PLANE GEOMETRY

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PREFACE

This text represents the experience and developing ideals of a quarter of a century of geometry teaching. The philosophy and the methods set forth in it are those of the author's own class-room. He believes that a pupil learns more of the subject when it is presented as in these pages, and he is certain that every pupil who is taught by this method under the guidance of a teacher who is himself interested in his subject, becomes inspired to learn and is willing to work to learn. An awakening of a love of the subject is a service that may be rendered to our pupils of every intellectual grade. The text is so arranged that it may be variously used.

A Short Course.—From the complete material presented in this text, what may be considered as the fundamentals, or as constituting a *minimum requirement*, has been designated by a BLACK FACE paragraph numeral. It constitutes a *Short Course.*

As much as possible of the remaining material should be included. This remaining material, which completes and rounds out the subject, is designated by a LIGHT FACE paragraph numeral.

A Laboratory Course.—Chapter 1 and the chapters which contain the *Experimental Determination of Principles* may be used as a laboratory course, omitting all of the chapters containing the *Classification and Explanation of Principles* except the most elementary portions.

A Complete Course.—The text is primarily intended to be used as a whole, without many omissions. It contains all that need be required of any class. The experimental chapters should be rigorously required of all, and the *accurate* or *free-hand construction* of all figures *should be insisted upon throughout* the portions devoted to *classification and explanation of principles.*

Exercises.—A sufficient number of exercises, carefully selected and graded, will be found in the list of *Additional Theorems*, *Exercises* and *Review Exercises* to enable the teacher to vary his class assignments from term to term for many years. No class should be expected to solve all.

For Exceptional Pupils.—The wealth of material presented affords a splendid means of developing the pupil of exceptional ability. Extra work for additional credit may be allowed every pupil. It is better to stimulate a pupil to greater interest and accomplishment, especially in a subject so vast and offering so great rewards as geometry, than to hurry an able pupil through the subject, almost always to his loss rather than to his profit.

Dedication.—The conception of this work runs far back in the years. It is the author's contribution in the service of his profession. It is dedicated to all those many students of the grand old science, who perhaps unknowingly, and to all those teachers who have keenly felt the need of a more sympathetic interpretation of its immortal truths.

In the hope that it will influence better teaching and learning, and will help to upbuild and uphold a generation which will reverence our beloved geometry for its own sake, as well as for its uses inseparable from a great constructive civilization—the author sends this little volume forth upon its mission.

C. Addison Willis.

Girard College, Philadelphia.

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CHAPTER I

FIRST PRINCIPLES

1. Note-book.—Preserve all drawings and written work in a loose leaf note-book with unruled pages. Number to correspond with the text. Draw large, accurate figures with distinct lettering.

2. Drawing Instruments.—Each pupil should be provided with one six-inch triangle with 30°, 60° and 90° angles, a six-inch scale divided decimally, a protractor, and a compass.

3. How Geometry Began.—Many ancient nations, the Chinese, Babylonians, Egyptians, Greeks, Romans, and others, constructed wonderful works, as roads, canals, aqueducts, temples, pyramids, and tombs. They laid out cities and gardens. These required appreciation of geometric principles. Herodotus, a Greek historian who wrote about 450 B.C., states that the Egyptians developed many geometric principles because the annual overflow of the Nile obliterated the boundary marks of the lands, and it became necessary to re-locate these boundaries. This was probably the earliest use of surveying for land measurement, and it is from this use of geometric principles that the name geometry, which means "earth measurement," is derived.

EXERCISES

1. What ruler first appointed surveyors in Egypt? Refer to an article on geometry in an encyclopedia.
2. Look up the derivation of the word *geometry*. Name some other familiar English words containing one or the other of these Greek root-words.

4. The Modern Value of Geometry.—Geometry forms the structure of many modern sciences and arts. Artists apply its principles in pictures and designs; architects use them in drawing plans; civil engineers and surveyors use geometry now as they did centuries ago; astronomers could not measure nor describe the motions and positions of the stars or planets without it; nor could navigators find their place at sea. Many artisans, also, and workers in machine shops, carpenters, stone cutters, and others, make constant use of geometric principles.

5. Familiar Use of Geometric Terms.—Every word which has to do with position, form or size, is geometric.

EXERCISES

1. How far is a point on the surface of the earth from the center of the earth? How tall are you?
2. What is the form of the earth? Of a piece of crayon?
3. What is the area of the floor of the class-room in square feet?
4. What is the volume of the class-room in cubic feet?
5. What kind of surface are the floor and walls?
6. What is the form of the uncut end of a lead pencil?
7. What is the form of a base-ball diamond?
8. What is the form of a tennis court?
9. What is the form of a Rugby football?
10. Write such formulas of mensuration (geometric measurement) as you can recall. Have these formulas practical use? Many of them are derived and explained in this text.

6. The Subject Matter.—Geometry considers the properties of solids, surfaces, lines and points. A *solid* is a definite portion of space. Thus the properties of the space, or room, occupied by an object belong to geometry, and not the physical properties of the object.

A *surface* is the boundary of a solid.

A *line* is the boundary of a surface.

A *point* is the end of a line, or divides a line.

7. Dimensions.—A dimension is extent of some definite kind.

A *point* has no dimensions; that is, it has no extent. It has position only.

A *line* has one dimension, length.

A *surface* has two dimensions, length and width (or breadth).

A *solid* has three dimensions, length, width, and height.

EXERCISES

1. Refer to a point in the room; where is it?
2. Refer to a line; how long is it?
3. Refer to a surface; how long and how wide is it?
4. Refer to a solid; estimate its three dimensions.

8. Points.—*A point is named* by a capital letter.

EXERCISES

1. Draw a point · or ×. Name it.
2. Draw a straight line. Mark a point on it 

9. Lines.—A *straight line* is the shortest line that can be drawn, or imagined, between two points.

A *curved line* is one no part of which is straight.

A *sect* is a straight line of definite length.

A *unit sect* is a sect of generally known or standard length, as the inch, meter, mile.

A *sect is measured* in terms of some unit sect. The measurement is a number which expresses how many times the given sect contains the unit sect.

A *line is named* by two capital letters, or by a single letter.

A sect is *bisected* when it is divided by a point or intersecting line into two equal parts.

EXERCISES

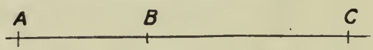
1. From what word is *straight* derived?
2. What instrument is used to draw a straight line?
3. What instrument contains a unit sect?
4. What instruments may be used to copy a given sect?
5. Draw a sect of 3 inches; one of 3.4 inches.
6. Draw a sect as long as the sect AB, or x.

7. Measure the sect AB.
8. What drawing instrument is used to draw a curved line? What kind of curved line may be drawn with it?
9. May this curved line be measured in the same units as a straight line?
10. Draw such a curved line; measure its length in inches.
11. Bisect (by judgment) each of the sects of Example 5 by a point.
12. Bisect the sect of Example 6 by a line.
13. Draw and trisect a sect.
14. Can an indefinite line be bisected?
15. Draw and define a *broken line.*
16. Draw and define a *mixed line.*

10. Positive and Negative Sects.—These terms refer to the direction in which a sect is drawn or measured. In general, sects are considered positive. Sometimes it is necessary, or convenient, to distinguish between a sect drawn, or measured, from left to right and one drawn, or measured, from right to left. The former direction is then generally considered positive, and the latter negative.

EXERCISES

1. Draw a sect and read it as a positive sect. Read it as a negative sect.
2. If the distance and direction of a point A from a point B are indicated on a surveyor's plot by $+785.46$ feet; how are the distance and direction of point B from point A indicated?
3. Read sect AC as the sum of two sects. Read AB as the (algebraic) sum of two sects, one positive and the other negative.



4. A man walks from point A, -250 feet to point B; and from point B, $+300$ feet to point C. Make a figure to show where point C is located with reference to point A.

11. Planes.—A *plane,* or a *plane surface,* is such a surface that a straight line between any two points of the surface lies wholly within it.

Plane geometry is the geometry of figures which are, or could be, drawn on a plane.

EXERCISES

1. How does a carpenter test the planeness of a board? Does the method agree with the definition of a plane?

2. Why is a carpenter's plane so called? Why is a certain machine tool called a planer?
3. Define a *plane curve.* Refer to, or form with a string or wire, a curve that is not plane.
4. Are two pencils with points in contact, in the same plane?
5. What kind of surfaces are cylindrical and conical surfaces?

12. Straight Lines in Relative Position.—Two straight lines in the same plane may be *intersecting, parallel* or *coinciding.* A *system of parallels* consists of three or more parallel lines.

Three or more lines are *concurrent* which have a common point of intersection. They form a *pencil* of lines.

EXERCISES

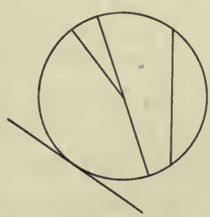
1. Draw two parallels; a system of parallels; two intersecting lines; four concurrent lines; two coinciding lines; a pencil of three lines.
2. Define parallel lines.
3. Hold two pencils so that they are not intersecting, parallel or coinciding. Are they in the same plane? Refer to two such lines in the room.

13. Generation by Motion.—The line, surface and solid may be regarded as generated by motion.

EXERCISES

1. When a point moves, what does it generate; that is, what does the path of the point represent?
2. If a line moves, what does it generate?
3. If a surface moves, what does it generate?

14. Circles.—A *circle* is a portion of a plane bounded by a curved line every point of which is equidistant from a fixed point within it, called the *center.* The *circumference* is the bounding line; which is also called a *circle.*



A *radius* is a straight line joining the center with a point on the circumference.

A *chord* is a straight line joining two points of the circumference.

A *diameter* is a chord which passes through the center.

A *tangent* is a straight line which touches the circumference in only one point however far it is extended.

An *arc* is a portion of a circumference.

A *unit arc* is an arc of standard length.

A *degree of arc* ($1°$) is $\frac{1}{360}$ of a circumference.

The degree is the *unit arc* which is most used.

The *measurement of an arc* is a number which expresses how many times the given arc contains the unit arc of the same circle.

A *minute of arc* is $\frac{1}{60}$ of a degree.

A *second of arc* is $\frac{1}{60}$ of a minute.

A *semicircle* is one-half of a circle.

A *quadrant* is one-quarter of a circle, or of a circumference.

Every circle contains $360°$ of arc, but the *size of a circle* is given by the length of its radius or diameter.

A circle is named by a capital letter at the center.

EXERCISES

1. Draw a circle. Draw a radius, diameter, chord, tangent, degree of arc, quadrant, semicircle, semicircumference.
2. Draw a curve which is not a circle.
3. What instrument is used to measure arcs of circles?
4. How many degrees of arc are there in a quadrant? In a semicircumference? How many quadrants are in a semicircumference or semicircle?
5. Draw a circle and mark arcs of approximately $60°$; $120°$; $45°$; $30°$; $1°$.
6. On a circle, lay off an arc $AB = +160°$; from point B, lay off an arc $BC = -100°$. Read the value of arc CA.

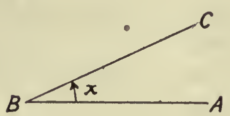
15. Relative Position of Two Circles.—Two circles of the same, or different, radii may be situated in relation to each other in three ways: (1) *out of contact;* (2) *tangent;* (3) *intersecting.*

Concentric circles have the same center and different radii.

EXERCISES

1. Draw two circles out of contact in two different ways (internally and externally out of contact).
2. Draw two circles tangent in two different ways (internally and externally).
3. Draw two intersecting circles.
4. Draw two concentric circles; three concentric circles.
5. What class includes concentric circles?

16. Angles.—An *angle* is a figure formed by two straight lines drawn from the same point.



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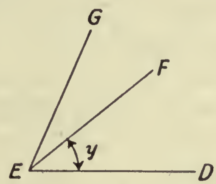

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The *parts of an angle* are the *sides* and the *vertex.*

An angle may be considered as having been *generated* by the rotation of one of its sides about the vertex from a position of coincidence with the other side. An angle may be positive or negative according to the direction of this rotation. The counter-clockwise direction is generally regarded as positive.

A *measurement arc* is often used to indicate the size or direction, or both, of an angle.

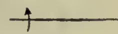
An *angle is named* (1) by three capital letters, reading from one side to the other with the vertex letter in the middle; (2) when no confusion can occur, by the vertex letter alone; (3) by a small letter, or number, or capital letter placed within the angle.

EXERCISES

1. Read the angles of Figures 1 and 2 in all possible ways.
2. Define the vertex and sides of an angle.
3. Draw two lines, *AB* and *CD*, intersecting at point *O*. Read all the angles which are formed.

17. Angles Named According to Size.

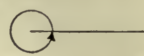
A *zero angle* is one whose sides coincide before rotation.



A *straight angle* is one whose sides lie in the same straight line in opposite directions from the vertex.

A *right angle* is one-half of a straight angle.

A *perigon* is an angle whose sides coincide after rotation.



An *acute angle* is less than a right angle.

An *obtuse angle* is greater than a right angle and less than a straight angle.

A *reflex angle* is greater than a straight angle and generally is not more than a perigon.

An *oblique angle* is any angle other than a multiple of a right angle.

Two angles whose sum is a right angle are *complementary.*

Two angles whose sum is a straight angle are *supplementary.*

Two angles whose sum is a perigon are *explementary.*

The angles are *complements, supplements, explements* of each other.

A *unit angle* is an angle of standard size.

A *degree of angle* is $\frac{1}{90}$ of a right angle.

The *measurement of an angle* is a number which expresses how many times the given angle contains the unit angle.

EXERCISES

1. Draw angles of each kind named above, with their measurement arcs.
2. Calculate the complement and supplement of 72°.
3. What angle has a negative complement and a positive supplement?
4. Estimate the angles x, y and DEG of paragraph **16**; in right angles, degrees and straight angles.
5. Draw angles with the protractor to the following measures: 1 degree, or 1°; 45°; 130°; $2\frac{1}{2}$ right angles; 180°; $1\frac{1}{2}$ straight angles.

6. What angle occurs most frequently in familiar objects?
7. Is the size of an angle determined by the length or by the position of its sides?

18. Angles Named According to Relative Position.—Unless otherwise stated, these names refer only to *two* angles.

Adjacent angles have a common side and vertex.

Opposite angles (or vertical angles) are the non-adjacent angles formed by two intersecting lines.

Supplementary-adjacent angles are both supplementary and adjacent.

A *transversal* is a line which intersects two or more other lines.

The names given to pairs of angles, one at each point of intersection, are:

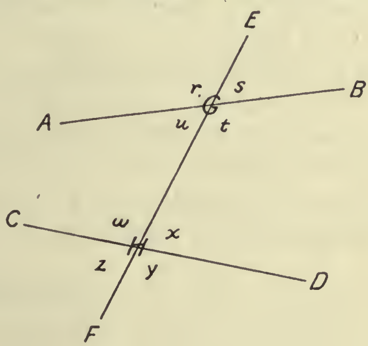
Corresponding; r and w.

Alternate-interior; u and x.

Alternate-exterior; r and y.

Consecutive-interior; t and x.

Consecutive-exterior; s and y.



EXERCISES

1. Draw two externally adjacent angles. Draw two internally adjacent angles. Draw four successively adjacent angles.
2. Draw two intersecting lines. Name all pairs of opposite angles.
3. Draw two supplementary-adjacent angles. Draw two supplementary angles that are not adjacent.
4. Name all other pairs of corresponding angles formed in the transversal figure; of alternate-interior angles; of alternate-exterior angles; of consecutive-interior angles; of consecutive-exterior angles.
5. Draw three lines cut by a transversal. Name a *set* of corresponding angles.

19. Other Angle Names.—A *perpendicular* is a straight line which forms a right angle with another straight line. The *foot of the perpendicular* is the point of intersection.

A *bisector* of an angle is a straight line which divides it into two equal angles.

EXERCISES

1. Draw AB perpendicular to CD at point A. Name the foot of the perpendicular AB. Is line CD also perpendicular to line AB?
2. Can an angle bisector be drawn otherwise than through the vertex?
3. Draw acute, right, obtuse, reflex and straight angles, and a perigon. Draw by judgment the bisector of each.
4. Draw and trisect an angle?

20. Distance.—The *distance between two points* is the length of the straight line joining the points.

The *distance from a point to a straight line* is the length of a perpendicular drawn from the point to the line.

The *distance between two parallels* is measured on a perpendicular drawn to both parallels.

The *distance from a point to a circle* is measured on a radius of the circle, or on a radius prolonged, which is drawn through the given point.

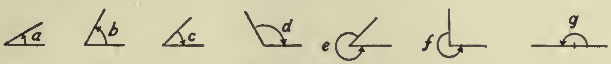
EXERCISES

1. Mark two points. Draw and measure the distance between them.
2. Mark a point and draw a straight line. Draw and measure the distance from the point to the line.
3. Draw two parallels. Draw and measure the distance between them.
4. Draw a circle and mark a point within it. Draw and measure the distance from the point to the circumference.
5. Mark a point outside the circle which is two inches from the circumference.

21. Review Exercises

1. Give the geometric names of various objects in the class-room.
2. What kinds of angles are formed by the corners of your drawing triangles? Measure them in degrees.

3. Name two streets that intersect at a right angle; two that intersect at an oblique angle; two streets that will not intersect no matter how far they are extended in either direction.
4. Can a sect be measured in degrees? Or an angle in inches? Has a degree of angle any equivalent in inches?
5. Can an arc be measured in inches?
6. If you walk from a given point A, $+250$ feet to a point B; and then walk from point B, -400 feet to a point C; how is point C situated with respect to point A? Make a drawing
7. If the terminal side of an angle rotates from the initial side AB, $+120$ degrees to the position AC; and then rotates from the position AC, -200 degrees to the position AD; how is the terminal side AD situated with respect to the initial side AB?
8. Judge the measure of each of these angles in terms of the right angle; also in terms of the degree.



9. Reduce $\frac{3}{7}$ of a right angle to degrees, minutes and seconds.
10. Reduce 22° 30′ (22 degrees and 30 minutes) to a fraction of a right angle.
11. How many degrees of angle has the minute hand of a clock described (generated) from twelve o'clock to one minute (of time) past twelve?
12. How many degrees of angle are between a line running from a certain point northeast, and a line from the same point due east?
13. Calculate and draw the supplement of 200 degrees; the complement of 100 degrees.
14. Write the complement of x degrees; the supplement. Write the complement of x right angles; the supplement.
15. Find the angle whose supplement equals three times the complement. *Help.* $180° - x = 3(90° - x)$.
16. Express algebraically the relation that x and y are complementary; supplementary.
17. One of two complementary angles is 10° less than twice the other angle. Find the angles.
18. The difference between two supplementary angles is 25°. Find the angles.
19. Upon what does sign depend in sects, angles and arcs?
20. What angle-bisector is a perpendicular to one of the sides of the angle?
·21. Measure the distance from this point to the nearest and farthest edges of this page.

22. What line is the distance from the center of a circle to the circumference?
23. On what line is the distance measured between two concentric circles?
24. On what line is the distance measured between the circumferences of two circles that are out of contact?
25. Define parallels by means of a property of the distance between them.
26. Show that in every use of the term *distance* between two objects, *the shortest distance* is referred to.
27. Draw two straight lines which do not coincide, which form a zero angle at infinity.

22. ABBREVATIONS AND SYMBOLS

adjacent	adj.	feet, inches	$'$ $''$
angle, angles	$\angle$, $\angle$s	is greater than	$>$
alternate-interior	alt.-int.	is less than	$<$
alternate-exterior	alt.-ext.	opposite	opp.
circle, circles	$\odot$, Ⓢ	parallel, is parallel to	$\parallel$
consecutive	cons.	parallels	$\parallel$s
corresponding	corres.	perpendicular, is perpendicular to	$\perp$
degree, minute, second	$^\circ$ $'$ $''$	perpendiculars	$\perp$s
equal, is equal to	$=$	point, points	pt., pts.
equals	$=$s	right	rt.
is unequal to	$\neq$	straight	st.
unequals	$\neq$s	supplementary-adjacent	sup.-adj.
exterior	ext.	paragraph reference	1,26,etc.
interior	int.		

CHAPTER II

STRAIGHT LINES, CIRCLES, SYMMETRY

Experimental Investigation of Principles

23. Geometrical Experiments.—The *facts* of geometry are all about us. We intend to inquire into nature's laws and to discover these facts through our own powers of observation. For this purpose a series of experiments will be undertaken; an *experiment* being *an investigation made for the purpose of ascertaining a fact.*

We shall make a drawing according to some *assigned* (or given) property, and shall *observe* some *new* property which is invariably associated with the given property. The finding of this new property is the *result of the experiment.*

Make all experiments for the general case, so that results shall not be limited in their application. State the result in the shortest and clearest expressions possible. Introduce nothing outside of the experiment. If an experiment is inconclusive, do not hesitate to so state the result. You are not to state what you think ought to be true, but exactly what the experiment shows to be true. Sometimes it is well to repeat an experiment one or several times.

EXERCISES

1. Are other sciences based upon experiment?
2. How may the equality or difference of two sects be tested by the use of instruments? Of two angles?
3. How may the perpendicularity, or lack of it, of two lines be tested?

24. Experiment I.—The number of straight lines that can be drawn through given points.

(*a*) *One given point.*

(*b*) *Two given points.*

(*c*) *Three or more given points.*

(*a*) Mark a point A. Draw a straight line through it. If possible, draw other straight lines through point A. Observe the number of straight lines which can be drawn through point A, and their directions.

RESULT.—Any number of straight lines can be drawn through a given point, and in any direction.

(*b*) Mark two points A and B. Draw a straight line through (containing) both points. If possible, draw other straight lines containing both points, in different positions from the line first drawn. Observe the number of straight lines which can be so drawn. State result.

(*c*) Mark three points at random. Observe the number of straight lines which can be drawn containing all the points. Can a straight line be drawn containing any three marked points? Make the experiment for four points.

RESULT.—In general, no straight line can be drawn through three or more given points. But the points may be so situated that one straight line shall contain them.

25. Experiment II.—Prolongation of a straight line.

Draw a (straight) line. Prolong (extend) it in both directions, using dotted lines. Observe how far it can be thus extended if the drawing is not limited by the extent of the paper.

RESULT.—

26. EXPERIMENT III.—**The number of points of intersection of straight lines and circles.**

(*a*) *Two straight lines.*

(*b*) *One straight line and one circle.*

(*c*) *Two circles* (*of the same or different radii*).

In stating the results, observe that in certain positions of the straight lines and circles there may be no points of intersection, and in other positions there may be one or more points of intersection or of contact. State the complete result for each part.

27. Constructions.—A *construction* is a drawing made for the purpose of producing a figure *with some required* *property.* The *method* of obtaining this required figure is the subject of experiment.

The different construction methods are:

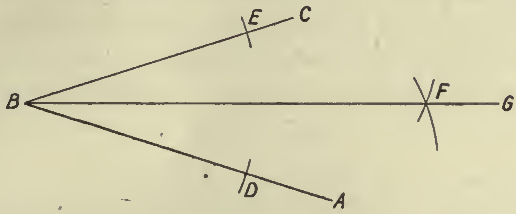
(1) Rule and compass.

(2) Measurement.

(3) Trial and error.

(4) Copying a standard.

28. Experiment IV.—To draw (or construct) the bisector of an angle. That is: An experiment to find a method of drawing the bisector of an angle.



$\angle ABC$ is the given angle;

Required to construct its bisector.

METHOD (*a*).—(1) Draw an arc, center at B, any radius, intersecting BA and BC in points D and E.

(2) Draw two arcs, centers at D and E, any equal radii greater than $\frac{1}{2}$ DE, intersecting each other at F.

(3) Draw BF, which is the required bisector of $\angle ABC$.

METHOD (*b*).—(1) Measure $\angle ABC$ with the protractor.

(2) Divide the measurement by 2.

(3) Lay out $\angle ABG$ equal to the quotient.

EXERCISES

1. After obtaining the bisector BG, how may $\angle$sABG and GBC be compared in order to test their equality?
2. Can any angle, straight, obtuse or reflex, be bisected by these methods?
3. Can another different bisector of $\angle ABC$ be drawn?
4. What methods as defined in **27**, have been employed in the preceding experiment?
5. Can an angle be divided into three, four, five, etc., equal angles by either of these methods?

29. Experiment V.—To bisect a sect.

Make the construction: (*a*) using the rule and compass method; (*b*) using the measurement method; (*c*) using the trial and error method.

AB is the given sect;

Required to bisect it.

METHOD (*c*).—(1) Lay off with the compass any length *AC* judged $= \frac{1}{2}AB$, and lay off $CC' = AC$.

(2) If C' does not coincide with *B*, set the compass to a different length as *AD*, and make $DD' = AD$.

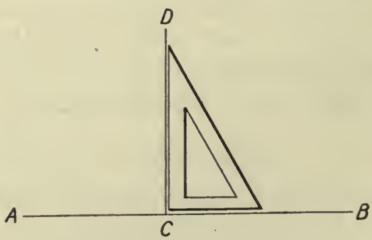
(3) Repeat until the middle point of *AB* is found, as at *E*.

EXERCISES

1. At how many points can a sect be bisected?
2. Can a sect be divided into any number of equal parts?
3. Which of the methods of the preceding experiment can be used to divide a sect into three, four, five, etc., equal parts?

30. Experiment VI.—To construct a perpendicular to a line at a point in the line.

Find (*a*) a method by rule and compass; (*b*) a method by measurement; (*c*) a method by copying a standard.



AB is the given line, and *C* is a point in *AB*;

Required to draw a perpendicular to *AB* at point *C*.

METHOD (*c*).—(1) Place a standard right angle with vertex at *C*, and one side coinciding with *CB*.

(2) Draw *CD*, which is the required perpendicular.

EXERCISES

1. Can a perpendicular be drawn (erected) at any point in the line?
2. How many perpendiculars can be erected at a given point as C, on one side of line AB?
3. Methods (*a*) and (*b*) in the preceding experiment are equivalent to bisecting what kind of angle?

31. Experiment VII.—To construct a perpendicular to a line from a point without (outside of) it.

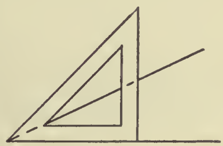
Draw and describe methods (*a*) by rule and compass; (*b*) by copying a standard.

32. Experiment VIII.—To construct an angle equal to a given angle; or, **to copy a given angle.**

Draw and describe methods (*a*) by rule and compass; (*b*) by measurement.

EXERCISES

1. Use these methods to draw an angle equal to a given obtuse angle.
2. Use the method indicated in the figure to copy a given angle. This is a modification of the measurement method.



33. Experiment IX.—**To copy a sect; addition and subtraction of sects.**

Draw two sects. Describe different methods of copying them, of drawing a sect equal to their sum, and of drawing a sect equal to their difference.

34. Experiment X.—**Addition and subtraction of angles.**

Draw two angles; etc.

35. Experiment XI.—**To draw the complement and the supplement of a given angle.**

Draw an angle. Find (*a*) a method by rule and compass; (*b*) a method by measurement; of drawing the required angles.

36. Experiment XII.—The relative size of opposite angles.

Draw two intersecting lines. Observe the size of opposite angles. State result.

37. Experiment XIII.—The relative position of two perpendiculars to the same straight line.

Draw the figure. State result.

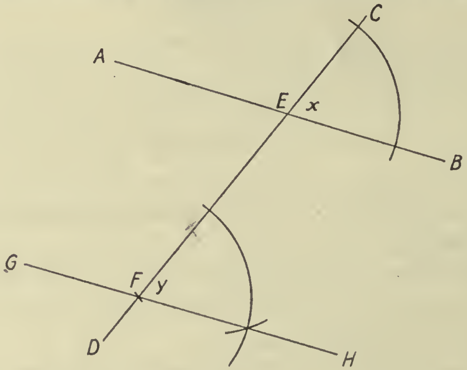
38. Experiment XIV.—**To construct a line through a given point parallel to a given line.**

Draw the given line and the given point. Use the principle of **37** to complete the required figure.

EXERCISES

1. Can a line be drawn through any given point parallel to any given line?
2. How many lines, in different positions, can be drawn through a given point parallel to a given line?

39. Experiment XV.—The position of two lines cut by a transversal making a pair of corresponding angles equal.



To Draw the Required Figure.—(1) Draw AB cut by transversal CD, forming $\angle x$.

(2) Mark point F on CD; construct $\angle y = \angle x$, and in such a position that it forms with $\angle x$ a pair of corresponding angles; using either the rule and compass method of copying the angle, as shown, or copying the angle with a protractor.

(3) Draw GH, the side of $\angle y$.

Observe the position of AB and GH. The result should be a complete statement of the property of the figure, thus:

Result.—Two straight lines which are cut by a transversal so as to make a pair of corresponding angles equal, are parallel.

EXERCISES

1. May any pair of corresponding angles be made equal with the same result?
2. Draw a figure in which the protractor is used to make $\angle y = \angle x$.

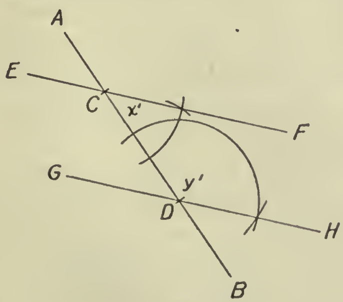
40. Experiment XVI.—The position of two lines cut by a transversal making a pair of alternate-interior angles equal.

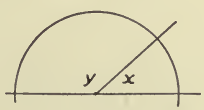
Draw the figure. State result.

41. Experiment XVII.—**The position of two lines cut by a transversal making a pair of alternate-exterior angles equal.**

Draw the figure. State result.

42. Experiment XVIII.—The position of two lines cut by a transversal making a pair of consecutive-interior angles supplementary.





To Draw the Figure.—(1) Draw supplementary angles x and y.

(2) Draw AB, mark points C and D.

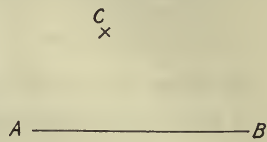
(3) Draw $\angle$s x' and y' equal to $\angle$s x and y respectively, and EF and GH, the sides of $\angle$s x' and y'. Or measure $\angle$s x' and y' with a protractor, so that their sum equals 180°. State result.

43. Experiment XIX.—**The position of two lines cut by a transversal making a pair of consecutive-exterior angles supplementary.**

Draw the figure. State result.

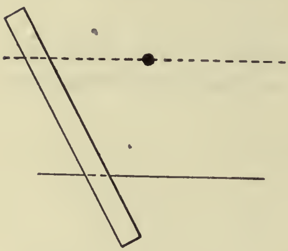
44. Experiment XX.—To construct a line through a given point parallel to a given line.

This figure has been constructed by means of two perpendiculars to the same line; **38.** Employ in the present construction the principles of **39, 40** and **41.**



45. Experiment XXI.—**To construct a line through a given point parallel to a given line, by using rule and triangle.**

This construction can be made by placing the rule and triangle in certain positions, and by then sliding the triangle along the rule. Find the best positions for the rule and triangle. This is an accurate and rapid method of drawing parallels and is the best method for general use. It is much used by draughtsmen.



EXERCISES

1. What represents the transversal in this construction? What pair of angles is made equal?
2. To which of the methods of **44** is this method equivalent?
3. The rule may be considered as a pair of standard parallels. Draw two parallel lines by this instrument alone. May a line be drawn through any given point parallel to any given line by this method?

46. Experiment XXII.—Relations between pairs of angles when two parallels are cut by a transversal.

Draw two parallels by any method.

(a) Draw a new transversal. Observe the relative size of each pair of corresponding, alternate-interior, etc. angles.

(*b*) Draw a perpendicular to one of the parallels and observe its position with respect to the other parallel.

State results for both (*a*) and (*b*).

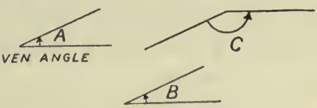
47. Experiment XXIII.—**The position of two parallels to the same line.**

Draw the figure. State result.

48. Experiment XXIV.—Relation of angles whose sides are parallel, each to each.

To Draw the Figure.—(1) Draw an oblique angle.

(2) Draw other angles whose sides are parallel, each to each, to those of the given angle. There are four positions in which such angles can be drawn, two of these positions being shown in the figure at *B* and *C*.



Observe the relation between the given angle and the constructed angles. State result.

49. Experiment XXV.—Relation of angles whose sides are perpendicular, each to each.

Draw the figure and state result.

50. Experiment XXVI.—**Properties of a perpendicular.**

(*a*) Observe the relative lengths of a perpendicular and an oblique line drawn from the same given point to the same given line. State result.

(*b*) Draw a line and a perpendicular to it; mark a point in the perpendicular; measure two equal distances on the given line from the foot of the perpendicular (one on each side of it); draw two oblique lines from the point in the perpendicular to the ends of the equal distances.

Observe how the lengths of the two oblique lines compare. State result.

(*c*) Make an experiment similar to part (*b*), in which the two distances measured on the given line from the foot of the perpendicular are unequal. State result.

Symmetry

51. The Symmetry of Plane Figures is of Two Kinds:—(1) A figure possesses symmetry *with respect to an*

axis, which is a straight line, when it can be folded upon the axis so that the part of the figure on one side of the axis coincides with the part on the other side. The axis is called the *axis of symmetry*.

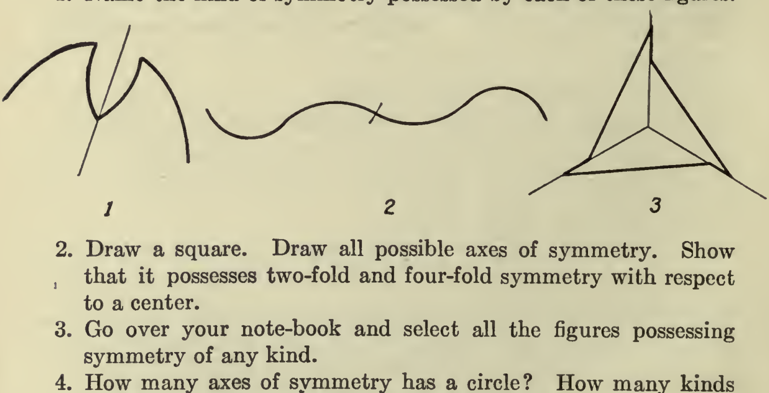
(2) A figure possesses symmetry *with respect to a center*, which is a point, when, after rotation about the center through any integral part of a perigon, it coincides with its original position. The point is called the *center of symmetry*.

Symmetry with respect to a center is two-fold, three-fold, etc., according to the integral division of a perigon through which it is rotated into coincidence with the original position.

Any one *element*, or symmetrical division, of a figure is the *symmetrical impression* of any other element.

EXERCISES

1. Name the kind of symmetry possessed by each of these figures.



2. Draw a square. Draw all possible axes of symmetry. Show that it possesses two-fold and four-fold symmetry with respect to a center.
3. Go over your note-book and select all the figures possessing symmetry of any kind.
4. How many axes of symmetry has a circle? How many kinds of symmetry with respect to a center has it?

52. Experiment XXVII.—**The law of symmetry of a point with respect to an axis.**

Draw a line AB and a point C without it. Fold the paper on AB and trace the symmetrical impression C', of point C. Unfold the paper; draw CC'.

Observe the relation of CC' to the axis AB.

State this relation as the result of the experiment.

53. Experiment XXVIII.—**To construct the symmetrical impression with respect to an axis:**

(*a*) *Of a sect.*

(*b*) *Of an indefinite straight line.*

(*c*) *Of an irregular curve.*

(*a*) Find the symmetrical impression of the two end points of the sect; join them by a straight line.

(*b*) Find the symmetrical impression of any two points of the given line, etc.

(*c*) Find the symmetrical impression of an indefinite number of points of the given curve; draw a curve containing them.

Test the accuracy of each construction by folding the paper on the axis.

EXERCISES

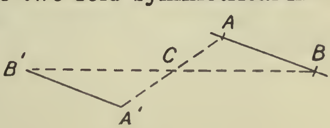
1. Construct the symmetrical impression of a perpendicular with respect to the line on which it stands, as an axis.
2. Construct the symmetrical impression of a line parallel to the axis.
3. Construct the symmetrical impression of a line intersecting the axis. Show also how this can be done by copying an angle.
4. Construct the symmetrical impression of a circle with respect to an axis outside the circle. What is the least number of points which will determine the symmetrical circle?

54. Experiment XXIX.—**The law of symmetry of a point with respect to a center:**

Mark two points A and B. Fix a piece of transparent paper with a pin at A; trace point B upon the transparent paper. Revolve the transparent paper through a straight angle; trace the new position of point B on the drawing at B'. Remove the transparent paper and draw BB'.

Observe the relation of BB' to the center of symmetry A. State result.

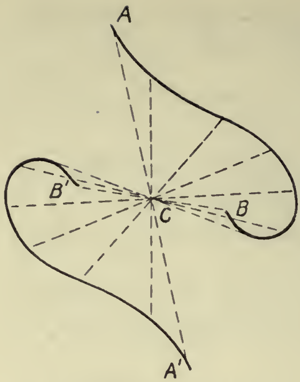
55. Experiment XXX.—**The two-fold symmetrical impression with respect to a center:**



(*a*) *Of a sect.*

(*b*) *Of an indefinite straight line.*

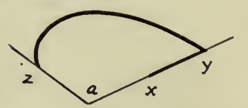
(*c*) *Of an irregular curve.*



Test the accuracy of each construction by tracing the figure on transparent paper and revolving on the center through a straight angle.

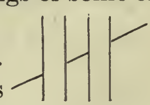
EXERCISES

1. Construct the symmetrical impression of a sect with respect to a center at one of its extremities.
2. Construct a design with three-fold symmetry from the given element *xyz*. What must be the measure of angle *a*?

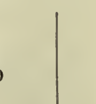


3. Are doorways, windows, mantlepieces, building outlines, symmetrical?
4. Are most pieces of furniture symmetrical? Name some with one or the other kind of symmetry.
5. Examine any pleasing outline of design, landscape gardening, architecture, etc. Do all possess symmetry or a modified symmetry?
6. Examine some good group pictures or statuary. Do they possess a certain balance of proportion and grouping that may be regarded as symmetry?
7. Are flowers, leaves, entire plants, symmetrical?

56. Experiment XXXI.—**Geometrical illusions.** The properties of some geometric figures are always judged incorrectly. Draw the following figures much enlarged. It is well to make blackboard drawings of some of them.

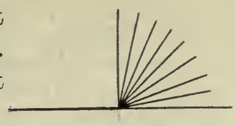


(*a*) Draw four vertical lines, equally spaced. Rule an oblique line as shown. Do the parts of the oblique line appear to lie in the same straight line?

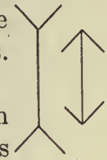


(*b*) Measure a perpendicular equal in length to the horizontal line. Do they appear equal?

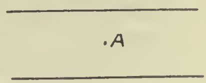
(*c*) Divide one of two adjacent right angles into a number of equal parts. State apparent size of the two right angles.



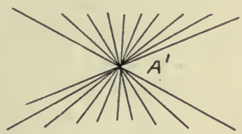
(*d*) Draw two equal parallel lines. Attach the "arrow-head" and "feather" marks at the ends. State result.



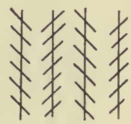
(*e*) Draw two horizontal parallel lines. From a point A midway between them, draw radial lines as shown at A', at about equal angles. State result.



(*f*) Draw four parallel lines. Draw sets of short oblique parallel lines across them. State result.



General Results.—Our judgment of geometrical figures may lead us into errors. The results of the experiments performed in this chapter have been stated on the assumption, which is probably a safe one, that nature prefers parallels, perpendiculars, equalities, universal laws and general simplicity.

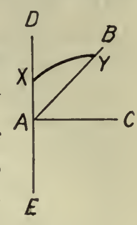


57. Review Exercises

1. Draw opposite angles; bisect one of them; prolong the bisector through the other angle of the pair. How does it divide the other angle? State result.
2. Divide a given sect into four equal parts.
3. Construct accurate angles of 90° and 45°, using rule and compass. Test these angle values of your protractor and triangles.
4. Divide a given obtuse angle into four equal parts.
5. Draw with the protractor an angle of 25°. Extend both sides through the vertex. Calculate and measure the other three angles thus formed.
6. Draw a triangle ABC and a point D without it. Construct the two-fold symmetrical impression of ABC with respect to the center D.
7. Draw a circle with a chord AB. Construct the symmetrical impression of the circle about the chord as an axis.
8. Which of the printed Gothic capitals, A, B, C, etc., possess

symmetry with respect to: (*a*) a vertical axis; (*b*) a horizontal axis; (*c*) both vertical and horizontal axes; (*d*) a center and not to any axes; (*e*) a center and also one or two axes; (*f*) oblique axes? (*g*) Which letters are asymmetrical (not symmetrical)?

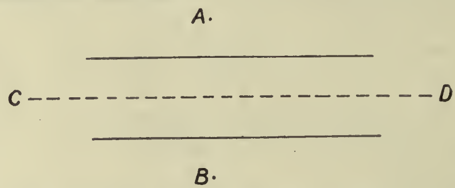
9. State all the methods that may be used to draw a line through a given point parallel to a given line.



10. Construct a design from the given element *xy*. Use: (1) *AB* as an axis; (2) *AC* as an axis; (3) *DE* as an axis. What kinds of symmetry will the complete design possess?

11. Mark a point on the floor. With a piece of string find the shortest length from the point to the wall. Draw this position of the string. What line will this position be?

12. Draw two sects *AB* and *CD*: (*a*) perpendicular bisectors of each other, but not equal; (*b*) so that *AB* is the perpendicular bisector of *CD*, but *AB* is not bisected; (*c*) so that each bisects the other; but not perpendicular; (*d*) so that they intersect but neither sect is bisected.



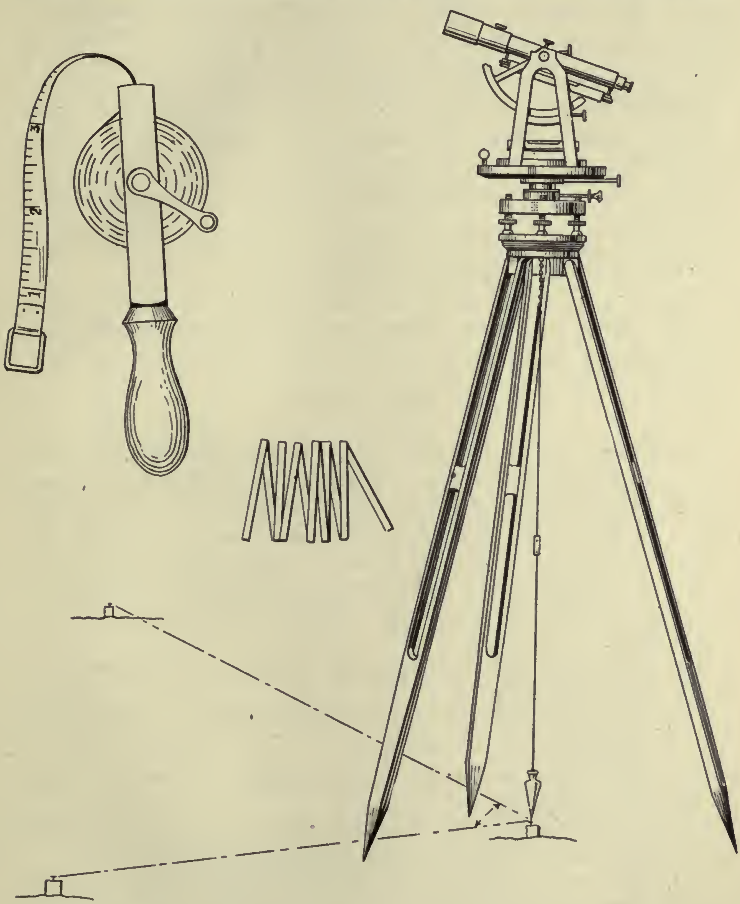
13. Draw two parallel lines as shown. Draw radial lines from points *A* and *B*, terminating in line *CD*. State the illusion result.

14. When a draughtsman draws parallels by sliding his T-square along the edge of his board, what construction principle is employed?

APPLICATIONS

58. Surveying, Building and Designing.—A surveyor uses a *tape* to measure or lay out lines (sects), and a *transit* to measure or lay out angles. A carpenter or mason uses a tape or *rule* to measure or lay out lines.

The tape is a flexible scale, usually 50 or 100 feet long. The transit is an accurate portable protractor. The transit consists of an accurately graduated horizontal circle, sometimes also a vertical circle or part of such a circle, a telescope provided with two intersecting cross-hairs at the focal point within the tube of the telescope, spirit levels and levelling screws for levelling the graduated circle, clamps and slow motion screws for clamping or rotating the gradu-



ated circle or the telescope, either separately or together, and a tripod for mounting the instrument.

The transit is set up over the vertex of the angle which is to be measured or laid out. The graduated circle is levelled and the telescope focussed for cross-hairs and distance. The telescope is then set with the "upper motion" so that the index reading of the circle is 0°. While clamped in this position, the circle and telescope are rotated together with the "lower motion" until the vertical cross-hair of the telescope cuts a point which marks one side of the angle to be measured or laid out. The upper clamp is loosened and the telescope alone is rotated into the position of the other side of the angle. The reading of the index on the graduated circle then gives the value of the angle. A text on surveying should be consulted for more complete instructions in the use of a transit, and for the best methods of measuring and laying out lines on the ground.

Some of the construction principles of the preceding experiments may be modified, or used without modification in practical problems.

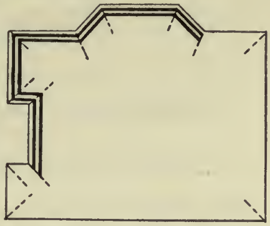
EXERCISES

1. Show how the angle at a corner of the room may be bisected by using a piece of string about ten feet long.
2. Show how the angle of a field may be bisected by using a tape. The tape must be so used that it is unnecessary to draw arcs on the ground. Points may be marked by stout wire marking pins, stuck in the ground. These should be made of about $\frac{3}{16}''$ wire and be about 12″ long.

3. How may a surveyor bisect the length of a curb between two street corners?
4. How may a surveyor erect a perpendicular to a line by using a tape?
5. How may a mason erect a perpendicular to a wall by using a piece of string?
6. Adapt the construction of **31** to the use of a tape; the point being nearer the line than the length of the tape.
7. How may a surveyor bisect an angle by means of a transit?
8. How may a surveyor erect a perpendicular by means of a transit?
9. How may a surveyor lay out in another place an angle equal to a given angle by using a transit?
10. How may a surveyor lay out a street parallel to another existing street by making use of the principles of **38** or **44**? Explain

at what point an angle must be measured with a transit, and at what other point an equal angle must be laid out.

11. A border for a hardwood floor is to be laid around a room as shown. How may the angle bisectors be obtained where the strips of wood are fitted together?



12. Sketch a symmetrical design for a colored glass window for a bathroom or stairway.
13. Sketch a symmetrical design for a decorated plate, book cover, lamp shade, rug, wall-paper pattern, or some other familiar object.

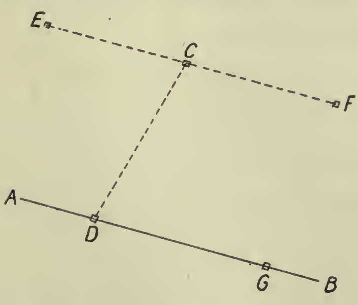
59. Standard Form Sheet for Recording Field Work.

Field Work

Department of Mathematics.————School

To lay out a line parallel to the curb of a road or path, and passing through a given point (at a distance of several hundred feet from the curb).

Geometrical Principle Employed.—If two straight lines are cut by a transversal making a pair of alternate interior angles equal, the lines are parallel.



AB is the curb and C is the given point.

(1) Mark any convenient point D, on the curb, and set a transit over point D; measure the angle BDC.

(2) Move the transit to point C; lay out the angle DCE equal to the angle BDC; drive stake E.

(3) CE is the required parallel.

Note.—If a building or other obstacle is in the way of the line CE, the angle DCF may be laid out equal to 180°-angle BDC. The work may be checked by repeating with another point G on AB.

CHAPTER III

PARALLELS, PERPENDICULARS AND ANGLES

Classification and Explanation of Principles

60. The Importance of Experimental Investigation.—The natural world about us is full of facts. We learn these facts from observation, either at large or in a laboratory. Many of the broader principles of Chapter I may be considered as being derived from general observation. We see types of lines, surfaces and solids, symmetry and motion, indefinitely repeated in natural objects. The experimental results which were obtained in Chapter II were those of a geometrical laboratory. Every science passes through a certain period of inception during which *facts* are accumulated, and this is true of chemistry, physics, psychology, biology, physical geography, astronomy, or any other science. A fact, or principle is *proved* by indefinitely repeating the experiment or observation.

61. The Importance of Analysis and Classification.—*Facts alone* are not sufficient to constitute a science. Experimental results may be incorrectly or imperfectly stated, due to errors of measurement or of judgment. If, however, a system of classification can be built up in which all experimental results are shown to agree with some general principles, and thus to agree among themselves, this very agreement is a further proof that these principles have been correctly determined by experiment. The greater the number of *independent principles* thus successfully analysed, the greater will be the certainty that the system of classification adopted is a true expression of nature's laws. We will now proceed with the *analysis* and *classification* of the facts or principles which we have noted in Chapter II.

62. The First Step in Classification.—This consists in the selection of the simplest, and therefore most reliable, principles as indicated by the experiments. These principles are either incapable of analysis in terms of still more elementary principles, or are not readily analyzed, or are so elementary that it is not worth while to analyze them.

63. Postulates.—These are the elementary principles selected as the basis of classification. Just what postulates are sufficient and necessary can only be determined as the analysis proceeds. It may be found necessary to increase the number originally selected or some of those selected may be found not to be required.

Postulates

1. Only one sect can be drawn between two given points (**24**(*b*)).

2. Two straight lines can intersect in only one point (**26**(*a*)).

3. A straight line and a circle, or two circles, can intersect in two and not more than two points (**26**(*b*) and (*c*)).

4. A straight line is the shortest distance between two points.

5. A sect or an angle can be divided into any number of equal parts, and in only one way (**28** and **29**).

6. All straight angles are equal.

7. All right angles are equal.

8. A right angle is one-half of a straight angle.

9. A perpendicular forms a right angle.

10. Only one perpendicular can be drawn to a straight line at a given point in the line, or from a given point without the line (**30** and **31**).

11. All radii of the same circle are equal.

12. Any definite figure can be constructed: as a straight line through a given point; a line through two given points; a sect of a given length; an angle equal to a given angle; a line passing through a given point and perpendicular to a given line; etc. (**28** to **35**, **38**.)

64. Axioms.—These are numerical relations which may also enter into the analysis of geometric principles. They are general in their application, not being limited to geometry, and are derived from reasoning of the most elementary kind.

Complete the statement of the axioms which are left incomplete.

AXIOMS

1. Things equal to the same thing are equal to each other.
2. If equals are added to equals, the sums are equal.
3. If equals are subtracted from equals,———.
4. If equals are multiplied by equals,———.
5. If equals are divided by equals,———.
6. If equals are added to unequals, the results are unequal in the same order.
7. If equals are subtracted from unequals,———.
8. If unequals are subtracted from equals,———.
9. If unequals are added to unequals, the greater to the greater and the less to the less,———.
10. If, of three unequals, the first is greater than the second, and the second is greater than the third,———.
11. The whole is equal to the sum of all its parts.
12. The whole is greater than any of its parts.
13. A quantity may be substituted for its equal in any expression.

65. The second step in classification is the preparation of a graded list of those experimental results, or principles, which can be analyzed in terms of other principles, and which are sufficiently important to require such analysis and explanation.

66. Theorems.—A *theorem* is a principle that is analyzed in more elementary terms.

A *corollary* is a theorem which is classified under another theorem.

The *order of arrangement* of theorems can be considerably modified. All that is necessary is to preserve their consecutive dependence.

67. Geometric Style.—A certain formal method has been developed as suitable to written geometrical analysis.

The *statement of a theorem* is divided into two parts:

(1) The *hypothesis*, which is a statement of the given, or constructed, properties of the figure.

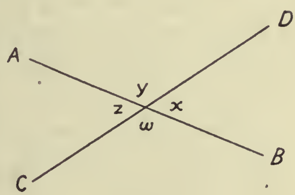
(2) The *conclusion*, which is a statement of the new, or resulting, properties of the figure.

The *analysis* (often called the *proof*) explains how the conclusion is worked out from more elementary principles. It consists of a *statement* of each separate logical step made, and the *reason* or explanation of the step.

68. Convenient Abbreviations.

analysis	anal.	hypothesis	hyp.
axiom	ax.	postulate	post.
conclusion	con.	theorem	theo.
corollary	cor.		

69. Theorem I.—*Opposite angles are equal* (**36**).



Hypothesis.—Straight lines AB and CD intersecting and forming pairs of opposite angles, x and z, y and w;

Conclusion.—$x = z$, $y = w$.

Analysis.

STATEMENT	REASON
1. $x + y =$ a straight $\angle$	Axiom 10
2. $y + z =$ a straight $\angle$	Axiom 11
3. $x + y = y + z$	Axiom 1
4. $x = z$	Axiom 3

Show also that the fact that $y = w$, is dependent upon these axioms.

EXERCISES

1. Draw the two straight angles separately which are considered as making up the composite figure of the theorem.
2. Does the analysis show that the fact that "opposite angles are equal" is not an independent principle?

3. Work out the same kind of analysis for $x = z$, in which $\angle$s x and z are combined with $\angle w$ instead of with $\angle y$.
4. In analyzing the second part, $y = w$, we may write $y + x =$ a straight $\angle$, $w + z =$ a straight $\angle$, $y + x = w + z$, $y = w$. Why may y and w be combined with different angles in the first two statements?
5. Upon what must the analysis of the first theorem depend? Upon what may that of the second theorem depend? Of other theorems?

Additional Theorems

70. These are theorems of minor importance, not assigned a theorem number.

(1) *Complements of equal angles are equal.*

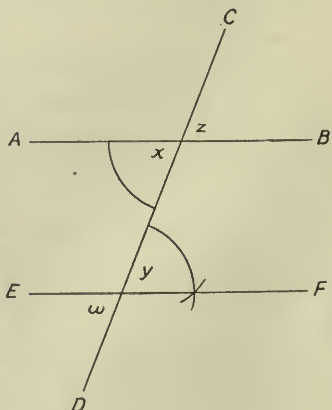
Helps.—Draw two equal angles; construct their complements. State the hypothesis and conclusion. Write the analysis to show how the equality of the complements depends upon Postulate 7 and Axioms 1, 3 and 11.

(2) *Supplements of equal angles are equal.*

71. Theory of Parallels.—The analysis of the principles of **37**, and **39** to **46**, cannot be made entirely in terms of the selected postulates and previous theorems. Two additional postulates will therefore be stated:

Postulate 13.—Two lines that are cut by a transversal making a pair of corresponding angles equal, are parallel (**39**).

Postulate 14.—If two parallel lines are cut by a transversal, any pair of corresponding angles is equal (**46**).



72. Theorem II.—*Two lines that are cut by a transversal making a pair of alternate-interior angles equal, are parallel* (**40**).

Hypothesis.—AB and EF cut by transversal CD making alternate-interior $\angle$s x and y equal:

Conclusion.—$AB \parallel EF$.

Analysis.

STATEMENT	REASON
1. $x = y$	Hypothesis (*i.e.*, made equal in drawing the figure.)
2. $x = z$	Theorem I (a principle previously analyzed.)
3. $y = z$	Axiom 1.
4. $AB \parallel CD$	Postulate 13.

EXERCISES

1. Draw separate figures showing the opposite angles and the corresponding angles which are considered as making up the figure of the theorem.
2. Make the analysis using the corresponding $\angle s$ x and w.
3. Construct a figure with the other pair of alternate-interior angles equal, and show that a pair of corresponding angles is also equal, and hence the lines are parallel.

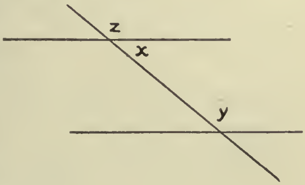
73. Theorem III.—*Two lines that are cut by a transversal making a pair of alternate-exterior angles equal, are parallel* (**41**).

Construct the figure; state the *hypothesis* and *conclusion*. Analyze (*a*) by using **Postulate 13**; and (*b*) by using **Theorem II.**

ADDITIONAL THEOREMS

74. (1) State the result of the experiment of **42** as a theorem. Analyze it in more than one way.

Helps.—(1) $x + y =$ a straight $\angle$; why?—(2) $x + z =$ a straight $\angle$; Ax.—(3) $z + y = x + z$; Ax.—(4) $y = z$;—(5) lines are $\parallel$.



(2) State the result of the experiment of **43** as a theorem. Analyze it in more than one way.

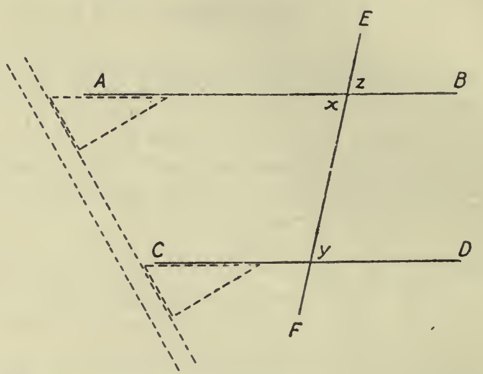
75. Theorem IV.—*Two lines perpendicular to the same line are parallel* (**37**).

Helps.—(1) The right angles are equal. What postulate?

(2) These form a pair of angles with the third line considered as a transversal.

76. A converse theorem is one in which the hypothesis and conclusion are respectively the conclusion and hypothesis of a previous theorem. The previous theorem is the *direct theorem.*

77. Theorem V. Converse of Theorem II.—*If two parallel lines are cut by a transversal, any pair of alternate-interior angles is equal* (**46**).



State the *hypothesis* and *conclusion.*

Analysis.

STATEMENT	REASON
1. $y = z$	Postulate 14.
2. $x = z$	What theorem?
3. $x = y$	What axiom?

EXERCISES

1. Draw separate figures showing how the figure of the theorem is made up of more elementary parts.
2. Make the analysis of Theorem V with some other pairs of alternate-interior and corresponding angles.

78. Theorem VI. Converse of Theorem III (46). State the theorem. Draw the figure; write out the analysis in more than one way.

Additional Theorems

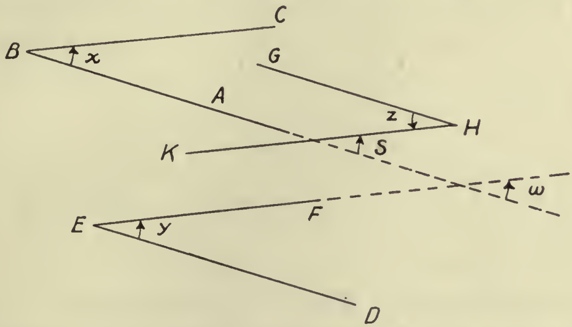
79. (1) State the **converse of Theorem 1, 74**. Analyze by the method of Theorem V.

(2) State the **converse of Theorem 2, 74.** Analyze by any method.

80. Theorem VII. Converse of Theorem IV.—*A line which is perpendicular to one of two parallels is perpendicular to the other also* (**46**).

Helps.—Consider the perpendicular as a transversal.

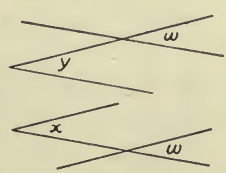
81. Theorem VIII.—*Two angles whose sides are parallel, each to each, are equal if both are acute or both obtuse; and are supplementary if one is acute and one obtuse* (**48**).



Hypothesis.—An acute $\angle x$, and acute $\angle$s y and z, with sides parallel to the sides of $\angle x$.

Conclusion.—$x = y$ and $x = z$.

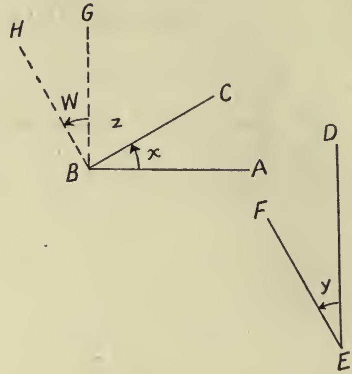
Helps.—(*a*) Prolong BA and EF to form $\angle w$. Notice that the figure now contains two pairs of parallel lines cut by transversals, and hence $\angle$s x and y are both equal to $\angle w$. Why? $\therefore x = y$. Show also that $x = z$.



(*b*) Draw an obtuse angle, and two other obtuse angles whose sides are parallel to the sides of the given angle. Show that these angles are each equal to the given angle.

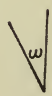
(*c*) Draw an acute angle, and two obtuse angles whose sides are parallel to the sides of the given angle. Show that these angles are supplementary to the given angle.

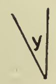
82. Theorem IX.—*Two angles whose sides are perpendicular, each to each, are equal if both are acute or both obtuse, and are supplementary if one is acute and the other obtuse* (**49**).



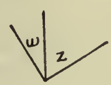
State the *hypothesis* and *conclusion*.

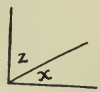
Analysis.





STATEMENT	REASON
1. Draw $BG \perp BA$ and $BH \perp BC$	What postulate?
2. $BG \parallel ED$ and $BH \parallel EF$	What theorem?
3. $w = y$	What theorem?
4. $w = x$	**70.** Theorem (1)
5. $\therefore x = y$	What axiom?





Analyze also for other angles as in Theorem VIII (**81**).

83. General directions to be observed in the analysis of theorems.

1. Draw large and distinct figures, with all necessary and no unnecessary lettering.

2. Construct the figure from its given properties.

3. Eliminate all useless statements.

4. Cultivate a correct style.

5. *Auxiliary lines* may be drawn with any assigned property. If the line possesses additional properties, it is part of the analysis to show that these exist. Such lines are drawn dotted.

Additional Theorems

84. (1) *The bisectors of two supplementary-adjacent angles are perpendicular.*

(2) *An angle ABC is bisected by BD; through the vertex B a line EF is drawn perpendicular to BD. Line EF makes equal angles with the sides of angle ABC.*

State this theorem in general terms.

(3) *In $\triangle ABC$, $AB > AC$; point D is taken on AB so that $AD = AC$. Then $BC > BD$.*

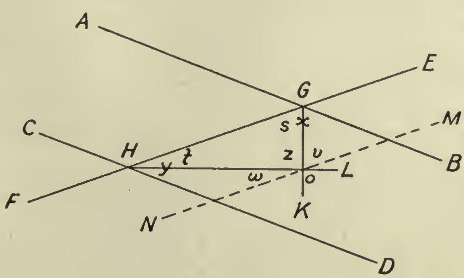
(4) *The bisectors of alternate-interior angles of parallel lines are parallel.*

(5) *The bisectors of corresponding angles of a system of parallels form another system of parallels.*

(6) *The bisectors of two pairs of opposite angles formed by two intersecting straight lines are perpendicular.*

(7) *The bisectors of a pair of consecutive-interior angles of parallels are perpendicular.*

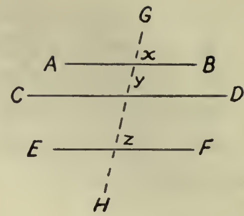
Helps.—(1) Draw $MN \| EF$.—(2) $x + y =$ a straight $\angle$;—(3) $s + t =$ a right $\angle$;—(4) $s = v$, $t = w$, etc.



(8) *Two lines parallel to the same line are parallel to each other* (**47**).

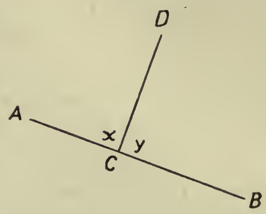
Helps.—Hyp.: AB and $EF \parallel CD$.

Analysis.—(1) Draw GH; (2) $x = y$, $y = z$; (3) $x = z$; (4) $AB \parallel EF$.



(9) *If one straight line meets another straight line so that adjacent angles are equal, the angles are right angles, and the lines are perpendicular.*

Helps.—(1) $x + y = 180°$;—(2) $x = y$;—(3) $2x = 180°$; —(4) $x = 90°$;—(5) $CD \perp AB$.

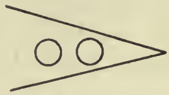


85. Review Exercises

1. Are we compelled to follow any certain order in making experiments? Or are the experiments, for the most part, completely independent of each other? Are the analyses of the results of these experiments also independent?
2. Mark two points on the surface of a globe, or ball. Will the straight line containing them lie wholly in the surface?
3. Draw a line on the paper by marking along an edge of your triangle. Turn the triangle over on this line as an axis. Does the drawn line still coincide with the edge of the triangle? If so, what kind of line is it? Has any other line this property?
4. How many straight lines are determined by three points?
5. How many points are determined by three straight lines?
6. Do two points in space determine a straight line?
7. Of what angle is the complement equal to the angle?
8. Of what angle is the supplement equal to the angle?
9. Of what angle is the supplement double the angle?
10. The difference of two supplementary angles is d degrees. Find the angles.
11. Given a line AB, a point C without it, and an angle x. Find a point D on AB such that CD makes an angle $CDA = x$. Use the rule and compass method.
12. x and $2\,x$ represent any pair of adjacent angles on a figure formed by a transversal cutting two parallels. Draw the figure and write on it the measures in degrees of the eight angles.
13. Two parallels are cut by a transversal so that x and y are

adjacent angles and x and $2y$ are corresponding angles. Sketch the figure and calculate all the angles in degrees.

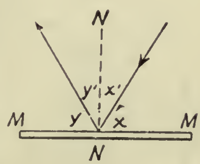
14. Draw the right angle of your triangle. Turn the triangle over using a perpendicular side as an axis. Draw the right angle again adjacent to the angle first drawn. How does this drawing test the accuracy of the right angle of the triangle?
15. The sum of two adjacent angles is 85°. Find the angle between their bisectors?
16. The angle between the bisectors of two adjacent angles is 80° 30′. What is the sum of the angles?
17. Reproduce this figure on a larger scale. In what does the illusion consist?



18. Measure the distance from the vertex of the right angle of your triangle to the hypotenuse.
19. A country road runs from southeast to northwest. In what direction would you cross the fields from a point to the west of the road, to reach it by the shortest way?
20. Would surveyors, architects and navigators be able to make use of the principles of geometry even if the analysis of experimental principles had never been developed?
21. In what two ways are axioms distinguished from postulates?
22. John and James each received two dollars for spending money during a holiday. John spent more of his money than James spent of his. Who has more remaining? What axiom is illustrated?
23. If the greater quantity of one pair of unequals is added to the less quantity of another pair of unequals, and the less quantity of the first pair is added to the greater of the second pair, which sum will be greater? Can such a result be stated as an axiom?
24. Express the axioms algebraically. Thus, Axiom 1; $a = b$, $a = c$, $\therefore b = c$.
25. Which of the axioms have been used in algebra: (1) In the reduction of equations, (2) in the processes of elimination in simultaneous equations, (3) in the reduction of inequalities?

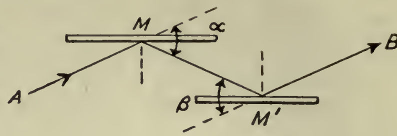
Applications

86. Light Reflected from Mirrors. *The law of reflected light:* It is proved experimentally that the angle of incidence, x, equals the angle of reflection, y; or $x' = y'$. NN is called the *normal* to the plane of the mirror MM.

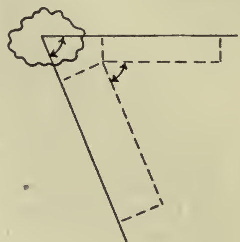


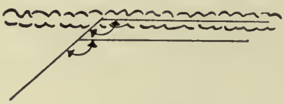
EXERCISE

Show that if two mirrors, M and M', are parallel, the incident ray A and the finally reflected ray B, are parallel. *Help.*—Show that $\alpha = \beta$.

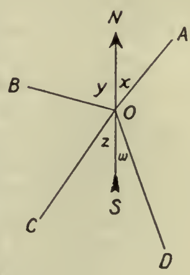


87. SURVEYING.—*Measurement of an inaccessible angle.* An angle in an inaccessible corner of a field may be measured as shown in the sketches. Explain how this is done and the principle involved.

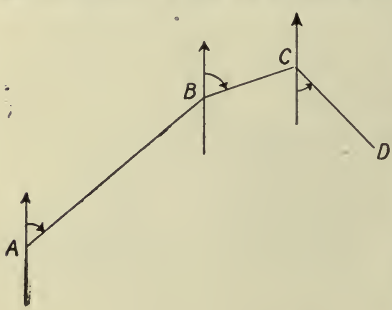




Bearings.—Point O is a point of observation; SN is a *meridian* through O; the angles x, y, z, w, are the *bearings* of points A, B, C, D, as seen from O. A is northeast of O, B is northwest, etc. If $x = 35°$, the bearing of line OA is read north 35 degrees east, and is written N 35° E. The length OA is the *distance* or *course.* A point can be located from a given point when its bearing and distance and also the direction of the meridian through the given point are known.



A surveyor may make a map of a piece of ground, or of a broken line, by measuring bearings and distances, first at point A, next at point B, etc.

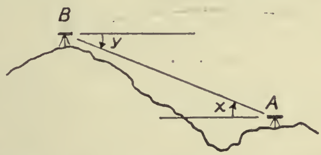


An *angle of elevation* is an angle measured in a vertical plane between a horizontal and an oblique line, the horizontal line being below the oblique line.

An *angle of depression* is an angle measured in a vertical plane between a horizontal and an oblique line, the horizontal line being above the oblique line

EXERCISES

1. An imaginary line is drawn from the top B of a hill to a point A in the plain that stretches away from its base. The angle of elevation x of the top of the hill as observed at A is 35°. What is the angle of depression y of the point A as observed at B? Why?



2. Are two vertical lines, determined by plumb-lines several feet or several hundred feet apart absolutely parallel? Why not?

3. Draw any meridian SN through a given point X. Locate a point Y whose bearing from X is S 50° W, and whose distance from X is 175 feet. Plot on a scale of 100 feet to the inch.
4. The bank of a river is located by a series of straight lines. Plot the river bank by drawing an irregular curved line agreeing closely with the broken line $ABCDEF$. Such a line is called in surveying a *traverse line.*

FIELD-NOTES

Beginning point of line	Bearing of line from beginning point	Length of line, feet
A	N 50° E	120
B	N 72° E	50
C	S 45° E	150
D	S 80° W	70
E	N 25° W	120

88. Latitude.—The latitude of a place is determined by measuring the angle of elevation *e*, of the sun when it is directly south, which occurs at, or about, noon.

The angle of elevation of the sun can be most accurately determined by using an *artificial horizon*. This is a mirror accurately levelled, or a shallow disk of mercury; *M*, Figure 2.

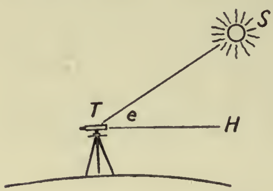


Fig. 1.

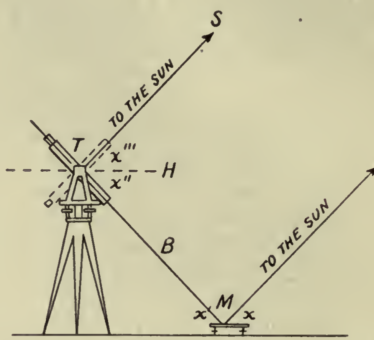


Fig. 2.

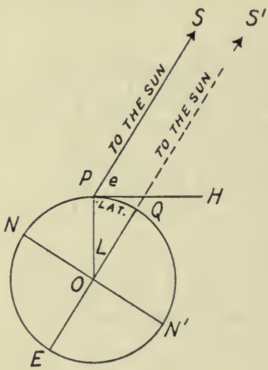


Fig. 3.

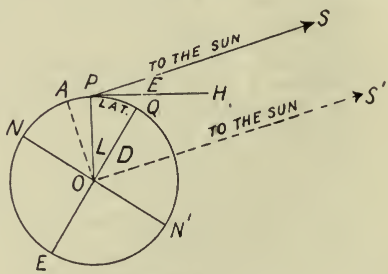


Fig. 4.

In Figure 3, *P* is the place on the earth of which the latitude is required, *EQ* is the equator, *NN'* is the axis of the earth, *PH* is the horizontal plane corresponding to *TH* of Figures 1 and 2. The angle *QOP* is the latitude of *P*. This figure shows the position of the sun with respect to the earth at the instant of *equinox*, which occurs at some place on the earth on March 21st and September 23d.

At all times between September 23d and March 21st, the sun is south of the equator, and the angle which a line OS', or the (approximately) parallel line PS, makes with the plane EQ of the earth's equator, is called the south or minus declination of the sun. This angle is QOS'.

At all times between March 21st and September 23d, the sun is north of the equator, and the angle QOS' is called its north or plus declination, Figure 5.

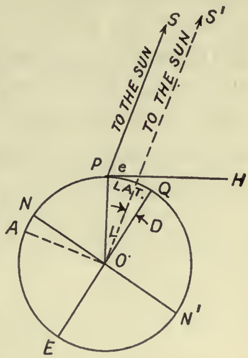


FIG. 5.

Commander Peary, Capt. Amundsen and Capt. Scott located the poles of the earth by this or a similar method.

EXERCISES

1. Show that one-half of the total angle STM, Figure 2, is the angle of elevation of the sun, STH.
2. How is the angle of elevation E, Figure 3, related to the latitude angle L, of place P?
3. In Figure 4, draw $OA \perp OS'$. Show that the declination angle D, of the sun, equals angle NOA; and that the angle of elevation E, of the sun, equals angle AOP. Then $L = 90° - (E + D)$; and if $\angle D$ is given its minus sign, $L = (90° - E) + D$.
4. In Figure 5, draw $OA \perp OS'$. Show that the declination angle $D = \angle NOA$; and that $\angle E = \angle AOP$. Then $L = 90° + D - E$; and if D is given its plus sign, $L = (90° - E) + D$, as in Exercise 3.
5. If the angle of elevation of the sun measured near the Old State House in Philadelphia, at an instant when the sun was exactly south, and when its declination was zero, was 50.05°; what is the latitude of this point?
6. The angle of elevation of the sun when its direction is south, measured at a place P, was 75° 10′. The declination at this time was +13° 45′. What is the latitude of the place of observation?

Another Method of Finding the Latitude of a Place.—The *measuring quadrant* illustrated in Figure 6 was first used by Tycho Brahe, a Danish astronomer (1546–1601), to measure the angles of elevation of stars. If the angle of elevation *HPS* of the pole star (Polaris) is measured at any time of the year, (Figures 6 and 7) the latitude *L* of the place *P* can be found. The reason of this is that the pole of the earth extended northwards to its point of intersection with the "celestial sphere" is very close to Polaris. Polaris, however, describes a small circle around the pole, and therefore the angle of elevation of the star must, in general, be corrected to obtain the angle of elevation of the pole. This

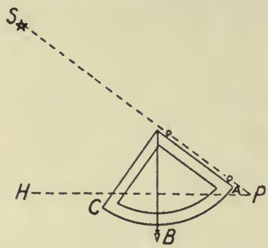


FIG. 6.

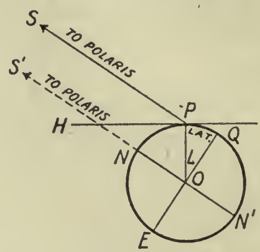


FIG. 7.

method is troublesome to apply because the crosshairs of the telescope must be illuminated, since the observation is made at night. The correction of the angle requires the use of certain tables published by the Government Printing Office at Washington, D. C.

EXERCISES

7. Show what arc of the quadrant, Figure 6, is read to give the value of angle *HPS*, and show what relation $\angle HPS$ has to the latitude angle *L*, of Figure 7.
8. The angle of elevation of Polaris is measured at a certain place $= 49° 18'$. The correction to be applied to obtain the angle of elevation of the pole is $- 1° 2'$. Find the latitude of the place of observation.

FIELD WORK

89. DEPARTMENT OF MATHEMATICS———SCHOOL.

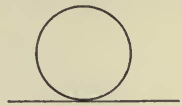
To determine the latitude of ———School.

Geometrical principle employed:

Two acute angles whose sides are perpendicular, each to each, are equal.

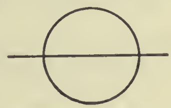
METHOD.—(1) Set up and accurately level the transit, about 15 minutes before noon on March 21st, or September 23d.

(2) Set the horizontal crosshair tangent to the lower edge of the sun; as the angle of elevation of the sun increases, follow it up with the crosshair, until the angle of elevation has reached its maximum.



(3) Read the vertical angle; level the telescope by the telescope spirit level and read the index error of the vertical circle.

(4) Add 0° 15′ to the value of the angle of elevation, to allow for parallax, refraction and the radius of the sun.



(5) Latitude = 90° − corrected angle of elevation of sun.

Note: Dark glasses must be fastened in front of the object glass of the telescope.

To determine the latitude of a place with accuracy within one or two minutes, the Nautical Almanac must be used.

The latitude of a place can be determined on any day of the year, if the *meridian altitude* of the sun is measured on that day, and the tables found in the Nautical Almanac for declination, parallax, refraction and semi-diameter of the sun, are consulted.

CHAPTER IV

TRIANGLES

PRINCIPLES DETERMINED EXPERIMENTALLY

90. Triangles Defined.—A triangle is a plane figure bounded by three straight lines.

The *sides* are the bounding lines.

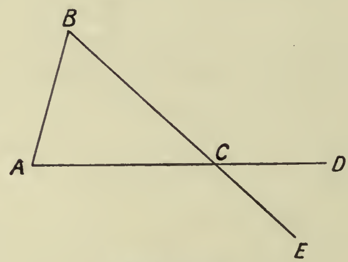
The *perimeter* is the sum of the sides.

The *interior angles,* or the *angles,* of the triangle are the angles formed by the sides.

The *vertices* of the triangle are the vertices of the angles.

An *exterior angle* is an angle formed by one of the sides extended and the adjacent side not extended.

The *parts of a triangle* are the three sides and the three angles.



EXERCISES

1. Name the triangle of the figure. Name the sides; the angles by a single letter; the vertices; the exterior angles which are drawn.
2. What angle of the figure is neither an interior nor an exterior angle? To which angle is it equal?
3. Draw a triangle with all possible exterior angles.
4. What is the relation between an interior and an exterior angle at the same vertex?

91. Triangles Classified.—*The classification is based upon:* (1) the relative length of sides; (2) the size of angles.

According to sides, a triangle is called:

Equilateral, when the three sides are equal.

Isosceles, when two sides are equal.

Scalene, when no two sides are equal.

According to angles, a triangle is called:

Equiangular, when the three angles are equal.

Right, when one angle is a right angle.

Obtuse, when one angle is obtuse.

Acute, when all angles are acute.

Oblique, when it is not a right triangle.

EXERCISES

1. Draw triangles of each of the eight kinds.
2. Draw triangles of every possible kind, combining a name from the first list with one from the second; as, obtuse-isosceles; etc.
3. What triangle is the *general type*?

92. Relations Between Triangles.—*Cóngruent triangles* are triangles which are alike in all respects, and which can be made to coincide.

Similar triangles have the same shape, or angles, but not the same size, or lengths of sides. They are considered in Chapters X and XI.

The *homólogous parts* of congruent or similar triangles are the correspondingly situated parts.

EXERCISES

1. Draw two congruent triangles. Name all the homologous sides, angles, vertices, and exterior angles.
2. Draw two similar right triangles. Name the homologous sides and angles.

93. Position of the Parts of a Triangle.

An angle is *included* by two sides.

A side is *included* by two angles.

Two angles are *adjacent* to each side.

Each side is *opposite* an angle.

Each angle is *opposite* a side.

EXERCISES

1. Draw a triangle *ABC;* letter the sides opposite the angles *A*, *B*, *C*, respectively *a*, *b*, *c*. This will be called the *standard lettering* of a triangle.
2. What angle is included by *a* and *c*? What sides include *C*? What angles are adjacent to *a;* to *b*? What angle is opposite *b*? What side is opposite *C*? What side is included by angles *A* and *C*?

Names of Special Parts of Triangles.

In an isosceles triangle:

The *base* is the unequal side.

The *vertex angle* is included by the *equal sides.*

The *base angles* are adjacent to the base.

In a right triangle:

The *perpendicular sides* include the right angle.

The *hypotenuse* is opposite the right angle.

EXERCISES

1. Draw an isosceles triangle. Name all the parts.
2. What angle of the triangle in Exercise 1 is opposite the base? What angles are opposite the equal sides?
3. Draw a right triangle. Name all the parts.
4. What angles of the right triangle include the hypotenuse? What angles are opposite the perpendicular sides?

In any triangle:

The *base* is the side on which it stands; but any side may be called the base.

An *altitude* is a line drawn from a vertex perpendicular to and meeting the opposite side.

A *median* is a line drawn from a vertex to the middle point of (bisecting) the opposite side.

EXERCISES

5. Draw a triangle with all possible altitudes.
6. In what kind of triangle does one altitude (or more) fall upon a side produced?
7. In what kind of triangle are some of the sides the altitudes?
8. Draw a triangle with all possible medians. Will the medians of a triangle ever fall outside the triangle?

Abbreviations.

altitude	alt.	isosceles	isos.
congruent, is congruent to	$\cong$	similar	$\sim$
homologous	hom.	triangle, triangles	$\triangle$, $\triangle\!\!\!\triangle$

94. Experiment I.—Properties of the angles of a triangle.

(*a*) The number of obtuse angles possible in a triangle.

(*b*) The number of right angles possible in a triangle.

(*c*) The number of acute angles possible in a triangle.

(*d*) The sum of the angles of a triangle.

(*e*) The relation between an exterior angle and the interior angles of a triangle.

Part (*a*).—(1) Draw a triangle with one obtuse angle. (2) Draw, if possible, a triangle with more than one obtuse angle. State result.

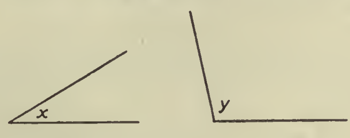
Part (*b*).—Make the same experiment for one or more right angles. Notice that an infinite triangle can be drawn with two right angles. State result.

Part (*c*).—How many acute angles may a triangle have? How many must it have? State result.

Part (*d*).—Measure the angles of several triangles of different shapes with the protractor. State result.

Part (*e*).—Measure the exterior angles of the triangle of part (*d*) and compare their values with the sum of certain interior angles. State result.

95. Experiment II.—**A method of finding the third angle of a triangle when two angles are known, without drawing the triangle.**



Method (*a*).—Draw the two given angles, x and y. Draw the angles x and y adjacent, and find the angle z which completes a straight angle. z is the third angle required.

Method (*b*).—Measure the given angles with a protractor, and find the third angle without drawing it.

EXERCISES

1. Two of the angles of a triangle are 75° and 38°. Calculate the third angle without a drawing.
2. Can a triangle be drawn with the angles 81°, 47°, 52°? With the angles 35°, 120°, 40°? With the angles 20°, 136°, 12°?
3. Can a triangle be drawn with two (interior) angles 75° and 45° and the exterior angle at the third vertex equal to 120°.
4. If the interior angles A, B, C of a triangle are 70°, 30° and 80°, calculate the three exterior angles A', B', C', adjacent respectively to angles A, B, C.
5. If all the angles of a triangle are equal, how large is each?
6. If one angle of a triangle is 126°, and the other two angles are equal, how large is each exterior angle?
7. If two angles of a triangle are each 72° 30′, calculate the third angle.
8. One angle of a triangle is double another angle, and the third angle is 20° more than the smallest angle. Calculate the angles and sketch the triangle.

96. Construction of Triangles from Given Parts.—In general, three given parts, sides or angles, are sufficient to construct a triangle. If less than three parts are given, the triangle is not determined. If more than three parts are given, the triangle cannot be constructed unless the additional parts agree by chance or by calculation with the triangle as determined by three given parts.

Different Combinations of Three Parts of a Triangle.

(1) Three sides.

(2) Two sides and one angle; of which there are three possible arrangements:

- (*a*) Two sides and the included angle.
- (*b*) Two sides and the angle opposite the greater of the two sides.
- (*c*) Two sides and the angle opposite the less of the two sides.

(3) One side and two angles; of which there are three possible arrangements:

- (*a*) The side included by the two angles.
- (*b*) The greater angle opposite the side and the less angle adjacent to it.

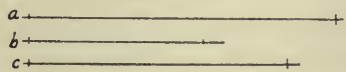
(*c*) The less angle opposite the side and the greater angle adjacent to it.

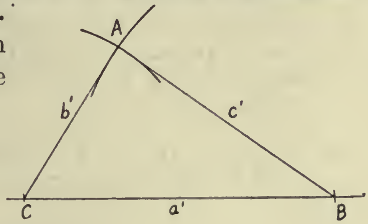
(4) Three angles.

Values of the Given Parts.—The lengths of sides may be given (1) equal to sects which are drawn, or (2) by numerical measure. The size of angles may be given (1) equal to drawn angles, or (2) by numerical measure.

97. Experiment III.—A method of drawing a triangle of which three sides are given.

Given.—Sects a, b, c, from which the sides of the triangle are to be measured.





METHOD.—(1) Measure $CB = a$.

(2) Draw two arcs, centers at C and B, with radii equal respectively to b and c, intersecting each other at A.

(3) Draw AC and AB.

(4) ABC is the required triangle.

Discussion.—(1) If $a = b + c$, the triangle becomes a straight line.

(2) If $a < b + c$, no triangle is possible.

EXERCISES

1. Does the order of placing the sides affect the shape or size of the triangle?
2. Construct a triangle whose sides are 3.2, 2.4, and 2.7 inches. Measure the first side from the scale, and measure the two radii from the scale. Measure the angles of the triangle.
3. Are triangles possible whose sides are (1) 12, 17, 15; (2) 106, 363, 257; (3) 93, 71, 50; (4) 28, 28, 15; (5) 80, 80, 80; (6) 29, 81, 46?
4. Construct an isosceles triangle whose sides are 3, 3 and 4 inches.
5. Construct an equilateral triangle whose sides are each 3.5 inches.

98. Experiment IV.—To draw a triangle of which two sides and the included angle are given.

Helps.—Draw two sects a and b, and an angle C, from which the given parts of the triangle are measured.

Describe the method as in **97.** Are there any special or impossible cases?

EXERCISES

1. Construct a triangle of which two sides are 2.9 and 1.5 inches, and the included angle is 57°. Measure the sides from the scale and the angle directly from the protractor. Measure the remaining side and angles.
2. Are triangles possible with the parts (1) $a = 150$, $b = 3$, $C = 86°$; (2) $a = 10$, $b = 10$, $C = 162°$; (3) $a = 7$, $c = 5$, $B = 215°$?
3. Construct an isosceles triangle whose equal sides are each 3.0 inches and whose vertex angle is 70°.
4. Construct a right triangle whose perpendicular sides are 4.2 and 1.7 inches.

99. Experiment V.—To draw a triangle of which two sides and the angle opposite the greater of the two sides are given.

Construct the figure, and describe as in **97.** Note any special or impossible cases.

EXERCISES

1. Construct a triangle of which two sides are 2.9 and 1.5 inches, and the angle opposite the 2.9-inch side is 57°. Measure the remaining parts.
2. Construct a right triangle of which the hypotenuse is 4.0″ and a perpendicular side is 2.3″.

100. Experiment VI.—To draw a triangle of which two sides and the angle opposite the less of the two sides are given.

This is the most complicated construction of a triangle from three given parts. Several cases occur.

(1) Use these given measurements. Construct two entirely different triangles (two figures) from the measurements, the required relative position being maintained in both figures.

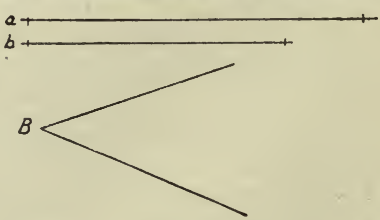


Fig. 1.

(2) Use these given measurements. What do you observe in this case as to the number of possible triangles and the kind of triangles obtained?

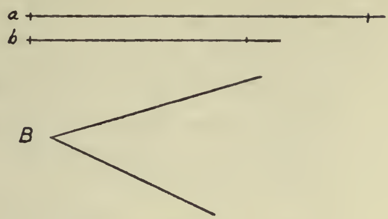


FIG. 2.

(3) Use these given measurements. What do you observe in this case?

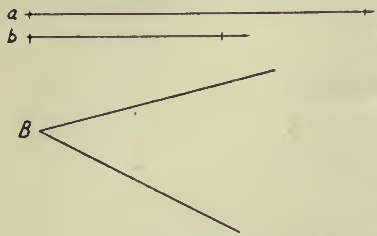


FIG. 3.

(4) Draw any two sects a and b, with $a>b$, and a right angle B. Construct the triangle, if possible.

(5) Draw any two sects a and b, with $a>b$, and an obtuse angle B. Construct the triangle, if possible.

In order to classify these results, draw a perpendicular in Figures (1), (2) and (3), from vertex C of the triangle, to side c, as in Figure 4. Call this the perpendicular distance, p, of the triangle.

RESULT.—When the given angle is acute and (1) when $b>p$, two different triangles can be constructed; (2) when $b = p$, one right triangle can be constructed; (3) when $b<p$, no triangle can be constructed.

When the given angle is right or obtuse (the angle being opposite the less of the two given sides), no triangle can be constructed.

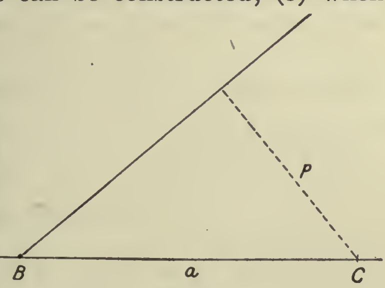


FIG. 4.

Because of the fact that two entirely different triangles can sometimes be constructed from the same measurements, this construction is called the *ambiguous case* in triangles.

EXERCISES

Construct triangles from the following measurements. Note which case arises in each construction. Measure the remaining parts in each figure (in some figures there are two sets of parts).

1. $a = 3.2''$, $b = 2.1''$, $B = 25°$	4. $a = 3.2''$, $b = 1.5''$, $B = 40°$
2. $a = 3.2''$, $b = 2.1''$, $B = 50°$	5. $a = 2.7''$, $b = 2.0''$, $B = 90°$
3. $a = 3.2''$, $b = 1.6''$, $B = 30°$	6. $a = 2.9''$, $b = 1.8''$, $B = 110°$

101. Experiment VII.—To draw a triangle of which two angles and the included side are given.

Construct the figure from given parts and observe special and impossible cases.

EXERCISES

1. Construct a triangle of which one side $= 2.8''$, and the two angles including the side are 42° and 70°.
2. Construct an isosceles triangle of which the base is $4.0''$ and the base angles are each 35°.
3. Which of the following triangles are (*a*) possible, (*b*) special, (*c*) impossible? (1) $a = 10$, $B = 50°$, $C = 75°$; (2) $a = 75$, $B = 130°$, $C = 65°$; (3) $b = 12$, $A = 170°$, $C = 10°$.
4. If the given side of a triangle is $2''$, and the two angles including the side are supplementary, what is the value of the third angle of the triangle?

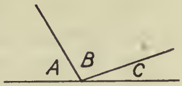
102. Experiment VIII.—To draw a triangle of which one side and two angles are given, either angle being adjacent to the given side and the other angle opposite the given side.

Helps.—(1) Find the third angle of the triangle; (2) proceed as in **101.** Notice that two possible arrangements of the given parts may be specified, but only one triangle can be drawn for each arrangement.

EXERCISES

1. Construct triangles from the following measurements: (1) $a = 3.4''$, $B = 72°$, $A = 85°$; (2) $a = 3.4''$, $A = 72°$, $B = 85°$; (3) $b = 1.5''$, $B = 87°$, $C = 135°$.
2. Construct a right triangle with the hypotenuse $= 3.5''$ and an acute angle $= 37°$.

103. Experiment IX.—**To draw a triangle of which the three angles are given.**



Helps.—(1) What relation must exist between the three given angles? (2) Observe the number of triangles that can be constructed with the same given angles.

EXERCISES

1. Show that when three angles of a triangle are known, no more is given than when two angles of the triangle are known.
2. Construct an equiangular triangle.

104. Experiment X.—Properties of an isosceles triangle.

Draw an isosceles triangle. Observe:

(1) The relative size of the base angles.

Bisect the vertex angle of the triangle. Observe:

(2) Where the bisector meets the base.

(3) The position of the bisector with respect to the base.

(4) The symmetry of the triangle.

Draw and bisect an exterior angle at the vertex. Observe:

(5) The position of the bisector.

Extend one of the equal sides of the triangle through the vertex by a distance equal to the length of a side; join the extremity of the part extended with the nearer end of the base. Observe:

(6) The position of the connecting line.

State results, including any other properties of the triangle which you discover.

105. Experiment XI.—**To draw a perpendicular to a given line through a given point, (*a*) when the point is in the line, and (*b*) when the point is without the line.**

Helps.—Draw a straight line and mark the given point. Adapt the figure of **104,** part (6), to both constructions. Thus in (*b*), the given line contains the base of the isosceles triangle, and the given point is at the extremity of the extended side of the triangle.

106. Experiment XII.—**To draw a line parallel to a given line and passing through a given point.**

Helps.—Draw a straight line and mark the given point. Adapt the figure of **104,** part (5). Describe method.

107. Experiment XIII.—Properties of an equilateral triangle.

108. Experiment XIV.—A triangle with two equal angles.

Helps.—Draw a side of the triangle; and make the two angles adjacent to the side, equal. Observe the kind of triangle that is formed.

109. Experiment XV.—A triangle with three unequal sides.

Helps.—Draw the triangle. Observe the size of the angles and how they are situated with respect to the sides.

EXERCISES

1. The sides of a triangle are $a = 5$, $b = 7$, $c = 5$. Into what parts, or segments, does the bisector of angle B divide side b?
2. The angles of a triangle are 70°, 55°, 55°. What kind of triangle is it?
3. The sides of a triangle are $a = 5$, $b = 6$, $c = 7$; the angles are about 44°, 78°, 58°. How are these angles situated in the triangle?

110. Experiment XVI.—A line joining the middle points of two sides of a triangle.

Helps.—Draw the triangle, mark the middle points of any two sides; join these points with a line. Observe (*a*) the position of this line, and (*b*) its length as compared with a side of the triangle.

111. Experiment XVII.—Concurrent lines of a triangle.

Draw acute, right and obtuse scalene triangles. In each triangle, draw:

(*a*) The three altitudes.
(*b*) The three medians.
(*c*) The three angle bisectors.
(*d*) The three perpendicular-bisectors of the sides.
(*e*) The bisectors of the three pairs of exterior angles.

Observe.—(*a*) The concurrence of the altitudes, the *orthocenter;* (*b*) the concurrence of the medians, and where on each median the point of concurrence is situated, the *centroid*; (*c*) the concurrence of the angle bisectors, and the special position of the point of concurrence with respect to the sides of the triangle, the *incenter;* (*d*) the concurrence of the perpendicular-bisectors of the sides, and the special position of the point of concurrence with respect to the vertices of the triangle, the *circumcenter;* (*e*) the concurrence of each two exterior angle bisectors with the bisector of the non-adjacent interior angle (three points), and the special position of these points of concurrence with respect to the sides and extended sides of the triangle, the *excenters*.

112. EXPERIMENT XVIII.—**To find the center of gravity of a triangle.**

Help.—This point is found as in **111,** part (*b*).

113. Experiment XIX.—Circles related to a triangle.

(*a*) The inscribed circle, tangent to the three sides.

(b) The circumscribed circle, containing the three vertices.

(c) The three escribed circles, tangent to one side and to two extended sides.

Help.—The centers of these circles are found as in **111,** parts (*c*), (*d*) and (*e*).

EXERCISES

1. Draw an isosceles triangle. Draw the altitudes, medians, angle-bisectors, perpendicular-bisectors of the sides, and bisectors of the exterior angles. Draw the inscribed, circumscribed and escribed circles.
2. Draw as in Exercise 1, for an equilateral triangle.

114. REVIEW EXERCISES

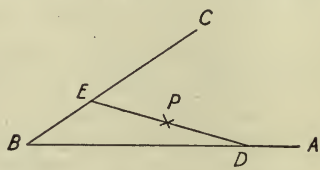
Construct the following triangles from the given parts:

1. A right isosceles triangle whose equal sides are given; that is, equal to a given sect or to a given measurement.
2. An isosceles triangle whose base and equal sides are given.
3. An isosceles triangle whose equal sides and base angles are given.
4. An isosceles triangle whose base and vertex angle are given.

5. A triangle whose altitude and base angles are given.
6. A triangle whose altitude, base, and one base angle are given.
7. An equilateral triangle, whose altitude is given.
8. An isosceles triangle whose base and altitude are given.
9. A triangle of which one side, the opposite angle and the altitude to another side are given.
10. A triangle of which a side, the altitude to the side and the median to the side are given.
11. (*a*) A triangle of which two sides are 3.0 and 2.3 inches, and the altitude to the shorter side is 1.5 in. (*b*) A triangle with the same given sides and the altitude to the longer side 1.5 in.
12. A triangle of which a side, the altitude to that side, and the radius of the circumscribed circle are given.
13. Are any of the constructions of examples 1 to 12 impossible for certain relative values of the given parts?
14. Construct accurately without a protractor:

(*a*) an angle of 60°; (*d*) an angle of 15°;
(*b*) an angle of 30°; (*e*) an angle of 75°;
(*c*) an angle of 120°; (*f*) an angle of 105°.

15. Draw a straight line through a given point, such that the perpendiculars to the line from two other given points shall be equal.
16. Draw an angle ABC and a point P within it. Find a line DE passing through P, such that the part of the line included by the sides of the angle is bisected at point P. *Help.*—Apply a principle similar to that of **110**.



17. Draw two sects a and b, and two unequal angles, C and C', of which $C>C'$. Construct a triangle having two sides and the included angle equal respectively to a, b, C. Construct a second triangle having two sides and the included angle equal respectively to a, b, C'. Observe the relative lengths of the third sides c and c' of the two triangles.
18. Draw four sects a, b, c, c'; of which $c>c'$. Construct a triangle having three sides equal respectively to a, b, c. Construct a second triangle having three sides equal respectively to a, b, c'. Observe the relative size of the angles C and C'.
19. If a triangle is drawn from the measurements, $a = 7$, $c = 12$, $B = 75°$, and a second triangle is drawn from the measurements $a' = 7$, $c' = 12$, $B' = 105°$, how will the sides b and b' of the two triangles compare in length?

Applications

115. Surveying and Engineering Measurements.—In each of the following problems make a sketch of the figure with the proper lettering, and write all given values on this sketch. Then select a suitable scale and construct the figure accurately by the methods of **97–102.** Find the required values by measurement from the completed figure.

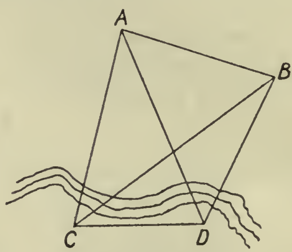
Accurate solutions of these problems may be made by *trigonometry*, which is a development of geometry and algebra for the purpose of calculating the required parts of triangles from the given parts. A surveyor or engineer would construct the figures to scale, as is required in these exercises, as a check on the more accurate calculated values.

The line measurements given are made with a steel tape and the angle measurements are made with a transit.

EXERCISES

1. A tree, C, is observed from two points, A and B, 1000 feet apart on a straight road. The angles CAB and CBA are respectively 70° and 45°. How far is the tree from each point? Use a linear scale either of 500 feet to the inch, or of 200 feet to the inch.
2. A farm is bounded by three straight roads, the sides of the farm being 850 feet, 1000 feet, 960 feet. What angles do the sides of the farm make at the three vertices?
3. A vertical pole, 22 feet high, casts a shadow at mid-day 35 feet long on a horizontal plane. What is the angle of elevation of the sun?
4. An aeroplane is directly over a field that is known to be $3\frac{1}{2}$ miles distant. The angle of elevation at the point of observation is 10°. How high above the ground is the aeroplane?
5. The distance, AB, across a pond is desired. A point C, is selected from which CA and CB can be measured 1300 ft. and 890 ft. respectively, and at which the angle ACB is measured 42°. Find AB.
6. The distance of a lighthouse A, from a point B, on shore is required. A line BC is laid off on shore 1500 ft. long. The angles between this line and the direction of the lighthouse are measured at points B and C, 73° and 81° respectivley. How far is the lighthouse from B?

7. Two steamers leave the same port at the same time. One sails northwest 15 miles an hour; the other east 19 miles an hour. How far apart are they after 5 hours?
8. A bridge is to span a river. The positions of the abutments on the banks of the river are marked by stakes A and B. A stake C is placed on the same side of the river as A, 2460 feet along the bank; the angles BAC and BCA are measured 80° and 57° respectively. Find distance AB across the river.
9. The height of a flagpole is desired. At a distance of 130 feet from the base the angle of elevation of its top is 50°. Find the height of the pole.
10. The distance between two buildings is 1500 feet, and the distance of a point of observation from the nearer building is 2400 feet. The angle between lines from the point of observation to the buildings is 20°. How far is the point of observation from the other building? The omission of what word would make this construction ambiguous?



11. A surveyor often uses methods involving several triangles, in order to find some distance that cannot be directly measured. A and B are two buoys in a harbor, whose locations are to be marked on a chart. A line CD is laid out on shore, and is also drawn to scale on the chart; C being nearer to A, and D nearer to B. CD is measured 500 feet; angles ACD, BCD, BDC, ADC, are measured respectively 115°, 45°, 110°, 60°. Plot accurately and measure AB.

CHAPTER V

TRIANGLES

Classification and Explanation of Principles

116. The First Step in Classification.—The most elementary principles relating to triangles, based upon the experimental results of Chapter IV, are:

Postulate 15.—Two triangles are congruent if the three sides of one triangle, are equal respectively to the three sides of the other triangle (**97**).

Postulate 16.—Two triangles are congruent if two sides and the included angle of one triangle, are equal respectively to the two sides and included angle of the other triangle (**98**).

Postulate 17.—Two triangles are congruent if two sides and the angle opposite the greater of the two sides of one triangle, are equal respectively to two sides and the angle opposite the greater of the two sides of the other triangle (**99**).

Postulate 18.—Two triangles are *not necessarily* congruent if two sides and the angle opposite the less of the two sides of one triangle, are equal respectively to two sides and the angle opposite the less of the two sides of the other triangle (**100**).

Postulate 19.—Two triangles are congruent if one side and any two angles of one triangle, are equal respectively to one side and the two homologous angles of the other triangle (**101, 102**).

Postulate 20.—Two triangles are *not necessarily* congruent if the three angles of one triangle are equal respectively to the three angles of the other triangle (**103**).

These postulates mean that if two triangles are known to have any three parts respectively equal (with two exceptions), they are alike in all respects; that is, the remaining parts are also respectively equal.

117. The Second Step in Classification.—Most of the other principles relating to triangles can be explained from the elementary principles, or Postulates, of **116,** and from the previously analyzed and classified principles of Chapter III.

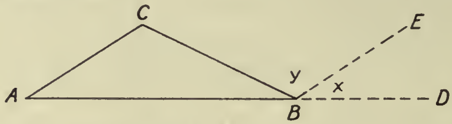
118. Theorem I (of triangles).—*The sum of two sides of a triangle is greater than the third side.*

Help.—Apply Postulate 4.

Corollary.—*The difference between any two sides of a triangle is less than the third side.*

Help.—Apply Axiom 3 to the conclusion of Theorem I.

119. Theorem II.—*The sum of the angles of a triangle is a straight angle* (**94**).



State the *hypothesis* and *conclusion.*

Analysis.—(1) Extend AB to D and draw $BE \| AC$; what postulate?—(2) x and y are equal to angles of the triangle; what postulate and theorem of parallels?—(3) $x+y+B$ = a straight $\angle$; what axiom?—(4) substitute.

Write the preceding analysis in the form of **69, 72,** etc. This is one of the most famous theorems, and is believed to have been analyzed by Pythagoras and Aristotle, as well as by Euclid.

Corollaries.—1. *A triangle has at least two acute angles* (**94**).

2. *When two angles of a triangle are known, the third angle can be found.*

3. *The two acute angles of a right triangle are complementary.*

4. *When two angles of one triangle are equal respectively to two angles of another triangle, the third angles are also respectively equal.*

120. Theorem III.—*An exterior angle of a triangle equals the sum of the two non-adjacent interior angles* (**94**).

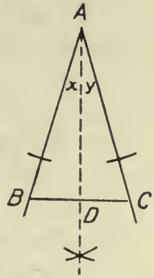
Corollary.—*An exterior angle of a triangle is greater than either of the non-adjacent interior angles.*

121. Theorem IV.—*The base angles of an isosceles triangle are equal* (**104**).

Hypothesis.—Isosceles triangle ABC.

Conclusion.—$B = C$.

Analysis.



Statement	Reason
1. Draw AD bisecting $\angle A$	Postulate 12
2. Compare triangles ABD and ACD	
3. $AB = AC$	Hypothesis
4. $AD = AD$	Identical
5. $\angle x = \angle y$	Made so in drawing AD
6. $\triangle ABD \cong ACD$	Postulate 16
7. $\therefore B = C$	Homologous parts of $\cong$ $\triangle$s

The method of analysis employed, is to draw an auxiliary line which will form two triangles with other lines of the figure. Three homologous parts of these two triangles are known to be equal from the method of drawing the figure. Therefore the triangles are congruent, or alike in all respects (with exceptions); and therefore any remaining homologous parts of the two triangles are equal (**116**). This method of analysis is much used. It will be found an aid in understanding congruence of triangles, if equal parts of the two triangles under consideration are overlaid with colored crayons. Thus AB and AC may be overlaid with a red crayon, AD for each triangle with a blue crayon, and angles x and y with a yellow crayon.

Corollaries.—1. *An equilateral triangle is also equiangular* (**107**).

Help.—Such a triangle is isosceles in several ways.

2. *Each angle of an equilateral triangle is two-thirds of a right angle, or* 60° (**107**).

122. Theorem V.—*The bisector of the vertex angle of an isosceles triangle bisects the base* (**104**).

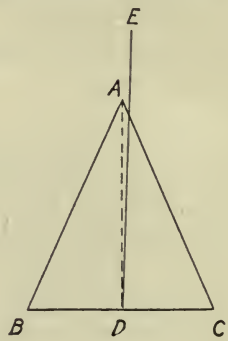
Helps.—Show that the two triangles formed in the figure are congruent by Postulate 16; and therefore two homologous sides of these triangles are equal.

123. Theorem VI.—*The bisector of the vertex angle of an isosceles triangle is perpendicular to the base* (**104**).

Helps.—Use the method of **121** and **122**. Certain angles of the two triangles are therefore equal. Then apply **84,** Theorem (9); or complete as that theorem is analyzed.

124. Theorem VII.—*The median drawn to the base of an isosceles triangle* (*a*) *bisects the vertex angle and* (*b*) *is perpendicular to the base* (**104**).

125. Theorem VIII.—*The altitude drawn to the base of an isosceles triangle* (*a*) *bisects the vertex angle, and* (*b*) *bisects the base* (**104**).



126. Theorem IX.—*The perpendicular-bisector of the base of an isosceles triangle* (*a*) *contains the vertex of the triangle, and* (*b*) *bisects the vertex angle* (**104**).

Hypothesis.—Isosceles $\triangle ABC$, with DE the perpendicular-bisector of BC.

Conclusion.—(*a*) DE contains point A; (*b*) DE bisects $\angle A$.

Analysis.

Part (*a*):

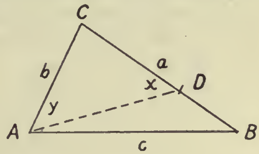
Statement	Reason
1. As DE may not contain point A, draw AD	Postulate 12
2. $AD \perp BC$	Theorem VII (**124**)
3. But $DE \perp BC$	Hypothesis
4. $\therefore DE$ coincides with AD	Postulate 10
5. $\therefore DE$ contains point A	Postulate 21 (see **127**)

127. A Postulate. *Postulate* 21.—If two straight lines coincide, any property of one of the lines belongs to the other line also.

128. A General Theorem of Isosceles Triangles.—Theorems V to IX may be summarized thus: In an isosceles triangle, the bisector of the vertex angle, median to the base, altitude to the base, and perpendicular-bisector of the base, all coincide in one line.

Corollary.—State a similar theorem for an equilateral triangle.

129. Theorem X.—*If two sides of a triangle are unequal, the angles opposite the sides are unequal, the greater angle being opposite the greater of the two sides* (**109**).

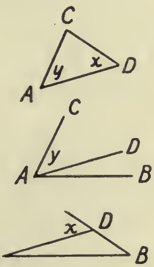


Hypothesis.—$\triangle ABC$ with $a > b$.

Conclusion.—$A > B$.

Analysis.

STATEMENT	REASON
1. Measure $CD = AC$ and draw AD	What postulate?
2. $x = y$	What theorem?
3. $A > y$	Axiom 12
4. $A > x$	Axiom 13
5. $x > B$	What theorem?
6. $A > B$	What axiom?



Corollary.—*If the three sides of a triangle are all unequal the three angles are all unequal, the size of the angles being in the same order as that of the sides to which they are opposite* (**109**).

130. Theorem XI.—*If two angles of a triangle are equal, the sides opposite these angles are equal and the triangle is isosceles* (**108**).

Helps.—Draw the triangle with two equal angles. Draw an auxiliary line bisecting the third angle. Show that the two triangles formed have three parts respectively equal and are congruent; etc.

Corollary.—*An equiangular triangle is also equilateral.*

EXERCISE

Of what theorem is Theorem XI the converse?

131. Additional Axioms.—Just as the statement of a new postulate was found necessary in **126** (stated in **127**), so some new axioms will be found necessary in Theorem XII (**133**), as there analyzed.

EXERCISES

1. If there are two possible places of concealment of an object, and one of these places has been examined without finding the object, where must the object be?
2. If there are three possible places of concealment of an object, and two of them have been examined without finding the object, where must it be?
3. How many possible conditions may exist as to the relative size of two objects, A and B? If, of these three possible conditions, two have been shown to be impossible, what may we conclude?

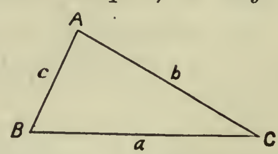
This elementary reasoning leads to the statement of the following axioms:

Axiom 14.—One of two quantities of the same kind may be larger than, equal to, or smaller than the other.

Axiom 15.—If, of three possible relations of size between two quantities of the same kind, two cannot exist, then the third relation exists.

132. Indirect Analysis.—An analysis in which a principle is established by disproving all other possible conditions, is called *indirect*. This method is applied for the first time in Theorem XII (**133**).

133. Theorem XII.—*If two angles of a triangle are unequal, the sides opposite these angles are unequal, the larger side being opposite the larger angle.*



Hypothesis.—$\triangle ABC$ with $B > C$.

Conclusion.—$b > c$.

Analysis.

STATEMENT	REASON
1. Three relations of size are possible between b and c. (a) $b > c$ (b) $b = c$ (c) $b < c$	Axiom 14

2. Relation (b), $b = c$, is impossible since then B would $= C$	Theorem IV
3. Relation (c), $b < c$, is impossible Since then B would be $< C$	Theorem X
4. $\therefore$ Relation (a) is the true one, or $b > c$	Axiom 15

Corollary.—State a corollary for three unequal angles, similar to the corollary of Theorem X.

EXERCISE

1. Of what theorem is Theorem XII the converse?

134. Converse Theorems of Isosceles Triangles.—*A triangle is isosceles, in which:*

(1) *The bisector of an angle is perpendicular to the opposite side.*

(2) *The bisector of an angle bisects the opposite side.*

(3) *A median bisects the angle from which it is drawn.*

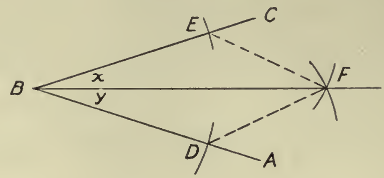
(4) *A median is perpendicular to the side to which it is drawn.*

(5) *The perpendicular-bisector of a side contains the opposite vertex.*

Help.—All of these theorems may be analyzed by showing that the two triangles into which the given triangle is divided, are congruent; therefore, etc.

135. The Position Assigned a Principle in the System of Analysis and Classification.—All the principles of Chapter IV which were discovered and proved experimentally, have now been analyzed and classified in this chapter, except the principles of **110–113.** These principles cannot be analyzed until the logical system has been carried to a more advanced stage. The system of classification has, however, now been developed to the point where certain principles of Chapter II, **28–32** inclusive, can be analyzed.

136. Theorem XIII.—*The rule and compass construction for bisecting an angle is correct* (**28**).

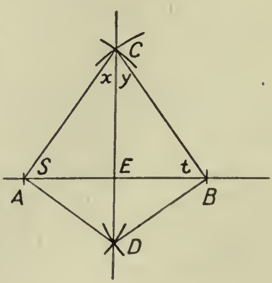


Hypothesis.— $\angle ABC$, $BD = BE$, $DF = EF$, BF drawn.
Conclusion.—BF bisects $\angle ABC$; that is $x = y$.
Analysis.

STATEMENT	REASON
1. Compare $\triangle BEF$ and BDF	
2. $BE = BD$	Hypothesis
3. $EF = DF$	Hypothesis
4. $BF = BF$	Identical
5. $\triangle$ are $\cong$	Postulate 15
6. $\therefore x = y$	Homologous parts of $\cong$ $\triangle$

137. Theorem XIV.—*The rule and compass construction for bisecting a sect is correct* (**29**).

Helps.—(1) Prove $\triangle CAD \cong \triangle CBD$; (2) $\therefore x = y$; (3) $\triangle CAB$ is isos.; (4) $\therefore CE$ bisects AB.



EXERCISES

1. Draw the two triangles separately whose congruence is first proved. Overlay the equal parts with colored crayons. Draw the isosceles triangle which is next considered, with the bisector of its vertex angle.
2. Why cannot $\triangle CAE$ and CBE be proved congruent at once from the three respectively equal parts, $CA = CB$, $CE = CE$, $s = t$?
3. After finding that $x = y$, because the $\triangle CAD$ and CBD are congruent, can the $\triangle CAE$ and CBE then be proved also congruent? Would this method of completing the analysis or that suggested be better?

138. Theorem XV.—*The rule and compass construction for erecting a perpendicular to a straight line at a point in the line is correct* (**30**).

Helps.—Use congruent triangles and **84,** Theorem (9.)

139. Theorem XVI.—State a theorem as indicated in **31.**

Help.—Analyze by a method similar to that of **137.**

140. Theorem XVII.—State the theorem indicated in **32.** Analyze it.

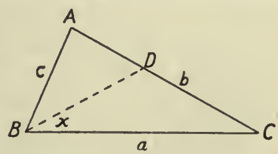
141. Variations in the Analysis of Theorems.—(1) Analyze Theorem II (**119**) by drawing an auxiliary line through $C \parallel AB$. This was probably the method employed by Pythagoras.

(2) Analyze Theorem II (**119**) by drawing lines parallel to AC and BC from any point in AB.

(3) Analyze Theorem III (**120**) by employing the two principles: the sum of the angles of a triangle is a straight angle; and, an exterior angle and adjacent interior angle are supplementary.

(4) Analyze Theorem VIII of Chapter III (**81**) by drawing an auxiliary line through B and E. Two pairs of corresponding angles are equal, and therefore angles B and E, which are the differences of these angles are equal. Similarly for angles B and H.

(5) Analyze Theorem IX of Chapter III (**82**) by extending lines BA, BC and EF until they intersect, thus forming two right triangles containing respectively the angles B and E. Relations between the angles of these triangles establishes the equality of angles B and E.



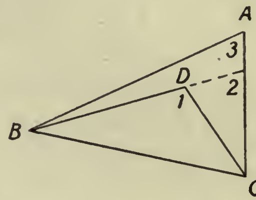
(6) Analyze Theorem XII (**133**), by a direct method, as follows: Draw BD so that $x = C$; then $BD + DA > c$: $\therefore b > c$.

142. The Third Step in Classification.—A system of classification in which all known experimental principles have been assigned a place in a proper logical sequence, may be extended into the realms of investigation and discovery. Many principles are less evident than those which form the framework of the system, and would probably never have become known without the development of the analytic, or logical method.

Some of the principles stated in **143** have probably been thus discovered; as Theorems (8), (14), (15), (16), (17), (18), (31), (42), and (47) to (53.)

Additional Theorems

143. (1) *The angle formed by two lines drawn from any point within a triangle to two vertices of the triangle, is greater than the third angle of the triangle.*

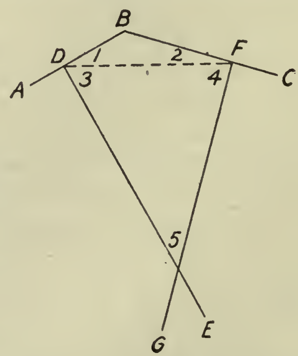


Helps.— $\angle 1 > \angle 2$; $\angle 2 > \angle 3$; $\therefore$ $\angle 1 > \angle 3$.
Give reasons.

(2) *The bisectors of any two angles of a triangle intersect.*

Helps.—Show that the angle formed by the bisectors is (*a*) less than a straight angle, and (*b*) greater than the third angle of the triangle. (The preceding theorem.)

(3) *Two lines drawn perpendicular respectively to two intersecting lines, also intersect.*



Helps.—$\angle s 1 + 2 > 0°$; $\therefore$ $\angle s 3 + 4 < 180°$; $\therefore$ $\angle 5 > 0°$; $\therefore$ DE and FG intersect.

(4) *A straight line is terminated by two parallel lines; through its middle point another straight line is drawn also terminated by the parallels. The second line is bisected by the first line.*

(5) *Two sects, AB and CD, are drawn bisecting each other. Then AC and BD are equal and parallel.* State a similar property of two other lines that may be drawn in the figure.

Helps.—(1) Use congruent triangles to show that $AC =$

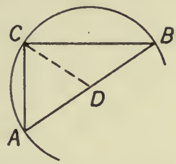
BD; (2) a pair of homologous angles of these triangles are equal, and therefore the lines are parallel by **72**.

(6) *If one of the equal sides of an isosceles triangle is produced through the vertex by a distance equal to the length of the side, and the extremity of the produced sect is joined to the nearer extremity of the base, this line is perpendicular to the base.*

Helps.—(1) Two isosceles triangles are formed, having equal angles (**121**); (2) two of these angles together form a right angle.

Where has this principle been established experimentally?

(7) *Two chords of a circle which are drawn from the same point on the circumference and which pass through the extremities of a diameter, are perpendicular.*



Help.—Identify the figure with Theorem (6).

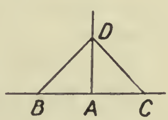
(8) *The shortest side of a right triangle whose acute angles are* 30° *and* 60°, *is one-half the hypotenuse.*

Helps.—Draw an auxiliary line dividing the right angle into two parts of 30° and 60°. One of the triangles formed is equilateral and the other is isosceles.

(9) *The exterior angles at the base of an isosceles triangle are equal.*

(10) *The hypotenuse of a right triangle is the longest side.*

(11) $AD \perp BC$, $AB = AC = AD$. *Then* $\triangle DBC$ is a *right isosceles triangle.*



(12) *The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.*

Where has this principle been established experimentally?

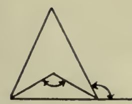
(13) State and analyze the converse of Theorem (12).

(14) *The exterior angle at the vertex of an isosceles triangle is double either (interior) base angle.*

(15) State and analyze the converse of Theorem (14).

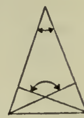
(16) *AB is the diameter of a circle with its center at O; AC is a chord; OC is a radius. Then* $\angle BOC = 2\angle BAC$.

Help.—Apply Theorem (14).



(17) *The bisectors of the base angles of an isosceles triangle form an angle equal to an exterior angle at the base of the triangle.*

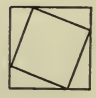
Help.—Apply **120.**



(18) *The angle formed by the altitudes drawn to the equal sides of an isosceles triangle, is supplementary to the vertex angle of the triangle.*

(19) *In a $\triangle ABC$, $\angle A$ is double $\angle B$. Then the bisector of $\angle A$ meets BC in a point D such that $AD = BD$.*

(20) *If equal distances are measured in succession on the sides of a square, and these points are joined, another square is formed.*



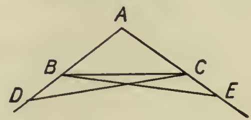
Helps.—(1) The figure has equal sides; (2) its angles are right angles.

(21) *One of the equal sides of an isosceles triangle is greater than one-half the base.*

(22) *The bisectors of the base angles of an isosceles triangle form another isosceles triangle with the base.*

(23) *The bisectors of the exterior angles at the base of an isosceles triangle form another isosceles triangle with the base.*

(24) *$\triangle ABC$ is isosceles with $AB = AC$; $BD = CE$. Then $\triangle BCD \cong \triangle CBE$.*



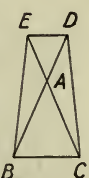
(25) *The medians drawn to the equal sides of an isosceles triangle are equal.*

Help.—Use congruent triangles (two ways).

(26) *The altitudes drawn to the equal sides of an isosceles triangle are equal.*

(27) *The bisectors of the base angles of an isosceles triangle extended to meet the opposite sides, are equal.*

(28) State a theorem concerning the lengths of the medians, altitudes, and angle-bisectors of an equilateral triangle.

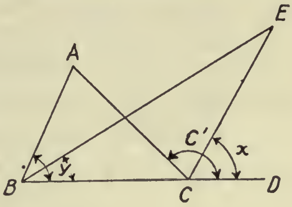


(29) *$\triangle ABC$ is isosceles with $AB = AC$; the sides are produced so that $AD = AE$; BE, CD and DE are drawn. Then $BE = CD$, and $DE \| BC$.*

(30) *Homologous medians, altitudes and angle-bisectors of two congruent triangles are equal.*

Help.—The smaller triangles formed within the given triangles, and containing the homologous lines as sides, may be proved congruent.

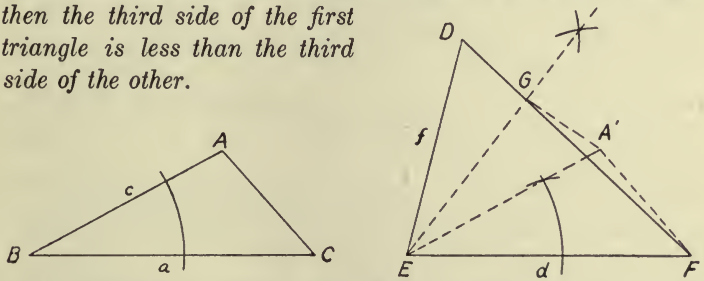
(31) *Triangle ABC has an exterior angle C′, at C. The bisectors of interior angle B and exterior angle C′, meet at E. Angle E = ½ angle A.*



Helps.—$\angle C' = \angle A + \angle B$; $\angle x = \angle E + \angle y$; $\angle C' = 2\angle x$; etc.

(32) *Two isosceles triangles are congruent if their vertex angles and equal sides are respectively equal.*

(33) *If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first triangle is less than the included angle of the other triangle, then the third side of the first triangle is less than the third side of the other.*



Hypothesis.—$\triangle$s ABC and DEF with $a = d$, $c = f$, $\angle B < \angle E$.

Conclusion.—$AC < DF$.

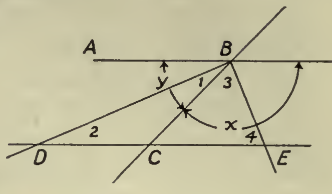
Analysis.—(1) Place $\triangle ABC$ in the position $A'EF$;—(2) draw EG bisecting $\angle DEA'$;—(3) $\triangle DEG \cong \triangle A'EG$;—(4) $\because$ $GD = GA'$;—(5) but $FG + GA' > A'F$;—(6) $\therefore DF > AC$.

(34) State the converse of Theorem (33.)

Helps.—Use the indirect method of analysis depending upon Axioms 14 and 15.

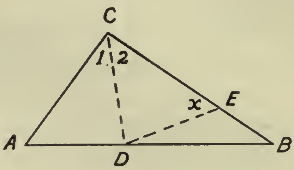
(35) *The symmetrical impression of a straight line with respect to a center of symmetry is parallel to the given line.*

Helps.—(1) Two triangles are congruent; (2) two angles are equal; (3) ∴ the lines are parallel.



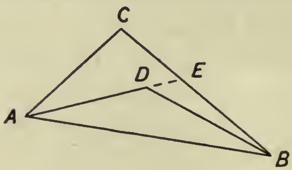
(36) *AB*$\|$*DE; any transversal BC; BD and BE bisecting* $\angle$s *x* and *y*. *Then CD = CE.*

Helps.—(1) $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$;—(2) ∴ $\triangle$s *BCD* and *BCE* are both isosceles;—(3) ∴ *CD* and *CE* are both equal to *BC*.



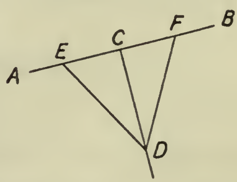
(37) *Analyze Theorem X* (**129**) *as follows:*—(1) *Draw CD bisecting* $\angle$ *C, and make CE = CA;*—(2) $\triangle CAD \cong \triangle CDE$;—(3) ∴ $\angle x = \angle A$;—(4) ∴ *it can be shown that* $\angle A > \angle B$.

(38) *The sum of two lines drawn from any point within a triangle to the extremities of one of the sides, is less than the sum of the other two sides of the triangle.*



Helps.—(1) In $\triangle ACE$, $AC + CE > AD + DE$;—(2) in $\triangle DEB$, $DE + EB > DB$;—(3) add these inequalities (Axiom 9);—(4) eliminate *DE* (Axiom 7);—(5) the conclusion follows.

(39) *Two oblique lines drawn from the same point in a perpendicular and cutting off equal distances from the foot, are equal* (**50**).



(40) State the converse of Theorem (39.)

(41) *A perpendicular is the shortest distance from a point to a straight line* (**50**).

Help.—Apply **133**.

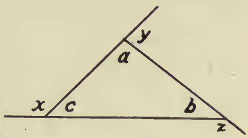
(42) *The sum of three lines drawn from any point within a triangle to the three vertices is* (*a*) *less than the sum of the sides, and* (*b*) *greater than one-half the sum of the sides.*

Helps.—(*a*) Apply Theorem (38) for three positions of the figure, and add the inequalities; (*b*) apply **118.**

(43) *In a quadrilateral ABCD, AB is the longest of the four sides and CD is the shortest.* Then $\angle C$ *is greater than* $\angle A$, *and* $\angle D$ *is greater than* $\angle B$.

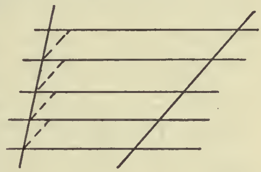
Help.—Draw a diagonal and apply **129.**

(44) *The sum of the three exterior angles of a triangle (one at each vertex) is a constant value.*



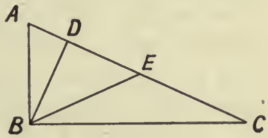
Helps.—State the value of each exterior angle in terms of interior angles; add; substitute the known value of $\angle$s $a + b + c$.

(45) *If equal segments are laid off on a straight line and parallels are drawn through the points of division, any other line cutting the parallels is also divided into equal segments.*

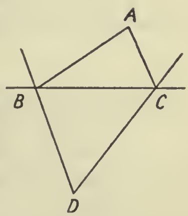


(46) *Any side of a triangle is less than one-half the perimeter.*

(47) *An altitude BD, and a median BE, are drawn to the hypotenuse of the right triangle ABC.* Then $\angle DBE = \angle A - \angle C$.



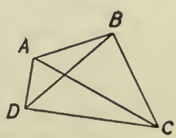
(48) *The bisectors of the exterior angles at B and C of* $\triangle ABC$, *meet at D.* $\angle D = 90° - \frac{1}{2} \angle A$.



(49) *An altitude drawn to any side of a triangle, divides the angle from whose vertex it is drawn into two angles whose difference equals the difference between the other angles of the triangle.*

(50) *The perimeter of any quadrilateral is greater than the sum of the diagonals.*

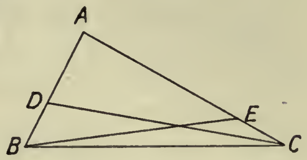
Helps.—Apply **118** to four triangles in such a way that the sides of the quadrilateral are placed on the greater sides of the inequalities; add and divide by 2.



(51) *The sum of the diagonals of any quadrilateral is greater than the sum of either pair of two opposite sides.*

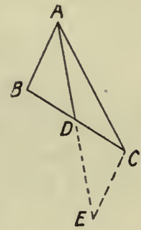
Helps.—Apply **118** to two of the small triangles in such a way that the segments of the diagonals are placed on the greater side of the inequalities; add.

(52) *In* $\triangle ABC$, $AC > AB$, $BD = CE$. *Then* $CD > BE$.



Helps.—(1) $\angle ABC > \angle ACB$; why?—(2) then apply Theorem (33) of this section, to the $\triangle$'s DBC and EBC.

(53) *The median drawn to a side of a triangle is less than one-half the sum of the other two sides.*



Helps.—(1) $AE <$ what two lines?—(2) $\triangle ABD \cong \triangle ECD$; etc.

144. The Fourth Step in Classification.—There is always an uncertainty as to the correctness of principles whose discovery has depended entirely upon analysis. For it must be remembered that they are only an outgrowth of some of the postulates and previously analyzed principles, or theorems. If, however, any errors exist in the system of analysis and classification up to this point, they can best be detected by putting such deduced principles to an experimental test.

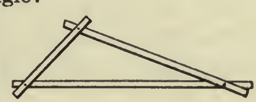
EXERCISES

1. Draw a right triangle whose acute angles are 30° and 60°. Measure the shortest side and the hypotenuse and test the relation of **143,** Theorem (8).
2. Draw an isosceles triangle, with an exterior angle at the vertex. Measure this exterior angle and the base angles of the triangle with the protractor, and test the principle of **143,** Theorem (14).
3. Draw a triangle by constructing (1) a base, (2) one of the angles adjacent to the base, (3) a side any length, (4) an exterior angle double the given base angle, (5) thus completing the triangle. Measure the sides and test the property of **143,** Theorem (15).
4. Test the principle of **143,** Theorem (16), by constructing the figure and measuring the angles with a protractor.

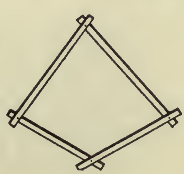
5. Test the principle of **143,** Theorem (17), by constructing the figure and measuring the angles with a protractor.
6. Test the principle of **143,** Theorem (18), by measurement.
7. Test the principle of **143,** Theorem (31), by measurement.
8. Test the principle of **143,** Theorem (42), by measuring the three sides of the given triangle and the three lines drawn from the point within it to the vertices. The sum of the three inner lines should lie between two certain values, wherever their point of concurrence is taken.
9. Test the principle of **143,** Theorem (47).
10. Test the principle of **143,** Theorem (48).
11. Test the principle of **143,** Theorem (49).
12. Test the principle of **143,** Theorem (50).
13. Test the principle of **143,** Theorem (51).
14. Test the principle of **143,** Theorem (53).

145. Review Exercises

1. One acute angle of a right triangle is 25° 36′. What is the other acute angle?
2. An exterior angle of a triangle is 98° and a non-adjacent interior angle is 57°. Find the interior angle at the third vertex.
3. Calculate the third exterior angle of a triangle two of whose exterior angles are 110° 32′ and 75° 41′.
4. The exterior angle at the vertex of an isosceles triangle is 142° 30′. Calculate the base angles.
5. May two sides of one triangle be equal respectively to two sides of another triangle, and still the two triangles be not congruent?
6. May two sides of a right triangle be equal respectively to two sides of another right triangle, and still the two triangles be not congruent?
7. Does the median of a triangle bisect its angle?
8. What is the value of the angle formed by the bisectors of the acute angles of a right isosceles triangle?
9. If three sticks are pinned together with a single rivet at each intersection, can the triangle formed be changed in shape without bending or breaking the sticks? Upon what geometric principle does the rigidity depend?



10. If four sticks are pinned together in the same way, is the frame rigid? How can another stick be added to make it rigid?



11. If any point is taken within the triangle whose sides are 7, 11, 15 inches, and lines are drawn from the point to the extremities of the 11 inch side, between what values will the sum of these two lines be found?
12. If lines are drawn from the point (Exercise 11) to the three vertices of the triangle, between what values will the sum of the three lines be found? Will the sum of the three interior lines vary with the position of the point?
13. Under which of the Postulates of **116** is each of the following principles of right triangles included? Two right triangles are congruent:
   (*a*) If the two perpendicular sides of one triangle are equal respectively to the two perpendicular sides of the other.
   (*b*) If the hypotenuse and one of the perpendicular sides of one triangle are equal respectively to the hypotenuse and a perpendicular side of the other.
   (*c*) If the hypotenuse and an acute angle of one triangle are equal respectively, etc.
   (*d*) If one of the perpendicular sides and the adjacent acute angle of one triangle are equal respectively, etc.
   (*e*) If one of the perpendicular sides and the opposite acute angle of one triangle are equal respectively, etc.
14. Under which of the Postulates of **116** is this statement included? Two isosceles triangles are congruent if the two equal sides and one of the base angles of the two triangles are respectively equal.
15. Why is it sometimes necessary to separate the classification of principles which evidently are closely associated and have been so considered when they were investigated experimentally? Give examples of such principles, some of which have been analyzed in Chapter III and some in Chapter V.

Applications

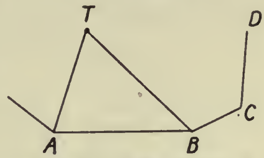
146. Surveying.—Land surveying is largely responsible for the development of geometric science, which in its turn has made scientific surveying possible.

EXERCISES

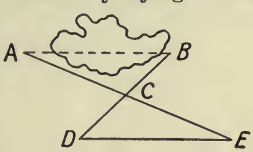
1. A surveyor measured the three angles of a triangular field, 37° 40.3′, 81° 55.0′, 60° 24.5′. What check exists as to the accuracy of the measurements?
2. It is often necessary to find the value of an angle which cannot be directly measured. In **115**, Exercise 6, calculate the value

of the angle formed by the two lines converging at the lighthouse.

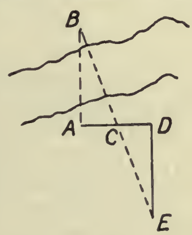
3. If the sides of a triangular field have been measured with a tape, may a map of the field be drawn to scale?
4. In what different ways may the least number of parts of a triangular field be measured so that a map of the field may be drawn?
5. AB is a known side of a field $ABCD$, etc.; T is a tree whose position is to be located on the map. What different methods may be employed to locate the tree, each giving sufficient data for the construction of the triangle ABT to scale?



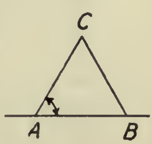
6. A farm is bounded by four straight lines. What measurements of lines only may be made in order to construct a map to scale?
7. The distance AB across a pond may be found by laying out on the ground a triangle congruent to the triangle of which one side is the required distance. Point C is any point from which AC and BC can be measured. Explain how the remaining points are located, what line equals the required line AB, and why the triangles are congruent.



8. This method may be used when the required distance AB is across a river. At A a right angle is turned (see **143,** Theorem (6), and any distance AC is measured on the perpendicular. An equal distance CD is measured. At D another right angle is turned. The surveyor finds a place E in the perpendicular, from which he sights C and B in a straight line, and drives stake E. What line is equal to the required line AB? What triangles are congruent and why?

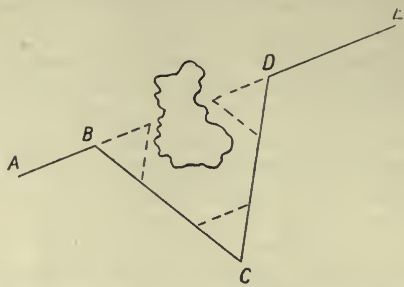


9. A triangle is sometimes used in surveying, formed by measuring AB along a given line $= 50$ feet, and then holding a 100-foot tape in the position ACB, with C at the 50-ft. mark. What is the value of angle A?

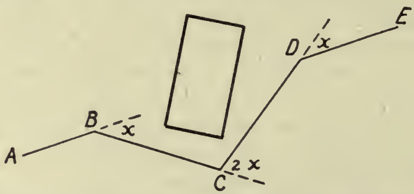


10. A surveyor may extend a line AB beyond an obstacle, as a building or pond, and also determine the unmeasured interval BD, by the method indicated in the figure. The small triangles are constructed as in Exercise 9. What

measurements are made to locate points C and D? Why is line DE an extension of AB? How is BD found?

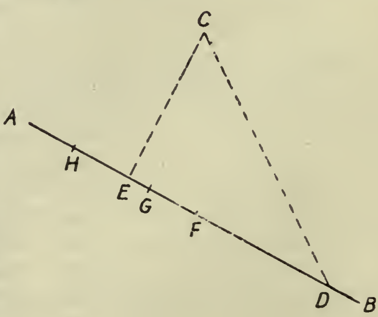


11. A line AB may also be extended beyond an obstacle by the method illustrated in the figure. What measurements must be taken in addition to the angular measurements indicated? The distance BD cannot be found by this method except by constructing the triangle BCD to scale on paper and measuring the length BD to the scale of the drawing.



The methods described in Examples 7, 8, 9 and 10 are not employed in high-grade work. More accurate methods are described in the examples of **115**, the required distances being found by trigonometric calculation.

12. A perpendicular to a line AB from a point C can be laid out very accurately as follows: Set any point D on line AB, or use the end point B if not too far away and visible from C. Set a transit at D and measure $\angle ADC$. Set the transit at C and lay off $\angle DCE = 90° - \angle ADC$; and set a stake at E on line AB.



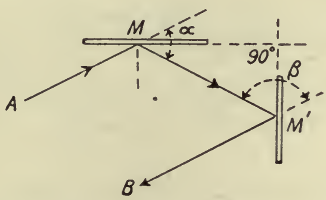
The setting of a stake at E may be done accurately by using two transits, one at B or D sighted on A, and the other at C sighted along CE; or as follows: When transit is set up at B or D set two or more stakes in line AB, as at F, G, H. Stretch a string over the tops of the stakes. Then stake E is set to this string, with the transit set up at C.

147. Light Reflected from Mirrors.—The cases of a single mirror, and of two parallel mirrors, were considered in **86.** The present problems involve principles of triangles.

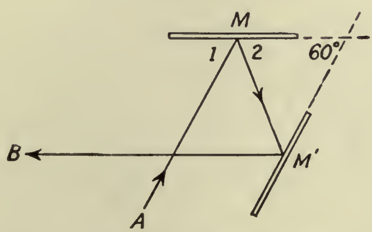
EXERCISES

1. Show that if two mirrors, M and M', are perpendicular, the incident ray A is parallel to the finally reflected ray B.

   Help.—Show that $\alpha + \beta = 180°$.

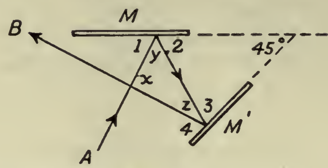


2. Show that if two mirrors, M and M', are set at an angle of 60° with each other, that the finally reflected ray B makes an angle of 60° with ray A.



3. Show that if two mirrors, M and M', are set at an angle of 45° with each other, the reflected ray B makes a right angle with the incident ray A.

Helps.—
$$
\begin{array}{llllll}
(1) & 2\angle 2 + y & & & = 180^\circ \\
(2) & & 2\angle 3 & + z & = 180^\circ \\
(3) & \angle 2 & + \angle 3 & & = 135^\circ \\
(4) & x + y & & + z & = 180^\circ
\end{array}
$$

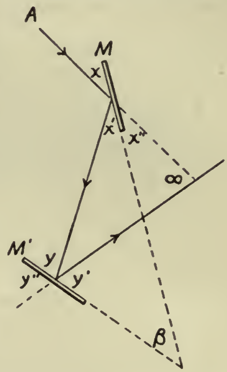


Eliminate $\angle 2$ and $\angle 3$ from Eq. (1), (2) and (3); substitute value of $y + z$ in Eq. (4).

4. Show that if two mirrors, M and M', are set at an angle β, the angle α between the directions of the incident and finally reflected rays is equal to 2β.

Helps.—(1) $x = x'$ and $y = y'$; (2) $\alpha = 2y - 2x$ (**120**); (3) $\beta = y - x$; (4) $\therefore \alpha = 2\beta$.

Notice the application of **143,** Theorem (31), to the triangles $\alpha M'M$ and $\beta M'M$.



5. Does the result of Exercise 4 agree with those of the three preceding exercises?

148. Instruments Which Measure Angles by the Use of Mirrors. *The Optical Square.*—This instrument (Fig. 1) is used for laying out a line perpendicular to a given line.

Fig. 1.

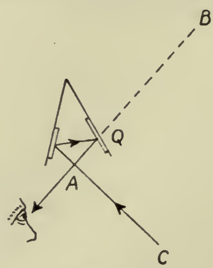


Fig. 2.

The instrument is held at a point A(Fig. 2 on line AB so that object B can be seen through the windows Q. At the same time the reflection of an object C lying on the perpendicular AC is seen in the mirror M', apparently in line with B. Upon which of the mirror formations of **147** does the operation of this instrument depend?

The Sextant.—This instrument is used to measure angles where rapidity is necessary rather than great accuracy, as in army reconnoitering. It is also used on boats since a transit cannot be used on account of the motion of the vessel. It is the only accurate angle-measuring instrument that can be so used. The instrument consists of a graduated arc of a circle of about sixty degrees (hence its name), about the center, O, of which moves an arm D. The arm D is shown at the zero reading of the arc (Fig. 3). In this position the mirrors M and M' are parallel.

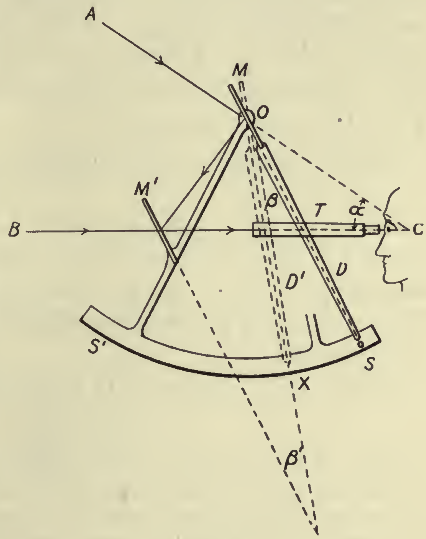


FIG. 3.

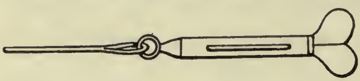
The mirror M' is fixed to the frame, and is so placed that a reflection of an object may be seen in it by looking through telescope T, at the same time that another object may be seen directly through the unsilvered part of M'. The mirror M is mounted on the movable arm D.

In order to read an angle BCA, or α, (Fig. 3), the sextant is held, either horizontally or vertically as the case may be, so that an object on one side of the angle is seen directly ahead at B. The arm D is then moved to position D' so that the object A on the other side of the angle is seen through the telescope by double reflection from mirrors

M and M', as if it also were straight ahead and coincided in position with B. Then by **147**, Exercise 4, angle $\beta' = \frac{1}{2}\alpha$. Since $\beta = \beta'$, the angle may be read on the scale ss', and $\alpha = 2$ arc sx. The scale ss' is divided into degrees of one-half size and thus the angle α is read in its true value.

The sextant is used on ship-board to measure the altitude angles of the sun, moon, planets or stars; that is, the angle from the horizon to the star or to the lower or upper edge of the sun or moon. From these angles the location of the vessel in latitude and longitude are calculated. Dark glasses are provided and must be turned into position when the sun is observed.

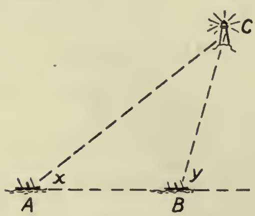
149. Navigation.—The location of a vessel at sea depends entirely upon geometric principles. The distance travelled by a vessel is measured directly by a *log*, which is dragged in the water by a cord and which registers the distance by the revolution of a propeller.

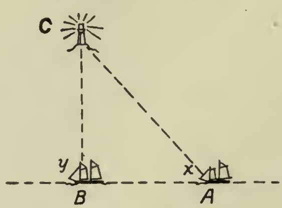


Angles are measured with a sextant (**148**).

EXERCISES

1. The distance of a ship from a lighthouse or other object is determined by the following method. At any place A, the angle x is measured; the ship's course is kept in a straight line until a position B is reached at which angle y is found to equal twice $\angle x$. The distance AB is measured meanwhile by a log. Prove that $BC = AB$. This theorem is contained in **143.**

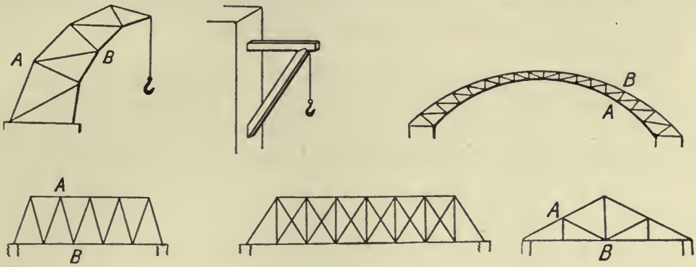




2. Another method of determining the distance at which a vessel passes an object C, is to observe position A, at which $\angle x = 45°$, and position B at which $\angle y = 90°$. What other measurement is made? How is BC found?

These methods are known to navigators as "Doubling the angle on the bow."

150. Engineering.—All engineering structures such as cranes, roof and bridge trusses, scaffolding, etc., where rigidity is necessary, are framed in a system of triangles.



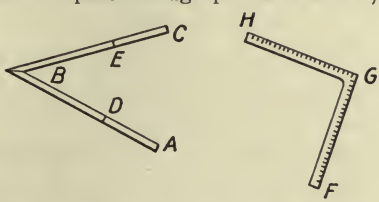
EXERCISE

If a single cross brace as AB, were to be omitted in a framed structure, would the structure be rigid? Could it then collapse without breaking a member? (See **145**, Exercises 9 and 10.)

151. Building and Carpentry.—Geometric principles are employed in all construction work.

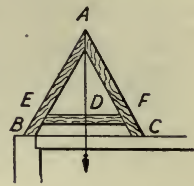
EXERCISES

1. A carpenter uses the following method to bisect an angle, ABC. Measure $BD = BE$; place a carpenter's square so that the edges GF and GH pass through points D and E, and so that the



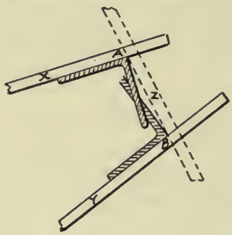
scale readings from G on the two arms are equal. Show that the method is correct. May the square be turned over so as to bring the vertex G nearer to point B?
2. A construction similar to that of Exercise 1 may be made by using your triangle, by laying off scales on the two perpendicular sides. Bisect an angle by this method.
3. A carpenter may use the following method to bisect angle ABC,

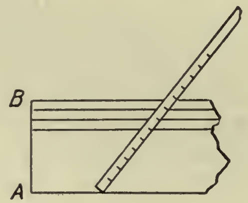
Exercise 1. Measure $BD = BE$; draw DE; bisect DE by measurement at K; draw BK. State the principle involved.



4. The builder's plumb level is sometimes used to set two points at the same level. Show that B and C are at the same level when the plumb line hangs at the middle point D, of the cross-piece. $\triangle AEF$ is isosceles, and $AB = AC$.

5. A carpenter is sometimes required to cut two timbers x and y, so that a third timber z shall make equal angles with both of them. Explain how he may use the two squares and why AB is the required line on which to cut x and y.





6. A board may be divided into any number of strips of equal width by the method shown. Explain why the perpendicular line AB is divided equally (**143,** Theorem (45)).

152. Problems for Field Work.—Suitable problems depending upon principles previously determined and classified are:

(1) To extend the curb of a road or path in the same straight line on the other side of a building that stands on the line of the road, using the method of **146,** Exercise 10, or of **146,** Exercise 11.

(2) To find the distance from an accessible point to a point on the other side of a stream, wall or street; (*a*) by the method of **146,** Exercise 8, or (*b*) by the method of **115,** Exercise 8.

In method (*a*) the required distance is measured on the ground, while in method (*b*) the plot is drawn to scale on paper and the required distance is measured on the drawing.

(3) The plot of a piece of ground bounded by straight

lines may be made by the method of **146,** Exercise 6. Trees, etc., may be located by any of the methods of **146,** Exercise 5.

(4) To lay out a tennis court, basket-ball court, etc.

(5) Measure the angles of a triangular field, in order to check the principle of **94**(*d*) and **119.**

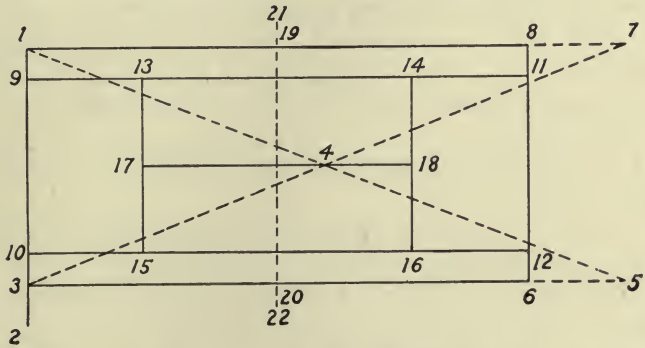
Field Work

153. Department of Mathematics ——— School

To Lay out a Tennis Court

Geometrical Principles Employed.—(1) If one of the equal sides of an isosceles triangle is extended through the vertex its own length and the end of the extended line is joined to the nearer end of the base, this line is perpendicular to the base (**104,** part (6); **143,** Theorem (6)).

(2.) Elementary properties of a rectangle.



Method.—(1) Set corner 1; set stake 2 so that 1–2 is parallel to boundary line; measure 1–3 = 36 feet.

(2) Erect ⊥'s 3 – 5 and 1 – 7; measure 3 – 6 = 1 – 8 = 78 feet.

(3) Check by measuring 8 – 6 = 36 feet.

(4) Set stakes 9, 10, 11, 12, so that 1 – 9 = 10 – 3 = etc. = 4′ 6″.

(5) Set stakes 13, 14, 15, 16, so that 9 – 13 = 18′ and 9 – 14 = 60′, etc.

(6) Set other stakes in order.

CHAPTER VI

POLYGONS

Principles Determined Experimentally

154. Polygons Defined.—A *polygon* is a plane figure bounded by straight lines.

Almost all general descriptive words are used in the same sense for polygons as for triangles, except that in a polygon having an even number of sides, two angles or two sides are *opposite,* while in a triangle and in polygons having an odd number of sides, a side and an angle are *opposite.*

A *diagonal* is a straight line joining two non-consecutive vertices.

A *convex* polygon is one in which no interior angle exceeds a straight angle.

A *concave* polygon is one in which at least one interior angle is reflex.

A *regular* polygon is both equilateral and equiangular. Their properties are considered in Chapters XIV and XV.

155. Polygons Named According to the Number of Sides.

A *quadrilateral* has four sides.

A *pentagon* has five sides.

A *hexagon* has six sides.

A *heptagon* has seven sides.

An *octagon* has eight sides.

A *decagon* has ten sides.

A *dodecagon* has twelve sides.

A *pentadecagon* has fifteen sides.

Other polygons may also be named from the Greek numeral signifying the number of sides.

156. Classification of Quadrilaterals.

quadrilateral	parallelogram	rectangle	square
			oblong
		oblique parallelogram	rhombus
			rhomboid
	trapezoid		
	trapezium		

Quadrilaterals are classified according to the parallelism of opposite sides:

A *parallelogram* has both pairs of opposite sides parallel.

A *trapezoid* has one pair of opposite sides parallel.

A *trapezium* has no sides parallel.

Parallelograms are classified according to the angles:

A *rectangle* has all right angles.

An *oblique parallelogram* has no right angles.

Rectangles are classified according to relative lengths of sides:

A *square* is equilateral.

An *oblong* has one pair of opposite sides longer than the other pair.

Oblique parallelograms are classified according to relative lengths of sides:

A *rhombus* is equilateral.

A *rhomboid* has one pair of opposite sides longer than the other pair.

157. Names of Special Parts of Quadrilaterals.

In a trapezoid:

The *bases* are the parallel sides.

The *median* connects the mid-points of the non-parallel sides.

The *altitude* is a perpendicular to the bases, included (comprehended) between them.

An *isosceles* trapezoid is one whose non-parallel sides are equal.

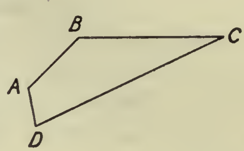
In a parallelogram:

The *bases* are the horizontal sides, if the figure is so drawn; but either pair of parallel sides may be called bases.

The *altitude* is a perpendicular to the bases, comprehended between them.

158. EXERCISES

1. Draw a trapezium; place capital letters at the vertices. Name the quadrilateral, reading the vertex letters consecutively. Draw all possible exterior angles; all possible diagonals. What parts are opposite?
2. Draw a pentagon, *ABCDE.* What angle is opposite side *AB?* What side is opposite $\angle E$? What angles are adjacent to *BC?* What angle is included by *BC* and *CD?*
3. Draw a polygon of the general shape of *ABCD.* Draw a polygon congruent to it; a polygon similar to it. Name one set of homologous sides of the three polygons; one set of homologous angles. Draw homologous diagonals in each polygon.



4. Draw convex polygons of 3 sides; 4 sides; 5 sides; 6 sides. Draw (if possible) concave polygons of 3; 4; 5; 6 sides. Draw all possible diagonals of these polygons. What special property has at least one diagonal of a concave polygon?
5. Draw (approximately) regular polygons of 3; 4; 5; 6 sides. Draw (if possible) equilateral polygons which are not equiangular, and also equiangular polygons which are not equilateral, of 3; 4; 5; 6 sides.
6. What is the least number of sides a polygon may have? the greatest number?
7. Draw the three kinds of quadrilaterals. Draw the four kinds of parallelograms.
8. What other names are given to a regular triangle? What other name has a regular quadrilateral?
9. Draw a trapezoid; the median; the altitude. Name the bases; the non-parallel sides.
10. Draw an isosceles trapezoid.
11. Draw a trapezoid one of whose non-parallel sides is the altitude.
12. Draw a parallelogram whose sides form its altitudes.
13. Draw the two altitudes of a rhomboid.

159. Experiment I.—The properties of a rhomboid. Draw a rhomboid by drawing two parallel lines cut by two other parallel lines at oblique angles and unequal intercepts. Draw the two diagonals. Observe:

(*a*) The relation between opposite sides.

(*b*) The relation between opposite angles.

(*c*) The relation between consecutive angles.

(*d*) The various pairs of congruent triangles formed.

(*e*) How the diagonals divide each other.

(*f*) The relative length of the diagonals.

State results.

160. Experiment II.—The properties of an oblong.

Observe, in addition to properties (*a*), (*b*), (*c*), (*d*), (*e*) of Experiment I:

(*a*) That there are four pairs of overlapping congruent triangles.

(*b*) The relative length of the diagonals.

State results.

161. Experiment III.—The properties of a rhombus.

Observe:

(*a*) The position of the diagonals with respect to each other.

(*b*) How the diagonals divide the angles of the rhombus.

162. Experiment IV.—The properties of a square.

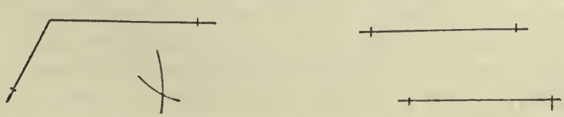
Observe the properties of the rhomboid, oblong, and rhombus which belong to a square.

163. Experiment V.—Methods of constructing a rhomboid.

Construct a rhomboid:

(*a*) By drawing two pairs of parallel sides, as in drawing the figure of Experiment I.

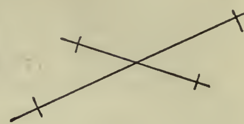
(*b*) By drawing two pairs of equal sides. Complete the figure as the other two sides fall.



(*c*) By drawing one pair of sides parallel and equal, and completing the figure as the other two sides fall.

(*d*) By drawing the two diagonals, oblique, unequal, and bisecting each other, and completing the figure as the four sides fall.

164. Experiment VI.—Methods of constructing.



From the diagonals:

(*a*) An oblong.

(*b*) A rhombus.

(*c*) A square.

165. Experiment VII.—Properties of a trapezoid.

Observe in any trapezoid:

(*a*) The position of the median.

(*b*) The length of the median compared with the lengths of the bases.

(*c*) How the median divides the diagonals.

Observe, in an isosceles trapezoid:

(*d*) Lengths of diagonals.

(*e*) Angle properties of the figure.

(*f*) Pairs of congruent triangles.

166. EXERCISES

1. Do the diagonals of a rhomboid or oblong bisect the angles of the figure?
2. Are the diagonals of all trapezoids equal?
3. Do the diagonals of any trapezoid bisect each other?
4. Two sides of a rhomboid are 21 and 36 feet respectively; what are the other sides? One of the angles is 70°; what are the other angles?
5. In what kind of quadrilaterals are the diagonals equal; equal and perpendicular; unequal and perpendicular; equal without bisecting the angles of the quadrilateral; perpendicular without bisecting the angles of the quadrilateral; axes of symmetry of the quadrilateral? In what kind may only one diagonal bisect the angles of the quadrilateral? What quadrilaterals have a center and axes of symmetry? A center of symmetry but no axes of symmetry?
6. The bases of a trapezoid are 7 and 15 inches; calculate the median.

167. Construction of Polygons from Given Parts.

EXERCISES

Construct the following figures:

1. A pentagon $ABCDE$; $AB = 1.5''$, $\angle B = 130°$, $BC = 1.0''$, $\angle C = 90°$, $CD = 2.0''$, $\angle D = 150°$, $DE = 1.6''$.
2. A pentagon $ABCDE$; $AB = 1.2''$, $BC = 2.8''$, $CD = 0.7''$, $DE = 3.0''$, $EA = 2.2''$; diagonals, $BE = 2.7''$, $CE = 3.4''$.
3. A hexagon in which each side $= 1.0''$, and each angle $= 120°$.
4. A parallelogram of which two sides and the included angle are to be equal respectively to two given sects and to one given angle.
5. A parallelogram of which two adjacent sides and a diagonal are given.
6. A parallelogram of which one side, one diagonal, and the angle included between the given side and diagonal are given.
7. A parallelogram of which two diagonals are respectively $3.0''$ and $1.7''$, and their included angle is $50°$.
8. A parallelogram of which one side, one diagonal, and the angle included between the two diagonals are given.
9. A parallelogram of which one side $= 1.6''$, a diagonal $= 4.0''$, the angle included by the two diagnals $= 45°$.
10. A rectangle of which two adjacent sides are given.
11. A rectangle of which one side and one diagonal are given.
12. A rectangle of which one diagonal and the angle between the diagonals are given.
13. A rectangle of which one diagonal and an angle between the diagonal and a side are given.
14. A square of which one side $= 2.5''$.
15. A square of which a diagonal $= 1.7''$.
16. A rhombus of which the two diagonals are respectively $3.0''$ and $2.4''$.
17. A rhombus of which one side and one diagonal are given.
18. A rhombus of which one side $= 2.0''$, and one diagonal $= 4.5''$.
19. A trapezoid of which the two bases and the altitude are given.
20. A trapezoid of which one base, the two adjacent angles, and altitude are given.
21. An isosceles trapezoid of which one base, one of the non-parallel sides, and an angle adjacent to the given base are given.
22. A trapezoid of which the median, lower base, and altitude are given.
23. A trapezium of which the four sides and their order, and a diagonal (and its position in the figure) are given

24. A trapezium of which the two diagonals and their included angle are given.
25. A trapezium of which the four sides and their order and one angle are given.
26. A trapezium of which the four sides in order are 2.0″, 1.5″, 2.4″, 3.0″, and the angle included by the 2.0″ and 3.0″ sides is 80°.
27. Which of the preceding constructions are ambiguous (two possible figures); which indefinite (any number of possible figures); which impossible?

CHAPTER VII

POLYGONS

Classification and Explanation of Principles

168. The Second Step in Classification.—No additional postulates are required in the analyses of the principles of Chapter VI, which depend upon the principles of parallels (Chapter III) and of congruent and isosceles triangles (Chapter V). The polygons of most importance (other than triangles) are parallelograms and trapezoids. In the Theorems which follow, the analysis of which depends upon congruence of triangles, draw the triangles separately and overlay the equal parts with colored crayons.

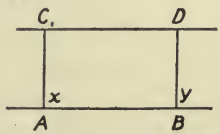
169. Theorem I.—*A diagonal of a parallelogram divides it into two congruent triangles.*

Helps.—Draw a rhomboid. Show that three parts of the triangles are respectively equal (**116**).

170. Theorem II.—(*a*) *Any two opposite angles of a parallelogram are equal;* (*b*) *any two consecutive angles are supplementary.*

Helps.—(*a*) This follows from **169,** or from **81.** (*b*) This follows from **79,** Theorem (1).

171. Theorem III.—*The opposite sides of a parallelogram are equal.*



Corollary.—*Parallels are everywhere equally distant.*

Helps.—(1) $AC \| BD$, why? (2) $AC = BD$, why?

172. Theorem IV.—*The diagonals of a parallelogram bisect each other.*

Help.—Be careful to select the triangles whose congruence will prove that the parts of the diagonals are respectively equal.

173. Theorem V.—*The diagonals of a rectangle are equal.*

Help.—The triangles selected for congruence must have the entire diagonals for their respective sides.

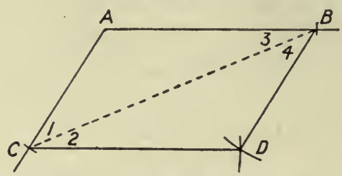
174. Theorem VI.—*The diagonals of a rhombus* (*a*) *are perpendicular, and* (*b*) *bisect the angles of the figure.*

Help.—Select two of the small adjacent triangles for congruence; or apply properties of an isosceles triangle.

175. Theorem VII.—*State and analyze all the properties of a square.*

Converse Principles

176. Theorem VIII.—*If both pairs of opposite sides of a quadrilateral are equal, the figure is a parallelogram* (**163,** part (*b*)).



Hypothesis.—Quadrilateral $ABDC$ with $AB = CD$ and $AC = BD$.

Conclusion.—$ABDC$ is a parallelogram; that is, $AB \| CD$ and $AC \| BD$.

Analysis.

Statement	Reason
1. Draw diagonal BC	Postulate
2. Compare $\triangle s\ ABC$ and BDC	
3. 4. 5. Find three respectively equal parts	Give reasons
6. $\therefore \triangle s$ are $\cong$	What postulate?
7. $\therefore \angle 1 = \angle 4$ and $\angle 3 = \angle 2$	Hom. parts of $\cong \triangle s$
8. $\therefore AC \| BD$ and $AB \| CD$	**72**

177. Theorem IX.—*If one pair of opposite sides of a quadrilateral are parallel and equal, the figure is a parallelogram.*

Helps.—Construct the figure as in **163,** part (*c*). The figure may be proved to be a parallelogram by showing that both pairs of sides are parallel, each to each, as in **176**; or by showing that both pairs of sides are equal, each to each, since we may apply the principle of **176**.

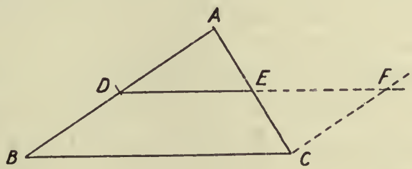
178. Theorem X.—*If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

Helps.—Both pairs of opposite sides may be proved parallel, or the principles of either **176** or **177** may be applied. Of these, the last mentioned makes the best analysis.

179. Theorem XI.—*If the diagonals of a quadrilateral bisect each other and are equal, the figure is a rectangle.*

Helps.—(1) The figure is a parallelogram (**178**); (2) prove two overlapping triangles are congruent; (3) two consecutive angles of the figure are equal; (4) these angles are also supplementary; (5) and are therefore both right angles.

180. Theorem XII.—*A line which bisects two sides of a triangle* (*a*) *is parallel to the third side; and* (*b*) *equals one-half the third side.*



Hypothesis.—

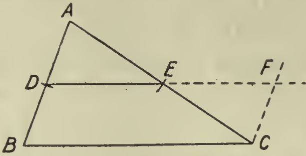
Conclusion.—(*a*).

(*b*).

Analysis.

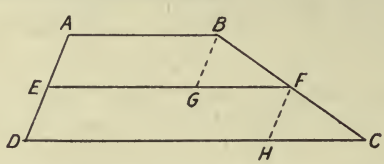
Statement	Reason
1. Extend DE; draw $CF \parallel BA$	
2. $\triangle ADE \cong \triangle EFC$	
3. $DB \parallel$ and $= CF$	
4. $DFCB$ is a parallelogram	
5. $\therefore DE \parallel BC$	
6. $DE = EF$	
7. $\therefore DE = \frac{1}{2} BC$	

181. Theorem XIII.—*A line which bisects one side of a triangle and is parallel to a second side, bisects the third side.*



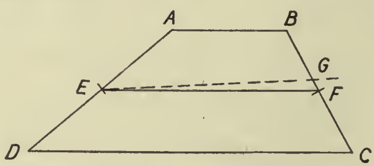
Helps.—(1) Extend DE and draw $CF \| BA$;—(2) $DFCB$ is a parallelogram;—(3) $BD = CF$;—(4) $\triangle ADE \cong \triangle EFC$:—(5) $\therefore AE = EC$.

182. Theorem XIV.—*A line which bisects one of the non-parallel sides of a trapezoid and is parallel to the bases bisects the other non-parallel side.*



Helps.—(1) Draw BG and $FH \| AD$;—(2) $\triangle BGF \cong \triangle FHC$;—(3) $\therefore BF = FC$.

183. Theorem XV.—*A line which joins the middle points of the two non-parallel sides of a trapezoid (i.e., the median) is parallel to the bases.*

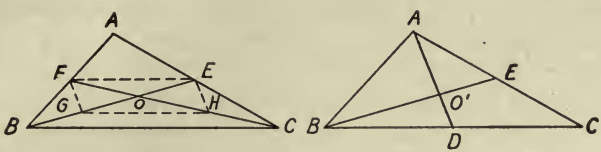


Helps.—(1) Draw $EG \| AB$ and DC;—(2) G is middle point of BC;—(3) F and G coincide (**63,** Postulate 5);—(4) $\therefore EF$ and EG coincide (**63,** Postulate 1);—(5) $\therefore EF \| AB$ and DC (**127,** Postulate 21).

184. Theorem XVI.—*The median of a trapezoid equals one-half the sum of the bases.*

Helps.—(1) Draw a diagonal; (2) the part of the median intercepted within each triangle equals $\frac{1}{2}$ a base; etc.

185. Theorem XVII.—*The medians of a triangle are concurrent in a point which is two-thirds the length of each median from its vertex.*



Helps.—Part I.—(1) BE and CF intersect in point O;—(2) Draw EF; draw GH bisecting BO and CO respectively;—(3) EF is $\parallel$ and $=$ to GH;—(4) draw FG and EH;—(5) $FEHG$ is a parallelogram;—(6) $\therefore OF = OH$ and $OE = OG$;—(7) $\therefore BO = \frac{2}{3} BE$ and $CO = \frac{2}{3} CF$.

Part II. (1) Similarly $BO' = \frac{2}{3} BE$ and $AO' = \frac{2}{3} AD$;—(2) $\therefore$ points O and O' coincide, and the medians are concurrent.

Notice that this is the first of the concurrent lines of a triangle (**111**, part (*b*)) which has been analyzed.

186. The Third Step in Classification.—The following properties of polygons (**187** and **188**) might never have been discovered if we were forced to rely upon experimental methods of investigation alone (see **142**).

187. Theorem XVIII.—*The sum of the interior angles of a polygon depends upon the number of sides of the polygon and is independent of its shape.*

Draw polygons of different numbers of sides and of different shapes; let the number of sides of each polygon be denoted by n.

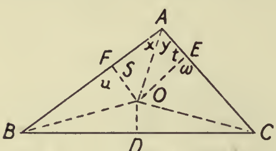
Helps.—(1) Mark any point P, within each polygon, and draw auxiliary lines from P to each vertex; (2) state the number of triangles formed in each polygon in terms of n; (3) state the sum of the angles of one triangle using the straight angle as the unit; (4) state the sum of the angles of all the triangles in terms of n; (5) deduct the sum of the angles around point P; (6) the resulting expression gives the sum of the interior angles of the polygon in terms of n, the number of sides. Express the result also in degrees.

188. Theorem XIX.—*The sum of the exterior angles of a polygon, one at each vertex, is a constant value and is independent of the number of sides of the polygon.*

Helps.—(1) State the value of the sum of an interior angle and an exterior angle at each vertex of the polygon, using the straight angle as the unit; (2) state the sum of all the interior and of all the exterior angles in terms of n; (3) state the sum of all the interior angles alone (**187**); (4) subtract; the difference is the value of all the exterior angles. Express the result also in degrees.

189.—THEOREM.—*Every triangle is isosceles.*

This theorem is found in *Mathematical Recreations and Problems*, by W. W. R. Ball; published by Macmillan and Co.



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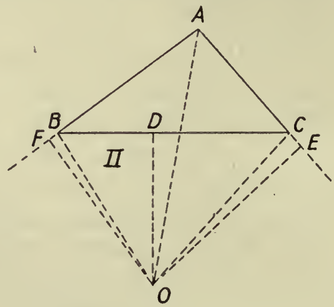


Hypothesis.—Any $\triangle ABC$.

Conclusion.—$\triangle ABC$ is isosceles.

Analysis.

STATEMENT	REASON
1. Draw DG the $\perp$-bisector of BC, AO bisecting $\angle A$, BO and CO, $OE \perp AC$, $OF \perp AB$.	Postulate
2. There are four possible cases: (*a*) DG and AO do not intersect,	

(*b*) DG and AO intersect at point D,
(*c*) DG intersects AO within $\triangle ABC$,
(*d*) DG intersects AO outside $\triangle ABC$.

Consider Case (*a*):

3. Then $DG \| AO$
4. $\therefore AO \perp BC$
5. $\therefore \triangle ABC$ is isosceles — **134.** Theorem (1)

Consider Case (*b*):

6. Then AO bisects BC
7. $\therefore \triangle ABC$ is isosceles — **134.** Theorem (2)

Consider Case (*c*) (Figure 1):

8. Compare $\triangle\!\!\triangle AOF$ and AOE
9. $AO = AO$
10. $x = y$
11. $s = t$
12. $\therefore \triangle AOF \cong \triangle AOE$ — Postulate 19
13. $\therefore AF = AE$ and $OF = OE$
14. Compare $\triangle\!\!\triangle BOF$ and COE
15. $OB = OC$
16. $OF = OE$
17. $u = w$
18. $\therefore \triangle BOF \cong COE$ — Postulate 17
19. $\therefore BF = CE$
20. $\therefore AF + BF = AE + CE$ — Axiom
21. Or $AB = AC$
22. $\therefore \triangle ABC$ is isosceles

Consider Case (*d*) (Figure 2):

23. $\triangle AOF \cong \triangle AOE$
24. $\therefore AF = AE$ and $OF = OE$
25. $\triangle BOF \cong COE$
26. $\therefore BF = CE$
27. $\therefore AF - BF = AE - CE$
28. Or $AB = AC$
29. $\therefore \triangle ABC$ is isosceles

Thus in every case $\triangle ABC$ is isosceles. It has therefore been proved that no triangle other than an isosceles triangle can be drawn.

190. The Fourth Step in Classification.—The principles of **187, 188** and **189,** which were discovered by making use of methods of analysis, and not by experiment, as were the principles of Chapters II, IV and VI, must now be tested experimentally, and verified or rejected as the result of such tests shall indicate.

EXERCISES

1. Draw polygons of 4 sides, 5 sides, 6 sides, 7 sides and 8 sides, respectively, of different shapes, both convex and concave. Draw one exterior angle at each vertex. Note that the exterior angle is negative at a vertex where the interior angle is reflex. Measure with the protractor each interior and each exterior angle of all the polygons. Find the sum of the interior and of the exterior angles for each figure, and enter the results in such a table as the following:

Number of sides of polygon	Sum of interior angles measured in straight angles	Sum of exterior angles measured in straight angles
4		
5		
6		
7		
8		

   Does the result which may now be stated for the general polygon of n sides, agree with the values deduced in **187** and **188**?

2. May triangles other than isosceles triangles be drawn? Does the conclusion of the theorem of **189** agree with experiment? Do you find any flaw in the logical analysis? Either we must discover wherein the logical error is concealed, or our entire logical system may be invalidated.

   The error will be found in having assumed incorrect, or impossible, positions for the point O. Such an assumption is a Postulate; which having been found to be false must be rejected or replaced by a correct Postulate. An accurate construction of the figure will reveal the nature of the error. State as a Postulate, thus discovered experimentally, the principle: If perpendiculars are drawn from the point of intersection of an angle-bisector and the perpendicular-bisector of the opposite side of a scalene triangle, to the other two sides of the triangle, these perpendiculars fall, etc.

Additional Theorems

191. (1) *If the diagonals of a quadrilateral are perpendicular-bisectors of each other, the figure is an equilateral parallelogram.*

(2) *If the diagonals of a quadrilateral are perpendicular-bisectors of each other, and equal, the figure is a square.*

(3) *If two straight lines are equally distant from each other at any two points (measured on perpendiculars) they are parallel.*

(4) Analyze Theorem XVII, **187,** by dividing the polygons into triangles by drawing diagonals, but so that no triangles overlap.

(5) *Each interior angle of an equiangular polygon can be expressed in terms of the number of sides of the polygon.*

(6) *Each exterior angle of an equiangular polygon can be expressed in terms of the number of sides of the polygon.*

(7) *The lines joining in order the middle points of the sides of any quadrilateral form a parallelogram.*

Helps.—Draw one or both diagonals of the given quadrilateral. Apply **180** (*a*) or (*b*). Note the three possible variations in the analysis.

(8) *The lines joining the middle points of the opposite sides of any quadrilateral bisect each other.*

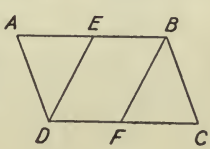
(9) *The lines joining the middle points of the opposite sides of an oblong form a rhombus.*

Help.—The new figure must be shown to be equilateral, and its diagonals to be unequal.

(10) *The diagonals of an oblique parallelogram are unequal.*

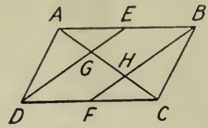
Help.—Apply **143,** Theorem (33).

(11) *A parallelogram whose diagonals are unequal is oblique.*



(12) *In the parallelogram ABCD, E and F are the mid-points of AB and DC. Then DE is parallel and equal to FB.*

(13) *In the figure of Theorem* (12), *the lines DE and FB trisect the diagonal AC.*



Helps.—(1) In $\triangle ABH$, GE bisects AH; (2) similarly FH bisects GC; (3) $\therefore AG = GH = HC$.

(14) *The bisectors of the angles of a quadrilateral form another quadrilateral whose opposite angles are supplementary.*

(15) *Corollary of Theorem* (14). *When the given quadrilateral is a parallelogram, that formed by the angle bisectors is a rectangle.*

(16) *A convex polygon cannot have more than three acute angles.*

Help.—If an interior angle is acute, what is the exterior angle at that vertex?

(17) *In a triangle ABC, AD is a median prolonged to E so that DE = AD. Then CE is parallel to AB.*

(18) *The angles adjacent to either base of an isosceles trapezoid are equal.*

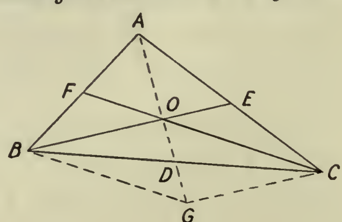
(19) *The diagonals of an isosceles trapecoid are equal.*

(20) *The sum of the angles at the points of a five-pointed star is a straight angle.*

(21) *The perpendiculars drawn from opposite vertices of a parallelogram to a diagonal are equal.*

(22) *Any line drawn through the point of intersection of the diagonals of a parallelogram, and terminated by the sides is bisected.*

(23) *If a diagonal of a parallelogram bisects an angle, the parallelogram is equilateral.*

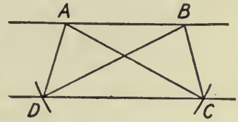


(24) Analyze Theorem XVII, **185,** as follows: Draw $\triangle ABC$ and medians BE and CF intersecting at O; draw AD through O. Prove that AD is a me-

dian; *i.e.*, $BD = BC$; and that O is a trisection point on each median.

Helps.—(1) Prolong AD so that $OG = AO$ and draw GB and GC;—(2) in $\triangle AGC$, $OE \| GC$;—(3) in $\triangle ABG$, $OF \| BG$;—(4) $\therefore$ $OCGB$ is a parallelogram;—(5) in $\triangle AGC$, $OE = \frac{1}{2} GC$;—(6) $\therefore$ $OE = \frac{1}{2} BO = \frac{1}{3} BE$;—(7) $BD = DC$.

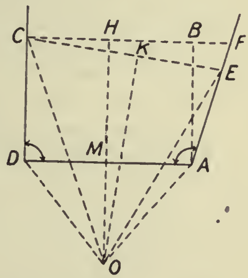
(25) *If the angles adjacent to either base of a trapezoid are equal, the trapezoid is isosceles.*



(26) *If the diagonals of a trapezoid are equal, the trapezoid is isosceles.*

Help.—Draw the altitudes from points A and B.

(27) *Any obtuse angle equals a right angle.*



Hypothesis.—A right $\angle CDA$ and an obtuse $\angle FAD$.

Conclusion.— $\angle CDA = \angle FAD$.

Helps.—(1) Construct rectangle $ABCD$, extend CB to F, lay off $AE = AB$, draw CE, draw HO perpendicular-bisector of CB and KO perpendicular-bisector of CE, draw OD, OC, OA, OE.

(2) HO intersects KO at O; **143,** Theorem (3.)

(3) $\triangle CDO \cong \triangle EAO$; 3 sides respectively equal; **116,** Postulate 15.

(4) $\therefore$ $\angle CDO = \angle EAO$.

(5) $\angle ADO = \angle DAO$; base $\angle$s of isosceles $\triangle$; **121.**

(6) $\therefore$ $\angle CDA = \angle FAD$; **64,** Axiom 3.

Find the error in the preceding theorem, which may be found in *Mathematical Recreations and Problems* by W. W. R. Ball; published by Macmillan and Co.

192. Review Exercises

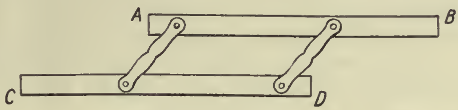
1. Why was it impossible to include the analyses of **180, 181** and **185** in the chapter on triangles (Chapter V)?
2. Two consecutive sides of a parallelogram are 10 and 12 inches respectively, and the included angle is 75°. Calculate the remaining sides and angles.

3. The bases of a trapezoid are 7 inches and 10 inches. Calculate the length of the median.
4. The median of a trapezoid is 13.5″ and one base is 7.4″. What is the other base? Write a formula for the median m, in terms of the bases b_1 and b_2.
5. Calculate the sum of the interior angles of a triangle, quadrilateral, pentagon, hexagon, octagon, decagon, using the formula of **187.**
6. Calculate the value of each interior angle and of each exterior angle of equiangular polygons of 3, 4, 5, 6, 8 and 10 sides.
7. If one of the interior angles of an equiangular polygon is 150°, how many sides has the polygon?
8. Calculate the number of sides of an equiangular polygon if one of the exterior angles is 18°.
9. How many sides has a polygon if the sum of the interior angles is ten straight angles? If the sum of the interior angles is 630°?
10. How many sides has a polygon the sum of whose interior angles equals the sum of the exterior angles?
11. How many sides has a polygon the sum of whose interior angles is three times the sum of the exterior angles? a times the sum of the exterior angles?
12. Five of the interior angles of a hexagon are 70°, 135°, 140°, 130°, 45° respectively. What is the other angle?
13. Under what circumstances should a postulate, which is supposed to state a geometric principle of the most elementary and most evident kind, be modified or rejected?
14. Can an argument based logically upon the accepted postulates and theorems of a science *prove* a principle which is found to be contrary to the *fact* as established by experimental investigation? Which possesses the higher degree of certainty, logically deduced or experimental laws?
15. Find out, if possible, what was generally accepted by the scientists of the world as the *law of falling bodies* before Galileo made his famous experiments at the Tower of Pisa about 1590. Was the logically deduced "*law*" confirmed or destroyed as the result of these experiments?
16. Construct a regular hexagon each side of which is 1 inch. Calculate the angles, and draw a side and measure an angle alternately.
17. In the story of Sir Galahad will be found a description of the "pentangle" which adorned his shield. Design the "pentangle" from the description.
18. Make a list of all theorems of Chapters III, V and VII in which the equality of two or more sects has been shown to be a neces-

sary consequence of the constructed property of the figure. Make a similar list concerning equal angles.

19. State theorems of congruent parallelograms and quadrilaterals, in terms of the required number and arrangement of respectively equal sides and angles.

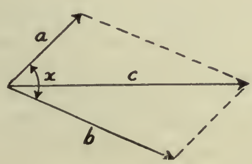
20. Show experimentally that the relation between the number of sides of a polygon and the total number of possible diagonals is expressed by the formula, $d = (n - 1)n - [3 + (3 + 4 + \ldots n)]$; where d = number of diagonals and n = number of sides.



21. Why is the edge AB of the *parallel ruler* shown in the figure, parallel to the edge CD in all positions?

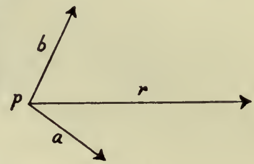
Applications

193. Physics and Engineering. *The Parallelogram of Forces.*—When several forces act simultaneously upon the same body they may be replaced by a single force which will produce an effect equivalent to the combined effect of the acting forces. This single equivalent force is the *resultant;* the acting forces are the *components.* The law of the relation of resultant and two component forces may be stated geometrically as follows: If the sides of a parallelogram are drawn in units representing the values of the component forces, the angle between the sides being the angle between the directions of the forces, then the length and direction of the diagonal of the parallelogram represents the value and direction of the resultant. This law is determined experimentally.



EXERCISES

1. What lines in the figure represent the component forces? What line represents the resultant?
2. Two forces of 10 and 20 lb. respectively, act at an angle of 45° to each other. Construct the parallelogram of forces to scale and find the resultant and the angle which the resultant makes with the direction of the larger force.
3. The resultant of two forces is 30 lb., and makes an angle of 30° with one of its components which has a value of 40 lb. Find the value of the other component and the angle between the components.
4. Show how the resultant of three given forces may be obtained by combining the resultant of two of them with the third. May this combination be made in any order?
5. Let the resultant, r, of three forces, and two of these forces, a and b, be given in value and direction. Find the value and direction of the third component.



The *parallelogram of velocities* is constructed in the same way as for forces.

6. A boat is travelling northward at a velocity of 20 miles per hour, and a passenger is walking across the deck towards the east at a velocity of 4 miles per hour. Find the actual direction and velocity of the passenger relative to a fixed object.
7. A mail bag is thrown from a train at a right angle to the track, at a velocity of 10 feet per second. The train is travelling 50 miles per hour. Find the direction and velocity with which the mail bag travels until it hits the ground.
   Help.—The velocity of the train is equivalent to 88 feet per second.
8. An aeroplane is travelling 80 miles per hour in a northeasterly direction, while the wind is blowing from the east at a velocity of 40 miles per hour. Find the direction and velocity in relation to the earth at which the aeroplane is travelling.
9. A vessel while moving at a rate of 30 miles per hour parallel to the shore line, fires a shot at a target which is exactly abeam. The projectile has a velocity of 200 feet per second. In what direction must the gun be aimed if the shot is to hit the target?

The Polygon of Forces.—Several forces acting simultaneously upon the same body may balance each other so that the resultant effect is zero. The forces are then said to be in *equilibrium*. The geometric condition for equilibrium of a system of forces is that the lines representing in length and direction the values and directions of the forces shall form the sides of a polygon when placed end to end in the same direction. The forces a, b, c in Fig. 1 are in equilibrium if they form a triangle as in Fig. 2.

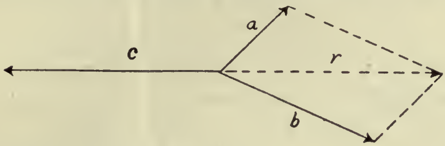


FIG. 1.

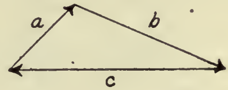
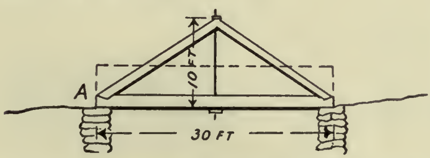


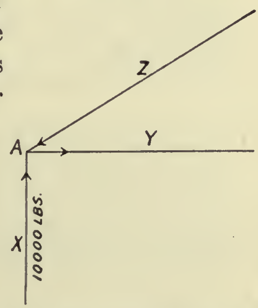
FIG. 2.

EXERCISES

10. It is evident that the force c balances the resultant r of forces a and b. Show by construction that force a balances the resultant of forces b and c, and also that force b balances the resultant of forces a and c.
11. Draw a point P and any four forces which are in equilibrium acting at the point P, no two forces being in the same straight line.

The principle of the polygon of forces is employed by engineers in determining the breaking stress in framed structures such as bridge and roof trusses, buildings and cranes, (see figures in **150**). From the stress as thus found, the size of the piece, or member, is calculated.





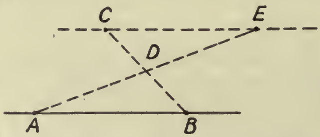
12. Let the bridge truss shown be designed to support a total uniform load of 40,000 lb. A force of 10,000 lb. may then be considered as supporting each end. The remaining 20,000 lb. rests upon the abutments and has no effect upon the truss stresses. Draw the polygon (triangle) of forces at the point A, where the value and direction of force x and also the direc-

tions of forces y and z are known. Determine the amount of stress in both the upper and lower bridge members.

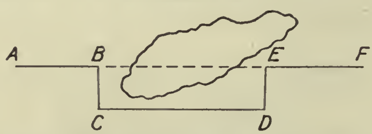
194. Surveying.—Many uses are found for the principles of polygons.

EXERCISES

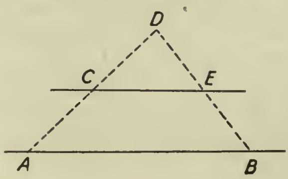
1. A surveyor has measured three angles of a four-sided field: 75° 30.5′, 128° 51.2′, 135° 3.7′. If these are correct, what is the fourth angle?
2. A surveyor has measured the angles of a five-sided field; 110° 21.7′, 141° 13.3′, 148° 54.0′, 47° 20.9′, 92° 10.6′. What total error has been made? Distribute the error equally among the measured angles to find the most probable correct values.
3. If the four sides of a quadrilateral field are measured with a tape may a map of the field be drawn to scale? What more must be measured? If no transit is available what other line measurement will determine the field?
4. Explain the construction of a line through point C parallel to line AB, by an adaptation of the principles of **163,** part (d), and **178.** In an out-of-doors problem line AB will be a street curb, fence, etc., and a stake is to be driven at a point E so that CE is parallel to AB.



5. A surveyor wishes to determine the line AF, point F being visible from point A. Explain the method. What parallelogram property is made use of in finding AF?



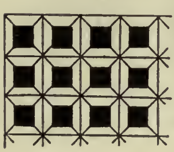
6. Another method of laying out a line through point C parallel to AB is shown by the figure. Explain how the measurements are made and upon what principle it depends.

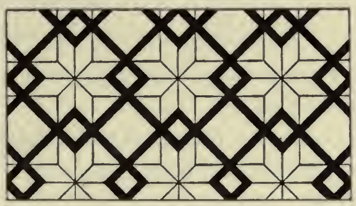


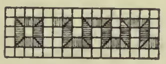
195. Designing.—Triangles, parallelograms and regular polygons appear in a great variety of artistic designs in which symmetry is also an essential feature.

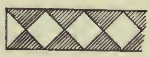
EXERCISE

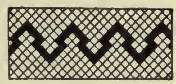
The following designs are for tiling, parquetry flooring, linoleums and embroidery. Draw some of these designs to scale, or copy or originate other similar designs. Suitable coloring may be used.

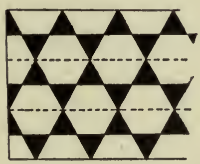


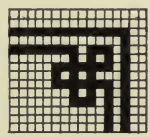


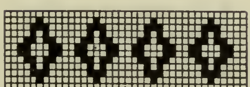


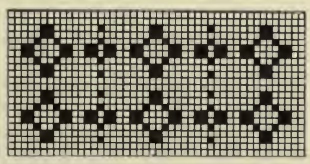












196. Problems for Field Work.—In these problems the principles of Chapters VI and VII are employed.

1. To lay out a line beginning at a given point, as a stake driven in the ground, and parallel to a curb, fence or side of a building.
   (*a*) Use the method described in **194,** Exercise 4.

(*b*) Use the method described in **194,** Exercise 6.

(*c*) The most accurate method is suggested in **58,** Exercise 11. This may be used as a check on the accuracy of methods (*a*) and (*b*).

2. Measure the angles of a four-sided field (or at four stakes driven in the ground), in order to verify the principle of **187.** Distribute the total error as in **194,** Exercise 21.

197. STANDARD FORM SHEET FOR RECORDING FIELD WORK.

FIELD WORK

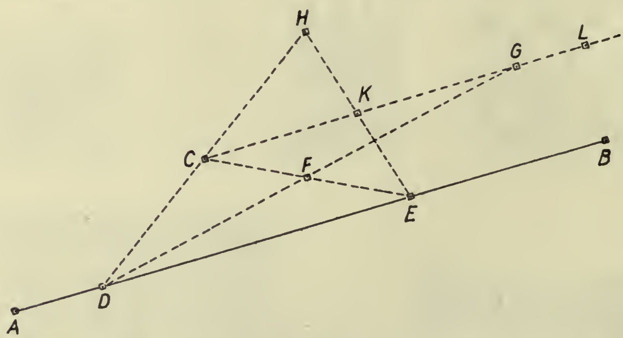
DEPARTMENT OF MATHEMATICS———SCHOOL

To lay out a line beginning at a given point, parallel to a given line:

Geometric Principles Employed.—(1) If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram (**178**).

(2) A line which bisects two sides of a triangle is parallel to the third side (**180**).

(3) Two lines which are cut by a transversal making a pair of alternate-interior angles equal, are parallel (**72**).



AB is a curb, fence, side of a building, etc.; *C* is a point on the required parallel line.

METHOD (*a*):

(1) Drive stakes at *D* and *E* on line *AB*.

(2) Measure *EC;* divide by 2; measure to middle point *F*.

(3) Measure *DF;* prolong until *FG* = *DF*.

METHOD (*b*):

(1) Measure DC; prolong until $CH = DC$.

(2) Measure HE; bisect at K.

METHOD (*c*).

(1) Measure $\angle DEC$.

(2) Lay out $\angle ECL = \angle DEC$.

Points C, K, G, and L should lie in the same straight line.

CHAPTER VIII

LOCI

Principles Determined Experimentally

198. A Locus Defined.—A line may be considered as being made up of an indefinite number of points. When all the points which compose a line possess some common property of position, the line is called a *locus*.

A *locus* is (in general) a line which is made up of all the points which satisfy a given condition. The line must contain all possible points which satisfy the condition and no other points. A point which moves according to a fixed law generates a locus or part of a locus.

A *locus* may also be a group of lines, a line and a point, a series of detached points, a surface, etc.

199. Experiment I.—To find the locus of points which are equidistant from two given points.

Draw two points A and B. To find a point of the required locus, draw two arcs with centers at A and B respectively, and with any equal radii, which intersect each other. Obtain about 8 or 10 points in this way. Draw the line which contains all the determined points. This line is the required locus, since if all possible points equidistant from points A and B were found, the points themselves would form the line.

Result.—The locus of points which are equidistant from two given points is the perpendicular-bisector of the sect joining the given points.

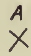
200. Experiment II.—To find the locus of points which are at a fixed (or constant) distance from a given point.

Mark the given point A and draw the given fixed distance BC. Construct the locus by finding a number of detached points and then drawing the line which an indefinite number of such points would form.

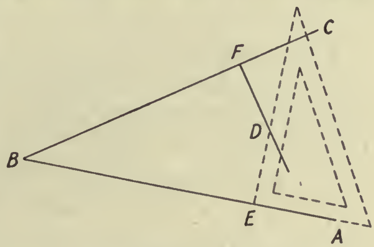
RESULT.—The locus of points which are at a fixed distance from a given point is a circle whose center is the given point and whose radius is the fixed distance.

201. Experiment III.—To find the locus of points which are at a fixed distance from a given straight line.

Help.—The locus is composed of two lines.



202. Experiment IV.—To find the locus of points within an angle which are equidistant from the sides of the angle.



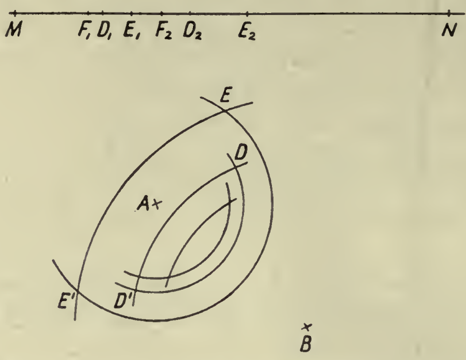
The perpendicular distances from each point of the locus to the sides of the angle are equal. Such a point may be found by trial by using the compasses to measure the equal distances, and placing them so that they measure a perpendicular to the side of the angle, the perpendicular being carefully judged, or a right triangle being used for accuracy.

203. EXPERIMENT V.—TO FIND THE LOCUS OF POINTS WHICH ARE TWICE AS FAR FROM ONE OF TWO GIVEN POINTS AS FROM THE OTHER.

Mark two points A and B.

To find a point of the locus, draw any straight line MN; mark a point D_1, on MN, and measure $D_1D_2 = MD_1$. Draw an arc with center at A and radius $= MD_1$; and a second arc with center at B and radius $= MD_2$; the points of intersection of these arcs, D and D', are points of the required locus.

Repeat with radii ME_1, and ME_2, which determine the points E and E' of the locus. Find enough points to completely determine the locus, which will be found to be a circle. A second circle may be found if the relative distances of the points of the locus from A and B are reversed.



The *result* may be stated if desired, by stating (1) the form of the locus, (2) the position of the center of the circle, (3) the radius of the circle. (2) and (3) are stated in terms of the sect AB.

204. Review Exercises

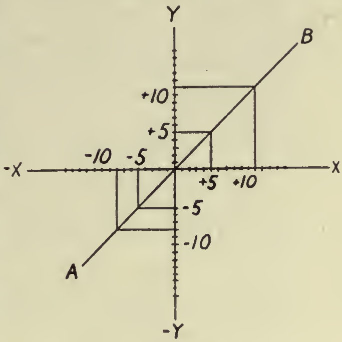
Construct the following loci by finding only sufficient points in each to determine the required locus.

1. The locus of points which are equidistant from the ends of a sect.
2. The locus of points which are equidistant from two parallel straight lines.
3. The locus of points which are equidistant from two intersecting straight lines.
4. The locus of the centers of circles which contain two given points.
5. The locus of the centers of circles of fixed radius tangent to a given straight line.
6. The locus of the centers of circles of fixed radius which contain a given point.
7. The locus of points which are equidistant from two concentric circles.

205. Algebraic Equations Considered as Loci.—Pupils who have studied graphs and plotting points from their coördinates, will understand the following geometric interpretation of algebraic equations in two variables.

EXERCISES

1. Plot a number of points for which $x = y$. If all possible points were plotted, they would make up a line bisecting one of the pairs of vertical angles formed by the coördinate axes. $x - y = 0$ is therefore the equation of this line, or locus.



2. What is the equation of the locus of points equidistant from the sides of the other pair of vertical angles formed by the coördinate axes?
3. Combine these equations into a single equation with a double ($\pm$) sign. Of what may this equation be considered the locus?
4. Write the equation of the locus of points which are at a distance $+5$ or -5 from the axis of abscissas.
5. Plot the locus $x - 2y = 12$; that is, the locus of points of which the abscissa exceeds twice the ordinate by 12. It is preferable to use squared paper to facilitate plotting.
6. Plot the locus $x^2 + y^2 = 100$. To what locus of this chapter does the graph correspond?

206. Constructions Depending upon Loci.—The location of a point which fulfils two independent given conditions may be determined by drawing the two loci which contain the required point. The intersection of these loci determines the point.

It often happens that more than one point fulfils the given conditions. It is understood that all such points are required.

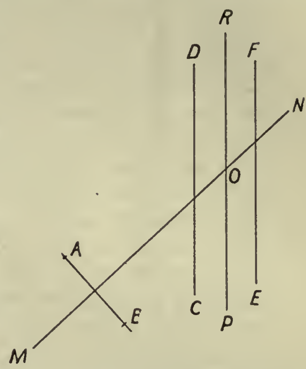
Certain peculiarities or special cases of the problem which may occur should also be noted.

EXERCISES

1. Find a point equidistant from two given points and also equidistant from two given parallel lines.

   Let A and B be the given points, and CD and EF be the given parallels.

The required point, in order to be equidistant from points A and B, lies on the perpendicular-bisector of the sect AB; and in order to be equidistant from the parallels CD and EF, it lies on the line PR, parallel to them and equidistant from them. The point O, which is the intersection of the two loci is therefore the required point.



Special Cases.—(*a*) If PR is parallel to MN, there are two points at infinity.

(*b*) If PR coincides with MN, every point of MN is a required point.

(*c*) There is no impossible case; unless we prefer to designate Case (*a*) as impossible.

2. Find a point equidistant from two given points and also equi-distant from the sides of a given angle.

   Help.—The sides of the angle may, if necessary, be extended through the vertex and the bisector of the angle may therefore also be extended through the vertex.

3. Find a point equidistant from two given points and situated upon a given straight line.

4. Find a point equidistant from two given points and also at a given (fixed) distance from a given straight line.

   Help.—There are in general two points which fulfil these conditions, and several special cases.

5. Find a point equidistant from two given points and situated on a given circumference.

   Help.—There are in general two points. Special cases occur (*a*) where there is one point, and (*b*) where there is no point.

6. Find a point equidistant from the sides of an angle and situated upon a given straight line.

7. Find a point equidistant from two given points and at a given distance from a third given point.

8. Find a point equidistant from three given points.

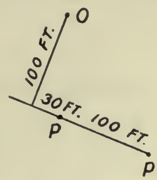
   Help.—Find a point equidistant from points (1) and (2), and also equidistant from points (2) and (3).

9. Find a point within a triangle and three points outside the triangle, which are equidistant from the three sides of the triangle (or sides extended).

10. Find a point equidistant from two intersecting straight lines and situated upon a third given straight line.

11. Find a point equidistant from two intersecting straight lines and situated upon a given circumference.
Help.—Besides the general case, there are five special cases.

12. A pirate buried his treasure 250 feet from an oak tree O, and 70 feet from the line marked by two poplars P and P'. A sailor who discovered the pirate's chart and directions failed to locate the treasure. What may have been his error?



13. A tree is 20 feet from a straight fence. A gardener directs his assistant to plant a bush 30 feet from both the tree and the fence. Are the directions sufficient? For what case would they be sufficient? Draw a sketch.

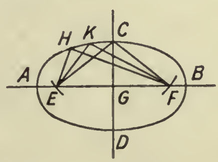
14. Find a point equidistant from two perpendicular axes, that is, on the locus $x \pm y = 0$; and also at a distance of 5 units from the vertical axis, that is, on the locus $x = \pm 5$.
Help.—There are four points.

Applications

207. Surveying.—*A point equidistant from three given points* may be located by the method of **206,** Exercise 8. The lines are bisected by measurement, and the perpendiculars are erected either by measuring a right angle with a transit or by some method in which a tape is used.

208. Higher Mathematics.—*Coördinate geometry,* described briefly in **205,** forms the basis of calculus and of a great deal of applied mathematics, such as engineering, military science and navigation. An understanding of loci is therefore a requisite for such professional studies.

209. The Ellipse.—This curve is the locus of points the sum of whose distances from two given points is a constant.



To construct an ellipse: (1) draw the axes AB and CD bisecting each other at right angles; (2) draw two arcs with center at C and radii equal to AG, intersecting AB at points E and F, which are called the foci; (3) tie two knots in a string at a distance apart equal to AB; fix the knots with pins at points E and F, and slip a pencil around the line $AHKCB$ keeping the string taut.

The locus is correctly described since $EA + AF = EH + HF = EC + CF =$ etc. $=$ a constant, which $= AB$.

EXERCISES

1. Construct an ellipse whose axes are 5 and 3 inches respectively.
2. Lay out an elliptical flower bed 24 feet by 12 feet.
3. Draw an elliptical window with symmetrical colored glass design.
4. Draw an elliptical picture-frame, platter, or embroidered doily or table-cover.
5. Observe occasional elliptical outlines in architecture as arches, ceilings, column bases, windows. These are, in general, more pleasing than circular arcs.

CHAPTER IX

LOCI

Classification and Explanation of Principles

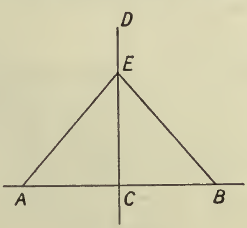
210. The Method Employed.—After it has been determined experimentally that a certain line (or lines) is made up of points having some given or assigned property; it is necessary to show, in order to explain this principle: (*a*) that every point in the determined line possesses the assigned property; and (*b*) that every possible point which possesses the assigned property is in the determined line.

If only property (*a*) were known about a determined line, the line still might be only a *part* of the locus. If only property (*b*) were known about the determined line, the entire line might be *more* than the required locus. If both properties (*a*) and (*b*) belong to the line, then the determined line is the *complete locus.*

Only two of the loci determined experimentally in Chapter VIII are important in advanced analysis. The four concurrent line properties of a triangle (**111**) which have not as yet been analyzed, are explained from the properties of loci (**213–216**).

211. Theorem I.—*The locus of points equidistant from the ends of a sect is the perpendicular-bisector of the sect.*

Part (*a*).—Any point on the perpendicular-bisector of a sect is equidistant from the ends of the sect.

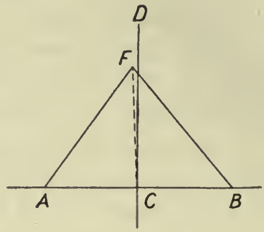


Hypothesis.—Sect AB with perpendicular-bisector CD, and a point E on CD.

Conclusion.—Point E is equidistant from points A and B; *i.e.*, $AE = BE$.

Analysis.—(1) $\triangle ECA \cong ECB$; (2) $\therefore AE = BE$. Write out the complete analysis in standard form.

Part (*b*).—Any point which is equidistant from the ends of a sect is on the perpendicular-bisector of the sect.



Hypothesis.—Sect AB with perpendicular-bisector CD, and a point F equidistant from A and B; *i.e.*, $AF = BF$.

Conclusion.—Point F is on CD.

Analysis.—(1) Draw CF;—(2) $\triangle FCA \cong \triangle FCB$;—(3) $\therefore$ $CF \perp AB$;—(4) $\therefore CF$ coincides with CD and F is on CD. Write out complete analysis in standard form.

Corollaries. 1.—*Two points equidistant from the ends of a sect determine the perpendicular-bisector of the sect.*

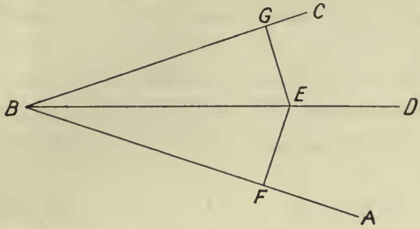
Helps.—(1) Each point is on the perpendicular-bisector; why? (2) $\therefore$ the line joining the two points coincides with the perpendicular-bisector.

2. *If a perpendicular to a sect contains a point that is equidistant from the ends of the sect, it bisects the sect.*

Helps.—This may be analyzed by principles of congruent triangles, or of isosceles triangles, or as follows:—(1) the perpendicular-bisector of the sect contains the given point; why?—(2) $\therefore$ the two perpendiculars coincide, etc.

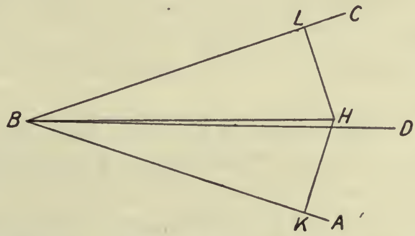
212. Theorem II.—*The locus of points within an angle equidistant from the sides is the bisector of the angle.*

State Part (a).



State the *hypothesis*, and the *conclusion* ($EF = EG$).

State Part (b).

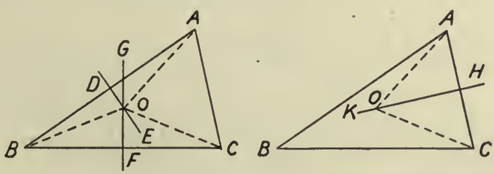


State *hypothesis* ($HK = HL$); and the *conclusion.*

Helps.—(1) Consider ⚠HLB and HKB;—(2) if $x = y$, BH bisects $\angle ABC$;—(3) $\therefore$ BH coincides with BD, and point H is on BD.

Corollary.—*One point equidistant from the sides of an angle determines with the vertex the bisector of the angle.*

213. Theorem III.—*The three perpendicular-bisectors of the sides of a triangle are concurrent.*



Hypothesis.—$\triangle ABC$ with perpendicular-bisectors of sides, DE, FG and HK.

Conclusion.—DE, FG and HK are concurrent.

Analysis.

Statement	Reason
1. DE and FG intersect at a point O	**143,** Theorem (3)
2. Pt. O is equidistant from pts. A and B; that is, $AO = OB$	Locus Theorem I, Part (*a*)
3. Pt. O is equidistant from pts. B and C; that is, $OB = OC$	Locus Theorem I, Part (*a*)
4. $\therefore$ Pt. O is equidistant from pts. A and C; that is, $OA = OC$	Axiom 1
5. $\therefore$ HK contains pt. O	Locus Theorem I, Part (*b*)
6. $\therefore$ DE, FG and HK are concurrent	

Corollary.—*Point O is equidistant from the three vertices of the triangle.*

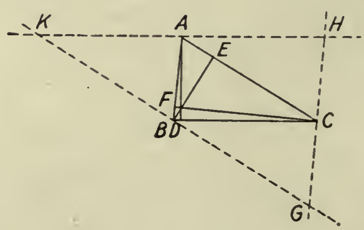
214. Theorem IV.—*The three angle-bisectors of a triangle are concurrent.*

Helps.—(1) Two of the angle-bisectors intersect at a point O; what theorem? (2) point O is in the third angle-bisector.

Write out the complete theorem using statements identical with those of Theorem III, except that point O is here equidistant from the sides of the angles.

Corollary.—*Point O is equidistant from the three sides of the triangle.*

215. Theorem V.—*The three altitudes of a triangle are concurrent.*

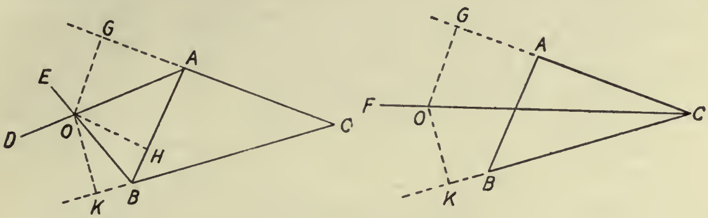


Helps.—(1) Draw GH, HK, $KG \| AB$, BC, CA respectively;—(2) $AD \perp KH$; why?—(3) $KACB$ is a parallelogram; $\therefore$ $KA = BC$;—(4) also $AH = BC$; why?—(5) $\therefore$ $KA =$

AH;—(6) ∴ AD bisects KH.—(7) Similarly BE and CF are perpendicular-bisectors of the sides KG and GH of $\triangle GHK$.—(8) Consider now that $\triangle ABC$ is removed, leaving $\triangle GHK$ and the three perpendicular-bisectors of its sides; they are concurrent; why?

Write out the complete analysis.

216. THEOREM VI.—*The bisectors of each two exterior angles of a triangle and of the non-adjacent interior angle are concurrent.*



State the *hypothesis* and the *conclusion*.

Helps.—(1) AD and BE intersect in a point O;—(2) which is equidistant from CA and CB;—(3) ∴ CF contains point O. Show that there are two other similar points.

State a corollary similar to the corollaries of Theorems III and IV.

ADDITIONAL THEOREMS

217. (1) If any point of the median to the base of an isosceles triangle is joined to the ends of the base, another isosceles triangle is formed.

Help.—The median is the perpendicular-bisector of the base.

(2) The middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

Help.—Show that the perpendicular-bisectors of two of the sides of the triangle are concurrent at this point.

(3) The circumcenter, incenter, orthocenter and centroid of an equilateral triangle coincide.

(4) The diagonals of a quadrilateral divide it into four

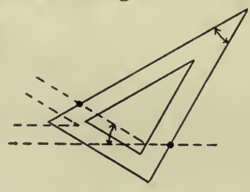
triangles whose circumcenters are at the vertices of a parallelogram.

(5) The perpendicular-bisector of a chord of a circle contains the center of the circle.

(6) The line of centers of two intersecting circles is a perpendicular-bisector of the chord joining their points of intersection.

218. Review Exercises

1. Sketch the locus of points which are twice as far from one of the sides of an angle as from the other: (*a*) when the points lie within the angle; (*b*) when they lie outside of the angle.
2. Construct by points, the locus of the vertex of an angle of constant size, whose sides pass through two given points.
   Help.—Use an angle of the drawing triangle. Place the triangle in different positions such that the sides forming the angle contain the two marked points.



3. If a locus is known to be a straight line, how many points are required to determine it?
4. Construct the locus of the centers of circles which are tangent to the two sides of an angle.
5. Find a point (or points) which is equidistant from two given points and also at a given distance from a given circumference.
6. Find a point which is equidistant from two intersecting straight lines and at a given distance from their point of intersection.
7. Plot the locus $xy = 12$, using both plus and minus values of the coördinates.
8. A stone wall across a farm runs north and south; a second wall crosses it east to west; 200 ft. east of the point of intersection a third wall crosses the second from S.E. to N.W. at an angle of 50°. The owner of the farm finds in an old paper: "trap-door to concealed passage leading to treasure vault, is at point equally distant from three walls." Draw map and locate the trap-door. In how many possible places may it be located?
9. The first mathematician to use the coördinate method of plotting algebraic equations was René Descartes (1596–1650). Coördinate geometry is also called *Cartesian geometry* in his honor. Find in an encyclopedia some facts concerning Descartes.

CHAPTER X

THE MEASUREMENT OF SECTS, AND SIMILAR TRIANGLES

Principles Determined Experimentally

219. In this chapter we propose to investigate more complex relationships of sects than those merely of equality or inequality, these relationships forming an entirely new group of geometric principles. Some of the more evident of these principles have certainly been known as early as 2000 B. C., and researches in this field were made by Pythagoras about 500 B. C.

220. Measurement of Sects.—A *ratio* is a quotient arising from the division of one quantity by another of the same kind; as $\frac{a}{b}$, $a \div b$, $a : b$.

A *proportion* is an equality of ratios; as $\frac{a}{b} = \frac{c}{d}$, $a \div b = c \div d$, $a : b = c : d$.

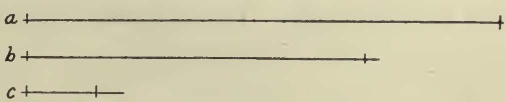
The *measurement* of a quantity is a ratio of the quantity measured, to the unit of the same kind. See also Chapter I.

The *ratio of two quantities* of the same kind is the ratio of their measures in terms of a common unit.

EXERCISES

1. What units are generally used to measure sects of about the length of this page? very short sects? the side of a room? geographical distances? astronomical distances?

2. Measure sects a and b with the unit c. What is the ratio of $\frac{a}{b}$?



3. Measure the edges of this page in inches. Draw two sects less than two inches whose ratio equals the ratio of the two edges.

221. Properties of a Proportion.—$\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$, represents any proportion.

The *terms* are the numbers a, b, c and d.

The *means* are the second and third terms, b and c.

The *extremes* are the first and fourth terms, a and d.

The four numbers (quantities, values) are *proportional.*

The fourth term of a proportion is a *fourth proportional* of the three preceding terms.

When the two means are the same, either is a *mean proportional* of (between) the first and fourth terms.

A *reciprocal* of a number is unity divided by the number. The numbers are *reciprocals,* as x and $\frac{1}{x}$.

EXERCISES

1. Name the means and extremes of the proportions, $\frac{7}{3} = \frac{14}{6}$; $5 : x = 15 : 2$; $x : x + 1 = 6 : 8$.
2. What is the fourth proportional of 5, 8 and 25?
3. What is the mean proportional of 2 and 8; of 1 and 25? Express the proportions.
4. Write four proportional numbers.
5. Write a series of three equal ratios of which the first ratio is $\frac{1}{2}$.
6. Write as many simple proportions (two ratios in each) as possible from the ratio series, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.
7. Combine these proportions into a series, $\frac{a}{b} = \frac{x}{y}$ and $\frac{a}{b} = \frac{s}{t}$. What axiom is used?
8. Write the reciprocal of 5; of $\frac{1}{3}$; of $\frac{2}{7}$; of .25.

222. Division of a Sect.—A sect is divided by a point, or intersecting line, into two parts or *segments.*

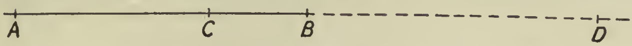
When the point is between the ends of the sect it divides the sect *internally;* when the point is in the sect extended it divides the sect *externally.* The segments extend from the point of division to each end of the sect.

A sect is divided in a given ratio when the ratio of the segments equals the given ratio.

Two sects are divided proportionally when the four segments are proportional.

EXERCISES

1. The sect AB is divided internally at C and externally at D.



Read the segments of AB formed by each point of division.

2. Draw a sect 3 inches long. Mark a point which divides it internally in the ratio of $1 : 2$. Mark a point which divides it internally in the ratio of $\frac{2}{3}$.
3. Divide the sect of Exercise 2 externally in the same ratios.
   Help.—Let x = segment DB; then $x + 3$ = segment DA; and $\frac{x}{x+3} = \frac{1}{2}$.
4. Draw sects of 3 and 4 inches. Divide them proportionally so that a segment of the 3-inch sect $= \frac{1}{2}$. inch.
5. A mile is divided into segments of 2000 and 3280 feet. Divide an inch proportionally, the segments being calculated to two decimals.
6. A sect is said to be divided *harmonically* when it is divided by two points, one internal and the other external, so that the ratio of the segments formed by one point of division equals the ratio of the segments formed by the other point.
   Divide a sect $AB = 2$ inches internally at C so that $BC = .5''$, and harmonically at D. Calculate DB.
   Help.—$\frac{DA}{DB} = \frac{CA}{CB}$.

223. Necessary Accuracy of Experimental Work.—In the experimental work of this chapter, measure sects to the nearest hundredth of an inch and angles to the nearest quarter degree. Drawings must be made very accurate in order to obtain reliable results.

224. Experiment I.—The ratio of two sects.

Draw two sects of different lengths, each of several inches.

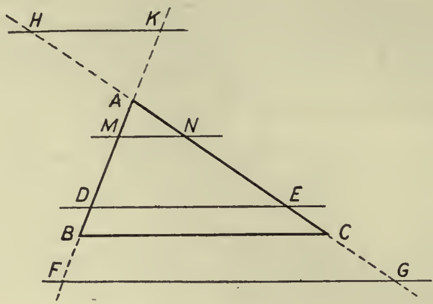
(*a*) Find their measures in inches. Calculate the ratio of their lengths as a decimal, to three places.

(*b*) Find their measures in centimeters or in any arbitrary unit as +——+ Calculate their ratio.

Observe the relation between the ratios.

Result.—The ratio of two sects is independent of the unit of measurement.

225. Experiment II.—A line parallel to a side of a triangle.



Draw any oblique scalene triangle ABC. Draw a line parallel to a side of the triangle, intersecting the other two sides. Measure the segments into which the two sides are divided by the parallel line. Calculate the ratios $\frac{AD}{DB}$ and $\frac{AE}{EC}$, each to three decimals. Compare these ratios. Repeat with other parallel lines, as FG, HK, MN

Result.—A line parallel to a side of a triangle divides the other two sides proportionally.

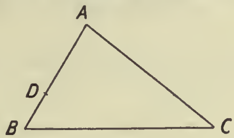
EXERCISES

1. The sides of $\triangle ABC$ are $AB = 2''$, $AC = 3''$, $BC = 2.4''$. If DE cuts AB so that $AD = 1.5''$, calculate AE.
2. In the triangle of Exercise 1, MN cuts AC so that $NC = 2.2''$; calculate MB.
3. In the same triangle, FG cuts AB prolonged so that $BF = 0.5$; calculate CG.
4. In the same triangle $AH = \frac{1}{2} AC$; calculate AK.
5. If point D is the middle of AB, to what previous simpler principle does this principle reduce?
6. The sides of a triangle are 100 ft., 250 ft., 280 ft. A line parallel to the 280 ft. side cuts the 250 ft. side 60 ft. from its intersection with the 100 ft. side. At what point does this line cut the 100 ft. side? From the same point in the 250 ft. side, a line runs par-

allel to the 100 ft. side. At what point does it cut the 280 ft. side?

7. Two sides of a parallelogram are 10″ and 15″, and a diagonal is 18″. At a point in this diagonal, 3 inches from one end, lines are drawn parallel respectively to the four sides of the parallelogram. Calculate the points at which these lines intersect the sides.

226. Experiment III.—The position of a line which divides two sides of a triangle proportionally.



Draw any triangle ABC; mark a point D on AB. Measure AD, AB and AC. Calculate the position of a point E on AC such that $\frac{AE}{AC} = \frac{AD}{AB}$. Measure AE and mark point E; draw DE. Observe the position of line DE.

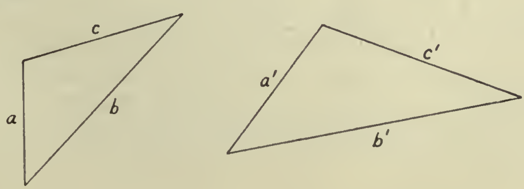
Result.—A line which divides two sides of a triangle proportionally, etc.

EXERCISES

1. The sides of a triangle are 4″, 6″, 8″. A line cuts the 4″ and 6″ sides at points 1″ and 1.5″ respectively from their point of intersection. Is this line parallel to the 8″ side?
2. Two adjacent sides of a rectangular room are 10 ft. and 15 ft. A line joins two points on the sides of the room, measured respectively 3 ft. 6 in. and 5 ft. 3 in. from a corner of the room. Is this line exactly or approximately parallel to a diagonal of the room.

227. Similar triangles are defined in **92.**

228. Experiment IV.—The relation of the sides of similar triangles.



Draw any oblique scalene triangle; measure the three angles with a protractor. Draw a second triangle of a different size whose angles are equal to the angles of the first triangle. Measure all the sides of both triangles. Calculate the ratio of each pair of homologous sides, $\frac{a}{a'}, \frac{b}{b'}, \frac{c}{c'}$, to three decimals. Observe the relation between these ratios. State result.

EXERCISES

1. The sides of a triangle are 3″, 5″, 5.4″. The side of a similar triangle, homologous to the 3″ side, is 6″. Calculate the other sides of the second triangle.
2. The sides of a $\triangle XYZ$ are $x = 27''$, $y = 31''$, $z = 22''$. The side y' of a similar $\triangle X'Y'Z'$ is 42″. Calculate sides x' and z'.

229. Experiment V.—Triangles whose sides are proportional.

Draw any triangle; measure the three sides. Multiply each side by the same number, as 0.73, or 1.5, etc. Construct a second triangle from a set of three sides thus obtained. Measure the three angles of both triangles. Observe the kind of triangles and state result.

EXERCISES

1. The sides of two triangles are respectively 7″, 5″, 3.4″, and 21″, 15″, 10.2″. Are the triangles similar?
2. The sides of two triangles are respectively 4.5 ft., 5.8 ft., 7.1 ft., and 5.4 ft., 6.96 ft., 9.94 ft. Are the triangles similar?
3. If a triangle is drawn with sides $a = 6''$, $b = 8''$, $c = 10''$, the angles will be found to measure approximately, $A = 37°$, $B = 53°$, $C = 90°$. Obtain the values of the angles of a triangle *DEF*, of which the sides are $d = 15''$, $e = 25''$, $f = 20''$, without constructing the triangle.

230. Experiment VI.—Triangles with two sides proportional and the included angles equal.

Draw any triangle; measure two sides and the included angle. Construct the second triangle. Measure the other two angles and the third sides of both triangles. Observe (*a*) the kind of triangles obtained, and (*b*) the ratio of the third sides.

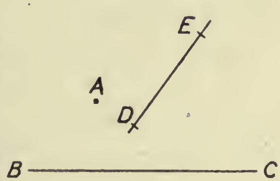
EXERCISES

1. If two triangles are constructed from the parts, respectively, $a = 5''$, $b = 7''$, $C = 40°$, and $a' = 10''$, $b' = 14''$, $C' = 40°$; are the triangles similar?
2. If $\triangle ABC$ is constructed from the values $a = 10''$, $b = 30''$, $C = 45°$, the remaining parts will be found to be approximately, $c = 24''$, $A = 17°$, $B = 118°$. In a $\triangle DEF$, $d = 15''$, $f = 45''$, $E = 45°$. Determine without constructing the triangle, the values of parts e, D and F.

231. Projection.—The *projection of a point* on a line is the foot of the perpendicular from the point to the line. The given line is the *line of projection.*

The *projection of a sect* on a given line is the sect included by the projections of its end points.

EXERCISES



1. Draw a point A and a line BC; project A upon BC. Draw a sect DE; project DE upon BC. Name the line of projection.
2. When does the projection of a sect become a point? When is it equal to the sect? Is it ever greater than the sect?
3. Draw a sect AB intersecting a line CD at A. Construct the projection of AB on CD.

4. Draw two intersecting sects; construct the projection of each on the other.
5. Draw a triangle: project each two sides on the third side.
6. In what kind of triangle do the projections of any two sides upon the other side fall within the side upon which the projection is made? In what kind of triangle do the projections of some sides fall on a side prolonged? In what kind of triangle is the projection of one or more sides a point?
7. In every triangle, how do the projections of any two sides upon the third side compare with the length of the third side?
8. Construct a sect $AB = 3''$, at such an angle to line CD that its projection on $CD = 2.2''$.

232. Experiment VII.—The Relation Between the Segments of a Sect and the Projections of the Segments.

Draw a sect; divide it into two or more unequal segments. Draw any other indefinite line, not parallel to the sect. Project the segments of the sect upon the other line. Measure the segments and their projections. Observe how certain ratios compare. State result.

EXERCISES

1. A sect, AB, 3″ long is divided at point C into segments AC = 1.2″ and CB = 1.8″. The segments are projected on a line DE such that the projection of AC = 0.8″. Calculate the projection of segment CB.
2. The projections of the segments AC and CB of a sect AB are 3.5″ and 6.2″. Sect AB = 12″. Calculate the segments of AB.

233. Experiment VIII.—The relation between the sides of a right triangle.

Draw one or more right triangles; measure the sides. Square the value of each side. Observe the relation between the squares of the three sides of each triangle. State result.

EXERCISES

1. The perpendicular sides of a right triangle are 3 and 4 inches. Calculate the hypotenuse.
2. The hypotenuse and base of a right triangle are respectively 13 and 5 inches. Calculate the altitude.
3. May the lengths 8, 15, 17 inches be taken as the sides of a right triangle? Why?

234. Experiment IX.—The condition for similar polygons.

Draw a trapezium (a polygon of four sides).

(*a*) Draw a polygon having its angles equal, each to each, to those of the given polygon, but its sides in no constant ratio to the sides of the given polygon.

(*b*) Draw a polygon having its sides proportional to the sides of the given polygon, but its angles in no relation to those of the given polygon.

(*c*) Draw a polygon having its angles equal, each to each, to the angles, and its sides proportional to the sides, of the given polygon.

Observe the following principles:

(*a*) May two polygons be mutually equiangular without being similar?

(*b*) May two polygons have their sides proportional without being similar?

(*c*) State the complete condition for similar polygons of more than three sides.

EXERCISES

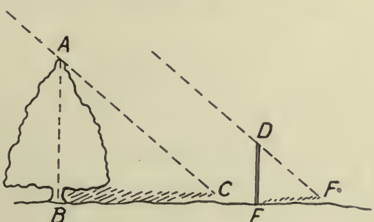
1. A rectangle has adjacent sides of 4″ and 2″. A second rectangle has homologous sides of 6″ and 3″, Are the rectangles similar?
2. The sides of two quadrilaterals are respectively, 3″, 5″, 4.8″, 2″; and (in the same order) 6″, 10″, 9.6″, 4″. Are they similar?
3. The sides of one of two similar pentagons are 7″, 10″, 5″, 8″, 6″. A side of the second pentagon, homologous to the 10″ side of the first, is 12″. Calculate the other sides of the second pentagon.

235. Abbreviation.—Similar, is similar to, $\sim$.

Applications

236. Forestry.—The following methods of determining the heights of trees with sufficient accuracy, are described in a bulletin issued by the U. S. Department of Agriculture.

(1) *The Shadow Method.*—Thales employed this method to find the height of the great pyramids of Egypt, to the admiration of native scholars.



EXERCISES

1. Are the sun's rays, AC and DF, substantially parallel?
2. Prove that the triangles, formed by tree, shadow and sun's ray

is similar to that formed by the vertical pole, shadow of pole and sun's ray.

3. The shadow of a tree is measured 130 feet at the same time that the shadow of a 12-foot pole is 15 feet. Calculate the height of the tree.
4. May this method be used to find the height of a flagpole, building, stack, etc.
5. Will the two triangles be similar if the ground is sloping? If the tree is leaning?
6. Find a few facts concerning Thales.

(2) *The Graduated Rod Method.*

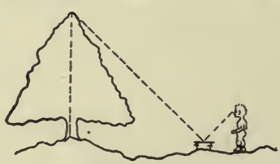
(*a*) The observer standing.

(*b*) The observer lying down.

EXERCISES

7. Explain both methods as illustrated.
8. Place letters on the figures and read the similar triangles.
9. What measurements are taken in each method?
10. Assume reasonable measurements and calculate the heights of the trees by both methods.
11. Is this method applicable if the ground is sloping?

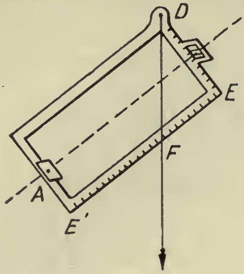
(3) *The Mirror Method.*

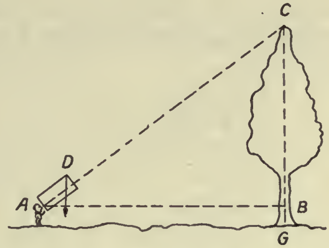


EXERCISES

12. Explain the method illustrated.
13. Where are the similar triangles?
14. Assume reasonable measurements and calculate the height of the tree.
15. Is this method applicable on sloping ground if the mirror is leveled?

(4) *Faustmann's Hypsometer.*—A simplified form of this instrument is shown. The instrument is sighted to the top





of the tree and the reading is taken at F on the scale EF. It is necessary to measure AB, which may be done by pacing, and to measure or estimate BG. This may be done by sighting through a hand level and noting the point B on the level line AB, and then measuring from B to the ground. After calculating CB, the distance BG is added to find the total height of the tree.

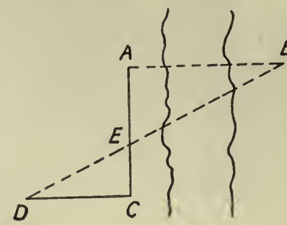
In the instrument as usually made, the point of attachment D of the plumb-line is on a movable scale, so that the distance AB in feet may be set off on the scale equal to ED. The graduations on scale EE' are made in the same units as employed on scale ED, and the height of the tree in feet is therefore equal to the reading of the scale distance EF. A narrow adjustable mirror hinged at point E is provided, and the observer may thus read the scale at F while sighting the instrument on C, the figures of the scale EE' being reversed.

EXERCISES

16. Name the two similar triangles. Prove that they are similar.
17. If $DE = 5''$, $EF = 6''$, $AB = 40'$, $BG = 7'$; calculate CG.
18. If $BG = 8.5$ feet, $AB = 120$ feet; and if ED is set at 120 units and EF reads 95 units; what is the value of BC in feet, and what is the height of the tree?

237. Surveying.—The principles of similar triangles afford some useful methods of determining distances which cannot be measured directly, and of laying out perpendiculars.

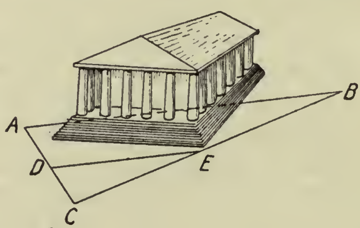
The Distance Across a River.—The distance AB is required. Turn a right angle at A; measure any distance AC; turn a right angle at C; measure any distance CD. Set a stake at point E on both lines DB and AC.



EXERCISES

1. Name the similar triangles of the figure. Prove that they are similar.
2. What proportion can be stated whose solution will give the distance AB?
3. What additional measurement, not mentioned above, must be made, in order to give sufficient data for the calculation of AB?
4. If $AC = 500'$, $CD = 320'$, $CE = 180'\ 6''$, calculate AB.
5. Upon what principle of this chapter does the calculation of AB depend?
6. How does this method differ from that of **146**, Exercise 8?

The Distance Between Two Points on Opposite Sides of a Building on Pond.—The distance AB is required. Set a stake at any point C from which both A and B can be seen; measure CA and CB. Mark any point E on CB such that a line DE will pass the building. Calculate the position of point D on CA such that DE is parallel to AB, and set a stake at D. Measure DE and calculate AB.

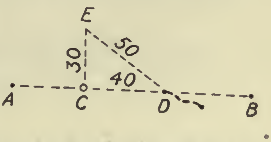


EXERCISES

7. How is the distance CD found, such that DE will be parallel to AB?
8. How is the distance AB calculated, after all measurements are completed?
9. Calculate AB if $CA = 240'$, $CB = 300'$, $CE = 200'$, $DE = 238'$.
10. Upon what principles of this chapter do the calculations of CD and AB depend?

11. How does this method differ from that of **146**, Exercise 7?
12. Is it possible to modify the method, by setting point E (*a*) on CB extended through B; (*b*) on CB extended through C? Sketch the figures in these cases.

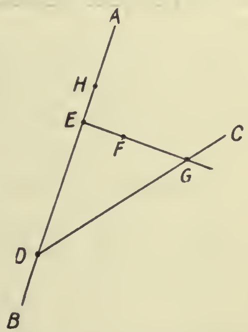
A Perpendicular to a Given Line at a Point in the Line.—This is a method quickly used and often sufficiently accurate. Measure $CD = 40'$; hold a 100-foot tape with one end at C and the 80-foot mark at D; take hold of the 30-foot mark and hold the tape so that both parts EC and ED are taut. Mark point E.



EXERCISES

13. Upon what principle of this chapter does the method depend?
14. What is the largest triangle that can be laid out in this way with a 100-foot tape, with exact foot-lengths for its sides?
15. Can the shorter side of the triangle be measured on CD?

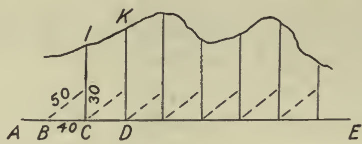
A Perpendicular to a Given Line from a Point Without the Line.—The perpendicular is to be laid out from C which is more than 100 feet from line AB. Place any stake D on line AB; measure CD; place a stake E on AB at about the foot of the required perpendicular, and measure DE; erect a perpendicular EF; place a stake at G on both EF and CD; measure GC.



EXERCISES

16. How is EH calculated such that CH will be parallel to EG?
17. After calculating EH, how is the required perpendicular marked on the ground?
18. Let $CD = 250.4'$, $DE = 175.8'$, $GC = 36.5'$; calculate EH.
19. May the point E be placed beyond H and point G in line CD extended? Draw a figure illustrating this and describe the method of calculating EH.
20. Upon what principle of this chapter does the method depend?

Locating a Curved Line.—A straight line AE is staked out near the curved line; and divided into 40-foot intervals. Perpendiculars, or offsets, CI, DK, etc., are measured.



EXERCISE

21. Plot the bank of a creek from the given field notes, to a scale of 100 feet to the inch. The points lie on a straight line at intervals of 40 feet. The offsets are measured to the center of the creek.

Pts.	Offsets at 40′	Pts.	Offsets at 40′	Pts.	Offsets at 40′	Pts.	Offsets at 40′
0	0	3	+ 60	6	+60	9	−90
1	+ 30	4	+100	7	+45	10	−35
2	+ 50	5	+ 95	8	−70		

CHAPTER XI

SIMILAR TRIANGLES AND SIMILAR POLYGONS

Classification and Explanation of Principles

238. The First Step in Classification.

Postulates

1. Mutually equiangular triangles are similar.

2. The homologous sides of similar triangles are proportional.

3. Two triangles are similar if two sides of one triangle are proportional to two sides of the other and if the included angles are equal.

4. Two triangles are similar if all the sides are proportional.

5. Two polygons are similar if they are mutually equiangular, and also if their homologous sides are proportional; and conversely.

239. The Second Step in Classification.—All other principles of similar triangles and similar polygons can be analyzed in terms of the postulates of **238** and other previously analyzed principles.

240. Theorem I.—*A proportion can be changed according to certain laws.*

(*a*) **Law of Products.**—*The product of the means equals the product of the extremes.*

Hypothesis.—Any proportion, $\frac{a}{b} = \frac{c}{d}$.

Conclusion.—$ad = bc$.

Help.—Clear of fractions. What axiom is used?

Corollary.—*If the product of two numbers equals the product of two other numbers, the numbers of one product may be made the means and those of the other product the extremes, of a proportion.*

Hypothesis.—$xy = st.$

Conclusion.—(1) $\frac{x}{s} = \frac{t}{y}$, (2) $\frac{x}{t} = \frac{s}{y}$, (3) $\frac{s}{x} = \frac{y}{t}$, etc.

Help.—(1) Divide by sy; etc.

(*b*) **Law of Inversion.**—*The ratios of a proportion may be inverted.*

Hypothesis.—$\frac{a}{b} = \frac{c}{d}.$

Conclusion.—$\frac{b}{a} = \frac{d}{c}.$

(*c*) **Law of Alternation.**—*The means, or the extremes, of a proportion may be interchanged.*

Help.—Apply law (*a*); divide by cd, etc.

(*d*) **Law of Composition.**—*The sum of the numerator and denominator of each ratio may be made the numerators of the ratios.*

Hypothesis.—$\frac{a}{b} = \frac{c}{d}.$

Conclusion.—$\frac{a+b}{b} = \frac{c+d}{d}.$

Help.—Add 1 to each ratio.

(*e*) **Law of Division.**

Conclusion.—$\frac{a-b}{b} = \frac{c-d}{d}.$

(*f*) **Law of Ratio Series.**

Hypothesis.—$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ etc.

Conclusion.—$\frac{a+c+e+---}{b+d+f+--} = \frac{a}{b} = \frac{c}{d} =$ etc.

Analysis.—(1) $\frac{a}{b} = r; \frac{c}{d} = r;$ etc.

(2) $a = br;\ c = dr;$ etc.

(3) $a + c + e + --- = br + dr + fr + ---$

(4) $r = \frac{a+c+e+---}{b+d+f+---} = \frac{a}{b} = \frac{c}{d} = ---$

What axioms are used?

EXERCISES

1. Solve these proportions for the unknown term: (1) $\frac{3}{7} = \frac{4}{x}$; (2) $\frac{5}{2} = \frac{x}{3}$; (3) $\frac{x}{9} = \frac{4}{3}$; (4) $\frac{5}{x} = \frac{31}{3}$.

   Help.—Apply law (*a*).

2. Solve for the unknown term: (1) $\frac{3}{x} = \frac{x}{7}$; (2) $\frac{x}{2} = \frac{32}{x}$.

   Help.—Apply law (*a*).

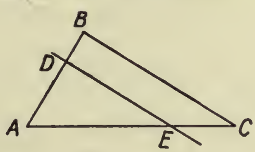
3. Change the proportion $\frac{7}{9} = \frac{21}{27}$; according to laws (*a*) to (*f*). Is each new proportion obtained a true equation?

4. Change the ratio series $\frac{2}{3} = \frac{4}{6} = \frac{10}{15} = \frac{1}{1.5}$; according to law (*f*).

5. Solve $x : x + 1 = 5 : 7$ for x, and check.

6. Show that the proportion $\frac{a}{b} = \frac{c}{d}$ is equivalent to: (1) $\frac{d}{b} = \frac{c}{a}$; (2) $\frac{a}{a+b} = \frac{c}{c+d}$; (3) $\frac{a}{a-b} = \frac{c}{c-d}$; (4) $\frac{a+c}{b+d} = \frac{a}{b}$; (5) $\frac{a+b}{c+d} = \frac{b}{d}$.

241. Theorem II.—*A line parallel to a side of a triangle divides the other two sides proportionally.*



Hypothesis.—$\triangle ABC$ with $DE \parallel BC$.

Conclusion.—$\frac{AD}{AB} = \frac{AE}{AC}$.

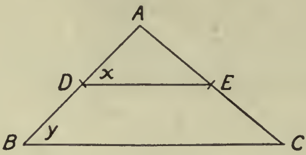
Helps.—Show that there are two similar triangles; use *Postulate* 2, **238.** Draw the triangles separately.

Corollaries.—(1) $\frac{DE}{BC} = \frac{AD}{AB} = \frac{AE}{AC}$.

(2) $\frac{AD}{AE} = \frac{AB}{AC}$; $\frac{AD}{DB} = \frac{AE}{EC}$; $\frac{AD}{AE} = \frac{DB}{EC}$; etc.

Helps.—If $\frac{AD}{AB} = \frac{AE}{AC}$; then $\frac{AB}{AD} = \frac{AC}{AE}$, $\frac{AB - AD}{AD} = \frac{AC - AE}{AE}$, $\frac{AD}{AB - AD} = \frac{AE}{AC - AE}$, $\frac{AD}{DB} = \frac{AE}{EC}$.

242. Theorem III.—*A line which divides two sides of a triangle proportionally is parallel to the third side.*

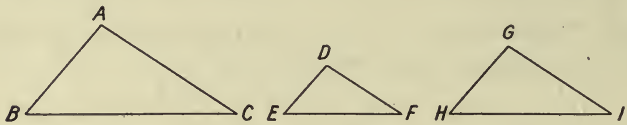


Hypothesis.—$\triangle ABC$ with DE drawn so that $\dfrac{AD}{AB} = \dfrac{AE}{AC}$.

Conclusion.—$DE \| BC$.

Helps.—(1) $\triangle ADE \sim \triangle ABC$; Postulate 3; (2) $\therefore x = y$; (3) $\therefore DE \| BC$; why? Draw the triangles separately.

243. Theorem IV.—*Two triangles similar to the same triangle are similar to each other.*



Hypothesis.—$\triangle$s DEF and $GHI \sim \triangle ABC$.

Conclusion.—$\triangle DEF \sim \triangle GHI$.

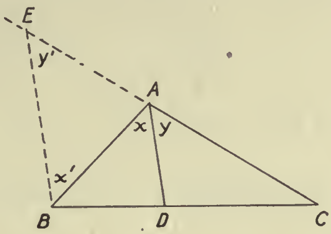
Helps.—$\angle A = \angle D$; $\angle A = \angle G$; $\therefore \angle D = \angle G$; etc.

244. The Third Step in Classification.—The principles which follow, **245** to **256,** are probably the result of the application of deductive reasoning, and, with the exception of **250** in an elementary form, were not discovered experimentally.

In making use of similar triangles in analysis it is advisable to mark the mutually equal angles of the two triangles under consideration, x and x', y and y', z and z'. The homologous sides are then readily identified as being opposite the homologous angles. In some figures it is also helpful to draw the triangles under consideration apart from the general figure, and to letter all angles and sides as they are lettered in the general figure. The use of colored crayons, overlaying homologous sides of similar triangles with the same color, will be found a great help.

Notice that, if two angles of one triangle are equal, each to each, to two angles of another triangle, the triangles are similar. Why?

245. Theorem V.—*The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*

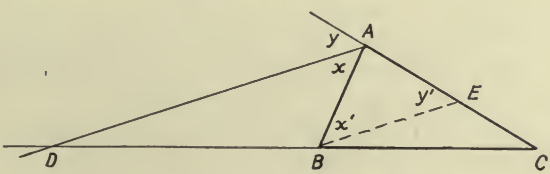


State the *hypothesis* and the *conclusion*.

Helps.—(1) How are the auxilliary lines drawn? (2) $\frac{CA}{AE} = \frac{CD}{DB}$; why? (3) $\triangle ABE$ is isosceles; how shown? (4) $\therefore AB = AE$; (5) $\therefore \frac{CA}{AB} = \frac{CD}{DB}$.

Draw the elementary figures which make up the figure of the thèorem.

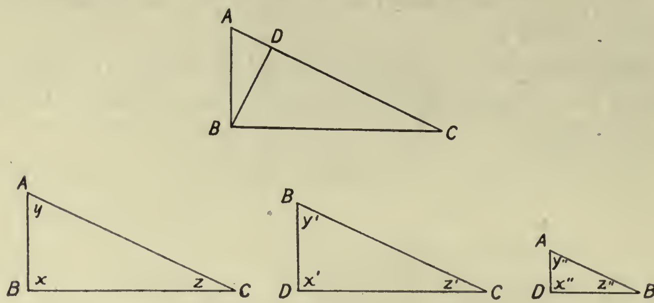
246. Theorem VI.—*The bisector of an exterior angle of a triangle divides the opposite side externally into segments, etc.* (complete the theorem after analysis).



Helps.—(1) Draw BE; (2) $\frac{CA}{AE} = \frac{CD}{DB}$; (3) $\triangle ABE$ is isosceles; (4) $AB = AE$; etc.

Draw the elementary figures as in **245.**

247. Theorem VII.—*The altitude drawn to the hypotenuse of a right triangle divides the triangle into two triangles which are similar to the given triangle and to each other.*



Helps.—The three triangles to be considered are ABC, BDC and ADB.

248. Theorem VIII.—*The altitude drawn to the hypotenuse of a right triangle is a mean proportional between the segments into which it divides the hypotenuse.*

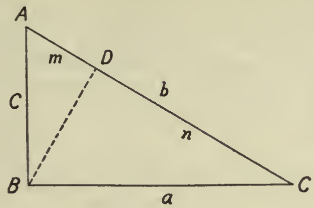
Helps.—Each pair of triangles of **247** affords three proportions. If these are stated in full, it will be found that a proportion with a repeated mean (or extreme) is found in each set of three. One of these mean proportionals is the required relation. A repeated *extreme*, $\frac{a}{b} = \frac{c}{a}$, may also be regarded as a *mean* proportional, since an equivalent proportion is $\frac{b}{a} = \frac{a}{c}$.

249. Theorem IX.—*Each perpendicular side of a right triangle is a mean proportional between the hypotenuse and the projection of the side upon the hypotenuse.*

Help.—The required relations are found among the proportions suggested in **248.**

Corollary.—*The square of a perpendicular side of a right triangle equals the product of the hypotenuse and the projection of the side upon the hypotenuse.*

250. Theorem X.—*The sum of the squares of the two* *perpendicular sides of a right triangle equals the square of the hypotenuse.*



Hypotenuse.—Right $\triangle ABC$.
Conclusion.—$b^2 = a^2 + c^2$.
Analysis.

STATEMENT	REASON
1. $a^2 = bn$ and $c^2 = bm$	Theorem IX, Corollary
2. $a^2 + c^2 = bn + bm$	Axiom
3. $a^2 + c^2 = b(n + m)$	Algebraic Axiom
4. $a^2 + c^2 = b.b$	Axiom
5. $\therefore a^2 + c^2 = b^2$	Algebraic Axiom

Corollaries.—(1) $a^2 = b^2 - c^2$; $c^2 = b^2 - a^2$.
(2) $a = \sqrt{(b + c)(b - c)}$; etc.

The relation of Corollary 2 is convenient in numerical calculation.

251. Historical Interest of Theorem X. (250).—This relation is supposed to have been known for simple number values (at least for the numbers 3, 4, 5) by the Egyptians as early as 2000 B. C., and also by the ancient Hindoos and Chinese. The general principle was discovered by Pythagoras (582–500 B. C.), whose method of analysis was probably somewhat like the method here presented. The theorem is known as the *Pythagoréan Theorem.* An entirely different method was given by Euclid (Chapter XVII). Numbers which possess the relation $a^2 + b^2 = c^2$ are called Pythagorean numbers; many such series exist in higher values.

252. Theorem XI.—*In any triangle, the square of a side opposite an acute angle equals the sum of the squares of the other two sides minus twice the product of one of those sides by the projection of the other side upon it.*

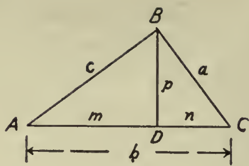


FIG. 1.

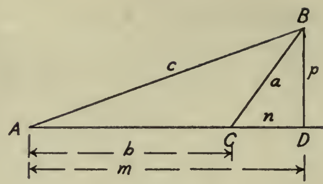
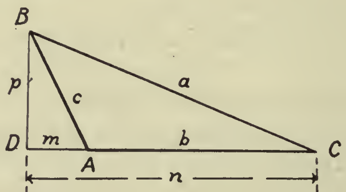


FIG. 2.

Hypothesis.—$\triangle ABC$ with an acute $\angle A$, and $BD \perp AC$.
Conclusion.—$a^2 = b^2 + c^2 - 2bm$.
Analysis.

STATEMENT	REASON
1. $m + n = b$, or $b + n = m$	Axiom 11
2. $n = b - m$ or $n = m - b$	Axiom 3
3. $n^2 = b^2 - 2bm + m^2$	Algebraic axiom
4. $n^2 + p^2 = b^2 - 2bm + m^2 + p^2$	Axiom 2
5. $a^2 = b^2 - 2bm + c^2$	**250** and Axiom 13

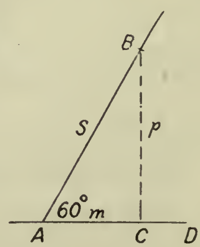
253. Theorem XII.—*In any obtuse triangle, the square of the side opposite the obtuse angle equals, etc.* (*Complete the theorem after analysis.*)



Write the *hypothesis* and *conclusion.*

Helps.—(1) $n - m = b$; (2) $\therefore n = b + m$. Complete as in **252.**

254. Theorem XIII.—*The projection of a sect can be calculated when the angle which the sect makes with the line of projection is* (*a*) 60°, (*b*) 30°, (*c*) 45°.



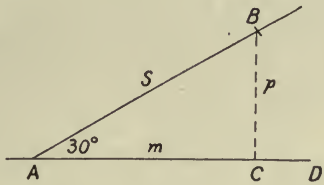
Part (*a*).

Hypothesis.—A sect s making an angle of 60° with the line of projection AD, and its projection m.

Conclusion.—$m = \frac{1}{2}s$.
Analysis.

STATEMENT	REASON
1. $\angle C = 90°$; $\angle B = 30°$	
2. $m = \frac{1}{2}s = .5s$	**143,** Theorem (8)

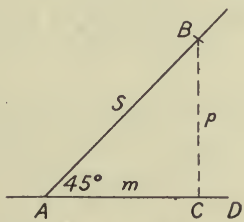
Part (*b*).



State *hypothesis* and *conclusion*.

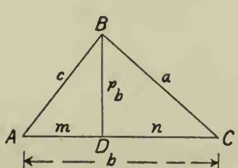
Helps.—(1) $p = \frac{1}{2}s$; (2) $m^2 = s^2 - p^2$; (3) $m^2 = \dfrac{3s^2}{4}$; (4) $m = \dfrac{s\sqrt{3}}{2} = .866s$.

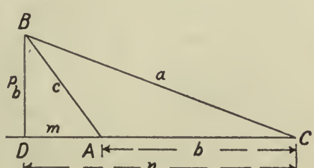
Part (*c*).



Helps.—(1) $m = p$; (2) $m^2 + p^2 = s^2$; (3) $m = \dfrac{s\sqrt{2}}{2} = .707s$.

255. Theorem XIV.—*An altitude of a triangle can be expressed in terms of the three sides.*





Analysis.

STATEMENT	REASON
1. In $\triangle BDA$, Fig. 1, $p_b^2 = c^2 - m^2$.	What theorem?
2. In $\triangle ABC$, $a^2 = b^2 + c^2 - 2bm$	What theorem?
3. From 2, $m = \dfrac{b^2 + c^2 - a^2}{2b}$	Axioms
4. Substituting from 3 in 1, $p_b^2 = c^2 - \left(\dfrac{b^2 + c^2 - a^2}{2b}\right)^2$	Axiom
5. $p_b^2 = \left(c + \dfrac{b^2 + c^2 - a^2}{2b}\right) \times \left(c - \dfrac{b^2 + c^2 - a^2}{2b}\right)$	Algebraic axiom
6. $= \dfrac{(+a + b + c)(-a + b + c)}{2b} \times \dfrac{(a - b + c)(a + b - c)}{2b}$.	Algebraic axiom
7. Let $a + b + c = 2s$ then	
$-a + b + c = 2s - 2a = 2(s - a)$	Algebraic axiom
$a - b + c = 2s - 2b = 2(s - b)$,	Algebraic axiom
etc.	
8. Substituting from 7 in 6 $p_b^2 = \dfrac{2s \cdot 2(s - a) \cdot 2(s - b) \cdot 2(s - c)}{4b^2}$	Axiom
9. $p_b = \dfrac{2}{b}\sqrt{s(s - a)(s - b)(s - c)}$	Algebraic axioms

Show that the same result is obtained in Fig. 2.

Helps.—(1) $p_b^2 = c^2 - m^2$; (2) $a^2 = b^2 + c^2 + 2bm$; (3) $m = \dfrac{a^2 - b^2 - c^2}{2b}$; etc.

Write the values of p_a and p_c by interchanging letters in the formula for p_b.

256. Theorem XV.—*A median of a triangle can be expressed in terms of the three sides.*

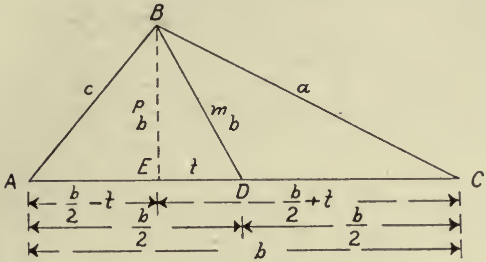
Analysis.

1. In $\triangle BED$, $p_b^2 = m_b^2 - t^2$

2. In $\triangle BEC$, $a^2 = p_b^2 + \left(\dfrac{b}{2} + t\right)^2 = p_b^2 + \dfrac{b^2}{4} + bt + t^2$

3. In $\triangle BEA$, $c^2 = p_b^2 + \left(\frac{b}{2} - t\right)^2 = p_b^2 + \frac{b^2}{4} - bt + t^2$

4. Adding 2 and 3; $a^2 + c^2 = 2p_b^2 + \frac{b^2}{2} + 2t^2$



5. Substituting value of p_b^2 from 1 in 4;

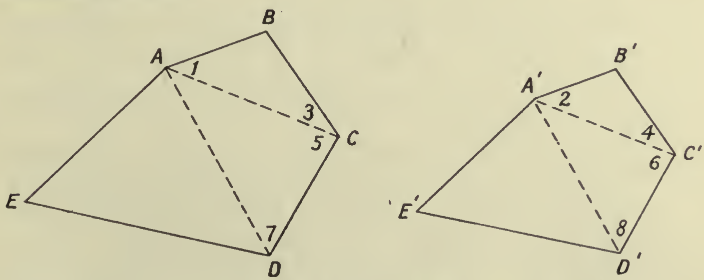
$$a^2 + c^2 = 2(m_b^2 - t^2) + \frac{b^2}{2} + 2t^2 = 2m_b^2 + \frac{b^2}{2}$$

6. Solving 5 for m_b^2; $m_b^2 = \dfrac{2(a^2 + c^2) - b^2}{4}$

7. $m_b = \frac{1}{2}\sqrt{2(a^2 + c^2) - b^2}$

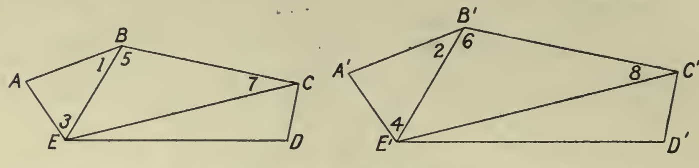
Show that the same result is obtained if $\angle A$ is obtuse. Write the values of m_a and m_c by interchanging letters in the formula for m_b.

257. Theorem XVI.—*Two similar polygons can be divided into the same number of triangles similar each to each and similarly placed.*



Helps.—(1) $\triangle ABC \sim \triangle A'B'C'$; **238,** Postulate 3;—(2) $\therefore$ $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$; and $\dfrac{AC}{A'C'} = \dfrac{AB}{A'B'} = \dfrac{CD}{C'D'}$;—(3) $\angle 5 = \angle 6$, why? and $\therefore$ $\triangle ACD \sim \triangle A'C'D'$; etc.

258. Theorem XVII.—*Two polygons that are composed of the same number of triangles similar each to each and similarly placed, are similar.*



Helps.—(1) $\angle A = \angle A'$; $\angle 1 = \angle 2$; $\angle 3 = \angle 4$;—(2) $\angle 5 = \angle 6$; $\angle 7 = \angle 8$; etc.,—(3) $\therefore$ polygons are mutually equiangular. —(4) $\frac{AB}{A'B'} = \frac{BE}{B'E'} = \frac{BC}{B'C'} = \frac{EC}{E'C'} =$ etc.—(5) $\therefore$ polygons have their homologous sides proportional.—(6) $\therefore$ the polygons are similar; **238,** Postulate 5.

259. The Fourth Step in Classification.—The principles of **245** to **256** cannot be considered as established beyond all doubt until they have been tested by actual measurements. This we will now proceed to do.

EXERCISES

1. Draw a triangle (full size) whose sides are 4, 5.6 and 6.2 inches. Draw the bisector of each interior angle. Calculate by **245** the segments into which each bisector divides the opposite side. Measure these segments and note the agreement of the measurements with the calculated values.

   Thus to calculate the segments of the largest side, calling the segments x and $6.2 - x$, we have $\frac{x}{4} = \frac{6.2 - x}{5.6}$, whence $x = 2.58''$, and $6.2 - x = 3.62''$. Calculate in the same way the segments of the other sides.

2. Draw the bisectors of each pair of exterior angles of the triangle of Exercise 1, and prolong each bisector until it meets the opposite side prolonged. Calculate by **246** the segments into which each bisector divides the opposite side externally. Measure these segments (from the external point of division to each end of the side), and note the agreement with the calculated values.

   Thus to calculate the segment of the largest side, calling the segments x (to the nearer end of the side) and $6.2 + x$ (to the farther end of the side), we have $\frac{x}{4} = \frac{6.2 + x}{5.6}$, whence derive values of x and $6.2 + x$. Calculate in the same way the segments of the other sides.

3. Draw a right triangle (full size) whose sides are 3, 4 and 5 inches. Draw the altitude to the hypotenuse. Calculate by **249** the projection of each perpendicular side upon the hypotenuse, *i.e.*, the segments into which the altitude divides the hypotenuse. Measure these projections (or segments) and note the agreement with the calculated values.

   Thus to calculate the smaller segment x, of the hypotenuse, we have $\frac{x}{3} = \frac{3}{5}$, whence $x = 1.8''$. Calculate in the same way the other segment y.

4. In the triangle of Exercise 3, calculate by **248** the length of the altitude drawn to the hypotenuse. Measure the altitude and compare with the calculated value.

5. Draw a sect 3 inches long and through one end of it a line making with the sect an angle of 60°. Project the sect on the second line. Calculate by **254**(a) the value of the projection. Measure the projection and compare with the calculated value.

6. Draw a sect of 4 inches, and a line making an angle of 30° with it. Project the sect on the other line. Calculate by **254**(b) the value of the projection. Measure and compare results.

7. Draw any sect and a line making an angle of 45° with it. Proceed as in Exercises 5 and 6.

8. Draw a triangle (full size) of which two sides are 5 and 3 inches and the included angle 60°. Calculate the third side by **252.**

   Thus: the projection of the $3''$ side on the $5''$ side is (by **254**(a)) $1.5''$.

   Then $a^2 = 3^2 + 5^2 - 2(5)(1.5)$. Complete the calculation. Measure the third side and compare with the calculated value.

9. Draw two triangles in each of which two sides are 5 and 3 inches respectively; in one triangle the included angle being 30°, and in the other 45°. Calculate the projections of the $3''$ side on the $5''$ side by **254**(b) and **254**(c), and complete calculations for the third side of each triangle as in Exercise 8.

10. Draw a triangle of which two sides are 5 and 3 inches and the included angle 120°. Calculate the third side by **253.**

    Thus: the projection of the $3''$ side on the $5''$ side is $1.5''$.

    Then $a^2 = 3^2 + 5^2 + 2(5)(1.5)$; complete the calculation. Measure the third side and compare with the calculated value.

11. Draw triangles with the sides as in Exercise 9, the included angle in one triangle being 150° and in the other 135°. Calculate the projections of the $3''$ side on the $5''$ side as in Exercise 9, and the values of the third side of each triangle by **253.** Measure and compare values.

12. Draw a triangle (full size) whose sides are 4, 5, and 6 inches. Draw the three altitudes. Calculate the lengths of the altitudes by **255**.

    Thus, to calculate the altitude to the longest side:

$$s = \frac{4+5+6}{2} = 7.5$$

$$s - 4 = 3.5,\ s - 5 = 2.5,\ s - 6 = 1.5$$

$$p = \frac{2}{6}\sqrt{(7.5)(3.5)(2.5)(1.5)}$$

    Complete the calculation. Measure the altitude and note the agreement with the calculated value. Calculate and measure the other altitudes in the same way.

13. Draw the medians in the triangle of Exercise 12. Calculate their lengths by **256**. Measure these lengths in the triangle and compare with the calculated values.
14. In every case in the preceding exercises do the calculated values show a sufficiently close agreement with the deduced principles as to confirm the accuracy of these principles?

260. Additional Theorems

(1) *Two isosceles triangles with equal vertex angles are similar.*

(2) *All equilateral triangles are similar.*

(3) *Homologous altitudes of similar triangles are proportional to any two homologous sides.*

Help.—Show that the triangles containing the altitudes as homologous sides are mutully equiangular.

(4) *Homologous medians of similar triangles are proportional to any two homologous sides.*

Help.—Use **238,** Postulate 3, to show that the triangles under consideration are similar.

(5) *Homologous angle-bisectors of similar triangles are proportional to any two homologous sides.*

(6) *In a triangle ABC, altitudes AD, BE and CF are drawn Show that $\triangle ADC$ is similar to $\triangle BEC$, and $\triangle ADB$ is similar to $\triangle CFB$, and that there is another pair of similar triangles formed.*

(7) *Two right triangles are similar if they have a pair of equal acute angles.*

(8) *Two isosceles triangles are similar if they have a base angle of each respectively equal.*

(9) *The projections of the segments of a sect are proportional to the segments* (**232**).

(10) *A system of parallel lines divides any two transversals proportionally.*

(11) *The diagonals of a trapezoid divide each other proportionally.*

(12) State another theorem relating to the proportionality of the segments into which the diagonals of a trapezoid divide each other.

(13) *A pencil of three concurrent lines divides two parallel lines which are cut by all the lines of the pencil, proportionally.*

Helps.—Use two pairs of similar triangles. The two proportions thus occurring have a common ratio.

(14) *The diagonal of a square equals the side multiplied by* $\sqrt{2}$; *or* $d = s\sqrt{2}$.

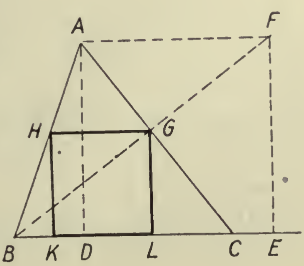
(15) *The squares of the perpendicular sides of a right triangle are proportional to their projections on the hypotenuse.*

Help.—Apply **249,** corollary.

(16) *The altitude of an equilateral triangle may be expressed in terms of the side; or* $p = \frac{s\sqrt{3}}{2}$.

(17) *The side of a square may be expressed in terms of the diagonal.*

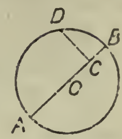
(18) *The side of an equilateral triangle may be expressed in terms of the altitude; or* $s = \frac{2p\sqrt{3}}{3}$.



(19) *Find a method of drawing a square HGKL inscribed in the triangle ABC. Prove that the method found is correct.*

Helps.—(1) $AF = FE$;—(2) $\frac{GL}{FE} = \frac{BG}{BF}$;—(3) $\frac{HG}{AF} = ?$; (4) $\therefore \frac{GL}{FE} = \frac{HG}{AF}$; etc.

(20) *A relation exists between a perpendicular on one side of a diameter of a circle and the segments into which it divides the diameter.*



Helps.—Draw DA and DB; see **143,** Theorem (7). Then apply one of the preceding theorems of this chapter relating to right triangles.

(21) *A relation exists between a chord of a circle, the diameter drawn through the end of the chord, and the projection of the chord on that diameter.*

(22) *A relation exists between the product of the perpendicular sides of a right triangle, the hypotenuse, and the altitude to the hypotenuse.*

(23) *Any angle of a triangle may be evaluated, whether acute, right or obtuse, by the relation of the squares of the three sides.*

(24) *Two triangles whose sides are parallel, each to each, are similar.*

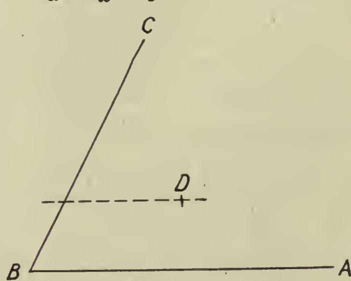
Helps.—From **81,** we may have; (a) $x + x' = 180°$, $y + y' = 180°$, $z + z' = 180°$; or (b) $x + x' = 180°$, $y + y' = 180°$, $z = z'$; or (c) $x + x' = 180°$, $y = y'$, $z = z'$; or (d) $x = x'$, $y = y'$, $z = z'$. Show that (d) is the only possible numerical relation of the angles of the two triangles.

(25) *A relation exists between two triangles whose sides are perpendicular, each to each.*

(26) *The segments into which the bisector of angle A of $\triangle ABC$ divides side a, are* $\frac{ab}{b+c}$ *and* $\frac{ac}{b+c}$.

Help.—If x is a segment of a, $\frac{x}{a-x} = \frac{b}{c}$.

(27) *Point D lies within $\angle ABC$. Find a method of drawing a sect through point Determinated by the sides of the angle and bisected by point D. Prove that the method found is correct.*



(28) *In the figure of Theorem* (27), *find a method*

of drawing a sect through point D so that it is divided by point D in any given ratio. Prove that the method found is correct.

(29) *The product of a side of a triangle multiplied by the altitude drawn to that side, equals the product of any other side of the triangle multiplied by the altitude drawn to it.*

(30) *Two polygons similar to the same polygon are similar to each other.*

Help.—It is necessary to prove both that the polygons are mutually equiangular and that their homologous sides are proportional.

(31) *Homologous diagonals of similar polygons are proportional to any two homologous sides.*

(32) *The perimeters of two similar polygons are proportional to any two homologous sides.*

Help.—(1) The sides $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$ = etc.; (2) apply **240** (*f*).

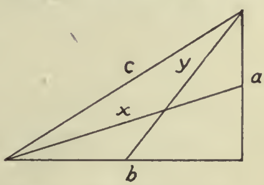
(33) *A sect is divided harmonically into segments of which x is one internal segment and y is the corresponding external segment, when* $y = \frac{sx}{s - 2x}$.

(34) *The bisectors of adjacent interior and exterior angles of a triangle divide the opposite side harmonically.*

(35) *The sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals.*

Help.—Express the value of each diagonal from **252** and **253,** and add.

(36) *Four times the sum of the squares of the medians drawn to the perpendicular sides of a right triangle, equals five times the square of the hypotenuse.*



Helps.—(1) Express x and y from **256**; (2) square these values and multiply out the coëfficients 2 and $\frac{1}{2}$; (3) add and reduce.

(37) *The difference of the squares of two sides of any triangle equals the difference of the squares of the projections of these sides upon the third side.*

Help.—Draw an altitude, thus forming two right triangles.

261. Review Exercises

1. The perpendicular sides of a right triangle are 12 and 16. Calculate the hypotenuse.
2. Calculate the diagonal of a square whose side is 10.
3. Calculate the side of a square whose diagonal is 12.
4. Two sides and the included angle of a triangle are 10, 15, 135°. Calculate the third side.
5. Two sides and the included angle of a triangle are 20, 45, 60°. Calculate the third side.
6. The perpendicular sides of a right triangle are 6 and 10. Calculate their projections on the hypotenuse.
7. To check the layout of a tennis court, 36 feet by 78 feet, calculate and measure the diagonals.
8. A derrick mast 50 feet high is guyed by a rope running from the top of the mast to a tree trunk 100 feet distant. Allow 5 feet for fastening, and calculate the length of rope required.
9. Calculate the resultant of two forces of 75 lb. and 32 lb., acting at an angle of 90°.
10. The resultant of two forces acting at an angle of 90° is 12 lb.; one of the forces is 3 lb. Calculate the other force.
11. Calculate the resultant of two forces of 10 lb. and 15 lb. acting at an angle of 45°; at an angle of 135°.
12. Calculate the resultant of three forces of 10, 12 and 15 lb. respectively, the first two forces acting at a right angle to each other, and the third force acting at a right angle to the resultant of these two.
13. Show that the values $a^2 + b^2$, $a^2 - b^2$, $2ab$ may represent the sides of a right triangle. Where is the right angle?
14. Pythagoras stated that the values $2n + 1$, $2n(n + 1)$, $2n^2 + 2n + 1$ are the sides of a right triangle. Prove that this is so.
15. Let $n = 1$ in the general values of Exercise 14; show that the sides of the triangle are in this case, 3, 4, and 5. Let $n = 2$, and find the sides of the right triangle from the formulas of Exercise 14. Similarly, let $n = 3, 4, 5$, etc.
16. One of the perpendicular sides of a right triangle is 2 more than the other; the hypotenuse is 10. Calculate the sides. Solve by forming an algebraic equation. Check.
17. One of the perpendicular sides of a right triangle is 20 inches and the hypotenuse is 10 more than the other side. Calculate side and hypotenuse.

18. An ancient Chinese problem which appears in many texts: In the middle of a circular well 10 feet in diameter, grows a reed which projects 1 foot above the surface. When the reed is bent over, remaining straight from its root, the top just reaches the edge of the well. How deep is the well?

19. A builder plans a barn with the ridge of the roof 10 feet above the loft floor, and the floor 24 feet wide. How long must he order the rafters, allowing 2 feet overhang at the eaves and 6 inches for trimming the ends?

20. Two chords of a circle, 8 and 12 inches respectively, are drawn from the same point in a circumference to the extremities of a diameter. Calculate the diameter.

21. A perpendicular to the diameter of a circle is 10 inches and one of the segments into which it divides the diameter is 5 inches. Calculate the diameter.

22. The altitude to the hypotenuse of a right triangle is 3 inches and the projection of one of the perpendicular sides upon the hypotenuse is 8 in. Calculate the three sides of the triangle.

23. The rise CB in 100 feet of slope AB is 17 feet. Calculate the horizontal distance in 100 ft. of slope.

24. A 40-foot ladder is placed against a building with its foot 10 feet from the wall. How high does it reach on the building?

25. Can a rod 40 inches long be placed within a trunk 24 inches by 30 inches, by 16 inches high; from an upper corner to a lower opposite corner?

26. The sides of a triangle are 10, 25, 30. Calculate the segments into which each angle-bisector divides the opposite side.

27. Calculate the position of the point where the bisector of the exterior angle opposite the 30 side of Exercise 26 intersects this side produced.

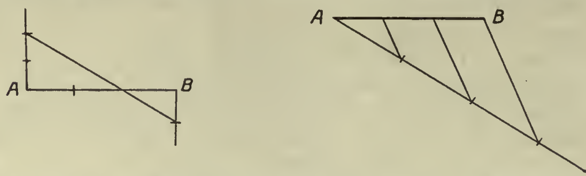
28. Calculate the length of the three altitudes of the triangle of Exercise 26.

29. Calculate the lengths of the three medians of the triangle of Exercise 26.

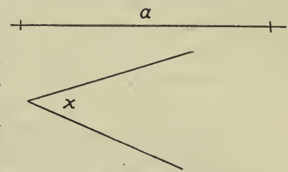
30. Two men are walking on opposite sides of a street. A tree near the curb is between them. The man on the farther side walks uniformly; must the man on the near side walk uniformly or otherwise, in order to keep the tree always between them?

31. Show how the methods of these figures can be used to divide a given sect AB. into any number of equal parts. Explain the

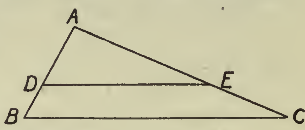
principles on which the result depends. Draw similar figures showing the division of the sect into 5 equal parts.



32. Construct a right triangle when the sum a, of the two perpendicular sides, and an acute angle x; are given.



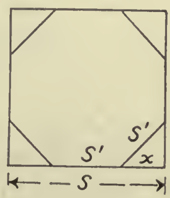
33. Construct a right triangle when the perimeter and a similar triangle are given.
34. Calculate the value of $\sqrt{35^2 - 24^2}$ by the method of **250,** Corollary 2.
35. In $\triangle ABC$, $AB = 10''$, $AC = 15''$. Line DE is parallel to BC and AE is $3''$ more than AD. Find the lengths AD and AE. Check the result.



36. In the triangle of Exercise 35, AE is $10''$ more than AD. Calculate AD and AE and check result.
37. Test the principle of **260,** Theorem (13), by accurate measurement.
38. Test the principles of **260,** Theorem (14), by measurement.
39. Test the principle of **260,** Theorem (15), by measurement.
40. Test the principle of **260,** Theorem (16), by measurement.
41. Calculate the diagonal of a square if the side $= 50'$.
42. Calculate the altitude of an equilateral triangle if a side is $120'$.
43. Calculate the side of a square if the diagonal is $24''$.
44. Calculate the side of an equilateral triangle if the altitude is $24''$.
45. Test the principle of **260,** Theorem (20), by accurate measurement.
46. The diameter of a circle is $24''$. Calculate the length of the perpendicular on one side of the diameter erected at a point $6''$ from one end.
47. The sides of a triangle are $10''$, $12''$, $16''$. Calculate the segments into which the bisector of the largest angle divides the opposite side, using the formulas of **260,** Theorem (26).
48. Determine which angles of the following triangles are acute, right or obtuse. (1) $a = 5, b = 12, c = 13$; (2) $a = 15, b = 10, c = 18$; (3) $a = 68, b = 51, c = 100$; (4) $a = 7, b = 25, c = 24$.

49. The sides of a right triangle are 3, 4, 5. Calculate (*a*) the altitude to the hypotenuse; (*b*) the segments into which the altitude divides the hypotenuse; (*c*) the length of the median to the hypotenuse; (*d*) the length of the bisector of the right angle; (*e*) the segments into which this angle-bisector divides the hypotenuse.
50. To what simpler formula do the formulas of Theorems XI and XII of similar triangles reduce if A is a right angle? What does the projection of c on b then become?
51. Calculate the median to the base of an isosceles triangle whose sides are 4″, 4″, 3″; (*a*) using the formula of **250**; (*b*) using the formula of **255**; (*c*) using the formula of **256**.
52. A carpenter lays out a perpendicular by measuring a string 24 feet long, with knots at 8 feet and 14 feet from one end. The ends are tied together and the string is stretched around three nails so that the knots are at the nails. Make a sketch and explain the principle upon which the method depends.
53. A man 5 feet 6 inches tall stands 8 feet from a lamp-post and casts a shadow on the ground 10 feet long. How high is the lamp-post?
54. Calculate the altitudes of a triangle, of which the sides are 102, 104 and 106 feet.
55. What is the diameter required, of a bar of round tool steel in order that a square bar $\frac{3}{8}$ inch on a side may be cut from it?
56. What is the largest square bar that can be cut from a round bar of steel $1\frac{1}{4}$ inches in diameter?
57. Calculate the chord of a circle whose radius is 12″, which subtends an arc of 90°.
58. Calculate the length of a tangent drawn to the circle of Exercise 57, from a point which is 30″ from the center of the circle.
59. Calculate the length of the shortest chord which can be drawn in the circle of Exercise 57, through a point 10″ from the center of the circle.
60. The radii of two concentric circles are 10″ and 12″. Calculate the length of a chord of the larger circle which is tangent to the smaller.
61. Show that the side of a regular octagon constructed by cutting the corners from a square is $s' = s(\sqrt{2}-1)$.

Helps.—$s = s' + 2x = s' + \dfrac{2s'\sqrt{2}}{2}$; solve for s'.



62. If an octagon of side s' is required, show that it may be drawn by first drawing a square of side s, such that $s = s'(\sqrt{2} + 1)$.

63. Find the side of a regular octagon formed by cutting the corners from a square whose side is 20.
64. What is the side of a square which must be first drawn, in order that the regular octagon formed by cutting off the corners shall have sides equal to 20?

Applications

262. Drawing Maps and Designs to Scale.—All reductions and enlargements of figures depend upon the following general property of similar figures: If all angular measurements of two figures are equal, all linear measurements are proportional; and conversely. Enlargements of stage scenery, wall decorations, architectural designs, etc., are made as illustrated in the following exercises.

EXERCISES

1. Measure the bearing of A from B on the map of Fig. 1. Measure the distance AB. Measure the approximate coast line of the island.

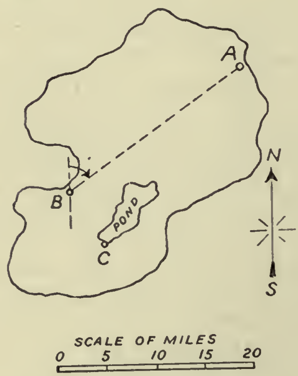


Fig. 1.

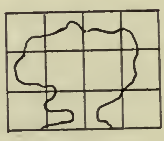
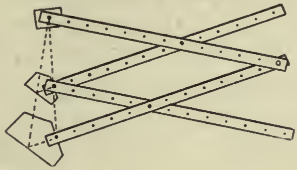


Fig. 2.

Fig. 3.

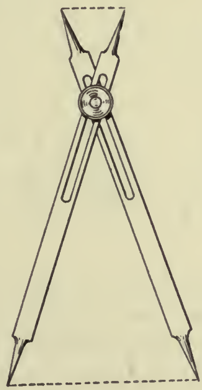
2. Enlarge Fig. 2 on a scale of 1 to 3. Draw a similar rectangle, divided into the same number of squares, and follow proportionally located points in the enlarged figure.
3. Enlarge the design of Fig. 3 on a scale of 1 to $2\frac{1}{2}$.

The pantograph is an instrument for enlarging or reducing drawings. A simplified form of the instrument is shown.



EXERCISES

4. Sketch the pantograph adjusted so as to enlarge in a ratio of 1 to 5.
5. How is the instrument used if a design is to be reduced to one-fifth the original size?



Proportional Dividers.—This instrument is used by draughtsmen to obtain a sect having a given ratio to a given sect. It can thus be used to transfer a drawing, one measurement after another, to a different scale.

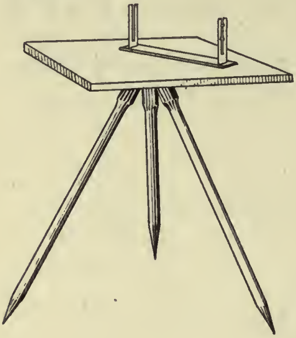
EXERCISE

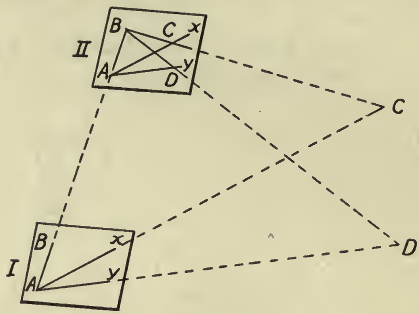
6. Explain the operation of the instrument.

263. THE PLANE TABLE.—This is an instrument used in surveying to locate points within or adjacent to a plot of ground, as corners of buildings, trees, fence corners, etc.,

upon a map. It consists of a drawing board mounted on a tripod. One line, as AB, of a plot is measured and drawn to scale on a sheet of paper which is fastened to the Plane Table. The table is set up in the field over each end in turn of the line AB. The lines are drawn on the paper in the directions of the points to be located. The angular values are thus preserved on the paper, and the resulting figure is similar to the true figure lying on the ground.

Objects in a lake or bay may be located with the Plane Table.

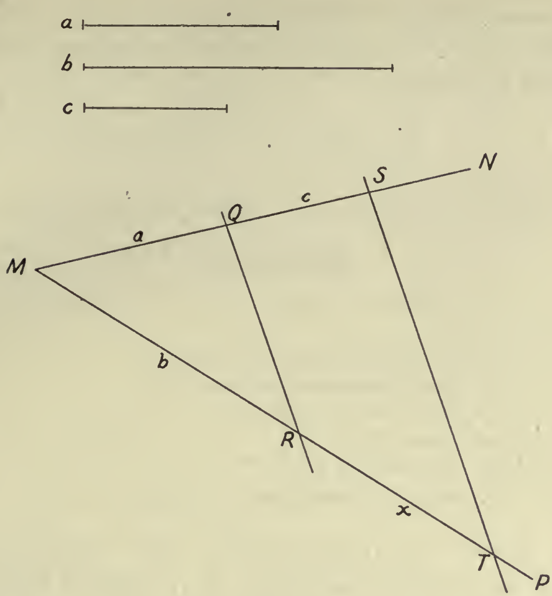




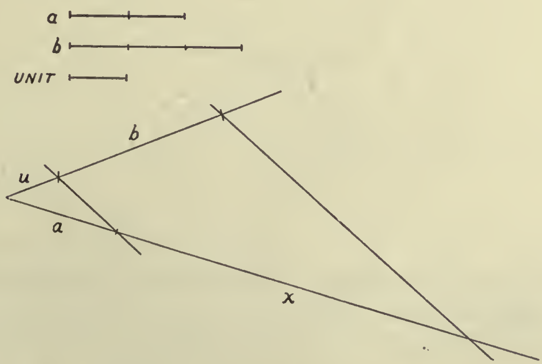
264. The Geometric Solution of Numerical Problems.—These methods are chiefly interesting because of their antiquity, since they were used by Greek mathematicians many centuries before arithmetic was sufficiently perfected and before algebra had been invented.

Any unit may be used in measuring the sects, the same unit being used throughout a problem.

A Fourth Proportional of Three Given Numbers.—Let the three given numbers be represented by sects a, b and c. Draw two intersecting lines MN and MP; lay off the sects a, b and c, as shown; draw QR and ST parallel to QR. Then x is the fourth proportional required, since $\frac{a}{b} = \frac{c}{x}$. In order to find the numerical value of the result, measure x with the same unit used in drawing the given sects, a, b, c.

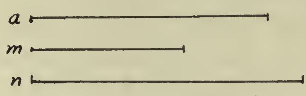


The Product of Two Numbers.—Let $ab = x$; then $ab = x \cdot 1$; and $\frac{1}{a} = \frac{b}{x}$.



The Quotient of Two Numbers.—Let $\frac{a}{b} = x$; then $\frac{a}{b} = \frac{x}{1}$. Draw the figure as in the preceding problems.

A Number Divided into Parts Proportional to Two Given Numbers.—Let it be required to divide line a into parts proportional to m and n.

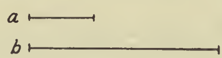


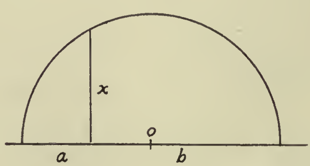
EXERCISES

Solve these problems by the geometric method and check the result arithmetically:

1. Find the fourth proportional of 2, 5, 8.
2. Find the third proportional of 2 and 3.
3. Find the product of 2.5 and 1.8.
4. Find the square of 2.6.
5. Find the quotient of 4.3 by 1.7.
6. Divide 7 into parts proportional to 2 and 3.

The Mean Proportional Between Two Numbers.—Let a and b be the given numbers. It is required to find the value of x in the proportion $\frac{a}{x} = \frac{x}{b}$.





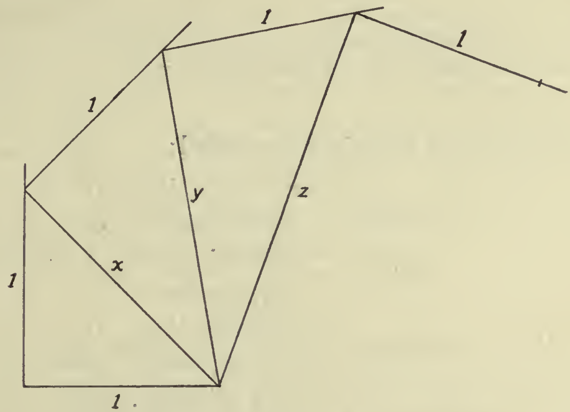
Show that this value is determined in the constructed figure (**260,** Theorem (20)).

The Square Root of a Number.—Let $x = \sqrt{a}$; then $x^2 = a$; and $x.x = a.1$; $\frac{1}{x} = \frac{x}{a}$. Construct the figure as in the preceding problem, assuming a unit of measurement u.

EXERCISES

7. Find the mean proportional of 2 and 8.
8. Find the square root of 4.
9. Find the mean proportional of 1.5 and 2.4.
10. Find the square root of 3.
11. Solve geometrically, $x = \sqrt{2.7 \times 4.5}$.

A Table of Square Roots.—The accompanying figure may be regarded as a geometric table of square roots. The diagonal lines give the square roots of consecutive integers.



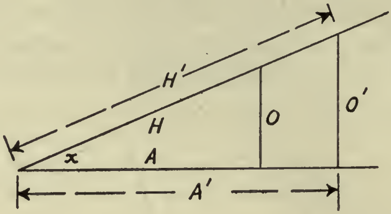
EXERCISES

12. Explain why $x = \sqrt{2}$; $y = \sqrt{3}$; etc.
13. Construct the figure as far as the square root of 6. Make a numerical table and compare with the known arithmetical values.

265. First Principles of Trigonometry.—This great science has been referred to in several places. It has been developed from the principles of similar triangles. If a sufficient number of lines and angles of a figure are known to permit the definite construction of the figure, then the remaining, or unknown parts, may be determined by measuring the completed figure. The object of trigonometry

is to furnish methods of calculating the unknown parts with any desired degree of accuracy.

The Functions of an Acute Angle.—Draw any acute angle x; draw a line O perpendicular to side A. In the triangle HOA, the ratio of any two sides is called a *function* of angle x.



Four of the six functions are called:

$$\text{sine } x = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{O}{H}$$

$$\text{cosine } x = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{A}{H}$$

$$\text{tangent } x = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{O}{A}$$

$$\text{cotangent } x = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{A}{O}$$

EXERCISES

1. Show that the value of a function is independent of the position of the perpendicular; *i.e.*, $\frac{O}{H} = \frac{O'}{H'}$, etc.

2. Draw angles of 10°, 20°, 30°, etc., to 80°. In each figure draw a perpendicular and measure the three sides of the right triangle formed. Calculate the values of the sine, cosine, tangent and cotangent of each angle, taking the results to two decimals. Thus, in the figure, $x = 20°$, $O = ?$, $A = ?$, $H = ?$, $\sin 20° = \frac{O}{H} = \text{———} = ?$, $\cos 20° = \frac{A}{H} = \text{———} = ?$, $\tan 20° = \frac{O}{A} = \text{———} = ?$, $\cot 20° = \frac{A}{O} = \text{———} = ?$

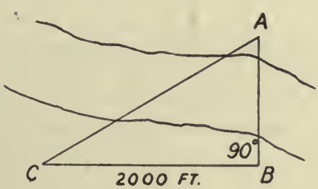
3. Make a table of the results as follows:

Angle	Sine	Cosine	Tangent	Cotangent
10°	.17			
20°	.34			
30°	.50			
40°				
50°				
60°				
70°				
80°				

Problems Solved by Trigonometry.—A triangle whose sides are many feet or miles in length, or so large that its sides reach from the earth to the sun or from earth to a star, may be similar to one of the triangles used in calculating a table of functions. The function values of the acute angles of these larger triangles are therefore equal to the values found from the constructed triangles.

EXERCISES

4. The distance AB across a river is required. A surveyor measures a distance BC at a right angle to the required line; and at point C he measures $\angle C = 30°$. To which of the triangles of Exercise 3 is this triangle (full size) similar?



To what function of 30° is $\frac{AB}{BC}$ equal?

Form the equation, $\frac{AB}{2000} = ?$, taking the proper value from the table. Solve for AB.

In solving the problems which follow, use the more exact function values given in the table at the end of this section.

5. The horizontal distance from a point A on the ground, to the base of a flagpole is 130 feet. At A the angle of elevation of the top of the pole is 50°. Find the height of the pole above the transit. Add 4 feet for the height of transit above the ground at the base of the pole (Compare **115,** Exercise 9).

6. A hillside slopes at an angle of 20° with the horizontal. What is the rise in 500 feet measured along the slope?
7. An aeroplane is observed over an aviation field that is known to be 3.5 miles distant, at an angle of elevation of 10°. What is the altitude of the aeroplane (Compare **115,** Exercise 4)?
8. Find the base of an isosceles triangle whose vertex angle is 40° and whose equal sides are 20 in.
   Help.—Draw the bisector of the vertex angle.
9. A pendulum 10 feet long swings from end to end of a straight line 3.4 feet long. Through what angle does the pendulum swing?
   Help.—Draw the bisector of the total angle.
10. The radius of a circle is 10 inches; radii drawn to the extremities of a chord form a central angle of 80°. Find the length of the chord.
11. Calculate the projection of a sect 5 inches long on a line which forms an angle of 40° with the sect.
12. Calculate the angles of the isosceles triangle of **97,** Exercise 4.
    Help.—Cosine of one of the base $\angle$s $= 2/3 = .6667$.
13. Calculate the base of the isosceles triangle of **98,** Exercise 3.
    Help.—$\frac{\frac{1}{2}\text{ base}}{3} = \text{sine } \frac{1}{2} \text{ vertex angle}, \therefore \text{ base} = 6 \text{ sine } 35°$.
14. A vertical pole 12 feet high casts a shadow on a horizontal plane 13.33 feet long. Find the angle of elevation of the sun.
15. The projection of a sect 20 inches long is 17.5 inches. Find the angle which the sect makes with the line of projection.
16. Calculate the functions of 30°, 45° and 60° from the values obtained in **254.**

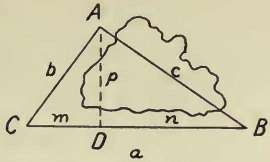
Oblique Triangles can also be solved by dividing them into two adjacent or overlapping right triangles.

EXERCISES

17. The distance AB, across a pond is required. A point C is selected from which the lines $CA = 280.5$ feet, and $CB =$

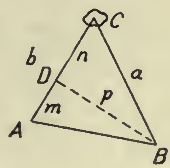
620.8 feet are measured. Angle C is measured 37°.

Helps.—(1) In $\triangle ACD$, calculate p and m;—(2) $n = a - m$; —(3) in $\triangle ABD$, calculate $\angle B$ and then side c; or, calculate $c = \sqrt{p^2 + n^2}$.

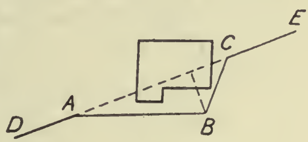


18. An island C is located on a map by selecting two known points A and B on the shore, measuring $AB = 1250$ feet, $\angle CAB = 67°$; $\angle CBA = 48°$. Find AC and BC.

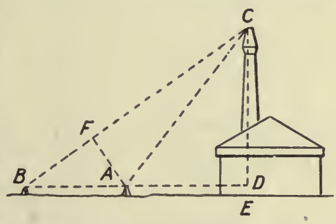
Helps.—(1) In $\triangle ABD$, calculate p and m; —(2) $\angle C = 180° - (A + B)$;—(3) in $\triangle BCD$, calculate n and a;—(4) $b = m + n$.



19. A line is to be extended past a building. A surveyor measures $\angle DAB = 153°$, $AB = 325$ feet, $\angle ABC = 105°$. Calculate BC so that point C shall be on the line DA extended, and $\angle BCE$ so that CE shall be the prolongation of line DAC.



20. The height of a stack is found as follows: A transit is set up at point A and the angle of elevation of point C is measured, $\angle DAC = 32°$; the height of the transit above the ground is measured on the wall of the building; $DE = 4.7$ feet. The telescope of the transit is turned over and a point B is located in the plane of $\triangle ACD$. We will suppose that the transit stands at the same height at B as at A. Angle ABC is measured 24°; AB is measured on the ground $= 125.4$ feet.



Helps.—Calculate (1) AF; (2) $\angle ACF$; (3) AC; (4) CD; (5) CE.

21. Solve the triangle of **99**, Exercise 1.

Helps.—(1) $\frac{p}{1.5} =$ sine 57°;—(2) $\frac{p}{2.9} =$ sine A;—(3) $\frac{m}{1.5} =$ cosine 57°;—(4) $\frac{n}{2.9} =$ cosine A;—(5) $c = m + n$;—(6) $C = 180° - (B + A)$.

Geometrical methods of calculation which are limited to special cases in geometry, may be *generalized* by trigonometry.

EXERCISES

22. Find a formula for calculating the projection m, of a sect s, on a given line for any angle x, between the sect and the line of projection (see **254**).

    Helps.—$\frac{m}{s}$ = cosine x.

    $\therefore m = s$. cosine x.

23. Calculate the projection of a sect of 24″ which makes an angle of 30° with the line of projection.

    $m = 24 \times$ cosine $30° = 24 \times .8660 =$ etc.

24. Calculate the projection of a sect of 38″ which makes an angle of 51° with the line of projection.

25. Find a general formula for the third side of a triangle when two sides and their included angle are given (see **252**).

    Helps.—$a^2 = b^2 + c^2 - 2bm$; $\frac{m}{c}$ = cosine A

    $\therefore a^2 = b^2 + c^2 - 2bc\cdot$cosine A

26. Calculate the third side of a triangle of which two sides and their included angle are respectively, $b = 1.5''$, $c = 2.9''$, and $A = 57°$ (see **98**, Exercise 1).

    Helps.—$a^2 = (1.5)^2 + (2.9)^2 - 2(1.5)(2.9) \cos 57°$.

27. Calculate the angles B and C of the triangle of Exercise 26.

    Helps.—(1) $\frac{m}{c}$ = cosine A;—(2)$n = b - m$;—(3) cosine $C = \frac{n}{a}$; (4) $B = 180° - (A + C)$

Some geometrical methods of calculation may be *simplified* by trigonometry.

EXERCISES

28. Find a formula for the diagonal of a square in terms of the side s (**260**, Theroem (14)).

    Helps.—(1) $\frac{s}{d}$ = sine 45°; (2) $d = \frac{s}{\text{sine } 45°}$

29. Calculate the diagonal of a square whose side is 50 (**261**, Exercise 41).

30. Calculate the distance from home plate to second base in a base ball diamond.

31. Find a formula for the altitude of an equilateral triangle in terms of the side of the triangle (**260**, Theorem (16)).

    Help.—$\frac{a}{s}$ = sine 60°.

32. Calculate the altitude of an equilateral triangle of which the side is 120 (**261**, Exercise 42).

Table of Functions

Angle	Sine	Cosine	Tangent	Cotangent	
0°	.0000	1.0000	.0000	......	90
1°	.0175	.9998	.0175	57.2900	89
2°	.0349	.9994	.0349	28.6363	88
3°	.0523	.9986	.0524	19.0811	87
4°	.0698	.9976	.0699	14.3007	86
5°	.0872	.9962	.0875	11.4301	85
6°	.1045	.9945	.1051	9.5144	84
7°	.1219	.9925	.1228	8.1443	83
8°	.1392	.9903	.1405	7.1154	82
9°	.1564	.9877	.1584	6.3138	81
10°	.1736	.9848	.1763	5.6713	80
11°	.1908	.9816	.1944	5.1446	79
12°	.2079	.9781	.2126	4.7046	78
13°	.2250	.9744	.2309	4.3315	77
14°	.2419	.9703	.2493	4.0108	76
15°	.2588	.9659	.2679	3.7321	75
16°	.2756	.9613	.2867	3.4874	74
17°	.2924	.9563	.3057	3.2709	73
18°	.3090	.9511	.3249	3.0777	72
19°	.3256	.9455	.3443	2.9042	71
20°	.3420	.9397	.3640	2.7475	70
21°	.3584	.9336	.3839	2.6051	69
22°	.3746	.9272	.4040	2.4751	68
23°	.3907	.9205	.4245	2.3559	67
24°	.4067	.9135	.4452	2.2460	66
25°	.4226	.9063	.4663	2.1445	65
26°	.4384	.8988	.4877	2.0503	64
27°	.4540	.8910	.5095	1.9626	63
28°	.4695	.8829	.5317	1.8807	62
29°	.4848	.8746	.5543	1.8040	61
30°	.5000	.8660	.5774	1.7321	60
31°	.5150	.8572	.6009	1.6643	59
32°	.5299	.8480	.6249	1.6003	58
33°	.5446	.8387	.6494	1.5399	57
34°	.5592	.8290	.6745	1.4826	56
35°	.5736	.8192	.7002	1.4281	55
36°	.5878	.8090	.7265	1.3764	54
37°	.6018	.7986	.7536	1.3270	53
38°	.6157	.7880	.7813	1.2799	52
39°	.6293	.7771	-8098	1.2349	51
40°	.6428	.7660	.8391	1.1918	50
41°	.6561	.7547	.8693	1.1504	49
42°	.6691	.7431	.9004	1.1106	48
43°	.6820	.7314	.9325	1.0724	47
44°	.6947	.7193	.9657	1.0355	46
45°	.7071	.7071	1.0000	1.0000	45
	Cosine	Sine	Cotangent	Tangent	Angle

Trigonometry has been developed so that it meets the needs of the most exact scientific work, such as surveying, engineering, navigation, military science, and astronomy. Seven place tables of the logarithms of the functions are published, and ten place tables are used in some calculations. There are also special formulas for solving oblique triangles, finding areas, and for many other uses. A text on trigonometry should be used if further study is desired.

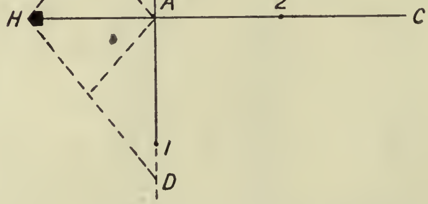
In using the table on page 175, angles from 0° to 45° are read in the *left hand column* and the names of the functions are found at the *top of the table;* angles from 45° to 90° are read in the *right hand column* and the names of the functions are found at the *bottom of the table.*

266. PROBLEMS FOR FIELD WORK.

1. The distance across a stream, or inaccessible street, etc., by the method of **237.**

2. The distance between two points on opposite sides of a building or pond, by the method of **237.**

3. Lay out a baseball diamond when the position of home plate and the direction of second base are given.



Let H be the home plate, and HC the direction of center field.

(1) Find the distance to second base, $H2 = \sqrt{90^2 + 90^2}$.

(2) Divide by 2 and measure to point A.

(3) Erect perpendiculars AB, AD.

(4) Measure $A1 = A3 = HA = ?$

(5) Check by measuring $H1 = 12 = 23 = 3H = 90$ feet.

4. Make a map of some curved path or road or border of a stream, by the method of **237**.

5. Find the distance from a position A to a place B as follows: First observer proceeds on a line perpendicular to line AB, to C; second observer proceeds along line BA to D, until observer at C signals him that the angle BCD measured by an optical square (**148**) is a right angle. The distances AC and AD are paced or measured with a tape. AB may then be calculated (**248**).

This method of estimating the distance to an enemy station, etc., is used in warfare.

6. Find the distance from a point inside the school grounds to an inaccessible point outside the grounds by the method of **265**, Exercise 4, or Exercise 18.

CHAPTER XII

CIRCLES

Principles Determined Experimentally

267. Circles Defined.—Refer to **14** for general definitions.

Congruent, or *equal*, *circles* are circles which have the same radii.

Two points on the circumference of a circle divide the circumference into a *minor* and a *major arc*.

A *central angle* is formed by two radii.

An *inscribed angle* is formed by two chords meeting in the circumference.

A chord is said to *subtend* the two arcs opposite the chord, and conversely. A central or an inscribed angle *subtends* or *intercepts* the included arc. *To subtend* means *to be opposite to.*

A *segment* is a part of a circle bounded by a chord and the subtended arc.

A *sector* is a part of a circle bounded by two radii and the intercepted arc.

An inscribed angle whose sides pass through the extremities of a chord is *inscribed in the segment* cut off by the chord.

Two circles may in some positions have a *common chord* or *common tangents.*

A *secant* is a straight line intersecting a circle in two points and extending outside the circle in one or both directions.

A *polygon is inscribed in a circle* when all its vertices lie in the circumference. The circle is then *circumscribed* about the polygon.

A *polygon is circumscribed about a circle* when all its sides are tangents of the circle. The circle is *inscribed* in the polygon.

EXERCISES

1. What part of a circumference is an arc of 60°; 45°; 25°; 72° 10′?
2. Define complementary, supplementary and explementary arcs (see **17**).
3. Draw a circle with a chord subtending an arc of (approximately) 75°. What major arc does the chord subtend?
4. What chord subtends equal arcs of a circumference?
5. What must be given in order to determine the position and size of a circle?
6. Draw a circle with a central angle subtending an arc of 90°. How many degrees are subtended by the explementary central angle?
7. Draw an inscribed angle subtending an arc of 90°.
8. Draw an angle inscribed in a segment of (approximately) 150°.
9. Define a semicircle as a segment.
10. Define a semicircle as a sector of a certain number of degrees.

268. Experiment I.—Relations between central angles and the subtended arcs.

(*a*) *Equal central angles in the same or in two equal circles.*

Draw a circle; draw several pairs of equal central angles. Observe the relative lengths of the subtended arcs of each pair of equal angles. State the result.

(*b*) *Unequal central angles in the same or in two equal circles.*

Draw several unequal central angles. Observe the relative lengths of the subtended arcs. Result.

(*c*) *The measure of the arc subtended by a central angle.*

Draw a central right angle. Bisect it; divide one of the parts (by trial) into three equal angles; divide one of these parts again into three equal angles; divide one of these parts into five equal angles. This will give a central angle of one degree (of angle). Observe the part of the circumference which is subtended by the central angle of one degree; the number of degrees of arc subtended by each central angle; the number of degrees of arc which will be subtended by any central angle. State result.

(*d*) *The measures of the arcs subtended by a central angle of two concentric circles.*

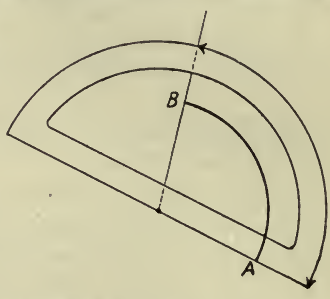
Draw two concentric circles; draw a central angle.

Observe the relative lengths of the two subtended arcs when measured in inches on the circumference; and their relative lengths when measured as fractional parts of the two circumferences, *i.e.*, in degrees of arc. Result.

EXERCISES

1. What arc measured in degrees of arc will a central angle of 75° subtend in a circle of 2″ radius? In a circle of 5″ radius?
2. If an arc of 30° measures 5″ in a given circle, what will an arc of 90° measure in a circle of double the radius?

269. The Protractor.—The principle of **268**(*d*) explains the use of a protractor, which can be used to (*a*) measure a given angle; (*b*) construct an angle of a given measure; (*c*) measure a given arc; (*d*) construct an arc of a given measure on a given circumference.



EXERCISES

1. Draw an arc AB of about one inch in radius. Measure it in degrees.
2. Construct an arc of 35° with a radius of 6 inches.

270. Experiment II.—Properties of chords.

(*a*) *The relative lengths of arcs subtended by equal chords, in the same or in two equal circles.*

(*b*) *The relative lengths of arcs subtended by unequal chords.*

(*c*) *The relative distances (measured on perpendiculars) of equal chords from the center of the circle.*

(*d*) *The relative distances of unequal chords from the center of the circle.*

(*e*) *The relation of a diameter perpendicular to a chord, to the chord and to the two subtended arcs.*

(*f*) *The property of a chord equal in length to the radius of the circle.*

In considering Part (*b*), observe whether the minor or major subtended arcs vary in the same way as the chords; also whether there is a simple numerical relation between the lengths of chords in inches and the subtended arcs in degrees.

In considering Part (*d*), observe whether there is a simple numerical relation between the length of a chord and its distance from the center of the circle.

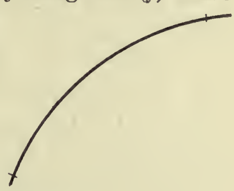
In considering Part (*f*), draw a number of chords equal in length to the radius of the circle, end to end around the circle.

State all results.

EXERCISES

1. Construct an equilateral triangle with its three vertices in the circumference of a circle.
2. Construct a symmetrical six-pointed star by drawing a circle and using Part (*f*) above.
3. Draw a circle and lay off an arc of 120° by using Part (*f*) above

271. Experiment III.—A method of bisecting an arc.



(*a*) *By rule and compass.*

(*b*) *By measurement.*

(*c*) *By trial and error.*

Part (*a*): Use the principle of **270** (*e*).

272. Experiment IV.—Properties of tangents.

(*a*) *The relative position of a tangent and the radius drawn to the point of contact (or point of tangency).*

(*b*) *The relative lengths of two tangents drawn to a circle from the same external point.*

(*c*) *The number of common tangents that can be drawn to two circles under different conditions.*

In considering Part (*c*), draw two circles of different radii, that are (1) externally out of contact, (2) externally tangent, (3) intersecting, (4) internally tangent, (5) internally out of contact. Observe the number of common tangents that can be drawn in each case. State all results.

EXERCISES

1. What position have two tangents drawn at the ends of a diameter?
2. In what position are two circles if three common tangents can be drawn to them?

3. How many common tangents have two concentric circles?
4. Invent a rule and compass construction for drawing a line through a given point parallel to a given line, based on a common tangent to two equal circles.
5. Draw a circle and mark a point on the circumference. Show how the direction of the tangent at this point can be accurately determined from a radius drawn to the point of tangency.

273. Experiment V.—Properties of the line of centers of two circles.

(*a*) *The relation between the line of centers of two intersecting circles and of their common chord.*

(*b*) *The position of the point of contact of two tangent circles with respect to the line of centers.*

(*c*) *The position of the common internal tangent of two tangent circles with respect to the line of centers.*

(*d*) *The position of the points of intersection of a pair of internal or external common tangents with respect to the line of centers.*

In considering Part (*a*), draw (1) two equal circles, and (2) two unequal circles.

In considering parts (*b*) and (*c*), draw (1) two externally tangent circles, and (2) two internally tangent circles.

In considering part (*d*), draw two circles that are (1) externally out of contact, (2) externally tangent, (3) intersecting. State results.

EXERCISES

1. Draw a circle of $1\frac{1}{2}''$ radius; mark a point on the circumference. Draw a second circle of $2''$ radius tangent externally to the first circle at this point.

   Helps.—Draw the radius of the first circle to the point of tangency and extend it outside the circle. Use Part (*b*) above.
2. Draw an arc of $3''$ radius and mark a point on it. Draw a second arc (or circle) of $1''$ radius tangent internally to the first arc at the given point.

274. Experiment VI.—Properties of inscribed angles.

(*a*) *The size of an angle inscribed in a semicircle.*

(*b*) *The size of an angle inscribed in a minor segment.*

(*c*) *The size of an angle inscribed in a major segment.*

(*d*) *The relative size of angles inscribed in the same segment.*

(*e*) *The measure of an inscribed angle in terms of the subtended arc.*

State results.

275. Experiment VII.—**A method of drawing a line perpendicular to a given line.**

(*a*) *When the point is in the given line.*

(*b*) *When the point is without the given line.*

Adapt Part (*a*) of **274** to both constructions.

276. Experiment VIII.—**The number of circles that can be drawn through given points.**

(*a*) *One given point.*

(*b*) *Two given points.*

(*c*) *Three given points.*

(*d*) *Four or more given points.*

(*e*) *The conditions that four or more given points shall lie on the same circle.*

In stating the results of Parts (*a*), (*b*), (*c*), state where the centers of the circles which contain the given points are found, as well as the number of such circles; that is, describe the locus of their centers.

Part (*d*): If the points are placed at random, will any circle contain them all? Result.—In general, etc.

Part (*e*): A certain set of lines must be concurrent in order that the points shall be on a circle (*i.e.*, concyclic).

EXERCISES

1. What is the locus of the centers of circles which contain the ends of a sect?
2. Can a circle be drawn containing the vertices of a square? Of an oblong? Of a rhombus? Of an isosceles trapezoid?
3. Are the vertices of an oblique scalene triangle concyclic?
4. How many points may be concyclic? How many points must be concyclic?
5. If three points chance to lie in the same straight line, where is the center of the circle which may be said to contain them?

277. Experiment IX.—The Number of Circles That Can be Drawn Tangent to Given Straight Lines.

(*a*) *One given line.*

(*b*) *Two given lines.*

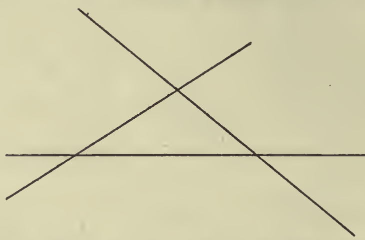
(*c*) *Three given lines.*

(*d*) *Four or more given lines.*

(*e*) *The condition that four or more given straight lines shall be tangent to the same circle.*

Part (*b*): Consider two cases, (1) the given lines intersecting, (2) the lines parallel.

Part (*c*): Consider the lines, (1) two lines parallel and the third line intersecting them, (2) the three lines parallel,



(3) the lines inclosing a triangle, (4) the lines concurrent. In (3), note that four circles can be drawn tangent to the three given lines.

Part (*e*): Find the set of lines that must be concurrent as the necessary condition.

EXERCISES

1. What is the locus of the centers of the circles in Part (*b*), (1) and (2)?
2. What are the tangent circles called in respect to the triangle enclosed by the three lines in Part (*c*), (3)?
3. Can a circle be drawn tangent to the sides of a square? Of an oblong? Of a rhombus? Of an isosceles trapezoid?

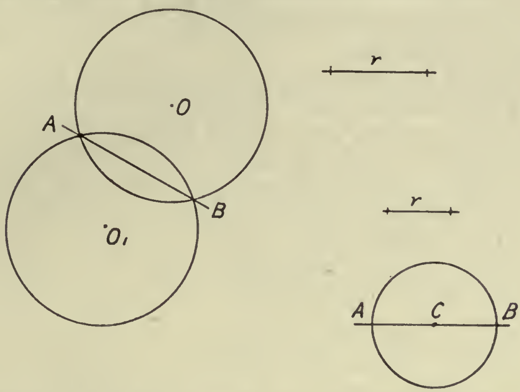
278. Experiment X.—Any Other Properties of Circles Which Can be Discovered Experimentally.

279. Construction of Circles from Given Data.—Find all possible solutions of each problem. Discuss special and impossible cases, which should be illustrated by sketches.

Use rule and compass or measurement methods, and avoid trial and error solutions.

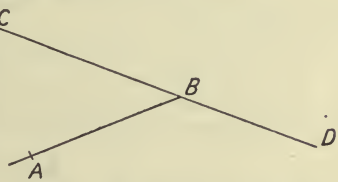
Construct a circle (or circles) of which there are given:

1. The length and position of a chord, and the radius.

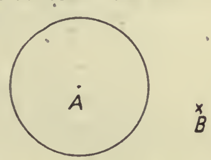


Helps.—AB is the given chord; r is the given radius.
In general two circles can be drawn.
Special cases: (1) When $r < \frac{1}{2} AB$, no circle is possible. (2) When $r = \frac{1}{2} AB$, one circle, as C, can be drawn.

2. The length and position of a chord and the position of a tangent through an end of the chord.



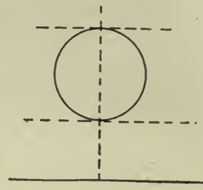
3. (The position of) a tangent, the point of tangency, and a point on the circumference.
4. Two intersecting tangents and the radius.
5. A chord and a straight line containing the center of the circle.
6. A tangent circle and the center of the required circle.



7. Two non-intersecting circles, both of which are tangent to the required circle, and the radius of the required circle.
8. A tangent circle and a point in the circumference of the required circle and the radius of the required circle.
9. The length and position of a chord and a circumference containing the center of the required circle.
10. A tangent circle and the point of tangency, and a point in the circumference of the required circle.

11. State other conditions for the construction of a required circle (or circles). Make the conditions definite but not conflicting.

280. Constructions Relating to Circles.—Construct by rule and compass methods:

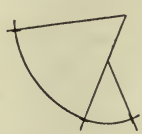


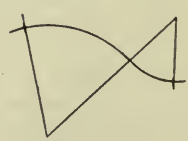
1. A tangent to a given circle parallel to a given straight line.
2. A tangent to a given circle perpendicular to a given straight line.
3. A chord in a given circle of given length and parallel to a given straight line.
4. A chord of a given circle passing through a given point within the circle and bisected at that point.
5. A right triangle when the lengths of the hypotenuse and of one of the perpendicular sides are given. Construct (1) by **274**(a); (2) by any other method.

6. A right triangle when the hypotenuse and the altitude to the hypotenuse are given.
7. A chord of given length in a given circle, which shall pass through a given point within the circle.
8. Four circles within a given square, each tangent to the sides of the square and to the two adjacent circles.
9. A chord of given length in a given circle, which shall pass (if produced) through a given point outside the circle.

281. Review Exercises

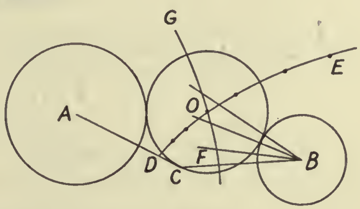
1. Draw two tangent arcs of radii 2″ and 3.5″, 45° and 60° respectively, in two positions of contact.
2. Construct the locus of the centers of circles tangent to a given circle at a given point on the circle.





3. Construct an arc of 60°, 6″ radius, without using a protractor.
4. Construct a circle (or circles) tangent to two given intersecting straight lines and having its center in a third given straight line.
5. Construct a circle tangent to a given circle and to a given straight line and having a given radius.

6. Two chords intersect on the circumference of a circle and have their other ends at the ends of a diameter of the circle. The chords are 5″ and 6″ long. Calculate the diameter.
7. A chord of a circle is 10″ and its distance from the center is 6″. Calculate the radius of the circle.
8. The perpendicular from the center of a circle to a chord subtending an arc of 120° is 3″. Calculate the radius and the length of the chord.
9. A chord subtending an arc of 60° is 5″ from the center. Calculate the radius and the chord.
10. The radius of a circle is 10″. Calculate the length of a chord subtending an inscribed angle of 45°.
11. The center sect of two circles is 10″; the radii are 4″ and 2″. Calculate the length of an internal common tangent.
12. Draw two externally tangent circles. Prove experimentally that the internal common tangent bisects the two external common tangents.
13. Draw a number of chords of a circle concurrent in a point of the circumference. What is the locus of their middle points?
14. Draw a number of secants of a circle concurrent in a point outside the circle. What is the locus of the bisection points of both the major and minor segments of the secants?
15. Make an experiment similar to that of Exercise 14 for a pencil of chords intersecting within the circle.
16. Draw a circle or an arc of more than 180° without marking the center. Find the center by using the right angle of the draughtsman's triangle (**274** (*a*)).
17. Construct a circle tangent to the two circles *A* and *B* and containing the point *C*. Draw (*a*) the locus of points equidistant from circle *A* and point *C*; (*b*) a similar locus for circle *B* and point *C*; (*c*) the required circle with center at *O*.



18. Exercise 17 is the geometric solution of the problem of locating the German "super-gun" which shelled Paris in 1917–18. See Mathematics Teacher for Dec., 1918.

Three stations, *A*, *B*, *C*, were located and plotted. A listening apparatus was installed at *C*, and observers were stationed at *A* and *B*. Each observer timed with accurate stop watches the interval between hearing the discharge of the gun at *C* and hearing it at his station, *A* or *B*. Let this interval be 1.7

seconds for the A observer, and 2.5 seconds for the B observer. Sound travels about 1100 feet per second. Draw the circle A with a radius 1.7×1100 feet, and circle B with radius 2.5×1100 feet. Circle O is found as in Exercise 17, or by trial; at the center, O, of which the gun is located.

Applications

282. Railroad and Road Curves.—A preliminary survey of a proposed railroad consists in staking out a series of straight lines, locating the points of intersection, measuring the lengths, AB, BC, etc., (Fig. 1) and the *intersection angles*, x, y, etc.; also in estimating the radii of the connecting circular arcs which will best conform to the surface of the ground.

The lines AB, BC, etc., are known as *tangents*.

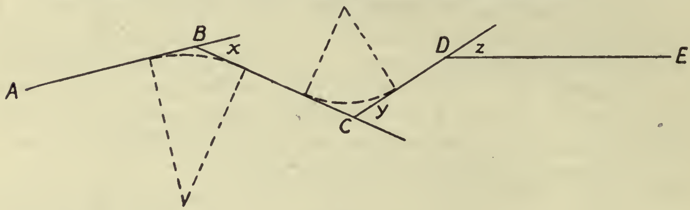


Fig. 1.

EXERCISES

1. Show that the central angle at O equals the intersection angle x (Fig. 2).

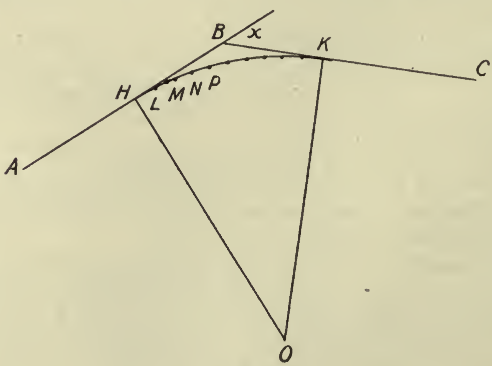


Fig. 2.

2. How do the tangent lengths BH and BK compare (Fig. 2)?
3. Plot the tangents on a scale of 1000 feet to the inch, from the following notes (Fig. 1):

Tangents	Lengths, feet	Intersection ∠s	Values	Direction of turn
AB........	2000	B	40°	To right
BC........	2500	C	90°	To left
CD........	1500	D	50°	To right
DE........	3000	E	35°	To right

To Lay Out a Circular Arc by Chords.—The necessary calculation are made by trigonometry. Thus if, in Fig. 2, $x = 40°$, $\angle HOK = 40°$, and the arc HK may be divided into 10 equal arcs of 4° each. The radius $HO = 1433$ feet, the tangents, $BH = BK = 522$ feet; and the chords $HL = LM = MN = 100$ feet. Set up a transit at H, sight on B, turn 2° and measure 100 feet to L; turn another 2° and measure $LM = 100$ feet; etc.

EXERCISES

4. Plot the curve described above on the drawing of Exercise 3.
5. Why are the angles $BHL = LHM = MHN =$ etc. $= 2°$ (Fig. 2)?

The Two-transit Method of Laying Out a Circular Arc.—The arc HK may also be laid out with two transits, without making any measurements with a tape.

After calculating BH and BK (Fig. 2) and locating points H and K, set a transit at each of these points. In order to locate points L, M, N, etc.; turn $\angle BHL = 2°$ with the H-transit, and an angle $BKL = 18°$ with the K-transit, and set a stake on both lines of sight; then turn $\angle BHM = 4°$ and $\angle BKM = 16°$, and set stake M as before; etc.

EXERCISE

6. Arrange the angles of each transit in a table:

Point on arc	∠ turned by *H*-transit	∠ turned by *K*-transit
L	2°	18°
M		
N		
Etc.		

Compound Curves.—Such curves are composed of two or more arcs successively tangent to each other.

A *reverse curve* is a compound curve composed of two arcs of equal radii, joining two parallel tangents.

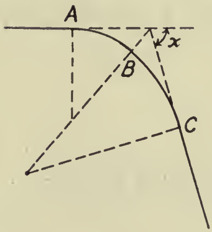


FIG. 1.

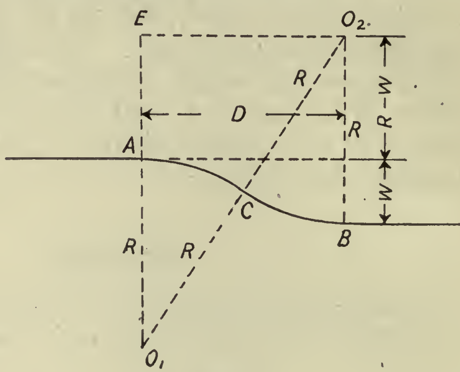


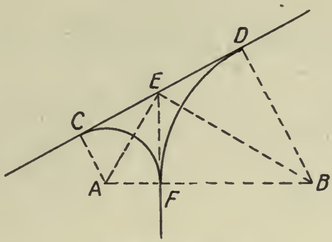
FIG. 2.

EXERCISES

7. Design, by trial, a compound curve in which arc AB (Fig. 1) has a radius of 300 feet and arc BC a radius of 500 feet, and in which the intersection angle $x = 90°$.
8. Design a reverse curve joining two parallel tangents the distance between which is 30 feet, the radius of both arcs being 30 feet.
9. Show that in Fig. 2, $R = \dfrac{D^2 + W^2}{4W}$.

   Help.—$O_1E = 2R - W$; $\overline{O_1O_2}^2 = \overline{O_1E}^2 + \overline{EO_2}^2$
10. Calculate R when $D = W$; also when $D = W = 30$ ft.; also when $D = 30$ ft. and $W = 20$ ft.

Y-curves.—Such curves connect two railroad tracks as shown, and are sometimes used to reverse the direction of an entire train.



EXERCISES

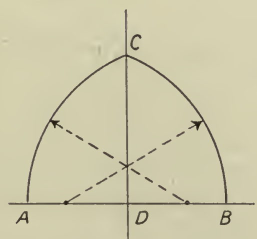
11. Show that $EC = EF = ED$.
12. Show that $EC = \sqrt{R \cdot r}$, where R and r are the radii of the curves.
    Helps.—$\triangle AEB$ is a right $\triangle$. Apply **248.**
13. Calculate EC when $R = r = 200$ ft. Also when $R = 200$ ft. and $r = 150$ ft.

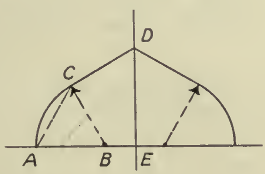
283. The Circle in Architecture.—Arches, columns, domes, doorways, windows, and other architectural features exhibit many beautiful and interesting uses of the circle.

The Gothic Arch.—AB is the span; CD is the altitude.

EXERCISES

1. Design a Gothic arch with as pan of 40 ft., and an altitude of 30 ft.
2. Design a Gothic arch with a span of 30 ft., and an altitude of 40 ft.



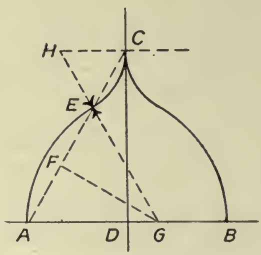


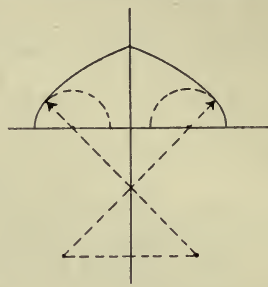
The Breccia Arch.—$\triangle$ ABC is equilateral.

3. Design a Breccia arch with a span of 40 ft. and an altitude DE *of* 30 ft.

The Persian (*ogee*) *Arch.*—Trisect AC at E and F; draw $FG \perp AC$; prolong GE to H; G and H are the arc centers. The arch may also be drawn with E at the middle of AC.

4. Design Persian arches with 30 ft. span and 40 ft. altitude; (*a*) with E at the trisection point; (*b*) with E at the bisection point.



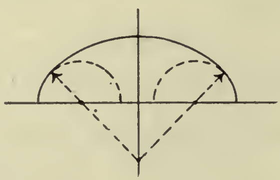


The Basket-handle Arch.

5. Design a basket-handle arch whose span = 60 ft., altitude = 25 ft.

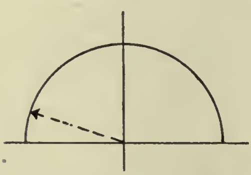
The Tudor Arch.

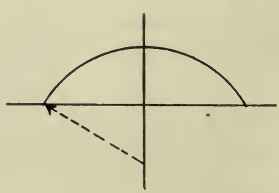
6. Design a Tudor arch whose span = 60 ft., altitude = 30 ft.



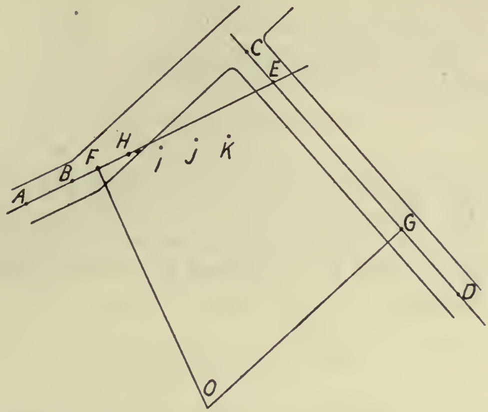
Circular Arches.

7. Design circular arches: (*a*) span = 120 ft., altitude = 60 ft.; (*b*) span = 120 ft., altitude = 40 ft.





284. A Problem for Field Work.—To lay out a circular arc tangent to two straight lines, by the two-transit



method (**282**). Select two straight curbs or center lines of paths, and mark such points as A, B, C, D, by driving stakes or with a chisel. Set transits over points A and D and sight on points B and C respectively, and set stake E on the two lines of sight. Set the transit at E and measure angle AED. Measure any equal distances EF and EG, say 200 feet. The central angle O of the arc connecting points F and $G = 180° - \angle AED = \angle FOG$. Divide $\angle FOG$ by $10 = \angle FOH$. Set the transits over points F and G. Sight the F-transit on point G and turn an angle $GFH = 9$ times $\frac{1}{2}\ \angle FOH$. Sight the G-transit on point F and turn an angle $FGH = \frac{1}{2}\angle FOH$. Set stake H on the line of sight of both transits. Proceed in this way until the 9 stakes H, I, J, K, etc., are located.

This problem is greatly simplified by selecting two perpendicular lines, or by laying out two perpendicular lines from any selected point E.

CHAPTER XIII

CIRCLES

Classification and Explanation of Principles

285. The first step in classification is the selection of the most elementary experimental properties of circles as a basis of analysis.

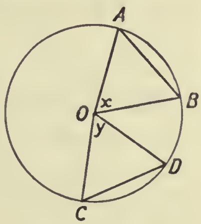
Circle Postulates

1. Any point within a circle is at a less distance from the center than the radius; and conversely.
2. Any point without a circle is at a greater distance from the center than the radius; and conversely.
3. If two or more points are equally distant from another point, a circle may be drawn with its center at that other point which will contain all of the equidistant points.
4. A circle is determined by (*a*) the center and radius; (*b*) by a diameter.
5. A diameter bisects a circle.
6. Equal chords of the same circle or of equal circles subtend equal arcs; and conversely.
7. Equal chords of the same circle or of equal circles are equally distant from the center; and conversely.
8. Unequal chords of the same circle or of equal circles subtend unequal arcs; the greater chord subtending the greater minor arc.
9. Unequal chords of the same circle or of equal circles are unequally distant from the center, the greater chord being at the less distance from the center.
10. A diameter is the greatest chord.

11. (*a*) A tangent is perpendicular to the radius drawn to the point of tangency. (*b*) A line perpendicular to a radius at its extremity is tangent to the circle.
12. (*a*) Every point of a tangent lies outside the circle except the point of contact. (*b*) If every point of a straight line lies outside a circle except one point, the line is a tangent.
13. Equal central angles of the same circle or of equal circles subtend equal arcs; and conversely.
14. A central angle has the same measure in degrees of angle as the subtended arc has in degrees of arc. Or, a central angle equals its subtended arc.
15. The line of centers of two tangent circles contains the point of tangency.

286. The second step in classification is the analysis of other experimental properties of circles in terms of more elementary properties.

287. Theorem I.—*In the same circle or in equal circles, equal chords subtend equal central angles; and conversely.*



Part I.

Hypothesis.—Circle O with equal chords AB and CD, subtending central angles x and y.

Conclusion.—$x = y$.

Analysis.

1. Arc AB = arc CD What postulate?
2. $\therefore \angle x = \angle y$ What postulate ?

Part II.

Hypothesis.—Circle O with equal central angles x and y, subtending chords AB and CD.

Conclusion.—Chord AB = chord CD.

Analysis.

1. Arc AB = arc CD What postulate?
2. $\therefore$ chord AB = chord CD What postulate?

Help.—The postulates used are 6 and converse, and 13 and converse.

288. Theorem II.—*The perpendicular bisector of a chord contains the center of the circle.*

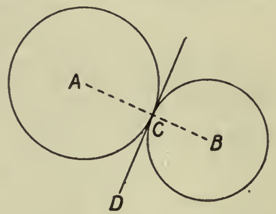
Help.—Use a locus theorem or a theorem of isosceles triangles. (See also **217,** Theorem (5).)

Corollaries.— (1). *The perpendicular bisectors of two non-parallel chords of a circle determine the center of the circle.*

2. *Three points on the circumference of a circle determine the center of the circle.*

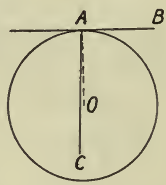
289. Theorem III.—*A diameter perpendicular to a chord bisects it and both major and minor subtended arcs.*

Corollary.—*A radius perpendicular to a chord bisects it and the subtended arc.*



290. Theorem IV.—*Two tangent circles have a common tangent at the point of tangency.*

Helps.—(1) Draw AB, which contains point C; Postulate? (2) Draw CD tangent to circle A; (3) $CD \perp AC$; why? (4) $\therefore$ CD is tangent to circle B; why? Analyze also for two internally tangent circles.



291. Theorem V.—*A perpendicular to a tangent at the point of tangency contains the center of the circle.*

Helps.—(1) Draw OA; (2) OA coincides with AC; etc.

292. Theorem VI.—*Two tangents drawn to a circle from an exterior point are equal.*

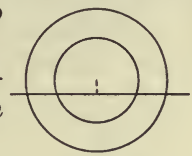
Helps.—(*a*) Draw a line from the point to the center of the circle, and radii to the points of tangency. Prove that the triangles are congruent. Or, (*b*) connect the points of tangency and draw radii as before. Use principles of isosceles triangles. Or, (*c*) draw a circle concentric to the given circle and containing the given exterior point. Prolong the tangents so that they shall be chords of this larger circle. Use circle Postulate 11(*a*) and 7 converse, and Theorem III or Corollary.

Additional Theorems

293. (1) *The line of centers of two intersecting circles is a perpendicular bisector of the common chord* (See also **217,** Theorem (6).)

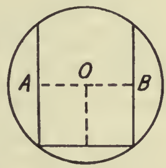
(2) *The line which joins the center of a circle with the center of a chord is perpendicular to the chord and bisects the arc subtended by the chord.*

(3) *A radius which bisects an arc bisects the chord subtended by the arc and is perpendicular to the chord.*



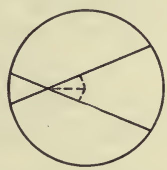
(4) *If a straight line intersects two concentric circles the two sects intercepted between the circles are equal.*

(5) *Two chords which are both perpendicular to the same chord are equal.*



Help.—Show that $OA = OB$.

(6) *Two equal chords which intersect within a circle divide each other so that the segments of one chord equal the segments of the other.*



(7) *A line drawn from the center of the circle to the point of intersection of two equal intersecting chords, bisects the angle formed by the chords.*

Analyze also when the chords intersect without the circle.

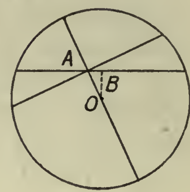
(8) *If two secants which intersect outside a circle make equal angles with a line drawn from the center of the circle to the point of intersection of the secants, the whole secants are equal and the segments into which they are divided by the circle are respectively equal.*

(9) *A side of an equilateral inscribed hexagon is equal to the radius of the circle.*

Helps.—Draw radii; the central angles are equal; each central angle is 60°; each triangle formed is equiangular; ∴ each triangle is equilateral.

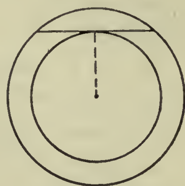
(10) *Two equal chords which intersect within a circle intercept one pair of equal opposite arcs.*

(11) *Two chords which make equal angles with a diameter drawn through their point of intersection are equal.*



(12) *A chord which is drawn through a given point within a circle perpendicular to the diameter through the given point, is shorter than any other chord drawn through the given point.*

Helps.—Draw OB; show that $OA > OB$; etc.

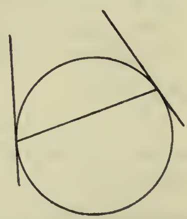


(13) *A chord of the larger of two concentric circles which is tangent to the smaller circle is bisected at the point of tangency.*

(14) *The line joining the point of intersection of two tangents with the center of the circle:* (*a*) *bisects the angle formed by the tangents;* (*b*) *bisects the central angle formed by the radii to the points of tangency;* (*c*) *bisects the arcs subtended by the tangents;* (*d*) *is a perpendicular bisector of the chord of contact of the tangents.*

(15) *The side of a circumscribed square equals the diameter of the circle.*

(16) *The tangents drawn at the extremities of a diameter are parallel.*



(17) *The tangents drawn at the extremities of a chord make equal angles with the chord.*

(18) *Chords of the larger of two concentric circles which are tangent to the smaller of the two circles, are equal.*

(19) *The sum of two opposite sides of a circumscribed quadrilateral equals the sum of the other two opposite sides.*

Help.—Apply **292.**

(20) *The central angle formed by radii which are drawn to the points of tangency of two tangents which intersect outside the circle, is supplementary to the angle formed by the tangents.*

(21) *The line of centers of two circles externally out of con-*

tact contains the points of intersection (a) of the internal common tangents, and (b) of the external common tangents.

Helps.—Draw lines from the point of intersection of the tangents to the centers of the circles. Show (*a*) that these lines lie in the same straight line; (*b*) that they coincide.

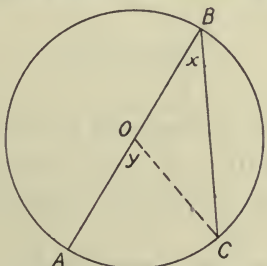
294. The Third Step in Classification.—The more important experimental results of Chapter XII have been classified, but there are many measurement properties of angles and sects associated with circles, which are not evident from the inspection of a constructed figure and can only become known by extending the methods of analysis into the realms of discovery.

295. Angle Properties of Lines Drawn to a Circle.—Any two intersecting lines which intersect or meet or touch a circle, form an angle whose measure is related to the measure of the subtended or intercepted arc or arcs. The simplest of these angle measures are stated in Postulates 13 and 14. Other similar properties will now be deduced.

296. Theorem VII.—*An inscribed angle has one-half the measure of the intercepted arc; or, it is measured by one-half the intercepted arc; or, it equals one-half, etc.*

This principle is best considered in three cases.

Case 1.—*When one side of the inscribed angle is a diameter.*



Hypothesis.—Inscribed $\angle x$, AB being a diameter.

Conclusion.—$\angle x = \frac{1}{2}$ arc AC.

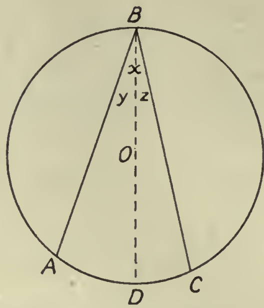
Analysis.—(1) Draw OC.

(2) $y =$ arc AC

(3) $x = \frac{1}{2}y$.

(4) $\therefore x =$ etc.

Case 2.—*When the center of the circle is within the inscribed angle.*



Analysis.—(1) Draw diameter BD.

(2) $y = \frac{1}{2}$ arc AD.

(3) $z = \frac{1}{2}$ arc DC.

(4) $\therefore x =$ etc.

Case 3.—*State and analyze this case.*

Corollaries.—(1) *Angles inscribed in the same segment are equal.*

2. *Inscribed angles subtending equal arcs are equal.*

3. *An angle inscribed in a semicircle is a right angle.* (See also **143,** Theorem (7).)

4. A principle relating to an angle inscribed in a major or a minor segment.

EXERCISES

1. What inscribed angle intercepts an arc of 95°?
2. What arc does an inscribed angle of 95° intercept?

297. Theorem VIII.—*The angle formed by a tangent and a chord drawn from the point of tangency has a measure in terms of subtended arcs.*

Helps.—There are two of these angles. Draw a diameter through the point of tangency; one of the angles to be considered is the *sum,* and the other is the *difference,* of a right angle and an inscribed angle. The measures of these angles in terms of arcs can be expressed from known principles. State the theorem discovered.

EXERCISES

1. A chord subtends an arc of 80°, and a tangent is drawn at one extremity of the chord. Calculate the value of the angle formed.
2. A chord meets a tangent at the point of tangency at an angle of 75°. What is the value of the arc subtended by the chord (or intercepted by the angle)?
3. A chord subtends an arc of 100°, and tangents are drawn at each end of the chord. What angle is formed by the tangents at their point of intersection?

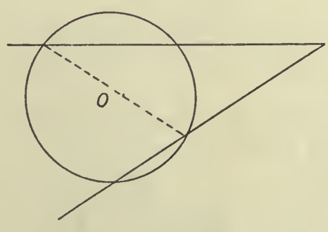
298. Theorem IX.—*The angle formed by two chords which intersect within a circle has a measure in terms of subtended arcs.*

Helps.—Draw a chord connecting extremities of the given chords, forming a triangle; the angle to be considered is related to angles of this triangle, which are measured by certain arcs as previously analyzed. Draw the elementary figures. State the theorem.

EXERCISES

1. Two chords which intersect within a circle intercept opposite arcs of 70° and 100°. What angle do the chords form?
2. Two chords intersect within a circle forming an angle of 30°; one of the two intercepted opposite arcs is 20°. What is the value of the other of the opposite arcs?

299. Theorem X.—*The angle formed by two secants which intersect outside the circle has a measure in terms of the two intercepted arcs.*



Helps.—Draw a chord as shown. The angle to be considered has a relation in terms of two inscribed angles. Draw the elementary figures. Analyze also by drawing a different chord from that shown in the figure.

300. Theorem XI.—*The angle formed by a secant and a tangent which intersect outside the circle has a measure in terms of the two intercepted arcs.*

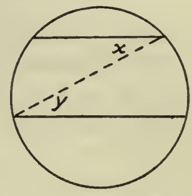
301. Theorem XII.—*The angle formed by two tangents which intersect outside a circle has a measure in terms of the two intercepted arcs.*

Corollary.—*The angle formed by two tangents which intersect outside a circle has a measure in terms of the smaller intercepted arc alone.*

Helps.—(1) $A = \frac{1}{2}(x - y)$; (2) $x + y = 360°$; (3) eliminate x.

EXERCISES

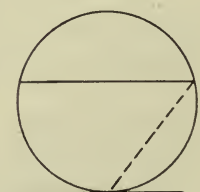
1. Two secants which intersect without a circle intercept arcs of 80° and 40°. What angle do the secants form?
2. A secant and a tangent which intersect without a circle intercept arcs of 100° and 40°. What angle do the secant and tangent form?
3. Two tangents which are drawn from the same exterior point intercept arcs of 200° and 160°. What angle do the tangents form?
4. Two secants meet at an angle of 35°; the nearer of the intercepted arcs is 50°. What is the value of the farther arc?
   Help.—$35° = \frac{1}{2}(x - 50°)$.



Additional Theorems

302. (1) *The arcs intercepted on a circle by two parallel chords are equal.*

(2) *The arcs intercepted on a circle by a tangent and a chord parallel to it are equal.*



(3) *The bisector of an inscribed angle bisects the subtended (or intercepted) arc.*

(4) *The bisectors of all the angles inscribed in the same segment are concurrent.*

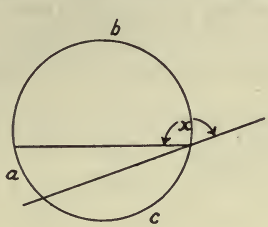
State where the bisectors are concurrent.

(5) *Opposite angles of an inscribed quadrilateral are supplementary.*

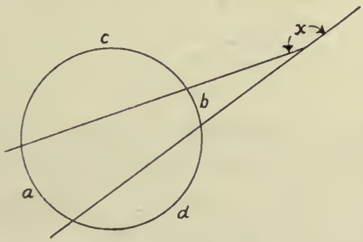
Help.—Consider the arcs which measure these angles.

(6) Corollary of Theorem (5). *The sum of one pair of opposite angles of an inscribed quadrilateral equals the sum of the other pair of opposite angles.*

(7) The value of angle x can be expressed in terms of arcs b and c. Investigate, and state the theorem discovered.

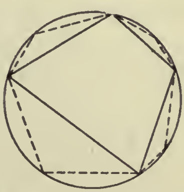


(8) The value of angle x can be expressed in terms of arcs b, c and d.



Help.—Show that $x = b + \frac{1}{2}(c + d)$.

(9) *An exterior angle of an inscribed quadrilateral equals the interior angle at the opposite vertex.*



(10) An angle is inscribed in each of the four outside segments of a circle, which are formed by the sides of an inscribed quadrilateral. Investigate the measure of the sum of these four angles.

(11) State a general theorem concerning the measure of an angle in terms of subtended arcs, according as the vertex is within, on, or without the circle.

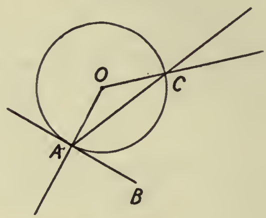
(12) An angle formed by two lines passing through the extremities of a diameter is acute, right, or obtuse, accord-

ing to the position of the vertex. Investigate and state the theorem.

(13) The center of the circumscribed circle of a triangle lies within the triangle, on a side, or without the triangle, according to the angle-shape of the triangle. Investigate and state the theorem.

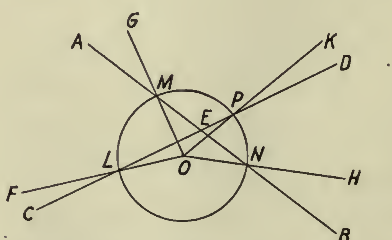
303. The Fourth Step in Classification.—Make an experimental test of the principles of circles which have been deduced from the postulates and other experimental principles.

1. *The angle formed by a tangent and a chord drawn from the point of tangency* (**297**).



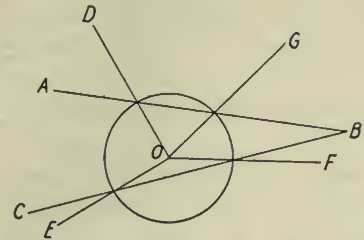
Draw the figure of **297** large enough, or extend the angle lines of the figure as shown, so that angles *BAC* and *AOC* can be measured with a protractor. Compare these measurements and check the principle of **297.**

2. *The angle formed by two chords which intersect within a circle* (**298**).



Measure angle *AEC*, and angles *FOG* and *HOK*. Note that the latter angles measure the subtended arcs *LM* and *NP*. Compare with the principle of **298.**

3. *The angle formed by two secants which intersect outside a circle* (**299**).



Measure $\angle ABC$, and $\angle$sDOE and FOG; etc.

4. *The angle formed by a secant and a tangent which intersect outside a circle* (**300**).

5. *The angle formed by two tangents which intersect outside a circle* (**301**).

6. The principle of **302,** Theorem (5).

7. The principle of **302,** Theorem (7).

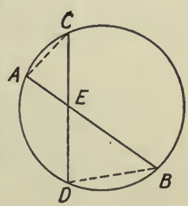
8. The principle of **302,** Theorem (8).

9. The principle of **302,** Theorem (9).

10. The principle of **302,** Theorem (10).

304. The Third Step in Classification. An Investigation of Sect Properties of Lines Drawn to a Circle.—The sects intercepted by a circumference on any two intersecting lines which intersect or meet a circle in two points or which are tangent to the circle, possess some relative or proportional values. The simplest of these sect properties are stated in **63,** Postulate 11, and in **292.** Other sect relations which are not evident from a constructed figure, will now be deduced from the principles of similar triangles.

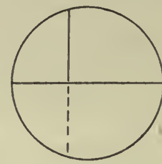
305. Theorem XV.—*A relation exists between the lengths of the segments of two chords which intersect within a circle.*



Helps.—(1) Draw AC and BD. (2) $\triangle AEC \sim \triangle BED$; because $\angle A = \angle D$ and $\angle C = \angle B$; why? (3) State the proportion between sides AE, CE, BE and DE. (4) Change the proportion into products.

Draw the two triangles under consideration separately, and overlay the equal angles with colored crayons. Then overlay the homologous sides of the triangles which enter into the required proportions, with colored crayons.

Corollaries.—(1) *The segments of all chords of a pencil of chords intersecting within a circle are related.*

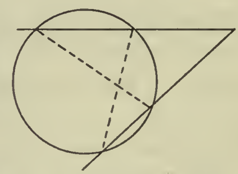


2. *A line drawn perpendicular to the diameter of a circle is a mean proportional between the segments of the diameter.* (See also **260,** Theorem (20).)

EXERCISES

1. Two chords 10″ and 12″ respectively, intersect at a point which is 4″ from the end of the 10″ chord. How far from the end of the 12″ chord is the point of intersection?
2. A third chord of the circle is concurrent with the two chords of Exercise 1, and intersects them at a point which is 6″ from its nearer end. What is the length of the chord?
3. The diameter of a circle is 400 feet and a perpendicular is erected to the diameter at a point which is 40 feet from one end of the diameter. Calculate the length of the perpendicular from the diameter to its point of intersection with the circle.

306. Theorem XVI.—*A relation exists between the lengths of two secants which intersect outside a circle and the external segments of the secants.*



Helps.—The overlapping triangles are similar. State the proportions in the form of equal products. Follow the suggestions given in **305.**

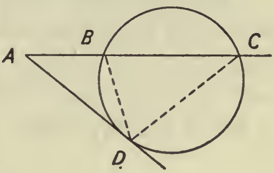
Corollary.—*The lengths of all secants of a pencil of secants intersecting outside of a circle, and of their external segments, are related.*

EXERCISES

1. Two secants are 20″ and 24″ measured from their point of intersection to the farther point of intersection with the circle. The 20″ secant intersects the circle 12″ from its point of intersection with the other secant. At what distance from their point of intersection does the 24″ secant cut the circle?

2. The secants of Exercise 1 meet a third secant at the same point. A length of 1″ is intercepted on this third secant between the two points of intersection with the circle. What is the total length of this secant?
Help.—$x(x + 1) = 240$.

307. Theorem XVII.—*A relation exists between the lengths of a secant and a tangent drawn to a circle from the same point outside a circle, and the external segment of the secant.*



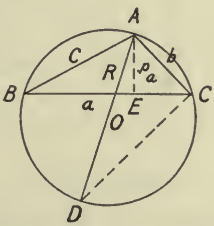
Helps.—$\triangle ABD \sim \triangle ACD$. State the required relation as a proportion. Follow the suggestions given in **305.**

Corollary.—*The square of the tangent equals, etc.*

EXERCISES

1. In the figure of Theorem XVII, the segment $AB = 10''$, and segment $BC = 12''$. Calculate the length of the tangent AD.
2. The length of a tangent $AD = 12''$, and the length of the secant $AC = 20''$. Calculate AB and BC.

308. Theorem XVIII.—*The radius of the circumscribed circle of a triangle can be expressed in terms of the three sides of the triangle.*



State the *hypothesis* and *conclusion*.
Helps.—(1) Draw *AE, OD, CD;* (2) $\triangle ABE \sim \triangle ACD$;

(3) $\dfrac{c}{p_a} = \dfrac{2R}{b}$; (4) $R = \dfrac{bc}{2p_a}$;

(5) but $p_a = \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$; (255);

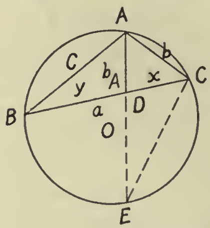
(6) $\therefore R = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$.

EXERCISES

1. Calculate the radius of the circumscribed circle of a triangle of which the sides are 10, 12 and 20.
2. Calculate the radius of the circumscribed circle of a triangle of which the sides are 20, 20 and 10.

309. Theorem XIX.—*The square of the bisector of an angle of a triangle equals the product of the sides including the angle; less the product of the segments into which the third side of the triangle is divided by the bisector.*

State the *hypothesis* and *conclusion.*



Helps.—(1) Draw the circumscribed circle; produce AD to E, draw CE; (2) $\triangle ADB \sim \triangle ACE$; (3) $\frac{c}{AE} = \frac{b_A}{b}$; (4) $xy = b_A \cdot DE$. In order to eliminate AE and DE; proceed as follows; (5) $AE = b_A + DE$; (6) Substitute from (5) in (3); $\frac{c}{b_A + DE} = \frac{b_A}{b}$; (7) Solve for DE in (6) and substitute in (4); $\frac{xy}{b_A} = \frac{bc - b_A^2}{b_A}$; (8) $\therefore b_A^2 = bc - xy$.

EXERCISES

1. The sides of a triangle are 10, 12 and 20. Calculate the length of the bisector of the angle opposite the 12 side.
   Help.—(1) Calculate the segments x and y by **245**; (2) calculate b_A by the above formula.
2. Calculate by the above formula; the length of the bisector of the vertex angle of an isosceles triangle of which the sides are 20, 20 and 10.

310. Theorem XX.—*The length of the bisector of an angle of a triangle can be expressed in terms of the three sides of the triangle.*

Helps.—(1) $b_A^2 = bc - xy$; (2) $x = \frac{ab}{b+c}, y = \frac{ac}{b+c}$, (**260,** Theorem (26); (3) substitute from (2) in (1);

$$b_A^2 = bc - \frac{a^2bc}{(b+c)^2}; \text{ (4) } \therefore b_A^2 = bc\left[1 - \frac{a^2}{(b+c)^2}\right]$$

$$= bc\left[\frac{(b+c)^2 - a^2}{(b+c)^2}\right] = bc\left[\frac{(a+b+c)(-a+b+c)}{(b+c)^2}\right]$$

$$= bc\left[\frac{2s \cdot 2(s-a)}{(b+c)^2}\right]; \text{ (5) } \therefore b_A = \frac{2}{(b+c)}\sqrt{bcs\,(s-a)}$$

EXERCISES

1. The sides of a triangle are 10, 12 and 20. Calculate the length of the bisector of the angle opposite the 12 side, using the above formula.
2. Calculate by this formula, the three angle bisectors of a triangle of which the sides are 20, 20 and 10.

311. The Fourth Step in Classification.

1. *Two chords which intersect within a circle* (**305**).

Measure the chord segments to the nearest hundredth of an inch. Test the proportion or the relation of products, as discovered.

2. The principle of **305,** Corollary 2.

3. *Two secants which intersect outside a circle* (**306**).

Measure the secants and their external segments. Test the relation discovered.

4. *A secant and a tangent which intersect outside a circle* (**307**).

5. *The radius of the circumscribed circle of a triangle* (**308**).

Draw any triangle and its circumscribed circle. Measure the sides of the triangle and the radius of the circle. Test the relation discovered.

6. Test the relation of **309** by measurement.

7. Test the relation of **310** by measurement.

312. The Position Assigned a Principle in Classification. Notice in what widely separated positions apparently closely related principles must be placed, if they are associated with the group of principles by which they are analyzed or explained. Thus the following special lines of a triangle have been expressed in terms of the three sides:

1. Under similar triangles; an altitude (**255**) and a median (**256**).

2. Under circles; an angle-bisector (**309** and **310**) and the radius of the circumscribed circle (**308**).

3. Under areas; the radius of the inscribed circle, which will be derived in Chapter XVII.

Additional Theorems

313. (1) Analyze Theorem VIII (**297**) by drawing radii perpendicular to the chord and to the tangent. The measure of the required angle can be obtained from that of the central angle thus formed.

(2) Analyze Theorem VIII (**297**) by drawing a chord through the farther extremity of the given chord parallel to the tangent. Apply **302,** Theorem (2).

(3) Analyze Theorem IX (**298**) by drawing an auxiliary chord through the extremity of one of the given chords parallel to the other given chord. Apply **302,** Theorem (2).

(4) Analyze Theorem X (**299**) by the method of Theorems (2) and (3).

(5) Analyze Theorem XI (**300**) by the method of the preceding theorems.

(6) Analyze Theorem XII (**301**) by the method of the preceding theorems.

(7) Analyze Theorem XI (**300**) by considering a tangent as a special case of a secant whose points of intersection coincide, thus deriving Theorem XI from Theorem X.

(8) Analyze Theorem II of Chapter V (**119**) by circumscribing a circle about the triangle.

(9) Show that the analysis of Theorem 8 is faulty logic because the theorem itself enters into the analysis through intermediate theorems.

(10) *A chord subtending an arc of* 120° *bisects a radius perpendicular to it.*

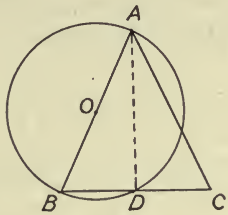
(11) *Tangents drawn from an exterior point a radius distant from the circumference, form an equilateral triangle with their chord of contact.*

(12) *Lines drawn from the vertices of a circumscribed quadrilateral to the center of the circle, form central angles of which each non-adjacent pair are supplementary.*

(13) *The angle between two tangents to a circle is double the angle between the chord of contact and a radius drawn to a point of tangency.*

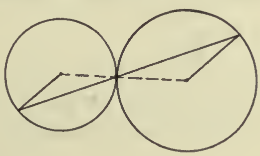
(14) *A circumference whose diameter is the hypotenuse of a right triangle, contains the vertex of the right angle.*

(15) *A circle drawn with one of the equal sides of an isosceles triangle as a diameter, bisects the base of the triangle.*



Helps.—Draw AD; $AD \perp BC$, why? $\therefore AD$ bisects BC.

(16) *A line is drawn through the point of contact of two tangent circles (externally or internally tangent). Radii drawn to the extremities of this line are parallel.*



Helps.—Draw the line of centers. Show that alternate-interior angles are equal.

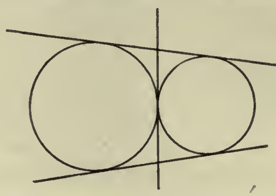
(17) *A line is drawn through the point of contact of two tangent circles (two cases). Tangents drawn at the extremities of this line are parallel.*

Helps.—(*a*) Draw radii as in Theorem (16); or (*b*) Draw a common tangent, and show that alternate-interior angles are equal because their arcs have the same measure.

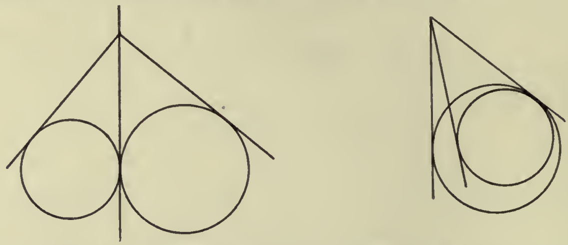
(18) *The perpendicular from the center of a circle to a side*

of an inscribed equilateral triangle equals one-half the radius of the circle.

(19) *The internal common tangent of two externally tangent circles bisects the external common tangents.*

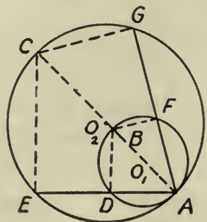


(20) *Tangents drawn to two tangent circles from a point on their internal common tangent are equal (two cases).*



(21) *A circle is drawn with the radius of another circle as its diameter. A chord of the larger circle drawn from the point of tangency of the two circles is bisected by the circumference of the smaller circle.*

(22) *All chords drawn from the point of tangency of two circles (internally or externally tangent) are divided proportionally by the circumference of the smaller circle.*

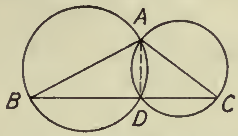


Helps.—Draw a common diameter; $\triangle ABF \sim \triangle ACG$ and $\triangle ABD \sim \triangle ACE$, etc.

(23) *Two (unequal) circles intersect in points A and B; a common secant is drawn through A intersecting the circles in C and D; chords BC and BD are drawn. Then* $\angle CBD$ *is the same for any position of the secant CD; or,* $\angle CBD$ *is constant.*

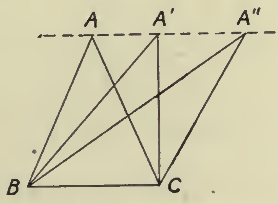
Helps.—Consider the measures of angles C and D and how $\angle CBD$ is related to these angles.

(24) *The circles described on two sides of a triangle intersect on the third side.*



Helps.—Draw $AD \perp BC$; show that each circle contains point D.

(25) *Of all triangles drawn with the same base and with equal altitudes, the isosceles triangle has the largest vertex angle.*



Helps.—Circumscribe a circle about the isosceles triangle ABC, what kind of line is $AA'A''$? $\therefore$ points A', A'', etc., are outside the circle.

(26) *A chord is a mean proportional between a diameter drawn through one of its extremities and its projection on the diameter.*

(27) *The sum of the three alternate angles of an inscribed hexagon equals the sum of the other three alternate angles.*

(28) Find in a way similar to that employed in the preceding theorem, that a relation exists between the sums of the two sets of alternate angles of any inscribed polygon of an even number of sides. State the theorem.

(29) *The point of intersection of any pair of common tangents of two circles divides the center sect into segments proportional to the radii.*

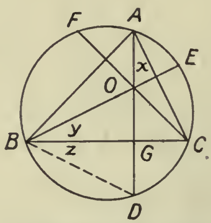
(30) Corollary of (29). *The points of intersection of the internal and external common tangents of two circles divide the center sect harmonically.*

(31) *A pencil of chords, AB, AC, AD,* etc., *is drawn from a point A in the circumference of a circle; a line parallel to the tangent at A, intersects the chords in points B', C', D', etc. Then $AB \cdot AB' = AC \cdot AC'$* = etc.

Helps.—Draw a diameter AX perpendicular to the tangent at A; draw BX, CX, DX, etc.; $\triangle AB'X' \sim \triangle ABX$, etc.

(32) *The common chord of two intersecting circles, if produced bisects their common tangents.*

(33) *If the altitudes of a triangle are produced to intersect the circumscribed circle, the sects intercepted between the orthocenter and the circumference are bisected by the sides of the triangle.*



Helps.—(1) $x = y$; (2) $x = z$; (3) $\therefore y = z$; (4) hence $OG = GD$.

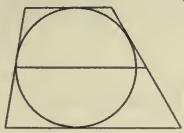
(34) *An internal common tangent of two equal circles, which are externally out of contact, bisects the center sect.*

(35) *Secants drawn from the point of intersection of the internal and external common tangents of two externally tangent circles, to the centers of the circles, are perpendicular.*

(36) *The perpendicular bisectors of the sides of an inscribed polygon are concurrent at the center of the circle.*

(37) *The bisectors of the angles of a circumscribed polygon are concurrent at the center of the circle.*

(38) *The perimeter of a circumscribed trapezoid equals four times the sect which joins the middle points of the two non-parallel sides.*



(39) *The sum of three alternate sides of a circumscribed hexagon equals the sum of the other three alternate sides.*

(40) Test the principles of Theorems (38) and (39) experimentally.

314. Locus Constructions.—Construct the following loci experimentally, by finding the positions of a sufficient number of points to indicate the form of the complete locus.

1. The locus of the middle points of a system of parallel chords of a circle.

2. The locus of the middle points of all equal chords of a circle.

3. The locus of the middle points of all chords of a circle which are concurrent in a point on the circumference.

4. The locus of middle points of all chords of a circle which are concurrent in a point within the circle.

5. The locus of the middle points of all secants of a circle which are concurrent in a point outside the circle. Plot the bisection points both of the entire secants and of their external segments.

6. The locus of the centers of all circles which are tangent to a given straight line at a given point.

7. The locus of the centers of all circles tangent to a given circle at a given point. Draw both externally and internally tangent circles.

8. From a given point on the given straight line of

Exercise 6, tangents are drawn to all the circles. Find the locus of the points of tangency.

9. The locus of the vertices of all angles of a given size, or value, whose sides pass through two given points. Mark two points several inches apart, and use the 30° angle of a drawing triangle for the given angle. In order to complete the locus it is necessary to find positions of the angle where one side produced through the vertex, contains one of the given points.

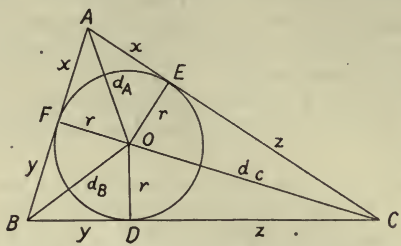
315. Review Exercises

1. A secant and a tangent intersect at an angle of 30°; the secant intercepts an arc of 100° between its points of intersection with the circle. Calculate the other two intercepted arcs.
2. Two angles of an inscribed triangle are 48° and 105°; calculate the arcs subtended by the sides.
3. The vertex angle of an inscribed isosceles triangle is 50°; calculate the arcs subtended by the sides.
4. Two secants intercept arcs of 150° and 95°; calculate the angle between them.
5. Two secants form an angle of 30°, the nearer intercepted arc being 50°; calculate the farther of the intercepted arcs.
6. The measure of the farther arc intercepted by two secants is four times the measure of the angle formed by the secants; the nearer arc is 30°; calculate the angle and remoter arc.
7. Two tangents form an angle of 25°; calculate the major and minor intercepted arcs.
8. An inscribed angle subtends an arc of 75°; calculate its value. What arc is subtended by an inscribed angle of 75°?
9. Two intersecting chords intercept, in order, the arcs 80°, 120°, 30°,130°; calculate the angles formed by the chords.
10. One of the angles formed by two intersecting chords is 55°; an arc opposite this angle is 78°, and an adjacent arc is 100°. Calculate the other arcs.
11. A chord is drawn from a point of tangency making an angle of 42° with the tangent; calculate the values of the two subtended arcs.
12. A chord subtends an arc of 75°; at what angle must a line intersecting it at an extremity be drawn in order to be a tangent to the circle?
13. Three consecutive sides of an inscribed quadrilateral subtend arcs of 72°, 90°, 58°, calculate the angles of the quadrilateral,

the angles between the diagonals, and the angles at which the diagonals meet the sides.

14. Two tangents are drawn at the extremities of a chord which subtends an arc of 155°; at what angle do the tangents intersect?
15. Two perpendicular chords intercept adjacent arcs of 40° and 70°; calculate the other intercepted arcs.
16. Two adjacent arcs AB and BC, measure 150° and 100°; a tangent is drawn at B, and a secant through AC; calculate the angle of intersection.
17. A chord 2″ long is drawn perpendicular to a diameter of a circle of 1.5″ radius; at what point does the chord intersect the diameter?
18. A chord of 1″ is bisected by a chord on which a segment of $\frac{1}{4}$″ is cut off; calculate the radius of the circle.
19. From the end of a tangent sect of 2″, a secant is drawn through the center of a circle of 3″ diameter; calculate the external segment of the secant.
20. The sides of an isosceles triangle are 4, 4, 3; calculate the three angle-bisectors, and the radius of the circumscribed circle.
21. Calculate the radius of the circumscribed circle of a triangle whose sides are 3, 4, 5.
22. The radius of a circle is 2″; calculate the chord subtending an arc of 120°.
23. Two circles of 8″ and 10″ radii have a center sect of 20″; calculate the tangent sects of internal and external common tangents.
24. In the formulas of **308** and **310**, let $a = b = c$. Show that $R = \frac{a\sqrt{3}}{3}$; $b_A = \frac{a\sqrt{3}}{2}$; and that these values agree with values for the altitude and median of an equilateral triangle, derived in a similar way from the formulas for p_a and m_a, **255** and **256** (see **260**, Theorem (16); and with **254**, Part (*b*).
25. Given an angle, and the length of a chord; construct a circle such that the segment cut off by the chord contains inscribed angles equal to the given angle.
26. Given a chord of a circle and the measure of its subtended arc; construct the circle.
27. Divide a circle into two segments such that the angle inscribed in one segment is double the angle inscribed in the other.
28. Construct a circle tangent to each of two parallel lines and to a third line intersecting the parallels.
29. Construct a circle tangent to each of two concentric circles. Write a formula for its radius.
30. Construct a circle tangent to each of two concentric circles and also tangent to a line intersecting one or both of the given circles.

31. Construct a circle of given radius, tangent to each of two given circles which are externally out of contact; discuss all cases.
32. The sides of a triangle are 5″, 6″, 8″. Calculate the segments into which the sides are divided by the points of tangency of the inscribed circle.

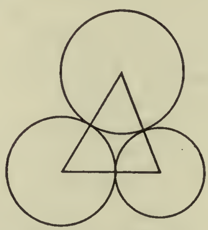


Helps.—$AF = AE$, etc. Form a system of equations, $x + y = 5$, etc.; solve for x, y, z.

33. Derive formulas for the segments into which the sides a, b, c, of a triangle are divided by the points of contact of the inscribed circle.
34. Calculate the distance of the center of the inscribed circle of the triangle of Exercise 32 from each vertex of the triangle.
Helps.—Calculate r; then d_A, etc.
35. Show that the general formula for d_A is,

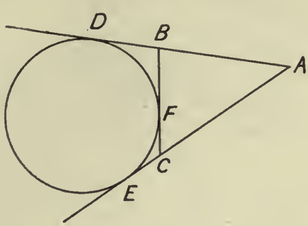
$$d_A = \sqrt{\frac{(s - a)[s(s - a) + (s - b)(s - c)]}{s}}.$$

36. Three circles are drawn externally tangent, each to the other two. The center sects are 10, 12 and 15 inches. Find the radii of the circles.

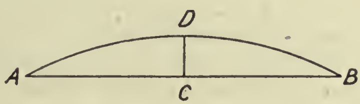


37. Two circles are externally tangent at A. One of the external common tangents touches the circles at points B and C, respectively. Show that a circle whose diameter is BC, contains point A.
38. Show that two equal externally tangent circles intercept equal chords on a common secant drawn through the point of tangency.

39. The triangle ABC is formed by two fixed tangents AD and AE and a movable tangent BC. Show that the perimeter of $\triangle ABC$ is constant, wherever tangent BC is drawn.

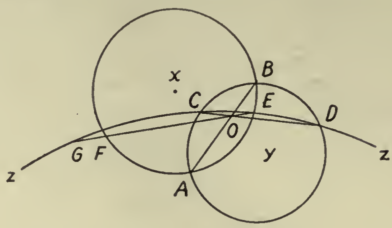


40. The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments of 10 and 40 inches respectively. Calculate the altitude.
41. Show that, if the perpendicular sides of a right triangle are in the ratio of 1 to 2, the altitude to the hypotenuse divides it in the ratio of 1 to 4.
42. A surveyor measured a chord AB of a circular railroad curve, equal to 240 feet, and the perpendicular distance CD at the middle of chord AB, equal 40 feet. Show how to calculate the radius of the curve.



43. The altitude to the hypotenuse of a right triangle of which one acute angle is 30°, divides the hypotenuse in the ratio of 1 to 3.
44. Two chords of 6 and 10 inches respectively, intersect so as to divide the shorter chord into segments of 2 and 4 inches. Find the segments of the longer chord.
45. A secant and a tangent are drawn from the same external point. The chord intercepted on the secant is 8 inches and the external segment of the secant is 14 inches. Find the length of the tangent.
46. The radius of a circle is 21 inches. Find the length of a tangent drawn from a point 35 inches from the center of the circle. Find also the length of a secant drawn from the same point if the length of its external segment is 20 inches.
47. Tangents drawn to two intersecting circles from any point on their common chord produced are equal.

48. The common chords of each pair of three intersecting circles are concurrent.



Helps.—(1) AB and CD intersect in O; draw chord EO which intersects circles x and z in F and G, respectively; (2) $AO \cdot OB = EO \cdot OF$, etc.; (3) $\therefore OF = OG$, and therefore points G and F coincide.

49. The center line of two circles, which are externally out of contact, intersects the circles in the successive points A, B, C, D, and intersects their common external tangent in point E. Show that $EA \cdot ED = EB \cdot EC$.

Helps.—(1) Mark the points of tangency, T and T'; draw chords AT, BT, CT', DT'; (2) $\frac{EA}{ET} = \frac{EC}{ET'}$, etc.

50. In Exercise 49, draw a secant from point E intersecting the circles in points F, G, H, J. Show that $EF \cdot EJ = EG \cdot EH$.

51. The radii of two circles are 10 and 4 inches, respectively, and their center sect is 20 inches. Find the points of intersection of the internal and external common tangents with the center sect.

Helps.— $\frac{4}{10} = \frac{x}{20 - x}$ and $\frac{4}{10} = \frac{y}{20 + y}$.

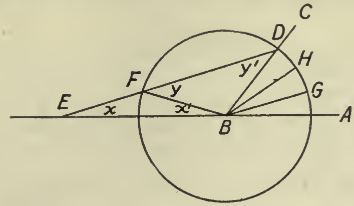
52. The radii of two concentric circles are 10 and 24 inches respectively. Find the length of a chord of the larger circle that is tangent to the smaller circle.

53. The radius of a circle is 24 inches. Find the length of the shortest chord that can be drawn through a point 20 inches from the center of the circle.

Applications

316. Famous Problems.—*The Trisection of an Angle.*—This is one of the famous problems of antiquity. There is no known solution by use of rule and compass alone. The following method requires the use of a scale, and would not be considered a solution by those old Greek geometers who

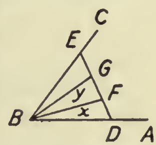
regarded the science chiefly as a splendid intellectual recreation.



$\angle ABC$ is the given angle. Draw any circle with center B; produce AB; find by trial, using a scale, a sect ED such that $EF = BF$. Then $x = \frac{1}{3}\angle ABC$, and can be laid off at ABG, etc.

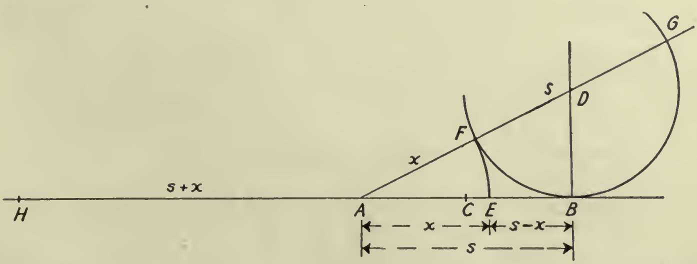
EXERCISES

1. Prove that $\angle x = \frac{1}{3}\angle ABC$.
2. A method of trisecting an angle, which has been frequently suggested by students is as follows: Draw $BD = BE$; draw DE; trisect DE by some purely "geometrical" method, as in **261**, Exercise 31; draw BF and BG. Prove that $\angle x$ does not equal $\angle y$ by showing that the bisector of $\angle DBG$ does not bisect DG, **245**.



3. Can the error in the construction of Exercise 2 be detected by measurement?

The Division of a Sect in Mean and Extreme Ratio.—This is such a division of the sect that one segment is a mean proportional between the given sect and the other segment. There are two points of division, one internal and the other external.



AB is the given sect. Draw circle D, tangent to AB at B, with radius $= \frac{1}{2} AB$; draw AD. Then E is the required internal point of division, if $AE = AF$; and H is the required external point if $AH = AG$.

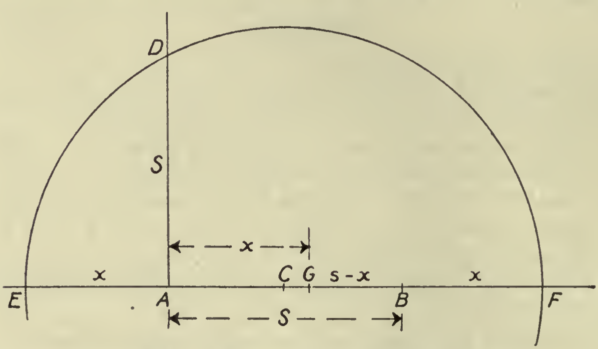
EXERCISES

4. Prove $\frac{s-x}{x} = \frac{x}{s}$.

Helps.—(1) $\frac{AF}{AB} = \frac{AB}{AG}$; (2) change by the Law of Division (**240**).

5. For the external point of division H, the segments of sect AB are $HA = s + x$ and $HB = 2s + x$. Prove $\frac{2s + x}{s + x} = \frac{s + x}{s}$.

6. Another construction for the division of a sect into mean and extreme ratio is shown. Prove $\frac{s-x}{x} = \frac{x}{s}$.



Helps.—(1) $\frac{EA}{AD} = \frac{AD}{AF}$ (**305**); (2) $\therefore \frac{s-x}{x} = \frac{x}{s}$ (**240**).

7. Calculate the values of the segments of a sect divided in mean and extreme ratio. Solve the equation of Exercises 4 and 6 by algebra. Show that $x = \frac{-1 \pm \sqrt{5}}{2} \cdot s = +.618s$ or $-1.618s$.

8. In Fig. 1, let $HA = x_1$; then $\frac{x_1 + s}{x_1} = \frac{x_1}{s}$. Solve for x_1; $x_1 = \frac{+1 \pm \sqrt{5}}{2} \cdot s$. Show that $x_1 = s + x$ or $-x$, and therefore gives the points of division H and E, which were found in Exercise 7.

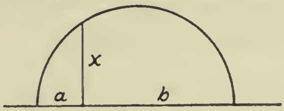
9. Measure the segments of the sects AB of the preceding figures. Do the measurements agree with the calculated values?

Artistic Value of Mean and Extreme Ratio.—This division of a sect possesses a remarkable artistic and psychological interest. It has been called the "divine ratio" or the "golden section."

EXERCISES

10. Draw several equal vertical lines, side by side, and divide them at different points. It will be found that many persons possess a marked preference for the "divine ratio," in which the point of division is above the center; that is, $x = .62s$.
11. Measure the most pleasing proportioned book covers, platters, "oval" (elliptical) picture frames, etc. It will be found that the width approximates .62 of the length.

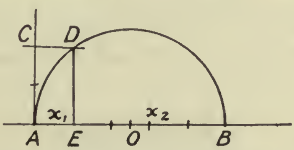
A Mean Proportional.



EXERCISES

12. Explain the construction of the figure and prove that x is the required sect.
13. Calculate the value of x and check the geometrical solution by measurement.

The Geometrical Solution of a Quadratic Equation.—



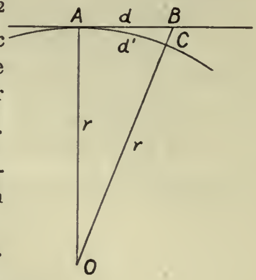
Let the given equation be $x^2 - 5x + 4 = 0$. Then $x(5 - x) = 4$; $\therefore \frac{x}{2} = \frac{2}{5 - x}\cdot$ Draw a circle with $AB = 5$ units as diameter; draw $AC \perp AB$, equal 2; draw $CD \| AB$, draw $DE \perp AB$. Then AE and EB are the roots of the equation.

EXERCISES

14. Solve the equation $x^2 - 9x + 14 = 0$, geometrically.
15. Solve the equation $x^2 - 10x + 25 = 0$, geometrically.
16. Solve the equation $x^2 - 6x = 0$, geometrically.
17. Check the geometrical solutions by algebraic solutions of the preceding equations.

317. Terrestrial Measurements.—*The curvature of a sphere* in a distance d, is expressed in terms of the intercept c, on a radius, included between the arc and the tangent.

EXERCISES



1. In the right triangle BAO, $\overline{OB}^2 = \overline{OA}^2 + \overline{AB}^2$. Substitute the lengths r, c and d; omit the term c^2 which will be very small in comparison with the other terms; solve for c. Show that $c = \dfrac{d^2}{2r}$. Solve also for r and d. In these formulas, d' may be substituted for d, when d is small in comparison with r.
2. Let $r = 100$ inches and $d = 20$ inches. Find c.
3. Let $d = 20$ inches and $c = 1$ inch. Find r.
4. Let $r = 120$ inches and $c = 4$ inches. Find d.

The Size of the Earth.—Careful measurements prove that the curvature of the earth is approximately 8 inches in 1 mile.

EXERCISE

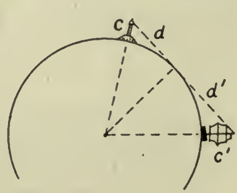
5. Find the radius of the earth from the preceding data, using the formula for r.

The Distance of the Horizon at Sea.—This depends upon the height of the observer above the mean surface of the water.

EXERCISES

6. Find the distance of the horizon to an observer who stands upon a boardwalk so that his eye is 30 feet above the water level of the ocean.

   Helps.—$c = \dfrac{30}{5280}$ miles; let $r = 3960$ miles. $d = \sqrt{2cr}$ miles.

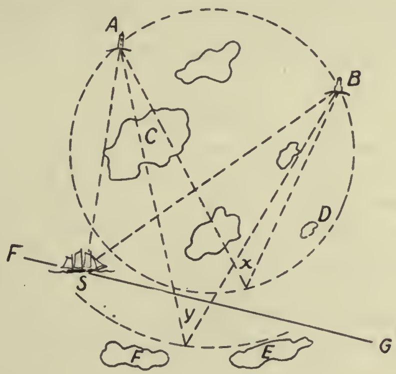


7. Find the distance at which the topmast of a vessel is visible to the observer of Exercise 6, if the topmast is 80 feet above the water line.

   Helps.—For the vessel, $c' = \dfrac{80}{5280}$. The distance required $= d$ of Exercise 6 $+ d'$ of Exercise 7.

8. How high must a war balloon ascend at the sea coast in order to sight an enemy fleet 100 miles distant?
   Help.—A more exact result will be obtained in this problem by solving the equation $(c + 3960)^2 - \overline{3960}^2 = \overline{100}^2$.
9. How far out at sea may a lighthouse be visible if the light is 120 feet above water level, and the watch on the vessel is 60 feet above the water level? The effective distance will be considerably less than that calculated except under most favorable weather conditions.

318. Navigation. *The "Danger Angle."*—The reefs or bars C, D, E, etc., are shown on a sailing chart, A and B



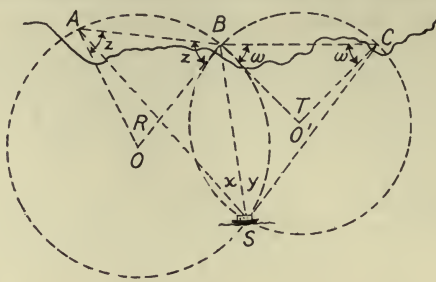
being the positions of two prominent objects also marked on the chart. The navigator draws circles passing through A and B (or they are already shown on the chart), in such a way as to include or exclude the danger region. He measures any inscribed angles x and y. If he then lays his course on the line FG so that the sextant reading of the angle ASB is less than x and greater than y, the course is safe. Angles x and y are the "danger angles."

EXERCISE

1. Explain why the method employed ensures a safe course.

The Three Point Problem.—This name is given to a method used for accurately locating a ship which is near enough to the shore to distinguish three prominent objects.

The angles x and y are measured with a sextant on the ship. The circles O and O' can then be constructed, their

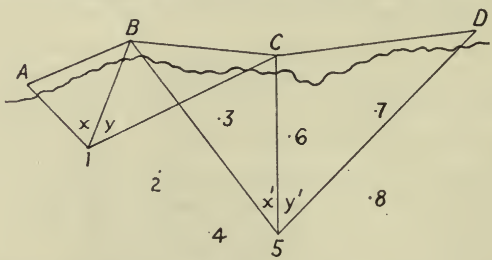


intersection on the chart determining the position of the ship at S.

EXERCISES

2. What relation exists between angles x and R; T and y?
3. What relation exists between angles z and R; w and T?
4. Prove therefore that $z = 90° - x$, and $w = 90° - y$.
5. Plot three points A, B, C; $AB = 20{,}000$ ft., $BC = 15{,}000$ ft., $\angle ABC = 160°$. Find the position of the ship S if the angle x is measured $40°$, and angle y is measured $35°$.

319. Surveying. *Location of Soundings.*—The "three point problem" is generally used to locate the boat which makes the soundings along a shore line, or within a lake or



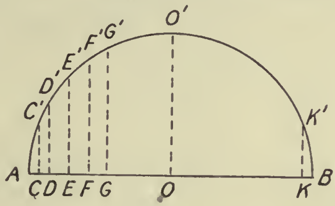
bay. The series of points A, B, C, D, etc., are located and plotted on the shore. At each observation point, 1, 2, 3, etc., angles x, y, x', y'; etc., are measured with a sextant, and simultaneously a sounding is taken. A record is kept, from which the soundings can be plotted on the map or chart.

Point	1st angle	2nd angle	Sounding, feet
1	AB 75° 10′	BC 42° 15′	12.5
2	AB 50° 5′	BC 61° 3′	24.3
3			
4			
5	BC 31° 52′	CD 35° 37′	27.8
Etc.			

EXERCISE

1. Plot the points 1, 2 and 5, from the record.

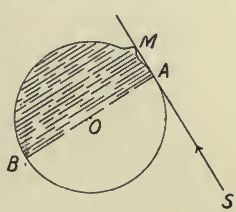
To Lay Out a Semicircular Curve by Ordinates.—



The diameter AB is divided into any number of equal parts, AD, DE, EF, etc., with a smaller part $AC = KB$ at the ends. The ordinates CC', DD', etc., are calculated by **30 5** Cor. 2.

EXERCISES

2. If $AB = 400$ feet, $AC = CD = 20$ feet, $DE = EF = FG =$ etc. $= 40$ feet, calculate CC', DD', etc., to OO'.
3. Plot the diameter and points of division, draw the ordinates, measure off the lengths of the ordinates. Test the accuracy of the calculated lengths by drawing a circle with center at O.



320. Astronomy. *The Height of the Moon's Mountains.*—Galileo, about the year 1600, measured the heights of some of the mountains on the moon by the following method. He measured the distance AM at the instant that the sun's ray SA, first iluminated the top of the mountain M. The diameter of the moon must also be known.

EXERCISE

1. If $AB = 2160$ miles, and AM is measured 120 miles, calculate the height of the mountain M.
   Help.—(*a*) Use the formula of **317,** Exercise 1; and (*b*) use the method of **317,** Exercise 8.

Eclipses of the Sun and Moon.—The eclipse of the sun is caused by the converging cone of total shadow of the moon falling upon the earth's surface at P (Fig. 1). The eclipse of the moon is caused by the entrance of the moon into the cone of total shadow of the earth (Fig. 2).

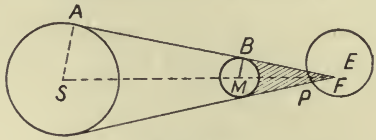


FIG. 1.

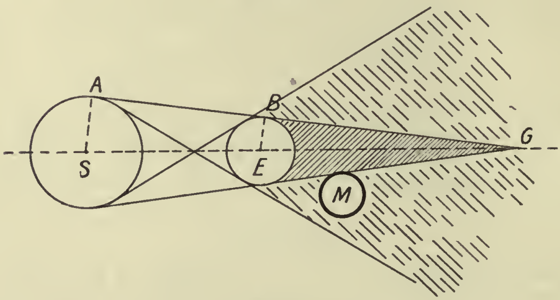


FIG. 2.

EXERCISES

2. Take the distance from center to center of the sun and earth as 93,000,000 miles; the distance from the earth to the moon as 241,500 miles; the radius of the sun 432,000 miles; the radius of the earth 3960 miles; the radius of the moon as 1080 miles. (The distances from the earth to the sun and to the moon are variable.) Calculate the length MF of the moon's shadow, Fig. 1. Does the moon's shadow reach the surface of the earth? Calculate the radius of the shadow at P.
3. In Fig. 2, show that $EG = \dfrac{SE \cdot EB}{SA - EB}$.
4. Calculate the length of the earth's total shadow, using the values given in Exercise 2 and the formula of Exercise 3.

5. Calculate the diameter of the earth's shadow, Fig. 2, at the distance at which the moon enters it. If the moon enters the shadow centrally, is the shadow sufficiently broad to entirely obscure the moon?

321. THE CIRCLE IN DESIGN.—Ornamental designs in-

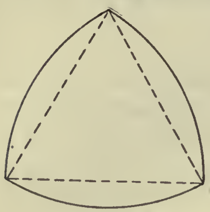


FIG. 1.

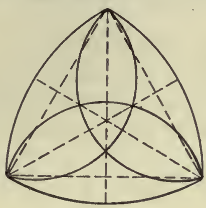


FIG. 2.

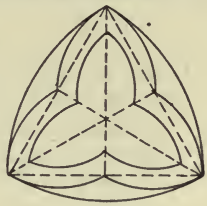


FIG. 3.

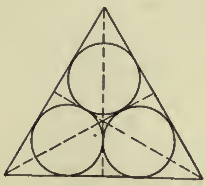


FIG. 4

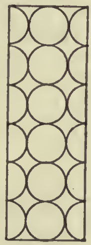


FIG. 5.

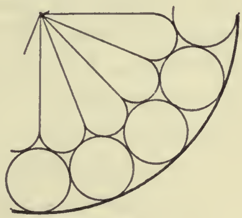


FIG. 6.

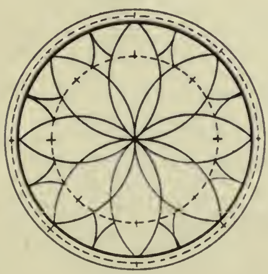


FIG. 7.

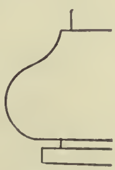


FIG. 8.

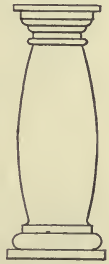


FIG. 9.

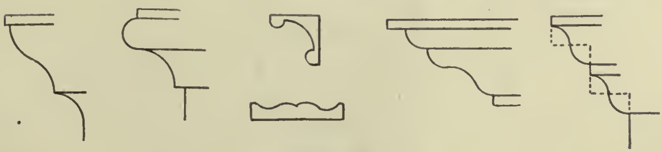


FIG. 10.

volving circles and arcs of circles are found in railings, brackets, cut glass, window glass, rugs and linoleums, and in innumerable other familiar objects. A few suggestive illustrations are given.

Figure 1 is called the *curved equilateral triangle.* Figures 5 and 6 are art glass designs; Fig. 8 is a column base; Fig. 10 shows various forms of mouldings and cornices.

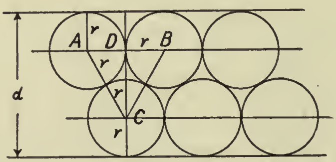
EXERCISES

1. Make other designs for some definite application, as a piece of scroll work to fill in the top of an arched doorway; a piece of embroidery or a sofa cushion; an arm of a built-in bench for a porch or hall; etc.
2. Name some familiar objects which are always or frequently circular in form.

Architectural Designs Containing Tangent Circles.—These forms occur in endless variety. Constructions may be made by trial, but interesting applications of algebraic calculation may be made, and should be made in laying out exact work. Construct the figures shown, from the calculated values.

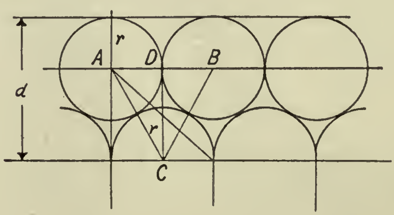
EXERCISES

3. Express the radius r, of one of the circles in terms of the distance, d.



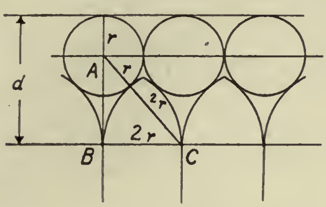
Helps.—$CD = r\sqrt{3}$; $d = CD + 2r = r(2 + \sqrt{3})$; $\therefore r = \dfrac{d}{2 + \sqrt{3}}$ $= d(2 - \sqrt{3}) = .27d.$

4. Express r in terms of d.



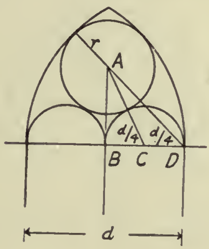
Helps.—$CD = r\sqrt{3}$; $d = r(1 + \sqrt{3})$; $\therefore r = \dfrac{d(\sqrt{3} - 1)}{2} = .366d.$

5. Express r in terms of d.



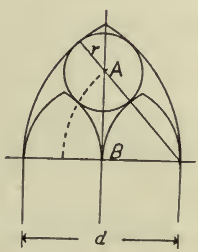
Help.— $r = \dfrac{d(\sqrt{5} - 1)}{4}$

6. Express r and AB in terms of d.



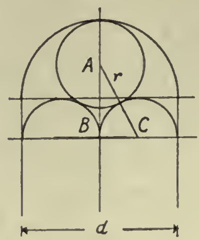
Helps.—In $\triangle ABD$, $\overline{AB}^2 = (d - r)^2 - \left(\dfrac{d}{2}\right)^2 = \dfrac{3d^2}{4} - 2dr + r^2.$ In $\triangle ABC$, $\overline{AB}^2 = \left(r + \dfrac{d}{4}\right)^2 - \left(\dfrac{d}{4}\right)^2 = r^2 + \frac{1}{2}\,dr$; $\therefore r = .3d$ and $AB = .49d$.

7. Express r and AB in terms of d.



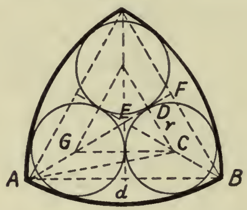
Helps.—$r = \dfrac{d}{4}$ and $AB = \dfrac{d\sqrt{5}}{4} = .56d.$

8. Express r and AB in terms of d.



Helps.—$r = \frac{9}{32}d$ and $AB = \frac{15}{32}d$.

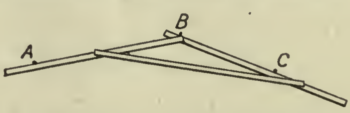
9. Express r and BC in terms of d.



Helps.—In $\triangle CDE$, $DE = \frac{r\sqrt{3}}{3}$; $AE = \frac{2}{3}AF = \frac{d\sqrt{3}}{3}$; $AD = AE + DE = \frac{(d+r)\sqrt{3})}{3}$; $AC = d - r$. In $\triangle ACD$, $(d - r)^2 = r^2 + \frac{(d+r)^2}{3}$; $2d^2 - 8dr - r^2 = 0$; $d = \frac{4 \pm 3\sqrt{2}}{2} \cdot r$; $\therefore r = \frac{2}{4 \pm 3\sqrt{2}} \cdot d = (-4 \pm 3\sqrt{2})\, d = .24d$. $BC = AG = AD - DG$; $AD = \sqrt{\overline{AC}^2 - \overline{CD}^2} = d\sqrt{.\overline{76}^2 - .\overline{24}^2} = .72d$; $DG = r\sqrt{3} = .42d$; $\therefore BC = .30d$.

NOTE.—The preceding problems are found in "A Source Book of Problems for Geometry," by Mable Sykes; Allyn and Bacon.

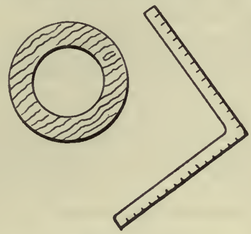
322. Various Useful Applications. *To Draw an Arc Through Three Given Points When the Radius Is Large.*—A



pencil point at B will describe the required arc if the arms

BA and BC slide along pins inserted at points A and C. Why?

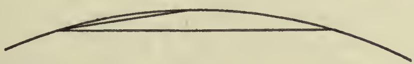
To Find the Diameter of a Column.—A carpenter may be



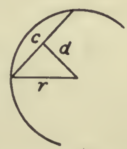
required to cut a circular column along a diametral plane. Explain how the diameter is determined by laying the square on the column as shown.

To Make a Core Box Truly Cylindrical.—Explain how a pattern maker uses his square, and why the vertex of the square must scrape the hollow as it is slid around.

To Find the Radius of an Arc.—Find the center by the



method of constructing a circle which will contain three given points.

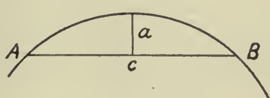


The Length of a Chord.—Show how r is expressed in terms of the chord c, and its distance d, from the center. When $d = \frac{1}{2}r$, show that $c = r\sqrt{3}$. Solve this last formula also for r in terms of c.

The Rise of an Arc.—The rise of the arc AB from the chord c, of the arc is a. Show that

$$a = \frac{2r \pm \sqrt{(2r + c)(2r - c)}}{2}$$

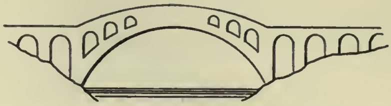
$$= \frac{d \pm \sqrt{(d + c)(d - c)}}{2}.$$



Solve also for r or d in terms of c and a. $r = \dfrac{c^2 + 4a^2}{8a}$.

EXERCISES

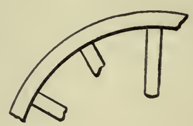
1. Find the length of a chord 6 inches from the center of a circle of radius 12 inches.
2. Find the diameter of a circle if a chord drawn 10 inches from the center is 20 inches long.
3. A circular arch bridge is to span a stream 52 feet wide. If the rise of the crown of the arch is 13 feet, what is the radius of the arch?



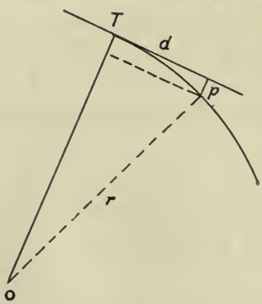
4. A doorway is 6 feet wide. The rise of the circular top is 1 foot above the straight sides. Calculate the radius of the arc.

5. A surveyor wishes to find the radius of a railroad curve. He measures a chord of the curve = 100 feet, and the offset at the center of the chord = 2 feet. What is the radius of the curve?

6. A chord measured on a piece of a broken wheel is 20 inches and the rise of the arc is 5 inches. What is the radius of the wheel?



To Lay Out a Circular Curve by Offsets from the Tangent

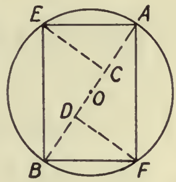


at the Beginning Point of the Curve.—Show that the value of the perpendicular offset p, at a distance d, from the point of tangency T, is $p = r - \sqrt{r^2 - d^2}$.

EXERCISE

7. Calculate the offsets from the tangent of a curve of 1000 feet radius, at distances from the point of beginning of the curve of 100, 200 and 300 feet.

The Strongest Rectangular Beam That Can Be Cut from a Circular Log.—The beam is laid out on the end of the log by drawing a diameter AB, and trisecting at points C and D, and erecting perpendiculars CE and DF, etc.



EXERCISES

8. Calculate the sides AE and BE of the beam in terms of the diameter d. Show that $\frac{AE}{BE} = \frac{5}{7}$ approximately.

9. What are the dimensions of the strongest beam that can be cut from a log 2 feet in diameter?

323. PROBLEMS FOR FIELD WORK.—(1) *To Lay Out a Circular Curve by Ordinates.*—Use the data of 319, Exercise 2, or calculate the ordinates for a curve of any other radius. Stake out points A, C, D, ——— B; erect perpendiculars by the method of **237**; measure on each perpendicular the length of the calculated ordinate; drive stakes at points C', D', E', etc.

2. *To Lay Out a Circular Curve by Offsets from the Tangent.*—Take the radius of the curve = 200 feet. Select a point A on a straight line where the curve is to begin. Calculate the offsets for values of d = 40, 80, 120 etc. feet. See **322,** Exercise 7. Stake out these distances along the tangent from point A, and erect perpendiculars to the tangent; measure on each perpendicular the length of the calculated offset until the curve has turned a quarter circle; extend the last offset which is the tangent at the other extremity of the curve.

CHAPTER XIV

REGULAR POLYGONS. CIRCLES ASSOCIATED WITH POLYGONS

Principles Determined Experimentally

324. Previous Mention.—Regular polygons have been defined and briefly considered in Chapter VI; inscribed and circumscribed polygons in Chapter XII.

325. Experiment I. Equilateral Inscribed Polygons.—Draw several circles; draw equilateral inscribed polygons of 4, 5, 6, 7 sides by dividing the circumferences by trial into equal arcs.

Observe that the polygons are equiangular.

EXERCISES

1. How can the division points of the circumference be found without trial in the case of the equilateral quadrilateral?
2. How can the division points be found without trial in the case of the hexagon?

326. Experiment II. Equiangular Circumscribed Polygons.—Draw several circles; divide the circumferences into 4, 5, 6, 7 equal arcs; draw polygons whose sides are tangent at the division points. Such polygons will be equiangular by **301.**

Observe that the polygons are also equilateral.

327. Experiment III.—Properties of Regular Polygons.—Calculate each interior angle of regular polygons of 4, 5, 6, 7 sides, by **191,** Theorem (5). Draw the polygons by drawing alternately a side and an angle of each polygon.

(*a*) Bisect all the interior angles of the polygons. Observe that the bisectors are concurrent.

(*b*) Draw the perpendicular-bisectors of all the sides of the polygons.

Observe that they are concurrent.

(*c*) Draw the inscribed and circumscribed circles of all the polygons.

Observe (1) that the circles of each polygon are concentric; (2) that the inscribed circle is tangent to each side at its middle point; (3) that the triangle formed by a half-side of a polygon, a radius of the circumscribed circle and a radius of the inscribed circle is a right triangle.

328. Definitions.—The common center of the inscribed and circumscribed circles of a regular polygon is the *center* of the polygon.

The radius of the circumscribed circle of a regular polygon is the *radius* of the polygon.

The radius of the inscribed circle of a regular polygon is the *apothem* of the polygon.

The right triangle formed by a half-side, radius and apothem is an *element* of the polygon.

EXERCISES

1. Calculate the central angle of an element of a regular polygon of 4, 5, 6, 7, sides.
2. Write a general formula for the central angle of an element of a regular polygon of n sides.
3. Write a formula for a half-side s, expressed in terms of the perimeter p.
4. Express a general relation between a half-side s, a radius r, and an apothem a.
5. Write a general formula for one of the equal angles of a triangle formed by two adjacent sides of the polygon and a diagonal joining the extremities of the two sides.

   Help.—(1) Write the value of one interior angle of the polygon; (2) subtract from 180°; (3) divide by 2. Value is $\frac{180°}{n}$.

329. Problems Relating to Regular Polygons.—In these problems only an element of the polygon is required, except in those which involve diagonals.

1. Find by measurement the radius and apothem of a regular pentagon of which the side is 1.4 inches.

2. Find by measurement the side and radius of a regular hexagon of which the apothem is 1.2 inches.

3. Derive a formula for the value of the side and radius of an equilateral hexagon, expressed in terms of the apothem.

Help.—Refer to **260,** Theorem (18).

4. Check the measured values of Exercise 2 by calculating the same values by the formula of Exercise 3.

5. Find by measurement the apothem and radius of a regular pentagon of which the perimeter is 30 feet.

6. Find by measurement the apothem and side of a regular decagon of which the radius is 2 inches.

7. Find the number of sides of a regular polygon of which the side is 2 inches and radius 2.61 inches.

Helps.—Construct the element; measure the central angle; divide it into 360°.

8. Find the number of sides of a regular polygon of which the side is 20 feet and the apothem 37.32 feet.

9. The least diagonal of a regular decagon is 2 inches. Find by measurement the side, apothem and radius.

Helps.—(1) Calculate the angles of the triangle formed by diagonal and two adjacent sides, by **328,** Exercise 5; (2) Construct this triangle. (3) Construct the element of the polygon on the side as thus determined.

10. The least diagonal of a regular polygon is 2.83 inches and a side is 2 inches. Find the number of sides of the polygon.

11. Calculate the radius, apothem and perimeter of a square of which the side is 20 inches.

12. Calculate the radius, apothem and perimeter of an equilateral triangle of which a side is 20 inches.

13. Calculate the apothem and radius of a regular octagon of which the side is 6 inches.

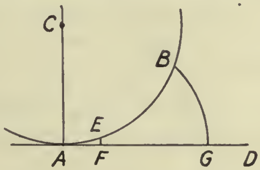
Helps.—(1) $6^2 = r^2 + r^2 - 2r(.707r)$ (**252** and **254**); (2) $.586r^2 = 36; r = \sqrt{61.43} = 7.84$; (3) $a^2 = (r + s)(r - s)$.

330. Experiment IV.—The Ratio of the Circumference to the Diameter of a Circle.

(*a*) Draw a number of circles of different size; say 1.0″, 1.5″, 2.0″, 2.5″, 3.0″ in diameter. Measure the circumferences as carefully as possible with a flexible scale. A narrow uniform strip of paper placed on edge may be used. Enter the values in column c' of the accompanying table.

d	c'	c''	c'''	Average value of c	$\frac{c}{d} = \pi$	Average value of π
1.0						
1.5						
2.0						
2.5						
3.0						

(*b*) *Rankine's Approximation for Finding the Length of a Circular Arc.*—AB is the given arc with center C. Draw

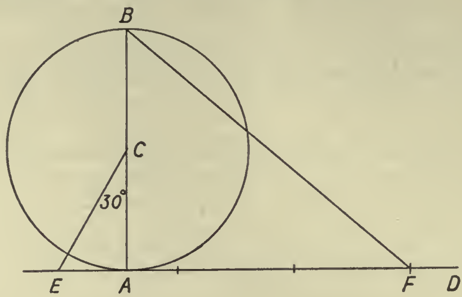


radius CA and tangent AD. Find $AE = \frac{1}{4}$ arc AB, and draw arc EF with center A. Draw arc BG with center F. Then sect AG = arc AB, approximately. The error in this method almost vanishes for small arcs and reaches $\frac{1}{900}$ of the length of the arc for an arc of 60°. It should not be used for arcs exceeding 90°.

Use the circles of part (*a*) and find the lengths of arcs of 60° for each circle. Multiply by 6 and enter the values in column c'' of the table.

(*c*) *Ceradini's Approximation.*—Draw diameter AB; tangent AD; $\angle ACE = 30°$; make $EF = 3r$; draw BF. Then

BF = arc AB = $\frac{1}{2}$ circumference. Find the values of the circumferences of the circles used in Parts (*a*) and (*b*) by this method. Enter in column c''' of the table.



Complete the table as indicated.

EXERCISES

1. Calculate the circumference of a circle of which the diameter is 20 feet.
2. Calculate the diameter of a circle of which the circumference is 50 feet.
3. What is the diameter of a cask if the circumference measures 10 feet? Will the cask pass through a door 3 feet wide?
4. Write a formula for the length of an arc of F degrees, in a circle whose diameter is d.

331. THE LENGTH OF AN ARC DETERMINED FROM ITS CHORD.—A formula that gives the length of an arc not exceeding 60°, with close approximation is $l = \frac{8c - C}{3}$; where C is the chord of the arc and c is the chord of $\frac{1}{2}$ the arc.

To use the formula, construct the arc to scale and measure the chords.

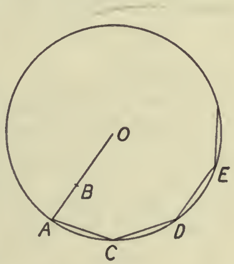
EXERCISES

1. Find the length of an arc of 45° in a circle whose radius is 20 inches.
2. Calculate the length of the arc of Exercise 1 by using the formula of **330**, Exercise 4.
3. The chords of Exercise 1 may be calculated as follows: (1) $C^2 = 20^2 + 20^2 - 2(20)(.707 \times 20)$; (2) $a^2 = 20^2 - \left(\frac{C}{2}\right)^2$, where a

is the apothem of a regular octagon whose side is C; (3) $\frac{40}{c} = \frac{c}{20 - a}$. Substitute these values in the formula $l = \frac{8c - C}{3}$, and solve for l.

332. Exact Constructions of Regular Inscribed Polygons. The problem of finding rule and compass constructions for regular inscribed polygons, has been a favorite one with mathematicians. Methods have been given in preceding chapters for the rule and compass construction of inscribed equilateral triangles, squares and regular hexagons. By successively bisecting the arcs subtended by the sides of these polygons, regular inscribed polygons of 8, 12, 16, 24, 32, etc., sides may be constructed.

A Rule and Compass Construction of a Regular Inscribed Decagon.—Draw a radius OA; divide it into mean and extreme ratio at point B (**316**); OB is the required side of the decagon. A regular inscribed pentagon, and regular polygons of 20, 40, etc., sides can be obtained from the decagon.



For all practical purposes, the trial and error method used in **325** is the best, except for polygons of 3, 6, 12, etc., and 4, 8, 16, etc., sides.

In 1796, Gauss, the greatest mathematician of the Eighteenth century, then nineteen years of age, discovered a method of constructing all regular inscribed polygons the number of whose sides is given by the formula

$$n = 2^q(2^{2^p} + 1)$$

where p and q are integers.

EXERCISE

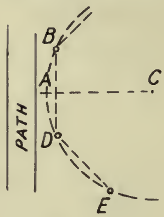
Find the value of n in Gauss' formula for $q = 0$, $p = 0$; $q = 0$, $p = 1$; $q = 1$, $p = 0$: $q = 1$, $p = 1$; etc., to $q = 3$, $p = 3$.

Applications

333. (1) An hexagonal nut, one inch on each side, is to be cut from a round bar of iron. What diameter of stock must be taken?

2. A square nut is to be made for a bolt $1\frac{1}{2}$ inches in diameter. The least thickness of metal around the hole equals the radius of the bolt. What diameter of round stock must be taken?

3. A gardener plans a summer-house to be laid out as a regular octagon. The radius is 10 feet. He drives a stake C, at the center and a second stake A at the center of the arc subtended by the side parallel to the path. Measure from a drawing of an element made to scale, the chords AB and (side) BD. The stakes B, D, E, etc., can then be placed.



4. In the description of Solomon's Temple (1 Kings, VII, 23) the dimensions of the "Molten sea," which appears to have been a great circular brazen vessel, are given as "ten cubits from the one brim to the other" (diameter), and "a line of thirty cubits did compass it round about" (circumference). What was the writer's estimate of the value of π?

CHAPTER XV

REGULAR POLYGONS AND CIRCLES

Classification and Explanation of Principles

334. Constants, Variables and Limits.—*A constant* is a quantity which possesses a fixed value.

A variable is a quantity whose value changes continually.

The limit of a variable is the constant which the variable continually approaches, but may never quite equal.

EXERCISES

1. One half of a pile of sand is removed in a day, one half of the remainder is removed the second day, and so on. The whole pile of sand is a constant value; the total amount removed up to any given time is a variable. What is the limit of the variable?
2. The sum of the infinite geometrical series, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$, etc., for n terms, is a variable. What is the limit of this variable as n is indefinitely increased?
3. What is the limit of the repeating decimal, .333 . . . . ?

Analysis by the Principles of Limits.—Some analyses involving metrical relationships (that is, where numerical values, or measures, are involved) are made by obtaining a relation under certain given conditions, and then considering the figure to undergo successive modifications until it merges into another figure. If the property which was shown to exist in the primary figure holds true during the successive modifications of that figure, it holds true in the final figure. This final figure is the limit of the changing or variable figure; the values of sects, angles, arcs, areas, ratios, in the final figure are the limits of the variable sects, angles, etc., of the variable figure.

EXERCISE

4. If a regular polygon possesses some property which remains unchanged in principle as the number of sides of the polygon is indefinitely increased, what limiting figure must also possess this same property?

Abbreviation.—Approaches as a limit, $\doteq$.

The First Step in Classification

335. Postulates of Regular Polygons and Circles.—(1) An equilateral inscribed polygon is regular.

2. A circumscribed polygon whose points of tangency divide the circumference of the circle into equal arcs, is regular.

3. The perimeter of a regular inscribed polygon (*a*) is less than the circumference of the circle; (*b*) increases as the number of sides is indefinitely increased; (*c*) approaches the circumference of the circle as a limit.

4. The perimeter of a regular circumscribed polygon (*a*) is greater than the circumference of the circle; (*b*) decreases as the number of sides is indefinitely increased; (*c*) approaches the circumference of the circle as a limit.

5. Two regular polygons of the same number of sides are similar (see **234**).

6. Corresponding sects of two regular polygons of the same number of sides are proportional.

Limit Axioms.—(1) If two variables are proportional to two constants, the limits of the variables are proportional to the constants.

2. If two variables approaching limits are always equal, their limits are equal.

3. If a variable is approaching a constant as a limit, the variable multiplied by or divided by any number approaches the constant multiplied by or divided by the same number, as a limit.

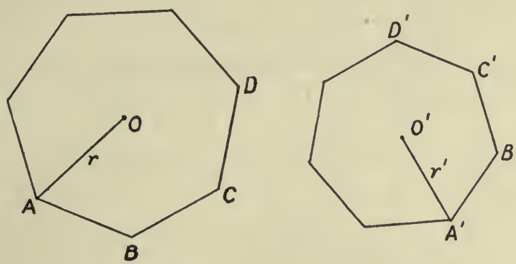
EXERCISES

1. How much less is the perimeter of a regular hexagon inscribed in a circle whose radius is 10 inches, than the circumference of the circle? Use the value of $\pi = 3\frac{1}{7}$.

2. How much greater is the perimeter of a regular quadrilateral circumscribed about a circle whose radius is 10 inches, than the circumference of the circle?
3. If the diagonal of a square whose side is 10, is 14.142, what is the diagonal of a square whose side is 20?
4. The apothem of a regular hexagon whose side is 10, is 8.66. What is the apothem of a regular hexagon whose side is 30? What is the side of a regular hexagon whose apothem is 34.64?

The Second Step in Classification

336. Theorem I.—*The perimeters of two regular polygons of the same number of sides are proportional to the radii of the polygons.*



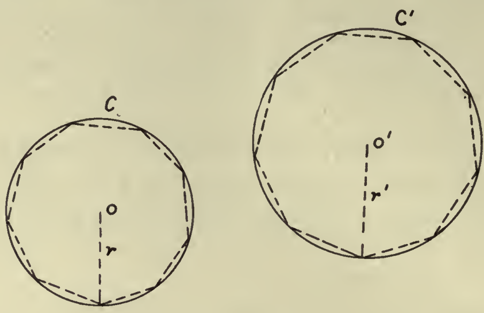
Hypothesis.—Regular polygons O and O', of the same number of sides.

Conclusion.—$\frac{p}{p'} = \frac{r}{r'}$.

Analysis.

Statement	Reason
1. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \text{etc.}$	**335**; Postulate 5
2. $\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'}$	**240**(f)
3. Or $\frac{p}{p'} = \frac{AB}{A'B'}$	**64**; Axioms 11 and 13
4. $\frac{r}{r'} = \frac{AB}{A'B'}$	**335**; Postulate 6
5. $\therefore \frac{p}{p'} = \frac{r}{r'}$	**64**; Axiom 1

337. Theorem II.—*The ratio of the circumference to the radius is constant for all circles.* Or: *Circumferences of circles are proportional to their radii.*



Hypothesis.—O and O' are circles with circumferences c and c', and radii r and r'.

Conclusion.—$\frac{c}{r} = \frac{c'}{r'}$.

Analysis.

Statement	Reason
1. Inscribe regular polygons of the same number of sides in the two circles, the perimeters being p and p'.	
2. The polygons are similar.	**335**; Postulates 1 and 5
3. Let the number of sides of the polygons be indefinitely increased by doubling the number of sides; then the successive polygons are regular, and similar at each step.	**335**; Postulates 1 and 5
4. $\frac{p}{p'} = \frac{r}{r'}$.	**336**; Theorem I
5. But $p \doteq c$, and $p' \doteq c'$.	**335**; Postulate 3
6. $\therefore \frac{c}{c'} = \frac{r}{r'}$, or $\frac{c}{r} = \frac{c'}{r'}$.	**335**; Limit Axiom 1

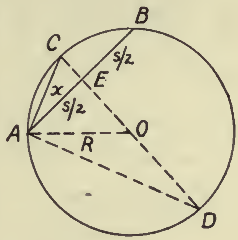
Corollaries.—(1) *The ratio of the circumference to the diameter is constant for all circles. That is;* $\frac{c}{d} = \frac{c'}{d'}$, = etc. = π.

2. *In any circle;* (*a*) $c = \pi d$; (*b*) $c = 2\pi r$.

3. *The length of an arc of n degrees of a circle whose radius is r, is* $a = \frac{n}{360} \cdot 2\pi r$.

The Third Step in Classification

338. Theorem III.—*The side of a regular inscribed polygon can be expressed in terms of the side and radius of the regular inscribed polygon of half the number of sides.*



Hypothesis.—AC is the side x, of a regular inscribed polygon of n sides; AB is the side s of a regular inscribed polygon of $\frac{n}{2}$ sides; R is the radius.

Conclusion.—x may be expressed in terms of s and R.

Analysis.

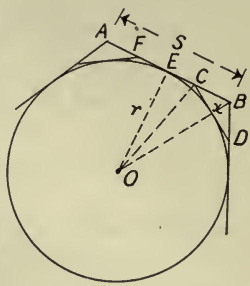
STATEMENT

1. Draw CD, OA, AD
2. $CD \perp AB$
3. $AE = \frac{s}{2}$.
4. $\angle CAD$ is a right $\angle$
5. $x^2 = CD \cdot CE$
6. $x^2 = 2R(R - OE) = 2R\left(R - \sqrt{R^2 - \left(\frac{s}{2}\right)^2}\right)$
7. $x^2 = R(2R - \sqrt{4R^2 - s^2})$
8. $x = \sqrt{R(2R - \sqrt{4R^2 - s^2})}$.

Corollary.—If $R = 1$, $x = \sqrt{2 - \sqrt{4 - s^2}}$.

EXERCISE

The side of a regular hexagon inscribed in a circle of unit radius is 1. Calculate the side of a regular dodecagon (12 sides) inscribed in the same circle.



339. Theorem IV.—*The side of a regular circumscribed polygon can be expressed in terms of the side and apothem of the regular circumscribed polygon of half the number of sides.*

Hypothesis.—FC, CD, are the sides x, of a regular circumscribed polygon of n sides; AB is the side s of a regular circumscribed polygon of $\frac{n}{2}$ sides; r is the apothem.

Conclusion.—x may be expressed in terms of s and r.

Analysis.

STATEMENT

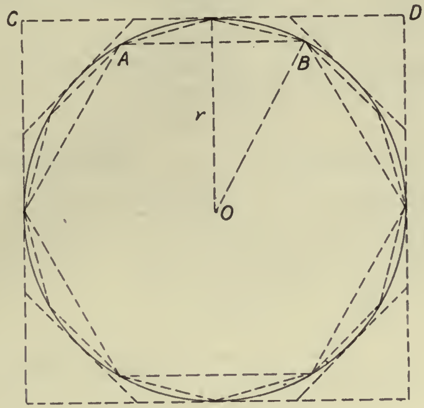
1. Draw OE, OC, OB
2. OC bisects $\angle EOB$.
3. $\dfrac{EC}{CB} = \dfrac{OE}{OB}$
4. $\dfrac{EC}{EB - EC} = \dfrac{OE}{\sqrt{OE^2 + EB^2}}$
5. $\dfrac{\frac{x}{2}}{\frac{s}{2} - \frac{x}{2}} = \dfrac{r}{\sqrt{r^2 + \left(\frac{s}{2}\right)^2}}$
6. $x = \dfrac{2rs}{2r + \sqrt{4r^2 + s^2}}$

Corollary.—*If* $r = 1$, $x = \dfrac{2s}{2 + \sqrt{4 + s^2}}$.

EXERCISE

The side of a regular quadrilateral (square) circumscribed about a circle of unit radius is two. Calculate the side of a regular octagon circumscribed about the same circle.

340. Theorem V.—*The value of π can be calculated with as close an approximation as desired.*



Hypothesis.—Circle O, with radius $r = 1$; circumference c; diameter $d = 2r = 2$.

Conclusion.—The value of $\pi = \frac{c}{d}$ or $\frac{c}{2r}$ can be calculated, etc.

Analysis.

STATEMENT	REASON
1. Draw a regular inscribed hexagon with side $AB = s$, and a circumscribed square with side $CD = s'$.	
2. Side $AB = s = r = 1$; side $CD = s' = 2r = 2$	
3. The perimeter of the hexagon $= 6$; the perimeter of the square $= 8$	
4. $\therefore$ $c > 6$ and < 8; $\therefore$ $\pi > \frac{6}{2}$ or 3 and $< \frac{8}{2}$ or 4	**335**; Postulates 3 and 4
5. The perimeter of a regular insribed polygon of 12 sides $= 12\left[\sqrt{2 - \sqrt{4 - 1}}\right] = 6.211657$	**338**; Theorem III
The perimeter of a regular cir-	

cumscribed polygon of 8 sides

$= 8\left[\dfrac{2 \cdot 2}{2+\sqrt{4+2^2}}\right] = 6.627418$ **339**; Theorem IV

6. $\therefore$ $c > 6.211657$ and < 6.627418

$\therefore \pi > \dfrac{6.211657}{2}$ or 3.105829 and $<$

$\dfrac{6.627418}{2}$ or 3.313709 **335**; Postulates 3 and 4

7. If the doubling of the sides of inscribed and circumscribed polygons is continued indefinitely, the following tables are obtained:

Regular Inscribed Polygons			Regular Circumscribed Polygons		
No. of sides	Perimeter	Ratio of perimeter to diameter of circle	No. of sides	Perimeter	Ratio of perimeter to diameter of circle
6	6	3	4	8	4
12	6.211657	3.105829	8	6.627418	3.313709
24	6.265257	3.132629	16	6.365196	3.182598
48	6.278700	3.139350	32	6.303450	3.151725
96	6.282064	3.141032	64	6.288226	3.144118
192	6.282905	3.141453	128	6.284448	3.142224
384	6.283115	3.141558	256	6.283500	3.141750
			512	6.283264	3.141632

8. The perimeters of both inscribed and circumscribed regular polygons approach the circumference c as a limit:

$\therefore \dfrac{p}{d}$ and $\dfrac{p'}{d} \doteq \dfrac{c}{d}$ or π **335**; Limit Axiom 3

9. $\pi > 3.141558$ and < 3.141632

$\therefore \pi = 3.1416$, approximately.

341.—The History of π.—The determination of π is one of the famous geometric problems. It is attempted in the MS. of Ahmes, now preserved in the British Museum. Other later attempts took the form of a determination of the area of a circle, or of the side of a square of equal area. Such a value involves that of π, and thus the problem has been known as "*squaring the circle.*"

Archimedes (250 B.C.), and Ptolemy (150 A.D.), obtained values of π almost as accurate as that calculated in **340.** Metrus and Romanus, of Holland, both belonging to the 16th century, calculated the value of π by the method of **340,** the latter using polygons up to 1,073,741,324 sides, and obtaining a value correct to sixteen decimals. Lambert (1750) proved π incommensurable, that is, having no exact value; and Lindemann (1882) proved it transcendental, that is, not expressible as a radical nor as a root of an algebraic equation with integral coefficients. There are other methods of calculating π, which belong to the realm of higher mathematics. Mr. Shanks, in 1873, carried the calculation to 707 decimals. Ten decimals are necessary for the most exact astronomical calculations, and four to seven decimals are sufficient for those of mechanics and engineering.

The establishment of the theorem of **340** may justly be regarded as one of the greatest triumphs of the logical system.

The number π enters into a great many important mathematical expressions, or formulas, which have nothing whatever to do with a circle. The fact that π equals the ratio $\frac{c}{d}$ is really not its definition, but should be regarded merely as an incidental—and the most elementary—property (Ball's "Mathematical Recreations and Problems").

Additional Theorems

342. (1) *The apothem and radius of a regular triangle can be expressed in terms of the side.*

Helps.—(a) $r = \dfrac{s\sqrt{3}}{6} = .2887s;$

(b) $R = \dfrac{s\sqrt{3}}{3} = .5774s.$

(2) *The side of a regular triangle can be expressed in terms of (a) the apothem, and (b) the radius.*

Helps.—(a) $s = 2r\sqrt{3} = 3.4642r$:
(b) $s = R\sqrt{3} = 1.7321R.$

(3) *The central angle subtended by a side of a regular polygon is double the angle between a side and a least diagonal.*

(4) *The perimeter of the circumscribed equilateral triangle of a circle has a fixed ratio to the perimeter of the inscribed equilateral triangle.*

(5) *Find the ratio of the sides and perimeters of the inscribed and circumscribed squares of a circle.*

(6) *Find the ratio of the sides and perimeters of the inscribed and circumscribed regular hexagons of a circle.*

(7) *The side of a regular decagon whose radius is r, is,* $s = \dfrac{r}{2}(\sqrt{5} - 1).$

Helps.—By **332,** $\dfrac{r - s}{s} = \dfrac{s}{r}\cdot$ See also **316.**

343. The Fourth Step in Classification.

EXERCISES

1. Draw a regular inscribed polygon of 5 sides, in a circle whose radius is about 3 inches. Bisect one of the arcs subtended by a side of the polygon and join the bisection point with an end of the nearest side,thus obtaining a side of the regular inscribed polygon of 10 sides. Measure a side of each polygon. Calculate the side of the decagon from the side of the pentagon by using the formula of **338.** Does the calculated result agree with the measured value?
2. Use the figure of Exercise 1 and draw a regular circumscribed pentagon and one side of the regular circumscribed decagon. Measure a side of each polygon. Calculate the side of the decagon from that of the pentagon by using the formula of **339.** Does the calculated result agree with the measured value?

3. Construct accurate figures for all the Theorems of **342** and test the correctness of the formulas and other principles of the section by careful measurements of the figures. The figure of Exercise 1 may be used for the test of Theorems (3) and (7).

344. REVIEW EXERCISES

1. If a string 6 feet long is laid so as to form an equilateral triangle, what are the radii of the inscribed and circumscribed circles?
2. Can a cask which measures 9 feet in circumference be taken through a 3-foot door?
3. Calculate the length of an arc of 75° in a circle whose radius is 1200 feet.

   Helps.—(*a*) Use the formula of **337**, Corollary 3; (*b*) use the formula of **331**, the chords C and c being measured from a figure drawn to scale.
4. The length of an arc of 60° is 12.35 inches. Calculate the diameter of the circle.
5. The circumference of a circle is 10 feet. What is the circumference of a circle of double the radius?
6. The chord C of an arc of 60° in a circle whose radius is 10 feet, is 10. Calculate the length of an arc of 60° by the formula of **337**, Corollary 3. Also (*a*) calculate the chord c of one-half the arc by the formula of **331**; and (*b*) calculate this chord by the formula of **338**; and (*c*) find some other method of calculating this chord.
7. Find the diameter of a baseball or tennis ball by measuring the circumference.
8. Calculate the side of an equilateral triangle of which the apothem is 30.
9. Find the length of an arc of 60° in a circle whose radius is 30 inches (**337**, Corollary 3).
10. Find the length of an arc of 45° in a circle whose radius is 20 inches.
11. Find the length of an arc of 115° 35′ in a circle whose diameter is 50 inches.

    Help.—$\dfrac{n}{360} = \dfrac{115\frac{35}{60}}{360} = \dfrac{6{,}935}{21{,}600}$
12. The chord C of an arc of 60° in a circle whose radius is 30 inches is 30 inches. Calculate the chord of $\frac{1}{2}$ the arc from this value and the result obtained for the length of the arc in Exercise 9; (*a*) using the formula of **331**; (*b*) using the formula of **338.**
13. If the diagonal of a square whose side is 5, is 7.07, what is the diagonal of a square whose side is 8? What is the side of a square whose diagonal is 8 (**335**, Postulate 5)?

Applications

345. The Design of a Running Track.—The best and usual shape of a running track is that of a rectangle with semi-circular ends (Fig. 1). The form of Fig. 2 may be used when the available ground space does not permit the use of the better form. The *standard line* on which the length of the track is calculated is indicated by the dotted line, and is $1\frac{1}{2}$ feet from the inner edge of the track.

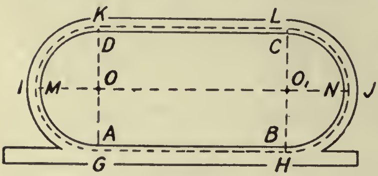


Fig. 1.

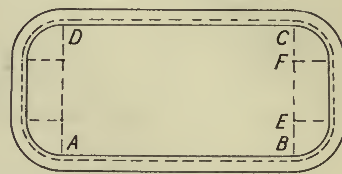


Fig. 2.

EXERCISES

1. Calculate the total length of a track if $AB = 200$ feet and $BC = 200$ feet (Fig. 1).
   Help.—The two ends form a complete circle 203 feet in diameter.
2. Calculate the total length of a track of the form of Fig. 2, if $AB = 200$ feet, $BC = 200$ feet, and the radius of the corner turns, measured to the standard line, = 50 feet.
   Helps.—$EF = 203 - 2 \times 50 \doteq 103$ feet.
3. What must be the length of AB for a quarter-mile track if GK (Fig. 1) = 200 feet?
4. What must be the value of BC for an eight-laps-to-the-mile track, in which $AB = 100$ feet (Fig. 1)? Find the over-all length, IJ, to the standard line.
5. The over-all length, IJ, of a quarter-mile running track is 400 feet. Find AB and GK (Fig. 1).
   Helps.—Let $x = GK$; then $AB = 400 - x$; $\therefore$ length of track is $2(400 - x) + 3.1416x = \frac{5280}{4}$.
6. How many feet more will a runner travel in one lap of the quarter-mile track of Exercise 3, if he runs 3 feet from the inner edge of the track, than he would travel on the standard line?

346. Power Transmission and Shop Practice.—The simplest forms of power transmission is by gear wheels and belts. The *velocity ratio* of the driving shaft A, and the

driven shaft B, is the ratio of the circumferences of the *pitch circles* of the two gear wheels; since the point of contact of the teeth of the gear wheels is at the point of tangency of these circles; Fig. 1. The velocity ratio of the two shafts C and D is the ratio of the circumference of the driving pulley C and the driven pulley D, around which the belt passes; Figs. 2 and 3.

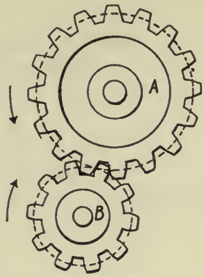


FIG. 1.

EXERCISES

1. The diameter of gear wheel A (Fig. 1) is 12 inches, and that of gear wheel B is 8 inches. The number of revolutions per minute of shaft A is 60. How many revolutions per minute does shaft B make?
2. What is the velocity ratio of the first and third shafts of a train of three gears of 10, 12 and 8 inches diameter, respectively?

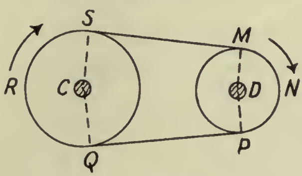


FIG. 2.

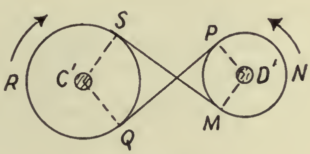


FIG. 3.

3. The driving gear of a bicycle, carried on the pedal shaft, is 8 inches diameter; the rear gear on the wheel axle is 3 inches diameter; the tire of the rear wheel is 28 inches diameter. How many revolutions per minute must the pedals make in order to drive the bicycle at the rate of 15 miles per hour? *Help.*—Number of revolutions per minute of pedal shaft =
$$\frac{5280 \times 15 \times 12}{60} \times \frac{1}{28 \times 3.1416} \times \frac{3}{8}.$$
4. If the diameters of the pulleys on both shafts in Fig. 2 are 24 inches, and the shafts are 10 feet apart, center to center, calculate the length of belt required.
5. In Fig. 2, the diameters of the pulleys are 36 inches and 18 inches, and the distance of the shafts is 8 feet, center to center. Draw the pulleys and the distance between centers to scale, measure the lengths $SM = PQ$, and the angles MDP and QCS, and

calculate the arcs *MNP* and *QRS* by the method of **337.** Find the total length of the belt.

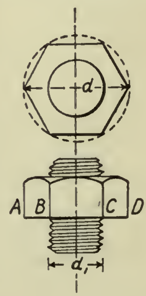
6. Use the dimensions of Exercise 5 to apply to the crossed belt of Fig. 3. Calculate the length of the belt.

The cutting speed of a tool or the velocity of some moving part of a machine must be known by those who design machinery.

EXERCISES

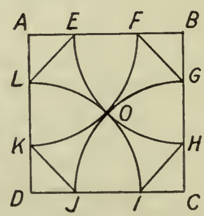
7. If the allowable cutting velocity of a tool when a piece of iron is turned in a lathe, is 40 feet per minute at the surface of the revolving piece, at how many revolutions per minute should a lathe be driven in order to produce this surface velocity for a piece of iron which is 5 inches in diameter? d inches in diameter?
8. At what velocity does the rim of a 10-foot diameter fly-wheel travel, if it makes 80 revolutions per minute?
9. If pulley C in Figs. 2 and 3 is 24 inches diameter and makes 50 revolutions per minute, what is the linear velocity of the belt in feet per minute?
10. If a belt is moving at a linear velocity of 500 feet per minute, and is driving a shaft by a 3-foot pulley, how many revolutions per minute is the driven shaft making?

Draughting and other applications.



EXERCISES

11. Show that in drawing a side view of a hexagonal nut, that $AB = CD = \frac{1}{4}d$, and $BC = \frac{1}{2}d$. In practice, d is made $2d_1$.
12. Draw a bolt whose diameter d_1 is $1\frac{1}{2}$ inches, and a hexagonal nut.
13. $ABCD$ is a square. With centers at A, B, C, D, arcs are drawn with a radius $= AO$, intersecting the sides in points E, F, G, etc. Show that the figure $EFGHIJKL$ is a regular octagon. This method is very useful in practice as a quick and accurate construction.



14. Draw a square side of which is 3 inches. Construct a regular octagon by the method of Exercise 13. Calculate the side of the octagon.

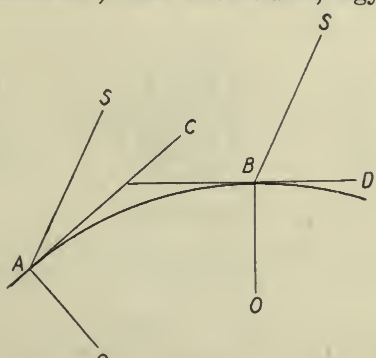
15. Show that a side of a square from which a regular octagon of side s, may be cut by the method of Exercise 13, is given by the formula, $s' = s(\sqrt{2} + 1)$. Also $s = s'(\sqrt{2} - 1)$.
16. Find the side of a square from which a regular octagon whose side is 2 inches may be cut.

347. Astronomical Problems.—All sections of the earth by a plane may be considered (except in the most accurate work) as circles.

EXERCISES

1. Taking the diameter of the earth as 41,780,000 feet, calculate the length in feet of one minute of arc on the equator or on a meridian, or on any "great circle."
2. The earth rotates on its axis in 24 hours. Calculate the velocity of a point on the equator in miles per hour.
3. The earth moves about the sun in an orbit that is approximately circular with a mean radius of 90,000,000 miles, in about 365 days. Calculate its velocity in the orbit in miles per hour.

The Size of the Earth.—The most accurate method of determining the size of the earth is by the measurement of an arc of a "great circle" on the earth's surface. A and B are two places on the same meridian; AC and BD are the horizontal planes at these points. Observations are made on a star S in the meridian plane of arc AB, the angles of elevation, CAS and DBS, being measured simultaneously. Eratosthenes made a fairly accurate calculation of the diameter of the earth by this method (using the sun as the object of observation) near Alexandria, Egypt, about 250 B. C.



EXERCISES

1. Show that $\angle AOB = \angle DBS - \angle CAS$, BS and AS being parallel.
2. If $\angle DBS$ is measured $67° 31'$, and $\angle CAS$ is measured $65° 43'$ find $\angle AOB$.
3. If arc AB is measured 124.5 miles, calculate the circumference and diameter of the earth in miles.

348. THE RADIAN.—This is a new unit of angle and arc. It is the central angle of a circle (or its arc) whose arc is the length of the radius. Do not confuse this with the central angle subtended by a chord equal to the radius. The Radian is not used in elementary geometry, but is an important unit in higher mathematics.

EXERCISES

1. Calculate the number of degrees in a radian.

   Help.—Radian $= \frac{360°}{2\pi}$.
2. What central angle is subtended by a chord equal to the radius?
3. How many radians are contained in an arc of 137°?

349. The Circle and Regular Polygon in Design.—Many ornamental designs for rugs, carpets, wall paper, china and glass-ware decorations, lace and embroidery, tiling, stone-tracery, etc., are based upon the regular polygon and its inscribed and circumscribed circles. Refer to a "Source Book of Problems for Geometry," by Sykes; Allyn and Bacon; for an analysis of numerous designs of this kind. See also **195** and **321.**



EXERCISES

1. Lay out some of these designs on squared paper, on a larger scale.
2. Make some original designs based upon regular polygons and circles.
3. Sketch some designs which you see in use. Draw the principle lines of the designs and omit details and scrolls.

Field Work

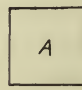
350. *Design and lay out a running track* which is 4, 6 or 8 laps to the mile. Refer to **345,** Fig. 1. The available rectangular plat should first be measured approximately and laid out on paper to scale. The shape and dimensions of the standard line is then determined and laid out on the plat. The point A is then located on the ground from a measurement scaled from the map; $AG = 1\frac{1}{2}$ feet. Lay out AB parallel to the side of the field. Erect a perpendicular to AB at point A, and set stakes at points O and O_1. In the same way set stakes at points D and C. The perpendiculars are most accurately laid out with a transit, but a tape may be used. Measure DC as a check. The half circles AMD and BNC are laid out by holding one end of the tape on stakes O and O_1 and carrying the end around, say, 4 paces at a time and driving stakes at each point on the curves thus found. Set stakes on lines AB and DC about 10 feet apart. The outside line of the track is laid out, say, 10 feet from the inside line. Points opposite A, B, C, D are first located. The arcs may be laid out from centers O and O_1; or if the radius is too great for the length of tape, stakes may be set by sighting the stakes which are already placed on the inner arcs, in line with the center hub stakes O and O_1.

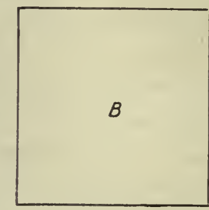
CHAPTER XVI

AREAS

MEASUREMENT OF AREAS. PRINCIPLES DETERMINED EXPERIMENTALLY

351. An area is a definite part of a surface. A *new unit* is required for the measurement of an area. *The unit of area* is the area, or surface, or extent, of a square, each side of which is a unit sect. Thus *unit A* is a *square centimeter*, and *unit B* is a *square inch.*





Equivalent areas contain the same area measure, but may be different in shape.

A *segment* and a *sector* of a circle are certain parts of the area of a circle. Refer to **267.**

An *annulus* is the area included between two concentric circles.

Only *plane areas* are considered in this text.

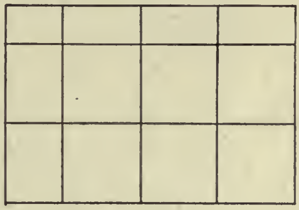
Abbreviation.—Equivalent; is equivalent to; $\Bumpeq$.

EXERCISES

1. What geometric units have been previously considered, for measuring: (*a*) sects; (*b*) angles; (*c*) arcs? Recall their definitions.
2. What unit, or units, are used to measure areas the size of this page? the area of a room? of a farm? of a country?
3. Write a table of equivalents between different area units.
4. Are equivalent areas necessarily congruent?
5. Draw a square, a circle, an oblong and an equilateral triangle which are approximately equivalent.
6. Add geometrically, a square 2 inches on a side and an oblong 2 inches by 3 inches.

7. Subtract geometrically, the areas of Exercise 6.
8. If two equal rectangles are added to or subtracted from two equal triangles, are the results equivalent? Are they necessarily congruent?
9. Draw an annulus of which the radii of the bounding circles are 1.2 and 1.7 inches, respectively.

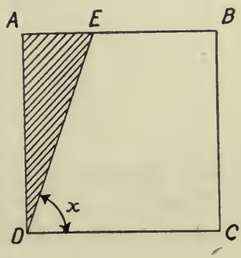
352. Experiment I.—The area of a rectangle.



Draw rectangles; (*a*) 2 inches by 3 inches, full size; (*b*) $2\frac{1}{2}$ inches by 3 inches; (*c*) $2\frac{1}{2}$ inches by $3\frac{1}{4}$ inches. Divide into square inches and fractions thereof; count the area measures in units and fractions. Observe the relation between the area measure of each rectangle and the sect measures of length and width. Squared paper may be used in this and other experiments in areas. State result.

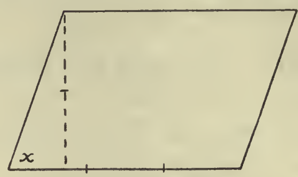
EXERCISES

1. State a formula for the area R of a rectangle, in terms of the width w, and the length l.
2. Solve the formula of Exercise 1 for w and for l.
3. Draw a square foot to one-half linear scale, and divide it into square inches to one-half linear scale. How many square inches are contained in the square foot?
4. Calculate the number of square inches (that is, the area) in a rectangle which is 15 inches by 8 inches.
5. How many square feet are in a rectangle 18 inches by 24 inches?
6. What is the length of a rectangle whose area is 750 square inches and width 15 inches?
7. Draw a square inch *ABCD;* cut it along a line *DE;* place the portion *DEA*, cut off, so that *AD* coincides with *BC*. Is the resulting figure equivalent to the original square inch? May this new figure also be called a *square inch*, since it contains an area equal to that of the given square inch?



353. Experiment II.—The area of an oblique parallelogram.

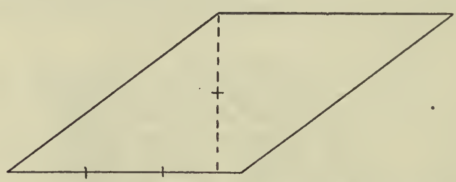
Draw rhomboids: (a) with base 3 inches and altitude 2 inches (full size); (b) with base $3\frac{1}{2}$ inches and altidude $2\frac{1}{2}$



inches. Make the angle x equal the angle x of **352,** Exercise 7. Divide the rhomboids into units and fractions thereof, of the shape of the unit referred to in **352,** Exercise 7. Observe the relation between the area measures and the dimensions of the figures. State result.

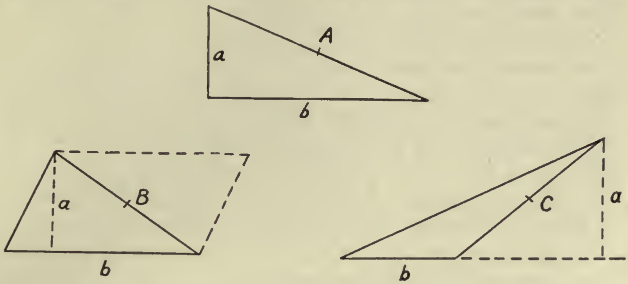
EXERCISES

1. State a formula for the area P of a rhomboid in terms of the base b, and the altitude a.
2. Solve the formula of Exercise 1 for a and b.
3. Calculate the area of an oblique parallelogram of which the base is 25 feet and altitude is 12 feet.
4. Perform Experiment II by cutting the rhomboid on some altitude, and placing the portion cut off in such a position as to form a rectangle. Obtain the area measure of this rectangle, which must equal that of the given parallelogram.
5. If a rhomboid is drawn with the sides including any other angle than the angle x selected for the figures of Experiment II, would the result be the same?



354. Experiment III.—The area of a triangle.

Draw triangles of different shapes. Trace on thin paper; revolve the traced figures on the points A, B, C, respectively, which are the middle points of one side of the triangles, and draw the symmetrical impressions of the triangles as indicated in the second figure. What figure is formed in each



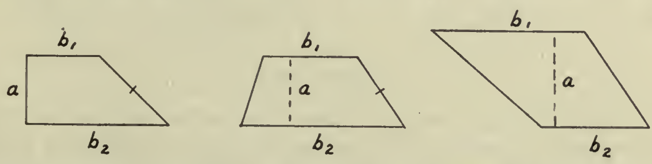
case by two adjoining congruent triangles? What is the area of the complete figure? What is the area of the single given triangle? State result.

EXERCISES

1. Write a formula for the area T, of a triangle. Solve for each letter.
2. Calculate the area of a triangle of which $a = 8''$, $b = 10''$.
3. Calculate the base and altitude of a triangle whose area = 40 square inches, and of which the base exceeds the altitude by 2 inches.
4. Two sides of a triangle are 20 and 36, and the included angle is 45°. Calculate the area. *Help.*—First find the altitude.
5. Perform Experiment III by dividing the given triangles into square units and fractions thereof. Use squared paper. This method employs no analysis and is purely experimental.

355. Experiment IV.—The area of a trapezoid.

Draw trapezoids; trace; revolve on the middle point of one of the non-parallel sides. What figure is formed by two



adjoining trapezoids? What is the area of this figure? What is therefore the area of the trapezoid?

EXERCISES

1. State the result of Experiment IV as a formula. Solve for each letter.
2. The dimensions of a trapezoid are $b_1 = 10$, $b_2 = 18$, $a = 8$. Calculate the area, S.
3. The bases of a trapezoid are 15 and 25; one of the non-parallel sides is 10; and its projection on a base is 6. Calculate the area.
4. In a trapezoid, $b_1 = 120$, $b_2 = 190$, *a side* $= 100$, the angle included between the 100 side and the 190 base $= 60°$. Calculate the area.
5. The area of trapezoid is 400 square units, the altitude is 30 units, one base is 10 units. Find the other base.
6. Perform Experiment IV by dividing the given trapezoids into square units and fractions thereof.

356. Experiment V.—The area of a circle.

Draw a circle about 4 inches in diameter and cut it out. Cut it on a diameter AB. Cut each semicircle into the same number of equal sectors; say into 8 sectors. Place the sectors as shown in Fig. 2. How may the resulting figure be made to approximate more and more closely to a rectangle as in Fig. 3? What are the base and altitude of this rectangle? What is therefore the area of a circle?

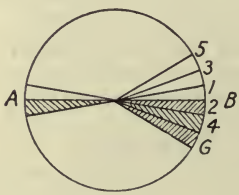


Fig. 1.

Fig. 2.

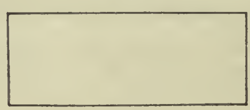
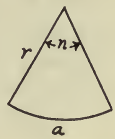


Fig. 3.

EXERCISES

1. State the area C as a formula in terms of r (or d) and c.
2. State the value of C in terms of r alone (except that π will enter); in terms of d; in terms of c. Refer to **337** Corollary 2.
3. Solve the formulas of Exercise 2 for r, d and c, respectively.
4. Calculate the area of a circle whose radius is 10″.
5. Calculate the area of a circle whose circumference is 10″.

6. Calculate the radius of a circle whose area is 7854 square inches.
7. Perform Experiment V by drawing a circle on squared paper and counting the area measure.
8. Show by a method similar to that employed in Experiment V, that the area of a sector equals one-half the product of the arc and the radius. Express this as a formula.
9. Express the area of a sector in terms of its central angle, n degrees, and its radius r.



   Help.—Arc $a = \frac{n}{360} \cdot 2\pi r$.
10. Calculate the area of a sector whose radius is 10″ and of which the length of arc is 12″.
11. Calculate the area of a sector whose radius is 20″ and central angle 70°.

357. EXPERIMENT VI.—THE RATIO OF TWO AREAS. Draw two rectangles having different bases and altitudes. Divide them into square inches and fractions thereof. Find the area measure of each rectangle and express as a ratio.

Divide the two rectangles into any other arbitrary square unit s, and again find their area measure in terms of this unit, and express as a ratio. Observe how the ratios compare. State result.

EXERCISES

1. A's farm contains twice as many square feet as B's farm contains. Does A's farm also contain twice as many square kilometers as B's farm?
2. If two figures are drawn to some part-size scale, will the ratio of the areas of the figures be the same as in the full-size figures?

358. EXPERIMENT VII.—THE RATIO OF THE AREAS OF SIMILAR FIGURES.

(a) Draw a rectangle. Draw other similar rectangles with ratios of similitude to the first rectangle of 2 : 1; 3 : 1; ½ : 1; respectively: that is, the length and width of the second rectangle are each double the corresponding dimensions of the first rectangle; etc. Divide these rectangles into smaller rectangles of the size and shape of the first rectangle, and fractions thereof. Observe how the ratios of the areas compare with the ratios of similitude of the figures.

(*b*) Draw similar triangles. Divide the larger triangles into smaller triangles of the size and shape of the smallest triangle of the series. With this triangle as a unit of measure, observe how the ratios of the areas of the triangles compare with their ratios of similitude. State the general result.

EXERCISES

1. State the result as a formula in terms of the areas S and S', and a ratio of similitude $\frac{a}{a'}$ or $\frac{b}{b'}$.
2. The area of a rectangle is 5 square inches. What is the area of a similar rectangle whose ratio of similitude to the first is 10 : 1?
3. The area of one rectangle is 36 times the area of a similar rectangle. The base of the larger rectangle is 12 feet. What is the base of the smaller rectangle?
4. The area of an equilateral triangle is 10 square inches. What is the side of an equilateral triangle whose area is 40 square inches, expressed in terms of the side s, of the first triangle?
5. Show by drawings that the law of the ratio of the areas of two similar figures holds for, (*a*) parallelograms, (*b*) trapezoids.
   Help.—In the figures of trapezoids, a cutting and re-uniting of the figure is necessary.

359. REVIEW EXERCISES

1. Is it correct to say that a number of inches multiplied by a number of inches results in a number of square inches? Can two denominate numbers be multiplied? If a denominate number is multiplied by an abstract number, what kind of number results?
2. Show that the curved (convex) surface of a circular cylinder, bounded by parallel planes perpendicular to the axis, is a rectangle. State a formula for the area of the convex surface. For the total area.
3. Show that the convex surface of a right circular cone is a sector. What is its arc? Its radius? State a formula for the area of the convex surface. For the total area.
4. The circumference of a tree is 80 inches. Find the area of cross-section.
5. A horse is tied to a stake by a rope 50 feet long. Over what area can he graze?

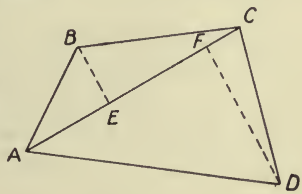
6. A horse is tied to a stake in the corner of a square court by a rope 50 feet long. Over what area can he graze?
7. A horse is tied to a stake at the corner of a square building. The rope is 100 feet long, and the building is 50 feet square. Over what area can he graze?
8. How many square feet will be enclosed by a rope 120 feet long, if it is laid down on the ground (*a*) in the form of a square; (*b*) in the form of an equilateral triangle; (*c*) in the form of a circle?
9. What may we conclude is the greatest (or maximum) area for a given perimeter, based upon the results of Exercise 8?
10. What part of a circle is the area of a sector of 110°? Of 86° 32′?

Applications

360. Surveying.—Land is generally valued and sold in terms of its area measure. City lots are sometimes rated in terms of the "front foot."

EXERCISES

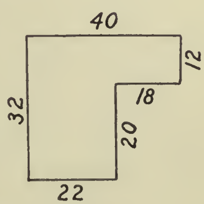
1. A town building lot is 120 feet by 80 feet. If it is sold for $1200, what price is obtained per square foot?
2. A farm is bounded by four straight lines. A diagonal AC is 560.75 feet; the distances BE and DF, perpendicular to AC, are respectively 93.44 feet and 612.20 feet. Find the area of the farm in square feet. (The figure is not drawn to scale.)



361. Estimating.—Estimates for painting, roofing, paving, flooring, wall paper, carpets, etc., are made in terms of the areas involved.

EXERCISES

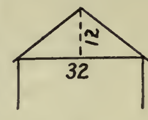
1. Find the cost of laying a cement pavement on the court of which the plan is given, at $2.50 a square yard. The dimensions are in feet.



2. A room is 20 feet long, 12 feet wide, and 10 feet high. Find the cost of plastering the walls and ceiling of the room at 20 cents a square foot, deducting 25 percent of the area for doors, windows and woodwork.

3. What is the cost of laying an asphalt pavement per mile of road, 14 feet wide, at $4.50 per square yard?

4. A barn roof is 32 feet wide at the eaves, 12 feet high to the ridge, and 40 feet long. Allow two feet for overhang at the eaves. Find the cost of roofing with tar paper at 5 cents a square foot, which allows for overlapping and waste.



5. How many sheets of paper 8 by 11 inches, can be cut from 100 sheets 24 by 44 inches?

362. Problems for Field Work.—(1) Find the area covered by a building all of the angles of which are right angles, by measuring all horizontal dimensions. Draw a plan and divide the area into rectangles.

2. Find the area covered by any existing or proposed paths, roads, pavements, etc. The cost may be estimated by procuring a trade price of the required pavement per square foot or square yard.

CHAPTER XVII

AREAS

Classification and Explanation of Principles

363. The First Step in Classification. *Area Postulates.*—(1) If an area is divided and the portions are placed in any other relative position without overlapping, the resulting area is equivalent to (that is, equal in measure to) the original area.

2. The unit of area is a unit of the same kind, and is a square (or its equivalent) each side of which is a unit sect.

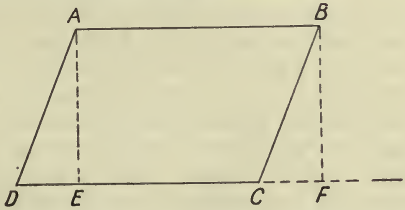
3. The area of a rectangle is the unit of measure multiplied by the product of the length and width of the rectangle; or, it equals the product of the length and width (or base and altitude).

4. The area of a circle is the limit which the areas of inscribed and circumscribed regular polygons approach, as the number of sides is indefinitely increased.

5. The areas of two similar rectangles are proportional to the squares of two homologous sides.

The Second Step in Classification

364. Theorem I.—*The area of an oblique parallelogram equals the product of the base and altitude.*



Hypothesis.—An oblique parallelogram $ABCD$, with base $DC = b$, and altitude $AE = a$.

Conclusion.—Area $ABCD = ab$.

Analysis.

STATEMENT	REASON
1. Extend DC and draw $BF \perp DC$	Postulate
2. $\triangle ADE \cong \triangle BCF$	Postulate: 2 sides and include angle =, etc.
3. $DC = EF = b$	Axiom.
4. Area $ABFE \approx$ Area $ABCD$	**363**; Postulate 1.
5. Area $ABFE = ab$	**363**; Postulate **3**.
6. $\therefore$ Area $ABCD = ab$	Axiom.

Corollaries.—(1) *The areas of two parallelograms which have the same base and different altitudes are proportional to the altitudes.*

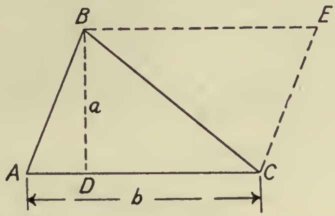
2. State a similar corollary for parallelograms which have different bases and the same altitude.

EXERCISES

1. Are all rectangles which have their bases and altitudes equal, each to each, congruent?
2. Are all oblique parallelograms having their bases and altitudes equal, each to each, congruent? Are they equivalent?
3. Write a formula expressing the ratio of the areas, P and P', of two rectangles or two oblique parallelograms, whose bases and altitudes are respectively b and a, and b' and a'.
4. Write a formula expressing the ratio of the areas of any two parallelograms whose bases are each equal to b, and whose altitudes are respectively a and a'.
5. Write a formula similar to that of Exercise 4 if the parallelograms have equal altitudes a, and bases b and b' respectively.
6. Write a formula for the area S, of a square whose side is s.
7. Write a formula for the side s of a square whose area is S.
8. The base and altitude of two parallelograms are respectively $12''$ and $36''$, and $12''$ and $18''$. What is the ratio of their areas?
9. Find the area of a rectangle of which the base is 10 inches and the diagonal is 12 inches.
10. The sides of an oblique parallelogram are $10''$ and $15''$; the projection of the shorter side on the longer side is $6''$. Calculate the area. Construct the parallelogram.

11. Show that the area of the oblique parallelogram of Theorem I equals the side AD multiplied by the altitude drawn to that side.
12. Prove by similar triangles that the product ab = the product of AD by the altitude drawn to AD.

365. Theorem II.—*The area of a triangle equals one-half the product of the base and altitude.*



Helps.—(1) Draw $BE \| AC$ and $CE \| AB$; (2) area $BECA = ab$; (3) $\therefore$ area $\triangle ABC = \frac{1}{2}\, ab$.

Corollaries.—(1) *The areas of two triangles which have the same base and different altitudes are proportional to the altitudes.*

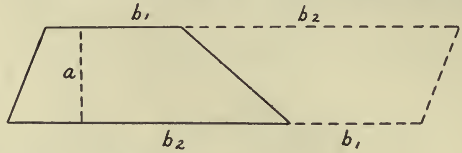
2. State the Corollary which corresponds to **364,** Corollary 2.

EXERCISES

1. Show that the area of the triangle ABC may be expressed as one-half the product of any side by the altitude drawn to that side.
2. Show that the area of a right triangle equals one-half the product of the perpendicular sides.
3. Write formulas expressing the ratios between the areas of two triangles T and T', the relations between the dimensions of which correspond to those stated for parallelograms in **364,** Exercises 3, 4 and 5.
4. Find the area of a triangle of which two sides are 12″ and 20″, and the projection of the shorter side on the longer is 8″.
5. The mathematical treatise of Ahmes gives the area of an isosceles triangle whose sides are 10, 10, 4, as 20. What formula did Ahmes employ? What is the correct area? When is Ahmes' formula approximately correct?
6. Find the area of a triangle of which two sides are 20 and 35 and the included angle is 90°.

7. Find the area of a triangle of which two sides are 20 and 35 and the included angle is 45°.
8. Find the area of a triangle of which two sides are 20 and 35 and the included angle is 135°.

366. Theorem III.—*The area of a trapezoid equals one-half the product of the sum of the bases by the altitude.*



Write out the analysis.

EXERCISES

1. Find the area of a trapezoid whose bases are 20″ and 35″ and whose altitude is 15″.
2. Find the other base of a trapezoid of which the area is 750 square feet, the known base 30 feet, and altitude 40 feet.
3. Find the altitude of a trapezoid of which the area is 1200 square feet, and the bases are 30 and 20 feet, respectively.
4. The method used by the ancient Egyptians for finding the area of a piece of land in the form of a trapezoid was to multiply half the sum of the parallel sides by one of the other sides. Is this method ever correct? When is it approximately correct?

The Third Step in Classification

367. Theorem IV.—*The area of a triangle can be expressed in terms of the three sides.*

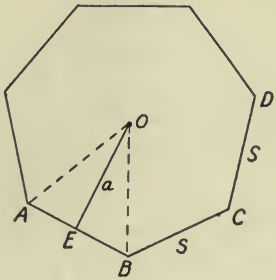
Helps.—Combine the result of Theorem II, **365**, and the value of an altitude, **255**.

$$T = \sqrt{s(s-a)(s-b)(s-c)}.$$

EXERCISES

1. Find the area of a triangle whose sides are 10, 12 and 14.
2. Find the area of the isosceles triangle of **365**, Exercise 5, by the formula of this section.
3. Find the area of a right triangle whose sides are 30, 40 and 50 by the formula of this section. Check by the shorter method of **365**.

368. Theorem V.—*The area of a regular polygon equals one half the product of the perimeter by the apothem.*



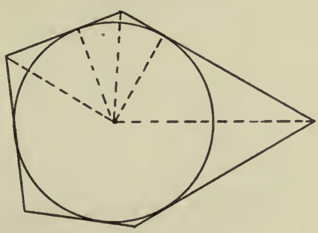
Write out the analysis.

Corollary.—*If* s = *a side*, n = *the number of sides*, a = *the apothem, then* $P = \frac{nas}{2}$.

EXERCISES

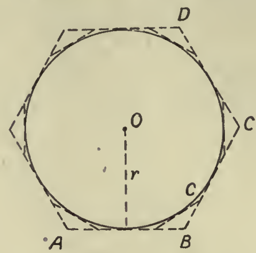
1. Find the area of a regular octagon of which a side is 24.85 inches and the apothem 30 inches.
2. Find the area of a regular hexagon whose side is 40.
3. Use the formula of Theorem V to find the area of a square whose side is 40. Check by the formula $S = ab$ (**363,** Postulate 3).

369. Theorem VI.—*The area of any circumscribed polygon equals one-half the product of the perimeter by the radius of the inscribed circle.*



Write out the analysis.

370. Theorem VII.—*The area of a circle equals one-half the product of the circumference and radius.*



Hypothesis.—Circle O, with radius r, and circumference c.

Conclusion.—Area of circle, $S = \frac{rc}{2}$.

Analysis.

Statement	Reason
1. Circumscribe any regular polygon $ABCD$ . . . , about circle O, with an area P, and a perimeter p.	
2. $P = \frac{rp}{2}$	**369**; Theorem VI.
3. Indefinitely increase the number of sides of polygon $ABCD$ . .	
4. Area $P \doteq$ Area S	**363**; Postulate 4
5. And $p \doteq c$	**335**; Postulate 4(c)
6. $\therefore \frac{rp}{2} \doteq \frac{rc}{2}$	**335**; Limit Axiom 3
7. But P always $= \frac{rp}{2}$, and $\therefore S = \frac{rc}{2}$	**335**; Limit Axiom 2

Corollaries.

1. $S = \pi r^2$ or $3.1416r^2$.
2. $S = \frac{\pi d^2}{4}$ or $.7854d^2$.
3. $S = \frac{c^2}{4\pi}$ or $.0796c^2$.

4. The areas of two circles are proportional to the squares of their radii, diameters or circumferences.

EXERCISES

1. The formula given by Ahmes for the area of a circle is $S = \frac{8}{9}d^2$. Show that this is equivalent to a value of $\pi = 3.16$.
2. Is it probable from the evidence of the formula itself that Ahmes arrived at this measure by comparing areas experimentally, or by deductive reasoning?
3. Calculate the radius, diameter, and circumference of a circle whose area is 78.54 square inches; of a circle whose area is 1200 square inches.
4. Find the area of a sector of 10° in a circle of radius 30.
5. Find the area of a sector whose arc equals twice the radius, in a circle of 12″ radius.

371. Theorem VIII.—*The radius of the inscribed circle of a triangle may be expressed in terms of the three sides.*

Hypothesis.—$\triangle ABC$; the inscribed circle with center O and radius r.

Conclusion.—r may be expressed in terms of the sides a, b, c, of the triangle.

Analysis.—(1) Draw OA, OB, OC; (2) Area $\triangle BOC = \frac{a \cdot r}{2}$, etc.; (3) Area $\triangle ABC = \frac{a+b+c}{2} \cdot r = s \cdot r$; (4) But area $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$; (5) $\therefore$ $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

EXERCISES

1. Calculate the radius of the inscribed circle of a triangle whose sides are 3, 4 and 5.
2. Calculate the area of the triangle of Exercise 1, using the formula $S = sr$; and the value of r found in Exercise 1.

Additional Theorems

372. (1) *Two triangles of which two sides are equal, each to each, and the included angles supplementary, are equivalent; that is, equal in area.*

(2) *A median divides a triangle into two equivalent triangles.*

(3) *The area of a trapezoid equals the product of the altitude by the median.*

(4) *The area of any quadrilateral equals one-half the product of a diagonal by the sum of the perpendiculars drawn to the diagonal from the two opposite vertices.* Write the formula.

(5) *If the diagonals of a quadrilateral are perpendicular, the area is* $S = \frac{d_1 d_2}{2}$, *where* d_1 *and* d_2 *are the diagonals.*

(6) *The area of a rhombus equals one-half the product of the diagonals.*

(7) *The area of a square is* $S = \frac{d^2}{2}$; *where* d *is the diagonal.*

(8) *The diagonal of a square may be expressed in terms of the area.*

(9) *The area of a circumscribed square is double that of the inscribed square of the same circle.*

(10) *The area of an equilateral triangle is* $T = \frac{s^2\sqrt{3}}{4}$; *where* s = *a side of the triangle.*

Helps.—(*a*) Derive the formula from the area formula, $T = \frac{ab}{2}$, (**365**); (*b*) Derive it from $T = \sqrt{s(s-a)(s-b)(s-c)}$, (**367**), where s = one-half the sum of the sides of the triangle.

(11) *The side of an equilateral triangle may be expressed in terms of the area.*

Help.—$s = \frac{2\sqrt{3T\sqrt{3}}}{3}$

(12) *The area of a regular hexagon is* $H = \frac{3s^2\sqrt{3}}{2}$; *where* s *is a side of the hexagon.*

(13) *The side of a regular hexagon may be expressed in terms of the area.*

(14) *The area of an annulus may be expressed in terms of the radii, or of the diameters, of the bounding circles.*

Helps.—(1) $S = S_1 - S_2$; (2) $S_1 = \pi r_1^2$ and $S_2 = \pi r_2^2$; (3) Subtract and factor; $S = \pi(r_1 + r_2)(r_1 - r_2)$.

(15) *The area of a triangle may be expressed in terms of the three sides and the radius of the circumscribed circle.*

Helps.—Refer to **308** and **367.**

$T = \dfrac{abc}{4R}$.

(16) *The radius of the circumscribed circle of a triangle may be expressed in terms of the area and sides of the triangle.*

(17) *The area of a minor segment of a circle is* $S = \dfrac{x}{360} \cdot \pi r^2 - \dfrac{pc}{2}$; *where x is the central angle of the segment, r the radius of the circle, c the length of the chord of the segment, and p the perpendicular upon the chord from the center of the circle.*

Help.—Area of segment = area of sector − area of triangle.

(18) *State the formula for the area of a major segment of a circle.*

(19) *The difference of the areas of a circle and the circumscribed square may be expressed in terms of the radius or diameter of the circle.*

(20) *The different of the areas of a circle and the inscribed square may be expressed in terms of the radius or diameter of the circle.*

373. EXERCISES

1. If the area of a triangle of which two sides and the included angle are respectively 37″, 51″ and 65°, is approximately 216 square inches. What is the area of a triangle of which two sides and the included angle are respectively 37″, 51″, and 115°?
2. Check the value given in Exercise 1 by constructing one of the triangles and measuring an altitude.
3. A diagonal of a quadrilateral is 27″ and the perpendiculars drawn to this diagonal from the other two vertices are respectively 10″ and 45″. Calculate the area of the quadrilateral.
4. The diagonals of a rhombus are 12″ and 20″. Calculate the area.

5. Find the diagonal of a square whose area is 3200 square inches.
6. The area of a square circumscribed about a circle is 240 square inches. Find the area of a square inscribed in the same circle.
7. Find the area of an equilateral triangle whose side is 20 inches, using the formula of **372**, Theorem (10.)
8. The area of an equilateral triangle is 100 square inches. Find the side of the triangle.
9. Find the area of a regular hexagon inscribed in a circle whose radius is 12 inches.
10. Find the area of an annulus whose radii are 10″ and 20″ respectively.
11. The radius of one of the bounding circles of an annulus is 24 inches and the area of the annulus is 1200 square inches. Find the radius of the other bounding circle (*a*) if the given circle is the smaller of the two circles; and (*b*) if the given circle is the larger of the two circles.
12. Find the area of the triangle of **308,** Exercise 1, using the formula of **372,** Theorem (15.)
13. Find the area of a segment of 120° in a circle whose radius is 12 inches.
14. Find the difference in the areas of a circle whose radius is 20 inches, and its inscribed square, using the formula of **372**, Theorem (20.)

The Fourth Step in Classification

374. EXERCISES

1. Construct the triangle of **367**, Exercise 1, to a smaller scale. Measure an altitude. Calculate the area by the formula $S = \frac{1}{2}ab$ (**365**). Compare the result with the value obtained in the Exercise referred to.

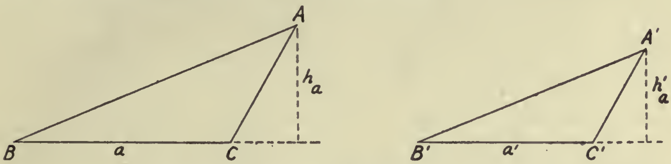
   Exercises 2 and 3, **367,** also are checks on the formula of **367.**
2. Construct the triangle of **371,** Exercise 1, and the inscribed circle. Measure the radius. Compare with the value calculated in the Exercise referred to.
3. Construct the inscribed and circumscribed squares of the same circle. Measure a side of each, and calculate the areas. Does the result agree with the principle of **372,** Theorem (9)?
4. Construct an equilateral triangle whose side is 20. Draw and measure an altitude and calculate the area by the formula of **365.** Does the result agree with the value obtained in **373,** Exercise 7?
5. Construct a triangle whose sides are 10, 12 and 14 (Exercise 1). Construct the circumscribed circle. Measure the radius. Cal-

culate the area by the formula of **372,** Theorem (15). Does the result agree with the values obtained in **367,** Exercise 1, and in Exercise 1 of this section?

Relations Between Areas

The Second Step in Classification

375. Theorem IX.—*The areas of two similar triangles are proportional to the squares of any two homologous sides; or, to the square of their ratio of similitude.*



Hypothesis.—Similar triangles ABC and $A'B'C'$.

Conclusion.—$\dfrac{\text{Area } \triangle ABC}{\text{Area } \triangle A'B'C'} = \dfrac{a^2}{a'^2} =$ etc.

Analysis.

Statement	Reason
1. Area $\triangle ABC = \dfrac{h_a \cdot a}{2}$	Theorem
2. Area $\triangle A'B'C' = \dfrac{h_a' \cdot a'}{2}$	Theorem
3. $\dfrac{\text{Area } \triangle ABC}{\text{Area } \triangle A'B'C'} = \dfrac{h_a \cdot a}{h_a' \cdot a'}$	Axiom
4. $\dfrac{h_a}{h_a'} = \dfrac{a}{a'}$,	Theorem of $\sim$ ⧍
5. $\dfrac{\text{Area } \triangle ABC}{\text{Area } \triangle A'B'C'} = \dfrac{a^2}{a'^2}$	Axiom

Corollaries.—(1) *The areas of two similar triangles are proportional to the squares of homologous altitudes, medians, or of any two homologous lines.*

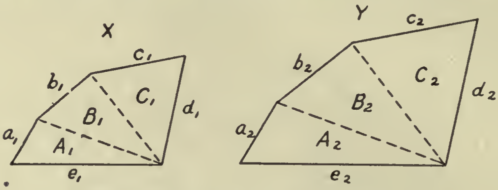
2. *Any homologous lines (sides, altitudes, etc.) of two similar triangles are proportional to the square roots of the areas of the triangles.*

EXERCISES

1. The sides of two triangles are respectively 7, 10 and 14, and $10\frac{1}{2}$, 15 and 21. What is the ratio of the areas of the triangles?
2. The area of a triangle is four times the area of a similar triangle. What is the ratio of two homologous sides of the triangles?

The Third Step in Classification

376. Theorem X.—*The areas of two similar polygons are proportional to the squares of two homologous sides.*



Hypothesis.—Similar polygons X and Y.

Conclusion.—$\dfrac{X}{Y} = \dfrac{a_1^2}{a_2^2} = \dfrac{b_1^2}{b_2^2} =$ etc.

Analysis.

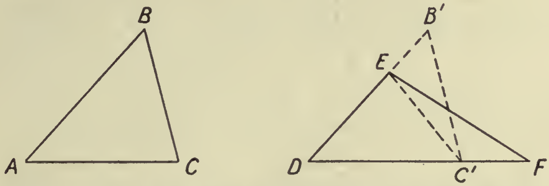
Statement	Reason
1. Divide X and Y into triangles similarly placed, A_1, B_1 . . . A_2, B_2 . . .	
2. $\triangle A_1 \sim \triangle A_2$; $\triangle B_1 \sim \triangle B_2$, . . .	Theorem of $\sim$ triangles.
3. $\dfrac{A_1}{A_2} = \dfrac{a_1^2}{a_2^2}; \dfrac{B_1}{B_2} = \dfrac{b_1^2}{b_2^2};$ . . .	**375**; Theorem IX.
4. $\dfrac{a_1^2}{a_2^2} = \dfrac{b_1^2}{b_2} =$ . . .	
5. $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} =$ . . .	
6. $\dfrac{A_1 + B_1 + \ldots}{A_2 + B_2 + \ldots} = \dfrac{A_1}{A_2} = \dfrac{a_1^2}{a_2^2} = \dfrac{b_1^2}{b_2} =$ . . .	
7. $\dfrac{X}{Y} = \dfrac{a_1^2}{a_2^2} = \dfrac{b_1^2}{b_2^2} =$ . . .	

State corollaries similar to those of **375**.

EXERCISES

1. The area of a pentagon is 50 square inches; the ratio of similitude of a second pentagon similar to the first pentagon, is 9:1. Find the area of the second pentagon.
2. The areas of two similar quadrilaterals are 50 and 100. A side of the first quadrilateral is 10; what is the homologous side of the second quadrilateral?

377. Theorem XI.—*The areas of two triangles which have an angle of one triangle equal to an angle of the other, are proportional to the products of the sides including the equal angles.*



Hypothesis.—$\triangle ABC$ and DEF with $\angle A = \angle D$.

Conclusion.—$\dfrac{\text{Area } \triangle ABC}{\text{Area } \triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$.

Analysis.

Statement	Reason
1. Place $\triangle ABC$ in the position $B'DC'$ and draw EC'.	
2. $\dfrac{\text{Area } \triangle DB'C'}{\text{Area } \triangle DEC'} = \dfrac{DB'}{DE} = \dfrac{AB}{DE}$	**365**; Corollary
3. $\dfrac{\text{Area } \triangle DEF}{\text{Area } \triangle DEC'} = \dfrac{DF}{DC'} = \dfrac{DF}{AC}$	**365**; Corollary
4. $\dfrac{\text{Area } \triangle ABC}{\text{Area } \triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$	Axiom

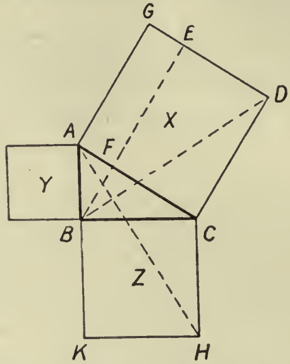
Corollary.—*The areas of two triangles which have an angle of one supplementary to an angle of the other, are proportional, etc.*

EXERCISES

1. The area of a triangle, two of whose sides are 10 and 12 and of which the included angle is 60°, is 103.92. What is the area of a triangle, two of whose sides are 5 and 30 and of which the included angle is 60°?

2. What is the area of a triangle, two of whose sides are 15 and 25 and of which the included angle is 120°?

378. Theorem XII.—*The area of the square drawn on the hypotenuse of a right triangle equals the sum of the areas of the squares drawn on the two perpendicular sides.*



State the *hypothesis* and the *conclusion.*

Helps.—(1) Rectangle $CDEF \approx 2$ times $\triangle BCD$; (2) Square $Z \approx 2$ times $\triangle ACH$; (3) $\triangle BCD \cong \triangle ACH$; (4) $\therefore$ Rectangle $CDEF \approx$ square Z; etc.

This *theorem* has been analyzed before under similar triangles, but there is an essential difference in the method employed here and that of the former analysis. The analysis here given was employed by Euclid.

More than three hundred different analyses of the theorem have been discovered.

The Fourth Step in Classification

379. EXERCISES

1. Draw an irregular hexagon; draw a second hexagon similar to the first, with the ratio of similitude 1 : 2. Divide the hexagons into triangles in any way, and measure a side and an altitude to that side in each triangle. Calculate the areas of each hexagon by adding the areas of the triangles into which it is divided. How does the ratio of the areas of the hexagons compare with their ratio of similitude? Does this agree with the Theorem of **376?**

2. Draw any oblique scalene triangle; draw a second triangle with an angle equal to an angle of the first triangle. Measure a side and an altitude to that side in each triangle and find the areas by the formula of **365.** Find also for each triangle the products of the sides including the equal angles. Compare the ratio of the areas with the ratio of these products, by reducing each ratio to a decimal. Does the result agree with the principle of **377?**

Additional Theorems

380.—(1) *The square on the base of a right isosceles triangle is equivalent to four times the triangle.*

(2) *The four triangles into which the diagonals divide a parallelogram are equivalent.*

(3) *Straight lines drawn from the middle point of a diagonal of a quadrilateral to opposite vertices divide it into two equivalent parts.*

(4) *The parallelogram formed by joining the middle points of each two adjacent sides of a quadrilateral is equivalent to one-half the quadrilateral.*

(5) *A line drawn through the middle point of the median of a trapezoid, cutting the bases, divides the trapezoid into two equivalent parts.*

(6) *The triangle formed by joining the middle point of one of the non-parallel sides of a trapezoid to the extremities of the other non-parallel side is equivalent to one-half the trapezoid.*

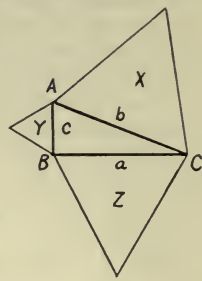
(7) *The sects joining the middle points of the three sides of a triangle divide it into four equivalent (congruent) triangles.*

(8) *The areas of two parallelograms having an angle of one equal to an angle of the other are proportional to the products of the sides including the equal angles.*

(9) *The sum of the areas of the squares on the sects drawn from any point within a rectangle to two opposite vertices equals the sum of the areas of the squares on the sects drawn from the point to the other two opposite vertices.*

Helps.—Draw lines through the point parallel to the sides of the rectangle, forming right triangles.

(10) *The area of the equilateral triangle drawn on the hypotenuse of a right triangle equals the sum of the areas of the equilateral triangles drawn on the other two sides.*



Helps.— (1) $\frac{Y}{X} = \frac{c^2}{b^2}$; $\frac{Z}{X} = \frac{a^2}{b^2}$; (2) $\frac{Y+Z}{X} = \frac{a^2+c^2}{b^2} = 1$; (3) $\therefore X = Y + Z$.

(11) State and analyze a general theorem for similar polygons, suggested by Theorem (10).

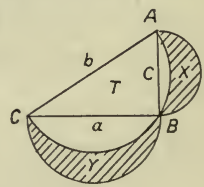
(12) *The area of a triangle of which one angle is* 30° *equals one quarter the product of the including sides.* Express as a formula.

(13) *The area of a triangle of which one angle is* 45° *equals one quarter the product of the including sides times the square root of two.* Express as a formula.

(14) Find the formula for the area of a triangle of which one angle is 60° and the including sides are b and c.

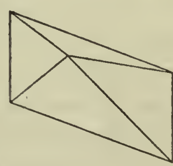
(15) Write the formulas for the areas of triangles of which two sides are b and c and the included angle is 150°, 135° or 120°.

(16) *If semicircles are drawn on the three sides of a right triangle, as in the figure, the sum of the crescents X and Y equals the area T of the given triangle.*



Helps.—(1) Write the formulas for the semicircles in terms of their diameters: (2) write the sum of the areas X and Y as an algebraic sum of T and the semicircles.

(17) *If lines are drawn from any point within a parallelogram to the four vertices, the sum of the areas of two opposite triangles thus formed is equal to the sum of the areas of the other two opposite triangles.*



(18) *A parallelogram is divided into two equivalent areas by a line drawn through the point of intersection of the diagonals.*

(19) *A trapezoid is divided into two equivalent areas by a line which joins the middle points of the bases.*

(20) *The area of a trapezoid equals the product of one of the non-parallel sides by the perpendicular drawn to that side from the middle point of the other of the non-parallel sides.*

(21) *Lines which join the middle points of the sides of a parallelogram, in order, form four equivalent triangles, and the sum of the areas of these triangles equals the area of the parallelogram formed by the four lines.*

(22) *Lines which are drawn through any point on a diagonal of a parallelogram, parallel to the sides of the parallelogram, form with the sides of the given parallelogram four smaller parallelograms, two of which are equivalent.*

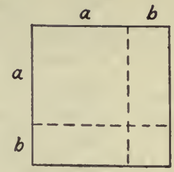
381. REVIEW EXERCISES

1. Find the non-parallel sides and the diagonals of an isosceles trapezoid of which the area is 144 square units and the altitude is 8 units and one base is 12 units.
2. Two sides of a triangle are 12″ and 15″ and the altitude to the longer of the two sides is 10″. Find the altitude to the other of the two sides.
3. The perpendicular sides of a right triangle are 4 and 6. Find the area of the triangle.
4. The hypotenuse of a right triangle is 20 and its base is 12. Find the area and the altitude to the hypotenuse.
5. Find the side of a rectangle of which the area is 120 square inches and the other side is 6 inches.
6. Find the diagonal of a rectangle of which the area is 48 square inches and base 8 inches.
7. The base and altitude of a rhomboid are 12″ and 10″ respectively. Find the area.
8. Two sides of a rhomboid are 8″ and 14″ and the altitude to the longer side is 7″. Find the altitude to the shorter side.
9. Two sides of a parallelogram are 50″ and 30″ and the included angle is 30°. Find the area.
10. Calculate the area of a quadrilateral of which one diagonal is 120, and the perpendiculars upon this diagonal from two opposite vertices are 50 and 75.
11. Two perpendicular diagonals of a trapezium are 81 and 44. Calculate the area.

12. The side of a rhombus is 10 and a diagonal is 16. Calculate the other diagonal and the area.
13. Calculate the area of a square whose diagonal is 24. Check by finding the area from a side.
14. Calculate the areas of the circumscribed and inscribed squares of a circle whose diameter is 8.
15. The area of a square is 72; calculate the diagonal without finding the side. Check by finding the side and then the diagonal.
16. The side of an equilateral triangle is 12. Calculate the area. Check by first finding the altitude.
17. Calculate the radius of the circle in which a regular inscribed hexagon has an area 180.
18. The difference of the areas of a circle and the circumscribed square is 85.84 square inches. Calculate the radius of the circle. Check by finding each area from the radius.
19. The sum of the four segments included between a circle and the inscribed square is 456.64 square inches. Calculate the radius of the circle.
20. The difference between the areas of a circle and the regular inscribed hexagon is 1000 square inches. Calculate the radius of the circle and its area.
21. The area of a circumscribed square is 25 square inches. What is the area of the square inscribed in the same circle?
22. Find the area of the bases of a cylinder which is 6 inches long and 2 inches in diameter.
23. How many square inches are in the convex surface of a cylinder 12 inches long and 10 inches diameter?
24. A cylindrical boiler is 10 feet long and 4 feet diameter. How many square feet of sheet metal are required to construct its convex surface?
25. Find the area of the base of a cone which is 6 inches in altitude and 2 inches in diameter at the base.
26. Calculate the total area of the surface of a cone 16 inches altitude and 24 inches diameter of base.

    Help.—Calculate the slant height; convex surface; area of base.
27. The sides of a rhombus and the shorter diagonal are each $10''$. Calculate the longer diagonal and the area.
28. Calculate and arrange in order of length, the circumference of a circle, the perimeters of a square, an equilateral triangle, a regular hexagon; all of the same area. Let the area = 1. In which figure is the ratio of perimeter to area a minimum?

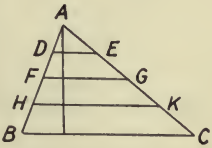
29. Show by a figure that $(a + b)^2 = a^2 + 2ab + b^2$.
Help.—Show the equivalence of the areas.



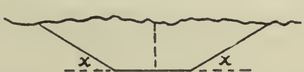
30. Show by a figure that $(a - b)^2 = a^2 - 2ab + b^2$.
31. Show by a figure that $(a + b)(a - b) = a^2 - b^2$.
32. Show by a figure that $(a + b)(a + c) = a^2 + ab + ac + bc$.
33. Show by a figure that $(a - b)(a - c) = a^2 - ab - ac + bc$.
34. Show by a figure that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.

Euclid gave the geometric construction for Exercises 29 to 31 in the "Elements." Algebra was almost unknown at that time, but many familiar algebraic relationships were discovered in geometric form.

35. If two polygons are similar, and one is four times the other in area, what is their ratio of similitude? What if one is double the other?
36. Divide the altitude of a triangle into four equal parts by lines parallel to the base. What parts of the entire triangle are the areas of the strips? *Help.*—Calculate the areas of $\triangle ADE$, AFG, etc., in terms of $\triangle ABC$.

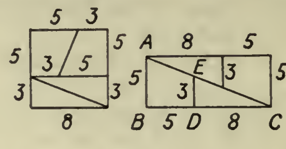


37. The homologous sides of two similar polygons are 5 and 10. Find the side of a similar polygon equivalent to their sum; to their difference.
38. The area of an equilateral triangle is 1200 square inches. Calculate the side.
39. The cross-section of a cut through a hill is a trapezoid, 20 feet



wide at the bottom, 10 feet deep, the angles of slope x, 45°. Calculate the area of cross-section.
40. Write a formula for the cross-section of Exercise 39, when the cut is w feet wide at bottom, c feet deep, slope 45°.
41. The scale of a map is 100 miles to the inch. What area is inclosed by a square inch? What area is inclosed by a rectangle $\frac{1}{2}$ inch by $1\frac{1}{2}$ inches?
42. If a rectangular area 2 by 3 inches is known to represent on a certain map an area of 15,000 square miles, what is the linear scale of the map?

43. The following is found in many popular lists of "trick" problems. Draw a square 8 inches on a side; cut it as shown; re-assemble the parts to form a rectangle 5 by 13 inches. How many square inches are apparently gained by the operation? If this were true what axiom of areas would be contradicted? Show the fallacy of the apparent result by calculating the length of DE if AC and BC are straight lines. If $DE = 3$ inches, can both these lines be straight?



44. The base, b, of a triangle is increased by an amount b'; how much must the altitude a be decreased in order that the area shall remain unchanged?

45. The altitude and base of a triangle are a and b; the altitude and base of a similar triangle are a' and b'. Calculate the altitude and base of a similar triangle equal to their sum.

Helps.—$a'' = \sqrt{\frac{a}{b}(ab + a'b')} = a\sqrt{1 + r^2}$; $b'' = \sqrt{\frac{b}{a}(ab + a'b')}$ $= b\sqrt{1 + r^2}$; where a'' and b'' are the required base and altitude, and r is the ratio of similitude of the given triangles.

46. The altitude and base of a triangle are 10 and 13; the altitude and base of a similar triangle are 20 and 26. Calculate the altitude and base of a similar triangle equal to their sum.

47. Calculate the area of an isosceles triangle whose equal sides are 24 and vertex angle is 30°; 60°; 135°; 90°.

48. Two similar fields contain together 579 square feet and their ratio of similitude is $\frac{7}{12}$. Calculate the area of each field.

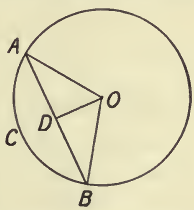
Helps $\frac{S_1}{S_2} = \frac{7^2}{12^2}$; $\frac{S_1 + S_2}{S_1} = \frac{193}{49}$; etc.

49. Calculate the area of a parallelogram whose diagonals are 75 and 200, and one of their included angles 30°.

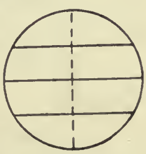
50. Show that $T = \frac{(a + b)\sqrt{ab}}{2}$; where T is the area of a right triangle, and a and b are the segments of the hypotenuse formed by the altitude. Calculate the area of the triangle if these segments are 4 and 6.

51. When Dido and her followers landed on the coast of Africa, "they asked of the natives only so much land as they could enclose with a bull's hide" (Bullfinch, "Age of Fable"). It is told that the hide was cut into strips. If this hide contained 50 square feet and the strips were $\frac{1}{10}$ inch wide, what area circle was obtained? What area square could be enclosed?

52. A square is 2″ on a side. Find the side of a second square whose area is $1\frac{1}{2}$ times the area of the first square.
53. Find the side of a square equivalent to the sum of two squares whose sides are 3″ and 5″ respectively.
54. Calculate the altitude of a triangle whose base is 20″, and which is equivalent to a triangle whose altitude is 12″ and base 16″.
55. Calculate the altitude of a triangle whose base is 160″ and which is equivalent to the sum of two triangles which measure respectively, altitude 50″, base 120″, and altitude 75″, base 60″.
56. Find the area of the segment ACB cut off by a chord which subtends an arc of 120°, in a circle whose radius is 20 inches.



Helps.—Area of segment ACB = area of sector $OACB$ − area of $\triangle OAB$. Altitude OD of $\triangle OAB = \frac{1}{2}$ radius; why? Find the base of $\triangle OAB$; etc.
57. A stake is driven on the farther edge of a stream 50 feet wide, and a horse on the nearer side of the stream is tied to the stake by a rope 100 feet long. Over what area can the horse graze without crossing the stream?
58. Find the area of the minor segment cut off by a chord which subtends an arc of 90° in a circle of 10 inches radius.
59. What fractional parts of the area of the circle are included between each pair of equidistant parallel chords in this figure?



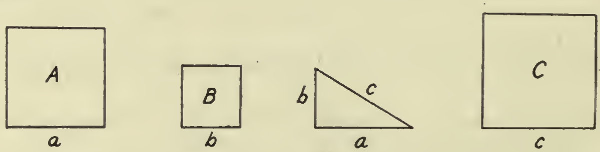
60. What area is included between a circle of 10 inches radius and the sides of an inscribed square?
61. Find the area of an annulus whose radii are 12 and 20 inches respectively?
62. Find the area of a triangle whose sides are 102, 104, 106.
Help.—Result is $S = 2^3 \cdot 3^2 \cdot 13 \cdot 5 = 4680$.
63. Find the side of a square which is equivalent to the sum of two triangles whose bases and altitudes are respectively 10 and 20 inches, and 8 and 20 inches.
64. Write a formula expressing the area of a parallelogram of which two sides are a and b and the included angle is 60°.
65. Write a formula expressing the area of a parallelogram of which two sides are a and b and the included angle is 45°.
66. The perimeter of a rectangle is 480 feet and the area is 8000 square feet. Find the dimensions of the rectangle.

67. Two homologous sides of two similar pentagons are 6 and 10 inches respectively. The area of the smaller pentagon is 720 square inches. What is the area of the larger pentagon?
68. If the areas of two similar quadrilaterals are 1620 and 288 square inches, respectively, what is the ratio of any two homologous sides of the quadrilaterals?
69. A line is drawn parallel to a side of a triangle and divides the other two sides into segments whose ratio is 2 to 3. What is the ratio of the areas into which the triangle is divided? There are two positions of the line.
70. Where must a line be drawn, parallel to one side of a triangle, in order to bisect the area of the triangle?

Applications

382. Construction Problems in Areas.—These constructions are made by the rule and compass method, and are chiefly interesting because of their association with the early development of geometry. Almost all problems in combining and dividing areas, equivalence of areas, etc., which occur in practice are better performed by geometrical or trigonometrical calculation. Such constructions as are here given may sometimes be valuable as checks on calculation.

1. *To construct a square equivalent to the sum of two given squares.*



A and B are the given squares. Prove that $C \approx A + B$.

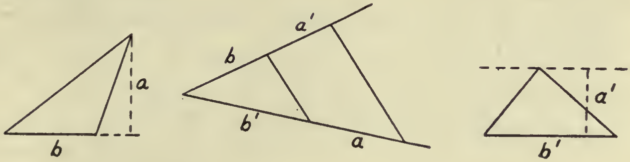
2. *To construct a square equivalent to the difference of two given squares.*

Show how this construction can be made.

3. *To construct a square equivalent to the sum of any number of given squares.*

Show how the required square is obtained.

4. *To construct a triangle equivalent to a given triangle and with a given base.*



Explain the construction.

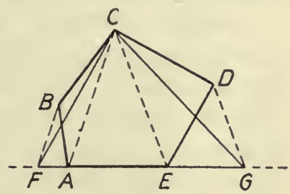
5. *To construct a triangle equivalent to a given rectangle and with a given base.*

Show how this construction can be made.

6. *To construct a square equivalent to a given rectangle.*

Help.—Use the principle of **316,** Exercise 12, to obtain the side of the required square.

7. *To construct a triangle equivalent to a given polygon.*



Show that $\triangle FCG \approx$ polygon $ABCDE$.

8. *To construct a square equivalent to a given triangle.*

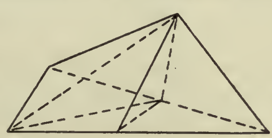
Helps.—Let a and b be the altitude and base of the given triangle, and s be the side of the required square.

Then $$\frac{ab}{2} = s^2, \text{ and } \frac{a}{s} = \frac{s}{\frac{b}{2}}$$

Use the method of Problem 6 to complete the construction.

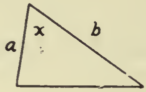
EXERCISES

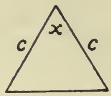
1. Construct a square equivalent to the sum of two given triangles.
2. Construct a circle equivalent to the sum of two given circles.
3. Construct an equilateral triangle equivalent to a given triangle.
4. Bisect a triangle by a line parallel to a side.
5. Bisect a quadrilateral by a line drawn from a given vertex.



6. Construct an isosceles triangle of which the vertex angle equals an angle of a given triangle, and which is equivalent to the given triangle.

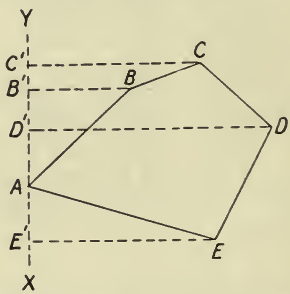
   Helps.—(1) $c^2 = ab$; (2) to find c, refer to **316,** Exercise 12, or Exercise 13.





383. Surveying.—When the shape of a piece of land is rectangular, triangular, circular, etc., its area is found by methods already given in this chapter. The shape of a parcel of land is generally irregular, and special methods have been devised for computing the areas of such plots from linear and angular measurements.

1. Areas Bounded by Straight Lines. (*a*) *Area by Coördinates.*—Take vertex A, as the origin of coördinates. By measuring the sides and interior angles of the field $ABCDE$, it is possible to calculate by trigonometry the coördinates of all the vertices. The lines $B'B$, $C'C$, $D'D$, $E'E$, and AB', $B'C'$, $C'D'$, $D'E'$, $E'A$, can then be found. These lines are the dimension lines of the part areas, ABB', $B'BCC'$, $C'CDD'$, $D'DEE'$, $E'EA$, and from them the values of these areas can be found. The polygon $ABCDE$ is an algebriac sum of the part areas.



EXERCISES

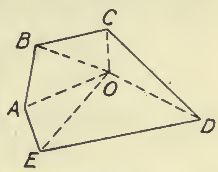
1. The coördinates (abscissa and ordinate) of the vertices of a field $ABCDE$, are A, (0, 0); B, (+100, +80); C, (+150, +90); D, (+175, +20); E,(+120, −50). Plot the vertices on squared paper and draw the field.
2. Find the values of the lines $B'B$, $C'C$, $D'D$ and $E'E$.
3. Find the values of the lines AB', $B'C'$, $C'D'$, $D'E'$ and $E'A$.
4. Find the areas ABB', $B'BCC'$, $C'CDD'$, $D'DEE'$ and $E'EA$.

5. Find the area of the field $ABCDE$ from the algebraic sum of the part areas.
Help.—Polygon $ABCDE = -\triangle ABB' -$ trapezoid $B'BCC' +$ trapezoid $C'CDD'$, etc.

(*b*) *Area by Triangles.*—By this method the field is divided into triangles, (*a*) by diagonals, (*b*) by lines drawn to every vertex from some interior point, (*c*) in any convenient way.

EXERCISES

6. The measurements of a field $ABCDE$ are, $\angle AOB = 60°$, $\angle BOC = 60°$, $\angle COD = 90°$, $\angle DOE = 60°$, $\angle EOA = 90°$; $OA = 120$ feet, $OB = 120$ feet, $OC = 80$ feet, $OD = 150$ feet, $OE = 100$ feet. Find the area by finding the sum of the areas of the triangles. Trigonometry is required for this calculation when the central angles are other than 30°, 45°, 60°, 90°, and their supplements.



7. Find the area of a four-sided field $ABCD$, of which the measurements are $AB = 400$ feet, $BC = 1500$ feet, $CD = 1200$ feet, $DA = 800$ feet, and diagonal $BD = 1000$ feet. Find the areas of triangles ABD and BCD by the formula of **367.**

2. Areas Bounded by Irregular Curved Lines.—The methods used give the area approximately, but with sufficient exactness. They apply to plots of either of these forms.

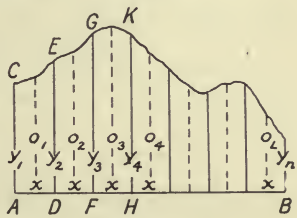


FIG. 1.

(*a*) *The Ordinate Method.*—Measurements are taken in the field, AD, DF, FH, etc., and perpendiculars AC, DE, etc. The field is then plotted accurately to scale, and the line AB is divided into equal parts and the ordinates O_1, O_2, etc., are drawn at the middle points of the equal segments. The area equals the average ordinate multiplied by the *uniform* segment x.

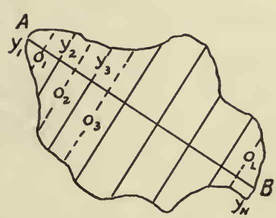


FIG. 2.

EXERCISES

8. The measurements of a field (Fig. 1) are $AD = DF = FH =$ etc. $= 40$ feet; ordinates y_1, y_2, etc., are 80, 90, 120, 140, 140, 110, 75, 70, 30 feet. Plot the field on squared paper.
9. Find the area of the field of Exercise 8 by the ordinate method.

(*b*) *The Trapezoid Method.*—Each part area, $ACED$, $DEGF$, etc., is approximately a trapezoid.

EXERCISES

10. Show that the area of a field of either form, when AD, DF, etc. are all equal to x, is given by the formula,

$$S = \frac{x}{2}\left[y_1 + y_n + 2(y_2 + y_3 + \;.\;.\;.\;.\; y_{n-1})\right]\cdot$$

11. Find the area of the field of Exercise 8 by the trapezoid method.

(*c*) *Simpson's Rule.*—This is considered the most accurate method. The formula for the area is,

$$S = \frac{x}{3}\left[y_1 + y_n + 4(y_2 + y_4 + \text{etc.}) + 2(y_3 + y_5 + \text{etc.})\right]$$

$$= \frac{x}{3}\left[A + 4B + 2C\right]\cdot$$

EXERCISES

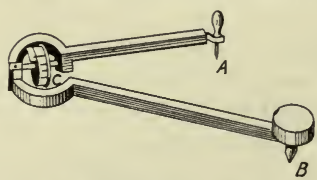
12. For what do the letters A, B, C, stand in the final formula?
13. Calculate the area of the field of Exercise 8 by Simpson's Rule.

3. Special Methods of Finding Areas. (*a*) *Area by Squared Paper.*—The field is plotted and the area measure is counted.

EXERCISE

14. Find the area of the field of Exercise 8 by this method.

(*b*) *Area by Planimeter.*—An area may be measured automatically by this instrument, by tracing the bounding



line with the point A while the point B is fixed. The revolving counter C records the area. Explanations of the planimeter may be found in texts on surveying and engineering.

(c) *Area by Approximate Rectangle, Triangle, Circle, etc.*—The area is plotted accurately, and straight lines are drawn which approximate as closely as possible to the bounding curve, while the form of the rectilinear figure is made a rectangle, triangle or trapezoid; or in some cases an approximate circle or semi-circle may be used. The dimensions of this approximate figure are scaled from the drawing.

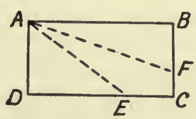
EXERCISE

15. Find the area of the field of Exercise 8 by drawing an approximate rectangle.

384. Dividing Land.—Problems of dividing land occur constantly in surveying. They are solved by methods of calculation, and may be checked by geometrical constructions.

EXERCISES

1. Divide a rectangular field 120 feet by 240 feet into three equivalent parts by lines parallel to the shorter side.
2. Divide a rectangular field 120 feet by 240 feet, into three equivalent parts by lines drawn from a vertex of the field. Find BF and DE.



3. The sides of a triangular field are AB = 150 feet, BC = 200 feet, CA = 210 feet. Divide it into two equivalent parts by a line drawn from the vertex A.
4. Divide the field of Exercise 3 into two equivalent parts by a line DE parallel to BC. Calculate AD and AE.
5. Divide the field of Exercises 3 and 4 into two equivalent parts by a line drawn from a point F, which is 20 feet from B on the side AB.

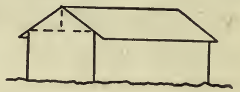
385. Estimating and Engineering.

EXERCISES

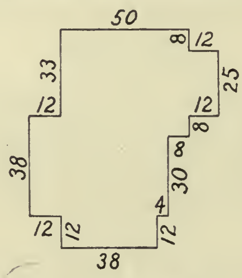
1. A room is 30 feet long, 20 feet wide, 10 feet high. Calculate the areas of the four walls and ceiling. There are 4 windows

$3\frac{1}{2}$ feet by 6 feet, and 2 doors 4 feet by $7\frac{1}{2}$ feet. What area is covered by plaster? Estimate its cost at 95 cents per square yard.

2. A barn is 40 feet wide, 70 feet long, and 24 feet high to the eaves; the rise of the roof is 12 feet. Estimate the cost of painting the barn two coats if 1 gallon of paint will cover 400 square feet one coat, and paint costs $2.75 per gallon.



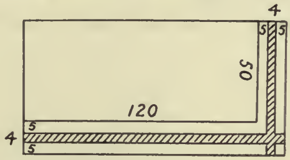
3. Shingles average 6 by 24 inches. When laid on a roof, $\frac{1}{4}$ of the shingle is exposed. Estimate the cost of shingling the roof of the barn of Exercise 2 at $20.00 per 1000 shingles.
4. A rectangular piece of sheet metal 4 inches by 5 inches, weighs $\frac{1}{2}$ a pound. What is the weight of a rectangular piece of the same sheet metal 10 inches by 12 inches?
5. Find the number of square yards covered by a rectangular building 120 feet by 55 feet. Each yard in depth of cellar excavation forms a cubic yard with a square yard of surface. Estimate the number of cubic yards excavated in digging the cellar 6 feet deep; and the cost at 75 cents per cubic yard.
6. Make a similar cost estimate for the excavation of a cellar 6 feet



deep for the building of which the floor plan is here given.

7. The number of gallons of water flowing past a given line across a river is found by measuring its "cross-section." Find the cross-section of a river, the soundings being taken from a bridge at intervals of 50 feet from shore to shore. Depths in feet: 10, 20, 25, 30, 20, 18, 15, 10, 5. Total width 500 feet. If the velocity of current is 200 feet per minute, how many cubic feet of water flow past the bridge per minute?
8. If 10 per cent of the water of Exercise 7 can be pumped from the river to supply a city, is there sufficient water for a city of 1,000,000 inhabitants, allowing 200 gallons per person per day of 24 hours?

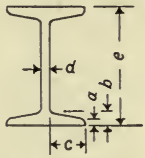
9. A playground is trapezoidal in shape, the parallel sides being 300 feet and 240 feet, and a third side perpendicular to them 250 feet. How many children can be accommodated if 200 square feet is the estimated allotment per child?



10. A cement walk 4 ft. wide is laid along two sides of a corner lot. Calculate the cost at 25 cents per square foot.
11. A cement walk 5 feet wide runs straight for 80 ft., then follows a circular curve of an inner radius of 50 feet for a quarter circle, then runs straight for 100 feet. Estimate the cost at 20 cents per square foot.
12. Parquet floors of white oak are to be laid in rooms as follows: two rooms 30 by 12 feet; three rooms 22 by 18 feet; one room 35 by 15 feet; one hall 6 by 24 feet. Estimate cost if flooring 2 inches wide costs $25 per 1000 linear feet, and an additional charge is made of 10 cents per square foot for laying and finishing.
13. A circular rod in a roof truss must sustain a safe tension of 20,000 pounds per square inch of cross-section. What must be its diameter if the total tension in the rod is 60,000 pounds? What if the tension is 12,000 pounds?
14. How many boiler tubes 3 inches in diameter are required to give a total cross-section equal to that of a 3-foot diameter chimney?
15. Three pipes, 10 inches, 12 inches, 8 inches, diameter, respectively, empty into a basin. What diameter must be given to a single discharging pipe whose cross-sectional area equals the combined areas of the three pipes?
16. What total pressure is exerted by the steam in the cylinder of a steam engine against a piston 12 inches in diameter if the pressure is 120 pounds per square inch? The horsepower of the engine is given by the formula h.p. $= \frac{p.l.a.n.}{33{,}000}$; in which p is the average steam pressure, l the length of stroke, a the area of the piston, n the number of strokes per minute. Find the horsepower of the engine considered if the length of stroke is 20 inches and number of strokes is 160 (number of revolutions 80) per minute.
17. A transit telescope magnifies 7 diameters. How many times is the area magnified?

18. Show that the formula for the area of an annulus may be expressed, $A = \pi \cdot \frac{d_1 + d_2}{2} \cdot t$; where d_1 and d_2 are the outer and inner diameters and t is the thickness (width of ring). This is often used as a shop formula.

19. The cross-section area of an I-beam is required in order to find



the weight per yard of length. Show that the area of cross-section is given by the formula $A = de + 2c(a + b)$.

20. Find the cross-section area of an I-beam if $a = 1''$, $b = 1\frac{1}{2}''$, $c = 1\frac{1}{2}''$, $d = \frac{1}{2}''$, $e = 10''$.

Problems for Field Work

386. (1) Find the area of a plot of ground bounded by straight lines, by measuring all of the sides of the plot and as many diagonals as are necessary to divide it into triangles. The diagonals should be selected so as to form as nearly as possible equilateral triangles. Use the formula of **367.**

2. Find the area of a piece of land entirely bounded by a curved line, or by straight and curved lines. Use the method of **383,** Part 2(*b*).

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