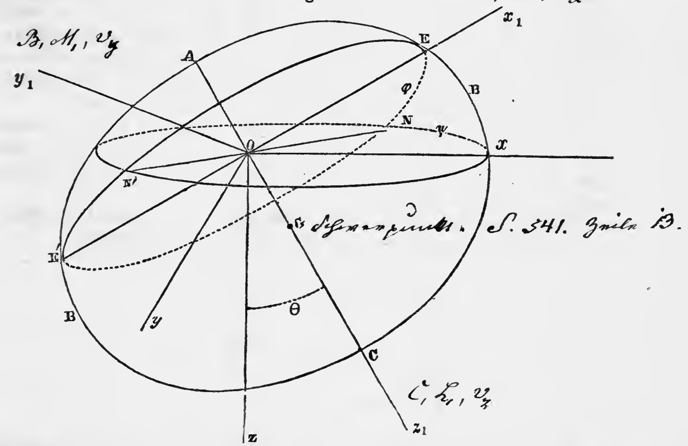
In the above expressions the rotating body (of any shape) $A\,B\,C\,D$ (fig. 1) is supposed retained by the *fixed* point (within or without its mass) $O$. $Ox$, $Oy$ and $Oz$ are the three co-ordinate axes, *fixed in space*, to which the motion of the body is referred. $Ox_1$, $Oy_1$, $Oz_1$, are the three *principal axes* belonging to the point $O$, and which, of course, partake of the body's motion. The position of the body at any instant of time is determined by those of the moving axes.

$A$, $B$ and $C$ express the several "moments of inertia" of the mass with reference, respectively, to the three principal axes $Ox_1$, $Oy_1$, $Oz_1$; $N_1$, $M_1$ and $L_1$ are the moments of the *accelerating forces,* and $v_x$, $v_y$, $v_z$, the *components of rotary velocity,* all taken with reference to these same axes.

Like lineal velocities, velocities of rotation may be decomposed —that is, a rotation about any single axis may be considered as the resultant of components about other axes (which may always be reduced to three rectangular ones): and by this means, about whatever axis the body, at the instant we consider, may be revolving, its actual velocity and axis are determined by a knowledge of its components $v_x$, $v_y$, $v_z$, about the principal axes $Ox_1$ $Oy_1$ $Oz_1$, these components being, as with lineal velocities, equal to the resultant velocity multiplied by the cosine of the angles their several rectangular axes make with the resultant axis.

As the true axis and rotary velocity may continually vary, so the components $v_x$, $v_y$, $v_z$, in equations (1) are variable functions of the time.

Fig. 1.




For the purpose of determining the axes $Ox_1$, $Oy_1$ and $Oz_1$, with reference to the (fixed in space) axes *Ox*, *Oy*, *Oz*, three auxiliary angles are used.

If we suppose the moving plane of $x_1$ $y_1$, at the instant considered, to intersect the fixed plane of $xy$ in the line $NN'$ and call the angle $xON=\psi$, and the angle between the planes $xy$ and $x_1 y_1$ (or the angle $zOz_1$)$=\theta$, and the angle $NOx_1=\varphi$, (in the figure, these three angles are supposed *acute* at the instant taken,) these three angles will determine the positions of the axes $Ox_1$, $Oy_1$, $Oz_1$, (and hence of the body) at any instant, and will themselves be functions of the time; and the rotary velocities $v_x$, $v_y$, $v_z$, may be expressed in terms of them and of their differential co-efficients.

For this purpose, and for use hereafter in our analysis, it is necessary to know the values, in terms of $\varphi$, $\theta$ and $\psi$, of the cosines of the angles made by the axes $Ox_1$, $Oy_1$ and $Oz_1$ with the fixed axes $Oz$ and $Oy$.

These values are shown to be (vide Bartlett's Mech., p. 172)

| | |
|---|---|
| $\cos x_1 Oz = -\sin\theta \sin\varphi$ | $\cos x_1 Oy = \cos\theta \cos\psi \sin\varphi - \sin\psi \cos\varphi$ |
| $\cos y_1 Oz = -\sin\theta \cos\varphi$ | $\cos y_1 Oy = \cos\theta \cos\psi \cos\varphi + \sin\psi \sin\varphi$ |
| $\cos z_1 Oz = \cos\theta$ | $\cos z_1 Oy = \sin\theta \cos\psi$ |

The differential angular motions, in the time $dt$, about the axes $Ox_1$, $Oy_1$, $Oz_1$, will be $v_x dt$, $v_y dt$, and $v_z dt$. We may determine the values of these motions by applying the laws of composition of rotary motion to the rotations indicated by the increments of the angles $\theta$, $\varphi$ and $\psi$.

If $\theta$ and $\varphi$ remain constant the increment $d\psi$ would indicate that amount of angular motion about the axis $Oz$ perpendicular to the plane in which this angle is measured. In the same manner $d\varphi$ would indicate angular motion about the axis $Oz_1$; while $d\theta$ indicates rotation about the line of nodes $ON$. In using these three angles therefore, we actually refer the rotation to the three axes $Oz$, $Oz_1$, $ON$, of which one, $Oz$, is fixed in space, another, $Oz_1$, is fixed in and moves with the body, and the third, $ON$, is shifting in respect to both.

The angular motion produced around the axes $Ox_1$, $Oy_1$, $Oz_1$, by these simultaneous increments of the angles $\varphi$, $\theta$ and $\psi$, will be equal to the sum of the products of these increments by the cosines of the angles of these axes, respectively, with the lines $Oz$, $Oz_1$ and $ON$.

The axis of $Oz_1$ for example makes the angles $\theta$, 0° and 90° with these lines, hence the angular motion $v_z dt$ is equal (taking the sum without regard to sign) to $\cos\theta d\psi + d\varphi$.

In the same manner (adding without regard to signs),

$$v_x dt = \cos x_1 Oz d\psi + \cos\varphi d\theta$$

and

$$v_y dt = \cos y_1 Oz d\psi + \cos(90° + \varphi) d\theta.$$

But if we consider the motion about $Oz_1$ indicated by $d\varphi$, positive, it is plain from the directions in which $\varphi$ and $\psi$ are laid off on the figure, that the motion $\cos\theta d\psi$ will be in the reverse direction and negative, and since $\cos\theta$ is positive $d\psi$ must be regarded as negative, hence

$$v_z dt = d\varphi - \cos\theta d\psi.$$

The first term of the value of $v_x dt$, $\cos x_1 Oz d\psi$ [since $\cos x_1 Oz$ $(= -\sin\theta \sin\varphi)$ is negative and $d\psi$ is to be taken with the negative sign] is positive. But a study of the figure will show that the rotation referred to the axis $Ox_1$, indicated by the first term of this value, is the reverse of that measured by a positive increment of $\theta$ in the second, and hence, (as $\cos\varphi$ is positive,) $d\theta$ must be considered negative. Making this change and substituting the values given of $\cos x_1 Oz$, $\cos y_1 Oz$, and for $\cos(90° + \varphi)$, $-\sin\varphi$, we have the three equations

$$\left.\begin{array}{l} v_x dt = \sin\theta \sin\varphi \, d\psi - \cos\varphi \, d\theta \\ v_y dt = \sin\theta \cos\varphi \, d\psi + \sin\varphi \, d\theta \\ v_z dt = d\varphi - \cos\theta \, d\psi \end{array}\right\} \quad (2.)^*$$

The general equations (1.) are susceptible of integration only in a few particular cases. Among these cases is that we consider, viz., that of a *solid of revolution* retained by a fixed point *in its axis of figure.*

Let the solid $ABCD$ (fig. 1) be supposed such a solid, of which $Oz_1$ is the axis of figure. It will be, of course, a principal axis, and any two rectangular axes in the plane, through $O$ perpendicular to it, will likewise be principal. By way of determining them, let $Ox_1$ be supposed to pierce the surface in some arbitrarily assumed $E$ point in this plane. Let $G$ be the center of gravity (gravity being the sole accelerating force). The moments of inertia $A$ and $B$ become equal, and equations (1.) reduce to

$$\left.\begin{array}{l} C dv_z = O \\ A dv_y - (C - A) v_z v_x dt = \gamma a M g dt \\ A dv_x + (C - A) v_y v_z dt = -\gamma b M g dt \end{array}\right\} \quad (3.)\dagger$$

in which the distance $OG$ of the point of support from the center of gravity is represented by $\gamma$, $g$ is the force of gravity, $M$ the mass, and $a$ and $b$ stand for the cosines $x_1 Oz$ and $y_1 Oz$ and of which the values are (p. 52)

$$a = -\sin\theta \sin\varphi, \qquad b = -\sin\theta \cos\varphi.$$

The first equation (3) gives by integration $v_z = n$, $n$ being an arbitrary constant; it indicates that the rotation about the axis of figure remains always constant.

Multiplying the two last equations (3) by $v_y$ and $v_x$ respectively and adding the products, we get

$$A(v_y dv_y + v_x dv_x) = \gamma Mg(a v_y - b v_x) dt.$$

From the values of $a$ and $b$ above, and from those $v_x$ and $v_y$ (equations 2) it is easy to find

$$(a v_y - b v_x) dt = -\sin\theta d\theta = d.\cos\theta;$$

substituting this value and integrating and calling $h$ the arbitrary constant

$$A(v_y^2 + v_x^2) = 2\gamma Mg \cos\theta + h \qquad (a)$$

* To avoid the introduction of numerous quantities foreign to our particular in vestigation and a tedious analysis, I have departed from Poisson and substituted the above simple method of getting equations (2.), which is an instructive illustration of the principles of the composition of rotary motions.

† See Bartlett's Mech. Equations (225) and (118) for the values of $L_1\ M_1\ N_1$: in the case we consider the extraneous force $P$ (of eq. 118) is $g$; the co-ordinates $x', y'$ of its point of application $G$ (referred to the axes $Ox_1$, $Oy_1$, $Oz_1$,) are zero and $z^1 = OG = \gamma$: cosines of $\alpha$, $\beta$ and $\gamma$ are $a$, $b$ and $c$: hence $L_1 = O$, $M_1 = \gamma a Mg$, $N_1 = -\gamma b Mg$.

g.

Multiplying the two last equations (3), respectively, by $b$ and $a$ and adding and reducing by the value just found of $d.\cos\theta$ and of $v_z$, we get

$$A(b\,d\,v_y+a\,d\,v_x)+(C-A)n\,d.\cos\theta=O \qquad (b)$$

Differentiating the values of $a$ and $b$ and referring to equations (2) it may readily be verified (putting for $v_z$ its value $n$) that

$$d\,b=(v_x\cos\theta-a\,n)\,d\,t$$
$$d\,a=(b\,n-v_y\cos\theta)\,d\,t$$

and multiplying the first by $Av_y$ and the second by $Av_x$, and adding

$$A(v_y\,d\,b+v_x\,d\,a)=A\,n(b\,v_x-a\,v_y)\,d\,t=-A\,n\,d.\cos\theta.$$

Adding this to equation ($b$), we get

$A\,d.(b\,v_y+a\,v_x)+C\,n\,d.\cos\theta=O$, the integral of which is
$A(b\,v_y+a\,v_x)+C n\cos\theta=l$ ($l$ being an arbitrary constant). ($c$)

Referring to equations (2) it will be found by performing the operations indicated, that:

$$v_x{}^2+v_y{}^2=\sin^2\theta\frac{d\,\psi^2}{d\,t^2}+\frac{d\,\theta^2}{d\,t^2}$$

$$b\,v_y+a\,v_x=-\sin^2\theta\frac{d\,\psi}{d\,t}$$

Substituting these values in equations ($a$) and ($c$), we get

$$C n\cos\theta-A\sin^2\theta\frac{d\,\psi}{d\,t}=l$$

$$A\left(\sin^2\theta\frac{d\,\psi^2}{d\,t^2}+\frac{d\,\theta^2}{d\,t^2}\right)=2\,M\,g\,\gamma\cos\theta+h$$

If, at the origin of motion, the axis of figure is simply deviated from a vertical position by an arbitrary angle $\alpha$, in the plane of $xz$, and an arbitrary velocity $n$ is imparted about this axis alone; then $v_x$ and $v_y$ will, at that instant, be zero, $\theta=\alpha$, and the substitution of these values in equations ($a$) and ($c$) will determine the values of the constants $l$ and $h$.

$$h=-2\,M\,g\,\gamma\cos\alpha$$
$$l=C\,n\cos\alpha,$$

which substituted in the above equations, make them

$$\left.\begin{aligned}\sin^2\theta\frac{d\,\psi}{d\,t}&=\frac{C n}{A}(\cos\theta-\cos\alpha)\\ \sin^2\theta\frac{d\,\psi^2}{d\,t^2}+\frac{d\,\theta^2}{d\,t^2}&=\frac{2\,M\,g\,\gamma}{A}(\cos\theta-\cos\alpha)\end{aligned}\right\}\ (4.)$$

These together with the last equation (2) which may be written, (substituting the value of $v_z$)

$$d\,\varphi=n\,dt+\cos\theta\,d\,\psi \qquad (5.)$$

will, (if integrated) determine the three angles $\varphi$, $\theta$ and $\psi$ in terms of the time $t$. They are therefore the differential equations of motion of the gyroscope.

Let $NEE'$ (fig. 1) be a section of the solid by the plane $x_1\,y_1$. This section may be called the *equator*. $E$ being some fixed point in the equator (through which the principal axis $Ox_1$ passes), the angle $\varphi$ is the angle $EON$.

If $N$ is the *ascending node* of the equator—that is, the point at which $E$ in its axial rotation *rises above* the horizontal plane, the angle $\varphi$ must increase from $N$ towards $E$—that is, $d\varphi$ (in equation 5) must be positive and (as the second term of its value is usually very small compared to the first) the angular velocity $n$ must be positive. That being the case the value of $d\varphi$ will be exactly that due to the constant axial rotation $n\,dt$, augmented by the term $\cos\theta\,d\psi$, which is the projection on the plane of the equator of the angular motion $d\psi$ of the node. This term is an *increment* to $n\,dt$ when it is positive, and the reverse when it is negative. In the first case, the motion of the node is considered *retrograde*—in the second, *direct.*

The first member of the second equation (4) being essentially positive, the difference $\cos\theta - \cos\alpha$ must be always positive—that is, the axis of figure $Oz_1$ can never rise *above* its initial angle of elevation $\alpha$. As a consequence $\frac{d\psi}{dt}$ [in first equation (4)] must be always positive. The node $N$, therefore, moves always in the direction in which $\psi$ is laid off positively, and the motion will be direct or retrograde, with reference to the axial rotation, according as $\cos\theta$ is negative or positive—that is, as the axis of figure is above or below the horizontal plane. In either case the motion of the node in its own horizontal plane is always progressive in the same direction. If the rotation $n$ were reversed, so would also be the motion of the node.

If this rotation $n$ is zero, $\frac{d\psi}{dt}$ must also be zero and the second equation (4) reduces at once to the equation of the compound pendulum, as it should. Eliminating $\frac{d\psi}{dt}$ between the two equations (4) we get

$$\sin^2\theta\frac{d\theta^2}{dt^2}=\frac{2Mg\gamma}{A}\left[\sin^2\theta-\frac{C^2n^2}{2AM\gamma g}(\cos\theta-\cos\alpha)\right](\cos\theta-\cos\alpha).$$

The length of the simple pendulum which would make its oscillations in the same time as the body (if the rotary velocity $n$ were zero) is $\frac{A}{M\gamma}$.* If we call this $\lambda$ and make for simplicity

* The length of the simple pendulum is (see Bartlett's Mech., p. 252) $\lambda=\frac{k_1^2+\gamma^2}{\gamma}$

The moment of inertia $A=M(k_1^2+\gamma^2)$; hence $\frac{A}{M\gamma}=\lambda$.

$\frac{C^2 n^2}{2 A^2 g} = \frac{2 \beta^2}{\lambda}$ the above equation becomes

$$\sin^2\theta \frac{d\theta^2}{dt^2} = \frac{2g}{\lambda} [\sin^2\theta - 2\beta^2 (\cos\theta - \cos\alpha)] (\cos\theta - \cos\alpha) \quad (6)$$

and the first equation (4) becomes

$$\sin^2\theta \frac{d\psi}{dt} = 2\beta \sqrt{\frac{g}{\lambda}} (\cos\theta - \cos\alpha). \quad (7.)$$

Equation (6) would, if integrated, give the value of $\theta$ in terms of the time; that is, the inclination which the axis of figure makes at any moment with the vertical; while eq. (7) (after substituting the ascertained value of $\theta$) would give the value of $\psi$ and hence determines the progressive movement of the body about the vertical *Oz*.

These equations in the above general form, have not been integrated;* nevertheless they furnish the means of obtaining all that we desire with regard to gyroscopic motion, and in particular that self-sustaining power, which it is the particular object of our analysis to explain.

In the first place, from eq. (6), by putting $\frac{d\theta}{dt}$ equal to zero, we can obtain the maximum and minimum values of $\theta$. This diff. co-efficient is zero, when the factor $\cos\theta - \cos\alpha = 0$, that is, when $\theta = \alpha$; and this is a *maximum*, for it has just been shown from equations (4) that $\theta$ cannot exceed $\alpha$. It will be zero also and $\theta$ a *minimum*,† when

$$\sin^2\theta - 2\beta^2 (\cos\theta - \cos\alpha) = 0$$

or

$$\cos\theta = -\beta^2 + \sqrt{1 + 2\beta^2 \cos\alpha + \beta^4} \quad (8.)$$

(The positive sign of the radical alone applies to the case, since the negative one would make $\theta$ a greater angle than $\alpha$.)

It is clear that ($\alpha$ being given) the value of $\theta$ depends on $\beta$ alone, and that it can never become zero unless $\beta$ is zero; and as long as the impressed rotary velocity $n$ is not itself zero (however minute it may be), $\beta$ will have a finite value.

Thus, however minute may be the velocity of rotation, it is sufficient to prevent the axis of rotation *from falling to a vertical position.*

The self-sustaining power of the gyroscope when very great velocities are given *is but an extreme case of this law.* For, if $\beta$ is very great, the small quantity $1 - \cos^2\alpha$ may be subtracted from the quantity under the radical (eq. 8) without sensibly altering its value, which would cause that eq. to become

$$\cos\theta = \cos\alpha.$$

* The integration may be effected by the use of elliptic functions: but the process is of no interest in this discussion.

† It is easy to show that this value of $\theta$ belongs to an actual minimum; but it is scarcely worth while to introduce the proof.

That is, when the impressed velocity $n$, and in consequence $\beta$ is very great, the minimum value of $\theta$ differs from its maximum $\alpha$ by an exceedingly minute quantity.

Here then is the result, analytically found, which so surprises the observer, and for which an explanation has been so much sought and so variously given. The revolving body, though solicited by gravity, *does not visibly fall.*

Knowing this fact, we may *assume* that the impressed velocity $n$ is very great, and hence $\cos\theta-\cos\alpha$ exceedingly minute, and on this supposition, obtain integrals of equations (6) and (7), which will express with all requisite accuracy the true gyroscopic motion. For this purpose, make

$$\theta=\alpha-u, \qquad d\theta=-du$$

in which the new variable $u$ is always extremely minute, and is the angular descent of the axis of figure below its initial elevation.

By developing and neglecting the powers of $u$ superior to the square, we have

$$\sin^2\theta=\sin^2\alpha-u\sin 2\alpha+u^2\cos 2\alpha \; *$$
$$\cos\theta-\cos\alpha=u\sin\alpha-\tfrac{1}{2}u^2\cos\alpha$$

substituting these values in eq. 6 we get

$$\sqrt{\frac{g}{\lambda}}\,dt=\frac{du}{\sqrt{2u\sin\alpha-u^2(\cos\alpha+4\beta^2)}}. \; \dagger$$

$\beta$ having been assumed very great, $\cos\alpha$ may be neglected in comparison with $4\beta^2$, and the above may be written

$$\sqrt{\frac{g}{\lambda}}\,dt=\frac{du}{\sqrt{2u\sin\alpha-4\beta^2u^2}}. \qquad (d)$$

Integrating and observing that $u=o$, when $t=o$, we have

---

* By Stirling's theorem,

$$f(u)=U+U'\frac{u}{1}+U''\frac{u^2}{1.2}, \text{ \&c.,}$$

in which $U$, $U'$, $U''$ &c. are the values of $f(u)$ and its different co-efficients when $u$ is made zero.

Making $f(u)=\sin^2(\alpha-u)$, and recollecting that $\sin 2u=2\sin u\cos u$ and $\cos 2u=\cos^2 u-\sin^2 u$, we get the value of $\sin^2\theta$; and making $f(u)=\cos(\alpha-u)-\cos\alpha$ the value in text of $\cos\theta-\cos\alpha$ is obtained.

† Eq. 6 may be written

$$\frac{\lambda}{g}\frac{d\theta^2}{dt^2}=2(\cos\theta-\cos\alpha)-4\beta^2\frac{(\cos\theta-\cos\alpha)^2}{\sin^2\theta}.$$

By substituting the values just found, of $d\theta$, $\sin^2\theta$ and $\cos\theta-\cos\alpha$ and performing the operations indicated, neglecting the higher powers of $u$, (by which $\frac{(\cos\theta-\cos\alpha)^2}{\sin^2\theta}$ reduces simply to $u^2$) and deducing the value $\sqrt{\frac{g}{\lambda}}\,dt$, the expression in the text, is obtained.

$$\sqrt{\frac{g}{\lambda}}\,.\,t=\frac{1}{2\beta}\,.\,\text{arc}\left[\cos=1-\frac{4\beta^2 u}{\sin\alpha}\right]*$$

$$u=\frac{\sin\alpha}{4\beta^2}\left(1-\cos\overline{2\beta\sqrt{\frac{g}{\lambda}}\,.\,t}\right)$$

or, (since $\cos 2a=1-2\sin^2 a$)

$$u=\frac{1}{2\beta^2}\sin\alpha\sin^2\overline{\beta\sqrt{\frac{g}{\lambda}}\,.\,t} \qquad (9.)$$

Putting $\alpha-u$ in place of $\theta$ (equat. 7) neglecting square of $u$, we get

$$\frac{d\psi}{dt}=\frac{1}{\beta}\sqrt{\frac{g}{\lambda}}\,.\sin^2\overline{\beta\sqrt{\frac{g}{\lambda}}\,.\,t} \qquad (10)\dagger$$

from which, observing that $\psi=0$, when $t=0$

$$\psi=\frac{1}{2\beta}\sqrt{\frac{g}{\lambda}}\,.\,t-\frac{1}{4\beta^2}\sin\left(2\beta\sqrt{\frac{g}{\lambda}}\,.\,t\right) \qquad (11)$$

These three expressions (9), (10), (11), represent the vertical angular depression—the horizontal angular velocity—and the

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* $\frac{du}{\sqrt{2u\sin\alpha-4\beta^2u^2}}$ may be put in the form $\frac{2\beta}{\sin\alpha}\cdot\frac{\frac{\sin\alpha}{4\beta^2}du}{\sqrt{2u\frac{\sin\alpha}{4\beta^2}-u^2}}$.

Call $\frac{\sin\alpha}{4\beta^2}=\text{R}$, and the integral of the 2d factor of the above is the arc whose radius is R and versed sine is $u$; or whose cosine is $\text{R}-u$; or it is R *times* the arc whose cosine $1-\frac{u}{\text{R}}$ with radius unity. Substituting the value of R in the integral and multiplying by the factor $\frac{2\beta}{\sin\alpha}$ we get the value of $\sqrt{\frac{g}{\lambda}}\,t$, of the text.

† In eq. (7) if we divide both members by $\sin^2\theta$, and, in reducing the fraction $\frac{\cos\theta-\cos\alpha}{\sin^2\theta}$, use the values already found and neglect the *square*, as well as higher powers $u$, (which may be done without sensible error owing to the minuteness of $u$, though it could not be done in the foregoing values of $dt$ and $t$, since the co-efficient $4\beta^2$ in those values, is reciprocally great, as $u$ is small) the quotient will be simply $\frac{u}{\sin\alpha}$.

Substituting the value of $u$ and dividing out $\sin\alpha$ we get the value of $\frac{d\psi}{dt}$ in the text.

The integral of $\sin^2\beta\sqrt{\frac{g}{\lambda}}\,t\,dt$ results from the formula $\int\sin^2\varphi\,d\varphi=\frac{1}{2}\varphi-\frac{1}{4}\sin 2\varphi$, easily obtained by substituting for $\sin^2\varphi$, its value $\frac{1}{2}-\frac{1}{2}\cos 2\varphi$

extent of horizontal angu
any time $t$.*

The first two will reach

when $\sin\beta\sqrt{\frac{g}{\lambda}}\,t=1$ and =

These values of $t$ in equati

$$\psi=\frac{\pi}{4\beta^2}.$$

Hence, counting from the $\frac{d\psi}{dt}$ and $\psi$ are all zero, we ponding values of these var

$$t=\frac{\pi}{2\beta}\sqrt{\frac{\lambda}{g}},\quad u=\frac{1}{2\beta^2}$$

which correspond to the mo and $\frac{d\psi}{dt}$ are maxima, and

$$t=\frac{\pi}{\beta}\sqrt{\frac{\lambda}{g}},\quad u=$$

when, it appears ($u$ being th gained its original elevatio destroyed.

All these values are (owin very minute. If we suppose 100 revolutions per second, t ment of ordinary proportion of arc, and the period of un

Hence the horizontal moti be exceedingly slow compare expressed by $n$.

If, in equations (9) and (10 find but a repetition of the s being recurring functions of

We see then the revolvin uniform unchanging elevatio port at a uniform rate, (as it ure generates what may be

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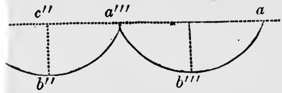
* The assumption that $\psi=0$ whe the node coincides with the fixed a analysis I suppose the initial positio to the above value of $\psi$, the const motion of the axis of figure is the sa

ılating curve (fig. 2) whose
are *cusps* lying in the same




; $c b$, $c' b'$, &c., are to the ampli-

sin $\alpha : \pi$. If the initial ele-

*liameter to the circumference of*
:es the *cycloid.*
·quations (9) and (10) will give,

$$d t = \frac{d \psi}{2\beta \sqrt{\frac{g}{\lambda}}\, u},$$

we get

$$= \frac{u\, d u}{\sqrt{\frac{1}{2\beta^2} u - u^2}};$$

*d* generated by the circle whose

)th the angles $u$ and $\psi$ are arcs
)oint of the axis of figure at a
to their minuteness may be con-

$= \frac{1}{2\beta^2} \sin \alpha$; but then, while
the arc described by the same
f a *small circle,* whose actual
in the ratio of $1 : \sin \alpha$. The
:ircumstances; and the axis of
ıed to the circumference of a
$\frac{1}{\beta^2} \sin \alpha$, which rolled along
the vertical through the point

moves with uniform velocity.
quation 11) is due to this uni-
*mean precession.*

extent of horizontal angular motion of the axis of figure after any time $t$.*

The first two will reach their respective maxima and minima when

$\sin\beta\sqrt{\frac{g}{\lambda}}t=1$ and $=0$; or when $t=\frac{\pi}{2\beta}\sqrt{\frac{\lambda}{g}}$ and $t=\frac{\pi}{\beta}\sqrt{\frac{\lambda}{g}}$.

These values of $t$ in equation (11) give

$$\psi=\frac{\pi}{4\beta^2}. \qquad \psi=\frac{\pi}{2\beta^2}.$$

Hence, counting from the commencement of motion when $t$, $u$, $\frac{d\psi}{dt}$ and $\psi$ are all zero, we have the following series of corresponding values of these variables

$$t=\frac{\pi}{2\beta}\sqrt{\frac{\lambda}{g}},\ u=\frac{1}{2\beta^2}\sin\alpha,\ \frac{d\psi}{dt}=\frac{1}{\beta}\sqrt{\frac{g}{\lambda}},\ \psi=\frac{\pi}{4\beta^2}$$

which correspond to the moment of greatest depression, when $u$ and $\frac{d\psi}{dt}$ are maxima, and

$$t=\frac{\pi}{\beta}\sqrt{\frac{\lambda}{g}},\ u=0,\ \frac{d\psi}{dt}=0,\ \psi=\frac{\pi}{2\beta^2};$$

when, it appears ($u$ being the zero), the axis of figure has regained its original elevation and the horizontal velocity is destroyed.

All these values are (owing to the assumed large value of $\beta$) very minute. If we suppose the rotating velocity $n=100\pi$ or 100 revolutions per second, the maximum of $u$ (with an instrument of ordinary proportions) would be a fraction of a minute of arc, and the period of undulation but a fraction of a second.

Hence the horizontal motion about the point of support will be exceedingly slow compared with the axial rotation of the disk expressed by $n$.

If, in equations (9) and (10), we increase t indefinitely, we will find but a repetition of the series of values already found, they being recurring functions of the time.

We see then the revolving body <u>does</u> <u>not</u> in fact maintain a uniform unchanging elevation, and move about its point of support at a uniform rate, (as it appears to do) But the axis of figure generates what may be called a <u>corrugated</u> <u>cone</u>, and any

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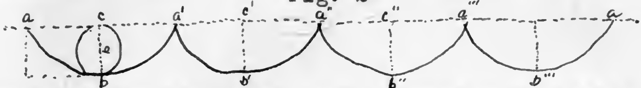
* The assumption that $\psi = 0$ when <u>t</u> is zero supposes that the initial position of the node coincides with the fixed axis of <u>x</u>. In my subsequent illustrations and analysis I suppose the initial position to be at 90° therefrom, which would require to the above value of $\psi$, the constant $\frac{1}{2}\pi$ to be added. The horizontal angular motion of the axis of figure is the same as that of the node.

point it would describe an undulating curve (fig.2) whose superior culminations a, a , a , &c., are cusps lying in the same horizontal plane, and whose

Fig. 2




sagittae cb, c' b' , &c., are to the amplitudes aa', a'a'', &c., as $\frac{\sin\alpha}{2\beta^2} : \frac{\pi}{2\beta^2} :: \sin\alpha : \pi$.

If the initial elevation $\alpha$ is 90°, this ratio is as the diameter to the circumference of the circle : a property which indicates the cycloid.

Assuming $\alpha$ =90° and $\sin\alpha$ =1, equations (9) and (10) will give, by elimination of $\sin^2\beta\sqrt{\frac{g}{\lambda}}\,t$,

$$\frac{d\psi}{dt} = 2\beta\sqrt{\frac{g}{\lambda}}\,u \qquad dt = \frac{d\psi}{2\beta\sqrt{\frac{g}{\lambda}}\,u},$$

substituting this value in eq.(d) we get

$$d\psi = \frac{2\beta u\,du}{\sqrt{2u-4\beta^2u^2}} = \frac{u\,du}{\sqrt{\frac{1}{2\beta^2}u-u^2}}\ ;$$

the differential equation of the cycloid generated by the circle whose diameter is $\frac{1}{2\beta^2}$.

In this position of the axis, both the angles u and $\psi$ are arcs of great circles described by a point of the axis of figure at a units distance from O, and owing to their minuteness may be considered as rectilinear co-ordinates.

If $\alpha$ is not 90°, the sagittae bc$=\frac{1}{2\beta^2}\sin\alpha$; but then, while the angular motion is the same, the arc described by the same point of the axis will be that

of a <u>small</u> <u>circle</u>, whose actual length will likewise be reduced in the ratio of 1 : sin $\alpha$. The curve is therefore a cycloid in all circumstances; and the axis of figure moves as if it were attached to the circumference of a minute circle whose diameter is

$\frac{1}{2\beta^2}$ sin $\alpha$, which rolled along the horizontal circle,

<u>aa′a″</u>, about the vertical through the point of support.
The centre <u>e</u> of this little circle moves with uniform velocity. The <u>first</u> <u>term</u> of the value of $\psi$ (equation 11) is due to this uniform motion : it may be called the <u>mean</u> <u>precession</u>.

The second term is due to the circular motion of the axis about this centre, and, combined with the corresponding values of $u$, constitutes what may be called the *nutation.*

These cycloidal undulations are so minute—succeed each other with such rapidity, (with the high degrees of velocity usually given to the gyroscope,) that they are entirely lost to the eye, and the axis seems to maintain an unvarying elevation and move around the vertical with a uniform slow motion.

It is in omitting to take into account these minute undulations that nearly all popular explanations fail. They fail, in the first place, because they substitute, in the place of the real phenomenon, one which is purely imaginary and *inexplicable*, since it is in direct variance with fact and the laws of nature;—and they fail, because these undulations—(great or small, according as the impressed rotation is small or great) furnish the only true clue to an understanding of the subject.

The fact is, that the phenomenon exhibited by the gyroscope which is so striking, and for which explanations are so much sought, is only a *particular and extreme phase* of the motion expressed by equations (6) and (7)—that the self-sustaining power is not *absolute*, but one of degree—that however minute the axial rotation may be, the body never will fall quite to the vertical;—however great, it cannot sustain itself without any depression.

I have exhibited the undulations, as they exist with high velocities,—when they become minute and nearly true cycloids; with low velocities, they would occupy (horizontally) a larger portion of the arc of a semi-circle, and reach downward approximating, more or less nearly, to contact with the vertical: and, *finally*, when the rotary velocity is zero, their cusps are in diametrically opposite points of the horizontal circle, while the curves resolve themselves into vertical circular arcs which coincide with each other, and the vibration of the pendulum is exhibited. All these varieties of motion, of which that of the pendulum is one extreme phase and the gyroscopic another, are embraced in equations (6) and (7) and exhibited by varying $\beta$ from 0 to high values, though, (wanting general integrals to these equations) we cannot determine, except in these extreme cases, the exact elements of the undulations. The minimum value of $\theta$ may however always be determined by equation (8).

If we scrutinize the *meaning* of equations (6) and (7), it will be found that they represent, the first, the horizontal angular component of the velocity of a point at units distance from $O$, and the second, the actual velocity of such point.*

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* In more general terms equations (4) express, the first, that the *moment of the quantity of motion* about the fixed vertical axis $Oz$ remains always constant: the *second* that the living forces generated in the body (over and above the *impressed* axial rotation) are exactly what is *due to gravity through the height, h.*

Both are expressions of truths that might have been anticipated; for gravity

For $\sin\theta \frac{d\psi}{dt}$ is the horizontal, and $\frac{d\theta}{dt}$ the vertical, component of this velocity. Calling the first $v_h$, and the second $v_v$, and the resultant $v_s$, and calling $\cos\theta - \cos\alpha$, (which is the true height of fall) $h$, those equations may be written

$$v_h = \frac{Cn}{A} \frac{h}{\sin\theta} \qquad (e)$$

$$(v_h{}^2 + v_v{}^2) = v_s{}^2 = \frac{2g}{\lambda} h \qquad (f)$$

This velocity $v_s$ (as a function of the height of fall) is exactly that of *the compound pendulum*, and is *entirely independent of the axial rotation n.* Hence, (as we might reasonably suppose) rotary motion has no power to impair the work of gravity *through a given height*, in generating velocity; but it does have power to *change the direction of that velocity.* Its effect is precisely that of a material undulatory curve, which, deflecting the body's path from vertical descent, finally directs it upward, and causes its velocity to be destroyed by the same forces which generated it. And it may be remarked, that, were the cycloid, we have described, *such a material curve*, on which the axis of the gyroscope rested, without friction and *without rotation*, it would travel along this curve by the effect of gravity alone, (the velocity of descent on the downward branch carrying it up the ascending one,) with *exactly the same velocity* that the rotating disk does, through the combined effects of gravity *and* rotation.

Equation ($a$) expresses the horizontal velocity produced by the rotation.

If we substitute its value in the second, we may deduce

$$v_v \text{ or } \frac{d\theta}{dt} = \sqrt{\frac{2g}{\lambda} h - \frac{C^2 n^2}{A^2} \frac{h^2}{\sin^2\theta}}$$

If we take this value at the commencement of descent, *and before any horizontal velocity is acquired*, (making $h$ indefinitely small), the second term under the radical may be neglected, and the first increment of descending velocity becomes $\sqrt{\frac{2g}{\lambda} h}$, precisely what is due to gravity, and *what it would be were there no rotation.*

Hence the popular idea that a rotating body offers any *direct* resistance to a change of its plane, is unfounded. It requires as little exertion of force (in the direction of motion) to move it

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cannot increase the moment of the quantity of motion about an *axis parallel to itself;* while its power of generating living force by working through a given height, cannot be impaired.

Had we considered ourselves at liberty to assume them, however, the equations might have been got without the tedious analysis by which we have reached them.

from one plane to another, as if no rotation existed; and (as a corollary) as little expenditure of work.

But deflecting forces are developed, by angular motion given to the axis, and normal to its direction, which are very sensible, and are mistaken for *direct* resistances. If the extremity of the axis of rotation were confined in a vertical circular groove, in which it could move without friction; or if any similar fixed resistance, as a material vertical plane, were opposed to the *deflecting* force, the rotating disk would vibrate in the vertical plane, as if no rotation existed. Its equation of motion would become that of the compound pendulum, $\frac{d\theta}{dt}=\sqrt{\frac{2g}{\lambda}h}$. What then is the resistance to a change of plane of rotation so often alluded to and described? A *misnomer* entirely.

The above may be otherwise established. If in equations (3) we introduce in the second member an indeterminate horizontal force, $g'$, applied to the centre of gravity, parallel to the fixed axis of $y$, and contrary to the direction in which, in our figure, we suppose the angle $\psi$ to increase, the projections of this force on the axes $Ox_1$, $Oy_1$, will be $a'g'$ and $b'g'$ and the last two of these equations will become, (calling cosines $x_1Oy$ and $y_1Oy$, $a'$ and $b'$,)

$$A\,dv_y-(C-A)\,n\,v_x\,dt=\gamma M(ag+a'g')\,dt$$
$$A\,dv_x+(C-A)\,n\,v_y\,dt=-\gamma M(bg+b'g')\,dt$$

Multiplying the first by $v_y$ and the second by $v_x$ and adding

$$A(v_y\,dv_y+v_x\,dv_x)=\gamma M[g(av_y-bv_x)\,dt+g'(a'v_y-b'v_x)\,dt].$$

But $(av_y-bv_x)dt$ has been shown (p. 53) to be $=d.\cos\theta$,—and by a similar process it may be shown that $(a'v_y-b'v_x)dt=$ $=d.(\sin\theta\cos\psi)$. (For values of $a'$ and $b'$, see p. 52.)

Let us suppose now that the force $g'$ is such that the axis of the disk may be always maintained in the plane of its initial position $xz$. The angle $\psi$ would always be 90°, $d\psi=0$, and $d.(\sin\theta\cos\psi)=0$. That is, the co-efficient of the new force $g'$ becomes zero; and the integral of the above equation is as before (p. 54),

$$A(v_y{}^2+v_x{}^2)=2\gamma Mg\cos\theta+h.$$

But the value of $v_y{}^2+v_x{}^2$ likewise reduces (since $\frac{d\psi}{dt}=0$) to $\frac{d\theta^2}{dt^2}$ and the above becomes the equation of the compound pendulum.

$$(g)\qquad \frac{d\theta^2}{dt^2}=\frac{2\gamma Mg}{A}\cos\theta+h=\frac{2g}{\lambda}(\cos\theta-\cos\alpha),\ (h\text{ being determined.})$$

This is the principle just before announced, that, with a force so applied as to prevent any *deflection* from the plane in which gravity tends to cause the axis to vibrate, the motion would be precisely *as if no axial rotation existed.*

To determine the force of $g'$; multiply the first of preceding equations by $b$, and the second by $a$, and add the two, and add likewise $A(v_y db + v_x da) = -And\cos\theta$ (see p. 54) and we shall get

$$Ad(bv_y + av_x) + Cnd\cos\theta = \gamma Mg'(a'b - ab')dt.$$

By referring to the values of $a$, $a'$, $b$, $b'$, and performing the operations indicated and making $\cos\psi = o$, $\sin\psi = 1$, the above becomes,

$$Ad(bv_y + av_x) + Cnd\cos\theta = \gamma Mg'\sin\theta dt.$$

But the value of $(bv_y + av_x)$ (p. 54) becomes zero when $\frac{d\psi}{dt} = o$.

Hence $$g' = \frac{Cnd\cos\theta}{\gamma M\sin\theta dt} = -\frac{Cn}{\gamma M}\frac{d\theta}{dt} \text{ *}$$

The second factor $\frac{d\theta}{dt}$ is the *angular velocity* with which the axis of rotation is moving.

Hence calling $v_s$ that angular velocity, *the value of the deflecting force*, $g'$ may be written (irrespective of signs),

$$g' = \frac{C}{\gamma M}nv_s: \qquad (h)$$

that is, it is directly proportional *to the axial rotation* $n$, and *to the angular velocity* of the axis of that rotation. By putting for $C$, $Mk^2$ (in which $k$ is the distance from the axis at which the mass $M$, if concentrated, would have the moment of inertia, $C$,) the above takes the simple form

$$g' = \frac{k^2}{\gamma}nv_s.$$

In the case we have been considering above, in which $g'$ is supposed to counteract the deflecting force of axial rotation, the angular velocity $v_s$, or $-\frac{d\theta}{dt}$ (equation $g$) is equal to $\sqrt{\frac{2g}{\lambda}(\cos\theta - \cos\alpha)}$.

But in the case of the *free* motion of the gyroscope, this deflecting force combines with gravity to produce the observed movements of the axis of figure.

If, *therefore*, we *disregard* the axial rotation and consider the body simply as fixed at the point $O$, and acted upon, at the center of gravity, by two forces—one of gravity, constant in intensity and direction—the other, the *deflecting* force due to an axial rotation $n$, whose variable intensity is represented by $\frac{C}{\gamma M}nv_s$,

* The effect of gravity is to diminish $\theta$ and the increment $d\theta$ is negative in the case we are considering. Hence the negative sign to the value of $g'$, indicating that the force is in the direction of the *positive* axis of $y$, as it should, since the tendency of the node is to move in the reverse direction.

and whose direction is always normal to the plane of motion of the axis; we ought, introducing these forces, and making the axial rotation $n$ zero, in general equations (3), to be able to deduce therefrom the identical equations (4) which express the motion of the gyroscope.

This I have done; but as it is only a verification of what has previously been said, I omit in the text the introduction of the somewhat difficult analysis.*

Equation (5) becomes (in the case we consider), by integration,

$$\varphi = nt + \psi \cos \alpha$$

which, with the values of $u$ and $\psi$ already obtained, determines completely the position of the body at any instant of time.

Knowing now not only the exact nature of the motion of the gyroscope, but the direction and intensity of the forces which

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* To introduce these forces in eq. (3) I observe, first, that as both are applied at $G$ (in the axis $Oz_1$) the moment $L_1$ is still zero and the *first* eq. becomes, as before,

$$Cdv_z = O \text{ or } v_z = \text{const.}$$

And as we disregard the impressed axial rotation, we make this constant (or $v_z$) zero.

The deflecting force $\frac{Cn}{\gamma M} v_s$ (taken with contrary sign to the *counteracting* force just obtained) resolves itself into two components $\frac{Cn}{\gamma M}\frac{d\theta}{dt}$ and $-\frac{Cn}{\gamma M}\frac{d\psi}{dt}\sin\theta$, the first in a horizontal, the second in a vertical plane, and both normal to the axis of figure.

The second is opposed to gravity, whose component normal to the axis of figure, is $g \sin\theta$.

Hence we have the two component forces (in the directions above indicated),

$$M.\frac{Cn}{\gamma M}\frac{d\theta}{dt} \text{ and } M\left(g - \frac{Cn}{\gamma M}\frac{d\psi}{dt}\right)\sin\theta.$$

These moments with reference to the axes of $y_1$ and $x_1$ will be

$$-\sin\varphi\gamma M\left(g - \frac{Cn}{\gamma M}\frac{d\psi}{dt}\right)\sin\theta - \cos\varphi\gamma M\frac{Cn}{\gamma M}\frac{d\theta}{dt}, \text{ and}$$

$$\cos\varphi\gamma M\left(g - \frac{Cn}{\gamma M}\frac{d\psi}{dt}\right)\sin\theta - \sin\varphi\gamma M\frac{Cn}{\gamma M}\frac{d\theta}{dt}.$$

Hence equations (3) (making $v_z$ zero, and putting for $M_1$ and $N_1$ the above values, and recollecting the values of $a$ and $b$, (p. 53) become

$$\left.\begin{array}{l} Adv_y = a\gamma Mgdt - aCn\frac{d\psi}{dt}dt - Cn\cos\varphi\frac{d\theta}{dt}dt \\ Adv_x = -b\gamma Mgdt + bCn\frac{d\psi}{dt}dt - Cn\sin\varphi\frac{d\theta}{dt}dt \end{array}\right\} (i)$$

Multiplying the equations severally by $v_y$ and $v_x$, adding and reducing (as on p. 53) we get

$$A(v_y dv_y + v_x dv_x) = \gamma Mgd.\cos\theta - Cn\frac{d\psi}{dt}d.\cos\theta - Cnd\theta(v_y\cos\varphi + v_x\sin\varphi)$$

But $v_y\cos\varphi + v_x\sin\varphi$ will be found equal to $\sin\theta\frac{d\psi}{dt}$ (by substituting the values produce it, it is not difficult to understand why such a motion takes place.

Fig. 1 represents the body as supported by a point *within* its mass; but the analysis applies to any position, in the axis of figure, within or without; and figs. 3 and 4 represent the more familiar circumstances under which the phenomenon is exhibited.

Let the revolving body be supposed (fig. 3, vertical projection), for simplicity of projection, an exact *sphere*, supported by a point in the axis prolonged, at $O$, which has an initial elevation $\alpha$ greater than 90°. Fig. 4 represents the projection on the horizontal plane $xy$; the initial position of the axis of figure (being in the plane of $xz$) is projected in $Ox$.

$Ox$, $Oy$, $Oz$, are the three (fixed in space) co-ordinate axes, to which the body's position is referred.

In this position, an initial and high velocity $n$ is supposed to be given about the axis of figure $Oz_1$, so that the visible portions move in the direction of the arrows $b$, $b'$, and the body is left subject to whatever motion about its point of support $O$, gravity may impress upon it. Had it no axial rotation, it would immediately fall and vibrate according to the known laws of the pendulum. Instead of which, while the axis maintains (apparently) its elevation $\alpha$, it moves slowly around the vertical $Oz$, receding from the observer, or from the position $ON''$ towards $ON$.

It is self-evident that the first *tendency* (and as I have likewise proved, the first effect) of gravity is to cause the axis $Oz_1$ to descend vertically, and to generate vertical *angular velocity*. But with this angular velocity, the *deflecting* force proportional to that velocity and normal to its direction, is generated, which pushes aside the descending axis from its vertical path.—But as the direction of motion changes, so does the direction of this force—always preserving its perpendicularity. It finally acquires

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of $v_y$ and $v_x$); hence the two last terms destroy each other, and the above equation becomes identical with equation ($a$) from which the 2d eq. (4) is deduced.

Multiplying the 1st equation ($i$) by $\cos\varphi$ and the second by $\sin\varphi$ and adding, we get,

$$A(\cos\varphi dv_y + \sin\varphi dv_x) = -Cnd\theta.$$

By differentiating the values of $v_y$ and $v_x$, performing the multiplications, and substituting for $d\varphi$ its value, $\cos\theta\, d\psi$, (proceeding from the 3d equation (2) when $v_z=0$) the above becomes

$$A\left(\sin\theta\frac{d^2\psi}{dt^2} + 2\cos\theta\frac{d\psi}{dt}\frac{d\theta}{dt}\right) = -Cn\frac{d\theta}{dt}.$$

Multiplying both members by $\sin\theta\, dt$, and integrating, the above becomes

$$\sin^2\theta\frac{d\psi}{dt} = \frac{Cn}{A}\cos\theta + l;$$

the same as the 1st equation (4) when the value of the constant $l$ is determined.

an intensity and upward direction adequate to neutralize the downward action of gravity; but the *acquired downward velocity* still exists and the axis *still* descends at the same time acquiring a constantly increasing horizontal component, and with it a still increasing upward deflecting force. At length the descending

Fig. 3.

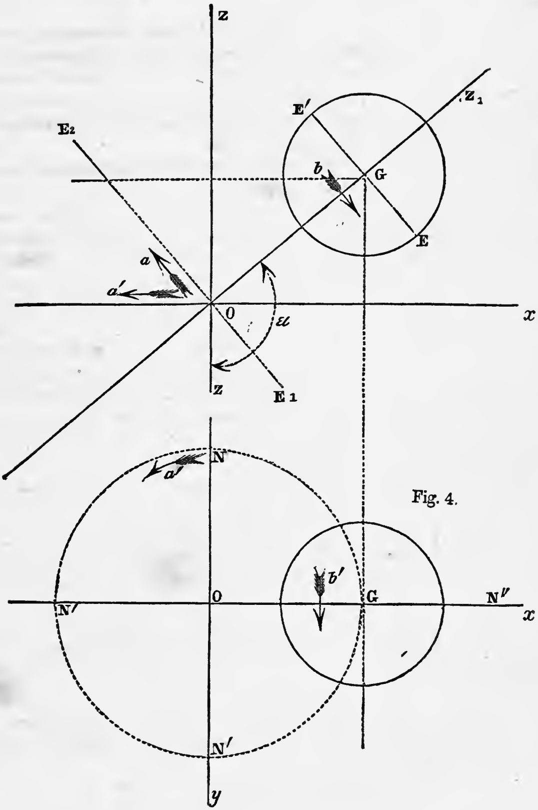



Fig. 4.

component of velocity is entirely destroyed—the path of the axis is horizontal; the deflecting force due to it acts directly contrary to gravity, which it exceeds in intensity, and hence causes the axis to commence rising. This is the state of things at the point $b$ (fig. 2). The axis has descended the curve $a\,b$, and has acquired a velocity due to its *actual* fall $a\,d$; but this velocity has been deflected to *a horizontal direction.* The *ascent* of the branch $b\,a'$ is precisely the converse of its descent. The *acquired* horizontal velocity impels the axis horizontally, while the deflecting force due to it (now at its maximum) causes it to commence ascending. As the curve bends upward, the normal direction of this force opposes itself more and more to the horizontal, while gravity is equally counteracting the vertical, velocity. As the *horizontal* velocity at $b$ was due to a fall through the height $a\,d$, so, through the medium of this deflecting force, it is just capable of restoring the *work* gravity had expended and *lifting* the axis back to its original elevation at $a'$, and the cycloidal undulation is completed, to be again and again repeated, and the axis of figure, performing undulations too rapid and too minute to be perceived, moves slowly around its point of support.

Referring to fig. 3, the *equator* of the revolving body (a plane perpendicular to the axis of figure and *through the fixed point* $O$,) would be an imaginary plane $E_1$ $E_2$. Its intersection with the horizontal plane of $xy$ would be the line of nodes $N$, $N'$. In the position delineated, the progression of the nodes is *direct.* For, at the *ascending* node $N$, any point in the imaginary plane of the equator (supposed to revolve with the body) would move *upwards* in the direction of the arrow $a$, while the node moves in the *same direction from* $O$ (of the arrow $a'$). Were the axis of

Fig. 5.

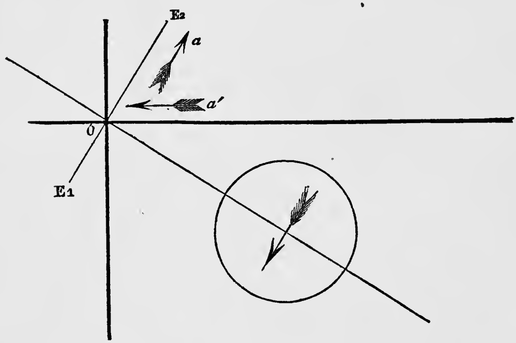
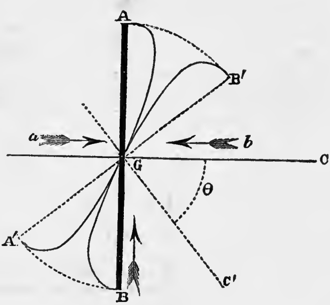



figure below the horizontal plane, (fig. 5) the upward rotation of the point would be from $O$ to $E_2$ (as the arrow $a$), while the progression of the node (in the same direction as before as the arrow $a'$) would be the reverse, and the motion of the node would be *retrograde*—yet in both cases the same in space.

As the deflecting force of rotary motion is the sole agent in diverting the vertical velocity produced by gravity from its downward direction, and in producing these paradoxical effects; and as the foregoing analysis while it has determined its value, has thrown no light upon its origin, it may be well to inquire how this force is created.

Popular explanations have usually turned upon the deflexion of the *vertical* components of rotary velocity by the vertical angular motion of the axis produced by gravity. In point of fact, however, *both* vertical and horizontal components are deflected, one as much as the other; and the simplest way of studying the effects produced, is to trace a vertical projection of the path of a point of the body under these combined motions. For this purpose conceive the mass of the revolving disk concentrated in a single ring of matter of a radius $k$ due to its moment of inertia $C = Mk^2$, (see Bartlett Mech. p. 178) and, for simplicity, suppose the angular motion of the axis to take place around the centre figure and of gravity $G$.

Fig. 6.




Let $AB$ be such a ring (supposed perpendicular to the plane of projection) revolving about its axis of figure $GC$, while the axis turns *in the vertical plane* about the same point $G$. Let the rotation be such that the visible portion of the disk moves upward through the semi-circumference, from $B$ to $A$, while the axis moves downward through the angle $\theta$ to the position $GC'$. The point $B$, by its *axial rotation* alone, would be carried to $A$; but the plane of the disk, by simultaneous movement of the axis, is carried to the position $A'B'$, and the point $B$ arrives at $B'$ instead of $A$, through the curve projected in $BGB'$. The equation of the projection, in circular functions, is easily made; but its general character is readily perceived, and it is sufficient to say, that it passes through the point $G$,—that its tangents at $B$ and $B'$ are perpendicular to $AB$ and $A'B'$,—and that its concavity, throughout its whole length, turned to *the right.* The point $A$ descends on the other, or remote side of the disk, and makes an exactly similar curve $AGA'$ with its concavity reversed.

The *centrifugal* forces due to the deflections of the vertical motions are normal to the concavities of these curves; hence, on the side of the axis *towards* the eye, they are to the *left*, and on the opposite or further side, to the *right*, (as the arrows $b$ and $a$.) Hence the joint effect is to press the axis $G\,C$ from its vertical plane $CGC'$, horizontally and towards the eye. Reverse the direction of axial rotation and the curves $AA'$ and $BB'$ will be the same, except that $AA'$ would be on the *near*, and $BB'$ on the *remote* side of the axis $G\,C$, and the direction of the resulting pressure will be reversed.

A projection on the horizontal plane would likewise illustrate this deflecting force and show at the same time that there is *no resistance in the plane of motion of the axis*, and that the whole effect of these deflexions of the paths of the different material points, is a mere *interchange of living forces between the different material points of the disk;* but it is believed that the foregoing illustration is sufficient to explain the *origin* of this force, whose measure and direction I have analytically demonstrated.

It may be remarked, however, that the intensity of the force will evidently be directly as the velocities *gained and lost* in the motion of the particles from one side of the axis to the other; or as the *angular velocity of the axis*, and as the distance, $k$, of the particles from that axis. It will also be as the *number of particles* which undergo this gain and loss of living force in a given time; or *as the velocity of axial rotation*. Considered as applied normally at $G$ to produce rotation about *any* fixed point $O$ in the axis, its intensity will evidently be *directly* as the arm of lever $k$, and *inversely* as the distance of $G$ from $O$ ($\gamma$). Hence the measure of this force already found, from analysis, $g' = \frac{k^2}{\gamma} n v_s$.

In the foregoing analysis, the entire ponderable mass is supposed to partake of the impressed rotation about the axis of figure $Oz_1$; and such must be the case, in order that the results we have arrived at may rigidly apply. Such, however, cannot be the case in practice. A portion of the instrument must consist of mountings which do not share in the rotation of the disk. It is believed the analysis will apply to this case by simply including the *whole mass*, in computing the moment of inertia $A$ and the mass $M$, while the moment $C$ represents, as before, that of *the disk alone.*

In this manner it would be easy to calculate what *amount of extraneous* weight (with an *assumed* maximum depression $u$), the instrument would sustain, with a given velocity of rotation.

The analogy between the minute motions of the gyroscope and that grand phenomenon exhibited in the heavens,—the "precession of the equinoxes"—is often remarked. In an ultimate analysis, the phenomena, doubtless, are identical; yet the immediate causes of the latter are so much more complex, that it is difficult to institute any profitable comparison.

At first sight, the undulatory motion attending the precession, known as "nutation" (nodding) would seem identical with the undulations of the gyroscope. But the identity is not easily indicated; for the earth's motion of nutation is mainly governed by the moon, with whose cycles it coincides; and the solar and lunar precessions and nutations are so combined, and affected by causes which do not enter into our problem, that it is vain to attempt any minute identification of the phenomena, without reference to the difficult analysis of celestial mechanics.

On a preceding page, I said that a horizontal motion of the rotating disk around its point of support, without descending undulations, was at variance with the laws of nature. This assertion applied however only to the actual problem in hand, in which no other external force than gravity was considered, and no other initial velocity than that of axial rotation.

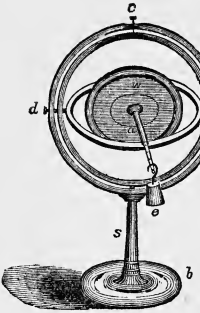
Analysis shows, however, that an initial *impulse* may be applied to the rotating disk in such a way that the horizontal motion shall be absolutely without undulation. An initial horizontal angular velocity such as would make its corresponding deflective force equal to the component of gravity, $g \sin \theta$, would cause a horizontal motion *without* undulation.

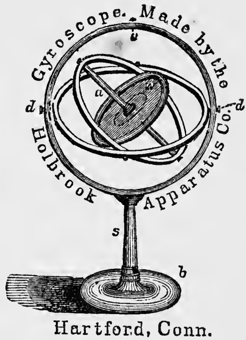
If the axial rotation $n$, as well as the horizontal rotation, is communicated by an impulsive force, analysis shows that it may be applied in *any plane* intersecting the horizontal plane *in the line of nodes;* but if applied in the plane of the equator (where it can communicate nothing but an *axial* rotation $n$), or in the horizontal plane, its intensity must be infinite.

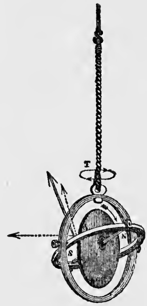
My announced object does not carry me further into the consideration of the gyroscope than the solution of this peculiar phenomenon, which depends solely upon, and is so illustrative of, the laws of rotary motion.

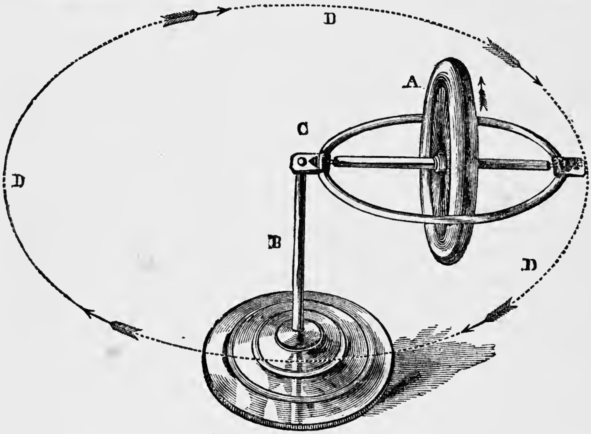
If I have been at all successful in making this so often explained subject more intelligible—in giving clearer views of some of the supposed effects of rotation, it has been because I have trusted solely to the *only* safe guide in the complicated phenomena of nature, *analysis*.

[The foregoing analysis of the phenomana of the Gyroscope, by Major Barnard, of the Corps of Engineers of the United States Army, and late Superintendent of the Military Academy at West Point, is inserted in this Journal, although it will also appear in the "*American Journal of Science and Art,*" for July, because many of our readers have become interested in the subject from the articles which have already appeared in our pages, and because we have been asked for a more scientific explanation of what has been called the self-sustaining power in the rotary disc. The length of the paper has crowded many articles of educational intelligence into the next number. ED.]



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# XVI. EDUCATIONAL MISCELLANY.

## ON THE MOTION OF THE GYROSCOPE AS MODIFIED BY THE RETARDING FORCES OF FRICTION, AND THE RESISTANCE OF THE AIR: WITH A BRIEF ANALYSIS OF THE TOP.

BY MAJOR J. G. BARNARD, A. M.

Corps of Engineers U. S. A.

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In a previous paper (see article in this Journal for June, 1857, to which this paper is intended to be supplementary,) I have investigated the "Self-sustaining power of the Gyroscope" in the light of analysis. From the general equations of "Rotary motion" I have deduced the laws of motion for the particular case of a *solid of revolution* moving about a fixed point in its axis of figure, (or the prolongation thereof). I have shown that such a body, having its axis placed in any degree of inclination to the vertical, and having a high rotary motion about *that axis,* will not, under the influence of gravity, *sensibly fall;* but that any point in the axis will describe "an undulating curve whose superior culminations are *cusps* lying in the same horizontal plane;" that this curve approaches more and more nearly to the cycloid, as the velocity of axial rotation is greater; that when this velocity is very great the undulations become very minute and "the axis of figure performing undulations too rapid and too minute to be perceived, moves slowly about its point of support." I have shown how the direction and velocity of this *gyration* are determined by the direction and velocity of axial rotation and the distance of the center of gravity of the figure from the point of support, and that the remarkable phenomenon exhibited by the gyroscope is but a *particular case* due to a *very high velocity* of axial rotation, of the general laws of motion of such a body as described, which embrace the motion of the pendulum in one extreme and that of the gyroscope in the other, and that intermediate between these two extreme cases (for moderate rotary velocities) the undulations of the axis, will be large and sensible.

I have likewise shown that whenever, to the axis of a rotating solid, an angular velocity is imparted, a force which I have called "*the deflecting force*" acting perpendicular to the plane of motion of that axis, is developed, whose intensity is proportional to this angular velocity, and likewise to the rotary velocity of the body; and that it is this *deflecting force* which is the immediate *sustaining agent,* in the gyroscope.

In the above deductions of analysis is found the full and complete solution of the "self-sustaining power of the gyroscope."

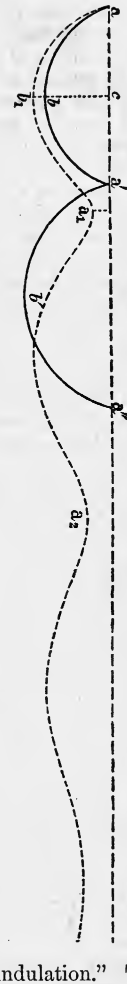
To make the character of the motion indicated by analysis, sensible to the eye, it is only necessary to attach to the ordinary gyroscope, in the prolongation of the axis, an arm of five or six inches in length, and having an universal joint at its extremity, and to swing the instrument as a pendulum; or, the extremity of an arm of such a length may be rested in the usual way, upon the point of the standard, when, with the centre of gyration removed at so great a distance from the point of support, the undulatory motion becomes very evident.

But it cannot fail to be observed that the motion preserves this peculiar feature but for a very short period. The undulations speedily disappear; instead of periodical moments of *rest* (which the theory requires at each *cusp*) the gyratory velocity becomes *continuous*, and nearly uniform and horizontal; and it increases as the axis (owing to the retarding influences of friction and the resistance of the air) slowly falls. In short, the axis soon seems to move upon a descending spiral described about a vertical through the point of support.

The experimental gyroscope, in its simplest form consists of two distinct masses, the rotating disk, and the *mounting* (or ring in which the disk turns). The point of support in the latter, though it gives free motion about a vertical axis, constrains more or less, the motion of the combined mass about any other. The rotating disk turns at the extremities of its axle, upon points or surfaces in the mass of the mounting, *with friction;* it is rare, too, that the point of support, of the mounting, is adjusted in the exact prolongation of the axis of the disk.

Without attempting to subject to analysis causes so difficult to grasp as these, I shall first attempt to show, by general considerations, what would be the *immediate* influence of the retarding forces of friction and the resistance of the air upon our theoretical solid; and then point out the further effect due to the discrepancies of figure, above indicated. Leaving out of consideration the minute effect of friction at the point of support, these forces exert their influence, mainly in retarding the *rotary velocity of the disk.* Friction—at the extremities of the axle of the disk, and the resistance of the air, at its surface, are powerful enough to destroy entirely in a very few minutes, the high velocity originally given to it. It is in this way, mainly, that they modify the motion indicated by analysis.

If the rotary velocity remained *constant* while the axis made *one* of the little cycloidal curves $aba'$, (fig. 1) the deflecting force would be just sufficient, as I have shown (p. 556 of the article cited) to lift the axis back to its original elevation $a'$, and to destroy, *entirely*, the velocity it had acquired through its fall $cb$. If, at $a'$, the rotary velocity $n$ underwent an *instantaneous* diminution, and remained constant through another undulation, a curve, of larger amplitude and sagitta $a'b'a''$ would be described, and the axis would *again* rise to its original elevation $a''$, and *again* be brought to rest. We might then, on casual consideration of the subject, expect to see the undulations become more and more sensible as the rotary velocity decreased. The reverse, however, is the case, as I have already stated. In fact, the above supposition would require the rotary velocity $n$ to be a *discontinuous* decreasing function of the time; whereas it is, really a *continuous* decreasing function. It is undergoing a gradual diminution between $a$ and $a'$. The *deflecting force*, which is constantly proportional to it, is *therefore* insufficient to keep the axis up to the theoretical curve $a b a'$, but a *lower* curve $a b_1 a_1$ is described; and when the culmination $a_1$ is reached, it is *below* the original elevation $a'$.




1.

But the 2d of our general equations for the gyroscope (4), [afterwards put under the simple form $\left(\text{eq. } (f)\right)$ $v_s{}^2 = \frac{2g}{\gamma} h$] which is *independent of n*, shows that the angular velocity of the axis will always be that due to its *actual fall h* below the initial elevation. On reaching the culmination $a_1$ therefore, the axis will not come to rest, but will have a horizontal velocity due to the fall $a' a_1$ and the curve will not form a *cusp* but an *inflexion* at $a_1$.

The axis will commence its second descent, therefore, with an *initial horizontal velocity.* It will not descend as much as it would have done had it started *from rest* with its diminished value of $n$; and, for the same reason as before, will not be able as again to rise high as its starting point $a_1$ but to a somewhat lower point $a_2$ and with an increased horizontal velocity. These increments of horizontal velocity will constantly ensue as the culminations become lower and lower, while on the other hand, the undulations become less and less marked, as indicated by the figure.

I have stated in my former paper (p. 559) that a certain *initial* horizontal angular velocity such as would "make its corresponding deflecting force equal to the component of gravity, $g \sin \theta$, would cause a horizontal motion without undulation." This horizontal velocity is rapidly attained through the agencies just described: or, at least, nearly approximated to, and the axis, as observation shows, soon acquires a continuous and uniform horizontal motion.

On the other hand, this sustaining power being directly proportional to the rotary velocity of the disk, as well as to the angular velocity of the axis, diminishes with the former, and as it diminishes, the axis must descend, acquiring angular velocity due to the height of fall: hence the rapid gyration and the descending spiral motion which accompanies the loss of rotary velocity.

A more curious and puzzling effect of the friction of the axle is presented, when we come to take into consideration, instead of our theoretical solid, the discrepancies of figure presented by the actual gyroscope. If, with a high initial rotation, the common gyroscope be placed on its point of support with its axis somewhat inclined *above* a horizontal position, it will soon be observed to *rise*. In my analytical examination (p. 543) I have stated as a deduction from the second equation (4), that "the axis of figure can never rise *above* its initial angle of elevation." That equation supposes that the rotary velocity *n remains unimpaired*, and is the expression of a fundamental principle of dynamics—that of "living forces" (so-called), which requires that the living force generated by gravity be directly proportional to the height *of fall*, and involves as a corollary that through the agency of its own gravity alone, the centre of gravity of a body can never rise above its initial height.* The anomaly observed, therefore, either requires the action of some *foreign force; or*, that the living force lost by the rotating disk, shall, through some hidden agency, be expended in performing this work of *lifting* the mass.

The discrepancy here exhibited between the motion proper to our theoretical solid of revolution and the experimental gyroscope is due to the division of the latter into two distinct masses, one of which rotates, *with friction,* upon points or surfaces in the other; and to the fact that at the point of support (in the latter) there is not *perfectly free motion* in all directions.

The friction at the extremities of the axle of the disk, tends to impress on the mass which constitutes the "mounting," a rotation in the same direction. Were the motion of the latter upon its fixed point of support *perfectly free*, the mounting and disk would soon acquire a *common rotatory velocity* about the axis of the disk. But the mounting *is* perfectly free to turn about the *vertical* axis through the point of support, though *not about any other.* If we decompose, therefore, the rotation which would be impressed upon the mounting into two components, one about this vertical, and the other about a horizontal axis—the first takes *full effect*, and the latter is destroyed at the point fo support. If the axis of the instrument is *above* the horizontal, this component of rotation is in the same direction as the *gyration* due to gravity, and *adds to it;* if the axis is *below* the horizontal, the component is the reverse of the natural gyration, and *diminishes it.*

* The first of these equations (as I have remarked in a note to p. 547) is the expression of another fundamental principle—more usually called the "principle of areas."

But I have shown that the axis soon acquires, independent of this cause, a gyration whose deflecting or sustaining force is just equivalent to the downward component of gravity. The *addition* to this gyratory velocity caused by friction when the axis is inclined *upwards* puts the deflecting force in *excess*, and the axis is raised; it is raised, as in all other cases in which *work* is done through acquired velocity—viz., by an expenditure of *living force;* but in this instance, through a most curious and complicated series of agencies.

The phenomenon may be best illustrated in the following manner. Let the outer extremity of the common gyroscope, having its axis inclined *above* the horizontal, be supported by a thread attached to some fixed point vertically above the point of support, so that gyration shall be free. Here gravity is eliminated, and the axis of our theoretical solid of revolution would remain perfectly motionless; but the gyroscope starts off, of itself, to gyrate in *the same direction* that it would were its extremity *free.* This gyration increases (if the rotary velocity is great) until the deflecting force due to it, lifts the outer extremity from its support on the thread, and it continues indefinitely to rise. Try the same experiment with the axis *below* the horizontal. The gyration will commence spontaneously as before, but in the *reverse* direction: it will increase until the *inner extremity is lifted from the point of support*, (the action of the deflecting force being here reversed,) the instrument supporting itself on the thread alone. If the experiment is tried with the axis perfectly horizontal, no gyration takes place, for the component of rotation, due to friction, is, in this position, zero.

The foregoing reasoning accounts, I believe, for all the observed phenomena of the experimental gyroscope, and shows how, from the theory of our imaginary solid of revolution, a consideration of the effects of the discrepancies of form, and of the actual disturbing forces, leads to their satisfactory explanation.

The great similarity between the phenomena of the top and gyroscope, renders it not uninteresting to compare the laws of motion of the two. If we conceive a solid of revolution terminated at its lower extremity by a *point* (the ordinary form of the top), resting upon a horizontal plane without friction, *and having a rotary motion about its axis of figure*, such a body will be subject to the action of two forces; *its weight*, acting at the centre of gravity, and the *resistance of the plane*, acting at the point vertically upwards.

According to the fundamental principles of dynamics, the centre of gravity will move as if the mass and forces were concentrated at that point, while the mass will turn about this centre as if it were fixed. Calling $R$ the resistance of the plane, $M$ the mass, and $Mg$ the weight of the top, and $z$ the height of the centre of gravity above the plane, we shall have for the equation of motion of the centre of gravity*

$$M\frac{d^2z}{dt^2}=R-Mg \qquad (1.)$$

As the angular motion of the body is the same as if the centre of gravity was fixed, and as $R$ is the only force which operates to produce rotation about that centre, if we call $C$ the moment of inertia of the top about its axis of figure, and $A$ its moment with reference to a perpendicular axis through the centre of gravity, and $\gamma$ the distance, $GK$ (fig. 2) of the point of support from that centre; the equations of rotary motion will become identical with equations (3) (p. 541), substituting $R$ for $Mg$

$$\left.\begin{array}{l} C\,dv_z=0 \\ A\,dv_y-(C-A)\,v_z v_x\,dt=\gamma\, a\, R\,dt \\ A\,dv_x+(C-A)\,v_y v_z\,dt=-\gamma\, b\, R\,dt \end{array}\right\} \qquad (2.)$$

The first of equations (2) gives us $v_z$ as for the gyroscope, equal a constant $n$.

Multiplying the 2d and 3d of equations (2) by $v_y$ and $v_x$ respectively, and adding and making the same reduction as on p. 53, we shall get

$$A\,(v_y\,dv_y+v_x\,dv_x)=R\gamma\, d.\cos\theta.$$

But $z$ (the height of the centre of gravity above the fixed plane) $=-\gamma\cos\theta$; hence $\gamma d.\cos\theta=-dz$; and equation (1) gives $R=M\left(\frac{d^2z}{dt^2}+g\right)$. Substituting these values of $R$ and $\gamma\, d.\cos\theta$ in the preceding equation, and integrating, we have

$$A\,(v_y^{\,2}+v_x^{\,2})+M\left(\frac{dz^2}{dt^2}+2gz\right)=h \qquad (3.)$$

From the 2d and 3d of equations (2) the equation ($c$) (of the gyroscope, p. 542) is deduced by an identical process.

$$A\,(b\,v_y+a\,v_x)+Cn\cos\theta=l,$$

and a substitution in the two foregoing equations of the values of the cosines $a$ and $b$, and of the angular velocities $v_x$ and $v_y$, in terms of the angles $\varphi$, $\theta$ and $\psi$ (see pp. 540, 541), and for $z$ and $\frac{dz}{dt}$ their values, $-\gamma\cos\theta$, and $\gamma\sin\theta\frac{d\theta}{dt}$, and a determination of the constants, on the supposition of an initial inclination of the axis $\alpha$, and of initial velocity of axial rotation $n$, will give us for the equations of motion of the *top:*

$$\left.\begin{array}{l} \sin^2\theta\dfrac{d\psi}{dt}=\dfrac{Cn}{A}(\cos\theta-\cos\alpha) \\ A\left(\sin^2\theta\dfrac{d\psi^2}{dt^2}+\dfrac{d\theta^2}{dt^2}\right)+M\gamma^2\sin^2\theta\dfrac{d\theta^2}{dt^2}=2Mg\gamma(\cos\theta-\cos\alpha) \end{array}\right\} (4.)$$

* As there are no horizontal forces in action, there can be no horizontal motion of the centre of gravity except from initial impulse, which I here exclude.

from which the angular motions of the top can be determined. The first is identical with the first equation (4) for the gyroscope. The second differs from the second gyroscopic equation only in containing in its first member the term $M\gamma^2 \sin^2\theta \frac{d\theta^2}{dt^2}$, or its equivalent $M\frac{dz^2}{dt^2}$, expressing the living force of vertical *translation* of the whole mass.

The second member (as in the corresponding equation for the gyroscope) expresses the *work of gravity*, and the first term of the first member expresses the living force due to the angular motion of the axis. Instead therefore of the work of gravity being expended (as in the gyroscope) *wholly* in producing angular motion, part of it is expended in vertical translation of the centre of gravity. The angular motion takes place not (as in the gyroscope) about the point of support (which in this case is not *fixed*), but about the centre of gravity (to which the moments of inertia $A$ and $B$ refer); and that centre, motionless horizontally, moves vertically up and down, coincident with the small angular undulations of the axis through a space which will be more and more minute as the rotary velocity $n$ is greater.

An elimination of $\frac{d\psi}{dt}$ between the two equations (4) and a study of the resulting equation, would lead us to the same general results, as the similar process, p. 544, for the gyroscope.

The vertical angular motion, expressed by the variation which the angle $\theta$ undergoes, becomes exceedingly minute (the maximum and minimum values of $\theta$ approximating each other) when $n$ is great, and the axis gyrates with slow undulatory motion about a vertical through the centre of gravity. It would be easy, likewise, to show by substituting for $\theta$ another variable, $u=\alpha-\theta$, always (in case of high values of $n$) extremely small, and whose higher powers may therefore be neglected, that the co-ordinates of angular motion, $u$ and $\psi$, approximate more and more nearly to the relation expressed by the equation of the cycloid as $n$ increases; though the approximation is not so rapid as in the gyroscope. All the results and conclusions flowing from the similar process for the gyroscope (see pp. 545, 546, 547, 548) would be deduced. As, however, the centre of gravity, to which these angular motions are referred, is not a *fixed point*, but is itself constantly rising and falling as $\theta$ increases or diminishes, the actual motion of the axis is of a more complicated character.

If $GK''$ (see fig. 2) is the initial position of the axis of the top, the motion of the centre of gravity will consist ina vertical falling and rising through the distance $GG'=GK''(\cos z_2\, G'G''-\cos z_1\, GG'')=\gamma(\cos\theta_1-\cos\alpha)$ (in which $\theta_1$ is the *minimum* value of $\theta$)

while the extremity of the axis or *point*, $K$, describes on the supporting surface and about the projection $G''$ of the centre of gravity, an undulating curve $a$, $b$, $a'$, $b'$, $a''$, &c., having *cusps* $a$, $a'$, &c., in the circle described about $G''$ with the radius $G''K''=\gamma \sin\alpha$, and tangent, externally, to the circle described with a radius $G'' K'=\gamma \sin\theta_1$. But, as in the case of the gyroscope, these little undulations speedily disappear through the retarding influence of friction and resistance of the air, and the point of the top describes a circle, more or less perfect, about $G''$.

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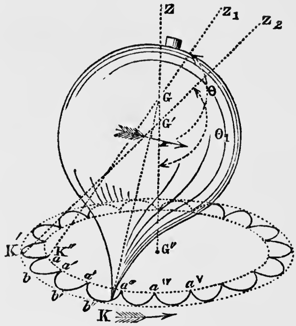


The *rationale* of the self-sustaining power of the top is identical with that of the gyroscope; the *deflecting* force due to the angular motion of the axis plays the same part as the sustaining agent, and has the same analytical expression. Owing to *friction*, the top likewise rises, and soon attains a vertical position; but the agency by which this effect is produced is not exactly the same as for the gyroscope.

If the extremity of the top is rounded, or is not a perfect mathematical point, it will *roll*, by friction, on the supporting surface along the circular track just described. This rolling speedily imparts an angular motion to the axis greater than the horizontal gyration due to gravity, and the deflecting force becomes in excess, (as explained in the case of the gyroscope,) and the axis rises until the top assumes a vertical position. Even though the extremity of the top is a very perfect *point*, yet if it happens to be slightly *out* of the axis of figure (and rotation) the same result will, in a less degree, ensue: for the point, instead of resting *permanently* on the surface, will *strike it*, at each revolution, and in so doing, propel the extremity along. The conditions of a *perfect point*, perfectly centered in the axis of figure, are rarely combined, or rather are *practically impossible;* but it is easy to ascertain by experiment that the more nearly they are fulfilled, and the harder and more highly polished the supporting surface, the less tendency to rise is exhibited; while the great *stiffness* (or tendency to assume a vertical position) of tops with rounded points, is a fact well known and made use of in the construction of these toys.

☞ The references throughout this paper are to my paper on the gyroscope in the June number of the Am. Journal of Education.

# XVIII. EDUCATIONAL MISCELLANY AND INTELLIGENCE.

## ON THE EFFECTS OF INITIAL GYRATORY VELOCITIES, AND OF RETARDING FORCES, ON THE MOTION OF THE GYROSCOPE.

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In one of the concluding paragraphs of my first paper on the Gyroscope (Am. Journal of Education, June, 1857,) I stated that "an initial impulse may be applied to the rotating disk in such a way that the horizontal motion shall be absolutely without undulation. An initial angular velocity such as would make its corresponding deflective force equal to the component of gravity $g \sin \theta$, would cause a horizontal motion *without* undulation."

The statement contained in the last sentence quoted, is not rigidly true; for *besides* the component of gravity, there is another force to be considered, viz., the centrifugal force due to the gyratory velocity, which acts either in conjunction with, or in opposition to, the component of gravity, according as the axis of the disk is above or below a horizontal.

In this last position this force is null (as regards its effects in sustaining or depressing the axis), and to *this* angular elevation of the axis the statement quoted is true without qualification. The assumption of an initial horizontal velocity requires only a new determination of constants for equations ($a$) and ($c$) (pp. 541, 542, June No.).

If we make, in those equations

$$\theta = \alpha,\ \varphi = 90°,\ \psi = 90°,\ a = -\sin\alpha,\ v_x = m,\ v_y = 0,\ v_z = n,$$

(in which $m$ is the assumed initial velocity) and determine the constants $h$ and $l$ therefrom, the equations of motion will become

$$\left.\begin{aligned} \sin^2\theta\frac{d\psi}{dt} &= \frac{Cn}{A}(\cos\theta - \cos\alpha) + m\sin\alpha \\ \sin^2\theta\frac{d\psi^2}{dt^2} + \frac{d\theta^2}{dt^2} &= \frac{2Mg\gamma}{A}(\cos\theta - \cos\alpha) + m^2 \end{aligned}\right\} \quad (1)$$

and from them we get

$$\sin^2\theta\frac{d\theta^2}{dt^2} = \left[\frac{2Mg\gamma}{A}\sin^2\theta - \frac{2Cmn}{A}\sin\alpha - \frac{C^2n^2}{A^2}(\cos\theta - \cos\alpha) \right.$$
$$\left. - m^2(\cos\theta + \cos\alpha)\right](\cos\theta - \cos\alpha) \quad (2)$$

From this we get $\frac{d\theta}{dt} = 0$ when $\cos\theta - \cos\alpha = 0$; and as $\frac{d\psi}{dt}$ is not zero for this initial elevation, it indicates, instead of a cusp, a tangency to the horizontal here.

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* This paper is intended to give a more rigidly mathematical demonstration of the effects of "retarding forces" than is given in (December No. p. 529,) of this Journal; and to give the theory of the "motions" of the Gyroscope a more general form, by the introduction of "Initial Gyratory Velocities."

If the curve described is horizontal without undulation, the other factor of the second member of eq. (2) should likewise become zero with $\theta = \alpha$: an effect which may ensue from a suitable value given to $m$.

The value of the deflecting force due to a given angular velocity $m$ is (p. 552, June number) $\frac{C}{\gamma M} m n$, and if we suppose this equal to the component of gravity $g \sin \alpha$, we shall have $m = \frac{M g \gamma}{C n} \sin \alpha$.

If we substitute this value of $m$ in the second member of equation (2) and assume $\alpha = 90°$ the factor in question becomes zero for $\theta = \alpha$, and the maximum and minimum values of $\theta$ are the same, indicating a horizontal motion without undulation.

For every other initial elevation than 90° a different value of $m$ is required to produce this result, in consequence of the influence of the centrifugal force of gyration at other elevations.

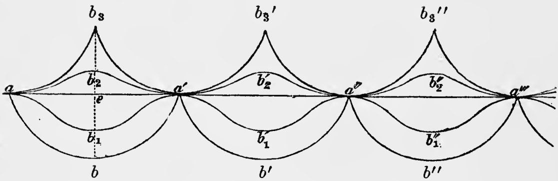
With $\alpha = 90°$, equation (2) becomes

$$\sin^2 \theta \frac{d \theta^2}{d t^2} = \left[ \frac{2 M g \gamma}{A} \sin^2 \theta - \frac{2 C m n}{A} - \frac{C^2 n^2}{A^2} \cos \theta - m^2 \cos \theta \right] \cos \theta \quad (3)$$

Placing the first factor of the second member equal to zero and solving with reference to $\cos \theta$ we get (recollecting the value given to $\beta$ in our former article)

$$\cos \theta = - \beta^2 - \frac{A m^2}{4 M g \gamma} + \sqrt{\left( \beta^2 + \frac{A m^2}{4 M g \gamma} \right)^2 + 1 - \frac{C m n}{M g \gamma}}. \quad (4)$$

1.




For $m = 0$, equation (3) expresses the cycloidal curve with cusps $a$, $a'$, $a''$, &c., as has been already shown in our former investigation. For $m > 0$ but $< \frac{M g \gamma}{C n}$ the minimum value of $\theta$ derived from equation (4) is greater than when $m$ is zero, while instead of a cusp (there is as has already been observed) a tangency at $a$, and the curve has the wave form $a b_1 \; a' b'_1$ (the points $b_1 \; b_1' \; b_1''$, &c. being higher than $b \, b' \, b''$).*

When $m = \frac{M g \gamma}{C n}$ the curve unites with the horizontal $a \, a' \, a'' \, a'''$ and there is no undulation; equation (4) giving $\cos \theta = 0$, or $\theta = 90°$.

* In reality, the *amplitudes*, $a \, a'$, $a' \, a''$, of the undulations become increased, at the same time that the sagittæ are diminished, but, for the sake of comparison, I have represented them the same for each variety of curve.

When $m > \frac{Mg\gamma}{Cn}$, $\frac{d\theta}{dt}$ becomes still zero with $\theta = \alpha = 90°$; but this instead of a maximum is now a *minimum* value of $\theta$, for the value of $\theta$ which satisfies equation (4) is greater than 90°, and the curve $a b_2 \ a' b_2'$, &c., undulates *above* the plane $a a' a''$.

Finally when $m = \frac{2\,Mg\gamma}{Cn}$, equation (4) will give $\cos\theta = -\frac{1}{2\beta^2}$ and a substitution of this in the first equation (1) (making $\alpha = 90°$), will give $\frac{d\psi}{dt} = 0$: showing that the curve makes cusps at its superior culminations, and that the common cycloidal motion is resumed. In fact the value of $\frac{d\psi}{dt} = \frac{1}{\beta}\sqrt{\frac{g}{\lambda}}$ (p. 547, June number) at the *lowest* point $b$ of the cycloid, is, (substituting the values of $\beta$ and $\lambda$) exactly equal to $\frac{2\,Mg\gamma}{Cn}$, and the value of the sagitta $u$ corresponding to $e b$ is what we have just found for $\cos\theta$, or $e b_3$, viz. $\frac{1}{2\beta^2}$.

If now, retaining $m$ constant at this value to which we have brought it, we increase the rotary velocity, $n$, or vice versa, a curve *with loops*, (fig. 2,) may be described, as it can be shown that, for the maximum value of $\theta$, $\frac{d\psi}{dt}$ becomes negative.*

2.

In my supplementary paper in the December number of this Journal I have endeavored to show how the theoretical cycloidal motion of a simple solid of revolution is modified by the retarding forces of friction and the resistance of the air, and that the theory explains all the phenomena observed in the ordinary gyroscope.

It may be objected however that the nature of the curve given in Fig. 1, (p. 531,) is in some degree *assumed*, and I therefore wish to show that it can be confirmed by mathematical demonstration.

The rotary velocity $n$ of the disk is supposed to be gradually destroyed through the retarding forces of friction at the extremities of the axle, and of the resistance of the air at the surface.

Without attempting to give analytical expressions for the retarding forces, it is sufficient to say that the rotary velocity, at the end of any

*If $m$ is made *negative* and small (i. e., a *backward* initial velocity given) a looped curve like the above, but lying *below* the plane $a a' a''$, results. All these curves ($n$ being always supposed very great) are but the different forms of the "cycloid" known as *prolate, common,* and *curtate* cycloids; the common—a *particular* case of the curve—corresponding to the *particular* case of the problem in which the initial gyratory velocity is either zero or has the *particular value* $\frac{2\,Mg\gamma}{Cn}$.

time $t$, counting from the commencement of motion, may be expressed thus

$$n-f(t)*$$

in which $n$ is the *initial* rotary velocity of the disk.

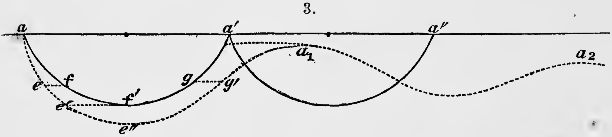
If we substitute this expression for $v_z$ in the last two equations (3) (p. 541, June No.,) and follow a similar process to that by which equations (4) of that paper are deduced, we shall get, for the equations of motion

$$\left.\begin{array}{l} \sin^2\theta\dfrac{d\psi}{dt}=\dfrac{Cn}{A}(\cos\theta-\cos\alpha)-\dfrac{C}{A}\displaystyle\int_o^t f(t)\,d\,.\cos\theta \\ \sin^2\theta\dfrac{d\psi^2}{dt^2}+\dfrac{d\theta^2}{dt^2}=\dfrac{2Mg\gamma}{A}(\cos\theta-\cos\alpha) \end{array}\right\} \quad (5)$$

For the sake of simplicity suppose the initial position of the axis be horizontal, or $\alpha=90$ and the above become

$$\left.\begin{array}{l} \sin^2\theta\dfrac{d\psi}{dt}=\dfrac{Cn}{A}\cos\theta-\dfrac{C}{A}\displaystyle\int_o^t f(t)\,d\,.\cos\theta \\ \sin^2\theta\dfrac{d\psi^2}{dt}+\dfrac{d\theta^2}{dt^2}=\dfrac{2Mg\gamma}{A}\cos\theta \end{array}\right\} \quad (6)$$

3.




If $aff'a'$ represents the cycloidal curve, and $aee'e''g'$ the curve in question, it will be observed that the angular velocity of the axis given by the 2nd equation (6) is the same for both, for equal values of $\theta$, while the value of *the horizontal component* of that velocity, $\sin\theta\dfrac{d\psi}{dt}$, is less than for the cycloidal curve, by the term $\dfrac{C}{A\sin\theta}\displaystyle\int_o^t f(t)\,d\,.\cos\theta$.

As $\theta$ diminishes, $d\cos\theta$ is positive and this term is subtractive and hence for any point $e$ or $e'$ on the descending branch, $\dfrac{d\psi}{dt}$ is less than for the corresponding point $f$ or $f'$ of the cycloid, and the branch $aee'e''$ will be *behind* the branch $aff'$, and will descend lower.

At $e''$ the term $\dfrac{C}{A\sin\theta}\displaystyle\int_o^t f(t)\,d.\cos\theta$, attains its *maximum*, for as the curve ascends, $\theta$ increases, and the increments of $\cos\theta$ become negative.

* When the retarding force is independent of the velocity, as in the case of friction, the $f(t)$ in the above expression is linear; when this force is dependent upon the velocity, as for the resistance of the air, $f(t)$ will, in general, be an infinite and diverging series in the powers of $t$; whether the force is due to either, or both combined, of these causes, the above expression for the velocity of rotation may however be used for the present purpose.

But as the values of $t$ on this branch of the curve are nearly double those or equal values of $\theta$ of the descending one, the integral $\int_0^t f(t)\,d.\cos\theta$ will become zero at some point $g'$, *before* $\theta$ has regained its initial value, at which point $\frac{d\psi}{dt}$ will be the same as for the corresponding point $g$ of the cycloid. Above the point $g'$ the term $\frac{C}{A\sin\theta}\int_0 f(t)\,d.\cos\theta$ becomes negative and (with its negative sign) becomes additive and therefore, above $g'$ the values of $\frac{d\psi}{dt}$ are always *greater* than for corresponding points of the cycloid. Hence the angular velocity of the axis can never become zero and consequently the axis cannot rise to its initial elevation and form a cusp, but must make an inflexion and culminate at $a$, *below* the initial elevation.

Commencing a second descent from $a'$ with an *initial velocity*, the succeeding wave will be *flattened* (as shown in treating the subject of "initial gyratory velocities"), the second culmination $a_2$ will not (as a similar train of reasoning to that just gone through for the first undulation proves) be as high as $a_1$: and *pari ratione*, each succeeding wave will be more flattened and extended than the preceding, until they soon virtually disappear, and the curve becomes a descending helix.

After these undulations have disappeared, as the descent is only due to loss of rotary velocity (and consequently loss of *deflecting force*) measured by $f(t)$, it is evident that the future character of the helix will be determined by this function.

In fact, as the descending velocity $\frac{d\theta}{dt}$ is then very minute compared with the horizontal velocity $\frac{d\psi}{dt}$, its square may be neglected in the 2nd equat., (6); and, equating the values of $\sin\theta\frac{d\psi}{dt}$ deduced from these two equations, we shall have

$$\frac{C}{A}\int f(t)\,d.\cos\theta = \frac{Cn}{A}\cos\theta - \sin\theta\sqrt{\frac{2Mg\gamma}{A}\cos\theta}.$$

By differentiating both members and making various reductions we get

$$\sqrt{\frac{Mg\gamma}{A}}\cdot\frac{3\sin^2\theta-2}{\sqrt{\sin\theta\sin 2\theta}} = \frac{C}{A}(n-f(t))$$

an equation which, *after the disappearance of the undulations*, gives the value of $\theta$ in terms of $t$.

As $f(t)$ increases $\theta$ diminishes in the first member, to the limit corresponding to $\sin^2\theta = \frac{2}{3}$ which makes the numerator of the fraction in the first member 0, and the denominator a maximum; showing, to that limit, a constant descent of the axis, or a descending helix for the curve.

As the values of $f(t)$ beyond $f(t) = n$ do not belong to the question, there can be no farther descent below that value of $\theta$ which reduces the first member to zero; or beyond $\sin^2\theta = \frac{2}{3}$.

At this elevation, as the *deflecting force* has vanished entirely with the rotary velocity, it is evident the elevation of the axis must be maintained by the *centrifugal force alone*, due to the gyratory velocity.

In fact, if we calculate directly the angle to which the axis must fall from a horizontal position, in order that the velocity generated shall be just sufficient, if deflected into horizontal gyration, to exert a centrifugal force adequate to maintain it, we shall find this same value, $\sin^2\theta = \frac{2}{3}$.*

In reality, the air resists gyration as well as rotation, and hence the descent will continue; but if a gyroscope could be placed in a *perfect vacuum*, and the slight friction at the point of support be entirely annulled, the axis would descend in a helix until it reached this limit, at which it would forever gyrate, though the rotation of the disk would soon by friction of the axle, entirely cease.

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* If the solid of revolution is of dimensions so small that it may be considered concentrated in its centre of gravity, it would require, in the fall of its axis through angle $90°-\theta$, the velocity $\sqrt{2g\gamma\cos\theta}$; and this velocity, deflected into horizontal gyration in a circle whose radius is $\gamma\sin\theta$, would create a centrifugal force $2g\frac{\cos\theta}{\sin\theta}$, whose component normal to the axis of figure is $2g\frac{\cos^2\theta}{\sin\theta}$. Equating to this the opposing component of gravity $g\sin\theta$, we get $\sin^2\theta = \frac{2}{3}$, as in the text.

For finite dimensions of the solid, the direct determination of the limit in question, is more complicated, and it is scarcely necessary to introduce it here.

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