Philosophy of
Arithmetic
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John Malcolm.

THE

PHILOSOPHY

OF

ARITHMETIC,

(Considered as a Branch of Mathematical Science)

AND THE

ELEMENTS OF ALGEBRA:

DESIGNED FOR THE USE OF SCHOOLS,

AND IN AID OF PRIVATE INSTRUCTION.

BY JOHN WALKER:

FORMERLY FELLOW OF DUBLIN COLLEGE.

" *Would you have a man reason well, you must use him to it betimes, and*
" *exercise his mind in observing the connexion of ideas, and following them in*
" *train. Nothing does this better than Mathematics; which therefore, I think,*
" *should be taught all those who have time and opportunity: not so much to make*
" *them Mathematicians, as to make them reasonable creatures.*"

LOCKE's Conduct of the Understanding.

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1812.

☞ Mr. WALKER *gives private instructions to individuals, or parties of six persons, of either sex, in the Subjects of this Treatise, in the Elements of Geometry, in Astronomy, and the other Mathematical Branches of Natural Philosophy; as well as in the Greek and Latin Classics.*

No. 73,
Lower Dorset-Street.

TO

MRS. *AGNES CLEGHORN*,

As to a Lady who is well qualified

To estimate the Execution of the following Work;

AND WHO EVIDENCES, BY HER EXAMPLE,

That superior intellectual Endowments,

Improved by more than ordinary Acquirements,

IN LITERATURE AND SCIENCE,

Are perfectly consistent

With the Retiredness of the Female Character,

With its attractive Graces,

AND WITH THE MOST EXEMPLARY DISCHARGE

OF

DOMESTIC DUTIES;

This Treatise is respectfully inscribed

By her faithful

And much obliged

Servant

THE AUTHOR.

PREFACE.

ARITHMETIC is one of the two great branches of MATHEMATICS; and, when scientifically treated, needs not fear a comparison with her more favoured sister, GEOMETRY, either in precision of ideas, in clearness and certainty of demonstration, in practical utility, or in the beautiful deduction of the most interesting truths.

In the order of instruction, ARITHMETIC ought to take precedence of GEOMETRY; and has, I conceive, a more necessary connection with it, than some are willing to allow. "Number," as Mr. Locke remarks, "is that which the mind makes use of, in measuring "all things that by us are measurable." And I question whether the doctrine of *ratio* in Geometry has not been needlessly obscured, by a vain attempt to divest it of numerical considerations. Upon this subject I have elsewhere expressed my views more at large.

But

But as generally taught, ARITHMETIC has been degraded from the rank of SCIENCE, and converted into an art almost mechanical; useful indeed in the compting-house, but affording more exercise to the fingers than to the understanding. It is commonly taught by persons, who are rather expert *Clerks* than men of *Science*, and are themselves strangers to the rational principles of the most common operations which they perform. The absurd questions current among them about *the product of money multiplied by money*, &c. afford a sufficient exemplification of this remark. Thus, while there are few things which children are more generally taught, than the technical art of calculation, perhaps there are few things of which men are more generally ignorant, than the *Science* of ARITHMETIC: and this ignorance indeed is betrayed by their contempt of it, as a branch of study beneath a scholar.

Yet, when rationally taught, it affords perhaps to the youthful mind the most advantageous exercise of its reasoning powers, and that for which the human intellect becomes most early ripe: while the more advanced parts of the science may try the energies of an understanding the most vigorous and mature.—Reduced also to a few comprehensive principles, and divested of that needless multiplicity of various *Rules*, by which the subject is commonly perplexed,—the knowledge of it may be communicated with unspeakably

ably greater facility and expedition; and, when once attained, will not be liable (as at present) to be soon forgotten.

To present ARITHMETIC in that scientific form, is the object of the following treatise; which, it is hoped, may prove beneficial to the young of both sexes, and not uninteresting to some of more advanced age.

The scientific principles of common ARITHMETIC are so coincident with those of ALGEBRA, (or *Universal Arithmetic*) that—to persons acquainted with the former—the Elements of the latter offer no serious difficulty. Of the *Elements* of ALGEBRA therefore I have given such a view, as may open that wide field of science to the Student, and enable him at his pleasure to extend his progress, by the aid of any of the larger works extant on the subject.

Having designed this work for the instruction of those, who come to it most uninitiated in Science, I have aimed at giving a clear and full explanation of the most elementary principles: I have endeavoured to be familiar and plain, yet without departing from the rigidness of demonstration. How far I have succeeded in this attempt, other judges must decide. I shall think myself compensated for my labour, if it should prove, in any degree, the occasion of rescuing the SCIENCE of ARITHMETIC from general neglect, and

of

of introducing this branch of MATHEMATICS into the system of liberal education.

Mr. Locke's remark, which I have prefixed as a motto to this treatise, is well worthy of attention. It is not so much the intrinsic dignity of *Mathematical Science*, —nor even its extensive application to the most important purposes of civil society,—that recommends it as an object of *general* study. In its influence on the mental character and habits, it possesses a still stronger claim for adoption into the course of general education. No study so much, as that of MATHEMATICS, contributes to correct precipitancy of judgment; to promote patience of investigation, clearness of conception, and accuracy of reasoning; to communicate the power of fixed attention, and closeness of thinking.

These are habits universally important; and to be formed in early life. Nor is it necessary, in order to derive these benefits from mathematical studies, that we should pursue the study to any great length, or become profound Mathematicians. Here it is of much less consequence how far we proceed, than that we make ourselves fully masters of the ground—as far as we proceed;—that whatever we learn, whether little or much, should be learned thoroughly. A smattering of half information about a variety of subjects, is calculated to excite that vanity and presumption

of

of knowledge, which is repressed by a radical acquaintance with the most elementary principles of some one science.

May I be allowed to express my opinion, that some degree of *mathematical* knowledge is no less useful to females, than to the other sex; and importantly adapted to counteract the tendency of an education, which too often enfeebles the judgment, while it excites the imagination? Indeed it is with satisfaction that I perceive that absurd and illiberal prejudice rapidly giving way, which would shut the door of solid information against those, on the formation of whose minds so much of the welfare of society must depend.

In bringing this Volume through the press, I have encountered difficulties, which might not be expected to occur in a City—the metropolis of IRELAND, and the seat of a learned UNIVERSITY. Some of those difficulties have been such, as necessarily make the price of the work higher, than is generally affixed to Volumes of an equal bulk:—though it may be remarked that, if a little more of the modern ART of printing had been employed, the Volume might easily have been swelled to twice its present size, without any increase of the matter.

Notwithstanding

Notwithstanding much pains bestowed on the correction of the press, I have to intreat the indulgence of the reader for the following errors; some of which escaped my eye, and others have been generated after the passages had undergone my last revision.

ERRATA.

Page
8. line 21. for "difference between 28 and 5," *read*, "difference between 23 and 5."
14. l. 19. *read*, "engage"
15. l. 23. for "3681 and 108," *read*, "3681 and 1080"
16. l. 15. for "§ 61 and 62," *read* "§ 62 and 63."
22. l. 34. *read*, "exercise."
34. l. 16. for "5 times 9 is 6," *read*, "5 times 9 is to 6."
12 35. l. 27. for "is equal to a," *read*, "is equal to c."
5 47. l. 9. *read*, "whose sum is $\frac{62}{45}$ or $1\frac{17}{45}$."
77. Ex. 9. The last term should be "$3ax^2$."
81. last line. for "$-3axy$," *read* "$-3axy^2$."
83. l. 2. for "xy^2-y^2" *read* "xy^2-y^3". ibid. l. 4. for "y^3", *read*, "y^2." ibid. l. 11. for "xy," *read*, "xy^2."
87. l. 25. for "$\frac{a}{m}=d$, and $\frac{b}{m}=c$," *read*, "$\frac{a}{m}=c$, and $\frac{b}{m}=d$." ibid. l. 28. for "$a=dm$, and $b=cm$," *read*, "$a=cm$, and $b=dm$."
88. l. 4. for "da or cb," *read*, "ca or db." ibid. l. 22. for "last," *read*, "least."
90. l. 18. and 20. for "100th." read, "1000th."
105. l. 8. from bottom. for "$\frac{3}{6}$," *read* "$\frac{6}{3}$."

119.

ERRATA.

Page

119. l. 2. for "$\frac{5}{2x}$", *read*, "$\frac{2x}{5}$."

155. l. 2. from bottom. for "$\overline{4}$," *read*, "$\frac{9}{4}$."

156. l. 25. for "57," *read*," "27."

157. l. 5. from bottom, and last line. for "£60," *read*, "£100."

162. l. 11. for "§ 284," *read*, "§ 290."

165. l. 18. for "4056," *read*, "7056."

168. l. 2. from bottom. After "+7." add, "+6+5+4."

THE *PHILOSOPHY* OF ARITHMETIC, &c.

CHAP. I.

Nature and Principles of the Arabic Numeral Notation. Its Advantages above the Greek and Roman. Insensibility to the Magnitude of high Numbers. Duodecimal Notation.

1. THE first thing in the subject of this treatise, which claims our attention, is our present method of numeral notation; or the method employed for designating numbers by the aid of written characters. For it, as well as some other most important improvements in Arithmetic, we are indebted to the Arabs. It was brought by the Moors into Spain; and John of Basingstoke, archdeacon of Leicester, is supposed to have introduced it into England about the middle of the 11th century. It is one of those inventions, of which we often enjoy the advantages, without duly estimating their importance. Simple, ingenious, and highly useful, it is yet so familiar to us from our childhood, that it fails of engaging our attention, or exciting our admiration.

2. We may be impressed however with a conviction of its ingenious simplicity, if we reflect on the endless varieties and indefinite magnitude of numbers; and then observe that we are enabled, by the aid of only ten characters

racters (the nine significant figures and the cypher) to designate any numbers whatsoever with the utmost facility and distinctness; and this, in a form which subjects them most conveniently to arithmetical computation. The important utility of the contrivance it may be sufficient for the present to illustrate by the following remark. Most children of a very young age can with ease multiply or divide the number 67,489 by the number 508. But let the same numbers be expressed by the Roman method of notation, by letters which prevailed in Europe before the introduction of the Arabic, thus—$\overline{\text{lxvii}}$.cccclxxxix and dviii;—a man will be puzzled to perform either operation. The Greeks employed a numeral notation similar to the Roman: and it is truly wonderful how their mathematicians (even with the aid of some mechanical contrivances) surmounted the difficulties, which they had to encounter in their arithmetical calculations; while we know that they were engaged in some of a very long and complicated nature.

3. Yet when we examine the fundamental principle of the Arabic notation, it becomes a matter of surprise that the invention was not of earlier discovery: for it proceeds on a principle extremely simple, and one that must have been employed in all ages, whenever there was a practical occasion of counting any very large number. We may illustrate the principle by supposing that we had to count a great heap of guineas. It is plain that unless we employ some check on our numeration, we shall be very apt to lose our reckoning, and get astray as we advance. What then is the most obvious method of securing accuracy in our reckoning? Is it not to count by tens, or some fixed number, beyond which we never shall proceed? Thus when we have reckoned ten guineas, we may lay them aside in one parcel; and proceed to count another parcel of ten. But to prevent the number of these parcels from accumulating so as to lead us astray, whenever we have counted ten such parcels we may make them up into a rouleau, containing therefore ten times ten guineas, or one hundred: and whenever we have ten such rouleaus, we may combine them into one set, consisting of ten hundred, or a thousand, guineas: and so on. And by this simple contrivance it would never be necessary to reckon beyond the number ten. Now it is precisely upon this principle that we proceed in designating numbers by the

Arabic

Arabic notation. The several columns of figures, from the right hand column, are the compartments in which we dispose the several combinations of ten. The first column on the right hand is the place for all odd units, below ten: the next to it on the left hand, or second column, is the place for all parcels of ten, below ten such parcels: the third column, for all parcels of a hundred (or ten times ten) below ten: the fourth, for all parcels of a thousand (or ten hundred) below ten: the fifth, for all parcels of ten thousand below ten: the sixth, for all parcels of a hundred thousand (or ten times ten thousand) below ten: the seventh, for all parcels of ten hundred thousand (or a million) below ten, &c.

4. Thus by the help of the nine significant figures and the cypher we are able to designate all numbers however great; and this, while each of the figures (called the ten *digits*, from the Latin word signifying *a finger*) always retains the same numeral significancy. For example, in the two numbers 57 and 570, the character 5 denotes in each the number five, and the character 7 the number seven: but in the former the 5 standing in the second column designates five parcels of ten each, or fifty; but in the latter, where it stands in the third column, it designates five parcels of a hundred each, or five hundred: and in the former, the 7 standing in the right hand column designates seven units; but in the latter, standing in the second column, designates seven tens, or seventy. And thus we see that the cypher, though it denote that there is *no* number belonging to its column, yet must be written; in order to bring the significant figures into their proper places. If therefore I want to express the number *four million and sixty-eight thousand and fifty-three;* the seventh column being the place of millions, the character 4 must be followed by six figures; and the fourth column being the place of thousands, the characters 68 must be followed by three figures: and thence I conclude that besides the significant figures 4, 68, and 53, a cypher must be interposed between the two latter, and another cypher between the two former: thus—4068053.

5. To facilitate numeration, we commonly mark off by a comma every period of six figures, commencing from the right hand, and often semi-periods of three figures. And as the name of a *million* is given to ten hundred thou-

sand, so ten hundred thousand millions are called a *billion*; the place of which therefore commences at the thirteenth column. In like manner the names of a *trillion*, *quadrillion*, &c. are given to ten hundred thousand billions, trillions, &c.

6. But here it is to be observed, that the facility with which we can designate the highest numbers, and perform every arithmetical calculation on them, has occasioned an insensibility to the enormous magnitude of the numbers of which we speak. One billion is very easily mentioned, and easily designated by an unit followed by twelve cyphers: thus—1,000000,000000. A child also can multiply or divide that number. But perhaps the reader will be surprised at the statement that there is not one billion of seconds in thirty thousand years: though there be 60 seconds in every minute, 60 minutes in every hour, 24 hours in every day, and in a solar year 365 days 5 hours 48 minutes and about 48 seconds. At that calculation, the precise number of seconds in 30,000 years is only 946707,840000; or above 50 thousand millions less than one billion. So that the number of seconds, which have passed since the creation of the world, is considerably less than the fifth part of one billion. In fact it is only by some such considerations that we can form any conception of numbers so immense.

7. From the view we have taken of the Arabic notation, it is plain that a cypher, wherever it occurs, increases tenfold the value of every figure standing on its left hand; but does not affect the value of the figures standing on its right hand. It appears also that the several columns may be conceived to be headed with their respective titles, as *parcels* of a thousand each, of a hundred, of tens, &c.

8. If the reader revert to the illustration adduced in § 3. he may observe that, instead of counting the heap of guineas by tens and combinations of tens, we might as well count by twelves and combinations of twelves; or by any other fixed number sufficiently low. And to the numeration by twelves, for instance, a notation similar to the Arabic may be applied, only introducing two new characters to designate the numbers *ten* and *eleven*. Then the figures 10 would denote the number *twelve*; for the 1, standing in the second column, would denote one parcel of twelve: and the figures 203 would denote the number two hundred and ninety-one; for the 2, standing in the third column, would

would denote two parcels of twelve times twelve each, that is, two hundred and eighty-eight. and the units being added mak

9. And certainly if this *duodecimal* notation had been originally adopted, and the language accommodated to it by affording distinct names for the several combinations of twelve, it would have possessed a considerable advantage above the *decimal* notation, which proceeds by combinations of ten. For the number twelve admitting four divisors, (namely 2, 3, 4, 6) while the number ten can be evenly divided only by 2 and 5, we should be much less frequently involved in fractional remainders than at present. And if all the divisions of measures, weights, coins, &c. ran in the same duodecimal progression, the practical advantages would be very great.

10. But it appears from the structure of all known languages that numeration by *tens* has been adopted by all nations in all ages, rather than numeration by *twelves*, or any other number. And this is obviously to be accounted for from the natural circumstance of the number of our fingers; the fingers being in the origin of society the readiest instrument to assist numeration, and still indeed frequently employed for that purpose by the rude peasantry. So that we may conclude that if nature had furnished men with twelve fingers instead of ten, the duodecimal numeration would have been as general, as the decimal now is; and languages would have abounded as much with names for the combinations of twelve, as they now do with names for the combinations of ten.

11. Observe that any two or more successive digits of a number may be considered as a number of the same denomination with the last of these digits Thus in the number 2345, the digits 34 may be considered as 34 tens; the digits 23 as 23 hundreds, &c.

CHAP. II.

Addition and Subtraction. Reason of proceeding from Right to Left. Methods of Proof. Examples for Practice. Signs +, −, =.

12. ON *addition* and *subtraction* little need be said. They are the two fundamental operations of Arithmetic, into which all others may be resolved. For whatever arithmetical

metical operation we perform, the change made on the given number must be either an increase or diminution of it, that is, an addition to it or subtraction from it. And accordingly we shall find that multiplication and division are but abridged methods of addition and subtraction.

13. In addition we want to find the total amount of ~~several~~ two or more given numbers; in subtraction, to find the difference between two given numbers, or the number remaining after taking the less from the greater. To perform either operation, it is necessary that the learner should be able to assign the sum of any given number and another not exceeding nine, or the difference between them.

14. In addition we successively take the sum of the digits standing in each column, and combine those sums into one total. The reason of commencing from the right hand column, or place of units, and proceeding from right to left, is that we may carry on the combination of the sums of the several columns as we proceed. Thus, in adding together 509 and 293, the sums of the numbers standing in the several columns are 12 units, 9 tens (or 90) and 7 hundreds, or 700. Now adding the one ten contained in the 12 units to the 9 tens (the sum of the second column) we have 10 tens, or 1 hundred; which added to the 7 hundreds (the sum of the third column) gives 8 hundreds; and these combined with the 2 units in the sum of the first column give 802 as the total. By proceeding from right to left, we are saved the trouble of writing the sums of the several columns separately, and afterwards combining them by a second addition. We write down under each column the right hand figure of its sum, and *carry* the other figures to the next column. But the same result will be obtained by repeated additions proceeding from left to right, or taking the sums of the columns in any order. And in this way the young scholar may advantageously be made to prove his work.

15. In arranging the numbers which we want to add it is obviously needful, that the digits of the corresponding columns of each number should be disposed in line exactly under each other: as it is necessary, in adding pounds, shillings and pence, to avoid placing a number denoting pence in the column appropriated to the numbers denoting shillings. And the scholar ought to be exercised in the due arrangement of the numbers for himself, and not have them given him arranged by the teacher.

16. In

16. In subtraction, the number which is to be subtracted from the other is called the *subtrahend*; the number from which the subtraction is to be made, the *minuend*. If we have to subtract 346 from 579, it is plain that we may subtract the units tens and hundreds of the subtrahend successively from the units tens and hundreds of the minuend; and that the sum of the remainders 233 is the remainder
when each of the figures of the subtrahend are less than the corresponding figur min
sought. And in such a case it matters not whether we proceed from left to right, or from right to left. But if any digit of the minuend be less than the digit in the corresponding column of the subtrahend, for instance if we have to subtract 279 from 546, as we cannot subtract 9 units from 6 units, nor 7 tens from 4 tens, we may suppose the minuend resolved into the parts 16, 130, and 400: and then subtract the 9 units from 16; the 7 tens from 13 tens; and the two hundreds from 4 hundreds. And thus when any digit of the minuend is less than the corresponding digit of the subtrahend, conceiving a unit prefixed to it and performing the subtraction, when we proceed to the next column we have to conceive the next digit of the minuend less by 1, on account of the one which has been already borrowed from it. But it affords the same result in practice, to conceive the next digit of the subtrahend increased by one, and the digit of the ~~subtrahend~~ minuend unaltered: as it obviously gives the same remainder to subtract 8 from 14, as to subtract 7 from 13. And hence appears the reason of what is called the *carriage* in subtraction; and the reason of proceeding from right to left: though the same result may be obtained by repeated subtractions proceeding from left to right. The carriage in subtraction may be accounted for on another principle, namely, that if the two numbers be equally increased, their difference will remain unvaried. Thus, in subtracting 19 from 56, when we take 9 from 16, we may conceive that we have added 10 to the minuend, and therefore must add 10 also to the subtrahend.

17. Besides the same attention to the arrangement of the numbers as is necessary in addition, the scholar ought to be exercised in performing the operation of subtraction whether the subtrahend be above or below the minuend.

18. The remainder found being the difference between the given numbers, or the number by which the minuend exceeds the subtrahend, it is plain that adding the remainder

mainder to the subtrahend must give a total equal to the minuend: or that subtracting the remainder from the minuend must give a remainder equal to the subtrahend. This affords two methods of proving subtraction. And in addition if we subtract any one of the numbers from the total, the remainder must be equal to the sum of all the other numbers.

19. The sign + interposed between two numbers denotes that the numbers are to be added: the sign — interposed between two numbers denotes that the latter is to be subtracted from the former. These signs are technically called *plus* and *minus*, from the two Latin words signifying more and less. Thus 23+5 (read 23 *plus* 5) denotes the sum of 23 and 5. And 23—5 (read 23 *minus* 5) denotes the remainder subtracting 5 from 23. The sign = interposed between any two numbers or sets of numbers denotes an *equality* between the number or set of numbers on the one side and on the other side of that sign: and such a statement is called an *equation*. Thus 23+5=28, and 23—5=18 are equations, denoting that the sum of 23 and 5 is equal to 28, and that the difference between 23 and 5 is equal to 18.

20. We shall have such frequent occasion for these signs and terms, that the young Arithmetician cannot too soon become familiar with them. A little patient explanation and illustration will soon make a child as familiar with them, as with the Arabic characters: and it is ridiculous to think how many have been deterred from attempting the study of Algebra, by the mere formidable appearance of its out-works, a number of strange symbols and terms, which they do not understand. But every thing the most simple is obscure till it is understood; and every term is alike unintelligible, till its meaning is explained.

21. In the following questions for exercise in addition and subtraction, the sum or difference of the numbers is to be supplied by the scholar after the sign of equality.

1 Ex. 5209+726+30874=

2 Ex. 5,678093+23,456789+908+4321+86=

Let the answers to these examples be proved by subtracting the numbers successively from the total; or by subtracting any one or more of them from the total, and comparing the remainder with the sum of the rest; or by adding two or more of the given numbers separately, and

their

their sum to the rest; or by repeated additions of the digits in the several columns proceeding from left to right.

3 Ex. 3456—508 = 4 Ex. 987654—109345 =

Let the answers to these examples be proved by adding the subtrahend to the remainder; and by subtracting the remainder from the minuend.

5 Ex. A man has five apple trees, of which the first bears 157 apples, the second 264, the third 305, the fourth 97, and the fifth 123. He sells 428 apples; 186 are stolen. How many has he left for his own use?

6 Ex. Out of an army of 57,068 men, 9503 are killed in battle; 586 desert to the enemy; 4794 are taken prisoners; 1234 die of their wounds on the passage home; 850 are drowned. How many return alive to their own country?

7 Ex. A man travelling from London to Edinburgh went the first day 87 miles, the second day 94 miles, the third day 115 miles, and going the fourth day 86 miles he was within 12 miles of Edinburgh. What is the distance between London and Edinburgh; and how far from the latter town was the traveller at the end of the third day?

8 Ex. A man at the beginning of the year finds himself worth £123,078. In the course of the year he gains by trade £8706; but spends in January £237, in February £301, and in each succeeding month as much as in the two preceding. What was the state of his affairs at the end of the year?

Chronology will furnish the teacher with an indefinite variety of examples. But it is to be observed in general, that pains should be taken to give the child a clear conception of the terms employed in a question, before he is called to solve it: and that the first illustrations of the use of Arithmetical rules should be borrowed from the objects with which the child is most familiar, and proposed in low numbers. The great advantage of an early application to Arithmetic is the exercise which it affords to the thinking faculty. And when a child is taught practically how to solve a question, the meaning of which he does not clearly understand, instead of any benefit accruing, a mental habit the most injurious is contracted, of resting in indistinct conceptions, and mistaking sounds or signs for knowledge. Here patience and judgment in the teacher are especially needful.

CHAP.

CHAP. III.

Nature and Principles of Multiplication. Sign ×. *Methods of Proof. Abbreviated Methods. Powers. Questions for Exercise.*

22. MULTIPLICATION is but an abridged method of addition, employed where we have occasion to add the same number repeatedly to itself. Of the two numbers multiplied together, and called by the common name of *factors*, the *multiplicand* is that number which we want to add repeatedly to itself; and the *multiplier* expresses the number of times that the former is to be repeated in that addition. The sum required is called the *product*. Thus, by the product of 6 multiplied by 4 we are really to understand the sum of four sixes, or $6+6+6+6$. The multiplication table, which is supposed to be committed to memory, furnishes us with all the products as high as 12 times 12, or the sum of 12 twelves: and the rule of Multiplication teaches us how to derive the higher products, where the factors (either or both of them) exceed twelve.

23. The product of any two numbers is the same, whichever of them be made the multiplier. For instance, if we multiply 8 by 5 we shall have the same product, as if we multiply 5 by 8. I have known many smile at the attempt to prove this, conceiving it so self-evident, as neither to admit nor require proof. But they are imposed on by their familiarity with the fact. It is by no means self-evident that the sum of 5 eights must be the same with the sum of 8 fives, or that $8+8+8+8+8=5+5+5+5+5+5+5+5$; which is the meaning of the proposition. However it admits a very easy proof from the following illustration. Suppose 5 rows of 8 counters regularly disposed under each other. Whatever way we count them, the total amount of the number must be the same. But counting them one way, we have 5 times eight; and counting them another way, it is plain that we have 8 times five counters. It is obvious that a similar proof would be applicable to any higher numbers.

24. The sign of multiplication is ×, or a St. Andrew's cross, interposed between the factors; and is to be carefully distinguished from the sign of addition +. Thus 12×8, or 8×12, denotes the product of 8 and 12.

25. The

25. The product of any two numbers is equal to the sum of all the products obtained, by multiplying all the parts, into which either is divided, by the other, or by each of the parts into which the other is divided. Thus, if we suppose 8 divided into the parts 4, 3, and 1; the product of 5 times 8 will be equal to the sum of the three products, 5 times 4, 5 times 3, and 5 times 1. And if we suppose the multiplier 5 also divided into the two parts 3 and 2; the product of 5 times 8 will be equal to the sum of the six products obtained by multiplying each of the three component parts of the multiplicand by each of the two component parts of the multiplier. The truth of this will appear very plain, by employing the same illustration that was adduced in the 23d section. In the 5 rows of 8 counters, aptly representing 5 times 8, let us suppose, first, two lines drawn downwards dividing each row of eight counters into the three parts 4, 3, and 1. It is then plain that the whole set of 5 times 8 counters is divided into three sets of 5 times 4, 5 times 3, and 5 times 1. Then supposing a line drawn across and dividing each row of 5 counters into 3 and 2, it is plain each of the 3 former sets will be divided into two, 3 times 4 and twice 4; 3 times 3, and twice 3; 3 times 1 and twice 1: so that the sum of these 6 sets is equal to the one set of 5 times 8 counters. This proof is exhibited to the eye in the subjoined scheme.

$$\begin{array}{c|c|c} 0000 & 000 & 0 \\ 0000 & 000 & 0 \\ 0000 & 000 & 0 \\ \hline 0000 & 000 & 0 \\ 0000 & 000 & 0 \end{array}$$

And it is plainly applicable to any other numbers, divided into any parts whatsoever. Thus, if we suppose 17 broken into the four parts, 6, 5, 4 and 2; and 9 broken into the three parts, 4, 3, and 2; the product of 9 times 17 must be equal to the sum of each of the twelve products obtained by multiplying each of the four parts of the multiplicand by each of the three parts of the multiplier: that is $17 \times 9 = \overline{24+20+16+8} + \overline{18+15+12+6} + \overline{12+10+8+4}$. With the principle brought forward in this section the student cannot be too familiar; as it is the foundation both of common multiplication and Algebraic, as well as fruitful in the most important inferences.

26. If

26. If our multiplier be the product of any two known numbers, we may employ a successive multiplication by the factors, of which the multiplier is the product. Thus, if we want to multiply any number by 54, we may multiply it by 9, and that product by 6: for 6 times 9 being 54, when we first find a number that is 9 times the multiplicand, and then multiply that number by 6, our product must be 6 times 9 times, or 54 times the multiplicand.

27. It appears from § 7 and § 25. that the product of any number multiplied by 10, 100, 1000, &c. is obtained at once by annexing one, two, three, &c. cyphers to the multiplicand on the right hand. Thus, the product of 327 multiplied by 1000 is 327,000: for each digit of the multiplicand is increased in value 1000 times. And combining the principle of the last section, it is plain that if our multiplier be 20, 300, 4000, &c. we may obtain the product by annexing one, two, three, &c. cyphers, and then multiplying by 2, 3, 4, &c. Thus $4296 \times 700 = 429{,}600 \times 7$.

28. From the principle stated in § 25. it is manifest that we can find the product of any two numbers: for however great the factors, they may be broken into parts not exceeding 12, the products of all which parts are furnished by the multiplication table. But when the factors, either or both of them, exceed twelve, the most convenient parts into which we can conceive them broken are those indicated by the digits. Thus, if I want to find the product of 537 multiplied by 9, I conceive the multiplicand divided into the parts 7, 30, and 500; and the product is by § 25. equal to the sum of the three products 9 times 7, 9 times 30, and 9 times 500; each of which the multiplication table furnishes. For 30 being 3 tens, 9 times 30 must be 27 tens, or 270; and 9 times 5 hundreds must be 45 hundreds, or 4500. The product sought therefore must be the sum of the three products, $63 + 270 + 4500$, that is, 4833. This addition of the successive products, by proceeding from right to left in taking the parts of the multiplicand, we are able to perform mentally, without writing the whole of each product separately.—Now if I want to find the product of 537 multiplied by 69, I suppose the multiplier also divided into the two parts 9 and 60; and having found the product of 9 times the multiplicand, I proceed to find the product of 60 times the multiplicand

tiplicand

tiplicand by § 27. writing the latter product (32220) under the former, preparatory to the addition of the products. It is plain therefore that the cypher annexed to the multiplicand for multiplying by 10 must stand in the column of units, and be preceded by the digits expressing the product of 6 times the multiplicand. But as that cypher will make no change in the subsequent addition, it is commonly omitted; taking care however to place the next digit in the column of tens. In like manner if my multiplier were 469, after having found the two former products, I proceed to multiply by 400, supposing two cyphers annexed to the multiplicand and then multiplying by 4, and writing this product (214800) under the second, preparatory to the addition of the three products.—The young Arithmetician should for some time be made to write the cyphers standing on the right hand of the successive products, that he may be convinced of the reason of the rule, which directs us to recede one figure towards the left hand in writing the several products obtained in multiplying by the successive digits of the multiplier.

29. The child should be taught to prove the accuracy of his work in multiplication by addition, so far as to convince him that the one is but an abridged method of performing the other; and by resolving either or both factors into other parts than those indicated by the digits.

30. We have noticed the reason of proceeding from the right hand to the left of the multiplicand. But it is generally indifferent in what order we take the digits of the multiplier: and it will sometimes afford a convenient abbreviation to depart from the usual order. Thus, if our multiplier be 945, instead of obtaining the product sought by three distinct products, two will be sufficient by commencing from the left hand of the multiplier; since having found the product of 9 times the multiplicand, 5 times that product will give us the product of 45 times the multiplicand. § 26. But when this method is employed, it is plain that the cyphers, which are usually omitted, ought to be expressed.

31. We have seen (§ 27.) the facility with which multiplication proceeds, when the multiplier consists of a significant figure followed by any number of cyphers. Now if our multiplier be within twelve of any such number, we may avail ourselves of a convenient abbreviation. For instance

instance if our multiplier be 4989, we observe by inspection that this number is within 11 of 5000. If then I take 5000 times the multiplicand, and subtract from that product 11 times the multiplicand, the remainder must be 4989 times the multiplicand; or must be the product sought. Such abridged methods of operation are useful for exercising youthful ingenuity: but ought not to be prematurely introduced. Rational theory, going hand in hand with practice, will soon make the student expert in discerning various advantages which may be taken. For example, if we have to multiply 123,456789 by 107988, the multiplier being within 12 of 108000, and 9 times 12 being 108, we may first find 12 times the multiplicand, and subtracting that product from 9000 times that product will give the remainder 13,331851,730532 for the product sought. But in general it is useless to occupy the learner's time in arithmetical operations on numbers so high, as scarcely ever occur in real practice. A much more advantageous exercise is to engage him in operating on low numbers *mentally*, without committing them to paper; for instance to find the product of 25 times 36. This is calculated to form a habit of fixed attention, and to strengthen the mental powers.

32. The product of any number multiplied by itself is called the *square* of that number, or its *second power*. The original number thus multiplied is called the *square root* of the product. Thus 64 is the square of 8: and 7 is the square root of 49. If the square of any number be multiplied by its root, the product is called the *cube*, or *third power*, of the original number. Thus, 64×8, or 512, is the cube of 8: and 8 the cube root of 512. And if the cube be multiplied by its root, the product is called its *fourth power*: and so on. Those powers of any number are often represented by annexing to the right hand of the root, and somewhat elevated, the figures 2, 3, 4, &c. which are called *indices* of the powers. Thus 8^2 expresses the square of 8; 2 being the *index* of the power. And 15^4 expresses the fourth power of 15, or $15 \times 15 \times 15 \times 15$.

33. Among the inferences flowing from the principle laid down in § 25. we may here proceed to state two, for which we shall have frequent occasion in Algebra. The square of any number is four times the square of its half. Thus $8^2 = 64$; and $4^2 = 16$. But $64 = 16 \times 4$. That this must

must be so is evident from § 25. For 8 being both multiplicand and multiplier, may be divided in both factors into the parts $4+4$: and the product 8×8 must be equal to the sum of four products, each of which is 4×4, or the square of 4.

34. Again, the square of the sum of any two numbers is equal to the sum of their squares together with twice their product. Thus, the sum of 4 and 3 is 7: and the square of 7 is equal to the sum of the squares of 4 and 3, together with twice the product of 4 and 3: that is $49=16+9+24$. This in like manner immediately appears by supposing the multiplicand and multiplier 7 divided into the two parts 4 and 3; or into any other two parts. On this principle, as we shall hereafter see, depends the extraction of the square root, and the reduction of quadratic equations in Algebra. It is of such frequent use, that the student cannot too soon become familiar with it: and it will afford a good exercise to calculate, without the pen, the square of any number within 100, by resolving the number into the two parts indicated by the digits. Thus, a child may be led to find the square of 69 mentally, if he only know that 60×60 is 3600, that twice 54 is 108, and can add mentally 3681 and 1080.

35. Any product is said to be a *multiple* of either factor; and either factor is called a *submultiple* of the product. Thus 96 is a multiple of 8 or of 12, because $12\times 8=96$; and 8 or 12 is a submultiple of 96.

36. In the following examples of multiplication let the young student write the required product after the sign of equality, $=$. And let him observe that, as 234×89 denotes the product of 234 and 89, so $\overline{234+6}\times 89$ denotes the product obtained by multiplying the sum of 234 and 6 (that is 240) by 89; and $\overline{234+6}\times\overline{89+11}$ denotes the product obtained by multiplying the sum of 234 and 6 (or 240) by the sum of 89 and 11, or 100. In like manner $\overline{234+6+7}\times\overline{89+11}$ is the same thing as 247×100: and that product is equal (§ 25) to the sum of the six products $\overline{234\times 89}+\overline{6\times 89}+\overline{7\times 89}+\overline{234\times 11}+6\times 11+\overline{7\times 11}$. Again $234\times 5\times 7$ denotes the product obtained by multiplying 234 by 5, and that product by 7; and is the same (§ 26.) with the product of 234×35. Again let it be observed that 10^5 (or the 5th power of 10, see § 32.) denotes the product of $10\times 10\times 10\times 10\times 10$.

1st. Ex.

1st. Ex. $\overline{1+2+3+4} \times \overline{5+6+7+8+9} =$

2d. Ex. $2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 =$

3d. Ex. $\overline{9+8+7+6} \times \overline{5+4+3+2+1} =$

4th. Ex. What are the cubes (or 3d. powers) of the numbers 4, 5, 6, 7, 8, 9, and 10?

5th. Ex. $3^5+4^5+9^4=$

6th. Ex. $345^3 \times 100^2=$

7th. Ex. $123456789 \times 9988=$ (See § 31.)

8th. Ex. $7539 \times 60054=$ See § 30.)

9th. Ex. How much does the square of 48 exceed the square of 24? (See § 33.)

10th. Ex. How much does the square of $\overline{57+28}$ (or $\overline{57+28}|^2$) exceed 57^2+28^2? (See § 34.)

For the method of proving multiplication, independently of division, see § 29. (For other methods see § 61 and 62.) Questions for exercise in the practical application of multiplication will be found in Chap. VI.

CHAP. IV.

Nature and Principles of Division. Sign ÷. Division of a smaller Number by a greater. Methods of Proof. Questions for Exercise.

37. DIVISION, in the primary view of it, is but an abridged method of subtraction. Here we enquire how often one number, called the *divisor*, may be subtracted from another number called the *dividend*. The *quotient* expresses the number of times, that the divisor may be subtracted from the dividend, or is contained in it. Thus, when I divide 96 by 12, the quotient is 8: for I may subtract the divisor 12 from the dividend 96 just 8 times. This might be ascertained by performing the successive subtractions, and reckoning the number of them: but is at once discovered by the multiplication table, which informs me that 96 is equal to 8 times 12, and therefore contains 12 in it exactly 8 times. If I divide 103 by 12, it is plain that after subtracting 12 from 103 eight times, there will remain 7: so that the quotient is still 8, but with 7 for a remainder. (See § 43.)

38. When

38. When one number is contained in another a certain number of times exactly, without leaving any remainder, the former number is said to *measure* the latter. Thus, 12 measures 96, but does not measure 103. The numbers 8 and 12 measure 24; 8 being contained in it exactly 3 times, and 12 exactly twice.

39. We often express division by writing the dividend above the divisor with a line interposed between them. Thus $\frac{84}{7}$ expresses the division of 84 by 7: and the following symbols $\frac{84}{7}=12$ express therefore that the quotient of 84 divided by 7 is equal to 12. The symbol $\div$ also is sometimes employed to express division, the dividend standing on the left hand of it, and the divisor on the right. Thus, $42 \div 6$ is another way of expressing the division of 42 by 6, as well as $\frac{42}{6}$.

40. If any quotient be made the divisor of the same dividend, the former divisor will be the new quotient, and the same remainder (if any) as before. Thus, dividing 103 by 12, the quotient is 8 with the remainder 7. Now if we divide 103 by 8, the quotient must be 12, leaving the same remainder. For the first division shews that the dividend contains 12 eight times and 7 over. Therefore it must contain 8 twelve times and 7 over; 8 times 12 and 12 times 8 being equal. (§ 23.) And thus also it is manifest that if any product be divided by either of the factors, the quotient must be the other factor: and that any dividend may be considered as the product of the divisor and quotient, with the remainder (if any) added.

41. In the view of division which has been hitherto proposed, the divisor must be conceived not greater than the dividend: else it would be absurd to enquire how often it is contained in the dividend. But there is another view of division, closely connected with the former, in which we may easily conceive the division of a smaller number by a greater. When we are called to divide 96, for instance, by 12, we may consider ourselves called to divide 96 into twelve equal parts, and to ascertain the amount of each. The quotient, found as before, is a number of that amount, or the twelfth part of 96. For since 96 contains in it just 8 twelves, it must contain just 12 eights; and therefore the quotient 8 is the twelfth part of 96.

And thus universally the quotient may be considered as that part, or submultiple, of the dividend which is denominated by the divisor; as the divisor may be considered that part, or submultiple, of the dividend which is denominated by the quotient. (Hitherto I suppose the divisor to measure the dividend.) Thus, dividing 64 by 4 the quotient is 16; for subtracting 16 fours from 64 there is no remainder. Therefore 4 is the sixteenth part of 64; and 16 is the fourth part of 64.

42. Now though it would be absurd to enquire how often 12 may be subtracted from 7, and therefore any division of 7 by 12 is inconceivable according to that view; yet it is not absurd to enquire what is the twelfth part of 7, or to speak of dividing 7 by 12 according to the latter view. For instance, I might have occasion to divide 7 guineas among 12 persons equally, or into 12 equal shares: and then it is plain that each person must get the twelfth part of seven guineas. The quotient, or twelfth part of 7, may be represented by the notation $\frac{7}{12}$: (§ 39.) and the child ought to be familiarized to this notation, previous to his entrance on the doctrine of fractions.

43. Let us now revert to the example of division introduced at the close of § 37. the division of 103 by 12. The quotient we saw is 8, but leaving a remainder of 7. Therefore 8 is not exactly the twelfth part of 103: for if I were dividing 103 guineas equally among 12 persons, after giving each of them 8 guineas there would be 7 guineas over: which 7 guineas I should proceed to divide equally among them; that is, I should give each of them the twelfth part of 7 guineas in addition to the 8 guineas he had received, in order to make the division accurate. Therefore the twelfth part of 103 is exactly $8+\frac{7}{12}$; or 8 and the twelfth part of 7. And so, wherever there is a remainder on a division, the student should be taught to correct the quotient by annexing to it that remainder divided by the divisor.

44. As to the practical method of performing division, the grounds of it are obvious from § 37. Let us first suppose that our divisor does not exceed 12: for instance let it be required to divide 5112 by 8. We immediately know from the multiplication table that 8 may be subtracted 600 times from the dividend, but not 700 times; since 600

times

times 8 (or 8 times 600) is 4800, but 8 times 700 is 5600, a number greater than the dividend. Subtracting therefore 4800 from 5112, there remains 312; and this one subtraction saves the trouble of 600 distinct subtractions of 8 from the dividend. We proceed now to the remainder 312, and consider, from the multiplication-table, what is the greatest number of times that 8 is certainly contained in it, or may be subtracted from it: and we immediately know as before that 8 is contained 30 times in 312, but not 40 times; 30 times 8 being 240, but 40 times 8 being 320, a number greater than 312. Subtracting therefore 30 times 8, or 240, from 312, there remains 72: in which remainder we see that 8 is contained just 9 times. Thus we have ascertained that from 5112, 8 may be subtracted 600 times, 30 times, and 9 times; or in all 639 times; which number is therefore the quotient, and the eighth part of 5112. If our dividend were 5119 it is plain that the quotient would be 639 with the remainder 7: and therefore that the eighth part of 5119 is $639 + \frac{7}{8}$. In practice, we perform the successive multiplications and subtractions mentally, as we proceed; attending only to that part of the dividend, which ascertains the successive digits of the quotient, and writing only those digits. But the learner ought to be exercised for some time in performing the operation at large, as I have described it; that he may be grounded in the rational principles upon which the practical contractions rest.

45. Let us now suppose that our divisor exceeds 12; for instance, that we have to divide 27783 by 49. We may at once conclude that the quotient must be less than 700, as 700 times 40 (or 28000) would exceed the dividend, and therefore much more 700 times 49. But the dividend does not contain the divisor even 600 times; for though 600 times 40 (or 24000) is less than the dividend, yet 600 times 49 will be found greater than the dividend. (Nothing but practice can make the student quick in perceiving this; and he may for a time have the trouble of trying numbers in the quotient, which he will find to be too great.) Subtracting therefore 500 times the divisor, or 24500, from the dividend, there remains 3283; from which we subtract 60 times the divisor, or 2940. In the remainder 343 we find that the divisor is contained just 7 times. So that the entire quotient is 567. In such in-

stances of what is called *long* division, it is necessary to write the successive remainders. But after the student has been grounded in the principles of the operation, it will be expedient that he should perform the subtractions without writing the successive products; subtracting the several digits composing them as he proceeds with the multiplication.

46. Thus it appears that we are enabled by the multiplication-table to determine the successive digits of the quotient from the left hand. But although the order of proceeding which we have described be the most convenient, I would have the young Arithmetician practised in resolving the dividend differently, and proceeding on similar principles, but in another order. Let us again take the last example, to illustrate my meaning. In dividing 27783 by 49, we first took 27000, a component part of the dividend, and finding that it contained 500 forty-nines and 2500 over, we incorporated the latter with 783 the other component part of the dividend, and proceeded in like manner to find the other component parts of the quotient. But the same result must be obtained by commencing with the latter component part of the dividend 783. Dividing it by 49 the quotient is 15 with the remainder 48. Adding that remainder to the other part of the dividend 27000, we may proceed in like manner to ascertain how many times 49 is contained in their sum, by commencing with the component part 7048. The quotient will be 143 with the remainder 41. And adding the remainder to the 20000 which has not yet been divided, 49 will be found to be contained in their sum 20041 just 409 times. Now the sum of the three quotients, $15+143+409$, is 567 as before. And thus the student may be taught to prove the accuracy of his work in division, not only by multiplying the divisor and quotient, (§ 40.) but also by resolving the dividend into any two or more parts, dividing each of them by the given divisor, and adding the quotients.

47. If the given divisor be the product of any two or more known factors, the quotient may often more expeditiously be obtained from successive divisions by those factors. Thus in the last example, 49 being 7 times 7, if we divide 27783 by 7, and again divide the quotient 3969 by 7, we shall have the result 567. Perhaps the child might here be advantageously introduced to the principle, for which we have such constant occasion in fractions, that the 7th. part of the 7th. must be the 49th. part, &c. (See

c. viii. § 8.) The principle admits such clear and familiar illustration, that I think any child who is capable of learning division may be convinced of its truth. But for establishing the present rule in division the following principles also ought to be employed, and will be sufficient.

48. The given dividend 27783 is 7 times the first quotient; and the first quotient 3969 is 7 times the second quotient. Therefore the given dividend is 49 times the second quotient: or 567 is the 49th. part of the given dividend. For (putting *a*, *b*, and *c* for three numbers) if *a* be 8 times *b*, and *b* 6 times *c*, then *a* must be 48 times *c*. Or thus:

49. The uumber 7, being 7 times less than 49, must be contained in the dividend seven times oftener. But 7 is contained in 27783 just 3969 times. Therefore 49 must be contained in it the 7th. part of 3969 times: or the quotient sought is the 7th. part of 3969.

50. But when this method is employed, we must carefully attend to the management of the remainders. Thus, dividing 5689 by 42, the quotient is 135 with the remainder 19: and if we employ a successive division by 7 and 6, the first quotient will be 812 with the remainder of 5, and on dividing that quotient by 6 we shall get the quotient 135 with the remainder of 2. But this 2 is to be considered as 2 sevens, or 14; which added to the former remainder gives 19 for the true remainder, as before. The reason of this will be plain from considering that by the first division we find that the dividend contains in it 812 sevens: so that any remainder on dividing that 812 must be regarded as of the denomination *sevens*. This may be made quite clear to the youngest student by suppposing that we wanted to divide 53 guineas by 12; that is, to find how many sets of 12 guineas are contained 53 guineas. Dividing 53 by 4, we find that it contains 13 sets of 4 guineas each and one over. Every three of this quotient will make a parcel of 12 guineas: and now to find their number, dividing 13 by 3, the quotient is 4, (four parcels of 12 guineas) and 1 over. But this 1 is plainly 1 set of 4 guineas: which added to the former 1 guinea gives 5 for the remainder and 4 for the quotient. Hence appears the reason of the rule, which directs us to multiply the remainder on the second division by the first divisor, and add the product to the remainder on the first division.

The

The same thing will appear from the doctrine of fractions.

§ 51. Any number is divided by 10—100—1000 &c. by cutting off as many digits from the right hand of the dividend, as there are cyphers in the divisor. The digits thus cut off express the remainder, and the remaining digits of the dividend the quotient. Thus, dividing 234567 by 1000, the quotient is 234 with the remainder 567. This is manifest from § 40. since the dividend is equal to 1000 times 234 with 567 added to the product. Hence it is plain that if our divisor consist of any significant figures followed by any number of cyphers, we may employ the method of division described in the last section. Thus if we want to divide 234567 by 7000, we may divide first by 1000 and then by 7; and the quotient will be 33 with the remainder 3567. For when we divide 234 by 7, the remainder of 3 is in fact 3 thousands, and is to be added to the first remainder 567. And we shall have the same result (though not so expeditiously) if we first divide by 7 and then by 1000,

52. When the given divisor is a submultiple of any of those last described, we may often abridge our work by multiplication. Thus if I have to divide 1234 by 25, I know at once that the quotient is 49 with the remainder 9. For 25 is the fourth part of 100, which is contained in the dividend 12 times with the remainder 34. Therefore (§ 49.) the dividend must contain 25 four times as often, that is, 48 times with the same remainder 34. But in this remainder 25 is contained once and 9 over. In like manner, 75 being the fourth part of 300, I know at once that the 75th. part of 1234 is 16 with the remainder 34, or that $\frac{1234}{75}=16+\frac{34}{75}$. Some other abbreviations of division, less commonly known, I shall point out in the following chapter. They may evercise the ingenuity of the student, and are calculated to develope very curious properties of certain numbers.

Examples for practice in division may be had from all the examples of multiplication at the end of Chap. III. In the following examples let the student supply the quotient after the sign of equality $=$.

1st. Ex. $123456789 \div 9000 =$

2d. Ex. $987654321 \div 125 =$ (See § 52.)

3d. Ex. $3933 \div 19 =$

4th. Ex. $31464 \div 19 =$

Let

Let the student observe in the two last examples that the dividend in the 4th. being 8 times the dividend in the 3d. the quotient in the 4th. is 8 times the quotient in the 3d.

5th. Ex. $3496 \div 19 =$

Here the dividend being the 9th. part of the dividend in the 4th. example, the quotient also is the 9th. part of the quotient in the 4th.

6th. Ex. $31464 \div 133 =$

Here the divisor being 7 times the divisor in the 4th. the quotient is the 7th. part of the quotient in the 4th.

7th. Ex. $180918 \div 3933 =$

8th. Ex. $180918 \div 437 =$

Here the divisor being the 9th. part of the divisor in the 7th example, the quotient is 9 times the quotient in the 7th.

9th. Ex. $5907^2 \div 9^3 =$

10th. Ex. $\overline{9^5 + 8^3} \div 7^4 =$

Besides the methods of proving division already pointed out, another method will be assigned in the next chapter, § 62 and 63.

CHAP. V.

Methods of abbreviated Operation, and of proving Division, continued. Properties of the Numbers 3, 9, 11, *&c.*

53. WE may arrive at the required quotient in division, by substituting for the given divisor any other whatsoever, either greater or less than the given one. To exhibit this, I shall first employ a number greater by 1 than the given divisor. Suppose for instance we have to divide 796 by 19. Dividing it by 20 the quotient is 39 with the remainder of 16. I say then that the required quotient must be 41 with the remainder 17. For I have found that 20 may be subtracted from the dividend 39 times: but for every time that I have subtracted 20 instead of subtracting 19, I have subtracted 1 too much; that is, I have subtracted in all 39 too much. Hence we may infer that the dividend besides containing 39 nineteens with a remainder of 16, contains also 39 units more: in which 39 there are 2 nineteens and 1 over.

1 over. Therefore the dividend contains in all 41 nineteens (39+2) with the remainder of 17 (16+1). Or, to give another illustration of the principle upon which this method proceeds; suppose we divide 796 guineas equally first among 20 persons: they will each get 39 guineas and the 20th. part of 16 guineas. But now finding that we were wrong in making the division among 20 persons, and that it ought to have been made only among 19, we take one person's share from him and divide it equally among the rest: so that each shall now get for his share 41 guineas and the 19th. part of 17 guineas. Thus if we have to divide 1234 by 99, we may know at once that the quotient is 12 with the remainder 46. For dividing by 100, the quotient is 12 with the remainder 34: but having thus subtracted 12 units too much, they must be added to 34 for the true remainder. And if the dividend be 12345, the correction will be made by adding 1 to the first quotient and 24 to the remainder; inasmuch as 123 contains 99 once and 24 over: so that the quotient sought is 124 with the remainder 69.

54. Now suppose we have to divide 123456789 by 99. Substituting 100 as our divisor, the quotient is 1234567 with the remainder 89. If we knew what number of times 99 is contained in that quotient, and with what remainder, the necessary correction would be made by adding that number to the quotient, and that remainder to the former remainder. Now this would be ascertained by dividing the first quotient by 99: but in place of this we may again substitute a division by 100, the result of which is to be similarly corrected. And thus continually dividing each successive quotient by 100, the sum of all the quotients and sum of all the remainders will furnish us with the true quotient and true remainder. Thus we have the quotient, by mere addition, $1234567 + 12345 + 123 + 1 = 1247036$. But the sum of all the remainders, $89+67+45+23+1 = 225$, containing 2 ninety-nines and 27 over, we add the 2 to the quotient: so that the true quotient is 1247038 with the remainder 27. At any period of the above process, when we see how often 99 is contained in the last quotient, we may discontinue the division by 100 and complete the corrections at one step.

55. Hitherto we have supposed that the substituted divisor exceeds the given divisor only by 1. But let us now suppose

suppose that we have to divide 1234 by 95. We may with equal facility conclude that the quotient is 12 with the remainder 94. For dividing by 100, the quotient is 12 with the remainder 34. But for every time that we have subtracted 100 instead of subtracting 95, we have subtracted 5 too much; that is, we have subtracted in all 60 too much, which 60 is therefore to be added to 34 the former remainder. (It is easy also to apply to this case the illustration adduced in § 53.) In like manner if we have to divide 1234567 by 7988, substituting 8000 which exceeds the given divisor by 12, we have the quotient 154 with the remainder 2567 to which remainder if we add 1848 (154×12) we shall have 4415 for the true remainder. It can scarcely however be advantageous to employ this method in practice, if the given divisor be much less than the substituted, which it is convenient to employ; and if the number of digits in the quotient be more than those in the divisor.

56. Hence it appears that if 9 measure the sum of the digits of any number, it will measure the number: and that the remainder left on dividing any number by 9 must be the same with the remainder on dividing the sum of its digits by 9. Thus 234, or 378, is evenly divisible by 9, because the sum of the digits $2+3+4$, or $3+7+8$, is so. For if instead of dividing 378 by 9, we substitute continued divisions by 10, the series of quotients will be $37+3+0$, and of remainders $8+7+3$; which latter sum containing just 2 nines, we carry 2 to the former quotients, and infer that the exact quotient is 42 without any remainder. In like manner it appears that the remainder on dividing 12345 by 9 must be 6, as that is the remainder on dividing the sum of the digits 15 by 9. And thus it is evident that any numbers written with the same digits, in whatever order, will give the same remainder on being divided by 9. A similar property of the numbers 99, 999, &c. may in like manner be inferred; only taking the digits by pairs, by threes, &c. from the right hand. Thus 12345 divided by 99 must give the remainder 69; because $45+23+1=69$: but 14652, or 15246 must be evenly divisible by 99, since $52+46+1=99$. Another demonstration will be found in § 59 and 60. for the property of the number 9.

57. Let

57. Let us now consider how division may be performed by the substitution of a divisor less than the given one. Suppose we have to divide 123456 by 101. Substituting 100 as our divisor, the quotient 1234 with the remainder 56 is manifestly too great. For every time that we have subtracted 100 instead of subtracting 101, we have subtracted 1 too little, that is we subtracted in all too little by 1234, which contains 101 twelve times with the remainder 22; as will appear on dividing 1234 by 101. The correction therefore would be made at once by subtracting 12 from the first quotient 1234, and 22 from the first remainder 56: which gives 1222 for the quotient sought, with the remainder 34. But instead of ascertaining the correction at once by dividing the first quotient by 101, let us again substitute a division by 100; and subtracting the quotient 12 and remainder 34 from the first quotient and first remainder, it is now plain that we have subtracted too much: and therefore the next correction must be made by addition. And thus when we successively employ a divisor less than the given one, our successive corrections must be made by alternate subtractions and additions: as we first subtract too much, then add too much, &c. Whereas when we employed a divisor greater than the given one, all our corrections proceeded by addition, as we were successively adding too little. If our substituted divisor be less than the given one by more than a unit, it appears as before that each quotient must be multiplied by the difference. Thus the quotient of 123456 divided by 5012 is 24 with the remainder 3168: for dividing by 5000 the quotient is 24 with the remainder 3456; from which remainder subtracting 288 (24×12) we have the true remainder.

58. Hence we may infer a property of the number 11, which shall be demonstrated from other principles in § 61, namely, that any number must be evenly divisible by 11, if the sum of the alternate digits from the last and the sum of the alternate digits from the penultimate be equal, or their difference evenly divisible by 11. Thus, 190817, or 718091, is evenly divisible by 11, since 11 measures the difference between $7+8+9$ and $1+0+1$. For if, instead of dividing by 11, we should investigate the quotient by successive divisions by 10, the successive digits would be the remainders; and these would be to be subtracted and added alternately.

59. The

59. The property of the number 9 stated in § 56. may be thus easily demonstrated. If from any number the sum of its digits be subtracted, the remainder must be evenly divisible by 9. For instance, if from the number 345 we subtract 12 $(3+4+5)$ the remainder 333 must be evenly divisible by 9. For the number 345 may be considered as made up of 100 threes, 10 fours, and 1 five. Let us now succesively subtract the digits, and observe the remainders. Subtracting 5, the remainder is 100 threes and 10 fours. Subtracting 4, the remainder is 100 threes and 9 fours. Finally subtracting 3, the remainder is 99 threes and 9 fours. But 9 must measure this number, as it is plain that it measures each of its component parts. And so putting the letters a, b, c, d, &c. for the digits of any number, *$1000a+100b+10c+d$ must be a just expression for *any* number written by four digits, that is, within 10,000. And if from this we subtract the sum of the digits $a+b+c+d$, the remainder, $999a+99b+9c$, must be evenly divisible by 9, inasmuch as 9 measures each of its component parts.

60. It immediately follows from the last section that the remainder on dividing any number by 9 must be the same with the remainder on dividing the sum of its digits by 9. For instance, 345 divided by 9 must give a remainder of 3, since that is the remainder on dividing 12 $(3+4+5)$ by 9. This is manifest from considering that 345 is equal to $333+12$; of which parts we have just seen that the former (333) must be evenly divisible by 9, and therefore the only remainder on dividing the whole by 9 must be that which occurs on dividing the latter part 12 (or the sum of its digits) by 9. And in like manner the same property is demonstrated to belong to the number 3.

61. By a similar process of reasoning it appears, that if from any number we subtract the sum of the alternate digits commencing from the last, and add to it the sum of the alternate digits commencing from the last but one, 11 must measure the resulting number. For let a number consisting of 4 digits be represented as before by $1000a+100b+10c+d$. Subtracting d and b, there will remain $1000a+99b+10c$. Now adding c and a, the resulting number, $1001a+99b+11c$, must be evenly divisible by 11. Hence given any number it is easy to know what the remainder must be on dividing it by 11. For instance, if the given number

* In this notation 1000 a stands for 1000 times a; 999 a for 999 times a; &c.

number be 91827, or 72819, the remainder must be 10; for subtracting 21 (the excess of the digits to be subtracted above those to be added) the remainder is evenly divisible by 11: therefore the remainder on dividing the whole by 11 must be that which occurs on dividing 21 by 11. But if the given number be 9182, the digits to be added exceed the digits to be subtracted by 14, that is $11+3$: whence we may infer that the given number wants 3 of being evenly divisible by 11; or that dividing it by 11 there will be a remainder of 8. We might enlarge upon other curious properties:—(for instance, if 11 measure any number consisting of an even number of digits, and consequently measure also the number consisting of the same digits in an inverted order, the sum of the digits in each quotient must be the same,)—but as they are of little practical importance, we shall rather pass to a useful method of proving multiplication and division.

62. Multiplication may be proved thus: divide both the factors by any number and (neglecting the quotients) mark the remainders; divide the product of those remainders by the same number and mark the remainder. This remainder must be the same with the remainder on dividing by the same number the product of the given factors. For instance, $648 \times 23 = 14904$. Now dividing 648 and 23 by 7, the remainders are 4 and 2; whose product 8 divided by 7 gives 1 for the remainder: and 1 must also be the remainder on dividing 14904 by 7; which may be easily demonstrated from the fundamental principle of multiplication. For breaking the factors into the parts $644+4$ and $21+2$, the former parts of each are evenly divisible by 7, and therefore also any multiples of those parts. Now the whole product 14904 is equal (§ 25.) to the sum of the four following products 644×21, and 4×21, and 644×2, and 4×2. Of these 7 measures the three first; and therefore the only remainder on dividing the whole product by 7 must be that which occurs on dividing the product of the remainders 4 and 2.

63. On this principle depends the common method of proving multiplication by what is called *casting out the nines*. It is in fact nothing but an application of the number 9 as a test, just as in the last example we applied the number 7: and the only advantage of the former is that we can ascertain the remainders without performing the divisions

by

by 9. It appears that if the remainder on the supposed product of the factors be not the same with that on the product of the remainders of the factors, we may conclude with certainty that there is an error in our work. But we cannot be equally certain that the work is right, if the remainders be the same. There is however a strong probability of it: which will amount to a moral certainty, if, after applying 9 as a test, we also apply 7 or 11. I would recommend the latter, from the facility with which the divisions may be performed, or the remainders calculated by § 61. It is to be observed that although any number may be employed as a test, yet there are some which would afford little or no evidence of the correctness of the work. For instance the application of 2 or 5 would only ascertain the correctness of the last digit of the product; all numbers ending with the same digit giving the same remainder when divided by 2 or by 5. It is evident that the same method of proof is applicable to division, considering the dividend *minus* the remainder as the product of the divisor and quotient.

64. We know by inspection whether a number may be evenly divisible by 2 or by 5, as the former measures all even numbers, and the latter all numbers ending with 5 or 0; and those alone. We have seen also that it is easily ascertained whether a number be evenly divisible by 3 or 9 or 11. The number 4 measures all numbers ending with 2 or 6, preceded by an odd digit; or ending with 4, 8, or 0, preceded by an even digit: or in short all numbers whose two last digits are evenly divisible by 4. Whether 6 measure a given number may be determined by observing whether it be evenly divisible by 3, and end with an even digit. If any number evenly divisible by 9 or by 3 end with 5, it must be evenly divisible by 45 or by 15. For dividing it by 9 or by 3 the last digit of the quotient must be 5, and therefore that quotient must be evenly divisible by 5. In like manner every even number that 9 measures must be evenly divisible by 18. Every even number which 11 measures must be evenly divisible by 22; and 55 must measure every number ending with 5 or 0 and evenly divisible by 11.

CHAP.

CHAP. VI.

Practical Application of Multiplication and Division. Questions for Exercise.

65. HERE the great object of a rational teacher should be, not to furnish the child with rules of operation, but to employ his reason in investigating the rules. From the first initiation of the youthful student into multiplication and division, he ought to be led to the practical use of these operations by familiar questions involving low numbers. For instance, he may be called to find how many apples are wanted in order to give 4 apiece to 16 persons; or called to divide 96 apples equally among 4 persons. And,—instead of learning what is called *Reduction*, ascending and descending, as distinct *rules*,—as soon as he can multiply and divide by 4, 12, and 20, he is capable of finding the number of farthings in 2£. and the number of pounds in 1920 farthings. Such tables as are needful for solving the following questions, will be found at the end of the volume.

Ex. 1. How many miles does a man travel in 6 days, who goes 87 miles a day?

Ex. 2. A man travels 465 miles in 5 days, and an equal distance each day. How many miles does he go in one day?

Ex. 3. How many hours in 365 days?

Ex. 4. How many weeks in 5824 days?

Ex. 5. A man spends 18*s.* a day. How much does he spend in the whole year?

Ex. 6. How much per day may a man spend, whose annual income is £1314?

Ex. 7. Supposing that a standard pint contains 9216 grains of wheat, how many grains in one gallon; and how many in one bushel?

Ex. 8. Supposing that one acre of land produces 30 bushels of wheat, how many acres would be necessary to produce 1844670 bushels?

Ex. 9. How many farthings in £8738 : 2 : 8?

Ex. 10. How many pounds, &c. in 16777215 farthings?

Ex. 11. How many inches in 25 English miles?

Ex. 12. How many bank notes, 8 inches long, would reach round the earth, supposing the distance to be 25000 miles?

Ex. 13. How many seconds are in a solar year; or 365 days, 5 hours, 48 minutes, 48 seconds?

Ex. 14. How many seconds are in a lunar month, or 29 days, 12 hours, 44 minutes, 3 seconds?

Ex. 15. How many Julian years (of 365 days 6 hours) would exceed an equal number of solar years by 7 days?

Let the young student observe that this question amounts to the enquiry how many times 11′ 12″ (that is, 11 minutes and 12 seconds) are equal to 7 days; and that the answer may therefore be obtained by dividing the number of seconds in 7 days by 672, the number of seconds by which one Julian year exceeds a solar. But the three last questions ought not to be proposed to a child without explaining the meaning of the terms employed in them:—that by a lunar month we mean the time which intervenes between one full moon and the next; by a solar year, the time which intervenes between one vernal equinox and the next; and by a Julian year, the time which Julius Cæsar in his regulation of the calendar assigned to the year, reckoning 365 days in ordinary years, but 366 days in every fourth, or leap year; which gives the average length of the Julian year 365 days 6 hours.

Ex. 16. How many English miles are equal to 11 Irish?

Ex. 17. How many pounds, &c. in 680314 grains?

Ex. 18. How many grains in 59 lb. 13 dwts. 5 gr?

Ex. 19. How many tons, &c. in 4114201 drams?

Ex. 20. How many drams in 35 ton 17 cwt. 1 qr. 23 lb. 7 oz. 13 dr.?

Ex. 21. An old lady observed that she had been for 52 years taking 2 oz. of snuff weekly, and that the snuff cost at an average 5d. per oz. What weight of snuff had she consumed, and how much had it cost her, reckoning the years Julian?

Ex. 22. Her husband remarked, that he for the same period had drank 1 quart of claret daily, and that the average price had been 35 guineas a hogshead. How much wine had he consumed, and what had it cost him?

CHAP.

CHAP. VII.

Doctrine of Ratio—direct—inverse—compound. Method of finding a fourth proportional. Abbreviations. Questions for Exercise.

66. WE have already remarked, that when any number is multiplied by another, the product is called a *multiple* of the multiplicand; and the latter is called a *submultiple* of the product. Thus, 54 is a multiple of 6, and 6 a submultiple of 54; because 54 is equal to 9 times 6. Thus again, 2 or 3 or 6 or 9 is a submultiple of 18. (Submultiples are otherwise called *aliquot parts.*) Now when two numbers are multiplied each by the same number, the products are called *equi-multiples* of the respective multiplicands; and the latter are called *equi-submultiples* of the products. Thus, 18 and 24 are equi-multiples of 3 and 4, or 3 and 4 equi-submultiples of 18 and 24: because 18 is 6 times 3, and 24 is 6 times 4.

67. By the *ratio* of two quantities we mean their relative magnitudes, or the magnitude of one in comparison of the other. Thus, although the absolute magnitude of a mile and 12 miles, is much ~~much~~ greater than that of an inch and a foot, yet the relative magnitude, or ratio, of the two former is just the same with that of the latter: or in other words, a mile is just as small a space in comparison of 12 miles, as an inch is in comparison of a foot.

68. A ratio is written by the aid of two dots interposed between the terms of the ratio; of which the former is called the *antecedent*, and the latter the *consequent*. And the ratio is called a ratio of *greater* or of *less inequality*, according as the antecedent is greater or less than the consequent. Thus, 3 : 5 expresses the ratio of 3 to 5; in which 3 is the antecedent, and 5 the consequent; and the ratio is a ratio of less inequality. But 7 : 5 is a ratio of greater inequality. The ratio of 5 to 7 is called the *reciprocal*, or *inverse* of the ratio of 7 to 5.

69. The ratio of any two numbers is the same with the ratio of any equi-multiples or equi-submultiples of those numbers. This is an important principle of very extensive application: and its truth will appear most manifest on a little consideration. Thus, if we take the ratio of 3 to 5,

and

and multiply both terms of it by 7: the products 21 and 35 are equi-multiples of 3 and 5; and the ratio of 3 to 5 must be the same with the ratio of those products, because it is evidently the same with the ratio of 3 times 7 to 5 times 7. Or, to take another instance, is it not evident that the ratio of 9 to 6 is the same with the ratio of 900 to 600, or of 90 to 60 (i. e. 9 tens to 6 tens) or in short of 9 times any number to 6 times the same number; that is, the same with the ratio of any equimultiples of 9 and 6? And is it not equally evident that the ratio of 9 to 6 is the same with the ratio of the third part of 9 to the third part of 6, that is of 3 to 2, or of any other equi-submultiples of 9 and 6? This indeed, if it were needful, might be deduced by necessary inference from the former; inasmuch as 9 and 6 are equi-multiples of 3 and 2, or of any equi-submultiples of 9 and 6; and therefore in the same ratio with them.

70. The equality or identity of two ratios is denoted by four dots interposed between the ratios. Thus, 9 : 6 : : 3 : 2 denotes that the ratio of 9 to 6 is the same with, or equal to, the ratio of 3 to 2; or, as we commonly more briefly express it, that 9 is to 6 as 3 to 2. Such a series is called a series of *proportionals*, or by one word, borrowed from the Greek language, an *analogy*. The first and fourth terms of such a series (i. e. the antecedent of the first ratio and consequent of the second) are called the *extremes*: the second and third terms (i. e. the consequent of the first ratio and antecedent of the second) are called the *means*. If the antecedent of the second ratio be the same with the consequent of the first, the terms are said to be in *continued proportion*. Thus, the numbers 3, 9, and 27 are in continued proportion; because 3 : 9 : : 9 : 27.

71. If any two ratios be equal, it is plain that their *reciprocals* must be equal; that is, that the consequent of the first ratio is to its antecedent as the consequent of the second ratio to its antecedent. Thus, since 9 : 6 : : 3 : 2, we may infer that 6 : 9 : : 2 : 3. For if 9 be as much greater in comparison of 6, as 3 is in comparison of 2, it follows that 6 is as much less in comparison of 9, as 2 is in comparison of 3.

72. Again, from any analogy we may infer that the first antecedent is to the second antecedent as the first consequent is to the second consequent. Thus, since 9 : 6 : : 3 : 2,

 we

we may infer that $9 : 3 :: 6 : 2$. For the two given ratios could not be equal, unless 9 were just as much greater in comparison of 3, as 6 is in comparison of 2. This may also be demonstrated from § 74. for the fourth proportional either to 9, 6 and 3, or to 9, 3 and 6 indifferently, is $\frac{6 \times 3}{9}$. To state the two last inferences generally, putting the letters a, b, c, d for any four proportional numbers, since $a : b :: c : d$, we may infer that $b : a :: d : c$; and that $a : c :: b : d$. The former inference is called *inversion*; the latter *alternation*, or *permutation*.

73. Again, from any given analogy we may infer that any equimultiples or equi-submultiples of the antecedents bear the same ratio to their respective consequents: and that the antecedents bear the same ratio to any equi-multiples or equi-submultiples of their consequents. Thus, since $9 : 6 :: 3 : 2$, we may infer that 5 times 9 is 6 as 5 times 3 to 2; or that the fifth part of 9 is to 6 as the fifth part of 3 to 2. For it is plain that if we increase or diminish the correspondent terms of equal ratios *proportionally*, the resulting ratios must still be equal. And from the same principle it appears that if we increase or diminish corresponding terms of each ratio by adding to them or subtracting from them the other terms, the resulting ratios must be equal: or in other words, that the sum or difference of the terms of the first ratio is to either of its terms as the sum or difference of the terms of the second ratio is to its correspondent term. For then correspondent terms of the equal ratios are increased or diminished proportionally. Thus, from the analogy $9 : 6 :: 3 : 2$ we may infer that 15 (the sum of 9 and 6) is to 6 as 5 (the sum of 3 and 2) to 2, &c. Or generally, from the analogy $a : b :: c : d$ we may infer that $\overline{a \pm b} : b :: \overline{c \pm d} : d$; or that $a : \overline{a \pm b} :: c : \overline{c \pm d}$; where the sign $\pm$ denotes the sum or difference of the terms between which it is interposed.—The inferences drawn in this section may be demonstrated also from the principles of § 76 and 77.

74. If we have given the three first terms of an analogy we may find the fourth, by taking the product of the second and third terms, and dividing that product by the first. Thus, suppose we want to find a fourth proportional to the numbers 3, 4, and 6; that is, such a number that the

the ratio of 3 to 4 shall be the same with the ratio of 6 to the fourth number found. Multiply 6 and 4, and divide their product 24 by 3: the quotient 8 is the fourth proportional sought. The truth of this result is evident in the present instance, 6 the antecedent of the second ratio being twice 3 the antecedent of the first; and therefore the ratio of 3 to 4 must be the same with the ratio of twice 3 to twice 4, that is, of 6 to 8. But suppose the three given terms are 3, 4, and 5. The fourth proportional is found by the same process: divide 20, the product of the given means, by 3 the first term; the quotient 6 and $\frac{2}{3}$ (or 6 and the third part of 2) is the fourth term sought: which we thus demonstrate. By the principle laid down in § 69. the ratio of 3 to 4 is the same with the ratio of their equi-multiples 5 times 3 to 5 times 4: or again, is the same with the ratio of the equi-submultiples of the latter, the third part of 5 times 3 to the third part of 5 times 4. But the third part of 5 times 3 is 5. Therefore 3 is to 4 as 5 to the third part of 5 times 4, that is to the quotient arising from dividing the product of the given means by the first term.—Let us now employ a general notation for exhibiting the same proof. Let the letters a, b, c, and x represent any four proportional numbers, of which we have given the three first, but want to find the fourth x. I say x is equal to $\frac{b \times c}{a}$, that is to the product of b and c divided by a.

For by § 69. $a : b :: a \times c : b \times c$, or $:: \frac{a \times c}{a} : \frac{b \times c}{a}$. But $\frac{a \times c}{a}$ is equal to $\overset{c}{\cancel{x}}$ (§ 40. latter part). Therefore $a : b :: c : \frac{b \times c}{a}$. Q. E. D.

75. Although the preceding demonstration involve no principle, but what must be sufficiently evident to a considerate mind, yet it may be satisfactory to some that another demonstration of the same thing should be exhibited. Let us then again suppose that we want to investigate a method for finding a fourth proportional to 3, 4, and 5. We know that 3 is to 4 as 1 (the third part of 3) to the third part of 4; or as the equi-multiples of the latter terms, 5 times 1, that is, 5 to 5 times the third part of 4. Thus we are landed in the same result as before; for 5

times the third part of 4, and the third part of 5 times 4 are equivalent, as the former must be three times less than 5 times 4, and therefore equal to its third part. This will be more fully shewn, when we come to the doctrine of fractions.

76. In any analogy the product of the extremes is equal to the product of the means. This immediately follows from what has been last demonstrated: since either extreme is equal to the product of the means divided by the other extreme. For instance; 5 : 7 : : 10 : 14. But we have seen that 14 is equal to the quotient arising from dividing the product of 7 and 10 by 5. Therefore (§ 40.) multiplying 14 by 5 must give a product equal to the product of 7 and 10. Or generally, putting the letters a, b, c, and d, for any four proportional numbers, we may infer that $a \times d = b \times c$. In like manner it appears that, if three numbers be in continued proportion, the product of the extremes is equal to the square of the mean. Thus 4 is to 6 as 6 to 9: and the product of 4 and 9 is equal to the square of 6.

77. We may also infer that, if two products be equal, their factors are *reciprocally proportional*; that is, that the multiplier of one is to the multiplier of the other, as the multiplicand of the latter to the multiplicand of the former. Thus, the product of 2 and 28 is equal to the product of 7 and 8: whence we may infer that 2 : 7 : : 8 : 28. And generally, employing letters to denote numbers, if $a \times b = x \times y$, we may infer that $a : x :: y : b$. For if to the three numbers a, x, and y we find a fourth proportional, it must by the last section be such a number that the product of it and a, shall be equal to the product of x and y; that is, it must be equal to b.

78. In any multiplication, unity is to either factor as the other factor to the product. Thus, the product of 6 and 5 is 30; and 1 : 6 : : 5 : 30. This immediately appears either from the last section, or from § 69. inasmuch as 5 and 30 are equimultiples of 1 and 6, and therefore in the same ratio.

79. In any division, the divisor is to unity as the dividend to the quotient. Thus dividing 36 by 4 the quotient is 9: and 4 : 1 : : 36 : 9. This appears from § 77. and from the principle that the dividend is always equal to the product of the divisor and quotient.

80. When

80. When we say that one quantity is *directly* as another quantity, it is to be understood that the one increases or diminishes in the same ratio in which the other increases or diminishes. But when one quantity increases in the same ratio in which another diminishes, or diminishes in the same ratio in which the other increases, we say that the one is *inversely* as the other. For example, if I purchase cloth at 20*s.* per yard, the amount of the cost depends upon the quantity purchased as in the first case, and is therefore said to be directly as that quantity. But if I have to ride a certain distance, the time requisite depends upon the speed employed as in the second case, and is therefore said to be inversely as that speed.

81. In multiplication, the product is directly as either factor when the other is given, or remains unvaried. Thus if I multiply 7 first by 3 and then by 5, the products 21 and 35 are as 3 to 5. (§ 69.) But in division, the quotient is directly as the dividend when the divisor is given; and inversely as the divisor when the dividend is given. Thus if I divide 24 and 27 by 3, the quotients 8 and 9 are in the ratio of 24 to 27. (§ 69.) But if I divide 24 first by 3 and then by 6, the quotients 8 and 4 are in the ratio of 6 to 3. (§ 77.)

82. Hence whenever any quantity so depends upon two others, that it is directly as each of them when the other is given, it must vary in the ratio of the product of two numbers taken proportional to those two quantities. Thus the distance to which a man rides depends upon the time for which he rides and the speed at which he rides, so as to be directly as either of them when the other is unvaried. If therefore A ride for three hours, and B for five hours, and A ride twice as fast as B, the number of miles which A rides must be to the number of miles which B rides as 6 : 5, the products of the numbers which are proportional to their times and speed. But whenever any quantity so depends upon two others, that it is directly as the first when the second is given, and inversely as the second when the first is given, it must vary as the quotient obtained by dividing the first by the second; that is, dividing numbers taken proportional to these quantities. Now if I ride a journey, the requisite time so depends on the distance which I have to ride and the speed which I employ. It is directly as the distance, and inversely as the speed.

speed. If therefore A has to ride 50 miles and B 40, and A ride twice as fast as B, the time in which A performs his journey must be to the time in which B performs his, as $\frac{50}{2}$ to $\frac{40}{1}$, that is as 25 to 40, or 5 to 8.

83. Any two products are said to be to each other in a ratio *compounded* of the ratios of their factors. Thus the ratio compounded of the ratios of 2 : 5 and 7 : 3 is the ratio of 14 : 15. Hence the ratio compounded of two equal ratios is, the ratio of the squares of the terms of either ratio. Thus the ratio compounded of the equal ratios 9 : 6 and 3 : 2 is the ratio 81 : 36, ($9^2 : 6^2$) or 9 : 4 ($3^2 : 2^2$). For since 9 : 6 : : 3 : 2, it follows (§ 73.) that multiplying both antecedents by 3 and both consequents by 2, 27 : 12 : : 9 : 4 ; or multiplying both antecedents by 9 and both consequents by 6, that 81 : 36 : : 27 : 12. But the ratio 27 : 12 is by definition the ratio compounded of the ratios 9 : 6 and 3 : 2. And thus it appears that, if any four numbers be proportional, their squares are proportional.

84. Hence also it is evident that the ratio compounded of any ratio and its reciprocal is a ratio of equality. Thus the ratio compounded of the ratios of 9 : 6 and 6 : 9 is the ratio of 54 : 54, i. e. a ratio of equality.

85. Again, any ratio being given us, we may conceive any number whatsoever interposed between its terms, and the given ratio as compounded of the ratios of the antecedent to the interposed number, and of the interposed number to the consequent. Thus the ratio of 9 : 6 may be considered as compounded of the ratios of 9 : 2 and 2 : 6. For 9 is to 6 as twice 9 to twice 6, which (by § 83.) is the compound ratio mentioned. In like manner we may conceive any two or more terms interposed, and the given ratio compounded of all the ratios taken in continuation. Thus, we may conceive the numbers 2, 5, and 7 interposed between 9 and 6 ; and the ratio of 9 to 6 will be compounded of the ratios of 9 to 2, 2 to 5, 5 to 7, and 7 to 6. For 9 : 6 : : $9 \times 2 \times 5 \times 7$: : $6 \times 2 \times 5 \times 7$. (§ 69.)

86. From what has been said it is manifest, that the problem of finding a fourth proportional to three given numbers will frequently admit of an abbreviated solution, by substituting lower numbers. For in the first place if the two first terms, or terms of the given ratio, admit of being divided evenly by the same number, we may substitute

ute for them the resulting quotients, as being in the same ratio. Thus, if it be required to find a fourth proportion to 27, 63, and 21, solving the problem at large according to the rule laid down in § 74. we should have to take the product of 63 and 21, and then divide that product 1323 by 27, which gives the quotient 49 as the fourth proportional required. But 3 and 7 being equi-submultiples of 27 and 63 are in the same ratio; (§ 69.) and operating with these lower numbers we find the same result. But secondly, whenever the first and third terms admit of being evenly divided by the same number, we may substitute the resulting quotients: for those equi-submultiples of the given antecedents must be proportional to the given consequent and the consequent sought. (§ 73.) Thus in the last example, after reducing the question to that of finding a fourth proportional to 3, 7, and 21, I may substitute for the first and third of these numbers their equi-submultiples 1 and 7; for putting x for the fourth proportional sought, inasmuch as $3 : 7 :: 21 : x$, the third part of 3 must be to 7 as the third part of 21 to x. And thus we at once arrive at the same result as before, that the number sought is 49.

87. Let it be required to find a number, to which a given number shall be in a ratio compounded of two or more given ratios. The ratio compounded of the given ratios is (by definition) the ratio of the products of their respective terms. Therefore this problem resolves itself into that of finding a fourth proportional to three given terms. Thus, if we want to find a number to which 6 shall be in a ratio compounded of $9 : 5$ and $15 : 36$, it is the same thing as if we were required to find a number to which 6 shall be in the ratio of $9 \times 15 : 5 \times 36$. But it is plain that both terms of this ratio are divisible by 9 and by 5, and that we may therefore substitute the ratio of the resulting quotients $3 : 4$; so that the number sought is $\frac{6 \times 4}{3}$ or 8. Hence it appears that, in solving this problem, if antecedent and consequent of either the same or different ratios admit of being evenly divided by the same number, we may substitute the resulting quotients: and that we therefore ought not to take the products of the corresponding terms of the ratios which we want to compound, till we have inspected them for the purpose of ascertaining whether they be capable of being thus reduced; nor till we have com-

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pared the antecedents of the given ratios with the given antecedent of the ratio whose consequent we seek. For in the last instance, after reducing the question to that of finding a fourth proportional to 3, 4, and 6, the terms may be reduced still lower by substituting for 3 and 6 their equi-submultiples 1 and 2. And thus a question, at first involving very high numbers and appearing to require a very tedious operation, may frequently admit a solution the most brief and facile.

88. The rule (§ 74.) for finding a fourth proportional is commonly called the Rule of Three; because we have three terms of an analogy given us to find the fourth. It may more justly be called the *rule of proportion.* Its very extensive practical application will be shewn in the 13th. Chapter. Meanwhile the young student may exercise himself in the principles of this chapter by solving the following questions; and may easily increase the number of the examples, at pleasure, by substituting any other numbers. Besides investigating the answer by performing the requisite operations of multiplication and division, I would strongly recommend that he should be accustomed to exhibit it by the aid of the symbols denoting those operations. Thus, if it be required to find a fourth proportional to the numbers 23, 24, and 25, the answer may be expressed by $\frac{24 \times 25}{23}$.

Ex. 1. Find a fourth proportional to 15, 40, and 24?

Ex. 2. The two first and the last terms of an analogy are 17, 9, and 234. What is the third term?

Ex. 3. The first and the two last terms of an analogy are 18, 126, and 17. What is the second term?

Ex. 4. What two numbers are in the ratio compounded of the ratios of 7 to 3, 4 to 5, and 11 to 13?

Ex. 5. What two numbers are in the ratio compounded of 7 : 3, and 6 : 14?

Ex. 6. What two numbers are in the ratio compounded of 17 : 3, 3 : 14, and 14 : 16?

Ex. 7. What is the ratio compounded of 17 : 3, and 6 : 34?

Ex. 8. From the analogy, 7 : 25 : : 21 : 75, what equation may be derived?

Ex. 9. From the equation $\overline{12 \times 7} = \overline{14 \times 6}$. what analogy may be inferred?

CHAP.

CHAP. VIII.

On the Nature of Fractions.

89. IF we divide any one whole thing, a foot, a yard, a pound, &c. into three equal parts, any one of them is one third of the whole; written thus— $\frac{1}{3}$. If we take two of them, we take two thirds of the whole, or $\frac{2}{3}$. Such expressions are called *fractions*; the number above the line is called the *numerator* of the fraction, and the number below the line the *denominator*. A *proper* fraction is that whose numerator is less than its denominator. If the numerator be equal to the denominator, or greater, the fraction is called *improper*.

90. The denominator always denotes the number of equal parts, into which the whole thing, or integer, is conceived to be divided. The numerator denotes the number of those parts, which are taken in the fraction. Thus the fraction $\frac{3}{7}$ intimates that the integer is divided into 7 equal parts, and that we take 3 of those parts in the fraction.

91. Hence any improper fraction whose numerator and denominator are equal, such as $\frac{7}{7}$, $\frac{4}{4}$ &c. is equivalent to the one integer which we suppose divided into equal parts. For if we divide a pound, for instance, into 7 equal parts, and take 7 of those parts, we just take the whole pound, neither more nor less. On the other hand it is manifest that $\frac{6}{7}$, or any proper fraction, is less than the whole; and that $\frac{8}{7}$, or any improper fraction whose numerator is greater than its denominator, is greater than the whole. Observe, that we consider and speak of the whole thing divided as *one* integer, whether it consist of a single pound, foot, yard &c. or of ever so many pounds, feet, yards &c.

92. According to the view which has hitherto been given of any fraction, such as $\frac{2}{3}$, we consider it as two thirds of one. But there is another view also, which it will be useful to attend to. It may be considered as the third part of two. This view arises immediately out of the former; for inasmuch as the third part of two is twice as great as the third part of one, it must be just equal to two thirds of one. In like manner the fraction $\frac{3}{7}$ may be indifferently considered either as three sevenths of one, or as the seventh part of 3: the latter being three times greater than the

the seventh part of one, and therefore just equal to three sevenths of one. Thus any fraction may be considered as a quotient, arising from the division of the numerator by the denominator. And hence the fractional notation is commonly employed to express division.

93. The value of any fraction varies directly as the numerator and inversely as the denominator. This appears at once from what has been last said, compared with § 80 and 81. The same thing also will appear from the first view given of a fraction, when we consider that if a whole thing be divided into a given number of equal parts, the greater the number we take of those parts the greater is the quantity we take and in the same ratio: but the greater the number of equal parts into which the whole thing is divided, the less is any one of them, or any given number. Thus $\frac{3}{4}$ is greater than $\frac{3}{7}$ in the ratio of 7 to 4. But $\frac{3}{7}$ is less than $\frac{5}{7}$ in the ratio of 3 to 5. Therefore $\frac{3}{4}$ is to $\frac{5}{7}$ in a ratio compounded of 3 : 5 and 7 : 4 (the direct ratio of the numerators and inverse ratio of the denominators) that is, as 21 : 20.

94. Any fraction is to 1, as the numerator of the fraction to its denominator. Thus $\frac{3}{7}$ is to 1 as 3 to 7. For 1 is equal to $\frac{7}{7}$: (§ 91.) But $\frac{3}{7}$ is to $\frac{7}{7}$ as 3 to 7. Here and throughout the subject when we speak of 1, it is to be understood in the sense explained at the end of § 91.

95. The value of any fraction will remain unaltered, if we multiply or divide both its terms by the same number; that value depending altogether on the *ratio* of its terms, and not their absolute magnitude. Thus the fraction $\frac{3}{4}$ is equal to the fraction $\frac{6}{8}$ or $\frac{9}{12}$ or $\frac{30}{40}$, &c. and the fraction $\frac{6}{12}$ is equal to the fraction $\frac{1}{2}$. For comparing, for instance, the fractions $\frac{3}{4}$ and $\frac{30}{40}$, in the latter the whole thing is conceived to be divided into 10 times as many equal parts as in the former; each of which therefore is 10 times less than each of the former: and consequently if we take 10 times as many of them as of the former, we shall take just the same quantity of the whole. And thus, the twelfth part of a foot being an inch, $\frac{6}{12}$ of a foot is 6 inches; but that is just equal to half a foot, or to the fraction $\frac{1}{2}$. The principles laid down in this section are so simple, that by a few familiar illustrations a very young child may be made to comprehend them; yet upon these simple principles the whole doctrine of fractions depends.

96. Hence

96. Hence we see how we may easily bring a given fraction to lower terms, if its numerator and denominator be capable of being divided evenly by the same number. As any number which evenly divides another is said to *measure* it; so a number which evenly divides two or more numbers is called a *common measure* of them. Numbers which admit no greater common measure than unity are said to be *prime* to each other: and if the terms of a fraction be prime to each other, it is in its lowest terms; as we cannot bring it to any equivalent fraction of lower terms. Thus the fraction $\frac{5}{6}$ is in its lowest terms; and the fraction $\frac{4}{6}$ may be brought to its lowest terms by dividing both numerator and denominator by 2 : for the equal fraction $\frac{2}{3}$ consists of numbers prime to each other.

97. Hence also it is easy to bring a given fractien (supposed to be in its lowest terms) to an equivalent one of another denominator, provided that other be some multiple of the given denominator. Thus, if it be required to bring $\frac{5}{6}$ to an equivalent fraction whose denominator shall be 18 : we observe that in changing the denominator from 6 to 18 we multiply it by 3 ; and therefore to maintain the equality of the two fractions, we must multiply the numerator by 3, so that the required fraction is $\frac{15}{18}$. And if it be required to bring the same fraction $\frac{5}{6}$ to another whose denominator shall be 162, we only want to ascertain by what number 6 must be multiplied in order to give the product 162, that we may multiply 5 the numerator by the same number. This is ascertained by dividing 162 by 6 ; and we thus find that 5×27 is the required numerator. Thus also $\frac{4}{6}$ may be brought to a fraction whose denominator is 15 ; because 15 (though not a multiple of 6) is a multiple of 3 the denominator of the equal fraction $\frac{2}{3}$.

98. To bring a given fraction to its lowest terms, it is only necessary to divide both its terms by their greatest common measure, that is by the greatest number which evenly divides them both. Thus, if we be given the fraction $\frac{21}{42}$, it is plain that both its terms are evenly divisible by 3, or by 7, or by 21. But of these common measures 21 is the greatest, and will therefore give the smallest quotients: so that the lowest terms of the fraction are $\frac{1}{2}$. But if the terms of the given fraction be high numbers, we may be unable to ascertain by inspection whether they be prime to each other; or if not, what number is their greatest common

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mon measure. We proceed therefore to state and demonstrate the method of discovering this.

99. Divide the greater number by the less: if there be no remainder, your divisor is the greatest common measure, inasmuch as no number greater than itself can measure the less of the two given numbers. Thus, if the two given numbers be 12 and 96, 12 must be their greatest common measure; for it measures 96, and no number greater than 12 can measure 12. But if there be a remainder on the first division, then divide your last divisor by that remainder; and so on, till you come to a remainder which will measure the last divisor. This remainder is the greatest common measure of the two given numbers: and therefore if you find no such remainder till you come to 1, the given numbers are prime to each other. Thus, if the two given numbers be 182 and 559; dividing the greater by the less we find the quotient 3 and the remainder 13: then dividing 182 by 13, we find the quotient 14, and no remainder. I say then that 13, the remainder which measures the first divisor, is a common measure of 182 and 559, and their greatest common measure. First, it is a common measure of them; for it measures 182, and therefore 3 times 182; and therefore $\overline{3 \times 182} + 13$, or the sum of 3 times 182 and 13. But that is equal to 559, as we saw by the first division. Therefore it is a common measure of 182 and 559.—But secondly, it is their greatest common measure. For suppose any greater number, for instance 17, to be a common measure of 182 and 559. Since it measures 182 it must also measure 3 times 182: and since 559, it measures $\overline{3 \times 182} + 13$, which is equal to 559. Inasmuch then as it measures both 3×182 and $\overline{3 \times 182} + 13$, it must measure 13; that is, a number greater than 13 must measure 13: which is absurd. Therefore 13 is the greatest common measure of 182 and 559. Q. E. D.

100. Let us propose the same proof in a general manner, putting letters for the numbers. Let the numbers, whose greatest common measure we want to find, be represented by the letters a and b, of which a is the less: and dividing b by a let the quotient be represented by x and the remainder by c. We may infer that $b = \overline{x \times a} + c$. Then dividing a by c, let the quotient be y and the remainder d. We may infer that $a = \overline{y \times c} + d$. Then dividing c by d, let

let the quotient be z without any remainder. I say d is the greatest common measure of a and b. For since it measures c, it must measure $\overline{y\times c} + d$, that is a. And since it measures a and c, it must measure $\overline{x\times a} + c$, that is b. But if we should suppose any greater number than d to be a common measure of a and b, since it measures a, it must measure $x\times a$; and since it measures both $x\times a$ and $\overline{x\times a} + c$ (i. e. b.) it must measure c, and therefore $y\times c$. And since it measures both $y\times c$ and $\overline{y\times c} + d$ (i. e. a) it must measure d; that is a number greater than d will measure d: which is absurd. And in like manner, if there be a remainder of 1 on the last division, we can prove that 1 is the greatest common measure of a and b; that is, that the numbers a and b are prime to each other.

Ex. 1. What is the value of the fractions $\frac{24}{7}$, $\frac{12}{4}$, $\frac{15}{4}$, and $\frac{29}{3}$? (See § 102. next chapter.)

Ex. 2. What fractions are equal to $5\frac{3}{7}$, $2\frac{3}{8}$, $7\frac{4}{5}$, and $12\frac{1}{3}$?

Ex. 3. Express 7, 8, 9, and 10 by fractions whose denominators shall be 4, 5, 6, and 7.

Ex. 4. What is the ratio of $\frac{5}{7}$ to $\frac{3}{7}$? of $\frac{5}{7}$ to $\frac{5}{11}$? and of $\frac{7}{8}$ to 1?

Ex. 5. Express $\frac{5}{6}$, $\frac{2}{3}$, and $\frac{7}{9}$, by equivalent fractions whose denominator shall be 18.

Ex. 6. Bring the fractions $\frac{63}{891}$, $\frac{121}{1573}$, and $\frac{15}{105}$ to their lowest terms?

Ex. 7. Bring the fractions $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{5}{6}$ to a common denominator? (See § 104.)

Ex. 8. Also the fractions $\frac{1}{3}$, $\frac{11}{12}$, and $\frac{4}{15}$?

Ex. 9. What is the ratio of $\frac{3}{7}$ to $\frac{4}{5}$? of $\frac{2}{8}$ to $\frac{3}{11}$? and of $\frac{7}{8}$ to $\frac{8}{9}$?

Ex. 10. A. and B. got legacies. A got £750; and his legacy was $\frac{5}{7}$ths of B's. What was B's?

Ex. 11. What is the greatest common measure of 153 and 493? of 336 and 1645? and of 133 and 459?

CHAP. IX.

On Addition and Subtraction of Fractions.

101. WHEN the fractions whose sum or difference we want to find have the same denominator, the method of performing

performing those operations is as obvious, as the addition or subtraction of integers. For it is as plain that the sum of two ninths and five ninths is seven ninths, and that their difference is three ninths, (i. e. that $\frac{2}{9}+\frac{5}{9}=\frac{7}{9}$ and that $\frac{5}{9}-\frac{2}{9}=\frac{3}{9}$) as that the sum of two shillings and five shillings is seven shillings, and their difference three shillings. Ninths in the former case, and shillings in the latter, are but the denomination of the numbers, which we add or subtract: and in place of the fractional notation, the column in which the numbers 2 and 5 stand might be headed with the denomination ninths, as it is commonly with the denomination shillings.

102. If the sum of the numerators exceed the common denominator, it is easy to ascertain what integral or mixed number it is equal to, by dividing the sum of the numerators by the denominator. Thus the sum of $\frac{8}{9}$, $\frac{7}{9}$, and $\frac{5}{9}$ is $\frac{20}{9}$. But since $\frac{9}{9}$ is equal to 1 (§ 91.) $\frac{18}{9}$ must be equal to 2; and therefore $\frac{20}{9}$ to $2\frac{2}{9}$. In like manner in the addition of pounds shillings and pence, if the sum of the numbers standing in the column of pence exceed 12, we divide it by 12, the number of pence in one shilling, &c. And in fact the operations, which the child is taught in addition and subtraction of what are called divers denominations, are all really fractional operations; 8 pence, for instance, being $\frac{8}{12}$ of a shilling, and 8 shillings $\frac{8}{20}$ of a pound. And in the case of half-pence and farthings, even the fractional notation is introduced.

103. In subtraction, if the fractional part of the subtrahend exceed the fractional part of the minuend, we combine with the latter a unit borrowed from the integral part of the minuend; and therefore have to conceive the right-hand integral digit of the minuend lessened by one. Thus, in subtracting $2\frac{7}{9}$ from $15\frac{4}{9}$, since we connot subtract $\frac{7}{9}$ from $\frac{4}{9}$, we subtract it from $1+\frac{4}{9}$, i. e. from $\frac{13}{9}$. The remainder is $\frac{6}{9}$: and we have then to subtract 2, not from 15, but from 14. But in place of this, carrying 1 to the subtrahend we subtract 3 from 15. See § 16.

104. If the fractions which we are required to add or subtract have different denominators, we must first bring them to equivalent fractions of the same denominator; and then proceed as before. Thus, if we have to find the sum or difference of $\frac{3}{7}$ and $\frac{7}{9}$, it is necessary to bring them both to the same denominator. Now we can bring $\frac{3}{7}$ to an

an equal fraction of any denominator, which is a multiple of 5, and $\frac{7}{9}$ to any which is a multiple of 9. (§ 97.) But the product of 5 and 9 being a multiple of both, we may bring the two given fractions to the denominator 45. And in doing this, in order to multiply both terms of the fraction by the same number (i. e. in order to keep it of the same value § 95.) we must multiply the numerator of each fraction by the denominator of the other; when they become $\frac{27}{45}$ and $\frac{35}{45}$; whose sum is $\cancel{\frac{32}{45}}$ ~~or~~ $\cancel{1\frac{32}{45}}$, and their $\frac{62}{45}$ or difference $\frac{8}{45}$. Hence, let there be ever so many fractions, of ever so various denominators, to be added, the reason is plain of the common practical rule, to take the product of all the denominators for a common denominator, and then to multiply the numerator of each fraction by all the denominators except its own: inasmuch as it is by these that we have multiplied its denominator.

105. But it is desirable to keep the terms of our fractions as low as possible; and we may often find a number less than the product of all the denominators, which is yet a multiple of them all, and will therefore answer for a common denominator. Thus, if we have to add $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{9}$; 36 the product of 9 and 4 being also a multiple of 3 will be a common denominator, and the fractions become $\frac{24}{36}$, $\frac{27}{36}$, and $\frac{20}{36}$. I forbear at present from bringing forward the rule for finding the least common multiple of two or more given numbers; (see Chap. 18.) as it would be hard to make the demonstration of it clearly intelligible without a little knowledge of Algebra.

Ex. 1. What is the sum of $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$?

Ex. 2. What is the sum of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$?

Ex. 3. What is the excess of $\frac{7}{8}$ above $\frac{3}{8}$? and of $\frac{7}{8}$ above $\frac{6}{7}$?

Ex. 4. What is the difference between the sum of $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ and the sum of $\frac{4}{5}+\frac{2}{3}+\frac{11}{15}$?

Ex. 5. A man left a legacy of 10,000£ among three sons, so that the eldest should have $\frac{1}{4}$ of it, and the second $\frac{1}{3}$ of it. What proportion of the legacy did the youngest receive?

CHAP. X.

On Multiplication and Division of Fractions.

106. FROM what has been said in the first section of the preceding chapter, it is evident that to multiply any fraction

fraction by an integer we need only multiply its numerator by the integer: and that any fraction will be divided by an integer, by dividing its numerator by the integer, whenever the integral divisor measures the numerator. For it is as plain that 3 times $\frac{9}{10}$ is $\frac{27}{10}$, or $2\frac{7}{10}$, and that the third part of $\frac{9}{10}$ is $\frac{3}{10}$, as that 3 times 9 is 27, and that the third part of 9 is 3.

107. But if we want to divide $\frac{7}{10}$ by 3, we cannot obtain the quotient by this process, as 3 does not measure 7. The third part of 7 is 2 and $\frac{1}{3}$: so that the third part of 7 tenths is $\frac{2}{10}$ and one third of a tenth. But we have still to enquire what fraction is equal to one third of a tenth, or what is the quotient of $\frac{1}{10}$ divided by 3. In such cases therefore we employ an operation always equivalent to the division of the numerator, namely the multiplication of the denominator. And accordingly the third part of $\frac{7}{10}$ is $\frac{7}{30}$. For if we suppose any whole thing divided first into 10 equal parts, and then into 30 equal parts, the latter being 3 times as many as the former must each of them be 3 times less than each of the former; and therefore 7 of them must be 3 times less than 7 of the former: or in other words $\frac{7}{30}$ is the third part of $\frac{7}{10}$. See § 93.

108. Thus the universal rule for dividing a fraction by an integer is, to multiply its denominator by the integer. And whenever we have to multiply a fraction by an integer which measures its denominator, the product is exhibited in lower terms by dividing the denominator by the integer, than by multiplying its numerator. Thus, 3 times $\frac{2}{9}$ is by the one process $\frac{6}{9}$; by the other $\frac{2}{3}$: results which we know are eqnal from § 95.

109. From the methods of multiplying and dividing a fraction by an integer, it is easy to pass to multiplication and division by a fraction. To multiply by a fraction, multiply by its numerator, and divide the product by its denominator. To divide by a fraction, divide by its numerator and multiply the quotient by its denominator. Thus, to multiply $\frac{5}{7}$ by $\frac{3}{4}$, multiply $\frac{5}{7}$ by 3; and divide the product $\frac{15}{7}$ by 4: the quotient $\frac{15}{28}$ is the product sought. For the multiplier $\frac{3}{4}$ being the fourth of 3, (§ 92.) the first product $\frac{15}{7}$ (obtained by multiplyiug $\frac{5}{7}$ by 3) is 4 times too great: and therefore its fourth part must be the true product sought. In like manner, if we have to divide $\frac{5}{7}$ by $\frac{3}{4}$, dividing by 3 the quotient $\frac{5}{21}$ is 4 times too small, as we have

have employed a divisor four times too great: and therefore the true quotient must be 4 times $\frac{5}{21}$ or $\frac{20}{21}$.

110. Hence appears the reason of the practical rule commonly given for multiplying a fraction by a fraction; namely, take the product of the numerators for the numerator of your product, and the product of the denominators for the denominator of your product. It appears that the latter operation is in fact a division of the fraction, in order to reduce the product to its just amount. Another proof of the operation may be derived from the principles laid down in § 78. and § 94. For if we want to multiply $\frac{5}{8}$ by $\frac{2}{3}$, unity must be to the multiplier as the multiplicand to the product. But $1 : \frac{2}{3} :: 3 : 2$. Therefore $3 : 2 :: \frac{5}{8}$ to the product, which fourth proportional must be obtained by multiplying $\frac{5}{8}$ by 2, and dividing the product by 3. (§ 74.)

111. The reason is equally evident of the practical rule commonly given for dividing a fraction by a fraction; namely to multiply by a fraction the reciprocal of the divisor. For it appears by comparing the operations, that to divide $\frac{5}{8}$ by $\frac{2}{3}$ is the same thing as to multiply $\frac{5}{8}$ by $\frac{3}{2}$. Another proof of the operation may be derived from the principle laid down in § 79. For if we have to divide $\frac{5}{8}$ by $\frac{2}{3}$ the divisor must be to unity as the dividend to the quotient. But $\frac{2}{3} : 1 :: 2 : 3$. Therefore $2 : 3 :: \frac{5}{8}$ to the quotient, which fourth proportional must be obtained, by multiplying $\frac{5}{8}$ by 3 and dividing the product by 2.

112. The same things are at once applicable to the multiplication or division of an integer by a fraction. The product of 7 multiplied by $\frac{3}{4}$ is $\frac{21}{4}$, or $5\frac{1}{4}$; the same as the product of $\frac{3}{4}$ multiplied by 7. The quotient of 7 divided by $\frac{3}{4}$ is $\frac{28}{3}$, or $9\frac{1}{3}$, the same as the product of 7 multiplied by $\frac{4}{3}$. Any integer indeed may be conceived as an improper fraction, whose denominator is 1. And here let it be observed that whenever our multiplier is a proper fraction the product must be less than the multiplicand; and whenever our divisor is a proper fraction, the quotient mnst be greater than the dividend. For when we talk of multiplying any thing by $\frac{3}{4}$, we really mean taking three fourths of the multiplicand; as when we talk of multiplying any thing by 1, we mean taking the multiplicand once. But $\frac{3}{4}$ (or any proper fraction) being less than one, three fourths of the multiplicand must be less

than the whole multiplicand. On the other hand in division, the less the divisor is the greater must be the quotient. Now if we divide any number by 1, the quotient is equal to the dividend. Therefore if we divide by a proper fraction, the quotient must be greater than the dividend. It is plain that since 1 is contained in 7 seven times, $\frac{3}{4}$ (which is less than 1) must be contained in 7 more than seven times.

113. Observe that if we multiply any fraction by its denominator the product is the numerator integral. Thus, the product of $\frac{3}{5}$ multiplied by 5 is 3, of $\frac{5}{7}$ multiplied by 7 is 4, &c. For $\frac{3}{5} \times 5$ is in the first place $\frac{15}{5}$; but to reduce this to its lowest terms, we should divide both terms by 5, when the result will be $\frac{3}{1}$ or 3. But we may save the trouble both of the multiplication and division, the latter just undoing the former. The same thing indeed at once appears from considering $\frac{3}{5}$ as the quotient of 3 divided by 5, (§ 92.) and from the principle that the product of the divisor and quotient is the dividend. It is evident also that any integer may be brought to a fractional form of any given denominator, by taking for the numerator of our fraction the product of the integer and given denominator. Thus, 7 is equal to $\frac{35}{5}$, to $\frac{42}{6}$, &c. It is plain indeed that since there are 5 fifths in 1, there must be 35 fifths in 7.

Ex. 1. $\frac{4}{9} \times 5 = ?$ $\frac{4}{9} \div 5 = ?$ $\frac{5}{12} \times 3 = ?$ $\frac{9}{11} \div 3 = ?$

Ex. 2. $\frac{4}{9} \times \frac{1}{2} = ?$ $\frac{4}{9} \div \frac{1}{2} = ?$ $\frac{5}{27} \times \frac{3}{5} = ?$ $\frac{5}{27} \div \frac{5}{3} = ?$

Ex. 3. $\frac{4}{9} \times 9 = ?$ $\frac{5}{27} \times 27 = ?$ $2 \times \frac{1}{2} = ?$ $2 \div \frac{1}{2} = ?$

Let the examples of multiplication be proved by division, and v. v.

Ex. 4. What fractional part of 3 is $\frac{2}{3}$ds. of 4?—On this question let the student observe that $\frac{2}{3}$ds. of 3 must be less than $\frac{2}{3}$ds. of 4 in the ratio of 3 : 4; and therefore the fractional part sought of 3 must be greater than $\frac{2}{3}$ds. of it, in the ratio 4 : 3. Whence the following analogy, as 3 : 4 :: $\frac{2}{3}$: $\frac{8}{9}$. And accordingly $\frac{8}{9}$ths. of $3 = 2\frac{2}{3}$; and $\frac{2}{3}$ds. of $4 = 2\frac{2}{3}$.

Ex. 5. What fractional part of 7 is $\frac{3}{7}$ths. of 5?

Ex. 6. A man spent $\frac{1}{4}$th. of a legacy in 5 months; $\frac{2}{3}$ds. of the remainder in 7 months; and then had £95 left. What was the amount of the legacy?

Here observe that, when he had spent $\frac{1}{4}$ of the legacy, $\frac{3}{4}$ths. were left. And when he had spent $\frac{2}{3}$ds. of this, he had spent in addition $\frac{1}{2}$ of the whole: for $\frac{2}{3}$ of $= \frac{6}{12} = \frac{1}{2}$.

Ex. 7.

Ex. 7. A man devised $\frac{2}{3}$ds. of his fortune to his eldest son; $\frac{2}{3}$ds. of the remainder to his younger; and the rest to his widow. The elder son's share exceeded the younger's by £750. How much had the widow? Here we are told that $\frac{2}{3}$ds.—$\frac{2}{9}$ths. of the fortune (or $\frac{4}{9}$ths. of it) amounted to 750£; whence we find the whole fortune, of which the widow had $\frac{1}{9}$th.

CHAP. XI.

On the Nature of Decimal Fractions.

114. After the doctrine of vulgar Fractions has once been mastered, decimal fractions can present no difficulty to the student. It is only necessary to take a clear view of the notation employed in them. In decimal fractions we use no other denominators than 10, 100, 1000, &c. and those denominators are not written, but intimated by the position of the *decimal point*; for we understand as many cyphers following a unit in the denominator, as there are digits standing on the right hand of the decimal point. Thus the decimal fraction .7 is equivalent with the vulgar fraction $\frac{7}{10}$; the decimal .037 with $\frac{37}{1000}$. And it appears that in decimal notation we write only the numerator, but have the understood denominator intimated by the decimal point: and that to write the vulgar fraction $\frac{37}{10000}$ decimally, we need only to omit the denominator and to prefix to the numerator the decimal point followed by two cyphers—thus .0037. It is necessary to prefix two cyphers, in order that four digits may stand on the right hand of the decimal point, as there are four cyphers following the unit in the denominator; and in order that the digits of the decimal fraction may yet express the numerator given.

115. From what has been said it appears, that annexing one or more cyphers to a decimal fraction on the right hand makes no change in the value of the fraction, inasmuch as for every cypher annexed both numerator and denominator are increased ten fold: but that prefixing one or more cyphers on the left hand, decreases the value of the fraction ten fold for every cypher prefixed, inasmuch

as

as the understood denominator is increased so many fold without any change in the numerator. Thus the decimals .7, .70, .700, &c. or their equivalent vulgar fractions $\frac{7}{10}$, $\frac{70}{100}$, $\frac{700}{1000}$ &c. are all of the same value (§ 95.): but .07 is ten times less than .7, .007 one hundred times less than .7, since $\frac{7}{100}$ is the tenth part of $\frac{7}{10}$, and $\frac{7}{1000}$ the hundredth part. (§ 107.) And in general any decimal fraction is multiplied by 10, 100, 1000, &c. by removing the decimal point one, two, three, &c. places towards the right hand; or divided, by removing it towards the left. Thus if I have to multiply the decimal .37 by 10, the product is 3.7; in which the 3 is integral, and only the 7 affected by the decimal point. For 10 times $\frac{37}{100}$ is $\frac{37}{10}$ (§ 108.) or $3\frac{7}{10}$ i. e. 3.7. Whereas 24.37 (in which the numbers on the left hand of the decimal point are integral) is divided by 10, by bringing another digit under the decimal point, removing it one place towards the left; which gives the quotient 2.437. For $24\frac{37}{100}$ is equal to $\frac{2437}{100}$ (24 being equal to $\frac{2400}{100}$): the tenth part of which is $\frac{2437}{1000}$, i. e. $2\frac{437}{1000}$; or 2.437.

116. In short there is no operation on decimals, which the student may not investigate by performing the same operation according to the notation of vulgar fractions; and then expressing the result decimally according to the simple rule of decimal notation. I shall proceed however to exhibit this investigation briefly in the following chapter: after premising the method of bringing any vulgar fraction to the decimal form.

117. Suppose then we are required to bring the fraction $\frac{3}{4}$ to the decimal form, or to find a decimal equal to the vulgar fraction $\frac{3}{4}$. This is in fact to bring $\frac{3}{4}$ to an equal fraction, whose denominator shall be some power of 10. Now we know that $\frac{3}{4}$ is equal to $\frac{30}{40}$, $\frac{300}{400}$, $\frac{3000}{4000}$, &c. or that annexing an equal number of cyphers to both numerator and denominator will not change the value of the fraction. But instead of 4 followed by any number of cyphers, we want that the denominator should be 1 followed by some number of cyphers. This change can be effected no otherwise than by dividing some of the former denominators by 4: and then to maintain the value of the fraction unaltered we must also divide the numerator by 4. Let us accordingly divide both terms of the fraction $\frac{300}{400}$ by 4; and it becomes $\frac{75}{100}$, a fraction capable of being written decimally

decimally; and the decimal sought is .75. Hence appears the reason of the practical rule, to annex any number of cyphers that may be necessary to the numerator, and divide by the denominator, and point off from the quotient as many decimal places as you have annexed cyphers. By this process in fact both terms of the given fraction are multiplied by the same power of 10, and divided by the denominator: and thus the value of the obtained fraction, which is capable of decimal notation, is the same with that of the given one. Thus $\frac{1}{2}=.5$; $\frac{1}{5}=.2$; $\frac{2}{25}=.08$: for $\frac{2}{25}=\frac{200}{2500}$ or $\frac{8}{100}$ or .08.

118. But in reducing vulgar fractions to decimals, we shall frequently find that, continuing the division ever so far, we can never arrive at an exact quotient, but shall at length come to a remainder the same with the given numerator or one of the former remainders; and therefore from that recurrence the same digits must continually recur in the quotient. (See § 123.) And this indeed must always be the case, except when the denominator of the given fraction reduced to its lowest terms is 2 or 5, or some power of these numbers, or some product of their powers. Thus in reducing $\frac{1}{3}$ to a decimal, we find it equal to .333, &c. and $\frac{1}{9}=.111$, &c. $\frac{7}{15}=.466$, &c. $\frac{41}{333}=.123123$, &c. Such are called *interminate* decimals, and are said to *circulate* through the figure or figures which continually recur. A method of calculating the vulgar fraction, which will produce any given circulating decimal, shall be assigned and demonstrated in another part of this work. (See c. 20.)

Ex. 1. Express $\frac{9}{10}$, $\frac{9}{100}$, $\frac{91}{100}$, $\frac{73}{1000}$ decimally.

Ex. 2. What vulgar fractions are equal to the decimals .75, .075, .024, .0015?

Ex. 3. Express $\frac{3}{8}$, $\frac{4}{25}$, $\frac{5}{16}$, $\frac{17}{32}$ decimally?

Ex. 4. Express $\frac{5}{6}$, $\frac{2}{7}$, $\frac{8}{9}$, $\frac{13}{18}$ decimally?

Ex. 5. Multiply .0015 by 1000?

Ex. 6. Divide .75 by 1000?

CHAP. XII.

Arithmetical Operations on Decimals.

119. SUPPOSE we have to find the sum or difference of the decimals .07 and .834. They are equivalent to the vulgar fractions $\frac{7}{100}$ and $\frac{834}{1000}$; which must be brought to the

the same denominator, before we can find their sum or difference; and then become $\frac{70}{1000}$ and $\frac{834}{1000}$: whose sum is $\frac{904}{1000}$, or .904, and their difference $\frac{764}{1000}$, or .764. Now if we write the given decimals thus—$\begin{array}{r}.834\\.07\end{array}$ — so as that the decimal points shall stand in line, we may understand a cypher after the 7 on the right hand, as it will make no change in the addition or subtraction; and then proceeding to take the sum or difference, we shall have the same results. And here we see that one advantage of decimal above vulgar fractions consists in the facility, with which they are brought to the same denominator. The decimal point of the sum or difference must also stand in line with the decimal points of the fractions, which we add or subtract; so that any digits standing on its left hand are integral. Thus if we add .9643 and .8, the sum is 1.7643. For brought to the same denomination in vulgar fractions the given fractions are $\frac{9643}{10000}$ and $\frac{8000}{10000}$; the sum of whose numerators is 17643. But the improper fraction $\frac{17643}{10000}$ is equal $1\frac{7643}{10000}$, that is to 1.7643.

120. Suppose we have to multiply .04 by .3; that is, the vulgar fraction $\frac{4}{100}$ by $\frac{3}{10}$: the product is $\frac{12}{1000}$, or .012. The number of cyphers in the denominator of the product being necessarily the sum of the cyphers in the denominators of the factors, the denominator of the product must be intimated by pointing off as many decimals in the product of the numerators (multiplied as integers) as the sum of the decimal places in both the factors. The same rule applies, where one of the factors is an integer, or one or both mixed numbers. Thus the product of 1.2 multiplied by .8 is .96. For $1.2=\frac{12}{10}$, and $.8=\frac{8}{10}$: the denominator of whose product is 100; and this product is intimated by pointing off two decimal places from the product of their numerators 96. In like manner the product of 12 multiplied by .8 is 9.6: but of 12 multiplied by .008 is .096.

121. Hence the rule of division is obvious. For since the dividend is always the product of the divisor and quotient, there must be as many decimal places in the dividend as the sum of the decimal places in the divisor and quotient. And if the given dividend have fewer decimal places than the divisor, we make the number equal by annexing decimal cyphers to the dividend on the right hand, which we have seen cannot alter the value of the dividend. (§ 115.)

(§ 115.) Thus dividing 1.2345 by .05 gives the quotient 24.69: for we must point off two decimal places from the quotient, that the decimal places in the dividend may equal the sum of those in the divisor and quotient. But 1.2345 divided by .0005 gives the 2469 integral: and 123.45 divided by .0005 gives the quotient 246900 integral: for we must annex two cyphers to the dividend in order to make the number of its decimal places equal to those in the divisor, and then there can be no decimal place in the quotient. It is plain that, in calculating the number of decimal places in the dividend, we must take into account every decimal cypher, which we have occasion to annex to the remainders for continuing the division. And that if the number of digits in the quotient be less than the number of decimal places requisite in it, we must supply decimal cyphers on the left hand. Thus dividing .25 by 4 integral gives the quotient .0625. For since there is no decimal place in the divisor, there must be as many in the quotient as in the dividend; and we have occasion to annex two decimal cyphers to the dividend in order to get a complete quotient. The truth of all these results will likewise appear by expressing our decimals as vulgar fractions. Thus .25 decimal is $\frac{25}{100}$ or $\frac{2500}{10000}$; the fourth part of which is $\frac{625}{10000}$ i. e. .0625.

122. In division of integers, when the divisor does not measure the dividend, it is common to continue the division decimally, annexing cyphers to the remainders, and pointing off as many decimal places from the quotient as we have annexed cyphers. For by this operation we in fact reduce to the decimal form the vulgar fraction which is part of the quotient. Thus in dividing 25 by 8, we have seen that the real quotient is $3\frac{1}{8}$, of which the fractional part may be turned into the decimal .125.

123. But here let the student observe, that it cannot be requisite for any practical purpose to continue this process as far, as might be necessary in order to obtain a perfectly accurate result. Thus if I divide 63 by 29, and continue the annexation of decimal cyphers, I find the quotient

2.1724137931034482758620689655l, &c.

the circulation of the same decimal digits not commencing till the 29th. place of decimals. But it would be quite useless in practice to continue the process so far. The three first decimal digits give us the fractional remainder

within

within less than $\frac{1}{1000}$th. part; (for we find it that it is somewhat more than $\frac{172}{1000}$, but less than $\frac{173}{1000}$): the four first, within less than $\frac{1}{10,000}$th. &c.—That the fraction $\frac{5}{29}$ reduced to the decimal form *must* at length circulate, will easily appear, if we consider—1st. that it cannot produce a terminate decimal, since there is no digit which multiplying 29 can give a product ending with a cypher:—2ly. that some one of the remainders must at length recur, since each remainder must be less than 29, and cannot be either 10 or 20; so that there are but 26 possible remainders.

Ex 1. What is the sum of 20.05 + 1.5 + .005?

Ex. 5. What is the difference between 3.75 and 375?

Ex. 3. What is the product of 375 × .5? of 3.75 × ·05? and of 3.75 × 10.5?

Ex. 4. What is the quotient of 3.75 ÷ 5? of 3.75 ÷ .15? and of 375 ÷ .15?

CHAP. XIII.

Practical Application of the Rule of Proportion.

124. IF I can purchase 4 yards Cloth for £2 : 15*s.* and want to know what quantity I ought to get, at the same rate, for £2 : 12*s.* it can be ascertained by the rule of proportion. For the quantities purchased at a given rate must be directly as the prices paid: therefore 4 yards, the quantity purchased for £2 : 15*s.* must be greater than the quantity purchased for £2 : 12*s.* in the same ratio in which the former sum of money is greater than the latter, or in the ratio of 55*s.* to 52*s.* or of the abstract numbers 55 : 52. Therefore 55 : 52 : : 4 yards to the quantity sought: which fourth proportional is found (§ 74.) by taking the product of the second and third terms and dividing it by the first; or is $\frac{4 \times 52}{55}$, or $3\frac{43}{55}$ yards, that is, 3 yards 9 inches and somewhat more than one third of an inch.

125. This example may serve to illustrate the following general rule for solving all such questions. 1st. Place as the third term of your analogy that given quantity, which is of the same denomination with the thing sought. Thus, in

in the last example, the question being what quantity of cloth can I get, the given quantity of cloth, or 4 yards, must be the third term of the analogy. 2ly. Consider from the nature of the question whether the answer must be more or less than that given quantity; and accordingly state the other two given terms in a ratio of less or greater inequality. Thus, in the last example, as it is plain that the answer must be less than 4 yards (that is, that 4 yards must be to the quantity sought in a ratio of greater inequality) the two given sums of money must be stated in a ratio of greater inequality; or the greater must be made the antecedent. 3ly. Having thus stated your terms, if the two first be mixed, or fractional, numbers, bring them to the same denomination; and then, altogether disregarding their denomination, proceed to find a fourth proportional by the rule given in § 74. availing yourself of any such abbreviations as the numbers admit. See § 86. Thus, in the last example, we brought both the sums of money to the denomination *shillings*, and then disregarded their denomination, as it is only the ratio of the numbers that is concerned.

126. We may now form another question to prove the correctness of our work in the last: viz. If I pay £2 : 12*s*. for $3\frac{43}{55}$ yards of cloth, what must I pay, at the same rate, for 4 yards? Here the thing sought being a sum of money, the given sum of money £2 : 12*s*. must be the third term of the analogy. And as the answer must be a greater sum of money, the two given quantities of cloth must be stated in a ratio of less inequality, that is, as $3\frac{43}{55}$ to 4. These terms, brought to the same denomination 55ths. become $\frac{208}{55}$ and $\frac{220}{55}$ whose ratio rejecting the common denominator is that of the numbers 208 and 220. So that, as 208 : 220 :: £2 : 12*s*. to the sum sought. The two first terms being both divisible by 4, we may substitute for them the ratio of the quotients 52 : 55; and we may then see by inspection that the fourth proportional sought is 55*s*. or £2 : 15, since there are 52*s*. in £2 : 12.

127. Let us apply our rule to another example, such as is commonly proposed as a question in the rule of Three *inverse*.

If a mason can build a wall in 6 days, working 7 hours a day, how many hours a day must he work in order to build it in 5 days? It is plain that he must work a greater number of hours each day; and therefore the fourth term of of the analogy must be greater than the third term, 7 hours: and hence the two first terms must be stated in a ratio of less inequality, thus—as 5 : 6 : : 7 hours to the number of hours sought. The answer therefore is $\frac{42}{5}$, or $8\frac{2}{5}$ hours; that is 8 hours and 24 minutes. The truth of this may be proved by forming another question in which this answer shall be one of the given terms, and any one of the former given terms shall be the term sought. Thus: if a mason, working 8 hours and 24 minutes a day, build a wall in 5 days, how many hours a day must he work in order to build it in 6 days? or—in how many days shall he build it working 7 hours a day? or lastly—if he build it in 6 days working 7 hours a day, in how many days shall he build it working each day 8 hours and 24 minutes? And thus whenever a question has been solved by the rule of proportion, the student may be profitably exercised in forming three other questions adapted to prove the truth of his answer: since we can find any one of the four terms of an analogy from having given the three others.

128. Those, who have learned Arithmetic according to the common systems, will perceive that I wholly disregard the distinction introduced in them between the Rule of Three *direct* and *inverse*. It is perfectly useless: and like all useless distinctions it is calculated only to perplex the learner and to render a simple subject complicated. They will also perceive that I place that as the third term of the analogy, which is commonly stated as the second. The common order never could have obtained such a currency, as to have been admitted even into some treatises written by men of science, unless Arithmetic had been degraded from the rank of science. Unimportant as the difference may appear to some in practice, the vulgar arrangement is mischievously calculated to conceal from view the principles of *ratio*, on which the solution proceeds: and is intrinsically absurd; as absurd, as if we spoke of the ratio between such heterogeneous quantities as 5lbs. of beef and 3 bars of music.

129. Hitherto we have supposed cases, in which the question is affected only by one given ratio: but there may be two ratios, or ever so many, concerned in the question. For instance: if 3 masons working 7 hours a day build a wall in 6 days, how many hours a day must 4 masons work in order to build it in 5 days? Here, if we consider only the

the decreased number of days, 7 hours would be less than the answer in the ratio of 5 : 6. And if we consider only the increased number of masons, 7 hours would be greater than the answer in the ratio of 4 : 3. Therefore 7 hours is to the answer in a ratio compounded of 5 : 6 and 4 : 3, that is, in the ratio of 20 : 18 (§ 83. 87.) or of 10 : 9. But 10 : 9 : : 7 : $\frac{63}{10}$. Therefore the answer is $6\frac{3}{10}$ hours, or 6 hours and 18 minutes. The general rule therefore for solving all such questions is this :—1st. determine the third term of the analogy as before. 2ndly. Consider how the answer would be affected by each of the ratios separately, and arrange the terms of each ratio accordingly, by the rule before given. 3rdly. Multiply the third term by the product of all the consequents and divide by the product of the antecedents. But here much trouble may frequently be saved by observing whether the terms of the given ratios may be reduced to lower, according to the rule given § 87.

130. That the student may be the more thoroughly convinced of the justice of the principles, on which we have proceeded in the solution of this question, let it be observed that the question might be resolved into two: first to find how many hours a day the same number of masons should work in order to build the wall in 5 days; and secondly, after having found this, to find how many hours a day 4 masons should work in order to build it in the same number of days. The first question would be solved by the analogy—as 5 : 6 : : 7 : $\frac{42}{5}$; and the second question by the analogy—as 4 : 3 : : $\frac{42}{5}$ to the answer. And thus we see that the answer would be obtained by multiplying 7 hours by the consequent of each of the given ratios, and dividing by the antecedent of each.

131. Let us now apply our rule to a question involving *three* distinct ratios. If a family of 13 persons spend £64 on butcher's meat, in 8 months when the meat is 6d. per lb. how much (at the same rate) should a family of 12 persons spend in 9 months, when the meat is $6\frac{1}{2}$ per lb? Here £64 is to the sum sought in a ratio compounded of the direct ratios of the number of consumers, the times of consumption, and the prices of the meat per lb. that is, in a ratio compounded of the ratios of 13 to 12, 8 to 9, and 6 to $6\frac{1}{2}$. But the last ratio being the same with that of 12 to 13, the terms of the first and last ratios may be

erased

erased (§ 87.) and therefore as 8 : 9 : : £64 to the answer, which is known by inspection to be £72.

132. Considering the different questions, to which have hitherto applied the rule of proportion, any person of common sense must see the absurdity of conceiving them solved by different rules;—must see that it would be absurd to talk of the question § 124. as solved by the *rule of cloth*, the question § 126. by the *rule of masonry*, &c. Yet this absurdity would not be a whit greater than that, which pervades all the common systems of Arithmetic, in presenting to the student as distinct rules the *Rule of Interest*, of *Exchange*, of *Fellowship* &c. &c. All these are but different applications of the one *Rule of Proportion:* and any student, acquainted scientifically with the principles of proportion, needs only to have the meaning of the terms employed in these different subjects distinctly explained to him, in order to be able to solve every question that can occur in them. We shall proceed to exemplify this in a few instances.

133. After explaining the meaning of the terms *Interest* and *per cent. per annum*—if it be asked, At 5 per cent. per ann. what is the interest of £275 : 10 for $3\frac{1}{2}$ years? it is plain that we are given the interest of 100£ for 1 year, in order to find the interest of £275 : 10 at the same rate for $3\frac{1}{2}$ years. The third term of the analogy therefore must be the given interest £5; and this must be to the interest sought in a ratio compounded of the ratios of the principals and times, that is in a ratio compounded of 100 : $275\frac{1}{2}$ and of 1 : $3\frac{1}{2}$, or of 200 : 551 and of 2 : 7, that is in the ratio of 400 : 3857. The answer therefore is $\frac{3857 \times 5}{400}$, or $\frac{3857}{80}$, that is £48 : 4 : 3. In this manner, though often not the most expeditious, the learner, ought for some time to calculate all questions in interest; and to prove his answer by such questions as the following: At what rate per cent. per annum, will the interest of £275 : 10 for $3\frac{1}{2}$ years be £48 : 4 : 3? or, At 5 per cent. per annum, what principal will gain £48 : 4 : 3 interest in $3\frac{1}{2}$ years? or—in what time will £275 : 10 gain £48 : 4 : 3 interest? And in some of those forms I have known persons, who have been for years calculating interest by the common technical rules, quite at a loss how to set about the solution; while children rationally taught

taught for a very few months have found no difficulty in the question.

134. But wherever the rate of interest is 5 per cent. per annum, the calculation is greatly facilitated by observing that 5£ being 100*s.* this is at the rate of a shilling for every pound: so that we at once know that at this rate the interest of £275 : 10 for 1 year is 275*s.* and 6*d.* or £13 : 15 : 6: which sum therefore multiplied by $3\frac{1}{4}$ gives the interest required. And when the interest is 6 or 4 per cent. per annum, it is often convenient to calculate it as at 5 per cent. then adding or subtracting a 5th. part. Various other advantages may be taken in particular cases, which are better left to the ingenuity of the student to discover.

135. As to *discount* it is but a species of interest; in the calculation of which however mercantile practice is at variance with scientific theory. If I hold a bill for £100 which will not be due for 31 days to come, and want ready money for it, it is plain that the person who should give me £100 in cash for the bill would be a loser of the interest for 31 days: and that he is therefore entitled to deduct part of the amount in cashing the bill for me. But it is as plain that if he retain the full interest upon £100 for 31 days, which is the mercantile practice, he retains too much and gives me too little: for he charges me with interest not only upon the principal which he advances, but also upon the interest which he keeps in his own hands. He ought equitably to give me the principal, which put to interest for 31 days would amount to £100.

136. The calculation of *Exchange* may be sufficiently illustrated, by considering the exchange between Great Britain and Ireland. A British shilling, or 12*d.* is equivalent to 13*d.* Irish currency; therefore 20*s.* British to £1 : 1 : 8 Irish; and £100 British to £108 : 6 : 8 Irish. Now Exchange is said to be at *par*, or at $8\frac{1}{3}$, whenever, I can get £100 British for £108 : 6 : 8, or £$108\frac{1}{3}$, Irish. It is said to be above or below *par*, when the premium to be paid is more or less than at this rate. For instance, Exchange is said to be $9\frac{1}{4}$, when for £100 British I must pay £109 : 5 Irish. The meaning of the terms being thus explained, all calculations are easy by the rule of proportion. For example:—At par what is the value in Irish currency of £275 : 10 British? The amount in Irish cur-

rency

rency must be greater, and in the ratio of 13 : 12. Therefore as 12 : 13 : : £275 : 10 to the answer. And this answer may be found most expeditiously by adding to £275 : 10 its 12th part. On the contrary Irish money may be changed into British *at par* by subtracting its 13th part. To calculate the amount in British currency of £275 : 10 Irish, exchange being $9\frac{1}{4}$, it is plain that the analogy must be — as $109\frac{1}{4}$: 100 : : £275 : 10 to the answer.

137. In calculations of *Fellowship* we are called to divide the profits of trade among several partners equitably, according to the time each has been in the trade and the capital he has invested in it. If they have had equal capitals in the trade and for the same time, it is plain that the profits must be divided equally between them. And universally each partner's share of the profits must be in a ratio compounded of his capital stock, and of the time it has been employed in the trade: for supposing either of these circumstances to be the same with all the partners, their shares will be directly as the other. The problem therefore resolves itself into this—To divide a given number into parts that shall be in any given ratios, or proportional to any given numbers: for instance, to divide 100 into 3 parts that shall be as 10, 8, and 7. Now $10+8+7=25$; and it is plain that the proportional parts of 100 must be greater than 10, 8, and 7, (the parts of 25) in the same ratio in which 100 is greater than 25, that is in the ratio of 4 : 1. Therefore the parts required are 40, 32, and 28. And universally the sum of the given numbers which assign the ratios of the parts is to the number to be divided, as the several given numbers to the proportional parts required. Now let us suppose that three partners, A, B, and C have had capitals of £2000, £3000, and £4000 in trade for 12, 9, and 7 months; and that at the end of the year they have to divide between them a profit of £2133. Their capitals are as 2, 3, and 4; their times as 12, 9, 7: and £2133 is to be divided between them into parts in the compound ratio of those numbers, that is, as 24, 27, and 28, the sum of which numbers is 79. Therefore as 79 : 2133 (or as 1 : 27) : : £24 to A's share, : : £27 to B's share, and : : £28 to C's share.

138. Although it be not the design of this treatise to enter into the minutiæ of practical Arithmetic, as applied

to

to mercantile transactions, yet I must not dismiss the subject without pointing out the application of the rule of proportion to another matter of frequent occurrence—*the equation of payments*. If A owe B £75 payable in 5 months, and £125 payable in 7 months, it is inquired at what time he should pay both sums together, without loss to either debtor or creditor. Now if the sums were equal, it appears obvious that the time sought must be exactly the middle period between the two times of payment, or 6 months: for thus each would lose the interest of one payment for a month, and gain the interest of an equal payment for the same time. But the sums due at the different times being unequal, it appears as obvious that A must withhold the payment of the £75 for a longer time than he anticipates the payment of the £125, in order to make the interest gained and lost equal; and that, in the ratio of 125 : 75, or of 5 : 3. We have only then to divide in that ratio the interval of 2 months (the distance between the two given dates of payment) and add the greater part, $1\frac{1}{4}$ months, to 5 months, in order to find the equated time of paying both sums: for thus the interest of £75 withheld for one month and a quarter is equal to the interest of £125 anticipated in the payment by $\frac{3}{4}$ths. of a month. Now if A should owe B a third sum, suppose £87 payable in 9 months, having combined the two former into one sum of £200 payable in $6\frac{1}{4}$ months, it is plain that by a similar process we may find the equated time of payment of the three, dividing $2\frac{3}{4}$ months $(9-6\frac{1}{4})$ into two parts in the ratio of 87 : 200, and adding the lesser part to $6\frac{1}{4}$ months: which gives the equated time for the payment of the three sums together $7\frac{24}{287}$ months, or what may be considered in practice 7 months and 3 days. Now let the student calculate the interest of £75 for $2\frac{24}{287}$ months, and of £125 for $\frac{24}{287}$ of a month, and £87 for $1\frac{263}{287}$ months: he will find the third, lost by A and gained by B, exactly equal to the sum of the two first gained by A and lost by B.

139. This operation however, which we have hitherto described at large in order to shew the scientific principles, would be altogether too tedious for mercantile practice: and it fortunately happens that it admits a most convenient abbreviation. Let us now return to the operation, by which we found $6\frac{1}{4}$ months as the equated time for payment of the two first sums. We first proceeded to divide 2 months

months $(7-5)$ in the ratio of $125:75$. Now this is done by the following analogy: (§ 137.)—as $125+75:2::125$ to the greater part, which is therefore $\frac{125\times 2}{125+75}$. But 125×2 is equal (§ 25.) to 125×7 *minus* 125×5. So that the fourth proportional may be thus express'd, $\frac{\overline{125\times 7}-\overline{125\times 5}}{125+75}$. We then added this fourth proportional to 5 months. In order to perform that addition let us bring 5 to the same denominator with the fourth proportional, and it becomes $\frac{125\times 5+75\times 5}{125+75}$. Now adding those two fractional expressions, the sum of their numerators is plainly 75×5 plus 125×7; (for on account of the subtraction of 125×5 in the one numerator and its addition in the other, that part must disappear) that is, the sum of the products of each payment multiplied by the time when it is payable: and the denominator, $125+75$ is the sum of the payments. And thus we arrive at the following practical rule:—multiply each payment by the time when it is due, and divide the sum of those products by the sum of the payments; the quotient is the equated time of payment sought. Accordingly proceeding by this rule to find the equated time of the three payments proposed in the last section, the answer is $\frac{\overline{75\times 5}+\overline{125\times 7}+\overline{87\times 9}}{75+125+87}$, or $\frac{375+875+783}{75+125+87}$, or $\frac{2033}{287}=7\frac{24}{287}$; as before.

140. I am aware that some have questioned the mathematical accuracy of this calculation, on the principle that a person paying money before it is due can justly be considered as losing only the *discount*, which is less than the interest. According to this idea the calculation is somewhat unfavourable to the creditor. But I confess that the principle upon which it is controverted appears to me palpably erroneous. If I owe £100 payable in three months, and have the money to pay it immediately, must it not as reasonably be supposed that I can gain the *interest* of £100 by delaying the payment till it become due, as it is supposed that my creditor will gain the interest by my paying him immediately? And if I have not the money, but wish to raise it for immediate payment, suppose by

issuing

issuing my note for 3 months, is it not equally plain that I must lose more than the discount of £100 for 3 months? For even according to the *theory* of discount (reckoning interest at 5 per cent. per annum) I must issue my note for £101 : 5*s.* in order to receive immediately £100. And is not this just the same thing as if I borrowed £100 for 3 months at 5 per cent. interest, in order to make immediate payment to my creditor? But according to the mercantile practice of discount I must issue my note for a still larger sum. It is not however worth while to pursue the discussion of this subject further. Those who know how much it has been contested will not wonder at my having said so much; and will be most ready to pardon me, if my ideas should be found incorrect.

Examples for practice.

Ex. 1. If $\frac{3}{4}$ of a yard of cloth cost 8*s.* 3*d.* what will 9 yards cost at the same rate?

Ex. 2. At the same rate, how many yards should I get for £4 : 19?

Ex. 3. If 7 horses eat a certain quantity of corn in 9 days, how many at the same rate will eat it in 7 days?

Ex. 4. If 75 workmen finish a piece of work in 12 days, in what time will 15 workmen finish it?

Ex. 5. A mason having built $\frac{2}{3}$ of a wall in 6 days, at the wages of 3*s.* 6*d.* per day, his employer agrees to pay him for the remainder at an increased rate of wages, in proportion as he shall increase his dispatch: and he finishes the wall in 2 days more. How much per day is he to receive?—Observe here that, if he had continued to work at the same rate, he would have taken 3 days to finish the wall, as $\frac{1}{3}$ of it remained to be built.

Ex. 6. If a man walk $7\frac{1}{4}$ miles in 2 hours and 10 minutes, how many miles will he walk at the same rate in 3 hours?

Ex. 7. At 5 per cent. per annum, what is the yearly interest of £725 : 15 : 6?

Ex. 8. — at 4, 6, and $4\frac{1}{2}$ per cent. per annum?

Ex. 9. Of what principal is £27 : 10 the yearly interest, at $5\frac{1}{2}$ per cent. per annum?

Ex. 10. What is the commission on goods bought by a factor to the amount of £576 : 15 : 8, at $2\frac{1}{2}$ per cent.?—*Commission* is an allowance of so much per cent. made to a factor for buying or selling for his employer. *Brokerage* is a similar allowance made to a broker, for assisting a merchant or factor in buying or selling goods.

Ex. 11. On what amount of goods is the brokerage £3 : 5 : 11$\frac{3}{4}$, at $\frac{3}{8}$ per cent. ?

Ex. 12. At what rate per cent. per annum will the interest of £100 for 5 years and 2 months amount to £24 : 10 : 10 ?

Ex. 13. At 4$\frac{3}{4}$ per cent. per annum, in what time will the interest of £100 amount to £34 : 16 : 8 ?

Ex. 14. Divide 79 into 5 parts that shall be in the ratio of 2, 3$\frac{1}{2}$, 5, 6$\frac{1}{2}$, and 8 ?

Ex. 15. Five partners A, B, C, D, and E joined in trade at the beginning of the year, putting in the respective capitals of £200, £350, £500, £650, and £800. Their joint profit at the end of the year was £790. What are their respective shares of it ?

Ex. 16. A. went into trade at the beginning of the year with a capital of £2576 : 10. On the 1st. of March he took B. into partnership with an equal capital : and on the 1st. of June they took C. into partnership with an equal capital. The joint profit at the end of the year is £1725. How is it to be divided between them ?

Ex. 17. Exchange being at par what is the amount in British currency of £217 : 15 : 6 Irish ? and in Irish currency of £217 : 15 : 6 British ?

Ex. 18. Ditto, Exchange being 9$\frac{3}{4}$; and Exchange being 10$\frac{1}{2}$?

Ex. 19. If A. can mow a field in 5 hours, and B. can mow it in 7 hours, in what time can A. and B. together mow it ?—On this and similar questions let it be considered that, if A. and B. worked with equal dispatch, they would together do the work in half the time that one of them would require to perform it alone : and if B.'s dispatch were twice as great as A.'s, they would together perform it in the third part of the time, which A. would require to perform it alone ; for A. and B. together would then be equivalent to three A.'s. Now according to the terms of the question B. working slower than A. in the ratio of 7 : 5, A. and B. are not equal to two A.'s, but only to A + $\frac{5}{7}$ of A. So that $\frac{12}{7}$ (or 1$\frac{5}{7}$) is to 1, or 12 is to 7, as 5 hours to the time sought.

Ex. 20. If A. can mow a field in 5 hours; and A. and B. together can mow it in three hours, in what time can B. mow

it

it alone? Here it is plain, from the observations on the last question, that 5 : 3 : : A+B : B. Therefore (§ 72.) 5—3 i. e. 2 : 3 : : A : B. But A.'s dispatch is as 5. Therefore 2 : 3 : : 5 to B.'s dispatch.

Ex. 21. If 9 bushels of corn serve 7 horses 10 days, how many bushels at the same rate will serve 20 horses 21 days?

Ex. 22. At the same rate, how many horses will eat 27 bushels in 3 days? And in what time will 21 horses eat 18 bushels?

Ex. 23. If a family of 19 persons expend £235 in 8 months, how much at the same rate will a family of 12 persons expend in 5 months?

Ex. 24. If 96 men working 9 hours a day for 10 days can dig a trench 400 yards long, 3 wide, and 2 deep, in how many days at the same rate can 108 men working 7 hours a day dig a trench of 175 yards long, 4 wide, and 3 deep?

Ex. 25. At $4\frac{1}{2}$ per cent. per annum, what is the interest of £575 : 15 for 7 years and 11 months?

Ex. 26. At what rate per cent. per annum will the interest of £1025 for 3 years and 5 months amount to £175 : 2 : 1?

Ex. 27. At 5 per cent. per annum, what principal will gain £350 : 4 : 2 interest in 10 years and 3 months?

Ex. 28. At $4\frac{1}{4}$ per cent. per annum, in what time will the interest of £375 : 10 amount to £4 : 15?

Ex. 29. A. began trade on the 1st. of January with a capital of £1000; and on the 1st. of March took in B. as a partner with a capital of £1500; and on the 1st. of May they admit C. as a partner with a capital of £2725. The joint profit at the end of the year is £1896. What are their respective shares?

Ex. 30. Three graziers, A, B, C, hold a piece of ground in common, for which they are to pay £75 a year. A. on the 1st. of January puts in 12 sheep, on the 1st. of March 8 sheep more, and on the 1st. of June draws 10 sheep. B. on the 1st. of January puts in 15 sheep, on the 1st. of February draws 6 sheep, and on the 1st of July puts in 12 sheep more. C. does not put in any sheep till the end of one month, and on the 1st. of February puts in 14; on the 1st. of April 4 sheep more; and on the 1st. of August draws 9 sheep. How much ought each to pay of the

the rent at the end of the year?—On this and similar questions in fellowship, where the capital of any partner varies during the partnership, let the student observe that the sum of all the products obtained by multiplying each capital by the time it has been employed must be proportional to his share in the partition of the profit, loss, &c. Just as we have seen that if A. had grazed 10 sheep for 12 months his share would be justly represented by 10×12, or 120; so when he grazes 12 sheep for 2 months, 20 sheep for 3 months, and 10 sheep for 7 months, his share must be represented by $\overline{12 \times 2} + \overline{20 \times 3} + \overline{10 \times 7}$, or 154.

Ex. 31. A. owes B. £25 to be paid in 1 month; £30 to be paid in 2 months; £45 to be paid in 3 months; and £15 to be paid in 4 months. What is the equated time for paying the whole? i. e. when should he pay him £115, so that it should be equivalent with the several distinct payments at the time specified?

Ex. 32. A. purchases goods from B. on the 15th. of January to the amount of £275: on the 1st. of February to the amount of £125: and on the 10th. of March to the amount of £312. He is allowed 3 months credit on each purchase: but wishes to give B. a bill for the whole amount at 31 days after date. When should it be dated?

CHAP. XIV.

Origin and Advantages of Algebra: Algebraic Notation: Definitions.

141. ALGEBRA is to be considered as but another method of Arithmetical computation, much more extensively applicable than the common, and much more powerful: while its fundamental principles are so coincident with those already stated, that no one who has made himself master of the former part of the subject can find any serious difficulty in the Elements of Algebra, so far as they are pursued in this treatise. The great advantage, which modern Mathematicians possess above the ancient, consists in their acquaintance with this art; which came to us originally from the Arabs, according to the testimony of Lucas de Burgo, who first published a treatise on it in Italian

Italian in the year 1494. That the Greek mathematicians, our masters in Geometry, were ignorant of Algebra, is certain; from their having in vain attempted to solve a problem, which with the aid of this science would have presented no serious difficulty. Yet it is not to be doubted that men so acute, and so conversant about numbers, must often unknowingly have employed a kind of Algebraic investigation: as it is common at this day to observe shrewd accountants, who have never learned Algebra, yet pursuing the solution of more complicated questions by a chain of reasoning perfectly Algebraic; while they labour indeed under much inconvenience and disadvantage from their unacquaintance with the notation and systematic rules of the art. Diophantus, a most ingenious mathematician of Alexandria, who lived in the fourth century, made wonderful advances in this method; insomuch that he is considered by some as the inventor of Algebra:—how justly, I shall not stop to inquire. It was certainly not from him, but from the Arabs, that we derived the art.

142. Algebra is also called *Universal Arithmetic*, from its employing general symbols instead of particular numbers, and affording us conclusions which form universal theorems. Thus putting the letters a and b for any two numbers whatsoever, $a+b$ expresses their sum, or the number produced by adding the number represented by b to the number represented by a: and $a-b$ represents their difference, or the number produced by subtracting b from a. Now if we add $a-b$ to $a+b$ algebraically, we shall find (as will appear in the next chapter) that the amount is twice a: and if we subtract $a-b$ from $a+b$ we shall find that the resulting number is twice b. And thus we are furnished with these general principles—that, if to the sum of any two numbers whatsoever we add their difference, the amount is twice the greater number: but if from the sum we subtract the difference, the remainder is twice the less. (These principles might be stated still more generally: but to do so at present would involve the student prematurely in the consideration of positive snd negative quantities.) Let the student try this in any numbers whatsoever, and he shall find it true: but he might often perform the same operations in common Arithmetic, adding for instance the difference between 19 and 5 (14) to their sum 24, or subtracting the former from the latter, without observing

observing even in that particular case that the sum was twice 19, and the difference twice 5. Whereas in the same Algebraic operations the results are obtained in a form, which at once presents those principles in their most universal extent to the attention.

143. From what has been said § 23. and 24. it appears, that the product of any two numbers represented by a and b may be expressed by $a \times b$ or $b \times a$: but it is more frequently and briefly expressed by ab or ba, writing the letters which denote the factors in continuation, without any sign interposed between them. Thus xyz, or zyx, or yxz expresses the product of the three factors denoted by the letters x, y, and z. In like manner $3a$ expresses three times a, or the product of a multiplied by 3; and $7xy$ expresses seven times xy. In such forms of expression the numbers prefixed to the letter or letters is called the numeral *coefficient*; and when no other numeral coefficient appears, 1 is understood to be prefixed.

144 According to what has been observed § 39. the division of a by b may be expressed thus $a \div b$; or (as is more usual) fractionally, thus $\frac{a}{b}$. Therefore $\frac{2\,mn}{3x}$ expresses the quotient arising from dividing $2mn$ by $3x$. And if we want to express $\frac{3}{4}$ths. of x algebraically, it is $\frac{3x}{4}$; for this expresses the 4th part of three times x, or $\frac{3}{4}$ths. of once x. See § 92. And if we be called to find a fourth proportional to three numbers represented by a, b, and c, the fourth proportional will be justly represented by $\frac{bc}{a}$; for this expresses the quotient arising from dividing the product of the given means by the given extreme. See § 74. And if we have this analogy $a : b :: c : d$, we may infer the equation $ad = bc$; or from the equation $ax = by$ we may infer the analogy $a : b :: y : x$. See § 76. and 77.

145. The square of a may be expressed by aa; its cube or 3rd. power by aaa; its 4th. power by $aaaa$, &c. (§ 143.) But they are more frequently denoted by *indices* or *exponents* of the powers, thus, a^2, a^3, a^4, &c. (See § 32.) And if I want to multiply any power of a by any other power, suppose the 7th. power by the 5th. power, the product will be the 12th. power, or a^{12}, its index being the

sum

sum of the indices of the factors; as is evident from § 143. by performing the operation according to the other notation. And as powers of the same root are multiplied by adding the indices of the factors, it is plain that they may be divided by subtracting the index of the divisor from the index of the dividend. Thus $\frac{a^{12}}{a^3} = a^9$.—The square root of a, or that number whose square is a, is denoted thus, $\sqrt[2]{a}$, or by the radical sign alone, $\sqrt{a}$; the cube root, or that, number whose cube is a, thus $\sqrt[3]{a}$. &c. Quantities with the radical sign $\sqrt{}$ prefixed are called *surds*. We otherwise write such surds by the aid of fractional exponents, of which the denominator indicates the root intended; thus, $a^{\frac{1}{2}}$, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, &c. And according to this notation $a^{\frac{2}{3}}$ expresses the cube root of the square of a, or that number of which a^2 is the cube. See Chap. 22.

146. We may here notice the facility with which many fractional expressions in Algebra may be reduced to lower terms. For instance, $\frac{2xyz}{3axy}$ may be at once reduced to $\frac{2z}{3a}$: for x and y being factors of both numerator and denominator, I may divide them both by xy; but this is done at once by erasing xy from both. For as the mere annexation of any letters expresses Algebraically the multiplication of the numbers which they represent, so the mere withdrawing of any letter must be equivalent to division by that letter. Thus if I want to divide abc by b, the quotient must be ac; since $ac \times b = abc$. (See § 40) Thus again, the fractional expression $\frac{3x^2yz}{2xy^2z} = \frac{3x}{2y}$; as will appear by writing the given fraction in the longer notation $\frac{3xxyz}{2xyyz}$, and dividing both numerator and denominator by the common factor xyz.

147. A *vinculum*, or line drawn over several terms of a compound quantity, is designed to give precision to the Algebraic expression. Thus $\overline{a+b} \times c$ denotes the multiplication of the sum of a and b by c; whereas $a+b \times c$ (without the vinculum uniting the terms $a+b$) might be understood as denoting the sum of a and the product of

b and

b and c, or $a+bc$. In like manner $c \times \overline{a-b}$ expresses the multiplication of c by the difference between a and b; whereas without the vinculum it might express the same thing as $ca-b$. And $\overline{a-b}|^2$ expresses the square of the difference between a and b; whereas $a-b^2$ would express the difference between a and the square of b. In place of the vinculum we often employ the mark of a parenthesis. Thus $(a+b) \div x$ expresses the division of $\overline{a+b}$ by x.

148. Propositions concerning the relative magnitude of quantities we commonly express in Algebra by equations. (See § 19.) Thus to express algebraically that a exceeds b by 7, we employ the equation $a=b+7$, or the equation $a-7=b$, or the equation $a-b=7$; any of which, according to the import of the notation as already explained, will be found to express the given relation between a and b. To express that half of a is less than two thirds of b by 4, we may employ this equation, $\frac{a}{2}+4=\frac{2b}{3}$. But more of this hereafter.

149. The observations in this chapter may be considered as the *grammar* of Algebra; and it is very desirable for the student to make himself expert in such exercises as the following.—Putting the letters x and y for any two numbers, express algebraically 1. the addition of twice y to three fifths of x; 2. the subtraction of half x from twice y; 3. the multiplication of their sum by their difference; 4. the quotient from dividing 25 by their difference; 5. the quotient from dividing their sum by three times x; 6. the subtraction of the square root of y from the cube of x; 7. that the product of their sum and difference is equal to the difference of their squares; 8. that the square of their sum exceeds the square of their difference by four times their product. On the other hand let the student exercise himself in interpreting such algebraic expressions as the following 1. $\overline{x+y}-\overline{x-y}=2y$; 2. $\frac{3x}{5}+2y=4y-\frac{x}{2}$; 3. $\overline{x+y}|^2 \times x-\frac{y}{2}=10xy$; 4. $\frac{\sqrt{x-y}}{x+y}+7=20-x$; 5. $\sqrt[3]{x}-\sqrt{y}=\frac{2xy}{3}$; 6. $\overline{x+y} \times \overline{x-y}=x^2-y^2$; 7. $\overline{x+y}|^2-4xy=\overline{x-y}|^2$; 8. $\overline{x-y}|^{\frac{2}{3}}=5$.

CHAP.

CHAP. XV.

Positive and Negative Quantities. Algebraic Addition and Subtraction.

150. Every quantity in Algebra is said to be *positive* or *negative* according as it is affected with the sign *plus* or *minus*, + or —: and whenever a quantity has not either of these signs prefixed, the sign + is understood, and the quantity is said to be positive. Thus 5, or +5, is positive; but —5 is negative Positive quantities are otherwise called *affirmative.* Some mathematicians, in treating this subject, have involved it in much perplexity, and plunged themselves into extravagant absurdities; talking of—5 as a quantity *less than nothing*, &c. to the disgrace of the science. But the student is to observe that—5 denotes just the same number as +5, but with the additional consideration that the former is to be subtracted, while the latter is to be added.

151. The simplest illustration of positive and negative quantities may be derived from a merchant's credits and debts. Five pounds are the same sum, whether it be due to him, or he owe it to another: but in the one case it may be considered as *positive* £5, for it is an addition to his property; and in the other as *negative* £5, for it is a subtraction from his property. And if the sum of his debts exceed the sum of his credits by £1000, the state of his affairs may be represented by—1000£, and undoubtedly is worse than if he had nothing and owed nothing. In such a case indeed, the man is often said even in mercantile language to be *minus* one thousand. Whereas if the sum of his credits exceed the sum of his debts by £1000, the state of his affairs may justly be represented by +1000£. These opposite signs then, without at all affecting the absolute magnitude of the quantities to which they are prefixed, intimate the additional consideration that those quantities are in contrary circumstances. Many other illustrations might be employed. Thus, if x, or $+x$, denote the force with which a body is moving in a certain direction,—x will denote an equal force in the contrary direction. But for younger students, I think it more expedient to confine their attention to the familiar illustration first adduced. When we talk of quantities of *contrary affections*, we mean quantities of which one is positive and the other negative. And by *the signs* we mean the signs + and —.

152. Let

152. Let us now consider the addition and subtraction of positive and negative quantities. And is it not plain from what we have said, that to add or subtract either kind of quantity must give the same result, as to subtract or add the same quantity with the contrary sign, or of the contrary affection? Thus, to add —5 is the same thing as to subtract $+5$: for is it not the same thing to add a *debt* of £5, as to subtract a *credit* of £5, or to take away £5 of positive property? On the other hand to subtract —5 must be the same thing as to add $+5$: just as it is the same thing to take away a debt and to add a credit of the same amount, or to give the person so much positive property. If a merchant's credits exceed his debts by £5000, and the state of his affairs be therefore $+5000$, it will just produce the same change in them, whether I cancel a debt of £1000 which he owes me, or another give him £1000. In either case alike the state of his affairs must become $+6000$. Hence if we have to add $+3$ to $+5$, the sum must be $+8$; but if —3 to —5, the sum must be —8: just as the sum of two credits of £3 and £5 is a *credit* of £8; but the sum of two debts of £3 and £5 must be a *debt* of £8. Again the sum of $+3$ added to —5 must be —2; and the sum of —3 added to $+5$ must be $+2$; just as if a merchant be *minus* £5000 (that is, if he owe £5000 more than he is worth) and I give him £3000, the state of his affairs becomes —£2000; but if the state of his affairs have been *plus* £5000, (that is, if he be worth £5000 more than he owes) and there be then added to him a debt of £3000, the state of his affairs becomes $+2000$. And thus we see that in the addition of numbers of the same affection, (both positive, or both negative) the sum of the numbers with the common sign is the sum sought: but that in the addition of numbers of contrary affections, (one positive and the other negative) the difference of the numbers with the sign of the greater is the sum sought.

153. Algebraic quantities are said to be *like*, when they consist of the same *literal* part, that is, are written with the same letters and having the same exponent. Thus, $2x$ and $-3x$ are like quantities; also $-3xy$ and $4xy$; also x^2y and $-2x^2y$; also $\sqrt{xy}$ and $3\sqrt{xy}$, or $\overline{xy}|^{\frac{1}{2}}$ and $\overline{3yx}|^{\frac{1}{2}}$. But $2x$ and $3y$ are *unlike* quantities, as also xy and x^2y. From what has been said and from the import of the signs

+ and

$+$ and $-$, it is plain that unlike quantities can be added only by annexing them together with their proper signs. Thus the sum of x and y is $x+y$; but the sum of x and $-y$ is $x-y$, or $-y+x$; the addition of $-y$ being the same thing as the subtraction of $+y$.

154. But *like* quantities may be further added by an incorporation of them into one sum; and the rule for their addition is now most simple. Add their numeral coefficients according to the rule given at the end of § 152, and annex the common letter or letters. Thus the sum of $3x$ and $5x$ is $8x$; the sum of $-3x$ and $-5x$ is $-8x$; the sum of $3x$ and $-5x$ is $-2x$; the sum of $-3x$ and $5x$ is $2x$; the sum of x^2y and $-3x^2y$ is $-2x^2y$. For in the last example, since there is no numeral coefficient expressed to the former quantity, we must understand the coefficient 1; and since there is no sign prefixed, we must understand the sign $+$. Then whatever quantity x^2y represent, since the sum of $+1$ and -3 is -2, it is plain that the sum of $+1x^2y$ and $-3x^2y$ must be $-2x^2y$. And thus the rule for adding like algebraic quantities, or incorporating them into one sum, is — take the sum of the numeral coefficients if they be of the same affection, prefixing the common sign; or the difference of the coefficients if they be of contrary affections, prefixing the sign of the greater; and in both cases annex the common literal part.

155. We have seen how to add *simple* Algebraic quantities, or those which consist of but one term. *Compound* quantities are those which consist of several terms, and called *binomial* if consisting of two terms (as the expression x^2-y^2); *trinomial* if consisting of three terms, as the expression $x^2-2xy+y^2$. Compound quantities are added, by adding separately the parts that are *like* and the parts that are unlike, according to the rules given in the two last sections. Thus the sum of the last binomial and trinomial exhibited is $2x^2-2xy$. If we have many quantities to add, let them be arranged as in the following example, placing like quantities under each other; and added according to the rule.

$$\begin{array}{lrrrrrr}
 & 5\sqrt{ab} & -\ abc & -\ 12bc & +\ b^2 & & \\
 & 3\sqrt{ab} & +\ 3abc & -\ 5bc & -\ 2b^2 & & \\
 & 7\sqrt{ab} & +\ 5abc & +\ 7bc & -\ 3b^2 & & \\
 & -\sqrt{ab} & -\ 7abc & -\ bc & -\ 4b^2 & +\ ab^2 & -\ a^2b \\
\hline
\text{Total} . & 14\sqrt{ab} & * & -11bc & -\ 8b^2 & +\ ab^2 & -\ a^2b
\end{array}$$

Observe

Observe that in algebraic operations we commonly proceed from left to right: and that when the leading term of any quantity is positive, the sign + is seldom prefixed. Now to incorporte any of the like terms in the preceding example into one sum, suppose the several sets of abc, we take the sum of all the positive terms and the sum of all the negative terms distinctly; and then incorporate these two sums. But the sum of $+5\,abc$ and $+3\,abc$ is $+8\,abc$; and the sum of $-7\,abc$ and $-abc$ is $-8\,abc$: so that we have to add $+8\,abc$ to $-8\,abc$; and their sum is 0, since the difference of their coefficients is nothing. Though such an example as the preceding is often proposed to the student, for the purpose of exercising him in the rules of addition, yet it is very rarely indeed that any such occurs in actual practice. The student who is expert in stating the sum of any two numbers, whether of the same or contrary affections, can find no difficulty in algebraic addition.

156. The rule of Subtraction is simple, and obvious from the principle mentioned in the beginning of § 152. Suppose the sign of the subtrahend changed to its contrary; (that is, if it be positive, suppose it negative, and if negative, suppose it positive:) then, instead of subtracting, add it to the minuend. Thus, if from $+2a$ I want to subtract $+2b$, the remainder is $2a-2b$. (The terms here being unlike cannot be further incorporated.) But if I subtract $-2b$, the remainder, or result, must be $2a+2b$; it being the same thing (§ 152.) to subtract $-2b$ and to add $+2b$. Thus again, it is evident that subtracting $3x$ from $10x$ the remainder is $7x$; but this is also the sum of $-3x$ added to $10x$. But $3x$ subtracted from $-10x$, gives for the remainder $-13x$, the sum of $-3x$ and $-10x$. Any longer example can now present no difficulty: for instance—

$$
\begin{array}{llllllll}
\text{From} & 5\sqrt{ab} & -\ abc & +\ 12bc & +\ b^2 & & & \\
\text{Take} & \underline{7\sqrt{ab}} & \underline{+\ 5abc} & \underline{+\ 7bc} & \underline{-\ 3b^2} & \underline{+\ ab^2} & \underline{-\ a^2b} & \\
\text{Remainder} & -2\sqrt{ab} & -\ 6abc & +\ 5bc & +\ 4b^2 & -\ ab^2 & +\ a^2b &
\end{array}
$$

And accordingly if to this remainder the subtrahend be added, the sum will be the minuend: or if the remainder be subtracted from the minuend, we shall have the subtrahend as the result.

157. The student should observe, that in Algebra we commonly talk of subtracting a greater number from a less: as in the leading terms of the preceding example we

subtract

subtract 7 from 5, and that by adding -7 to 5. And it appears that in the general expression $x-y$, if x denote a quantity less than y, the value of the expression $x-y$ is negative; just as $7-5=+2$, but $5-7=-2$.

158. But it may be objected, "is not the subtraction of "7 from 5 an unintelligible operation? and is the art of "Algebra only an art of jugglery, to enable us to do "strange things, without our understanding what we mean "by doing them?" It must be acknowledged that the science has been too often disfigured by writers, who have put it forward in some such form *, and have seemed to forget that to talk an unintelligible language is to talk nonsense. But there is a sense in which we may easily comprehend the subtraction of 7 from 5; namely by considering 5 as equivalent with the compound expression $7-2$. Now from this binomial $7-2$ we may subtract 7; and the remainder is evidently -2. Thus again, if the state of a merchant's affairs be $+10,000$£, he may lose or have subtracted from him 15,000£, and the state of his affairs becomes -5000£; so that -5 justly expresses the remainder on subtracting 15 from 10, or from the equal binomial $15-5$. In like manner subtracting -15 from -10, or from the equal binomial $-15+5$, the remainder must be $+5$: and since $+10=25-15$, subtracting -15 from $+10$ must give the remainder $+25$; while subtracting $+15$ from -10 (or from its equal $-25+15$) must give the remainder -25.

In the following examples let the questions in addition be proved by subtraction; and v. v.

Ex. 1. What is the sum of $5x$ and $3x$?

Ex. 2. Of $-5x$ and $-3x$?

Ex. 3. Of $-5x$ and $3x$?

Ex. 4. Of $5x$ and $-3x$?

Ex. 5. Of $\overline{3x+5}$ and $\overline{3x-5}$?

Ex. 6. Of $\overline{3x^2-2xy+y^2}$ and $-5x^2+5xy-y^2-5$?

Ex. 7. Subtract $2ay-b$ from $2ay+b$?

Ex. 8. $5a^2-7b$ from $-a^2+8b$?

Ex. 9. $a^3-3a^2x+3ax^2-x^3$ from a^3+3a^2x+ ~~$3ax^2$?~~ $3ax^2$

* It was with regret and with surprise that I met with some instances of this in a late Edition of Euler's Algebra, which has come into my hands since these pages were written. It is full time for such absurdities to be exploded, as the multiplication of nothing by infinity, &c. &c. See Vol. I. p. 34.

CHAP. XVI.

Algebraic Multiplication.

159. WE have seen that the product of any two simple quantities, as x and y, is expressed by xy or yx. But we have now to regulate the sign of the product. The practical rule is simple, viz. *if the factors be of the same affection, the product is positive; but negative, if the factors be of contrary affections:* that is, the product either of $x \times y$, or of $-x \times -y$ is $+xy$; but the product of $x \times -y$ or of $-x \times y$ is $-xy$.

160. The truth of this rule is sufficiently evident from the nature of multiplication, where the multiplier is positive. To multiply any quantity by x is in fact to add the multiplicand as many times as are represented by x. (§ 22.) Suppose x stand for the number 5, and the multiplicand be $+y$, representing a positive quantity, suppose an article of credit in mercantile accounts. The sum of that quantity added 5 times must be positive, or a *credit* of 5 times that amount. But if the multiplicand be $-y$, representing a negative quantity, suppose a debt, then the sum of that quantity added 5 times must be negative, or a *debt* of 5 times that amount. And thus it is plain that $+y \times x = +xy$, but $-y \times x = -xy$.

161. Let us now consider the case where the multiplier is negative $-x$, or -5. And first, suppose we have to multiply $+y$ by -5. Some might be willing to conclude that the product must be $-5y$, from the principle that it is indifferent in multiplication which of the factors be made the multiplier; and we have already seen that the product of -5 multiplied by $+y$ is $-5y$. Others have drawn the same inference from the consideration, that the multipliers -5 and $+5$, must give products just of contrary affections; and since the product of $+y$ multiplied by $+5$ is $+5y$, the product of $+y$ multiplied by -5 must be $-5y$. But although such arguments may render the conclusion probable from analogy, they do not amount to a convincing proof satisfactory to the reason. This must be derived from considering what we mean by multiplying any thing by a *negative* multiplier. Now as multiplying any thing by $+5$ imports an *addition* of the multiplicand 5 times, so

multiplying

multiplying it by -5 must import a *subtraction* of the multiplicand 5 times. But we have seen that the subtraction of $+y$ is the same thing as the addition of $-y$: (§ 152.) and therefore to subtract $+y$ five times, or to multiply $+y$ by -5, is the same thing as to add $-y$ five times, or to multiply $-y$ by $+5$; that is, the product must be $-5y$. The same consideration leads us at once to a view of the principle, which has appeared mysterious to many; namely, that the product of two negative quantities is positive. For instance, the product of $-y$ multiplied by -5 must be $+5y$, since the subtraction of $-y$ five times is the same thing as the addition of $+y$ five times.

162. After the multiplication of simple quantities, there remains no difficulty in the multiplication of compound. The principle on which it is performed is just the same as in common Arithmetic: (See § 25.)—*multiply each part of the multiplicand by each part of the multiplier, and add all the products thus obtained.* (Proceed in the operation regularly from left to right of each factor, lest you should omit any of the products.) Thus the product of $2x+3y$ multiplied by 5 is $10x+15y$; but multiplied by -5 is $-10x-15y$. The product of $2x+3y$ multiplied by $5-y$ must be the sum of four parts, namely $10x+15y$ (or $\overline{2x+3y}\times 5$) and $-2xy-3y^2$ (or $\overline{2x+3y}\times -y$.) The product sought is therefore $10x+15y-2xy-3y^2$. If any of the products be *like* quantities, write them one under the other, to prepare for the addition: as in the following example.

$$\begin{array}{lrrrr}
\text{Multiply} & x^2 & -\,2xy & +\,y^2 & \\
\text{by} & & x & -\,y & \\
\hline
 & x^3 & -\,2x^2y & +\,xy^2 & \\
 & & -\,x^2y & +\,2xy^2 & -\,y^3 \\
\hline
\text{Product} & x^3 & -\,3x^2y & +\,3xy^2 & -\,y^3
\end{array}$$

163. If the student multiply $x-y$ by $x-y$, he will find the product $x^2-2xy+y^2$, which is therefore the square of the binomial $x-y$, of which consequently the product exhibited in the preceding example is the cube, or third power. And here we may see another instance of the nature and use of Algebra, or Universal Arithmetic. The binomial $x-y$ is a general expression for the difference between

tween any two numbers. If we take any two numbers, for instance 7 and 3, we may by common Arithmetic multiply their difference 4 by itself, and the product 16 is the square of that difference. But here the product appears in a form which does not enable us to observe its relation with the factors. But performing the same operation algebraically, and comparing the product $x^2-2xy+y^2$ with the factors, we at once observe that the square of $x-y$ consists of the sum of the squares of x and y (x^2+y^2) *minus* twice the product of x and y $(-2xy)$: whence we are immediately furnished with this *universal* truth, that the square of the difference between any two numbers is equal to the sum of their squares *minus* twice their product; or is less than the sum of their squares by twice their product. (Thus $\overline{7-3}|^2=16=49+9-42=58-42$. In like manner $\overline{x+y}|^2$ (or the square of the sum of any two numbers) $=x^2+2xy+y^2$, or is equal to the sum of their squares *plus* twice their product; as we have before observed. (§ 34.) Again if we multiply $x+y$ by $x-y$, we shall find the product x^2-y^2; for of the four products which compose it $x^2+xy-xy-y^2$, the second and third when added together disappear. But this presents to us the general principle that the product of the sum and difference of any two numbers is equal to the difference of their squares. Thus, the product of $7+3$ (or 10) and $7-3$ (or 4) is 40; but this is the difference between the square of 7 and the square of 3.

164. Since, according to the rule of the signs in multiplication, the square of either $+3$ or -3 is $+9$, no number can be assigned for the square root of -9: and therefore the square root of -9 is an *impossible* quantity. In like manner $\sqrt{-a^2}$ is an expression that indicates an impossible quantity. But the square root of a^2 may be either $+a$ or $-a$; since either of these roots multiplied by itself gives $+a^2$ for the product. And therefore every positive quantity in Algebra is considered as having two square roots, one positive and the other negative.

Let the student now employ himself on the following questions for exercise in multiplication.

Ex. 1. $\overline{x+y}\times 2a=?$

Ex. 2. $\overline{x+y}\times -2a=?$

Ex. 3.

Ex. 3. $\overline{x-y} \times 2a = ?$
Ex. 4. $\overline{x-y} \times \overline{-2a} = ?$
Ex. 5. $\overline{12ax + 2y} \times \overline{x-3y} = ?$
Ex. 6. $\overline{x^3-3x^2y+3xy^2-y^3} \times \overline{x+y} = ?$
Ex. 7. What is the 6th power of $a+b$?
Ex. 8. What is the 6th power of $a-b$?

CHAP. XVII.

Algebraic Division: Resolution of Fractions into infinite Series.

165. IF the divisor and dividend be simple quantities, and the divisor be not any factor of the dividend, the quotient is expressed fractionally. Thus, the quotient of ab divided by x is $\frac{ab}{x}$: the quotient of x divided by $-ab$ is $\frac{x}{-ab}$: and $\frac{-\sqrt{2}}{a}$ expresses the quotient arising from dividing $-\sqrt{2}$ by a. And any quotient may be thus expressed.

166. If the divisor be a factor of the dividend, the quotient is obtained, as we have already observed, (§ 146.) by expunging that factor from the dividend: and *the sign of the quotient must be* $+$, *if the dividend and divisor be of the same affection; but* $-$, *if they be of contrary affections;* as is evident from the consideration that the dividend is the product of the divisor and quotient. Thus $2abc$ divided by b, or $-2abc$ divided by $-b$, gives the quotient $2ac$; since $2ac \times b$, or $-2ac \times -b$, gives the product $2abc$. But $-2abc$ divided by b, or $2abc$ divided by $-b$, gives the quotient $-2ac$; since $-2ac \times b = -2abc$, and $-2ac \times -b = 2abc$. In like manner, if the divisor and dividend have any common factors, but others not common, the division is performed by expunging the common factors from both, and writing the remaining terms fractionally with their proper signs. Thus, a^2bc divided by $-2ac$ gives the quotient $\frac{ab}{-2}$: and $-3axy^2 \div 3bxy = \frac{-ay}{b}$. This is in fact but

 reducing

reducing the origional fraction $\frac{-3axy^2}{3bxy}$ to lower terms, by dividing both numerator and denominator by $3xy$.

167. If the dividend be compound, but the divisor simple and a factor of each term of the dividend, the division is performed by expunging that factor from each term of the dividend, observing the former rule of the signs: for thus each part of the dividend is divided by the divisor. For instance $x^2-2xy\div x=x-2y$: and $ax-a\div -a=-x+1$. And if the simple divisor have other factors not found in each term of the dividend, after expunging the common factors, the quotient is expressed fractionally. Thus $\overline{x^2-2xy}\div 3xa=\frac{x-2y}{3a}$: and $\overline{ax-a}\div -ab=\frac{-x+1}{b}$, or $\frac{x-1}{-b}$. In the first form of the quotient we have divided both dividend and divisor by $-a$; in the second form by $+a$. And here it may be observed, that in any fractional expression, or in any division, we may change all the signs of the numerator and denominator, or of the dividend and divisor, without altering the value of the fraction or quotient.

168. If the divisor be compound, the quotient is often most conveniently expressed fractionally. But not unfrequently also we may obtain the quotient in a simpler form by an operation perfectly analogous to long division in numbers: only it is needful in the first instance to arrange the terms of both dividend and divisor according to the powers of some one letter. Thus if we have to divide $3xy^2-3x^2y+x^3-y^3$ by $x-y$, arranging the terms of the dividend according to the powers of the letter x, it becomes $x^3-3x^2y+3xy^2-y^3$. Now divide x^3, the first term of the dividend, by x, the first term of the divisor; and set down the quotient x^2 as the first term of your quotient. Then multiply the divisor $x-y$ by x^2, the first term found of the quotient: and subtract the product x^3-x^2y from the dividend. The remainder is $-2x^2y+3xy^2-y^3$. In like manner divide $-2x^2y$, the first term of this remainder, by x; and set down the quotient $-2xy$ as the second term of your quotient: by which multiplying the divisor $x-y$, and subtracting the product $-2x^2y+2xy^2$, the second remainder is xy^2-y^3. Finally repeating the operation

tion on this remainder, the third term of your quotient is $+y^2$, the product of which and $x-y$ is xy^2-y^3, which is equal to the last remainder: and therefore the quotient sought is $x^2-2xy+y^3$, without any remainder. Let us now exhibit the work at large.

$$
\begin{array}{rl}
x-y) & x^3-3x^2y+3xy^2-y^3 \;(x^2-2xy+y^2 \\
& \underline{x^3-\;x^2y\qquad\qquad\quad} \\
& 0-2x^2y+3xy^2-y^3 \\
& \underline{\;\;-2x^2y+2xy^2\qquad} \\
& \qquad\quad 0+xy^2-y^3 \\
& \qquad\quad \underline{+xy\;\;-y^3} \\
& \qquad\qquad 0\quad 0
\end{array}
$$

169. We may prosecute any algebraic division by this method, whatever be the terms of the dividend and divisor, provided the divisor be compound. But obviously it must often happen, that we shall never arrive at an exact quotient without a remainder: but, as in the case of common division, the exact quotient may be exhibited by annexing to the quotient the remainder divided by the divisor fractionally (§ 43.); and this may be done at any period of the division. For instance, taking the same dividend as in the last example, but the divisor $x+y$, we shall find the three first terms of the quotient to be $x^2-4xy+7y^2$, but with the remainder $-8y^3$. Therefore the quotient may be completed by annexing to it $\frac{8y^3}{x+y}$: thus—

$$
\begin{array}{rl}
x+y) & x^3-3x^2y+3xy^2-y^3 \;\left(x^2-4xy+7y^2-\frac{8y^3}{x+y}\right. \\
& \underline{x^3+x^2y\qquad\qquad\quad} \\
& -4x^2y+3xy^2-y^3 \\
& \underline{-4x^2y-4xy^2\qquad\quad} \\
& \qquad\quad +7xy^2-y^3 \\
& \qquad\quad \underline{+7xy^2+7y^3} \\
& \qquad\qquad\quad -8y^3
\end{array}
$$

170. Accordingly if we multiply the three first terms of the quotient by the divisor $x+y$, and add $-8y^3$ to the product, we shall find the dividend. But instead of terminating the division at the remainder $-8y^3$, we may con-

tinue

tinue the same process of division as long as we please: only let the student recollect that any fraction is multiplied by an integer either by multiplying the numerator or dividing the denominator; and on the other hand is divided by an integer either by multiplying the denominator or dividing the numerator. (See § 106. 107. 108.) Let us now continue to divide the last remainder $-8y^3$ by $x+y$.

$$x+y)\ -8y^3\ \left(-\frac{8y^3}{x}+\frac{8y^4}{x^2}-\frac{8y^5}{x^3}, \text{ \&c.}\right.$$

$$-8y^3-\frac{8y^4}{x}$$

$$+\frac{8y^4}{x}$$

$$+\frac{8y^4}{x}+\frac{8y^5}{x^2}$$

$$-\frac{8y^5}{x^2}$$

$$-\frac{8y^5}{x^2}-\frac{8y^6}{x^3}$$

$$+\frac{8y^6}{x^3}$$

171. We need not continue the process of division further; for it is now manifest by what law the series proceeds, namely that the signs of the terms are alternately *plus* and *minus*, and that each successive term is produced by multiplying the last term by y and dividing by x, or multiplying the last term by $\frac{y}{x}$. Such a series is called an *infinite series*, because it may be continued without end: and at any period of it, in order to complete the true quotient, we must discontinue the series, and annex the last remainder divided by the divisor. And by this method of actual division we may resolve any fraction into an infinite series: for even if the given denominator be simple, we may consider and express it as the sum or difference of two numbers. Thus let $\frac{a}{b}$ express any fraction, we may put $c+1$ for b, and performing the actual division of a by $c+1$

$c+1$ we obtain the series $\frac{a}{c}-\frac{a}{c^2}+\frac{a}{c^3}-\frac{a}{c^4}$, &c. where the law of continuation is obvious. Or, if we arrange the terms of our divisor differently and divide a by $1+c$, the series becomes $a-ac+ac^2-ac^3$ &c. Or if we put $c-1$ for b, the terms of the series will be the same, but all positive.

172. The doctrine of infinite series is very interesting in its nature, and important in its application: but in this elementary treatise we cannot enlarge on it further, than a simple illustration of it by another example. Let it be required to resolve $\frac{a+b}{a-b}$ into an infinite series.

$$
\begin{array}{l}
a-b\,\big)\,a+b\,\Big(1+\frac{2b}{a}+\frac{2b^2}{a^2}, \text{ \&c.} \\
\qquad\ \ a-b \\
\qquad\ \ \overline{\ \ +2b\ \ } \\
\qquad\qquad 2b-\frac{2b^2}{a} \\
\qquad\qquad \overline{\qquad +\frac{2b^2}{a}}
\end{array}
$$

Now suppose $a=10$ and $b=1$, then $\frac{a+b}{a-b}=\frac{11}{9}=1{,}222$, &c. (§ 118.) or $1+\frac{2}{10}+\frac{2}{100}+\frac{2}{1000}$, &c. the terms of which infinite series coincide in value with the terms found by the Algebraic division. And as we say that $\frac{2}{9}$ is equal to the circulating decimal .222, &c. so the Algebraic fraction $\frac{2b}{a-b}$ is equal to the infinite series $\frac{2b}{a}+\frac{2b^2}{a^2}+\frac{2b^3}{a^3}$, &c. If we terminate either the decimal or the algebraic series at any period, we must annex the last remainder divided by the given divisor in order to obtain the exact value of the given fraction. Thus $\frac{2}{9}=.222+\frac{2}{9000}$, and $\frac{2b}{a-b}=\frac{2b}{a}+\frac{2b^2}{a^2}+\frac{2b^3}{a^3}+\left(\frac{2b^4}{a^3}\right)\div(a-b)=\frac{2b}{a}+\frac{2b^2}{a^2}+\frac{2b^3}{a^3}+\frac{2b^4}{a^4-a^3b}$.

173. In resolving the fraction $\frac{a+b}{a-b}$, if we discontinue the division at the first term of the quotient, we find that the complete quotient is $1+\overline{2b\div a-b}$; which points out this general theorem, that dividing the sum of any two numbers

numbers by their difference gives a quotient greater by 1 than the quotient of twice the less divided by the difference. Thus the quotient of $7+3\div 7-3=2\frac{1}{2}$; and $6\div 4=1\frac{1}{2}$.

Each of the examples in multiplication at the end of the last chapter will afford the student an exercise in division: and let him resolve into infinite series the fractions $\frac{1}{1-a}$; $\frac{1}{1+a}$; $\frac{x}{x-y}$; $\frac{x+y}{x-y}$; $\frac{x}{9}$.

CHAP. XVIII.

Algebraic Operations on Fractional Quantities. Method of finding the least common Multiple.

174. EVERY rule here is exactly the same with that for the corresponding operation in common Arithmetic. After referring the student therefore to chap. 8. 9. and 10. it is only needful to illustrate the several operations by examples. Let it then be required to add $\frac{x}{y}$ to $\frac{a}{y}$. Here the fractions having the same denominator, we add their numerators and subscribe the common denominator. Therefore the sum required is $\frac{x+a}{y}$. And in like manner $\frac{x-a}{y}$ is the remainder, subtracting $\frac{a}{y}$ from $\frac{x}{y}$. But if the fractions to be added or subtracted have different denominators, they must (in order to incorporate the sum or difference into one fraction) be brought to equivalent fractions of the same denominator: and the product of the several denominators must always afford a common denominator, to which they may all be brought. (§ 104.) Thus $\frac{x}{y}+\frac{a}{b}=\frac{xb}{yb}+\frac{ay}{yb}=\frac{xb+ay}{yb}$. That $\frac{x}{y}=\frac{xb}{yb}$ appears from the consideration that the value of a fraction remains unaltered, if we multiply or divide both numerator and denominator by the same quantity. Thus again, $\frac{x}{y}-\frac{y}{x}=\frac{x^2-y^2}{xy}$.

175. Since

175. Since any fraction is multiplied by an integer, either by multiplying the numerator or dividing the denominator by that integer, it follows that $\frac{x}{y}\times a=\frac{ax}{y}$; and $\frac{a+x}{ay}\times a=\frac{a+x}{y}$ or $=\frac{a^2+ax}{ay}$. And thus $\frac{x}{y}\times y=x$; for $x=\frac{x}{1}$. (See § 113.) And since any fraction is divided by an integer, either by dividing the numerator or multiplying the denominator by that integer, it follows that $\frac{x}{y}\div a=\frac{x}{ay}$; and $\frac{a+ax}{y}\div a=\frac{1+x}{y}$, or $=\frac{a+ax}{ay}$.

176. In Algebra as in common Arithmetic, to multiply by a fraction we multiply by the numerator and divide by the denominator: and to divide by a fraction we divide by the numerator and multiply by the denominator, or (which amounts to the same thing) we multiply by the reciprocal of the given divisor. Thus $\frac{x}{y}\times\frac{a}{b}=\frac{ax}{by}$; and $\frac{x}{y}\div\frac{a}{b}=\frac{bx}{ay}$; and $\frac{x}{y}\times\frac{y}{x}=1$; since $\frac{xy}{xy}=1$.

177. Let the student recollect that any integral expression may be brought to a fractional of any given denominator; multiplying the integer by that denominator. (§ 113.) Thus, $x=\frac{5x}{5}=\frac{ax+xy}{a+y}=\frac{x\sqrt{c}}{\sqrt{c}}$. Therefore $\frac{x+y}{a}+2a=\frac{x+y+2a^2}{a}$.

178. We may now propose and demonstrate the rule for finding the least common multiple of two or more numbers. And first, let any two numbers a and b be given, and let m be their greatest common measure; (§ 98. 99.) and let $\cancel{\frac{a}{m}=d}$ and $\cancel{\frac{b}{m}=c}$. Then I say that the $\frac{a}{m}=$
product of c, d, and m is the least common multiple of $\frac{b}{m}=$
a and b. And 1st. it is a common multiple of them; for since $\cancel{a=dm}$, and $\cancel{b=cm}$, it is plain that both a and b measure cdm. But 2dly. it is their least common multiple; for let any other common multiple n be assumed, and let $ya=n$, and $xb=n$. Then $ya=xb$; and therefore $x:y::a:b$ (§ 77.) But $a:b::c:d$ (§ 81.) Therefore $x:y::c:d$. $a=c$ $b=d$

Now

Now m being the greatest common measure of a and b, it is plain that c and d are the lowest numbers in that ratio. Therefore x is greater than c, and y greater than d. Therefore ya or xb (that is n) is greater than da, or cb, that is than m. Thus, if I want to find the least common multiple of 15 and 20, I bring those numbers to the lowest terms in the same ratio, 3 and 4, by dividing them both by their greatest common measure 5; and the product of 3, 4, and 5 (or 60) is the least common multiple of 15 and 20. (See § 181.)

179. Any other common multiple of a and b must also be a multiple of m. For suppose that n is a common multiple of a and b and not measured by m, but that m is contained in n x times, leaving a remainder y, less than m. Then $n = xm + y$. Now since both a and b measure m, they must measure xm; and by hypothesis they measure n, or $xm + y$. Since then they measure xm and $xm + y$, they must both measure y; and y, a number less than m, will be a common multiple of a and b: which is contrary to the hypothesis.

180. Now suppose three numbers given, a, b, and c; to find their least common multiple. Let m be the least common multiple of a and b. Let n be the least common multiple of m and c. Then I say that n is the least common multiple of a, b, and c. For since (as we have just shewn) any common multiple of a and b, must also be a multiple of m, it is evident that any common multiple of a, b, and c, must be a common multiple of m and c; and therefore n the least common multiple of the two latter must also be the least common multiple of the three former. It is plain that, how many numbers soever be given, we can find their least common multiple by a similar process.

181. What has been demonstrated in § 178. may perhaps appear more clearly, if proposed in the following form. Let a and b represent any two numbers prime to each other, and therefore the lowest in the same ratio. Then ab, their product, must be their least common multiple; for if there were any lower the quotients of it divided by a and b $\left(\frac{x}{b} \text{ and } \frac{x}{a}\right)$ would be numbers less than a and b, and in the same ratio: which is absurd. Now let ma and mb represent any two numbers not prime to each other,

other, of which m is the greatest common measure, and therefore a and b the lowest numbers in the same ratio. Then mab must be the least common multiple of ma and mb; for if there were any less, the quotients of it divided by mb and ma would be less than the quotients of mab divided by the same, i. e. than a and b; and would be in the same ratio: which is absurd.

Ex. 1. What is the sum, and what the difference, of the two fractions $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$?

Ex. 2. ... Do. of the two fractions $\frac{1}{x+y}$ and $\frac{1}{x-y}$?

Ex. 3. $\frac{x}{x+y}\times\frac{y}{x-y}=?$

Ex. 4. $\frac{x+3}{y+5}\times\overline{y+5}=?$

Ex. 5. $\frac{x+y}{x}\div\frac{y}{x-y}=?$

Ex. 6. $\frac{mn-2mx}{3y}\div\frac{m}{3y}=?$

Ex. 7. Find the least common multiple of the numbers 15, 20, 25 and 35?

CHAP. XIX.

Arithmetical Progression.

182. QUANTITIES are said to be in *Arithmetical progression,* when they increase or decrease by a common difference. Thus, the series of natural numbers, 1, 2, 3, 4, 5, &c. increasing by the common difference 1; the series 7, 10, 13, 16, &c. increasing by the common difference 3; the series 19, 15, 11, 7, 3, decreasing by the common difference 4. It will be sufficient to consider the constitution and properties of an increasing series; as every thing said upon that kind will be easily applicable to the other: for by taking the terms of an increasing series in the contrary order we shall have a decreasing series.

183. Now if we put a for the first term of such a series and d for the common difference, the increasing series in Arithmetical

Arithmetical progression must be justly represented by a, $a+d$, $a+2d$, $a+3d$, &c. For as the second term of the series is generated by adding the common difference to the first term, and is therefore $a+d$, so the third term is generated by adding the common difference to the second term, and is therefore $a+2d$: and so on. Hence it is manifest that any term of such a series consists of the first term *plus* the common difference multiplied by a number one less than the number of that term. For instance, the 100th. term must be the sum of the first term and 99 times the common difference: just as the *second* term is the sum of the first term and *once* the common difference. And universally if we put n for the number of the term, the nth. term of such a series must be $a+\overline{dn-d}$; for $dn-d = \overline{n-1} \times d$. And thus, if we have given the first term and common difference, it is easy to find any proposed term of the series. For instance, let it be required to find the 100th. term of an increasing series in Arithmetical progression whose first term is 12 and the common difference 3; that is, of the series 12, 15, 18, &c. The 100th. term 100
must be $12+\overline{999 \times 3}$, that is $12+2997=3009$.

184. In any such series *the sum of the extremes* (that is, of the first and last terms) *is equal to the sum of any two terms equally remote from the extremes*; for instance, of the second term and last but one, or of the third term and last but two, &c. For whatever pair of terms equally remote from the extremes you take, one of them must be just as much greater than the first term as the other is less than the last; and therefore their sum must be just equal to the sum of the first and last. Thus, in the series a, $a+d$, $a+2d$, $a+3d$, $a+4d$, $a+5d$, consisting of 6 terms, the sum of the extremes, a and $a+5d$, is $2a+5d$: but the same is the sum of the 2d. and 5th. terms, or of the 3d. and 4th. terms. Or in the numerical series 5, 7, 9, 11, 13, 15, the sum of the first and last is 20, which is equal to $7+13$ or to $9+11$. And in like manner it is evident, that if the series consist of an odd nnmber of terms, the sum of the extremes is equal to twice the mean, or middle term. Thus the 7th. term of the last series is 17; and $5+17=22=11\times 2$, or twice the fourth term.

185. Hence it is easy to find the sum of all the terms of such a series, by multiplying the sum of the extremes by half

half the number of terms in the series. For let the series consist for instance of 6 terms, all the terms may be combined into 3 equal pairs of terms, the sum of each pair being equal to the sum of the extremes; and therefore 3 times any one of these pairs must be equal to the sum of all the terms. Thus the sum of the series 5, 7, 9, 11, 13, 15 is equal to $20\times 3=60$; and if continued to another term, the sum of the series is $22\times 3\frac{1}{2}=77$. And thus, if we have given the extremes and the number of terms in the series, we can at once find the sum of the series. For instance the sum of the natural numbers from 3 to 100 inclusive is $103\times 49=5047$; for it is plain that the number of terms in the series is 98.

186. Hence if we have given the first term, the common difference and the number of terms we can easily find the sum of the series; since we can find the last term (§ 183.) by adding to the first term the product of the common difference and a number less by one than the number of terms. Thus let the first term of a series in Arithmetical progression be 3, and the common difference 4; the 17th. term of that series must be $3+\overline{4\times 16}=3+64=67$; and the sum of the series continued to 17 terms must therefore be $\overline{3+67}\times\frac{17}{2}=70\times 8\frac{1}{2}=595$. And universally, let a be the first term, d the common difference, n the number of terms. Then the last term must be $a+d\times\overline{n-1}$ $=a+dn-d$; and the sum of the series must be $\overline{2a+dn-d}$ $\times\frac{n}{2}=\frac{2an+dn^2-dn}{2}$; which is therefore a general expression for the sum of any series in Arithmetical progression. When the common difference is equal to the first term, this expression becomes $\frac{an+an^2}{2}$. And when the number of terms also is equal to the first term, the expression becomes $\frac{a^2+a^3}{2}$.

187. As we may find the last term of a series from having given the first term, the common difference, and the number of terms; so we may find the common difference from having given both extremes and the number of terms: namely, by subtracting the first term (a) from the last, ($a+dn-d$) and dividing the remainder ($dn-d$) by $n-1$. Thus, let it be required to constitute a series of 8 terms in Arithmetical

Arithmetical progression, whose first term shall be 3 and the last term 30. The common difference must be $\frac{27}{7}=3\frac{6}{7}$. Accordingly the series is 3, $6\frac{6}{7}$, $10\frac{5}{7}$, $14\frac{4}{7}$, $18\frac{3}{7}$, $22\frac{2}{7}$, $26\frac{1}{7}$, 30. Or if, instead of the last term, we are given the sum of the series 132, dividing that sum $\left(\frac{2an+dn^2-dn}{2}\right)$ by 4 $\left(\frac{n}{2}\right)$ and subtracting 6 $(2a)$ from the quotient, the remainder divided by $n-1$ affords the same result.

188. If there be a series of 3 numbers in arithmetical progression, which may therefore be represented by a, $a+d$, $a+2d$; the product of the extremes (a^2+2ad) evidently is less than the square of the mean, $\overline{a+d}|^2$ (or $a^2+2ad+d^2$) by the square of the common difference. But we have seen (§ 76.) that the same product is equal to the square of a geometrical mean between the same extremes. Thus in the arithmetical series 2, 10, 18, the product of 2 and 18 is less than the square of 10 by 64, the square of the common difference 8. But in the geometrical series 2, 6, 18, the product of 2 and 18 is equal to the square of 6. And an arithmetical mean must always be greater than a geometrical between the same extremes.

Ex. 1. What is the 17th. term of the Arithmetical series 5, 9, 13, &c.? And what is the sum of the series?

Ex. 2. What is the common difference, and what is the sum of the Arithmetical series, whose first term is 5, and the 10th. term 15?

Ex. 3. The common difference of a decreasing Arithmetical series is $3\frac{1}{4}$; the first term $12\frac{1}{4}$. What is the 10th. term?

Ex. 4. If I spend 5*s.* in the first week of the year, and each succeeding week 1*s.* more than in the preceding, how much shall I spend in the whole year?

Ex. 5. If 100 eggs be laid in a right line, 1 yard asunder, and a man be placed at a basket 1 yard from the first egg, in what time can he put the eggs one by one into the basket, supposing him to go at the rate of 5 English miles an hour, including all delays?

Ex. 6. What is the sum of the even numbers from 2 to 1000 inclusive?

Ex. 7. Do. of the odd numbers from 1 to 999 inclusive?

CHAP.

CHAP. XX.

Geometrical Progression.

189. TERMS, which are in continued proportion, (§ 70.) or each of which bears the same ratio to the next, are said to be in *geometrical progression.* As in *arithmetical* progression the terms of a series have a common *difference*, so in *geometrical* they have a common *ratio.* Thus 2, 6, 18, 54, &c. are in geometrical progression, since each term is 3 times the preceding: and 3 is called the *denominator*, or *exponent*, of the common ratio. So again, 3, 6, 12, 24, &c. where the denominator of the common ratio is 2; each pair of adjacent terms being in the ratio of 1 : 2. And it is plain that any such series may be continued by multiplying the term last found by the denominator of the common ratio. If therefore the first term of the series be 3, and the denominator of the ratio $2\frac{1}{2}$, the series will be 3, $3 \times 2\frac{1}{2}$, $3 \times 2\frac{1}{2} \times 2\frac{1}{2}$, &c. or 3, $7\frac{1}{2}$, $18\frac{3}{4}$, &c. And as the 3rd. term is the product of 3 and the *square* of $2\frac{1}{2}$, so the 4th. term must be the product of 3 and the *cube* of $2\frac{1}{2}$; the 5th. term the product of 3 and the *fourth* power of $2\frac{1}{2}$.

190. Thus we see that any term of a *geometrical* series is the *product* of the first term and that *power* of the *denominator* of the common ratio whose *index* is less by 1 than the number of the term: just as we have seen that any term of an *arithmetical* series is the *sum* of the first term and that *multiple* of the common *difference* whose coefficient is less by 1 than the number of the term. For instance, in the geometrical series 3, 6, 12, 24, &c. the 24th. term must be the product of 3 and the 23rd. power of 2, or 3×2^{23}; as in the arithmetical series 3, 5, 7, &c. the 24th. term is $3 + \overline{23 \times 2}$. In finding the 23rd. power of 2, to avoid the tediousness of successive multiplications by 2, we square the 5th. power, which gives us the 10th. and square the 10th. power, which gives us the 20th. This multiplied by the cube of 2 gives us the 23rd. power. Thus $2^5 = 32$; $2^{10} = 32 \times 32 = 1024$; $2^{20} = 1024 \times 1024 = 1{,}048576$; and $2^{23} = 1{,}048576 \times 8 = 8{,}388608$.

191. We may now easily express such a series algebraically. Putting a for the first term, and d for the denominator of the common ratio, the 2nd. term must be ad; the

the 3rd. term ad^2; the 4th. term ad^3, &c. And let n be the number of terms in the series; then the index of d in the last term must be $n—1$; that is, the last term must be ad^{n-1}. Any geometrical series therefore is justly represented by a, ad, ad^2, ad^3......ad^{n-1}. And the product of the extremes is evidently equal to the product of any two terms equally remote from the extremes; as is true of the *sums* of the terms in an Arithmetical series.

192. Let it now be proposed to find the sum of such a series, continued (suppose) to 5 terms; and put s for that sum. We know that $s=a+ad+ad^2+ad^3+ad^4$: and if we multiply these equals by d, the products must be equal. But the product of s multiplied by d is sd; and the product of $a+ad+ad^2+ad^3+ad^4$ multiplied by d is $ad+ad^2+ad^3+ad^4+ad^5$. Now, since subtracting equals from equals the remainders must be equal, if we subtract s from sd, and the value of s from the value of sd, we shall have equal remainders. Let us perform the operation, and observe the result. Thus—

$$\begin{array}{rl} \text{From} & sd=ad+ad^2+ad^3+ad^4+ad^5 \\ \text{Take} & \;\;s=a+ad+ad^2+ad^3+ad^4 \\ \hline \text{Remainder} & sd—s=ad^5—a \end{array}$$

In this operation the student will observe, that to subtract s from sd we annex it to sd with the sign—; and that in subtracting the value of s from the value of sd, all the terms disappear except the last term of the minuend ad^5 and the first of the subtrahend a, which is therefore subtracted from ad^5 by annexing it with the sign—. Therefore we are certain that $sd—s=ad^5—a$: and now if we divide both of these equals by $d—1$, the quotients must be equal; that is, $\frac{sd—s}{d—1}=\frac{ad^5—a}{d—1}$. But $\frac{sd—s}{d—1}=s$, as appears by performing the division, or by observing that $\overline{d—1}\times s=sd—s$. Hence it follows that $s=\frac{ad^5—a}{d—1}$: and therefore the sum of the series, a, ad, ad^2, ad^3, ad^4, is found by continuing it to one term more, (or multiplying the last term ad^4 by d) subtracting the first term a, and dividing the remainder by a number less by 1 than the denominator of the common ratio. And universally, whatever be the number

number of terms, $s=a+ad+ad^2 \ldots +ad^{n-1}$; and multiplying both sides of that equation by d, $sd=ad+ad^2+ad^3 \ldots +ad^n$; and from these equals subtracting the former equals, $sd-s=ad^n-a$; and dividing these equals by $d-1$, $s=\frac{ad^n-a}{d-1}$.

193. Thus we see that the sum of any geometrical series is found by the following rule:—multiply the first term by that power of the denominator of the common ratio whose index is the number of terms in the series; from this product subtract the first term; and divide the remainder by the denominator of the ratio *minus* 1. For instance, let it be required to find the sum of the series 2, 6, 18, &c. continued to 8 terms. The denominator of the common ratio is 3; therefore the sum of the series is $\frac{\overline{2\times 3^8-2}}{3-1}$ $=\frac{\overline{2\times 6561-2}}{2}=\frac{13120}{2}=6560$. When the denominator of the common ratio is 2, since $2-1=1$, we are saved the trouble of the division. Thus the sum of the series 3, 6, 12, &c. continued to 10 terms is $\overline{3\times 2^{10}-3}$, or $\overline{3\times 1024-3}$ $=3069$. The same calculation is obviously applicable to the sum of a decreasing series, as 54, 18, 6, 2, by taking the terms in an inverted order; or always subtracting the *least* term from the product of the *greatest* and the denominator of the ratio considered as a ratio of less inequality. [And this method is less apt to perplex tiros, than the consideration of $\frac{1}{3}$ as the denominator of the ratio. By inverting the series the sum is $\frac{\overline{54\times 3-2}}{2}$; in the other method $(\frac{2}{3}-54)\div-\frac{2}{3}$. The two expressions are equivalent: for in dividing by $\frac{2}{3}$ we should multiply the dividend by 3 and divide by 2; so that the expression becomes $\frac{2-\overline{54\times 3}}{-2}$; in which fractional expression both numerator and denominator being negative, the value is positive and the same with $\frac{54\times 3-2}{2}$.]

194. From the nature of the Arabic notation it is evident that any number written by a repetition of the same digit, as 3333, or 77777, may be considered as the sum of

of a geometrical series, in which the denominator of the common ratio is 10: for $7777=7+70+700+7000$. And accordingly the sum of this series calculated according to the rule given in the last section, or $\frac{70000-7}{9}$, is 7777.

And so $\frac{30000-3}{9}=3333$, &c.

195. It is observable how rapidly numbers increase in geometrical progression. *One billion* is the 13th term of a decuple progression whose first term is unity: and we have already noticed (§ 6.) the enormous magnitude of that number. The inventor of the game of Chess, which is played on a board divided into 64 squares, is said to have been offered by an Eastern Monarch any reward he might desire. He desired only 1 grain of corn for the first square of the board, 2 for the second, 4 for the third; and so on in geometrical progression to the 64th. square. But it was found that not only all the corn in his majesty's dominions would not be sufficient to pay him, but not all that could be produced in 8 years on the surface of the terraqueous globe, if it were all arable land, and under cultivation. The number of grains demanded was $2^{64}-1$. We have already seen (§ 190.) that $2^{20}=1,048576$ and $2^{40}=1,048576^2=10,99511,627776$: $2^{60}=1048576^3$; and $2^{64}=1048576^3\times 2^4=1,152921,504606,846976\times 16=18,446744,073709,551616$, or less than 18 *trillions* and a half. Now supposing a bushel of corn to contain 600,000 grains, (i. e. supposing a standard pint to contain 9375 grains) and supposing an acre of land to produce in a year 30 bushels of corn, it would require one billion of acres to produce 18 trillions of grains. But the whole surface of the terraqueous globe amounts to little more than the 8th. of 1 billion of acres.

196. Let us now suppose a *decreasing* series in Geometrical progression, for instance 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c. The 22d. term of this series must be $\frac{1}{2^{20}}$ or $\frac{1}{1048576}$; and therefore the sum of the series $=4-\frac{1}{2^{20}}$. And if we continue the series to 66 terms, the sum must be $4-\frac{1}{2^{64}}$; that is, less than 4 by a fraction so small that, although subject to numerical

numerical calculation, it baffles all conception. But there is no limit to our power of continuing the series; and the further we continue it, the nearer must the sum approach to 4: while, continued ever so far, the sum of all the terms never can exceed 4. For if we continued the series to 1000 terms, the sum would be 4 *minus* a fraction whose numerator is 1 and the denominator the 998th. power of 2. Hence we may say that 4 is the exact sum of that series continued *in infinitum*: by which we mean that, let the series be continued ever so far, the sum of all the terms never can exceed 4; and that it may be continued so far as that the sum shall exceed any number ever so little less than 4, or that is less than 4 by a fraction ever so small. In like manner the sum of the *infinite* series 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, &c. is $\frac{9}{2}$ or $4\frac{1}{2}$. For by § 193. the sum of the finite series 3, 1, $\frac{1}{3}$, $\frac{1}{9}$ is $(9-\frac{1}{9}) \div 2$: and let the series be continued ever so far, the sum would be found by subtracting the last, or *least*, term from 9, and dividing the remainder by 2. But when the series is considered *infinite*, or continued without end, there is no *least* term to be subtracted, and therefore the sum is $\frac{9}{2}$. And universally let a, $\frac{a}{x}$, $\frac{a}{x^2}$, &c. represent a decreasing infinite series. The sum of that series is $\frac{ax}{x-1}$. For, continued to n terms, its sum is $\left(ax-\frac{a}{x^{n-1}}\right)$ $\div(x-1)$. But if the series be continued without end, there is no fraction $\frac{a}{x^{n-1}}$ to be subtracted from ax.

197. I have generally observed, that on the first discussion of this very curious subject there remains in the mind a suspicion of some latent fallacy in the reasoning. But let us bring its accuracy to a particular test. We know that the vulgar fraction $\frac{2}{3}$, turned into the decimal form, produces the circulating decimal .666, &c. (§ 118.) Now this circulating decimal is in fact the sum of an infinite decreasing series in geometrical progression; for it is equal to $\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}$, &c. (See § 114.) Let us then calculate its value according to the principles of the last section. The greatest term is $\frac{6}{10}$; the denominator of the common ratio 10. Therefore the sum of the series $=(\frac{6}{10}\times 10)\div\overline{10-1}=\frac{6}{9}$. But this fraction being equivalent to $\frac{2}{3}$, we have a confirmation that the principles are

just

just, which we have laid down for calculating the sum of an infinite decreasing series. Thus again $\frac{2}{9}=.222$, &c. and $\frac{2}{10}+\frac{2}{100}$, &c. $=(\frac{2}{10}\times 10)\div 9=\frac{2}{9}$. In like manner .999, &c. $=1$.

198. Upon these principles we can easily find the vulgar fraction, which produces any given circulating decimal. For instance, .212121, &c. $=\frac{21}{100}+\frac{21}{10000}$, &c. where the denominator of the common ratio is 100. Therefore the sum of the series is $\frac{21}{99}=\frac{7}{33}$: and accordingly $\frac{7}{33}$ reduced to the decimal form produces the given circulate. Let it be required to find the vulgar fraction, which shall circulate through the ten digits in regular order. The denominator of the ratio being the tenth power of 10, the sum of the series is $\frac{1234567890}{9999999999}=\frac{137174210}{1111111111}$.

199. Upon the principles brought forward in § 196. we may detect the sophism, by which Zeno pretended to prove that the swift-footed Achilles could never overtake a tortoise, if they set out together, and the tortoise were at first any distance before Achilles. "If," said he, "the tortoise at setting off be a furlong before Achilles, though the latter runs 100 times faster than the tortoise crawls, yet, when he has run a furlong, the tortoise will be the 100th. part of a furlong before him: and when Achilles has advanced that small space, the tortoise will still be before him by the 100th. part of it, and so on *for ever*." Now it is very true that, if we take the spaces or times decreasing in that geometrical ratio of 100 : 1, we cannot assign among them—(how far soever we continue the progression)—any one, at which Achilles will have overtaken the tortoise. But it is altogether false, that the sum of those spaces or times will be an infinite quantity, as is implied in Zeno's conclusion: for the sum of the infinite series 1, $\frac{1}{100}$, &c. is exactly $\frac{100}{99}$ or $1\frac{1}{99}$. And accordingly that gives us the precise spot where Achilles will overtake the tortoise: for when he has gone $\frac{100}{99}$ths. of a furlong, the tortoise, moving 100 times slower, will have gone $\frac{1}{99}$th. that is, they will be just together. And this affords another confirmation, to prove the truth of our calculation of the sum of an infinite decreasing series.

Ex. 1. What is the 8th. term of the Geometrical series 4, 12, 36, &c. and what is the sum of the series?

Ex. 2.

Ex. 2. What is the 9th. term, and what is the sum, of the Geometrical series a, a^2, a^3, &c.?

Ex. 3. What is the sum of the decreasing series 18, 6, 2, &c. continued *in infinitum?*

Ex. 4. Do. of the decreasing series a^9, a^8, a^7, &c.?

Ex. 5. What vulgar fraction will produce the circulating decimal .102102, &c.?

Ex. 6. If a man spend 1 farthing in the first week of the year, and each succeeding week twice as much as in the preceding, how much will he spend in the whole year?

Ex. 7. In how many minutes after 6 o'clock will the minute hand of a watch overtake the hour hand?

Ex. 8. If two men at opposite points of a circle set out at the same time and in the same direction, with velocities that are as 7 : 6, how many times must the quicker go round the circle before he overtakes the slower?

Ex. 9. If a courier ride at the rate of 6 miles an hour, and in $\frac{3}{4}$ of an hour after he has set out a second courier be dispatched to recall him, and ride at the rate of $7\frac{1}{2}$ miles an hour, at what distance will the second overtake the first?

CHAP. XXI.

Extraction of the Square Root.

200. TO extract the square root of a number is to find a number, whose square is the given number: and the multiplication table enables us to assign the root of any square number as far as 144. Many fractional numbers may have their square roots assigned with equal facility. Thus the square root of $\frac{16}{81}$, or $\sqrt{\frac{16}{81}}$, is $\frac{4}{9}$, because $\frac{4}{9}\times\frac{4}{9}=\frac{16}{81}$. And the square root of $\frac{8}{18}$ is $\frac{2}{3}$; for, although we cannot extract the square root of 8 or of 18, they not being square numbers, yet $\frac{8}{18}=\frac{4}{9}=\overline{\frac{2}{3}}|^2$: so that before we conclude that the square root of a fraction cannot be exactly assigned, the fraction should be brought to its lowest terms.

201. It is equally easy to assign the square root of any simple Algebraic quantity, which is a perfect square. Thus $\sqrt{9a^2}=3a$; for $3a\times 3a=9a^2$: and $\sqrt{\frac{12a^2}{27b^2}}=\frac{2a}{3b}$;

for

for $\frac{12a^2}{27b^2}=\frac{4a^2}{9b^2}$: and $\sqrt{a^2b^2}=ab$; for $ab\times ab=a^2b^2$. From this last example we may observe, that *the square root of any product is equal to the product of the square roots of its factors.* Thus $4\times16=64$, and $\sqrt{64}=8=2\times4=\sqrt{4}\times\sqrt{16}$. And hence it follows, that the product of any two square numbers must be a square number; for its square root is the product of the roots of the two factors.

202. The operation by which we extract the root of higher square numbers proceeds on the principle that the square of the binomial $a+b$ is $a^2+2ab+b^2$, and that if we divide the two latter terms $2ab+b^2$ by $2a+b$ the quotient is b. Now suppose we want to find the square root of 5476. We know that the square of 70 is 4900, and that the square of 80 is 6400. Therefore the square root sought is less than 80 but more than 70. Considering therefore the root sought as a binomial $(a+b)$ of which we now know one part (a) we subtract the square of 70 (4900) from 5476. The remainder 576, corresponding with $2ab+b^2$, must contain twice the product of 70 and the other part *plus* the square of the latter; and therefore if divided by twice 70 (140) plus the other part must give that other part for the quotient. And thus we find that the second part of the binomial root is 4; for $\overline{140+4}\times4=576$. The root sought therefore is $70+4$, or 74.

203. Let it now be required to extract the square root of 225625. We know at once that the root sought must be greater than 400 and less than 500: for $400^2=160000$, but $500^2=250000$. Subtracting therefore 400^2 from 225625, there remains 65625; which contains indeed 800 (400×2) above 80 times, but does not contain $800+80$ (or 880) so often as 80 times. The remaining part of the root therefore is less than 80, but more than 70; for multiplying 870 by 70 the product 60900 is less than 65625 by the remainder 4725. We have now however ascertained the second of the three digits of which the root must consist; and only want to find the last which stands in the place of units: for the root sought is above 470, but below 480. If then, considering 470 as the first part of a binomial root, we subtract its square from the proposed number 225625, the remainder divided by twice 470 *plus* the last digit of the root must give that last digit for the quotient. But we may save ourselves the trouble of squaring

squaring 470, observing that the subtraction of its square has been already performed. For $470^2 = 400^2 + 70^2 + \overline{800 \times 70}$. Now in our first operation we subtracted 400^2 from the given square: and 60900 which we subtracted from the remainder is 870×70, that is $\overline{800 \times 70} + \overline{70 \times 70}$. If therefore we divide the last remainder 4725 by 940 (twice 470) *plus* the last digit of the root, the quotient must be that last digit. But $\frac{4725}{945} = 5$. Therefore the root sought is 475. And in like manner we find that the square root of 6953769 is 2637; for the third remainder in the operation is 36869: but that is the remainder after subtracting 2630^2 from the given number; and therefore divided by twice 2630 *plus* the 4th. digit of the root must give that 4th. digit for the quotient:—just as the quotient of $2ab + b^2$ divided by $2a + b$ is b. And so, let there be ever so many digits in the root, they may be successively discovered.

204. In practice, we begin with the *first* or the *two first* digits of the proposed square, according as the number of its digits is *odd* or *even*; and subtracting from it the square number next below it, (afforded us by the multiplication table) writing its root as the first digit of our root, we annex to the remainder the next pair of digits in the proposed square. And so on, successively dividing all the digits of each completed remainder, except the last digit, by twice the digits of the root found; and thus ascertaining the next digit. Then annexing that digit to our divisor, we multiply the completed divisor by the digit of the root last found, and subtract the product from the last completed remainder. Let us annex the operation performed at large, and according to the abbreviated method; that a comparison of them may make their identity manifest.

$$
\begin{array}{rrl}
 & 6953769 & (\ 2000 \\
 & 4000000 & (\ +600 \\
4000) & \overline{2953769} & \quad +30 \\
+600) & 2760000 & \quad\ \ +7 \\
5200) & \overline{193769} & \\
+30) & 156900 & \\
5260) & \overline{36869} & \\
+7) & 36869 & \\
 & \overline{\ldots\ .} &
\end{array}
\qquad\qquad
\begin{array}{rl}
 & \dot{6}9\dot{5}3\dot{7}6\dot{9}\,(2637 \\
 & 4 \\
46) & \overline{295} \\
 & 276 \\
523) & \overline{1937} \\
 & 1569 \\
5267) & \overline{36869} \\
 & 36869 \\
 & \overline{\ldots\ldots}
\end{array}
$$

205. It is plain by inspection of these two operations that the only difference between them is, that in the shorter method we neglect writing the cyphers, and attend only to the significant figures concerned in each part of the process. But let us trace the several steps of the operation in that example. The number of digits in the proposed square being odd, we first attend to the single digit on the left hand, 6. The square number next below it is 4, whose root 2 we write as the first digit of our root; and subtracting 4 from 6 there remains 2; to which we annex the two next digits of our proposed square, 95. Then dividing 29 by twice 2, or 4, we might conceive that 7 should be the next digit of the root. But 7 times 47 being more than 295, we fix upon 6 as the next digit of the root; and annexing it to 4 we subtract 6 times 46, or 276, from 295, and to the remainder 19 we annex the two next digits of the proposed square 37. Then doubling 26, or adding 6 to the last divisor 46, we observe that 52 is contained in 193 three times. Therefore writing 3 as the next digit of the root, and annexing it to 52, we subtract 3 times 523, or 1569, from 1937, and to the remainder 368 we annex the two last digits of the proposed square, 69. Then doubling 263, or adding 3 to the last divisor 523; and observing that 526 is contained in 3686 seven times, we write 7 as the next digit of the root, annex it to 526, and subtract 7 times 5267 from 36869, when nothing remains: so that the proposed number is a complete square whose root is 2637. The proposed number is commonly pointed off by pairs of digits from the right hand, to ascertain the pairs which are to be annexed to the successive remainders, and whether we are to begin with the first or the two first digits on the left hand.

206. To explain the reason of the rule, by which we determine whether we are to begin with the first digit of the proposed number or with the two first digits; let it be observed that the number of digits in any square cannot exceed double the number of digits in the root, and cannot fall short of that by more than 1. Thus if there be 3 digits in the root, there must be at least 5 in the square, and there cannot be more than 6: if there be 9 or 10 digits in the square there must be 5 digits in the root. For take the greatest number consisting, for instance, of 3 digits,

namely

namely 999. Its square must be less than the square of 1000, that is, less than 1000000. Therefore the number of digits in the square of 999 cannot exceed 6. Now take the least number written with 3 digits, namely 100; and its sqnare consists of 5 digits. By the same mode of reasoning it is manifest that, if the root begin with any digit except 1, 2, or 3, (whose squares consist of a single digit) the square must consist of twice as many digits as the root. Since therefore the square proposed in the last example, 6953769 consists of 7 digits, the root must consits of 4 digits, and its first digit must be less than 4. Therefore we begin with inquiring the nearest square number to 6, not the nearest to 69: for this would give 8 for the first digit of the root.

207. If we find any remainder after the last subtraction, we conclude that the proposed number is not a complete square; but by annexing decimal cyphers in pairs, and thus continuing the process of extraction, we may approximate to the root at pleasure. In such a case it is evidently impossible ever to arrive at the exact root; since there is no significant digit whose square ends with a cypher: but we may approach nearer it than any assignable difference. Thus, if we desire to find a number which shall be nearer the root than by the millionth part of unity, we need only continue the process of extraction to 6 places of decimals, for which purpose we must have annexed 6 pairs of decimal cyphers. For even if the root could circulate from that in 9's, the remaining part would only be equal to $\frac{1}{1000000}$. But the root in this case can never circulate: for the value of every circulating decimal may be exactly assigned in a finite fraction (§ 198.) and we have seen that the exact root of such a number as we have supposed never can be assigned. We annex the decimal cyphers in pairs, because for every digit in the root after the first there must be two digits in the square. If the proposed number be partly integral and partly decimal, we must point off the integral part distinctly, and make the number of decimal places even, by annexing a cypher if necessary. Thus in extracting the square root of 27.345, the first digit of the root is, not 1, but 5.

208. From what has been said it appears that we may either express the square root of 2, for instance, as a surd,

surd,—thus $\sqrt{2}$ or $2^{\frac{1}{2}}$, or else proceed to extract it within any degree of accuracy that may be required:—thus.

$$\begin{array}{r|l}
 & 2.00(1.414213 \\
 & 1 \\
24) & 1.00 \\
281) & 400 \\
2824) & 11900 \\
28282) & .60400 \\
282841) & .383600 \\
2828423) & 10075900 \\
 & 1590631, \&c.
\end{array}$$

Now the square of 1.41 is 1.9881 or within .0119 of 2. The square of 1.414 is 1.999396, or within .000604 of 2; and so on.

209. The square roots of compound algebraic squares are extracted exactly in the same manner; first arranging the terms of the proposed square according to the powers of some one letter. For example let it be required to extract the square root of

$$\begin{array}{r|l}
 & a^4-4a^3b+8ab^3+4b^4(a^2-2ab-2b^2 \\
 & a^4 \\
2a^2-2ab) & -4a^3b+8ab^3+4b^4 \\
 & -4a^3b+4a^2b^2 \\
2a^2-4ab-2b^2) & \quad -4a^2b^2+8ab^3+4b^4 \\
 & \quad -4a^2b^2+8ab^3+4b^4 \\
 & \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

And accordingly if we multiply the trinomial $a^2-2ab-2b^2$ by itself, the product will be the proposed quantity. It is manifest in this example that the second remainder has been found by subtracting the square of a^2-2ab from the given quantity: for $\overline{a^2-2ab}|^2=a^4-4a^3b+4a^2b^2$.

210. We may here remark that 4 times the product of any two numbers differing by unity, *plus* 1, gives the square of their sum. For let a represent the less; then $a+1$ will represent the greater; and $2a+1$ their sum. But $\overline{2a+1}|^2=4a^2+4a+1$; and $4a^2+4a=4\times a\times\overline{a+1}$.

Thus

Thus $\overline{10+9}|^2=361=\overline{4\times 90}+1$. And if we add to any number its square $+\frac{1}{4}$, the sum must be a square number: for $a^2+a+\frac{1}{4}$ is the square of $a+\frac{1}{2}$. Thus $9+81+\frac{1}{4}$ is the square of $9\frac{1}{2}$ or $\frac{19}{2}$. Lastly the sum of any two numbers differing by unity is the difference of their squares. For $\overline{a+1}|^2-a^2=2a+1$.

Ex. 1. Extract the square roots of 6889? of 38416? and of 3?

Ex. 2. Extract the 4th. root of 4096? Since $x^2=\sqrt{x^4}$ and $x=\sqrt{x^2}$, it is plain that the 4th. root sought must be the square root of the square root of 4096.

CHAP. XXII.

Fractional and Negative Indices. Calculations of Surds.

211. IT is evident that the square of a^2 is a^4, and that the square of a^3 is a^6; since $aaa\times aaa=aaaaaa=a^6$. In like manner the cube of a^2 is a^6; since $aa\times aa\times aa=aaaaaa=a^6$. And putting n for the index of any power of a, the square of a^n is a^{2n}, its cube a^{3n}, its fourth power a^{4n}, &c. So that a^n is raised to any power by only multiplying its index n by the index of that power. It follows that a^n is the square root of a^{2n}, the cube root of a^{3n}, the fourth, or biquadrate, root of a^{4n} &c. So that we may express any root of a given quantity by dividing its index by the denominator of that root: just as the cube root of a^3 is a, or a^1, and the cube root of a^6 is a^2. For $\frac{3}{3}=1$, and $\frac{6}{3}=2$. Hence the origin of expressing roots by fractional exponents: for thus the square root of a^1 is justly expressed by $a^{\frac{1}{2}}$, its cube root by $a^{\frac{1}{3}}$, &c. In like manner the square root of a^3 is $a^{\frac{3}{2}}$; the cube root of a^2 is $a^{\frac{2}{3}}$, &c. And universally, putting n and m for any numbers whatsoever, the nth. root of a^m is $a^{\frac{m}{n}}$. And this mode of notation has many advantages above the expression by the radical sign $\sqrt[n]{a^m}$.

212. Since

212. Since $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}$, &c. and $\frac{1}{3}=\frac{2}{6}=\frac{3}{9}$, &c. therefore $a^{\frac{1}{2}}$ (or $\sqrt{a}$)$=a^{\frac{2}{4}}=a^{\frac{3}{6}}$, &c. and $a^{\frac{1}{3}}$ (or $\sqrt[3]{a}$) $=a^{\frac{2}{6}}=a^{\frac{3}{9}}$, &c. it follows that, as we can bring any two numbers integral or fractional to fractions of the same denomination, we may easily reduce any two quantities to equivalent expressions of the same radical sign. For instance, let it be required to bring $a^{\frac{2}{3}}$ and $b^{\frac{1}{2}}$ to the same radical sign: we have only to reduce the fractional indices $\frac{2}{3}$ and $\frac{1}{2}$ to equivalent fractions with the same denominator, and the expressions become $a^{\frac{4}{6}}$ and $b^{\frac{3}{6}}$ or $aaaa^{\frac{1}{6}}$ and $bbb^{\frac{1}{6}}$, or $\sqrt[6]{a^4}$ and $\sqrt[6]{b^3}$. Now we have observed (§ 201) that the square root of any product is equal to the product of the square roots of its factors: whence it follows that $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$. And universally the product of the *n*th. roots of any factors is equal to the *n*th. root of the product of the factors: or $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$. A similar principle must evidently be applicable to division $\sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{\frac{a}{b}}$. And since we can transform any two given surds of different radical signs into surds of the same radical sign, it is plain that we can thus express the product or quotient of any two given surds under one radical sign. Thus $\sqrt[n]{a}\times\sqrt[m]{b}=\sqrt[nm]{a^m b^n}$: for $\sqrt[n]{a}=a^{\frac{m}{nm}}$, and $\sqrt[m]{b}=b^{\frac{n}{nm}}$; but $\sqrt[nm]{a^m}\times\sqrt[nm]{b^n}=\sqrt[nm]{a^m b^n}$. In like manner $3\div\sqrt[3]{2}=\sqrt[3]{\frac{27}{2}}$.

213. Let it also be remembered that powers of the same quantity are multiplied by adding their indices, (e. gr. $a^4\times a^3=a^7$) and divided by subtracting the index of the divisor from the index of the dividend. (e. gr. $\frac{a^5}{a^2}=a^3$.) Now suppose we have to multiply $\sqrt{a}$ by a: the product may be expressed by prefixing a as a coefficient, thus, $a\sqrt{a}$. But since $a=\sqrt{a^2}$, the product may also be expressed by $\sqrt{a^3}$; for $\sqrt{a}\times\sqrt{a^2}=\sqrt{a^3}$. But we may at once arrive at the same conclusion by adding the indices of the factors $a^{\frac{1}{2}}$ and a; for $a^{\frac{1}{2}}\times a^{1}=a^{\frac{3}{2}}$, since $\frac{1}{2}+1=\frac{3}{2}$. In like manner

ner $\sqrt[3]{a^2}\times\sqrt{a}$, or $a^{\frac{2}{3}}\times a^{\frac{1}{2}}=a^{\frac{7}{6}}$, or $\sqrt[6]{a^7}$, since $\frac{2}{3}+\frac{1}{2}=\frac{7}{6}$. Thus also $\frac{a}{\sqrt{a}}=a^{1-\frac{1}{2}}=a^{\frac{1}{2}}$ and $\frac{\sqrt[3]{a^2}}{\sqrt{a}}=a^{\frac{2}{3}-\frac{1}{2}}=a^{\frac{1}{6}}$.

214. Now we know that $\frac{a}{a}=1$. But it may also be expressed by a^{1-1}, or a^0. And in like manner $\frac{a}{a^2}$, or $\frac{1}{a}$, may be expressed by a^{-1}; since $1-2$, or $0-1$, $=-1$. And thus $\frac{1}{a^2}=a^{-2}$; $\frac{1}{a^3}=a^{-3}$, &c. Thus we see that a^{-n} is a just expression for the reciprocal of $\frac{a^n}{1}$ or of a^n. We have observed that the product of any quantity and its reciprocal is 1: e. gr. $\frac{m}{n}\times\frac{n}{m}=\frac{mn}{mn}=1$. And accordingly $a^n\times a^{-n}=a^0=1$.

215. Though we cannot add or subtract surds by incorporation, unless they have the same irrational part, and otherwise must denote the addition or subtraction by the sign $+$ or $-$; yet it often happens that unlike surds may be transformed into like by resolving one or both of them into a rational part and an irrational. Thus $\sqrt{2}$ and $\sqrt{8}$ are unlike surds, and their sum or difference is $\sqrt{8}\pm\sqrt{2}$. But since $\sqrt{8}=\sqrt{4}\times\sqrt{2}$, and $\sqrt{4}=2$, therefore $\sqrt{8}=2\sqrt{2}$: and $2\sqrt{2}$ and $\sqrt{2}$ being like surds may be incorporated; their sum being $3\sqrt{2}$ and their difference $\sqrt{2}$. Thus also $\sqrt[3]{24}+\sqrt[3]{81}=2\sqrt[3]{3}+3\sqrt[3]{3}=5\sqrt[3]{3}$. And universally $\sqrt[n]{a^nx}\pm\sqrt[n]{b^nx}=\overline{a\pm b}\times\sqrt[n]{x}$. It is plain that the product of any two quadratic surds which are like, or may be transformed into like surds, must be rational. Thus $\sqrt{2}\times\sqrt{8}=\sqrt{16}=4$: and $\sqrt{a^2x}\times\sqrt{b^2x}=abx$. Otherwise the product of any two quadratic surds must be irrational. And as we may sometimes take one part of a given surd from under the radical sign and prefix it as a rational coefficient; so, whenever we have a surd with a rational coefficient, we may bring it under the radical sign: since $a\sqrt[n]{x}=\sqrt[n]{a^nx}$.

216. Any fraction with a binomial denominator, one or both of whose terms is a surd, may be transformed into an equivalent

equivalent fraction whose denominator shall be rational; upon the principle that the product of the sum and difference of any two quantities is equal to the difference of their squares. (§ 163.) Thus the fraction $\frac{a}{\sqrt3+\sqrt2}$, by multiplying both numerator and denominator by $\sqrt3-\sqrt2$, becomes $\frac{a\sqrt3-a\sqrt2}{3-2}$, or $a\sqrt3-a\sqrt2$. For when we multiply $\sqrt3+\sqrt2$, (the *sum* of $\sqrt3$ and $\sqrt2$) by $\sqrt3-\sqrt2$, (their difference), the product must be the difference of their squares. And in like manner, if the denominator consist of three or more parts, we may by successive multiplications render it rational. e. gr. Let the denominator be $2+\sqrt2-\sqrt a$: we may consider $\overline{\sqrt2-\sqrt a}$ as one term, and 2 as the other term composing the denominator; and if we multiply both numerator and denominator of the fraction by $2-\overline{\sqrt2-\sqrt a}$, the new denominator $2+2\sqrt{2a}-a$ will have in it but one irrational term, since there is but one irrational term in the square of the binomial surd $\sqrt2-\sqrt a$. And now considering the new denominator as consisting of the two parts $\overline{2-a}$, and $2\sqrt{2a}$, if we multiply both numerator and denominator by $2-a-2\sqrt{2a}$, the denominator must be the difference of the squares of $\overline{2-a}$ and $2\sqrt{2a}$, or $4-12a+a^2$. And thus the irrationality is removed from the denominator to the numerator. Pursuing this process the student will find that the fraction $\frac{8-5\sqrt2}{3-2\sqrt2}=4\sqrt2$: and accordingly $\overline{4+\sqrt2}\times\overline{3-2\sqrt2}=8-5\sqrt2$.

217. The square root of any binomial $a\pm b$ may justly be representted by the following expression, $\sqrt{\frac{a+\sqrt{a^2-b^2}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b^2}}{2}}$; for the square of this expression is $\frac{2a}{2}\pm\frac{2\sqrt{b^2}}{\sqrt4}$, that is $a\pm b$. By performing the operation the student will find that the square of that binomial is what we have assigned; and from the following considerations he may be convinced that it must be so. We know that the square of any binomial is composed of the sum of the

the squares of its parts, *plus* or *minus* twice the product of the parts. Now the parts of that binomial surd are squared by throwing off the radical signs prefixed to them; and therefore their squares are $\frac{a+\sqrt{a^2-b^2}}{2}$ and $\frac{a-\sqrt{a^2-b^2}}{2}$, and the sum of these two quantities is $\frac{2a}{2}=a$. Again let us consider what must be the product of the two binomial surds $\sqrt{\frac{a+\sqrt{a^2-b^2}}{2}}$ and $\sqrt{\frac{a-\sqrt{a^2-b^2}}{2}}$. It will be found by taking the product of the numerators and the product of the denominators, and prefixing to each the radical sign $\sqrt{}$. Therefore the denominator of the product must be $\sqrt{4}$, or 2. But since the numerators are the sum and difference of the same quantities a and $\sqrt{a^2-b^2}$, their product must be the difference of the squares of those quantities: (§ 163.) that is the difference between a^2 and a^2-b^2, which difference is b^2. Therefore the numerator of the product of the two binomial surds is $\sqrt{b^2}$ or b, and their product is $\frac{b}{2}$; and twice that product is b. And thus we see that the square of the assigned binomial surd must be $a\pm b$. Let us exemplify the truth of this in numbers. We know that $\sqrt{10+6}=4$. But it is also equal to $\sqrt{\frac{10+\sqrt{100-36}}{2}}+\sqrt{\frac{10-\sqrt{100-36}}{2}}$; for $\sqrt{100-36}$ $=\sqrt{64}=8$: and therefore $\sqrt{\frac{10+\sqrt{100-36}}{2}}=\sqrt{\frac{18}{2}}$ $=\sqrt{9}=3$; and $\sqrt{\frac{10-\sqrt{100-36}}{2}}=\sqrt{\frac{2}{2}}=\sqrt{1}=1$. Again $\sqrt{10-6}=2=\sqrt{\frac{10+\sqrt{100-36}}{2}}-\sqrt{\frac{10-\sqrt{100-36}}{2}}$ $-\sqrt{\frac{18}{2}}-\sqrt{\frac{2}{2}}=3-1$.

218. This mode of expressing the square root of a binomial has its principal use in some binomial surds, which often occur in practice. (See § 238.) For instance if we want to express the square root of $a\pm\sqrt{b}$, it may be designated

signated by prefixing the radical sign to the binomial surd; thus $\sqrt{a \pm \sqrt{b}}$, or $\overline{a \pm \sqrt{b}}|^{\frac{1}{2}}$. But whenever $a^2 - b$ is a square number, let us put r for the square root of that number, and we may express the square root of the given binomial by $\sqrt{\frac{a+r}{2}} \pm \sqrt{\frac{a-r}{2}}$. Thus the square root of $11 + 6\sqrt{2}$ (or $11 + \sqrt{72}$) is $\sqrt{\frac{11+\sqrt{49}}{2}} + \sqrt{\frac{11-\sqrt{49}}{2}}$ $= \sqrt{\frac{11+7}{2}} + \sqrt{\frac{11-7}{2}} = \sqrt{9} + \sqrt{2} = 3 + \sqrt{2}$: which is a simpler expression than $\sqrt{11+6\sqrt{2}}$. Again $\sqrt{7+2\sqrt{6}}$ may be more simply expressed, since $49 - 24 = 25$, a square number. Therefore the square root of $7 + 2\sqrt{6} = \sqrt{\frac{7+5}{2}}$ $+ \sqrt{\frac{7-5}{2}} = \sqrt{6} + 1$, or $1 + \sqrt{6}$. And accordingly $\overline{1+\sqrt{6}}|^2$ $= 7 + 2\sqrt{6}$.

Ex. 1. $\sqrt{x^3} \times \sqrt[3]{x^2} = ?$ $\sqrt[3]{2a^2x} \times \sqrt{3a^3x^5} = ?$

Ex. 2. $\sqrt{x^3} \div \sqrt[3]{x^2} = ?$ $\sqrt{3a^3x^5} \div \sqrt[3]{2a^2x} = ?$

Ex. 3. $x^2y \times a^0 = ?$ $\sqrt{x} \times x^{-\frac{1}{2}} = ?$

Ex. 4. $\sqrt{125x} \pm \sqrt{4x} = ?$ $\sqrt{80x^4y} \pm \sqrt{20x^4y}$?

Ex. 5. Reduce the fraction $\frac{3}{\sqrt{5}+\sqrt{8}}$ to an equivalent fraction with a rational denominator?

Ex. 6. Also the fraction $\frac{x}{1+\sqrt{x}-\sqrt{y}}$?

Ex. 7. What is the simplest value of $\sqrt{9+\sqrt{45}}$?

Ex. 8. ...of $\sqrt{19-\sqrt{261}}$?

CHAP. XXIII.

Reduction of Algebraic Equations, Simple and Quadratic.

219. TO reduce an equation is to discover the value of the unknown quantity in it, which has been represented by

by one of the final letters of the alphabet. Thus, if we have proposed to us the equation $5x - 34 = 57 + \frac{2x}{3}$; we may by a very short and easy process discover what number x stands for. Now, according to the import of the Algebraic symbols as already explained, the proposed equation expresses this fact, that the subtraction of 34 from 5 times the number represented by x gives a remainder equal to the sum of 57 and $\frac{2}{3}$rds of the number represented by x. And therefore whenever we shall have ascertained the value of x, this property must belong to the number found; so that if we substitute the number found for x, in each expression where that letter occurs in the proposed equation, the amount of the terms at one side must be equal to the amount of the terms at the other side of the equation. Thus, by reducing the equation $5x - 34 = 57 + \frac{2x}{3}$, we shall find that $x = 21$: and the truth of this result will appear by substituting 5×21 for $5x$; and $\frac{2}{3} \times 21$ for $\frac{2x}{3}$. For $5 \times 21 = 105$; and $\frac{2}{3} \times 21 = 14$: but $105 - 34 = 71$; and $57 + 14 = 71$. Such an equation as $5x - 34 = 57 + \frac{2x}{3}$ is called a simple equation, because the unknown quantity x does not rise in any term of it beyond the 1st. power.

220. The process of reducing such equations depends upon the following simple principles; that if to equal quantities we add the same or equal quantities the sums will be equal; or if from equal quantities we subtract the same or equal quantities the remainders will be equal; and that if we multiply or divide equal quantities by the same number, the products or quotients will be equal. From the former of these principles it follows, that we may transpose any term of an equation from one side of it to the other, changing its sign. Thus in the proposed equation $5x - 34 = 57 + \frac{2x}{3}$, we may bring over 34 from the left side of the equation to the right with the sign $+$; and infer that $5x = 57 + 34 + \frac{2x}{3}$. For this is in fact an addition of 34 to *both* sides of the equation; the sum

sum of -34 and $+34$ being 0. But we may also bring over $\frac{2x}{3}$ from the right side of the equation to the left with the sign $-$; and infer that $5x-\frac{2x}{3}=57+34=91$. For this is in fact but a subtraction of $\frac{2x}{3}$ from both sides of the equation; since $\frac{2x}{3}-\frac{2x}{3}=0$. From the same principle it follows, that we may at pleasure change the signs of all the terms at both sides of an equation. Thus from the equation $24-2x=-10$, we may infer that $-24+2x$ (or $2x-24)=10$: for this is in fact but a subtraction of the affirmative terms, and an addition of the negative to both sides of the equation.

221. Let us now take the equation $5x-\frac{2x}{3}=91$; in which we have brought over to one side of the equation all the terms in which x (the unknown quantity) occurs, and have only the amount of known numbers at the other side. We may now infer, that 3 times the one side is equal to 3 times the other side of the equation. But 3 times the binomial $5x-\frac{2x}{3}$ is equal to $15x-2x=13x$: for when we multiply the fraction $\frac{2x}{3}$ by its denominator 3, the product is the numerator $2x$ integral. (§ 113.) Therefore $13x=91\times 3=273$. And now we may divide both sides of this equation, $13x=273$, by 13, and infer that the quotients will be equal. But the quotient of $13x$ divided by 13 is x; which is therefore equal to $\frac{273}{13}=21$. And thus we have ascertained the value of x; and the reduction of the equation $5x-34=57+\frac{2x}{3}$ is completed. Let us exhibit the steps, which we have taken in one view.

$$5x-34=57+\frac{2x}{3}$$

$+34-\frac{2x}{3}$	$5x-\frac{2x}{3}=57+34=91$
$\times 3$	$(15x-2x=)\ 13x=91\times 3=273$
$\div 13$	$x=273\div 13=21$

222. The

222. The marks on the left hand of the derived equations denote the operation, by which each equation is derived from the preceding; 1st. the addition of $34-\frac{2x}{3}$ to both sides, or the transposition of those terms with their signs changed: 2ndly. the multiplication of both sides by 3; 3rdly. the division of both sides by 13. And it may be useful to the student at first to adopt that practice, of marking in the margin the operation by which he proceeds to derive each equation; although this will afterwards become unnecessary. In the first step of the preceding example, both the terms 34 and $\frac{2x}{3}$ are transposed by one operation; and ever so many terms may be transposed at once, only taking care to change the signs. But for a time it may be better for the student to transpose the terms one by one.

223. After the first step, we might have completed our reduction by one inference, observing that $5x-\frac{2x}{3}$ is the product of $\overline{5-\frac{2}{3}}\times x$. If therefore we divide both sides by $5-\frac{2}{3}$, that is by $4\frac{1}{3}$ or $\frac{13}{3}$, we shall at once have the equation $x=91\div\frac{13}{3}=\frac{273}{13}=21$. But in the second step of the reduction, as exhibited at the end of § 221. the student should well observe, how an equation may be cleared of any fractional expression, by multiplying both sides of the equation by the denominator of that fraction. And let there be ever so many fractional expressions in an equation, they may be all removed either successively, or by one operation. For instance, if we have this equation $\frac{x}{2}+\frac{2x}{3}+\frac{3x}{4}=5$, successive multiplications of both sides by 2, by 3, and by 4, would remove the several fractions, producing successively the equations $x+\frac{4x}{3}+\frac{6x}{4}=10$, and $(3x+4x$ i. e.$)$ $7x+\frac{18x}{4}=30$, and $(28x+18x$ i. e.$)$ $46x=120$.
Hence it is plain that the same result must be afforded by one multiplication of both sides by the product of 2, 3, and 4, or by 24. But it will answer the same purpose, and keep our numbers lower, to multiply both sides by 12 the

least

least common multiple of 2, 3, and 4: for each of the three fractional expressions might be brought to an equivalent fraction of that denominator. Multiplying then both sides of the given equation by 12, we derive this equation ($6x+8x+9x$ i. e.) $23x=60$: whence dividing both sides by 23, we obtain the value of x, namely $x=\frac{60}{23}=2\frac{14}{23}$. But to this value we might have arrived at once, by dividing both sides of the given equation by $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$. For $\frac{x}{2}+\frac{2x}{3}+\frac{3x}{4}=\overline{\frac{1}{2}+\frac{2}{3}+\frac{3}{4}}\times x$. Therefore $x=5\div\overline{\frac{1}{2}+\frac{2}{3}+\frac{3}{4}}=5\div\frac{6}{12}+\frac{8}{12}+\frac{9}{12}=5\div\frac{23}{12}=\frac{60}{23}$.

224. The rule therefore for reducing any simple equation of this kind may be thus proposed: Bring over by transposition to one side of the equation all the terms, in which the unknown quantity (whose value you are investigating) occurs; and all the other terms to the other side. Then divide both sides of the equation by such a divisor as, if multiplied by the unknown quantity, would give the former side for the product. Let us exemplify this rule by other instances. Let the given equation be $3x+\frac{2x}{3}+24=49-2x$. Now by transposition we infer that ($3x+\frac{2x}{3}+2x$ i. e.) $5x+\frac{2x}{3}=49-24=25$: and dividing both sides of this equation by $5+\frac{2}{3}$, we find that $x=25\div5\frac{2}{3}=25\div\frac{17}{3}=\frac{75}{17}=4\frac{7}{17}$. And accordingly if $4\frac{7}{17}$ be substituted for x in the original equation, we shall find the resulting number the same on both sides. That the divisor, which will give on one side x for the quotient, is $5+\frac{2}{3}$, appears from the consideration that this divisor multiplied by x gives for the product $5x+\frac{2x}{3}$. And if there be ever so many terms on one side, in each of which x (or the letter denoting the unknown quantity) appears as a factor, it is easy from the principle proposed in § 167 to assign the divisor which will give x for the quotient. Thus if both sides of the equation $\frac{3x}{5}+4x-\frac{4x}{3}-\frac{x}{2}=1$ be divided by $\frac{3}{5}+4-\frac{4}{3}-\frac{1}{2}$, the quotient on the left side will be x, and on the right side the value of x in a known number.

225. Again,

225. Again, if we proceed to reduce the equation $\frac{3x-4}{5}+20=20+\frac{2x+3}{7}$ according to the rule proposed in the beginning of the last section, we must observe that $\frac{3x-4}{5}$ is the same thing as $\frac{3x}{5}-\frac{4}{5}$; and that $\frac{2x+3}{7}$ is the same thing as $\frac{2x}{7}+\frac{3}{7}$. So that after the necessary transpositions the equation will be $\frac{3x}{5}-\frac{2x}{7}=\frac{3}{7}+\frac{4}{5}$; which gives $x=\overline{\frac{3}{7}+\frac{4}{5}}\div\overline{\frac{3}{5}-\frac{2}{7}}=\frac{43}{35}\div\frac{11}{35}=\frac{43}{11}$. Observe also, that wherever the same quantity stands on both sides of an equation with the same sign, (as $+20$ in the last proposed example) it may be expunged from both sides. For this is only a subtraction of the same quantity from two equals. But let us exhibit the same equation reduced, by first clearing it of fractions, (after expunging the $+20$ from both sides) and let the student observe the correspondence of the operations, aud sameness of the results.

$$\frac{3x-4}{5}=\frac{2x+3}{7}$$

$\times 5$	$3x-4=\frac{10x+15}{7}$
$\times 7$	$21x-28=10x+15$
$+28-10x$	$11x=43$
$\div 11$	$x=\frac{43}{11}=3\frac{10}{11}$

226. If we be given such an equation as $\frac{1}{x}+\frac{2}{x}+\frac{3}{x}=4$, where x appears not as a factor but as a divisor, it is evident that multiplying both sides by x will bring it to the other form; giving the equation ($1+2+3$ i. e.) $6=4x$. But if in an equation the unknown quantity x appear in one term as a factor and in another term as a divisor, we shall find produced a *quadratic* equation, in which x will rise to the second power or square. Thus, if the given equation be $\frac{1}{x}+\frac{x}{2}=4$, the multiplication of both sides by x

gives

gives the quadratic equation $1+\frac{x^2}{2}=4x$; the method of reducing which we shall deliver in the 231st. and following sections.

227. On the contrary many equations that appear in the form of quadratic, cubic, &c. may be easily brought to the form of simple equations. Thus the quadratic equation $5x^2-\frac{2x}{3}=7x$, by dividing both sides of it by x, becomes $5x-\frac{2}{3}=7$. And the cubic equation $5x^3+\frac{2x^2}{3}=7x^3-4x^2$, by dividing both sides by x^2, becomes $5x+\frac{2}{3}=7x-4$. And here we may observe that, if all the terms in an equation have any common factor or divisor, we ought in the first instance to divide both sides of the equation by that common factor, or multiply both sides by that common divisor: that is, the common factor or divisor ought to be expunged from all the terms.

228. If the unknown quantity in any term of an equation be affected with a radical sign, we may free the equation from irrationality, and often bring it to the form of a simple equation by bringing that term to one side, and then raising both sides to such a power as will make that term rational. Thus if we be given the equation $\sqrt{x}-3=7$. we first infer by transposition that $\sqrt{x}=10$; and then squaring both sides we have $x=10^2=100$. The ground of this inference is obvious; namely, that if two quantities be equal, their squares, cubes, &c. must be equal. And thus from the equation $\sqrt[3]{x}-3=7$, we may infer that $x=10^3=1000$: and from the equation $\sqrt{\frac{x-3}{2}}+5=7$, we infer first that $\sqrt{\frac{x-3}{2}}=2$, and then $\frac{x-3}{2}=2^2=4$; whence $x-3=8$, and $x=11$. If we have the equation $\sqrt{5+x}=1+\sqrt{x}$, two such operations will be necessary. For first, squaring both sides, we have $5+x=1+2\sqrt{x}+x$; whence, expunging x from both sides $5=1+2\sqrt{x}$: in which equation there is but one surd, to be removed as before. Thus, $2\sqrt{x}=5-1=4$, or $\sqrt{x}=2$; and squaring both sides, $x=4$.

229. As

229. As such equations are reduced upon the principle that the squares of equal numbers are equal, so an equation in which the unknown quantity appears, in every term where it occurs, in its second power, may be reduced upon the principle that the square roots of equal numbers are equal: and such equations may as reasonably as the former be reckoned simple. Thus if we have the equation $\frac{x^2}{3}-2=10$, after reducing it to the form $x^2=\overline{10+2}\times 3=36$, we infer that the square root of one side is equal to the square root of the other, that is, $x=6$. And from the equation $2x^2=40-\frac{x^2}{2}$ we infer first that $\frac{5x^2}{2}=40$; then that $x^2=40\div\frac{5}{2}=16$: and lastly, by extracting the square root of each side, that $x=4$.

230. Upon just the same principle, if one side of our equation be the perfect square of a binomial, of which x is one term, we may arrive at the value of x by extracting the square root of both sides. For instance the square of $x+3$ is x^2+6x+9: and therefore if we have this equation $x^2+6x+9=25$, we may infer that $x+3=\sqrt{25}=5$, and therefore that $x=5-3=2$. Or if we have the equation $x^2-6x+9=25$, we may infer that $x-3=5$, and therefore that $x=8$; since x^2-6x+9 is the square of $x-3$. And here let the student recollect, what has been shewn in § 34. and 163. that the square of any binomial consists of the sum of the squares of each term of the root, *plus* or *minus* twice their product, according as the terms of the binomial root are connected with the signs $+$ or $-$. Such an equation as $x^2=25$ is called a *pure* quadratic, the unknown quantity appearing only in the second power. But if in another term it appear also in the first power, as in the example $x^2\pm 6x+9=25$, the equation is called a *mixed*, or *affected* quadratic. Simple, quadratic, cubic, &c. equations are otherwise called equations of the first, second, third &c. degree.

231. Now suppose the equation $x^2\pm 6x=16$ were proposed to us: it is plain, from what we have seen in the last section, that by only adding 9 to each side we shall have an equation reducible by the mere extraction of the square root. And that operation of adding 9 to each side is called *completing the square*; for by that addition we

we render one side the complete square of the binomial root $x \pm y$.

232. Every mixed quadratic equation may be reduced by a similar process. Suppose we are given $x^2+3x=18$. Let us consider x^2+3x as the two first terms of the square of a binomial root, whose first term is x. Now I say that the other term of the root is $\frac{3}{2}$, and that the square will be completed by adding to both sides the square of $\frac{3}{2}$, or $\frac{9}{4}$. For $3x$ is the double product of x and the other term of the binomial root: therefore $\frac{3x}{2}$ is the simple product of the two terms of the root; that is, x and $\frac{3}{2}$ are the terms of the binomial root. Accordingly $x^2+3x+\frac{9}{4}$ is the complete square of $x+\frac{3}{2}$. And adding $\frac{9}{4}$ also to the other side of the equation, we have $x^2+3x+\frac{9}{4}=18+\frac{9}{4}=\frac{81}{4}$. And now extracting the square root of both sides, (which is always the operation to be employed after completing the square) we have $x+\frac{3}{2}=\sqrt{\frac{81}{4}}=\frac{9}{2}$, and therefore $x=\frac{9}{2}-\frac{3}{2}=\frac{6}{2}=3$.

233. But in reducing mixed quadratic equations, we must often employ some other steps to prepare for completing the square. And the steps previously necessary are sufficiently obvious, when we consider what object we propose; namely to arrive at an equation of which one side shall be the complete square of a binomial, whose first term is x. At that side x^2 must stand in the first place, affirmative, and without any coefficient different from unity. It may always be made affirmative, if necessary, by changing the signs of all the terms in the equation: (§ 220.) and it may be divested of any coefficient different from unity by a division or multiplication of both sides. In the second place must stand at that side, with its proper sign, the term in which x appears in its simple power; which term is the double product of x and the other part of the binomial root. Now when we have brought these two terms to one side, (which may always be done by transposition) and the remaining terms to the other side of the equation, we are prepared for completing the square. And it is completed by adding to both sides the square of half the coefficient of x in the second term.

234. Let

234. Let us now exhibit all the necessary steps in another example. Let the proposed equation be $6-x=\frac{15}{x}-\frac{5}{2x}$, shd b which appears simple, but will produce a quadratic; and is thus reduced.

$$6-x=\frac{15}{x}-\frac{2x}{5}$$

$$\begin{array}{r|l}
 \times x & 6x-x^2=15-\frac{2x^2}{5} \\
 +\frac{2x^2}{5} & 6x-\frac{3x^2}{5}=15 \\
 & \frac{3x^2}{5}-6x=-15 \\
 \div\frac{3}{5} & x^2-10x=-25 \\
 +25 & x^2-10x+25=-25+25=0 \\
 \surd & x-5=0 \\
 +5 & x=5
\end{array}$$

Here the student will observe that after the second step, in which we have brought to one side all the terms involving the unknown quantity, we then change the signs of all the terms on both sides, in order to make x^2 affirmative, and place that term first in which x^2 appears. As this term has the coefficient $\frac{3}{5}$, we next throw off that coefficient by dividing both sides by $\frac{3}{5}$; and are then ready for completing the square. Now the second term being $10x$, the square is to be completed by adding such a number, that $10x$ shall be twice the product of the terms of the binomial root. Whence it is plain that $5x$ is the product of those terms; and x being one of them, 5 must be the other; the square of which therefore, or 25, we add to both sides. Lastly, $10x$ or the double product of the parts being negative, the binomial root must be $x-5$, not $x+5$.

235. The student should now make himself expert at the process of completing the square, by taking a variety of examples in which x in the second term shall be affected with various coefficients. Thus, let $x^2+x=21$; the square will be completed by adding $\frac{1}{4}$ to both sides; for the binomial root must be $x+\frac{1}{2}$, the coefficient of x in the second term being 1. Let $x^2-\frac{5x}{3}=14$, the binomial root must

be

be $x-\frac{5}{6}$; and therefore the square will be completed by adding $\frac{25}{36}$ to both sides. This gives us $x^2-\frac{5x}{3}+\frac{25}{36}=14+\frac{25}{36}=\frac{529}{36}$, and $x-\frac{5}{6}=\sqrt{\frac{529}{36}}=\frac{23}{6}$; 23 being found by extraction to be the square root of 529. Hence $x=\frac{23}{6}+\frac{5}{6}=\frac{28}{6}=4\frac{2}{3}$. But since every positive quantity has two square roots, (§ 164.) one positive and the other negative, we may assign another value for x. For $\sqrt{\frac{529}{36}}$ may be either $\frac{23}{6}$ or $-\frac{23}{6}$. And the latter value will give $x=-\frac{23}{6}+\frac{5}{6}=-\frac{18}{6}=-3$. And accordingly, if in the given equation $x^2-\frac{5x}{3}=14$ we substitute for x either $4\frac{2}{3}$ or -3, we shall find the result 14. And hence in reducing quadratic equations, we may commonly arrive at two distinct values for the unknown quantity: of which more in the next chapter.

236. If after having prepared our quadratic equation for completing the square, we put a for the coefficient of x in the second term on the left hand, and b for the number on the right hand, it is plain that $x^2\pm ax=\pm b$ will be a general expression for such an equation so prepared. And every affected quadratic equation may be brought to that form. To complete the square we add to both sides the square of $\frac{a}{2}$; whence we have the equation $x^2\pm ax+\frac{a^2}{4}=\pm b+\frac{a^2}{4}$: whence by extracting the square root of both sides we have $x\pm\frac{a}{2}=\pm\sqrt{\pm b+\frac{a^2}{4}}$ and therefore $x=\mp\frac{a}{2}\pm\sqrt{\pm b+\frac{a^2}{4}}$. With this general expression for the value of the unknown quantity in an affected quadratic equation, the student ought to make himself very familiar.

And

And in it let him observe the nature and ground of the ambiguous sign $\mp$ prefixed to $\frac{a}{2}$. If the binomial root in the former step be $x+\frac{a}{2}$, then upon transposing $\frac{a}{2}$ it becomes negative. But if the binomial root be $x-\frac{a}{2}$, then $\frac{a}{2}$ becomes affirmative in the assigned value of x. Let him also observe that the ambiguity of the sign prefixed to the surd $\sqrt{\pm b+\frac{a^2}{4}}$, arises from the circumstance that the square root of any number may be indifferently affirmative or negative. (§ 164.) In that surd, the term $\frac{a^2}{4}$ is always affirmative, as it is the term which has been *added* to both sides for the purpose of completing the square. The term b is affirmative or negative, according to the sign it has annexed in the given equation $x^2\pm ax=\pm b$.

237. A biquadratic equation, or an equation of the 4th. degree, may be reduced just as a quadratic, if the unknown quantity x appear only in the 4th. power or only in the 4th. and 2nd. powers of it. Thus if $x^4=81$, the value of x is found by two extractions of the square root: for $x^2=\sqrt{81}=9$, and $x=\sqrt{9}=3$. And if $x^4-5x^2=36$, let us substitute y for x^2 in that equation, and it becomes $y^2-5y=36$, which is an affected quadratic, and affords $y=\frac{5}{2}\pm\sqrt{36+\frac{25}{4}}=\frac{5}{2}\pm\sqrt{\frac{169}{4}}=\frac{5}{2}\pm\frac{13}{2}=+9$ or -4. Now having the value of y or x^2, a second extraction gives us the value of x, since $x=\sqrt{y}=\sqrt{9}=\pm 3$. For as to the expression $\sqrt{-4}$ it is an impossible or imaginary quantity. (§ 164.) It is plain that any equation, in which the index of the unknown quantity in one term is half of its index in the other term, may in like manner be treated as an affected quadratic.

238. If we here employ the same general notation as in § 236. then y, or x^2, $=\mp\frac{a}{2}\pm\sqrt{\pm b+\frac{a^2}{4}}$; and therefore $x=\pm\sqrt{\frac{a}{2}\pm\sqrt{\pm b+\frac{a^2}{4}}}$. Now let us put c for the

value

value of $\pm b+\frac{a^2}{4}$: and the expression for the value of x in the supposed biquadratic equation becomes $=\pm\sqrt{\frac{a}{2}\pm\sqrt{c}}$. But we have seen in § 218. that this surd is capable of being expressed more simply, whenever $\frac{a^2}{4}-c$ is a square number.

239. Let the student exercise himself in reducing the following equations; of which the first seven are examples of simple equations, but those at the end of § 240. of quadratics.

Ex. 1. $5x-8=3x+20$.

Ex. 2. $3x+\frac{2x}{3}=\frac{5x}{2}+4$.

Ex. 3. $\frac{9}{5x}+12=7+\frac{12}{x}$.

Ex. 4. $\sqrt{251+x^2}-3=x$.

Ex. 5. $5x^2-12x=17x-3x^2$.

Ex. 6. $\sqrt{12+x}=2+\sqrt{x}$.

Ex. 7. $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}$.

Here the letter a denotes any known number, and the object of the reduction is to find the value of x, which denotes the unknown. The steps by which the reduction may be completed are—1st. $\times\sqrt{a+x}$ (that is, multiply both sides by $\sqrt{a+x}$, in order to remove the irrationality from the denominator) 2ndly. $-a-x$ (that is, subtract $a+x$ from both sides, or transpose, in order to have the surd alone at one side) 3rdly. square both sides; 4thly. $-x^2$ (that is, subtract x^2 from both sides, or expunge $+x^2$): when we get the simple equation $ax=a^2-2ax$, or $3ax=a^2$, and $x=\frac{a}{3}$. And this is a *general* value for x, whatever number be substituted for a in the given equation. Thus, if $\sqrt{x}+\sqrt{3+x}=\frac{6}{\sqrt{3+x}}$ (where we have substituted 3 for a) then $x=\frac{3}{3}=1$. If $\sqrt{x}+\sqrt{5+x}=\frac{10}{\sqrt{5+x}}$: (where we have substituted 5 for a,) then $x=\frac{5}{3}=1\frac{2}{3}$. And accordingly if we calculate the value of each side of the equation by by substituting $\frac{5}{3}$ for x, we shall find the amount of each side to be $\sqrt{15}$. For then $\sqrt{x}+\sqrt{5+x}=\sqrt{\frac{5}{3}}+\sqrt{\frac{20}{3}}$ $=\frac{\sqrt{5}+2\sqrt{5}}{\sqrt{3}}$ (§ 215.) $=\frac{3\sqrt{5}}{\sqrt{3}}=\sqrt{3}\times\sqrt{5}=\sqrt{15}$. And $\frac{10}{\sqrt{5+x}}=10\div\sqrt{\frac{20}{3}}=\frac{10\sqrt{3}}{\sqrt{20}}=\frac{10\sqrt{3}}{2\sqrt{5}}=\frac{5\sqrt{3}}{\sqrt{5}}=\sqrt{5}\times\sqrt{3}$ $=\sqrt{15}$. Let the student exercise himself in similar calculations, substituting 7, 8, &c. for a.

240. Let us now change the numeral coefficient of a in the numerator of the fraction on the right side of the proposed equation; and observe how the value of x will vary. Thus, let $\sqrt{x}+\sqrt{a+x}=\frac{3a}{\sqrt{a+x}}$. Reducing this equation, we find $x=\frac{4a}{5}$. And if $\sqrt{x}+\sqrt{a+x}=\frac{4a}{\sqrt{a+x}}$, then we find $x=\frac{9a}{7}$. We now perceive the law of variation in the values of x. For having successively employed 2, 3, and 4 as the coefficients of a in the numerator, we have found x equal to the product of a multiplied successively by the fractions $\frac{1}{3}$, $\frac{4}{5}$, and $\frac{9}{7}$; of which fractions the numerators are the squares of $\overline{2-1}$, $\overline{3-1}$, $\overline{4-1}$; and the denominators are $\overline{2\times2-1}$, $\overline{2\times3-1}$, and $\overline{2\times4-1}$. And accordingly if $\sqrt{x}+\sqrt{a+x}=\frac{5a}{\sqrt{a+x}}$, we shall find $x=a$ $\times\frac{\overline{5-1}|^2}{10-1}=\frac{16a}{9}$. But we may at once arrive at a general formula for the value of x, by reducing the equation $\sqrt{x}+\sqrt{a+x}=\frac{ma}{\sqrt{a+x}}$, in which m the literal coefficient denotes any number whatsoever, integral or fractional. For we shall find $x=a+\frac{\overline{m-1}|^2}{2m-1}$. From this formula we may easily calculate the value of x, whatever numbers be substituted for a and ma. Thus let $\sqrt{x}+\sqrt{9+x}$ $=\frac{12}{\sqrt{9+x}}$, then $m=\frac{4}{3}$, $\overline{m-1}|^2=\frac{1}{9}$, $2m-1=\frac{5}{3}$, $\frac{\overline{m-1}|^2}{2m-1}$ $=$

$=\frac{1}{9}\div\frac{5}{3}=\frac{1}{15}$, and $x=a\times\overline{\frac{m-1}{2m-1}}\Big|^2=\frac{9}{15}=\frac{3}{5}$. And accordingly if $\frac{3}{5}$ be substituted for x in the preceding equation, the value of each side will be found to be $\sqrt{15}$. These observations might be pursued farther: but enough has been said to call the attention of the student to the advantages of employing *literal* equations, in which we designate by letters known, as well as unknown, quantities.

Ex. 8. $x^2-40=2x-5$.

Ex. 9. $7x-x^2+5=11+2x$.

Ex. 10. $3x^2+7=\frac{3x}{5}+4x^2-21$.

Ex. 11. $x=\sqrt{x}+6$.

Ex. 12. $\frac{2x^4}{3}+\frac{3x^2}{2}=\frac{10x^2-45}{2}$.

Ex. 13. $\frac{3mx^2}{2a}+ax+\frac{2bx}{2a}=7ax^2+5cx-15$.

Here after having prepared for completing the square by bringing the equation to this form, $x^2+\frac{10acx-2a^2x-2bx}{14a^2-3m}=\frac{30a}{14a^2-3m}$ we consider the second term (the fractional trinomial in each part of which x appears) as the product of x multiplying $\frac{10ac-2a^2-2b}{14a^2-3m}$; the half of which therefore, or $\frac{5ac-a^2-b}{14a^2-3m}$, is the other part of the binomial root, whose square we want to complete. The square will therefore be completed by adding to both sides $\overline{\frac{5ac-a^2-b}{14a^2-3m}}\Big|^2$;

& $x=\pm\sqrt{\frac{30a}{14a^2-3m}+\overline{\frac{5ac-a^2-b}{14a^2-3m}}\Big|^2}-\frac{5ac+a^2+b}{14a^2-3m}$.

CHAP. XXIV.

On the Forms and Roots of Quadratic Equations. Method of exterminating the Second Term.

241. WE have seen (§ 236.) that all mixed quadratic equations may be brought to this form $x^2\pm ax=\pm b$. But of

of these four varieties, $x^2+ax=-b$ may be disregarded, as it is manifest that the value of x^2+ax cannot be negative unless when x is a negative quantity; and this form therefore really coincides with $x^2-ax=-b$. We distinguish therefore only three forms of quadratic equations; the first $x^2+ax=b$; the second $x^2-ax=b$; the third $x^2-ax=-b$.

242. The first of these forms when reduced gives $x=\pm\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$; in which $\sqrt{b+\frac{a^2}{4}}$ being necessarily a greater quantity than $\frac{a}{2}$ or $\sqrt{\frac{a^2}{4}}$ the value $+\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ is necessarily affirmative; while the other value $-\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ is necessarily negative. The second form when reduced gives $x=\pm\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$: and here also the value of $-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ is necessarily negative, as the negative part of it exceeds the affirmative; while the other value $+\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ is evidently affirmative.

243. But in the third form $x^2-ax=-b$, where reduction gives us $x=\pm\sqrt{-b+\frac{a^2}{4}}+\frac{a}{2}$, both values of x will be impossible, or imaginary, if b exceed $\frac{a^2}{4}$: for then the value of $-b+\frac{a^2}{4}$ will be negative, and its square root $\sqrt{-b+\frac{a^2}{4}}$, an impossible quantity. (§ 164.) If $b=\frac{a^2}{4}$ then it is plain that the two values of x coincide, and become $=\frac{a}{2}$, since the expression $\pm\sqrt{-b+\frac{a^2}{4}}$ becomes $=0$. But if b be less than $\frac{a^2}{4}$, then both the values of x must be affirmative, since the value of $-b+\frac{a^2}{4}$ is affirmative, and in

in the expression $-\sqrt{-b+\frac{a^2}{4}}+\frac{a}{2}$ the negative part is less than the affirmative.

244. By the *root*, or *roots*, of an equation we mean the value, or values, of the unknown quantity. And we have seen that every quadratic equation has two roots. For this we have hitherto accounted from the ambiguous sign of every square root. But the same thing may be illustrated from other principles. If we bring all the terms of an equation to one side by transposition, we shall have 0 at the other side. Thus, the quadratic equation of the first form, $x^2+4x=21$, may become, by the transposition of 21, $x^2+4x-21=0$. And the roots of that equation are therefore the numbers, which substituted for x give 0 as the value of the trinomial $x^2+4x-21$. Now if we multiply any two binomials, each of which has x for the first term, and for their second terms quantities into which x does not enter, we shall have a trinomial product, whose first term is x^2, the second term the product of x and the sum of the second terms of the binomial factors, and the third term the product of the second terms of the binomial factors. Thus the product of $x+a$ multiplied by $x-b$, or $x^2+ax-bx-ab$, may be considered as a trinomial, by considering $ax-bx$ as one quantity; and we see that it is the product of x and $a-b$; while the third term $-ab$ is the product of a and $-b$. If then we put $\pm s$ for the sum of any two quantities denoted by a and b, and p for their product, then the product of $x\pm a$ multiplied by $x\pm b$, must be justly represented by $x^2\pm sx\pm p$;—the general formula for a quadratic in which all the terms are brought to one side, and therefore 0 standing on the other side.

245. Hence it appears that any such quadratic may be considered as generated by the multiplication of two such binomials $x\pm a$ and $x\pm b$. But their product will become equal to 0, if *either* of the factors be equal to 0; that is if $x=\mp a$, or $=\mp b$. So that there must be two values of x in the quadratic $x^2\pm sx\pm p=0$, or two roots of that equation. And we have seen that the coefficient of x in its second term is the sum of those roots with their signs changed, and that its third term is their product. In like manner the quadratic equation $x^2+4x-21=0$ must have two roots, whose sum, when we change their signs, is $+4$ and

and their product -21. And accordingly reducing that equation, we have $x=\pm\sqrt{21+4}-2$, $=\pm 5-2$; that is $+3$ or -7: and multiplying the binomial factors $x-3$ and $x+7$, their product is the given trinomial $x^2+4x-21$, in which $+4$ the coefficient of the second term is the sum of -3 and $+7$, and the third term -21 is their product. If $x=+3$, then the binomial factor $x-3$ is equal to 0, and therefore the trinomial; as it must also if $x=-7$, and therefore $x+7=0$.

246. A quadratic of the first form, x^2+sx-p, will be generated by the multiplication of the binomial factors $x+a$ and $x-b$, if the sum of $+a$ and $-b$ be affirmative: that is if a, the negative root, exceed b, the affirmative. But if the sum of those roots be negative, that is, if a be less than b, the equation generated will be of the second form x^2-sx-p. And thus also we see, that in the first and second forms one of the roots must be affirmative and the other negative. But it is plain that a quadratic of the third form, x^2-sx+p, cannot be generated but by the multiplication of such binomial factors as $x-a$ and $x-b$; for thus alone the product of the two roots will be affirmative and at the same time their sum negative. Hence also it is plain that in this form both the roots must be affirmative, when they are possible. We saw in § 243. that when (in the equation x^2-sx+p) p exceeds $\frac{s^2}{4}$, both the roots are impossible; and that appeared from the impossibility of the square root of a negative quantity. But the same thing also appears, and more satisfactorily, from the consideration that s is the sum of two numbers, whose product is p. For it is impossible that the product of any two numbers should exceed the square of half their sum: which may be thus proved.

247. Let a and b represent any two numbers; then the square of their difference $a-b$, will be represented by $a^2-2ab+b^2$; which must be positive in its value, whether the value of $a-b$ be positive or negative. (§ 164.) Therefore the negative part $2ab$ cannot exceed the affirmative a^2+b^2: and adding $2ab$ to both, $4ab$ cannot exceed $a^2+2ab+b^2$; that is, four times the product of a and b cannot exceed the square of their sum: and therefore their product cannot exceed the fourth part of the square of their

their sum, or the square of half their sum.—Otherwise, if we put a for the smaller of two numbers and d for their difference, $a+d$ will represent the greater. But the product of a and $a+d$ is a^2+ad: and their sum is $2a+d$. Therefore half their sum is $a+\frac{d}{2}$, the square of which is $a^2+ad+\frac{d^2}{4}$. But a^2+ad is less than $a^2+ad+\frac{d^2}{4}$ by $\frac{d^2}{4}$: that is, the product of any two numbers is less than the square of half their sum by the square of their difference. If the two numbers be equal, that is, if d vanishes, then the product becomes equal to the square of half their sum, or to the square of either number. But in no case can the former quantity exceed the latter.

248. The second term of any affected quadratic equation may be exterminated, and the equation may therefore be brought to the form of a pure quadratic, by substituting for x in the given equation y *minus* or *plus* half the coefficient of x in the second term, according as the sign of that term is *plus* or *minus*. Thus suppose we be given the affected quadratic $x^2+ax-b=0$. Let us substitute $y-\frac{a}{2}$ for x. Then $x^2=y^2-ay+\frac{a^2}{4}$; and $ax=ay-\frac{a^2}{2}$. Therefore $x^2+ax=y^2-\frac{a^2}{4}$ and $x^2+ax-b=y^2-\frac{a^2}{4}-b=0$. But $y^2-\frac{a^2}{4}-b=0$, or $y^2=\frac{a^2}{4}+b$, is a pure quadratic, which gives $y=\pm\sqrt{\frac{a^2}{4}+b}$. And since we supposed that $x=y-\frac{a}{2}$, it follows that $x=\pm\sqrt{\frac{a^2}{4}+b}-\frac{a}{2}$; the very same values that we arrive at by completing the square. If our given equation be $x^2-3x-5=0$, then substituting $y+\frac{3}{2}$ for x, we have $y^2-\frac{29}{4}=0$, or $y=\pm\sqrt{\frac{29}{4}}$; and $x=\pm\sqrt{\frac{29}{4}}+\frac{3}{2}$.

CHAP.

CHAP. XXV.

Reduction of two or more Equations, involving several unknown Quantities.

249. IF we have given two simple equations, involving two unknown quantities, for instance $x+y=7$, and $x-y=2\frac{1}{3}$, we may derive from them an equation involving but one unknown quantity, the reduction of which will afford us its value, and thence the value of the other unknown quantity. There are always three methods, by which this may be effected. For 1st. we may find from each of the equations an expression for the value of one of the unknown quantities, and then state the equality of those expressions in a new equation, involving only the other unknown quantity. Thus, given the equations $x+y=7$ and $x-y=2\frac{1}{3}$, from the former equation we find $x=7-y$; and from the latter $x=2\frac{1}{3}+y$. Therefore $7-y=2\frac{1}{3}+y$; which equation reduced gives $y=\frac{7-2\frac{1}{3}}{2}=2\frac{1}{3}$. Now by substituting $2\frac{1}{3}$ for y in either of the given equations, we find the value of x. Thus since $x+y=7$, it follows that $x+2\frac{1}{3}=7$, which gives $x=7-2\frac{1}{3}=4\frac{2}{3}$.

250. Or, 2ndly. finding from one of the given equations an expression for the value of one of the unknown quantities, we may substitute that expression in the other equation for that unknown quantity, and so derive an equation involving only the other unknown quantity. Thus from the equation $x+y=7$, we have $x=7-y$: and substituting $7-y$ for x in the equation $x-y=2\frac{1}{3}$, we have $7-y-y$ (i. e. $7-2y$) $=2\frac{1}{3}$. Therefore $2y=7-2\frac{1}{3}=4\frac{2}{3}$, and $y=2\frac{1}{3}$: which affords us also the value of x as before.

251. Or, 3rdly. when we have the same unknown quantity appearing in one term of each equation, and affected with the same coefficient, we may by subtracting one equation from the other (if the signs of those terms be the same) or by adding the one to the other (if the signs be contrary) exterminate that unknown quantity, and derive an equation involving only the other. Thus in the equations $x+y=7$ and $x-y=2\frac{1}{3}$, subtracting $x-y$ from $x+y$

 the

the quantity x disappears, and the remainder $2y$ must be equal to the remainder obtained by subtracting $2\frac{1}{3}$, the value of $x-y$, from 7, the value of $x+y$: that is, $2y=4\frac{2}{3}$. Or, adding $x-y$ to $x+y$, the quantity y will disappear, and the sum $2x$ must be equal to the sum of 7 and $2\frac{1}{3}$; that is $2x=9\frac{1}{3}$. $\therefore x = 4\frac{5}{6}$

252. With this method, as being the most generally expeditious and convenient, the student ought to make himself very familiar, and expert at the preparatory operations, which are often necessary. Thus, if we have the equations $2x+3y=15$ and $3x-\frac{5y}{4}=12$, and want to exterminate x, we must prepare for the subtraction of one equation from the other by giving x the same coefficient in both. This might evidently be done by dividing the former by 2, and the latter equation by 3: for then the coefficient of x in both would be 1. Or it might be done by multiplying the former equation by 3, and the latter by 2: for then the coefficient of x in both would be 6. But it may at once be done by multiplying the former equation by $\frac{3}{2}$, or the latter by $\frac{2}{3}$. By the one process the coefficient of x in both will be 3, since $2x\times\frac{3}{2}=3x$; and by the other process will be 2, since $3x\times\frac{2}{3}=2x$. Thus again, if the coefficient of x in one of the given equations be 5 and in the other 7, multiplying the former equation by $\frac{7}{5}$, or the latter equation by $\frac{5}{7}$, will give the same coefficient in both. Or, if the coefficient of x in one or both equations be fractional, it is equally easy to determine the number, which multiplying either equation will make the coefficient of x in it the same as in the other. For it resolves itself into this question— what number multiplying $\frac{a}{b}$ will give $\frac{c}{d}$ for the product? The number required must be $\frac{bc}{ad}$ $\left(\text{or } \frac{c}{d}\div\frac{a}{b}\right)$, since any product divided by either factor gives the other factor for the quotient. Or thus, since any number multiplied by its

reciprocal

reciprocal gives 1 for the product, $\frac{a}{b} \times \frac{b}{a} \times \frac{c}{d}$ must equal $1 \times \frac{c}{d}$, that is must equal $\frac{c}{d}$.

253. If then we have given us the two equations $\frac{ax}{b} + \frac{by}{a} = ab$, and $\frac{bx}{a} + \frac{ay}{b} = \frac{b}{a}$, to find the values of x and y; we are to remember that the first object is to derive from the two given equations another equation involving only one of the unknown quantities. And this may be effected by the first method described in § 249. thus. From the first of the given equations, we find that $x = b^2 - \frac{b^2y}{a^2}$. (For $\frac{ax}{b} = ab - \frac{by}{a}$; whence, multiplying both sides by $\frac{b}{a}$, $x = b^2 - \frac{b^2y}{a^2}$.) And from the second of the given equations, $x = 1 - \frac{a^2y}{b^2}$. (For $\frac{bx}{a} = \frac{b}{a} - \frac{ay}{b}$; whence, multiplying both sides by $\frac{a}{b}$, $x = 1 - \frac{a^2y}{b^2}$.) Therefore equating the two values of x, we have $b^2 - \frac{b^2y}{a^2} = 1 - \frac{a^2y}{b^2}$; which equation involves only the unknown quantity y, whose value is found by reduction; namely $y = \frac{a^2b^4 - a^2b^2}{b^4 - a^4}$. And substituting for y, in either of the given equations, this its value, we have an equation involving only the unknown quantity x, which by reduction gives $x = \frac{b^4 - b^2a^4}{b^4 - a^4}$.—Or, pursuing the second method described in § 250. we derive from either of the equations, suppose from the first of them, an expression for the value of x, namely $x = b^2 - \frac{b^2y}{a^2}$, and substitute this expression for x in the second of the given equations; whence we have the equation $\frac{b^3}{a} - \frac{b^3y}{a^3} + \frac{ay}{b} = \frac{b}{a}$, involving only the unknown quantity y, and affording by reduction the same value for y as before.—Or lastly, pursuing the third method described in § 252. we may multiply

both

both sides of the first given equation by $\frac{b^2}{a^2}$, in order that x may be affected with the same coefficient in both; when we have $\frac{bx}{a}+\frac{b^3y}{a^3}=\frac{b^3}{a}$: from which subtracting the second of the given equations we have $\frac{b^3y}{a^3}-\frac{ay}{b}=\frac{b^3}{a}-\frac{b}{a}$. And this reduced gives the same value of y as before. For, multiplying both sides by a^3b, we have $b^4y-a^4y=a^2b^4-a^2b^2$: whence, dividing by b^4-a^4, we have $y=\frac{a^2b^4-a^2b^2}{b^4-a^4}$.—If in the given equations we put d for $\frac{b}{a}$, they will become $\frac{x}{d}+dy=a^2d$, and $dx+\frac{y}{d}=d$; and we shall have $x=\frac{d^4-a^2d^2}{d^4-1}$, and $y=\frac{a^2d^4-d^2}{d^4-1}$. Or if we put d for $\frac{b}{a}$, and p for ab, we shall have $x=\frac{d^4-dp}{d^4-1}$, and $y=\frac{d^3p-d^2}{d^4-1}$.—After thus reducing the literal equations, let the student substitute any numbers whatsoever for a and b, and prove the truth of the literal formulæ for the values of x and y, by calculating their numeral values according to them. It is plain that if $a=1$, the values of x and y must coincide, and become $\frac{d^2}{d^2+1}$. And if $a=d$, the value of x vanishes, or is $=0$.

254. If three equations be given us, involving three unknown quantities, we may by methods very similar successively ascertain the value of each. Thus, if we be given the equations $x+y=5$, and $x+z=7$, and $y+z=8$; equating the expressions for x afforded by the two first of these equations, we have $5-y=7-z$; from which and the third of the given equations $y+z=8$, we find $z=5$, and $y=3$: and therefore $x=2$. Or, if we be given the equations $x+y+z=6$, and $x+2y+3z=10$, and $3x+2y-z=12$; subtracting the first of these equations from the second, we have $y+2z=4$; and subtracting the third of them from three times the first, we have $y+4z=6$. But from the two equations $y+2z=4$ and $y+4z=6$, we find as before $z=1$, and $y=2$. And substituting these numbers

bers for y and z in any of the given equations, we find $x=3$. Thus the student will observe, that when we are given three equations involving three unknown quantities, we proceed to derive from them two equations involving only two unknown quantities: which we then reduce by the rules before laid down. And in like manner if we be given four equations involving four unknown quantities, we may always derive from them three equations involving only three unknown quantities, and so on.

255. But here let it be remarked, that the number of given independent equations must always be equal to the number of unknown quantities, else the values of these cannot be ascertained. Thus, if we be given the equation $x=8-\frac{y}{2}$, it is impossible from this alone to determine the values of either x or y: for we may suppose either of them to be any number *whatsoever*, and may then find by reduction a numeral value for the other which shall answer the condition of the given equation. Or if, along with that equation, we be given the equation $2x+y=16$, it will not at all assist us in the discovery of x and y: for the latter equation may be derived from the former, by doubling both sides, and transposing y; so that it affords us no information in addition to what the first gave us. Therefore we say that the given equations must be *independent*. But neither ought their number to *exceed* that of the unknown quantities. For if we have given us, for instance, three equations involving only two unknown quantities, we have seen that the value of these qnantities is absolutely determined and may be ascertained from any two of the given equations. The third equation therefore, if not deducible from the former, must be inconsistent with them. Thus, if we be given $x+y=8$, and $x-y=2$, and $2x+\frac{y}{2}=a$; from the two former equations we find that $x=5$, and $y=3$. Therefore $2x+\frac{y}{2}=10+1\frac{1}{2}=11\frac{1}{2}$; and if in the third given equation $a=11\frac{1}{2}$, it is an equation deducible from the two former; but if a be a number greater or less than $11\frac{1}{2}$, the condition expressed in that equation is inconsistent with the conditions expressed in the two former.

CHAP.

CHAP. XXVI.

The Application of Algebra to the Solution of Arithmetical Problems.

256. WHEN an arithmetical question is proposed, to be solved algebraically, the first thing to be done, after clearly understanding its terms, is to express the conditions of it in the symbolic language of Algebra. And here, in the first place, we represent the number or numbers which we want to discover by some of the final letters of the alphabet; and then we express in the form of an Algebraic equation what we are told in the question about each of these unknown numbers. (See § 148. and 149.) After we have thus accurately *translated* the proposed question into the language of Algebra, no more difficulty can remain to the student who is acquainted with the doctrine of the two last Chapters; since by merely reducing the given equations the value of the unknown quantities is discovered.—Thus, let it be required to find such a number, that multiplying it by 3, and dividing it by 3, the former product shall exceed the latter quotient by 3: or in other words) to find a number, whose third part is less than three times the number by 3. Let us put x for the number sought. Then $\frac{x}{3}$ expresses its third part; and $3x$ expresses three times the number. Now we are told that $\frac{x}{3}$ is less than $3x$ by 3, which is to be expressed by an equation. But the equation $\frac{x}{3}+3=3x$, or the equation $3x-3=\frac{x}{3}$, or the equation $3x-\frac{x}{3}=3$, accurately expresses what we want; for the first expresses that adding 3 to $\frac{x}{3}$ the sum is equal to $3x$; the second expresses that subtracting 3 from $3x$ leaves a remainder equal to $\frac{x}{3}$; and the third expresses that subtracting $\frac{x}{3}$ from $3x$ leaves a remainder

equal

equal to 3: all which are propositions equivalent with each other, and with the conditions of the question. It now only remains to reduce any of these equations, according to the rules already given. Thus, from the equation $3x - \frac{x}{3} = 3$, dividing both sides by $3 - \frac{1}{3}$, or by $\frac{8}{3}$ (that is, multiplying both sides by $\frac{3}{8}$) we find $x = 3 \times \frac{3}{8} = \frac{9}{8}$; that is, we discover that the number required is $\frac{9}{8}$. And accordingly three times $\frac{9}{8}$, or $\frac{27}{8}$, exceeds the third part of $\frac{9}{8}$, or $\frac{3}{8}$, by $\frac{24}{8}$, that is by 3. [If we propose a question perfectly similar to the last, only substituting the number 4 for the number 3, we shall find the answer to be $\frac{16}{15}$. And we may obtain a general formula for the answer to all such questions, by putting a for any number whatsoever, and inquiring what number is that, which multiplied by a, and divided by a, gives the former product exceeding the latter quotient by a? For then by the terms of the question $ax - \frac{x}{a} = a$; whence we have $x = \frac{a^2}{a^2 - 1}$.]

257. Again, if it be required to find two numbers, whose difference is 5, and the third part of their sum is 7: we may put x for the greater of the numbers sought and y for the less. Then the equation $x - y = 5$ expresses what we are told of their difference, that it is 5; and the equation $\frac{x+y}{3} = 7$ expresses what we are told of their sum, that its third part is 7. And here let it be recollected that, where there are two unknown numbers, there must be two equations afforded us by the terms of the question, in order to ascertain them. (See § 255.) Now reducing the equations $x - y = 5$, and $\frac{x+y}{3} = 7$, by any of the three methods described in the last chapter, we may find the numbers sought. Thus, from the second of those equations we may infer that $x + y = 21$; and from this equation subtracting the first, we find that $2y = 16$, and therefore $y = 8$; or adding to it the first equation, we find that $2x = 26$, and therefore $x = 13$. So that the numbers sought are 13 and 8: and accordingly their difference is 5, and the third part of their sum is 7.

258. In like manner, if we be required to find two such numbers, that two thirds of their sum shall be equal to six times their difference, and two thirds of their product shall

be

be equal to six times the quotient of the greater divided by the less: putting x for the greater and y for the less, their sum is $x+y$, and two thirds of this sum is $\frac{2x+2y}{3}$. Their difference is $x-y$, and six times this difference is $6x-6y$: and by the terms of the question $\frac{2x+2y}{3}=6x-6y$. Again, their product is xy, and two thirds of it is $\frac{2xy}{3}$: the quotient of the greater divided by the less is $\frac{x}{y}$, and six times this quotient is $\frac{6x}{y}$: and by the terms of the question $\frac{2xy}{3}=\frac{6x}{y}$. Nothing now remains but to reduce the two equations $\frac{2x+2y}{3}=6x-6y$, and $\frac{2xy}{3}=\frac{6x}{y}$; from the latter of which, dividing both sides by x, (see § 227.) we infer that $\frac{2y}{3}=\frac{6}{y}$; and thence that $2y^2=18$, and $y^2=9$, and $y=3$. Then substituting 3 for y in the first of the given equations it will stand $\frac{2x+6}{3}=6x-18$; whence $2x+6=18x-54$, and $16x=60$, and $x=\frac{60}{16}=\frac{15}{4}$. So that we find the greater of the numbers sought to be $3\frac{3}{4}$, and the less to be 3. And accordingly two thirds of their sum, $6\frac{3}{4}$, is equal to 6 times their difference, $\frac{3}{4}$ths. for $\frac{2}{3}$rds of $\frac{27}{4}$ is $\frac{18}{4}$, and 6 times $\frac{3}{4}$ths. is $\frac{18}{4}$: and two thirds of their product, $\frac{45}{4}$, is equal to six times the quotient of the greater divided by the less, $\frac{5}{4}$ths. for $\frac{2}{3}$rds. of $\frac{45}{4}$ is $\frac{30}{4}$, and 6 times $\frac{5}{4}$ths. is $\frac{30}{4}$. [Although the second of the given equations, $\frac{2xy}{3}=\frac{6x}{y}$, seem at first to involve two unknown numbers x and y, yet from the disappearance of x, we may infer that y alone is really concerned in it, and is equal to 3. And accordingly, if any number whatsoever be multiplied by 3, and divided by 3, two thirds of the product must be equal to 6 times the quotient; since two thirds of 3 times a is $2a$, and $2a=\frac{6a}{3}$.]

259. But

259. But it often happens, that a question apparently involving two unknown quantities may be treated most advantageously, as if it involved only one. For after designating one of them by one of the final letters of the alphabet, we may express the other by the aid of this letter and some given number. Thus, if it be required to find two numbers such, that their *difference* is 7, and their *ratio* that of 5 to 3; putting x for the less, it is plain that $x+7$ is a just expression for the greater; so that we need not introduce another letter to designate the greater. And now, having derived this expression for the greater, $x+7$, from one of the things told us in the question about the two numbers, we proceed to express algebraically the other circumstance told us, namely that the ratio of the greater to the less is that of 5 to 3. But this is expressed thus (§ 70.) $x+7 : x :: 5 : 3$. But from this analogy (§ 76.) we may derive the equation, $3x+21=5x$; whence we have $2x=21$, and therefore $x=\frac{21}{2}$; and $x+7$ (or the greater of the two numbers sought) $=\frac{21}{2}+7=\frac{35}{2}$. Accordingly, the difference between $\frac{35}{2}$ and $\frac{21}{2}$ is 7; and their ratio (§ 93.) is that of $35 : 21$, or $5 : 3$. Which of the two unknown quantities we shall employ the letter originally to designate, is often indifferent: but in general it is more convenient to employ it for designating the smaller of the two, and thence to derive an expression for the greater.

260. In like manner, if it be required what two numbers they are, whose *difference* is 5, and the *difference of their squares* 45: putting x for the less, $x+5$ expresses the greater; whose square is $x^2+10x+25$. Now we are told that the difference between this and x^2 (the square of the less) is 45. That is, $10x+25=45$; whence we have $10x=20$, and $x=2$; and therefore $x+5=7$. (See also the 5th of the Questions for Exercise.) Accordingly, the difference of 7 and 2 is 5; and the difference of their squares (49—4) is 45. If we investigate a general solution for all such questions as the last, by putting a for the given difference of the numbers, and b for the given difference of their squares, then x designating the less of the two numbers, $x+a$ expresses the greater; from whose square, $x^2+2ax+a^2$, subtracting x^2, we have $2ax+a^2=b$: whence $2ax=b-a^2$; and $x=\frac{b-a^2}{2a}$. [As long as x and a are any positive numbers, it is plain that b must exceed a^2, else the

value

value of $\frac{b-a^2}{2a}$ would be negative: that is, it appears that the difference of the squares of any two numbers must exceed the square of their difference. And from the equation $2ax=b-a^2$, it appears that the difference of the squares of any two numbers exceeds the square of their difference by twice the product of the less and difference.—The Geometrical Student may with advantage compare many such Algebraic results with the principles in the second book of Euclid's Elements.]

261. If it be required to find two numbers whose *sum* is 10, and the *difference of their squares* 40: putting x for the less, we may express the greater by $10-x$, according to the first of the given conditions; and then the second condition is expressed by the equation, $\overline{10-x}|^2-x^2=40$, that is, $100-20x=40$: whence we have $20x=60$, and $x=3$. Therefore the greater, or $10-x$, is 7. Or putting a for the given sum, and b for the given difference of the squares, we have $a^2-2ax=b$, and thence $2ax=a^2-b$, and $x=\frac{a^2-b}{2a}$. And from this literal notation we are furnished with the general theorem, that the square of the sum of any two numbers exceeds the difference of their squares by twice the product of the smaller and the sum. Or if we put x for the greater and $a-x$ for the less, then we have $2ax-a^2=b$; and thence $2ax=a^2+b$, and $x=\frac{a^2+b}{2a}$: which equations afford the general theorem, that twice the product of the greater of any two numbers and their sum is equal to the square of their sum *plus* the difference of their squares. Thus let the numbers be 8 and 5; their sum is 13, its square is 169; the difference of the squares of 8 and 5 is 39; and $8=\frac{13^2+39}{26}$, and $5=\frac{13^2-39}{26}$. But tho' we have given these methods of solution, there is a much better and readier solution of this problem, by dividing the difference of the squares by the sum of the numbers: for the quotient is the difference of the numbers. § 163.

262. Let us now investigate what two numbers they are, whose *sum* is 12, and their *product* $33\frac{1}{4}$. Putting x for either of the numbers, the other is represented by $12-x$; and therefore their product is $\overline{12-x}\times x$, or $12x-x^2$, whose amount

amount we are told is $33\frac{3}{4}$. Therefore $12x-x^2=33\frac{3}{4}$, or $x^2-12x=-33\frac{3}{4}$. This is a quadratic equation of the third form; which reduced, by the rules given in the 231st. and following sections, gives $x=\pm\sqrt{-33\frac{3}{4}+36}+6=\pm\sqrt{\frac{9}{4}}+6=6\pm1\frac{1}{2}$, that is, $7\frac{1}{2}$ or $4\frac{1}{2}$: which are the numbers that solve the problem; for their sum is 12, and their product $\frac{135}{4}$, that is $33\frac{3}{4}$. And universally putting s for the sum of two numbers, p for their product, and x for either of them, the other is expressed by $s-x$, their product by $sx-x^2$; and the equation $sx-x^2=p$, when reduced, gives $x=\pm\sqrt{\frac{s^2}{4}-p}+\frac{s}{2}$. (For then $x^2-sx=-p$; and completing the square, $x^2-sx+\frac{s^2}{4}=\frac{s^2}{4}-p$; and extracting the root of each side, $x-\frac{s}{2}=\pm\sqrt{\frac{s^2}{4}-p}$; whence $x=\pm\sqrt{\frac{s^2}{4}-p}+\frac{s}{2}$.) We have seen (§ 243.) that both these values will be impossible if p exceed $\frac{s^2}{4}$; and accordingly it is impossible that there should be any two numbers, whose product exceeds the square of half their sum. (See § 247.) If $p=\frac{s^2}{4}$, the numbers sought are equal, and each of them half the given sum. The student may exercise himself in observing the varieties in the solution of this problem, when the given sum, or product, is *negative*, or both of them.

263. But we have now to remark that the same problem may be solved, without the introduction of a quadratic equation. For, if we subtract 4 times the given product from the square of the given sum, the remainder must be equal to the square of the *difference* between the numbers sought. (For let a and b stand for any two numbers, the square of $a+b$, their sum, is $a^2+2ab+b^2$; and subtracting from this $4ab$, or 4 times their product, the remainder is $a^2-2ab+b^2$. But this is the square of $a-b$, the difference of the numbers. See § 163.) Hence therefore in the proposed problem we know, that $12^2-\overline{4\times33\frac{3}{4}}$ is equal to the square of the difference between the numbers sought; that

that is, that the square of their difference is 144—135, or 9; and therefore their difference is $\sqrt{9}$, or 3. So that the problem resolves itself into that of finding two numbers, whose sum is 12, and their difference 3. Or generally, putting s for the given sum, p for the given product, and d for the difference between the numbers; $s^2-4p=d^2$, and therefore $\pm\sqrt{s^2-4p}=d$. But given s the sum of two numbers, and d their difference, the greater of the numbers $=\frac{s}{2}+\frac{d}{2}=\frac{s}{2}+\sqrt{\frac{s^2}{4}-p}$, and the less $=\frac{s}{2}-\frac{d}{2}=\frac{s}{2}-\sqrt{\frac{s^2}{4}-p}$; the same expressions which we arrived at (§ 262.) by the reduction of the quadratic.

264. Let it now be required to find two numbers, whose *difference* is $4\frac{1}{3}$, and their *product* $25\frac{5}{9}$. Putting x for the less, $x+4\frac{1}{3}$ will express the greater: and their product therefore is $\overline{x+4\frac{1}{3}}\times x$, or $x^2+\frac{13x}{3}$. But by the terms of the question $x^2+\frac{13x}{3}=25\frac{5}{9}$. Now the reduction of this quadratic of the first form will give us the value of x, and therefore of $x+4\frac{1}{3}$. Thus:—completing the square by adding the square of $\frac{13}{6}$ to both sides, we have.

$$x^2+\frac{13x}{3}+\frac{169}{36}=25\tfrac{5}{9}+\frac{169}{36}=\frac{1089}{36}=\frac{121}{4}$$

$$\text{Therefore } x+\frac{13}{6}=\sqrt{\frac{121}{4}}=\frac{11}{2}$$

$$\text{and } x=\frac{11}{2}-\frac{13}{6}=\frac{33}{6}-\frac{13}{6}=\frac{20}{6}=3\tfrac{1}{3}$$

And therefore $x+4\frac{1}{3}=7\frac{2}{3}$. So that the numbers sought are $7\frac{2}{3}$ and $3\frac{1}{3}$: whose difference accordingly is $4\frac{1}{3}$, and their product $25\frac{5}{9}$. If we adopt the negative value for the square root of $\frac{121}{4}$, the resulting numbers will be the same, but negative. But though we have exhibited the most obvious solution of this question, as producing a quadratic equation, yet it appears from the observations in the last section, that it may be more expeditiously and elegantly solved, by adding 4 times the given product to the square of the given difference: which affords us the square of the sum, and therefore the sum. Thus, the square of the given

given difference is $\frac{169}{9}$; and 4 times the given product is $\frac{920}{9}$. Therefore $\frac{169}{9}+\frac{920}{9}$, or $\frac{1089}{9}$, is the square of the sum; whose square root therefore, or $\frac{33}{3}$, is the sum of the numbers sought. Therefore $\frac{33}{6}+\frac{13}{6}$, or $7\frac{2}{3}$, is the greater of the numbers; and $\frac{33}{6}-\frac{13}{6}$, or $3\frac{1}{3}$, is the less. Universally, let d be the given difference, p the given product: then d^2+4p is the square of the sum, and therefore $\pm\sqrt{d^2+4p}$ is the sum. Hence the numbers sought are $\frac{\sqrt{d^2+4p}+d}{2}$, and $\frac{\sqrt{d^2+4p}-d}{2}$.

265. Let us now inquire, what two numbers they are whose *sum* is $10\frac{1}{2}$, and the *sum of their squares* $61\frac{1}{4}$. If we put x for either number, the other must $=\frac{21}{2}-x$, whose square is $\frac{441}{4}-21x+x^2$: to which adding x^2, we have the sum of the squares $2x^2-21x+\frac{441}{4}=61\frac{1}{4}=\frac{245}{4}$. This affected quadratic the student may proceed to reduce; and he will find $x=\frac{21}{4}\pm\sqrt{-\frac{392}{16}+\frac{441}{16}}=\frac{21}{4}\pm\sqrt{\frac{49}{16}}=\frac{21}{4}\pm\frac{7}{4}$. So that the numbers sought are $\frac{28}{4}$ and $\frac{14}{4}$; that is, 7 and $3\frac{1}{2}$. But this problem also we may solve with more simplicity and elegance, by proceeding at once to investigate the *difference* of the numbers. Now, if we subtract the given sum of their squares from the square of their given sum, the remainder must be twice the product of the numbers: (since $\overline{a+b}|^2-\overline{a^2+b^2}=2ab$) and we have seen that subtracting 4 times the product (or twice this remainder) gives the square of the difference. Thus in the present instance, $\frac{441}{4}-\frac{245}{4}=\frac{196}{4}$, or 49, is twice the product of the numbers; and therefore $\frac{441}{4}-\frac{392}{4}\left(=\frac{49}{4}\right)$ is the square of their difference; which difference therefore is $\frac{7}{2}$. The half of this added to half the given sum affords us the greater, and subtracted from half the given sum affords the less. See remarks on the 7th. of the questions for exercise. [And universally

universally let a be the given sum of the numbers, b the given sum of their squares: then $a^2—b$ is twice the product of the numbers; and therefore $a^2—\overline{2a^2—2b}$, or $2b—a^2$, is the square of the difference. So that the greater of the numbers is $\frac{a}{2}+\sqrt{2b—a^2}$, and the less is $\frac{a}{2}—\sqrt{2b—a^2}$. And it appears that the square of the sum of any two numbers cannot exceed twice the sum of the squares; (else $\sqrt{2b—a^2}$ would be an impossible quantity. § 164.) and that if these two quantities be equal, the numbers must be equal: for then $\sqrt{2b—a^2}=0$.]

266. In like manner, given the *difference* of two numbers 3, and the *sum of their squares* 29, we may proceed to investigate the sum of the numbers, instead of solving the question by a quadratic equation. For subtracting the square of the given difference from the given sum of the squares, we have twice the product of the numbers: which added to the sum of the squares gives us the square of the sum: since $a^2+b^2+2ab=\overline{a+b}|^2$. Thus $29—3^2=20$; and $29+20=49$, the square of the sum; which sum is therefore $\sqrt{49}$, or 7: and the numbers sought $\frac{7+3}{2}$ and $\frac{7—3}{2}$, that is, 5 and 2. [And universally putting a for the given difference of the numbers, and b for the given sum of their squares; $b—a^2$ is twice their product; and therefore $b+\overline{b—a^2}$, or $2b—a^2$, is the square of their sum; and $\sqrt{2b—a^2}$ is their sum. Whence the numbers sought are $\frac{\sqrt{2b—a^2}+a}{2}$ and $\frac{\sqrt{2b—a^2}—a}{2}$.]

267. Let it now be required to find two numbers, whose *product* is 24 and the *sum of their squares* 73. Adding twice the product to the sum of the squares, we have the square of the sum; which is therefore $73+48=121$. And subtracting twice the product from the square of the sum, we have the square of the difference; which is therefore $73—48=25$. Whence we have the sum 11, and the difference 5: so that the numbers are $\frac{11+5}{2}$ and $\frac{11—5}{2}$, or 8 and 3. [And universally, putting a for the given product, and

and b for the given sum of the squares; $b+2a$ is the square of the sum, and $b-2a$ the square of the difference. Whence the numbers are $\frac{\sqrt{b+2a}+\sqrt{b-2a}}{2}$, and $\frac{\sqrt{b+2a}-\sqrt{b-2a}}{2}$. And it appears that twice the product of any two numbers cannot exceed the sum of their squares; and cannot be equal to it except when the numbers themselves are equal.]

268. But though we have given this solution, as the most facile and scientific, the student ought to exercise himself in the other method of solving the question by a quadratic equation. Thus, putting x for either of the numbers, the other will be represented by $\frac{24}{x}$. The sum of their squares therefore is $x^2+\frac{576}{x^2}$, which by the terms of the question is equal to 73. Let us now reduce the equation

$$x^2+\frac{576}{x^2}=73$$

Therefore $x^4+576=73x^2$ (See § 237.)

and $x^4-73x^2=-576$

Therefore $x^4-73x^2+\overline{\frac{73}{2}}\Big|^2=-576+\overline{\frac{73}{2}}\Big|^2$

that is $x^4-73x^2+\frac{5329}{4}=-576+\frac{5329}{4}=\frac{3025}{4}$

Therefore $x^2-\frac{73}{2}=\pm\sqrt{\frac{3025}{4}}=\pm\frac{55}{2}$

and $x^2=\frac{73}{2}\pm\frac{55}{2}=64$ or 9

Therefore $x=\sqrt{64}$ or $\sqrt{9}$; that is, 8 or 3.

269. But let us now in like manner solve the *general* problem—To find two numbers whose product is a, and the sum of their squares b. Putting x for either of them, the other is expressed by $\frac{a}{x}$. Then

$$x^2+\frac{a^2}{x^2}=b$$
$$x^4+a^2=bx^2$$
$$x^4-bx^2=-a^2$$
$$x^4-bx^2+\frac{b^2}{4}=\frac{b^2}{4}-a^2$$
$$x^2-\frac{b}{2}=\pm\sqrt{\frac{b^2}{4}-a^2}$$
$$x^2=\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-a^2}$$
$$x=\sqrt{\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-a^2}}$$

Now in § 267. we found the general expression for x to be, $x=\frac{\sqrt{b+2a}\pm\sqrt{b-2a}}{2}$. But it appears from § 217. and 218. that the two expressions are equivalent. For in the binomial $\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-a^2}$, the square of the rational part *minus* square of the irrational is equal to a^2, whose root is a. Therefore, as we have shewn in § 218. the square root of that binomial is equal to $\sqrt{\frac{b+2a}{4}}\pm\sqrt{\frac{b-2a}{4}}$. See also § 238.]

270. If it be required to find two numbers, whose *product* is $8\frac{3}{4}$, and the *difference of their squares* 6: putting x for the greater, the less will be expressed by $8\frac{3}{4}\div x$, that is by $\frac{35}{4x}$, whose square is $\frac{1225}{16x^2}$. Therefore $x^2-\frac{1225}{16x^2}=6$; and $x^4-\frac{1225}{16}=6x^2$; and $x^4-6x^2=\frac{1225}{16}$. Hence completing the square, $x^4-6x^2+9=\frac{1225}{16}+9=\frac{1369}{16}$; and extracting the root, $x^2-3=\sqrt{\frac{1369}{16}}=\frac{37}{4}$. Therefore $x^2=\frac{37}{4}+3=\frac{49}{4}$; and $x=\sqrt{\frac{49}{4}}=\frac{7}{2}$; and $\frac{35}{4x}=\frac{35}{14}=2\frac{1}{2}$. So that the numbers sought are $3\frac{1}{2}$ and $2\frac{1}{2}$. [But let us now pursue the same investigation generally, putting a for the given

given product, b for the given difference of the squares, x for the greater of the numbers sought, and therefore $\frac{a}{x}$ for the less. Then

$$x^2-\frac{a^2}{x^2}=b$$

Therefore $x^4-a^2=bx^2$
and $x^4-bx^2=a^2$
Therefore $x^4-bx^2+\frac{b^2}{4}=\frac{b^2}{4}+a^2$
and $x^2-\frac{b}{2}=\pm\sqrt{\frac{b^2}{4}+a^2}$
Therefore $x^2=\frac{b}{2}\pm\sqrt{\frac{b^2}{4}+a^2}$
and $x=\sqrt{\frac{b}{2}\pm\sqrt{\frac{b^2}{4}+a^2}}$

But this expression for the value of x cannot be simplified as in the last problem; for if we attempt it, we shall be involved in the impossible quantity $\sqrt{-a^2}$.]

271. If it be required to find two numbers, whose *sum* or *difference* is a, and the *sum* or *difference of their square roots* is b; putting x and y for the square roots of the numbers sought, the numbers will be represented by x^2 and y^2. So that the problem resolves itself into that of finding two numbers, whose sum or difference is b, and the sum or difference of their squares is a: the solution of which we have seen in § 260. 261. 265. 266. If we be given the product of two numbers equal to a, and the product of their square roots equal b, the conditions are insufficient to ascertain the numbers, since the product of their square roots must be equal to the square root of their product, (§ 201.) and the two conditions given are therefore not independent. See § 255. If we be given the *sum* or *difference* of two numbers $=a$, and the *product of their square roots* $=b$; this resolves itself into the problem of finding two numbers whose sum or difference is given, and their product: (see § 262. 263. 264.) since the given product of their square roots is the square root of their product, and its square is therefore the product of the numbers sought.

 If

If we be given the *product* of two numbers, and the *sum* or *difference of their square roots;* the square root of the former being the product of their square roots, this also is the same thing as if we were given the sum or difference of two numbers and their product, to find the numbers. And in like manner we may find two numbers whose *sum* or *difference* is given and the *product of their squares.*

272. Hitherto we have exemplified the application of Algebra to questions purely numerical, and in which the Algebraic expression of their conditions is very obvious. But when more than abstract numbers are concerned in the problem, the translation of it into the language of Algebra will often exercise the ingenuity of the student. For instance let the following question be proposed—A gentleman, mounting his horse, was asked by a schoolmaster, what o'clock it was? He replied, *I must be at a friend's house in the country against* 5 *o'clock: now if I ride at the rate of* 10 *miles an hour, I shall have* 5 *minutes to spare; but if at the rate of* 9 *miles an hour, I shall be* 8 *minutes too late.* What was the hour? Here we are told in fact, that the time it would take to ride a certain distance at the rate of 10 miles an hour is 13 minutes less, than the time it would take to ride the *same* distance at the rate of 9 miles an hour: but we are not told either of these times, nor the distance. Let us put x for the distance or number of miles, which the man had to ride. Now will not $\frac{x}{10}$ and $\frac{x}{9}$ be just expressions for the times, in which he would ride that distance at the rate of 10 and of 9 miles an hour? (Thus, if a man has to ride 50 miles, and ride at the rate of 10 miles an hour, he will ride it in 5 hours, or $\frac{50}{10}$. If he ride at the rate of 8 miles an hour, he will ride it in $\frac{50}{8}$ of an hour, that is, in $6\frac{1}{4}$ hours.) But we are told that the former time, $\frac{x}{10}$, is less than the latter, $\frac{x}{9}$, by 13 *minutes*, that is, by $\frac{13}{60}$ of an hour: which is expressed by this equation, $\frac{x}{9}-\frac{x}{10}=\frac{13}{60}$. And this reduced

gives

gives us $x = \frac{13}{60} \times 90 = \frac{117}{6}$, or $19\frac{1}{2}$. So that the distance he had to ride was $19\frac{1}{2}$ miles: and $\frac{x}{10}$, or the time in which he would ride this distance at the rate of 10 miles an hour, is $\frac{117}{60}$ of an hour, that is 117 minutes, or an hour and 57 minutes. It must therefore have wanted 2 minutes of 3 o'clock when he was setting out, since we are told that at this rate of riding he would arrive at his destination 5 minutes before 5 o'clock. And accordingly riding $19\frac{1}{2}$ miles at 9 miles an hour, it would take him $\frac{13}{6}$ of an hour, or two hours and 10 minutes: and setting off 2 minutes before 3 o'clock he would not arrive at his destination till 8 minutes past 5. Or, we may put x for the interval (in minutes) between the time of his setting out and 5 o'clock: then $x-5$ and $x+8$ are the times he would take to ride the same distance at the rate of 10 and of 9 miles an hour; which times must be as $9:10$. Therefore $10x-50=9x+72$. And thus we find more directly that it wanted 122 minutes, or two hours and two minutes, of 3 o'clock, when he set out.

273. From the solution of this question the student may observe, that a problem apparently very complicated, and at first view seeming to present inextricable difficulties, may yet admit the shortest and most easy solution from the facilities afforded us by Algebraic notation. But let him also observe the care and attention requisite in forming Algebraic expressions for the quantities concerned in the problem, and stating the equations which its conditions afford. The slightest error here must affect all our subsequent operations, and lead us astray. Thus, for instance, we wanted to express by an equation, that the time represented by $\frac{x}{10}$ is less by 13 *minutes* than the time represented by $\frac{x}{9}$. If we attempted to do that by the equation $\frac{x}{9}-\frac{x}{10}=13$, we should be involved in a completely false result. For $\frac{x}{10}$ and $\frac{x}{9}$ express time in the denomination of *hours*; while 13 expresses time in the denomination of *minutes*. This error

 therefore

therefore must be avoided by bringing them both to the same denomination; that is, by expressing 13 minutes as a fractional part of an hour, $\frac{13}{60}$, or by bringing $\frac{x}{10}$ and $\frac{x}{9}$ to minutes, that is, multiplying them both by 60. For the equation $\frac{60x}{9}-\frac{60x}{10}=13$ expresses the same thing as $\frac{x}{9}-\frac{x}{10}=\frac{13}{60}$.—Again let it be observed that, instead of proceeding immediately to investigate the thing which the problem requires us to find, it is often necessary, and oftener convenient, to proceed to the investigation of some other quantity, upon which the determination of that thing depends. Thus we were required to find *what o'clock it was*, when the man was setting out: but from the nature of the question it appears that this must depend upon the *distance* he had to ride; which distance therefore (in the first method) we proceed to investigate.—Lastly let it be observed, that the utmost precision is necessary at the commencement in fixing the import of the letters x, y, &c. or determining for what quantities they are designed to stand: and that this must be distinctly recollected at the conclusion, when we have reduced our equation. It is therefore expedient, that the young Algebraist should for some time mark in writing the designed import of each letter, and of each Algebraic expression, which he employs. The following question will exemplify the importance of this rule.

274. A man, being asked his age, replied—*Ten years ago I was eight times as old as my son: and if we both live, till he be twice as old as he is now, I shall then be twice as old as he.* What are their present ages? Here are six different quantities, any one of which we might proceed to investigate, and for each of which we ought to have expressions:—the present ages of the father and son; their ages ten years ago; and their ages hereafter, when the son shall be twice as old as he is now. But all these are so connected by the terms of the question, that the determination of any one of them will determine all the rest. Now the first thing we should do is—not to look for an equation prematurely—but to fix on expressions for all the quantities concerned in the question. Thus—let

$x=$

x = son's age 10 years ago.
Then $x+10$ = his present age.
and $2x+20$ = his age when twice as old as now.
But $8x$ = father's age 10 years ago.
Therefore $8x+10$ = father's present age.
and $4x+40$ = his age when he shall be twice as old as his son.

The last of these expressions has been formed by doubling the third of them. But we may have another expression for the father's age at that time, by adding to his present age the same number of years, which we added to the son's present age for expressing his age at that time: as it is plain that the father and son must be older then than they are now by the *same* number of years. Now we doubled the son's present age; that is, we added $x+10$ to his present age. So that adding $x+10$ to the father's present age $8x+10$, we have $9x+20$ for another just expression of his future age, when he shall be twice as old as his son. Now equating these two expressions for the same age, $4x+40=9x+20$, and reducing the equation, we have $x=4$. That is, ten years ago the son was 4 years old, and the father 32: and therefore the son is now 14, and the father 42. Accordingly, when the son shall be 28, the father will be 56, or twice as old as his son.—We shall now propose various other examples of Arithmetical Problems, and exhibit their Algebraic solution.

275. *What fraction is that, which will become equal to* 1 *by adding* 3 *to the numerator, but equal to* $\frac{1}{4}$ *by adding* 3 *to the denominator?* Putting x for the numerator, and y for the denominator of the fraction, we are told that $\frac{x+3}{y}=1$, and that $\frac{x}{y+3}=\frac{1}{4}$. From the first of these equations $x+3=y$, and $x=y-3$. From the second, $x=\frac{y+3}{4}$. Therefore $y-3=\frac{y+3}{4}$; and $4y-12=y+3$; and $3y=15$; and $y=5$. Therefore x (or $y-3$) $=5-3=2$. And the fraction required is $\frac{2}{5}$. Accordingly adding 3 to the numerator, it becomes $\frac{5}{5}$ or 1; and adding 3 to the denominator, it becomes $\frac{2}{8}$ or $\frac{1}{4}$. [If we now generalize this problem, thus—*To find two numbers,* x *and* y, *such that adding* a

a *to* x *the quotient of* x+a *divided by* y *shall be* m; *but adding* a *to* y *the quotient of* x *divided by* y+a *shall be* $\frac{m}{n}$? then we have the equations

$$\frac{x+a}{y}=m$$

$$\text{and } \frac{x}{y+a}=\frac{m}{n}$$

From the first of which $x=my-a$

and from the second $x=\frac{my+ma}{n}$

$$\text{Therefore } my-a=\frac{my+ma}{n}$$

$$\text{and } mny-an=my+ma$$

$$\text{and } mny-my=ma+an$$

$$\text{and } y=\frac{ma+an}{mn-m}$$

Therefore x (or $my-a$)$=\frac{ma+an}{n-1}-a=\frac{ma+a}{n-1}$. The fraction required therefore is $\frac{ma+a}{n-1}\div\frac{ma+an}{mn-m}$; which reduced to its lowest terms (multiplying both dividend and divisor by $n-1$, and dividing them both by a) becomes $m+1\div\frac{m+n}{m}=\frac{m^2+m}{m+n}$. And this is a general expression for the value of $\frac{x}{y}$. And accordingly the two equations $\frac{m^2+m+a}{m+n}=m$, and $\frac{m^2+m}{m+n+a}=\frac{m}{n}$, give the same value for a: namely $a=mn-m$. Thus assuming for m and n any numbers whatsoever, suppose 7 and 5, the fraction $\frac{7^2+7}{7+5}$, or $\frac{56}{12}$, is such that adding 7×4 (or 28) to the numerator, it becomes equal to 7; but adding 28 to the denominator, it becomes equal to $\frac{7}{7}$.]

276. *A merchant's property consists of goods, bills, and cash. The value of his goods is equal to the amount of his bills and cash together: the amount of his cash is equal to twice the amount of his bills and half his goods together: and if*

if he had not lost the third part of his goods by a fire, the amount of his property would have been £12,000. What is the value of his goods, of his bills, and of his cash?—Putting x for his bills and y for his cash, $x+y$ will express the present amount of his goods, according to the first of the conditions; and by the second of the conditions $y=2x+\frac{x+y}{2}$. But $x+y$, the present amount of his goods, being $\frac{2}{3}$ of their former amount, (as we are told that he had lost $\frac{1}{3}$ of his goods by fire) $\overline{x+y}\times\frac{3}{2}$, or $\frac{3x+3y}{2}$, will express their former amount, or what would be their amount were it not for the loss by fire. By the third of the conditions therefore $x+y+\frac{3x+3y}{2}=12000$. So that we have now two equations given us for finding x and y. By the first of them $y=5x$; and this value substituted for y in the second, gives us $15x=12000$. Therefore $x=800$, the amount of his bills, and y, or $5x$, $=4000$, the amount of his cash; and $x+y=4800$, the present amount of his goods. But he had lost by fire 2400: and were it not for this, his property would have been $800+4000+4800+2400$, that is £12,000.

277. *A person buying a set of books was asked* 4*s. a volume: but finding that he had not enough of money by* 3*s. to pay for them at that price, he cheapened them to* 3*s.* 8*d. a volume; and after paying for them found he had* 6*s.* 4*d. left. How many volumes were there?*—Putting x for the number of volumes, we are furnished with two different expressions for the money, which he had. For at 4*s.* a volume, the cost of the books would be $4x$, and his money therefore was $4x-3$. But at 3*s.* 8*d.* a volume, the cost of the books is $3\frac{2}{3}\times x$, or $\frac{11x}{3}$; and another expression therefore for his money is $\frac{11x}{3}+6\frac{1}{3}$. (See § 273.) Therefore equating these two expressions for his money, we have $4x-3=\frac{11x}{3}+6\frac{1}{3}$: whence $12x-9=11x+19$; and $x=28$. Accordingly 28 volumes at 4*s.* would cost 112*s.* and his money was 3*s.* less than this sum, that is 109*s.* Now the cost of the 28 volumes at 3*s.* 8*d.* a volume was 102*s.* 8*d.* and

and after paying for them at this price, he had left 6*s.* 4*d.* —But we may more expeditiously arrive at the value of x, by observing that we are given 3*s.*+6*s.* 4*d.* or $9\frac{1}{3}s$, as the difference of the cost of x volumes at 4*s.* and at 3*s.* 8*d.* a volume: the difference of which prices is 4*d.* or $\frac{1}{3}s.$ per volume; and therefore the whole difference of cost is justly expressed by $\frac{1}{3}\times x$, or $\frac{x}{3}$. Therefore $\frac{x}{3}=9\frac{1}{3}$; and $x=28$, as before.

278. *A grocer, having two kinds of tea, which stand him in* 8*s.* *and* 7*s.* *per lb. desires to mix them so, that the compound may stand him in* 7*s.* 2*d.* *per lb. In what proportion must they be mixed?* Put x for the number of *lbs.* of the dearer tea in the compound, and y for the number of *lbs.* of the cheaper; then $x+y$ expresses the number of *lbs.* in the whole compound, which at 7*s.* 2*d.* or $\frac{43}{6}s.$ per *lb.* costs $\frac{43x+43y}{6}$. But the part of the compound represented by x costs $8x$, and the part represented by y costs $7y$. Therefore $8x+7y=\frac{43x+43y}{6}$; whence $48x+42y=43x+43y$; and $5x=y$. Resolving this equation into an analogy (§ 77.) we have $x:y::1:5$; that is, with every pound of the dearer tea 5*lbs.* of the cheaper are to be mixed; and so in proportion for any greater or smaller quantities. Accordingly mixing 1*lb.* of the dearer with 5*lbs.* of the cheaper, the cost of the whole 6*lbs.* is 8*s.*+35*s.* or 43*s.*: and this divided by 6 gives $7\frac{1}{6}$, or 7*s.* 2*d.* for the cost of the mixture per *lb.*—The same result appears from common Arithmetical principles. If the teas were mixed in equal quantities, it is plain that the cost of the compound would be 7*s.* 6*d.* per *lb.* or the cost of the compound would be found by dividing 1*s.* (the difference of the prices) into two equal parts, and adding the half to the smaller price, or subtracting it from the greater. If a smaller proportion of the dearer tea be in the compound, the cost of the mixture per *lb.* will be less, and would be ascertained by adding a *proportionally smaller* part of 1*s.* to the price of the cheaper tea. Now we are told that the cost of the compound is to be 7*s.* 2*d.* per *lb.* that is, 1*s.* the difference of the prices is divided in the ratio of 2 : 10, or 1 : 5; which

which therefore must be the ratio in which the quantity of the dearer tea is less than the quantity of the cheaper. And in like manner, if the cost of the compound was to be 7*s.* 7*d.* per *lb.* the quantity of the dearer tea in the mixture must exceed the quantity of the cheaper in the ratio of 7 : 5; or with every 5*lbs.* of the cheaper tea 7*lbs.* of the dearer must be mixed. If the prices of the teas instead of 8*s.* and 7*s.* were 8*s.* 3*d.* and 7*s.* 5*d.* the difference of the prices would be 10*d.* and in order that the compound should cost 7*s.* 7*d.* per *lb.* the quantity of the dearer should be to that of the cheaper tea as 2 : 8, or 1 : 4.

279. *How much brandy at* 8*s. per gallon, and British spirits at* 3*s. per gallon, must be mixed together, so that in selling the compound at* 9*s. per gallon, the distiller may clear* 30 *per cent.?* Here, in the first place, the student ought to form a distinct conception of the meaning of the expression, *clearing* 30 *per cent.* And if he set out without accurately understanding this, he would probably be involved in error. It does not mean, that on what he sells for £100 he is to have a profit of £30, or that he is to sell for £100 what costs him but £70: but it means, that what costs him £100 he is to sell for £130; and so in proportion on any other quantities. Instead therefore of proceeding to calculate the quantity of brandy and spirits in what shall cost £100, and be sold for £130 at 9*s.* per gallon; we may advantageously calculate the quantities in what shall cost 10*s.* and be sold at the rate specified for 13*s.* Now putting x for the number of gallons of brandy in the compound, their cost is $8x$; and putting y for the number of gallons of spirits in the compound, their cost is $3y$: and we have the equation $8x+3y=10$. But $x+y$ is the number of gallons in the whole compound, and their selling price at 9*s.* per gallon is $9x+9y$: and we have the equation $9x+9y=13$. Reducing these two equations we find $x=\frac{17}{15}$ and $y=\frac{14}{45}$. (For multiplying both sides of the first given equation by 3, we have $24x+9y=30$; from which subtracting the second given equation we have $15x=17$, and therefore $x=\frac{17}{15}$; which number substituted for x in either of the given equations affords us $y=\frac{14}{45}$.) These numbers afford the precise quantities of the brandy and spirits which would cost 10*s.* and at 9*s.* per gallon be sold for 13*s.* Mixing them therefore in the ratio of $\frac{17}{15} : \frac{14}{45}$, or 51 : 14, the required profit will be had at that selling

selling price.—We might arrive at the same conclusion by common Arithmetic, from the principles stated in the last section; first finding the *cost* of a gallon of the compound by the analogy $130 : 100 :: 9s. : \frac{90}{13}s$, or $6\frac{12}{13}s$. The difference of the cost prices is $5s$. and the excess of the cost price of the brandy above the cost price of the compound is $\frac{14}{13}s$. but the excess of the cost price of the compound above the cost price of the spirits is $\frac{51}{13}s$.: from which we collect as before that to every 51 gallons of brandy 14 gallons of spirits are to be added.

280. *Two couriers set out at the same time in contrary directions, 525 miles asunder. The one travels 40 miles the first day, and every succeeding day goes 4 miles farther than the preceding. The other travels 50 miles the first day, and every succeeding day 5 miles less than the preceding day. When will they meet?* It is plain that the principles of Arithmetical progression are applicable to this question; as the number of miles that each courier has travelled when they meet is the sum of a series in Arithmetical progression, the terms of the one increasing by the common difference 4, and the terms of the other decreasing by the common difference 5. Putting x therefore for the number of days at which they meet, this will also be the number of terms in each series. The first term in one series is 40, and the last term is $40+4\times\overline{x-1}$, or $36+4x$. Therefore the sum of that series is $\overline{40+36+4x}\times\frac{x}{2}$, or $\frac{76x+4x^2}{2}$. The first term of the other series is 50, and its last term is $50-5\times\overline{x-1}$, or $55-5x$. Therefore the sum of this series is $\overline{50+55-5x}\times\frac{x}{2}$, or $\frac{105x-5x^2}{2}$. We now have expressions for the distance, which each courier has travelled when they meet; and we are told that the sum of those distances is 525 miles, which gives us this equation:

$$\frac{181x-x^2}{2}=525$$

Therefore $x^2-181x=-1050$

and $x^2-181x+\frac{32761}{4}=-1050+\frac{32761}{4}=\frac{28561}{4}$

Therefore $x-\frac{181}{2}=\pm\sqrt{\frac{28561}{4}}=\pm\frac{169}{2}$

But

But the nature of the question marks that the *positive* value of the root cannot afford the answer. Adopting therefore the negative value, we have $x=\frac{181}{2}-\frac{169}{2}=\frac{12}{2}=6$. And accordingly calculating the distance that each courier has gone in 6 days, we shall find the sum of the distances 525 miles.—But we may arrive more expeditiously at the equation $\frac{181x-x^2}{2}=525$, by considering the completion of 525 miles by the two couriers, travelling at the rates specified, as equivalent with the completion of the same distance by one courier, travelling at a rate compounded of the two rates, that is, going 90 miles the first day, and one mile less every successive day. So that we have to find the number of terms in a decreasing Arithmetical series, whose first term is 90, the common difference 1, and the sum of the series 525. Putting x therefore for the number of terms, the last term is $\overline{90-x-1}$, or $91-x$; and the sum of the terms is expressed by $\overline{90+91-x}\times\frac{x}{2}$, or $\frac{181x-x^2}{2}$.

281. *A company wanting to make up a contribution of £80. find that they must each pay £1. 6s. 8d. more, than if there were three more contributors. What is the number in company?* Putting x for the number in company, the quota of each must be represented by $\frac{80}{x}$. If there were three more in company, the number would be $x+3$, and the quota of each $\frac{80}{x+3}$. Now we are told that the former quota exceeds the latter by $1\frac{1}{3}$£. that is,

$$\frac{80}{x+3}+\frac{4}{3}=\frac{80}{x}$$

$$\text{Therefore } \frac{4x^2+12x}{3}=240$$

$$\text{and } x^2+3x=180$$

$$\text{Therefore } x^2+3x+\frac{9}{4}=180+\frac{}{4}=\frac{729}{4}$$

$$\text{and } x=\frac{27}{2}-\frac{3}{2}=\frac{24}{2}=12$$

Accordingly

Accordingly the twelfth part of £80 is £6. 13*s.* 4*d.* but the fifteenth part of £80 is £5. 6*s.* 8*d.* less than the former by £1. 6*s.* 8*d.*—[If we generalize the problem, by putting a for the total sum to be contributed, b for the supposed additional number of contributors, and c for the difference of the quotas; then $x=\sqrt{\frac{ab}{c}+\frac{b^2}{4}}-\frac{b}{2}$: and it appears that the problem is impossible in fact, unless c measure ab, and unless $\frac{ab}{c}+\frac{b^2}{4}$ be a square number.]

282. *What number is that, which divided by the product of its digits gives* 2 *for the quotient; and if* 27 *be added to the number, the digits will be inverted?* Here it is to be understood that the number sought is written with two digits, or is less than 100; as may be collected from the latter condition. And let the student form a clear conception of the meaning of that condition; namely, that the sum of 27 and the number sought is a number written with the same digits, but in an inverted order. Now putting x for the left hand digit of the number and y for the right hand digit, we have seen (§ 59.) that the number sought will be expressed by $10x+y$; as the number written with the same digits inverted will be expressed by $10y+x$. But we are told that $\overline{10x+y}$ divided by xy gives 2 for the quotient; and that the sum of $\overline{10x+y}$ and 27 is $10y+x$; that is

$$\frac{10x+y}{xy}=2$$
$$10x+y+27=10y+x$$

From the latter of these equations we have $y=x+3$; and substituting for y in the former equation this its value, we have $\frac{11x+3}{x^2+3x}=2$: whence $x=\sqrt{\frac{49}{16}}+\frac{5}{4}=\frac{7}{4}+\frac{5}{4}=3$. Therefore y (or $x+3$) $=6$. So that the number required is 36. And accordingly $\frac{36}{3\times 6}=2$; and $36+27=63$.—From the general equation $10x+y+a=10y+x$, we may derive the equation $a=9y-9x$; from which we may infer the general principle that if to any number written with two digits, of which the left hand digit is less than the right hand digit, 9 times the difference of the digits be added, the sum

sum will be a number written with the same digits, but inverted. And in like manner, if the right hand digit of the number be greater than the left hand digit, *subtracting* from the number 9 times the difference of the digits will give a similar remainder.—In like manner, if there be a number written with three digits, adding to it, or subtracting from it (according as the right hand digit is less or greater than the left hand digit) 99 times the difference of the first and last digits, must give a sum or remainder written with the same digits, but inverted: as appears from the equation $100a+10b+c\pm x=100c+10b+a$. The student may pursue this investigation at his pleasure.

283. *Two partners A. and B. gained £140 by trade. A.'s money was 3 months in trade, and his gain was £60 less than his stock: and B.'s money, which was £50 more than A.'s, was in trade 5 months. What were their respective stocks and profits?* This question differs from any of the common questions in Fellowship (§ 137.) only in this, that we are not told the stock of either partner, but must investigate their stocks as well as profits. Putting x for A.'s stock, $x+50$ will express B.'s stock; $x-60$ A.'s gain; and therefore B.'s gain must be $140-\overline{x-60}$, that is $200-x$. But we know that their gains are in the ratio compounded of their stocks and times, or in a ratio compounded of the ratios of $x:x+50$, and of $3:5$, that is in the ratio of $3x:5x+250$. So that we have the analogy $3x:5x+250::x-60:200-x$; and thence the equation

$$600x-3x^2=5x^2-50x-15000$$

$$\text{or}\quad 8x^2-650x=15000$$

$$\text{Therefore}\quad x^2-\frac{325x}{4}+\frac{105625}{64}=\frac{15000}{8}+\frac{105625}{64}=\frac{225625}{64}$$

$$\text{and}\quad x=\sqrt{\frac{225625}{64}}+\frac{325}{8}=\frac{475}{8}+\frac{325}{8}=100$$

So that A.'s stock having been £100, his gain was £40; B.'s stock was £150, and his gain ~~£60.~~ £100 Accordingly, calculating the division of the joint profit between them at those capitals and the given times (that is, dividing £140 into two parts in the ratio of $10\times3:15\times5$, or $2:5$) we shall find the shares £40 and ~~£60.~~ £100 —

234. *Sold*

284. *Sold a piece of cloth for £24, and gained as much per cent. as the cloth cost me? What was the price of the cloth?* Putting x for the price of the cloth, the absolute profit is $24-x$. Now this being the profit on what costs £x the profit on what would cost £100 (or the gain *per cent.* see § 279.) is determined by the following analogy; as $x : 100 :: 24-x : \frac{2400-100x}{x}$. But we are told that this gain per cent. is equal to x; so that we have the equation $x=\frac{2400-100x}{x}$; or $x^2+100x=2400$. Therefore $x^2+100x+2500=4900$; and $x=70-50=20$. Accordingly, £4 profit on £20 is at the rate of 20 *per cent.*

285. *A grazier bought as many sheep as cost him £60, out of which he reserved* 15 *sheep; and selling the remainder for £54, he gained 2s. a head by them. How many sheep did he buy?* Putting x for the number bought, $x-15$ expresses the number sold for £54, and therefore $\frac{54}{x-15}$ expresses the selling price per head. But $\frac{60}{x}$ expresses the purchasing price per head; and we are told that the former exceeds the latter by 2*s.* or £$\frac{1}{10}$. Therefore we have

$$\frac{60}{x}+\frac{1}{10}=\frac{54}{x-15}$$
$$600+x=\frac{540x}{x-15}$$
$$585x+x^2-9000=540x$$
$$x^2+45x=9000$$
$$x^2+45x+\frac{2025}{4}=9000+\frac{2025}{4}=\frac{38025}{4}$$
$$x=\frac{195}{2}-\frac{45}{2}=75$$

Accordingly 60 sheep (75—15) sold for £54 give the selling price of 18*s.* per head: and 75 sheep bought for £60 give the purchasing price of 16*s.* per head.

286. *What two numbers are they whose sum is* 8, *and the sum of their cubes* 152? Here if we employ the notation, which might probably first occur to the student, we shall put

put x and y for the numbers sought; and we have the two equations $x+y=8$, and $x^3+y^3=152$: and our object must now be to reduce the cubic equation to one of a lower order. Cubing therefore both sides of the first equation, we have $x^3+3x^2y+3xy^2+y^3=8^3=512$; and from this equation subtracting the second of the given equations, we have $3x^2y+3xy^2=512-152=360$. Now dividing one side of this equation by $3x+3y$, and the other side by its equal 8×3, or 24, we have $xy=15$; and the problem therefore resolves itself into that of finding two numbers whose sum is 8 and their product 15. (See § 262. and 263.) The numbers required are 3 and 5.

287. But we may frequently obtain a more facile and elegant solution for a problem, by employing for the numbers sought a designation borrowed from the principle, that the greater of any two numbers is equal to half their sum *plus* half their difference, and the less equal to half their sum *minus* half their difference. (This appears from reducing the equations $x+y=a$, and $x-y=b$. See also § 142.) Let us now resume the solution of the last problem. We are told that 4 is half the sum of the numbers sought. Therefore putting x for half their difference, the greater will be expressed by $4+x$, and the less by $4-x$. The cube of the greater, or $\overline{4+x}|^3$, is $64+48x+12x^2+x^3$. —The cube of the less is $64-48x+12x^2-x^3$. And the sum of these cubes is $128+24x^2$; which sum we are told is equal to 152. Therefore $24x^2=152-128=24$; and $x^2=1$; and $x=1$. The numbers sought therefore are $4+1$ and $4-1$, or 5 and 3.—[Generalizing this solution by putting a for half the given sum, x for half the difference of the numbers, and b for the given sum of their cubes, the numbers sought are expressed by $a+x$ and $a-x$, whose cubes are $a^3+3a^2x+3ax^2+x^3$ and $a^3-3a^2x+3ax^2-x^3$. But the sum of these cubes is $2a^3+6ax^2$. Therefore $2a^3+6ax^2=b$; and $6ax^2=b-2a^3$; and $x^2=\frac{b-2a^3}{6a}$.

Therefore $x=\sqrt{\frac{b-2a^3}{6a}}$; or $\sqrt{\frac{b}{6a}-\frac{a^2}{3}}$.]

288. *What two numbers are they, whose sum is* 6 *and the sum of their* 4*th powers* 272? Putting x (as in the last section) for half their difference, the numbers sought are expressed by $3+x$ and $3-x$. But $\overline{3+x}|^4+\overline{3-x}|^4=162$ +

$+108x^2+2x^4=272$. Therefore $x^4+54x^2=55$: and completing the square $x^4+54x^2+729=784$; and $x^2+27=28$. Therefore $x^2=1$; and $x=1$; and the numbers sought are $3+1$ and $3-1$, or 4 and 2.]—Universally putting a for half the given sum, and b for the sum of the biquadrates, $\overline{a+x}|^4+\overline{a-x}|^4=2a^4+12a^2x^2+2x^4=b$: whence $x^4+6a^2x^2=\frac{b}{2}-a^4$; and $x^4+6a^2x^2+9a^4=\frac{b}{2}+8a^4$. Therefore $x^2+3a^2=\sqrt{\frac{b}{2}+8a^4}$; and $x=\sqrt{-3a^2+\sqrt{\frac{b}{2}+8a^4}}$.—By the aid of a similar notation we can find two numbers, whose sum is given and the sum of their fifth powers. For the 5th. power of $a+x$ is $a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$: and the 5th power of $a-x$ is $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$. But the sum of these 5th. powers is $2a^5+20a^3x^2+10ax^4=b$; a biquadratic equation of that form which we can reduce as a quadratic.]

289. *To find four numbers in Arithmetical progression, whereof the product of the extremes is* 54, *and that of the means* 104*?* Putting x for the smaller extreme, and y for the common difference, the series is expressed by x, $x+y$, $x+2y$, and $x+3y$. The product of the extremes is $x^2+3xy=54$: the product of the means is $x^2+3xy+2y^2=104$: from which subtracting the former equation, we have $2y^2=50$; and $y^2=25$. Therefore the common difference $y=5$; and substituting this number for y in the equation $x^2+3xy=54$, we have $x^2+15x=54$: which gives us $x=-\frac{15}{2}+\sqrt{\frac{441}{4}}=-\frac{15}{2}+\frac{21}{2}=3$. So that the numbers sought are 3, 8, 13, and 18.—We see in the solution of this problem, that when four numbers are in Arithmetical progression the product of the means exceeds the product of the extremes by twice the square of the common difference: as, if three numbers be in Arithmetical progression, the square of the mean exceeds the product of the extremes by the square of the common difference.

290. *Given the sum of three numbers in Arithmetical progression* $=24$, *and the sum of their squares* $=210$, *to find the numbers?* Employing the same notation as in the last section, the numbers are expressed by x, $x+y$, and $x+2y$: and their squares by x^2, $x^2+2xy+y^2$, and $x^2+4xy+4y^2$. The sum of the numbers is $3x+3y=24$: the

the sum of their squares is $3x^2+6xy+5y^2=210$. Squaring the first of these equations, and multiplying the second by 3, we have $9x^2+18xy+9y^2=576$, and $9x^2+18xy+15y^2=630$. Subtracting the former of these equations from the latter we have $6y^2=54$; and $y^2=9$. Therefore the common difference $y=3$; and substituting this number for y in the first of the given equations, we have $3x+9=24$; and $x=\frac{15}{3}=5$. So that the numbers sought are 5, 8, and 11.—In like manner, if we be given the sum, and sum of the squares, of *four* numbers in Arithmetical progression, we have the equations $4x+6y=a$, and $4x^2+12xy+14y^2=b$. Squaring both sides of the former, and multiplying both sides of the latter by 4, we have $16x^2+48xy+36y^2=a^2$, and $16x^2+48x+56y^2=4b$. Whence $20y^2=4b-a^2$.—In like manner if there be *five* terms in the series, we shall find $50y^2=5b-a^2$. If there be *six* terms in the series, we shall find $105y^2=6b-a^2$. If there be *seven* terms in the series we shall find $196y^2=7b-a^2$.—In all this investigation let it be remembered that a denotes the given sum of the terms, b the given sum of their squares, and y the common difference of the terms. And we find that the coefficient of b is always the number of terms in the series; but the coefficients of y^2 are found to be successively 1, 6, 20, 50, 105, 196, according as the number of terms in the series is 1, 2, 3, 4, 5, 6, or 7.

291. [We might now proceed to investigate the law of continuation in the series of coefficients of y^2; so as to be able to calculate the coefficient of y^2, when the number of terms in the series is 10, or any other assigned number; and this without being at the trouble of discovering it by the same operation, by which we have ascertained the first seven terms. But as the investigation lies rather beyond the elementary subject of this treatise, and, if minutely detailed, would lead us too far away from our present object; I shall content myself with pointing out to the curious student some of the steps and the ultimate result. Observing the series 1, 6, 20, 50, 105, 196, we find the differences of the successive terms to be 5, 14, 30, 55, 91. Observing this series, we find the differences of its successive terms (called the *second differences* of the terms of the former series) to be 9, 16, 25, 36, or the squares of the numbers 3, 4, 5, 6: so that in the series of the first differences 5,

14, 30, 55, 91, the first term 5 is the sum of 2^2+1^2; the second term $14=3^2+2^2+1^2$; the third term $30=4^2+3^2+2^2+1^2$; and so on. This may lead us to the constitution of the series 1, 6, 20, 50, &c. whose law of continuation we investigate. Its first term is unity: its second term $6=2^2$ + twice 1^2. Its third term $20=3^2$ + twice 2^2 + three times 1^2. Its fourth term $50=4^2$ + twice 3^2 + three times 2^2 + four times 1^2: and so on. Now 50, the *fourth* term of that series, is the coefficient of y^2 when the number of terms in the Arithmetical series is *five*. (§ 284.) Suppose then that the number of terms in the Arithmetical series is 10. The coefficient of y^2, in the equation $10b-a^2=ny^2$, will be the sum of the following numbers $9^2+\overline{2\times 8^2}+\overline{3\times 7^2}+\overline{4\times 6^2}+\overline{5\times 5^2}+\overline{6\times 4^2}+\overline{7\times 3^2}+\overline{8\times 2^2}+\overline{9\times 1^2}$; or will be 825. And accordingly taking any series of ten terms in Arithmetical progression, it will be found that 825 times the square of the common difference = 10 times the sum of the squares of the terms *minus* the square of the sum of the terms.]

292. [But we still need to simplify the calculation of the coefficient of y^2. Suppose then that the Arithmetical series consists of 5 terms: and let $n=5$. We have seen that the coefficient of y^2 will be the sum of the following terms, $\overline{n-1}|^2+2\times\overline{n-2}|^2+3\times\overline{n-3}|^2+4\times\overline{n-4}|^2$; that is, the sum of the following terms,

$$\begin{array}{r} n^2-2n+1 \\ 2n^2-8n+8 \\ 3n^2-18n+27 \\ 4n^2-32n+64 \\ \hline 10n^2-60n+100 \end{array}$$

In this expression, $10n^2-60n+100$, the coefficient of n^2 is the sum of $1+2+3+4$. The last term 100 is the sum of $1^3+2^3+3^3+4^3$. The coefficient of n in the second term is $\overline{1^2\times 2}+\overline{2^2\times 2}+\overline{3^2\times 2}+\overline{4^2\times 2}=\overline{1^2+2^2+3^2+4^2}\times 2$. Now from the doctrine of Arithmetical progression we can easily calculate the sum of any of the natural numbers ascending from unity. We therefore only want to know a facile method of calculating the sum of their *squares*, and the sum of their *cubes*. The latter is easily calculated from the following curious property—*that the*

sum

sum of the cubes of any of the natural numbers commencing with unity is equal to the square of their sum: as in the preceding instance $1+2+3+4=10$; and $10^2=100=1^3+2^3+3^3+4^3$. And the sum of the squares of the terms of such a series is equal to the 6th. part of the sum of the highest term + three times its square + twice its cube. Therefore twice the sum of the squares is equal to the third part of the latter sum. Accordingly in the preceding instance $\overline{1^2+2^2+3^2+4^2}\times 2=\frac{4+\overline{3\times 4^2}+\overline{2\times 4^3}}{3}=\frac{180}{3}=60.$]

293. [Let it be recollected that in the trinomial, $10n^2-60n+100$, (expressing the coefficient of y^2 when the number of terms in the Arithmetical series is *five*) the series of natural numbers from unity, of which 10 is the sum, 100 the sum of the cubes, and 60 twice the sum of the squares, is 1, 2, 3, 4; its highest term being one less than the number of terms in the series whose common difference is y. Whatever therefore be the number of terms in this series, represented by n, the series of natural numbers, from which the terms of the trinomial formula are to be determined, is $1, 2, 3 \ldots n-1$. Now the sum of this is (§ 185.) $n\times\frac{n-1}{2}=\frac{n^2-n}{2}$. Therefore the first term of the trinomial formula is universally expressed by $\frac{n^2-n}{2}\times n^2=\frac{n^4-n^3}{2}$. The third term also is universally expressed by $\overline{\frac{n^2-n}{2}}\Big|^2=\frac{n^4-2n^3+n^2}{4}$; for we have remarked in the last section that the sum of the cubes of $1, 2, 3 \ldots n-1$, is equal to the square of their sum. In the second term of the trinomial formula (which term is to be subtracted from the sum of the first and third) the coefficient of n is by the last section universally expressed by $\frac{\overline{n-1}+3\times\overline{n-1}|^2+2\times\overline{n-1}|^3}{3}=\frac{2n^3-3n^2+n}{3}$: which multiplied by n gives $\frac{2n^4-3n^3+n^2}{3}$ for the universal expression of the second term in the trinomial formula. In order to subtract this from the sum of the two former, let

us bring them all to the common denominator 12. The two former become $\frac{6n^4-6n^3}{12}$ and $\frac{3n^4-6n^3+3n^2}{12}$: whose sum is $\frac{9n^4-12n^3+3n^2}{12}$. From which subtracting $\frac{8n^4-12n^3+4n^2}{12}$ (the value of the other term) we have left $\frac{n^4-n^2}{12}$ for the universal expression of the coefficient of y^2 in the equation $nb-a^2=my^2=\frac{n^4-n^2}{12}\times y^2$; where y represents the common difference of any Arithmetical progression, a the sum of the series, n the number of terms, and b the sum of their squares.]

294. [This investigation originated in the problem proposed § 290. to find a series in Arithmetical progression from having given us the sum of its terms, and the sum of their squares. But we may now reverse the problem, and easily find the sum of the squares of the terms of any given Arithmetical progsession. For from the last equation we arrived at, $nb-a^2=\frac{n^4y^2-n^2y^2}{12}$, we find $b=\frac{n^3y^2-ny^2}{12}+\frac{a^2}{n}$. Therefore putting s for the sum of any Arithmetical series, d for the common difference, and n for the number of terms, the sum of the squares of the terms is equal to $\frac{d^2}{12}\times n^3-n+\frac{s^2}{n}$. Thus, if the Arithmetical series 3, 5, 7, &c. be continued to ten terms, the 10th. term is 21; the sum of the terms is 120; its square is 14400; and therefore $\frac{s^2}{n}=1440$. But $\frac{d^2}{12}=\frac{1}{3}$; and $n^3-n=990$. Therefore $\frac{d}{12}\times n^3-n=\frac{990}{3}=330$. And the sum of the squares of the terms, $3^2+5^2+7^2....+21^2$, $=330+1440=1770$.]

295. *To find four numbers in Arithmetical progression the sum of whose squares shall be* 214, *and the continued product of the numbers* 880? Here putting $x-3y$ for the smaller extreme, and $x+3y$ for the greater, from the nature of Arithmetical progression $2y$ will be the common difference; and the two means will be expressed by $x-y$ and

and $x+y$: so that the Arithmetical series is $x-3y$, $x-y$, $x+y$, and $x+3y$. (If the student should attempt to express the series by x, $x+y$, $x+2y$, $x+3y$, he would find himself involved in considerable difficulties: and he may observe how the notation we have adopted tends to simplify the equations, which express the conditions of the problem.) The sum of the squares of these four terms is $4x^2+20y^2=214$. Their continued product, or $\overline{x-3y} \times \overline{x+3y} \times \overline{x-y} \times \overline{x+y}$, or $\overline{x^2-9y^2} \times \overline{x^2-y^2}$, is $x^4-10x^2y^2+9y^4=880$. From the former of these equations $x^2=\frac{107-10y^2}{2}$; and therefore $x^4=\overline{\frac{107-10y^2}{2}}\Big|^2=\frac{11449-2140y^2+100y^4}{4}$. Substituting for x^4 and x^2 in the second of the given equations these their values derived from the first, we have $\frac{11449-2140y^2+100y^4}{4}-\frac{1070y^2-100y^4}{2}+9y^4=880$; whence $336y^4-4280y^2=3520-11449=-7929$; and $y^4-\frac{535y^2}{42}=-\frac{7929}{336}=-\frac{2643}{112}$. Therefore $y^4-\frac{535y^2}{42}+\frac{286225}{7056}=\frac{286225}{7056}-\frac{2643}{112}=\frac{119716}{7056}$; and $y^2-\frac{535}{84}=\pm\sqrt{\frac{119716}{4056}}=\pm\frac{346}{84}$; and $y^2=\frac{535}{84}-\frac{346}{84}=\frac{189}{84}=\frac{9}{4}$. Therefore $y=\frac{3}{2}$; and $2y$, or the common difference of the series, $=3$. But we have seen that $x^2=\frac{107-10y^2}{2}=\frac{107-22\frac{1}{2}}{2}=\frac{169}{4}$. Therefore $x=\sqrt{\frac{169}{4}}=\frac{13}{2}$; and the Arithmetical series $x-3y$, $x-y$, $x+y$, $x+3y$, is 2, 5, 8, 11.

296. In pursuing the solution of this problem, I have retained the given numbers 214 and 880, which are the assigned values of the sum of the squares and continued product of the four nnmbers sought. And it is important that the student should acquire a readiness and accuracy in performing the numerical calculations thus occasioned. But except for promoting this object, it is much preferable to

to substitute for the given numbers (when they are so large) some of the initial letters of the alphabet. Thus the given equations in the last problem may be stated generally, $4x^2+20y^2=a$, and $x^4-10x^2y^2+9y^4=b$.—If instead of the *sum of the squares* and continued product, there be *given the common difference and continued product of four numbers in Arithmetical progression*, the series is found still more easily. For putting $2a$ for the given common difference, the four numbers may be expressed as in the last section by $x-3a$, $x-a$, $x+a$, and $x+3a$: the continued product of which terms is $x^4-10a^2x^2+9a^4=b$. Whence, completing the square, we have $x^4-10a^2x^2+25a^4=b+16a^4$: and therefore $x^2-5a^2=\pm\sqrt{b+16a^4}$; and $x^2=5a^2\pm\sqrt{b+16a^4}$. Now if the given common difference be 3, and therefore $a=\frac{3}{2}$, and if $b=880$, then $5a^2\pm\sqrt{b+16a^4}=\frac{45}{4}\pm\sqrt{961}=\frac{45}{4}\pm31=$ (taking the positive value of the root) $\frac{169}{4}$. And therefore $x=\sqrt{\frac{169}{4}}=\frac{13}{2}$, as before. We fix on the *positive* value of the root, because it affords a positive square number, for the value of x^2: whereas the negative value of $\sqrt{961}$ would give $x^2=\frac{45}{4}-31=-\frac{79}{4}$, which is impossible.—It is to be remarked, that although we can find *four* numbers in Arithmetical progression from the data assigned in either of the last sections, yet to find *three* such numbers from similar data would necessarily involve us in a *cubic* equation; the management of which does not come within the subject of the present treatise.

297. *To find three numbers in geometrical progression whose sum shall be* 26, *and the sum of their squares* 364? Putting x and y for the two first terms, the third will be expressed by $\frac{y^2}{x}$; and we have given us the equations $x+y+\frac{y^2}{x}=a$, and $x^2+y^2+\frac{y^4}{x^2}=b$. In the first of these equations transposing y, we have $x+\frac{y^2}{x}=a-y$; & squaring both sides of this equation we have $x^2+2y^2+\frac{y^4}{x^2}=a^2$

$-2ay$

$-2ay+y^2$. Therefore $x^2+y^2+\frac{y^4}{x^2}=a^2-2ay=$ (by the second of the given equations) b. Therefore $y=\frac{a^2-b}{2a}$ $=\frac{676-364}{52}=\frac{312}{52}=6$. Thus we have found that the mean, or second term of the series is 6. Therefore the sum of the extremes is $a-6=20$; and their product is $y^2=36$: so that we now have given us the sum and product of the extremes; from which they are found to be 2 and 18; and the series required is 2, 6, 18.

298. *To find four numbers in geometrical progression, such that the difference of the extremes shall be* 52, *and the difference of the means* 12? Putting x for half the sum of the means, and a for half their given difference, the two means will be expressed by $x-a$, and $x+a$. (See § 287.) Then from the nature of geometrical progression the smaller extreme must be $\frac{\overline{x-a}|^2}{x+a}$, and the greater extreme $\frac{\overline{x+a}|^2}{x-a}$: for $x\pm a : x\mp a :: x\mp a : \frac{\overline{x\mp a}|^2}{x\pm a}$. Therefore the difference of the extremes is $\frac{\overline{x+a}|^2}{x-a}-\frac{\overline{x-a}|^2}{x+a}=52=b$: and multiplying both sides of this equation by $x-a$ and by $x+a$, we have

$$\overline{x+a}|^3-\overline{x-a}|^3=b\times x^2-a^2=bx^2-a^2b.$$

But subtracting $x^3-3x^2a+3xa^2-a^3$ (or $\overline{x-a}|^3$) from $x^3+3x^2a+3xa^2+a^3$ (or $\overline{x+a}|^3$) the remainder is $6x^2a+2a^3=bx^2-a^2b$. Therefore $bx^2-6x^2a=2a^3+a^2b$. Whence $x^2=\frac{2a^3+a^2b}{b-6a}=\frac{432+1872}{52-36}=\frac{2304}{16}$: and $x=\sqrt{\frac{2304}{16}}=\frac{48}{4}=12$. So that the two means are 6 and 18, and the extremes 2 and 54.

[299. If we have the *sum* and *product* of any two numbers, we may thence derive expressions for the *sum of their squares*, *cubes*, *biquadrates*, &c. For putting x and y for the numbers; s for their sum and p for their product.— in the first place $x^2+y^2=s^2-2p$, since the square of the sum

sum is equal to the sum of the squares + twice the product. —In the second place, multiplying the equation $x^2+y^2=s^2-2p$ by the equation $x+y=s$, we have $x^3+y^3+xy^2+yx^2=s^3-2sp$. But $xy^2+yx^2=\overline{x+y}\times xy=s\times p$. Therefore $x^3+y^3+sp=s^3-2sp$; and $x^3+y^3=s^3-3sp$.—In the third place, multiplying both sides of the last equation by the equation $x+y=s$, we have $x^4+y^4+xy^3+yx^3=s^4-3s^2p$. But $xy^3+yx^3=\overline{x^2+y^2}\times xy=\overline{s^2-2p}\times p=s^2p-2p^2$. Therefore $x^4+y^4+s^2p-2p^2=s^4-3s^2p$; and $x^4+y^4=s^4-4s^2p+2p^2$.—In like manner if we proceed to calculate the value of x^5+y^5, it will be found by multiplying the value of x^4+y^4 by s, and subtracting from the product the value of x^3+y^3 multiplied by p; whence $x^5+y^5=s^5-5s^3p+5sp^2$. And again multiplying this value of x^5+y^5 by s, and subtracting from the product the value of x^4+y^4 multiplied by p, we have the value of $x^6+y^6=s^6-6s^4p+9s^2p^2-2p^3$. We may now remark on these expressions for the sums of the powers, 1st. that the signs of the terms are alternately affirmative and negative: 2ndly. that the number of terms is always half of the even number next above the index of the power; (for instance, the expression for the value $x^{10}+y^{10}$, or for $x^{11}+y^{11}$, will consist of 6 terms, but for $x^{12}+y^{12}$ of 7 terms). 3rdly. Putting n for the index of the power, the first term of the expression for the value x^n+y^n will be s^n, and in every succeeding term the index of s decreases bp 2, and the index of p increases by 1. 4thly. the coefficient of the second term is n; and if n be an odd number, the coefficient of the last term also is n; but if n be an even number, the coefficient of the last term is 2, and the literal part p with the index $\frac{n}{2}$: 5thly. the coefficient of the third term is the sum of the natural numbers from 2 to $n-2$, or is $\frac{n^2-3n}{2}$: 6thly. the coefficient of the fourth term is the sum of all the coefficients of the third terms of the preceding powers from the *last but one*; the coefficient of the fifth term is the sum of all the coefficients of their fourth terms: and so on. Thus in the expression for the sum of the 12th. powers of x and y, the coefficient of the third term will be $10+9+8+7+3+2$, or the sum of the coefficients of the second terms in all the expressions of the preceding powers from the tenth: and the coefficient of the 6th. term will be the sum of the coefficients of the 5th. terms of all the preceding powers from the tenth; that is, of the tenth, ninth, and eighth powers, as it is in the sum of the 8th. powers of x and y that a *fifth* term first appears. Hence we may derive the following expressions for the coefficients of the terms in the expression for the value of x^n+y^n. The coefficient of the third term is $n\times\frac{n-3}{2}$; of the fourth term is $n\times\frac{n-4}{2}\times\frac{n-5}{3}$; of the fifth term is $n\times\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}$, &c. &c. And thus calculating the value of $x^{12}+y^{12}$ it is found to be $s^{12}-12s^{10}p+54s^8p^2-112s^6p^3+105s^4p^4-36s^2p^5+2p^6$.]

300. *To find two numbers, whose product shall exceed their sum by* 11, *and the sum of whose squares shall be* 58? Putting x for the sum and y for the product of the two numbers, the sum of their squares is expressed by x^2-2y; as we have seen in the beginning of the last section: so that we have the two equations $y-x=11$, and $x^2-2y=58$. Adding twice the former equation to the latter, we have $x^2-2x=58+22=80$. Therefore $x=\sqrt{81}+1=10$; and $y=11+10=21$. Having thus ascertained the sum of the numbers $=10$, and their product $=21$, we find that the numbers required are 3 and 7. (§ 263.)

[301. *To find four numbers in geometrical progression whose sum shall be* (*a*) 80, *and the sum of their squares* (*b*) 3280? Putting x and y for the two means, the extremes will be expressed by $\frac{x^2}{y}$ and $\frac{y^2}{x}$. (For $x:y::y:\frac{y^2}{x}$; and $y:x::x:\frac{x^2}{y}$.) Now putting s for the sum of the means and p for their product, (which is also the product of the extremes) we have the sum of the extremes, or $\frac{x}{y}+\frac{y^2}{x}=a-s$. But by § 299. $x^2+y^2=s^2-2p$; and in like manner the sum of the squares of the extremes, or $\frac{x^4}{y^2}+\frac{y^4}{x^2}$, is equal to the square of their sum *minus* twice their product, that is, $=\overline{a-s}|^2-2p$. Hence, adding the sum of the squares

squares of the means to the sum of the squares of the extremes, we have the equation $s^2+\overline{a-s}|^2-4p=3280=b$. —Again, from the equation $\frac{x^2}{y}+\frac{y^2}{x}=a-s$, we have $x^3+y^3=\overline{a-s}\times xy=\overline{a-s}\times p=ap-sp$. But by § 299. $x^3+y^3=s^3-3sp$. Therefore $s^3-3sp=ap-sp$, and $s^3=ap+2sp$; whence $p=\frac{s^3}{2s+a}$. Now substituting for p this expression of its value in the equation $s^2+\overline{a-s}|^2-4p=b$, we have $s^2+\overline{a-s}|^2-\frac{4s^3}{2s+a}=b$; that is, $2s^2-2sa+a^2-\frac{4s^3}{2s+a}=b$; whence, multiplying both sides by $2s+a$, we have $-2s^2a+a^3=2sb+ab$. Therefore $s^2+\frac{sb}{a}=\frac{a^2-b}{2}$; & completing the square $s^2+\frac{sb}{a}+\frac{b^2}{4a^2}=\frac{a^2-b}{2}+\frac{b^2}{4a^2}$. Whence we have $s=\sqrt{\frac{a^2-b}{2}+\frac{b^2}{4a^2}}-\frac{b}{2a}=\sqrt{1560+420\frac{1}{4}}-20\frac{1}{2}=\frac{89}{2}-20\frac{1}{2}=24$. Thus we have ascertained that the sum of the two means is 24. But we have before found $p=\frac{s^3}{2s+a}$. Therefore the product of the means $=\frac{24^3}{48+80}=\frac{13824}{128}=108$. Hence the means are found to be 6 and 18; and therefore the extremes 2 and 54.]

In the following questions for exercise, lest any difficulty should remain to the student, I have either referred to a preceding section where a similar question has been solved, or have exhibited the translation of the question into the language of Algebra. Yet I would strongly recommend, that he should not apply to these aids, until he has attempted to solve the questions without them.

Questions for Exercise.

1. What two numbers are they, whose sum is 7 and their difference $2\frac{1}{7}$? (§ 287.)

2. Divide £20 between A. and B. so that A. shall have 10*s*. 6*d*. more than B.? (§ 287.)

3. —— so that $\frac{2}{3}$rds. of A.'s share shall exceed $\frac{3}{4}$ths of B.'s by 6*s.* 8*d.* ?—Putting x for A.'s share and $20-x$ for B.'s, we have the equation $\frac{2x}{3}-\frac{60-3x}{4}=\frac{1}{3}$: from which A.'s share will be found $10\frac{4}{7}$£. and B.'s share $9\frac{3}{7}$£. But let the student receive a caution in the reduction of that equation. After multiplying both sides by 12, it will stand—not $8x-180-9x=4$, but $8x-180+9x=4$. For in the fraction $\frac{60-3x}{4}$ the mark of division, or line separating the numerator and denominator, acts as a *vinculum* on the terms of the numerator: and therefore after the multiplication by 12, we have to subtract $\overline{180-9x}$ from $8x$, that is, to add $-180+9x$.

4. What two numbers are they, whose ratio is that of 7 : 5, and whose sum is 7? —or—whose difference is 5? (§ 259.)

5. —— whose difference is 3, and the difference of their squares 18? (§ 260.) Dividing the difference of the squares by the difference of the numbers the quotient is the sum of the numbers. (§ 163.)

6. —— whose sum is 3, and the difference of their squares $5\frac{1}{3}$? (§ 261.)

7. —— whose sum is 3, and the sum of their squares $6\frac{1}{2}$? (§ 265.) Putting x for half the difference of the numbers, the greater is expressed by $\frac{3}{2}+x$, and the less by $\frac{3}{2}-x$. Therefore the sum of their squares is expressed by $\frac{9}{2}+2x^2=6\frac{1}{2}$; which gives $x^2=1$: and therefore the difference of the numbers is 2. This method of denoting two numbers is frequently of the greatest advantage.

8. —— whose difference is 2, and the sum of their squares $13\frac{1}{4}$? (§ 266.)

9. —— whose sum is 15, and their product $31\frac{1}{4}$? (§ 262. 263.)

10. —— whose difference is 10, and their product $31\frac{1}{4}$? (§ 264.)

11. —— whose product is $8\frac{1}{6}$, and the sum of their squares $17\frac{25}{36}$? (§ 267.)

12. —— whose product is 18, and the difference of their squares 27? (§ 270.)

13. —— whose sum is $4\frac{1}{36}$ (or difference $\frac{17}{36}$) aud the sum of their square roots $2\frac{5}{6}$? or — the difference of their square roots $\frac{1}{6}$? (§ 271.)

—whose

14. —— whose sum is $4\frac{1}{16}$ (or their difference $\frac{17}{16}$) and the product of their square roots 2 ? (§ 271.)

15. —— whose product is 4, and the sum of their square roots $2\frac{5}{6}$? or the difference of their square roots $\frac{1}{6}$ (§ 271.)

16. —— whose sum is 5, and the product of their squares 36 ? (§ 271.)

17. —— whose difference is 1, and the product of their squares $2\frac{1}{4}$?

18. —— whose product is 7, and their ratio that of 7 : 4 ? $\left(x : \frac{7}{x} :: 7 : 4.\right)$

19. To find a fraction such, that if you add 8 to the numerator it shall become equal to 2 ; but if you add the numerator to the denominator it shall become equal to $\frac{2}{5}$? (Putting $\frac{x}{y}$ for the fraction, we have $\frac{x+8}{y}=2$, and $\frac{x}{y+x}=\frac{2}{5}$.)

20. To find a fraction which shall be to its reciprocal as 4 : 9, and whose denominator exceeds its numerator by 3 ? (We have $\frac{x}{x+3} : \frac{x+3}{x} :: 4 : 9$; and therefore $\frac{9x}{x+3}=\frac{4x+12}{x}$.)

21. A man riding from his own house to Dublin went at the rate of $7\frac{1}{2}$ miles an hour. Returning home he came at the rate of $6\frac{1}{2}$ miles an hour, and was 8 minutes longer on the road. What was the distance ? (§ 272.)

22. A.'s age is to B.'s as 4 : 3 ; and three years ago it was as 3 : 2. What are their ages ? (§ 274.)

23. A man left in his will £10,000 to be equally divided among his children. Three of them died before their father, and the survivors in consequence got £750 a-piece more than they would have got, if all had lived. What was the number of children ? (§ 281.)

24. There are two silver cups and one cover for both. The first cup with the cover weighs 14*oz*. The second cup with the cover weighs $\frac{2}{3}$rds. of the first cup without the cover ; but without the cover weighs $\frac{1}{2}$ of the first cup. What are the weights of each ?—(Putting x for the weight of the cover, we have $14-x$ for the weight of the first cup, and therefore $7-\frac{x}{2}$ for the weight of the second cup.

Adding

Adding x to this, we have $7+\frac{x}{2}$ for the weight of the second cup and cover together, which we are told is $\frac{2}{3}$rds. of $14-x$, that is $=\frac{28-2x}{3}$.)

25. A journeyman was engaged for 40 days, at the wages of 3*s.* 6*d.* a day for every day he worked; but to forfeit 2*s.* 6*d.* for every day he absented himself. At the end of the period he received 4£. 6*s.* How many days did he work, and how many was he absent? (Putting x for the former number, the amount of his wages is $\frac{7}{2}s. \times x = \frac{7x}{2}$; and the amount of his forfeitures is $\frac{5}{2}s. \times \overline{40-x} = \frac{200-5x}{2}$. This subtracted from $\frac{7x}{2}$ gives a remainder equal to 86*s.* whence we have $12x-200=172$.)

26. A market woman bought a certain number of eggs at 2 a penny, and as many at 3 a penny: and selling them at the rate of 5 for 2*d.* she lost 4*d.* on the whole. What number of eggs had she? (Putting x for the number of each sort, we have $\frac{x}{2}+\frac{x}{3}$ for the whole cost, and $2x\div\frac{5}{2}$, or $\frac{4x}{5}$, for the whole selling amount; which is less than $\frac{x}{2}+\frac{x}{3}$ by 4.)

27. A person desiring to give 3*d.* a-piece to some beggars, found he had not money enough in his pocket by 8*d.* but giving them 2*d.* a-piece, he had 3*d.* remaining. How many beggars were there? (§ 277.)

28. There is a fish whose tail weighs 9*lb.* his head weighs as much as his tail and half his body; and his body weighs as much as his head and tail. What is the weight of the fish? (Putting x for the weight of the body, the weight of the head is $9+\frac{x}{2}$; and we are told that $x=9+\frac{x}{2}+9=18+\frac{x}{2}$.)

29. A bill of £70. 12*s.* was paid in guineas and crown pieces: and the number of pieces of both sorts was 100. How many were there of each? (Putting x for the number of

of guineas, $100-x$ is the number of crowns. The amount of the former, at 21*s.* is $21x$; and of the latter at 5*s.* is $500-5x$: so that $21x+500-5x=1412$, the number of shillings in £70. 12*s.*)

30. A person bought a chaise, horse, and harness, for £60. The horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness. What did he give for each? (Putting x for the price of the harness, $2x$ is the price of the horse, and $6x$ the price of the chaise. But $6x+2x+x=60$.)

31. A. saves $\frac{1}{5}$th. of his income yearly. B. with the same income spends yearly £50 more than A. and at the end of 4 years finds himself £100 in debt. What is their income? (Putting x for the income, $\frac{4x}{5}$ is A.'s yearly expenditure, and therefore $\frac{4x}{5}+50$ is B.'s yearly expenditure; which in 4 years amounts to $\frac{16x}{5}+200$: and this exceeds $4x$ by 100.)

32. To divide 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other. (Putting x for half of the first part, that part is $2x$, the second part $3x$, and the third part $4x$. But $2x+3x+4x$, or $9x=36$.)

33. A footman, hired at the wages of £8 a year and a livery, was turned away at the end of 7 months, and received only £2. 13*s.* 4*d.* and his livery. What was its value? (Putting x for the value of the livery, $8+x$ is the amount of what he should have received for 12 months service. What he receives for 7 months service is $2\frac{2}{3}+x$, which therefore is to $8+x :: 7 : 12$.)

34. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3: but two of the greyhound's leaps are as much as three of the hare's. How many leaps must the greyhound take to catch the hare? (Putting x for the number of leaps taken by the *hare* before she is overtaken, it is plain that the greyhound must go over a space of ground equal to $x+50$ of the hare's leaps: and this he will do in a smaller number of leaps than $x+50$, and smaller in the ratio of $3:2$, that is in $\frac{2x+100}{3}$ of his own

leaps. But the number of leaps taken by the hare is to the number taken by the greyhound in the same time as $4:3$. Therefore $x : \frac{2x+100}{3} :: 4 : 3$; whence $x = 400$; and $\frac{2x+100}{3}$, or the number of leaps taken by the greyhound to overtake the hare, $=300$.)

35. A person in play lost $\frac{1}{4}$ of his money, and then won 3*s.* after which he lost $\frac{1}{3}$ of what he then had, and then won 2*s.* lastly he lost $\frac{1}{7}$ of what he then had, and found he had but 12*s.* remaining. What had he at first? (Putting x for the number of shillings which he had at first, $\frac{3x}{4}+3$ expresses what he had after his first loss and first winning; $\frac{2}{3}$rds. of this, or $\frac{x}{2}+2$ expresses what he had after his second loss, and therefore $\frac{6}{7}$ths. of $\frac{x}{2}+4$ expresses what he had after his third loss, or when he had 12*s.* left.)

36. A. gives to B. as much money as B. has already: B. returns to A. as much as A. has left: A. returns to B. as much as B. has then left: and lastly B. returning to A. as much as A. has then left, it is found that they have each 16*s.* How much had each originally? (Putting x for the number of shillings which A. had originally, and y for the number which B. had, their numbers after the successive changes are expressed by $x-y$ and $2y$, $2x-2y$ and $3y-x$, $3x-5y$ and $6y-2x$, $6x-10y$ and $11y-5x$. So that we have the two equations $6x-10y=16$, and $11y-5x=16$; whence we find $x=21$ and $y=11$.)

37. What two numbers are they, whose sum is twice their difference, and whose product is 12 times their difference? (Putting x for the less, $3x$ must express the greater (as appears from the equation $y+x=2y-2x$) and $3x^2$ their product. Therefore $3x^2=24x$, and $x=8$.)

38. What number is it (written with 2 digits) which is equal to 4 times the sum of its digits; and to which if 18 be added, the digits will be inverted? (§ 282.)

39. To find four numbers such, that the first with half the rest, the second with $\frac{1}{3}$ of the rest, the third with $\frac{1}{4}$ of the rest, and the fourth with $\frac{1}{5}$ of the rest may each of them equal 10? (Putting v, x, y, z for the numbers we have

have $v+\frac{x+y+z}{2}$, and $x+\frac{v+y+z}{3}$, and $y+\frac{v+x+z}{4}$, and $z+\frac{v+x+y}{5}$, all equal to each other: Therefore subtracting twice the first of these from 3 times the second, from 4 times the third, and from 5 times the fourth, we have $2x-v$, and $3y-v$, and $4z-v$, each equal to 0; and therefore $x=\frac{v}{2}$, $y=\frac{v}{3}$, $z=\frac{v}{4}$. Hence, substituting these values for x, y, z, in the equation $v+\frac{x+y+z}{2}=10$, we have $v+\frac{v}{4}+\frac{v}{6}+\frac{v}{8}=10$, or $\frac{37v}{24}=10$.)

40. To divide the number 90 into 4 such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2; the sum, difference, product, and quotient shall be all equal to each other? (Putting x for the first part, $x+4$ will express the second part, (for $y-2=x+2$) and $\frac{x}{2}+1$ the third part, (for $2z=x+2$) and $2x+4$ the fourth part, since $\frac{v}{2}=x+2$. And the sum of these four expressions $=90$)

41. If A. and B. together can perform a piece of work in 8 days; A. and C. together in 9 days; and B. and C. in 10 days; how many days will it take each person to perform the same work alone? (Putting x for the time in which A. would perform it alone, the times in which B. and C. would perform it alone are expressed by $\frac{8x}{x-8}$, & $\frac{9x}{x-9}$ according to the two first conditions. (For $x-8:8::x:\frac{8x}{x-8}$; and $x-9:9::x:\frac{9x}{x-9}$. See remarks on questions 19. and 20. page 66.) But B.'s time being $\frac{8x}{x-8}$, C.'s time according to the third condition is also expressed by $\frac{80x}{80-2x}$. (For $\frac{8x}{x-8}-10$, or $\frac{80-2x}{x-8}$, is to $10::\frac{8x}{x-8}:\frac{80x}{80-2x}$.) Therefore $\frac{80x}{80-2x}=\frac{9x}{x-9}$; whence we have $x=14\frac{34}{49}$, $\frac{8x}{x-8}=17\frac{23}{41}$, and $\frac{9x}{x-9}=23\frac{7}{31}$.)

22. A person

42. A person bought a number of oxen for £80; and if he had bought 4 more for the same money, he would have paid £1 less for each. How many did he buy? (Putting x for the number, we are told that $\frac{80}{x}$ exceeds $\frac{80}{x+4}$ by 1.)

43. What two numbers are they whose sum, product, and difference of their squares are all equal to each other? (Since their sum is equal to the difference of their squares, dividing the latter by the former must give 1 for the quotient, which is therefore equal to the difference of the numbers. § 163. Therefore putting x for the less, $x+1$ is the greater, $2x+1$ their sum, and x^2+x their product. So that $x^2+x=2x+1$: whence $x=\frac{\sqrt{5}}{2}+\frac{1}{2}$.)

44. To divide 6 into two such parts, that their product may be to the sum of their squares as 2 to 5? (Putting x and $6-x$ for the parts, their product is $6x-x^2$, and the sum of their squares is $2x^2-12x+36$: so that $6x-x^2 : 2x^2-12x+36 :: 2 : 5$.)

45. To find two numbers whose difference is 3, and the difference of their cubes 117. (Dividing x^3-y^3 by $x-y$, the quotient is x^2+xy+y^2.)

46. To find two numbers whose difference is 15, and half their product is equal to the cube of the smaller number? (Putting x and $x+15$ for the numbers, we have $\frac{x^2+15x}{2}=x^3$, which is depressed to a quadratic by dividing both sides by x.)

47. A person bought a number of sheep for £18. 15*s.* and selling them again at 30*s.* a-piece, gained by the bargain as much as 3 sheep had cost him. What was their number? (Putting x for the number, the amount of the sale was $30x$, and the profit $30x-375$. The cost of each sheep was $\frac{375}{x}$, and therefore of 3 sheep was $\frac{375\times 3}{x}=30x-375$.)

48. What number is it (written with two digits) which divided by the sum of its digits gives 8 for the quotient, and if 5 times the sum of the digits be subtracted from it, the digits will be inverted? (§ 282.)

 49. To

49. To find a number written with 3 digits in Arithmetical progression, and such that if divided by the sum of its digits the quotient is 59; and if 396 be subtracted from it, the digits will be inverted. (By the last of the conditions we know that 396 is 99 times the excess of the first digit above the last. See latter part of § 282. Therefore that excess is 4, and the difference of the series is 2, and the digits will be represented by x, $x-2$, and $x-4$; whose sum is $3x-6$, and the number written with those digits is expressed by $100x+\overline{10x-20}+\overline{x-4}=111x-24$. Therefore $\frac{111x-24}{3x-6}=59$.)

50. What two numbers are they, whose sum multiplied by the greater is equal to 77, and whose difference multiplied by the less is equal to 12? ($x^2+xy=77$, and $xy-y^2=12$.)

51. To find a number such, that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21? ($\overline{10-x}\times x=21$.)

52. To divide 24 into two such parts, that their product may be equal to 35 times their difference? ($\overline{24-x}\times x=\overline{24-2x}\times 35$.)

53. A. and B. having 100 eggs between them, and selling at different prices, received each the same sum for his eggs. If A. had sold as many as B. he would have received 18*d.* If B. had sold no more than A. he would have received only 8*d.* How many eggs had each? (Putting x for the number of A.'s eggs, B.'s number will be $100-x$. Now if A. had sold $100-x$ at the price he got, the amount would have been 18*d.*: therefore as $100-x : x :: 18 : \frac{18x}{100-x}$ the sum which A. received. In like manner the analogy, as $x : 100-x :: 8 : \frac{800-8x}{x}$, gives a just expression for the equal sum which B. received. Therefore $\frac{18x}{100-x}=\frac{800-x}{x}$.)

54. One bought 120 pounds of pepper, and as many of ginger, and had one pound of ginger more for a crown than of pepper; and the whole price of the pepper exceeded that of the ginger by 6 crowns. How many pounds of pepper had he for a crown, and how many of ginger? (Putting

(Putting x for the number of pounds of pepper which he had for a crown, the number of pounds of ginger will be $x+1$: the number of crowns which the pepper cost will be expressed by $\frac{120}{x}$, and which the ginger cost by $\frac{120}{x+1}$; the former of which exceeds the latter by 6.)

55. To find 4 numbers in Arithmetical progression, whose sum is 18, and the sum of their squares 86? (§ 290.)

56. A. sets off from Dublin to Belfast at the same time that B. sets off from Belfast to Dublin. Each travels uniformly the same road: but A. arrives at Belfast 4 hours after they have met, B. 9 hours after they have met. In what time did each perform his journey? (Putting x for the number of hours in which A. performed it, $x+5$ is B.'s number. Therefore the part which B. performs in 9 hours A. had performed in a shorter time, and that in the ratio of $x+5 : x$. Therefore A. had performed that part in $\frac{9x}{x+5}$ hours: and in 4 hours more he arrived at Belfast. Hence we have $x=\frac{9x}{x+5}+4$.)

57. What two numbers are they whose sum is $4\frac{1}{2}$, and the sum of their cubes $33\frac{3}{4}$? (§ 286. 287.)

58. —— whose sum is 5, and the sum of their 4th. powers 87? (§ 288.)

59. —— whose sum is $3\frac{1}{2}$, and the sum of their 5th. powers $39\frac{12}{32}$?

60. To find four numbers in Arithmetical progression, whereof the product of the extremes is 25, and the product of the means $49\frac{1}{2}$? (§ 289.)

61. To find 3 numbers in Arithmetical progression, whose sum is 9, and the sum of their squares $27\frac{1}{2}$? (§ 290.)

62. —— the sum of whose squares shall be 84, and their continued product 105? (§ 295.)

63. —— whose common difference is 3, and their continued product 308? (§ 296.)

64. To find 3 numbers in geometrical progression, whose sum is 13, and the sum of their squares 91? (§ 297.)

65. To find 4 numbers in geometrical progression, whereof the difference of the extremes shall be 78, and the difference of the means 18? (§ 298.)

66. To find 4 numbers in geometrical progression, whose sum shall be 15, and the sum of their squares 85? (§ 301.)

 CHAP.

CHAP. XXVII.

On Permutations and Combinations.

302. THE doctrine of *permutation*, or *alternation*, teaches us to find all the varieties of order, in which any number of different things may be arranged. Thus, the five first letters of the alphabet, (a, b, c, d, e) may be arranged in 120 different ways. For it is plain that any two of them, as a and b, may be arranged in two ways, either ab or ba. Therefore I say that any three of them, as a, b, and c, may be arranged in six (2×3) ways; for beginning the arrangement with any one of the three, the other two may follow in two different orders: thus, abc and acb, bac and bca, cab and cba. In like manner it appears that any four of them, as a, b, c, and d, may be arranged in 24 ($2 \times 3 \times 4$) different ways: for beginning with a, the other three may follow in 6 different orders; and we shall equally have six different arrangements beginning with b, or c, or d; therefore in all 24 different arrangements of the four letters. And just in the same way it is manifest that the five letters, a, b, c, d, and e, admit five times 24 different arrangements or permutations. And thus we see that the number of permutations of 5 different things is the continued product of 5, 4, 3, and 2; or 120:—of 6 different things is $6 \times 5 \times 4 \times 3 \times 2 = 720$: and universally that the number of permutations of n different things is $n \times \overline{n-1} \times \overline{n-2}$, &c. $\times 2$; or is the continued product of all the natural numbers from 2 to n. And thus it will be found that on a set of 10 bells there may be rung 3,628800 changes. And if we suppose ten changes to be rung in one minute, it would require 252 days to ring all the changes on 10 bells. But it will be found that all the changes on 12 bells could not be rung in 91 years.

303. Hitherto we have supposed all the terms, whose permutations we enquire, to be different. But let us now suppose that any of the terms are alike: for instance, let us enquire in how many different orders we may arrange the digits of the number 232234, among which six digits there are three 2.'s and two 3.'s. Here the rule for ascertaining the number of permutations is this: calculate as before what the number of permutations would be if all the

the terms were different; then the number of permutations which each set of like terms would admit if they were different; divide the former number by the product of the latter numbers, and the quotient will be the number of permutations sought. Thus in the present example, six different digits would admit 720 permutations: but of the six given digits there are three 2.'s, and three different terms admit 6 permutations; there are two 3.'s, and two different terms admit 2 permutations: therefore $\frac{720}{6\times 2}$, or 60, is the number of permutations which the digits of the number 232234 admit; or, with these digits we may express 60 different numbers. We shall proceed to exhibit the truth of this rule in a sufficient variety of instances, to establish it by induction.

304. If there be any number of terms all alike, as three or four or five a's, it is plain that they admit of but 1 arrangement; that is, the number of permutations which so many *different* letters would admit is to be divided by itself on account of their being all the same.—If we have any number of terms all of which but one are the same, they will admit just as many different arrangements as the number of the terms. Thus, four a's and one b, will admit five permutations; for we may begin or end with b, or interpose b among the a's in three different places;—*baaaa*, *aaaab*, *abaaa*, *aabaa*, *aaaba*. Now the number of permutations which 5 *different* letters admit is $5\times 4\times 3\times 2 = 120$; but we find that on account of *four* of the letters being the same, this number is to be divided by $4\times 3\times 2$, that is, by the number of permutations which four *different* letters admit.—Again, if we have any number (n) of terms all of which but two are the same, they will admit a number of permutations equal to $n\times\overline{n-1}$. Thus, four a's, one b, and one c (or 6 terms, of which four are alike) will admit 30 (6×5.) permutations. For we have proved that the four a's and the b admit 5 permutations: but in each one of these 5 arrangements (as *aaaab*) c may take 6 different positions, either in the beginning, or end, or four intermediate places. Therefore 6 times 5 must be the total number of permutations. But 6 *different* letters admit a number of permutations equal to $6\times 5\times 4\times 3\times 2$: and on account of 4 of the letters being the same we see that

that this number must be divided by $4 \times 3 \times 2$, that is, by the number of permutations which 4 *different* letters admit. —A similar reasoning will establish the rule, where all but three of the terms are alike; and in every case of this kind.—And we may hence infer the truth of the rule, where we have different sets of like terms: as if we have three *a*'s, two *b*'s, and one *c*. For on account of the three *a*'s being like terms, we have seen that the total number of permutations which six *different* letters would admit must be divided by 3×2; and that on account of the two *b*'s being like terms, it must be divided by 2: therefore on both these accounts together it must be divided by $3 \times 2 \times 2$.

305. We have hitherto in each permutation included *all* the given terms. But let us now enquire how many permutations may be formed, out of any number of given terms, in sets consisting each of some lower number: for instance, how many sets of 3 letters variously arranged we may form out of the 8 first letters of the alphabet. The number is $8 \times 7 \times 6 = 336$; or is the product of the natural numbers decreasing from 8 to three terms. And universally, let m be the number of different things given, and n the number to be taken at a time in each set, the number of different sets consisting each of n terms which may be formed out of m things is $m \times \overline{m-1} \times \overline{m-2}$, &c. continued to n terms. Let us now establish the truth of this rule. And first suppose there be 8 different letters, and each permutation is to consist of 2 letters. Any permutation may begin with any one of the 8 letters, and this may be followed by any one of the *remaining* 7 letters. Therefore the number of permutations in all is 8×7, or $m \times \overline{m-1}$. Then suppose that each set is to consist of 3 letters. It may begin, as before, with any one of the 8 letters, and this may be followed by as many different *sets* of 2 letters as can be formed out of the remaining 7 letters. But the latter number we have seen is 7×6. Therefore the number of sets of 3 letters variously arranged which can be formed out of 8 different letters is $8 \times 7 \times 6$, or $m \times \overline{m-1} \times \overline{m-2}$. And just in the same way it may be proved, that the number of sets of 4 letters each, which may be formed out of 8 different letters, is $8 \times 7 \times 6 \times 5 = 1680$, or the product of the terms of the series $m \times \overline{m-1}$, continued to

4 terms:

4 terms: for beginning with any one of the 8 letters, it may be followed by as many different sets of 3 letters as can be formed out of the remaining 7; and this number is, by the last case, $7 \times 6 \times 5$.—If the number in each set is to be only 1 less than the total number of given things, the number of sets will be the same with the number of permutations of the total number of things: or the number of sets consisting each of $m—1$ things, which may be formed out of m things, is the same with the number of permutations of m things.

306. As the *permutations* of any given things are the different orders in which they may be arranged, so the *combinations* of any given things are the different collections which can be formed out of them, without regarding the order of arrangement. Here no two sets are to consist of precisely the same things; but we do not consider a different arrangement of the same things as a distinct *combination*. Thus, let it be required to find how many combinations of 4 different letters may be formed out of the first 6 letters of the alphabet. Each combination, as *abcd*, admits 24 ($4 \times 3 \times 2$) *permutations*. (§ 302.) Therefore the total number of combinations must be the 24th. part of the total number of permutations of 4 letters which can be formed out of 6 different letters. But this latter number is $6 \times 5 \times 4 \times 3$. (§ 305.) Therefore the number of combinations sought is $\frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2} = 15$. And universally let m be the total number of different things given, n the number of them in each combination, the number of *permutations* consisting each of n things which may be formed out of m things is the product of the terms of the series m, $m—1$, &c, continued to n terms: and if this product be divided by $2 \times 3 \times 4 \ldots \times n$ (the number of permutations which n things admit) the quotient will be the number of combinations sought.

Questions for Exercise.

1. How many different numbers may be written with all the significant figures?

2. How often may a club of 7 persons place themselves at dinner in a different order?

3. How

3. How many different numbers may be written with two units, three 2.'s, four 3.'s, and five 4.'s?

4. How many numbers are there consisting each of four different digits?

5. How many changes may be rung with 3 bells out of 10?

6. Out of the letters a, b, c, d, e, x, y, z, how many different products may be obtained by the multiplication of two, of three, and of four factors?

CHAP. XXVIII.

On the Binomial Theorem. Extraction of the Cube and higher Roots.

307. WE have seen that the square of the binomial $x \pm a$ is $x^2 \pm 2xa + a^2$: and that its cube is $x^3 \pm 3x^2a + 3xa^2 \pm a^3$. If we multiply this by $x \pm a$, we shall have the 4th. power of that binomial root, and shall find it to be $x^4 \pm 4x^3a + 6x^2a^2 \pm 4xa^3 + a^4$. Multiplying this again by $x \pm a$, we find the 5th. power to be $x^5 \pm 5x^4a + 10x^3a^2 \pm 10x^2a^3 + 5xa^4 \pm a^5$. And in like manner the 6th. power of $x \pm a$ is found to be

$$x^6 \pm 6x^5a + 15x^4a^2 \pm 20x^3a^3 + 15x^2a^4 \pm 6xa^5 + a^6.$$

To find the higher powers by this process of continued multiplication would be very tedious; and in the powers already ascertained there are obvious circumstances appearing, which encourage us to investigate the law of their generation:—so much so indeed that I cannot but wonder the discovery was not earlier made. For 1st. we may observe that the number of terms in each series is one more than the index of the power: 2ndly. that in the powers of $x-y$ the signs are alternately *plus* and *minus*; while it is only in this circumstance they differ from the powers of $x+y$: 3rdly. that the first and last terms of each series are the correspondent powers of x and a; and that in the intermediate terms, consisting of combinations of x's and a's, the powers of x continually decrease, and the powers of a increase, by unity; so that in each term the sum of the indices

indices of x and a is equal to the index of the power of the binomial: 4thly. that in all the powers the coefficient of the first and last terms is 1, and the coefficient of the second and penultimate is the same with the index of the power; 5thly. that the series of coefficients proceeding from left to right and from right to left is the same. And 6thly. it may be remarked that the sum of the coefficients in any of the powers is equal to the corresponding power of 2. Thus in the square of $x+a$, the sum of the three coefficients is $4=2^2$: in the cube of $x+a$, the sum of the four coefficients is $8=2^3$, &c.

308. Thus it appears that the only thing remaining to be determined is—the coefficients of the intermediate terms between the second and penultimate. Returning now to the 6th. power of $x+a$, the two first terms are x^6+6x^5a. The coefficient of the second term, 6, or $\frac{6}{1}$, or $\frac{1\times6}{1}$, is the product of the coefficient of the first term multiplied by the index of x in that first term, and divided by the index of a in the second term. Now, in like manner, the coefficient of the third term $15x^4a^2$ is the product of the coefficient of the second term multiplied by the index of x in that second term, and divided by the index of a in the third term. For $\frac{6\times5}{2}=15$. And again, 20 the coefficient of the 4th. term is equal to $\frac{15\times4}{3}$, or is obtained by multiplying the coefficient of the 3rd. term by the index of x in it, and dividing the product by the index of a in the 4th. term. And this rule we shall find hold good in every other instance.

309. Let us now raise $x+a$ to the 7th. power, according to the principles which we have noticed. The literal parts of the eight terms must be

$$x^7+x^6a+x^5a^2+x^4a^3+x^3a^4+x^2a^5+xa^6+a^7$$

The coefficient of the 1st. term must be 1; of the 2nd. term $\frac{1\times7}{1}=7$; of the 3rd. term $\frac{7\times6}{2}=21$; of the 4th. term $\frac{21\times5}{3}=35$; of the 5th. term $\frac{35\times4}{4}=35$; of the 6th.

term

term $\frac{35\times 3}{5}=21$; of the 7th. term $\frac{21\times 2}{6}=7$; of the 8th. term $\frac{7\times 1}{7}=1$. But we need not have prosecuted the discovery of the coefficients beyond the 4th. term; as we have seen that the coefficients of the four latter terms must be the same with those of the first four in an inverted order. And thus we ascertain that the 7th. power of $x+a$ is

$$x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7$$

And this result will be found the same with that, which is obtained by multiplying the 6th. power of $x+a$ by $x+a$. In like manner the 7th. power of $x-a$ consists of precisely the same terms, but the signs of the 2nd. 4th. 6th. and 8th. terms negative.

310. We may now employ a general formula, putting n for the index of the power to which we want to raise the binomial $x+a$. The nth. power of $x+a$ will consist of $\overline{n+1}$ terms: of which the literal parts will be

$$x^n+x^{n-1}a+x^{n-2}a^2, \text{ \&c.} \ldots xa^{n-1}+a^n.$$

The numeral coefficients, or (as they are called) the *unciæ* of the terms will be 1, $\frac{n}{1}$, $\frac{n\times\overline{n-1}}{1\times 2}$, $\frac{n\times\overline{n-1}\times\overline{n-2}}{1\times 2\times 3}$, $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{1\times 2\times 3\times 4}$, &c. And this is the celebrated *binomial theorem* discovered (or first brought to perfection) by Sir Isaac Newton: according to which the *uncia*, or numeral coefficient of the mth. term will be ascertained by taking the continued product of the natural numbers decreasing from n and continued for $\overline{m-1}$ terms, and dividing that product by the continued product of the natural numbers decreasing from $\overline{m-1}$ to 2, or to unity. The literal part of the mth. term will be $x^{n-m+1}\times a^{m-1}$: and if the binomial root be $x-a$, the sign of the mth. term will be *minus* or *plus*, according as m is an even or an odd number. Thus in the 10th. power of $x-a$, the literal part of the 5th. term is x^6a^4, and its coefficient is $\frac{10\times 9\times 8\times 7}{4\times 3\times 2}$ $=210$.

$=210$. Therefore the 5th. term of $\overline{x-a}|^{10}$ is $+210x^6a^4$; but its 6th. term is $-252x^5a^5$.

311. After having thus explained the rule, and exhibited its truth in a sufficient number of instances to establish it by induction; let us now endeavour to investigate the reason, why things *must be* as we have seen they are. Now if we multiply together the 5 binomial factors, $x+a$, $x+b$, $x+c$, $x+d$, $x+e$, I say that the terms of the product must include every *combination* of 5 letters out of those 10, and no other combinations of letters. For if any one of those combinations, as *xxbde*, did not appear in the product, it is plain that one necessary term of it would be omitted: for the product may be considered as produced by multiplying $\overline{x+a}\times\overline{x+c}$ by $\overline{x+b}\times\overline{x+d}\times\overline{x+e}$; and it is plain that in the product of the two former factors *xx* is a necessary term, and that in the product of the three latter factors *bde* is a necessary term: therefore in the product of the five factors we must have the product of *xx* multiplied by *bde*; or *xxbde* is a necessary term. It is equally evident that no combination of fewer letters than 5, nor of more than 5, can appear in the product. Let us now suppose the second term of each binomial factor to be the same, that is, that each of the 5 factors is $x+a$; it is plain that all the possible combinations of 5 letters which can be formed out of these are *six*, viz. 1. the combination of five x.'s; 2. of four x.'s and one a; 3. of three x.'s and two a.'s; 4. of two x.'s and three a.'s; 5. of one x and four a.'s; 6. of five a.'s. And thus it appears that in the 5th. power of $x+a$, the number of terms must be 6, and that their literal parts proceed as we have described in § 307. the indices of x decreasing by unity from the index 5, and the indices of a similarly increasing.—Further, if each binomial factor be $x-a$, (instead of $x+a$) then the sign of the second, fourth, and sixth terms must be *minus*; since in these terms the index of a is an odd number, and any odd power of a negative root is necessarily negative. —The student will observe that all the same reasoning, which we employ for determining the *fifth* power of $x\pm a$, is equally applicable to any other power.

312. Let us now return to the continued multiplication of the 5 binomial factors $x+a$, $x+b$, $x+c$, $x+d$, $x+e$. We see that the first term of the product will consist of a combination

combination of 5 x.'s, or will be x^5. This will be followed by all the possible combinations of 4 x.'s with some one of the 5 letters a, b, c, d, e. But is plain that the number of these combinations is *five*. These will be followed by all the possible combinations of 3 x's with some two of the 5 letters a, b, c, d, e. But the number of these is $\frac{5\times 4}{2}=10$; for (by § 306.) this is the number of combinations of 2 that can be formed out of those 5 letters. We shall next have all the possible combinations of 2 x.'s with some three of the five letters a, b, c, d, e. But the number of combinations of 3 letters which can be formed out of these 5 being $\frac{5\times 4\times 3}{3\times 2}=10$, the same must be the number of those terms of the product in which only two x.'s are combined with three other letters. These in like manner will be followed by all the possible combinations of one x with some four of the other 5 letters: and it appears from the same principles of § 306. that the number of these is $\frac{5\times 4\times 3\times 2}{4\times 3\times 2}=5$. And lastly we shall have one combination of the 5 letters $abcde$. Now when the second term in each of the binomial factors is the same, or where all the factors are $x+a$, the *five* combinations in which 4 x.'s appear become each of them x^4a: and therefore $5x^4a$ must be the second term in the 5th. power of $x+a$. The *ten* terms in which 3 x.'s appear become each of them x^3a^2: and therefore the third term must be $10x^3a^2$. And in like manner it appears that the three following terms are $10x^2a^3$, $5xa^4$, and a^5.—By a perfectly similar process of reasoning, putting n for the index of the power, it appears that the first term of the nth. power of $x+a$ is x^n; the second term $nx^{n-1}a$; the third term $\frac{n\times \overline{n-1}}{2}x^{n-2}a^2$, &c. For in the third term, for instance, the literal part must consist of a combination of a number of x.'s less by 2 than n with two a.'s; and the number of these combinations, or the numeral coefficient of the third term, must be equal to the number of combinations of *two* which can be formed out of n things. But this by § 306. is $\frac{n\times \overline{n-1}}{2}$.— Lastly, we have seen that the sum of the coefficients of the 5th. power

power of $x+a$ is equal to the number of all the different terms composing the product of $\overline{x+a}\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}\times\overline{x+e}$. But from the nature of multiplication the number of terms in that product must be 2^5 or 32. For the two first factors must give a product consisting of 4 (2^2) terms; and that multiplied by the third factor must give a product consisting of 8 (2^3) terms; and the product of this multiplied by the fourth must consist of 16 (2^4) terms; and this multiplied by the fifth factor must give a product consisting of 32 (2^5) terms.—In like manner it appears that the sum of the coefficients in the nth. power of $x+a$ must be 2^n.

313. We have thus strictly demonstrated the *binomial theorem* for raising a binomial $x+a$ to any power, as far as we have hitherto applied it; namely, where the index of the power is integral and affirmative. But what is most striking and importantly useful in this theorem is, that it is applicable also to those powers whose indices are *fractional* or *negative*. This part of the subject we cannot attempt to treat minutely in the present elementary treatise: but we shall just present it to the attention of the student by a few examples. Let it be recollected that the square root of $x+a$, or $\sqrt{x+a}$, may be expressed as the power of $x+a$, whose index is $\frac{1}{2}$, thus $(x+a)^{\frac{1}{2}}$; and that the expressions x^{-1}, x^{-2}, &c. are equivalent with $\frac{1}{x}$, $\frac{1}{x^2}$, &c. See Chap. 22. Now if we apply the binomial theorem for determining the power of $x+a$ whose index is $\frac{1}{2}$, we shall find produced an infinite series, which continually *approximates* in value to the square root of $x+a$. According to the formula, or the principles laid down in § 310. the first term of the series must be $x^{\frac{1}{2}}$, or $\sqrt{x}$. The coefficient of the second term must be $\frac{1}{2}$, and its literal part the product of a into that power of x whose index is $\frac{1}{2}-1=-\frac{1}{2}$. But $x^{-\frac{1}{2}}=\frac{1}{\sqrt{x}}$. Therefore the second term is $\frac{a}{2\sqrt{x}}=\frac{a\sqrt{x}}{2x}$. The coefficient of the third term must be $\frac{1}{2}\times-\frac{1}{2}\div2$, that is $-\frac{1}{8}$; and its literal part the product of a^2 into that power of x whose index is $\frac{1}{2}-2=-\frac{3}{2}$; that is

a^2

$\frac{a^2}{\sqrt{x^3}}$, or $\frac{a^2\sqrt{x}}{x^2}$. Therefore the third term is $-\frac{a^2\sqrt{x}}{8x^2}$. The coefficient of the 4th. term must be $-\frac{1}{8}\times\overline{-\frac{3}{2}}\div 3=\frac{1}{16}$; and of the fifth term must be $\frac{1}{16}\times\overline{-\frac{5}{2}}\div 4=-\frac{5}{128}$. The literal parts of the fourth and fifth terms must be the products of a^3 and a^4 into those powers of x whose indices are $\frac{1}{2}-3$ $(=-\frac{5}{2})$ and $\frac{1}{2}-4$ $(=-\frac{7}{2})$; that is, must be $\frac{a^3}{\sqrt{x^5}}$ and $\frac{a^4}{\sqrt{x^7}}$, or $\frac{a^3\sqrt{x}}{x^3}$ and $\frac{a^4\sqrt{x}}{x^4}$. Therefore the fourth term is $+\frac{a^3\sqrt{x}}{16x^3}$; and the fifth term is $-\frac{5a^4\sqrt{x}}{128x^4}$: and so on. It is plain that the series can never terminate, as the negative values of $n-1$, $n-2$, &c. continually increase: but the farther we continue the series the more nearly we approximate to the value of the square root of $x+a$. Further, that we have not been led astray by any fanciful analogy in considering that square root as the power whose index is $\frac{1}{2}$, and applying the binomial theorem to expand that power into the form of a series, we may be convinced by proceeding to extract the square root, according to the rule given in § 209. For continuing that process, we shall find precisely the same series

$$\sqrt{x}+\frac{a}{2\sqrt{x}}-\frac{a^2}{8\sqrt{x^3}}+\frac{a^3}{16\sqrt{x^5}}-\frac{5a^4}{128\sqrt{x^7}}, \text{ \&c.}$$

$$\text{or, } \sqrt{x}+\frac{a\sqrt{x}}{2x}-\frac{a^2\sqrt{x}}{8x^2}+\frac{a^3\sqrt{x}}{16x^3}-\frac{5a^4\sqrt{x}}{128x^4}, \text{ \&c.}$$

314. It appears from the latter form of the series that, if x be a square number, all the terms of the series will be rational. Suppose $x=4$, and $a=1$: then $x+a=5$; $\sqrt{x}=2$; and all the powers of $a=1$. Therefore $\sqrt{5}=2+\frac{2}{8}-\frac{2}{128}+\frac{1}{512}-\frac{10}{128\times 256}$, &c. Now the square of the two first terms exceeds 5 by $\frac{1}{16}$: but the square of the three first terms is less than 5 only by the fraction $\frac{31}{4096}$. But instead of seeking greater accuracy in our root by summing up a greater number of terms in the series, it is better to change our numeral substitutions for x and a, by taking a square number nearer to 5 than 4 is. Now the square of $2\frac{1}{4}$ (the two first terms of the last series) or $\frac{81}{16}$ is only $\frac{1}{16}$th.

greater

greater than 5. Resolving 5 therefore into $\frac{81}{16}-\frac{1}{16}$, and expanding the square root of this binomial into a series by the binomial theorem, or the formula at the end of the last section, the two first terms of the series are $\frac{9}{4}-\frac{2}{144}$, or $\frac{162}{72}-\frac{1}{72}=\frac{161}{72}$. Now this fraction is so near the square root of 5, that its square exceeds 5 only by $\frac{1}{5184}$, or is true to the fifth place of decimals. And if we wish for greater accuracy, it may be attained by resolving 5 into $\left.\frac{161}{72}\right|^2-\frac{1}{5184}$, or $\frac{25921}{5184}-\frac{1}{5184}$. And as any number may be divided into two parts, one of which shall be a square number, it is plain that we may thus approximate to the square root of any number whatsoever: tho' the facility of continuing the process of extraction decimally makes it superfluous to apply the binomial theorem in practice to this purpose.

315. But let us now by a similar process investigate the cube root of $x+a$, or $(x+a)^{\frac{1}{3}}$. Here the first term of the series is $x^{\frac{1}{3}}$; and the coefficient of the second term is $\frac{1}{3}$. The index of x in the second term is $\frac{1}{3}-1=-\frac{2}{3}$; and the index of a is 1. Therefore the second term is $\frac{a}{3\sqrt[3]{x^2}}$ or $\frac{a\sqrt[3]{x}}{3x}$. The coefficient of the third term is $\overline{\frac{1}{3}\times-\frac{2}{3}}\div 2=-\frac{1}{9}$; and its literal part is the product of a^2 into that power of x whose index is $\frac{1}{3}-2=-\frac{5}{3}$. Therefore the third term is $-\frac{a^2\times\sqrt[3]{x}}{9x^2}$. The coefficient of the fourth term is $\overline{-\frac{1}{9}\times-\frac{5}{3}}\div 3=\frac{5}{81}$; and its literal part is the product of a^3 into that power of x whose index is $\frac{1}{3}-3=-\frac{8}{3}$. Therefore the fourth term is $\frac{5a^3\times\sqrt[3]{x}}{81x^3}$. And so on. Now from this formula

$$\sqrt[3]{x}+\frac{a\times\sqrt[3]{x}}{3x}-\frac{a^2\times\sqrt[3]{x}}{9x^2}+\frac{5a^3+\sqrt[3]{x}}{81x^3}, \text{ \&c.}$$

(in

(in which all the terms will be rational if x be a cube number) we may approximate to the cube root of any number whatsoever. Thus if we want to extract the cube root of 5, we must divide it into two parts, one of which shall be a cube number, and as near as we can obtain it in value to 5. Now the cube root of 5 evidently lying between 1 and 2, and nearer to 2 than to 1, we may put $\sqrt[3]{x}=\frac{17}{10}$, and therefore $a=5-\overline{\frac{17}{10}}\big|^3=\frac{5000-4913}{1000}=\frac{87}{1000}$. Then the 2nd. term $\frac{a\times\sqrt[3]{x}}{3x}=\overline{\frac{87}{1000}\times\frac{17}{10}}\div\frac{4913\times3}{1000}=\frac{87\times17}{3\times49130}=\frac{493}{49130}$. And the sum of the two first terms $\frac{17}{10}+\frac{493}{49130}=\frac{83521+493}{49130}=\frac{84014}{49130}$; which exceeds the true root by less than .00006. We might approximate still nearer at pleasure, either by calculating the value of more terms of the series, or by putting $\sqrt[3]{x}=\frac{84014}{49130}$ or = the nearly equivalent fraction $\frac{84}{49}$. And in this manner we may approximate to the 4th. 5th. or any of the higher roots of any assigned number. But for this approximation to the cube root another and much more convenient formula will be assigned in § 319.

316. The binomial theorem may similarly be applied to the calculation of powers whose indices are negative. Thus $(x+a)^{-1}=\frac{1}{x+a}=\frac{1}{x}-\frac{a}{x^2}+\frac{a^2}{x^3}-\frac{a^3}{x^4}$, &c. this being the series into which the fraction $\frac{1}{x+a}$ is expanded by actual division. See Chap. 17. But we shall have the very same series, if we calculate the value of $(x+a)^{-1}$ by the binomial theorem. For then the first term of the series must be $x^{-1}=\frac{1}{x}$. The coefficient of the second term must be -1; and its literal part the product of a into x^{-2}, or into $\frac{1}{x^2}$. Therefore the second term must be $-\frac{a}{x^2}$. The coefficient of the third term must be $\frac{-1\times-2}{2}=1$; and its literal

part

part the product of a^2 into x^{-3}, or into $\frac{1}{x^3}$. Therefore the third term is $+\frac{a^2}{x^3}$. The coefficient of the fourth term must be $\frac{1\times-3}{3}=-1$. And so on.

317. In like manner if we expand $(x+a)^{-2}$ into an infinite series by the binomial theorem, the first term is $x^{-2}=\frac{1}{x^2}$. The coefficient of the second term is -2; and its literal part the product of a into x^{-3}, or into $\frac{1}{x^3}$. Therefore the second term is $-\frac{2a}{x^3}$. The coefficient of the third term is $\frac{-2\times-3}{2}=3$; and its literal part is $\frac{a^2}{x^4}$: so that the third term is $+\frac{3a^2}{x^4}$. In like manner the fourth and fifth terms are found to be $-\frac{4a^3}{x^5}$ and $\frac{5a^4}{x^6}$: and so on. And universally expanding $(x+a)^{-n}$ into an infinite series by the binomial theorem we find the series

$$*\ \frac{1}{x^n}-\frac{na}{x^{n+1}}+\frac{n\times\overline{n+1}\times a^2}{2x^{n+2}}-\frac{n\times\overline{n+1}\times n+2\times a^3}{2\times 3x^{n+3}},\ \&c.$$

And by this formula we may calculate the value of any such fractions as $\frac{1}{(x+a)^n}$, or $\frac{1}{\sqrt[n]{x+a}}$, or $\frac{b}{\sqrt[n]{(x+a)^m}}$.—The truth of this formula may be thus established. Since any fraction multiplied by its reciprocal gives 1 for the product, unity must be the product of $\frac{1}{(x+a)^n}\times(x+a)^n$. The latter

* In *Euler*'s Algebra Vol. I. p. 179 (2nd. Ed. Lond. 1810) there is a material error in the delivery of this formula. In the numerators of the terms, instead of $n+1$, $n+2$, &c. they are given $n-1$, $n-2$, &c. which neither corresponds with the result of the binomial theorem, nor with the particular cases before exhibited.

 by

by the binomial theorem is equal to

$$x^n + nx^{n-1}a + \frac{n \times \overline{n-1}}{2} \times x^{n-2}a^2, \text{ \&c.} =$$

$$x^n + nax^{n-1} + \frac{n^2a^2 - na^2}{2} \times x^{n-2}, \text{ \&c.}$$

Now if we multiply the terms of this formula for $(x+a)^n$ by the terms of the formula for its reciprocal, we shall find the product of the two first terms to be 1, and the several products of the other terms successively destroying each other. Let us exhibit this in a trinomial of each formula, as it will afford a useful praxis to the student: and let him recollect that powers of the same root are multiplied or divided by adding or subtracting their indices.

$$\begin{array}{lllllll}
\text{Multiply} & \frac{1}{x^n} & - \frac{na}{x^{n+1}} & + \frac{n^2a^2}{2x^{n+2}} & & & \\
\text{by} & x^n & + nax^{n-1} & + \frac{n^2a^2x^{n-2}}{2} & & & \\
\hline
& 1 & - \frac{na}{x} & + \frac{n^2a^2}{2x^2} & & & \\
& & + \frac{na}{x} & - \frac{n^2a^2}{x^2} & + \frac{n^3a^3}{2x^3} & & \\
& & & + \frac{n^2a^2}{2x^2} & - \frac{n^3a^3}{2x^3} & + \frac{n^4a^4}{4x^4} & \\
\hline
& 1 & * & * & * & + \frac{n^4a^4}{4x^4} &
\end{array}$$

Thus all the terms have disappeared except 1, and the product of the two last terms of the trinomial factors: which would in like manner be destroyed by the following terms, if we took another term of each formula.

318. The rule commonly given in the systems of Arithmetic for the extraction of the cube root directs to an operation so extremely tedious and troublesome, that it is of little or no practical utility. It may be needful however to make a few remarks on the grounds of the operation. It depends upon the constitution of the cube of the binomial $a+x$, namely $a^3+3a^2x+3ax^2+x^3$. The cube root of the first term of this is the first term of the root; and

3 times

3 times its square dividing the second term, $3a^2x$, gives the second term of the root. If there be more terms than two terms in the root, for instance if we have to extract the cube root of

$$x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6$$

—after determining the first term of the root x^2, we divide the second term $6x^5a$ by $3x^4$. The quotient $2xa$ is the second term of the root. Now considering the two terms found, x^2+2xa, as the ascertained part of the root, we subtract the cube of that binomial, $x^6+6x^5a+12x^4a^2+8x^3a^3$, from the given cube. The remainder is $3x^4a^2+12x^3a^3$, &c. the first term of which we divide by $3x^4$, and the quotient a^2 is the third term of the root. And the extraction is complete, since the cube of $x^2+2xa+a^2$ is found to be just equal to the assigned cube. By a similar process we may proceed in the extraction of the 4th. root of any assigned quantity, (arranged according to the powers of some one letter) by taking the 4th. root of its first term for the first term of the root, and dividing the second term by 4 times the cube of this, for finding the second term of the root. Subtracting then the 4th. power of the two parts of the root found from the given quantity, we divide the first term of the remainder by 4 times the cube of the first term of the root for determining the third term of the root. And we may proceed similarly in extracting any higher roots.

319. But to extract the cube root of 5, for instance, to 6 decimal places by such a process would be insufferably tedious: and we may effect the object with little comparative trouble by the following formula. Let a be any number, whose cube root we desire to extract. Assume r^3 a perfect cube, as near as may be to a, either greater or less. Then $\dfrac{2a+r^3}{a+2r^3}\times r=\sqrt[3]{a}$ nearly. Suppose we want to find the cube root of 5: we are in the first place to assume a perfect cube number as near as may be to 5; and the nearer we approximate to 5 in our substitution for r^3 the more accurate will be the result of the formula. Now the cube root of 5 lying between 1 and 2, we might try $\frac{3}{2}$, $\frac{5}{3}$, and $\frac{7}{4}$, as approximations to its root: but of these $\frac{7}{4}$ is the

 nearest.

nearest. (For the cube of $\frac{3}{2}=\frac{27}{8}$, less than 5 by $\frac{13}{8}$; the cube of $\frac{5}{3}=\frac{125}{27}$, less than 5 by $\frac{10}{27}$; the cube of $\frac{7}{4}=\frac{343}{64}$, more than 5 by $\frac{23}{64}$.) Assuming then $r=\frac{7}{4}$, $r^3=\frac{343}{64}$, $a=5$ $=\frac{320}{64}$; we have $\frac{2a+r^3}{a+2r^3}\times r=\frac{983}{1006}\times\frac{7}{4}=\frac{6881}{4024}=1.709990$, which is the true root to the 5th place of decimals, and exceeds the true root by little more than $\frac{1}{50000}$. If instead of $\frac{7}{4}$, we put $\frac{69}{40}$ for r (which is $\frac{1}{40}$ less than $\frac{7}{4}$, though its cube be still somewhat greater than 5) the same formula would afford us $\sqrt[3]{5}=\frac{66827121}{39080721}=1.709976$; which is accurately true at least to the 7th. place of decimals.—Again, to extract the cube root of 131, putting $r^3=125$, $r=5$, we have $\frac{262+125}{131+250}\times 5=\frac{387}{381}\times 5=\frac{1935}{381}=5.0787\bar{4}$ = nearly the cube root of 131, being true to the 5th. place of decimals. And we may approximate nearer by putting $r=\frac{193}{38}$, or rather $=\frac{507}{100}$.—But I would recommend that the number originally assumed for r^3 should be taken sufficiently near the given number, to prevent the necessity of a repeated operation. Thus instead of assuming r^3 $=5^3=125$, which is less by 6 than the given number, let us assume $r=\frac{51}{10}$, and therefore $r^3=\frac{132651}{1000}$, which exceeds 131, or $\frac{131000}{1000}$, only by $\frac{1651}{1000}$; and we shall find the result of the formula in one operation to be $\sqrt[3]{131}=$ 5.0.78753, which is true to at least the 7th. place of decimals.

320. Having thus shewn how we may approximate at pleasure to the cube root of any assigned number not a perfect cube; I shall only add that the roots of perfect cube numbers, up to *one billion*, may be ascertained with much facility in the following manner. We at once know the number of digits in the root, by pointing off the number in periods of 3 figures from the right hand, and reckoning as a period the left hand digits thus cut off, whether they be one, two, or three. Thus, if the cube consist of 4, 5, or 6 digits, its cube root must consist of 2 digits; if the cube consist of 7, 8, or 9 digits, its root must

must consist of 3 digits; if the cube consist of 10, 11, or 12 digits, its root must consist of 4 digits. (The reason of this will appear just as we ascertained the number of digits in the *square root*. § 206.) The first period of the cube determines the first digit of the root to be that, whose cube is next below that period. The last digit of the cube determines the last digit of the root to be that, whose cube ends with that digit: for there are no two digits whose cubes end with the same digit. Thus if 300,763 be proposed as a perfect cube, we at once know that its root is 67, as the cube of 6 (or 216) is the nearest cube number below 300, and 7 is the only digit whose cube ends with 3. But we might otherwise determine the first digit of this root to be 6: thus—Subtract the penultimate digit of 7^3 (or 343) from 6 the penultimate digit of the given cube. The remainder is 2. Then consider what multiple of 7, the last digit of 3×7^2, ends with 2: and 42 being 6 times 7, this determines the *penultimate* digit of the root to be 6. Thus again 5,451,776 being proposed as a cube number, the first digit of its root is necessarily 1, and the last 6; and the penultimate digit is necessarily 7: for subtracting 1, the penultimate digit of 216 (6^3) from 7 the penultimate digit of the given cube, the remainder is 6; but 3×6^2 ends with 8; and 7 or 2 is the only digit which multiplying 8 gives a product ending with 6. We fix upon 7, as the root sought is evidently nearer 200 than 100.—From the two last digits of the root being 76, we might determine that the first, or antepenultimate, digit is 1. For $76^3 = 438976$: and subtracting 9, the antepenultimate digit of this number from 7 (or 17) the corresponding digit of the given cube, the remainder is 8; and 1 (or 6) is the only digit which multiplying 8 (the last digit of 3×6^2) gives a product ending with 8.—Thus again, 3,086,626,816 being proposed as a cube number, the last digit of the root is 6, and subtracting 1, the penultimate digit of 6^3, from 1, the remainder is 0; and 5 being the only digit which multiplying 8 (the last digit of 3×6^2) gives a product ending with 0, 5 must be the penultimate digit of the root. Now $56^3 = 175616$; and subtracting 6 the penultimate digit of 56^3 from 8 the corresponding digit of the given cube, the remainder is 2. Therefore 4 is the antepenultimate digit of the root, as 4 is the only digit which multiplying 8 gives a product ending with 2. But 1 is the first digit of the root. Therefore the cube root sought is 1456.

TABLES

TABLES

I. *Of English Money.*

4 Farthings = 1 Penny. 4 Pence = 1 Groat. 12 Pence = 1 Shilling. 5*s.* = 1 Crown. 20 Shillings = 1 Pound Sterling. 21 Shillings = 1 Guinea. (6*s.* 8*d.* = 1 Noble. 10*s.* = 1 Angel. 13*s.* 4*d.* = 1 Mark.—In Ireland the value of the Penny is less in the ratio of 13 : 12.—Scots Money is divided in the same manner as English; but has one twelfth of its value. Thus a Pound Scots = 1*s.* 8*d.*

II. *Some Foreign Coins, or Denominations of Money reduced to English.*

A Florin = 1*s.* 6*d.* a Ducat = 9*s.* 3*d.* a Guilder (= 20 Stivers) = 1*s.* 9*d.* a Rix-dollar (= 50 Stivers) = 4*s.* 4½*d.* a Ruble (= 100 Copecs) = 4*s.* 6*d.* a Sol (= 12 Deniers) = ½*d.* a Livre Tournois (=20 Sols) = 10*d.* a French Pistole (= 10 Livres) = 8*s.* 4*d.* a Louis d'Or (= 24 Livres) = 1£. a Milre = 5*s.* 7½*d.* a Spanish Dollar (= 10 Rials) = 4*s.* 6*d.* a Spanish Pistole (= 36 Rials) = 16*s.* 9*d.* a Sequin = 7*s.* 6*d.* a Rupee = 2*s.* 6*d.* a Gold Rupee (= 4 Pagodas) = £1. 15*s.*

III. *Some ancient Coins, or Denominations of Money, reduced to English.*

Drachma (= 6 Oboli) = 7¾*d.* a Mina (= 100 Drachmæ) = £3. 4*s.* 7*d.* a common, or Attic Talent (= 60 minæ) = £193. 15*s.*—(Note—the Mina and Talent are properly denominations of *weight.*)—a Golden Stater (= 25 Drachmæ) = 16*s.* 1¾*d.* A Denarius (= 10 Asses = 4 Sestertii) = 7¾*d.*

IV. *English Weights—Avoirdupois.*

1 Ounce = 16 Drams. 16 oz. = 1 Pound. 28 lbs. = 1 Quarter. 112 lbs. (= 4 Qrs.) = 1 Hundred. 20 Cwt. = 1 Ton.

V. *Troy*

V. *Troy Weight—used for weighing Gold, Silver, Jewels, Silk, and all Liquors.*

24 Grains = 1 Penny-weight (dwt.) 20 dwts. = 1 Ounce. 12 oz. = 1 Pound.—The following also used by Apothecaries in compounding their medicines, 20 Grains = 1 Scruple. 3 Scruples = 1 Dram. 8 Drams = 1 Ounce. —Note,—the troy Pound is to the Avoirdupois Pound nearly as 88 : 107. The Troy Ounce is to the Avoirdupois Ounce nearly as 80 : 73.

The Paris Pound = 1 lb. 3 oz. 15 dwts. Troy.

The Paris Ounce = 19 dwts. $16\frac{1}{2}$ gr. Troy.

The Roman Libra (= 12 Unciæ) = 10 oz. 18 dwts. 14 gr. Troy, nearly.

The Roman Uncia = the English Avoirdupois Ounce.

The Attic Drachma = 2 dwts. 17 gr. nearly.

The Attic Mina (= 100 Drachmæ) = 1 lb. 1 oz. 10 dwts. 10 gr.

The Attic Talent (= 60 Minæ) 67 lb. 7 oz. 5 dwts. Troy.

VI. *Measures of Length.*

12 Inches = 1 Foot. 3 Feet = 1 Yard. 2 Yards = 1 Fathom. $5\frac{1}{2}$ Yards = 1 Pole. 40 Poles (= 220 Yards) = 1 Furlong. 8 Furlongs (= 1760 Yards) = 1 Mile. 3 Miles = 1 League.—The Irish Mile = 2240 Yards; and therefore is to the English as 14 : 11.

The Roman Foot = $11\frac{3}{5}$ Inches nearly.

5 Roman Feet = 1 Passus. 125 Passus = 1 Stadium. 8 Stadia (= 1000 Passus) = 1 Milliare: which was therefore to the English Mile as 967 : 1056; or nearly as 23 : 25.

The Grecian Foot exceeded the English by nearly $\frac{1}{12}$ of an Inch.—The Persian Parasang = 30 Stadia.

A French League = $2\frac{1}{4}$ English Miles nearly.

A Toise = 6 French Feet, or $6\frac{2}{5}$ English Feet nearly.

A German Mile = 4 English.—A Russian Verst = $\frac{3}{4}$ Do.

In measuring Cloth, &c. $2\frac{1}{4}$ Inches = 1 Nail; and therefore 4 Nails = 1 Quarter of a Yard. 3 Quarters = 1 Ell Flemish. 5 Quarters = 1 Ell English. 4 Quarters, $1\frac{1}{5}$ Inch. = 1 Ell Scots.

In

In Land-measuring, a Perch = $16\frac{1}{2}$ Feet in Length: of which 40 in Length and 4 in Breadth make an English Statute Acre = 43560 Square Feet = 4840 Square Yards = 160 Square Poles = 4 Roods.—The Irish Acre exceeds the English by 2 Roods $19\frac{1}{4}$ Perches nearly.—The French *arpent* contains $1\frac{1}{4}$ English Acre.

VII. *Measures of Capacity—for Liquids.*

2 Pints = 1 Quart. 4 Quarts = 1 Gallon.—In *Ale* and *Beer*, 36 Gallons = 1 Barrel. $1\frac{1}{2}$ Barrel (= 54 Gallons) = 1 Hogshead. 2 Barrels = 1 Puncheon. 2 Hogsheads = 1 Butt. 2 Butts = 1 Tun.—In *Wine*, *Spirits*, &c. 42 Gallons = 1 Tierce. $1\frac{1}{2}$ Tierce (= 63 Gallons) = 1 hogshead. 2 Tierces = 1 Puncheon. 2 Hogsheads = 1 Pipe. 2 Pipes = 1 Tun.—Note—the *Ale* Gallon contains 282 cubic Inches; the *Wine* Gallon 231.

The Roman Cyathus = $\frac{1}{12}$ Pint, Wine Measure: the Hemina (= 6 Cyathi) = $\frac{1}{2}$ Pint: the Sextarius = 1 Pint: the Congius = 7 Pints: the Urna = 3 Gallons $4\frac{1}{2}$ Pints: the Amphora = 7 Gallons 1 Pint.

The Attic Cyathus = $\frac{1}{12}$ Pint: the Cotyle = $\frac{1}{2}$ Pint.

VIII. *Dry Measure.*

2 Pints = 1 Quart. 2 Quarts = 1 Pottle. 2 Pottles = 1 Gallon. 2 Gallons = 1 Peck. 4 Pecks = 1 Bushel. 8 Bushels = 1 Quarter. 5 Quarters = 1 Wey. 2 Weys = 1 Last.—Note—the Winchester Bushel contains 2250 Cubic Inches.

The Roman Modius = 1 Peck, or 2 Gallons.

The Attic Chœnix = 1 Pint: the Medimnos = 1 Bush. 3 Quarts.

IX. *Time.*

60 Seconds = 1 Minute. 60 Minutes = 1 hour. 24 Hours = 1 Day. 7 Days = 1 Week. $365\frac{1}{4}$ Days = 1 Julian Year = 52 Weeks, 1 Day, 6 Hours:—The Solar Year = 365 Days, 5 Hours, 4 minutes, 48 Seconds.

FINIS.

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CHAP.

*CHAP.

☞ The Chapters marked with an Asterisk may be omitted by the Student, in the first reading; as well as the passages included between Crotchets [ ].

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J. H.