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PLANE GEOMETRY

ABRIDGED AND APPLIED
COLLEGE PREPARATORY

M. AUERBACH
C. B. WALSH

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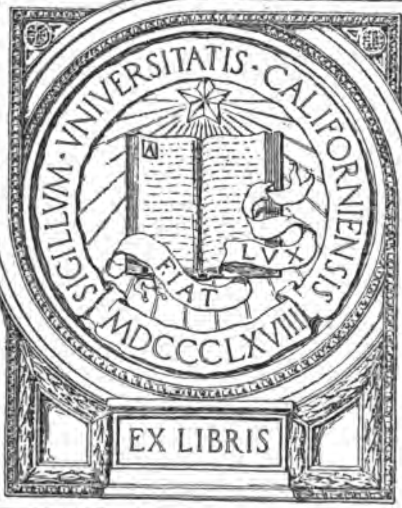

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PLANE GEOMETRY

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LIPPINCOTT'S SCHOOL TEXT SERIES
EDITED BY WILLIAM F. RUSSELL, PH.D.
DEAN, COLLEGE OF EDUCATION, STATE UNIVERSITY OF IOWA

PLANE GEOMETRY

I. ABRIDGED AND APPLIED
II. COLLEGE PREPARATORY

BY

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J. B. LIPPINCOTT COMPANY

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PREFACE

In separating this text book on Plane Geometry into two parts, the Authors have followed what appears to them to be a normal and logical requirement essential to a proper presentation of the subject, and the most appropriate reference to these divisions would seem to be a designation of them as a First Study and a Second Study. In the former, the objective is to afford a general view of the subject, with emphasis on applications, the study being intended for the use of all high school pupils, and the material is so presented as to make it available also for use in junior high schools. The Second Study is devoted to a more intensive treatment of Plane Geometry, with special emphasis on theoretical work, and is addressed particularly to Regents' and college entrance requirements. In the entire text the inductive method is followed as far as practicable, and simplicity is gained rather by the means of scientific accuracy than at its expense.

In view of the fact that the purpose and method of the two parts of the book differ somewhat, a separate consideration of each of these divisions seems desirable.

Part I

The Authors feel confident that the First Study will serve a fourfold purpose.

First: That it will contribute to a solution of the question as to how much mathematics shall be required of high school pupils who do not intend to enter college, and with this objective, an effort has been made to plan a course adapted to the needs and interests of pupils meeting minimum requirements in mathematics. The Authors believe that it has been customary in many schools to meet this situation by a study of only a portion—three or four books of the geometry in the required course, thus giving students merely an intensive knowledge of a part of the subject, instead of a broadly comprehensive view. The course here outlined covers

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not only all that is really important in the five books—although the syllabus contains fewer propositions than are now comprised in the first three books of most available texts—but also work in the application of three of the trigonometric functions.

Second: That it will suggest a course in elementary geometry so thoroughly adapted to the mental development of pupils in the ninth or tenth school year that it may be profitably used to supersede the conventional course in formal geometry.

The course here outlined will not in any respect restrict the preparation of the student for college; on the contrary, will find him much more ready and willing to proceed to the collegiate preparatory work with his knowledge of the subject enriched by application and vitalized with interest.

Third: That it will open the eyes of the pupil to the relation of geometry to the activities and necessities of every-day life, and emphasize the practical application of the science, both in specific professions and trades, and in the affairs of daily life.

Fourth: That it will arouse in the pupil a conception of the dignity and power of the subject. To this end the Authors have treated it scientifically, endeavoring to develop gradually in the mind of the pupil a natural assumption of this treatment; and it has been their purpose, by departing from formal methods, to lead the pupil to reason rather than merely to remember.

The following means have served in the attainment of the ends just stated:

Revision of the Syllabus

In the First Study the NUMBER OF PROPOSITIONS has been reduced to approximately half of those given in the standard texts. Propositions have been retained or selected on three bases:

First: Those that are rich in application.

Second: Those of peculiar interest to the young student.

Third: Those essential to the sequence of the study.

The *wording of the propositions* retained has departed materially from the traditional phraseology in an effort to avoid the formidable and stilted qualities of the latter, while retaining its scientific correctness. In fact, the language of the students in the classroom has suggested many of these changes: *e.g.* "Three

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sides determine a triangle" to replace "Two triangles are congruent if three sides of one are respectively equal to three sides of the other." The Authors feel that some of these changes in wording are desirable scientifically as well as from a practical standpoint: *e.g.*, "If the ratio of the sides of one triangle to those of another is constant, the triangles are similar," to replace "Two triangles are similar if the sides of one are respectively proportional to the sides of the other;" and "An angle whose vertex is outside a circle is measured by half the difference of the intercepted arcs," to replace three statements.

A transition, more gradual than usual, from the geometry of the grade school to the more scientific work of the secondary school has been secured. The proofs given in the first section would be wholly convincing but the forms and detail less conventional than those demanded in more rigorous demonstrations, some of which are preceded by explanations. *This does not mean that the text is one of Concrete or Observational Geometry. The bare essentials* are retained from the outset, and subsequently there is a gradual *introduction and demand for all the rigor* obtainable in secondary school work. This *gradual transition* tends to prevent the discouragement so often manifested during the beginning of the study.

Minor Details.

First: No authorities are required for auxiliary construction—consistent reference to them makes a proof unduly formidable—no other reference should be permitted.

Second: Nothing is introduced in the text until it is required—thus avoiding long lists of definitions, axioms, postulates, and the like.

Third: Throughout the text *emphasis is laid upon the idea of classification;* and by means of proper grouping of definitions, postulates, and developed facts, the student is trained to regard the subject, not as a miscellany of isolated facts, but as a framework of interrelated *sub-topics.* A few reference books are mentioned. The use of these has been found so exceedingly helpful and inspiring in the classes which have been led by the Authors that they feel that the omission of some such lists would be a great calamity. The Authors recall many instances in which new life

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has been given to the subject through such references; in fact there are instances where a pupil who might never have discovered himself mathematically has developed true mathematical enthusiasm and ability by browsing through suggested supplementary reading.

In using the text, the Authors earnestly suggest that *actual writing of proofs be deferred* until the class is quite ready to fall in naturally with a more or less set form of proof.

Let the need of a form become apparent to the student who is trying to write a proof unaided by conventions before insisting upon the adoption of one in written work. Indeed the Authors feel that the entire first chapter of this text could well be developed before the necessity arises for a single written proof from the student. Sufficient material for written work will be available from the exercises during this period.

Part II

In addition to a review of the First Study, the Authors desire to direct attention to some details of this Second Division of the work which seem worthy of special mention.

First: *The size of the syllabus.* The number of propositions developed in the First Study has been considerably augmented to include all the demands of *College Entrance and Regents' Examinations.* This has been accomplished by inserting the additional theorems between the theorems of the original syllabus, thus preserving the *sequence of the former Study.* A separate syllabus of construction problems is given in the chapter entitled "Methods of Attacking Problems" (page 306).

Second: *The grouping of the syllabus.* To facilitate the retention of the frame-work of the subject, the propositions are collected in groups by topics, so far as the sequence permits, and such grouping has necessitated certain departures from the traditional arrangement by books.

Third: *The type of exercises.* In this part emphasis is placed on theoretical exercises, as contrasted *with special reference* to practical exercises in the First Study; and a large collection of college entrance papers, together with a still larger selection of isolated exercises of this character, is included.

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FOURTH: *A chapter on methods of proof.* More scientific habits of work are fostered through a discussion and a careful classification of methods of proof with illustrative exercises.

FIFTH: *A chapter on methods of attacking problems.* This chapter is largely similar in purpose to the chapter just mentioned, except that it deals with construction work. A special syllabus of what might be termed fundamental constructions is given, including those presented in standard texts, and those propositions are of such character as to form a necessary and sufficient basis for the work required by colleges. This chapter groups many typical constructions and methods employed, and definite reference throughout the book is made to this, as well as to the chapter on methods of proof.

SIXTH: *Suggestions for club or other additional work.* The chapter entitled "Suggestions," and exercises preceded by the letter "d"—frequent in the book—may be omitted (not constituting a requirement for college preparation) without impairing the integrity of the course. They are included to give additional interest and breadth to the subject where time and the ability of the student permit. This chapter contains in the suggestions for club work a list of topics suitable for discussion by students or teacher in a mathematics club of high school grade, and appended to this list will be found a bibliography of appropriate references.

In summary, then, it may be said that this Second Study is intended to enlarge upon the course outlined in the First Study, not only in that it answers the requirements of college entrance examinations, but in that it also makes possible at the same time a richer and fuller course for those students whose interest and ability warrant it.

In closing, the Authors desire to acknowledge three distinct sources of assistance.

Realizing the present eclectic tendency of teachers in the matter of exercises, as evidenced by the general use of typewritten lists of problems, the Authors have availed themselves frequently of many of the standard texts in the selection of material of this kind. To Mr. Lewi Tonks, a former pupil of the Authors, who read the proof and criticised the contents of the book, the Authors feel

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especially indebted, and wish at the same time to acknowledge the assistance of Mr. H. W. Smith, of the Ethical Culture School, and of Mr. P. S. Clarke, of Pratt Institute, Brooklyn, whose suggestions for revision of the English of the text were of value.

Finally, the Authors take pleasure in this opportunity to record their deep appreciation of the endorsement and encouragement they have received from the authorities of the Ethical Culture School—Prof. Felix Adler, Rector; Franklin C. Lewis, Superintendent; and Dr. Henry A. Kelly, High School Principal. Their approval made possible the experimental work in the School which has developed and justified the course here presented, and the Authors feel that their kindly sympathy and intelligent coöperation in the growth of the experiment have contributed in large measure to its success.

THE AUTHORS.

OCTOBER, 1919.

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SYMBOLS AND ABBREVIATIONS

$=$	is equal, or equivalent, to
$\neq$	is not equal, or equivalent, to
$\backsim$	is similar to
$\cong$	is congruent to
$\doteq$	approaches as a limit
$\propto$	varies as
$\underline{\propto}$	is measured by
$>$	is greater than
$\ngtr$	is not greater than
$<$	is less than
$\nless$	is not less than
$+$	plus, or increased by
$-$	minus, or diminished by
$\div$, —	divided by
$\times$,( ),$\cdot$	multiplied by
$\parallel$	a parallel, or is parallel to
$\nparallel$	not a parallel, or is not parallel to
$\parallel$s	parallels
$\perp$	a perpendicular, or is perpendicular to
$\not\perp$	not a perpendicular, or is not perpendicular to
$\perp$s	perpendiculars
$\frown$	arc
$\overline{AB}$	straight line AB
$\odot$ Ⓢ	circle, circles
$\triangle$, $\triangle$s	triangle, triangles
▱, ▱s	parallelogram, parallelograms
$\angle$, $\angle$s	angle, angles
$\because$	since
$\therefore$	therefore
....	and so on
( ), { }, [ ], $\overline{\quad}$	signs of aggregation
$\sqrt[n]{\ }$	the nth root of
π	3.14159......

adj.	adjacent
alt.	alternate
ap.	apothem
approx.	approximately
ax.	axiom
circf.	circumference
comp.	complement, complementary
con.	conclusion
cong.	congruent
const.	construction
cor.	corollary
corres.	corresponding
def.	definition
diff.	difference
ex.	exercise
ext.	exterior
fig.	figure
ht.	height, or altitude
hom.	homologous
hy.	hypotenuse
hyp.	hypothesis
int.	interior
isos.	isosceles
lat.	lateral
peri.	perimeter
pl.	plane
pt., pts.	point, points
n-gon	polygon of n sides
post.	postulate
prob.	problem
proj.	projection
prop.	proposition
rect.	rectangle
reg.	regular
rt.	right
sec.	sector
seg.	segment
sq.	square
st.	straight
subst.	substitute
sup.	supplement, supplementary
sym.	symmetrical
th.	theorem
vert.	vertical

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PART I

FIRST STUDY

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PLANE GEOMETRY

UNIV. OF CALIFORNIA

CHAPTER I

INTRODUCTION

NOTE.—For pupils who have had no intuitional geometry section A of the Introduction may be postponed until the work in Areas has been completed—p. 77.

A. A FEW FACTS CONCERNING THE EARLY DEVELOPMENT OF GEOMETRY

In a cabinet in the British Museum there is a piece of clay somewhat over an inch thick and perhaps fifteen inches square which might be referred to as the first book about geometry. Near it, on a roll of papyrus, yellowed by age, is a collection of notes containing instructions for finding the contents of areas and solids. When we reflect that this clay tablet and the manuscript are considerably over thirty-five hundred years old, we can see that the study of geometry is by no means a modern development.

The tablet and the manuscript represent respectively the earliest available records of the geometric knowledge of the Babylonians and the Egyptians. Centuries ago these two races found it necessary to devise some means for accomplishing what today seems a very simple undertaking. Perhaps the necessity was forced upon the Egyptians for a reason that does not seem very apparent at first. The River Nile, as we know, rises twice a year and inundates the country bordering on it for many miles. Naturally this flood produces changes in the line of the river banks, and new turns and curves give the adjacent land a very different appearance on each occasion. A farmer whose land bordered the river might therefore find himself one year in possession of a good deal of property, and the next year with much less. This condition, we are told by the historian Herodotus, caused Rameses II, who was king of Egypt about 1350 B.C., to declare a law: "This king divided the land among all Egyptians so as to give each a quadrangle of equal size,

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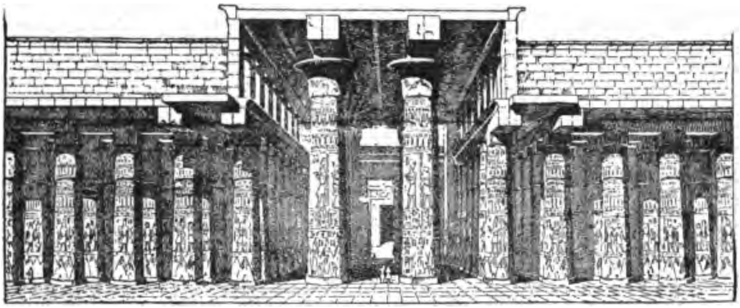
and to draw from each his revenues by imposing a tax to be levied yearly; but everyone from whose part the river tore away anything had to go to him and notify him of what had happened; he then sent the overseers who had to measure out by how much the land had become smaller, in order that the owner might pay on what was left in proportion to the entire tax imposed." As a result of this it became necessary for the Egyptians to employ surveyors who should determine the areas of the land lost or gained, and these surveyors put into practical use such rudimentary knowledge as was then available.



Restoration of the Great Hall of Karnak

Seven centuries before this Egyptian kings had undertaken large construction operations, the very nature of which showed that their contractors and builders understood the elementary principles of the science of mensuration. Menes, the first Egyptian king, built a large reservoir and two temples at Phthah and Memphis, the ruins of which are still in existence, and under Amenembat III, a later king, the Egyptians designed and constructed a very large irrigating system covering considerable territory and requiring a careful calculation of areas, water flow, gradients, etc. Students of Egyptian art and religion find frequent evidence that this race had a crude knowledge of geometrical principles. The pavements of the temples show designs of triangles, squares, five-pointed stars, and rectangles, and the locations of the buildings themselves show geometric knowledge, for their temples were supposed to be constructed with reference to a certain fixed point. The Egyptians were sun-worshippers, and their temples were

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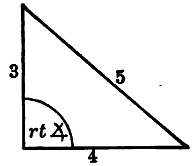
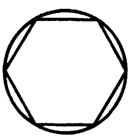
designed to receive sunlight through the doorways at certain times of the day, as a part of the religious ceremonies. It is interesting to note that the movement of the North star has been in a measure demonstrated to our later-day astronomers by the fact that Egyptian temples, built three or four thousand years ago and designed to face the North star, are no longer in the perpendicular to it. The Egyptians were astronomers, and in locating their temples used the sun and the North star to establish base lines. The surveyors, called the "harpedonaptæ" or "rope-stretchers," fixed the right angle to the north-south line by stretching a rope knotted in three places around pegs. The distances between the knots were in the ratio of 3–4–5, showing that they knew this to be the ratio of sides of a right triangle. Our present day surveyors are still following the same method and have improved upon the method of the Egyptians only by substituting a steel tape for the rope.



We mentioned the ancient manuscript of the Egyptians now in the British Museum. The writer of this manuscript was called Aah-mesu (The Moon-born), an Egyptian scribe commonly called Ahmes. The original from which he copied it was probably in existence about 2300 B.C., but has never been discovered. The commercial value of the document is shown by the fact that it contained rules and formulas for finding the capacity of the wheat warehouses constructed in ancient Egypt, as well as a treatise of considerable length on a crude algebraic system. A temple built for the worship of the god Horus on the island of Edfu has on its walls hieroglyphics describing the land which the priests of the temple owned, and the formulas for finding the areas of these plots.



Less is known about the Babylonians in these particulars, but so far as we can learn, their geometrical knowledge was used more in the arts than for practical purposes. Their monuments, found in the ruins of Babylon, show geometrical designs, such as a regular hexagon in a circle, and the pictures of their chariots show the wheels divided into sixths. The Babylonians appear to have followed this division into sixths in their arrangement of the calendar,

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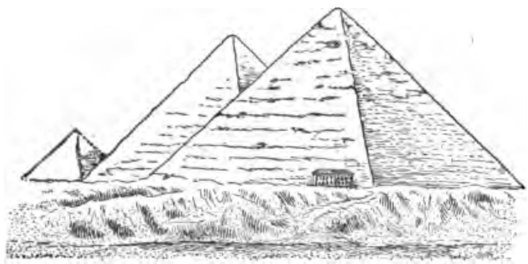
for their year consisted of three hundred sixty days, and they divided the circle into three hundred sixty degrees, on the theory that each degree represented the supposed revolution of the sun round the earth.

Although the geometric knowledge of the Egyptians and Babylonians may seem to us somewhat crude and simple, we must remember that, as compared with the savage races which surrounded them, these people represented the greatest advancement in civilization and scientific knowledge. We see that much of this was due to the very necessities of life; that to build public works, levy taxes, determine boundaries, required a knowledge of the science of mensuration. In the case of the Egyptians, their requirements, so far as we are able to estimate them, were even broader than those of the Babylonians. The construction of the pyramids shows clearly a geometrical design, executed scientifically, and this work, as well as the erection of wheat warehouses and storage reservoirs, necessitated what was doubtless to them a very advanced conception of the principles of solid geometry.



Egyptian Pyramids

Strangely enough, however, we are indebted to neither of these races for the development of this knowledge into a science, but to a race whose place in history is much later. The Greeks in the ancient world occupied a position in some respects similar to that which America has held in the modern world. They were a people much given to exploitation and expansion, as well as to scientific and philosophical pursuits, and in addition to this, prided themselves on their high degree of adaptability. Plato said, "Whatever we Greeks receive we improve and perfect." They did not originate ideas so much as they adopted those of other races and improved upon them to a degree which causes history to associate the Greeks themselves with the original conception. The Greeks were travelers

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and traders, interested in the arts and sciences, and a distinguishing characteristic of the race was their desire to learn and experiment with new things. Seven hundred years before Christ, Greek merchants began sending their ships across the Mediterranean to Egypt. Travelers began to bring back accounts of this other great nation, and the Greeks were immediately interested in the reports of what the Egyptians had done and were doing.

Thales (640–546 B.C.), a merchant of Miletus, was among those who became acquainted with the Egyptians as a result of commercial intercourse. Thales was at heart a student, and the geometrical theories and practices of the Egyptians interested him. In later life he terminated his business activities and opened a school in his native city, Miletus, where he began teaching the principles of geometrical science as it was then known. The problems with which his pupils concerned themselves would seem elementary to us. They dealt merely with finding the heights of objects or the distances of ships from the shore, but his school, which has come down to us under the name of the Ionic school (so called from the Greek province in which Miletus was situated), was the first intelligent effort to systematize the study of geometry.

One of the students in the school of Thales was a noteworthy successor of the first Greek geometrician. Pythagoras (580–501 B.C.) founded a school of mathematics at Crotona in the southern part of Italy. His plan was much more elaborate than that of Thales. Pythagoras felt that the study and character of the school would create a deeper impression if it were organized as a secret society. The watchword was "Silence," and its members were pledged to secrecy as to the nature of the work which was done. The Greek government felt that the secret methods of the school might be used to conceal harmful activities, and finally ordered the institution closed. This circumstance, and the pledge of secrecy

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imposed upon the members of the school, have prevented our learning much about it. The Pythagorean proposition, which states that the square on the hypothenuse of a right triangle is equal to the sum of the squares on the other two sides, bears the name of the school, although the fact was known for a long time before Pythagoras proved it to be true. "Pythagoras changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom and investigated its propositions in an immaterial and intellectual manner."

Pythagoras

Archytas (430–365 B.C.), who followed Pythagoras, was not so much interested in speculative or geometrical subjects as he was in the application of the science to practical uses. He invented several mechanical toys operated on geometrical principles. Very few sailors realize that the ability of one man to move a tremendous weight of sail was made possible by the discovery of this Greek mathematician who lived over twenty centuries ago, for it was Archytas who worked out and applied the principles of the pulley. He is believed to have been the first student to find a solution of the problem called the "duplication of the cube," that is, to find the dimensions of a cube the volume of which shall be twice that of a given cube.

Plato

Plato (429–348 B.C.) was a contemporary of Archytas, and his name is associated even more generally with geometrical science than that of his compatriot. Plato called his school the "Academy," and the underlying prin-

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ciple of his theory was the abstract and systematic development of geometric science. Plato insisted that the only instruments needed for the study of the subject were the straight edge and the compasses, and the history of the science has demonstrated the accuracy of his conclusions, as they are the only scientific tools needed for all elementary work in the science.

Eudoxus (408–355 B.C.) studied under Plato for a time, and subsequently did some independent work in the science. He directed his attention chiefly to the principles of proportion and certain methods of proof, to which we shall make reference later. He was the first scientist to begin to put into book form the mathematical knowledge of his time, and may properly be considered the logical forerunner of the mathematician Euclid. Euclid, who was a teacher in a school of mathematics in Alexandria, Egypt, about 300 B.C., was the author of what is probably the most famous book on geometry. He collected and arranged all the knowledge of the science down to his own time, and his book still stands today in many respects as a final authority and the background of the entire science. Despite the fact that the Egyptians and Babylonians first developed a crude knowledge of the subject, it is perhaps appropriate that the Greeks, whose generations of scientists, culminating in Euclid, gave so much study to it, should have furnished the name by which we call it. The Greek word "ge" meaning the earth, and "metron" to measure, are the roots which formed our name for the science

Euclid

From its first crude beginning in the necessity for measuring the destruction wrought by an ancient river, its instruments, crude pegs and a knotted rope, developed and applied as a science by the mathematicians of five centuries before the Christian era, supplemented and enlarged by the observations and discoveries of nearly twenty centuries of research, the science by which the surveyors of Egypt located their boundaries is today the method used for determining the power of a battleship or the contents of a mountain range.

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SUMMARY

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BIBLIOGRAPHY

ALLMAN, G. J.: "Greek Geometry from Thales to Euclid": pp. 2, 3, 5, 7, 15, 16, 22, 29, 41, 139–140, 143 (footnote) and 154.

BALL, W. W. R.: "A Primer of the History of Mathematics": pp. 3–6, 8–14, 32, 42–48.

BALL, W. W. R.: "A Short History of Mathematics": pp. 5–8, 15–16, 25–27, 40, 54–63, 69–72.

FINK, K.: "Brief History of Mathematics": pp. 190–214 (with omissions).

GOW, J.: "A History of Greek Mathematics": pp. 176–178.

BOYER, J. F.: "Histoire des Mathematiques": Chapters I–V.

CANTOR, MORITZ: "Vorlesungen über Geschichte der Mathematik": Vol. I, pp 17–52, 90–114, 134–146, 170–187, 201–277.

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EXERCISES. SET I. BASED UPON HISTORIC FACTS

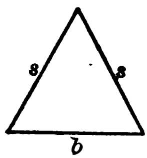
1. From the derivation of the word geometry, can you suggest any studies or professions in which geometry may be applied?

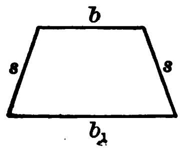
2. What gave rise in the first place to the art and eventually to the science of geometry?

3. From the little told you in the foregoing paragraphs and any references you may have read, what would you judge to be the essential difference between the geometry of the Egyptians and the geometry of the Greeks?

4. Judging from the character of the Roman, would you expect him to do much to advance the science of geometry?

5. Among the formulas given in Ahmes Papyrus for determining areas are the following: I. The area of an isosceles triangle equals half the product of the base and one of the equal sides. II. The area of an isosceles trapezoid equals half the product of the sum of the bases, and one of the equal sides.





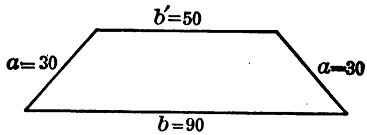
a. Using b, b_1, for the bases, and s for each of the equal sides write an algebraic formula for each of these areas, A.

b. What is the error in each of these formulas?

c. Draw figures to show that at times this error would not matter much.

d. Draw figures showing cases where the error would make a considerable difference.

e. In the accompanying diagram find just what error is made (correct to tenths) by using the Egyptian formula.



6. Another formula given in Ahmes Papyrus is that for finding the area of a circle. It tells you to multiply the square of the radius by $\frac{16}{9}$. What value must the Egyptian then have assigned to π?

7. What is meant by saying that a 3–4–5 triangle is a right triangle?

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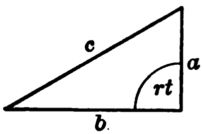
8. Show by knotting a piece of cord so that the parts have the ratio 3 to 4 to 5 how the Egyptian "rope-stretchers" obtained their east-west line. (Stretch the cord around pins on a board and after it is in place test the accuracy of the method with your right triangle.)

9. Plato and his school interested themselves in the so-called Pythagorean numbers. Such numbers are those that would represent the lengths of the sides of a right triangle.



In this kind of triangle they must be such that $a^2+b^2=c^2$. The school of Plato found that $[(\frac{1}{2}n)^2+1]^2=n^2+[(\frac{1}{2}n)^2-1]^2$, and hence that $(\frac{1}{2}n)^2+1$, n, and $(\frac{1}{2}n)^2-1$ were Pythagorean numbers.

a. Verify the statement.

b. Find ten sets of Pythagorean numbers.

10. Pythagoras himself found that n, $\frac{1}{2}(n^2-1)$, and $\frac{1}{2}(n^2+1)$ were numbers such as described in the last exercise. Verify this statement.

11. Bramagupta, a Hindu writer of the seventh century, gave p, $\frac{1}{2}(\frac{p^2}{q}+q)$, and $\frac{1}{2}(\frac{p^2}{q}-q)$ as Pythagorean numbers.

a. Give various values to p and q to test his statement.

b. Verify his statement.

12. In the Culvasutras, a Hindu manuscript, directions for constructing a right angle are as follows: Divide a rope by a knot into parts 15 and 39 units in length respectively, and fasten the ends to a piece 36 units in length.

a. Draw a diagram to show what is meant by this.

b. Check to see whether these are Pythagorean numbers.

c. Is it true that all numbers having the ratio of these three are Pythagorean numbers?

13. Archimedes proved that the value of π lay between $3\frac{1}{7}$ and $3\frac{10}{71}$. How does this compare with the value we use to-day?

B. A FEW ILLUSTRATIONS OF GEOMETRIC FORM

Before we begin a systematic study of geometry, let us see if we can find any illustrations of the kind of forms about which we hope to learn something. Do we not find such forms in nature? We

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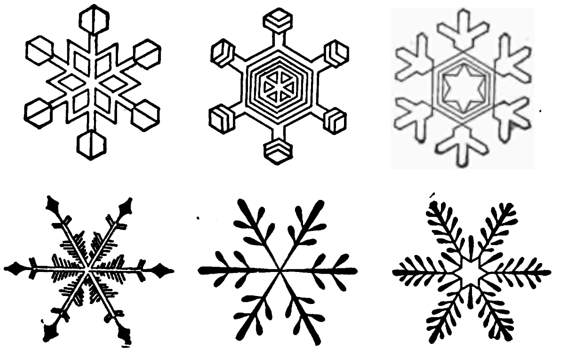
recall symmetrical trees and conical mountains; we think of a circular moon, spherical raindrops and crystals of many forms.

Symmetrical tree

Conical mountain



Crystals

EXERCISES. SET II. ILLUSTRATIONS OF GEOMETRIC FORMS

14. Make a list of some geometric forms you have found in nature, under the following heads: spherical, conical, cylindrical, prismatic, circular, etc.

Aside from natural objects, geometric forms continually appear in the works of man. The building and room in which we are, the furniture, windows, doorways, all are geometric in form. The familiar objects of our daily life—coins, boxes, cylindrical tubes, balls—all illustrate the application of geometrical principles.

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15. As in the preceding exercise, make a list of some geometric forms you can find in the works of man.

C. MEANING OF GEOMETRIC FORMS

If we note these forms carefully, we see that they are various combinations of the simple elements—points, lines, surfaces and solids. At the outset, therefore, we should be sure that our ideas about these elements are correct.

First let us consider a **geometric solid.** We have seen cones made of wood, and others of ice cream; we have looked into a well and said it was cylindrical; we have watched a soap bubble and called it spherical. So we see that it is the shape or form, and not the substance of which an object is made, to which we refer in speaking of a geometric solid. When we mention a sphere, we mean the space which it occupies or its shape without reference to its physical properties or the material of which it is made.

If, as we have just noted, a solid is a limited portion of space, what limits it? How is a solid separated from the rest of space? The boundaries of a solid are **surfaces,** and since a solid is identified, not by its material, but only by its shape, so must its boundary be identified by its shape. A chalk box, for instance, is in the form of a prism, *i.e.*, the space it occupies is a geometric prism. The boundaries of this prism are called surfaces, and they divide it from the rest of space.

The surfaces meet and form **lines.** The edges of the chalk box are referred to as the union of its sides. Now if we think of the geometric prism—the space occupied by the box—the intersection of the surfaces are lines.

It is evident, then, that lines crossing form **points.** *A limited portion of space is called a solid, the boundaries of a solid are called surfaces, the intersections of surfaces are called lines, and the places where lines cross are called points.*

We see thus that a point is a place or position, and can, therefore, have no length, breadth, or thickness. For convenience, we represent a point by a dot of lead, ink, or chalk. Such a dot is obviously not a point, because it has some size, however small it may be, but it marks a location, which is the real point.

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If we could imagine a point to move, we would call its path a line. A line, then, would have no width, but it would have length. If we could now imagine a line to move, not along itself, we would say it generated a surface. Again, the surface would have only length and width, but, of course, no depth. If we consider a surface to move, not along itself, it would form a solid, which would have three dimensions.

In our study of geometry we shall have to deal with straight and curved lines. Let us note, then, that a **straight line** *is one which is fixed (or determined) by any two of its points*, and a **curved line** *is one no part of which is straight.* Throughout the book, it is to be understood that when the word line is used without a qualifying adjective, a straight line is designated. A line is of indefinite length, so that when we wish to refer to *a limited portion of a line we shall call it a* **sect.** All the facts with which we shall be concerned for a time will be those relating to a single plane. A **plane** *is a surface such that if any two points in it be connected by a straight line, that line lies wholly within the surface.*

EXERCISES. SET III. MEANING OF GEOMETRIC FORMS

16. If a series of 600 points were put within an inch would they form a line?

17. A machine has been manufactured which will rule 10,000 distinct lines within the space of one inch. Are these lines geometric?

18. Fold over a piece of paper. What will the crease represent?

19. If oil is poured on water, of what material is the surface formed?

20. Put your foot in a heap of snow and quickly withdraw it. Is the impression that is left a physical or geometric solid?

21. If I place a piece of red paper on a blue one, what is the color of the surface between them?

22. If 1000 geometric surfaces were placed one on top of the other would a geometric solid be formed?

23. Is a cake of ice a geometric solid?

24. (*a*) Make a list of some things in life which are referred to as points. (*b*) How many of these are geometric points?

25. (*a*) Make a list of some things in life which are referred to as lines. (*b*) How many of these are geometric lines?

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D. SUGGESTIONS OF A FEW USES OF GEOMETRY

We have reviewed briefly the historical development of geometry and have called to mind illustrations of geometric forms, both in nature and in manufactured articles and have clarified our ideas of these forms.

Let us now consider a few of the uses of geometry. The subject grew out of the need of land-measuring. Hence, historically at least, surveying is the first known use of geometry. The following illustrations show some of the things that we ought soon to be able to do. Laying out boundaries of property so that the owner shall have his just share, or finding the areas of pieces of ground, are problems requiring practical mensuration

Diagram I

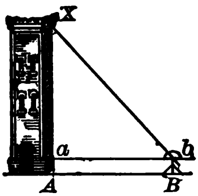
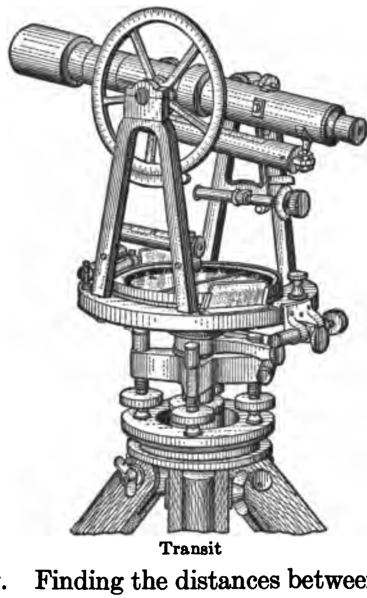


Diagram II

Diagram III



Transit

about which we shall soon study. Finding the distances between inaccessible points, as from X to Y in diagram I, across rivers and

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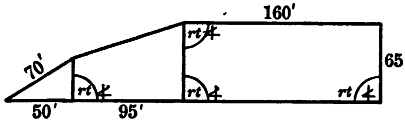
over swamps, as in Exercises 111–113, and heights of objects as in diagrams II and III, are all geometric problems such as are included in surveying. We have all seen men in our streets with transits on tripods. (A transit is an instrument for measuring angles.) Geometry is necessary to solve the problems for which they are getting the data.

EXERCISES. SET IV. MENSURATION

26. Make a list of mensuration formulas with which you are already familiar.

27. Find the area of the following piece of property (three lots). The measurements taken by a surveyor are noted on the diagram.

Other surveying problems will be found later in the book. We do not yet know enough geometry to solve many such problems.



Another use of geometry that quickly comes to the mind is designing. We mark the use of geometric design in parquet floors, linoleums, tilings, wall and ceiling papers, grill-work, stained-glass windows, arches, and in similar objects. By learning how to make five fundamental constructions (Exercises 28–32, inc.), we shall be able to combine them into many geometric designs, and thus get a clearer idea of one of the uses of geometry. The reasons why these constructions are correct, and more elaborate work in design, must be postponed until later in the text.

EXERCISES. SET V. CONSTRUCTIONS—DESIGNING

All the constructions in these exercises are to be made with the use of compasses and unmarked straight edge only. The pupil is reminded that unfamiliar technical terms will be found by referring to the index.

In general, in geometry, auxiliary lines (those needed only as aids) are indicated by dotted lines, preferably light.

28. From a given point on a given straight line required to draw a perpendicular to the line.

Let AB be the given line and P be the given point.

It is required to draw from P a line perpendicular to AB.

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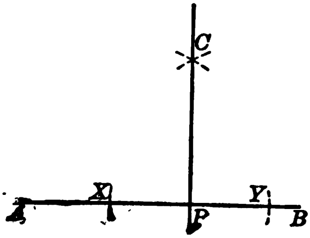
With P as center and any convenient radius strike arcs cutting AB at X and Y.



With X as center and XY as radius strike an arc, and with Y as center and the same radius strike another arc, and call one intersection of the arcs C.

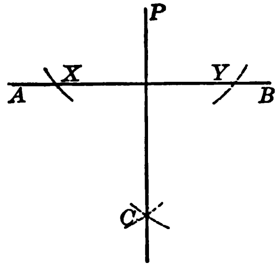
With a straight edge draw a line through P and C, and this will be the perpendicular required.

29. From a given point outside a given straight line required to let fall a perpendicular to the line.

Let AB be the given straight line and P be the given point.

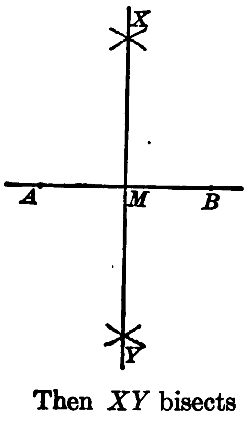
It is required to draw from P a line perpendicular to AB.

With P as center and any convenient radius describe an arc cutting AB at X and Y.



With X as center and any convenient radius describe an arc, and with Y as center and the same radius describe another arc, and call one intersection of the arcs, C.

With a straight edge draw a straight line through P and C, and this will be the perpendicular required.



It is interesting to test the results in Exercises 28 and 29 by cutting the paper and fitting the angles together.

30. Required to bisect a given sect.

Let AB be the given sect.

It is required to bisect AB.

With A as center and AB as radius describe an arc, and with B as center and the same radius describe another arc.

Call the two intersections of the arcs X and Y.

Draw the straight line XY.

Then XY bisects the sect AB at the point of intersection M.

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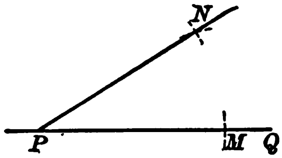
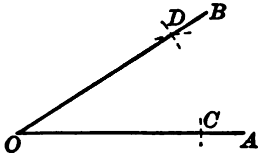
31. From a given point on a given line required to draw a line making an angle equal to a given angle.



Let P be the given point on the given line PQ, and let angle AOB be the given angle.

What is now required?

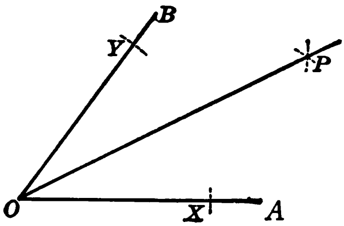
With O as center and any radius describe an arc cutting AO at C and BO at D.



With P as center and OC as radius describe an arc cutting PQ at M.

With M as center and CD as radius describe an arc cutting the arc just drawn at N, and draw PN.

Then angle MPN is the required angle.



32. Required to bisect a given angle.

Let AOB be the given angle.

It is required to bisect the angle AOB.

With O as center and any convenient radius strike an arc cutting OA at X and OB at Y.

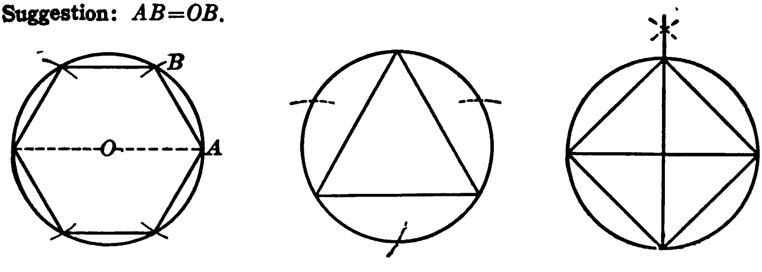
With X as center and the sect XY as radius strike an arc, and with Y as center and the same radius strike an arc, and call one point of intersection of the arcs P.

Draw the straight line OP.

Then OP is the required bisector.

33. Make constructions similar to the following:

Suggestion: $AB = OB$.



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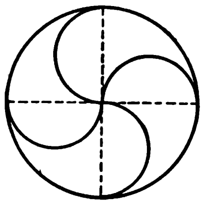
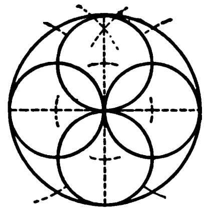
34. Draw the following figure.

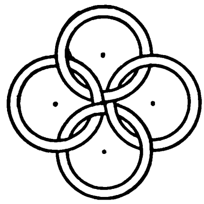


35. Make a construction similar to *a*.



a

d36.* Copy *b*.



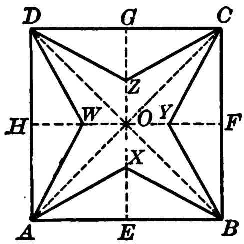
b

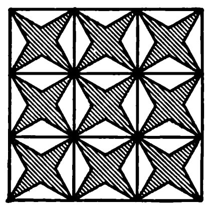
37. From a study of Exercise 28, suggest how to erect a perpendicular at the end of a sect.

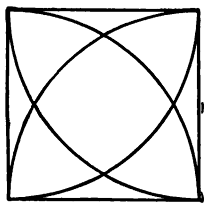
38. On a given sect construct a square.

39. These figures show a parquet floor design, and one of the units of the design enlarged. Construct figures similar to these.

ABCD is a square, and *X*, *Y*, *Z*, and *W* are the mid-points of the semidiameters *OE*, *OF*, *OG*, *OH*, respectively.





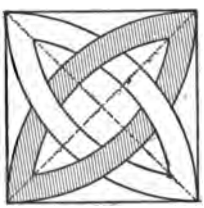


40. Make a construction similar to the adjoining figure.

The vertices of the square are used as centers for four of the arcs.

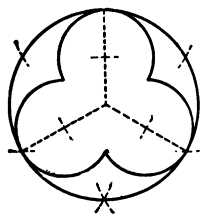
The radius equals one side of the square.

41. Make a construction similar to *a*.



a

42. Make a construction similar to *b*.



b

* As here, d will be prefixed to any exercise which the student is likely to find difficult at this stage.

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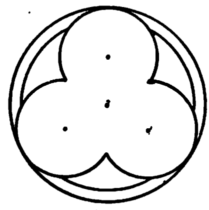
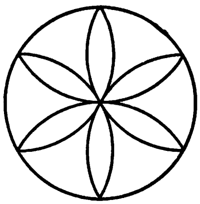
d43. Make a construction similar to the following:

The figure is based on an equilateral triangle, the centers of the interior arcs being the midpoints of radii drawn to the vertices of the equilateral triangle inscribed in a circle (*i.e.*, having its vertices on the circle).

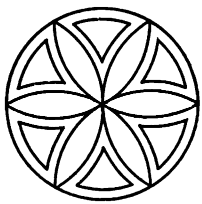


Note: See Fig. 2, exercise 33.

N. B.—The design shown in Mable Sykes, "Source-Book of Problems for Geometry," page 160, II, 5 (Fig. 138a), shows a good application of exercise 43.



a

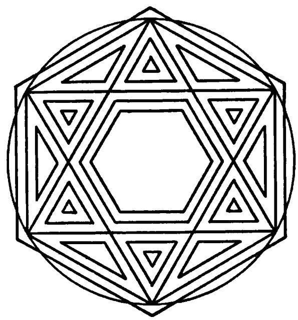


b

44. Make a construction similar to *a*.

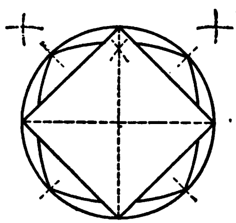
d45. Make an ornamental design similar to *b*. The circle is divided into how many equal arcs? How many degrees in each central angle?

What kind of triangle is formed by two consecutive radii and the sect joining their ends? What other method does this suggest of dividing a circle into six equal arcs?



46. Make an ornamental drawing similar to the one in the accompanying figure. Describe the construction.

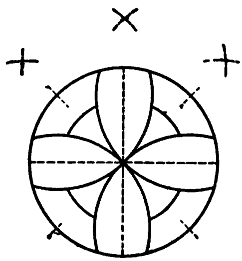
Suggestion: First draw an equilateral polygon with six sides in a circle.



47. Construct a six-pointed star.

48. Bisect each of the four right angles formed by two lines intersecting each other at right angles.

49. Make constructions similar to the following:



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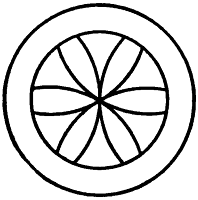
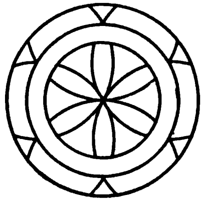
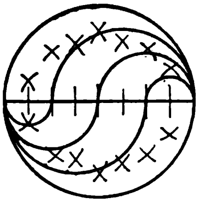
50. Make constructions similar to the following:





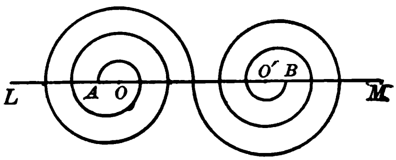
In such figures artistic patterns may be made by coloring various portions of the drawings.



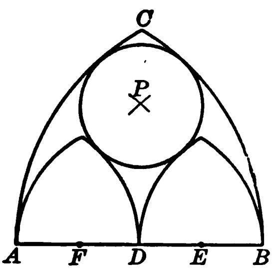
In this way designs are made for stained-glass windows, oil-cloth, colored tiles, and other decorations.

51. Draw a sect of any convenient length, and upon it construct a design similar to the one in the figure.

52. On a line *LM* take a sect *AB*. Divide it into 8 equal parts. With your compasses make an ornamental scroll as shown in the diagram.



53. Using the hints given, make a copy of the accompanying outline drawing of a Gothic window. The arc *BC* is drawn with *A* as center and *AB* as radius. The small arches are described with *A*, *D*, and *B* as centers and *AD* as radius. The center *P* is found by taking *A* and *B* as centers and *AE* as radius. How may the points *D*, *E*, and *F* be found?



54. In many different machines, such as the sewing machine, printing press, etc., there is a wheel called a cam, which is used to modify the motion of the machinery. Cams are constructed in various shapes and dimensions, depending upon the use for which they are designed. The figure shows the method of drawing the pattern of a heart-shaped or "uniform-motion" cam. Let

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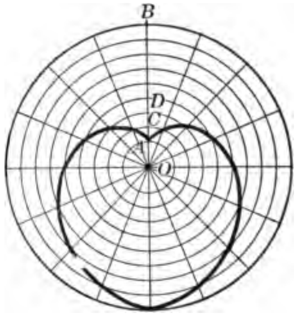
the "throw" be AB and the center O. Divide AB into eight equal parts at C, D, etc. Through $A, C, D, \ldots, B$, draw circles with centers at O. Draw sects dividing the angular magnitude around O into sixteen equal parts. Beginning at A, mark the points where the consecutive circles and consecutive sects intersect, and through these points draw a smooth curve, as in the figure.



Draw such a cam with AB equal to a given sect m, and OA equal to a given sect n.

(Taken with modifications from Stone-Millis, Elementary Plane Geometry.)

55. Select and copy some geometric design.

56. Make an original design based on the fundamental constructions given in exercises 28–32.

Pupils particularly interested in this part of the work are referred to: Sykes, Mabel, "Source Book of Problems in Geometry." (Pub. Allyn and Bacon.)

Geometry is used in architecture. Whether the architect is drawing the plans for an ordinary dwelling-house or a massive cathedral, he is constantly concerned with geometric forms and constructions. Consider for a moment what problems of this character must have confronted the architect of some large building in our community.

The list of the direct uses of geometry would be very long if complete. In a few sentences let us, therefore, simply enumerate a few more miscellaneous suggestions for its uses. Problems scattered throughout this book show more concretely how geometry is used in the cases enumerated. In making all kinds of diagrams, reducing and enlarging maps, the principles of geometry are applied. In engineering, geometry is needed for such matters as laying out railroads, and planning the constructions of machines, bridges, and tunnels; and in astronomy, ascertaining the altitude of stars and similar problems require geometric principles.

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EXERCISES. SET VI. SOME OTHER USES OF GEOMETRY

57. State any other uses of geometry which you know.

58. At the entrance to New York Harbor is a gun having a range of 12 mi. Draw a line inclosing the range of fire, using any convenient scale.

59. Two forts are placed on opposite sides of a harbor entrance, 13 mi. apart. Each has a gun having a range of 10 mi. Draw a plan showing the area exposed to the fire of both guns, using any convenient scale.

60. Make an accurate diagram of a tennis court or a foot-ball field noting all lime lines.

d61. Draw to a convenient scale a plan of the ground floor of your school building.

E. THE BASIC PRINCIPLES OF GEOMETRY

As in all scientific work of an exact nature, the discoveries in geometry rest upon a few basic principles. These may be classified under three heads: *definitions*, *axioms*, and *postulates*.

You all can probably recall having heard people argue most heatedly about some question and reach no conclusion at all. This is often the case simply because when two people argue, they very often do so without having clearly in mind the conditions about which they are arguing. In all debates or discussions it is essential that we start with the same premises, and our work in geometry should help us to learn to collect our premises in orderly fashion.

The premises upon which the early parts of geometry rest are to a great extent definitions, and it is therefore very necessary that we have a clear image and definition of each new technical term we meet. The wording of our definitions may differ, but the content must be the same. Every good definition should include all that may fall under a particular class, and clearly exclude all that does not fall under that class. Suppose, for instance, we want to define the word botany. We might say, to begin with, that it is a science—but we have not differentiated it from the physical sciences, so we say it is a natural science. But, then, so is zoölogy. Hence it is necessary to differentiate still further, and say it is the natural science which deals with plant life. Now have we fully

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and finally defined it? Can you possibly think of any science—now that it is thus defined—with which it can be confused? Make all possible tests, and if you find that it cannot be confused with any other science, well and good,—then we have found an acceptable definition.

Throughout this book no word will be defined until we are ready to make use of it, but then it will be our duty to see the word in its full meaning.

From the knowledge we have of algebra, we already know what some of the **axioms** are, and some uses to which they may be put, though we may not have defined the word "axiom." Some of the axioms for which we shall have immediate use are:

1. *The sums of equals added to equals are equal.*

Example

If $5=5$

and $\underline{a=b}$

then $5+a=5+b$.

2. *The remainders of equals subtracted from equals are equal.*

Example

If $a=c$

and $\underline{b=d}$

then $a-b=c-d$

3. *The products of equals multiplied by equals are equal.*

Example

If $a=x$

and $\underline{b=y}$

then $ab=xy$.

4. *The quotients of equals divided by equals are equal.*

Example

If $x=y$

and $\underline{m=p}$

then $\frac{x}{m}=\frac{y}{p}$

Cases in which the divisor is zero will not be considered in this text.

5. *A quantity may be substituted for its equal in a statement of equality or inequality.*

Example

If $x=5$	If $a=b$
and $\underline{x+y=7}$	and $\underline{2a+5>a+2}$
then by substitution $5+y=7$	then $2b+5>a+2$

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6. *Two quantities which are equal to equal quantities, are equal to each other.*

Example If $a = b$
$b = c$
and $c = d$,
then $a = d$

7. *The whole is equal to the sum of its parts.*

Example $\frac{a}{2}+\frac{a}{3}+\frac{a}{6}=a$

NOTE: "part" is here used in the sense of a common or vulgar fraction.

Other axioms will be stated as we need them. Thus we see that an *axiom is the statement of a general mathematical truth which is granted without any proof.*

A **postulate** *is simply a geometric axiom.* That is, it is the statement of a geometric truth which is granted without any proof.

We shall now attempt to formulate a few such truths.

EXERCISES. SET VII. ILLUSTRATIONS OF POSTULATES

62. Why is it shorter to cut across a field than to go around it?

g63.* How many pairs of roots are there when two simultaneous linear equations are solved? What is the graphic explanation of this?

g64. Why is it that in making the graph of a linear equation, such as $x+y=13$, we need to plot but two points, and that a third point may be used to check the correctness of our work?

65. Why is it that the Panama Canal is a great advantage over the route formerly used to reach a point on the western coast of South America from the West Indies?

66. Why is it that in putting up a croquet set all one needs to do to get the wickets in line with the stakes is to tie a string tightly to one stake and stretch it and fasten it to the other stake?

* As here, g will be prefixed to any exercise in this text which presupposes an acquaintance with the graph. The pupil is here referred to M. Auerbach, An Elementary Course in Graphic Mathematics (Allyn and Bacon), pp. 29–31, for review, and previous pages in the same if the subject is new to him.

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67. What does the bricklayer do to get a row of bricks in a straight line? Why? Does the gardener do anything similar to this?

68. Point out which of the following postulates upholds each of your answers to exercises 62–67.

POSTULATES OF THE STRAIGHT LINE NEEDED IN PROOFS

1. *Two intersecting straight lines determine a point.*
2. *Two points determine a straight line.*
3. *A straight line is the shortest distance between two points.*

69. How often can two straight lines intersect?

70. Use your answer to the last question to state postulate 1 in another way.

71. How many straight lines can be drawn between two points?

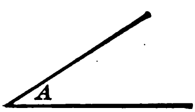
72. Use your answer to the last question to state postulate 2 in another way.

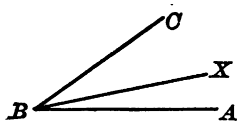
73. Give at least one good illustration of how each of the three postulates mentioned may be used in practical life.

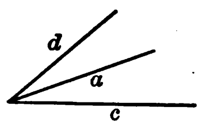
We have used several other postulates in making some of the constructions on pages 13 to 18. They are: (1) *A sect may be produced indefinitely.* (2) *A circle may be described with any point as center and any sect as radius.* (3) *A point and direction determine a straight line.* But these are so exceedingly obvious that we shall not feel obliged to quote them.

DEFINITIONS

An **angle** *is the opening between two lines. The lines are called the* **sides** *of the angle, and the point at which they meet the* **vertex.**







An angle may be named in any one of three ways as $\measuredangle A$ in Fig. 1 where there is no danger of confusion, or as in Fig. 2 $\measuredangle ABC$, $\measuredangle ABX$, $\measuredangle XBC$ where there are several angles (the vertex always being read second), or again as in Fig. 3 $\measuredangle ca$, $\measuredangle cd$, $\measuredangle ad$.

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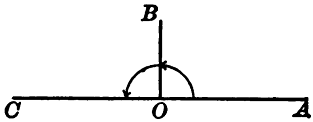
1. Kinds of angles defined according to individual size.

A **straight** *angle is one whose sides run in opposite directions so as to form a straight line.*

$\measuredangle AOB$ is a straight angle.

A **right** *angle is one-half a straight angle.*



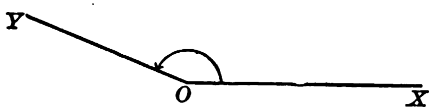
$\measuredangle AOB$ and $\measuredangle BOC$ are right angles.

An **acute** *angle is less than a right angle.*

$\measuredangle RST$ is an acute angle.

An **obtuse** *angle is greater than a right angle but less than a straight angle.*

$\measuredangle XOY$ is an obtuse angle.



2. Kinds of angles defined according to sums.

Complementary *angles are two whose sum is a right angle.*

Supplementary *angles are two whose sum is a straight angle.*

EXERCISES. SET VIII. SUMS OF ANGLES IN PAIRS

74. Construct: (*a*) two complementary angles whose ratio is 1 to 3; 3 to 5.

(*b*) Two supplementary angles whose ratio is 1 to 3; 3 to 5.

MEASUREMENT OF ANGLES

There are three systems of measurement of angles. The one probably known to most of us is the sexagesimal system, and was mentioned on page 6. It divides the entire angular magnitude about a point into 360 parts, each of which is called a degree; these again are divided into sixtieths, each of which is called a minute, and each minute is again divided into sixtieths, each of which is called a second.

The size of an angle then depends upon the amount of opening between its sides. The amount of opening depends upon the

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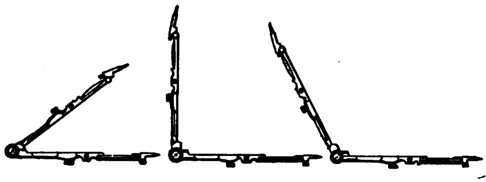
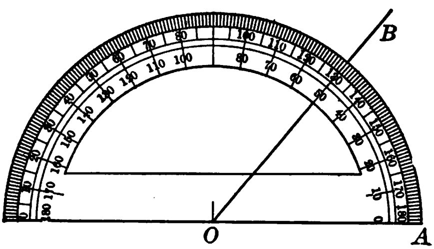
amount that one side has to revolve to bring it into the position of the other, and the greater that amount the greater the angle. Thus in these compasses the first angle is smaller than the second, which is also smaller than the third. The length of the sides has nothing to do with the size of the angle.



A special instrument, called a protractor, is frequently used for angle measurements. The figure given below represents one form of the protractor. By joining the notch O of the protractor to each graduation mark we obtain a set of angles at O, each usually representing an angle of one degree.

To measure a given angle with the protractor, place the notch of the protractor at the vertex of the angle, and the base line along one side of the angle. The other side of the angle then indicates on the protractor the number of degrees in the angle. Thus the angle AOB contains 50°.



To draw an angle of a given number of degrees, place the base of the protractor along a straight line and mark on the line the position of the notch O. Then place the pencil at the required graduation mark, and (after removing the protractor) join the point so marked to O.

EXERCISES. SET IX. MEASUREMENT OF ANGLES

75. Show how a fan can be used to illustrate the idea of angular magnitude.

76. What kinds of angles are formed by the hands of a clock at (1) two o'clock, (2) three o'clock, (3) four o'clock, (4) six o'clock?

77. What kind of angle is equal to (1) its complement, (2) its supplement?

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78. What kind of angle is less than its supplement?

79. What angle is 7° less than one-third its complement?

d80. State as a formula (1) the number of degrees in the complement of the supplement of any angle *a*, (2) the number of degrees in the supplement of the complement of angle *a*.

81. Two supplementary angles are in the ratio of 7 to 2. Find the number of degrees in each.

82. If four lines *a*, *b*, *c*, *d*, are drawn from a point *O* in the order given, so that *a* is perpendicular to *c*, and *b* is perpendicular to *d*, find $\sphericalangle ad$ if $\sphericalangle bc$ is 60°. (See definition, page 31.)

83. Three angles together make up the angular magnitude about a point. The first is 10° greater than the second, and the second is 17° greater than one-half the third. How many degrees in each?

84. Find that angle whose supplement is eight times its complement. Is it possible to find one whose supplement is one-eighth of its complement?

85. How many degrees in the angle which exceeds one-third its complement by 15°?

86. Find the number of degrees in the angle whose excess over its complement is one-fourth the difference between its complement and itself.

87. Construct two straight angles. Cut one out and place it on the other. What can you say of them? Do you think it is true of all straight angles?

88. Construct two right angles. (*a*) How is each related to a straight angle? (*b*) How are they related to each other? Why?

89. Using a protractor (*a*) construct three angles each of which is the complement of 20°. (*b*) Construct two angles each of 30°. Construct their complements. (*c*) What conclusions can you draw from (*a*) and (*b*)? (*d*) Why?

90. Do the same as you did in exercise 89 for the supplements of 50° and 60°.

Corollary. *A truth that is directly derived from another is called a corollary.* The conclusions drawn in exercises 88–90 are called corollaries, since they are all directly derived from the conclusion drawn in exercise 87.

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POSTULATES OF THE ANGLES

1. *All straight angles are equal.*

Cor. 1. ***All right angles are equal.*** Why?

Cor. 2. ***Complements of the same angle or equal angles are equal.*** Why?

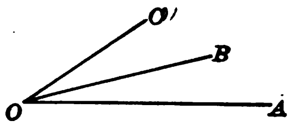
Cor. 3. ***Supplements of the same angle or equal angles are equal.*** Why?

Perpendicular. *A perpendicular is a line that meets another at right angles.*

3. Kinds of angles defined according to relative position.

Adjacent angles *are two that have a common vertex and a common side lying between them.*

∢*AOB* and ∢*BOC* are adjacent.

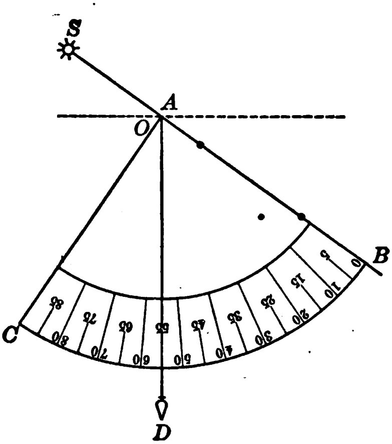


EXERCISES. SET X. RELATIVE POSITION OF ANGLES

91. Why is it that ∢*AOB* and ∢*AOC* are not adjacent angles?

92. Can you name other angles in the diagram which are not adjacent?

93. Tycho Brahe (1546–1601), a Danish nobleman who built and operated the first astronomical observatory, in his earliest observations used a quadrant for measuring the altitudes of stars, or their angular distances above the horizon. Show that when the instrument was held in a vertical plane, and the sights *A* and *B* aligned with the star *S*, the altitude of the star was determined by observing the angle *CAD*.



(Taken from Stone Millis, Elementary Plane Geometry.)

94. Make such a quadrant of cardboard or wood and

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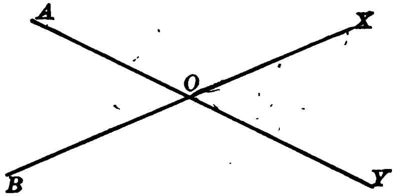
use the method of exercise 93 to find the elevations of objects in the neighborhood such as trees, hills, steeples, telephone-poles, etc.



Vertical angles *are those which are so placed that the sides of each are the prolongations of the sides of the other.* OX is the prolongation of BO, and OY is the prolongation of AO. Therefore $\measuredangle YOX$ and $\measuredangle AOB$ are vertical angles.

EXERCISES. SET X (continued)

95. Name two other angles in the preceding figure which are vertical, and tell why they are vertical.

96. Classify angles according to (1) individual size, (2) relative size, (3) relative position.

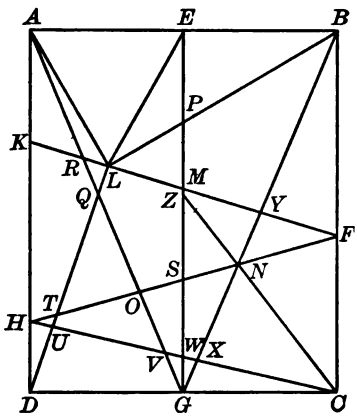
97. (*a*) Draw two complementary-adjacent angles.

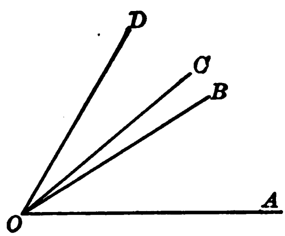
(*b*) Draw two angles of the same size as those in (*a*) but not adjacent.

(*c*) Draw two supplementary-adjacent angles.

(*d*) Draw two non-adjacent supplementary angles.

98. In the accompanying diagram select those angles which are straight, right, acute, obtuse, complementary, supplementary, adjacent, and vertical.





99. Read the angle which is equal to:

(*a*) $\measuredangle AOB + \measuredangle BOC$.

(*b*) $\measuredangle AOC + \measuredangle COD$.

(*c*) $\measuredangle AOB + \measuredangle BOC + \measuredangle COD$.

(*d*) $\measuredangle AOC + \measuredangle COD - \measuredangle BOD$.

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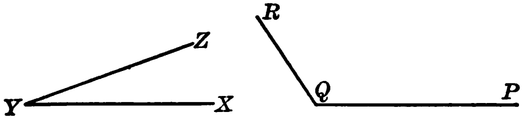
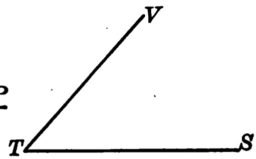
100. Construct an angle equal to:





(*a*) The sum of these three angles.

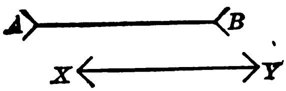
(*b*) The sum of ∢PQR and ∢XYZ less ∢STV

101. Show that sects bisecting two complementary-adjacent angles form an angle of 45°.

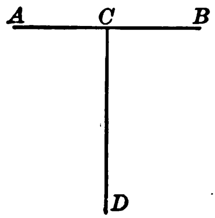
102. What kind of angle do sects bisecting two supplementary-adjacent angles form? Prove your answer.

103. The following will illustrate the unreliability of observation and the need of logical proof.

(Subdivisions (*a*) through (*f*) were taken from Wentworth-Smith, Plane Geometry, and (*g*) through (*i*) from Hart and Feldman, Plane Geometry.)

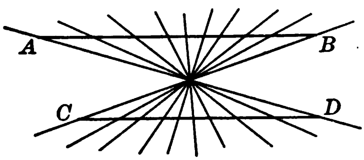


(*a*) Estimate which is the longer sect, $\overline{AB}$ or $\overline{XY}$, and how much longer. Then test your estimate by measuring with the compasses or with a piece of paper carefully marked.



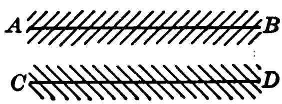
(*b*) Estimate which is the longer sect, $\overline{AB}$ or $\overline{CD}$, and how much longer. Then test your estimate by measuring as in (*a*).

(*c*) Look at this figure and state whether



AB and CD are both straight ines. If one is not straight, which one is it? Test your answer by using a ruler or the folded edge of a piece of paper.

(*d*) Look at this figure and state whether $\overline{AB}$ and $\overline{CD}$ are the same distance apart at A and C as at B and D. Then test your answer as in (*a*).



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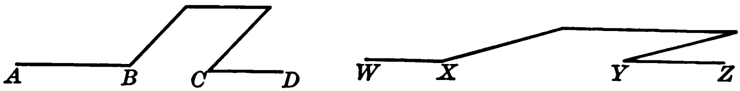
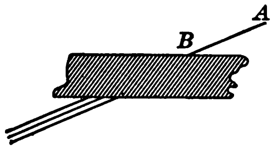
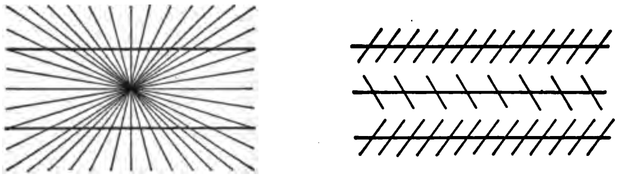
(*e*) Look at this figure and state whether AB will, if prolonged, lie on CD. Also state whether WX will, if prolonged, lie on YZ. Then test your answer by laying a ruler along the lines.



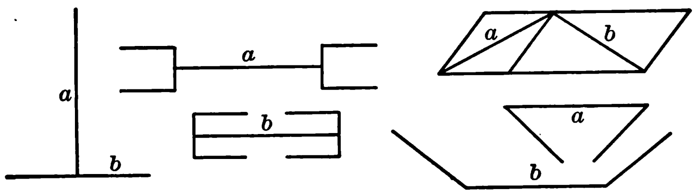


(*f*) Look at this figure and state which of the three lower lines is AB prolonged. Then test your answer by laying a ruler along AB.

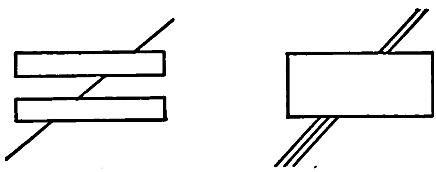
(*g*) In the figures below, are the lines everywhere the same distance apart? Test your answer by using a ruler or a slip of paper.



(*h*) In the diagrams given below, tell which sect of each pair is the longer, *a* or *b*, and test your answer by careful measurement.



(*i*) In the figures below, tell which lines are prolongations of other lines. Test your answers.



104. (*a*) Draw two unequal supplementary-adjacent angles.

(*b*) Extend the common side of these angles through the vertex, and call the angles thus formed α, β, γ, δ.

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(*c*) What relation exists between α and β?

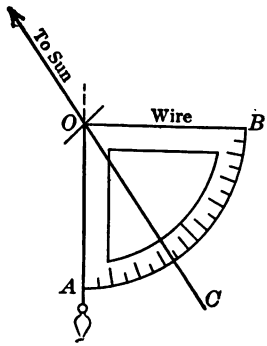
(*d*) What relation exists between β and γ?

(*e*) What further relation do you notice that is based upon the relatons stated in (*c*) and (*d*)?

(*f*) Do the same, using angles β, γ, δ.

(*g*) Do the same, using angles γ, δ, α.

(*h*) What conclusion can you draw?



105. If a plumb-line is fastened to a horizontal wire nail at the vertex of the angle of a quadrant, and the quadrant is turned so that the plumb-line falls along 90° (here indicated by *OA*), by noting where the shadow of the nail strikes the quadrant the angular altitude of the sun may be obtained. Explain why *OC* in the diagram gives the angular altitude of the sun.

INSTRUMENTS FOR MEASURING ANGLES

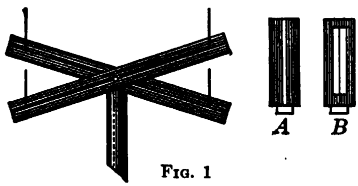


FIG. 1

Surveyors and engineers employ for measuring angles costly instruments called theodolites.* An inexpensive substitute for a theodolite is shown in the accompanying figure 1. It consists of

* The identical theodolite with which the historic Mason-Dixon line, between Maryland and Pennsylvania was run, settling a controversy of a century growing out of the overlapping charters of Charles I to Lord Baltimore and Charles II to William Penn, has lately become the possession of the Royal Geographical Society, of London, through Edward Dixon, descendant of Jeremiah Dixon, who used it. Mason and Dixon had observed, for the Royal Society, the transit of Venus at the Cape of Good Hope in 1761, and did their American work two years later. When the line was resurveyed 150 years later, by the Coast and Geodetic Office of Washington, it was proved to be exceptionally accurate, with no errors of latitude of more than two or three seconds—certainly a creditable result for the time and the primitive instrument with which the work was done. The Mason-Dixon theodolite has two sights, a large compass in the center of its horizontal plate, and is adapted for measuring either horizontal circles or magnetic bearings. The graduated circle is twelve inches in diameter, divided into five minutes, and read by a single vernier.

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two pieces of wood shaped like rulers mounted on a vertical axis, by a pin driven through their exact centers. The vertical needles inserted near the end of the rulers are used for sighting. In place of the needle nearest the eye, it is better to employ a thin strip of wood, A, having a fine vertical slit; and in place of the other needle, a vertical wire fixed in a light frame, B. By the help of this instrument, and a protractor, one can measure with considerable accuracy an angle on the ground; for instance, the angle MON (figure 2).

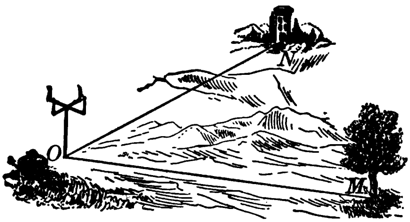
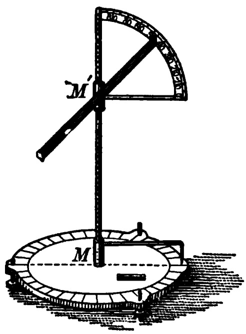


FIG. 2

The following is a simple substitute for the theodolite. By means of it angles may be measured in both horizontal and vertical planes. The vertical rod MM' is free to revolve in the socket at M, carrying a horizontal pointer which indicates readings on the horizontal circle divided into degrees. These divisions must be marked. The pointer at M' is provided with sights and is free to move in a vertical circle around M'. By sighting along this pointer, vertical angles may be measured on the quadrant.



(This instrument was suggested in Betz and Webb, Plane Geometry.)

EXERCISES. SET XI. INSTRUMENTS FOR MEASURING ANGLES

106. Construct an instrument such as that shown in figure 1 of the preceding section or an astrolabe or a good substitute by means of which angles may be measured in vertical and horizontal planes.

F. THE DISCOVERY OF SOME FACTS AND THEIR INFORMAL PROOF

A **theorem** *is the statement of a fact which is to be proved.*

The fact which you discovered if you worked exercise 104 and applied in 105 is one which was known to Thales about 600 B.C.

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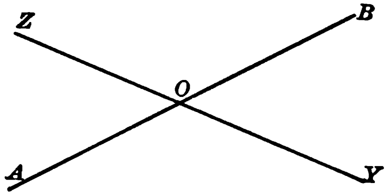
It is very important, so we shall attempt to prove it again. It is the first theorem in our syllabus.

Theorem 1. ***Vertical angles are equal.***

In the accompanying diagram, what angle is the supplement of $\measuredangle BOZ$?



What other angle is the supplement of $\measuredangle BOZ$?

What postulate can you quote to prove the equality of these two angles which are the supplements of $\measuredangle BOZ$?

In similar fashion prove that $\measuredangle BOZ \equiv \measuredangle AOY$.

A plane **polygon** *is a portion of a plane whose boundaries are straight lines. The lines which bound the polygon are called its* **sides**; *the intersections of its sides are the* **vertices** *of the polygon; and the angles formed by its sides, its* **angles.**

A **triangle** *is a polygon of three sides.* Triangles may be classified in at least two ways.

1. CLASSIFICATION BASED UPON SIDES

A **scalene** *triangle is one having no two sides equal.*

An **isosceles** *triangle is one having two equal sides.*

An **equilateral** *triangle is one having all three sides equal.*

2. CLASSIFICATION BASED UPON ANGLES

An **acute** *triangle is one in which all the angles are acute.*

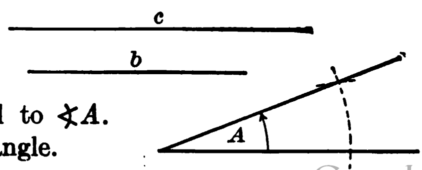
A **right** *triangle is one which has one of its angles a right angle.*

An **obtuse** *triangle is one which has one of its angles an obtuse angle.*

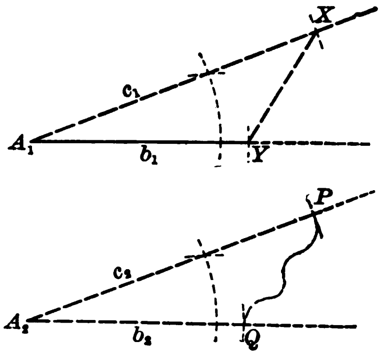
The following experiment will enable us to solve such problems as the one in Ex. 110, in which we would like to find the distance between two points separated by marshland, but for which we lack the necessary information.

EXPERIMENT

(*a*) Construct a triangle having two of its sides equal to sects b and c, and the angle whose sides they form equal to $\measuredangle A$. Construct a second such triangle.



(*b*) Cut out and tear off part of one triangle as indicated by the ragged dotted line between Q and P in the diagram, and place it upon the other so that the equal angles coincide, and so that the equal sides b_1* and b_2 fall along each other.



(*c*) Describe what happens.

(*d*) What parts of these triangles are you sure will fit on each other to start with?

(*e*) Where did P fall and where did Q fall when you had placed them as you were told to?

(*f*) What is it that finally determines that the triangles may be made to coincide throughout?

(*g*) Can you quote a postulate to uphold your statement in (*f*)?

(*h*) Repeat the process of constructing, cutting, and placing when the given $\measuredangle A$ is a right angle.

(*i*) Repeat again when the given $\measuredangle A$ is an obtuse angle.

(*j*) What conclusion can you draw?

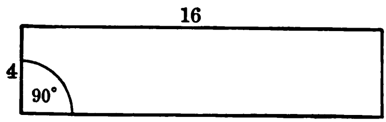
Such triangles as A_1XY and A_2PQ in the figure, which may be made to coincide throughout are said to be *congruent*.

Congruent *polygons are those which may be made to coincide throughout.* Hence they have not only the same shape but the same size. The symbol used to denote congruence is $\cong$, which is simply the sign of equality in addition to the initial letter of the word *similis* (Latin for similar) thrown down on its side.

Superposition Postulate. It may be noted that in the preceding experiment we have assumed the following fact. *Any geometric figure may be moved about without changing its size or shape.*

EXERCISES. SET XII. MEANING OF CONGRUENCE AND CLASSIFICATION OF TRIANGLES

107. Draw a polygon having the same area as the accompanying figure, but not the same shape.



*Read "b sub-one," "b sub-two," indicating that these sects equal the original sect b.

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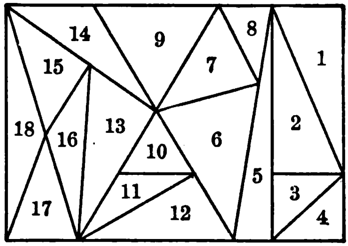
108. Draw a polygon having the same shape as that in exercise 107, but one-fourth its area.

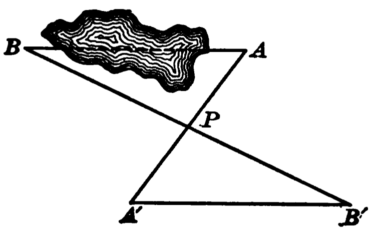
109. Classify triangles 1 through 18 in the following diagram (*a*), according to sides and (*b*) according to angles.



Just as no matter how many straight lines are drawn between two points they will all coincide with one another, so, no matter how many triangles are constructed with two sides and the angle between them, equal each to each, they may be made to coincide. We said that two points determine a straight line. Likewise we can state the second theorem of our syllabus as follows:

* **Theorem 2.** ***Two sides and the included angle determine a triangle.***

EXERCISE. SET XIII. APPLICATION OF CONGRUENCE OF TRIANGLES



110. Show how the distance from A to B can be found when because of some obstruction it cannot be measured directly.

Suppose a commander of an army wished to find the distance across a stream. He would have a problem different from that in Ex. 110, and hence that case in the congruence of triangles wouldn't help him. The following experiment would enable him to solve the problem, however, as outlined in Ex. 111.

* It is suggested that theorems marked in this way be developed in the classroom only. The proofs are introduced rather for the sake of letting the pupils realize that the sequence is a perfect chain than as a test of the pupil's power. In order to avoid any possible danger of memory work, the authors believe it wiser to omit entirely the proof of such propositions as appear to be beyond the power of the pupil to develop alone.

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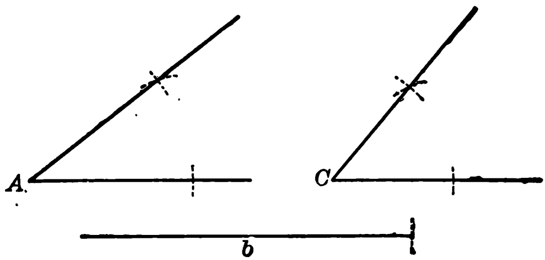
EXPERIMENT

(*a*) Construct a figure having two of its angles equal to $\measuredangle A$ and $\measuredangle C$ respectively, and the side common to the two angles equal to the sect b, as in the accompanying diagram. Construct a second such figure.

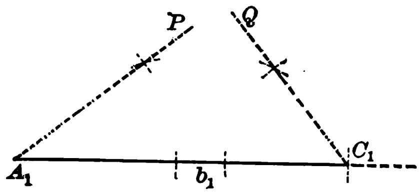


(*b*) Cut one of them out and place it upon the other so that the equal parts coincide.

(*c*) What would happen if $\overline{A_1P}$ and $\overline{C_1Q}$ were produced?

(*d*) What would happen if the lines similarly placed in the other figure were produced?

(*e*) What conclusion can you draw?



(*f*) Would all triangles having two angles and the side between them equal each to each be congruent?

(Test for cases where $\measuredangle C$ is a right angle or an obtuse angle.)

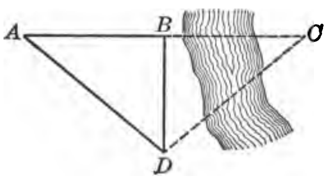
If the test is satisfactory we can state our third theorem as follows:

Theorem 3.** ***Two angles and the included side determine a triangle.

NOTE.—The discovery of Theorems 2 and 3 is attributed to Thales.

EXERCISES. SET XIII (continued)

111. If it is necessary for a commander of an army to know the distance from B to the inaccessible point C across a stream, he may find it as follows: Run a line BD at right angles to BC. Prolong CB. From D locate a point A in the prolongation of CB so that $\measuredangle BDA \equiv \measuredangle CDB$. Measure AB. Show that $\overline{CB} \equiv \overline{AB}$, and hence that $\overline{BC}$ may be found.



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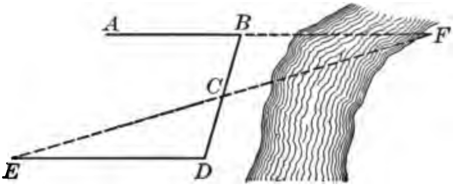
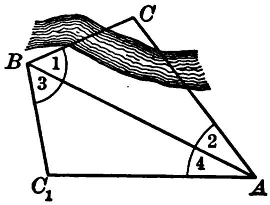
112. To measure the distance from B to the inaccessible point F, run BD in any convenient direction. Locate C, the mid-point of BD. From D, with an instrument, run DE so that $\measuredangle CDE \equiv \measuredangle FBC$, and locate E in line with C and F. Show that $BF \equiv ED$, and hence may be determined.



Such problems as 111 suggest methods that could be used in spy work, since only crude instruments need be employed.

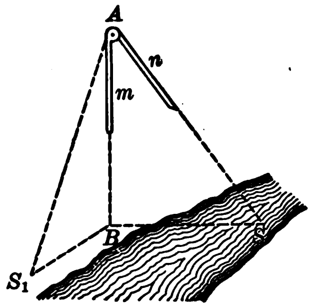
113. To find the distance $\overline{AC}$, when C is inaccessible, let B be a convenient point from which A and C are visible. Lay out a triangle ABC_1 making $\measuredangle 3 = \measuredangle 1$ and $\measuredangle 4 = \measuredangle 2$. Show that the distance $\overline{AC}$ may be found by measuring $\overline{AC_1}$.



This is a method attributed to Thales for finding the distance of a ship from shore.

114. Thales of Miletus is said to have invented another way of finding the distance of a ship from shore. This method may have been as follows:

Two rods, m and n, are hinged together at A. One arm m is held vertically while the other n is pointed at the ship S. Then the instrument is revolved about m as an axis until n points at some familiar object S_1 on the shore. Explain why $\overline{BS_1} = \overline{BS}$.



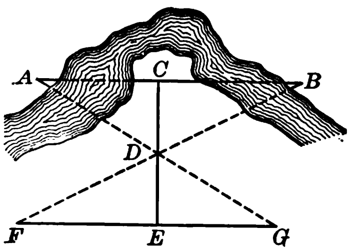


Fig. 1

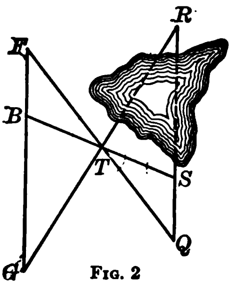


Fig. 2

d115. Tell what measurements to make to obtain the distance between two inaccessible points, A and B, in Figure 1.

d116. Explain the method suggested by the diagram in Figure 2 for finding the distance from S to the inaccessible point R.

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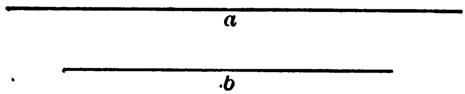
SOME PROPERTIES OF THE ISOSCELES TRIANGLE

Before we can solve many more practical problems it will be necessary for us to collect a number of geometric facts. This we will do *first* in a simple fashion—by experimenting—and *second* by actual proof, for though many of us may already believe them to be true we must be ready to convince others.

EXPERIMENT

(*a*) Construct an isosceles triangle having sect *b* as its base, and each of its equal sides equal to sect *a*. Bisect its vertex angle (*i.e.*, the angle included by the equal sides).



(*b*) Cut the triangle out, crease along the bisector of the vertex angle and see what happens.

(*c*) Are there any congruent triangles? If so can you tell why they are congruent?

(*d*) Make a list of three other facts you have thus discovered concerning the isosceles triangle.

(*e*) Test to see if these facts are all true when the base *b* is greater than the equal sides.

By the bisectors of the angles of a triangle, those sects of the bisectors are indicated which are terminated by the opposite sides.

The facts discovered in the last experiment may be stated as corollaries to the fact that the bisector of the vertex angle of an isosceles triangle divides it into two congruent triangles. We find that the parts of those triangles similarly placed with respect to the parts known to be equal to begin with, are equal. Such parts are called **homologous.** Homologous parts of congruent polygons are always equal.

Theorem 4. ***The bisector of the vertex angle of an isosceles triangle divides it into congruent triangles.***

Cor. 1. ***The angles opposite the equal sides of a triangle are equal.***

Cor. 2. ***The bisector of the vertex angle of an isosceles triangle bisects the base, and is perpendicular to it.***

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SOME PROPERTIES OF THE EQUILATERAL TRIANGLE

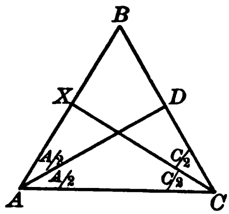
When we study the equilateral triangle we find more corollaries to theorem 4.

EXERCISES. SET XIV. EQUILATERAL TRIANGLES

117. What fact concerning the angles of an equilateral triangle can you base on the fact concerning the angles opposite the equal sides of an isosceles triangle?

118. How would you state a corollary concerning the equilateral triangle corresponding to corollary 2 under theorem 4?

119. Construct an equilateral triangle and bisect any two of its angles as in the accompanying diagram.



120. Can you prove triangles *ABD* and *CBX* congruent?

121. In these triangles what side of *CBX* is homologous to $\overline{AD}$ in *ABD*?

122. What conclusion can you draw concerning these sects?

123. Try to prove the same fact, using triangles *ACD* and *CAX*.

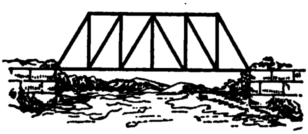
124. Could you use two other triangles to prove the same fact?

The results of exercises 117–124 may be stated as follows:

Theorem 4. Cor. 3. ***An equilateral triangle is equiangular.***

Cor. 4. ***The bisectors of the angles of an equilateral triangle bisect the opposite sides and are perpendicular to them.***

Cor. 5. ***The bisectors of the angles of an equilateral triangle are equal.***



The following experiment will help us to understand why it is that a long span of a bridge in which the truss is made with queen-posts and diagonal rods, as shown in the diagram, is sufficiently supported.

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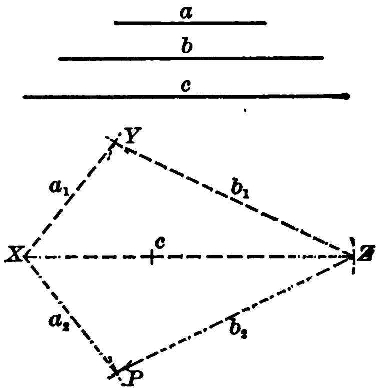
EXPERIMENT

(*a*) Using three sects of different lengths construct two triangles, XYZ and XPZ and place them as in the accompanying diagram so that the longest sides c are coincident, and $XY \equiv a \equiv XP$ are next to each other.



(*b*) Draw $\overline{YP}$. What kind of triangles are ΔXYP and ΔZYP? Why?

(*c*) Use this fact to prove $\angle XYZ \equiv \angle XPZ$.

(*d*) In (*c*) what axiom did you have to apply to prove $\angle XYZ \equiv \angle XPZ$?

(*e*) Now what parts do you know to be equal in ΔXYZ and ΔXPZ?

(*f*) What conclusion can you draw concerning these triangles?

(*g*) Test to see whether you could prove ΔXYZ and ΔXPZ congruent by placing $\overline{XY} \equiv a \equiv \overline{XP}$ (the shortest sides) together.

(*h*) If you placed the triangles as suggested in (*g*) what axiom would you have to apply to prove two angles of the triangles equal?

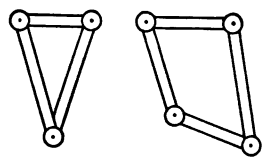
(*i*) Is the conclusion you drew in (*f*) true here? If so, we may state

Theorem 5. ***A triangle is determined by its sides.***

EXERCISES. SET XV. FURTHER APPLICATIONS OF CONGRUENCE OF TRIANGLES

125. Why is it that the last case in the congruence of triangles is separated from the first two cases by a theorem on the isosceles triangle?

126. (*a*) Three iron rods are hinged at the extremities, as shown in the diagram. Is the figure rigid? Why?



(*b*) Four iron rods are hinged, as shown in the diagram. Is the figure rigid? If not, how many rods would you add to make it rigid, and where would you add them?

NOTE. This experiment can be tried conveniently with the Mecano toy.

127. How many diagonal braces are needed to support a crane? Why?

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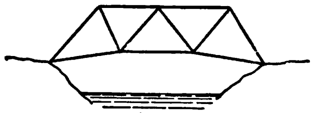
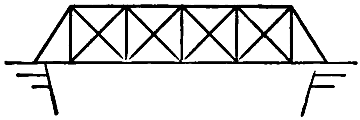
128. How many tie-beams connecting each pair of rafters are needed to brace a two-sided roof sufficiently? Why?



129. Note how the girders of the bridge in the accompanying diagram are fastened. Why cannot the bridge collapse?



130. Why is the long span of the bridge represented in the diagram sufficiently supported?

131. Prove by means of congruent triangles that the directions given on page 19 for the duplication of an angle are sound.

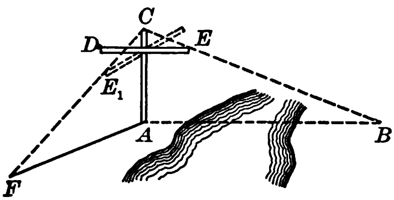
132. Could you have duplicated $\measuredangle AOB$ in that exercise if $OD > OC$?

133. Prove the directions given for bisecting an angle correct.

134. Prove that the directions for erecting a perpendicular to a line at a given point in it are correct. Can you suggest any variation in this construction such as is pointed out in exercise 132?

d135. Prove that the directions for bisecting a sect perpendicularly are correct. Compare this construction with the bisection of any angle.

136. Prove that the directions for dropping a perpendicular from a point to a line are correct.



d137. In the sixteenth century, the distance from A to the inaccessible point B was found by use of an instrument consisting of a vertical staff AC, to which was attached a horizontal cross bar DE that could be moved up and down on the staff. Sighting from C to B, DE was lowered or raised until C, E and B were in a straight line. Then the whole instrument was revolved, and the point F at which the line of sight CE_1 struck the ground again was marked, and FA measured. Show that $\overline{FA} = \overline{AB}$.

(This exercise is taken with modifications from Stone-Millis, Plane Geometry.)

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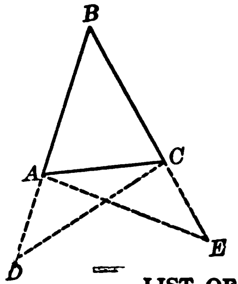
d138. Cor. 1, theorem 4, is known as the "Pons asinorum," or "Bridge of Asses." Its discovery is attributed to Thales. The proof suggested in this exercise, however, is due to Euclid. He produced BA and BC, the equal sides of the triangle to D and E, so that $BD \equiv BE$. Then he proved (1) $\triangle BCD \cong \triangle BAE$, (2) $\triangle ACD \cong \triangle CAE$, and so, by subtracting $\sphericalangle ACD$ from $\sphericalangle BCD$, and $\sphericalangle CAE$ from $\sphericalangle BAE$ found $\sphericalangle BAC \equiv \sphericalangle BCA$. Give the details of the proof.



LIST OF WORDS DEFINED IN CHAPTER I

Solid, surface, line, point; straight, curved line, sect; plane. Angle, vertex, sides; straight, right, acute, obtuse angles; complementary, supplementary angles; adjacent, vertical angles; perpendicular. Polygon, vertices, sides, angles; triangle; scalene, isosceles, equilateral triangles; acute, right, obtuse triangles. Congruent, homologous. Theorem, corollary, axiom, postulate.

SUMMARY OF AXIOMS IN CHAPTER I

1. The sums of equals added to equals are equal.
2. The remainders of equals subtracted from equals are equal.
3. The products of equals multiplied by equals are equal.
4. The quotients of equals divided by equals are equal.
5. A quantity may be substituted for its equal in a statement of equality or inequality.
6. Two quantities which are equal to equal quantities are equal to each other.
7. The whole is equal to the sum of its parts.

SUMMARY OF POSTULATES IN CHAPTER I

Straight Line

1. Two intersecting straight lines determine a point.
2. Two points determine a straight line.
3. A straight line is the shortest distance between two points.

Angle

4. All straight angles are equal.

   Cor. 1. All right angles are equal.

   Cor. 2. Complements of the same angle or equal angles are equal.

   Cor. 3. Supplements of the same angle or equal angles are equal.

Superposition

5. Any geometric figure may be moved about without changing its size or shape.

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SUMMARY OF THEOREMS PROVED IN CHAPTER I

1. Vertical angles are equal.
2. Triangles are determined by two sides and the included angle.
3. Triangles are determined by two angles and the included side.
4. The bisector of the vertex angle of an isosceles triangle divides the triangle into two congruent triangles.
   - Cor. 1. The angles opposite the equal sides of a triangle are equal.
   - Cor. 2. The bisector of the vertex angle of an isosceles triangle bisects the base, and is perpendicular to it.
   - Cor. 3. An equilateral triangle is equiangular.
   - Cor. 4. The bisectors of the angles of an equilateral triangle bisect the opposite sides and are perpendicular to them.
   - Cor. 5. The bisectors of the angles of an equilateral triangle are equal.
5. Triangles are determined by their sides.

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CHAPTER II

THE PERPENDICULAR, THE RIGHT TRIANGLE AND PARALLELS

A. THE PERPENDICULAR

We have been studying the congruence of triangles in general, and as a necessary and interesting part of that topic we have considered some properties of the isoscéles triangle. Now we are to give our attention to another special kind, the right triangle, but before doing so, we need to know more than we do about perpendiculars.

Two facts that we should note at the beginning are sufficiently obvious to permit our accepting them without proof, *i.e.*, postulating them.

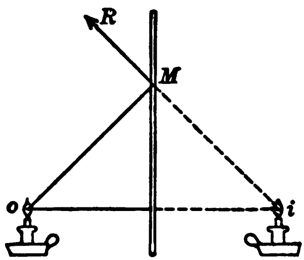
Postulates of Perpendiculars. 1. *At a point in a line only one perpendicular can be erected to that line.*

2. *From a point outside a line only one perpendicular can be drawn to that line.*

Because of this property of perpendiculars, by the **distance from a point to a line** is meant *the length of a perpendicular from the point to the line.*

The less we postulate and the more we prove, the more scientific is our work. Hence, later in our study of geometry, we shall prove these postulates of perpendiculars.

EXERCISE. SET XVI. DISTANCE FROM A POINT TO A LINE



139. Prove the familiar fact that the image of an object in a mirror appears to be as far behind the mirror as the object is in front of it.

Hints: (a) It is proved in physics that a ray of light striking a plane surface is reflected from it at the same angle as it strikes it. Assume that fact here. (b) *i*, *M*, *R* lie in a straight line. See the diagram. (c) Prove triangles congruent.

Before proving another important property of perpendiculars we must add to our list of axioms.

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Axioms of Inequality. 1. *If unequals are operated on by positive equals in the same way, the results are unequal in the same order.*

In other words, if we add a positive number to each of two unequal numbers the sums will be unequal in just the same way that the original numbers were unequal, *i.e.*, the greater number increased will still be greater than the smaller number increased; *e.g.*, $7>5$ and $7+2>5+2$. Would the same be true if we started with $7>-5$, or $-7<-5$? Give your reason.

If instead of adding equal numbers to the two unequal numbers, we had subtracted equal numbers from each, or divided or multiplied each by equal numbers, the remainders, quotients, or products would have been unequal in the same order as the original numbers.

2. *If unequals are subtracted from equals, the remainders are unequal in the reverse order.*

Example.

$$\begin{array}{rl} & 10=10 \\ \text{and} & \underline{6>3} \\ \therefore & 4<7 \end{array}$$

EXERCISES. SET XVII. NUMERICAL INEQUALITY

140. State in algebraic symbols the above axioms of inequality.

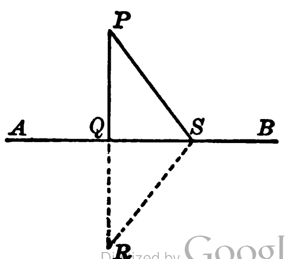
141. (*a*) Why is it that if $-2<-1$, $(-2)^{2n}>(-1)^{2n}$?

(*b*) Why is this not an exception to our axiom?

We are now ready to prove another important fact about perpendiculars which will give us another reason for measuring distance from a point to a line by means of the perpendicular.

Theorem 6. ***The perpendicular is the shortest sect that can be drawn from a point to a line.***

To show that from P a point outside of $\overline{AB}$, the shortest sect that we can draw to $\overline{AB}$ is the perpendicular $\overline{PQ}$, let us draw any other sect, say $\overline{PS}$ to $\overline{AB}$, and then show that $\overline{PQ}<\overline{PS}$. So far in our study of geometry we have no statement which will lead us to this conclusion directly.



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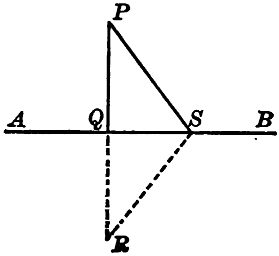
The axiom of inequality which we have just discussed might suggest, however, that we can derive the conclusion we wish by showing that $2\overline{PQ} < 2\overline{PS}$.



Then, we also recall that we have listed one fact concerning the inequality of sects and that is the postulate that a straight line is the shortest distance between two points. Thus we are led to make the construction shown in the diagram, *i.e.*, produce $\overline{PQ}$ to R so that $\overline{QR} = \overline{PQ}$ and connect R and S.

Now $\overline{PQ}+\overline{QR}$ (or $2\overline{PQ}$) $< \overline{PS}+\overline{RS}$. Why?

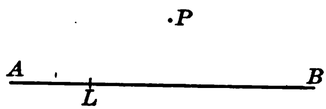
$\overline{PS}+\overline{RS} \equiv 2\overline{PS}$ if $\overline{PS} \equiv \overline{RS}$.

Try to prove $\overline{PS} \equiv \overline{RS}$.

How do we prove sects equal? Finish the proof.

EXERCISES. SET XVIII. INEQUALITY OF SECTS

142. Given the point P outside AB and L in the line. Which is shorter, the distance from P to L or the distance from P to AB?



143. Which is the longest side of a right triangle? Give a reason for your answer.

B. THE RIGHT TRIANGLE

We are now ready to go on with the study of the congruence of triangles by noting two cases of the congruence of right triangles.

EXPERIMENT

Construct a right triangle given the *side opposite the right angle*, called the **hypotenuse,** and an angle adjacent to it. Let us see if we have data sufficient to determine the triangle.

(*a*) What is the only side of the triangle that is known?

(*b*) Where then will you have to start the construction?

(*c*) Is the direction of a second side fixed? Why?

(*d*) Is the direction of the third side then fixed? Why?

We are thus led to expect:

Theorem 7.** ***The hypotenuse and adjacent angle determine a right triangle.

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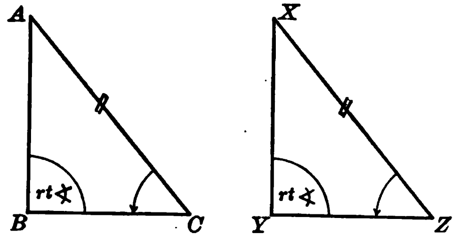
Let us convince ourselves of the truth or falsity of this conclusion by actual proof.



Given: $\triangle ABC$ and $\triangle XYZ$ with $\angle B$ and $\angle Y$ each a rt. $\angle$ and $\angle C = \angle Z$ and $\overline{AC} = \overline{XZ}$.

To prove: $\triangle ABC \cong \triangle XYZ$

Outline of proof: Place $\triangle ABC$ on $\triangle XYZ$ so that $\overline{AC}$ coincides with its equal $\overline{XZ}$ and $\overline{CB}$ falls along $\overline{ZY}$. What right have we to do this?

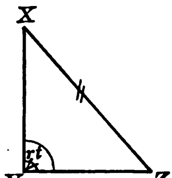
Then $\overline{AB}$ will fall along $\overline{XY}$. Why? Finally B will coincide with Y. Why?

The other case of congruence of right triangles is:

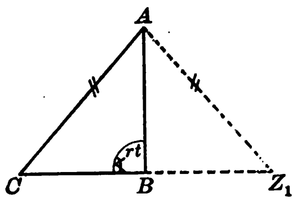
Theorem 8. ***The hypotenuse and another side determine a right triangle.***

Given: $\triangle ABC$ and $\triangle XYZ$ with $\overline{AB} = \overline{XY}$, $\overline{AC} = \overline{XZ}$ and $\angle B$ and $\angle Y$ each a rt. $\angle$.

To prove: $\triangle ABC \cong \triangle XYZ$.



Suppose we place $\triangle XYZ$ next to $\triangle ABC$ (as in the following diagram) with $\overline{XY}$ coinciding with $\overline{AB}$. (Do we know that we can?)



(1) Why will the figure formed be a triangle?

(2) What kind of triangle will be formed?

(3) Can you now throw this theorem back to the previous one?

C. PARALLELS

Parallels, or *parallel straight lines, are* **coplanar** *lines* (*lines lying in the same plane*) *which never meet.* Do you see any reason for emphasizing the fact that the lines must lie in the same plane? Take two pencils and hold them so that they would neither meet if continued nor be parallel.

Draw a line on a piece of paper and erect two perpendiculars to it. Do these perpendiculars appear to be parallel? Since they lie in the same plane they must either be parallel or meet. Can they meet? Give a reason for your answer. This leads us to state another theorem for our syllabus, one which is frequently used in mechanical drawing.

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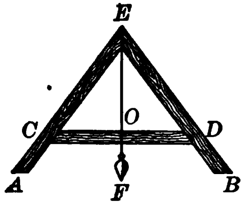
Theorem 9. ***Lines perpendicular to the same line are parallel.***

EXERCISES. SET XIX. PARALLELS

144. What principle is a carpenter using when he lays off parallel lines on a board by moving one arm of his square along a straight edge of the board, and marking along the other arm?

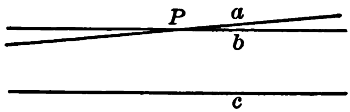


145. What is the principle involved in the use of the T-square for drawing parallels?

d146. The accompanying picture shows a carpenter's plumb-level, the forerunner of the spirit-level. AE and EB are strips of wood of equal length. $\overline{CE}=\overline{ED}$ and O is the midpoint of $\overline{CD}$. A and B rest on the points to be levelled, and they are found to be level when EF passes through O. Explain.

Before proceeding with our study of parallels, we need the

Postulate of Parallels. *Through a given point only one line can be drawn parallel to a given line.*

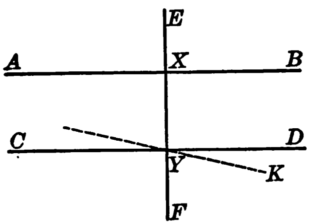


Not both lines a and b can be parallel to c. Why?

Cor. 1. ***Lines parallel to the same line are parallel to one another.***

Suggestion for proof: If lines a, b, both parallel to c, should meet how would the postulate of parallels be violated?

Theorem 10.** ***A line perpendicular to one of a series of parallels is perpendicular to the others.



Given: $\overline{AXB} \parallel \overline{CYD}$†. $\overline{EXYF} \perp \overline{AXB}$.

To prove: $EF \perp CD$.

Suggestions: Draw $YK \perp EF$.

What relation will exist between YK and AB?

Why will YK and CD coincide?

What relation exists between CD and EF? Why?

Why is it necessary to consider only two parallels?

---

* When naming a line by more than two of its points it is necessary to use a bar over the letters. In the case of two points it is immaterial. Why?

† How do the statements given show that EF cuts AB and CD?

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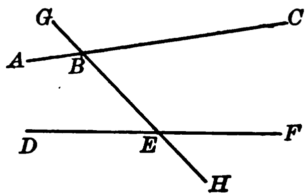
EXERCISE. SET XIX (continued)

147. Representing a series of lines by $p, q, r, \ldots$ construct the following figures, stating in each case all possible relative positions of the first and last line of the series:

(*a*) $p \parallel q$, $p \parallel r$. (*b*) $p \parallel q$, $q \parallel r$. (*c*) $p \perp q$, $q \perp r$. (*d*) $p \perp q$, $q \parallel r$. (*e*) $p \perp q$, $q \parallel r$, $r \perp s$. (*f*) $p \parallel q$, $q \perp r$, $r \parallel s$.

ANGLES FURTHER DEFINED ACCORDING TO RELATIVE POSITION



If, as in the accompanying diagram, $\overline{ABC}$ and $\overline{DEF}$ are cut by $\overline{GBEH}$, which is called a **transversal** (since it cuts across), certain sets of angles are formed to which for brevity we give the following names:

$\measuredangle ABE$ and $\measuredangle FEB$ are called **alternate-interior** angles.

$\measuredangle GBA$ and $\measuredangle HEF$ are called **alternate-exterior** angles.

$\measuredangle EBC$ and $\measuredangle FEB$ are called **consecutive-interior** angles.

$\measuredangle GBA$ and $\measuredangle DEH$ are called **consecutive-exterior** angles.

$\measuredangle CBG$ and $\measuredangle FEB$ are called **corresponding** angles.

EXERCISES. SET XX. RELATIVE POSITION OF ANGLES

148. Only one pair of each kind of angles is mentioned in the last paragraph, though there are two pairs of each except the last kind. Name the second pair in each case, and three remaining pairs of corresponding angles.

149. Explain the meaning of alternate as used here.

150. Explain the meaning of consecutive as used here.

151. Explain the meaning of interior as used here.

152. Explain the meaning of exterior as used here.

153. What kind of angles with regard to relative position are formed in the letter *Z*? In the letter *A*? In the letter *E*? *H*? *N*?

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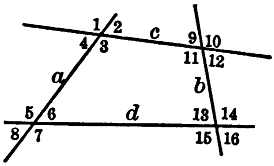
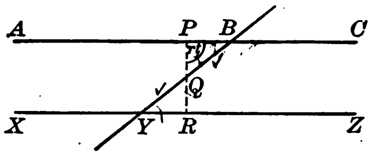
154. In the accompanying diagram select angles under each class you know (including adjacent and vertical) using:



(*a*) *c* and *d* with *a* as transversal.
(*b*) *c* and *d* with *b* as transversal.
(*c*) *a* and *b* with *c* as transversal.
(*d*) *a* and *b* with *d* as transversal.

PROPERTIES OF PARALLELS

Theorem 11. ***If, when lines are cut by a transversal, the alternate-interior angles are equal, the lines thus cut are parallel.***



Given: $\overline{ABC}$, $\overline{XYZ}$ cut by BY.
$\angle ABY = \angle BYZ$.
To prove: $\overline{ABC} \parallel \overline{XYZ}$.

PROOF

Take Q in BY so that $QB = QY$.
Draw $QP \perp AB$ cutting AB at P.
Extend PQ to R in XZ.
Then in $\triangle PQB$ and $\triangle RQY$,

(1) $QB = QY$	(1) Construction.
(2) $\angle PBQ = \angle RYQ$	(2) Data.
(3) $\angle PQB = \angle RQY$	(3) Vertical angles.
(4) $\therefore \triangle PQB \cong \triangle RQY$	(4) Two angles and the included side equal each to each.
(5) $\therefore \angle BPQ = \angle YRQ$	(5) Homologous parts of congruent triangles.
(6) But $\overline{PQR} \perp AB$	(6) Construction.
(7) $\therefore \angle BPQ$ is a rt. $\angle$	(7) Definition of perpendicular.
(8) $\therefore \angle YRQ$ is a rt. $\angle$	(8) Quantities equal to the same quantity are equal to each other.
(9) $\therefore \overline{PQR} \perp \overline{XRZ}$	(9) Definition of perpendicular.
(10) $\therefore \overline{ABC} \parallel \overline{XYZ}$	(10) Lines perpendicular to the same line are parallel.

The student's attention is called to the form and arrangement of this demonstration, as it is the first formal proof given in the text. Note that after the general statement of the theorem following the words "given" and "to prove,"* specific statements are given

* In place of the word "given" either "data" or "hypothesis" is frequently used, and in place of "prove" the word "conclusion."

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referring to the particular diagram drawn. These statements should be as brief as possible, and such, that were the diagrams erased, it could be reconstructed. The steps of the proof and the reasons for them are arranged in parallel columns. The convenience of such an arrangement is at once apparent if it be compared with a proof written in essay form. Write the proof that way and draw your own conclusions as to which you would prefer to use, giving your reasons.

Show that the following corollaries are true by showing that a pair of alternate-interior angles are equal.

Cor. 1. ***If the alternate-exterior angles or corresponding angles are equal when lines are cut by a transversal, the lines thus cut are parallel.***

Cor. 2. ***If either the consecutive-interior angles or the consecutive-exterior angles are supplementary when lines are cut by a transversal, the lines thus cut are parallel.***

EXERCISES. SET XXI. CONSTRUCTION OF PARALLELS

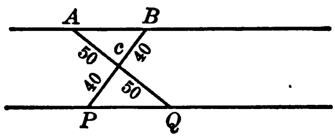
155. Parallels may be constructed by using a T-square and a triangle. Explain.

156. Draw three pairs of parallel lines using successively each of the three sides of one of your triangles against a ruler.

157. By using your knowledge of corresponding angles, draw a line through a given point, and parallel to a given line.

158. (*a*) Practice drawing parallels with compasses and ruler until you can draw them accurately. (*b*) Test your work by drawing any transversal, and measuring a pair of angles that should be equal. (*c*) Which is more likely to be in error, your drawing or your test?

159. The diagram suggests a method of running one line parallel to another when you are on a field without a transit. Explain and justify the procedure.



Presently we shall prove a theorem which is closely related to theorem 11. Before doing so, however, we shall want to see why it need be proved at all.

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EXERCISES. SET XXII. RELATED STATEMENTS

160. (*a*) Is it true that if a triangle is equilateral, it is also isosceles?

(*b*) Is it true that if a triangle is isosceles, it is also equilateral?

161. (*a*) Is it true that if two angles are right angles, they are equal?

(*b*) Is it true that if two angles are equal, they are right angles?

162. (*a*) Is it true that all men are bipeds?

(*b*) Is it true that all bipeds are men?

163. (*a*) Is it true that if a man lives in Cincinnati, he lives in Ohio?

(*b*) Is it true that if a man lives in Ohio, he lives in Cincinnati?

164. Explain how each of the statements (*a*) and (*b*) in each of the four preceding exercises is related to the other: that is, how can (*a*) in each case be formed from (*b*), and how can (*b*) be formed from (*a*)?

165. Make a statement related to each of the following as (*b*) is related to (*a*) in each of exercises 160 to 163, and tell whether or not your statements are true.

(*a*) If a man lives, he breathes.

(*b*) If a polygon is a triangle, it has three sides.

(*c*) If it rains, the ground is wet.

166. From exercises 160 to 163 and 165 what can you conclude as to the truth or falsity of two statements related in this way? (*a*) May both of them be true? (*b*) May one of them be true and the other false?

167. If, then, we have proved that a statement is true, is it necessary to prove a statement related to it as the second is to the first in each of those exercises, or may we take it for granted that the second will be true without proof?

Statements related as those in the last set of exercises, are called **converse** statements. We saw that each of two converse statements could be formed from the other by *interchanging the data or hypotheses with the conclusion or conclusions;* that is, the two statements were so related that what was given in each was what was supposed to follow in the other. From the fact that it is so much

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easier to make a statement whose converse is absurd than one whose converse is true, it appears that we should never claim that the converse of a theorem in geometry is true without having proved it so.

EXERCISES. SET XXII (continued)

168. Make a statement of something in life which you know to be true, but whose converse is false.

169. Make a statement of something in life which you know to be true, but whose converse is true.

170. Do the same as you were requested to do in the last two exercises, but take the statements from geometry.

171. Select from the theorems already proved two that are converse statements.

172. The converse of a definition is always true. Test your definitions of Chapter I from this point of view. (See lists at end of Chapter I, page 46.)

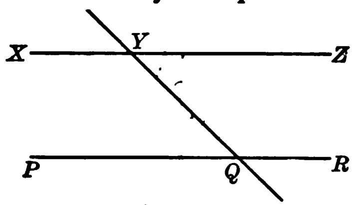
173. State what was given us in theorem 11.

174. State what we proved in theorem 11.

175. State what would be given in the converse of theorem 11.

176. State what would have to be proved in the converse of theorem 11.

Theorem 12. ***Parallels cut by a transversal form equal alternate-interior angles.***



Fill in all the blank spaces and answer the questions in the following:

Given:

To prove:

PROOF

(1) If through the mid-point of $\overline{YQ}$ we were to draw a line perpendicular to $\overline{XZ}$ what other fact would we know about that line?	(1) Why?
(2) What parts have we then equal in the triangles thus formed?	(2) How do you know each of these pairs are so related?
(3) By what method could we then prove the fact we wish to prove?	

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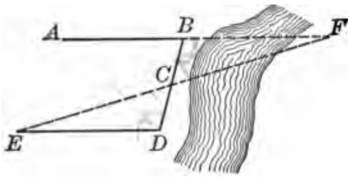
Cor. 1. ***Parallels cut by a transversal form equal corresponding and equal alternate-exterior angles.***

Cor. 2. ***Parallels cut by a transversal form supplementary consecutive-interior, and supplementary consecutive-exterior angles.***

EXERCISES. SET XXIII. APPLICATIONS OF PARALLELISM

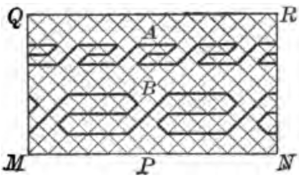


177. The accompanying diagram suggests a convenient method of measuring the distance from B to an inaccessible point F. Explain.

Note DE || FB.

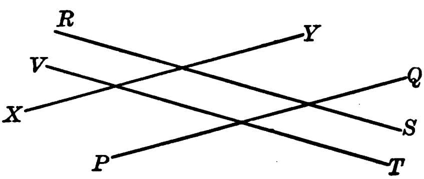
How can this be done on the ground?

178. The "square network" shown in the figure is used in designing for drawing a great variety of patterns. The patterns A and B drawn upon it are examples of Arabian frets. The best way to rule the square network is to draw a horizontal line MN and mark off equal divisions on it. At each point of division, by use of the triangle, draw two lines, such as PQ and PR, each making an angle of 45° with MN.

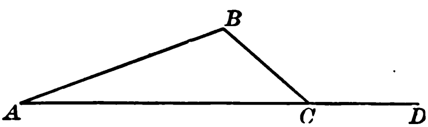


Draw such a network, then upon it construct a pattern, either an original design or a copy of these Arabian frets.

(Taken from Stone-Millis, Plane Geometry.)



179. In the annexed diagram if $XY \parallel PQ$ and $RS \parallel VT$ how many angles would you need to know in order to find the remaining angles?



180. If the side AC of a triangle ABC is extended, as in the annexed diagram, how could a line be drawn through C to make an angle equal to the $\measuredangle B$? Would any other angles be equal? Why?

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LIST OF WORDS DEFINED IN CHAPTER II

Distance (point to a line). Hypotenuse. Parallels, coplanar, transversal; alternate-interior, alternate-exterior, consecutive-interior, consecutive-exterior, corresponding angles. Converse.

SUMMARY OF AXIOMS IN CHAPTER II

Inequality

1. If unequals are operated upon by positive equals in the same way, the results are unequal in the same order.
2. If unequals are subtracted from equals, the remainders are unequal in the reverse order.

SUMMARY OF POSTULATES IN CHAPTER II

Perpendiculars

1. At a point in a line only one perpendicular can be erected to that line.
2. From a point outside a line only one perpendicular can be drawn to that line.

Parallels

3. Through a given point only one line can be drawn parallel to a given line.

Cor. 1. Lines parallel to the same line are parallel to one another.

SUMMARY OF THEOREMS IN CHAPTER II

6. A perpendicular is the shortest sect that can be drawn from a point to a line.
7. The hypotenuse and an adjacent angle determine a right triangle.
8. The hypotenuse and another side determine a right triangle.
9. Lines perpendicular to the same line are parallel.
10. A line perpendicular to one of a series of parallels is perpendicular to the others.
11. If when lines are cut by a transversal the alternate-interior angles are equal the lines thus cut are parallel.

Cor. 1. If the alternate-exterior angles or corresponding angles are equal when lines are cut by a transversal, the lines thus cut are parallel.

Cor. 2. If either the consecutive-interior angles or the consecutive-exterior angles are supplementary when lines are cut by a transversal, the lines thus cut are parallel.

12. Parallels cut by a transversal form equal alternate-interior angles.

Cor. 1. Parallels cut by a transversal form equal corresponding, and equal alternate-exterior angles.

Cor. 2. Parallels cut by a transversal form supplementary consecutive interior, and supplementary consecutive-exterior angles.

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CHAPTER III

ANGLES OF POLYGONS AND PROPERTIES OF PARALLELOGRAMS

A. ANGLES OF POLYGONS

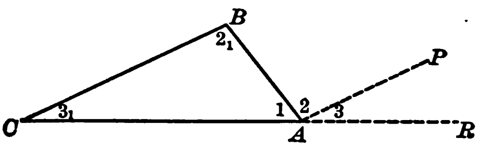
In attempting to develop a formula for the sum of the angles of a polygon, it is best for us to begin with the simplest polygon, the triangle.

EXERCISE. SET XXIV. SUM OF ANGLES OF A TRIANGLE.

181. By reproducing the angles of a given triangle, place these angles adjacent to one another. What does the sum of the angles appear to be in this case?

Theorem 13. ***The sum of the angles of a triangle is a straight angle.***

Prove this proposition, using the hints given by the following diagram and notes:



Produce CA and draw $AP \parallel CB$.

What is the sum of ∢s 1, 2, and 3?

What is the relation of ∢2 to ∢2_1? Of ∢3 to ∢3_1?

Cor. 1. ***A triangle can have but one right or one obtuse angle.***

Cor. 2. ***Triangles having two angles mutually equal are mutually equiangular.***

Cor. 3. ***A triangle is determined by a side and any two homologous angles.***

An **exterior angle** *of a polyyon is one formed by a side of the polygon and the prolongation of an adjacent side.*

In the preceding diagram ∢RAB is an exterior angle of the ΔABC.

Cor. 4. ***An exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.***

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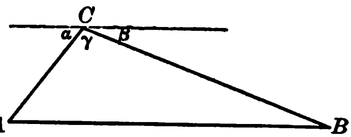
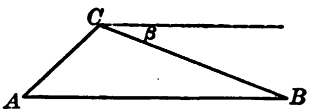
EXERCISES. SET XXIV (continued)

182. (*a*) The theorem that the sum of the angles of a triangle equals a straight angle may be proved by drawing a line through a vertex parallel to the opposite side. Give proof.

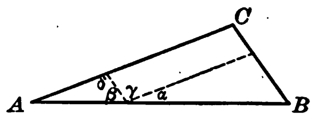


NOTE.—This proof is attributed to the Pythagoreans.

(*b*) Prove the same fact by drawing a sect from a vertex parallel to the opposite side.



(*c*) Prove the same theorem by drawing through any point on one side lines parallel to the other sides of the triangle.



183. Prove this same fact another way by erecting perpendiculars to one side at its extremities and dropping a perpendicular to the same side from the opposite vertex.

184. Show how the following procedure may be used to test accuracy with which you measure angles with an instrument. Select the three positions not in a straight line; call them stations A, B, and C. From station A measure the angle between the directions to B and C; at B, measure the angle between the directions to C and A; at C, the angle between the directions to A and B. Would your measurements be accurate, and if not, what error would there be, if you found the angles to be respectively: 100° 27′, 23° 13′, and 56° 23′?

185. If one angle of an isosceles triangle is 60°, find the other angles.

186. (*a*) If the vertex angle of an isosceles triangle is $v°$ write an expression for each of the other angles.

(*b*) If a base angle of an isosceles triangle is $b°$ write an expression for each of the other angles.

187. What angle do the bisectors of the acute angles of a right triangle form?

188. Construct the following angles: 60°, 30°, 120°, 75°, 150°, 105°.

The pupil is again reminded that the use of instruments is

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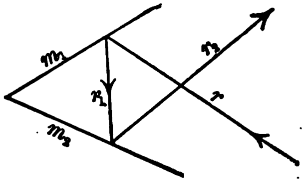
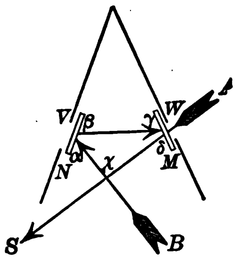
restricted to the straight edge and the compasses in scientific geometric constructions.

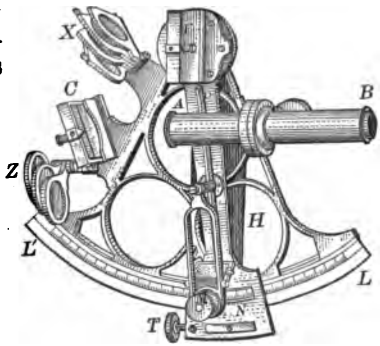


d189. (*a*) Two mirrors, m_1 and m_2, are set so as to form an acute angle with each other. A ray of light is reflected by m_1 so as to strike m_2. The ray is again reflected by m_2 and crosses its first path. Prove that $\measuredangle rr_2 \equiv 2 \measuredangle m_1m_2$.

(*b*) How should these mirrors be placed in order that the ray in the second case may be parallel to the original ray?

(*c*) Part (*a*) of this exercise is the principle underlying several important optical instruments such as the "optical square' and the "sextant." A description of them may be found in any good encyclopædia or in Gillespie's





Surveying, p. 61, and Stone-Millis' Plane Geometry, p. 40, Ex. 14. Look them up and make a crude optical square either of pasteboard or of wood.

A **diagonal** *of a polygon is a sect connecting any two non-consecutive vertices.*

EXERCISES. SET XXV. SUMS OF ANGLES OF POLYGONS

190. Find the sum of the angles of a quadrilateral. (Can you so divide it into triangles that the sum of the angles of the triangles formed will be the sum of the angles of the quadrilateral?)

191. Draw a five-sided polygon.

(*a*) How many diagonals can be drawn from one vertex?

(*b*) How many triangles are formed by drawing these diagonals?

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(*c*) What is the sum of the angles of the triangles thus formed?

(*d*) What is the sum of the angles of a 5-gon?

Give a reason for each of your answers. Check your conclusion concerning the sum of the angles by the use of the protractor.

A polygon is said to be **convex** *if each of its angles is less than a straight angle.* Only convex polygons will be considered in the early study of geometry. If a polygon has *four* sides it is called a **quadrilateral**; if *five* sides, a **pentagon**; *six* sides, a **hexagon**; *seven* sides a **heptagon**; *eight* sides, an **octagon**; *nine* sides, a **nonagon**; *ten* sides, a **decagon**; etc.

Theorem 14. ***The sum of the angles of a polygon is equal to a straight angle taken as many times less two as the polygon has sides.***

(1) Draw all the diagonals possible from one vertex of the polygon.

(2) If the figure has n sides how many triangles will you form by drawing these diagonals?

Give the proof following the suggestions given in (1) and (2).

The pupil will find the following (1) and (2) also give hints leading to an equally desirable proof.

(1) Connect any point inside the polygon with each vertex.

(2) If the figure has n sides how many triangles will you thus form?

A **regular** *polygon is one which is both equilateral and equiangular.*

Can you draw a polygon which is equilateral and not equiangular?

Can you draw a polygon which is equiangular and not equilateral?

Cor. 1. ***Each angle of a regular polygon of n sides equals the*** $\frac{n-2}{n}$***th part of a straight angle.***

If the sides of a polygon are produced in turn forming one exterior angle at each vertex, these angles are called **the exterior angles** of the polygon.

Could two exterior angles be formed at each vertex? How would two such angles be related?

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Cor. 2.* ***The sum of the exterior angles of a polygon is two straight angles.***

(1) What would be the sum of the adjacent interior and exterior angle at each vertex?

(2) What would be the sum of all the interior and exterior angles together?

Cor. 3. ***Each exterior angle of a regular polygon of n sides is equal to the $\frac{2}{n}$th part of a straight angle.***

EXERCISES. SET XXV (continued)

192. Fill in the blank spaces in the accompanying table:

No. of sides of polygon	No. deg. in sum of int. angles	No. deg. in each angle of regular n-gon
3		
4		
5		
6		
7		
8		
9		
10		

193. How many sides has a regular polygon if each angle is (*a*) 150°, (*b*) 144°, (*c*) 170°, (*d*) 175°?

Hints on solution: $\frac{n-2}{n}(180)=150$; or $\frac{2}{n}(180)=30$. Solve for n.

194. In surveying an hexagonal field the angles were found to be 118°, 124°, 116°, 129°, 130°, 112°.

(*a*) What error was made?

(*b*) Before making a drawing of the survey, the engineer has to distribute this error proportionately over all the angles so as to increase or decrease all the angles according as the sum of the angles measured was too small or too great. Distribute the error correctly in this case.

* Theorem 14 and this corollary were proved in their general form by Regiomontanus (1436–1476), although the facts were known to earlier mathematicians, and were proved by them for special cases.

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195. In surveying a pentagonal field, the angles were found to be $A = 103° 15'$, $B = 110° 37'$, $C = 99° 45'$, $D = 122° 40'$, and $E = 102° 15'$.

(*a*) What error was made?

(*b*) Distribute the error proportionately.

196. (*a*) Make a list of regular polygons of the same number of sides that may be used to cover a plane surface with a geometric design.

(*b*) For what purposes are such designs used?

(*c*) What combinations of regular polygons have you seen used?

(*d*) Sketch some of these designs.

(*e*) Why is it these combinations are possible?

197. Make as many constructions as possible of each of the polygons mentioned, using only rule and compasses for the purpose.

198. Construct accurately a design based upon each kind of regular polygon or a combination of regular polygons mentioned in Ex. 196.

d199. Mabel Sykes, Source Book of Problems for Geometry, p. 16, par. 23, Ex. 2.

d200. Ibid., p. 16, par. 23, Ex. 4.

d201. Ibid., pp. 16–17, par. 24, Ex. 1.

d202. Ibid., pp. 16–17, par. 24, Ex. 4.

203. Prove the proposition concerning the sum of the angles of a polygon, according to the following suggestions:

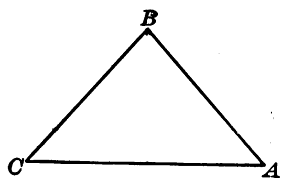
(*a*) By connecting a point on one of the sides of the polygon with each vertex.

(*b*) By connecting a point outside of the polygon with each vertex.

The converse of a proposition concerning isosceles triangles has many interesting and important applications, and we are now ready to prove it.

Theorem 15. ***If two angles of a triangle are equal, the sides opposite them are equal.***

Hint: Can you draw a line to cut $\triangle ABC$ into two triangles in such a way that the construction itself will make an angle and a side equal respectively in the two triangles?



Cor. 1. ***Equiangular triangles are equilateral.***

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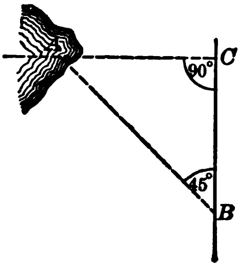
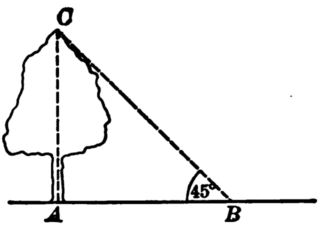
EXERCISES. Set XXVI. SIDES AND ANGLES OF A TRIANGLE

204. (*a*) Show that a person may find the distance (AC) at which he passes an object A, when going in the direction BC, if he notes when his course makes an angle of 45° with the direction of the object and again when it is at right angles to the object taking account of the distance (BC) which he traversed between the observations.



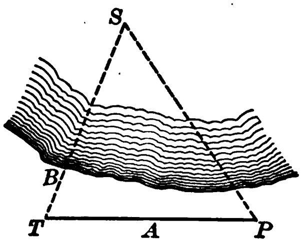
(*b*) How could this method be applied to the problem of finding at what distance from a lighthouse a ship passes it?

205. To ascertain the height of a tree or of the school building, fold a piece of paper so as to make an angle of 45°. Then walk back from the tree until the top is seen at an angle of 45° with the ground (being, therefore, careful to have the base of the triangle level). Then the height AC will equal the base AB, since ABC is isosceles. A paper protractor may be used for the same purpose. Can you suggest a better method than that of measuring from the ground?



206. The distance of a ship at sea may be measured in the following manner:

Make a large isosceles triangle out of wood, and standing at T, sight to the ship and along the shore on a line TA, using the vertex angle of the triangle. Then go along TA until a point P is reached, from which T and S can be seen along the sides of a base angle of the triangle. Then $TP = TS$. By measuring TB, BS can be found.



207. Distance can easily be measured by constructing a large equilateral triangle of heavy pasteboard, and standing pins at the vertices for the purpose of sighting.

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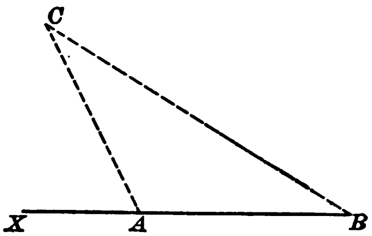
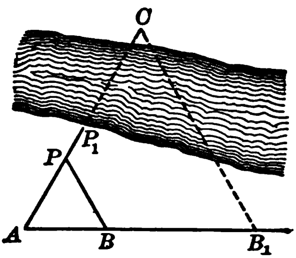
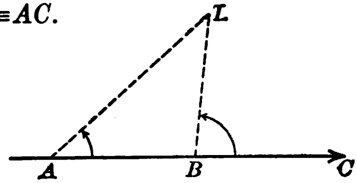
To measure P_1C, stand at some convenient point A, and sight along APC, and also along AB. Then walk along AB until a point B_1 is reached, from which B_1C makes with B_1A an angle of the triangle (60°). Then prove that $AC=AB_1$. Also, since AP_1 can be measured, find P_1C.





208. Measure the angle CAX, either in degrees with a protractor, or by sighting along a piece of paper and marking down the angle. Then go along XA produced until a point B is reached from which BC makes with BA an angle equal to half of angle CAX. Then show that $AB \equiv AC$.



NOTE.—A navigator uses the principle involved in the foregoing exercises when he "doubles the angle on the bow" to find his distance from a lighthouse or other object. If he is sailing on the course ABC and notes a lighthouse L when he is at A, he takes the angle A; and if he notices when the angle that the lighthouse makes with his course is just twice the angle noted at A, then $BL \equiv AB$. He has AB from his log (an instrument that tells how far a ship goes in a given time), so he knows BL. He has "doubled the angle on the bow" to get this distance.

B. PARALLELOGRAMS

A **parallelogram** *is a quadrilateral whose opposite sides are parallel.*

Theorem 16. ***Either diagonal of a parallelogram bisects it.***

Suggestion: In the proof note that the diagonal is a transversal of the parallel sides.

Cor. 1. ***The parallel sides of a parallelogram are equal, and the opposite angles are equal.***

The sects of common perpendiculars included by parallels are called the **distances** *between the parallels.*

Cor. 2. ***Parallels are everywhere equidistant.***

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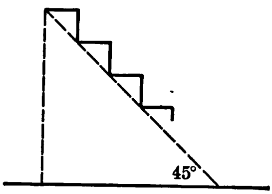
EXERCISES. SET XXVII. PARALLELOGRAMS

209. One angle of a parallelogram is 20° more than three times another. Find all the angles.

210. (*a*) Establish a relation between consecutive angles of a parallelogram.

(*b*) How many angles of a parallelogram must be known in order to determine the others?

211. A stairway inclined 45° to the horizontal leads to a floor 15′ above the first. What is the length of the carpet required to cover it if each step is 10″ high? If each is 12″ high? If each is 9″?



Can this problem be solved without knowing the height of the steps? Is it necessary to know that the steps are of the same height?

(Taken from Slaught and Lennes, Plane Geometry.)

212. Is the converse of the fact that a diagonal bisects a parallelogram true or false? Give a reason for your answer.

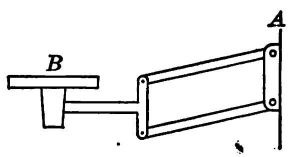
Theorem 17. ***A quadrilateral whose opposite sides are equal is a parallelogram.***

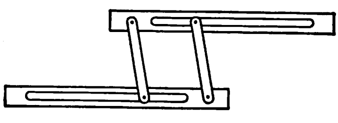
(1) What fact would we need to know about these opposite sides?
(2) By what method can we obtain this fact?

EXERCISES. SET XXVII (continued)

213. An adjustable bracket such as dentists often use, is outlined in the figure. It is fastened to the wall at *A*, and carries a shelf *B*. Why is it that as the bracket is moved so that *B* is raised and lowered, the shelf remains horizontal?

(Taken from Stone-Millis, Plane Geometry.)





214. The accompanying figure is a diagram of the "parallel ruler," which is used by designers for drawing parallel lines.

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(*a*) Upon what principle of parallelograms must its construction depend?

(*b*) Make such an instrument.

A **rectangle** *is a parallelogram one of whose angles is a right angle.*

Cor. ***Each angle of a rectangle is a right angle.*** Why?

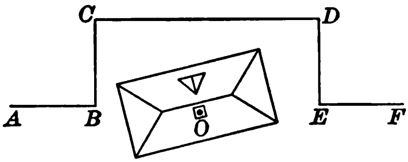
EXERCISE. SET XXVII (continued)

215. Prove that a parallelogram whose diagonals are equal is a rectangle. (This fact is used as a check by carpenters and builders. Could it be used in laying out a tennis court?)

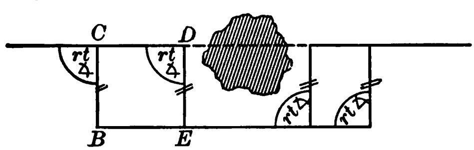
Theorem 18. ***A quadrilateral having a pair of sides both equal and parallel is a parallelogram.***

EXERCISES. SET XXVII (continued)

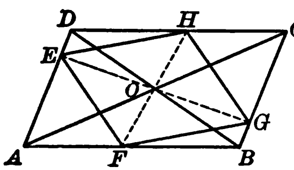
216. Justify the following method used by surveyors for prolonging a line beyond an obstacle; that is, show that in the diagram EF is AB prolonged beyond O. BC is run at right angles to AB; $CD \perp BC$; $DE \perp CD$ and $DE \equiv CB$; $EF \perp DE$.



217. The accompanying diagrams show another way of extending a line beyond an obstacle. (*a*) By reference to diagram, state the procedure in words. (*b*) Show that it is correct. (*c*) Compare this method with that of Ex. **216.** Note, for example, why two lines (CB and DE) are used in Ex. 217, and only one (CB) in Ex. 216. Under what conditions would you use each method?



218. If the vertices of one parallelogram are in the sides of another, the diagonals of the two parallelograms pass through the same point (called the **center** of the parallelograms).



Suggestions: Call the intersection of the diagonals AC and BD, O. Draw OE, OF, OG, OH and prove EOG and FOH straight lines.

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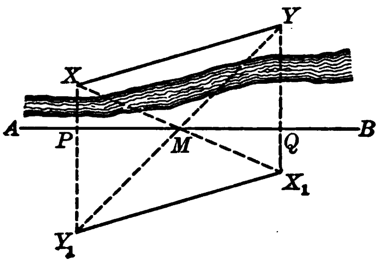
d219. An interesting outdoor application of the theory of parallelograms is the following: Suppose that you are on the side of this stream opposite to XY, and wish to measure the length of XY. Run a line AB along the bank. Then take a carpenter's square, or even a large book, and walk along AB until you reach P, a point from which you can just see X and B along two sides of the square. Do the same for Y, thus fixing P and Q. Using the tape, bisect PQ at M. Then walk along YM produced until you reach a point Y_1 that is exactly in line with M and Y, and also with P and X. Then walk along XM produced until you reach a point X_1 that is exactly in line with M and X, and also with Q and Y. Then measure Y_1X_1 and you have the length of XY. For since $YX_1 \perp PQ$, and $XY_1 \perp PQ$, $YX_1 \parallel XY_1$. And since $PM \equiv MQ$, therefore $XM \equiv MX_1$ and $Y_1M \equiv MY$. Therefore Y_1X_1YX is a parallelogram. Give the reason for each of these steps.



*** Theorem 19.** ***A parallelogram is determined by two adjacent sides and an angle,*** or two parallelograms are congruent if two adjacent sides and an angle are equal each to each.

Methods of proof.

(1) Congruence of triangles, or

(2) Superposition, or

(3) Properties of parallelograms.

NOTE.—If (1) or (3) is used it must be shown that the parts proved equal are arranged in the same order in the two parallelograms.

EXERCISES. SET XXVII (continued)

220. Construct a parallelogram, given

(*a*) Two adjacent sides and an included angle.

(*b*) Two adjacent sides and a non-included angle.

(*c*) A side, a diagonal, and the angle between them.

221. In physics it is shown that if two forces (such as a push and a pull) are exerted in different directions upon the same object, they have the same effect as a single force called their **resultant.**

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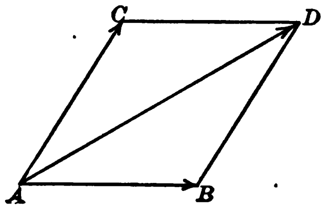
How is this fact illustrated by the bean-shooter?

Referring to the accompanying diagram, if the directions and magnitudes of two forces working on object A are represented by the sects AB and AC, the direction and magnitude of the resultant will be represented by the sect AD, which is the diagonal of the parallelogram, with AB and AC as adjacent sides. In physics, such a diagram is, for obvious reasons, called the **Parallelogram of Forces.**



A force of 50 pounds is exerted upon a body pulling it in one direction, and at the same time another force of 100 lbs. pulls it in a direction at an angle of 45° with the first. Show by the parallelogram of forces the effect on the body.

NOTE: (*a*) Use only compasses and ruler in solving. Represent 50 lbs. by any given sect as unit; draw the forces to scale, and find the resultant.

(*b*) Use protractor and marked edge in reading result.

(*c*) Could you read the resultant to any degree of accuracy by any other method?

222. Two forces, 250 lbs. and 400 lbs. respectively, are exerted upon a body at right angles with each other. Find their resultant as in Ex. 221. Check the result by computation. Why was such a check not available for you in Ex. 221?

223. Find the resultant of two forces exerted upon a body at an angle of 150° with each other, one of 50 lbs., the other of 60 lbs.

224. When a train is approaching a station at a velocity of 40 ft. per second, a mail bag is thrown at right angles from the car with a speed of 20 ft. per second. Find the actual direction and speed of the moving bag.

(Taken from Stone-Millis, Plane Geometry.)

225. The resultant of three forces may be found by getting the resultant of two of the original forces and then finding the resultant of that and the third original force. This process may be continued to obtain the resultant of several forces.

Find the resultant of three coplanar forces of 200 lbs., 150 lbs., and 175 lbs. respectively, acting on the same body at the same

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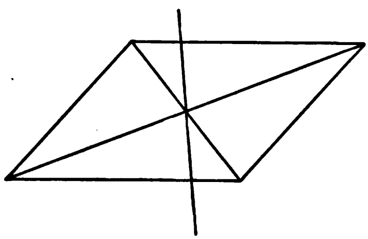
time. The angle between the first and second force is 45°, and that between the second and third is 60°. Why note that the forces are coplanar?

226. A force of 100 lbs. makes an angle of 60° with a second force of 120 lbs. exerted on the same body, and makes an angle of 90° with a third force of 140 lbs., and an angle of 120° with a fourth force of 160 lbs. If the forces are coplanar and act simultaneously, find their resultant.

Theorem 20. ***The diagonals of a parallelogram bisect each other.***

EXERCISES. SET XXVII (continued)



227. Draw any line through the intersection of the diagonals of a parallelogram.

(*a*) Give a list of the pairs of congruent triangles formed. Give reasons for your assertions.

(*b*) Give a list of the pairs of congruent quadrilaterals. Verify your statements.

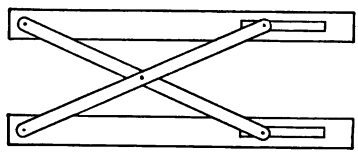
228. Cut a parallelogram out of cardboard. Placing a pin at the intersection of the diagonals, try to balance the parallelogram. Why would you expect it to balance?

The intersection of the diagonals is called the **center of gravity** of a parallelogram. Why?

Theorem 21. ***A quadrilateral whose diagonals bisect each other is a parallelogram.***

EXERCISE. Set XXVII (continued)

229. The same principle is often used in the construction of iron gates that was employed in the making of a parallel ruler used in the eighteenth century (see diagram). What is the principle?



The student is encouraged to make such an instrument.

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EXERCISES. SET XXVIII. PARALLELS

230. Classify quadrilaterals.

231. Summarize ways of proving:

(*a*) Sects equal.

(*b*) Angles equal.

(*c*) Lines parallel.

LIST OF WORDS DEFINED IN CHAPTER III

Exterior angle and angles of a polygon. Convex polygon, diagonal; quadrilateral, pentagon, hexagon, heptagon, octagon, nonagon, decagon; regular polygon. Parallelogram, rectangle. Distance between parallels.

SUMMARY OF THEOREMS DEVELOPED IN CHAPTER III

13. The sum of the angles of a triangle is a straight angle.

Cor. 1. A triangle can have but one right or one obtuse angle.

Cor. 2. Triangles having two angles mutually equal are mutually equiangular.

Cor. 3. A triangle is determined by a side and any two homologous angles.

Cor. 4. An exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.

14. The sum of the angles of a polygon is equal to a straight angle taken as many times less two as the polygon has sides.

Cor. 1. Each angle of a regular polygon of n sides equals the $\frac{n-2}{n}$th part of a straight angle.

Cor. 2. The sum of the exterior angles of a polygon, made by producing each of its sides in succession, is two straight angles.

Cor. 3. Each exterior angle of a regular polygon is the $\frac{2}{n}$th part of a straight angle.

15. If two angles of a triangle are equal, the sides opposite them are equal.

Cor 1. Equiangular triangles are equilateral.

16. Either diagonal of a parallelogram bisects it.

Cor. 1. The parallel sides of a parallelogram are equal, and the opposite angles are equal.

Cor 2. Parallels are everywhere equidistant.

17. A quadrilateral whose opposite sides are equal is a parallelogram.

18. A quadrilateral having a pair of sides both equal and parallel is a parallelogram.

19. A parallelogram is determined by two adjacent sides and an angle.

20. The diagonals of a parallelogram bisect each other.

21. A quadrilateral whose diagonals bisect each other is a parallelogram.

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CHAPTER IV

AREAS

A. INTRODUCTION. REVIEW OF FRACTIONS *

In dealing with areas, we are largely concerned with ratios. A **ratio** *is a fraction*, and therefore our work in this chapter presupposes a familiarity with fractions. For those of us who need a review of this topic the following will be helpful.

A **fraction** *is an indicated quotient*, the *dividend* of which is the **numerator** and the *divisor* the **denominator.**

Since this is review, let us summarize without discussion the fundamental facts which we need to recall about fractions.

PRINCIPLES

I. *The value of a fraction is not changed if both numerator and denominator* (*i.e.*, both **terms** of the fraction) *are multiplied or divided by the same quantity.* Why?

This statement includes **cancellation,** for that *is the process of dividing both numerator and denominator by a common factor.*

Illustrations: 1. $\frac{a}{b}=\frac{5a}{5b}=\frac{na}{nb}$ 2. $\frac{64a^3b}{128ab^5}=\frac{a^2}{2b^4}$

3. $\frac{a^2-4}{2a^2-8a+8}=\frac{a+2}{2(a-2)}$

II. *Considering as the signs of a fraction that of the numerator, that of the fraction, and that of the denominator, any two may be changed without changing the value of the fraction.* Why?

Recall that the numerator and denominator of a fraction are treated as if each were enclosed in a parenthesis.

Illustrations: 1. $\frac{a}{b}=-\frac{-a}{b}=-\frac{a}{-b}=\frac{-a}{-b}$ 2. $\frac{a-2}{a-5}=\frac{2-a}{5-a}=\ldots\ldots$

3. $\frac{abc}{def}=\frac{a(-b)c}{de(-f)}=\frac{-abc}{-def}=\ldots\ldots$

* Those who have not studied algebraic fractions are advised to take up this topic at this time. Any standard algebra text will furnish sufficient material.

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III. *Fractions may be added or subtracted by changing them to equivalent fractions having the same denominators and adding or subtracting their numerators, putting the sum or difference over the common denominator.* Why?

Changing to a same denominator depends for its validity upon preceding principle I. The same or common denominator may be found by getting the Lowest Common Multiple of the denominators.

Illustration: $\dfrac{m-2n}{m^2+mn+n^2}-\dfrac{m^2-3n^2}{m^3-n^3}+\dfrac{3m-n}{m-n}$

The L. C. M. of m^2+mn+n^2, m^3-n^3, and $m-n$ is m^3-n^3.

$$\therefore \text{the sum} = \frac{(m-2n)(m-n)}{m^3-n^3}-\frac{(m^2-3n^2)}{m^3-n^3}+\frac{(3m-n)(m^2+mn+n^2)}{m^3-n^3}$$

$$=\frac{m^2-3mn+2n^2-m^2+3n^2+3m^3+2m^2n+2mn^2-n^3}{m^3-n^3}$$

$$\text{or } \frac{2m^3+m^2+2m^2n-3mn+2mn^2+5n^2-n^3}{m^3-n^3}$$

IV. *Fractions may be multiplied by multiplying their numerators and denominators separately, obtaining thus the numerator and denominator respectively of the product.* Why?

Illustration: $\dfrac{5c^2-20d^2}{c^3+8d^3}\cdot\dfrac{c^2-2cd+4d^2}{25cd^4}$

$$=\frac{5(c+2d)(c-2d)}{(c+2d)(c^2-2cd+4d^2)}\cdot\frac{c^2-2cd+4d^2}{\underset{5}{25cd^4}}$$

$$=\frac{c-2d}{5cd^4}$$

V. *Fractions may be divided by inverting the divisor and proceeding as in multiplication.* Why?

Illustration: $\dfrac{a^2-7a+12}{a-1}\div\dfrac{a^2-16}{1-a^2}$

$$=\frac{(a-4)(a-3)}{a-1}\cdot\frac{\overset{-1}{(1-a)}(1+a)}{(a-4)(a+4)}=\frac{(1+a)(3-a)}{4+a}$$

EXERCISES. SET XXIX. FRACTIONS

232. Would the value of a fraction be changed if both numerator and denominator were squared? Illustrate and give a reason for your answer.

233. Why in dividing fractions do we invert the divisor and multiply?

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234. Would the result of cancellation ever be zero? Illustrate and give a reason for your answer.

235. Write in four ways the fraction $\frac{a-7}{a-5}$.

236. Find the difference between

(*a*) $\frac{1}{n}+\frac{1}{m}$ and $\frac{1}{n+m}$ (*b*) $\frac{1}{n}-\frac{1}{m}$ and $\frac{1}{n-m}$ (*c*) $\frac{n+1}{3}$ and $\frac{n}{3}+1$.

237. Are any or all of the expressions in the following groups identical?

(*a*) $\frac{bh}{2}, \frac{1}{2}bh, \frac{b}{2}h$, and $b\frac{h}{2}$ (*b*) $\frac{b+B}{2}, \frac{b}{2}+\frac{B}{2}$, and $\frac{1}{2}(b+B)$

Justify your answers.

238. State, with reasons, whether or not the following are identities:

(*a*) $\frac{1}{(b-a)^2} \equiv \frac{1}{(a-b)^2}$ (*b*) $\frac{1}{(b-a)^3} \equiv \frac{1}{(a-b)^3}$

(*c*) $\frac{(a-x)^2}{a^2} \equiv \left(1-\frac{x}{a}\right)^2$

239. Add: $\frac{3a-b}{a^2-b^2}-\frac{2}{b-a}$. Why is it correct to refer to such a combination as addition?

Simplify (which means *do whatever is indicated by the symbols*).

240. $\frac{a}{4c^2}\left(\frac{c}{2a}\right)^3 .16$

241. $\frac{x^3-8}{6xy}\left(3-\frac{4}{2-x}\right)\cdot\frac{18x^2y^2}{x^2+4+2x}$

242. $\left(\frac{-a^2}{x}\right)\div\left(\frac{-a}{x^2}\right)^5\cdot\left(\frac{a}{x}\right)^3\left(\frac{-1}{x^2}\right)^3$

243. $\frac{4a^2-4ab-3b^2}{8a^2x}\div\left(a-\frac{9b^2}{4a}\right)$

244. $\frac{2-x}{1-2x}-\frac{x+2}{2x+1}+\frac{6x}{4x^2-1}$

245. $\frac{1}{y-1}+\frac{2}{y-2}-\frac{3}{y-3}+\frac{4y-3}{(y^2-y)(y-2)}$

246. $\left(\frac{a}{x}+\frac{x}{a}-2\right)\left(\frac{a}{x}+\frac{x}{a}+2\right)\div\left(\frac{a}{x}-\frac{x}{a}\right)^2$

247. $\left(\frac{1+p}{1+p^2}-\frac{1+p^2}{1+p^3}\right)\div\left(\frac{1+p^2}{1+p^3}-\frac{1+p^3}{1+p^4}\right)$

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248. (a) $x+\frac{x+1}{a}\cdot\frac{x-1}{a+b}$ (b) $\left(x+\frac{x+1}{a}\right)\frac{x-1}{a+b}$

(c) $x+\left[\frac{x+1}{a}\cdot\frac{x-1}{a+b}\right]$

249. $a-b-\frac{a^2+b^2}{a-b}$

250. $\frac{m^2-5m-84}{a^2m^2-a^2m-56a^2}$

251. $\frac{(x^2-49)(x^2-16x+63)}{(x^2-14x+49)(x^2-2x-63)}$

252. $\frac{2}{m+2}-\left(2+\frac{m}{m-3}\right)\left(\frac{9-m^2}{4-m^2}\right)\left(\frac{m+2}{m^2+m-6}\right)$

B. AREAS. DEVELOPMENT OF FORMULAS

Just as a sect is measured by finding the number of linear units it contains, so a surface is measured by finding the number of square units it contains, or better, its *ratio to the unit square*, which is known as the **area** *of the surface.* A square unit is a square each side of which is a linear unit. For example, if we were to measure the length and the width of this page (taking the inch as a linear unit) a square inch would be the corresponding square unit, and the area would be found in square inches. The selection of the unit in practical measurements is just a matter of convenience. Why, for instance, select an inch instead of a mile in measuring the length of this page?

In comparing areas or sects, we compare the abstract numbers which express the ratio of their measures in terms of a common unit. When we say that two sects compare as 5 to 4 we refer to the fact that when measured in terms of the same unit their measures would compare as 5 to 4—say one was 5″ and the other 4″. Similarly, when we say two rectangles compare as 6 to 5 we mean that if the area of one were 6 sq. ft. the area of the other would be 5 sq. ft.

Such quantities as we have just referred to are said to be **commensurable** *because they can be measured in terms of the same unit.* In comparing two sects it is sometimes impossible to get a common unit of measure, that is, to select a unit so that it will be contained an exact number of times in both. Such sects are said to be **incom-**

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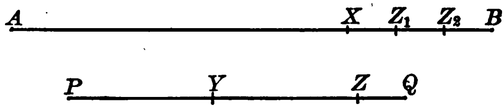
mensurable. It is known from experimental work in mensuration that a circumference and its diameter are incommensurable, for if the diameter were 1 inch, the circumference would be 3.14159+ inches (π inches).

EXERCISES. SET XXX. COMPARISON OF SECTS

253. Assuming the fact that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides, show that the diagonal of a square and a side of the square are incommensurable.

254. When two sects are commensurable a common measure can be found as follows:



If the sects are $\overline{AB}$ and $\overline{PQ}$. Lay off $\overline{PQ}$ on $\overline{AB}$. It will be contained once with a remainder $\overline{XB}$. Lay off $\overline{XB}$ on $\overline{PQ}$. It will be contained twice with a remainder $\overline{ZQ}$. Lay off $\overline{ZQ}$ on $\overline{XB}$. It will be contained just 3 times with no remainder. Then $\overline{Z_2B}$, is a common measure (indeed the greatest common measure) of $\overline{AP}$ and $\overline{PQ}$. Any part of $\overline{Z_2B}$ would also be a common measure of $\overline{AB}$ and $\overline{PQ}$. Call $\overline{Z_2B}$, $\bar{u}$.

How many $\bar{u}$'s does $\overline{XB}$ contain?
How many $\bar{u}$'s does $\overline{YZ}$ contain?
How many $\bar{u}$'s does $\overline{AX}$ contain?
Show that $\bar{u}$ is a common measure of $\overline{AB}$ and $\overline{PQ}$.

We shall refer to the *consecutive sides of a rectangle* as its **dimensions,** calling one the **altitude,** and the other the **base.**

For the sake of brevity the expressions "the ratio of two sects" or "the products of two sects" will be used to indicate the ratio or the product of the abstract numbers expressing their lengths in terms of the same unit of measure. For the same reason "rectangle," "parallelogram," etc., will be used for the "area of rectangle," "area of parallelogram," etc. Therefore, the expression "two rectangles compare as the products of their dimensions" is a conventionally abbreviated form of "the areas of two rectangles compare as the products of the lengths of their dimensions expressed in terms of the same linear unit."

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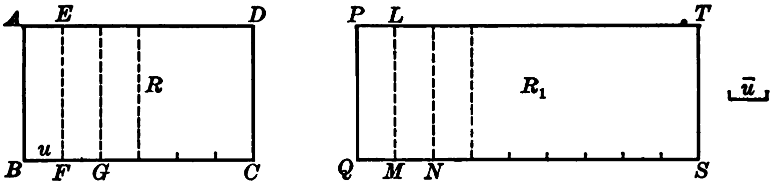
Theorem 22.* ***Rectangles having a dimension of one equal to that of another compare as their remaining dimensions***



Given: Rectangles $ABCD$ (call it R) and $PQST$ (call it R_1) with $AB = PQ$, and BC and QS commensurable.

To prove: $\frac{R}{R_1} = \frac{\overline{BC}}{\overline{QS}}$

Suggestions for proof:

Let $\bar{u}$ be a common measure of $\overline{BC}$ and $\overline{QS}$.

Lay $\bar{u}$ off on $\overline{BC}$ and $\overline{QS}$ and erect $\perp$s at points of division $F, G, \ldots$, $M, N, \ldots$.

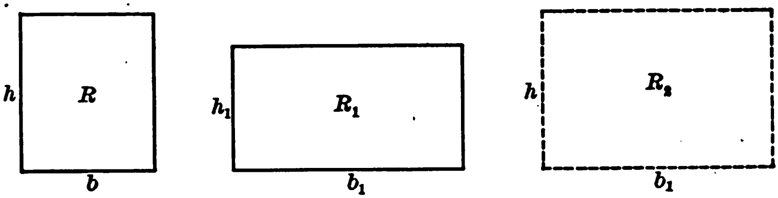
If $\bar{u}$ is contained m times in $\overline{BC}$ and n times in $\overline{QS}$ what is the ratio $\frac{\overline{BC}}{\overline{QS}}$?

What kind of figures are $ABFE, \ldots, PQML, \ldots$?

Are they congruent? Why?

If R and R_1 are respectively composed of m and n congruent parallelograms what is the ratio of $\frac{R}{R_1}$?

Theorem 23. ***Any two rectangles compare as the products of their dimensions.***



Given: Rectangles R and R_1 with dimensions h, b, and h_1, b_1 respectively.

To prove: $\frac{R}{R_1} = \frac{bh}{b_1h_1}$

Suggestions for proof:

Construct a rectangle (R_2) having as dimensions h and b_1.

$\frac{R}{R_2} = ?$ $\qquad \frac{R_1}{R_2} = ?$ $\qquad \therefore \frac{R}{R_1} = ?$

---

*Although the proof here suggested applies only to those cases where the sects are commensurable, the theorem is always valid.

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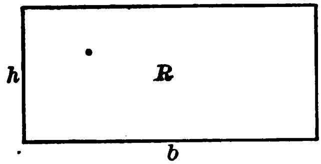
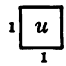
Theorem 24. *The area of a rectangle is equal to the product of its base and altitude.*





Given: Rectangle R with base b and altitude h.

To prove: Area of $R \equiv bh$.

Suggestions for proof:

What does area mean?

Call the unit of surface u. u will be a rectangle with base and altitude 1. Why? $\frac{R}{u} \equiv \frac{bh}{1 \cdot 1}$. Why? For what does $\frac{R}{u}$ stand?

EXERCISES. SET XXXI. AREAS OF RECTANGLES

255. What per cent of surface is allowed for joints and waste, if 120 rectangular sheets of tin, 14″ by 18″, are just sufficient to cover a roof 165 sq. ft.?

256. A map is drawn to a scale of 1″ to 1000 miles. What actual area would be represented on the map by a foot square?

257. From a rectangular sheet of paper cut a strip ⅛ of the sheet in width. What part of the sheet is left? Then from the same sheet cut off ⅛ of its length. What part of the original sheet is now left?

258. Prove, by letting a and b represent the lengths of two sects, that $(a \pm b)^2 \equiv a^2 + b^2 \pm 2ab$.

259. How long would a rectangular strip of paper 1 sq. ft. in area be if it were .01″ wide?

260. Both a square and a rectangle three times as long as it is wide have a perimeter of 64 ft. Compare their areas.

261. Thucydides (430 B.C.), a Greek historian, estimated the size of the Island of Sicily by the time it took him to sail around it, knowing how long it took him to sail around a known area. Was his method correct? Give a reason for your answer.

262. 144 sq. ft. is the area of a square and also of a rectangle four times as long as wide. How do their perimeters compare?

If one side of a parallelogram is selected as its **base,** *the distance between it and the opposite side is called its* **altitude.**

Would it make any difference at what point its altitude was measured? Why?

How many altitudes has a parallelogram?

Is this definition consistent with what we have referred to as an altitude in the case of the rectangle?

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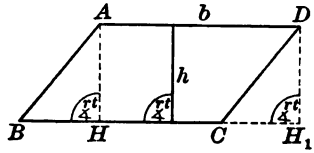
Theorem 25. ***The area of a parallelogram is equal to the product of its base and altitude.***



Given: ▱$ABCD$ with base b and altitude h.
To prove: Area of ▱$ABCD = bh$.
Suggestions for proof:

Draw $\overline{AH}$ and $\overline{DH_1} \perp \overline{BC}$. What kind of figure is AHH_1D?
What is the area of AHH_1D.
Compare ⊿s ABH and DCH_1.
How do the areas of AHH_1D and $ABCD$ compare?
What is the base of each, and what the altitude of each?
Suppose AH cut BC produced, will the theorem still hold true?

EXERCISE. SET XXXII. AREAS OF PARALLELOGRAMS

263. If one side of your parallel ruler is held fixed while the opposite side is raised and lowered to various positions, will the areas of the various parallelograms be changing? If so, what will be the greatest area obtainable, and what the least?

Any side of a triangle may be taken as its **base,** *and the perpendicular from the opposite vertex to that side will be its* **altitude.**

EXERCISE. SET XXXIII. ALTITUDES OF TRIANGLES

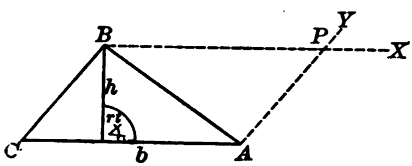
264. (*a*) How many altitudes has a triangle? Illustrate.

(*b*) Does your answer to (*a*) hold for a right triangle? Illustrate.

(*c*) Will the altitudes of a triangle always fall within the triangle?

(*d*) What fact have we already proved about the altitudes of an equilateral triangle?

Theorem 26. ***The area of a triangle is equal to half the product of its base and altitude.***



Given: △ABC with b as base and h as altitude.
To prove: Area of △$ABC = \frac{1}{2}bh$.
Suggestions for proof:

Draw $\overline{BX} \parallel \overline{CA}$ and $\overline{AY} \parallel \overline{CB}$, meeting at P.

What kind of figure is $APBC$?
What is the area of $APBC$?
How is the area of △ABC related to that of $APBC$?

Cor. 1. ***Any two triangles compare as the products of their bases and altitudes.***

Cor. 2. ***Triangles having one dimension equal compare as their remaining dimensions.***

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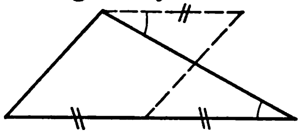
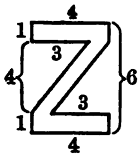
EXERCISES. SET XXXIV. AREAS OF TRIANGLES

265. Prove theorem 26, using the suggestion given by the accompanying diagram.





266. Calculate the area of the letter *Z* shown in the figure, the dimensions being indicated in centimeters.

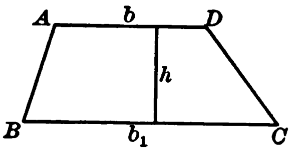
267. With a marked edge draw a triangle, and taking necessary measurements find its area, using in turn its three bases and altitudes.

268. Where do the vertices of all triangles having the same base and the same area lie? Give reasons for your answer.

A **trapezoid** *is a quadrilateral with one, and only one, pair of parallel sides. The parallel sides are called its* **bases,** *and the distance between them is called its* **altitude.**

Why do the words "and only one" need to be included in the definition?

Theorem 27. ***The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.***

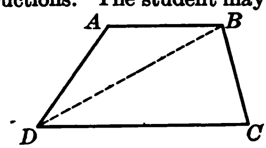


Given: Trapezoid *ABCD* with bases b and b_1 and altitude h.

To prove: Area of trapezoid equals $\frac{1}{2}h\,(b+b_1)$.

Notes on proof:

This fact may be proved by making constructions that will divide the figure into rectangles, parallelograms, or triangles, or combinations of them. Why? The following diagrams show some such constructions. The student may use one of them or suggest another. It is interesting as an exercise to show that the same formula for the area of a trapezoid may be derived from each of the following diagrams. Which gives the simplest derivation?



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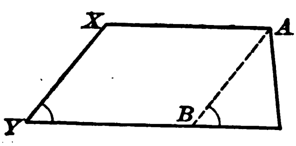
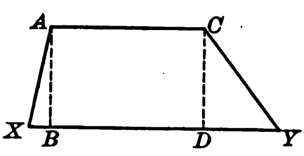


2. Draw $\overline{AB} \parallel \overline{XY}$.



3. Draw $\overline{AB} \perp \overline{XY}$ and $\overline{CD} \perp \overline{XY}$.

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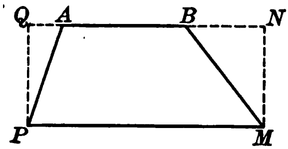
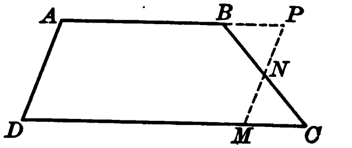
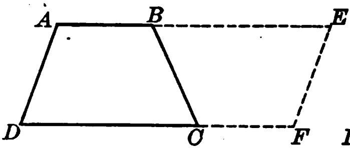


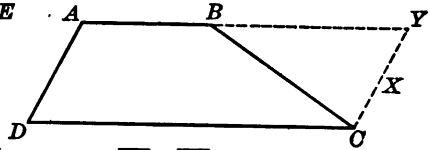
4. Draw $\overline{PQ}$ and $\overline{MN} \perp \overline{AB}$ produced.



5. Through N, the midpoint of $\overline{BC}$, draw $\overline{MN} \parallel \overline{DA}$ cutting $\overline{AB}$ produced in P.

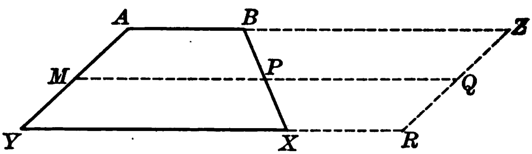


6. Extend $\overline{AB}$ by $\overline{DC}$ to E and $\overline{DC}$ by $\overline{AB}$ to F.



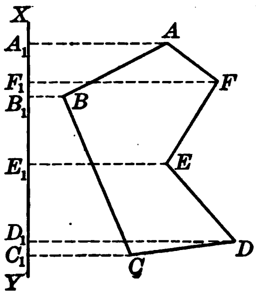
7. Draw $\overline{CX} \parallel \overline{DA}$ cutting $\overline{AB}$ produced in Y.

8. Extend $\overline{YX}$ to R so that $\overline{XR} = \overline{AB}$. Draw $\overline{RZ} \parallel \overline{YA}$. Bisect $\overline{AY}$ in M and draw $\overline{MP} \parallel \overline{YX}$ cutting $\overline{RZ}$ in Q.



EXERCISES. SET XXXV. AREAS OF TRAPEZOIDS

269. The diagram shows how the area of an irregular polygon may be found, if the distance of each vertex from a given *base line*, as XY, is known. These distances AA_1, BB_1, etc., are called *offsets*, and are the bases of trapezoids whose altitudes are A_1B_1, B_1C_1, C_1D_1, etc. The area $ABCDEF$ may now be found by the proper additions and subtractions.



On cross-section paper plot the points whose coördinates are given below, join them in order, draw FA, and find the inclosed area in each case:

Case	A	B	C	D	E	F
(*a*)	5, 7	4, 5	4, 1	7, 0	5, 3	6, 3
(*b*)	2, 7	3, 3	6, 0	5, 2	6, 6	8, 7
(*c*)	2, 5	3, 6	2, 4	4, 2	6, 5	4, 7

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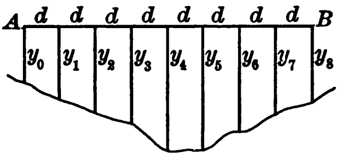
270. In order to determine the flow of water in a certain stream, soundings are taken every 6 ft. on a line AB at right angles to the current. A diagram may then be made to represent a vertical cross-section of the stream. If the area of this cross-section and the speed of the current are known, it is possible to determine the amount of water flowing through the cross-section in a given time.



The required area is often found approximately by joining the extremities of the offsets y_0, y_1, y_2, etc., by straight lines, and finding the sum of the trapezoids thus formed. That is, the strips between successive offsets are replaced by trapezoids. This gives the *Trapezoidal Rule* for finding an area. It may be stated as follows: To half the sum of the first and last offsets add the sum of all intermediate offsets, and multiply this result by the common distance between the offsets.

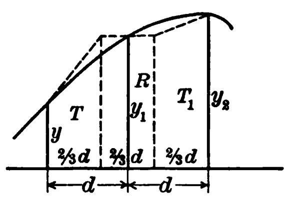
(*a*) State this as an algebraic formula.

(*b*) Verify the formula.

(*c*) Find the area of the cross-section of a stream if the soundings taken at intervals of 6 ft. are respectively 5 ft., 6.5 ft., 11 ft., 14.5 ft., 16 ft., 9 ft., and 6.5 ft.

(*d*) In the midship section of a vessel the widths taken at intervals of 1 ft. are successively 16, 16.2, 16.3, 16.4, 16.5, 16.7, 16.8, 15, 10, 4, 0, measurements being in feet. Find the area of the section. (Use the line drawn from the keel ⊥ to the deck as base line.)

d271. A third rule for finding plane areas, known as *Simpson's Rule*, usually gives a closer result than the Trapezoidal Rule. In proving Simpson's rule two consecutive strips are replaced by a rectangle and two trapezoids as follows: Divide $2d$ into three equal parts, erect ⊥*s* at the points of division, and complete the rectangle whose altitude is the middle offset y_1, as in the figure. Join the extremities of y and y_2 to the nearer upper vertex of this rectangle.



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Then if the areas of the trapezoids are T and T_1, and if the area of the rectangle is R,

$$T = \frac{1}{2}(y+y_1)\cdot\frac{2}{3}d,\ R = y_1\cdot\frac{2}{3}d,\ T_1 = \frac{1}{2}(y_1+y_2)\cdot\frac{2}{3}d.$$

$$\therefore\ T+R+T_1 = \frac{1}{3}d\,(y+4y_1+y_2).$$

If, now, the number of strips is *even*, and if the offsets are lettered consecutively $y_0, y_1, y_2, \ldots, y_n$, the addition of the areas of successive double strips, found by the above formula, gives the result:

$$\text{Area} = \frac{1}{3}d\,(y_0+4y_1+2y_2+4y_3+\ldots+2y_{n-2}+4y_{n-1}+y_n).$$

In words: To the sum of the first and last offsets add twice the sum of all the other even offsets and four times the sum of all the odd offsets, and multiply by one-third the common distance between the offsets.

(*a*) Verify this rule.

(*b*) By Simpson's Rule find the area of the stream in Ex. 270 (*c*).

(*c*) By Simpson's Rule find the area of the section of the vessel in Ex. 270 (*d*).

LIST OF WORDS DEFINED IN CHAPTER IV

Ratio, fraction, numerator, denominator, terms of fraction, cancellation, simplify. Area, commensurable, incommensurable; dimensions, base and altitude of rectangle, parallelogram, triangle, trapezoid.

SUMMARY OF THEOREMS PROVED IN CHAPTER IV

22. Rectangles having a dimension of one equal to that of another compare as their remaining dimensions.

23. Any two rectangles compare as the products of their dimensions.

24. The area of a rectangle is equal to the product of its base and altitude.

25. The area of a parallelogram is equal to the product of its base and altitude.

26. The area of a triangle is equal to half the product of its base and altitude.

Cor. 1. Any two triangles compare as the products of their bases and altitudes.

Cor. 2. Triangles having one dimension equal compare as the remaining dimensions.

27. The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.

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CHAPTER V

ALGEBRA AS AN INSTRUMENT FOR USE IN APPLIED MATHEMATICS

A. LOGARITHMS

Although logarithms are introduced at this point, as a part of a chapter on Algebra, it is not essential that they be studied at this time. They might well be omitted until there is a feeling of necessity on the part of the pupils in the solution of the more difficult problems based upon similarity and the trigonometric functions. In fact, those schools wishing to omit such problems under Ratio, Proportion, Variation and Similarity need not include the topic at all. For this reason, when logarithms are desirable in the solution of problems, this fact has been indicated. The integrity of the course will not be injured in the least by the omission of the topic of logarithms or any of these exercises.

I. INTRODUCTION

If $8^2=64$, then 2 is called the **logarithm** of 64 to the **base** 8. This is written $\log_8 64=2$. Again $3^4=81$ or $\log_3 81=4$. Hence we see that the logarithm of a number is simply *the exponent of the power to which another number* (called the **base**) *must be raised to obtain that number*. Thus, in the last example 4 is the exponent of the power to which the base 3 must be raised to obtain the number 81.

EXERCISES. SET XXXVI. MEANING OF LOGARITHMS

272. Write in logarithmic form:

(*a*) $5^3=125$ (*c*) $7^2=49$ (*e*) $10^{1.30103}=20$ (*g*) $10^{3.47712}=3000$
(*b*) $2^7=128$ (*d*) $10^1=10$ (*f*) $10^3=1000$ (*h*) $10^{5.4771}=300{,}000$
(*i*) $10^6=1{,}000{,}000$ (*j*) $100^{1.5}=1000$

273. Write in exponential form:

(*a*) $\log_6 216=3$ (*d*) $\log_{10} 2=.3010$ (*g*) $\log_{10} .01 = -2$
(*b*) $\log_9 81=2$ (*e*) $\log_{10} 1=0$ (*h*) $\log_{10} .001= -3$
(*c*) $\log_{11} 1331=3$ (*f*) $\log_{10} .1= -1$ (*i*) $\log_{10} 500=2.6990$

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Before studying logarithms, their principles and applications, it will be well for us to recall the laws concerning exponents in multiplication, division, involution and evolution. We have learned to multiply, divide, raise to powers and extract roots of simple expressions when the exponents have been positive integers, so that we shall here only summarize what we already know, adding the statement that the laws governing positive integral exponents govern all exponents, both fractional and negative

a. PRINCIPLES OF EXPONENTS

1. *The exponent of the product of any number of factors of like base is equal to the sum of the exponents of the factors.*

Illustrations: (1) $a^n \cdot a^p \cdot a^q \equiv a^{n+p+q}$

(2) $a^{\frac{3}{5}} \cdot a^{\frac{4}{5}} \equiv a^{\frac{7}{5}}$

(3) $b^{-n} \cdot b^{-b} \equiv b^{-n-p}$

(4) $b^{+\frac{4}{5}} \cdot b^{-\frac{3}{10}} \equiv b^{\frac{4}{5}-\frac{3}{10}} \equiv b^{\frac{5}{10}} \equiv b^{\frac{1}{2}}$

2. *The exponent of the quotient of two quantities of like base is equal to the exponent of the dividend diminished by that of the divisor.*

Illustrations: (1) $x^a \cdot x^{-b} \equiv x^{a-b}$

(2) $x^a \div x^{-b} \equiv x^{a+b}$

(3) $m^{-\frac{p}{q}} \div m^{\frac{r}{s}} \equiv m^{\frac{-ps-rq}{qs}}$

3. *The exponent of any power of a quantity is equal to the product of the exponent of the base and the index of the power.*

Illustrations: (1) $(b^p)^q \equiv b^{pq}$

(2) $\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} \equiv a^{\frac{pr}{qs}}$

4. *The exponent of any root of a quantity is equal to the quotient of the exponent of the base and the index of the root when that index is not zero.*

Illustrations: (1) $\sqrt[-m]{b^x} \equiv b^{-\frac{x}{m}}$

(2) $\sqrt[\frac{p}{q}]{x^{-a}} \equiv x^{-\frac{aq}{p}}$

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EXERCISES. SET XXXVII. DRILL IN APPLICATION OF LAWS OF EXPONENTS

Perform the operations indicated in the following problems:

274. $x^4x^{-7}x^{-8}$

275. $\frac{3}{8}a^{\frac{1}{2}}\cdot\frac{5}{6}a^{\frac{1}{3}}$

276. $k^{n+2}\div 3k^2$

277. $p^{\frac{m}{n}}p^{\frac{n}{m}}p^{\frac{1}{nm}}$

278. $(rs^at^b)\left(-qs^{\frac{-7}{a}}t^{-\frac{4}{b}}\right)$

279. $4^{\frac{1}{2}}\cdot 8^{\frac{4}{5}}$
(Hint: Express 4 and 8 as powers of 2 so as to have a common base).

280. $25^{\frac{3}{2}}.125^{\frac{2}{3}}$

281. $25^{-\frac{1}{2}}\cdot 5^{-2}$

282. $10^{3.4362}\cdot 10^{.7856}$

283. $10^{-\frac{1}{2}}\cdot 10^{\frac{8}{4}}\div 10^{\frac{7}{2}}\cdot\sqrt[3]{10^5}$

284. $10^{-\frac{1}{2}}\cdot 10^{\frac{8}{4}}\div\left(10^{\frac{7}{2}}\cdot\sqrt[3]{10^5}\right)$

285. $(10^{5.6723})^7$

286. $\left(d^{\frac{1}{x+y}}\right)^{(x^2-y^2)^2}$

287. $\dfrac{16^{\frac{1}{2}}\cdot 125^{-\frac{1}{3}}}{25^{\frac{1}{2}}\cdot 32^{\frac{2}{5}}}$

288. $\dfrac{3^n(3^{n-1})^n}{3^{n+1}(3^{n-1})(9^{-n})}$

289.* $\sqrt[4]{\dfrac{10^{-2}(10^7)(10^{\frac{3}{5}})^{10}}{(10^{-\frac{2}{3}})^6(10^{\frac{4}{5}})^{10}}}$

---

*NOTE: No doubt some of us are by this time curious to know the meaning of the zero, negative and fractional exponent.

(I) *Meaning of the zero exponent.*

Let $a^0=x$

Then $x\cdot a^p=a^0\cdot a^p$

$=a^{0+p}$

$=a^p$

$\therefore x=\dfrac{a^p}{a^p}=1$

$\therefore a^0=1$ Why?

Quote the axiom applied.

(II) *Meaning of the negative exponent.*

Let $a^{-p}=x$

Then $a^{-p}\cdot a^p=x\cdot a^p$ Why?

But $a^{-p}\cdot a^p=1$ Why?

$\therefore x\cdot a^p=1$ Why?

$\therefore x=\dfrac{1}{a^p}$ Why?

$\therefore a^{-p}=\dfrac{1}{a^p}$ Why?

(III) *Meaning of the fractional exponent.*

The fourth law given in the text expresses the meaning of the fractional exponent since $\sqrt[r]{x^p}=x^{\frac{p}{r}}$, but we may make the meaning still clearer by the following:

Since $x^{\frac{1}{2}}\cdot x^{\frac{1}{2}}=x$, $x^{\frac{1}{2}}$ must by definition of square root be the square root $(\sqrt{x})$.

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Let us see how the annexed table may be used to simplify certain calculations.

TABLE OF POWERS OF 2

$2^1=2$	$2^6=64$	$2^{11}=2048$	$2^{16}=65536$
$2^2=4$	$2^7=128$	$2^{12}=4096$	$2^{17}=131072$
$2^3=8$	$2^8=256$	$2^{13}=8192$	$2^{18}=262144$
$2^4=16$	$2^9=512$	$2^{14}=16384$	$2^{19}=524288$
$2^5=32$	$2^{10}=1024$	$2^{15}=32768$	$2^{20}=1048576$

1. Suppose we wished to multiply 1024 by 512
   $\because 1024=2^{10}$ and $512=2^9$ and $2^{10}\cdot 2^9=2^{19}$,
   $\therefore 1024\cdot 512=2^{19}=524288.$
2. Suppose we wished to divide 1,048,576 by 32768.
   $\because 1048576=2^{20}$ and $32768=2^{15}$ and $2^{20}\div 2^{15}=2^5$,
   $\therefore 1048576\div 32768=2^5=32.$
3. Suppose we wished to raise 64 to the third power.
   $\because 64=2^6$ and $(2^6)^3=2^{18}$ and $2^{18}=262144$,
   $\therefore 64^3=(2^6)^3=262144.$
4. Suppose we wished to find the 5th root of 1,048,576.
   $\because 1048576=2^{20}$ and $\sqrt[5]{2^{20}}=2^{\frac{20}{5}}=2^4$
   $\therefore \sqrt[5]{1048576}=\sqrt[5]{2^{20}}=2^4=16$

In similar manner a table of the powers of any number may be computed, and these four operations (multiplication, division, involution, evolution), reduced to the operations of addition, subtraction, multiplication, and division of exponents.

Similarly $x^{\frac{1}{s}}\cdot x^{\frac{1}{s}}\cdot x^{\frac{1}{s}}\cdot$ ....to s factors

$=x^{\frac{1}{s}+\frac{1}{s}+\frac{1}{s}}$....to s terms. Why?

$=x^{\frac{s}{s}}=x$ Why?

$\therefore x^{\frac{1}{s}}=\sqrt[s]{x}$ Why?

Still more generally, $x^{\frac{p}{q}}=x^{\frac{1}{q}}\cdot x^{\frac{1}{q}}\cdot x^{\frac{1}{q}}$........to p factors.

$[=(\sqrt[q]{x})^p]$

But $\sqrt[q]{x^p}$ also means $(x^p)^{\frac{1}{q}}=(x\cdot x\cdot x\cdot$.........to p factors$)^{\frac{1}{q}}$.

$=x^{\frac{1}{q}}\cdot x^{\frac{1}{q}}\cdot x^{\frac{1}{q}}\cdot$ ......to p factors.

$=x^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+}$....to p terms.

$=x^{\frac{p}{q}}$

EXERCISES. SET XXXVIII. USE OF TABLES OF POWERS

Using the given table of 2's, find the values of the following:

290. 512×2048

291. $\dfrac{256 \times 16384}{262144}$

292. $(32)^3$

293. $(512)^2$

294. $\sqrt{131072}$

295. $\sqrt[3]{262144}$

296. $\sqrt[3]{65536}$

297. $\sqrt[4]{\dfrac{(256)^5(524288)^2}{(32)^6}}$

298. $\left[(2^2)^2\right]^2$

299. $2^{2^{2^2}}$

TABLE OF POWERS OF 16

No.	Power	No.	Power
1	0.00	64	1.50
2	0.25	256	2.00
4	0.50	1024	2.50
8	0.75	4096	3.00
16	1.00	65536, etc.	4.00
32	1.25		

EXERCISES. SET XXXVIII (concluded)

From the foregoing table compute the following:

300. 16×4096

301. 8^4

302. $65536 \div 4096$

303. $\sqrt{\dfrac{65536(256)^2}{64(1024)}}$

304. $16 \times \sqrt[3]{\dfrac{(65536 \times 1024)^2}{256^2}}$

b. HISTORICAL NOTE

There is a large amount of computation necessary in the solution of some of the practical applications of mathematics. The labor of making extensive and complicated calculations can be greatly lessened by employing a table of logarithms. About the year 1614 a Scotchman, John Napier (1550–1617), Baron of Merchiston, invented a system by which multiplication can be performed by addition, division by subtraction, involution by a single multiplication, and evolution by a single division. From Henry Briggs (1556–1631), who was a professor at Gresham College, London, and later at Oxford, this invention received modifications which made it more convenient for ordinary practical purposes.

Laplace, the great French astronomer, said: "The employment of logarithms by reducing to a few days the labors of months, doubles, as it were, the life of an astronomer, besides freeing him from the errors and disgust inseparable from long calculations."

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The logarithms now in general use are known as common logarithms, or as Briggs' logarithms, in order to distinguish them from another system, also a modified form of Napier's system. The logarithms of this other modified system are frequently employed in higher mathematics, and are known as natural or hyperbolic logarithms.

II. PRINCIPLES OF COMMON LOGARITHMS

For practical purposes, the exponents of the powers to which 10, the base of our decimal system, must be raised to produce various numbers are put in table form. That is, the logarithms of numbers to the base 10 are tabulated. For the positive integral powers of 10 we would need no tables, for those we can find by inspection. But exponents may be negative and they may be fractional. For the negative integral powers of 10 as we shall see presently, we would need no tables either. But fractional exponents or the fractional parts of exponents we cannot readily find, and hence for them we need tables.

We all know that

$\because 10^3 = 1000, \therefore$ *log $1000 = 3$.

$\because 10^2 = 100, \therefore \log 100 = 2$.

$\because 10^1 = 10, \therefore \log 10 = 1$.

Similarly it can be shown that

$\because 10^{-1} = \frac{1}{10}, \therefore \log .1 = -1$

$\because 10^{-2} = \frac{1}{10^2}, \therefore \log .01 = -2$

$\because 10^{-3} = \frac{1}{10^3}, \therefore \log .001 = -3$

$$\because \frac{10^1}{10^1} = 10^{1-1} = 10^0, \text{ and } \frac{10^1}{10^1} = 1; \therefore 10^0 = 1, \therefore \log 1 = 0.$$

$10^{.47712}$ or $10^{\frac{47712}{100000}}$, that is, the one-hundred-thousandth root of $10^{.47712}$ is nearly 3. $\therefore \log 3 = .47712$ nearly.

Although log 3 can never be expressed exactly as a decimal fraction, it can be found to any required degree of accuracy. In this book logarithms are given to four decimal places. These are sufficient for ordinary computations.

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* When the base 10 is used the base is not indicated in writing the logarithms of numbers. Thus we write $\log 3 = .47712$, not $\log_{10} 3 = .47712$.

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EXERCISES. SET XXXIX. COMMON LOGARITHMS

305. What are the logarithms of the following to the base 10:

(*a*) 10000? (*b*) $\frac{1}{10000}$? (*c*) .0001? (*d*) 10^9? (*e*) 10^{-9}? (*f*) $\sqrt[3]{10}$? (*g*) $10^{\frac{1}{2}}$? (*h*) $10^{\frac{1}{7}}$?

306. Between what two consecutive integers does the logarithm of each of the following numbers lie? Why?

(*a*) 600 (*b*) 5728 (*c*) 13 (*d*) 496,287 (*e*) 46923 (*f*) 9

307. Between what two consecutive negative integers does the logarithm of each of the following numbers lie? Why?

(*a*) .06 (*b*) .007 (*c*) .0008 (*d*) .0625 (*e*) .00729 (*f*) .00084 (*g*) 0.5

308. What is meant by saying that:

(*a*) log 880 = 2.94448? (*b*) log 92.12 is 1.96435? (*c*) log 4.37 is .64048?

Since 3585 lies between 1000 and 10,000, its logarithm lies between 3 and 4. It has been calculated as 3.55449. *The integral part* 3 *is called the* **characteristic**, and *the decimal part* .55449, the **mantissa** *of the logarithm.*

$\because$ 358.5 = 3585 ÷ 10, $\therefore$ log 358.5 = log (3585 ÷ 10) = log 3585 − log 10 = 3.55449 − 1 = 2.55449.

That is, since log 3585 = 3.55449
log 358.5 = 2.55449 and similarly it can be shown
that log 35.85 = 1.55449
log 3.585 = 0.55449
log .3585 = .55449 − 1*
log .03585 = .55449 − 2.

Thus we see that

(*a*) *The characteristic can be found by inspection in all cases.*

$\because$ the number 589 lies between 100 and 1000, log 589 lies between 2 and 3. $\therefore$ log 589 = 2 + some mantissa.

---

* Log 0.3585 = .55449 − 1 may be written in two other ways as follows: $\bar{1}$.55449 or 9.55449 − 10. The last method is the most practical, as we shall see as we proceed.

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(*b*) *The mantissa is the same for any given succession of digits, wherever the decimal point may be.*

(See last table of numbers with their logarithms.)

(*c*) *As a result of* (*a*) *and* (*b*) *only a table of mantissas need be given.*

EXERCISES. SET XXXIX (continued)

309. What is the characteristic of the logarithm of:

(*a*) 384? (*c*) .297? (*e*) A number of n integral places?

(*b*) 5286? (*d*) Any number of millions?

(*f*) Any decimal fraction whose first significant digit is in the first decimal place?

(*g*) In the second decimal place? (*h*) In the third decimal place?

(*i*) In the seventh decimal place? (*j*) In the n^{th} decimal place?

310. From the last exercise formulate a principle by means of which the characteristic of the logarithm of any positive number may be found.

III. THE FUNDAMENTAL THEOREMS OF LOGARITHMS

(*a*) *The logarithm of the product of two numbers equals the sum of their logarithms to the same base.*

1. Let $a \equiv 10^{l_1}$, then $\log a \equiv l_1$
2. Let $b \equiv 10^{l_2}$, then $\log b \equiv l_2$
3. $\therefore ab \equiv 10^{l_1+l_2}$, and $\log ab \equiv l_1 + l_2 \equiv \log a + \log b$.

(*b*) *The logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor, all to the same base.*

1. Let $a \equiv 10^{l_1}$ then $\log a \equiv l_1$
2. Let $b \equiv 10^{l_2}$, then $\log b \equiv l_2$
3. $\therefore \frac{a}{b} \equiv \frac{10^{l_1}}{10^{l_2}} = 10^{l_1 - l_2}$, and $\log \frac{a}{b} \equiv l_1 - l_2 \equiv \log a - \log b$.

(*c*) *The logarithm of the n^{th} power of a number equals n times the logarithm of that number.*

1. Let $a \equiv 10^{l}$, then $\log a \equiv l$
2. $\therefore a^n \equiv 10^{ln}$, and $\log a^n \equiv nl \equiv n \log a$.

(*d*) *The logarithm of the n^{th} root of a number equals $\frac{1}{n}$th of the logarithm of the number.*

1. Let $a \equiv 10^{l}$, then $\log a \equiv l$
2. $\therefore a^{\frac{1}{n}} \equiv 10^{\frac{l}{n}}$, and $\log a^{\frac{1}{n}} \equiv \frac{l}{n} \equiv \frac{1}{n} \log a$.

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Th. (*c*) might have been stated more generally, so as to include Th. (*d*) thus: $\text{Log } a^{\frac{x}{y}} = \frac{x}{y} \log a$. The proof would be substantially the same as in Ths. (*c*) and (*d*).

EXERCISES. SET XXXIX (concluded)

Given log 2 = 0.3010, log 3 = 0.4771, log 5 = 0.6990, log 7 = 0.8451, and log 514 = 2.7110, find the following:

311. Log 6. **312.** Log 14. **313.** Log 7^{10}. **314.** Log $\sqrt{2}$.
315. Log 42. **316.** Log $5^{\frac{1}{4}}$. **317.** Log 105. **318.** Log 1.05.
319. Log $\sqrt{514}$. **320.** Log 514^2. **321.** Log 1542. **322.** Log 257.
323. Log 1799 [$= \log(\frac{1}{2} \cdot 514 \cdot 7)$]. **324.** Log $\sqrt[7]{3^4}$. **325.** Log $\sqrt{21}$.
326. Show how to find log 5, given log 2.

IV. USE OF THE TABLE

(*a*) *Given a number, to find its logarithm.*

In the table on p. 103 only the mantissas are given. For instance, in the row beginning 43, and in columns headed 0, 1, 2, 3, ... , 9 will be found:

N	0	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425

This means that the mantissa of log 430 is .6335, of log 431 is .6345, and so forth, to log 439.

Therefore log 431 = 2.6345, log 434 = 2.6375, log 43.7 = 1.6405, log 4.39 = 0.6425, log .438 = .6415 − 1, log .0433 = .6365 − 2. Now, since 437.8 is .8 of the way from 437 to 438, ∴ log 437.8 is about .8 of the way from log 437 to log 438. ∴ log 437.8 = log 437 + .8 of the difference between log 438 and log 437. ∴ log 437.8 = 2.6405 + .8 of .0010 = 2.6405 + .0008 = 2.6413.

This process of *finding the logarithm of a number lying between two tabulated numbers is called* **interpolation.** This is not wholly accurate, since the numbers do not vary as their logarithms, but it is sufficiently accurate for most practical purposes. If greater accuracy is desired, tables of five or six or even more places are used. The mantissas here given are correct to .0001. This will give a result which is correct to three figures in general, and an approximation to four figures, which will be sufficiently accurate for the computations in this book.

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EXERCISES. SET XL. USE OF TABLE

Find by using the table:

327. Log 49.
328. Log 723.
329. Log 1580.
330. Log 4285.
331. Log 14.5.
332. Log 14.7.
333. Log 14.73.
334. Log 5.93.
335. Log .00432.
336. Log 1.672.
337. Log .00002376.
338. Log $\sqrt{29}$.
339. Log 5.692^3.
340. Log $\sqrt[3]{36.54}$.
341. Log $.0057^6$.

(*b*) *Given a logarithm to find the corresponding number.*

The number corresponding to a given logarithm is called its **antilogarithm.** Example: $\because$ 0.4771 = log 3, $\therefore$ antilog 0.4771 = 3.

N	0	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425

Here we see that antilog 4.6345 = 43100, antilog 0.6395 = 4.36, antilog $\bar{3}.6405$ = .00437.

Now suppose we wished to find antilog $\bar{2}.6417$.

$\because$ $\bar{2}.6417$ is .2 the way from $\bar{2}.6415$ to $\bar{2}.6425$.

$\therefore$ antilog $\bar{2}.6417$ is about .2 the way from antilog $\bar{2}.6415$ to antilog $\bar{2}.6425$.

$\therefore$ antilog $\bar{2}.6417$ is about .2 the way from .0438 to .0439.

$\therefore$ antilog $\bar{2}.6417$ = .04382.

The following form is a good one for use.

Required: Antilog 2.7361. Antilog 2.7364 = 545   2.7361
Antilog 2.7356 = 544   2.7356
Tabular diff. = 8   Diff. = 5

$\therefore$ Antilog 2.7361 = $544\frac{5}{8}$ = 544.6.

EXERCISES. SET XL. (concluded)

Find from the table:

342. Antilog 1.9321.
343. Antilog 2.9049.
344. Antilog $\bar{2}.7813$.
345. Antilog 3.0354.
346. Antilog $\bar{1}.0354$.
347. Antilog $\bar{3}.1628$.
348. Antilog $\bar{4}.8393$.
349. Antilog 10.5843.
350. Antilog 0.6923 − 2.
351. Antilog 8.6923 − 2.
352. Antilog 7.5194 − 10.
353. Antilog 9.2490 − 10.
354. Antilog 10.4687 − 10.
355. Antilog 3.5357.
356. Antilog 0.3471.

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(c) *Computation by logarithms.*

Since many errors occur because of failure to arrange work carefully, the pupil is advised to arrange all work in as compact and neat a form as possible. A few examples worked out in full may be suggestive, therefore we append the following:

1. In how many years will $600 double itself at 3% interest compounded annually?

Solution: Let the number of years be n.

At the end of one year the amount will be 1.03 of $600, at the end of the second year it will be 1.03 of 1.03 of $600, or 1.03^2 of $600, and so forth.

$\therefore$ at the end of n years it will be 1.03^n of $600.

$\therefore 1.03^n \times 600 = 1200$ or $1.03^n = 2$

$\therefore n \log 1.03 = \log 2$ and $\therefore n = \dfrac{\log 2}{\log 1.03}$

$$\left.\begin{aligned}\log 2 &= .3010\\ \log 1.03 &= .0128\end{aligned}\right\} \therefore n = \frac{.3010}{.0128}$$

$\therefore \log n = \log .3010 - \log .0128$

$\log .3010 = 9.4786 - 10$

$\log .0128 = 8.1072 - 10$

$\therefore \log n = 1.3714$ antilog $1.3714 = 23.5+$

$\therefore n = 23.5+$, or the sum will double itself in **24** years.

2. Required the value of $\dfrac{5286 \times \sqrt{427}}{\sqrt[3]{1754 \times 3292}}$.

Solution:

$\log 529 = 2.7235$

$\log 528 = 2.7226$

Tab. diff. $= 9$

$.6$

$5.4 \quad \therefore \log 528.6 = 2.7231$

$\log 427 = 2.6304 \quad \therefore \log \sqrt{427} = 1.3152$

$\therefore$ log numerator $= 4.0383 \ldots\ldots 14.0383 - 10$

$\log 1760 = 3.2455$

$\log 1750 = 3.2430$

Tab. diff. $= 25$

$.4$

10.0

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$\therefore$ log 1754 = 3.2440 $\therefore$ log $\sqrt[3]{1754} = 1.0813$;

log numerator = 14.0383–10

log 3300 = 3.5185

log 3290 = 3.5172

Tab. diff. = 13

.2

2.6 or 3 $\therefore$ log 3292 = 3.5175

$\therefore$ log denominator = 4.5988 . . . . 4.5988

antilog 9.4409 − 10 = .276 $\therefore$ log fraction (log numerator − log denominator) = 9.4395 − 10

antilog 9.4393 − 10 = .275 9.4393 − 10

Diff. for 1 = 16 2

$\therefore$ antilog 9.4395 − 10 = $.275\frac{2}{16} = .275\frac{1}{8}$.

$\therefore$ Fraction = .2751

As soon as the pupil is able to interpolate mentally the left column may be omitted.

Often in problems involving the process of evolution difficulties may arise owing to the fact that the characteristic may be negative and not a multiple of the divisor, while the mantissa is always positive. For instance, it may be desired to find $\sqrt[3]{0.03}$.

log $\sqrt[3]{0.03} = \frac{1}{3}$ log 0.03

$= \frac{1}{3}(\bar{2}.4771)$. Why is this a decidedly impracticable form?

$= \frac{1}{3}$ (8.4771 − 10). Why is this an inconvenient form?

$= \frac{1}{3}$ (28.4771 − 30). Why is this the form which is adopted?

= 9.4924 − 10

$\therefore \sqrt[3]{0.03} = 0.3107$

Again, suppose $\sqrt[7]{0.3^4}$ is called for.

log 0.3 = $\bar{1}.4771$

$\therefore \log \sqrt[7]{0.3^4} = \frac{4}{7}(9.4771 - 10) = \frac{37.9084 - 40}{7}$. Now, 40 is not a multiple of 10 and 7, hence we change to the equivalent

$$\log \sqrt[7]{0.3^4} = \frac{67.9084 - 70}{7} = 9.7012 - 10.$$

$\therefore$ antilog 9.7012 − 10 = $\sqrt[7]{0.3^4}$.

7

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In place of a table of logarithms engineers often use an instrument called a "slide rule." This is really a mechanical table of logarithms arranged ingeniously for rapid practical use. Results can be obtained with such an instrument far more quickly than with an ordinary table of logarithms, and that without recording or even thinking of a single logarithm. A "slide rule" ten inches long gives results correct to three figures. In work requiring greater accuracy a larger and more elaborate instrument which gives a five-figure accuracy is used.*

EXERCISES. SET XLI. COMPUTATION BY LOGARITHMS

357. If the hypotenuse of a right triangle and one leg are known, the other leg may be found by means of logarithms, for if h = hypotenuse, l_1 = one leg, then $l_2 \equiv \sqrt{h^2 - l_1^2} \equiv \sqrt{(h+l_1)(h-l_1)}$.

$\therefore \log l_2 \equiv \frac{1}{2}[\log(h+l_1) + \log(h-l_1)]$.

If the hypotenuse of a right triangle is 587, and one leg is 324, what is the other leg?

358. The area of an equilateral triangle whose side is s, is $\frac{s^2}{4}\sqrt{3}$.

(*a*) Find the area of an equilateral triangle whose side is 15.38 units.

(*b*) Find the side of an equilateral triangle whose area is 89.5 square inches.

359. The formula for the area of a triangle in terms of its sides is $A \equiv \sqrt{s(s-a)(s-b)(s-c)}$ where a, b, c are the sides of the triangle and s the semi-perimeter. Find the area of a triangle whose sides are 436, 725.4, 951.8 units respectively.

This is often referred to as Heron of Alexandria's formula, and will be proved later.

a360.† Show that the amount of P dollars at interest compounded annually for n years is $P\left(1+\frac{r}{100}\right)^n$; compounded semiannually is $P\left(1+\frac{r}{200}\right)^{2n}$.

---

* It is suggested that pupils will find an inexpensive slide rule of great use in rapid calculation. Such an instrument, called the "Favorite," can be purchased of Keuffel and Esser, New York, N. Y.

† As here, the letter *a* preceding the number of an exercise indicates that algebra beyond the solution of simple linear equations is required to solve the problem.

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361. In how many years will $1.00 double itself at
(*a*) 3% interest compounded annually?
(*b*) 4% interest compounded annually?
(*c*) 5% interest compounded annually?
(*d*) 6% interest compounded annually?

362. In how many years will $1.00 double itself at
(*a*) 3% interest compounded semi-annually?
(*b*) 4% interest compounded semi-annually?
(*c*) 5% interest compounded semi-annually?
(*d*) 6% interest compounded semi-annually?

363. Find the amount at compound interest, compounded annually of:
(*a*) $150 for 7 years at $5\frac{1}{2}\%$. (*b*) $1850 for 5 years at 4%.
(*c*) $10 for 50 years at 5%.

364. To find the present value A_0 of an annuity (a fixed sum of money, payable at equal intervals of time) of s dollars to continue for n years at $R\%$ compound interest, the formula

$$A_0 \equiv \frac{s}{R\%}\left[1-\frac{1}{(1+R\%)^n}\right] \text{ is used.}$$

Find the present value of an annuity (*i.e.*, the amount which, if put at compound interest for the given time and rate, will amount to the given sum).
(*a*) Of $1000 for 10 years, at 4% compound interest.
(*b*) Of $1200 for 10 years, at 4% compound interest.
(*c*) Of $1500 for 10 years, at 4% compound interest.
(*d*) Of $500 for 5 years, at 5% compound interest.

365. What annuity can be purchased for $3000, if it is to run for 15 years, at 5% compound interest compounded annually?

366. The diameter in inches of a connecting rod depends upon the diameter D of the engine cylinder, l the length of the connecting rod, and P the maximum steam pressure in pounds per sq. inch, according to Mark's formula $d \equiv 0.02758\sqrt{D \cdot l \cdot \sqrt{P}}$.

What is d when $D=30$, $l=75$, and $P=150$?

367. If fluid friction be used to retard the motion of a flywheel making V_0 revolutions per minute, the formula $V \equiv V_0 e^{-kt}$ gives the number of revolutions per minute, after the friction has been

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applied t seconds. If the constant $k=0.35$, the value of e being 2.718, how long must the friction be applied to reduce the number of revolutions from 200 to 50 per minute?

a368. The pressure, P, of the atmosphere in pounds per sq. inch, at a height of z feet, is given approximately by the relation $P_o \equiv P_o e^{-kz}$, where P_o is the pressure at sea level and k is a constant, the value of e being 2.718. Observations at sea level give $P_o=14.72$, and at a height of 1122 feet, $P=14.11$. What is the value of k?

369. If a body of temperature T_1 be surrounded by cooler air of temperature T_o, the body will gradually become cooler; and its temperature, T, after a certain time, say t minutes, is given by Newton's law of cooling, that is $T \equiv T_o+(T_1-T_o)e^{-kt}$, where k is a constant and $e=2.718$. In an experiment a body of temperature 55° C. was left to itself in air whose temperature was 15° C. After 11 minutes the temperature was found to be 25°. What is the value of k?

370. How many ciphers are there between the decimal point and the first significant figure in $(0.0504)^{10}$?

371. The loss of energy E through friction of every pound of water flowing with velocity v through a straight circular pipe of length l ft. and diameter d ft. is given by $0.0007v^2l \div d$.

Given $v=8.5$ ft. per sec., $l=3000$ ft., $d=6$ inches, find E.

372. A man bequeaths \$500, which is to accumulate at compound interest until the interest for one year at 5% will amount to at least \$300, after which the yearly interest is to be awarded as a scholarship. How many years must elapse before the scholarship becomes available, assuming that the original bequest is made to earn 5% compound interest?

373. In 1624 the Dutch bought Manhattan Island from the Indians for about \$24. Suppose that the Indians had put their money out at compound interest at 7%, and had added the interest to the principal each year, how large would be the accumulated amount in 1910?

Ans. In round numbers \$6,000,000,000. The actual valuation of Manhattan and Bronx real and personal property in 1908 was \$5,235,399,980.

374. The population of the State of Washington in 1890 was

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349,400, and in 1900 it was 518,100. What was the average yearly rate of increase? Assuming the rate of increase to remain the same, what should have been the population in 1910?

375. The founder of a new faith makes one new convert each year, and each new convert makes another convert each year, and so on. How long would it require to convert the whole earth to the new faith, assuming that the population of the world is 1,500,000,000?

Ans. Between 30 and 31 years.

376. The combined wealth of the United States and Europe was estimated (1908) to amount to about $450,000,000,000. Let us assume that the entire wealth of the world amounts to $\$10^{12}$. How long would it take $1.00 put out at compound interest at 3% to equal or exceed this amount?

Ans. 935 years.

The following problems should be solved by means of five-place tables:

377. The circumference of a circle is $2\pi r$ (r being radius). (Use $\pi=3.1416$.)

(*a*) Find the circumference of a circle whose radius is 143.7.

(*b*) Find the radius of a circle whose circumference is 528.45 units.

378. The area of a circle is πr^2.

(*a*) Find the area of a circle whose radius is 12.34″.

(*b*) Find the radius of a circle whose area is 243.5 sq. ft.

379. The area of the surface of a sphere is $4\pi r^2$.

(*a*) The radius of the earth is 3959 miles. What is its surface?

(*b*) What is the length of the equator?

(*c*) A knot is the length of one degree measured along the equator. How many miles in a knot?

380. The volume of a sphere is $\frac{4}{3}\pi r^3$. What is the weight in tons of a solid cast-iron sphere whose radius is 5.343 feet, if the weight of a cubic foot of water is 62.355 pounds, and the specific gravity of cast-iron is 7.154?

381. The stretch of a brass wire when a weight is hung at its free end is given by the relation:

$$S=\frac{mgl}{\pi r^2 k},$$

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where m is the weight applied, $g=980$, l is the length of the wire, r is its radius, and k is a constant. Find k for the following values: $m=944.2$ grams, $l=219.2$ centimeters, $r=0.32$ centimeters, and $S=0.060$ centimeters.

382. The weight P in pounds which will crush a solid cylindrical cast-iron column is given by the formula:

$$P=98{,}920\frac{d^{3.55}}{l^{1.7}},$$

where d is the diameter in inches, and l the length in feet. What weight will crush a cast-iron column 6 feet long and 4.3 inches in diameter?

383. The weight W of one cubic foot of saturated steam depends upon the pressure in the boiler according to the formula:

$$W=\frac{P^{0.941}}{330.36},$$

where P is the pressure in pounds per sq. inch. What is W if the pressure is 280 pounds per sq. inch?

384. The number, n, of vibrations per second made by a stretched string is given by the relation:

$$n=\frac{1}{2l}\sqrt{\frac{Mg}{m}},$$

where l is the length of the string, M the weight used to stretch the string, m the weight of one centimeter of the string, and $g=980$. Find n, when $M=6213.6$ grams, $l=84.9$ centimeters, and $m=0.00670$ gram.

385. If p is the pressure and u the volume in cubic feet of 1 lb. of steam, then from $pu^{1.0646}=479$ find u when p is 150.

The practical problems 366–369, 380–384, were taken from Rietz and Crathorne's College Algebra (Henry Holt and Co.).

The interesting problems 372–376 were taken from White's Scrapbook of Mathematics (Open Court Pub. Co.).

The student is referred to such texts if his interests or needs require further work in logarithms.

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N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

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N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

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B. RATIO, PROPORTION AND VARIATION

I. RATIO AND PROPORTION

On page 74 a ratio was defined as a fraction, and a fraction as an indicated quotient. Hence *the* **ratio** *of two numbers is their indicated quotient.* We recognize in this definition that a ratio and a fraction are identical, *i.e.*, the ratio of 3 to 4 and the fraction $\frac{3}{4}$ are the same. Thus we see at the very beginning that we are not dealing with a new subject, but that, having learned how to work with fractions, we already know a good deal about ratios. The symbols for expressing ratio we are familiar with as the symbols for expressing division, viz., often used in Europe to express division, –, the fractional form which we shall find most convenient, and shall use exclusively, and ÷, really a combination of the two symbols previously given.

The ratio of concrete numbers can be found only when the quantities are of the same denomination. When a common unit of measure can be found for two such quantities their ratio is the quotient of the numbers expressing their measures in terms of that unit. To find the ratio of 10 hours to 5 days, one must express both in terms of a common unit. Here, calling 5 days 120 hours, the ratio is $\frac{10}{120}$, or $\frac{1}{12}$. It is not always possible to get the ratio of two quantities, *e.g.*, 10 minutes and 5 bushels can have no common measure. Again, it is not always possible to get an exact ratio, as in the case of the circumference of a circle and its diameter. As we know, this ratio is called π, and its value we may express more or less accurately, but can never calculate exactly.

In a ratio the first number is called the **antecedent,** *and the second the* **consequent.** It is evident, then, that the antecedent corresponds to the numerator, and the consequent to the denominator, when we think of a ratio as a fraction.

Since a ratio is a fraction, one fundamental property of ratios is apparent, that is, both antecedent and consequent may be multiplied or divided by the same number without changing the value of the ratio. How could you state this fact algebraically? What principle of fractions verifies it?

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EXERCISES. SET XLII. RATIO

386. What is the ratio of 3 ft. 8 in. to 4 in.?

387. What is the ratio of $37\frac{1}{2}\%$ to $87\frac{1}{2}\%$?

388. What is the ratio of $5x^2$ to $(5x)^2$?

389. Simplify the ratio $\frac{x^2-1}{(x+1)^2}$.

390. Simplify the ratio $\frac{(s^3)^4}{s^8s^4}$.

391. Simplify the ratio of $\frac{a^2-b^2}{(a+b)^2}$ to $\frac{(b-a)^2}{a^2+b^2}$.

392. Find the ratio of a to b if $6a-7b=3a+4b$.

393. Does a ratio always remain the same if a constant is added to both antecedent and consequent? Discuss in detail.

394. What ratio is implied in the statement that the death-rate of a certain town for the month is 4 out of each 1000?

The statement of the equality of two ratios is called a **proportion.** In other words, a proportion is a fractional equation each member of which is a single fraction (or ratio); *e.g.*, $\frac{a}{b}=\frac{c}{d}$ is a proportion.

The first antecedent and the second consequent of a proportion are called the **extremes,** and *the first consequent and the second antecedent are called the* means *of a proportion.* In $\frac{a}{b}=\frac{c}{d}$, a and d are the extremes, and b and c the means.

The antecedents and consequents in a proportion are called the **terms of the proportion.** *If the first consequent equals the second antecedent, each of those terms is called a* **mean proportional** *between* the first and the last terms of the proportion; *the whole expression is referred to as a* **mean proportion.** In the mean proportion $\frac{a}{b}=\frac{b}{c}$, b is said to be a mean proportional between a and c. If we solve this equation for b two values would be obtained, $\pm\sqrt{ac}$.

EXERCISES. SET XLIII. PROPORTION

395. Find the value of y in the proportions

(*a*) $\frac{y}{4}=\frac{y-3}{5}$. (*b*) $\frac{5}{y-3}=\frac{7}{2y+6}$.

396. Find the mean proportionals between

(*a*) 16 and 4. (*b*) $\frac{1}{3}$ and $\frac{1}{27}$ (*c*) $a+b$ and $a-b$.

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397. What number added to each of the numbers 3, 7, 15, and 25 will give results which are in proportion?

398. In sterling silver, the amount of silver is .925 of the entire weight of the metal. (*a*) How many ounces of pure silver are needed to make 500 oz. sterling silver? (*b*) 500 oz. of pure silver will make how many ounces of sterling silver? (Form a proportion and solve this problem by means of it.)

399. The volumes of two similar solids have the same ratio as the cubes of any two homologous dimensions. The diameter of the first of two bottles which have the same shape is three times the diameter of the second. If the first holds 5 ounces, how much does the second hold?

Many properties of a proportion may be derived from its definition and the fundamental laws of algebra. We shall now suggest proofs for three theorems of proportion which are especially important because of their application to geometry.

Theorem 28. ***Any proportion may be transformed by* alternation, *i.e., the first term is to the third as the second is to the fourth.***

Given: $\frac{a}{b}=\frac{c}{d}$. **To prove:** $\frac{a}{c}=\frac{b}{d}$.

Suggestion for proof: By what must we multiply $\frac{a}{b}$ to get $\frac{a}{c}$?

Theorem 29. ***In any proportion, the terms may be combined by* addition (usually called composition); *i.e., the ratio of the sum of the first and second terms to the second term (or first term) equals the ratio of the sum of the third and fourth terms to the fourth term (or third term).***

N.B.—Addition and sum are used in the algebraic sense.

Given: $\frac{a}{b}=\frac{c}{d}$.

To prove: (1) $\frac{a\pm b}{b}=\frac{c\pm d}{d}$ or $\frac{b\pm a}{b}=\frac{d\pm c}{d}$

or (2) $\frac{a\pm b}{a}=\frac{c\pm d}{c}$ or $\frac{b\pm a}{a}=\frac{d\pm c}{c}$.

Suggestions for proofs: (1) $\frac{a}{b}\pm 1=\frac{c}{d}\pm 1$. Authority?

(2) If $\frac{a}{b}=\frac{c}{d}$ is $\frac{b}{a}=\frac{d}{c}$? Why?

If $\frac{a-b}{b}=\frac{c-d}{d}$ why does $\frac{b-a}{b}=\frac{d-c}{d}$?

Complete the proofs.

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Theorem 30. ***In a series of equal ratios, the ratio of the sum of any number of antecedents to the sum of their consequents equals the ratio of any antecedent to its consequent.***

Given: $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots\cdots=\frac{k}{l}=\cdots\cdot$.

To prove: $\frac{a+c+e+\cdots}{b+d+f+\cdots}=\frac{k}{l}=\cdots\cdot$.

Proof: Let $\frac{a}{b}=r$

$\therefore a=br,\ c=dr,$ etc.

$\therefore a+c+e+\cdots=r(b+d+f+\cdots)$

Finish the proof.

EXERCISES. SET XLIV. APPLICATIONS OF PROPORTION

400. In any proportion the product of the means is equal to the product of the extremes.

Hint. What axiom do we use in clearing an equation of fractions?

State a corollary of this theorem which will express the mean proportional as a function of the other terms of the proportion.

401. If the product of two numbers is equal to the product of two other numbers, the factors of either product may be made the means of a proportion of which the factors of the other are the extremes.

How are Exs. 400 and 401 related?

402. (*a*) If $\frac{a}{b}=\frac{c}{d}$ prove that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

(*b*) State this fact as a theorem in proportion (sometimes referred to as **addition and subtraction** or **composition and division**).

403. (*a*) Prove that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$.

(*b*) State this fact as a theorem in proportion. This transformation is usually referred to as **inversion.**

404. (*a*) State a brief way of testing the correctness of a proportion.

(*b*) Is the following a true proportion $\frac{976}{843}=\frac{874}{742}$?

405. *A* and *B* are in business, and their respective shares of the business are in the ratio of $\frac{2}{3}$. If the profits of a certain year are $16000, and during the year *A* draws $1200 and *B* $1000, at the end of the year how much of the profits does each receive?

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406. Is the validity of a proportion impaired by adding the same number to all the terms? Prove your answer to be correct.

407. By the term *specific gravity* of a substance is meant the ratio of the weight of a volume of that substance to the weight of an equal volume of some other substance taken as a standard. In practice that standard is water. A cubic foot of water weighs 62.4 lbs.

(*a*) What is the specific gravity of steel if a cubic foot of it weighs 490 lbs.?

(*b*) What is the specific gravity of ice if a cubic foot of it weighs 57.5 lbs.?

(*c*) The specific gravity of sea water is 1.024. What is the weight of 5 gallons?

408. What is the weight in tons of a solid cast-iron sphere whose radius is 5.433 ft. if the weight of a cubic foot of water is 62.355 lbs., and the specific gravity of cast-iron is 7.154? (See Ex. 380.)

409. What number added to a ratio whose antecedent is 5 and consequent is 8 will give the ratio $\frac{3}{4}$?

410. What number added to the terms of the ratio $\frac{7}{12}$ will give the ratio $\frac{2}{3}$?

Many of the important applications of proportion are made in physics. The exercises in proportion which follow depend upon three well-known facts in physics. The facts are:

(1) *Law of the Inclined Plane.* If F represents the force applied along an inclined plane, W the weight of the body, h the height of the plane and l the length of the plane, then

$$\frac{F}{W}=\frac{h}{l}.$$

Illustrative Problem. How heavy a weight could a force of 300 lbs. pull up an incline 75 ft. high and 400 ft. long?

Substituting: $F=300$, $h=75$, $l=400$; calculate W.

(2) *Boyle's Law.* If P_1 be the pressure of a gas of volume V_1, and P_2 the pressure of the same gas of volume V_2, the temperature remaining constant, we have $\frac{V_1}{V_2}=\frac{P_2}{P_1}$.

Illustrative Problem. Keeping the temperature the same, if 200 cu. cm. of gas exerting a pressure of 2 lbs. per sq. cm. be allowed to expand to a volume of 300 cu. cm. what pressure will it exert?

Substituting: $V_1=200$, $P_1=2$, $V_2=300$; calculate P_2.

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(3) *Charles's Law.* When the pressure is constant, if V_1 is the volume of a gas of temperature t_1+273 centigrade (called absolute temperature) and V_2 the volume of the same gas of temperature t_2+273, then
$$\frac{V_1}{V_2}=\frac{t_1+273}{t_2+273}.$$

Illustrative Problem. If 200 cu. cm. of gas is 300° C. (absolute temperature) keeping the pressure the same, what will be the temperature if the gas is expanded to 250 cu. cm.?

Substituting: $V_1=200$, $V_2=250$, $t_1=300-273$; calculate t_2.

EXERCISES. SET XLIV (concluded)

411. Find the force which must be exerted to draw a sled weighing 240 lbs. up a hill which is 300 ft. long and 50 ft. high.

412. A bladder holds 40 cu. in. of air under a pressure of 15 lbs. per sq. in. What is the size of the bladder when the pressure is reduced to 12 lbs. per sq. in.?

413. A certain mass of gas occupying a volume of 160 cu. cm. at a temperature of 47° C. is cooled to 17° C. Find the volume at the lower temperature.

414. A boy is able to exert a maximum force of 80 lbs. How long an inclined plane must he use to push a truck weighing 320 lbs. up to a doorway which is $3\frac{1}{2}$ ft. above the level of the ground?

II. VARIATION

If one variable is a constant number of times a second variable, the first quantity is said to **vary** *as the second.* If $x=ky$, then x is said to vary as y, and is written $x \propto y$. The circumference of a circle varies as its diameter because it is equal to a constant (π) times its diameter, *i.e.* $C=\pi D$ or $C \propto D$.

The illustration cited is called **direct variation,** but often *two quantities are said to vary* **inversely** when *an increase in the one causes a proportional decrease in the other, e.g.*, the time it takes to go from America to Europe varies inversely as the speed of the vessel. If we call the time t and the speed s we might write $t \propto \frac{1}{s}$. Again, we might illustrate this inverse variation by the apparent height of objects and our distance from them. If a building appears 6″ high when we are 200 feet from it, how high will it appear when we are only 100 feet from it?

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A quantity is said to vary **jointly** *as several others if it is equal to a constant number of times the product of the others; i.e., a* varies jointly as c, d, and e ($a \propto cde$), if $a \equiv kcde$. This relation may be illustrated by the area of a triangle which varies jointly as the base and altitude. In this case $k = \frac{1}{2}$. Again, we might illustrate this relation by a man's wages which vary jointly as the number of days he works and the pay he receives for one day's work. The constant in this as in many other cases is unity.

EXERCISES. SET XLV. APPLICATION OF VARIATION

415. The velocity of a falling body varies directly as the time during which it falls. (*a*) State this fact as a formula. (*b*) If the velocity of a body is 160 feet per second after falling 5 seconds, what will the velocity be after 12 seconds?

Notes on Solution: Such problems may be solved by first solving for the constant or by throwing them into the form of a proportion.

Here: By the first method $160 = 5k \therefore k = 32 \therefore v = 32 \cdot 12 = 384$, or by the second method $\frac{5}{160} = \frac{12}{v} \quad \therefore v = \frac{12.160}{5} = 384$.

(*c*) How long will a body fall before acquiring a velocity of 520 feet per second?

416. The distance through which a body falls from rest varies as the square of the time during which it falls.

(*a*) State this fact as a formula.

(*b*) If a body falls 576 ft. in 6 secs., how far does it fall in 10 secs.?

(*c*) How far will a body fall in 12 seconds?

(*d*) How far will a body fall during the twelfth second?

(*e*) How long will it take a body to fall a mile?

417. The pressure of wind on a flat surface varies jointly as the area of the surface and the square of the wind's velocity.

(*a*) State this fact as a formula.

(*b*) The pressure of the wind on 1 sq. ft. is 0.9 lb. when the velocity of the wind is 15 miles per hour. What is the pressure of the wind against the side of a house 120 feet deep and 70 feet high when the wind is blowing 40 miles an hour?

(*c*) What is the pressure on the same house when the wind is blowing 60 miles per hour?

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418. The heat one derives from a stove varies approximately inversely as the square of one's distance from the stove. If I move my position from 10 feet away from a stove to 35 feet away, what part of the original heat will I then receive?

419. The *law of gravitation* states that the weight of a body varies inversely as the square of its distance from the center of the earth.

(*a*) If a body weighs 10 lbs. on the surface of the earth what will it weigh 5 miles above the surface? (Consider the radius of the earth to be 4000 miles.)

(*b*) How high would a body have to be raised above the surface of the earth to lose half its weight?

420. The intensity of light varies inversely as the square of the distance from its source.

(*a*) How much farther from a lamp 20 feet away must a piece of paper be moved to receive half as much light?

(*b*) What is the relation of the intensity of light 15 feet from an electric light and 37 feet from the same light.

1421. Kepler proved that the squares of the times of revolution of the planets about the sun vary as the cubes of their distances from the sun. The earth is 93,000,000 miles from the sun, and makes a revolution in approximately 365 days. How far is Venus from the sun if it makes one revolution in 226 days? (Use logs.)

422. The strings of a musical instrument produce sounds by vibrating. The number of vibrations in any fixed interval of time varies directly as the length of the string, if the strings are alike in other particulars.

A *C* string 42″ long vibrates 256 times per second. A *G* string, like the *C* string except for length, vibrates 384 times per second. How long must it be?

423. The relation between the time of oscillation of a pendulum and its length is given by the following formula:

If two pendulums are of lengths L and l respectively and the number of oscillations per second are T and t respectively, then:

$$\frac{T^2}{t^2}=\frac{L}{l}.$$

(*a*) A pendulum which makes 1 oscillation per second is 39.1″ long. How often will a pendulum 156.4″ long vibrate per second?

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(*b*) How long would a pendulum have to be to oscillate once a minute?

1424. The relation between Q, the quantity of water in cubic feet per second passing over a triangular gauge notch, and H, the height in feet of the surface of the water above the bottom of the notch, is given by

$$Q \propto H^{\frac{5}{2}}.$$

When H is 1, Q is found to be 2.634. What is the value of Q when H is 4?

If the area of the reservoir supplying the notch is 80000 sq. ft., find the time in which a volume of water 80000 sq. ft. in area and 3 inches in depth will be drawn off when H remains constant and equal to 4 ft.

(The relation between Q and H may be written $Q \equiv kH^{\frac{5}{2}}$ where k is a constant.)

425. In steam vessels of the same kind it is found that the relation between H, the horse-power; V, the speed in knots; and D, the displacement in tons, is given by

$$H \propto V D^{\frac{2}{3}}.$$

Given $H=35640$, $V=23$, and $D=23000$, find the probable numerical value of H when V is 24.

426. Some particulars of steam vessels are given. Assuming in each case the relation H.P. $\propto V^3 D^{\frac{2}{3}}$ to hold, where H.P. denoted the horse-power at a speed of V knots and displacement D in tons, find in each case the probable H. P. necessary to give the indicated speed.

Name	H. P.	V	D
(i) Paris.............	20000	20.25	15000
(ii) Teutonic..........	.....	19.50	13800
(iii) Campania.........	.....	22.10	19000
(iv) Kaiser............	.....	22.62	20000
(v) Oceanic...........	.....	20.50	28500
(vi) Communipaw	.....	23.00	23000

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g427.* Assuming that the circumference of a circle is $3\frac{1}{7}$ times its diameter, make a graph showing that the circumference varies as the diameter.

Some of the numerous applications of Ratio, Proportion, and Variation to geometry will be given or suggested in the pages of this book.

LIST OF WORDS DEFINED IN CHAPTER V

Logarithm, base, characteristic, mantissa, interpolation, antilogarithm. Ratio, proportion, antecedent, consequent, terms, extremes, means, mean proportional, mean proportion, addition or composition, alternation, subtraction or division, inversion. Variation, direct, inverse, joint.

SUMMARY OF THEOREMS PROVED IN CHAPTER V

28. Any proportion may be transformed by alternation.
29. In any proportion the terms may be combined by addition.
30. In a series of equal ratios, the ratio of the sum of any number of antecedents to the sum of their consequents equals the ratio of any antecedent to its consequent.

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* As here, "*g*" preceding the number of an exercise indicates that its solution involves a knowledge of graphs.

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CHAPTER VI

SIMILARITY

A. INTRODUCTORY THEOREMS

EXPERIMENT I

1. Construct a scalene triangle.

(*a*) Divide one side into 4 equal sects, and through the first point of division construct a line parallel to a second side of the triangle. Compare the lengths of the sects thus cut on the third side of the triangle.

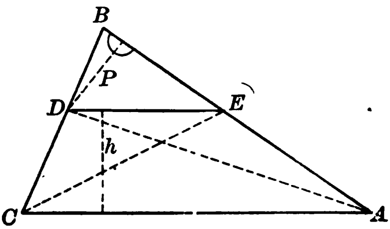
(*b*) Repeat the work of (*a*), dividing the first side of the triangle into 5 equal sects.

(*c*) Repeat the work of (*a*), dividing the first side of the triangle into 9 equal parts and drawing the line through the fourth point of division.

2. Repeat the work of 1, using an equilateral triangle.

Theorem 31. ***A line parallel to one side of a triangle, and cutting the other sides, divides them proportionally.***

(External division of the sides will be considered later.)



Given: $\triangle ABC$, D in $\overline{BC}$ and E in $\overline{AB}$, so that $\overline{DE} \parallel \overline{CA}$.

Prove: $\dfrac{AE}{EB} = \dfrac{CD}{DB}$

PROOF

(1) $\dfrac{\triangle DEA}{\triangle DBE} = \dfrac{\overline{EA}}{\overline{EB}}$.	(1) Triangles having one dimension equal compare as their remaining dimensions.
(2) Similarly it can be shown that $\dfrac{\triangle EDC}{\triangle DBE} = \dfrac{\overline{CD}}{\overline{DB}}$.	
(3) But if $\triangle DCE = \dfrac{h \cdot \overline{DE}}{2}$, h being the distance between $\overline{DE}$ and $\overline{CA}$, $\triangle DEA = \dfrac{h \cdot \overline{DE}}{2} \because \overline{DE} \parallel \overline{CA}$.	(3) Data and $\parallel$s are everywhere equidistant.

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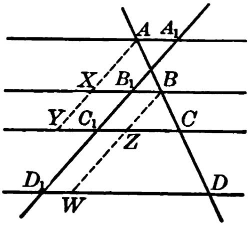
(4) and $\triangle EDC = \triangle DEA$.	(4) Quantities equal to the same quantity equal each other.
(5) $\therefore \frac{\triangle DEA}{\triangle DBE} = \frac{\triangle EDC}{\triangle DBE}$.	(5) Quotients of equals divided by equals are equal.
(6) $\therefore \frac{\overline{EA}}{\overline{EB}} = \frac{\overline{CD}}{\overline{DB}}$.	(6) See (4).

Cor. 1. *One side of a triangle is to either of the sects cut off by a line parallel to a second side, as the third side is to its homologous sect.*

Suggestion: If $\frac{\overline{AE}}{\overline{EB}} = \frac{\overline{CD}}{\overline{DB}}$ why would $\frac{\overline{AE}+\overline{EB}}{\overline{EB}} = \frac{\overline{CD}+\overline{DB}}{\overline{DB}}$

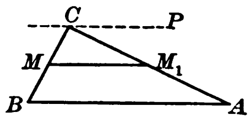


Cor. 2. *A series of parallels cuts off proportional sects on all transversals.*

Hints: $\frac{\overline{AB}}{\overline{AX}} = \frac{\overline{BC}}{\overline{XY}}$ and $\frac{\overline{BC}}{\overline{BZ}} = \frac{\overline{CD}}{\overline{ZW}}$, $\overline{AX} = \overline{A_1B_1}$, $\overline{BZ} = \overline{XY} = \overline{B_1C_1}$, $\overline{ZW} = \overline{C_1D_1}$, if $\overline{AXY}$ and $\overline{BZW}$ are how drawn?

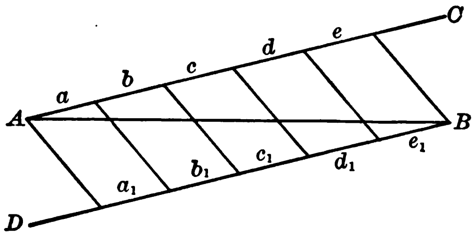
Cor. 3. *Parallels which intercept equal sects on one transversal, do so on all transversals.*

Cor. 4. *A line which bisects one side of triangle, and is parallel to the second, bisects the third.*



EXERCISES. SET XLVI. PROPORTIONAL SECTS

428. A sheet of ruled paper is useful in dividing a given sect into equal parts. (*a*) Explain. (*b*) Make such an instrument by using a sheet of tracing paper and drawing at least twenty parallels. (What must you be sure that these parallels do?)



429. Draughtsmen and designers sometimes divide a given sect into any required number of equal parts by the following method: To divide AB into 5 equal

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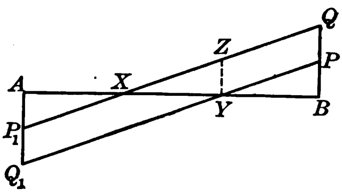
parts, draw AC at any convenient angle with AB. Draw BD parallel to AC. Beginning at A, mark off on AC five equal sects a, b, c, etc., of any convenient length. Beginning at B, mark off on BD five sects equal in length to those on AC, a_1, b_1, c_1, etc. Join their extremities as in diagram. These lines divide AB into 5 equal sects. Prove that this is a correct method.

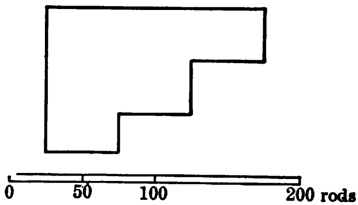
430. Among the applications of the propositions on parallel lines is an interesting one due to Arab Al-Nairizi (ca. 900 A. D.). The problem is to divide a sect into any number of equal parts. He begins with the case of trisecting a sect AB.

Make BQ and AQ_1 perpendicular to AB, and make $BP = PQ = AP_1 = P_1Q_1$. Then $\triangle XYZ$ is congruent to $\triangle YBP$, and also to $\triangle XAP_1$. Therefore $AX = XY = YB$. In the same way we might continue to produce BP, until it is made up of n lengths BP, and so for AP_1, and by properly joining points we could divide AB into $n+1$ equal parts. In particular, if we join P and P_1, we bisect the sect AB. Prove the truth of these statements.



Divide a sect into seven equal parts by this method.

431. Find the cost of fencing the field represented in the diagram. Field is drawn to scale indicated, and the fence costs $2.75 per rod.



432. If DE is parallel to BC in triangle ABC, compute the sects left blank from those given in the following table:

AD	DB	AE	EC	AB	AC
24	28	18	..	..	..
14	56	..	42	..	..
..	112	12	418	..	..
27	..	18	342	..	..

433. Divide a sect 11 units long into parts proportional to 3, 5, 7, and 9.

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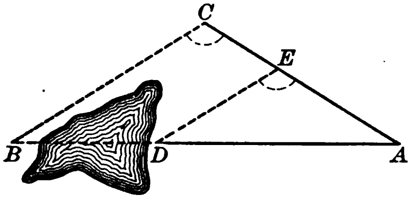
First compute the lengths of the required sects, then construct them, and measure the sects so obtained. Compare the results. Which method is more convenient? Which is more accurate?



434. The accompanying diagram suggests a method for finding the distance from a point A to a second point B, visible from but inaccessible to the first point A.

Hints: C is selected, from which both A and B are visible. $CB \parallel ED$.

435. If DE is parallel to BC in triangle ABC, prove that $\frac{\overline{AD}}{\overline{AE}} = \frac{\overline{BD}}{\overline{EC}}$.

436. Under the same conditions show that $\frac{\overline{AD}+\overline{AE}}{\overline{BD}+\overline{EC}} = \frac{\overline{AD}}{\overline{BD}}$.

437. State Exs. 435–436 in words.

EXPERIMENT II

1. Construct a scalene triangle.

(*a*) Divide two sides of the triangle into 8 equal sects. Join the corresponding points of division. By comparing certain angles, establish a relation between the lines just drawn and the third side of the triangle.

(*b*) Repeat (*a*), dividing the sides into 7 equal sects.

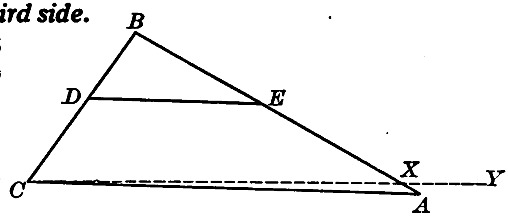
2. Repeat 1, using an equilateral triangle.

Theorem 32. ***A line dividing two sides of a triangle proportionally is parallel to the third side.***

Given: $\triangle ABC$ and $\frac{\overline{BD}}{\overline{DC}} = \frac{\overline{BE}}{\overline{EA}}$, ($D$ in $\overline{BC}$, E in $\overline{AB}$).

To prove: $\overline{DE} \parallel \overline{CA}$.

Proof: Details to be supplied by the student.



Draw $\overline{CY} \parallel \overline{DE}$ cutting $\overline{BA}$ in X. Then $\frac{\overline{DB}}{\overline{DC}} = \frac{\overline{BE}}{\overline{EX}}$. Why?

Show why X must coincide with A.

Cor. 1. ***A line dividing two sides of a triangle so that those sides bear the same ratio to a pair of homologous sects is parallel to the third side.***

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B. IDEA OF SIMILARITY

Earlier in the book it has been noted that two figures are called similar if they have the same shape. The symbol ($\backsim$), due to Leibnitz, for "is (or are) similar to," it has been pointed out, is an S thrown on its side. The S was doubtless used because it is the initial letter of the word "*similis*" (Latin for "like"). Before developing the subject we need, however, a more careful definition of similarity, for shape is only a vague notion and not a scientifically defined term.

Before defining similar figures we must note the meaning of **similar sets or systems of points.** The points A_1, B_1, C_1, ... and A_2, B_2, C_2, ... are said to be similar systems if they can be so placed that all the sects joining corresponding points, A_1A_2, B_1B_2, C_1C_2, ..., pass through the same point, and are divided by that point into sects having the same ratio.

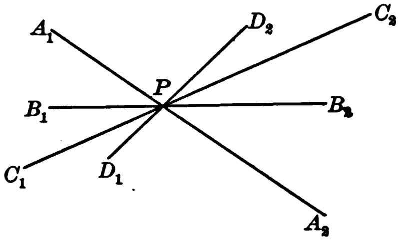

FIG. 1.

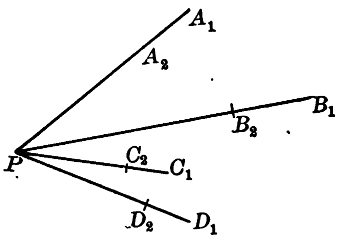

FIG. 2.

In Figs. 1 and 2, points A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 are similar systems. A_1A_2, B_1B_2, C_1C_2, and D_1D_2 pass through P and $\frac{A_1P}{PA_2} = \frac{B_1P}{PB_2} = \frac{C_1P}{PC_2} = \frac{D_1P}{PD_2} = r$. In Fig. 2 where P lies on the prolongation of sects A_1A_2, B_1B_2, etc., it is said to divide these sects externally. The topic "*External Division of a Sect*" will be further developed in the "Second Study."

The point P is called the **center of similitude,** and the ratio r is called the **ratio of similitude.**

Similar figures *are those which can be placed so as to have a center of similitude.*

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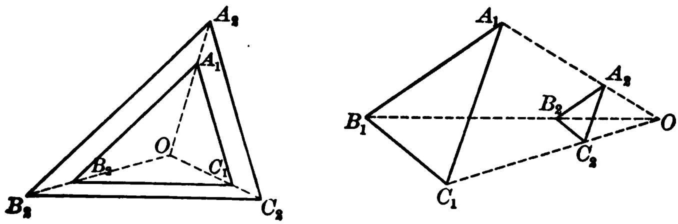
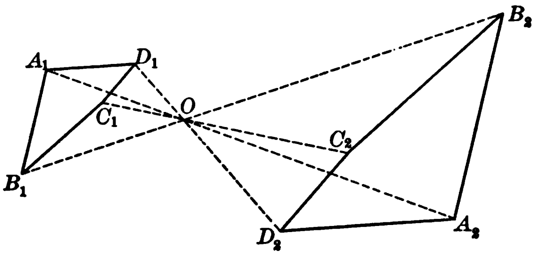
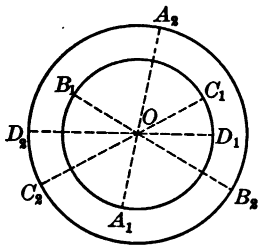
The following are illustrations of similar figures:



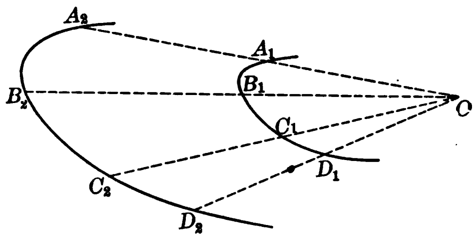
(*a*) Triangles.



(*b*) Other polygons.



(*c*) Circles.



(*d*) General Curvilinear

We are now in a position to prove our right to the use of the double symbol ($\cong$) for congruence.

Cor. ***Congruent figures are similar.***

If n-gon $ABC \ldots . n \cong A_1B_1C_1 \ldots . n_1$, they may be made to coincide. Then any point O may be taken as the center of similitude, and $\dfrac{\overline{OA}}{\overline{OA_1}} = \dfrac{\overline{OB}}{\overline{OB_1}} = \ldots . \dfrac{\overline{On}}{\overline{On_1}} = 1.$

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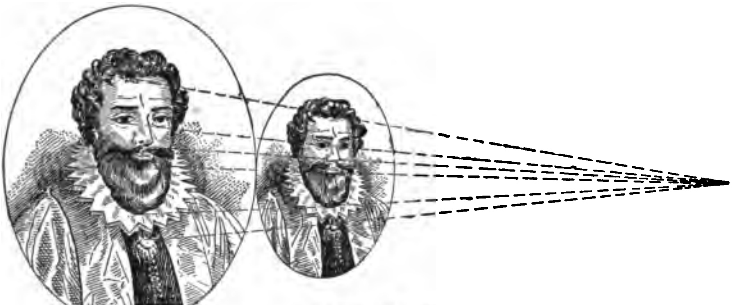
EXERCISES. SET XLVII. MEANING OF SIMILARITY

438. In (*a*) what is the ratio of similitude? What is the center of similitude? If $\overline{OA_1} = \overline{A_1A_2}$ and $\overline{OB_2} = 0.5''$, what does $\overline{B_1B}$ equal?

439. In (*a*) draw a sect through O which will be divided by sides of triangles in the ratio of $\overline{OA_1}$ to $\overline{OA_2}$. How many such lines can be drawn?

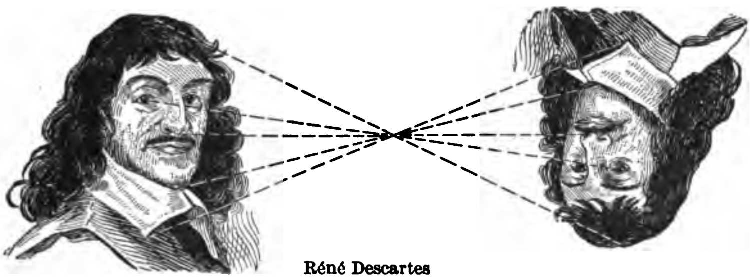
440. In (*c*) if $\overline{OB_1}=5$ cm. and $\overline{OB_2}=1$ dm. what is the ratio of $\overline{OA_1}$ to $\overline{OA_2}$? Can you mention any other sects in this figure having the same ratio?

441. In (*d*) if $\overline{OC_1}$ is ^{2} of $\overline{C_1C_2}$ what is the ratio of similitude? Under the same conditions what is $\overline{OA_2}$ if $\overline{OA_1}$ is $\frac{3}{4}''$?



François Vieta

Shadows furnish familiar illustrations of similar figures. In such cases the source of light is the center of similitude.



Réné Descartes

The lens of the camera gives a figure similar to the object in front of it with the image inverted.

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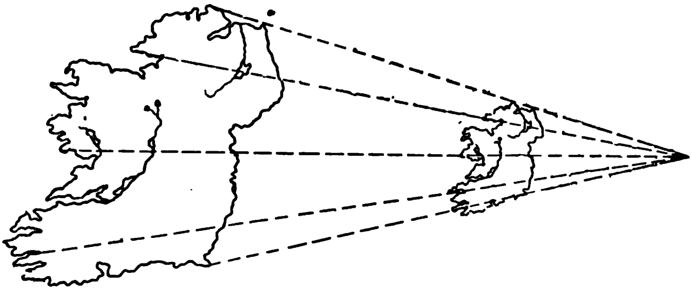
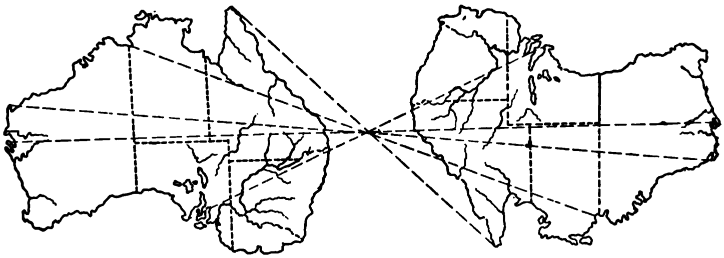
In reducing or enlarging maps we have another familiar illustration of the application of the principle of similarity.



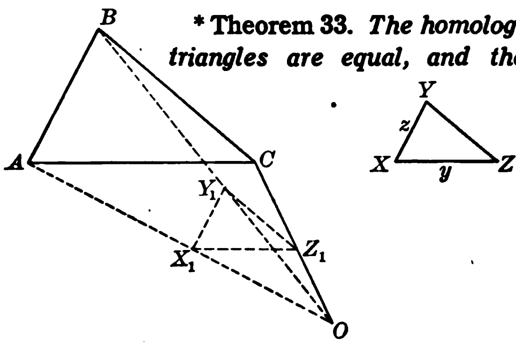


EXERCISES. SET XLVII (concluded)

442. State any illustrations or applications of similarity with which you are familiar.

443. If the ratio of similitude is 1, what relation between the figures exists besides similarity.

C. SIMILARITY OF TRIANGLES



*** Theorem 33.** ***The homologous angles of similar triangles are equal, and the homologous sides have a constant ratio.***

Given: $\triangle XYZ \backsim \triangle ABC$.

Prove: (I) $\angle X = \angle A$, $\angle Y = \angle B$, $\angle Z = \angle C$.

(II) $\frac{\overline{XY}}{\overline{AB}} = \frac{\overline{YZ}}{\overline{BC}} = \frac{\overline{ZX}}{\overline{CA}}$.

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PROOF

(1) $\therefore \triangle XYZ \backsim \triangle ABC$, it may be placed in the position of $\triangle X_1Y_1Z_1$ with O a center of similitude of it and $\triangle ABC$. — (1) Data and def. of sim. figs.

(2) $\therefore \dfrac{\overline{OX_1}}{\overline{OA}} = \dfrac{\overline{OY_1}}{\overline{OB}} = \dfrac{\overline{OZ_1}}{\overline{OC}}$ — (2) Def. of center of similitude.

(3) $\therefore \overline{X_1Y_1} \parallel \overline{AB}$
$\overline{Y_1Z_1} \parallel \overline{BC}$
$\overline{Z_1X_1} \parallel \overline{CA}$ — (3) Why?

(4) $\therefore \angle X_1Y_1O = \angle ABO$
$\angle OY_1Z_1 = \angle OBC$ and
$\angle OX_1Y_1 = \angle OAB$
$\angle OX_1Z_1 = \angle OAC$ — (4) Why?

(5) $\therefore$ What two $\angle$s of $\triangle X_1Y_1Z_1$ or $\triangle XYZ$ are equal to what two $\angle$s of $\triangle ABC$? — (5) Why?

(6) Are the third $\angle$s equal? — (6) Why?

(7) $\because \angle X = \angle A$ how can $\triangle XYZ$ be placed with respect to $\triangle ABC$? — (7) Why?

(8) What proportion will this give us? — (8) Why?

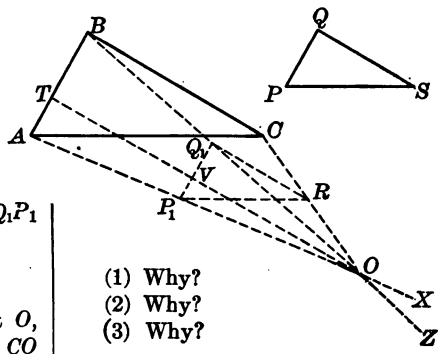
(9) In how many ways will you have to superpose $\triangle XYZ$ in order to prove (II)?

Complete the proof.

Theorem 34.** ***Triangles are similar when two angles of one are equal each to each to two angles of another.

Given: $\angle P = \angle A$, $\angle Q = \angle B$ in $\triangle$s PQS and ABC.

Prove: $\triangle PQS \backsim \triangle ABC$.



Suggestions for proof:

Draw $P_1Q_1 \parallel AB$ and equal to PQ.

Draw $\overline{AP_1X}$ and $\overline{BQ_1Z}$.

(1) If $\overline{AP_1X} \parallel \overline{BQ_1Z}$, then ABQ_1P_1 is a $\square$. — (1) Why?

(2) and $AB = PQ$ — (2) Why?

(3) $\therefore \triangle PQS \cong \triangle ABC$. — (3) Why?

If $\overline{AP_1X}$ intersects $\overline{BQ_1Z}$ at O, draw $P_1R \parallel AC$ and intersecting CO at R. Draw Q_1R.

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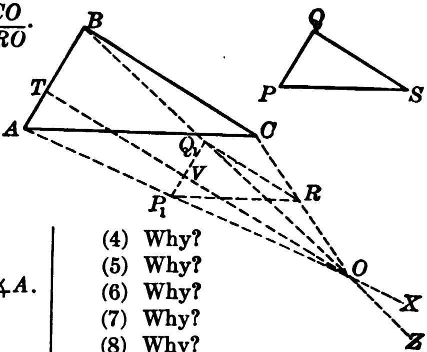


(4) Then $\frac{BO}{Q_1O}=\frac{AO}{P_1O}$ and $\frac{AO}{P_1O}=\frac{CO}{RO}$.

(5) $\therefore \frac{BO}{Q_1O}=\frac{CO}{RO}$.

(6) Any sect through O cutting P_1Q_1 in V and AB in T is divided so that $\frac{TO}{VO}=\frac{AO}{P_1O}$.

(7) $\therefore \triangle P_1Q_1R \backsim \triangle ABC$.

(8) $\therefore \angle Q_1 = \angle B$ and $\angle P_1 = \angle A$.

(9) $\therefore \triangle P_1Q_1R \cong \triangle PQS$

(10) $\therefore \triangle PQS$ may be made to coincide with $\triangle P_1Q_1R$.

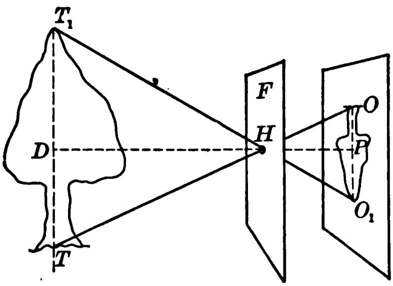
(11) $\therefore \triangle PQS \backsim \triangle ABC$.

(4) Why?
(5) Why?
(6) Why?
(7) Why?
(8) Why?
(9) Why?
(10) Def. of congruent figures.
(11) Def. of similar figures.

Discussion: Consider the instance in which $PQ = AB$. Note that the figures may be so placed that the center of similitude lies between them. Is a proof necessary for this case?

EXERCISES. SET XLVIII. SIMILARITY OF TRIANGLES

444. State a necessary and sufficient condition for the similarity of (*a*) right triangles, (*b*) isosceles triangles, and (*c*) equilateral triangles.



445. If rays of light from a tree (TT_1) pass through a hole (H) in a fence (F) and strike a wall, an inverted outline (O_1O) of the tree will be seen on the wall.

(*a*) Explain why this should be.

(*b*) If the distance $\overline{HD}$ is 35 ft. and $\overline{HP}$ is 9 ft. and the height of the image ($\overline{O_1O}$) is 8′ 8″, find the height of the tree.

(*c*) Under what conditions, if ever, will the image be the height of the tree?

d446. The location of the image A_1 of a point A, formed in a photographer's camera, is approximately found by drawing a

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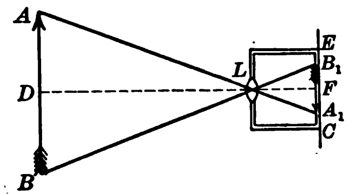
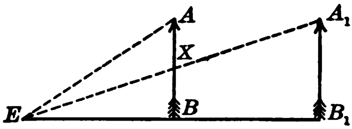
straight line AA_1 through the center of the lens L. If CE is the position of the photographic plate, then A_1B_1 is the image of AB. How large is A_1B_1 if $AB=6$ ft., $LD=12$ ft., and $LF=6$ in.?

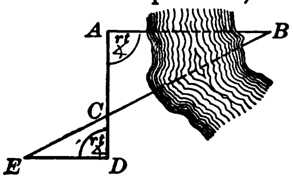


447. Show that an object which appears of a certain height, will, when moved twice as far away, appear to be comparatively of only one-half the height.



Hint: Show that $\overline{XB} = \frac{1}{2}\overline{AB}$.

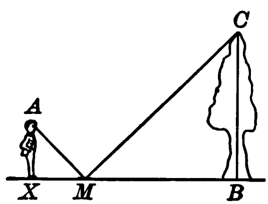
448. To measure indirectly from an accessible point A to an inaccessible point B, construct AD perpendicular to the line of sight from A to B, and ED perpendicular to AD. Let C be the point on AD which lies in line with E and B. By measurement, ED is 100 ft., CD 90 ft., and CA 210 ft. What is the distance across the river?



449. A man is riding in an automobile at the uniform rate of 30 miles an hour on one side of a road, while on a footpath on the other side a man is walking in the opposite direction. If the distance between the footpath and the auto track is 44 ft., and a tree 4 ft. from the footpath continually hides the chauffeur from the pedestrian, does the pedestrian walk at a uniform rate? If so, at what rate does he walk?

450. A mirror (referred to as a "speculum") has been used for crudely measuring the height of objects, such as trees.

In the diagram a mirror is placed horizontally on the ground at M. The observer takes such a position that the top of the tree (C) is visible in the mirror. What distances must he measure to be able to compute the height of the tree ($\overline{BC}$)?



NOTE.—Light is reflected from the surface of a mirror at an angle equal to the angle at which it strikes it.

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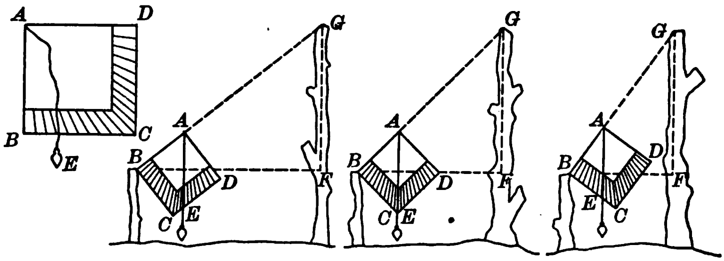
451. The accompanying diagrams show a simple device for measuring heights, using a square (such as *ABCD*), with a plumb-line (*AE*) suspended from one corner. *BC* and *DC* are divided into equal parts (say 10, 100, or 1000).

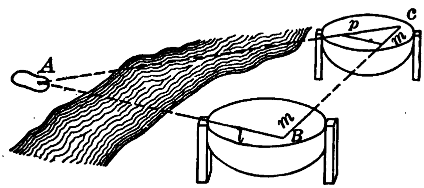


Study the diagrams, and show how to use the square in each case.

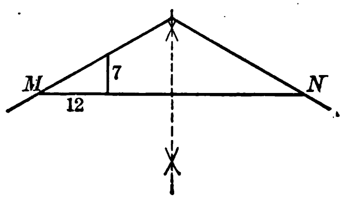
452. Look up a description of the hypsometer, and construct one of wood, stiff pasteboard, or any material that can be used practically.

(See D. E. Smith, "Teaching of Geometry." Ginn & Company.)

453. The distance from an accessible point *B* to an inaccessible one, *A*, was measured in the sixteenth century by the use of drumheads. On a drumhead placed at *B* a sect *l* was drawn toward *A*,



and another *m* toward an accessible point *C*. $\overline{BC}$ was measured. The drumhead was then placed at *C* with *m* in the direction $\overrightarrow{CB}$. The ratio of $\frac{m}{BC}$ was noted. A sect *p* was drawn in the direction $\overrightarrow{CA}$. Would any further measurements be necessary to make it possible to compute $\overline{AB}$? Explain.



454. To a convenient scale draw a symmetrical roof, pitch 7 inches to the foot, on *MN*, which is to represent 30′ 6″. The figure suggests the construction.

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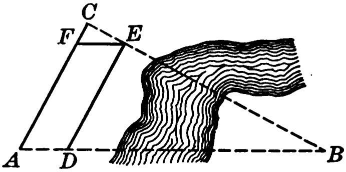
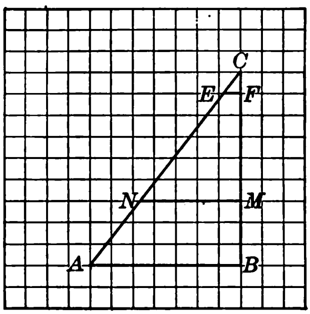
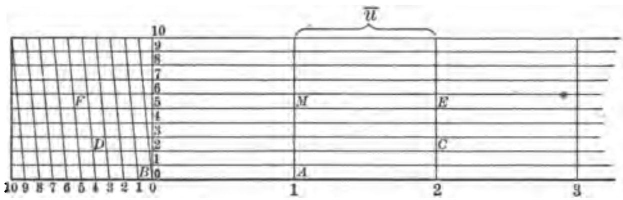
455. Another way of determining the distance from A to an inaccessible point B is to align A, D, and B. Run DE at random. Run AC parallel to DE. Align C, E, and B. Measure off FA equal to ED. Measure CF, EF, CA. Show how to compute AB and justify the method.



456. The accompanying diagram shows how coördinate (squared) paper may be used to divide a given sect into a number of equal parts (here 9). If the sect $\overline{AB}$ is laid off on one of the horizontal lines, the vertical lines will be perpendicular to it ($\overline{CB} \perp \overline{AB}$). Draw AC. At the first division on CB from C draw $FE \parallel BA$. $\triangle CFE \backsim \triangle CBA$. (*a*) Why, then, is $\overline{EF} = \frac{\overline{AB}}{9}$? (*b*) What part of $\overline{AB}$ is $\overline{NM}$? (*c*) Can you find a sect equal to $\frac{4}{9}$ ($\overline{AB}$) in the diagram?



457. The principle of the diagonal scale is the same as that underlying the division of a sect by the method of Ex. 456. In the accompanying diagram of a diagonal scale the unit is marked $\bar{u}$. What is the length of $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, and $\overline{MF}$?



458. Show how by the use of the diagonal scale to measure $0.3\bar{u}$, $0.56\bar{u}$, $0.75\bar{u}$, $1.8\bar{u}$, $3.1\bar{u}$, $0.35\bar{u}$, $0.82\bar{u}$, $2.67\bar{u}$.

459. Draw a triangle and measure the lengths of the sides to hundredths of an inch by the use of a diagonal scale in which $\bar{u} = 1$ inch. In measuring adjust the dividers to the side of the triangle, then apply them to the diagonal scale.

460. Make a diagonal scale on tracing paper, or of stiff cardboard or of wood, and with its aid measure correct to .01 in. (*a*)

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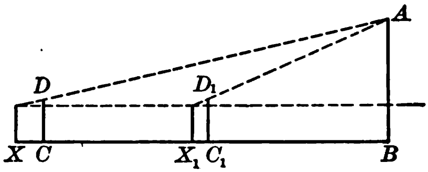
the hypotenuse of a right triangle whose legs are 1 in. and 2 in.; (*b*) the diagonal of a square of side 1 in.; (*c*) the altitude of an equilateral triangle of side 2 in.; (*d*) Verify each measurement by computation, assuming the Pythagorean Theorem.

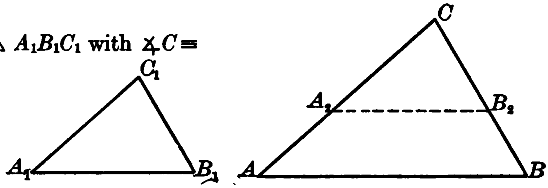
d461. To find the height of an object AB: Place the rod CD in an upright position. Stand at X and sight over D to A. Move the rod any convenient distance, so that it takes the position D_1C_1, and sight over D_1 to A. What measurements are necessary, and how may the height of the object be determined from them?



Theorem 35. ***Triangles which have two sides of one proportional to two sides of another and their included angles equal are similar.***

Given: $\triangle ABC$ and $\triangle A_1B_1C_1$ with $\angle C = \angle C_1$ and $\frac{\overline{A_1C_1}}{\overline{AC}} = \frac{\overline{B_1C_1}}{\overline{BC}}$.

To prove: $\triangle ABC \sim \triangle A_1B_1C_1$.



Suggestions for proof:

Place $\triangle A_1B_1C_1$ in position of A_2B_2C. How do you know this is possible?

Why will $\overline{A_2B_2}$ be parallel to $\overline{AB}$?

What follows about $\angle B_2A_2C$ and $\angle BAC$?

Can you throw this theorem back to the one immediately preceding?

EXERCISES. Set XLVIII (continued)

462. Extend your arm and point to a distant object, closing your left eye and sighting across your finger tip with your right eye. Now keep your finger in the same position and sight with your left eye. The finger will then seem to be pointing to an object some distance to the right of the one at which you were pointing. If you can estimate the distance between these two objects, which can often be done with a fair degree of accuracy, especially when there are buildings of which we can judge the width intervening, then you will be able to tell approximately the distance of your finger from the objects by the distance between the objects, for it will be ten times the latter. Find the reason for this.

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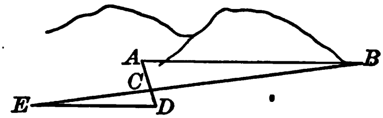
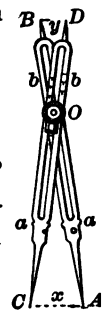


463. Explain how the accompanying figure can be used to find the distance from A to B on opposite sides of a hill. $\overline{CE} \equiv \frac{1}{4}\overline{BC}$, $\overline{CD} \equiv \frac{1}{4}AC$. $\overline{ED}$ is found by measurement to be 125 ft. What is the distance $\overline{AB}$?

464. The accompanying picture shows a pair of proportional compasses. Note that rods AB and CD are of equal length and pivoted together at O.

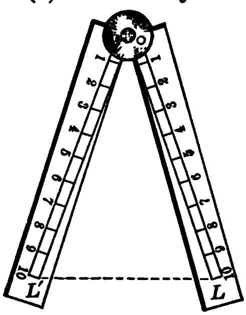


(*a*) Prove $\triangle AOC \triangle \backsim BOD$.

(*b*) Prove $\frac{x}{y} = \frac{a}{b}$.

(*c*) How may such an instrument be used to divide a sect?

(*d*) How may such an instrument be used to construct a triangle similar to a given triangle?



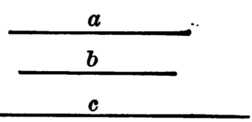
c*465. This picture shows a pair of sector compasses. It can be used in much the same way as the proportional compasses. Show how by means of it to get any part of a given sect.

Hint: To bisect a sect, open the compasses so that the distance from 10 to 10 is equal to the given sect. Then the distance from 5 to 5 equals one-half the given sect.

What part of the given sect would the distance from 6 to 6 be?

c466. Show how by using the sector compasses to divide a given sect into 10 equal parts.

c467. The sector compasses may be used to find the fourth proportional to three given sects as follows: From center O on OL mark off OA equal to a. Open the sector until the transverse distance at A equals b. Then if OB be marked off on OL equal to c, the transverse distance at B is the required fourth proportional. Why?



* c is prefixed to the numbers of exercises better suited to class discussions than to written or home work.

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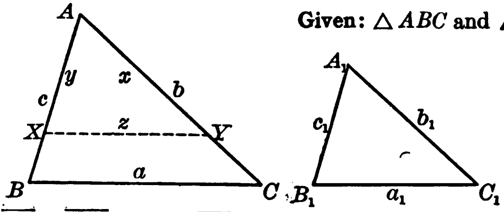
Theorem 36. ***If the ratio of the sides of one triangle to those of another is constant, the triangles are similar.***



Given: $\triangle ABC$ and $\triangle A_1B_1C_1$ with $\frac{a_1}{a} = \frac{b_1}{b} = \frac{c_1}{c}$.

To prove: $\triangle ABC \sim \triangle A_1B_1C_1$.

Suggestions for proof:

On AB lay off $\overline{AX} = \overline{A_1B_1}$, and on AC lay off $\overline{AY} = \overline{A_1C_1}$. Draw $\overline{XY}$.

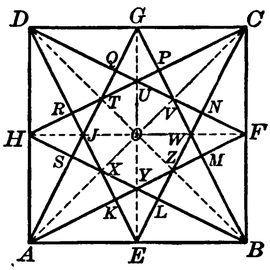
Prove $\triangle ABC \sim \triangle AXY$.

Prove $\triangle AXY \cong \triangle A_1B_1C_1$.

NOTE: $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$ (given) and $\frac{a}{z} = \frac{b}{x} = \frac{c}{y}$ (Why?) but $y = c_1$.

EXERCISES. SET XLVIII (concluded)

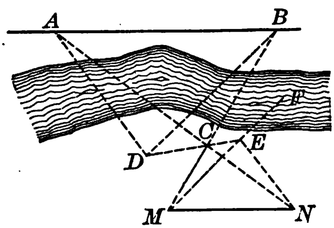
d468. In the figure, E, F, G, and H are the mid-points of the sides of the square $ABCD$, and the points are joined as shown.



Show that the following triangles are similar:

(*a*) EBC, ELB, ELY, and EUN; (*b*) BLZ, BST, and BXO; (*c*) EYZ and EUC; (*d*) YLZ, YMB, and BHP; (*e*) BOY and BHD; (*f*) BEZ and ATB; (*g*) BYZ and BHT; (*h*) BYF and BHC.

d469. The following gives a procedure used in surveying for running a line through a given point parallel to a wholly inaccessible line. Study the diagram and notes and then justify the method.



NOTES: Take C in sect MB. Select D at any convenient place. Run $MF \parallel DB$. Find E in MF in line with D and C. Run $EN \parallel AD$, meeting AC at N. Then $MN \parallel AB$.

Why is it that we can run parallels through M and D, whereas we cannot run the one through M directly?

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d470. The pantograph, invented in 1603 by Christopher Scheiner, is an instrument for drawing a plane figure similar to a given plane figure, and is, hence, useful for enlarging and reducing maps and diagrams.

The pantograph, shown in two positions, consists of four bars so pivoted at B and E that the opposite bars are parallel. Pencils are carried at D and F, and A turns upon a fixed pivot. BD and DE may be so adjusted as to make the ratio of $\frac{AB}{AC}$ (and hence $\frac{EF}{CF}$ and $\frac{AD}{AF}$) whatever is desired. So if F traces a given figure, D will trace a similar one, the ratio of similitude being the fixed ratio $\frac{AD}{AF}$.

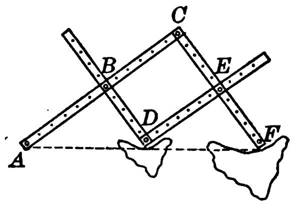

FIG. 1

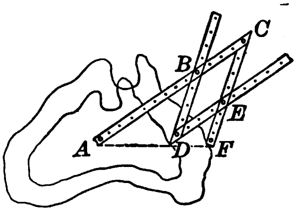

FIG. 2

(*a*) Prove that A, D, and F are always in the same straight line.

(*b*) Prove that $\frac{AD}{AF}$ is constant and equal to $\frac{AB}{AC}$.

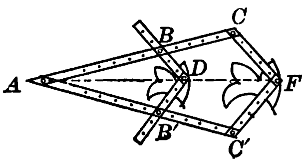

FIG. 3

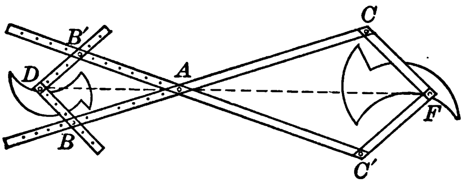

FIG. 4

(*c*) Make a pantograph. A crude one can be made of stiff cardboard and brass brads.

NOTE: Figs. 3 and 4 show interesting elaborations of the pantograph.

471. Summarize the conditions necessary and sufficient to make triangles similar.

472. How may four sects be proved proportional?

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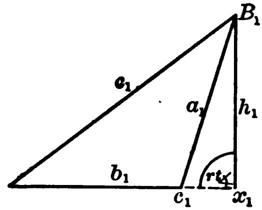
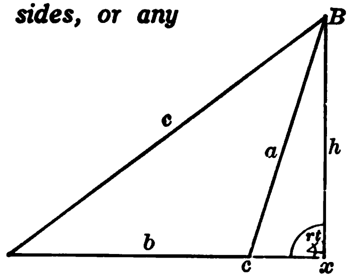
D. PERIMETERS AND AREAS OF SIMILAR TRIANGLES

Theorem 37. *The perimeters of similar triangles are proportional to any two homologous sides, or any two homologous altitudes.*





Given: $\triangle abc \backsim \triangle a_1b_1c_1$ with $h \perp b$ and $h_1 \perp b_1$.

To prove: $\dfrac{a+b+c}{a_1+b_1+c_1} = \dfrac{a}{a_1}$ (or $\dfrac{b}{b_1}$ or $\dfrac{c}{c_1}$) $= \dfrac{h}{h_1}$.

Suggestions for proof:

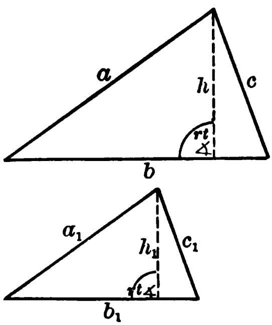
$\dfrac{a}{a_1} = \dfrac{b}{b_1} = \dfrac{c}{c_1}$. Why?

$\therefore \dfrac{a+b+c}{a_1+b_1+c_1} = \dfrac{a}{a_1}$. Why?

Prove $\dfrac{h}{h_1} = \dfrac{a}{a_1}$ by showing $\triangle CBX \backsim \triangle C_1B_1X_1$.

Cor. 1. *Any two homologous altitudes of similar triangles have the same ratio as any two homologous sides.*

Theorem 38. *The areas of similar triangles compare as the squares of any two homologous sides.*



Given: $\triangle abc \backsim \triangle a_1b_1c_1$.

To prove: $\dfrac{\triangle abc}{\triangle a_1b_1c_1} = \dfrac{a^2}{a_1^2}\left(\text{or } \dfrac{b^2}{b_1^2} \text{ or } \dfrac{c^2}{c_1^2}\right)$

Suggestions for proof:

$\dfrac{\triangle abc}{\triangle a_1b_1c_1} = \dfrac{hb}{h_1b_1} = \dfrac{h}{h_1} \cdot \dfrac{b}{b_1}$. Why?

$\dfrac{a}{a_1} = \dfrac{b}{b_1} = \dfrac{c}{c_1} = \dfrac{h}{h_1}$. Why?

$\therefore \dfrac{\triangle abc}{\triangle a_1b_1c_1} = \dfrac{a^2}{a_1^2}\left(\text{or } \dfrac{b^2}{b_1^2} \text{ or } \dfrac{c^2}{c_1^2}\right)$. Why?

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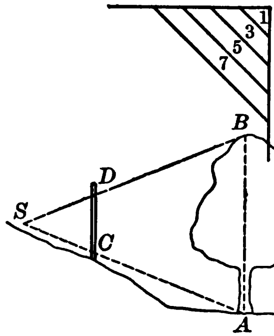
EXERCISES. SET XLIX. AREAS OF SIMILAR TRIANGLES

473. Draw a triangle. Construct a second one similar to it, having an area nine times as great.

474. Connect the mid-points of two adjacent sides of a parallelogram. What part of the area of the whole figure is the triangle thus formed?

475. Fold a rectangular sheet of paper from one corner as shown. The successive creases are to be equally distant from each other and parallel. Prove that the ratio of the successive areas between creases is 1 to 3 to 5 to 7, etc.



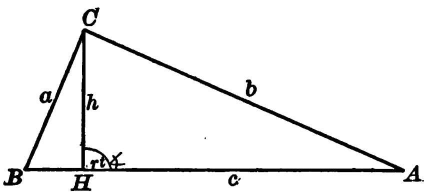
476. When, in forestry, shadows cannot be used, justify the following method of getting the height of a tree. A staff is planted upright in the ground. A man sights from S to the top and foot of the tree. His assistant notes where his line of sight crosses the staff.

(*a*) What measurements does he need to take?

(*b*) Assume a reasonable set of data and calculate AB.

E. APPLICATIONS OF SIMILAR TRIANGLES

Theorem 39. ***The altitude upon the hypotenuse of a right triangle divides the triangle into triangles similar each to each and to the original.***



Given: $\triangle abc$ with $a \perp b$ and $h \perp c$.

To prove: $\triangle BCH \backsim \triangle CHA \backsim ABC$.

Suggestions for proof:

∡B is common to the two right triangles BCH and ABC.

∡A is common to the two right triangles CHA and ABC.

I. PROJECTIONS

The word "projection" has a variety of meanings in general use. We refer to projecting ourselves into a situation; or to projecting a picture on a screen by means of a lantern. In geometry the word

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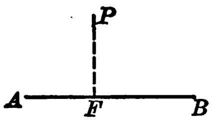
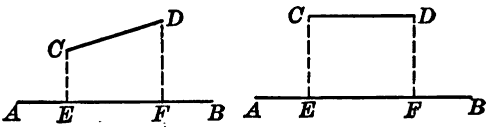
has a technical significance which we exemplify in the following little experiment. Place in the sunlight a table covered with a white cloth. Hold over the table (parallel to it) a plate of thin glass on which small figures of dark paper have been pasted. The sun will project these diagrams on the cloth. Upon reflection you will note a certain relationship between the point or sect represented on the glass and its manifestation on the cloth.

To use scientifically the idea of projection we need exact definitions rather than these vague assumptions. Let us express these ideas in geometrical terms.

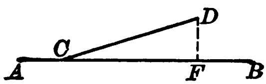
The intersection of a perpendicular to a line with that line is called the **foot of the perpendicular.** *The* **projection of a point** *on a line is the foot of the perpendicular from the point to the line. The* **projection of a sect** *on a line is the sect cut off by the projections of its extremities on that line.* For example:



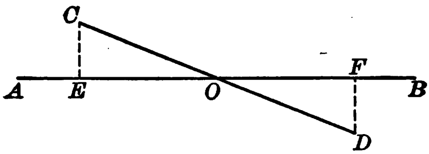
F is projection of P on $\overline{AB}$.



$\overline{EF}$ is projection of $\overline{CD}$ on $\overline{AB}$.



$\overline{CF}$ is projection of $\overline{CD}$ on $\overline{AB}$.



$\overline{EF}$ is the projection of $\overline{CD}$ on $\overline{AB}$.

Would the projection ever be as long as the original sect? Ever longer?

NOTE: These projections are sometimes referred to as orthogonal to distinguish them from other types met with in higher mathematics.

The notion of projections was originally obtained from that of shadows. The projection of a circle in one plane on another plane was its shadow. It is evident that a scientific study of shadows becomes very complicated. Consider, for instance, the effect on the shadow caused by the various relative positions of the planes and the positions of the light. Projective geometry, an advanced

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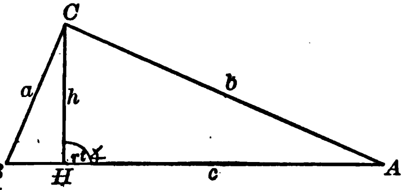
study, concerns itself with the more complex phases of the subject. In elementary geometry, we refer only to one very small instance of shadow geometry, reduced, as you note, to geometric definitions. These projections we refer to as right or *orthogonal.*



Cor. 1. ***Each side of a right triangle is a mean proportional between the hypotenuse and its projection upon the hypotenuse.***

NOTE: $\triangle BCH \backsim \triangle CBA$. Form a proportion involving $\overline{BC}$, $\overline{BA}$, $\overline{BH}$.

Cor. 2. ***The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.***

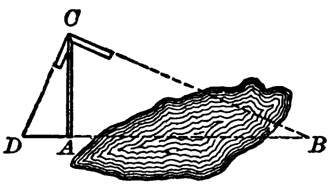
NOTE: $a^2 = c \cdot BH$, and $b^2 = c \cdot HA$. Why?
$\therefore a^2 + b^2 = c \cdot \overline{BH} + c \cdot \overline{HA} = c(\overline{BH} + \overline{HA}) = c^2$.

This fact is very important in geometry, and has an interesting history. The first proof of the theorem is attributed to Pythagoras about 500 B. C., although the fact was known much earlier. Reference has already been made to the history of the theorem in Chapter I. Its later development consists of numerous proofs worked out by later mathematicians. In the following exercises specimens of such will be found, and further interesting proofs are contained in Heath's Monograph, "The Pythagorean Theorem."

EXERCISES. SET L. PROJECTIONS. PYTHAGOREAN RELATION

477. What would be the projection of a sect 10″ long on a line with which it makes an angle of (*a*) 30°? (*b*) 45°? (*c*) 60°?

478. In the sixteenth century the distance from A to the inaccessible point B was determined by means of an instrument called the "squadra." The squadra, like a modern carpenter's square, consisted of two metallic arms at right angles to each other. To measure AB the squadra was supported, as in the figure, on a vertical staff AC. One arm was pointed toward B, and the point D on the ground, at which the other arm



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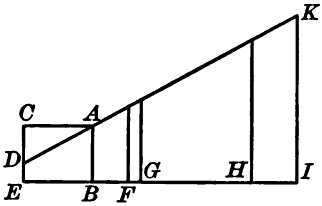
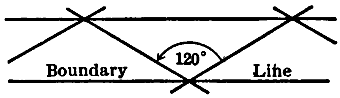
pointed, was noted. By measuring AD and AC, show how AB may be computed.

d479. Two stakes are set on a hillside whose slope is 20% (*i.e.*, 20 ft. rise in 100 ft. measured along the slope). The distance between the stakes, measured along the slope, is 458 ft. What is the horizontal distance between them?



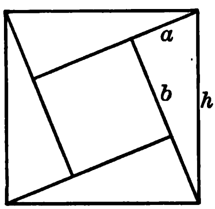
480. The accompanying drawing represents a plot of land divided as indicated. $DE=22'8''$, $EB=100'$, $BF=50'$, FG (alley) $=16'$, $GH=150'$, $HI=66'$, and $IK=232'6''$. Find the length of AB in feet, and the area of triangle ACD in square rods.

d481. The figure shows a ground plan of a zigzag or "worm" fence. The rails are 11 feet long, and a lap of 1 foot is allowed at each corner. Stakes, supporting the rider rails, are set along the boundary line. Find the amount of ground wasted by the construction of such a fence 100 rods long.



How much more fence is needed in this zigzag fence than in a straight one?

d482. Let ABC be any right-angled triangle, right-angled at C, and let the square $ABDE$ be described on the hypotenuse AB, overlapping the triangle. Prove that the perpendicular from E upon AC is of length b, and hence that the area of the triangle ACE is $\frac{1}{2}b^2$. Similarly, prove that the area of the triangle BCD is $\frac{1}{2}a^2$. Notice that these two triangles have equal bases c and total height c. Hence prove that $a^2+b^2=c^2$.



483. The great Hindu mathematician, Bhaskara (born 1114 A. D.), proceeds in a somewhat similar manner. He draws this figure, but gives no proof. It is evident that he had in mind this relation:

$$h^2 \equiv 4\cdot\frac{ab}{2} + (b-a)^2 \equiv a^2+b^2.$$

Give a proof.

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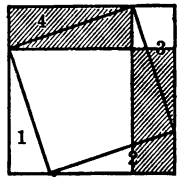
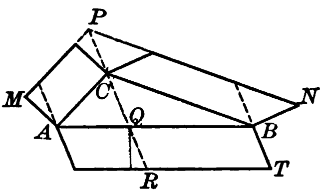
d484. A somewhat similar proof can be based upon the following figure:



If the four triangles, 1+2+3+4, are taken away, there remains the square on the hypotenuse. But if we take away the two shaded rectangles, which equal the four triangles, there remain the squares on the two sides. Therefore the square on the hypotenuse equals the sum of these two squares. Give details of the proof.

d485. This exercise makes the Pythagorean Theorem a special case of a proposition due to Pappus (fourth century A. D.), relating to any kind of triangle.

Somewhat simplified, this proposition asserts that if ABC is any kind of triangle, and MC, NC are parallelograms on AC, BC, the opposite sides being produced to meet at P; and if PC is produced making $QR=PC$; and if the parallelogram AT is constructed, then $AT \equiv MC+NC$.

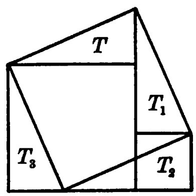


For $MC \equiv AP \equiv AR$, having equal altitudes and bases.

Similarly, $NC \equiv QT$. Adding, $MC+NC \equiv AT$.

If, now, ABC is a right triangle, and if MC and NC are squares, it is easy to show that AT is a square, and the proposition reduces to the Phythagorean Theorem. Show this.

d486. The Arab writer, Al-Nairizi (died about 922 A.D.), attributed to Thabit ben Korra (826–901 A.D.) a proof substantially as follows:



The four triangles T can be proved congruent. Then if we take from the whole figure T and T_1, we have left the squares on the two sides of the right angle. If we take away the other two triangles instead, we have left the square on the hypotenuse. Therefore the former is equivalent to the latter. Give details of proof.

d487. A proof attributed to the great artist, Leonardo da Vinci (1452–1519), is as follows:

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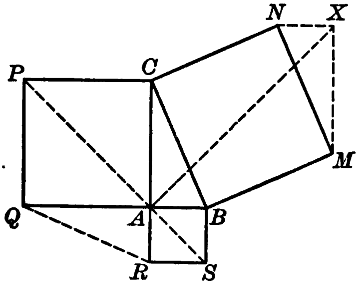
The construction of the following figure is evident. It is easily



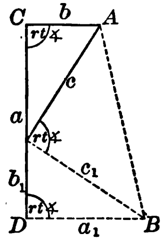
shown that the four quadrilaterals $ABMX$, $XNCA$, $SBCP$, and $SRQP$ are congruent.

$\therefore ABMXNCA$ equals $SBCPQRS$ but is not congruent to it, the congruent quadrilaterals being differently arranged.

Subtract the congruent triangles MXN, ABC, RAQ, and the proposition is proved.

Give details of proof.

d488. A proof attributed to President Garfield is suggested by the accompanying diagram. Work it out.



NOTE: $a \equiv a_1$, $b \equiv b_1$, $c \equiv c_1$. $ABDC$ is a trapezoid. What is the altitude of the trapezoid? Its bases? Its area? How else may the area of the trapezoid be found?

d489. Show that if $AB \equiv a$ (in Fig. 3),

(a) $EC \equiv \frac{a}{2}\sqrt{5}$. (c) $LB \equiv \frac{a}{5}\sqrt{5}$. (e) $OZ \equiv \frac{a}{6}\sqrt{2}$. (g) $ZL \equiv \frac{a}{15}\sqrt{5}$.

(b) $EL \equiv \frac{a}{10}\sqrt{5}$. (d) $LY \equiv \frac{a}{20}\sqrt{5}$. (f) $ZB \equiv \frac{a}{3}\sqrt{2}$. (h) $ZY \equiv \frac{a}{12}\sqrt{5}$.

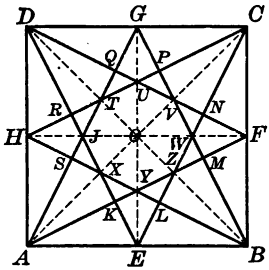
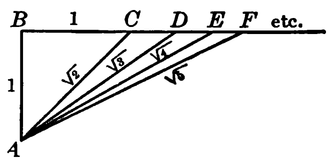


FIG. 3.
Moorish Design, from Mabel Sykes' Source Book of Problems for Geometry.

490. In the middle of a pond 10 ft. square grew a reed. The reed projected 1 ft. above the surface of the water. When blown aside by the wind, its top part reached to the mid-point of a side of the pond. How deep was the pond? (Old Chinese problem.)

491. Show that the following diagrams illustrate methods of representing the square roots of integers.



$AB = BC = 1$.
$AC = BD = \sqrt{2}$.
$AD = BE = \sqrt{3}$, etc.

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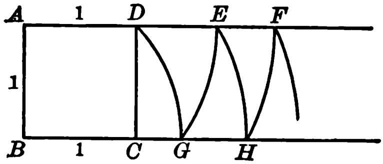
AC is a square.
$BD = BG = \sqrt{2}$.
$AG = AE = \sqrt{3}$, etc.

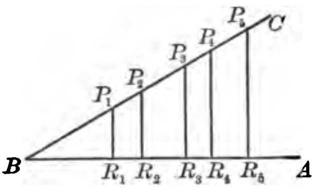


NOTE: Such methods would not be practical for large numbers Why?

The first of these methods is used on the "line of squares" on the sector compasses.

492. The Hindus said that triangles having the following sides are right triangles. How is the assumption they apparently made related to Theorem 40, Cor. 2? (*a*) 5, 12, 13. (*b*) 15, 36, 39. (*c*) 8, 15, 17. (*d*) 12, 35, 37.

II. TRIGONOMETRIC RATIOS



Let ABC be an acute angle. Drop a series of perpendiculars to $\overline{BA}$ from any points on $\overline{BC}$.

It will readily be noted that the right triangles formed, BP_1R_1, BP_2R_2, etc., are similar, having the angle B in common. The equality of the following ratios will result:

(1) $\dfrac{\overline{R_1P_1}}{\overline{BP_1}} = \dfrac{\overline{R_2P_2}}{\overline{BP_2}} = \dfrac{\overline{R_3P_3}}{\overline{BP_3}} = \dfrac{\overline{R_4P_4}}{\overline{BP_4}} = \dfrac{\overline{R_5P_5}}{\overline{BP_5}}$. (2) $\dfrac{\overline{BR_1}}{\overline{BP_1}} = \dfrac{\overline{BR_2}}{\overline{BP_2}} = \dfrac{\overline{BR_3}}{\overline{BP_3}} = \dfrac{\overline{BR_4}}{\overline{BP_4}} = \dfrac{\overline{BR_5}}{\overline{BP_5}}$ and (3) $\dfrac{\overline{R_1P_1}}{\overline{BR_1}} = \dfrac{\overline{R_2P_2}}{\overline{BR_2}} = \dfrac{\overline{R_3P_3}}{\overline{BR_3}} = \dfrac{\overline{R_4P_4}}{\overline{BR_4}} = \dfrac{\overline{R_5P_5}}{\overline{BR_5}}$. Why?

If we think of $\measuredangle ABC$ as being generated by sect BC revolving counterclockwise from the position BA to BC we may call BA the **initial** side and BC the **terminal** side. These perpendiculars from points on the terminal side may be thought of as **projectors** forming on the initial side the projections of sects of the terminal side. We may then summarize the facts given as ratios in what preceded as follows: For any acute angle if perpendiculars be dropped to the initial side from any points on the terminal side the ratios (1) *of the projector to the sect of the terminal side*, (2) *of the projection to the sect of the terminal side*, and (3) *of the projector to the projection of a sect of the terminal upon the initial side* are constants. These ratios are given the names **sine, cosine,** and **tangent,** respectively.

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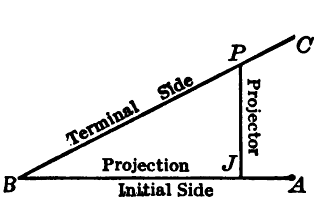
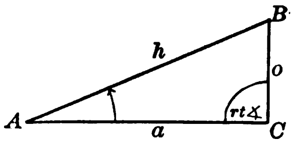


Thus (1) the sine of angle ABC is $\frac{\overline{JP}}{\overline{BP}}$.

(2) The cosine of angle ABC is $\frac{\overline{BJ}}{\overline{BP}}$.

(3) The tangent of angle ABC is $\frac{\overline{JP}}{\overline{BJ}}$.

These are referred to as **trigonometric ratios** of an angle.

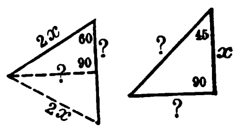


If in the fixed $\triangle ABC$, right angled at C, the side opposite A is called o, the side adjacent, a, and the hypotenuse, h, fill in the following:

$\sin A = ?$ $\cos A = ?$ $\tan A = ?$

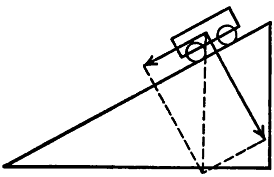
EXERCISES. SET LI. TRIGONOMETRIC RATIOS

493. Make a table of the values of the sines, cosines, and tangents of angles of 30°, 45°, and 60°.



494. Show by means of similiar triangles that the sine of 60° is the same as the cosine of 30°.

d495. When a wagon stands upon an incline, its weight is resolved into two forces, one the pressure against the incline, the other tending to make it run down the incline. Show that the force along the incline is to the weight of the wagon as the height of the incline is to its length. If the incline makes a 30° angle with the horizontal, with what force does a loaded wagon, weighing three tons, tend to run down the incline, *i.e.*, disregarding friction, what force must a team exert to pull it up the slope?



496. Fill out the following table:

Polygon	Dimensions	Perimeter	Area
Parallelogram	Base 18	300	. . .
	Angle 60°	. . .	300
Rectangle	Base 18	300	. . .
		. . .	300
Rhombus	Angle 60°	300	. . .
		. . .	300
Square	. . .	300	. . .
		. . .	300

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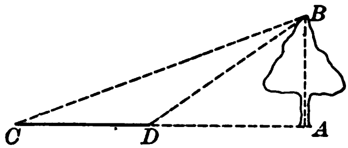
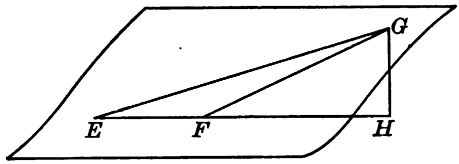
497. To measure the height of an object AB by drawing to scale:

Measure a distance CD towards A. Measure the angle ACB and the angle ADB. Then draw a plan thus: Representing, say, 100 ft. by a sect an inch long, draw EF to represent CD in the plan, and draw the angle HFG equal to the angle ADB, and the angle FEG equal to the angle ACB, and draw GH at right angles to EF prolonged. Measure GH. Show how to compute AB.





(*a*) If $CD = 175$ ft., the angle $ADB = 45°$, and angle $DCB = 30°$, compute AB.

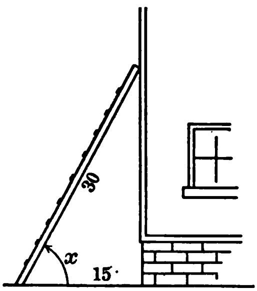
(*b*) Draw a diagram to scale and compare the result with that obtained from your calculations.

(*c*) Which of these results is more accurate? Why?

498. If a survey is made, using a 100 ft. tape, and on a hill, the lower chainman holds his end of the tape 2 ft. too low:

(*a*) What error will be caused in one tape length?

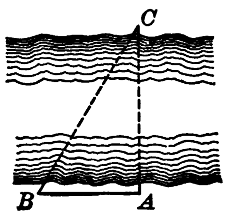
(*b*) If the distance between two stations on the hillside is recorded as 862 ft., what is the actual distance?



(*c*) If the problem is to lay off a distance of 900 ft., what is the actual distance laid off?

499. A ladder 30 ft. long leans against the side of a building, its foot being 15 ft. from the building. What angle does the ladder make with the ground?

500. In order to find the width of a river, a distance AB was measured along the bank, the point A being directly opposite a tree C on the other side. If the angle ABC was observed to be 60°, and AB 100 ft., find the width of the river.



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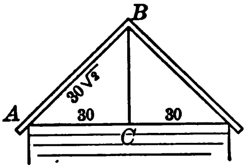
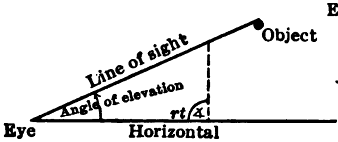
501. A house 30 ft. wide has a gable roof whose rafters are 20 ft. long. What is the pitch of the roof? (The pitch is the angle between a rafter and the horizontal.)

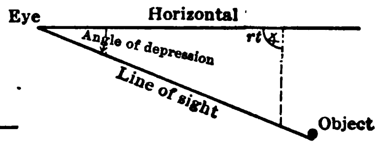
502. A barn 60 ft. wide has a gable roof whose rafters are $30\sqrt{2}$ ft. long. What is the pitch of the roof, and how far above the eaves is the ridgepole?



The **angle of elevation** *is the angle between the ray of light from the object to the eye and the horizontal line in the same plane, when the object is above the horizontal line. When the object observed is below the horizontal line, the angle is called the* **angle of depression.**

For example:





EXERCISES. SET LI (continued)

503. At a point 200 ft. in a horizontal line from the foot of a tower the angle of elevation of the top of the tower is observed to be 60°. Find the height of the tower.

504. The vertical central pole of a circular tent is 20 ft. high, and its top is fastened by ropes 40 ft. long to stakes set in the ground. How far are the stakes from the foot of the pole, and what is the inclination of the ropes to the ground?

505. At a point midway between two towers on a horizontal plane the angles of elevation of their tops are 30° and 60° respectively. Show that one tower is three times as high as the other.

506. A flagstaff 25 ft. high stands on the top of a house. From a point on the plane on which the house stands, the angles of elevation of the top and the bottom of the flagstaff are observed to be 60° and 45° respectively. Find the height of the house.

507. A man walking on a straight road observes at one milestone a house in a direction making an angle of 30° with the road, and at the next milestone the angle is 60°. How far is the house from the road?

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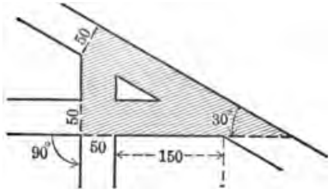
508. Find the number of square feet of pavement required for the shaded portion of the streets shown in the figure, all the streets being 50 ft. wide.



It is not possible to determine the trigonometric ratios of angles other than 30°, 45°, and 60° by elementary plane geometry. By the use of the protractor any acute angle can be drawn, and with a ruled edge the sects needed may be measured and approximations may be made for the ratios. The values correct to many decimal places have been scientifically worked out and tabulated. A table correct to four places follows for use in subsequent problems. Corrections for fractions of minutes may be made as in the case of logarithmic tables.

Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent
1	.0175	.9998	.0175	31	.5150	.8572	.6009	61	.8746	.4848	1.8040
2	.0349	.9994	.0349	32	.5299	.8480	.6249	62	.8829	.4695	1.8807
3	.0523	.9986	.0524	33	.5446	.8387	.6494	63	.8910	.4540	1.9626
4	.0698	.9976	.0699	34	.5592	.8290	.6745	64	.8988	.4384	2.0503
5	.0872	.9962	.0875	35	.5736	.8192	.7002	65	.9063	.4226	2.1445
6	.1045	.9945	.1051	36	.5878	.8090	.7265	66	.9135	.4067	2·2460
7	.1219	.9925	.1228	37	.6018	.7986	.7536	67	.9205	.3907	2.3559
8	.1392	.9903	.1405	38	.6157	.7880	.7813	68	.9272	.3746	2.4751
9	.1564	.9877	.1584	39	.6293	.7771	.8098	69	.9336	.3584	2.6051
10	.1736	.9848	.1763	40	.6428	.7660	.8391	70	.9397	.3420	2.7475
11	.1908	.9816	.1944	41	.6561	.7547	.8693	71	.9455	.3256	2.9042
12	.2079	.9781	.2126	42	.6691	.7431	.9004	72	.9511	.3090	3.0777
13	.2250	.9744	.2309	43	.6820	.7314	.9325	73	.9563	.2924	3.2709
14	.2419	.9703	.2493	44	.6947	.7193	.9657	74	.9613	.2756	3.4874
15	.2588	.9659	.2679	45	.7071	.7071	1.0000	75	.9659	.2588	3.7321
16	.2756	.9613	.2867	46	.7193	.6947	1.0355	76	.9703	.2419	4.0108
17	.2924	.9563	.3057	47	.7314	.6820	1.0724	77	.9744	.2250	4.3315
18	.3090	.9511	.3249	48	.7431	.6691	1.1106	78	.9781	.2079	4.7046
19	.3256	.9455	.3443	49	.7547	.6561	1.1504	79	.9816	.1908	5.1446
20	.3420	.9397	.3640	50	.7660	.6428	1.1918	80	.9848	.1736	5.6713
21	.3584	.9336	.3839	51	.7771	.6293	1.2349	81	.9877	.1564	6.3138
22	.3746	.9272	.4040	52	.7880	.6157	1.2799	82	.9903	.1392	7.1154
23	.3907	.9205	.4245	53	.7986	.6018	1.3270	83	.9925	.1219	8.1443
24	.4067	.9135	.4452	54	.8090	.5878	1.3764	84	.9945	.1045	9.5144
25	.4226	.9063	.4663	55	.8192	.5736	1.4281	85	.9962	.0872	11.4301
26	.4384	.8988	.4877	56	.8290	.5592	1.4826	86	.9976	.0698	14.3006
27	.4540	.8910	.5095	57	.8387	.5446	1.5399	87	.9986	.0523	19.0811
28	.4695	.8829	.5317	58	.8480	.5299	1.6003	88	.9994	.0349	28.6363
29	.4848	.8746	.5543	59	.8572	.5150	1.6643	89	.9998	.0175	57.2900
30	.5000	.8660	.5774	60	.8660	.5000	1.7321	90	1.0000	.0000	∞

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In trigonometry a more extended study of these ratios will be given. The ratios of angles of more than 90° will be considered, and other ratios which are constant will be developed.

Work involving calculations with the trigonometric ratios is often simplified by the use of tables of logarithmic functions. For this purpose, and for greater facility in the use of logarithms in general, and in the use of the natural functions, it would be to the pupil's advantage to procure a compact volume of tables. An excellent book for this purpose, costing only twenty cents, is Prof. A. Adler, Fünfstellige Logarithmen (Sammlung Göschen).

EXERCISES. SET LI (concluded)

509. The sect AB 15 inches long makes an angle of 35° with the line $\overleftrightarrow{OX}$. Find its projection on $\overleftrightarrow{OX}$. Find its projection on the line $\overline{OY}$ perpendicular to $\overline{OX}$ and in the same plane as $\overline{OX}$ and $\overline{AB}$.

510. What is the angle of the sun's altitude if the shadow of a telegraph pole 30 ft. high is 40 ft. long?

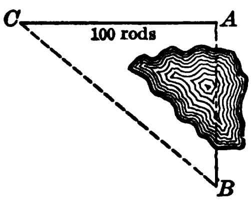
511. A tower is 615 ft. high. How large an angle does it subtend at a point which is $1\frac{1}{2}$ mi. away and on the same horizontal plane as its base?

512. A mariner finds that the angle of elevation of the top of a cliff is 16°. He knows from the location of a buoy that his distance from the foot of the cliff is half a mile. How high is the cliff?

513. At 40 ft. from the base of a fir tree the angle of elevation of the top is 75°. Find the height of the tree.

514. A flagstaff 75 ft. high casts a shadow 40 ft. long. Find the angle of elevation of the sun above the horizon.

515. To find the distance across a lake from a point A to a point B, a man measured 100 rods to a point C on a line perpendicular to the line AB, and found that the angle ABC was 50°. How could he find the distance across the lake? What is the distance?



516. What is the angle of slope of a road bed that has a grade of 5 per cent? One with a grade of 25 hundredths per cent? (By "a grade of 5 per cent" is meant a rise of five feet in a horizontal distance of one hun-

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dred feet. By "the angle of the slope" of such a grade is meant the angle whose tangent is 0.05.)

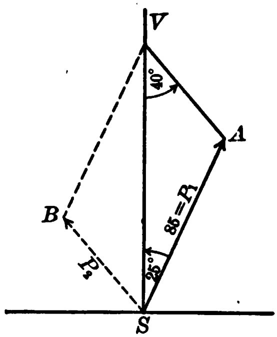
517. A steamer is moving in a southeasterly direction at the rate of 25 miles an hour. How fast is it moving in an easterly direction? In a southerly direction?

518. A balloon of diameter 50 ft. is directly above an observer and subtends a visual angle of 4°. What is the height of the balloon?

d519. The angle of elevation of a balloon from a point due south of it is 60°, and from another point 1 mile due west of the former, the angle of elevation is 45°. Find the height of the balloon.

520. Wishing to determine the width of a river, I observed a tree standing directly across the bank. The angle of elevation of the top of the tree was 32°; at 150 ft. back from this point, and in the same direction from the tree, the angle of elevation of the top of the tree was 21°. Find the width of the river.

521. A tree is standing on a bluff on the opposite side of the river from the observer. Its foot is at an elevation of 45°, and its top at 60°. (*a*) Compare the height of the bluff with that of the tree (*i.e.*, find the ratio). (*b*) What measurement would you use to find the height of the tree? (*c*) The height of the bluff? (*d*) The width of the river?



d522. Two men are lifting a stone by means of ropes. As the stone leaves the ground one man is pulling with a force of 85 lbs. in a direction 25° from the vertical, while the other man is pulling at an angle of 40° from the vertical. Determine the weight of the stone.

523. A 50 ft. pole stands on the top of a mound. The angles of elevation of the top and the bottom of the pole are respectively 35° and 62°. Find the height of the mound.

524. From the top of a mountain 1050 ft. high two buildings are seen on a level plane, and in a direct line from the foot of the mountain. The angle of depression of the first is 35°, and of the second is 21°. Find the distance between the two buildings.

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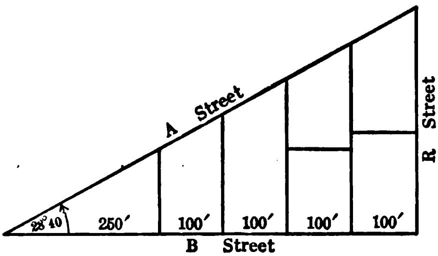
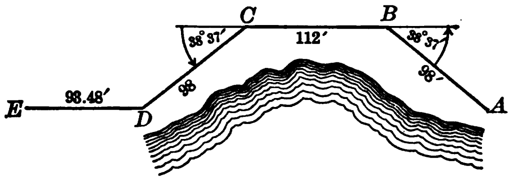
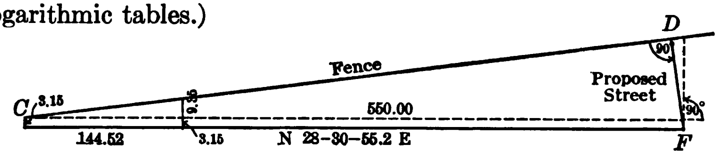
525. Certain lots in a city are laid out by lines perpendicular to *B* Street and running through to *A* Street, as shown in the figure. Find the widths of the lots on *A* Street if the angle between the streets is 28° 40′.



526. In surveying around an obstacle measurements were taken as shown in the figure. Find the distance on a straight line from *A* to *E*. (Use logarithmic tables.)



527. With data, find the length and bearing of *DF*, a proposed street. (Use logarithmic tables.)



d528. Look up and explain the principle of the Vernier. (Lock and Child, Trigonometry for Beginners (Macmillan), is a good reference for this point—pp. 120–126).

529. The shadow of a vertical 10 ft. pole is 14 ft. long. What is the angle of elevation of the sun?

530. The tread of a step on a certain stairway is 10″ wide; the step rises 7″ above the next lower step. Find the angle at which the stairway rises.

531. The width of the gable of a house is 34 ft. The height of the house above the eaves is 15 ft. Find the length of the rafters and the angle of inclination of the roof.

532. Find the angle between the rafter and horizontal in the following pitch of roof: two-thirds, one-half, one-third, one-fourth.

533. Two trees *M* and *N* are on opposite sides of a river. A line *NP* at right angles to *MN* is 432.7 ft. long, and the angle *NPM* is 52° 20′. What is the distance from *M* to *N*? (Use logarithmic tables.)

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534. In an isosceles triangle one of the base angles is 48° 20′, and the base is 18″. Find the legs, the vertical angle, and the altitude drawn to the base.

535. To find the height of a tower, a distance of 311.2 ft. was measured from the foot of the tower, and the angle of elevation of the tower was found to be 40° 57′. Find the height of the tower. (Use logarithmic tables.)

536. Find the shorter altitude and the area of a parallelogram whose sides are 10′ and 25′ when the angle between the sides is 74° 33′.

d537. The angle of elevation of the top of a spire from the third floor of a building was 35° 12′. The angle of elevation from a point directly above, on the fifth floor of the same building, was 25° 33′. What is the height of the tower and its horizontal distance from the place of observation, if the distance between consecutive floors is 12 ft., and the first floor rests on a basement 5 ft. above the level of the street?

538. (*a*) What size target at 33′ from the eye subtends the same angle as a target 3′ in diameter at 987 yds.?

(*b*) Find the angle it subtends.

539. The summit of a mountain, known to be 14,450 feet high, is seen at an angle of elevation of 29° 15′ from a camp located at an altitude of 6935 feet. Compute the air-line distance from the camp to the summit of the mountain. (Use logarithmic tables.)

d540. Two towns, *A* and *B*, of which *B* is 25 miles northeast of *A*, are to be connected by a new road. Ten miles of the road is constructed from *A* in the direction N. 23° E. What must be the length and direction of the remainder of the road, assuming that it follows a straight line?

541. A car track runs from *A* to *B*, a horizontal distance of 1275′ at an incline of 7° 45′, and then from *B* to *C*, a distance of 1585′. *C* is known to be 509′ above *A*. What is the average inclination of the track from *B* to *C*? (Use logarithmic tables.)

d542. On a map on which 1″ represents 1000′, contour lines are drawn for differences of 100′ in altitude. What is the actual inclination of the surface represented by that portion of the map at which the contour lines are $\frac{1}{4}$″ apart?

d543. The description in a deed runs as follows: Beginning at a stone (*A*), at the N. W. corner of lot 401; thence east 112′ to a

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stone (B); thence S. 36.5° W. 100′; thence west parallel with AB to the west line of said lot 401; thence north on the west line of said lot to the place of beginning. Find the area of the land described.

LIST OF WORDS DEFINED IN CHAPTER VI

Similar systems (or sets) of points, center of similitude, ratio of similitude, similar figures. Projection of a point, and of a sect, on a line; projector. Initial and terminal side of an angle. Trigonometric ratios; sine, cosine, tangent of an angle. Angle of elevation, angle of depression.

SUMMARY OF THEOREMS PROVED IN CHAPTER VI

31. A line parallel to one side of a triangle, and cutting the other sides, divides them proportionally.

Cor. 1. One side of a triangle is to either of the sects cut off by a line parallel to a second side, as the third side is to its homologous sect.

Cor. 2. A series of parallels cuts off proportional sects on all transversals.

Cor. 3. Parallels which intercept equal sects on one transversal, do so on all transversals.

Cor. 4. A line which bisects one side of a triangle, and is parallel to the second, bisects the third.

32. A line dividing two sides of a triangle proportionally is parallel to the third side.

Cor. 1. A line dividing two sides of a triangle so that those sides bear the same ratio to a pair of homologous sects is parallel to the third side.

33. The homologous sides of similar triangles have a constant ratio, and their homologous angles are equal.

34. Triangles are similar when two angles of one are equal each to each to two angles of another.

35. Triangles which have two sides of one proportional to two sides of another and the included angles equal are similar.

36. If the ratio of the sides of one triangle to those of another is constant, the triangles are similar.

37. The perimeters of similar triangles are proportional to any two homologous sides, or any two homologous altitudes.

Cor. 1. Homologous altitudes of similar triangles have the same ratio as homologous sides.

38. The areas of similar triangles compare as the squares of any two homologous sides.

39. The altitude on the hypotenuse of a right triangle divides the triangle into triangles similar to each other and to the original.

Cor. 1. Each side of a right triangle is a mean proportional between the hypotenuse and its projection upon the hypotenuse.

Cor. 2. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

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CHAPTER VII

THE LOCUS

A. REVIEW OF THE IDEA OF LOCUS AS MET WITH IN ALGEBRA

I. REVIEW AND SUMMARY OF ESSENTIAL POINTS IN THE INTRODUCTION TO GRAPHIC MATHEMATICS

a. Location of Points.

In locating places on a map we are accustomed to noting their longitude and latitude, which means that we refer to their distances north or south of the equator, and east or west of some meridian. So we may locate points on a piece of paper by stating their distance up or down from some fixed line of reference, and to the right or left of some other line of reference at right angles to the first.

These lines of reference are called the axes, the distances up or down are called the **ordinates,** *and those to the right and left the* **abscissas** *of the points. The ordinate and the abscissa of a point are together called its* **coördinates.** Paper ruled off in squares is used for convenience in counting, and in locating points. Such paper is called *coördinate paper*.

The abscissa of a point is given first, followed by the ordinate. Plus or minus are used in the case of the abscissa to denote distance to the right or left of the so-called **y-axis;** plus or minus, in the case of the ordinate, denote distance above or below the so-called **x-axis.** *The intersection of the axes is called their* **origin.**

The coördinates of a point are written in a parenthesis with a comma between them; *e.g.* (5,–2) refers to a point 5 units to the *right* of the y-axis and 2 units *below* the x-axis.

EXERCISES. SET LII. LOCATION OF POINTS

544. With reference to a single pair of axes, plot the following points on a sheet of coördinate paper:

(4, 5), (–2, 5), (–2, –5), (5, –2).

545. On the same sheet plot also the points: $(3, \frac{1}{2})$, $(-3, -^{3})$, $(6, -\frac{6}{5})$, $(-3, -\frac{1}{3})$.

546. Locate the points: (2, 0), (–5, 0), (0, 5), $(0, -\frac{2}{5})$, (0, 0).

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547. (*a*) All the points on the x-axis have what ordinate?

(*b*) All the points on the y-axis have what abscissa?

548. Construct the triangle whose vertices are (1, 1), (2, –2), (3, 2).

549. Construct the quadrilateral whose vertices are (2, –1), –4, –3), (–3, 5), (3, 4).

550. Construct the rectangle whose vertices are (–3, 4), (4, 4), –3, –2), (4, –2), and find its area.

551. Construct the triangle whose vertices are (–3, –4), (–1, 3), (2, –4), and find its area.

b. The Graph and its Applications.

The line connecting a series of points plotted as explained is called a **graph.** Graphs are useful for giving information quickly, in making estimates, and in the solution of many problems such as those involving time and distance. We have all seen the charts of trained nurses, and newspaper and magazine reports given in graphic form.

Among the numerous applications of the graph, then, we may list (1) records of statistics, (2) ready reckoners which furnish bases of interpolation or give convenient diagrams, (3) representations of formulas which make quick approximations possible.

EXERCISES. SET LIII. APPLIED PROBLEMS IN GRAPHIC MATHEMATICS

552. Observe the readings of the same thermometer at the same hours daily for a week, and record the results of your observations graphically.

553. A boy who can throw a stone from a sling shot with a velocity of 80 ft. per second is experimenting. He finds that when he throws it in a direction making an angle of 16° with the ground it pitches 35 yds. away. This and other results are given in the table below:

Angle (in degrees)...........	16	24	32	40	48	56	64	72	80
Distance (in yards)..........	35	50	60	65	66	61	52	39	22

(*a*) Draw a graph to represent these facts.

(*b*) Find (1) how far he can throw when the angle is 60°.

(2) what angle will produce a throw of 57 yds.

(3) what is the greatest distance the boy can throw, and what angle will produce this.

554. The vertices of a pentagonal field are located by the following points, $A = (-20,\ 15)$, $B = (10,\ 20)$, $C = (23,\ -20)$, $D = (-10,\ -30)$, $E = (-30,\ -10)$.

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(*a*) Draw the outline of the field.

(*b*) Give new values to A, B, C, D, E, so that the area shall remain the same, but the diagram lie wholly in the first quadrant, with E on the north-south axis, and D on the east-west axis.

(*c*) Find the area of the field.

555. The boiling-point of water on a Centigrade thermometer is marked 100°, and on a Fahrenheit 212°. The freezing-point on the Centigrade is zero, and on the Fahrenheit is 32°. Consequently a degree on one is not equal to a degree on the other.

(*a*) Show that the correct relation is expressed by the equation $C \equiv \frac{5}{9}$ (F−32), where C represents degrees Centigrade, and F degrees Fahrenheit.

(*b*) Construct a graph of this equation. Can you, by means of this graph, express a Centigrade reading in degrees Fahrenheit, and *vice versa?*

(*c*) By means of the graph express the following Centigrade readings in Fahrenheit readings, and *vice versa:* (1) 60° C.; (2) 150° F.; (3) −20° C.; (4) −30° F.

(*d*) What reading means the same temperature on both scales?

c. The Graph of Equations.

If in such an equation as $x+y=10$ various values of x are taken as abscissas of points whose ordinates are the corresponding values of y, and the points are joined, we have what is known as the **graph of the equation.** It is a fact which is proved in more advanced mathematics that the graph of an equation of the first degree is always a straight line.

Thus, if we represent graphically such a system of equations as $x+y=10$, and $x-y=4$, we have two straight lines. The coördinates of their intersection will give the solution of the equation. Why?

EXERCISES. SET LIV. GRAPHIC SOLUTION OF EQUATIONS

Solve the following systems of equations graphically.

556. $x+4y=11$
$2x-y=4$

557. $2x+3y=19$
$7x-2y=4$

558. $x+5y=-3$
$2x-3y=20$

559. $2x-9y=23$
$5x+y=-13$

560. $x+5y=0$
$3x+9y=-6$

561. $7x+2y=14$
$5x-3y=-21$

562. $2x-3y=7$
$5x-7y=14$

563. $6x-3y=15$
$2x+7y=45$

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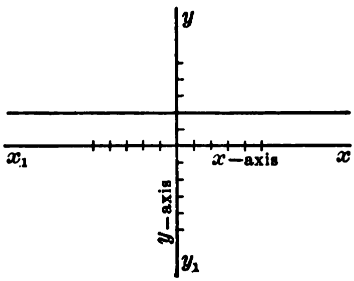
II. APPLICATION OF ELEMENTARY GRAPHIC MATHEMATICS TO GEOMETRY

Since we have studied the graphic solution of simultaneous equations, the idea of *locus* (plural, *loci*) is not an entirely new one. In our graphic work we found that the locus, obeying the law expressed by a linear equation, was a straight line.*

In our graphic work we find that all points +2 units from the x-axis are to be found on a line parallel to the x-axis and 2 units above it; likewise we found that all points in this line, no matter how far it may be extended, will be +2 units from the x-axis. Another way of expressing these facts is to say that the path of all points the y-value of which is 2, is the line parallel to the x-axis and 2 units above it, and next, that the y-value of every point in the line parallel to the x-axis and 2 units above it is +2. This is stated algebraically by means of the equation $y=2$.



EXERCISES. SET LV. THE EQUATION AS THE STATEMENT OF A LOCUS

564. Where are all points −2 units from the x-axis to be found? (Answer in a complete sentence.)

565. What can you say of all points in the line described in your answer to the last question? (Answer in a complete sentence.)

566. State the law which is obeyed by the line described in exercise 564 by means of an algebraic equation.

567. Where are all points +10 units from the y-axis to be found?

568. What can you say of all points in the line described in your answer to the last question?

569. State this law by means of an algebraic equation.

570. Answer the last three questions, inserting the following words in place of "+10 units from the y-axis":

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* In this work, those for whom it is not review will find Auerbach, An Elementary Course in Graphic Mathematics, Chapter I and Chapter III, pp. 22, 23, and 28–31, helpful.

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(*a*) -15 units from the y-axis. (*b*) $+12$ units from the x-axis.
(*c*) -18 units from the x-axis. (*d*) -7 units from the y-axis.
(*e*) $+13$ units from the y-axis.

571. What is the y-value of every point in the line parallel to the x-axis and $+6$ units from it? -6 units from it?

572. What is the x-value of every point in the line parallel to the y-axis and $+17$ units from it? -17 units from it?

573. What is the path of every point whose y-value is $+18$? -18? $+20$? -3?

574. What is the path of every point whose x-value is 30? -27? $+16$? -8?

575. Make a list of the equations expressing the facts stated, in order, in the last four questions.

576. (*a*) What is the y-value of every point in the x-axis?
(*b*) What is the path of every point whose y-value is this?
(*c*) What, then, is the equation of the x-axis?

577. $x=19$ expresses algebraically what two facts?

578. What is the equation of the y-axis? Why?

579. What is the equation of the parallel to the y-axis through the point $(-5, 7)$?

The arrangement of points that completely fulfills a given geometric condition is called the **locus** *of that condition.* This arrangement usually gives rise to a line or group of lines either straight or curved. For instance, the locus of the condition expressed by the equation $x=7$ is the line drawn parallel to the y-axis at a distance 7 units to the right of it. This is a brief way of saying that (1) all points in this line are 7 units to the right of the y-axis and (2) all points 7 units to the right of the y-axis lie in this line.

Because of the idea of motion involved, another acceptable definition of the word **locus** would be: *The complete path of a point that moves in accordance with some specified geometric condition.* For instance, the complete path of a point that moves so that its distance from the y-axis is -7 is the line 7 units to the left of the y-axis and parallel to it. Hence this line is called the locus of the point which moves so as to remain constantly 7 units to the left of the y-axis.

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EXERCISES. SET LV (concluded)

580. What is the locus of points:

(*a*) 3 units from the x-axis? (*b*) -5 units from the y-axis?

(*c*) -6 units from the x-axis? (*d*) 17 units from the y-axis?

581. What two facts do you imply in the answer to each of the parts in the last question?

582. Give the equation expressing the condition which determined each of the loci in exercise 580.

583. What is the locus of the condition expressed in each of the following equations?

(*a*) $x=15$ (*b*) $y=-9$ (*c*) $x=-12$ (*d*) $y=20$

584. What locus is represented by the equation $x^2=25$?

585. What locus is represented by the equation

(*a*) $x=y$? (*d*) $x-y=10$? (*g*) $2x+3y=20$?

(*b*) $x=-y$? (*e*) $x=-3y$? (*h*) $3x-5y=12$?

(*c*) $x+y=10$? (*f*) $x=3y$? (*i*) $x^2+y^2=25$?

586. Give the equation of the locus of a point:

(*a*) Just as far from the x-axis as from the y-axis.

(*b*) Three times as far from the x-axis as from the y-axis.

(*c*) Three times as far from the y-axis as from the x-axis.

(*d*) Minus five times as far from the y-axis as from the x-axis.

(*e*) Minus seven times as far from the x-axis as from the y-axis.

(*f*) Such that the sum of its distances from the axes is -11.

(*g*) Such that three times its distance from the x-axis increased by 9 times its distance from the y-axis is 26.

(*h*) Such that five times its distance from the x-axis diminished by twice its distance from the y-axis is 7.

(*i*) Such that the sum of the squares of its distances from the axes is 49.

(*j*) Such that four times the square of its distance from the x-axis increased by the square of its distance from the y-axis is 144.

Check the answer to each part of the last two questions by plotting the equation.

587. The theorem of Pythagoras is employed to find the "equation of a circle" about the origin as a center.

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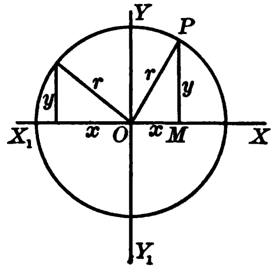
Take any point P in a circle about the origin O. Draw the ordinate MP. Let $OM \equiv x$, and $MP \equiv y$. Then $\overline{OM}^2 + \overline{MP}^2 \equiv \overline{OP}^2$. If the radius $OP \equiv r$, this becomes $x^2+y^2 \equiv r^2$. This equation holds for the coördinates of any point on the circle, and is called the *equation of the circle*, r being any known number.



Form the equation of the circle with the origin as center and (a) 7 as radius, (b) $7\sqrt{2}$ as radius.

B. THE PECULIARITY OF THE PROOF OF A LOCUS PROPOSITION

When we say that the locus of points on this page just one inch from its right edge is a line parallel to that edge, and one inch in from it, we really imply three facts. First, that any point on that line is one inch from that edge; second, that any point which is one inch from that edge and on the page is on that line; and third, that any point not on that line and on the page is not one inch from that edge. If the first of these three facts be called the direct statement, we already know that the second is its converse. The third is known as its **opposite.**

In symbols, three theorems so related might be stated as follows:

Direct theorem.	If $a=b$, then $c=d$.
Converse theorem	If $c=d$, then $a=b$.
Opposite theorem.	If $a \neq b$, then $c \neq d$.

Hence we see that while the converse of a fact simply interchanges its data or what is given with its conclusion, *the opposite of a fact negates both the data and conclusion.*

Now let us discover what we can as to the truth or falsity of converse and opposite theorems when the direct theorem is true.

EXERCISES. SET LVI. DIRECT, CONVERSE, OPPOSITE

The following exercises will help us in this task:

588. Form (1) the converse. and (2) the opposite of each of the following facts:

(*a*) If a man lives in Boston, he lives in Massachusetts.

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(*b*) If it rains, the ground is wet.

(*c*) If two lines meet at right angles they are perpendicular to each other.

(*d*) All vertical angles are equal.

(*e*) The supplements of equal angles are equal, or,
If two angles are equal, their supplements are equal.

(*f*) All men are bipeds.

589. (*a*) Of which of the six facts mentioned in the last exercise are the converse facts true?

(*b*) Of which of them are the opposite facts true?

(*c*) Of which of them are the converse facts false—or at least not necessarily true?

(*d*) Of which of them are the opposite facts either false or not necessarily true?

590. (*a*) Can you draw any conclusion as to the truth or falsity of converse and opposite theorems?

(*b*) Test this conclusion with several more instances.

Though a statement is true:

(1) Its converse may or may not be true.

(2) If its converse is true, its opposite is also true.

(3) If its converse is false, its opposite is also false.

The proof of this fact follows:

Given: That when $a=b$, $c=d$, and when $c=d$, $a=b$.
Prove: That when $a\neq b$, $c\neq d$.
Proof: Suppose $c=d$.
Then what follows? Why?
What conclusion can you draw?

Let us see how these conclusions help us to decide just how much must be proved in order to establish the truth of a locus proposition.

Since the converse of a fact is not necessarily true, in order to prove a line a required locus, not only must we prove (1) that any point in it fulfills the required conditions, but also (2) that any point that fulfills the required conditions is in the line. It is, however, unnecessary to prove the opposite, since if the converse has been proved true, we know the opposite is also true. In short, if we know (1) and (2) are true, we know without further proof that any point not in the line does not fulfill the required conditions.

Suppose we wished to prove

Theorem 40. ***The locus of points equidistant from the ends of a sect is the perpendicular bisector of the sect.***

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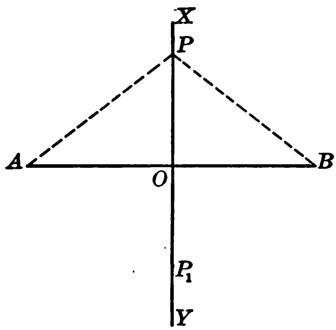
Given: $\overline{XOY} \perp \overline{AOB}$ so that $\overline{AO} \equiv \overline{OB}$.
Prove: $\overline{XY}$ is the locus of points equidistant from A and B.

If we proved first that any point P in $\overline{XY}$ is equidistant from A and B, and second that point P_1 which is equidistant from A and B lies in line $\overline{XY}$, what would we know about any point not in $\overline{XY}$?



∴ (1) Prove: $\overline{PA} \equiv \overline{PB}$, given: $\overline{AO} \equiv \overline{OB}$ and P in $\overline{XY} \perp \overline{AB}$ at O; and (2) prove: $\overline{OP_1} \perp \overline{AB}$, given: $\overline{P_1A} \equiv \overline{P_1B}$ and $\overline{AO} \equiv \overline{OB}$.

To prove (1):

What parts of △ AOP and △ BOP do you know are equal?

Are the △s congruent?

To prove (2):

What parts of △ AOP_1 and △ BOP_1 do you know are equal?

Are the △s congruent?

What fact must you prove to show that $\overline{OP_1} \perp \overline{AB}$?

Is there any other converse which we might have proved in place of the one here proved? Why are there two converses in this case?

Cor. 1. ***Two points each equidistant from the ends of a sect fix its perpendicular bisector.***

How many points determine a straight line?

EXERCISES. SET LVII. APPLICATIONS OF LOCUS

591. Show why a circle may be defined as the locus of points at a fixed distance from a given point.

Describe without proof:

592. The locus of the tip of the hand of a watch.

593. The locus of a point on this page and just 3″ from the upper right corner.

594. The locus of the center of a hoop as it rolls along the floor in a straight line.

595. The locus of the edges of the pages of a book as it is opened.

596. The locus of the handle of a door as it is opened.

597. The locus of the end of a swinging pendulum.

598. The locus of places described as 1 mile from where you are standing.

599. The locus of points 1′ above a given shelf.

d600. The locus of points 1′ from a given shelf.

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601. The locus of the center of a circle as it rolls around another circle, its circumference just touching that of the other circle.

602. The locus of the center of a ball as it rolls around another ball, its surface just touching that of the other ball.

603. The locus of one side of a rectangle as it revolves about the opposite side as axis.

604. The locus of the entire rectangle in the last exercise.

605. The locus of a point at 3″ from a fixed point P.

606. The locus of a point 3″ from a given line.

607. The locus of a point equidistant from two parallel lines.

608. The locus of a point equidistant from two given points.

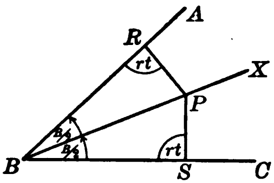
609. The locus of a point equidistant from two intersecting lines.

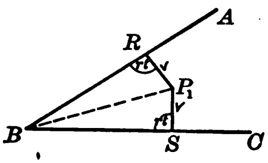
610. The locus of a point the distance $\leftarrow d \rightarrow$ from a given line l.

611. The locus of a point the distance $\leftarrow d \rightarrow$ from a given point P.

612. The locus of a point the same distance from the center and the circumference of the circle c.

We shall now prove one more very important fact concerning loci.





Theorem 41. ***The locus of points equidistant from the sides of an angle is the bisector of the angle.***

I. **Given:** $\angle CBA$, $\overline{BX}$ so that $\angle CBX = \angle XBA$. P any point in BX. $\overline{PR} \perp \overline{AB}$ cutting AB at R. $\overline{PS} \perp \overline{BC}$ cutting BC at S.

To prove: $\overline{PR} = \overline{PS}$.

(Proof left to the student).

II. **Given:** P_1 a point within the $\angle CBA$, so that $\overline{P_1R}$ (the $\perp$ to $\overline{AB}$) $= \overline{P_1S}$ (the $\perp$ to $\overline{BC}$).

To prove: $\overline{P_1B}$ bisects $\angle CBA$.

(Proof left to the student.)

Cor. 1. ***The locus of a point equidistant from two intersecting lines is a pair of lines bisecting the angles.***

Hint: At what angle do the bisectors of any two adjacent angles formed by the pair of intersecting lines meet each other? At what angle, then, do the bisectors of the vertical angles meet each other?

EXERCISES. SET LVII (concluded)

613. If a gardener is told to plant a bush 10′ from the north fence does he know exactly where to plant it? If not, state another direction which might be given him that he may know just where to plant it.

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614. Is the second direction given the gardener the only one you could give him to have the bush definitely located? If not, state other directions which might have answered.

615. How many loci are needed to locate a point on the floor of the room? On any one of its walls? On the ceiling?

616. (*a*) Where would all points that are two feet from the floor of a room lie?

(*b*) Where would all points 3′ from the front wall lie?

(*c*) Where would all points that are both 2′ from the floor and 3′ from the front wall lie?

(*d*) Suppose a point was described as being 2′ from the floor, 3′ from the front wall, and 4′ from a side wall. On how many loci is it? Exactly where is it?

(*e*) How many conditions are needed to fix a point in a room?

617. A man wants to build his home at the same distance from two railroad stations. (*a*) Is the location of his home fixed? (*b*) If at the same time he wishes to build a half mile from the bank of a river which runs parallel to and four miles from the road connecting the stations, is the location of his home fixed? Make an accurate construction showing how many locations answer the description.

618. Prove theorems 40 and 41 by means of direct and opposite.

619. What is the locus of the vertices of triangles which have a common base and equal areas?

620. What is the locus of points dividing sects which connect a given point and a given line in the ratio of 5 to 7?

621. What is the locus of the vertices of triangles resting on a common base and having fixed areas in the ratio of 5 to 7?

LIST OF WORDS DEFINED IN CHAPTER VII

Locus, opposite.

SUMMARY OF THEOREMS PROVED IN CHAPTER VII

40. The locus of points equidistant from the ends of a sect is the perpendicular bisector of the sect.

Cor. 1. Two points each equidistant from the ends of a sect fix its perpendicular bisector.

41. The locus of points equidistant from the sides of an angle is the bisector of the angle.

Cor. 1. The locus of points equidistant from two intersecting lines is a pair of lines bisecting the angles.

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CHAPTER VIII

THE CIRCLE

We have not only used the word "circle" very freely throughout the text, but have also used our compasses for the construction of the circle, or any part of it, whenever necessity arose. This has been due to the fact that the idea of a circle seems to be one with which all of us have grown up. But we have now reached the time to consider the idea scientifically, and add to our stock of facts concerning it.

Up to the present we have thought of the circle as a portion of a plane, and the curve bounding it as its circumference. This is not the sense in which the word circle is used as we advance in mathematics, so we shall have to revise our notion of it. The word circle is used to refer to both the portion of a plane and the curve which bounds it—the one to which it refers being determined by the context, but the definition covers only the boundary. When there is any danger of ambiguity the word circumference will be used in this text.

A **circle** *is a plane curve which contains all points at a given distance from a fixed point in the plane, and no other points.** Thus we see that a locus definition may be given as a corollary to this one; namely, *a circle is the locus of points at a given distance from a fixed point.** *The fixed point is called the* **center,** and *the given distance* (the distance from the center to any point on the circle) *is called its* **radius** (plural, radii). *The sect through the center and terminated by the circle is called its* **diameter.**

At this point we may state several corollaries to these definitions. It will be left to the student to verify them.

Cor. 1. ***All radii of equal circles are equal.***

Cor. 2. ***Circles of equal radii are equal.***

Cor. 3. ***All diameters of equal circles are equal.***

---

* Refer to Exs. 585 (*i*), 586 (*i*), 588, 591, and 611 for previous illustrations of this definition.

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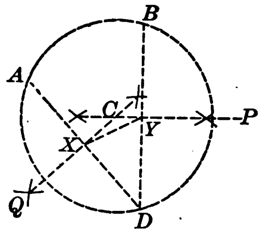
Cor. 4. ***A point is inside, on, or outside a circle, according as its distance from the center is less than, equal to, or greater than the radius.***

Cor. 5. ***A point is at a distance less than, equal to, or greater than the radius from the center according as it is inside, on, or outside the circle.***

A. PRELIMINARY THEOREMS

Theorem 42. ***Three points not in a straight line fix a circle.***



Given: Points A, B, D, not in a straight line.

To prove: (1) A circle can be passed through A, B, D.

(2) Only one circle can be passed through A, B, D.

Suggestions for proof: What is the locus of the centers of all circles passing through A and D? Why? Through B and D?

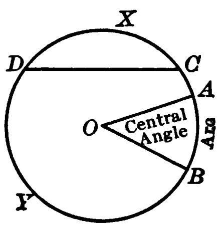
By means of the transversal XY in the diagram, show that $\overline{PY}$ and $\overline{QX}$ are not parallel, and that hence point C exists

Why can no second point such as C_1 exist?

EXERCISE. SET LVIII. THE CIRCLE AS A LOCUS

622. An amusement park is to be located at the same distance from each of three villages V_1, V_2, and V_3. V_2 is 5 miles from V_1, V_3 is 6 miles from V_1, and V_3 is 8 miles from V_2. Show by an accurate construction the actual location of the park. Can you think of any locations of V_1, V_2, and at the same time V_3 which would make it impossible to have a park so located? (Use sect $V_2 V_3$ to represent 8 mi.) V_2——————————V_3

In the accompanying diagram, $\measuredangle BOA$ is known as a **central angle.** Define such an angle.



Any portion of a circle, such as BA is known as an **arc.** When referring to any definite arc, such as BA or CD, we write it thus: $\overset{\frown}{BA}$, $\overset{\frown}{CD}$. Any two points on a circle (unless they are the ends of a diameter) are the ends of two arcs known as **minor** and **major** arcs. For instance, $\overset{\frown}{CXD}$ is the minor $\overset{\frown}{CD}$, and $\overset{\frown}{DYC}$ is the major $\overset{\frown}{DC}$. *The shorter of two arcs cut off by any two points on*

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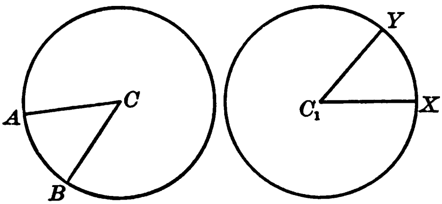
a circle is known as the **minor** *arc, and the longer is known as the* **major.** When not otherwise stated, the minor arc is referred to.

The sect CD is known as a **chord.** Define the word chord. The diameters of a circle are simply chords. Why?

Angles in a circle are said to **intercept** arcs, and chords are said to subtend arcs. Intercept comes from the Latin "inter," meaning "between," and "capio," meaning "to take," hence the angle intercepts or "takes the arc between its sides." Subtend comes from the Latin "sub," meaning "under," and "tendere," meaning "to stretch." Thus, as we see, the chord is the straight line which "stretches under the arc." Arcs are considered passive, and are referred to as being "intercepted by" angles, and "subtended by" chords, although in engineering the expression "the central angle subtended by an arc of $n°$" is very common. If the expression is used in this text it will only be when referring to engineering problems.

Theorem 43. ***In equal circles equal central angles intercept equal arcs, and conversely.***



I. Direct.

Given: $\odot C = \odot C_1$, $\angle ACB = \angle XC_1Y$.

Prove: $\overset{\frown}{AB} = \overset{\frown}{XY}$.

II. Converse.

Given: $\odot C = \odot C_1$, $\overset{\frown}{AB} = \overset{\frown}{XY}$.

Prove: $\angle ACB = \angle XC_1Y$.

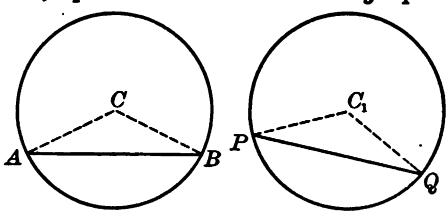
Method of proof, superposition. In superposing, which parts will you make coincide in proving the direct? Which in proving the converse?

B. THE STRAIGHT LINE AND THE CIRCLE

Theorem 44. ***In equal circles, equal arcs are subtended by equal chords, and conversely.***

Suggestion: Prove $\triangle ACB \cong \triangle PC_1Q$.

What means have you of proving the triangles congruent in the direct? What in the converse?



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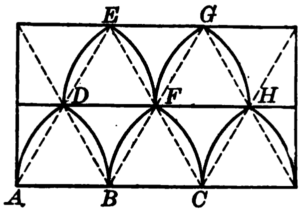
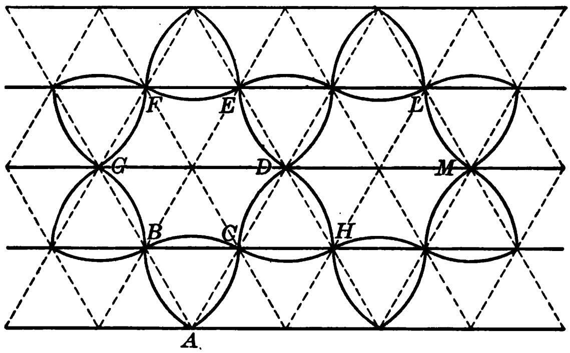
EXERCISES. SET LIX. CONGRUENCE OF CURVILINEAR FIGURES

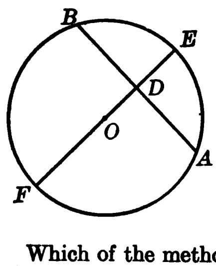
623. Prove that the curved figures *BDEF* and *CFGH* are congruent. The figure is based on a network of equilateral triangles. The vertices are the centers, and the sides the radii for the arcs.



624. In the accompanying figure prove that the curved triangles *ABC*, *CHD*, etc., are congruent. Also *BCDEFG* and *DHML*.



Theorem 45. ***A diameter perpendicular to a chord bisects it and its subtended arcs.***



Given: $\odot O$, diameter $EF \perp$ chord AB at D.
Prove: $\overline{AD} = \overline{DB}$. $\widehat{AE} = \widehat{EB}$. $\widehat{BF} = \widehat{FA}$.
Proof: $\widehat{AE} = \widehat{EB}$ if what angles are equal?
These angles are equal if what triangles are congruent? Write a complete proof.
By means of what angles can you prove $\widehat{BF} = \widehat{FA}$?

Cor. 1. ***A radius which bisects a chord is perpendicular to it.***

Which of the methods of proving lines perpendicular can be applied here?

Cor. 2. ***The perpendicular bisector of a chord passes through the center of the circle.***

Hints: Draw the radius which bisects the chord and prove the given bisector coincident with it, or treat as a locus.

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EXERCISES. SET LX. CONSTRUCTIONS BASED UPON CIRCLES

625. Give the construction for Fig. 1.

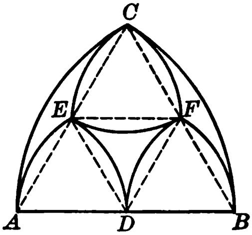


FIG. 1.

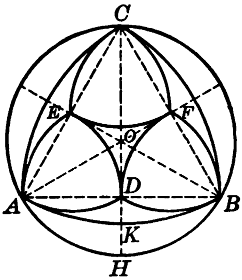


FIG. 2.

From Carlisle Cathedral.
(Figs. 1 and 2 applied.)

626. Inscribe Fig. 1 in a given circle. (See Fig. 2.)

627. Give the construction for the design shown in Fig. 3.

Suggestion: Construct the equilateral △ABC. Divide each side into three equal parts and join the points as indicated. The intersections are the centers, and AM is the radius for the arcs drawn as indicated.

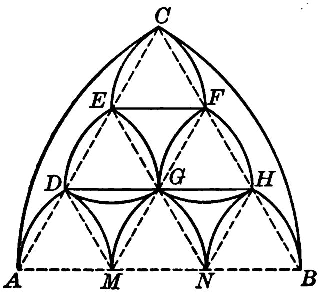
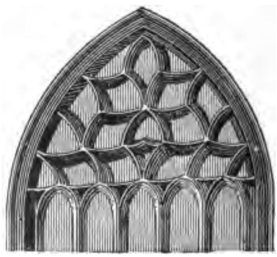


FIG. 3.



From Exeter Cathedral.
(Fig. 3 applied.)

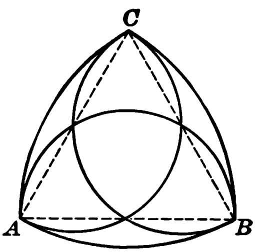


FIG. 4.

628. Construct Fig. IV.

Suggestion: Construct the equilateral $\triangle ABC$. A, B, and C are the centers for $\overset{\frown}{CB}$, $\overset{\frown}{CA}$, and $\overset{\frown}{AB}$. The semicircles are constructed on the sides of the linear triangle as diameters.

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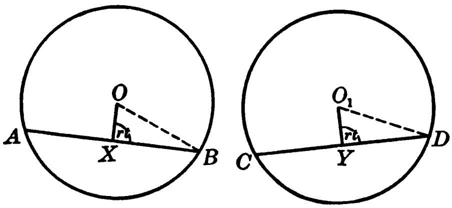
629. A civil engineer wishes to continue the circular track AB for some distance. Suggest how he can do it.

630. From the measurements of a piece of broken wheel a new wheel is to be cast of the same size. Show how to find the radius of the new wheel.

Theorem 46. ***In equal circles, equal chords are equidistant from the center, and conversely.***

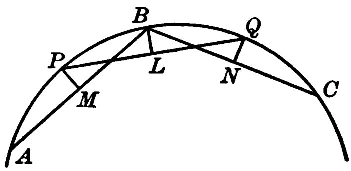


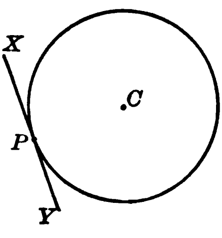
I. In the direct what parts are known to be equal in $\triangle OBX$ and $\triangle O_1DY \equiv$?

II. State and prove the converse.

EXERCISE. SET LXI. EQUAL CHORDS

631. The following method of locating points on an arc of a circle that is too large to be described by a tape is used by engineers. If part of the curve APB is known, take P as the mid-point. Then stretch the tape from A to B and draw PM perpendicular to it. Then swing the length AM about P, and PM about B, until they meet at L, and stretch the length AB along PL to Q. This fixes the point Q. In the same way fix the point C. Points on a curve can thus be fixed as near together as we wish. Why is this method correct?





A straight line is said to be **tangent** *to a circle when it touches it once, and only once.* Thus $\overline{XY}$ is tangent to $\odot C$ if it touches $\odot C$ at pt. P, and nowhere else.

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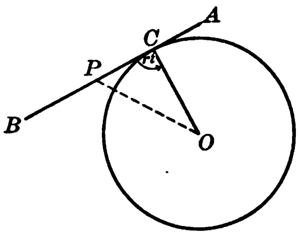


Theorem 47. ***A line perpendicular to a radius at its outer extremity is tangent to the circle.***

Given: $\odot O$, $\overline{AB} \perp$ radius OC at C.

Prove: AB tangent to $\odot O$.

PROOF

(1) $\overline{AB} \perp$ radius at C.	(1) Given.
(2) $\therefore \overline{AB}$ touches $\odot O$ at C.	(2) Authorities left to the student to insert.
(3) It remains to prove that no other point such as P in $\overline{AB}$ touches $\odot O$. $\therefore$ Draw sect OP.	
(4) $\overline{OC} \perp \overline{AB}$.	(4)
(5) $\therefore \overline{OP} \not\perp \overline{AB}$.	(5)
(6) $\therefore \overline{OP} > \overline{OC}$.	(6)
(7) $\therefore P$ lies outside of $\odot O$.	(7)
(8) $\therefore C$ and only C in $\overline{AB}$ is in $\odot O$.	(8)
(9) $\therefore \overline{AB}$ is tangent to $\odot O$.	(9)

The first three corollaries following are partial converses of this theorem.

We are not yet ready to prove the converse of Theorem 6. It will be proved later under the topic of inequalities. For the present, then, we shall add it as a postulate.

Postulate, the third postulate of perpendiculars:

The shortest distance from a point to a line is the perpendicular to that line.

Cor. 1. ***A tangent to a circle is perpendicular to the radius drawn to the point of contact.***

Given: $\overline{XY}$ tangent to $\odot O$ at P, radius $\overline{OP}$.

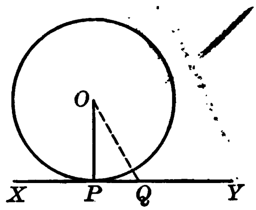
Prove: $\overline{XY} \perp \overline{OP}$.

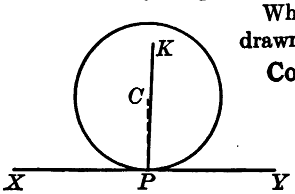
Suggestions for proof: Draw OQ to any point Q in $\overline{XY}$.

Where does Q lie?

How does $\overline{OQ}$ compare with $\overline{OP}$?

What conclusion can be drawn with respect to $\overline{OP}$?





Cor. 2. ***The perpendicular to a tangent at the point of contact passes through the center of the circle.***

Hint: Show that $\overline{KP}$ coincides with the radius CP, and $\therefore C$ lies on $\overline{KP}$.

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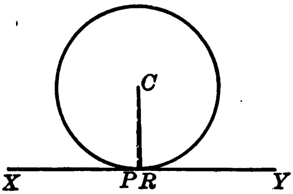
Cor. 3. ***A radius perpendicular to a tangent passes through the point of contact.***



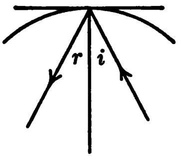
Hint: Draw the radius to the point of contact P, and show that CR coincides with CP.

Cor. 4. ***Only one tangent can be drawn to a circle at a given point on it.***

Hint: How many $\perp s$ can be erected to $\overline{CR}$ at R?

EXERCISES. SET LXII. THE TANGENT AND THE CIRCLE

632. When a ray of light strikes a spherical mirror (represented in cross section by the arc of a circle), the angle of incidence is found by drawing a tangent to the circle at the point of incidence, and erecting a perpendicular to the tangent at that point. In this case the perpendicular (called the *normal*) is a diameter. Why?



d633. The line of propagation of a sound wave also follows the law of reflection of a ray of light, namely, that the angle of incidence is equal to the angle of reflection.

The circular gallery in the dome of St. Paul's in London is known as a whispering gallery, for the reason that a faint sound produced at a point near the wall can be heard around the gallery near the wall, but not elsewhere. The sound is reflected along the circular wall in a series of equal chords. Explain why these chords are equal.

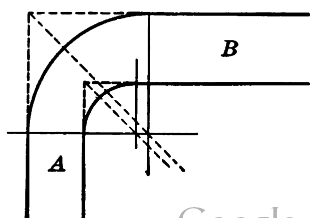
634. What is the locus of the centers of a number of hoops of different sizes (one inside the other) tied together at one point?

635. What is the locus of the centers of all circles tangent to a given line at a given point?

636. What is the locus of the center of a wheel as it rolls straight ahead along level ground? Prove this fact.

637. What is the locus of the centers of all circles tangent to both sides of an angle?

638. Two straight roads of different width meet at right angles. A is the narrower, B the wider. It is desired to join them by a road the sides of which are arcs of circles tangent to the sides of the straight roads. What construction lines are necessary? Draw a figure.



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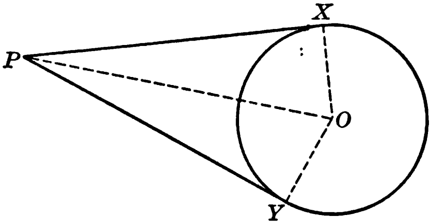
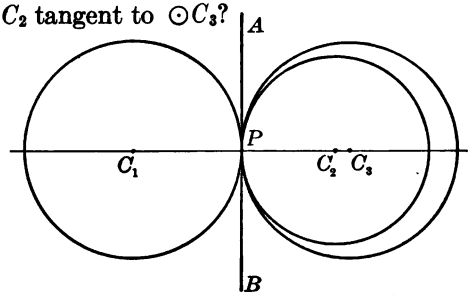
639. How would you construct a tangent to a given circle at a given point P on the circle?



Theorem 48. ***Sects of tangents from the same point to a circle are equal.***

Why are $\triangle$ OPX and OPY congruent?

Circles *are said to be* **tangent** *to each other when they are tangent to the same line at the same point.* Why is $\odot C_1$ tangent to $\odot C_2$? Why to $\odot C_3$? Why is $\odot C_2$ tangent to $\odot C_3$?



Circles are **externally** *tangent when their centers lie on opposite sides of the common tangent.* Name two pairs of circles that are externally tangent in the diagram.

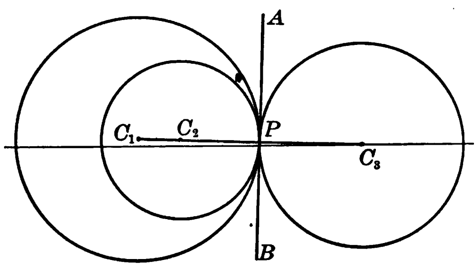
Circles are **internally** *tangent when their centers lie on the same side of the common tangent.* Which circles in the diagram are internally tangent?

EXERCISES. SET LXIII. TANGENT CIRCLES

640. (*a*) How are the hoops mentioned in Ex. 634 related to one another?

(*b*) How is the line in which their centers lie related to their common tangent? Why?

(*c*) Is this fact true of the centers of two cog-wheels when they mesh? Why?



The **line of centers** *of two circles is the sect connecting their centers.*

Theorem 49. ***The line of centers of two tangent circles passes through their point of contact.***

(For suggestions see Ex. 634, p. 167, and Ex. 640, p. 168.)

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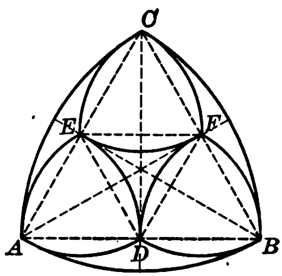
The **common chord** *of two intersecting circles is the sect connecting their points of intersection.*

EXERCISES. SET LXIII (concluded)

641. How is the line of centers of two intersecting circles related to their common chord? Prove your answer.

642. If two cog-wheels mesh, show that the point where they mesh is in a straight line with the centers of the wheels.

643. (*a*) Show how to construct an equilateral Gothic arch. (See the accompanying diagram.)



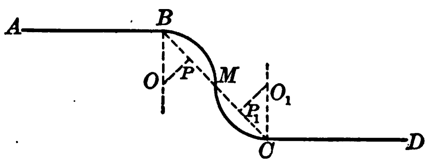
Suggestion: Construct the equilateral triangle ABC. With A and B as centers and AB as radius, construct the arcs BC and AC.

(*b*) If E, F, and D are the mid-points of the lines AC, CB, and AB, respectively, prove that equal equilateral triangles are formed.

(*c*) Construct the equilateral arches ADE and DBF and the curved triangle EFC.

Suggestion: Points D, E, C, F, B, and A are the centers and AD is the radius for the arcs drawn as indicated.

d644. An application of geometry to engineering is seen in cases where two parallel streets or lines of track are to be connected by a "reversed curve." If the lines are AB and CD, and the connection is to be made from B to C, as shown, we may proceed as follows: Draw BC and bisect it at M. Erect PO the perpendicular bisector of BM, and BO perpendicular to AB. Then O is one center of curvature. In the same way fix O_1. The curves may now be drawn, and they will be tangent to AB, to CD, and to each other. Prove that the curve BMC is a reversed curve tangent to $\overline{AB}$ and $\overline{CD}$; *i.e.*, prove (*a*) $\widehat{BM}$ tangent to $\overline{AB}$ at B, $\widehat{CM}$ tangent to $\overline{CD}$ at C; (*b*) $\widehat{BM}$ tangent to $\widehat{CM}$ at M; (*c*) $\widehat{BM} = \widehat{CM}$.



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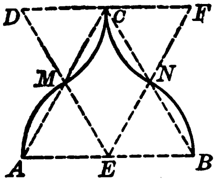
645. The figure represents a *Persian arch.* Triangles ABC and DEF are congruent and equilateral. The centers of the upper arcs MC and NC are respectively D and F; while the lower arcs are drawn with the center E. Prove that the area of the arch equals the area of triangle ABC.



646. The trefoil $ADBF$, etc., is constructed from circles described on the semisides of $\triangle ABC$. The points D, E, and F are the centers for the arcs which are tangent to the sides of $\triangle ABC$, and which form the trefoil $HYKZGX$. If PD and RF in Fig. 1 are radii for the arcs ZK and YK, prove that $PD = FR$.

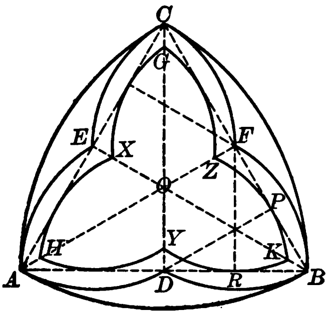


FIG. 1.

FIG. 1. applied.

d647. The semicircle $AGHB$ in Fig. 2 is constructed on AB as diameter, and CD is perpendicular to AB at its mid-point.

(*a*) Construct arcs CH and CG tangent to the line CD at point C, and to the semicircle.

Suggestion: Make $KC = AD$. Draw NE, the perpendicular bisector of KD, meeting CK extended at E. E is the center for the arc CH. What general problem in construction of circles is involved here?

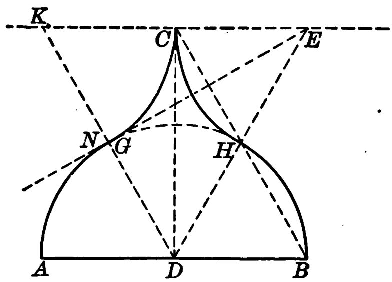


FIG. 2. Three-centered ogee arches.

(*b*) If the arcs CH, drawn with E as a center, and HB drawn with D as a center, are tangent at H, prove that the points D, H, and E are collinear.

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(*c*) Prove that C, H, and B are collinear.

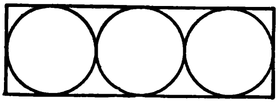
Suggestion: Join C and H, and B and C, and prove that each is parallel to DK.

(*d*) If $AB=8$, and $CD=8$, find the length of CE.

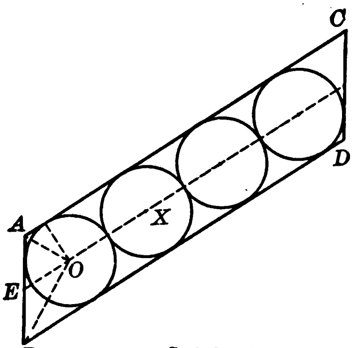
Suggestion: $KD=4\sqrt{5}$. Compare the sides of the similar triangles KCD and KNE. Find KE and hence CE.

(*e*) If $AB=s$ and $CD=h$, find the length of CK.

648. What must be the relation between the length and width of the rectangle $ABCD$ in order that the tangent circles may be inscribed as shown.



649. $ABCD$ is a parallelogram with four tangent circles inscribed.



Stair Railing.

(*a*) If the lines AC and BD are supposed to be indefinite in extent, show how to construct circle O tangent to the lines AC, AB, and BD, and circle X tangent to lines AC and BD and to circle O.

(*b*) If E is the mid-point of AB, and O and X are the centers of the circles, prove that the points E, O, and X are collinear.

650. Fig. 1 shows a trefoil formed of the three circles X, Y, and Z tangent to each other at the points T, S, and R. It is inscribed in the circle as shown.

(*a*) Show how to construct the figure.

Solution: Circumscribe an equilateral triangle about the circle. Connect each vertex with the center. Inscribe a circle in each of the triangles *FOG*, *GOE*, and *EFO*.

(*b*) Prove that the small circles are tangent to the large circle and to each other.

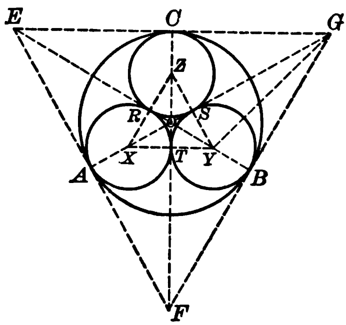


Fig. 1. Trefoil formed of tangent circles.

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651. (*a*) Construct the quadrifoil of tangent circles inscribed in a square so that each circle is tangent to two sides of the square and to two other circles.

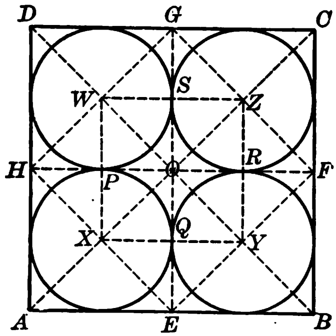


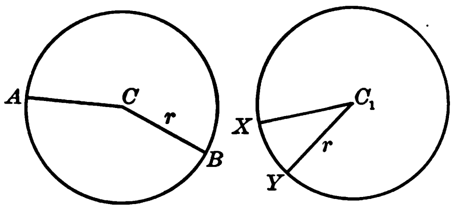
FIG. 2.


Inlaid tile design.
(FIG. 2 applied.)

(*b*) Prove that the lines joining the centers of the circles form a square.

C. THE ANGLE AND ITS MEASUREMENT

√ **Theorem 50.** ***In equal circles central angles have the same ratio as their intercepted arcs.***



Given: $\odot C \equiv \odot C_1$, $\measuredangle ACB$ and $\measuredangle XC_1Y$, $\widehat{AB}$ and $\widehat{XY}$ commensurable.

Prove: $\dfrac{\measuredangle ACB}{\measuredangle XC_1Y} = \dfrac{\widehat{AB}}{\widehat{XY}}$.

Suggestions for proof: Select a unit of measure for $\widehat{AB}$ and $\widehat{XY}$.

Divide these arcs into such units and connect the points of division with the centers C and C_1.

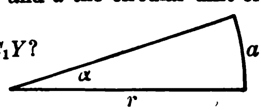
What can you say of all the central angles thus formed?

How do $\measuredangle s$ BCA and YC_1X compare?

How does $\widehat{BA}$ compare with $\widehat{YX}$?

Cor. 1. ***A central angle is measured by its intercepted arc.***

Suppose $\measuredangle \alpha$ were the angular unit of measure, and a the circular unit of measure in a circle of radius r.



What would be the numerical measure of $\measuredangle XC_1Y$?

What would be the numerical measure of $\widehat{XY}$?

How do these numerical measures compare?

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Up to the present time we have emphasized the fact that a magnitude can be measured by a unit of the same kind only. We must then justify a statement such as that given in Cor. 1, Theorem 50. This corollary should be stated as follows: The ratio of a central angle of a circle to the angular unit is equal to the ratio of its intercepted arc to the circular unit; or: The numerical measure of a central angle of a circle is equal to the numerical measure of its intercepted arc. But since this fact is one frequently referred to, and the correct statement of it is so lengthy, mathematicians have agreed to the abbreviated statement given in Cor. 1. The symbol "$\underline{\propto}$" is used for "is measured by." It suggests the ideas of both equality and variation.

A **secant** *is a straight line which intersects the circle.*

EXERCISES. SET LXIV. SECANT AND CIRCLE

g652. (*a*) Show by a graphic solution of the equations $x^2+y^2=49$, and $x=3$, that a secant cuts a circle in two points.

(*b*) What numbers would have to replace 3 in the second equation to change the equation to one of a tangent?

Theorem 51. ***Parallels intercept equal arcs on a circle.***

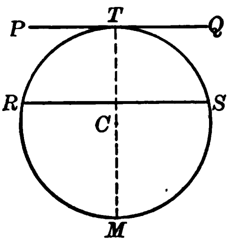

Fig. 1.

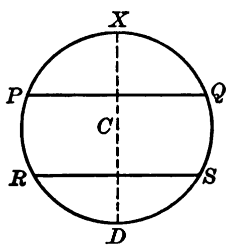

Fig. 2.

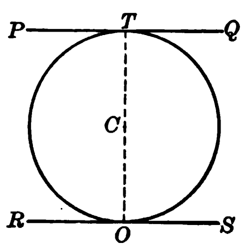

Fig. 3.

Case I. When the parallels are a secant and a tangent. (Fig. 1.)

Given: *PQ* tangent to circle *C* at *T*, secant *RS* || *PQ*, cutting *C* at *R* and *S*.

Prove: $\widehat{ST} \equiv TR$.

Suggestions for proof: Draw diameter through *T*, cutting ⊙*C* in *M*.

What relation exists between $\overline{TCM}$ and $\overline{PQ}$?

Then what relation exists between $\overline{TCM}$ and $\overline{RS}$?

What follows as to $\widehat{ST}$ and $\widehat{TR}$?

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Case II. When both parallels are secants. (Fig. 2.)

(Proof left to the student.) **Suggestion:** Draw the diameter perpendicular to one secant.

Case III. When both parallels are tangents. (Fig. 3.)

Draw the diameter through T, and give the proof in full.

EXERCISES. SET LXV. CIRCLES

653. Does it make any difference in what order the cases under Theorem 51 are proved, if the proofs are given as suggested?

654. Can you suggest methods of proof for Cases II and III which depend upon Case I? Which methods do you prefer? Why?

655. Construct a diagram consisting of:

(a) Two concentric circles whose radii are in the ratio of 1 to 3.

(b) Six circles lying between them, tangent to them, and each tangent to two others.

656. Make a diagram of the mariner's compass, putting in sixteen points of the compass.

Angles such as ABC in Figs. 1, 2, 3, Theorem 52, are called **inscribed angles.** *Their vertices are not only in the circle, but their sides are chords. An angle is said to be inscribed in a semicircle when its sides intercept a semicircle, in less than a semicircle when its sides intercept a major arc, and in more than a semicircle when its sides intercept a minor arc.*

Since measurement is but a numeric relation, axioms may be converted into authorities for statements concerning measurement by simple changes such as those illustrated by the following:

The measure of the sum of two magnitudes of the same kind is equal to the sum of their measures. *E.g.:* $\left.\begin{array}{r}\text{If } x \stackrel{\frown}{=} a\\ \text{and } y \stackrel{\frown}{=} b\end{array}\right\} x+y \stackrel{\frown}{=} a+b.$

The measure of any multiple of a magnitude is equal to that multiple of its measure. *E.g.:* If $x \stackrel{\frown}{=} a$, then $mx \stackrel{\frown}{=} ma$.

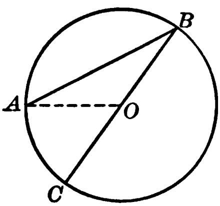

FIG. 1.

Theorem 52. ***An inscribed angle or one formed by a tangent and a chord is measured by one-half its intercepted arc.***

Given: $\odot O$, $\measuredangle ABC$ so that B is on the circle.

To prove: $\measuredangle ABC \stackrel{\frown}{=} \frac{1}{2}$ its intercepted arc.

Suggestions: Fig. 1. Compare $\measuredangle ABC$ with $\measuredangle AOC$. What is the measure of $\measuredangle ABC$?

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Figs. 2 and 3. By drawing the diameter through B, reduce these cases to that of Fig. I.

Fig. 4. Draw $\overline{ZAY} \parallel DC$.

What is the measure of $\measuredangle YAB$?

What arc may be substituted for $\widehat{YB}$? Why?

What, then, is the measure of $\measuredangle DBA$? Of its supplement $\measuredangle ABC$?

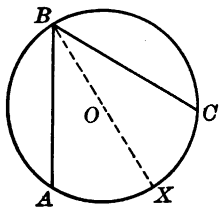


FIG. 2.

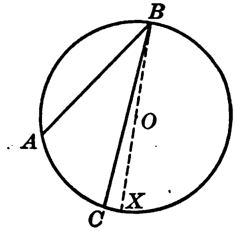


FIG. 3.

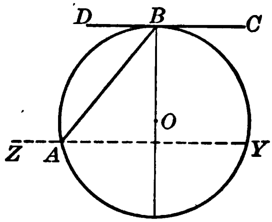
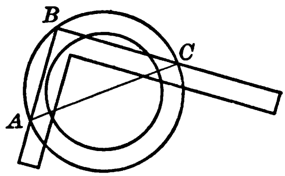


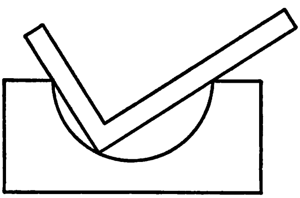
FIG. 4.

EXERCISES. SET LXVI. INSCRIBED ANGLES

657. What kind of angle is inscribed in a semicircle? In less than a semicircle? In more than a semicircle?

658. In carpentry, circular pieces of molding for door panels, etc., are sometimes turned out in the form of rings on a lathe; then these are cut into pieces according to the places in which they are to be used. To cut such a ring into two equal parts, place a carpenter's square upon it with the heel at the edge of the ring, and mark the points A and C where the arms of the square cross the edge of the ring. Show that ABC is half of the ring.





659. Pattern-makers and others use the carpenter's square as follows to determine if the "half-round hole" is a true semicircle: The square is placed as in the figure. If the heel of the square touches the bottom of the hole in all positions of the square, while the sides rest against the edges of the hole, the hole is a true semicircle. Justify this test.

660. Prove that the semicircles intersect in pairs on the sides of the triangle in Ex. 628.

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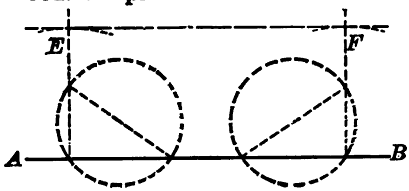
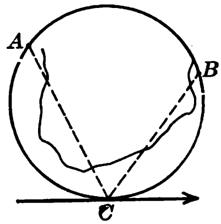
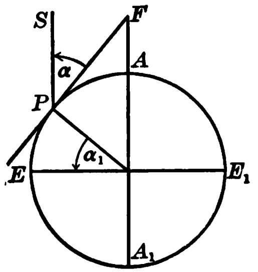
661. In practical work a line EF is sometimes drawn parallel to a given line AB, as shown in the figure. Explain the construction, and prove that $EF \parallel AB$.



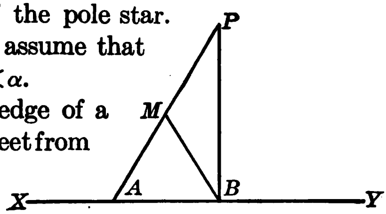
662. A ship is steered past a known region of danger as follows: A chart is made in which a circle is drawn through two points A and B, which can be seen from the ship, and with sufficient radius to inclose the danger region. The inscribed angle ACB is measured. Observations of A and B are made from the ship from time to time, and the course of the ship directed so that the angle between the directions to A and B never becomes greater than $\measuredangle ACB$. Justify this method.



663. Show by a diagram how you can use a protractor with a plumb-line attached to determine the horizontal line AC while sighting the top B of a building.



664. The diagram shows how the latitude of a place may be determined by observation of the pole star. Let EE_1 represent the equator, AA_1 the axis of the earth, P the place whose latitude is to be found, PF a plane (the horizon) tangent to the earth at P, and PS the line of observation of the pole star. Then $\measuredangle \alpha$ represents the latitude, and $\measuredangle \alpha$ is called the altitude of the pole star. For practical purposes we may assume that $AA_1 \parallel PS$. Prove that $\measuredangle \alpha_1 = \measuredangle \alpha$.



665. Suppose XY to be the edge of a sidewalk, and P a point in the street from which we wish to lay a gas pipe perpendicular to the walk. From P swing a cord or tape, say 50 feet long, until it meets XY at A. Then take M, the mid-point of PA, and swing MP about M, to meet XY at B. Then B is the foot of the perpendicular. Verify this.

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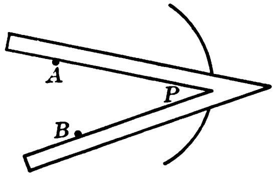
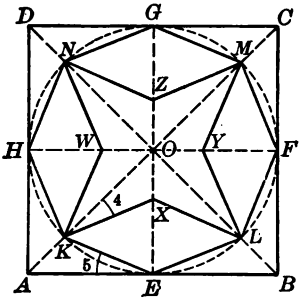
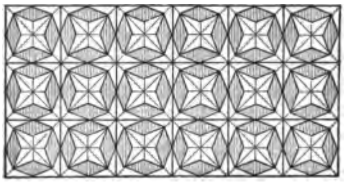
666. By driving two nails into a board at A and B, and taking an angle P made of rigid material, a pencil placed at P will generate an arc of a circle if the arms slide along A and B. Why? Try the experiment.



d667. Circle O is tangent to the sides of square $ABCD$ at points E, F, G, and H, and cuts the diagonals at points K, L, M, and N. Point X on OE is so chosen that $KX \equiv KE$. $OX \equiv OY \equiv OZ \equiv OW$, and the points are joined as indicated.



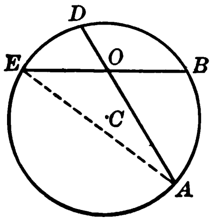


Parquet flooring. Arabic and Roman.

(*a*) Prove that $KELFM$, etc., is a regular octagon, and that $KELX$, $LYMF$, and $MGNZ$ are congruent rhombuses.

(*b*) Prove that $XO \equiv AK$.

Suggestion: Compare $\triangle s\ KXO$ and AEK. $\angle 4 \equiv \angle 5 \equiv \frac{1}{4}$ rt. $\angle$.



Theorem 53. ***An angle whose vertex is inside the circle is measured by half the sum of the arcs intercepted by it and by its vertical.***

Given: $\odot C$, chords AD and BE intersecting at O.

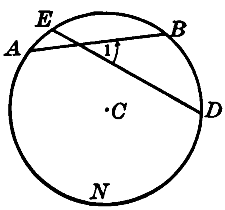
Prove: $\angle AOB \underline{\propto} \dfrac{\widehat{AB}+\widehat{DE}}{2}$.

Hint: How is $\angle AOB$ related to $\angle s\ A$ and E?

EXERCISES. SET LXVII. MEASUREMENT OF ANGLES

668. Fill out the blank spaces in the table:

$\angle 1$	$\widehat{AE}$	$\widehat{BD}$	$\widehat{EB}$	$\widehat{AND}$
35°	40°		80°	
48°	50°			216°
40°		50°	60°	
60°		54°		190°
		45°	90°	180°
	34°		108°	164°



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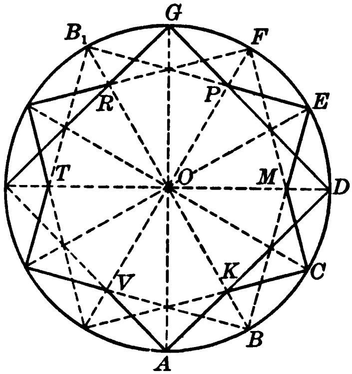


d669. (*a*) Prove that (1) $\overline{AB} \equiv \overline{AK} \equiv \overline{KC} \equiv \overline{CM}$, etc. (2) $\sphericalangle KCM \equiv \sphericalangle MEP$, etc. (3) $\sphericalangle AKC \equiv \sphericalangle CME$, etc.

(*b*) Find the number of degrees in each angle mentioned in (*a*).

Theorem 54. ***An angle whose vertex is outside the circle is measured by half the difference of its intercepted arcs.***

Given: $\sphericalangle ABC$ with vertex outside $\odot O$, intercepting in Fig. 1 $\overset{\frown}{AC}$ and $\overset{\frown}{ED}$; in Fig. 2 $\overset{\frown}{AC}$ and $\overset{\frown}{EA}$; in Fig. 3 $\overset{\frown}{AC}$ and $\overset{\frown}{CA}$.

Prove: $\sphericalangle ABC \underline{\propto} \frac{\overset{\frown}{AC} - \overset{\frown}{ED}}{2}$ in Fig. 1, $\frac{\overset{\frown}{AC} - \overset{\frown}{EA}}{2}$ in Fig. 2, and $\frac{\overset{\frown}{ARC} - \overset{\frown}{CA}}{2}$ in Fig. 3.

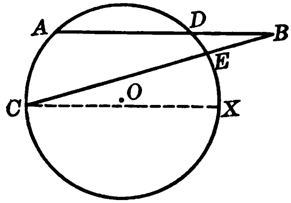


FIG. 1.

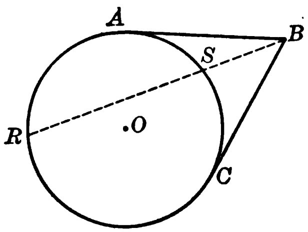


FIG. 3.

Hint: In Figs. 1 and 2 draw $\overline{CX} \| \overline{AB}$, and reduce case 3 to case 2 by drawing any secant *BSR*.

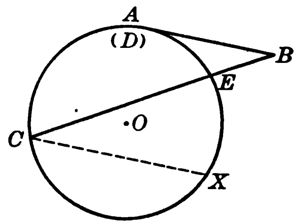


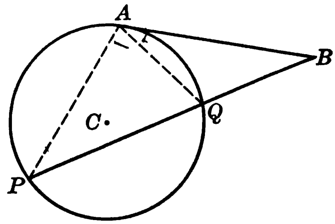
FIG. 2.

Theorem 55. ***A tangent* is the mean proportional between any secant and its external sect, when drawn from the same point.***

Given: $\odot C$, *AB* tangent at *A* and secant *BP* cutting $\odot C$ at *Q* and *P*.

Prove: $\frac{\overline{BP}}{\overline{AB}} \equiv \frac{\overline{AB}}{\overline{BQ}}$

Suggestion: Prove $\triangle ABQ \sim \triangle PBA$.



* By "tangent" in such cases is meant the sect from the point to the point of tangency.

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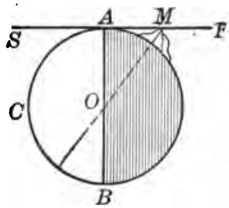
EXERCISES. SET LXVIII. TANGENT AND SECANT

670. (*a*) Assuming that the diameter of the earth is 8000 mi., how far can a man see from the top of a building 200 ft. high? (The height of the building is measured along the prolongation of the diameter.)

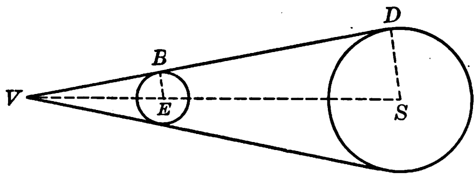
(*b*) How far can one see from the top of a mountain 1000 ft. high? 2000 ft. high? 3000 ft. high? 4000 ft. high?

(*c*) How far can one see from a balloon 1500 ft. above the sea?

d671. Galileo (1564–1642) measured the heights of the mountains on the moon, some of which are as much as 6 mi. high, as follows: ACB was the illuminated half of the moon just as the peak of the mountain M caught the beam SM of the rising or setting sun. He measured the distance AM from the half-moon's straight edge AB to the mountain peak M. Then by using the known diameter of the moon, show how he was able to compute the height of the mountain.



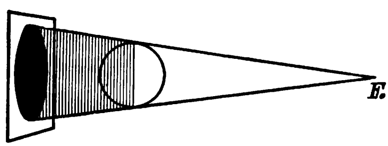
d672. Since the earth is smaller than the sun, it casts a conical shadow in space (umbra), from within which one can see no portion of the sun's disk. If S is the center of the sun, E the center of the earth, and V the end or vertex of the shadow, prove that the length of the shadow, $VE = \dfrac{ES \times EB}{SD - EB}$. Approximately, $ES = 92{,}900{,}000$ miles, $SD = 433{,}000$ miles, and $EB = 4{,}000$ miles. Compute VE.



673. Look up the terms "umbra" and "penumbra," and answer the following questions:

(*a*) How are the umbra and penumbra affected by a change in the distance apart of the luminous and opaque bodies?

(*b*) If a golf ball is held between the eye and the sun, is there any penumbra? Explain.



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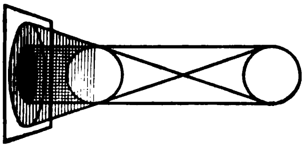
(*c*) What may be said of the umbra if the luminous body and the opaque body are of the same size?

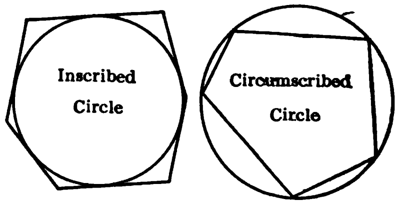


(*d*) What is the shape of the umbra if the luminous body is larger than the opaque body, as in the case of the sun and the earth?

(Exs. 672–673 are taken from Betz and Webb, Plane Geometry.)

D. MENSURATION OF THE CIRCLE

We now come to a very important section of our plane geometry —the section which develops the formula for *the length of the circle* and for *the area enclosed by the circle.* The latter is known as the **area of the circle.**



A polygon is said to be **circumscribed** *about a* **circle** *when each of its sides is tangent to the circle, and to be* **inscribed** *in a circle when each of its sides is a chord of the circle. In the first case, the* **circle** *is said to be* **inscribed** *in the polygon, and in the second case, the* **circle** *is said to be* **circumscribed** *about the polygon.*

Theorem 56. ***A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.***

Given: The regular n-gon $PQRST\ldots$

To prove: 1. A circle may be circumscribed about $PQRST\ldots$

2. A circle may be inscribed in $PQRST\ldots$

Hints for proof 1:

Let O be the center of the circle determined by P, Q, and R.	Three non-collinear points determine a circle.

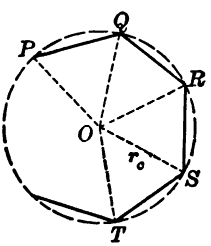


FIG. 1.

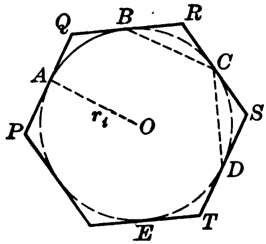


FIG. 2.

Draw $\overline{OP}, \overline{OQ}, \overline{OR}, \overline{OS}$.

By means of $\triangle$ OPQ and ORS prove $\overline{OS} = \overline{OP}$.

To what triangle can $\triangle OST$ be proved congruent?

Would it make any difference how many sides the original polygon had?

Give the details of the proof.

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Hint for proof 2: If O is the center of the circumscribed circle, what are PQ, QR, RS, etc.?

Cor. 1. ***An equilateral polygon inscribed in a circle is regular.***

Hint: In Fig. 1 what is the measure of each of the angles P, Q, R, etc.?

Cor. 2. ***An equiangular polygon circumscribed about a circle is regular.***

Suggestion: In Fig. 2 connect consecutive points of contact B, C, D, etc. Prove $\triangle BRC$, $\triangle CSD$, etc., congruent isosceles triangles.

What can then be said of the sums of any two of the equal sides of the triangles? (Such as $RC+CS$, $SD+DT$, etc.)

An angle such as $\measuredangle POQ$ *is called a* **central angle of the regular polygon.**

EXERCISES. SET LXIX. REGULAR POLYGONS AND CIRCLES

674. Make a design for tiling or linoleum patterns based upon inscribed equilateral hexagons.

675. Make a copy of the accompanying design of a rose window of six lobes. (Fig. 1.)

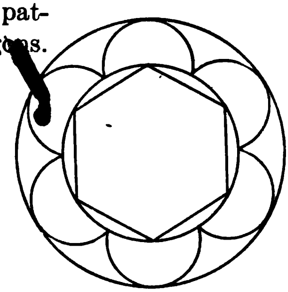

Fig. 1.

676. Make an accurate construction of the accompanying design representing a Gothic window. (Fig. 2.)

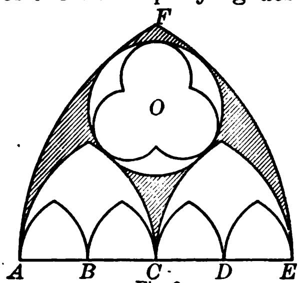

Fig. 2.

d677. Fig. 3 shows a star inscribed in a given square with all of its vertices on the sides of the square.

(a) Construct a figure in which points K, L, M, N, etc., shall be the mid-points of AE, EB, BF, FC, etc.

(b) Show that a circle can be circumscribed about the star constructed as in (a) and find its radius, if $AB=a$.

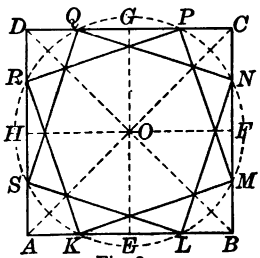

Fig. 3.

678. If a series of equal chords are laid off in succession on a circle, what relation exists between:

(a) the arcs subtended by the chords?

(b) the central angles intercepting the arcs?

(c) the inscribed angles formed by any two successive chords?

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679. If the series of equal chords mentioned in the last exercise were such as finally to form an inscribed polygon, what kind of polygon would it be? Why?

680. How are the central angles of a regular polygon related?

681. Make a table showing the number of degrees in the central angle of a regular inscribed polygon of 3, 4, ... 20 sides.

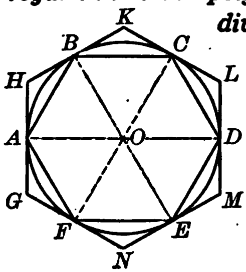
682. Write a formula by means of which we can obtain the number of degrees in the central angle of any regular polygon.

683. Which of the angles found for the table in Ex. 681 can we construct by means of compasses and straight edge?

684. Inscribe in a given circle each of the regular polygons you can by the means mentioned in the last exercise. (Do not go beyond the sixteen-sided polygon.)

685. Express as a formula the number of sides of the regular polygons you can inscribe in a circle up to this point.

Theorem 57. *If a circle is divided into any number of equal arcs, the chords joining the successive points of division form a regular inscribed polygon; and the tangents drawn at the points of division form a regular circumscribed polygon.*



Given: $\odot O$ with $\widehat{AB} = \widehat{BC} = \ldots\ldots = \widehat{FA}$.
(1) Chords AB, $BC \ldots\ldots FA$.
(2) GH, HK, etc., tangent to $\odot O$ at A, B, ... F respectively.

To prove: (1) $ABC \ldots . F$ a regular inscribed polygon.
(2) $GHK \ldots . N$ a regular circumscribed polygon.

Hints: To prove (1) use Cor. 1, Theorem 56.
To prove (2) use Cor. 2, Theorem 56.

Cor. 1. *Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.*

How do the vertices of the regular inscribed polygon divide the circle?

Cor. 2. *Lines drawn from each vertex of a regular polygon to the mid-points of the adjacent arcs subtended by the sides of the polygon form a regular inscribed polygon of double the number of sides.*

Hint: Show that the polygon is equilateral. Why? Or throw the corollary back to the proposition.

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Cor. 3. ***Tangents at the mid-points of the arcs between consecutive points of contact of the sides of a regular circumscribed polygon form a regular circumscribed polygon of double the number of sides.***

How does the corollary rest on the theorem?

Cor. 4. ***The perimeter of a regular inscribed polygon is less than that of a regulat inscribed polygon of double the number of sides; and the perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.***

(Proof left to the student.)

HISTORICAL NOTE.—This theorem presupposes the possibility of dividing the circle into any number of equal arcs. In the table found in answer to Exercise 681 it was probably seen that the number of equal arcs into which we are at this time able to divide the circle is very limited, but we have not yet learned to divide the circle exactly in as many ways as possible by means of elementary geometry. Some other methods will be discussed later.

As early as Euclid's time it was known that the angular magnitude about a point (and hence a circle) could be divided into 2^n, $2^n \cdot 3$, $2^n \cdot 5$, and $2^n \cdot 15$ equal parts. In 1796 it was discovered by Karl Friedrich Gauss, then only sixteen years of age, that $2^n \cdot 17$ equal parts of the circle could be found by the use of only the straight edge and compasses. Gauss also showed that in general it is possible to construct all regular polygons having (2^n+1) sides, when n is an integer and (2^n+1) is a prime number. He went still further and proved that regular polygons, having a number of sides equal to the product of two or more different numbers of this series, can be constructed.

EXERCISES. SET LXIX (continued)

686. Show that according to Gauss's formula, regular polygons of 3, 5, 17, and 257 sides can be constructed.

687. Inscribe a square in a circle, and by means of it a regular octagon, a regular 16-sided, and a regular 32-sided polygon.

688. What is the perimeter of the square in terms of the diameter of the circle in the last exercise? How do the perimeters of the octagon, hexadecagon, and 32-sided polygons compare with it? With the circle?

689. Repeat exercises 687 and 688 with respect to the inscribed equilateral triangle.

690. Repeat exercises 687 and 680 with respect to the regular circumscribed square and triangle.

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691. Between what two values will the length of the circle always lie?

692. Between what two values will the area of the circle always lie?

From Cor. 4, Theorem 57, and Exercises 688–692, it is seen that though the perimeter of the inscribed polygons increases as the number of its sides increases, it is always less than the length of the circle; and that while the perimeter of the circumscribed polygon decreases under the same conditions, it is always greater than the length of the circle.

In the first case the perimeter and likewise the area of the regular inscribed polygon are **increasing variables.** They are always less than the perimeter and the area of the circle, which are fixed or **constant.**

In the second case, the perimeter and the area of the regular circumscribed polygon are **decreasing variables** which are always greater than the perimeter and the area of the circle.

In the first case we say that the perimeter and the area approach as **superior limits** the circle and its area, while in the second we say that the perimeter and the area approach as **inferior limits** the circle and its area.

Thus, if p_n represents the perimeter of a regular inscribed polygon of n sides, a_n, its area, P_n the perimeter of a regular circumscribed polygon of the same number of sides, A_n, its area, and c the circle in and about which they are inscribed and circumscribed, and C its area, we say that as the number n is increased, p_n approaches c as its limit, P_n approaches c as its limit, a_n approaches C as its limit, and A_n approaches C as its limit. These statements are briefly written as follows:

$$p_n \doteq c, \qquad P_n \doteq c, \qquad a_n \doteq C, \qquad A_n \doteq C.$$

POSTULATES OF LIMITS

1. *The circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach if the number of sides of the polygon is indefinitely increased.*

2. *The area of the circle is the limit which the areas of regular inscribed and circumscribed polygons approach as the number of sides of the polygon is increased.*

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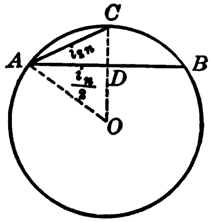
Theorem 58. ***A regular polygon the number of whose sides is $3 \cdot 2^n$ may be inscribed in a circle.***

(Proof left to the student.)

EXERCISE. SET LXIX (continued)

693. Inscribe a regular polygon of $3 \cdot 2^4$ sides.

Theorem 59. *If i_n represent the side of a regular inscribed polygon of n sides, i_{2n} the side of one of $2n$ sides, and r the radius of the circle, $i_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - i_n^2}}$.*



Given: $\odot O$ of radius r, AB (i_n) the side of a regular inscribed n-gon, AC (i_{2n}) the side of the regular inscribed $2n$-gon.

Prove: $i_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - i_n^2}}$.

PROOF

Draw radius OC cutting AB at D.	
Draw $\overline{AO}$.	
(1) Then $OC \perp AB$ and $AD = \frac{i_n}{2}$	(1) Why?
(2) $\therefore \overline{AC}^2 = (i_{2n})^2 = \left(\frac{i_n}{2}\right)^2 + \overline{CD}^2$	(2) Why?
(3) But $\overline{CD} = r - \overline{OD}$. $= r - \sqrt{r^2 - \left(\frac{i_n}{2}\right)^2}$	(3) Why? (Note $\triangle OAD$.)
(4) $\therefore (i_{2n})^2 = \frac{i_n^2}{4} + \left(r - \sqrt{r^2 - \left(\frac{i_n}{2}\right)^2}\right)^2$ $= r(2r - \sqrt{4r^2 - i_n^2})$	
(5) $\therefore i_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - i_n^2}}$	

EXERCISES. SET LXIX (continued)

694. Given a circle of radius 1 unit, compute:

(*a*) The length of a side of the inscribed square.

(*b*) The length of a side of the inscribed regular octagon.

(*c*) The length of a side of the inscribed regular 16-sided polygon.

(*d*) The perimeters of each of these polygons.

695. Do the same as you did in the last exercise for the regular hexagon, dodecagon, and 24-sided polygon.

696. Do the same as you did in Ex. 694, using a diameter of 1 unit.

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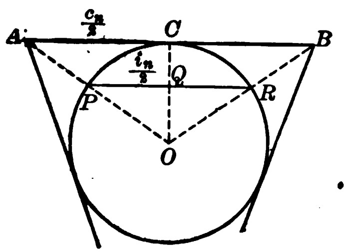
697. Do the same as you did in Ex. 695, using a diameter of 1 unit.

698. (*a*) Which computation is simpler—that using a radius of 1 unit or a diameter of 1 unit?

(*b*) If the radius is one unit what must be done to the perimeter found for each of the polygons in order that it be expressed in terms of the diameter?

Theorem 60. *If i_n represent the side of a regular inscribed polygon of n sides, c_n that of a regular circumscribed polygon of n sides, and r the radius of the circle,* $c_n = \frac{2ri_n}{\sqrt{4r^2 - i_n^2}}$.



Given: AB, a side c_n of a regular circumscribed polygon of n sides, tangent to $\odot O$ of radius r at point C; i_n the side of a regular inscribed polygon of n sides.

Prove: $c_n = \frac{2ri_n}{\sqrt{4r^2 - i_n^2}}$.

Suggestions for proof: Draw $\overline{AO}, \overline{BO}, \overline{CO}$. Let AO cut $\odot O$ at P, and BO cut it at R, and CO cut $\overline{PR}$ at Q.

Show that $PR = i_n$.
Note that OC is an altitude of $\triangle AOB$ and OQ of $\triangle POR$.
Why is $\triangle AOB \sim \triangle POR$?

(1) Then $\frac{c_n}{i_n} = \frac{r}{OQ}$.	(1) Why?
(2) In $\triangle OPQ$, $\overline{OQ} = \sqrt{r^2 - \left(\frac{i_n}{2}\right)^2}$	(2) Why?

Substitute (2) in (1) and complete the proof.

EXERCISE. SET LXIX (concluded)

699. Repeat the computations made in Exs. 694 and 695, or 696 and 697 for the sides and perimeters of regular circumscribed polygons.

The radius of its circumscribed circle is called the **radius of a regular polygon,** *and the radius of its inscribed circle is called the* **apothem** of a regular polygon.

Cor. *The apothem of a regular polygon is perpendicular to its side.*

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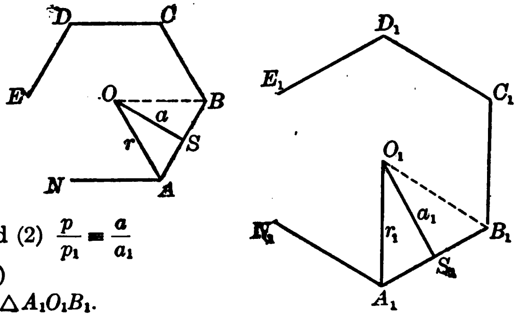
Theorem 61. ***The perimeters of regular polygons of the same number of sides compare as their radii, and also as their apothems.***

Given: Polygons $ABCD \cdots N$ and $A_1B_1C_1D_1 \cdots N_1$, regular, and each of n sides, and with perimeters p and p_1, centers O and O_1, radii r and r_1, and apothems a and a_1.



To prove: (1) $\frac{p}{p_1} = \frac{r}{r_1}$ and (2) $\frac{p}{p_1} = \frac{a}{a_1}$

Suggestions: Proof of (1)

Show that $\triangle AOB \backsim \triangle A_1O_1B_1$.

If $\frac{\overline{AB}}{A_1B_1} = \frac{r}{r_1}$ why is $\frac{p}{p_1} = \frac{r}{r_1}$?

Proof of (2)

What are a and a_1 in $\triangle AOB$ and $\triangle A_1O_1B_1$?

EXERCISE. SET LXX. PERIMETERS OF REGULAR POLYGONS

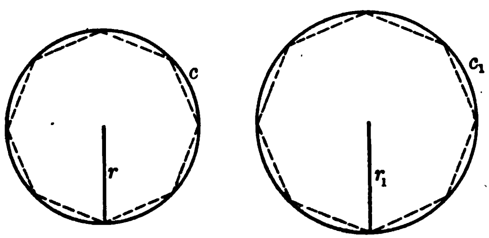
700. Construct a regular hexagon whose perimeter is $\frac{4}{5}$ of the perimeter of a given regular hexagon.

In the proofs that follow, two more assumptions are made, namely:

AXIOMS OF VARIABLES

1. *If any variable approaches a limit, any part of that variable approaches the same part of its limit.* That is, if $v_1 \doteq l_1$, then $\frac{v_1}{n} \doteq \frac{l_1}{n}$.

2. *If two variables are always equal, the limits which they approach are equal.* That is, if $v_1 \doteq l_1$, and $v_2 \doteq l_2$, and $v_1 = v_2$ then $l_1 = l_2$.



Theorem 62. ***Circumferences have the same ratio as their radii.***

Given: Circles c and c_1 of radii r and r_1.

Prove: $\frac{c}{c_1} = \frac{r}{r_1}$.

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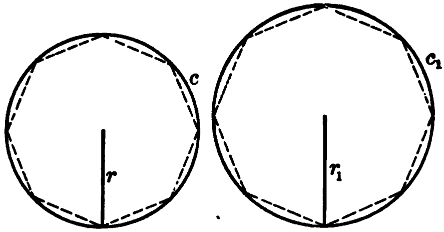
Proof

Inscribe in each circle a regular polygon of n sides, and let p and p_1 be their perimeters.



(1) Then $\frac{p}{p_1} = \frac{r}{r_1}$.

(2) or $\frac{p}{r} = \frac{p_1}{r_1}$.

Let n increase indefinitely.

(3) Then $p \doteq c$ and $p_1 \doteq c_1$ and

(4) $\therefore \frac{p}{r} \doteq \frac{c}{r}$ and $\frac{p_1}{r_1} \doteq \frac{c_1}{r_1}$.

(5) $\therefore \frac{c}{r} = \frac{c_1}{r_1}$.

(6) $\therefore \frac{c}{c_1} = \frac{r}{r_1}$.

(1) The perimeters of regular polygons of the same number of sides compare as their radii.

(2) By alternation.

(3) Postulate of limits for inscribed polygons.

(4) If any variable approaches a limit, any part of that variable approaches the same part of its limit as limit.

(5) If two variables are always equal, the limits which they approach are equal.

(6) By alternation.

Cor. 1. ***The ratio of any circle to its diameter is constant.***

$\frac{c}{c_1} = \frac{2r}{2r_1}$. Why? $\therefore \frac{c}{2r} = \frac{c_1}{2r_1}$. Why?

Cor. 2. ***Since the constant ratio*** $\frac{c}{2r}$ ***is denoted by the Greek letter*** π* (which is the initial letter of the Greek word "periphery") ***in any circle,*** $c = 2\pi r$.

EXERCISE. SET LXXI. VALUE OF π

701. Show that the second value given to π by Brahmagupta is another form of that given by Ptolemy. (See historical note.)

Theorem 63. ***The value of*** π ***is approximately*** **3.14159.**

If p_n stands for the perimeter of the regular inscribed polygon, and P_n for the perimeter of the regular circumscribed polygon in

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* Historical Note.—"Although this is a Greek letter, it was not used by the Greeks to represent this ratio. Indeed, it was not until 1706 that an English writer, William Jones, used it in this way."

Professor D. E. Smith further tells us that "probably the earliest approximation of the value of π was 3." In I Kings, vii, 23, we read: "And he made a molten sea, ten cubits from one brim to the other; it was round all about—

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a circle of diameter 1, the following table can be derived by using the formulas of the last two theorems:

(1) $i_{2n} \equiv \sqrt{2r^2 - r\sqrt{4r^2 - i_n^2}}$ and (2) $c_n \equiv \frac{2ri_n}{\sqrt{4r^2 - i_n^2}}$ in which r, i_{2n}, i_n, and c_n retain their original meanings.

(The student might verify a few of the values in the table by using logarithms wherever possible, and for convenience basing the calculations on a circle of diameter 2.)

No. of sides.	p_n < c <	P_n
6	3.0000000	3.4641016
12	3.1058285	3.2153903
24	3.1326286	3.1596599
48	3.1393502	3.1460862
96	3.1410319	3.1427146
192	3.1414524	3.1418730
384	3.1415576	3.1416627
768	3.1415838	3.1416101
1536	3.1415904	3.1415970

EXERCISES. SET LXXII. CIRCUMFERENCE

g702. Construct a graph by means of which the circumference of a circle of any given diameter may be obtained.

703. Find the size of the largest square timber which can be cut from a log 24 in. in diameter.

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and a line of thirty cubits did compass it round about." Again in ii Chronicles iv, 2, we read a similar sentence. And again in the Talmud is found the sentence: "What is three hand-breadth around is one hand-breadth through."

The following list of various other values given to π may be of interest to some of us.

Value	Attributed to
3.106	Ahmes (c. 1700 B.C.)
$3\frac{10}{71} < \pi < 3\frac{1}{7}$	Archimedes (287–212 B.C.)
$3\frac{17}{120}$	Ptolemy of Alexandria (87–165 A.D.)
$\frac{62832}{20000}$ or 3.1416	Aryabhatta (c. 500 A.D.)
$\frac{3927}{1250}$ and $\frac{754}{240}$ or $\sqrt{10}$	Brahmagupta (c. 600 A.D.)
$\frac{355}{113}$	Metius of Holland (1571–1635)
Computed to the equivalent of over 30 decimal places	Ludolph von Ceulen (1540–1610)
To 140 decimal places	Vega (1756–1802)
To 200 decimal places	Dase (1824–1861)
To 500 decimal places	Richter (1854)
To 707 decimal places	Shanks (1854)

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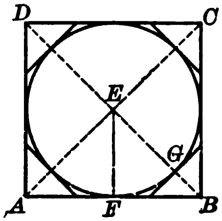
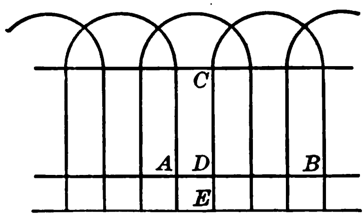
704. The following problems illustrate the type of problem that is suggested by books on carpentry.



To lay off an octagon on the end of a square piece of timber $ABCD$, draw diagonals AC and BD. With radius EF (the apothem of the square) draw arc cutting BD at G. Square out from G. Make a similar construction at each of the other corners of the square. Justify the rule.

705. How much belting does it require to make a belt to run over two pulleys, each 30 in. in diameter, the distance between their centers being 18 ft.?

706. The annexed figure represents a small wire fence used to protect flower beds. How many feet of wire are needed per running foot of fence if $AB = 1$ ft., $BD = CD = 9$ in., and $DE = 3$ in.?



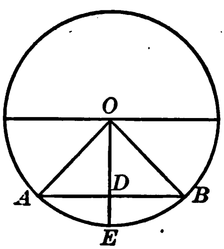
d707. Using 4000 miles as the radius of the earth, find the number of feet in the length of one minute at the equator. Use logarithms.

(This distance is commonly called a "knot.")

708. (*a*) If a cable were laid around the earth at the equator, how many feet would have to be added if the cable were raised 10 ft. above the surface of the earth?

(*b*) If the same were done around a gas-tank whose diameter is 100 ft.?

(*c*) In which case is the increase proportionally larger?



d709. Carpenters and other tradesmen frequently wish to know the circumference of a circle of given radius. The accompanying graphic method is given in some of the self-education books as a substitute for computation:

Draw radii AO and BO at right angles. Draw chord AB and line OE perpendicular to AB, meeting circle at E and chord at D.

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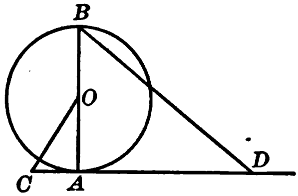
To 6 times the radius add the sect DE. The resulting sect is approximately the length of the circumference.

Compute the approximate per cent of error, using $\pi = 3.14159$.

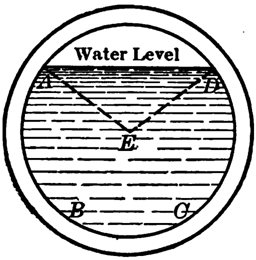
710. The central angle whose arc is equal to the radius is often used as the unit of measure of angles. It is called a *radian*. Find the number of degrees in a radian.



711. Many attempts have been made to construct a sect equal in length to a circle. The following approximate construction is one of the simplest. It is due to Kochansky (1685). At the extremity A of the diameter AB of a given circle of radius r draw a tangent CD, making $\angle COA = 30°$ and $CD = 3r$. Taking BD as semi-circumference is equivalent to taking what value for π? Carry work to four decimal places.

712. A running track consists of two parallel straight stretches, each a quarter of a mile long, and two semi-circular ends, each a quarter of a mile long at the inner curb. If two athletes run, one 5 ft. from the inner curb and the other 10 ft. from it, by how much is the second man handicapped?

713. Show how to go into the field and lay out a running track of the dimensions given in the last exercise.



d714. A conduit for carrying water is circular in form and is 10′ in diameter.

(*a*) Find the length $ABCD$ of the portion of the circular outline which is wet when the water reaches AD and $\angle AED$ is 120°.

(*b*) What is the length of $ABCD$ if $\angle AED$ is 60°?

This so-called "wetted perimeter" is of the greatest importance in determining friction, and therefore the resistance of the pipe to the water flow.

d715. Find the length of the forty-second parallel of latitude, assuming the radius of the earth to be 4000 miles.

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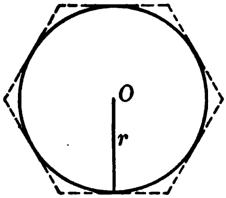

Theorem 64. ***The area of a circle is equal to one-half the product of its radius and its circumference.***

Given: ⊙ O of radius r, area C, and perimeter c.
Prove: $C = \frac{1}{2}cr$.

PROOF

Circumscribe about ⊙O a regular polygon of n sides and let P be its perimeter and A its area.

(1) Show that $A = \frac{1}{2}rP$.

Let n increase indefinitely.

(2) Then $A \doteq C$ and $P \doteq c$.

(3) $\therefore \frac{1}{2}rP = \frac{1}{2}rc$.

(4) $\therefore C = \frac{1}{2}rc$.

Write out the proof, giving all authorities.

Cor. 1. ***The area of a circle is equal to π times the square of its radius.***

(Substitution left to the student.)

Cor. 2. ***The areas of circles compare as the squares of their radii.***

(Proof left to the student.)

A portion of the area of a circle enclosed by two radii and their intercepted arc is a **sector.**

EXERCISES. SET LXXIII. AREA OF CIRCLE

g716. Construct a graph by means of which the area of a circle of any given diameter may be obtained.

717. Show that the area of a circle is equal to that of a triangle, whose base equals the length of the circumference of the circle and altitude equals the radius. This was proved by Archimedes.

718. Fill in the blanks in the following table:

	Perimeter P_1	Area A_1	Perimeter P_2	Area A_2
Square	300			300
Circle	300			300

719. Show by means of a carpenter's square how to find the diameter of a circle having the same area as the sum of the areas of two given circles.

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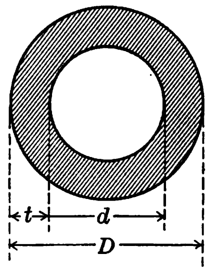
d720. If a one-inch pipe will empty 2 barrels in 15 min., how many barrels will an 8-in. pipe empty in 24 hrs.? (Make no allowance for friction.)

d721. In putting up blower pipes, two circular pipes 11 in. and 14 in. in diameter respectively join and continue as a rectangular pipe 14 in. in width. Find the length of the cross-section of the rectangular pipe.

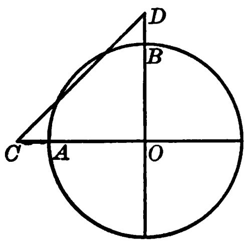


722. A convenient formula used in practical work for finding the area of a "hollow circle" or ring is:

$$A = \frac{\pi t(D+d)}{2}$$

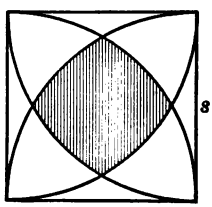
Establish this formula.

723. A horse tied by a rope 25 ft. long at the corner of a lot 50 ft. square, grazes over as much of the lot as possible. The next day he is tied at the next corner, the third day at the third corner, and the fourth day at the fourth corner. Draw a plan showing the arcs over which he has grazed during the four days, using a scale of $\frac{1}{4}$ inch to 5 feet. Calculate the area.

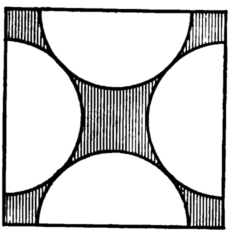


724. Justify the following rule used by sheet-metal workers or show the per cent of error if it is incorrect:

Draw radius $AO \perp OB$. Extend each one-fourth its own length to C and D. Then the sect CD is the side of the square of the same area as that of the circle.



d725. Construct a square with a side s. With the vertices as centers, and s as radius, construct arcs as in the figure. Find the perimeter and the area of the shaded portion bounded by the arcs.



d726. In the design shown in this figure, the side of the square is s. The inscribed semicircles are tangent to the diagonals. Find the perimeter and the area of the shaded portion of the figure.

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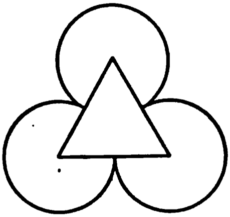
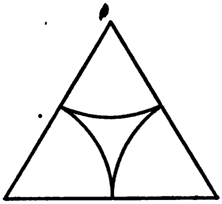
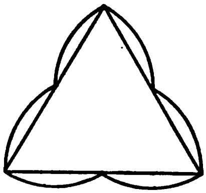
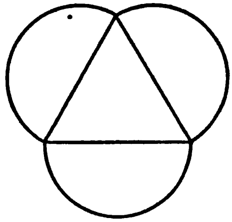
727. Construct an equilateral triangle. With each vertex as center, and with one-half a side as radius, describe arcs as indicated in the diagrams. Let $2r$ represent the length of a side of the equilateral triangle. Find the perimeter and the area of the figure bounded by the arcs.



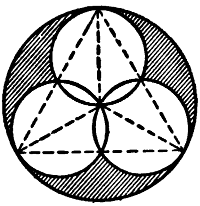


728. Modify the preceding exercise by using the mid-point of the sides as centers, as indicated in the diagrams.





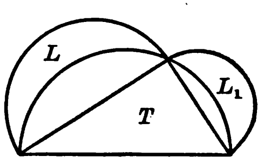
729. Inscribe an equilateral triangle in a circle of radius $2r$. Using the mid-point of each radius of the triangle as center and r as radius, describe circles. Find the perimeter and the area of the trefoil and of the shaded part of the resulting symmetric pattern. (Using S for the area of the shaded portion, C for that of the large circle, A for that of the small circle, T for that of the triangle, give the formula for S in terms of r, T, C, and A.)



730. In the papyrus of Ahmes, an Egyptian, the area of a circle was found by subtracting from the diameter one-ninth of its length and squaring the remainder This was equivalent to using what value of π?

731. In the Sulvasutras, early semi-theological writings of the Hindus, it is said: "Divide the diameter into 15 parts and take away 2; the remainder is approximately the side of the square, equal to the circle." From this compute their value of π.

d732. The proposition of the so-called lunes of Hippocrates (ca. 470 B.C.) proved a theorem that asserts in somewhat more general form, that if three semicircles be described on the sides of a right triangle as



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diameter, the lunes L and L_1 as shown in the diagram are together equivalent to the triangle T. Prove it.

d733. A problem of interest is one that Napoleon is said to have suggested to his staff on his voyage to Egypt: To divide a circle into four equal parts by the use of circles alone.

LIST OF WORDS DEFINED IN CHAPTER VIII

Circle, center, radius, diameter; central angle, arc (minor, major); chord, intercept, subtend; tangent to a circle, secant, tangent circles (internally, externally); line of centers, common chord, inscribed angle; inscribed, circumscribed regular polygons; center, central angles, radius, apothem of regular polygon; sector. Constant, variables (increasing, decreasing); limits (inferior, superior).

AXIOMS OF VARIABLES IN CHAPTER VIII

1. If any variable approaches a limit, any part of that variable approaches the same part of its limit, as limit.
2. If two variables are always equal, the limits which they approach are equal.

POSTULATE OF PERPENDICULARS (third) IN CHAPTER VIII

1. The shortest distance from a point to a line is the perpendicular to that line.

SUMMARY OF THEOREMS PROVED IN CHAPTER VIII

42. Three points not in a straight line fix a circle.

43. In equal circles equal central angles intercept equal arcs, and conversely.

44. In equal circles, equal arcs are subtended by equal chords, and conversely.

45. A diameter perpendicular to a chord bisects it and its subtended arcs.

Cor. 1. A radius which bisects a chord is perpendicular to it

Cor. 2. The perpendicular bisector of a chord passes through the center.

46. In equal circles, equal chords are equidistant from the center, and conversely.

47. A line perpendicular to a radius at its outer extremity is tangent to the circle.

Cor. 1. A tangent to a circle is perpendicular to a radius drawn to the point of contact.

Cor. 2. The perpendicular to a tangent at the point of contact passes through the center of the circle.

Cor. 3. A radius perpendicular to a tangent passes through the point of contact.

Cor. 4. Only one tangent can be drawn to a circle at a given point on it.

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48. Sects of tangents from the same point to a circle are equal.

49. The line of centers of two tangent circles passes through their point of contact.

50. In equal circles central angles have the same ratio as their intercepted arcs.

Cor. 1. A central angle is measured by its intercepted arc.

51. Parallels intercept equal arcs on a circle.

52. An inscribed angle or one formed by a tangent and a chord is measured by one-half its intercepted arc.

Cor. 1. An angle inscribed in a semicircle is a right angle.

53. An angle whose vertex is inside the circle is measured by half the sum of the arc intercepted by it and by its vertical angle.

54. An angle whose vertex is outside the circle is measured by half the difference of the intercepted arcs.

55. A tangent is the mean proportional between any secant and its external sect, when drawn from the same point.

56. A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.

Cor. 1. An equilateral polygon inscribed in a circle is regular.

Cor. 2. An equiangular polygon circumscribed about a circle is regular.

57. If a circle is divided into any number of equal arcs, the chords joining the successive points of division form a regular inscribed polygon, and the tangents drawn at the points of division form a regular circumscribed polygon.

Cor. 1. Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.

Cor. 2. Lines drawn from each vertex of a regular polygon to the mid-points of the adjacent arcs subtended by the sides of the polygon form a regular inscribed polygon of double the number of sides.

Cor. 3. Tangents at the mid-points of the arcs between consecutive points of contact of the sides of a regular circumscribed polygon form a regular circumscribed polygon of double the number of sides.

Cor. 4. The perimeter of a regular inscribed polygon is less than that of a regular inscribed polygon of double the number of sides, and the perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.

58. A regular polygon the number of whose sides is $3 \cdot 2^n$ may be inscribed in a circle.

59. If $\mathbf{i}_n$ represent the side of a regular inscribed polygon of $\mathbf{n}$ sides, and $\mathbf{i}_{2n}$ the side of one of $2\mathbf{n}$ sides, and $\mathbf{r}$ the radius of the circle, $\mathbf{i}_{2n} = \sqrt{2\mathbf{r}^2 - \mathbf{r}\sqrt{4\mathbf{r}^2 - \mathbf{i}_n^2}}$.

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60. If i_n represent the side of a regular inscribed polygon of **n** sides, c_n that of a regular circumscribed polygon of **n** sides, and **r** the radius of the circle, $c_n = \frac{2ri_n}{\sqrt{4r^2 - i_n^2}}$.

61. The perimeters of regular polygons of the same number of sides compare as their radii, and also as their apothems.

62. Circumferences have the same ratio as their radii.

Cor. 1. The ratio of any circle to its diameter is constant.

Cor. 2. In any circle $c = 2\pi r$.

63. The value of π is approximately 3.14159.

64. The area of a circle is equal to half the product of its radius and its circumference.

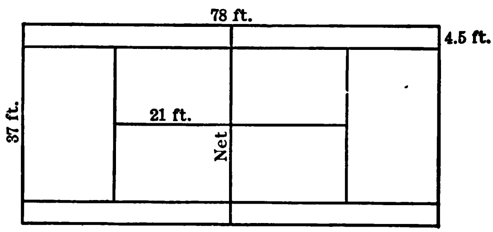
Cor. 1. The area of a circle is equal to π times the square of its radius.

Cor. 2. The areas of circles compare as the squares of their radii.

EXERCISES. SET LXXIV. MISCELLANEOUS

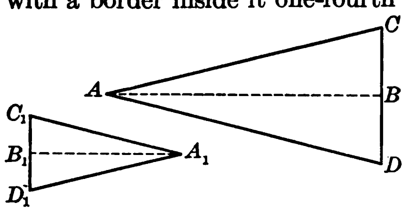
The problems in this set are miscellaneous in that their solution may depend upon any part of the text; but they are arranged, in general, in the order of difficulty. Those problems requiring the application of trigonometric ratios are preceded by "t."

734. Find the number of feet of lime-line of a tennis-court, as represented.



735. Find the number of yards of lime-line for a football field, which is 300 ft. by 160 ft., including all the ten-yard lines. How long would it take a runner to cover the total distance if he can make 40 feet in 12 seconds?

736. Construct an accurate diagram of a rectangular garden with a border inside it one-fourth the width of the garden.



737. A designer, in making a pennant, must make one in the same proportion as a given one, but larger. Find AB, if $CD = 36''$, $A_1B_1 = 43''$, and $C_1D_1 = 20''$.

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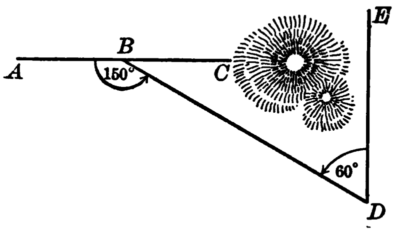
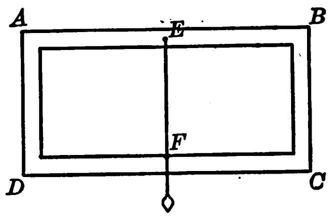
738. By use of the steel square lay out an angle of 45°.

739. By use of steel squares lay out an angle of 60°.

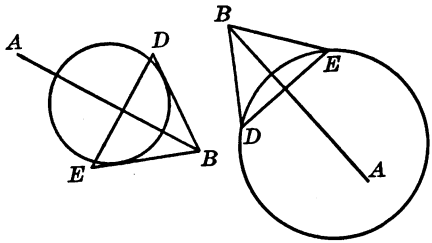
740. A straight railroad AB strikes a mountain at C, and a tunnel is to be driven at C in the direction $\overline{ABC}$. It is desired to commence the work at the other side of the mountain at the same time as the work is progressing from C. $\measuredangle ABD$ is made 150°, $BD = 3$ miles, and $\measuredangle D = 60°$. How far from D in DE must the tunnel be driven, and in what direction?



741. An instrument for leveling consists of a rectangular frame $ABCD$. E and F are the mid-points of AB and DC, respectively. A plumb-line is suspended from E. Show that when the plumb-line coincides with the mark F, DC is horizontal.

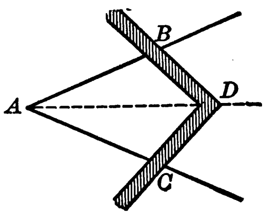


This instrument is shown in French books.



742. The accompanying diagrams represent an instrument for locating the center of circular discs. Three pieces of metal are so joined that AB bisects the angle formed by BD and BE, which two sects are equal. Prove that AB passes through the center of the circle.

743. A carpenter bisects an angle by the following rule: Lay off $AB \equiv AC$. Place a steel square so that $BD \equiv CD$ as shown in the diagram. Draw the line AD. Is this method correct? Give proof. Would this method be correct if the steel instrument did not have a right angle at D?



744. Answer the following questions without proof:

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(*a*) Are all equilateral polygons equiangular?

(*b*) Are the diagonals of a parallelogram equal?

(*c*) A diagonal of a parallelogram divides the figure into two congruent triangles. Is this proposition conversely true?

(*d*) Do the diagonals of a parallelogram bisect its angles?

(*e*) Under what circumstances do two chords bisect each other?

(*f*) An arc of 30° is subtended by a chord of 5″. In the same circle will an arc of 60° be subtended by a chord equal to, greater than, or less than 5″?

(*g*) Is it possible to inscribe a parallelogram in a circle?

745. In an ideal honeycomb the cells are 6 sided. Why is this? In what other regular form might they be built and yet fit snugly?

746. The wheel of an automobile makes 110 revolutions per minute. If it measures 2 ft. 6 in. in diameter, find the speed of the machine.

747. Walking along a straight road a traveler noticed at one milestone that a house was 30° off to the right. At the next milestone the house was 45° off to the right. How far was the house from the road? Is there more than one solution?

748. Calculate the diameter of the circle of water visible to an observer at sea, (*a*) when seated in a small boat, his eyes being 4 ft. above the surface of the sea, (*b*) when on the bridge of a steamer 25 ft. above the surface, (*c*) when at the masthead 60 ft. above the surface, (*d*) when on the top of a mountain 3000 ft. above sea level. (*e*) How far above sea level does the elevation of the observer begin to make a perceptible difference?

a749. In finding the diameter of a wrought-iron shaft that will transmit 90 horse-power when the number of revolutions is 100 per minute, using a factor of safety of 8, we have to find the diameter d from the formula $d = 68.5\sqrt[3]{\dfrac{90}{100 \times \dfrac{50000}{8}}}$. Find d.

750. Draw any quadrilateral *ABCD*. Take such measurements of your figure as you consider necessary and sufficient, and from your measurements construct the quadrilateral a second time. State what measurements you make, and how you draw the second figure. Cut out the two figures and fit one upon the other.

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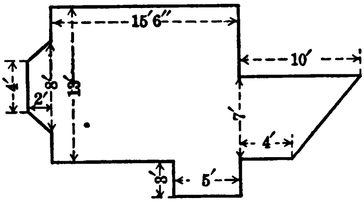
(*b*) Discuss a number of other sets of measurements that you could use to reproduce the quadrilateral $ABCD$. How many measurements must there be in every set?



(*c*) Construct a quadrilateral similar to $ABCD$, but 50% greater in area.

751. Find the number of square feet in the floor of the room shown in the accompanying plan.

752. Find the area of the accompanying polygon by filling out the following table, assuming reasonable values for necessary dimensions. (Fig. 1).

Parts of Polygon	Factors		Products
	Bases, or sums of parallel sides	Altitudes	
$\triangle ABB_1$	$BB_1 = 6.8$	$AB_1 = 5.6$	38.08
.	.	.	.
.	.	.	.
			2) ______
		Polygon $ABCDEFGH =$	

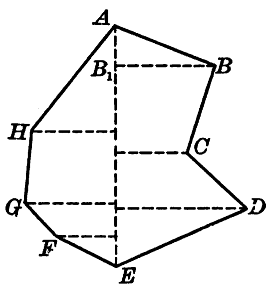


FIG. 1.

753. Show how to find the area of polygon $ABCDEFGHK$, assuming the shaded portion to be inaccessible.

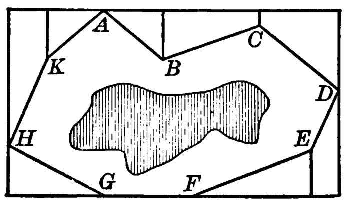


FIG. 2.

754. Using goods 20 in. wide, how many strips will it take, cut on true bias, to put a band 12 in. wide around a skirt 3 yds. wide?

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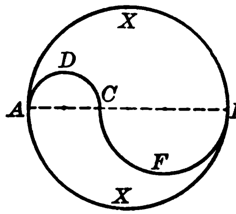
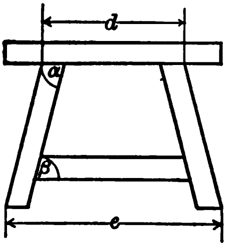
755. In the accompanying diagram how do (*a*) the perimeters and (*b*) the areas of the circle and the curvilinear figures *ADCFBX* compare? (*c*) Use this figure as a suggestion and show how, by means of arcs, to divide a circle into any number of equal areas.

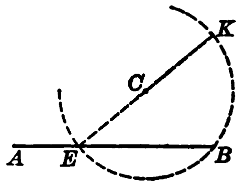




756. This is the cross-section of a foot-stool, in which the width of the top is to be 12 in., *d* 8 in., *e* 12 in., and the lengths of the legs 8 in. In making the stool, angle α and angle β are first laid out on paper. Show how, from the required dimensions, to lay out these angles on paper.

757. To find the diagonal of a square, multiply the side by 10, take away 1% of this product, and divide the remainder by 7. Test the accuracy of this rule of thumb used by some carpenters.

758. Construct a perpendicular at the end of a sect without producing the sect.



Hints: Let *AB* be the given sect. With any point *C*, between *A* and *B*, but outside the sect, as center, and radius *CB*, describe a major arc intersecting *AB* at *E*. Draw the diameter *EK*. *KB* is the required perpendicular.

Prove that this construction is correct.

759. The resultant of two forces acting upon a body is 400 lbs. One of the forces is 250 lbs. What are the limiting values for the other force?

760. A man decided to buy some numerals and make the face of a grandfather's clock, but when he came to divide the face into minutes he found that he was not able to do it without guessing. How could it be done accurately?

t761. A kite string is 250 ft. long, and makes an angle of 40° with the level ground. Find (approximately) the height of the kite above the ground, disregarding the sag in the string.

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762. Fill out the blank spaces in the table by referring to the diagram. (Fig. 1.)

$\angle 3$	$\angle 1$	$\angle 4$	$\angle 5$	$\overset{\frown}{EA}$	$\overset{\frown}{DB}$	$\overset{\frown}{AND}$	$\overset{\frown}{AF}$	$\overset{\frown}{BE}$	$\overset{\frown}{AG}$	$\overset{\frown}{FH}$
$\angle 1$	35°	...	$\angle 1$	40°	...	...	...	80°	...	30°
...	$\angle 3$	...	$\angle 3$	30°	50°	190°	...	...	105°	...

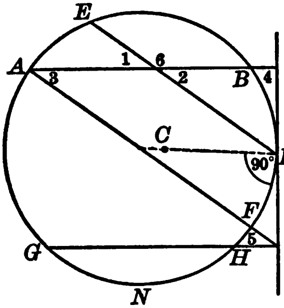


Fig. 1

763. The accompanying drawing is one of the earliest of Gothic tracery windows. The arch ACB is based on an equilateral triangle. A is the center and AB the radius for the arc CB, and B the center and AB the radius for arc AC. D is the mid-point of span AB. The arches AED and DFB are drawn on the half span by similar construction. Find the center and the radius of the circle with center O that shall be tangent to the four arcs, DE, DF, AC, CB.

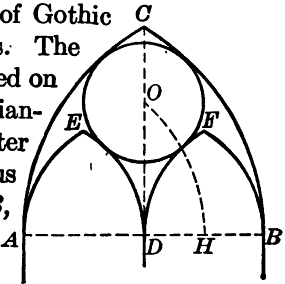
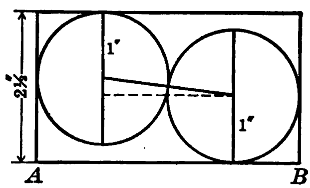


Fig. 2

Suggestions: With A as center and AH as radius cut the altitude CD at O. H is the mid-point of DB.

764. How many degrees are there in each of the angles of the Pythagorean badge?

765. From a strip of metal $2\frac{1}{2}''$ wide it is desired to cut off a rectangle from which two circular disks $2''$ in diameter can be cut. What length AB must be cut off?



766. In taking soundings to make a chart of a harbor it is necessary at each sounding to determine the position of the boat. This is sometimes done by measuring with a sextant the angles between lines from the observer to three range poles on the shore. When a chart is made, the position on the chart of each sounding is sometimes found as follows: The points A, B, and C represent the positions of the three range poles. Suppose the angles read

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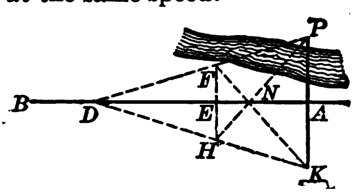
by the sextant were 50°, and 35° (50° between lines to *A* and *B*, and 35° between lines to *B* and *C*). On the chart lay off at *A* and *B*, angles *BAM* and *ABM*, each equal to 40°, and find the point *M*. At *B* and *C*, lay off angles *CBN* and *BCN* each equal to 55° and find *N*. With *M* and *N* as centers, draw circles passing through *B*. The other common point of these circles is the position of the sounding. Prove the correctness of this construction. Make the construction to scale.

767. The resistance offered by the air to the passage of a bullet through it varies jointly as the square of its diameter and the square of its velocity. If the resistance to a bullet whose diameter is .32 in., and whose velocity is 1562.5 ft. per second, is 67.5 oz., what will be the resistance to a bullet whose diameter is .5 in., and whose velocity is 1300 ft. per second?

768. Roman surveyors, called *agrimensores*, are said to have used the following method of measuring the width of a stream: *A* and *B* were points on opposite sides of the stream, in plain view from each other. The distance *AD* was then taken at right angles to *AB*, and bisected at *E*. Then the distance *DF* was taken at right angles to *AD*, such that the points *B*, *E*, and *F* were in a straight line. Make a drawing illustrating the above method, show what measurement affords a solution, and prove that this is so.

769. A student lamp and a gas jet illuminate a screen equally when it is placed 12 ft. from the former and 20 ft. from the latter. Compare the relative intensities of the two lights.

770. An endless knife runs on pulleys 48″ in diameter at the rate of 180 revolutions per minute. If the pulleys are decreased 18″ in diameter, how many revolutions per minute will they have to make to keep the knife traveling at the same speed?



771. In surveying, to determine a line from the inaccessible point *P* perpendicular to *AB*, lay off *FE* perpendicular to *AB* at an arbitrary point and of any length. Make $EH = EF$. Obtain point *D* in lines *PF* and *AB*; next, point *N* in *HP* and *AB*; next, *K* in *DH* and *FN*. *PK* is perpendicular to *AB*. Why?

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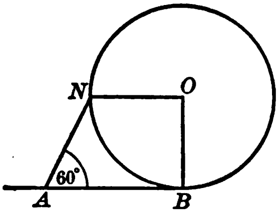
772. Make the construction shown in the diagram, and AB will be approximately the quadrant of the circle. Find the per cent by which it differs from the correct value.



773. The steel square may be tested by measuring across from the 9-inch point of the tongue to the 12-inch point of the blade. If this distance is exactly 15 inches, the square is true. Why?

774. Four of the largest possible equal sized pipes are enclosed in a box of square cross-section 18 in. on an edge. What part of the space do the pipes occupy?

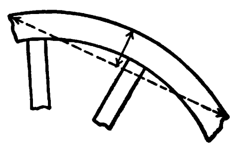
dt775. A boy pulls a sled with a force of twenty-five pounds by means of a rope ten feet long, and with his hands three feet from the ground. Find the component of his force effective in pulling the sled forward.

776. A camp kettle weighing 20 lbs. is suspended on a wire from two trees 10 feet apart. The wire is 20 feet long, and the kettle is suspended on it at a point midway between the trees. Find the tension on each strand of the wire.

777. A running track having two parallel sides and two semicircular ends, each equal to one of the parallel sides, measures exactly a mile at the inner curb. Two athletes run, one at the inner curb, and the other 10 ft. from this curb. By how much is the second man handicapped?

778. The last row of seats in a circular tent is 30 ft. away from the central pole, which is 20 feet high, and which is to be fastened by ropes from its top to stakes driven in the ground. How long must these ropes be in order that they may be 6 feet above the ground over the last row of seats, and at what distance from the center must the stakes be driven?

779. A wheelwright is given a part of a broken wheel to make a duplicate. To do this he needs the diameter. He measures the chord of the arc given him, 24″; the height of the segment is 4″. How large is the wheel?



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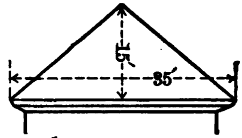
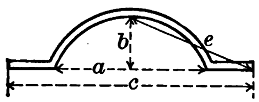


dt780. The width of the gable of a house is 35 ft. The height of the house above the eaves is 15 ft. Find the length of the rafters and the angle of inclination of the roof.

781. An aeroplane travels 1000 ft. upwards, $3\frac{1}{2}$ miles due west, and $2\frac{1}{2}$ miles due north. Find its distance from the starting-point.

782. The sides of a floor are 10 ft. and 6 ft. A man wishes to tile it with tiles in the shape of a hexagon whose sides are 6 in. How many tiles will it take?

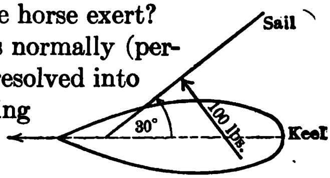
783. A flat circular sheet of metal is to be stamped into the form of a spherical segment with a flange. The figure shows a cross-section of the resulting piece of metal, *a* being the width of the spherical segment, *b* the depth or altitude of the segment, and *c* the outer diameter of the flange. The problem is to determine the size to cut the sheet metal in order that when stamped the piece may have these dimensions. Show that the required radius of the circular sheet equals *e* in the figure.



784. To lay off the length of a brace with the steel square: Suppose that the post is 4 ft. and the beam 3 ft. Apply 3 times to the timber from which the brace is to be cut, the distance across the square from the 12 in. point of the tongue to the 16 in. point of the blade. Why will this give the required length of the brace?

t785. In railroad construction and mining the material is sometimes hauled in a tram pulled by a horse. If the pull of the tram in the direction of the track is, say, 200 lbs., and if the horse walks at the side of track so that its pull is exerted at an angle of 25° with the track, what pull must the horse exert?

786. The force that the wind exerts normally (perpendicularly) on the sail of a boat is resolved into two components: one useless, in pushing the boat sidewise in the water in spite of the keels; and one useful component driving the boat directly forward. If, as in the following diagram, the sail is at an angle of 30° to the keel, and a force of wind of 100 lbs. acts on the sail (considered as applied at one point), find the effective value of the two components mentioned above.



NOTE.—A similar problem can be applied to the aeroplane.

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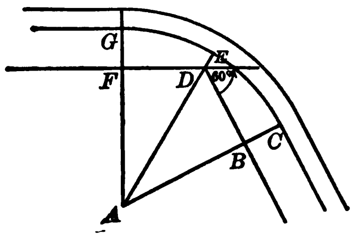
787. An angle inscribed in a circle varies directly as the intercepted arc. Show that in this case $k = \frac{1}{2}$.

788. A street-car track is 12′ from the curb ($GF = BC = 12'$). In passing the corner of two streets which deflect through an angle of 60° the rail must be 5′ ($DE = 5'$) from the corner. (*a*) Find the radius of the curve. (*b*) Find the length of the tangents from G and C to their point of intersection. (*c*) Find the length of arc GC; also the length of the outer arc. The width of the track is $4'8\frac{1}{2}''$. (All these curves are arcs of circles.)



789. Where two straight streets intersect, each corner is usually "rounded off." Show that the problem of laying out the corner arc is a simple one where the streets made an angle of 90°. Show by a plan how to lay out a corner where the angle is 45°, and the radius for the curbing is to be 20 ft. Find the total length of curved curbing needed.

dt790. A given regular polygon has n sides. How many measurements of the figure are both necessary and sufficient to determine it in size and shape?

dt791. A flagstaff is seen in a direction due north of a station A at an elevation of 17°, and from a station B 120 ft. due east of A the flagstaff bears 23° west of north. The two stations and the foot of the flagstaff being at the same level, determine the height of the flagstaff.

792. A straight street intersects one which is curved (with a large radius, such as 200 ft.), and the corner is to have a small radius, say 15 ft. Show in a plan how to find the center for the corner arc by means of the intersection of two loci. (Fig. 1.)

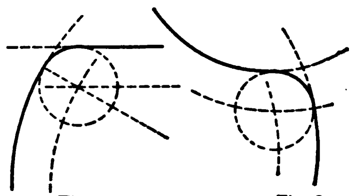


Fig. 1. Fig. 2.

How would you get, in actual field work, the curved locus? (Note its large radius.) (Fig. 2.)

793. Solve in another plan the corresponding problem where both streets are curved.

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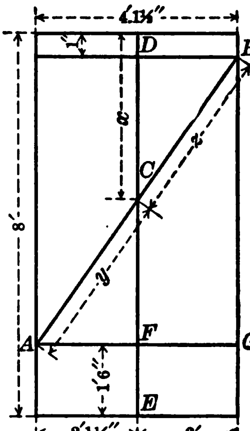
794. The "gear" of a bicycle is the diameter (in inches) of a wheel whose circumference would equal the distance gone forward with one revolution of the pedals. Find the gear if the diameter of the rear wheel is 28″ and the front and rear sprockets have 22 and 8 teeth, respectively.

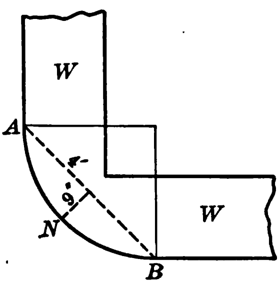
795. A belt runs over two pulleys, one of which is 4 ft. in diameter, and driven by an engine at the rate of 100 revolutions a minute. What must be the diameter of the other pulley if it is to turn a fan at the rate of 400 revolutions a minute?

796. The figure is the diagram of a part of the side of a bridge. The point C must be located on AB and on DE, where the holes must be bored to fasten the brace AB to the upright DE. Required to find the lengths of AB, x, y, and z, in order to locate C.



Suggestion: After finding AB, compare triangles ACF and ABG.

797. It is desired to construct a subway under a river. The bank on one side has a 30% slope from the edge of the river to the river bed. The maximum effective grade of a subway is 5%. On the surface it is five hundred feet from the bank of the river to the bed of the river. Determine the necessary length of the subway from the bank to the river bed.



798. WW is a wall with a round corner of dimensions as given in the figure from A to B, on which a molding, gutter, or cornice is to be placed. Find the radius of the circle of which the arc ANB is a part.

799. A, B are two beacons on a coast-line; S is a shoal off the shore, and the angle ASB is known to be 120°. Show that a vessel V sailing along the coast will keep outside the shoal if the angle AVB is always less than 110°.

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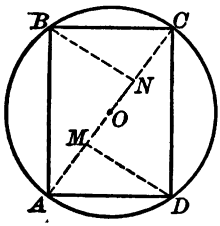
Prove a property of the circle that you used in proving that the ship will clear the shoal.

800. The method given by Galileo for finding the strongest rectangular beam that can be cut from a round log is as follows:

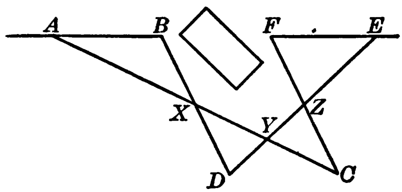


Let the circle $ABCD$ represent the end of the log, and let AC be a diameter. Divide the diameter into three equal parts at the points M and N, and from these points erect perpendiculars intersecting the circumference at points D and B. Draw AD, DC, CB, and BA. The rectangle thus formed is the cross-section of the strongest rectangular beam. Show that the dimensions of the rectangle are in the ratio of 1 to $\sqrt{2}$.

dt801. The resultant of two forces is 300 lbs. One of the forces acting at an angle of 37° with the resultant is 100 lbs. Find the other force, if the forces act at an angle of 65° with each other.

dt802. The radius of a circle is 7 ft. What angle will a chord of the circle 10 ft. long subtend at the center?

803. To prolong a line through an obstacle and to measure the distance along the line through the obstacle.



A and B are two points on the given line. Take X any point. Measure AX. Take $XC = AX$ and in line with AX. Mark Y on the mid-point of XC. Make $DX = BX$ and in line with BX. Make $ZY = YD$. Make $ZE = ZD$ and in line with ZD. Make $ZF = ZC$ and in line with ZC. Prove FE in line with AB. What line of the figure equals BF?

804. The cross-section of the train-shed of a railroad station is to have the form of a pointed arch, made of two circular arcs, the centers of which are on the ground. The radius of each arc equals the width of the shed, or 210 ft. How long must the supporting posts be made which are to reach from the ground to the dome of the roof?

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g805. Show by a graph that the area of a triangle having a fixed altitude varies as the base, and that one having a fixed base varies as the altitude.

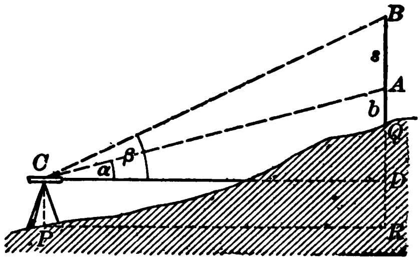
g806. Construct a graph showing the relation between the areas and the sides of equilateral triangles.

g807. Construct a graph showing the relation between a side and the area of a regular hexagon. By means of it find the area of a regular hexagon whose sides are 6. Compare with the computed area.

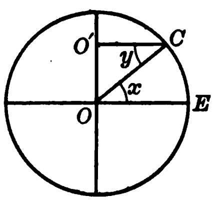
g808. If a side of a given regular polygon is a and its perimeter p, graph the relation of the perimeters and the corresponding sides of polygons similar to the given polygon.

g809. Given a polygon with area A and a side a. Construct a graph showing the relation between the areas and the sides corresponding to a in polygons similar to the one given.

dt810. The figure shows a method for determining the horizontal distance PR, and the difference of level QR between two points P and Q. A rod with fixed marks A and B upon it is held vertical at Q, and the elevation of these points ACD $(=\alpha)$ and BCD $(=\beta)$ are read by a telescope and divided circle at C, the axis of the telescope being at a distance CP $(=a)$ above the ground at P. If $QA=b$ and $AB=s$, write down expressions for PR $(=x)$ and QR $(=y)$.



Find x and y when $\alpha=6°\ 10'$ and $\beta=7°\ 36'$, the values of a, b, and s being 5 ft., $2\frac{1}{2}$ ft., and 5 ft. respectively.,



dt811. Find the radius of a parallel of latitude passing through Portland, Me. (43° 40′ N. lat.), if the radius of the earth is taken as 4000 mi.

(Note that in the figure $\measuredangle x$ equals $\measuredangle y$. Why?)

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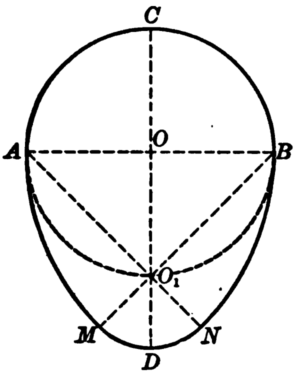
812. The oval in the figure is a design used in the construction of sewers. It is constructed as follows: In the circle O let CD, the perpendicular bisector of AB, meet $\widehat{AB}$ at O_1. Arcs AM and BN are drawn with AB as radius and with centers B and A respectively.



The chords BO_1 and AO_1 meet these arcs in M and N respectively.

The arc MDN has the center O_1 and radius O_1M.

(*a*) Make an accurate construction of the design.

(*b*) Is arc ACB tangent to arc AM and arc BN at A and B respectively? Why?

(*c*) Is arc MDN tangent to arc AM and arc BN at M and N respectively? Why?

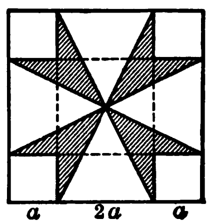
(*d*) If AB equals 8 ft., find BO_1, and hence O_1M, and finally CD. That is, if the sewer is 8 ft. wide, what is its depth?

d (*e*) If the width of the sewer is a ft., show that its depth is $\frac{a}{2}(4-\sqrt{2})$.

d (*f*) If the depth of the sewer is d ft., show that its width is $\frac{d}{7}(4+\sqrt{2})$.

d (*g*) Compute to two places of decimals the width of a sewer whose depth is 12 ft.

813. The circumference of the earth is approximately 25000 miles. Suppose an iron band 25000 miles long fits tightly around it. If you cut the band and put in three feet, will the band then be raised any appreciable distance from the earth? As much as $\frac{1}{4}$ inch, for example? $\frac{1}{100}$ inch? Consider the band, when enlarged, to be raised an equal distance from the earth at all points.



814. In this design, the side of the square upon which it is constructed is $4a$. Find the area of the shaded portion.

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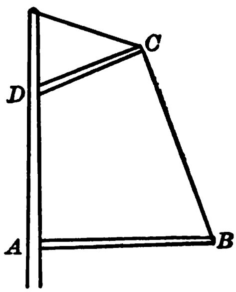
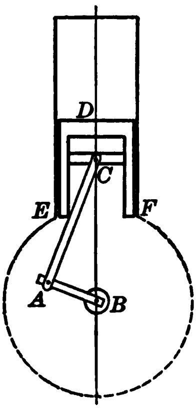
dt815. In constructing a sail, the amount of surface of canvas *ABCD* is known, and the lengths of *AB*, *AD*, and *DC* are given. The angle *A* is a right angle. Show how to construct the angle between *DC* and *DA*.



dt816. Find the number of square yards of cloth in a conical tent with a circular base and the vertex angle 72°, the center pole being 12 ft. high.

817. A girder to carry a bridge is in the form of a circular arc. The length of the span is 120 ft., and the height of the arc is 25 ft. Find the radius of the circle.



dt818. In the side of a hill which slopes upward at an angle of 32°, a tunnel is bored sloping downwards at an angle of 12° 15′ with the horizontal. How far below the surface of the hill is a point 115 ft. down the tunnel?

819. In constructing a gas engine the piston *D*, which is in the form of an inverted cup, is 5 in. in inside diameter; the crank *AB* is 5 in. between the centers of the pivots, and the connecting rod *AC* is 17 in. between the centers of the pivots. How far from the mouth of the cup must the pin *C* be adjusted in order that the connecting rod may just clear the edge of the cup at *E* and *F*, the diameter of *AC* being 1 in.?

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PART II

SECOND STUDY

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SUGGESTIONS FOR A REVIEW OF THE FIRST STUDY OF PLANE GEOMETRY

Before beginning our Second Study of Plane Geometry, it might be well for us to review the First Study. The following material furnishes a brief, suggestive outline for such a review.

A. Congruence—Theorems 1–8

I. *Define:* *

Sect, polygon, axiom, postulate, corollary, adjacent angles, congruent, homologous, perpendicular.

II. *Summarize:*

a. Conditions under which triangles are congruent in general;

b. Special conditions under which right triangles are congruent;

c. Facts about perpendiculars.

III. *Family Trees of Propositions:*

To trace a proposition back to its sources, that is, back to the definitions, postulates, and axioms upon which it rests, will be found an interesting and profitable form of review. A convenient arrangement is to make a "family tree" of a theorem, the branches of which are the authorities quoted in proving the proposition. Each branch should be followed down as in the main proposition until it ends in a postulate, an axiom, or a definition.

Such a tree of Theorem 6 is given as an illustration (Plate 1). The student is advised to make a tree of Theorem 5.

B. Parallels—Theorems 9–12

I. *Define:*

Parallels, transversal, alternate-interior angles.

II. *Classify angles* according to:

a. Individual size;

b. Relative size;

c. Relative position.

III. *Summarize:*

a. Conditions under which lines are parallel;

b. Methods of proving sects equal;

c. Methods of proving angles equal.

IV. *Family Tree:*

A family tree of Theorem 11, Cor. 2, is appended (Plate 2). The student is advised to study it and make one of Theorem 12.

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* Consult the First Study only when necessary.

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PLATE 1

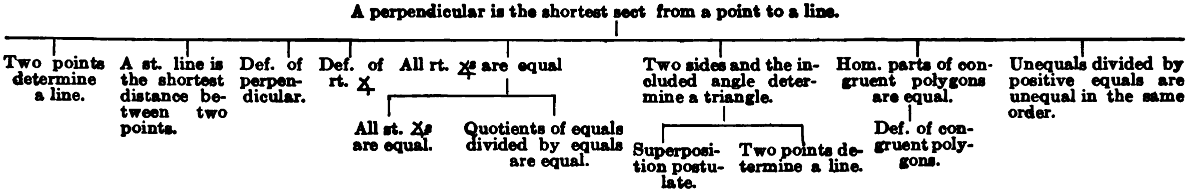


PLATE 2

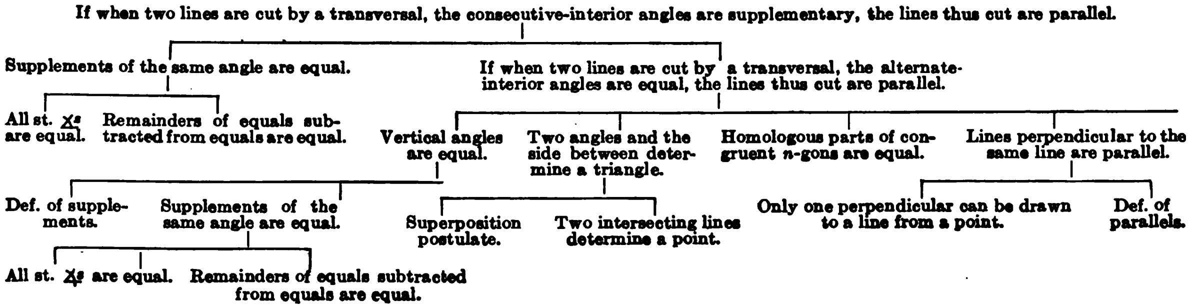
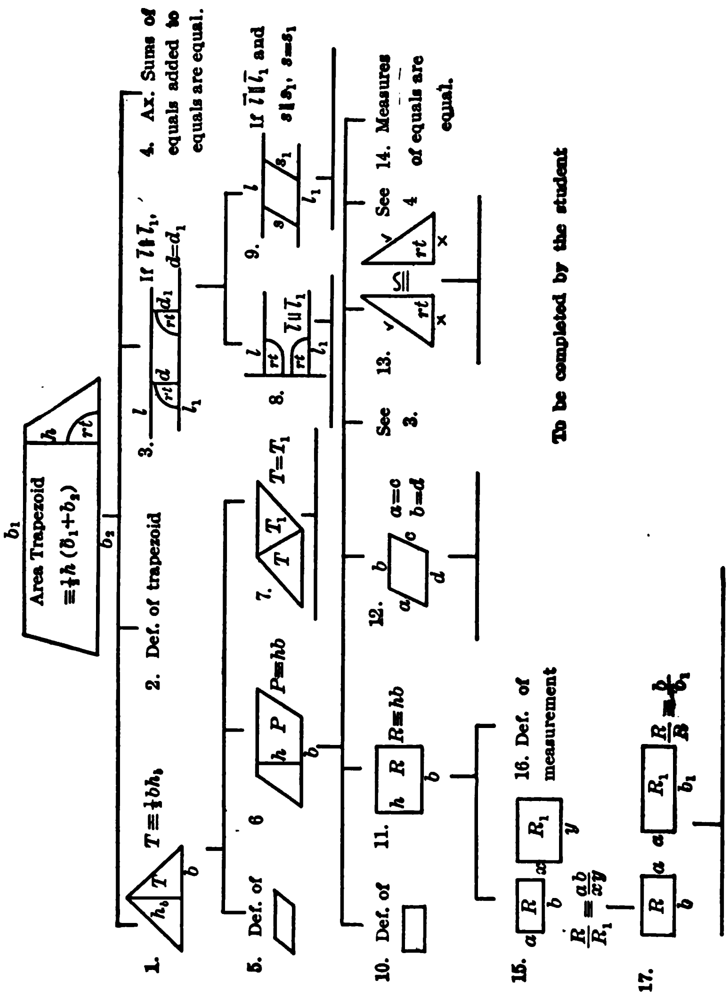


PLATE 3


Area Trapezoid
$=\frac{1}{2}h(b_1+b_2)$
1. $T=\frac{1}{2}bh_b$
2. Def. of trapezoid
3. If $l \parallel l_1$, $d=d_1$
4. Ax. Sums of equals added to equals are equal.
5. Def. of
6. $P=hb$
7. $T=T_1$
8. $l \parallel l_1$
9. If $l \parallel l_1$ and $s \parallel s_1$, $s=s_1$
10. Def. of
11. $R=hb$
12. $a=c$ $b=d$
See 3.
13.
See 4
14. Measures of equals are equal.
15. $\frac{R}{R_1}=\frac{ab}{xy}$
16. Def. of measurement
17. $\frac{R}{R_1}=\frac{b}{b_1}$
To be completed by the student

C. Sums of Angles—Theorems 13–15

I. *Define:*

Exterior angle.

II. *Summarize:*

The facts about the sums of the angles, interior and exterior, of an n-gon.

III. Make a *family tree* of Proposition 14.

D. Parallelograms—Theorems 16–21

I. *Define:*

Parallelogram, diagonal.

II. *Classify quadrilaterals.*

Is it possible to classify them in more than one way? If so, is there any preference?

III. *Summarize:*

a. Conditions under which a quadrilateral is a parallelogram;
b. Properties of a parallelogram.

IV. Make a *family tree* of Proposition 17.

E. Areas—Theorems 22–27

I. *Define:*

Commensurable, measurement, area.

II. *Summarize* the facts about areas by giving formulas for the areas of figures studied.

III. Complete the *family tree* of Theorem 27 as appended (Plate 3).

F. Similarity—Theorems 28–39

I. *Define.*

Ratio, proportion, center of similitude, similar figures.

II. *Summarize:*

a. Properties arising from similarity of triangles;
b. Conditions under which triangles are similar.

III. A *family tree* of Theorem 39, Cor. 2, is begun (Plate 4). The student is advised to make a tree of Proposition 36.

G. Locus—Theorems 40–41

I. *Define:*

Locus of a point.

II. *State* the facts of which proofs are necessary and sufficient to establish a locus theorem.

III. *State* the two locus theorems proved in the syllabus.

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PLATE 4

Th. 39, Cor. 2. The square of the hypotenuse of a right triangle is equal to the sum of the squares of its legs.

- The altitude upon the hypotenuse of a right triangle divides it into triangles similar to each other and to the original.
    - Comps. of equal $\angle$s are equal.
    - Triangles are similar if two angles of one are equal to those of another.
- The homologous sides of similar triangles have a constant ratio.
    - The homologous angles of similar triangles are equal.
    - Post. of superposition.
    - If corresponding angles are equal, lines cut by a transversal are parallel.
    - A line parallel to the base of a triangle cuts the remaining sides so that they are proportional to either pair of homologous sects.
    - Quantities equal to the same quantity are equal to each other.
- The products of equals multiplied by equals are equal.
- The sums of equals added to equals are equal.
- The whole is equal to the sum of its parts.

(To be completed by the student.)

H. The Circle and Straight Line—Theorems 42–49

I. *Define:*

Circle, chord, secant, tangent.

II. Summarize the facts proved in the syllabus about:

a. Chords;
b. Tangents;
c. Arcs.

III. Make a *family tree* of Proposition 48.

I. The Circle and Angle Measurement—Theorems 50–55

I. *Define:*

Central angle, inscribed angle.

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II. *State* the method of measuring:

- *a.* Central angles;
- *b.* Inscribed angles;
- *c.* Angles formed by a tangent and a chord;
- *d.* Angles with vertices outside the circle;
- *e.* Angles with vertices inside the circle.

III. A *family tree* of Proposition 55 is begun (Plate 5). The student should make one of Theorem 53.

J. Mensuration of the Circle—Theorems 56–64

I. *Define:*

Sector, apothem.

II. *State* formulas for:

- *a.* Area of a regular polygon;
- *b.* Circumference of a circle;
- *c.* Area of a circle.

III. *Summarize* the methods now known to you of proving:

- *a.* Sects equal;
- *b.* Angles equal;
- *c.* Lines perpendicular;
- *d.* Lines parallel;
- *e.* Triangles congruent;
- *f.* Triangles similar;
- *g.* Arcs equal;
- *i.* Chords equal.

PLATE 5

Th. 55. A tangent from a point to a circle is the mean proportional between any secant and its external sect, from the same point to the circle.

A. An inscribed angle, or one formed by a tangent and a chord is measured by one-half its intercepted arc.	Any two quantities of the same kind compare as their numeric measures.	*B.* Triangles are similar when two angles of one are equal each to each to two angles of another.	*C.* Homologous sides of similar triangles have a constant ratio.

Note.—If the student is in need of further review, he is advised to trace branches *A*, *B*, and *C* of this family tree back to their sources.

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CHAPTER I

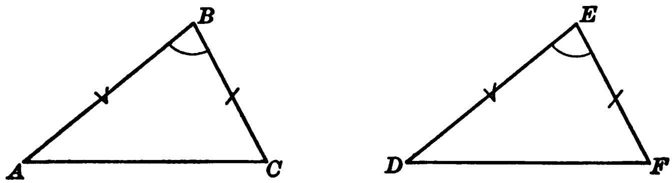
FUNDAMENTALS. RECTILINEAR FIGURES

Theorem 1. ***Vertical angles are equal.***

The pupil is earnestly advised not to refer to the First Study* for suggestions as to proofs of the theorems there taken up if it is possible for him to work them out without any help in the review. For this reason those theorems are stated in this part.

Theorem 2. ***Two sides and the included angle determine a triangle.***

Given: $\triangle ABC$ and $\triangle DEF$; $AB = DE$; $\measuredangle B = \measuredangle E$; and $BC = EF$.
Prove: $\triangle ABC \cong \triangle DEF$.



PROOF

(1) Place $\triangle DEF$ on $\triangle ABC$ so that $\measuredangle E$ coincides with $\measuredangle B$ and ED falls along BA.	(1) Superposition post.
(2) Then EF falls along BC. $\because \measuredangle E = \measuredangle B$. $\because ED = BA$, D will fall on A. $\because EF = BC$, F will fall on C.	(2) Data.
(3) $\therefore FD$ coincides with CA.	(3) Two points fix a straight line.
(4) $\therefore \triangle DEF \cong \triangle ABC$.	(4) Def. of congruence.

* Throughout "A Second Study of Plane Geometry" reference will be made to "A First Study of Plane Geometry" as "First Study," and no proofs will be given in this part of propositions contained in the former except in the case of the congruence of triangles. In this instance, one proof by the method of superposition will be given in full, since cutting and pasting were used in the First Study owing to the difficulty of the usual proof for the beginner.

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Theorem 3. *Two angles and the included side determine a triangle.*

Apply the method of proof used in Theorem 2.

What parts of the triangles will you first make coincident in superposing in this case?

What postulate is needed in order to clinch this proof?

Theorem 4. *The bisector of the vertex angle of an isosceles triangle divides it into two congruent triangles.*

Cor. 1. *The angles opposite the equal sides of an isosceles triangle are equal.*

Cor. 2. *The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.*

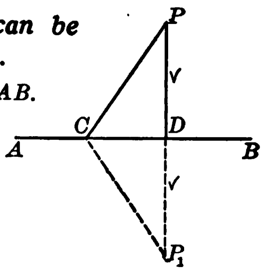
Cor. 3. *An equilateral triangle is equiangular.*

Cor. 4. *The bisectors of the angles of an equilateral triangle bisect the opposite sides and are perpendicular to them.*

Cor. 5. *The bisectors of the angles of an equilateral triangle are equal.*

Theorem 5. *A triangle is determined by its sides.*

Theorem 5a. *Only one perpendicular can be drawn through a given point to a given line.*



Given: (I) Point P in $\overline{AB}$; (II) Point P outside AB.

Prove: Only one line through P is perpendicular to AB.

Suggestion: (I) When P is in $\overline{AB}$, in how many positions of $\overline{PD}$ will the st. $\angle APB$ be divided into two right $\angle s$?

PROOF (II)

(1) Suppose $\overline{PD}$ meets $\overline{AB}$ at rt. $\angle s$ at D, and $\overline{PC}$ is a line drawn to any other point C in $\overline{AB}$.	
(2) Extend PD to P_1 so that $P_1D = PD$.	
(3) PCP_1 is not a straight line.	(3) Why?
(4) $\therefore \angle PCP_1$ is not a st. $\angle$.	(4) Why?
(5) $\triangle PCD \cong \triangle P_1CD$.	(5) Why?
(6) $\therefore \angle PCD = \angle P_1CD$.	(6) Why?
(7) $\therefore \angle PCD$ is not a rt. $\angle$.	(7) Why?
(8) $\therefore \overline{PC} \perp \overline{AB}$.	(8) Why?

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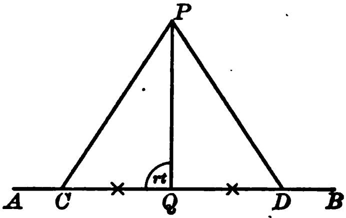
Theorem 5b. ***Two sects drawn from a point in a perpendicular to a given line, cutting off on the given line equal sects from the foot of the perpendicular, are equal and make equal angles with the perpendicular.***



Given: $PQ \perp AB$ at Q; P any point in PQ; $QC = QD$; C and D in AB.
Prove: $PC = PD$; $\angle CPQ = \angle DPQ$.
(Proof left to the student.)

In our first study, propositions dealing with inequalities were with one exception omitted. The following group of propositions supplies this omission, and before proving them it will be necessary to study a set of axioms dealing with inequalities.

1. *If unequals are operated on in the same way by positive equals, the results are unequal in the same order.*

e.g., If $a > b$ and $c = d$, where c and d are positive quantities, $a+c > b+d$, $a-c > b-d$, $ac > bd$, $\frac{a}{c} > \frac{b}{d}$; while $a^c > b^d$ and $\sqrt[c]{a} > \sqrt[d]{b}$ under certain conditions which do not affect work in elementary geometry.

Test these statements, substituting for the symbols the following values:

(1) $a=5$, $b=3$, $c=d=4$. (2) $a=\frac{1}{3}$, $b=\frac{1}{5}$, $c=d=4$.
(3) $a=2$, $b=-2$, $c=d=4$. (4) $a=\frac{1}{8}$, $b=-\frac{1}{8}$, $c=d=3$.
(5) $a=-\frac{1}{27}$, $b=-\frac{1}{8}$, $c=d=3$.

What conclusions can you draw?

What meaning is attached to the phrase "in the same order"?

2. *The sums of unequals added to unequals in the same order* (*or the same sense*) *are unequal in the same order.*

e.g., If $a > b$, and $c > d$, then $a+c > b+d$.

Show why this statement could not be made for subtraction of unequals.

3. *The differences of unequals subtracted from equals are unequal in the reverse order.*

Illustrate this fact.

4. *If the first of three quantities is greater than the second, which in turn is greater than the third, all the more then is the first greater than the third.*

Illustrate this fact.

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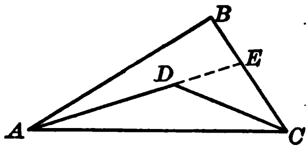
Theorem 5c. *The sum of two sects drawn from any point inside a triangle to the ends of one of its sides is less than the sum of its remaining sides.*

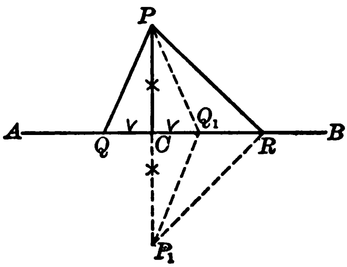


Given: $\triangle ABC$; D any point inside.
Prove: $DA+DC<BA+BC$.

PROOF

Extend AD to E in BC.	
(1) $DA+DE<$what part of $AB+BC$? $DC<DE+$what part of $AB+BC$?	(1) Why?
(2) $\therefore DA+DE+DC<$what?	(2) Why?
(3) $\therefore DA+DC<BA+BC$.	(3) Why?

Theorem 5d. *Two sects drawn from a point in a perpendicular to a given line and cutting off unequal distances from the foot of the perpendicular are unequal in the same order as those distances, and conversely.*



I. Direct.
Given: $PC\perp \overline{AQCRB}$; $CR>CQ$.
Prove: $PR>PQ$.
Suggestions: Where will Q_1 lie with respect to C and R if $CQ_1=CQ$? Why?
Why is it that whatever you prove true of PQ_1 is true of PQ?
Why is it that whatever is true of $PQ_1+P_1Q_1$ and $PR+P_1R$ is true of PQ_1 and PR?
Write a complete proof of this theorem.

II. Converse.
Given: $PC\perp \overline{AQCRB}$; $PR>PQ$.
Prove: $CR>CQ$.
Suggestions: Why can CR not be less than CQ?
Why can CR not equal CQ?
What remains for the relation of CR to CQ?

For convenience $PQ_1+Q_1P_1$ is at times referred to as PQ_1P_1 and is called a **broken line.** *Such a line is always composed of two or more sects.*

The method of proof here outlined is known as the method of *exclusion* or *elimination.* Any two quantities of the same kind (a and b) must bear one of the following relations to each other:

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$a>b$, $a=b$, or $a<b$. If any two of these relations can be shown to be false, the remaining relation must therefore be true. For further discussion and illustration see p. 299.

Cor. 1. ***All possible obliques from a point to a line are equal in pairs, and each pair cuts off equal sects from the foot of the perpendicular from that point to the line.***

(I) Under what condition will obliques from a point to a line be equal?

Why, then will all possible obliques from a point to a line be equal in pairs?

(II) Use the method of exclusion to prove the second part of the corollary.

Theorem 6. ***The perpendicular is the shortest sect from a point to a line.***

Suggestion: Use Theorem 5d to give a much simpler proof than was possible in the First Study.

Theorem 6*a*. ***The shortest sect from a point to a line is perpendicular to it.***

Hint: Show that this is a special case of Theorem 5d (converse)

Theorem 7. ***The hypotenuse and adjacent angle determine a right triangle.***

Theorem 8. ***The hypotenuse and another side determine a right triangle.***

EXERCISES. SET LXXV. TRIANGLES

Numeric

820. The perimeter of an isosceles triangle is 13, and the ratio of one of the equal sides to the base is $1\frac{2}{3}$. Find the three sides.

Theoretic

821. In proving triangles congruent, two methods have been used.

(*a*) When what elements are given equal can superposition be used?

(*b*) When must *juxtaposition* be used? Answer in a single brief sentence.

822. The sects of any bisector of a given sect cut off by perpendiculars erected at the ends of the given sect are equal.

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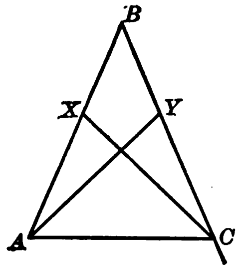
823. The sects of a perpendicular to the bisector of an angle at any point in it and limited by the sides of the angle are equal.

824. Sects joining any point in the bisector of an angle to points on the sides equidistant from the vertex are equal.



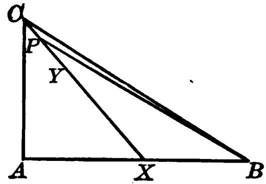
825. If in the accompanying diagram, $\angle BAC = \angle BCA$, and $\angle BAY = \angle BCX$, sect XC = sect YA.

826. If the sides of an equilateral triangle are prolonged in turn by equal lengths, and the extremities of these sects are joined, another equilateral triangle is formed.

827. Two isosceles triangles are congruent if one of the equal sides and the altitude upon that side are equal each to each.

828. Triangles are congruent if two sides and the altitudes upon one of them are equal each to each.

829. A triangle is determined by a side and the median* and the altitude to that side.



d830. If in the accompanying diagram $\angle A$ is a right angle, $YX = AC$, $CP = PY$, show that $PB + PX > CB + CA$.

831. The sum of the distances from any point inside a triangle to the vertices is greater than its semiperimeter, but less than its perimeter.

832. The sum of the diagonals of a quadrilateral is greater than the sum of either pair of opposite sides.

833. The sum of the diagonals of a quadrilateral is less than its perimeter, but greater than its semiperimeter.

834. The sum of the medians of a triangle is less than one and a half times its perimeter. (Prove this and the following exercise without assuming the concurrence of the medians).

835. The sum of the medians of a triangle is greater than its semiperimeter.

836. Each altitude of a triangle is less than half the sum of the adjacent sides.

* A *median* is the sect between any vertex of a △ and the midpoint of the side opposite that vertex.

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837. The sum of the altitudes of a triangle is less than the perimeter.

838. Show that the bisector of the vertex angle of an isosceles triangle is coincident with the altitude to its base.

Construction *

839. Bisect a *reflex* angle.

d840. To construct a triangle, having given a side, the median and the altitude to a second side.

d841. To construct a triangle having given one side, the corresponding median, and the altitude to another side.

Theorem 9. ***Lines perpendicular to the same line are parallel.***

Theorem 10. ***A line perpendicular to one of a series of parallels is perpendicular to the others.***

Theorem 11. ***If when lines are cut by a transversal the alternate-interior angles are equal, the lines thus cut are parallel.***

Cor. 1. ***If the alternate-exterior angles or corresponding angles are equal when lines are cut by a transversal, the lines thus cut are parallel.***

Cor. 2. ***If either the consecutive-interior angles or the consecutive-exterior angles are supplementary when lines are cut by a transversal, the lines thus cut are parallel.***

Theorem 12. ***Parallels cut by a transversal form equal alternate-interior angles.***

Cor. 1. ***Parallels cut by a transversal form equal corresponding angles and equal alternate-exterior angles.***

Cor. 2. ***Parallels cut by a transversal form supplementary consecutive-interior angles and supplementary consecutive-exterior angles.***

---

* For a discussion of methods of attacking problems in construction see Chapter VII, p. 306, where additional exercises will also be found. At this point, for instance, the topics, "The Synthetic Method of Attacking a Problem" (p. 309) and "The Formal Analysis of a Problem" (p. 318) should be studied.

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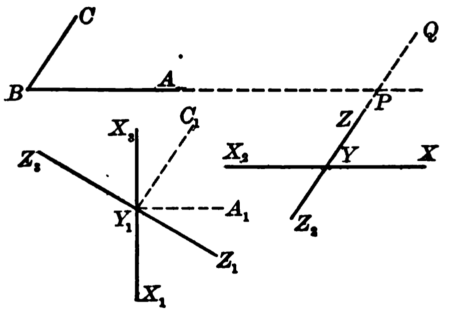
Theorem 12a. ***Two angles whose sides are parallel each to each or perpendicular each to each are either equal or supplementary.***



Given: (I) $\overline{X_2YX} \parallel BA$; $\overline{Z_2YZ} \parallel BC$.
(II) $\overline{X_3Y_1X_1} \perp BA$; $\overline{Z_3Y_1Z_1} \perp BC$.

Prove: (I) $\angle XYZ = \angle ABC = \angle X_2YZ_2$; $\angle ZYX_2 = 180° - \angle ABC = \angle Z_2YX$.
(II) $\angle X_1Y_1Z_1 = \angle ABC = \angle X_3Y_1Z_3$; $\angle Z_1Y_1X_3 = 180° - \angle ABC = \angle Z_3Y_1X_1$.

Suggestions: Using the construction lines prove (I).

(II) If $Y_1C_1 \parallel BC$ and $Y_1A_1 \parallel BA$, what relation exists between $\angle ABC$ and $\angle A_1Y_1C_1$?

How are Y_1C_1 and Y_1Z_1 related? How Y_1A_1 and Y_1X_1?

Then how are $\angle$s $A_1Y_1C_1$ and $Z_1Y_1A_1$ related?

Then how are $\angle$s $X_1Y_1Z_1$ and $Z_1Y_1A_1$ related?

Draw conclusions.

Theorem 13. ***The sum of the angles of a triangle is a straight angle.***

Cor. 1. ***A triangle can have but one right or one obtuse angle.***

Cor. 2. ***Triangles having two angles mutually equal are mutually equiangular.***

Cor. 3. ***A triangle is determined by a side and any two homologous angles.***

Cor. 4. ***An exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.***

Theorem 14. ***The sum of the angles of a polygon is equal to a straight angle taken as many times less two as the polygon has sides.***

Cor. 1. ***Each angle of an equiangular polygon of n sides equals the*** $\frac{n-2}{n}$***th part of a straight angle.***

Cor. 2. ***The sum of the exterior angles of a polygon is two straight angles.***

Cor. 3. ***Each exterior angle of an equiangular polygon of n sides is equal to the*** $\frac{2}{n}$***th part of a straight angle.***

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Theorem 15. ***If two angles of a triangle are equal, the sides opposite them are equal.***

Cor. 1. ***Equiangular triangles are equilateral.***

EXERCISES. SET LXXVI. PERPENDICULARS. PARALLELS. SUMS OF ANGLES OF POLYGONS

Numeric

842. Find the angles of an isosceles triangle if a base angle is double the vertex angle.

843. If the vertex angle of the isosceles triangle is 30°, find the angle formed by the bisectors of the base angles. If the vertex angle is B?

844. The bisector of the base angle of an isosceles triangle makes with the opposite leg an angle of 53° 17′. Find the angles of the triangle.

845. Find the angles of an isosceles triangle if the altitude is one-half the base.

846. If the angle at the vertex of an isosceles triangle is 36°, the bisector of a base angle divides the triangle into two isosceles triangles. Find the lengths of all the sects in the diagram if a leg of the given triangle is a and the base is b.

847. What is the sum of the angles of (*a*) a hexagon, (*b*) a heptagon, (*c*) an octagon, (*d*) a nonagon, (*e*) a decagon, (*f*) a polygon of 18 sides, (*g*) a polygon of 24 sides, (*h*) of 30 sides?

848. Find each angle of an equiangular (*a*) hexagon, (*b*) heptagon, (*c*) octagon, (*d*) nonagon, (*e*) decagon.

849. In what polygon is the sum of the angles three times as great as in a pentagon.

850. How many sides has a polygon if

(*a*) the sum of the interior angles equals 4 rt. ∡s? 3 st. ∡s? 6 rt. ∡s? 8 st. ∡s? 20 rt. ∡s?

(*b*) the sum of the interior angles is 2 (or 3, or 4, or 5, or 6) times as large as the sum of the exterior angles?

(*c*) the sum of the interior angles exceeds the sum of the exterior angles by 4 rt. ∡s? 3 st. ∡s? 9 st. ∡s?

(*d*) the ratio of each interior angle to its adjacent exterior angle is 2 to 1? 3 to 2? 5 to 1? a to b? May a and b have any values whatever?

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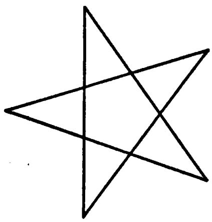
(*e*) each exterior angle contains 40°? 30°? 20°? 120°?

(*f*) each interior angle is $\frac{1}{3}$ st. ∢? $\frac{3}{5}$ st. ∢? a rt. ∢? $\frac{3}{2}$ rt. ∢? $\frac{8}{5}$ rt. ∢? $\frac{7}{8}$ st. ∢?

Only what kind of polygon is considered in (*d*), (*e*), and (*f*)?

d851. If the sides of a polygon are extended until they intersect, a *star polygon* results. A five-pointed star is called a *pentagram*, and is of historic interest as the badge chosen by the followers of Pythagoras. Star polygons may also be formed by chords of circles or by certain combinations of polygons. What is the sum of the vertex angles of a five-pointed star? A six-pointed star? An *n*-pointed star? (Make your solution general.)



d852. If A, B, C, denote the angles of a triangle, h_a, h_b, the altitudes upon BC, and AC, t_a, t_b, the bisectors of A and B, find:

(*a*) $\sphericalangle t_a t_b$. (*b*) $\sphericalangle h_a h_b$. (*c*) $\sphericalangle t_a h_a$.

853. The number of all diagonals of a polygon of n sides is $\frac{n(n-3)}{2}$ Test this statement in several instances.

Theoretic

854. The bisectors of a pair of consecutive-interior angles of parallels cut by a transversal are perpendicular to each other.

855. The bisectors of a pair of alternate-interior angles of parallels cut by a transversal are parallel to each other.

856. The bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base, and conversely.

857. An exterior angle at the vertex of an isosceles triangle is double a base angle.

858. State and prove the converse of Ex. 857

d859. If a leg of an isosceles triangle is produced through the vertex by its own length, and its extremity joined to the extremity of the base, the joining line is perpendicular to the base.

d860. The bisectors of the angles of a quadrilateral form a quadrilateral the sum of whose opposite angles is equal to two right angles.

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Theorem 16. *Either diagonal of a parallelogram bisects it.*

Cor. 1. *The parallel sides of a parallelogram are equal, and the opposite angles are equal.*

Cor. 2. *Parallels are everywhere equidistant.*

Theorem 17. *A quadrilateral whose opposite sides are equal is a parallelogram.*

Theorem 18. *A quadrilateral having a pair of sides both equal and parallel is a parallelogram.*

Theorem 19. *A parallelogram is determined by two adjacent sides and an angle; or parallelograms are congruent if two adjacent sides and an angle are equal each to each.*

Theorem 20. *The diagonals of a parallelogram bisect each other.*

Theorem 21. *A quadrilateral whose diagonals bisect each other is a parallelogram.*

EXERCISES. SET LXXVII. PARALLELOGRAMS

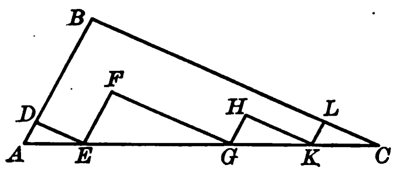
Theoretic

861. If in the accompanying diagram, $DE \parallel FG \parallel HK \parallel BC$ and $KL \parallel GH \parallel EF \parallel AB$.

(*a*) What is the sum of AD, DE, EF, FG, GH, HK, KL, and LC?

(*b*) How, if at all, does the length of AC affect the solution?

(*c*) How, if at all, does the number of parallels affect the solution?



d862. The sum of the perpendiculars from any point in the base of an isosceles triangle to the legs is constant, and equal to the altitude upon a leg.

863. The sum of the perpendiculars from any point inside an equilateral triangle to the three sides is equal to the altitude.

The following group of six propositions further supplies facts relating to inequalities of sects and angles omitted in the First Study.

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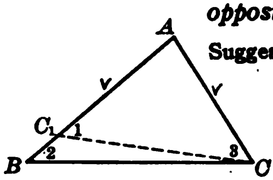
Theorem 21*a*. ***The difference between any two sides of a triangle is less than the third side.***

Hint: If $c<a+b$, how can we express a in terms of b and c?

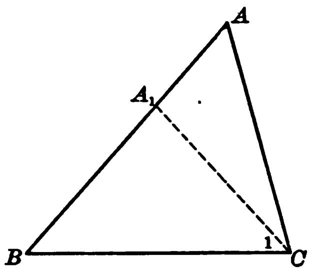
Theorem 21*b*. ***If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.***



Suggestions: Make $AC_1=AC$. Justify this construction.

What relation exists between ∡1 and ∡2?
What relation exists between ∡2 and ∡3?
What relation exists between ∡3 and ∡C?

Theorem 21*c*. ***If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.***

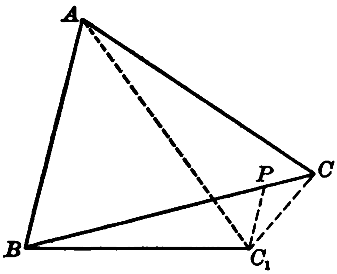


Suggestions: Make ∡1 = ∡B. Why?

What relation exists between AB, AA_1+A_1C, and AC?

NOTE.—Since Prop. 21*c* is the converse of Prop. 21*b* an indirect proof would have been possible. Why? Can you see any advantage in giving a direct proof where convenient?

Theorem 21*d*. ***If two triangles have two sides equal each to each but the included angles unequal, their third sides are unequal in the same order as those angles.***



Suggestions: Why is it desirable and why possible that the triangles should be placed together as suggested by the accompanying diagram with AB ($<AC$) in coincidence in the two triangles?

If we could break the sect BC into two sects such as $BP+PC_1$ it would be evident that $BP+PC_1>BC_1$.

Then our problem is to so locate P that $PC_1=PC$.
What kind of triangle, therefore, is CPC_1?
How, then, shall we draw C_1P?

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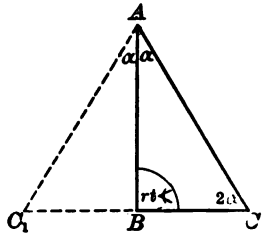
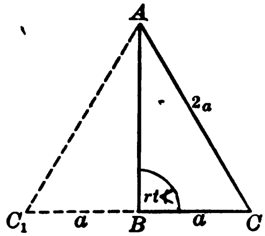
Theorem 21*e*. ***If two triangles have two sides equal each to each, but the third sides unequal, the angles opposite those sides are unequal in the same order.***



Suggestions: Use method of exclusion.

Theorem 21*f*. ***If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg, and conversely.***



Suggestions: (*a*) Direct.

What part of a straight angle is 2α?
What kind of triangle is ACC_1?
How are the sects CB, CC_1, and AC related?

(*b*) Converse.

How is AC_1 related to AC?
What kind of triangle is ACC_1?
How are the angles C_1AC, BAC, and ACC_1 related?

EXERCISES. SET LXXVIII. INEQUALITIES

Numeric

864. Two sides of a triangle are 10″ and 13″. Between what limits must the third side lie?

865. If two angles of a triangle are respectively 55° and 65°, which is the longest, and which the shortest side of the triangle?

866. If one angle of a triangle is one-third of a straight angle, and a non-adjacent exterior angle of the triangle is five-eighths of a straight angle, which side of the triangle is the longest and which is the shortest?

Theoretic

867. Either leg of an isosceles triangle is greater than a sect connecting the vertex with any point in the base.

868. The sect joining the vertex of an isosceles triangle to any point in the prolongation of its base is greater than either leg.

869. If one leg AB of an isosceles triangle ABC is produced beyond the base BC to a point D, then $\angle ACD$ is greater than $\angle D$.

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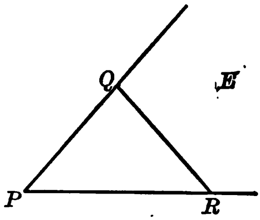
870. If $PQ \equiv QR$ in the accompanying diagram, prove $EP > ER$.



871. If no median of a triangle is perpendicular to the side to which it is drawn, the triangle is not isosceles.

872. If any point in the prolongation of a leg produced through the vertex of an isosceles triangle whose base is shorter than a leg is joined to the extremity of the base, a scalene triangle will be formed.

873. If a vertex of an equilateral triangle is joined to any point in the prolongation of the opposite side, a scalene triangle is formed. Which is the longest and which the shortest side of each of the triangles thus formed?

d874. Prove that the sect joining an extremity of the base of an isosceles triangle to any point in the opposite leg is greater than one sect cut off on that leg. Is it ever greater than either sect? Under what condition is it less than one of the sects?

875. The diagonals of a *rhomboid* intersect obliquely, and the greater angle formed by them lies opposite the greater side of the parallelogram. (A rhomboid is an oblique ▱ in which the adjacent sides are unequal.)

876. If a triangle is not isosceles the median to any side is not perpendicular to it, and the larger angle which it forms with that side lies opposite the greater of the remaining two sides.

d877. Any point not on the perpendicular bisector of a sect is unequally distant from the ends of the sect.

LIST OF WORDS DEFINED IN CHAPTER I

Exclusion, elimination, broken line.

SUMMARY OF AXIOMS IN CHAPTER I

1. If unequals are operated on in the same way by positive equals, the results are unequal in the same order.
2. The sums of unequals added to unequals in the same order (or the same sense) are unequal in the same order.
3. The differences of unequals subtracted from equals are unequal in the reverse order.
4. If the first of three quantities is greater than the second, which in turn is greater than the third, all the more then is the first greater than the third.

(For a summary of theorems see Chapter VIII, p. 324.)

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CHAPTER II

AREAS OF RECTILINEAR FIGURES

Theorem 22. ***Rectangles having a dimension of one equal to that of another compare as their remaining dimensions.***

Theorem 23. ***Any two rectangles compare as the products of their dimensions.***

Theorem 24. ***The area of a rectangle is equal to the product of its base and altitude.***

Theorem 25. ***The area of a parallelogram is equal to the product of its base and altitude.***

Cor. 1. ***Any two parallelograms compare as the products of their bases and altitudes.***

In proving Cors. 1, 2, 3 of this theorem and the next, express the areas as algebraic formulas and apply axioms. Why is such a procedure both convenient and natural?

Cor. 2. ***Parallelograms having one dimension equal compare as the remaining dimensions.***

Cor. 3. ***Parallelograms having equal bases and equal altitudes are equal.***

Theorem 26. ***The area of a triangle is equal to half the product of its base and its altitude.***

Cor. 1. ***Any two triangles compare as the products of their bases and altitudes.***

Cor. 2. ***Triangles having one dimension equal compare as their remaining dimensions.***

Cor. 3. ***Triangles having equal bases and equal altitudes are equal.***

Theorem 26*a*. ***The square on the hypotenuse of a right triangle equals the sum of the squares on the two legs.***

NOTE.—In this text the conventional phraseology will be followed, and the usual distinction between such expressions as "square on" and "square of" will be observed. **Square on** *will mean the area of the square constructed on the given sect*, and **square of,** *the square of the numeric measure of the given sect.*

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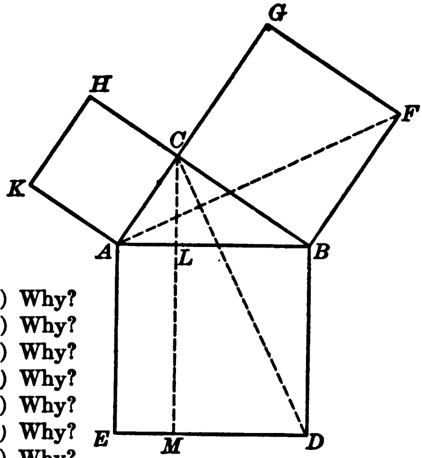
Given: $\triangle ABC$; $\measuredangle BCA =$ rt. $\measuredangle$; squares $CBFG$, $ABDE$, and $CAKH$.

Prove: Sq. $ABDE =$ Sq. $CBFG +$ Sq. $CAKH$.

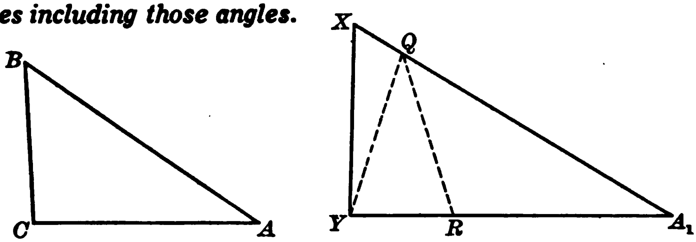
Proof: (Some steps which the student can readily supply have been purposely omitted.)



Draw $CM \perp \overleftrightarrow{EMD}$.	
Draw CD and AF.	
(1) $\therefore CM \parallel BD$ and AE.	(1) Why?
(2) ACG is a st. line.	(2) Why?
(3) $\triangle ABF = \frac{1}{2}(\overline{BF} \cdot \overline{CB})$.	(3) Why?
(4) $\triangle ABF = \frac{1}{2}$ (Sq. $CBFG$).	(4) Why?
(5) $LMDB$ is a rect.	(5) Why?
(6) $\triangle CDB = \frac{1}{2}(\overline{BD} \cdot \overline{LB})$.	(6) Why?
(7) $\triangle CDB = \frac{1}{2}$ (Rect. $LMDB$).	(7) Why?
(8) $BD = AB$, $CB = BF$.	(8) Why?
(9) $\measuredangle DBC = \measuredangle FBA$.	(9) Why?
(10) $\therefore \triangle ABF \cong \triangle CBD$.	(10) Why?
(11) $\therefore$ Sq. $CF =$ Rect. LD.	(11) Why?

Draw the necessary construction lines and show that sq. AH equals rect. AM. Finish the proof.

Theorem 26*b*. ***The areas of two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including those angles.***



Given: $\triangle ABC$ and $\triangle A_1XY$ with $\measuredangle A = \measuredangle A_1$.

To Prove: $\dfrac{\triangle ABC}{\triangle A_1XY} = \dfrac{\overline{AB} \cdot \overline{AC}}{\overline{A_1X} \cdot \overline{A_1Y}}$.

Suggestions: Proof being left to the student.

Place $\triangle ABC$ in the position of $\triangle A_1QR$. Draw QY.

$\dfrac{\triangle A_1RQ}{\triangle A_1YQ} = \dfrac{A_1R}{A_1Y}$, and $\dfrac{\triangle A_1QY}{\triangle A_1XY} = \dfrac{A_1Q}{A_1X}$.

Could the case arise in which QR intersects XY between X and Y? If so, show that this proof still holds or give one that does hold.

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Theorem 27. ***The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.***

EXERCISES. SET LXXIX. AREAS

Numeric

878. Find the expense of paving a path 4′ wide inside a square piece of ground the side of which is 50′ if the price is 18 cents per square yard. What would be the cost if the path were outside the piece of ground?

a879.* Find the dimensions of a rectangle given:

(*a*) Area 216 sq. ft., perimeter 60 ft.

(*b*) Area 600 sq. ft., difference of the sides 10 ft.

(*c*) Area 756, sides in the ratio $\frac{7}{3}$.

(*d*) Area 340 sq. ft., sum of squares of two consecutive sides 689 sq. ft.

880. Find the change in the area of a triangle of base a and altitude h in the following cases:

(*a*) If a and h are increased by m and n, respectively.

(*b*) If a and h are diminished by m and n, respectively.

(*c*) If a is increased by m, and h diminished by n.

(*d*) If a is diminished by m, and h increased by n.

881. A line of division is drawn between two sides of a triangle, dividing it into a triangle and a quadrilateral. What parts are these two figures, respectively, of the entire triangle if the line of division cuts off the following parts of the two sides, reckoned from the intersection of the sides?

(*a*) $\frac{1}{2}$ and $\frac{1}{3}$. (*b*) $\frac{2}{3}$ and $\frac{3}{4}$. (*c*) $\frac{1}{2}$ and $\frac{1}{2}$.

(*d*) $\frac{1}{5}$ and $\frac{1}{5}$. (*e*) $\frac{1}{m}$ and $\frac{1}{n}$ (*f*) $\frac{1}{n}$ and $\frac{1}{m}$.

882. Find the area of a rhombus, given:

(*a*) The diagonals 18 and 12 units.

(*b*) The sum of the diagonals 12, and their ratio $\frac{3}{5}$.

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* As here, "a" precedes a problem which calls for the solution of an affected quadratic equation.

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Theoretic

883. (*a*) Prove, geometrically, the algebraic formula,

$$a(b+c) \equiv ab+ac.$$

(*b*) Prove, geometrically, the algebraic formula,

$$a(b-c) \equiv ab-ac.$$

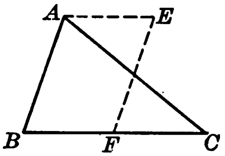
(*c*) Prove, geometrically, the algebraic formula,

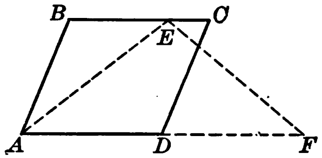
$$a^2-b^2=(a+b)(a-b).$$

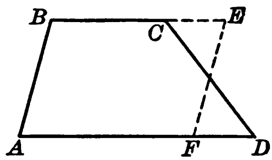
884. The area of a rhombus is equal to half the product of its diagonals.

885. If lines are drawn from any point inside a parallelogram to the four vertices, the sum of either pair of triangles with parallel bases is equal to the sum of the other pair.

886. The accompanying figures show easy methods of transforming (*a*) a triangle into a parallelogram, (*b*) a parallelogram into a triangle, (*c*) a trapezoid into a parallelogram. Explain. Can you give more than one explanation? If so, upon what does your explanation depend?







For problems in construction based upon this chapter see Chapters VII and VIII.

TERMS DEFINED IN CHAPTER II

Square on (a sect), square of (a quantity), medians of a triangle, rhomboid.

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CHAPTER III

SIMILARITY

The following are terms with which the student should now be familiar:

*Antecedent, consequent, extremes, means, mean proportion, mean proportional, continued proportion, inversion, composition, division, composition and division.**

Theorem 28. ***Any proportion may be transformed by alternation.***

Theorem 29. ***In any proportion the terms may be combined by composition or division.***

Theorem 30. ***In a series of equal ratios, the ratio of the sum of any number of antecedents to the sum of their consequents equals the ratio of any antecedent to its consequent.***

Theorem 31. ***A line parallel to one side of a triangle divides the other sides proportionally.***

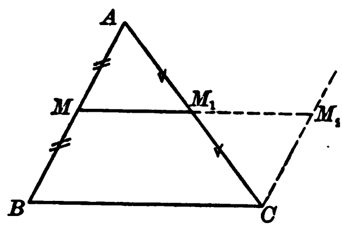
Cor. 1. ***One side of a triangle is to either of the sects cut off by a line parallel to a second side as the third side is to its homologous sect.***

Cor. 2. ***Parallels cut off proportional sects on all transversals.***

Cor. 3. ***Parallels which intercept equal sects on one transversal do so on all transversals.***

Cor. 4. ***A line which bisects one side of a triangle, and is parallel to the second, bisects the third.***

Cor. 5. ***A sect which bisects two sides of a triangle is parallel to the third side and equal to half of it.***



Suggestions: What means have you for proving lines parallel?

How may one sect be proved half of another?

What kind of figure would you like to have?

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* The student will find an alphabetical index of definitions on p. 379.

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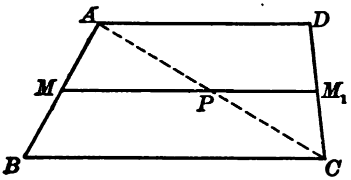
The sect joining the mid-points of the non-parallel sides of a trapezoid is called its **median.**



Cor. 6. ***The median of a trapezoid is parallel to the bases and equal to one-half their sum.***

Suggestions: Why draw a diagonal? Show that a line through $M \parallel \overline{BC}$ will bisect AC and therefore DC, and hence coincide with $\overline{MM_1}$.

Cor. 7. ***The area of a trapezoid equals the product of its median and altitude.***

What numerical relation exists between the median and the bases of a trapezoid?

Theorem 32. ***A line dividing two sides of a triangle proportionally is parallel to the third side.***

Cor. 1. ***A line dividing two sides of a triangle so that these sides bear the same ratio to a pair of homologous sects is parallel to the third side.***

DIVISION OF A SECT

A point in a sect is said to divide it **internally,** *and a point in the prolongation of a sect is said to divide it* **externally,** *and in both cases the divisions of the sect are reckoned from one extremity to the point of division and from that point to the other extremity of the original sect.*

I divides sect AB internally in the ratio $\frac{\overline{AI}}{\overline{IB}}$.

E divides sect AB externally in the ratio $\frac{\overline{AE}}{\overline{EB}}$.

NOTE.—When neither "internal" nor "external" is used to qualify the division, internal is understood.

Is there any ratio into which a sect cannot be divided externally?

A sect is said to be divided **harmonically** *when it is divided internally and externally in the same ratio.*

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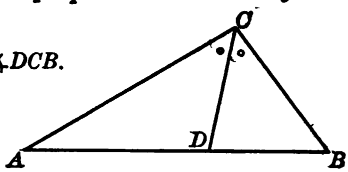
In the foregoing illustration AB is divided harmonically if $\frac{\overline{AI}}{\overline{IB}}=\frac{\overline{AE}}{\overline{EB}}$.

Is there any ratio into which a sect cannot be divided harmonically?

Discuss Theorems 31 and 32 from the point of view of external division of a sect.

Theorem 32a. ***The bisector of an angle of a triangle divides the opposite side into sects which are proportional to the adjacent sides.***

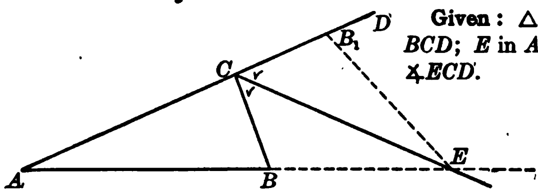


Given: $\triangle ABC$; D on AB and $\measuredangle ACD = \measuredangle DCB$.

To prove: $\frac{AD}{BD}=\frac{AC}{BC}$.

Suggestions: Compare $\triangle ADC$ and $\triangle BDC$ in two ways.

Theorem 32b. ***The bisector of an exterior angle of a triangle divides the opposite side externally into sects which are proportional to the adjacent sides.***



Given: $\triangle ABC$ with exterior $\measuredangle BCD$; E in AB produced; $\measuredangle BCE = \measuredangle ECD$.

To prove: $\frac{AE}{BE}=\frac{AC}{BC}$.

Suggestions: Compare $\triangle AEC$ first with $\triangle BEC$, and second with $\triangle B_1EC$

How should B_1 be taken so that $\triangle B_1EC$ may be substituted for $\triangle BEC$? Consider special cases where $a=b$, $a>b$.

Cor. 1. ***The bisectors of an adjacent interior and exterior angle of a triangle divide the opposite side harmonically.***

(Proof left to the student. Discuss special cases.)

EXERCISES. SET LXXX. RATIO. PROPORTION. PARALLELS

Numeric

887. Find the value of n if (a) $\frac{5}{n}=\frac{3}{8}$, (b) $\frac{a}{n}=\frac{b}{c}$.

888. If $\frac{a^3}{b^3}=\frac{216}{125}$, find $\frac{a}{b}$.

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889. If $\dfrac{\sqrt[3]{a}}{\sqrt[3]{b}}=\dfrac{1}{2}$, find $\dfrac{a}{b}$.

890. If $\dfrac{\sqrt[3]{a}}{3}=\dfrac{\sqrt[3]{b}}{5}$, find $\dfrac{a}{b}$.

91. Form all possible proportions involving a, b, p, and q if $ab=pq$.

892. Find the mean proportionals between:

(*a*) 2 and 50. (*c*) 3 and 21. (*e*) a and b.
(*b*) 2 and 75. (*d*) 3 and 19. (*f*) $a+b$ and $a-b$.

893. Find the third proportional to (*a*) 5 and 6, (*b*) a and b.

894. Transform the proportion $\dfrac{a}{b}=\dfrac{c}{d}$ so that b becomes the third term.

895. If $\dfrac{a+b}{b}=\dfrac{9}{5}$, what is the ratio of a to b?

896. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\dfrac{9}{11}$, find $\dfrac{a+c+e}{b+d+f}$.

897. Find a fourth proportional to:

(*a*) a, ab, c. (*b*) $a^2, 2ab, 3b^2$. (*c*) $x^3, xy, 5x^2y$.

898. Find a third proportional to:

(*a*) a^2b, ab. (*b*) $x^3, 2x^2$. (*c*) $3x, 6xy$. (*d*) $1, x$.

899. Find mean proportionals between:

(*a*) a^2, b^2. (*b*) $2x^3, 8x$. (*c*) $12ax^2, 3a^3$. (*d*) $27a^2b^3, 3b$.

900. If a, b, c, be in mean proportion, show that:

(*a*) $\dfrac{a}{a+b}=\dfrac{a-b}{a-c}$. (*b*) $(b^2+bc+c^2)(ac-bc+c^2)=b^4+ac^3+c^4$.

d901.* If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that: (*a*) $\dfrac{ab+cd}{ab-cd}=\dfrac{a^2+c^2}{a^2-c^2}$.

(*b*) $\dfrac{a^2+ac+c^2}{a^2-ac+c^2}=\dfrac{b^2+bd+d^2}{b^2-bd+d^2}$. (*c*) $\dfrac{a}{b}=\dfrac{\sqrt{3a^2+5c^2}}{\sqrt{3b^2+5d^2}}$.

(*d*) $\dfrac{\dfrac{a}{p}+\dfrac{b}{q}}{a}=\dfrac{\dfrac{c}{p}+\dfrac{d}{q}}{c}$. (*e*) $\dfrac{\dfrac{b}{a}+\dfrac{a}{b}}{\dfrac{ab}{a^2+b^2}}=\dfrac{\dfrac{d}{c}+\dfrac{c}{d}}{\dfrac{cd}{c^2+d^2}}$.

* For guidance in proofs such as this exercise requires see p. 304.

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a902. Solve the equations:

(a) $\frac{3x-1}{6x-7}=\frac{7x-10}{9x+10}$. (b) $\frac{x-12}{y+3}=\frac{2x-19}{5y-13}=\frac{5}{14}$.

(c) $\frac{x^2-2x+3}{2x-3}=\frac{x^2-3x+5}{3x-5}$. (d) $\frac{2x-1}{x^2+2x-1}=\frac{x+4}{x^2+x+4}$.

Hint: Transform. In doing so do you lose any roots?

903. Prove that a, b, c, d are in proportion if $(a+b-3c-3d)(2a+2b-c+d)=(2a+2b-c-d)(a-b-3c+3d)$.

904. If b is a mean proportional between a and c, show that $4a^2-9b^2$ is to $4b^2-9c^2$ in the ratio of a^2 to b^2.

905. If a, b, c, d are in continued proportion, *i.e.*, if $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $b+c$ is a mean proportional between $a+b$ and $c+d$.

d906. If $\frac{a+b}{b+c}=\frac{c+d}{d+a}$, prove that $a=c$, or $a+b+c+d=0$.

d907. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each of these ratios is equal to

$$\sqrt[3]{\frac{2a^2c+3c^3e+4e^2c}{2b^2d+3d^3e+4f^2d}}$$

908. Two numbers are in the ratio of $\frac{3}{4}$, and if 7 be subtracted from each, the remainders are in the ratio of $\frac{2}{3}$; find them.

909. What number must be taken from each term of the ratio $\frac{27}{35}$ that it may become $\frac{2}{3}$?

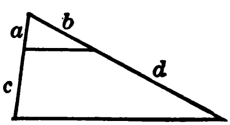
910. What number must be added to each term of the ratio $\frac{37}{29}$ that it may become $\frac{8}{7}$?

911. If $\frac{p}{b-c}=\frac{q}{c-a}=\frac{r}{a-b}$, show that $p+q+r=0$.

912. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a-b}$, show that $x-y+z=0$.

d913. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\sqrt{\frac{a^6b-2c^5e+3a^4c^3e^2}{b^7-2d^5f+3b^4cd^2e^2}}=\frac{ace}{bdf}$.

914. In the accompanying diagram,



(a) If $a=3$, $b=4$, $c=7$, find d.

(b) If $a=5$, $c=9$, $d=10$, find b.

(c) Find each sect in terms of the other three.

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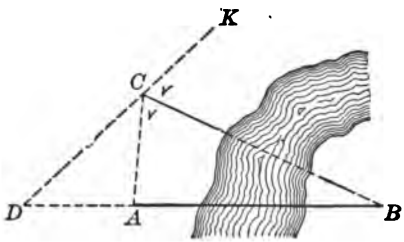
915. To measure indirectly the distance from an accessible point A to an inaccessible point B, run CK through C, a point from which A and B are both visible, making $\angle KCB = \angle BCA$. Sight D in line with K and C and also in line with A and B. What sects must now be measured in order to compute AB?

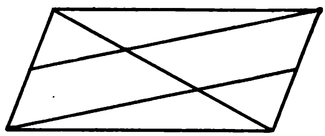


916. Find the area of a trapezoid, given the median m, and the altitude h.

Theoretic

917. The perpendiculars dropped from the mid-points of two sides of a triangle to the third side are equal.

918. Lines joining the mid-points of two opposite sides of a parallelogram to the ends of a diagonal trisect the other diagonal.



919. A line bisecting one of the non-parallel sides of a trapezoid and parallel to the base bisects the other non-parallel side.

920. The sects joining the mid-points of the consecutive sides of any quadrilateral form a parallelogram.

d Consider whether there are any modifications of this fact in the case of (*a*) the parallelogram, (*b*) the rectangle, (*c*) the rhombus, (*d*) the square.

d921. The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals determine the vertices of a parallelogram.

d922. The sects joining the mid-points of the opposite sides of a quadrilateral and the sects joining the mid-points of the diagonals are concurrent.

d923. If perpendiculars are drawn from the four vertices of a parallelogram to any line outside the parallelogram, the sum of the perpendiculars from one pair of opposite vertices equals the sum of those from the other pair.

924. The triangle formed by two lines drawn from the mid-point of either of the non-parallel sides of a trapezoid to the opposite vertices is equivalent to half the trapezoid.

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925. State and prove the converse of proposition 32*a*.

926. State and prove the converse of proposition 32*b*.

927. If a sect *PQ* is divided harmonically at *R* and *S*, then sect *RS* is divided harmonically at *P* and *Q*.

Construction

928. Construct a third proportional to two given sects.

929. Construct a fourth proportional to three given sects.

930. If *a*, *b*, *c* are given sects, construct (*a*) sect *d*, so that $\frac{a}{b}=\frac{d}{c}$, (*b*) $d=\frac{ab}{c}$.

931. Divide a sect (*a*) internally into sects proportional to two given sects, (*b*) externally into sects proportional to two given sects, (*c*) harmonically in the ratio of two given sects.

932. Divide a given sect (*a*) internally in a given ratio without the use of parallels, (*b*) externally, (*c*) harmonically.

d933. Construct two sects given (*a*) their sum and their ratio, (*b*) their difference and their ratio.

d934. Through a given point *P* draw a line meeting the sides of an angle *A* in the points *B* and *C* so that (*a*) $\overline{AB}=\overline{AC}$, (*b*) $\overline{BC}=2\overline{AC}$.

Theorem 33. ***The homologous angles of similar triangles are equal, and their homologous sides have a constant ratio.***

Theorem 34. ***Triangles are similar when two angles of one are equal, each to each, to two angles of another.***

Cor. 1. ***Triangles which have their sides parallel or perpendicular each to each are similar.***

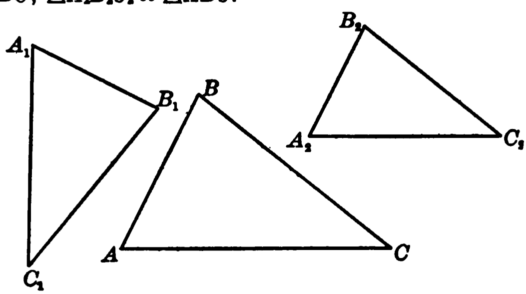
Given: $A_1B_1 \perp AB$; $A_2B_2 \parallel AB$; $B_1C_1 \perp BC$; $B_2C_2 \parallel BC$; $C_1A_1 \perp CA$; $C_2A_2 \parallel CA$.
Prove: $\triangle A_1B_1C_1 \backsim \triangle ABC$; $\triangle A_2B_2C_2 \backsim \triangle ABC$.

Suggestions: Show that:

(1) Three angles of one triangle cannot be supplementary to three angles of the other.

(2) Two angles of one triangle cannot be supplementary to two of the other.

(3) What, then, is the fact?



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Theorem 35. ***Triangles which have two sides of one proportional to two sides of another and the included angles equal are similar.***

Theorem 36. ***If the ratio of the sides of one triangle to those of another is constant, the triangles are similar.***

EXERCISES. SET LXXXI. SIMILARITY OF TRIANGLES

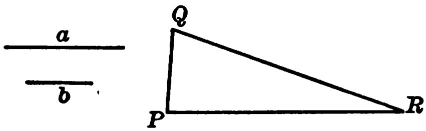
Numeric

935. If the sides of a triangle are 3, 7, and 8, find the sides of a similar triangle in which the side homologous to 7 is 9.

936. If the sides of a triangle are a, b, and c, find the sides of a similar triangle in which the side homologous to a is p.

Construction

937. The sides of a triangle are 5, 6, and 7. Construct a triangle similar to the original, having the ratio of similitude 3 to 2.



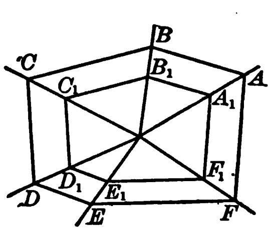
938. Construct a triangle similar to the accompanying triangle with the ratio of similitude equal to that of the two given sects a and b.

Theoretic

939. Two isosceles triangles are similar if an angle of one is equal to the homologous angle of the other.

940. Prove that the altitudes of a triangle are inversely proportional to the sides to which they are drawn.

941. If the altitudes AD and BE in $\triangle ABC$ are drawn, prove that $\frac{AC}{BC}=\frac{DC}{EC}$. Are the altitudes directly or inversely proportional to DC and EC?



942. If a spider, in making its web, makes $A_1B_1 \parallel AB$, $B_1C_1 \parallel BC$, $C_1D_1 \parallel CD$, $D_1E_1 \parallel DE$, and $E_1F_1 \parallel EF$, and then runs a line from $F_1 \parallel FA$, will it strike the point A_1? Prove your answer.

943. If D is taken in the leg AB of an isosceles triangle ABC, so that $CD=AC$ (the base), then $\overline{AC}^2=AD \cdot AB$.

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944. Isosceles or right triangles ABC and PQR are similar if $\frac{h_a}{h_p}=\frac{\overline{AB}}{\overline{PQ}}$.

945. The diagonals of a trapezoid divide each other proportionally.

946. If in triangle ABC altitudes AD and BE meet at O, then:
(a) $\overline{BD}\cdot\overline{DC}=\overline{DO}\cdot\overline{AD}$; (b) $\overline{BD}\cdot\overline{AC}=\overline{BO}\cdot\overline{AD}$.

947. If in a parallelogram $PQRS$, a sect QT is drawn cutting the diagonal PR in V, the side RS in L and the prolongation of PS in T, then $\overline{VQ}^2=\overline{VL}\cdot\overline{VT}$.

948. In similar triangles homologous angle bisectors are directly proportional to the sides of the triangle.

949. In a quadrilateral $ABCD$, right-angled at B and D, perpendiculars PE and PF from any point P in AC to the sides BC and AD are such that $\frac{\overline{PE}}{\overline{AB}}+\frac{\overline{PF}}{\overline{CD}}=1$.

d950. Every straight line cutting the sides of a triangle (produced when necessary) determines upon the sides six sects, such that the product of three non-consecutive sects is equal to the product of the other three.

The line XYZ must cut either (a) two sides of the triangle and the third side produced (Fig. 1), or (b) all three sides produced (Fig. 2). The proof in both instances is the same.

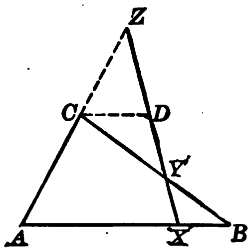

Fig. 1

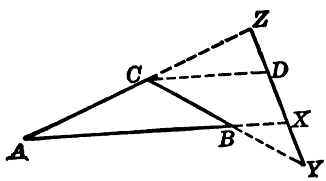

Fig. 2

Draw $CD \parallel AB$. From the similar triangles

$$\frac{\overline{AX}}{\overline{CD}}=\frac{\overline{AZ}}{\overline{CZ}} \text{ and } \frac{\overline{BY}}{\overline{CY}}=\frac{\overline{BX}}{\overline{CD}},$$

therefore $$\frac{\overline{AX}\cdot\overline{BY}}{\overline{CY}\cdot\overline{CD}}=\frac{\overline{AZ}\cdot\overline{BX}}{\overline{CZ}\cdot\overline{CD}},$$

whence $\overline{AX}\cdot\overline{BY}\cdot\overline{CZ}=\overline{AZ}\cdot\overline{BX}\cdot\overline{CY}$.

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This theorem was discovered by Menelaus of Alexandria about 80 B.C.

d951. Prove the converse of the last theorem.

Let XY produced cut AC produced in a point P.

Then $\overline{AX}\cdot\overline{BY}\cdot\overline{CP}\equiv\overline{AP}\cdot\overline{BX}\cdot\overline{CY}$.

But, by hypothesis, $\overline{AX}\cdot\overline{BY}\cdot\overline{CZ}\equiv\overline{AZ}\cdot\overline{BX}\cdot\overline{CY}$;

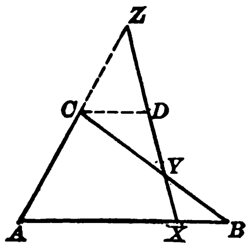

Fig. 1

whence $\dfrac{\overline{CP}}{\overline{CZ}}\equiv\dfrac{\overline{AP}}{\overline{AZ}}$;

whence $\dfrac{\overline{AP}-\overline{CP}}{\overline{AZ}-\overline{CZ}}\equiv\dfrac{\overline{AP}}{\overline{AZ}}$;

or $\dfrac{\overline{AC}}{\overline{AC}}\equiv\dfrac{\overline{AP}}{\overline{AZ}}$.

$\therefore\ \overline{AP}\equiv\overline{AZ}$;

that is, P coincides with Z.

d952. Lines drawn through the vertices of a triangle, and passing through a common point, determine upon the sides six sects, the product of three non-consecutive sects being equal to the product of the other three.

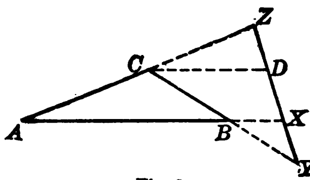

Fig. 2

The common point O may lie either inside or outside the triangle (Figs. 1 and 2). In both cases apply Ex. 950 to the $\triangle ACD$ and sect BOF and to the $\triangle BCD$ and sect AOE, then multiply the results.

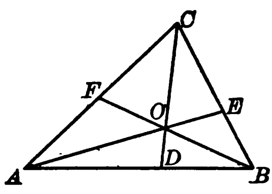

Fig. 1

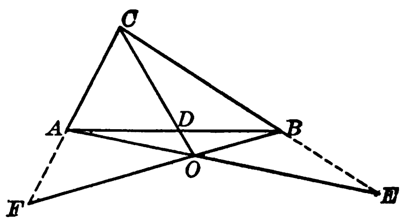

Fig. 2

This theorem was first discovered by Ceva of Milan, in 1678.

d953. Conversely, if three lines drawn through the vertices of a triangle determine upon the sides six sects, such that the product

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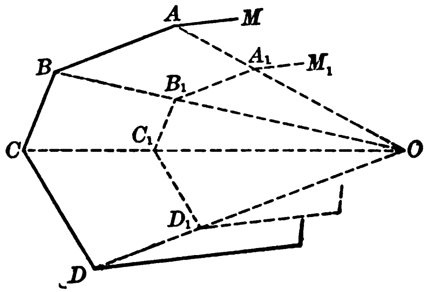
of three non-consecutive sects is equal to the product of the other three, the lines pass through the same point.

The proof is similar to that of Ex. 951.

Theorem 36*a*. ***The homologous angles of similar polygons are equal, and their homologous sides have a constant ratio.***

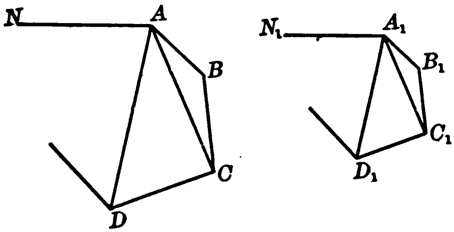


Suggestions: Why is it that the n-gons may be placed as in the accompanying diagram? To prove the second part use the similar triangles thus formed and show that the ratio of the homologous sides is equal to the ratio of similitude of the polygons.

Give the complete proof.

Cor. 1. ***If the ratio of similitude of polygons is unity, they are congruent.***

Cor. 2. ***The homologous diagonals drawn from a single vertex of similar polygons divide the polygons into triangles similar each to each.***



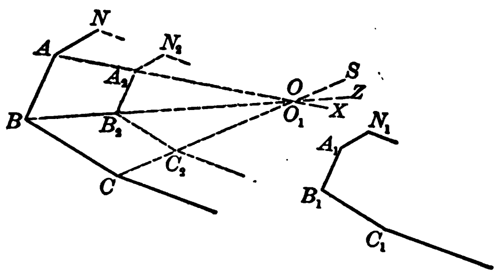
Why is $\triangle A_1B_1C_1 \backsim \triangle ABC$?

Having proved this can you prove $\triangle A_1C_1D_1 \backsim \triangle ACD$, and so forth?

Why is it that only these two sets of $\triangle$s need be proved similar?

Write the proof in full.

Theorem 36*b*. ***Polygons whose homologous angles are equal and whose homologous sides have a constant ratio are similar.***



Given: Polygons $ABC \ldots N$ and $A_1B_1C_1 \ldots N_1$ in which $\angle A_1 = \angle A$, $\angle B_1 = \angle B, \ldots \angle N_1 = \angle N$, and $\frac{A_1B_1}{AB} = \frac{B_1C_1}{BC} = \ldots = \frac{N_1A_1}{NA}$.

Prove: $A_1B_1C_1 \ldots N_1 \backsim ABC \ldots N$.

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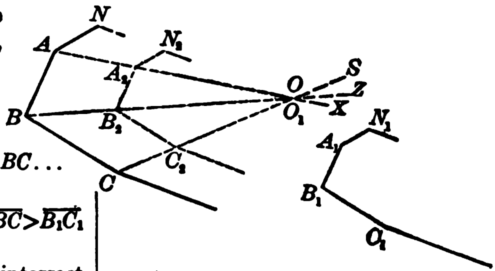
Proof



Place $A_1B_1C_1 \ldots N_1$ as in the diagram so that $A_2B_2 \parallel AB$, and draw $\overline{AA_2X}$ and $\overline{BB_2Z}$.

(1) $\because \dfrac{\overline{A_1B_1}}{\overline{AB}} = \dfrac{\overline{B_1C_1}}{\overline{BC}} = \ldots .$	(1) Data.
(2) If $\overline{AB} = \overline{A_1B_1}$ then $ABC \ldots N \backsim A_1B_1C_1 \ldots N$.	(2) Congruent polygons are similar.
(3) If $\overline{AB} > \overline{A_1B_1}$, then $\overline{BC} > \overline{B_1C_1}$ etc. and ABB_2A_2 is not a $\square$.	(3) The opposite sides of a parallelogram are equal, and conversely.
(4) and AA_2X and BB_2Z intersect at some point O.	(4) Non-parallel coplanar lines intersect.
(5) $\triangle AOB \backsim \triangle A_2OB_2$.	(5) Two angles equal each to each, or sides respectively parallel.
(6) $\therefore \dfrac{\overline{A_2B_2}}{\overline{AB}} = \dfrac{\overline{B_2O}}{\overline{BO}}$.	(6) Homologous sides of similar triangles have a constant ratio.
(7) $\because \measuredangle A_2B_2O = \measuredangle ABO$.	(7) Corresponding angles of parallels.
(8) and $\measuredangle A_2B_2C_2 = \measuredangle ABC$	(8) Data.
(9) $\therefore \measuredangle OB_2C_2 = \measuredangle OBC$	(9) The differences of equals less equals are equal.
(10) and $B_2C_2 \parallel BC$.	(10) Corresponding $\measuredangle^s$ equal.
(11) Draw $\overline{CC_2S}$ cutting $\overline{BB_2Z}$ in O_1	(11) See (1), (2), (3), and (4).
(12) Then $\triangle BO_1C \backsim \triangle B_2O_1C_2$.	(12) See (5).
(13) $\therefore \dfrac{\overline{B_2C_2}}{\overline{BC}} = \dfrac{\overline{B_2O_1}}{\overline{BO_1}}$	(13) See (6).
(14) $\therefore \measuredangle B_2C_2O_1 = \measuredangle BCO_1$	(14) See (7).
(15) $\dfrac{\overline{A_2B_2}}{\overline{AB}} = \dfrac{\overline{B_2C_2}}{\overline{BC}}$	(15) Data.
(16) $\therefore \dfrac{\overline{B_2O}}{\overline{BO}} = \dfrac{\overline{B_2O_1}}{\overline{BO_1}}$	(16) Quantities equal to equal quantities are equal to each other.
(17) $\therefore \overline{B_2O} = \overline{B_2O_1}$, and O_1 coincides with O.	(17) By division and the products of equals multiplied by equals are equal.
(18) $\therefore BO_1 = BO$ and CC_2 passes through O.	(18) Two points determine a straight line.
(19) Any $\overline{ORL}$ cutting sides of the polygons in R and L respectively is divided so that $\dfrac{\overline{OR}}{\overline{OL}} = \dfrac{\overline{A_1C_1}}{\overline{AB}}$.	(19) See (6).

In the same way it may be proved that $DD_2, \ldots NN_2$, pass through O and therefore $A_1B_1C_1 \ldots N_1 \backsim ABC \ldots N$ by the definition of similar figures.

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Cor. 1. *If homologous diagonals drawn from a single vertex of two polygons divide them into triangles similar each to each and similarly arranged, the polygons are similar.*

(Proof is left to the student.)

Theorem 37. *The perimeters of similar triangles are proportional to any two homologous sides, or any two homologous altitudes.*

Cor. 1. *Homologous altitudes of similar triangles have the same ratio as homologous sides.*

Cor. 2. *The perimeters of similar polygons have the same ratio as any pair of homologous sides or diagonals.*

Apply propositions 36*a*, 37, and the appropriate law concerning equal ratios.

Theorem 38. *The areas of similar triangles compare as the squares of any two homologous sides.*

Cor. 1. *The areas of similar polygons compare as the squares of any two homologous sides or diagonals.*

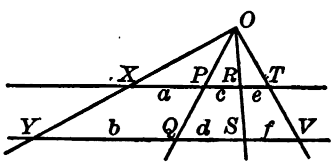
Proof similar to that of Theorem 37, Cor. 2. Give it in detail.

Cor. 2. *Homologous sides or diagonals of similar polygons have the same ratio as the square roots of their areas.*

Apply suitable axioms to the results obtained in Theorem 38, Cor. 1.

Concurrent lines *are those which pass through a common point.*

Theorem 38*a*. *If two parallels are cut by concurrent transversals, the ratio of homologous sects of the parallels is constant.*



(Proof left to the student.)

Discuss the case where O lies between the parallels.

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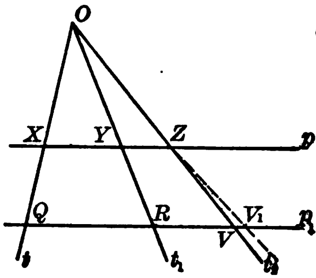

Theorem 38b. ***If the ratio of homologous sects of two parallels cut by three or more transversals is constant, the transversals are either parallel or concurrent.***

Given: $P \parallel P_1$ cut by transversals t, t_1, t_2, in X, Y, Z and Q, R, V respectively, so that $\frac{\overline{XY}}{\overline{QR}}=\frac{\overline{YZ}}{\overline{RV}}$

Prove: t, t_1, t_2 concurrent.

Proof

(1) If $XY = QR$, $t \parallel t_1$ and similarly since $\frac{\overline{XY}}{\overline{QR}}=\frac{\overline{YZ}}{\overline{RV}}$, $t_1 \parallel t_2$, etc.	(1) Why?
(2) If $XY < QR$, $t \nparallel t_1$ and will intersect it at some point O.	(2) Why?
(3) Then $\frac{\overline{XY}}{\overline{QR}}=\frac{\overline{OY}}{\overline{OR}}$.	(3) Why?
(4) Suppose OZ cuts P_1 in V_1. Then $\frac{\overline{YZ}}{\overline{RV_1}}=\frac{\overline{OY}}{\overline{OR}}$.	(4) Why?
(5) But $\frac{\overline{XY}}{\overline{QR}}=\frac{\overline{YZ}}{\overline{RV}}$.	(5) Why?
(6) Then $\frac{\overline{YZ}}{\overline{RV_1}}=\frac{\overline{YZ}}{\overline{RV}}$.	(6) Why?

Complete the proof.

EXERCISES. SET LXXXII. SIMILARITY OF POLYGONS

Numeric

954. The corresponding bases of two similar triangles are 11 in. and 13 in. The altitude of the first is 6 in. Find the corresponding altitude of the second.

955. The perimeter of an equilateral triangle is 51 in. Find the side of an equilateral triangle of half the altitude.

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956. The bases of a trapezoid are 20 in. and 12 in., and the altitude is 4 in. Find the altitudes of the triangles formed by producing the sides until they meet.

957. The perimeters of two similar polygons are 76 and 69. If a side of the first polygon is 4, find the homologous side of the other.

958. The sides of a polygon are 4, 5, 6, 7, and 8, respectively. Find the perimeter of a similar polygon if the side corresponding to 5 is 7.

959. The sides of a polygon are 2 in., $2\frac{1}{2}$ in., $3\frac{1}{4}$ in., 3 in., and 5 in. Find the perimeter of a similar polygon whose longest side is 7 in.

960. The diameter of the moon is approximately 3500 miles, and it is approximately 250,000 miles from the earth. At what distance from the eye will a two-inch disk exactly obscure the moon?

Construction

d961. Through a given point P_1 draw a line such that its distances from two given points P_2 and P_3 shall be in the ratio of (*a*) 3 to 5, (*b*) sect *b* to sect *c*.

d962. In the prolongation of the side *AB* of triangle *ABC*, find point *P* so that $\overline{AP}\cdot\overline{BP}=\overline{CP}^2$.

963. Construct a triangle similar to a given triangle and having a given altitude.

964. From a given rectangle cut off a similar rectangle by a line drawn parallel to one of its sides.

965. Construct a triangle having given:

(*a*) *a*, *b*, and the ratio of *b* to *c*.

(*b*) *a*, and the ratios of *a* to *b*, and *a* to *c*.

(*c*) *a*, and the ratios of *a* to *b*, and *b* to *c*.

(*d*) *a*, $b+c$, and the ratio of *b* to *c*.

d(*e*) *B*, the ratio of *a* to *c*, and h_c.

966. Given two sects *AB* and *CD*, and a point *P*. Draw a line *XY* through *P*—without producing *AB* and *CD* to meet—such that *AB*, *XY*, and *CD* would be concurrent if produced.

967. To draw a parallel to one side of a triangle, cutting off another triangle of given perimeter.

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Theoretic

968. The line joining the mid-points of the bases of a trapezoid is concurrent with the legs of the trapezoid.

969. Two triangles are similar if an angle of one is equal to an angle of the other and the altitudes upon the including sides are proportional.

970. The line bisecting the bases of a trapezoid passes through the intersection of its diagonals.

Suggestion: Prove that line coincident with the line joining the mid-point of one base and the intersection of the diagonals.

d971. Any sect drawn through the mid-point of one side of a triangle and limited by the parallel to that side through the opposite vertex, is divided harmonically by the second side and the prolongation of the third side.

972. If two triangles have equal bases on one of two parallel lines, and their vertices on the other, the sides of the triangles intercept equal sects on any line parallel to these lines and lying between them.

Reword the theorem (*a*) when one base is half the other. (*b*) When the bases have any given ratio.

Theorem 39. ***The altitude upon the hypotenuse of a right triangle divides it into triangles similar to each other and to the original.***

Cor. 1. ***Each leg of a right triangle is a mean proportional between the hypotenuse and its projection upon the hypotenuse.***

Cor. 2. ***The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.***

Cor. 3. ***The diagonal and the side of a square are incommensurable.***

Cor. 4. ***The altitude upon the hypotenuse of a right triangle is a mean proportional between the sects it cuts off on the hypotenuse.***

(The proof is left to the student.)

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Theorem 39a. ***In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides by the projection of the other upon it.***

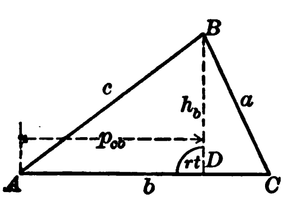


FIG. 1.

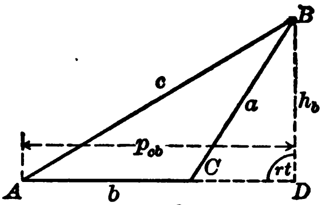


FIG. 2.

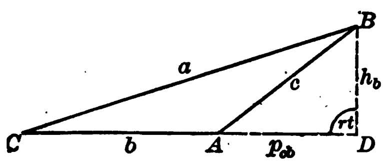
Given: In $\triangle ABC$, $\measuredangle A < 90°$, $\overline{AD} = p_{cb}$, the projection of c upon b.
Prove: $a^2 = b^2 + c^2 - 2b \cdot p_{cb}$.

PROOF

(1) $\triangle$s BCD and ABD are rt. $\triangle$s.	(1) Data and def. of projection.
(2) $\therefore$ in $\triangle BCD$, $a^2 = h_b^2 + \overline{DC}^2$.	(2) Why?
(3) But $h_b^2 = c^2 - p_{cb}^2$ in $\triangle ABD$.	(3) Why?
(4) and in Fig. 1, $\overline{DC}^2 = (b - p_{cb})^2$ while in Fig. 2, $\overline{DC}^2 = (p_{cb} - b)^2$ which are identical.	(4) Why?
(5) $\therefore a^2 = c^2 - p_{cb}^2 + (b - p_{cb})^2$ $= c^2 - p_{cb}^2 + b^2 - 2b \cdot p_{cb} + p_{cb}^2$ $= c^2 + b^2 - 2b \cdot p_{cb}$.	(5) Why?

Theorem 39b. ***In an obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides by the projection of the other upon it.***

Given: In $\triangle ABC$, $\measuredangle A > 90°$, $\overline{DA} = p_{cb}$, the projection of c upon b.
Prove: $a^2 = b^2 + c^2 + 2b \cdot p_{cb}$.
(Proof left to the student.)



Theorem 39c. ***I. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side, increased by twice the square of the median to it.***

This theorem is attributed to Appollonius of Perga (c. 225 B.C.).

II. The difference between the squares of two sides of a triangle is equal to twice the product of the third side and the projection of the median upon it.

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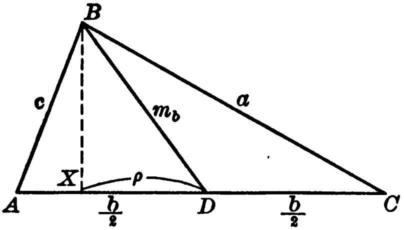
Given: $\triangle ABC$ in which $a>c$, m_b is the median to b and $\overline{XD}\equiv p$ is its projection upon b.



Prove: I. $a^2+c^2\equiv 2\left(\frac{b}{2}\right)^2+2m_b^2$.

II. $a^2-c^2\equiv 2bp$.

Outline of proof, which the student is asked to write.

(1) Show that $\angle BDC$ is obtuse and $\angle BDA$ is acute by comparing $\triangle$ ABD and BDC.

(2) Use propositions 39*a* and 39*b* to give the values of a^2 and c^2.

(3) Combine these values as indicated in I and II.

Consider this proposition when $a\equiv c$.

Cor. 1. ***If m_b represents the length of the median to side b of the triangle whose sides are a, b, c, then***

$$m_b\equiv \tfrac{1}{2}\sqrt{2(a^2+c^2)-b^2}.$$

Solve proposition 39*c* I for m_b.

Theorem 39*d*. ***If h_a represents the altitude upon side a of a triangle whose sides are a, b, c, and s represents its semiperimeter, i.e., $s\equiv\frac{a+b+c}{2}$, then***

$$h_a\equiv\frac{2}{a}\sqrt{s(s-a)\,(s-b)\,(s-c)}.$$

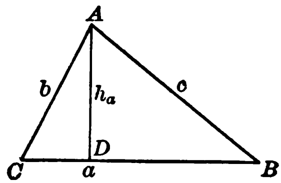


Fig. 1.

PROOF

(1) At least one of the angles B or C is acute. Suppose C is acute (Figures 1 and 2).

(2) Then $c^2\equiv a^2+b^2-2a\cdot\overline{CD}$.

(3) $\therefore \overline{CD}\equiv\frac{a^2+b^2-c^2}{2a}$

(4) But $h_a^2\equiv b^2-\overline{CD}^2$.

(5) $\therefore h_a^2\equiv b^2-\left(\frac{a^2+b^2-c^2}{2a}\right)^2$

or $h_a^2\equiv\left(b+\frac{a^2+b^2-c^2}{2a}\right)\left(b-\frac{a^2+b^2-c^2}{2a}\right)$

$\equiv\left(\frac{a^2+2ab+b^2-c^2}{2a}\right)\left(\frac{c^2-(a^2-2ab+b^2)}{2a}\right)$

$\equiv\frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}$

All authorities to be given by the student.

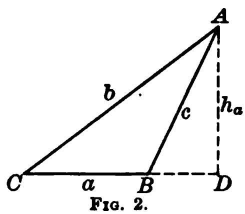


FIG. 2.

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$$\therefore h_a = \frac{1}{2a}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

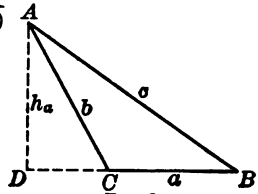

Fig. 3.

If $s = \frac{a+b+c}{2}$, then $a+b+c=2s$

$$a+b-c=2s-2c$$
$$a+c-b=2s-2b$$
$$b+c-a=2s-2a$$

$$\therefore h_a = \frac{1}{2a}\sqrt{2s\cdot 2(s-c)2(s-b)2(s-a)}$$
$$= \frac{4}{2a}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

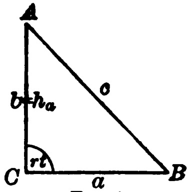

Fig 4.

Show that this proof holds for Figures 3 and 4, *i.e.*, when C is obtuse or right.

Cor. 1. ***If A stands for the area of a triangle whose sides are a, b, c, and whose semiperimeter is s, then***

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

By what must the value of h_a be multiplied to obtain the value of A?

This formula is known as Heron of Alexandria's, to which attention was called on p. 98, Ex. 359 of the First Study.

The Principle of Continuity. By considering both positive and negative properties of quantities, a theorem may frequently be stated so as to include several theorems. For instance, Theorems 39, Cor. 2, 39*a* and 39*b* may be stated as a single theorem if we take into consideration the direction of the projection of c upon b.

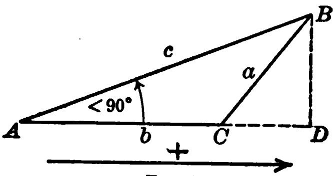

Fig. 1.

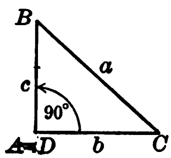

Fig. 2.

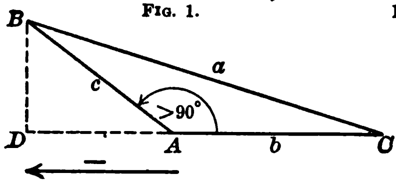

Fig. 3.

If $\overline{AD}$, the projection of c upon b in Fig. 1 be considered positive, $\overline{AD}$ in Fig. 3 will be negative. Therefore we may say in general that $a^2 = b^2+c^2-2b\cdot p_{cb}$, for in Fig. 1 that means $a^2=b^2+c^2-2b(+\overline{AD})$, or $a^2=b^2+c^2-2b\cdot p_{cb}$, while in Fig. 2 it means $a^2 = b^2+c^2-2b(0)$, or $a^2=b^2+c^2$, and in Fig. 3 it means $a^2=b^2+c^2-2b(-\overline{DA})$, or $a^2=b^2+c^2+2b\cdot p_{cb}$.

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EXERCISES. SET LXXXIII. METRIC RELATIONS

Theorems 39*a* through 39*d* enable us to calculate the altitudes and medians of a triangle in terms of its sides, and the length of the projection of any side upon any other, as well as to determine whether a triangle is acute, right or obtuse.

973. If in solving the identity $a^2 \equiv b^2 + c^2 - 2b \cdot p_{cb}$ for p_{cb} for any particular values of a, b, and c, we find (*a*) $p_{cb} = 0$, then $\measuredangle A = 90°$, (*b*) $p_{cb} > 0$, then $\measuredangle A < 90°$, (*c*) $p_{cb} < 0$, then $\measuredangle A > 90°$. Explain.

974. Give the formulas for b and c corresponding to those given in propositions 39*a* and 39*b* for a.

975. From the three formulas obtained in Ex. 974 derive formulas for p_{cb}, p_{ab}, p_{ca} or for p_{bc}, p_{ba}, p_{ac}.

976. Give a formula for m_a, for m_c, for h_b, for h_c.

977. In a right or an obtuse triangle, the greatest side is opposite the right or the obtuse angle. Hence if a is the greatest side of a triangle, show that if $a^2 > b^2 + c^2$ the triangle is acute, if $a^2 \equiv b^2 + c^2$ the triangle is right, and if $a^2 < b^2 + c^2$ the triangle is obtuse.

dt978. Show that $p_{cb} \equiv c \cos A$ and hence that the general formula might be $a^2 \equiv b^2 + c^2 - 2bc \cos A$.

Theorem 39*e*. ***If similar polygons are constructed on the sides of a right triangle, as homologous sides, the polygon on the hypotenuse is equal to the sum of the polygons on the other two sides.***

If P_1, P_2, and P_3 be similar polygons constructed on a, b, and c respectively, as homologous sides, when $\measuredangle C$ is a right angle in $\triangle ABC$, $\frac{P_1}{P_3} \equiv ?$, $\frac{P_2}{P_3} \equiv ?$, $\therefore \frac{P_1 + P_2}{P_3} \equiv ?$

Give the complete proof.

EXERCISES. SET LXXXIII (concluded)

Numeric

979. The base of an isosceles triangle is 48 in. Find the altitude if each arm equals 50 in.

980. Let ABC be a right triangle. The two sides about the right angle C are respectively 455 and 1,092 feet. The hypotenuse AB is divided into two sects AE and BE by the perpendicular upon it from C. Compute the lengths of AE, BE, and CE.

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981. (*a*) If two sides of a triangle equal 15 and 25, respectively, and the projection of 15 upon 25 equals 9, what is the value of the third side?

(*b*) Is the triangle right, acute, or obtuse?

982. The altitude of a triangle is 20 in. A line parallel to the base and 12 in. from the base cuts off a triangle that is what part of the given triangle?

983. If the side of an equilateral triangle equals 10 in., what is the length of the projection of one side upon another?

984. Find the projection of AB upon a line XY, if AB and XY include an angle of 45°, and $AB=2$.

985. Find the side a of the square equal to an equilateral triangle whose side is s. Solve the equation for s in terms of a.

986. Two sides of a triangle are 5 and 8, respectively, and include an angle of 30°. Find the area.

987. Find the area of an equilateral triangle of which

(*a*) The side is 30,

(*b*) The altitude is 34,

(*c*) The side is a,

(*d*) The altitude is h.

988. Find the area of a trapezoid, given:

(*a*) The median m, and altitude h;

(*b*) The median m, one leg l, and the angle between this and the base 30°;

(*c*) Bases b_1 and b_2, and legs each l.

989. In a trapezoid, given the two bases a, b, and the altitude h. The legs are divided into three equal parts by lines parallel to the bases. Find in terms of a, b, and h, the areas of the three parts into which the trapezoid is divided.

990. Find the side of a rhombus composed of two equilateral triangles and equal to another rhombus whose diagonals are 12 and 18.

991. ABC is a triangle and AD the altitude upon BC. If $AD=13$, and the length of the perpendiculars from D to AB and AC are 5 and $10\frac{2}{5}$, respectively, find the area of the triangle.

992. Find the area of a square in terms of (*a*) its perimeter p, (*b*) its diagonal d.

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993. Find the dimensions of a rectangle given:

(*a*) Its perimeter p and area a:

(*b*) Its length l and diagonal d;

(*c*) Its diagonal d and the ratio of its length to its width r.

994. Find the projection of AB upon XY, if $AB = m$, and the two lines include an angle (*a*) of 60°, (*b*) of 30°.

995. In triangle abc, $a = 8$, $b = 15$, and the angle opposite c equals 60°. Find c.

996. In triangle abc, $a = 3$, $b = 5$, and the angle opposite c equals 120°. Find c.

997. In triangle abc, $a = 7$, $b = 8$, and the angle opposite c equals 120°. Find c.

998. Two sides of a triangle are 20 and 30, respectively, and include an angle of 45°. Find the third side.

999. Two sides of a triangle are 16 and 12 in., respectively, and include an angle of 60°. Find the third side.

1000. Find the area of a rectangle in terms of its length l, and diagonal d.

1001. In triangle abc, $a = 20$, $b = 15$, and $c = 7$. Find the projection of b upon c. Is the triangle obtuse or acute?

1002. In a quadrilateral $ABCD$, $AB = 10$, $BC = 17$, $CD = 13$, $DA = 20$, and $AC = 21$. Find the diagonal BD.

1003. Two sides of a triangle are 17 and 10; the altitude upon the third side is 8. What is the length of the third side?

1004. The sides of a triangle are 7, 8, and 9, respectively. Find the length of the median to 8.

1005. The sides of a triangle are 10, 5, and 9, respectively. Find the length of the median to 9.

1006. The sides of a triangle are 22, 20, and 18, respectively. Find the length of the median to 18.

1007. The sides of a triangle are 9, 10, and 17, respectively. Find the three altitudes.

1008. The sides of a triangle are 11, 25, and 30, respectively. Find the three altitudes.

1009. The sides of a triangle are 12, 14, and 15 respectively. Find the three altitudes.

1010. (*a*) Find the altitude of an equilateral triangle with side s. (*b*) Find the side of an equilateral triangle with altitude h.

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1011. Find the area of a triangle whose sides are respectively (*a*) 13, 14, 15, (*b*) 9, 10, 17, (*c*) 11, 25, 30.

d1012. The sides of a triangle are as 8 to 15 to 17. Find the altitudes if the area is 480 sq. ft.

1013. Find one diagonal of a parallelogram, given the sides *a*, *b*, and the other diagonal *g*.

Construction

1014. Given any sect as unit, construct a sect which is $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ units.

1015. In a given sect *AB* find a point *P* such that (*a*) $\frac{PA}{PB}=\frac{1}{\sqrt{2}}$, (*b*) $\frac{PA}{PB}=\frac{1}{\sqrt{3}}$.

1016. Construct an equilateral triangle equal to (*a*) the sum of two given equilateral triangles, (*b*) their difference.

1017. Construct a polygon similar and equal to (*a*) the sum of two given similar polygons, (*b*) their difference.

1018. Construct a square equal to the sum of 3, 4, 5 given squares.

(For other construction problems based on this character, see Chapter VII.)

Theoretic

1019. The median drawn from the extremities of the hypotenuse of the right triangle *ABC* are *BE*, *CF*; prove that $4\overline{BE}^2+4\overline{CF}^2=5\overline{BC}^2$.

d1020. In a certain triangle *ABC*, $\overline{AC}^2-\overline{BC}^2=\frac{1}{2}\overline{AB}^2$; show that a perpendicular dropped from *C* upon *AB* will divide the latter into sects which are to each other as 3 to 1.

1021. If *ABC* is a right triangle, *C* the vertex of the right angle, *D* any point in *AC*, then $\overline{BD}^2+\overline{AC}^2=\overline{AB}^2+\overline{DC}^2$.

1022. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of its diagonals.

1023. If in the parallelogram *ABCD* $\measuredangle A=60°$, $\overline{AC}^2=\overline{AB}^2+\overline{BC}^2+\overline{AB}\cdot\overline{BC}$.

1024. The sum of the squares of the medians of a triangle is three-fourths the sum of the squares of its sides.

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d1025. The sum of the square on the difference of the legs of a right triangle, and twice the rectangle whose sides are the legs of the triangle, is equal to the square on the hypotenuse.

State this exercise in algebraic form and prove it.

1026. One-half the sum of the squares on the sum and difference of the legs of a right triangle is equal to the square on the hypotenuse.

State and prove this exercise algebraically.

d1027. Two similar parallelograms are to each other as the products of their diagonals.

1028. If in triangle ABC, $AB = BC$ and altitudes AD and BE intersect at O, then $\frac{BC}{AO} = \frac{BE}{AE}$.

1029. If in a triangle the ratio of the squares of two sides is equal to the ratio of their projections upon the third side, the triangle is a right triangle.

d1030. The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of the diagonals increased by four times the square of the sect joining their mid-points.

d1031. If perpendiculars are drawn to the sides of a triangle from any point within it, the sum of the squares of three alternate sects cut off on the sides is equal to the sum of the squares of the three remaining sects.

1032. If $ABCD$ is a sect such that $AB = BC = CD$, and P is any other point, prove that $\overline{PA}^2 + 3\overline{PC}^2 = \overline{PD}^2 + 3\overline{PB}^2$.

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CHAPTER IV

LOCUS

Theorem 40. ***The locus of points equidistant from the ends of a sect is the perpendicular bisector of the sect.***

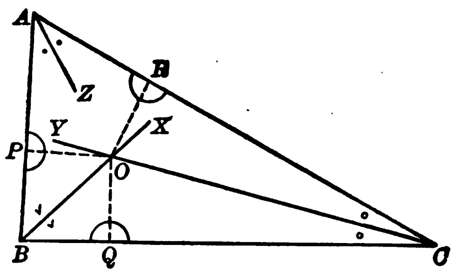
Cor. 1. ***Two points equidistant from the ends of a sect fix its perpendicular bisector.***

Theorem 41. ***The locus of points equidistant from the sides of an angle is the bisector of the angle.***

Cor. 1. ***The locus of points equidistant from two intersecting lines is a pair of lines bisecting the angles.***

(Concurrent *lines are those which pass through a common point.***)**

Theorem 41*a*. ***The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides of the triangle.***



Given: $\triangle ABC$; $\measuredangle BAZ = \measuredangle ZAC$, $\measuredangle ACY = \measuredangle YCB$, $\measuredangle CBX = \measuredangle XBA$.
To prove: AZ, CY, and BX are concurrent in a point equidistant from the sides.
Suggestions for proof: If BX were parallel to CY what relation would exist between $\measuredangle YCB$ and $\measuredangle CBX$?
If O is a point in YC, how is it located with regard to BC and AC?
If O is a point in BX, how is it located with regard to BC and AB?
Why, then, must O be on AZ?

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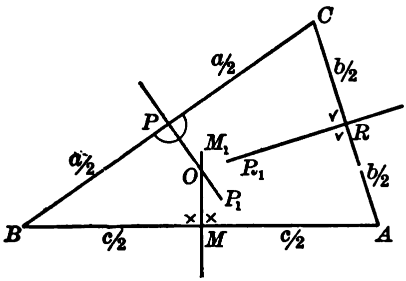
Theorem 41b. ***The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.***



Given: $\triangle ABC$; $MM_1 \perp \overline{AMB}$, $RR_1 \perp \overline{ARC}$, $PP_1 \perp \overline{CPB}$; $AM = MB$, $AR = RC$, $CP = PB$.

To prove: MM_1, PP_1, RR_1 are concurrent in a point equidistant from A, B, and C.

Suggestions for proof: If MM_1 were parallel to PP_1 what relation would exist between MM_1 and CB?

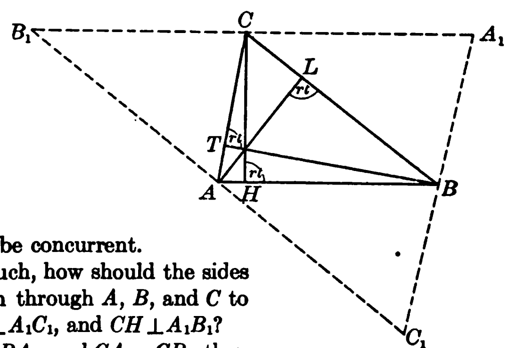
Therefore, what relation would exist between CB and BA?

If O is a point in MM_1 how is it located with regard to A and B?

If O is a point in PP_1 how is it located with regard to C and B?

Why, then, is O a point in RR_1?

Theorem 41c. ***The altitudes of a triangle are concurrent.***



Given: $\triangle ABC$; $CH \perp \overline{AHB}$, $BT \perp \overline{CTA}$, $AL \perp \overline{CLB}$.

To prove: AL, BT, CH are concurrent.

Analysis of proof: If AL, BT, CH were the perpendicular bisectors of the sides of $\triangle A_1B_1C_1$ they would be concurrent.

If they are to be such, how should the sides of $\triangle A_1B_1C_1$ be drawn through A, B, and C to make $AL \perp B_1C_1$, $BT \perp A_1C_1$, and $CH \perp A_1B_1$?

If $AC_1 = AB_1$, $BC_1 = BA_1$, and $CA_1 = CB_1$, then this construction would make possible a proof.

Give a synthetic proof.*

NOTE.—Where an obtuse triangle is involved, show that no separate proof is necessary.

---

* For notes on various types of proof see Chapter VI, p. 297.

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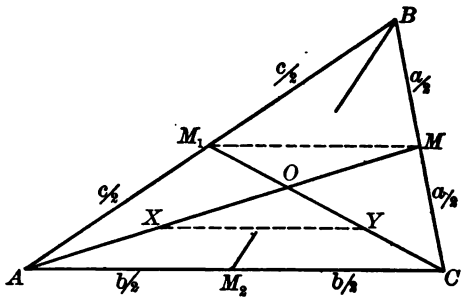
Theorem 41*d*. ***The medians of a triangle are concurrent in a point of trisection of each.***

Given: $\triangle ABC; BM = MC$, $CM_2 = M_2A$, $AM_1 = M_1B$; M in BC, M_1 in BA, M_2 in AC.

To prove: AM, BM_2, CM_1 meet in a point O such that $\overline{AO} = 2\overline{MO}$, $\overline{BO} = 2\overline{M_2O}$, and $\overline{CO} = 2\overline{M_1O}$.



Notes on proof: Why cannot CM_1 be parallel to AM?

Bisect AO at X and CO at Y.

How is XY related to AC? (Consider $\triangle AOC$.)

How is MM_1 related to AC? (Consider $\triangle ABC$.)

Therefore, how is MM_1 related to XY?

Prove $\triangle XOY \cong \triangle MOM_1$, and hence $XO = OM$ and $M_1O = OY$.

Therefore, $\overline{CO} = 2\overline{M_1O}$ and $\overline{AO} = 2\overline{MO}$.

Consider, now, medians AM and BM_2. Can they be parallel, and, if not, why would their point of intersection O_1 coincide with O?

EXERCISES. SET LXXXIV. LOCUS *

1033. Find the locus of the mid-points of sects drawn from a common point to a given line.

1034. Find the locus of the points in which the sects mentioned in Ex. 1033 are divided in the ratio 5 to 8.

1035. Find the locus of the points in which the sects mentioned in Ex. 1033 are divided in the ratio of two given sects a and b.

1036. Find the locus of the mid-points of sects connecting points on two parallels.

1037. Lines are drawn parallel to one side of a triangle and are terminated by the other two sides. What is the locus of their mid-points?

---

* Various terms are used in stating locus exercises. We shall follow the most usual interpretations, which are: (1) No locus exercise need be proved unless a proof is definitely called for, indicated by "prove"; (2) an accurate construction is called for when the terms "plot" or "construct" are used; (3) the terms "describe" or "find" are used in calling for a statement of what the locus is.

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1038. Parallel sects are drawn with their extremities in the sides of an angle. Find the locus of their mid-points.

1039. What is the locus of the vertices of triangles having (*a*) a given base and a given altitude? (*b*) a given base and a given area?

d1040. Find a point within a triangle such that the lines joining it to the vertices shall divide the triangle into three equal parts.

1041. If AB be a fixed sect, find the locus of a point which moves so that its distance from the nearest point in AB is always equal to a given sect c.

How does this locus differ from the one obtained if for the word "sect" we substitute "line"?

1042. If $PQRS$ be a rhombus, such that Q and S lie on two fixed lines through P, find the locus of R.

1043. If $PQRS$ be a parallelogram of constant area and given base PS, find the loci of R and Q.

1044. If A be a fixed point, $\overline{BC}$ a fixed line, n any integral number, P any point in BC, and Q a point in AP or PA produced so that $\overline{AQ} \equiv n \cdot \overline{AP}$, find the locus of Q.

1045. Find the locus in the last exercise if $\overline{AP} \equiv n \cdot \overline{AQ}$.

1046. If in the $\triangle PQR$ a sect QS be drawn to any point in the base, find the locus of a point T on this sect such that the ratio $\frac{QT}{TS}$ is constant.

Justify the two expressions "the locus of points" and "the locus of a point."

1047. If from the intersection of the diagonals of a parallelogram sects are drawn to the perimeter, find the locus of the point in these sects such that the ratio of the parts into which the sect is divided is (*a*) constant, (*b*) equal to a given ratio $\frac{m}{n}$, or (*c*) equal to the ratio of two given sects a and b.

1048. Given a square with side 3 in. Construct the locus of a point P such that the distance from P to the nearest point of the square is 1 in.

1049. Upon a given base is constructed a triangle, one of whose base angles is double the other. The bisector of the larger base angle meets the opposite side at the point P. Find the locus of P.

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d1050. What is the locus of points, the distances of which from two intersecting lines are to each other as m to n?

d1051. Find the locus of points the sum of whose distances from two given parallel lines is equal to a given length. Discuss all possible cases.

d1052. Find the locus of points the difference of whose distances from two given parallel lines is equal to a given length. Discuss.

d1053. Find the locus of points the sum of whose distances from two given intersecting lines is equal to a given length.

d1054. Find the locus of points the difference of whose distances from two given intersecting lines is equal to a given length.

1055. The vertex A of a rectangle $ABCD$ is fixed, and the direction of the sides AB and AD also are fixed. Plot the locus of the vertex C if the area of the rectangle is constant.

d1056. Plot the locus of a point if the product of its distances from two perpendicular lines is constant.

d1057. Plot the locus of a point P such that the sum of the squares of its distances from two fixed points is constant.

d1058. Plot the locus of a point such that the difference of the squares of its distances from two fixed points is constant.

d1059. Given the base of a triangle in magnitude and position and the difference of the squares of the other two sides, plot the locus of the vertex.

1060. Given a square $ABCD$. Let E be the mid-point of CD, and draw BE. A line is drawn parallel to BE and cutting the square. Let P be the mid-point of the sect of this line within the square. Construct the locus of P as the line moves, always remaining parallel to BE.

Other locus exercises will be found in the chapter on "Circles," pp. 273, 275, 276, 283, 284, 288, as well as in the chapter on "Methods of Attacking Problems," p. 306, *et seq.*

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CHAPTER V

THE CIRCLE

Theorem 42. *Three points not in a straight line fix a circle.*

Theorem 43. *In equal circles, equal central angles intercept equal arcs,* and conversely.*

Theorem 43a. *In equal circles, the greater of two central angles intercepts the greater arc, and conversely.*

Suggestion: Lay off the smaller central angle on the greater to prove the direct. What may be done in the case of the converse?

Theorem 44. *In equal circles, equal arcs are subtended by equal chords, and conversely.*

Theorem 44a. *In equal circles, unequal arcs are subtended by chords unequal in the same order, and conversely.*

Suggestions: If radii are drawn, what do we know of the triangles formed?
Then what method of proving sects unequal may be used in the proof of the direct?
In the proof of the converse, what is the only method you are ready to use in order to prove arcs unequal?
Write the proofs of both parts of this theorem.

Theorem 45. *A diameter perpendicular to a chord bisects it and its subtended arcs.*

Cor. 1. *A radius which bisects a chord is perpendicular to it.*

Cor. 2. *The perpendicular bisector of a chord passes through the center of the circle.*

Theorem 46. *In equal circles, equal chords are equidistant from the center, and conversely.*

Theorem 46a. *In equal circles the distances of unequal chords from the center are unequal in the opposite order, and conversely.*

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* Such arcs are actually congruent, but we are following custom in using the word "equal."

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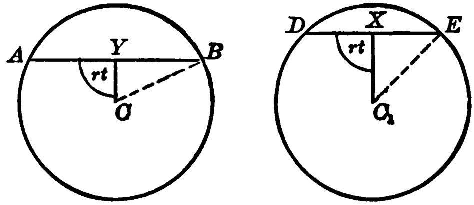

Axioms of Inequality (*continued*). 5. *Squares of positive unequals are unequal in the same order.* Illustrate.

Given: $\odot C = \odot C_1$; chord $AB >$ chord DE; $\overline{CY} \perp \overline{AYB}$, $\overline{C_1X} \perp \overline{DXE}$.

Prove: $CY < C_1X$.

PROOF OF THE DIRECT

$\overline{CB}^2 = \overline{YB}^2 + \overline{YC}^2$. $\overline{C_1E}^2 = \overline{XE}^2 + \overline{XC_1}^2$. $\therefore \overline{YB}^2 + \overline{YC}^2 = \overline{XE}^2 + \overline{XC_1}^2$. But $\overline{AB} > \overline{DE}$, $\therefore \overline{YB} > \overline{XE}$; $\therefore \overline{YB}^2 > \overline{XE}^2$; $\therefore \overline{YC}^2 < \overline{XC_1}^2$, and $\therefore \overline{YC} < \overline{XC_1}$.	Authorities left for the student to supply.

Can the same method be used for the proof of the converse? Give the proof in full.

NOTE.—Why is it better to use a direct method rather than the method of exclusion in the proof of the converse?

Theorem 47. *A line perpendicular to a radius at its outer extremity is tangent to the circle.*

Cor. 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

Cor. 2. *The perpendicular to a tangent at the point of contact passes through the center of the circle.*

Cor. 3. *A radius perpendicular to a tangent passes through the point of contact.*

Cor. 4. *Only one tangent can be drawn to a circle at a given point on the circle.*

Theorem 48. *Sects of tangents from the same point to a circle are equal.*

Theorem 49. *The line of centers of two tangent circles passes through their point of contact.*

Theorem 49a. *The line of centers of two intersecting circles is the perpendicular bisector of their common chord.*

What is the locus of points equally distant from the ends of a sect? Where, then, do the centers of these circles lie?

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EXERCISES. SET LXXXV. THE STRAIGHT LINE AND THE CIRCLE

1061. What methods can you now add to those known before this chapter of showing:

(*a*) Sects equal? (*b*) Angles equal? (*c*) Sects unequal? (*d*) Angles unequal? (*e*) Sects perpendicular?

1062. Can you now mention certain relations of a new kind of element? If so, what are they?

Numeric

1063. Two parallel chords of a circle are 4 and 8 units in length, and their distance apart is 3 units. What is the radius?

1064. Two parallel chords of a circle are d and k in length, and their distance apart is f. What is the radius?

1065. Find the length of a tangent from a point 15″ from the center of a circle whose radius is 5″.

1066. Find the radius of a circle if the length of a tangent from a point 23″ from the center is 16″.

1067. Find the length of the longest chord and of the shortest chord that can be drawn through a point 1′ from the center of a circle whose radius is 20″.

1068. The radius of a circle is 5″. Through a point 3″ from the center a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord, and the distance (to two decimal places) from one end of the chord to the ends of the diameter.

1069. The span (chord) of a bridge in the form of a circular arc is 120′, and the highest point of the arch is 15′ above the piers. Find the radius of the arc.

1070. The line of centers of two circles is 30. Find the length of the common chord if the radii are 8 and 26 respectively.

1071. Two circles touch each other, and their centers are 8″ apart. The radius of one of the circles is 5″. What is the radius of the other? (Two solutions.)

1072. If the radii of two concentric circles are denoted by a and b, respectively, find the radius of a third circle which shall touch both given circles and contain the smaller.

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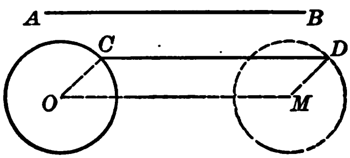
Locus

1073. Find the locus of the center of a circle which has a given radius and is tangent to a given circle.

1074. Find the locus of the extremity of a tangent of given length drawn to a given circle.

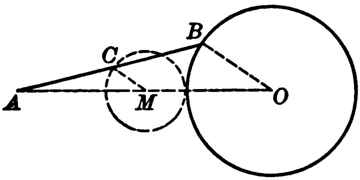
1075. Two equal circles are tangent to each other externally. Find the locus of the centers of all circles tangent to both.

1076. A sect so moves that it remains parallel to a given line, and so that one end lies on a given circle. Find the locus of the other end. Does the accompanying diagram give the complete locus?



1077. What is the locus of the mid-points of parallel chords of a circle? Prove the correctness of your statement.

1078. From a point outside how many tangents are there to a circle? Prove.



1079. Find the locus of the mid-point of a sect that is drawn from a given external point to a given circle.

1080. A straight line 3 in. long moves with its extremities on the perimeter of a square whose sides are 4 in. long. Construct the locus of the mid-point of the moving line.

1081. A circular basin 16 in. in diameter is full of water, and upon the surface there floats a thin straight stick 1 ft. long. Shade that region of the surface which is inaccessible to the mid-point of the stick, and describe accurately its boundary.

1082. The image of a point in a mirror is apparently as far behind the mirror as the point itself is in front. If a mirror revolves about a vertical axis, what will be the locus of the apparent image of a fixed point 1 ft. from the axis?

1083. In the rectangle $ABCD$ the side AB is twice as long as the side BC. A point E is taken on the side AB, and a circle is drawn

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through the points C, D, and E. Plot the path of the center of the circle as E moves from A to B.

1084. Find the locus of a point P such that the ratio of its distances from two fixed points is equal to the constant ratio m to n.

Construction *

1085. Find the center of a given circle.

1086. Inscribe a circle in a given triangle.

1087. Circumscribe a circle about a given triangle.

1088. Escribe circles about a given triangle. (See p. 359.)

1089. Through a given point in a circle draw the shortest possible chord.

1090. Inscribe a circle in a given sector. VII.

1091. With its center in a given line construct a circle which shall be:

(*a*) Tangent to another given line at a given point.

(*b*) Tangent to two other given lines.†

(*c*) Tangent at a given point to a given circle. VII.

1092. Construct a circle of given radius r, which shall:

(*a*) Pass through a given point and be tangent to a given line;

(*b*) Pass through a given point and be tangent to a given circle;

(*c*) Be tangent to a given line and a given circle;

(*d*) Be tangent to two given circles. VII.

1093. Construct a circle tangent to two given lines and having its center on a given circle. VII.

1094. An equilateral triangle ABC is 2 in. on a side. Construct a circle which shall be tangent to AB at the point A and shall pass through the point C. VII.

1095. To a given circle draw a tangent that shall be parallel to a given line.

1096. Draw two lines making an angle of 60°, and construct all the circles of ½ in. radius that are tangent to both lines.

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* While some more or less difficult construction problems have been inserted at this point, they have been primarily inserted for the benefit of those pupils who wish to test their power, and when found too difficult may well be omitted until Chapter VII has been studied. Such problems will be followed by the Roman number "VII."

† See p. 311 for the section dealing with "The Discussion of a Problem."

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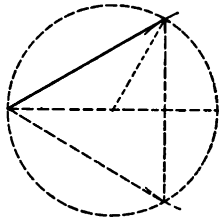


d1097. Construct an equilateral triangle, having given the radius of the circumscribed circle. VII.

1098. Construct a circle, touching a given circle at a given point, and touching a given line. VII.

1099. In a given square inscribe four equal circles, so that each shall be tangent to two of the others, and also tangent to two sides of the square.

d1100. In a given square inscribe four equal circles, so that each shall be tangent to two of the others, and also tangent to one and only one side of the square. VII.

d1101. In a given equilateral triangle inscribe three equal circles tangent each to the other two, each circle being tangent to two sides of the triangle.

1102. Draw a tangent to a given circle such that the sect intercepted between the point of contact and a given line shall have a given length. VII.

Theoretic

1103. If two chords intersect and make equal angles with the diameter through their point of intersection, they are equal.

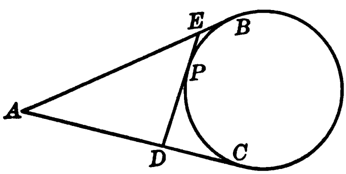
1104. The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the inscribed circle.

1105. If two common external tangents or two common internal tangents are drawn to two circles, the sects intercepted between the points of contact are equal.

1106. If two circles are tangent externally, the common internal tangent bisects the two common external tangents.

1107. A line tangent to two equal circles is either parallel to the sect joining their centers or bisects it.

1108. A coin is placed on the table. How many coins of the same denomination can be placed around it, each tangent to it and tangent to two of the others? Prove your answer.



1109. If through any point in the convex arc included between two tangents a third tangent is drawn, a triangle will be formed, the peri-

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meter of which is constant and equal to the sum of the two tangents.

1110. If a triangle is inscribed in a triangle ABC, whose semiperimeter is s, the sects of its sides from the vertices to the points of contact are equal to $s-a$, $s-b$, and $s-c$.

1111. The perimeter of an inscribed equilateral triangle is equal to half the perimeter of the circumscribed equilateral triangle.

1112. The radius of the circle inscribed in an equilateral triangle is equal to one-third of the altitude of the triangle.

d1113. In a circumscribed quadrilateral the sum of two opposite sides is equal to the sum of the other two sides, and a circle can be inscribed in a quadrilateral if the sum of two opposite sides is equal to the sum of the other two sides.

d1114. In what kinds of parallelograms can a circle be inscribed? Prove.

1115. The diameter of the circle inscribed in a right triangle is equal to the difference between the sum of the legs and the hypotenuse.

1116. All chords of a circle which touch an interior concentric circle are equal, and are bisected at the points of contact.

Theorem 50. ***In equal circles central angles have the same ratio as their intercepted arcs.***

Cor. 1. ***A central angle is measured by its intercepted arc.***

Theorem 51. ***Parallels intercept equal arcs on a circle.***

Theorem 52. ***An inscribed angle, or one formed by a tangent and a chord is measured by one-half its intercepted arc.***

Theorem 52*a*. ***The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.***

What point in the circumscribed circle is the mid-point of the hypotenuse?

Theorem 53. ***An angle whose vertex is inside the circle is measured by half the sum of the arcs intercepted by it and its vertical.***

Theorem 54. ***An angle whose vertex is outside the circle is measured by half the difference of its intercepted arcs.***

Theorem 54*a*. ***The opposite angles of a quadrilateral inscribed in a circle are supplementary.***

(Proof left to the pupil.)

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Cor. 1. ***A quadrilateral is inscriptible if its opposite angles are supplementary.***

Suggestion: Show, by the method of exclusion, that the fourth vertex of the quadrilateral lies on the circle passing through three of its vertices.

EXERCISES. SET LXXXVI. MEASUREMENT OF ANGLES

Numeric

1117. Find the value of an angle which (*a*) is inscribed, and intercepts an arc of 160°, (*b*) is inscribed in a segment of 250°.

1118. If the tangents from a point to a circle make an angle of 60°, what are the values of the arcs they intercept? What if the angle is a right angle?

1119. Find the angle whose sides are tangents drawn from a point whose distance from the center of the circle is the diameter of that circle

1120. An angle between two chords intersecting inside a circle is 35°, its intercepted arc is 25° 18′; find the arc intercepted by its vertical.

1121. A triangle is inscribed in a circle, and another triangle is circumscribed by drawing tangents at the vertices of the inscribed triangle. The angles of the inscribed triangle are 40°, 60°, and 80°. Find all the other angles of the figure.

1122. The arcs subtended by three consecutive sides of a quadrilateral are 87°, 95°, 115°; find the angles of the quadrilateral; the angles made by the intersection of the diagonals, and the angles made by the opposite sides of the quadrilateral when produced.

1123. Three consecutive angles of an inscribed quadrilateral are 140° 30′, 80° 30′, and 39° 30′. Find the numbers of degrees in the arcs subtended by the four sides if the arc intercepted by the largest angle is divided into parts in the ratio of 4 to 5.

1124. Three consecutive angles of a circumscribed quadrilateral are 85°, 122°, 111°. Find the number of degrees in each angle of the inscribed quadrilateral made by joining the points of contact of the sides of the circumscribed quadrilateral.

1125. The points of tangency of a quadrilateral, circumscribed

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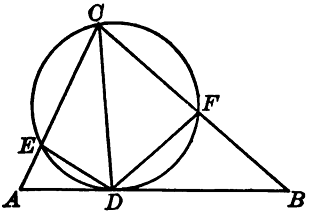
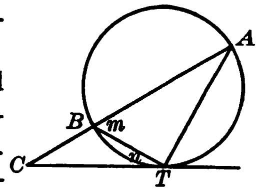
about a circle, divide the circumference into arcs, which are to each other as 4, 6, 10, and 16. Find the angles of the quadrilateral.

1126. If the sides AB and BC of an inscribed quadrilateral $ABCD$ subtend arcs of 60° and 130°, respectively, and the diagonals form $\angle AED = 70°$, find the number of degrees in (*a*) $\widehat{AD}$, (*b*) $\widehat{DC}$, (*c*) each angle of the quadrilateral.



1127. In this figure $\angle B = 41°$, $\angle A = 65°$, and $\angle BCD = 97°$. Find the number of degrees in each of the other angles, and determine whether or not CD is a diameter.

1128. In this figure $\angle m = 62°$ and $\angle n = 28°$. Find the number of degrees in each of the other angles, and determine whether or not AB is a diameter.



1129. At the vertices of an inscribed quadrilateral tangents are drawn to the circle, forming a circumscribed quadrilateral. The arcs subtended by the sides of the inscribed quadrilateral are in the ratio of 3 to 4 to 5 to 8.

(*a*) Find the angles of each quadrilateral.

(*b*) Find the angles between the diagonals of the inscribed quadrilateral.

(*c*) Find the angles between the opposite sides of the inscribed quadrilateral produced to intersect.

(*d*) Find the angles between the sides of the inscribed and those of the circumscribed quadrilateral.

1130. The vertices of a quadrilateral inscribed in a circle divide the circumference into arcs which are to each other as 1, 2, 3, and 4. Find the angles between the opposite sides of the quadrilateral.

1131. The sides of an inscribed quadrilateral subtend arcs in the ratio (*a*) 1 to 2 to 3 to 4, (*b*) 3 to 5 to 7 to 9. How many degrees in each angle of the quadrilaterals in (*a*) and (*b*)?

1132. The bases of an inscribed isosceles trapezoid subtend arcs of 100° and 120°. How many degrees in each angle of the trapezoid (*a*) if the bases are on the same side of the center, (*b*) if they are on opposite sides of the center?

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Theoretic

1133. The angle formed by two tangents is equal to twice the angle between the *chord of contact* * and the radius drawn to a point of contact.

1134. If the tangents drawn from an exterior point to a circle form an angle of 120°, the distance of the point from the center is equal to the sum of the tangents.

1135. An isosceles trapezoid is inscriptible; that is, a circle can be circumscribed about it.

1136. If in a circle two chords are drawn, and the mid-point of the arc subtended by one chord is joined to the extremities of the other chord, the two triangles thus formed are mutually equiangular, and the quadrilateral thus formed is inscriptible.

1137. If A, B, C, A_1, B_1, C_1 are six points in a circumference, such that AB is parallel to A_1B_1 and AC is parallel to A_1C_1, then BC_1 is parallel to B_1C.

1138. Let A be any point of a diameter, B the extremity of a radius perpendicular to the diameter, P the point in which BA meets the circumference, C the point in which the tangent through P meets the diameter produced. Prove that $AC=PC$.

1139. If two circles touch internally, and the diameter of the smaller is equal to the radius of the larger, the circumference of the smaller bisects every chord of the larger which can be drawn through the point of contact.

d1140. Two circles touch internally in the point P, and AB is a chord of the larger circle touching the smaller in the point C. Prove that PC bisects the angle APB.

1141. If two circles intersect at the points A and B, and through A a variable secant be drawn cutting the circles in C and D, the angle CBD is constant for all positions of the secant.

1142. If two circles are tangent externally, the corresponding sects of two lines drawn through the point of contact and terminated by the circles are proportional.

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* By the **chord of contact** is meant the sect joining the points of contact of a pair of tangents.

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1143. If two circles are tangent to each other and a sect be drawn through the point of tangency, terminating in the circles, the diameters from the extremities of this sect are parallel.

Case I. Circles tangent externally.

Case II. Circles tangent internally.

1144. If two circles are tangent to each other and a sect be drawn through the point of tangency and terminating in the circles, tangents at the extremities of this sect are parallel.

Case I, the circles tangent externally; and Case II, tangent internally.

d1145. If two sects OA and OB, not in a straight line, are divided in C and D, respectively, so that $\overline{OA}\cdot\overline{OC}=\overline{OB}\cdot\overline{OD}$, then A, B, C, D are *concyclic*, that is, lie on the same circle.

d1146. The altitudes of a triangle bisect the angles of the triangle determined by their feet, *i.e.*, the angles of the *pedal triangle*.

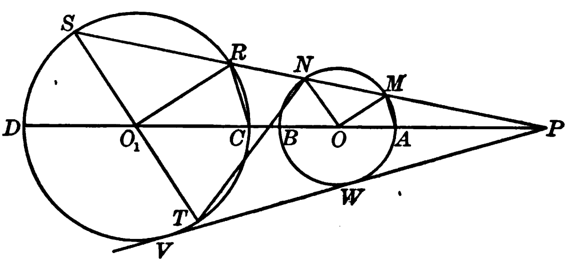
d1147. The feet of the medians and the feet of the altitudes of a triangle are concyclic.

Hint: Pass a circle through the feet of the medians, then prove that the foot of any one of the three altitudes will lie on this circle.

1148. If two circles are tangent externally, and a secant is drawn through the point of contact, the chords formed are proportional to the radii.

1149. If C is the mid-point of $\widehat{AB}$, and chord CD cuts chord AB in E, $\frac{\overline{CE}}{\overline{CA}}=\frac{\overline{CA}}{\overline{CD}}$

d1150. If two circles are tangent externally, the common tangent is a mean proportional between the diameters.



d1151. The line joining the extremities of two parallel radii of two circles passes through the direct center of similitude if the radii have the same direction, and through the inverse center if the radii have opposite directions.

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d1152. Taking any point as a center of similitude of two circles, the two radii of one of them, drawn to its points of intersection with any other line passing through that center of similitude, are parallel, respectively, to the two radii of the other, drawn to its intersections with the same line.

Hint: Use an indirect proof depending upon Ex. 1151.

d1153. All secants, drawn through a direct center of similitude P of two circles, cut the circles in points whose distances from P, taken in order, form a proportion.

d1154. If in the last exercise, the line of centers cuts the circles in points A, B, C, D, and any other secant through P cuts the circle in points M, N, R, S, prove that $\overline{PN} \cdot \overline{PR}$ is constant and equal to $\overline{PB} \cdot \overline{PC}$.

d1155. The common external tangents to two circles pass through the direct center of similitude, and the common interior tangents pass through the inverse center of similitude.

What method of drawing the common tangents to two circles may be derived from this fact?

1156. ABC is an isosceles triangle inscribed in a circle, BD a chord drawn from its vertex cutting the base in any point E. Prove $\frac{\overline{BD}}{\overline{AB}} = \frac{\overline{AB}}{\overline{BE}}$

1157. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller.

1158. If two circles touch at M, and through M three lines are drawn meeting one circle in A, B, C, and the other in D, E, F, respectively, the triangles ABC and DEF are similar.

Locus

1159. An angle of 60° moves so that both of its sides touch a fixed circle of radius 5 ft. What is the locus of the vertex?

1160. Find the locus of the mid-point of a chord drawn through a given point within a given circle.

1161. Through a point A on a circle chords are drawn. On each one of these chords a point is taken one-third the distance from A to the end of the chord. Find the locus of these points.

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1162. The locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms, with the base, a segment capable of containing the given angle.

1163. Find the locus of the points of contact of tangents drawn from a given point to a given set of concentric circles.

1164. A variable chord passes, when prolonged, through a fixed point outside a given circle. What is the locus of the mid-point of the chord?

1165. Upon a sect AB a segment of a circle containing 240° is constructed, and in the segment any chord CD subtending an arc of 60° is drawn. Find the locus of the intersection of AC and BD, and also of the intersection of AD and BC.

1166. The locus of the centers of circles inscribed in triangles having a given base and a given angle at the vertex is the arc which forms with the base a segment capable of containing a right angle plus half the given angle at the vertex.

1167. The locus of the intersections of the altitudes of triangles having a given base and a given angle at the vertex is the arc forming with the base a segment capable of containing an angle equal to the supplement of the given angle at the vertex.

d1168. Find the locus of a point from which two circles subtend* the same angle.

1169. If A and B are two fixed points on a given circle, and P and Q are the extremities of a variable diameter of the same circle, find the locus of the point of intersection of the lines AP and BQ.

d1170. The lines l_1 and l_2 meet at right angles in a point A. O is any fixed point on l_2. Through O draw a line meeting l_1 in B. P is a varying point on this line such that $\overline{OB}\cdot\overline{OP}$ is constant. Plot the locus of P as the line swings about O as a pivot.

Theorem 55. ***A tangent is the mean proportional between any secant and its external sect, when drawn from the same points to a circle.***

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* *If tangents from the same point to two circles form equal angles, the circles are said to subtend equal angles from that point.*

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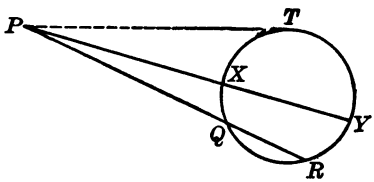


Cor. 1. ***The product of a secant and its external sect from a fixed point outside a circle is constant.***

Hint: Draw tangent PT.

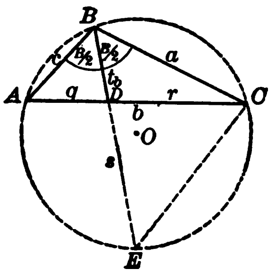
What is the constant in this corollary?

State the corollary in another way.

Theorem 55*a*. ***If chords intersect inside a circle, the product of their sects is constant.***

Prove by means of similar triangles.

Applying the principle of continuity, Theorem 55, Cor. 1, and Theorem 55a can be stated as a single theorem. State them so.



Theorem 55*b*. ***The square of the bisector of an angle of a triangle is equal to the product of the sides of this angle, diminished by the product of the sects made by that bisector on the third side.***

Given: $\triangle ABC$ with bisector t_b cutting AC at D into sects q and r.

Prove: $t_b^2 = ac - qr$.

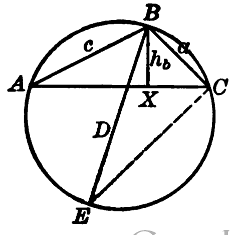
PROOF

Circumscribe $\odot O$ about ABC, and extend BD to E in $\odot O$ and draw chord CE.

Let $DE = s$.

(1) $\triangle BDA \backsim \triangle BCE$	(1) Why?
(2) $\therefore \dfrac{c}{t_b + s} = \dfrac{t_b}{a}$	(2) Why?
(3) $\therefore ac = t_b^2 + t_b s$	(3) Why?
(4) But $t_b s = qr$	(4) Why?
(5) $\therefore ac = t_b^2 + qr$	(5) Why?
(6) $\therefore t_b^2 = ac - qr$	(6) Why?

Theorem 55*c*. ***In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle and the altitude on the third side.***



Hint: Prove $\triangle ABX \backsim \triangle EBC$.

Consider the special case where $\angle B$ is a right angle and evolve a formula for h_b.

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Cor. 1. ***If R denote the radius of the circle circumscribed about a triangle whose sides are a, b, c, and semiperimeter s, then***

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

In the figure for Theorem 55*c*, $ac = h_b D$.

$$\therefore D = \frac{ac}{h_b} \quad \therefore R = \frac{ac}{2h_b}$$

But $h_b = \frac{2}{b}\sqrt{s(s-a)(a-b)(s-c)}$

$$\therefore R = \frac{ac}{\frac{4}{b}\sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

EXERCISES. SET LXXXVII. METRIC RELATIONS

Numeric

1171. A point P is 10 in. from the center of a circle whose radius is 6 in. Find the length of the tangent from P to the circle.

1172. The length of a tangent from P to a circle is 7 in., and the external sect of a secant is 4 in. Find the length of the whole secant.

1173. A point P is 8 in. from the center of a circle whose radius is 4. Any secant is drawn from P, cutting the circle. Find the product of the whole secant and its external sect.

1174. From the same point outside a circle two secants are drawn. If one secant and its external sect are 24 and 15, respectively, and the external sect of the other is 7, find that secant.

1175. Two chords intersect within a circle. The sects of one are m and n and one sect of the other is p. Find the remaining sect.

1176. If a tangent and a secant drawn from the same point to a circle measure 6 in. and 18 in., respectively, how long is the external sect of the secant?

1177. Two secants are drawn from a common point to a circle. If their external sects are 12 and 9, and the internal sect of the first is 8, what is the length of the second?

1178. The radius of a circle is 13 in. Through a point 5 in. from the center a chord is drawn. What is the product of the two

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sects of the chord? What is the length of the shortest chord that can be drawn through that point?

1179. One sect of a chord through a point 3.5 units from the center of a circle is 2 units in length. If the diameter of the circle is 12 units, what is the length of the other sect of the chord?

1180. The radius of a circle is 2 units. If through a point P, 4 units from the center, secant PQR is drawn, and QR is one unit, what is the length of PQ?

1181. $\triangle ABC$ is inscribed in a circle of radius 5 in. Find the altitude to BC if AB is 4, and AC is 5 in.

1182. The sides of a triangle are 4, 13, and 15. respectively. Find the radius of the circumscribed circle.

1183. In $\triangle abc$, $a=20$, $b=15$, and the projection of b upon c (p_{bc}) is 9. Find the radius of the circumscribed circle.

1184. In $\triangle abc$, $a=9$ and $b=12$. Find c if the diameter of the circumscribed circle is 15.

1185. The sides of a triangle are 18, 9, and 21, respectively. Find the angle bisector corresponding to 21.

1186. The sides of a triangle are 21, 14, and 25, respectively. Find the angle bisector corresponding to 25.

1187. The sides of a triangle are 22, 11, and 21, respectively. Find the angle bisector corresponding to 21.

1188. The sides of a triangle are 6, 3, and 7, respectively. Find the angle bisector corresponding to 7.

1189. In a triangle the sides of which are 48, 36, and 50, where do the bisectors of the angles intersect the sides? What are the lengths of the angle bisectors?

1190. In each of the Exercises 1181 to 1189 what kind of triangle is involved?

Construction

1191. Construct the mean proportional between two given sects, using in turn the methods suggested by the following propositions: (*a*) 39 Cor. 1, (*b*) 39 Cor. 4, (*c*) 55.

1192.* Construct a square equal in area to that of a given: (*a*) rectangle; (*b*) triangle; (*c*) trapezoid.

* For discussion and illustrations of the type of analysis applicable see pp. 318 to 321, and Problems 17, 19, 21, 22, 26, 27, Chapter VII.

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1193. Draw through a given external point P a secant PAB to a given circle so that $\overline{AB}^2=\overline{PA}\cdot\overline{PB}$. VII.

d1194. Draw through one of the points of intersection of two given intersecting circles a common secant of given length. VII.

d1195. From a point outside a circle draw a secant whose external sect is equal to one-half the secant.

Locus

d1196. Given the fixed base of a triangle and the sum of the squares of the other two sides, describe the locus of the vertex.

d1197. Repeat Ex. 1196, given the difference of the squares of the other two sides.

d1198. Through P any $\overline{PMN}$ is drawn, cutting a circle K in M and N, and P moves so that the product of the sects $\overline{PM}\,\overline{PN}$ has the constant value k^2. Find the locus of P.

Theoretic

1199. If chord AB bisects chord CD, either sect of chord CD is a mean proportional between the sects of AB.

1200. If two circles intersect, their common chord produced bisects the common tangents.

1201. In the diameter of a circle points A and B are taken equally distant from the center, and joined to a point P in the circumference. Prove that $\overline{AP}^2+\overline{BP}^2$ is constant for all positions of P.

1202. If a tangent is limited by two other parallel tangents to the same circle, the radius of the circle is the mean proportional between its sects.

1203. The tangents to two intersecting circles, drawn from any point in their common chord produced, are equal.

d1204. The sum of the squares of the diagonals of a trapezoid is equal to the sum of the squares of the legs plus twice the product of the bases.

1205. In an inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides. (Ptolemy's Theorem.)

d1206. If the opposite sides of an inscribed hexagon intersect, they determine three collinear points. ("Mystic Hexagram," discovered by Pascal when he was 16 years of age.)

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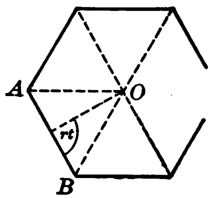
d1207. If a circumference intersects the sides a, b, c, of a $\triangle ABC$ in the points A_1 and A_2, B_1 and B_2, C_1 and C_2, respectively, then $\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}\cdot\frac{AC_2}{C_2B}\cdot\frac{BA_2}{A_2C}\cdot\frac{CB_2}{B_2A}\equiv 1$. (Carnot's Theorem.)

Theorem 56. ***A circle may be circumscribed about, and inscribed within, any regular polygon***

Cor. 1. ***An equilateral polygon inscribed in a circle is regular.***

Cor. 2. ***An equiangular polygon circumscribed about a circle is regular.***

Cor. 3. ***The area of a regular polygon is equal to half the product of its apothem and perimeter.***



Suggestion: What is the area of $\triangle AOB$?

Theorem 57. ***If a circle is divided into any number of equal arcs, the chords joining the successive points of division form a regular inscribed polygon; and the tangents drawn at the points of division form a regular circumscribed polygon.***

Cor. 1. ***Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.***

Cor. 2. ***Lines drawn from each vertex of a regular inscribed polygon to the mid-points of the adjacent arcs subtended by its sides form a regular inscribed polygon of double the number of sides.***

Cor. 3. ***Tangents at the mid-points of the arcs between consecutive points of contact of the sides of a regular circumscribed polygon, form a regular circumscribed polygon of double the number of sides.***

Cor. 4. ***The perimeter of a regular inscribed polygon is less than that of a regular inscribed polygon of double the number of sides; and the perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.***

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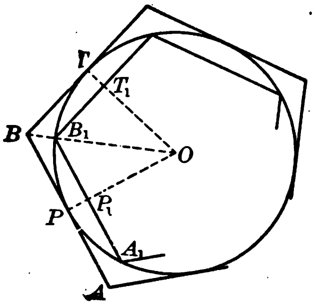
Cor. 5. *Tangents to a circle at the mid-points of the arcs subtended by the sides of a regular inscribed polygon, form a regular circumscribed polygon, of which the sides are parallel to those of the original polygon and the vertices lie on the prolongations of the radii of the one inscribed.*



Suggestions: Show that AB and A_1B_1 are both perpendicular to OP and are, therefore, parallel.

Show ∴ that $\angle A = \angle A_1$, $\angle B = \angle B_1$, ...

What kind of n-gon is then circumscribed?

$BP = BT$ and $B_1P_1 = B_1T_1$ and, therefore, B and B_1 lie on bisector of $\angle POT$.

Radius OB_1 bisects $\angle POT$.

Theorem 58. *A regular polygon the number of whose sides is $3 \cdot 2^n$ may be inscribed in a circle.*

Theorem 59. *If i_n represent the side of a regular inscribed polygon of n sides and i_{2n} the side of one of $2n$ sides and r the radius of the circle, $i_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - i_n^2}}$.*

Theorem 60. *If i_n represent the side of a regular inscribed polygon of n sides, c_n that of a regular circumscribed polygon of n sides, and r the radius of the circle, $c_n = \dfrac{2ri_n}{\sqrt{4r^2 - i_n^2}}$.*

Theorem 61. *The perimeters of regular polygons of the same number of sides compare as their radii and also as their apothems.*

Theorem 62. *Circumferences have the same ratio as their radii.*

Cor. 1. *The ratio of any circumference to its diameter is constant.*

Cor. 2. *In any circle $c = 2\pi r$.*

Theorem 63. *The value of π is approximately* 3.14159.

Theorem 64. *The area of a circle is equal to one-half the product of its radius and its circumference.*

Cor. 1. *The area of a circle is equal to π times the square of its radius.*

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Cor. 2. ***The areas of circles compare as the squares of their radii.***

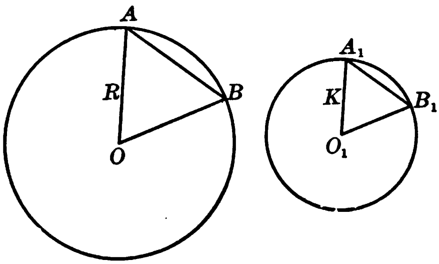
Cor. 3. ***The area of a sector is equal to half the product of its radius and its arc.***

(Proof left to the student.)

NOTE.—Cor. 3 does not suggest the most convenient method of determining the area of a sector. Suggest a more convenient one.

A **segment** *of a circle is a portion of it bounded by an arc and its subtending chord.* **Similar sectors** *and* **similar segments** *are those of which the arcs contain the same number of degrees.*

Cor. 4. ***Similar sectors and similar segments compare as the squares of their radii.***



Suggestions: How do circles O and O_1 compare in area? How, then, would like parts of them compare?

Why are similar sectors like parts?

Why is $\triangle AOB \backsim \triangle A_1O_1B_1$? How, therefore, do the triangles compare in area?

Justify the following algebraic statements which form the basis of proof here:

$$\frac{\text{sec. } AOB}{\text{sec. } A_1O_1B_1} = \left(\frac{R^2}{r^2}\right) = \frac{\triangle AOB}{\triangle A_1O_1B_1} \quad \therefore \frac{\text{sec. } AOB}{\triangle AOB} = \frac{\text{sec. } A_1O_1B_1}{\triangle A_1O_1B_1}$$

$$\therefore \frac{\text{seg. } AOB}{\triangle AOB} = \frac{\text{seg. } A_1O_1B_1}{\triangle A_1O_1B_1} \text{ or } \frac{\text{seg. } AOB}{\text{seg. } A_1O_1B_1} = \frac{\triangle AOB}{\triangle A_1O_1B_1} = \frac{R^2}{r^2}.$$

EXERCISES. SET LXXXVIII. MENSURATION OF THE CIRCLE

Numeric

1208. Find the circumferences of circles with diameters as follows:

(*a*) 9 in.	(*c*) 5.9 in.	(*e*) $2\frac{1}{2}$ ft.	(*g*) 29 centimeters
(*b*) 12 in.	(*d*) 7.3 in.	(*f*) $3\frac{1}{8}$ in.	(*h*) 47 millimeters

1209. Find the diameters of circles with circumferences as follows:

(*a*) 15	(*c*) $2\pi r$	(*e*) 188.496 in.	(*g*) 3361.512 in.
(*b*) π^2	(*d*) $7\pi a^2$	(*f*) 219.912 in.	(*h*) 3173.016 in.

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1210. Find the diameter of a carriage wheel that makes 264 revolutions in going half a mile.

1211. The diameter of a bicycle wheel is 28 in. How many revolutions does the wheel make in going 10 mi.?

1212. Find the radii of circles with circumferences as follows:

(*a*) 7π	(*c*) 15.708 in.	(*e*) 18.8496 in.	(*g*) 345.576 ft.
(*b*) $3\frac{1}{3}\pi$	(*d*) 21.9912 in.	(*f*) 125.664 in.	(*h*) 3487.176 in.

1213. Find the radius of a circle whose circumference is m units.

1214. An arc of a certain circle is 100 ft. long and subtends an angle of 25° at the center. Compute the radius of the circle correct to one decimal place.

1215. The circumference of a circle is 10. Find the circumference of one having twice the area of the original.

1216. Find the central angle of a sector whose perimeter is equal to the circumference.

1217. Find the areas of circles with diameters as follows:

(*a*) $16ab$	(*c*) 2.5 ft.	(*e*) $3\frac{2}{3}$ yd.	(*g*) 3 ft. 2 in.
(*b*) $24\pi^2$	(*d*) 7.3 in.	(*f*) $4\frac{5}{8}$ yd.	(*h*) 4 ft. 1 in.

1218. Find the area of circles with radii as follows:

(*a*) $5x$	(*c*) 27 ft.	(*e*) $3\frac{1}{8}$ in.	(*g*) 2 ft. 6 in.
(*b*) 2π	(*d*) 4.8 ft.	(*f*) $4\frac{5}{8}$ in.	(*h*) 7 ft. 9 in.

1219. Find the radii of circles with areas as follows:

(*a*) πa^2b^4	(*c*) π	(*e*) 12.5664	(*g*) 78.54
(*b*) $4\pi m^4n^6$	(*d*) 2π	(*f*) 28.2744	(*h*) 113.0976

1220. Find the areas of circles with circumferences as follows:

(*a*) 2π	(*c*) πa	(*e*) 18.8496 in.	(*g*) 333.0096 in.
(*b*) 4π	(*d*) $14\pi a^2$	(*f*) 329.868 in.	(*h*) 364.4256 in.

1221. Find the area of a circle whose circumference is C.

1222. Find the area of a sector whose radius is 5 and whose central angle is 40°.

1223. Find the area of a fan that opens out into a sector of 120°, the radius of which is $9\frac{3}{8}$ in.

1224. The arc of a sector of a circle $2\frac{1}{4}$ in. in diameter is $1\frac{3}{4}$ in. What is the area of the sector?

1225. Find the central angle of a sector whose area is equal to the square of the radius.

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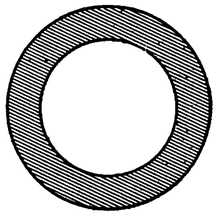
1226. Find the circumference of a circle whose area is S.

1227. A circle has an area of 60 sq. in. Find the length of an arc of 40°.

1228. Find the radius of a circle equivalent to a square the side of which is 6.

1229. The circumferences of two concentric circles are 30 and 40, respectively. Find the area bounded by the two circumferences by the shortest method you know.

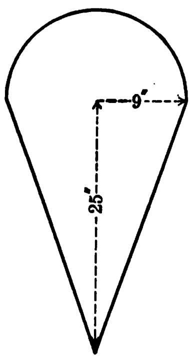


1230. In an iron washer here shown, the diameter of the hole is $1\frac{3}{8}$ in., and the width of the washer is $\frac{3}{8}$ in. Find the area of one face of the washer.

1231. The area of a fan which opens out into a sector of 111° is 96.866 sq. in. What is the radius? (Use $\pi = 3.1416$. Why?)

1232. The radius of a circle is 10 ft. Two parallel chords are drawn, each equal to the radius. Find that part of the area of the circle lying between the parallel chords.

1233. A square is inscribed in a circle of radius 10. Find the area of the segment cut off by a side of the square.



1234. Find a semicircle equivalent to an equilateral triangle whose side is 5.

1235. A kite is made as shown in the diagram, the semicircle having a radius of 9 in., and the triangle a height of 25 in. Find the area of the kite.

1236. Two circles are tangent internally, the ratio of their radii being 2 to 3. Compare their areas, and also the area left in the larger circle with that of each of the circles.

1237. A reservoir constructed for irrigation purposes sends out a stream of water through a pipe 3 ft. in diameter. The pipe is 1000 ft. long. How many times must it be filled if it is to discharge 10,000 acre-feet of water? (An acre-foot of water is the amount required to cover 1 acre to a depth of 1 ft.)

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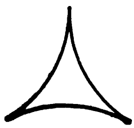
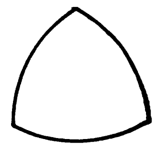
1238. Each side of a triangle is $2n$ centimeters, and about each vertex as center, with radius of n centimeters, a circle is described. Find the area bounded by the three arcs that lie outside the triangle, and the area bounded by the three arcs that lie inside the triangle.

1239. From a point outside a circle whose radius is 10, two tangents are drawn. Find the area bounded by the tangents and the circumference, if they include an angle of 120°. Find both results.

1240. Upon each side of a square as a diameter semicircles are described inside the square. If a side of the square is s, find the sum of the areas of the four leaves.



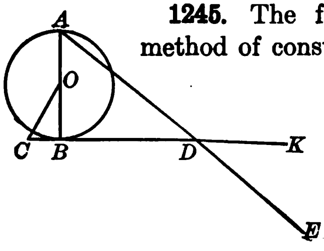
1241. Find the area bounded by three arcs each of 60° and radius 5 if the convex sides of the arcs are turned toward the area.



1242. Find the area bounded by three arcs each of 60° and radius 5 if the concave sides of the arcs are turned toward the area.

dt1243. The flywheel of an engine is connected by a belt with a smaller wheel driving the machinery of a mill. The radius of the flywheel is 7 ft., and of the driving wheel is 21 in. (*a*) How many revolutions does the smaller wheel make to one of the larger wheel? (*b*) The distance between the centers is 10 ft. 6 in. What is the length of the belt connecting the two wheels if it is not crossed? (*c*) If it is crossed?

1244. Given a circle whose radius is 16, find the perimeter and the area of the regular inscribed octagon.



1245. The following is Ceradini's approximate method of constructing a sect equal in length to a circle: Draw diameter AB and tangent BK at B. Draw OC making $\angle COB = 30°$. Make $\overline{CD} \equiv 3\overline{OB}$. Draw AD, and prolong it, making $\overline{AE} \equiv 2\overline{AD}$. Then AE is the required sect. Determine the accuracy of this construction by computing the ratio of AE to AB.

Suggestion: Let r = radius. Compute AB and AE in terms of r, then divide.

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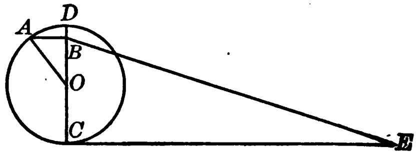
1246. Another method of finding the approximate value of the circle is as follows: Draw diameter CD. Make central angle $AOB = 30°$. Draw $AB \perp CD$. Draw CE tangent at C and equal to $3\overline{CD}$. Draw BE. Then BE equals the circle. Determine the accuracy of this construction.



1247. In making a drawing for an arch it is required to mark off on a circle drawn with a radius of 5 in. an arc that shall be 8 in. long. This is best done by finding the angle at the center. How many degrees are there in this angle?

1248. The perimeter of the circumscribed equilateral triangle is double that of the similar inscribed triangle.

1249. Squares are inscribed in two circles of radii 2 in. and 6 in., respectively. Find the ratio of the areas of the squares, and also of the perimeters.

1250. Squares are inscribed in two circles of radii 2 in. and 8 in. respectively, and on their sides equilateral triangles are constructed. What is the ratio of the areas of these triangles?

1251. A log a foot in diameter is sawed so as to have the cross-section the largest square possible. What is the area of this square? What would be the area of the cross-section of the square beam cut from a log of half this diameter?

1252. If r denotes the radius of a regular inscribed polygon, a its apothem, s a side, A an angle, and C the angle at the center, show that:

(*a*) In a regular inscribed triangle $s=r\sqrt{3}$, $a=\frac{1}{2}r$, $A=60°$, $C=120°$.

(*b*) In a regular inscribed quadrilateral $s=r\sqrt{2}$, $a=\frac{1}{2}r\sqrt{2}$, $A=90°$, $C=90°$.

(*c*) In a regular inscribed hexagon $s=r$, $a=\frac{1}{2}r\sqrt{3}$, $A=120°$, $C=60°$.

(*d*) In a regular inscribed decagon $s=\frac{1}{2}r(\sqrt{5}-1)$, $a=\frac{1}{4}r\sqrt{10+2\sqrt{5}}$, $A=144°$, $C=36°$.

1253. If r is the radius of a circle, a the apothem of a regular inscribed n-gon, and i_n one of its sides, i_{2n} a side of a regular inscribed

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$2n$-gon, c_n a side of a regular circumscribed n-gon; A_n the area of a regular inscribed n-gon, and A_{2n} that of a regular inscribed $2n$-gon, fill out the accompanying table:

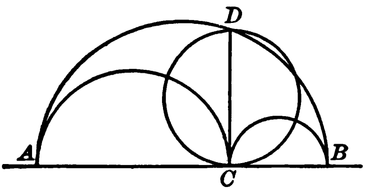
Given	Required	Given	Required
1. r, i_n	a	15. $i_n, n=3$	A_n
2. r, i_n	i_{2n}	16. $r, n=3$	A_n
3. r, i_{2n}	i_n	17. $i_n, n=6$	A_n
4. r, i_n	c_n	18. $r, n=6$	A_n
5. $r, n=3$	i_n, a	19. $i_n, n=12$	A_n
6. $r, n=6$	i_n, a	20. $r, n=20$	A_n
7. $r, n=12$	i_n, a	21. $i_n, n=12$	A_n
8. $r, n=4$	i_n, a	22. $r, n=12$	A_n
9. $r, n=8$	i_n, a	23. $i_n, n=4$	A_n
10. $r, n=10$	i_n, a	24. $r, n=4$	A_n
11. $r, n=5$	i_n, a	25. $i_n, n=8$	A_n
12. i_n, n, a	A_n	26. $r, n=8$	A_n
13. i_n, n, r	A_n	27. $r, n=5$	A_n
14. i_n, n, r	A_{2n}	28. $r, n=10$	A_n

Theoretic

1254. The area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

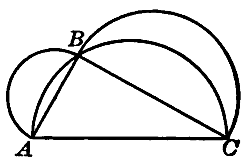
d1255. An equilateral polygon circumscribed about a circle is regular if the number of its sides is odd.

d1256. An equiangular polygon inscribed in a circle is regular if the number of its sides is odd.



1257. If C be a point in the straight line AB, the three semicircles, drawn respectively upon sects AB, AC, and CB as diameters, bound an area equal to a circle of which the diameter is the perpendicular CD, D being in the largest semicircle.

1258. If upon three sides of a right triangle semicircles be drawn as indicated in the diagram, the area of the right triangle is equal to the sum of the two crescent-shaped areas, bounded by the semicircles. (Hippocrates' Theorem.)



1259. Give a simpler proof for Ex. 859. (*b*) Generalize this fact.

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CHAPTER VI

METHODS OF PROOF

There are two general methods of proving theorems, the *direct* or *synthetic*, and the *indirect* method. Each of these methods of proof may be in its nature geometric or algebraic.

Further, indirect proofs, whether geometric or algebraic, may take different forms. Thus a theorem may be proved indirectly, either by means of *exclusion* or *reduction to an absurdity*, or by *analysis*.

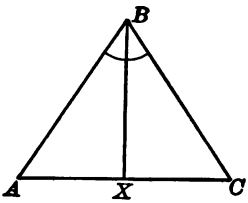
It is the object of this chapter to give an illustration of each of these methods of proof, together with several exercises that will be most naturally proved by that method.

A. THE DIRECT OR SYNTHETIC METHOD

In this method we employ either superposition or start with the data, and combining them with known truths proceed step by step until we arrive at the desired conclusion.

I. GEOMETRIC PROOF.*

Illustration: The bisector of the vertex angle of an isosceles triangle bisects the base.



Given: $\triangle ABC$, $AB = BC$, X in AC so that $\measuredangle ABX = \measuredangle CBX$.
Prove: $AX = XC$.

PROOF

In $\triangle ABX$ and $\triangle CBX$.	
(1) $AB = BC$, $\measuredangle ABX = \measuredangle CBX$.	(1) Data.
(2) $BX = BX$.	
(3) $\therefore \triangle ABX \cong \triangle CBX$.	(3) Two sides and the included angle determine a triangle.
(4) $\therefore AX = CX$.	(4) Hom. sides of cong. $\triangle$ are equal.

* The method of superposition should be very rarely used. For illustrations of it, recall the proofs of the fundamental theorems in congruence of triangles.

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EXERCISES. SET LXXXIX. SYNTHETIC METHODS OF PROOF

Give a synthetic proof of each of the following:

1260. The bisector of the vertex angle of an isosceles triangle is perpendicular to the base.

1261. If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles.

1262. Any point in the bisector of the vertex angle of an isosceles triangle is equidistant from the ends of the base.

II. ALGEBRAIC PROOF.

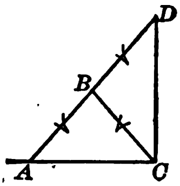

Illustration: If one leg of an isosceles triangle is extended through the vertex by its own length, the sect joining its end to the end of the base is perpendicular to the base.

Given: ABD a st. line, $AB = BD = BC$.
Prove: $DC \perp AC$.

PROOF

(1) $\angle A + \angle D + \angle DCA = 180°$.	(1) The sum of the $\angle$s of a $\triangle$ is a st. $\angle$.
(2) $\because AB = BC$ and $BC = BD$.	(2) Data.
(3) $\angle A = \angle BCA$, $\angle D = \angle DCB$.	(3) Base $\angle$s of an isosceles $\triangle$ are equal.
(4) $\therefore$ substituting (3) in (1), $\angle BCA + \angle DCB + \angle DCA = 180°$.	(4) Quantities may be substituted for their equals in an equation.
(5) But $\angle BCA + \angle DCB = \angle DCA$.	(5) The whole equals the sum of all its parts.
(6) $\therefore 2\angle DCA = 180°$.	(6) See (4).
(7) $\therefore \angle DCA = 90°$.	(7) Quotients of equals divided by equals are equal.
(8) $\therefore \angle DCA$ is a rt. $\angle$.	(8) The numeric measure of a rt. $\angle$ is 90°.
(9) $\therefore DC \perp AC$.	(9) By def. of $\perp$.

This method is especially adapted to the proof of numerical relations between angles or sects.

The following procedure is generally used in applying the algebraic synthetic type of proof.

1. Observation of the numeric relations that immediately follow from the data.
2. Statement of these relations in algebraic form—the equality or the inequality.
3. Reduction of these algebraic relations by the help of axioms until the desired conclusion is reached.

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EXERCISES. SET LXXXIX (concluded)

Give an algebraic synthetic proof of each of the following:

1263. The bisectors of two supplementary adjacent angles are perpendicular to each other.

1264. If the bisectors of two adjacent angles are perpendicular to each other, those angles are supplementary.

1265. If two sides of a triangle are unequal the angles opposite them are unequal in the same order.

1266. The sum of the altitudes of a triangle is less than its perimeter.

1267. The angle whose sides are the altitude from and the bisector of an angle of a triangle is equal to one-half the difference between the remaining angles of the triangle.

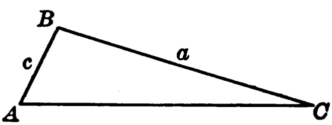
B. THE INDIRECT METHODS

I. GEOMETRIC; or II. ALGEBRAIC.

a. By the Method of Exclusion.

Two magnitudes of the same kind may bear one of three relations to each other. The first may be less than, equal to, or greater than the second. If it can be shown that two of these relations are false, the third is of necessity true. Similarly the position of a point may be fixed by the method of exclusion.

Illustration 1, *Theorem* 21c*:* If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.



Given: $\triangle ABC$ in which $\measuredangle A > \measuredangle C$.
Prove: $\bar{a} > \bar{c}$.

PROOF

$\bar{a} > \bar{c}$, $\bar{a} = \bar{c}$, or $\bar{a} < \bar{c}$.	Authorities left to the student.
Suppose $\bar{a} < \bar{c}$.	
Then $\measuredangle A < \measuredangle C$.	
But this contradicts the data.	
$\therefore \bar{a} \nless \bar{c}$.	
Suppose $\bar{a} = \bar{c}$.	
Then $\measuredangle A = \measuredangle C$.	
But this contradicts the data.	
$\therefore \bar{a} \neq \bar{c}$.	
$\therefore \bar{a} > \bar{c}$.	

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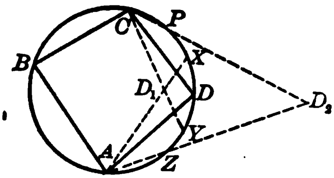

Illustration 2, *Theorem* 54*a*, *Cor.* 1: A quadrilateral whose opposite angles are supplementary is inscriptible.

Given: Quadrilateral $ABCD$ in which $\angle A + \angle C = 180°$, $\angle B + \angle D = 180°$.

Prove: A, B, C, D concyclic.

PROOF

Authorities left to the student.

(1) A circle may be passed through A, B, C.
(2) D lies outside, inside, or on this circle.
(3) Suppose D lies inside $\odot ABC$ in the position D_1.
Then $\angle B \overset{\propto}{=} \frac{1}{2}\overset{\frown}{AXC}$ and $\angle D_1 \overset{\propto}{=} \frac{1}{2}(\overset{\frown}{CBA} + \overset{\frown}{XY})$.
(4) $\therefore \angle B + \angle D_1 \overset{\propto}{=} \frac{1}{2}(\overset{\frown}{AXC} + \overset{\frown}{CBA} + \overset{\frown}{XY}) = \frac{1}{2}(360° + \overset{\frown}{XY}) > 180°$.
(5) $\therefore D$ cannot lie inside the circle.
(6) Suppose D lies outside $\odot ABC$ in the position D_2.
Then $\angle B \overset{\propto}{=} \frac{1}{2}\overset{\frown}{APC}$ and $\angle D_2 \overset{\propto}{=} \frac{1}{2}(\overset{\frown}{CBA} - \overset{\frown}{ZP})$.
(7) $\therefore \angle B + \angle D_2 \overset{\propto}{=} \frac{1}{2}(\overset{\frown}{APC} + \overset{\frown}{CBA} - \overset{\frown}{ZP}) = \frac{1}{2}(360° - ZP) < 180°$.
(8) $\therefore D$ cannot lie outside the circle.
(9) $\therefore D$ lies on the circle and $ABCD$ is inscriptible.

EXERCISES. SET XC. PROOF BY THE METHOD OF EXCLUSION

Prove the following facts by means of the method of exclusion:

1268. Knowing (1) that equal chords subtend equal arcs, and (2) that unequal chords subtend arcs unequal in the same order, prove the converse of each of these facts by the method of exclusion.

1269. In a fashion similar to that used to prove Ex. 1268, show that if $\left\{\begin{array}{l}(1) \text{ when } a = b,\ c = d; \\ (2) \text{ when } a > b,\ c > d; \\ (3) \text{ when } a < b,\ c < d,\end{array}\right\}$ then the converse of each of these facts is true. (This is known as the *Law of Converses.*)

b. By Reduction to an Absurdity. (Reductio ad absurdum.)

This method is similar to that of exclusion in that it makes an assumption which results in a contradiction of the data, but differs from it in that but one such assumption is made before a final conclusion is reached. Briefly, in proving a proposition by reduction to an absurdity, we do so simply by proving that the theorem which contradicts the conclusion of the original* is false.

* Such a theorem is called the contradictory of the direct.

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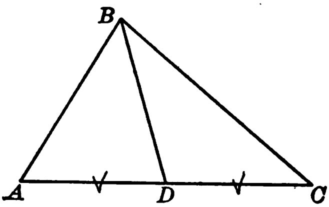


Illustration: If the median to the base of a triangle meets it obliquely, the remaining sides are unequal.

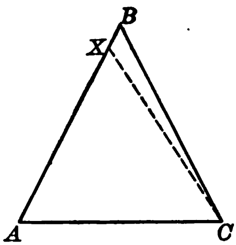
Given: D in AC in $\triangle ABC$; $AD = DC$, $\angle ADB \neq \angle BDC$.
Prove: $AB \neq BC$.

PROOF

(1) Suppose $AB = BC$.
Then $\triangle ABD \cong \triangle BDC$.
(2) $\therefore \angle BDA = \angle CDB$.
(3) But this contradicts the data and $\therefore$ the assumption is absurd.
(4) $\therefore AB \neq BC$.

Authorities left to the student.

EXERCISES. SET XCI. PROOF BY REDUCTION TO AN ABSURDITY



Prove the following exercises by the method of reduction to an absurdity:

1270. Prove that if two angles of a triangle are equal, the sides opposite them are equal.

Hint: Suppose $AB > BC$ and take $AX = BC$.

1271. If upon a common base an isosceles and a scalene triangle are constructed, the line joining their vertices does not bisect the vertex angle of the isosceles triangle.

Hint: Assume that it does bisect it.

1272. If perpendiculars are drawn to the sides of an acute angle from a point inside the angle, they enclose an oblique angle.

1273. Prove that if two triangles resting on a common base have a second pair of sides equal, and the third vertex of each outside the other triangle, their third sides are unequal.

c. By the Method of Analysis.

The method of analysis, which is attributed to Plato, was undoubtedly used by Euclid, but was probably emphasized by the former.

The analysis of a theorem is a course of reasoning, whether conscious or unconscious, by means of which a proof is discovered.

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It consists of discovering the immediate condition under which the conclusion would be true, and continuing to do this with each new condition until one known to be true is reached.

Analysis always takes the following form: Suppose we wish to prove that $A \equiv B$ if $C \equiv D$.

We start by saying:	$A \equiv B$ if $x \equiv y$.
But	$x \equiv y$ if $m \equiv n$.
and	$m \equiv n$ if $C \equiv D$.
But	$C \equiv D$ by data.

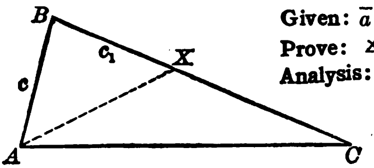
The proofs of theorems are usually put in the synthetic form, but they are derived analytically and then rearranged by retracing the steps taken.

The analytic method might also be called the method of *reduction*, or of *successive substitutions.*

If we wish to discover how to prove a theorem, we should always use the analytic method. It is much more likely to suggest those helpful auxiliary lines which are frequently needed before any relation between the theorem to be proved and a known theorem is apparent.

On the contrary, nothing in the synthetic method suggests the use of suitable auxiliary lines, and though we may continue to make deductions as they occur to us, we often waste time and energy without getting any nearer the conclusion.

Illustration 1, *Theorem* 21*b:* If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.



Given: $\bar{a} > \bar{c}$ in $\triangle ABC$.

Prove: $\angle A > \angle C$.

Analysis: $\angle A > \angle C$ if part of $\angle A > \angle C$.

Part of $\angle A > \angle C$ if that part can be made an exterior $\angle$ of a $\triangle$ in which $\angle C$ is a non-adjacent interior $\angle$.

This cannot be done as the $\angle$s are now placed.

How, then, can we find an $\angle$ which is equal to part of $\angle A$ and placed as desired?

Since $c < a$, an isos. $\triangle$ can be formed by laying off $BX = c$ on a.

Hence the auxiliary line AX is suggested, and $\angle A > \angle C$ if $\angle BAX > \angle C$.

$\angle BAX > \angle C$ if $\angle BXA > \angle C$.

But $\angle BXA > \angle C$.

$\therefore$ We may reverse the steps and give a brief synthetic proof if desired.

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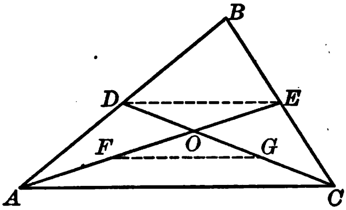
The analytic method does not always lead at once to the shortest proof of a theorem, though it is far more likely to do so than the synthetic method. At times an analysis suggests several methods of proof, and our selection will depend upon which seems the shortest. Skill in selection can be acquired only by practice.

Illustration 2: If one median in a triangle is intersected by a second, the sect between the vertex and the point of intersection is double the other sect.



Given: Medians CD and AE intersecting at O in $\triangle ABC$.

Prove: $\overline{AO} = 2\overline{OE}$, and $\overline{CO} = 2\overline{OD}$.

Analysis: One sect may be proved double another by proving (1) half the longer = the shorter, or (2) double the shorter = the longer.

The first method suggests that we take F and G in AO and CO so that $AF = FO$ and $CG = GO$.

Now $\overline{AO} = 2\overline{OE}$ and $\overline{CO} = 2\overline{OD}$ if $FO = OE$ and $GO = OD$.

$FO = OE$ and $GO = OD$ if GD and FE are diagonals of a ▱.

FE and GD are diagonals of a ▱ if $DE = FG$ and $DE \parallel FG$.

$DE = FG$ and $DE \parallel FG$ if DE bisects AB and BC in $\triangle ABC$, and if FG bisects AO and CO in $\triangle AOC$, for then $DE = \frac{1}{2}(AC) = FG$ and $DE \parallel AC \parallel FG$.

∴ a synthetic proof can now readily be given.

Success in this type of work very often depends on the selection of suitable auxiliary lines. Those which are most often of use are discovered by

(*a*) joining two points,
(*b*) drawing a line parallel to a given line,
(*c*) drawing a line perpendicular to a given line,
(*d*) bisecting given sects as in the last illustration,
(*e*) producing a sect by its own length, as might have been done in the last illustration.

Since we have not proved, even if it be true, that the algebraic processes employed in the proof of theorems in proportion are reversible, only a synthetic proof is valid. Of course, to suggest the synthetic line of argument it is desirable to give an analysis first if needed.

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Illustration:

Given: $\frac{a}{b}=\frac{c}{d}$.

Prove: $\frac{a^2}{b^2}=\frac{a^2+c^2}{b^2+d^2}$.

ANALYSIS

		Operation performed.
$\frac{a^2}{b^2}=\frac{a^2+c^2}{b^2+d^2}$	(1)	
if $a^2b^2+a^2d^2=a^2b^2+b^2c^2$	(2)	(1) $\cdot b^2(b^2+d^2)$
(2) is true if $a^2d^2=b^2c^2$	(3)	(2) $-a^2b^2$
(3) is true if $ad=bc$	(4)	$+\sqrt{(3)}$
(4) is true if $\frac{a}{b}=\frac{c}{d}$	(5)	(4) $\div bd$

But $\frac{a}{b}=\frac{c}{d}$ by data.

$\therefore$ the following synthetic proof

	$\frac{a}{b}=\frac{c}{d}$	(1)	(1) Given.
(1) $\cdot bd$,	$\therefore ad=bc$	(2)	(2) Products of equals multiplied by equals are equal.
(2) ^{2},	$\therefore a^2d^2=b^2c^2$	(3)	(3) Like powers of equals are equal.
(3) $+a^2b^2$,	$\therefore a^2d^2+a^2b^2=b^2c^2+a^2b^2$	(4)	(4) Sums of equals added to equals are equal.
(4) $\div b^2(b^2+d^2)$, $\therefore$	$\frac{a^2}{b^2}=\frac{a^2+c^2}{b^2+d^2}$	(5)	(5) Quotients of equals divided by equals are equal.

EXERCISES. SET XCII. ANALYTIC METHOD OF PROOF

Give an analysis of each of the following exercises, and follow it by a concise synthetic proof.

1274. Prove the second illustration under the analytic method by

(*a*) proving $\triangle DOE \cong \triangle FOG$ or $\triangle DOF \cong \triangle GOE$ (p. 303).

(*b*) Doubling DO and OE. Use each of the three methods suggested by the following figures.

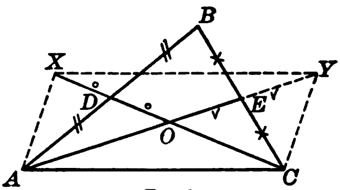

FIG. 1

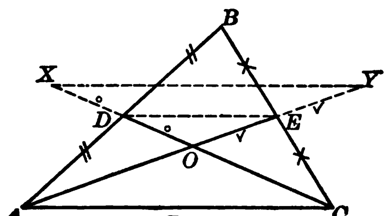

FIG. 2

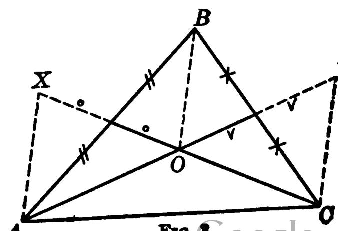

FIG. 3

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1275. Prove Theorem 31, Cor. 4, analytically.

1276. Prove Theorem, 31 Cor. 5, analytically.

1277. If one acute angle of a right triangle is double the other, the shorter leg is one-half the hypotenuse. Prove first by drawing auxiliary lines outside the triangle, and then by drawing them inside the triangle.

1278. A median of a triangle is less than half the sum of the adjacent sides.

1279. Prove the theorem given under *A II* analytically (p. 298).

1280. Prove the theorem given under *B I a* analytically (p. 299).

1281. Prove Ex. 1270 analytically.

1282. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$ prove that $\frac{a}{b}=\frac{c}{d}$.

1283. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^2-b^2}{a^2-3ab}=\frac{c^2-d^2}{c^2-3cd}$.

1284. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a^2+2ab}{3ab-4b^2}=\frac{c^2+2cd}{3cd-4d^2}$.

1285. If $\frac{a}{b}=\frac{c}{d}$, then $\dfrac{a^2-ab+b^2}{\frac{a^3-b^3}{a}}=\dfrac{c^2-cd+d^2}{\frac{c^3-d^3}{c}}$.

While practice alone can give skill in the proof of theorems, the following suggestions may be of help to the student.

First. Make as general a figure as possible.

If a fact is to be proved concerning triangles in general the figure should be that of a scalene triangle, since many facts are true of isosceles or equilateral that are not true of scalene triangles. Again, if a fact is to be proved concerning quadrilaterals in general, it might be misleading to draw a parallelogram or even a trapezoid.

Second. Always have clearly in mind what is given and what is to be proved.

Third. If the proof is not readily seen, resort to analysis.

Fourth. Give a proof by the method of reduction to an absurdity only as a last resort.

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CHAPTER VII

CONSTRUCTIONS. METHODS OF ATTACKING PROBLEMS

The solutions of the following fourteen problems and the corollaries to them are typical of one class of solution of construction problems, namely, those solutions which are at once found to rest directly upon some known theorem, and in addition, at times, upon some known construction.

Problem 1.* ***Draw a perpendicular to a given line from a given point*** (*a*) ***outside the line,*** (*b*) ***on the line.***†

The constructions rest directly upon Theorem 40, Cor. 1, the fact that two points equidistant from the ends of a sect determine the perpendicular bisector of that sect.

Problem 2. ***Bisect a given*** (*a*) ***sect,*** (*b*) ***angle,*** (*c*) ***arc.***

The construction of (*a*) rests immediately upon Problem 1.

The construction of (*b*) rests immediately upon Theorem 5, the fact that three sides determine a triangle.

The construction of (*c*) rests immediately upon Theorem 43, the fact that equal central angles intercept equal arcs, and Problem 2 (*b*).

Problem 3. ***Reproduce a given angle.***

The construction rests directly upon Theorem 5.

Problem 4. ***Draw a line through a given point, and parallel to a given line.***

The construction *may* rest directly upon Theorem 11, the fact that if when lines are cut by a transversal the alternate-interior angles are equal, the lines thus cut are parallel, and Problem 3.

* The first thirteen problems of the syllabus were taken up as exercises in the First Study, but are repeated here (with only suggestions for their construction) as an integral part of a syllabus of constructions.

† According to Proclus, this problem was first investigated by Œnopides, a Greek philosopher and mathematician of the 5th century B.C. Proclus speaks of such a line as a "gnomon"—the common name for the vertical piece on a sundial.

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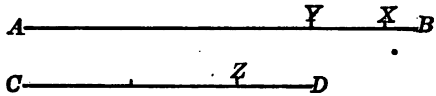
Problem 5. ***Construct a triangle, given any three independent parts; (a) two angles and the included side, (b) two sides and the included angle, (c) three sides, (d) the hypotenuse and a leg of a right triangle.***

(What can you say in case (*c*) if one side is equal to or greater than the sum of the other two sides?)

Problem 6. ***Divide a sect into n equal parts.***

The construction may rest upon Theorem 31, Cor. 3, the fact that parallels which intercept equal sects on one transversal do so on all transversals, and Problem 3. For further help see Ex. 429, p. 116.

Problem 7. ***Find a common measure of two commensurable sects.***



Given: $\overline{AB}$ and $\overline{CD}$ commensurable sects.
Required: A common measure *m*.
Solution: Lay off $AY \equiv CD$ on AB.
Then $AB \equiv CD + YB$.
Lay off $CZ \equiv 2\overline{YB}$ on CD.
Lay off $\overline{YX} \equiv \overline{ZD}$ on YB.
Lay off XB on ZD.
It is found to be contained exactly in ZD.

Then XB is the greatest common measure of AB and CD, and any integral part of $\overline{XB}$ is a common measure of them. Prove it.

Problem 8. ***Pass a circle through three non-collinear points.***

The construction rests directly upon Theorem 41*b*.

Cor. 1. ***Circumscribe a circle about a triangle.****

Problem 9. ***Divide a given sect into parts proportional to n given sects.***

The construction rests directly upon Theorem 31, Cor. 2.

Problem 10. ***Divide a given sect harmonically in the ratio of two given sects.***

Use (1) the method suggested in Problem 9, or (2) the method suggested by Theorem 32*b*, Cor. 1.

---

* The center of a circle circumscribed about a polygon is called its circumcenter.

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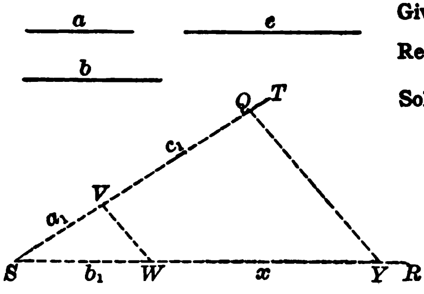
Problem 11. *Find a fourth proportional to three given sects.*

The construction rests directly upon Theorem 31. The solution is given, but the proof is left to the pupil.



Given: Sects a, b, c.

Required: Sect x, so that $\frac{\bar{a}}{\bar{b}} = \frac{\bar{c}}{\bar{d}}$.

Solution: Construct any angle RST.

On ST, lay off $SV = a$.
On SR, lay off $SW = b$.
On VT, lay off $VQ = c$.
Draw $\overline{VW}$.
Draw $\overline{QY} \parallel \overline{VW}$ and cutting SR at Y.
Then $\overline{WY}$ is the required sect x.

Cor. 1. *Find a third proportional to two given sects.*

What is the only modification needed in the construction of Problem 11 in order to find x, so that $\frac{\bar{a}}{\bar{b}} = \frac{\bar{b}}{\bar{x}}$?

Problem 12. *Upon a given sect as homologous to a designated side of a given polygon construct another similar to the original polygon.*

The construction rests directly upon Theorem 36*b* and Problem 11.

Problem 13. *Construct a square equal to the sum of two or more given squares.*

Use the Pythagorean Theorem.

Cor. 1. *Construct a square equal to the difference of two given squares.*

In this case the larger of the given squares will be the square on which side of the right triangle?

The construction therefore rests upon Problem 5 (d)

Cor. 2. *Construct a polygon similar to two given similar polygons and equal to* (a) *their sum,* (b) *their difference.*

How do the areas of similar polygons compare?

Problem 14. *Inscribe in a circle, regular polygons the number of whose sides is* (a) $3 \cdot 2^n$, (b) $4 \cdot 2^n$.

(a) The construction rests directly upon Theorem 58 and the construction of an equilateral triangle given its side (here the radius of the circle) in order to obtain a central angle of 120°.

(b) What is the central angle in the case of the square?

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THE SYNTHETIC METHOD OF ATTACKING A PROBLEM

The preceding type of construction problem is the simplest, and the solution of one of that nature is usually so readily seen, that without further explanation (except for one or two suggestions appended to the first exercises) the pupil is asked to solve the following set of exercises.

EXERCISES. SET XCIII. SYNTHETIC SOLUTIONS

1286. Trisect a right angle. (We know that each angle of an equilateral triangle is two-thirds of a right angle. What construction does this therefore suggest?)

1287. Divide an equilateral triangle into three congruent triangles. (We know that the bisectors of the angles of a triangle are concurrent, and that triangles are determined by two angles and the included side.)

1288. Construct an equilateral triangle with a given sect as altitude. (What fact about the altitude of an equilateral triangle suggests the construction?)

1289. Construct a square having given its diagonal.

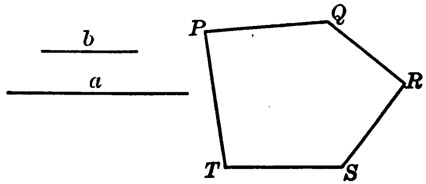
1290. Through two given points draw straight lines which shall make an equilateral triangle with a given straight line.

1291. On a given sect construct a rhombus having each of one pair of opposite angles double each of the other pair.

1292. On a given base construct a rectangle equal to (*a*) a given square, (*b*) another given rectangle, (*c*) a given triangle, (*d*) a given trapezoid.

1293. The sides of a polygon are 5, 7, 9, 11, 13. Construct one similar to it having the ratio of similitude 3 to 5.

1294 Construct a polygon similar to the accompanying polygon, having the ratio of similitude equal to that of the two given sects *a* and *b*.



1295. Construct a polygon similar to two (or more) given similar polygons and equivalent to their sum (or difference).

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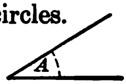
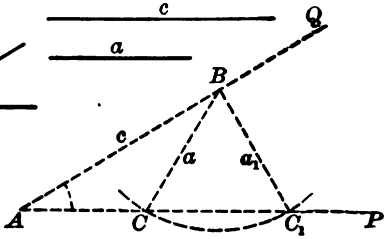
1296. Construct a circle equal to the sum of two given circles.

1297. Construct a circle equal to the difference of two given circles.





Problem 15. ***Construct a triangle given two sides and the angle opposite one of them.***

Given: Sides $\bar{a}$ and $\bar{c}$; $\measuredangle A$.

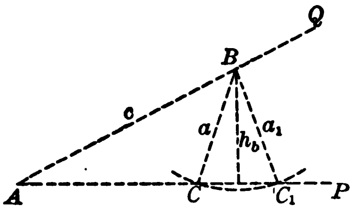
Required: $\triangle ABC$.

Construction: Construct $\measuredangle PAQ = \measuredangle A$.

On AQ, lay off $\overline{AB} = \bar{c}$.

With B as center and $\bar{a}$ as radius, strike an arc cutting AP in C and C_1, touching it in only C, or not at all, according as $a > h_b$, $a = h_b$, or $a < h_b$.

Then $\triangle ABC$ and $\triangle ABC_1$ fulfill the required conditions.



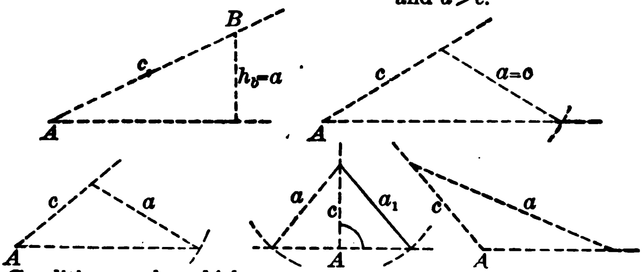
Discussion: I. Conditions under which there are two solutions.

(1) $\measuredangle A$ acute, $\bar{a} < \bar{c}$, but $> h_b$, the perpendicular from B to $\overline{AP}$.

II. Conditions under which there is but one solution.

(1) $\measuredangle A$ acute, $\bar{a} = h_b$ or c.

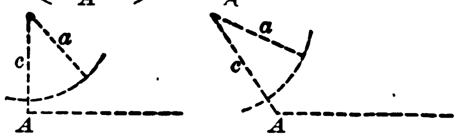
(2) $\measuredangle A$ acute, right, or obtuse, and $\bar{a} > \bar{c}$.



III. Conditions under which there is no solution.

(1) $\measuredangle A$ right or obtuse and $\bar{a} < c$.

(2) $\measuredangle A$ acute and $\bar{a} < h_b$.



THE DISCUSSION OF A PROBLEM

As in the case of Problem 15, many exercises in construction call for a *discussion*, by reason of the fact that the number of solutions of the problem varies under different conditions.

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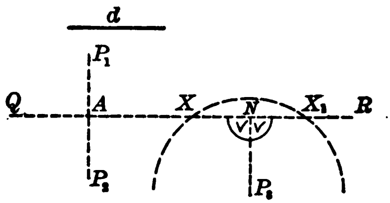
It will be noticed that the discussion of Problem 15 has been arranged under three heads, based upon the number of possible solutions. The discussion might have been arranged under entirely different headings. To show this, the pupil is asked to fill in the discussion under the following heads.

I. $\sphericalangle A$ acute.	II. $\sphericalangle A$ right.	III. $\sphericalangle A$ obtuse.
(1) $a < h_b$.	(1) $a < c$.	(1) $a < c$.
(2) $a \equiv h_b$.	(2) $a \equiv c$.	(2) $a \equiv c$.
(3) $a > h_b$ but $< c$.	(3) $a > c$.	(3) $a > c$.
(4) $a \equiv c$.		
(5) $a > c$.		

or,

I. $a < c$. II. $a \equiv c$. III. $a > c$. (Fill in all the necessary subheads.)

Further illustration of what is meant by the discussion of a problem in construction: To find a point X which shall be equidistant from points P_1 and P_2 and at a given distance d from P_3.



Given: Points P_1, P_2, P_3; sect d.
Required: Point X such that $P_1X = P_2X$ and $P_3X \equiv d$.
Construction: (1) Draw $\overline{P_1P_2}$ and bisect it at A.
(2) Erect $\overline{QAR} \perp \overline{P_1P_2}$.
(3) With P_3 as center and $\overline{d}$ as radius describe a circle cutting $\overline{QR}$ at X, X_1.
Then X, X_1 are the required points.

PROOF

(1) $P_1X \equiv P_2X$ and $P_1X_1 \equiv P_2X_1$ $\because$ X, X_1 lie on QR.	(1) The locus of points equidistant from the ends of a sect is the perpendicular bisector of that sect.
(2) $\because$ X and X_1 are on $\odot P_3$, $P_3X \equiv P_3X_1 \equiv \overline{d}$.	(2) Const., and the locus of points at a given distance from a given point is a circle of which the center is the given point, and the radius the given distance.

Discussion: Draw $\overline{P_3N} \perp \overline{QR}$.

I. Two solutions: If $\overline{d} > \overline{P_3N}$, for then a portion of $\overline{QR}$ will be a chord of the circle P_3.

II. One solution: If $\overline{d} = \overline{P_3N}$, for then $\overline{QR}$ will be tangent to OP_3 at X.

III. No solution: If $\overline{d} < \overline{P_3N}$, for then all points in $\overline{QR}$ will lie further from the center P_3 than the length of the radius.

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LOCI AND PROBLEMS SOLVED BY THE METHOD OF THE INTERSECTION OF LOCI

In many problems we are asked to find the position of a point which satisfies two given conditions. Each of the conditions determines a locus on which the point lies, and the solution of the problem is therefore the point or points common to both loci.

The last exercise illustrates this type of problem. The points X and X_1 fulfill two conditions. The first condition determined the locus of points equidistant from P_1 and P_2, the second determined the locus of points at a given distance d from P_3. The solution of the problem was those points common to both loci.

EXERCISES. SET XCIV. INTERSECTION OF LOCI

Give a full discussion of each of the exercises in this set. To find a point X such that:

1298. X shall be at the distance d_1 from point P_1, and d_2 from point P_2.

1299. X shall be equidistant from two parallel lines and at the distance d from point P.

1300. X shall be equidistant from two intersecting lines, and also equidistant from points P and P_1.

In line l find point X so that:

1301. X shall be at distance d from P.

1302. X shall be equidistant from P and P_1.

1303. X shall be at distance d from l_1.

1304. X shall be equidistant from two given parallel lines.

1305. Draw a circle of given radius to touch two given lines.

Draw a circle with a given radius:

1306. Passing through two given points.

1307. Passing through a given point and touching a given line.

1308. Passing through a given point and touching a given circle.

1309. Touching a given line and a given circle.

1310. Touching two given circles.

1311. Describe a circle touching two parallel lines and passing through a given point.

1312. Construct a right triangle, having given the hypotenuse and the altitude on the hypotenuse.

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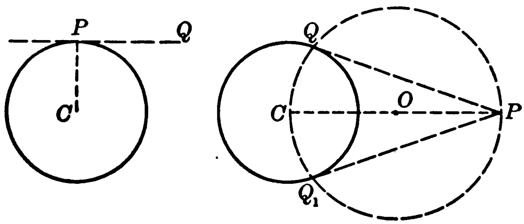
1313. Construct a right triangle, given sects of hypotenuse made by the bisector of the right angle.

1314. Construct a triangle, given an altitude and the sects made by the altitude upon the opposite side.

1315. Construct a right triangle, given the sects of the hypotenuse made by the altitude upon the hypotenuse.

Problem 16. ***Through a given point draw a tangent to a given circle, (a) when the point is on the circle, (b) when the point is outside the circle.***



Given: *Pt.* P (*a*) on $\odot C$, (*b*) outside $\odot C$.

Required: $\overline{PQ}$ tangent to $\odot C$.

Construction: For (*a*)

(1) Draw $\overline{CP}$.

(2) Draw $\overline{PQ} \perp \overline{CP}$.

Then PQ is the required tangent.

For (*b*)

(1) Draw sect CP.

(2) On CP as diameter describe $\odot O$ cutting $\odot C$ in Q and Q_1.

(3) Draw $\overline{PQ}$ and $\overline{PQ_1}$, each of which is the required tangent.

Proofs

For (*a*)

Since $\measuredangle CPQ$ is a rt. $\measuredangle$, and $\overline{CP}$ a radius, PQ is tangent to $\odot C$. Why?

For (*b*)

Draw $\overline{CQ}$ and $\overline{CQ_1}$.

(1) Then $\triangle$ PCQ and PCQ_1 are right $\triangle$. Why?

(2) $\therefore PQ$ and $PQ_1 \perp CQ$ and CQ_1 respectively. Why?

(3) $\therefore PQ$ and PQ_1 are tangent to $\odot C$. Why?

Discussion: When the point is on the circle there is but one solution. Why?
When it is outside the circle there are two and only two possible solutions. Why?

The solution of all construction problems calls for at least four parts. Namely:

1st—The statement of what is given.

2nd—The statement of what is required.

3rd—The construction of what is required.

4th—The proof of the correctness of this construction.

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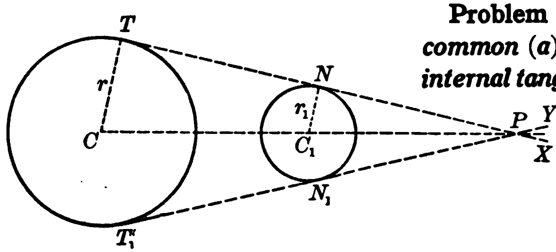
We have already seen that many problems call for a discussion, and in many of the problems in the syllabus an analysis will be given.

Omitting data and what is required, one might say then that the systematic solution of a problem consists of four parts: (1) The analysis, (2) the construction, (3) the proof, and (4) the discussion.



Problem 17. ***Construct a common (a) external and (b) internal tangent to two circles.***

Given: $\odot$s C and C_1 of radii r and r_1.

Required: The common (*a*) external tangents TN and T_1N_1, and (*b*) internal tangents TN and T_1N_1.

Analysis of (*a*): Suppose one of the required tangents, TN, drawn.

Then if $r \neq r_1$, $\overline{TNX}$ will intersect $\overline{CC_1Y}$ at P.

Then $\triangle CTP \backsim \triangle C_1NP$.

Then $\frac{CP}{C_1P} = \frac{r}{r_1}$.

But r and r_1 are known.

$\therefore$ to find P divide CC_1 externally in the ratio of $\frac{r}{r_1}$.

$\therefore$ to find tangent TN, draw PN tangent to $\odot C_1$ and show that if produced it will be tangent to $\odot C$.

The completion of the problem is left to the student, who should discuss the case where $r = r_1$.

Analysis of (*b*): The analysis in this case is similar to that in case (*a*) except that CC_1 will be divided internally in the ratio of r to r_1.

Construction and proof left to the student.

Problem 18. ***Inscribe a circle in a triangle.****

The center is determined by the intersection of what two loci?

Cor. 1. ***Find the centers of the escribed circles of a triangle.***†

The excenters are determined by the intersections of what loci?

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* *The center of the circle inscribed in a polygon is called the* **incenter** *of the polygon.*

† *The circles which are tangent to one side of a polygon and to the two consecutive sides produced, are called the* **escribed** *circles, and their centers are called the* **excenters** *of the polygon.*

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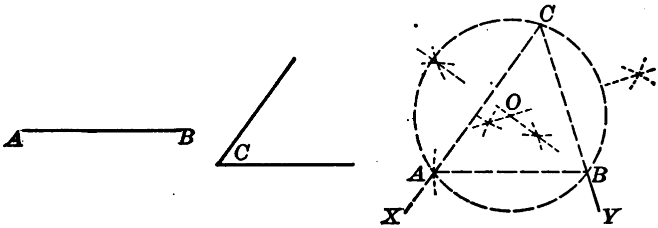
Problem 19. ***Construct upon a given sect the segment of a circle capable of containing a given angle.***



Given: Sect AB; $\measuredangle C$.

Required: Segment of $\odot O$ resting on AB as chord such that the inscribed $\measuredangle = \measuredangle C$.

Analysis: If the construction were complete ABC would be any one of a number of triangles of base AB and a vertex $\measuredangle C$.

The circle circumscribed about any one $\triangle ABC$ would give the required locus.

$\therefore$ the following:

Construction: Const. $\measuredangle XCY = \measuredangle C$.

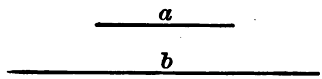
With B, any convenient pt. in CY as center, and radius = sect AB, strike an arc cutting CX in A.

Circumscribe a circle O about $\triangle ABC$.

Then segment ABC is the required segment.

(Proof left to the student.)

Problem 20. ***Construct a mean proportional between two given sects.***



Given: Sects a and b.

Required: Sect c, so that $\frac{\overline{a}}{c} = \frac{\overline{c}}{b}$.

Hint: Can you word Proposition 39, Cor. 4, to relate to the sects of a diameter of a circle?

Extreme and Mean Ratio. *If sect AB is so divided by the pt. P that* $\frac{\overline{AP}}{PB} = \frac{\overline{PB}}{AB}$, *the sect is said to be divided in* **extreme and mean ratio.** That is, one part of the sect is a mean proportional between the entire sect and the other part. This division of a sect is known as the "golden section." As in harmonic division, a sect may be divided internally and externally into extreme and mean ratio.

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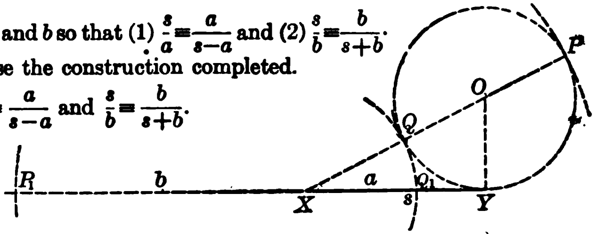
Problem 21. ***Divide a given sect into mean and extreme ratio.****

Given: Sect s.

Required: Sects a and b so that (1) $\frac{s}{a}=\frac{a}{s-a}$ and (2) $\frac{s}{b}=\frac{b}{s+b}$.

Analysis: Suppose the construction completed.



Then (1) $\frac{s}{a}=\frac{a}{s-a}$ and $\frac{s}{b}=\frac{b}{s+b}$.

These suggest a tangent as the mean proportional between a secant and its external sect or the leg of a rt. △ as the mean proportional between the hypotenuse and its projection upon the hypotenuse. As the proportions now stand there is but one known term.

But by composition we get (1) $\frac{s+a}{s}=\frac{s}{a}$ and (2) $\frac{b-s}{s}=\frac{s}{b}$.

This suggests s as the tangent to a circle, $\overline{s+a}$ as the entire secant, and $\bar{a}$ as its external sect on the one hand, and $\bar{b}$ as the entire secant and $\overline{b-s}$ as its external sect on the other hand.

The most natural secant to select is that which passes through the center.

Hence the following:

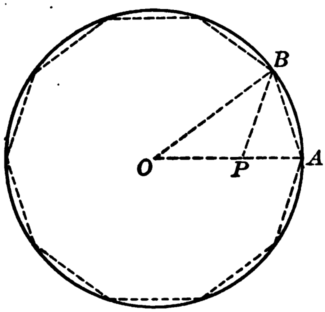
Construction: (1) Const. ⊙O of diameter s tangent to XY. ($XY=\bar{s}$) at Y.

(2) Draw secant $XQOP$.

(3) On XY lay off $XQ_1=XQ$, and on YX produced lay off $XP_1=XP$.

(4) Then Q_1 and P_1 are the required points. (Proof left to the student.)

Problem 22. ***Inscribe in a circle a regular decagon.***



Given: ⊙O.

Required: AB, the side of the regular inscribed decagon.

Analysis: If AB = side of reg. decagon, $\angle AOB=36°$.

$\therefore \angle OAB=\angle OBA=\frac{180°-36°}{2}=72°$

and if PB bisects $\angle ABO$, $\angle PBO=36°=\angle O$ and $\angle APB=180°-(36°+72°)=72°$.

$\therefore \triangle APB \sim \triangle ABO$ and $\triangle OPB$ is isosceles.

$\therefore \frac{AP}{AB}=\frac{AB}{OA}$. But $AB=PB=OP$.

$\therefore \frac{AP}{OP}=\frac{OP}{OA}$. $\therefore$ to find $OP=AB$ divide OA, the radius of ⊙O, into extreme and mean ratio, and use the mean as AB. (Proof left to student.)

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* A painting is said to be most artistically arranged when its center of interest is so placed that it divides the width of the picture into extreme and mean ratio. If the student is especially interested in the topic, he will find references in Chapter X.

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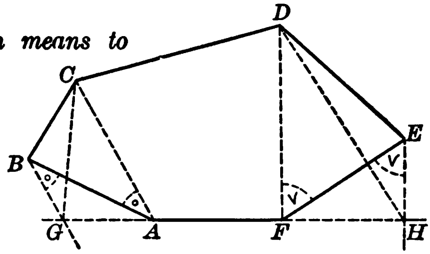
Cor. 1. ***Inscribe in a circle regular polygons the number of whose sides is (1) $5\cdot 2^n$, (2) $15\cdot 2^n$.***

(1) Left wholly to the student.

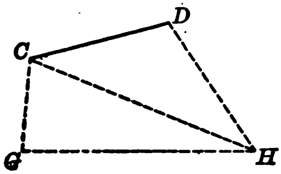
(2) **Hint:** $\frac{1}{15}=\frac{1}{6}-\frac{1}{10}$.

To **transform** *a polygon means to change its shape but not its area.*



Problem 23. ***Transform a polygon into a triangle.***

Given: Polygon $ABCDEF$.
Required: $A\triangle = ABCDEF$.
Hint: With CA as base, what is the locus of the vertices of triangles whose area = area $\triangle ABC$?

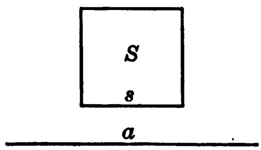


To eliminate side AB, by what particular $\triangle$ shall we then replace $\triangle ABC$?
Similarly, what $\triangle$ is to replace $\triangle DEF$?
Similarly, how dispose of side DH?

Problem 24. ***Construct the square equal to a given (a) parallelogram, (b) triangle, (c) polygon.***

Problem 25. ***Construct a parallelogram equal to a given square having given (a) the sum of base and altitude equal to a given sect, (b) the difference of base and altitude equal to a given sect.***

Given: Square S; sect a.
Required: $A\square = S$ with (a) base + alt. $= \bar{a}$, (b) base − alt. $= \bar{a}$.



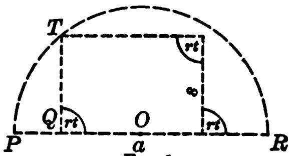


FIG. 1

(a) From Fig. 1 show that any parallelogram with QR as base and PQ as altitude = square S, and that it also fulfills the second condition.

(b) From Fig. 2 show that any parallelogram with QR as base and PQ as altitude = square s, and that it also fulfills the second condition.

This problem solves geometrically the algebraic problems, given (a) $x+y=a$ and $xy=s$, (b) $x-y=a$ and $xy=s$, find x and y.

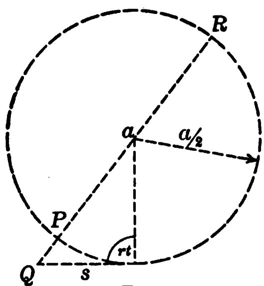


FIG. 2

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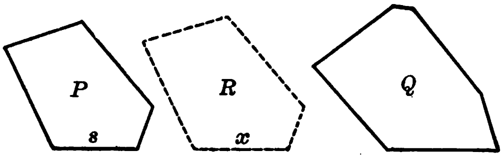
Problem 26. ***Construct a polygon similar to one and equal to another of two given polygons.***



Given: Polygons P and Q.

Required: Polygon R so that $R \backsim P$ and $R = Q$.

Analysis: If $R \backsim P$ and x homologous to s, then $\frac{P}{R} = \frac{s^2}{x^2}$. Why?

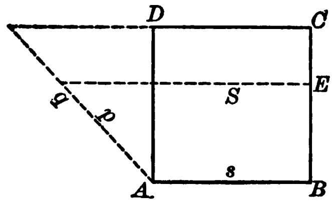
If P and Q equal m^2 and n^2 respectively, and $Q = R$, then $\frac{m^2}{n^2} = \frac{s^2}{x^2}$.

$\because \frac{P}{Q} = \frac{m^2}{n^2}$ and $\frac{P}{R} = \frac{s^2}{x^2}$

$\therefore \frac{m}{n} = \frac{s}{x}$, $\therefore x$ is the fourth proportional to the side of a square $= P$, the side of a square $= Q$, and the side in P homologous to x.

(Construction and proof left to the student.)

Problem 27. ***Construct a square which shall have a given ratio to a given square.***



Given: Square S of side s, ratio $\frac{p}{q}$.

Required: Square R of side r so that $\frac{R}{S} = \frac{p}{q}$.

Analysis: If a rectangle were to be found so that $\frac{\text{rect. } AE}{\text{square } S} = \frac{p}{q}$, the alt. s of square S would simply have to be divided so that $\frac{\text{alt. of rect.}}{s} = \frac{p}{q}$.

Then it remains to convert rect. AE into a square.

(Construction and proof left to the student.)

Cor. 1.* ***Construct a polygon equal to any part of a given polygon and similar to it.***

(Construction left to the student.)

FORMAL ANALYSIS OF A PROBLEM

When problems (as is usually the case) are of such nature that the application of known theorems and problems to their solution is not at once apparent, the only way to attack them is by a method

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* This corollary is included in the syllabus because of its practical value in drafting, since it is by this method that the draftsman finds his scale, in enlarging or reducing a diagram of any sort.

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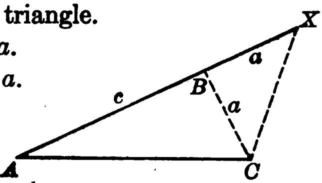
resembling the analytic method of proof of a theorem. This method is called the *analysis* of the problem.

In brief, the directions for following this method are:

1. Suppose the construction completed.
2. Draw a figure showing all the parts concerned. (Given parts in heavy lines, and required parts in dotted lines.)
3. Study the relations of the parts, and try to find some relation which will suggest a possible construction.
4. If a first attempt fails, introduce new relations by means of auxiliary lines, and continue the study of relations until a clue to the correct construction is derived.
5. Look for that clue in a rigid part of the figure—usually a triangle.

A few more illustrations to show just what is meant by these directions will be helpful.

1. Construct a triangle, having given the base, a base angle, and the sum of the remaining sides.



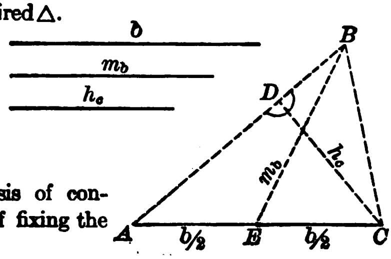
Suppose ABC to be the completed triangle.
In it we know $AC \equiv b$, $\angle A$, and $c+a$.
$\therefore$ produce AB to X so that $BX = a$.
Then $\triangle BCX$ is an isos. $\triangle$.
But $\triangle AXC$ is determined $(\bar{b}, \angle A, \overline{a+c})$.
But $\angle XCB \equiv \angle X \because BX \equiv BC$.
$\therefore \triangle ABC$ is determined.

2. Construct a triangle, having given one side, the median to that side, and the altitude to a second side.

Given: b, m_b, h_c.

Analysis: Suppose ABC to be the required $\triangle$.

Then the known parts are $AC \equiv b$,



$\therefore AE \equiv EC \equiv \frac{b}{2}$, $CD \equiv h_c$, $BE \equiv m_b$, and $\angle$ at $D \equiv$ rt. $\angle$s.

$\triangle DAC$ is determined. Why?

Using this triangle as the basis of construction, what means have we of fixing the vertex B?

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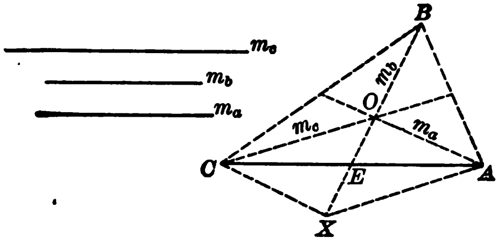
3. *Construct a triangle having given its medians.*



Given: m_a, m_b, m_c.
Required: $\triangle ABC$.
Analysis: Suppose ABC to be the required $\triangle$.

Evidently no $\measuredangle$ within the $\triangle$ is fixed by the medians.

But it is known that $OC = \frac{2}{3}m_c$, $OE = \frac{1}{3}m_b$, $OA = \frac{2}{3}m_a$, and that OE bisects AC.

$\therefore$ if OE is produced to X, so that $XE = EO$, a parallelogram is determined whose sides and one diagonal are known.

Hence the following:

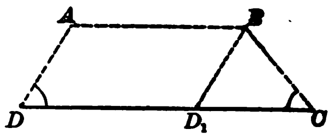
Construction: (1) Trisect m_a, m_b, m_c.

(2) Construct $\triangle CXO$ with $\frac{2}{3}m_c$, $\frac{2}{3}m_a$, and $\frac{2}{3}m_b$ as its sides.

(3) Complete the $\square OCXA$.

(4) Produce EO by $\frac{2}{3}m_b$, and vertex B is determined.

4. *Construct a trapezoid given the bases and the base angles.*



Analysis: Suppose $ABCD$ to be the completed trapezoid.

We know AB, DC, $\measuredangle D$, and $\measuredangle C$.

Is, then, any part of the figure determined?

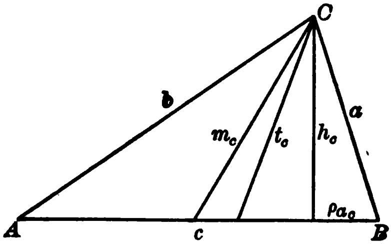
If we draw $BD_1 \parallel AD$, then ABD_1D will be a parallelogram and $DD_1 = AB$.

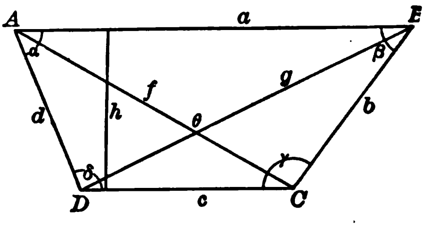
$\measuredangle BD_1C = \measuredangle ADC$ and $\overline{D_1C} = \overline{DC} - \overline{AB}$ and $\triangle BD_1C$ is thus determined.

(Construction, proof and discussion are left to the student.)

The preceding exercises are illustrative of analysis geometric in form, but the analysis of a problem in construction may also be algebraic in form, as, for example, in Problems 21, 26, and 27.

NOTE 1.—These diagrams show the symbolism used in reference to triangles and trapezoids throughout this text.





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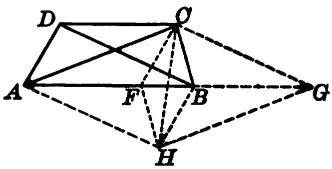
Note 2.—Construction of trapezoids.

The relations given by the following constructions are often useful. Let $ABCD$ be any trapezoid. Draw CF and $BH \parallel AD$, CG and $AH \parallel DB$. Join H (the intersection of AH and BH) with F, C, G. The figures $ACGH$, $ADBH$, $FCBH$, are parallelograms. $BF \equiv a - c$, $AG \equiv AB + BG = a + c$, $\sphericalangle ACG \equiv \theta$, $\sphericalangle CAH \equiv \sphericalangle CGH \equiv 180° - \theta$, $\sphericalangle CFH \equiv \sphericalangle CBH \equiv \alpha + \beta$. $\sphericalangle FCB \equiv \sphericalangle FHB \equiv 180° - (\alpha + \beta)$.



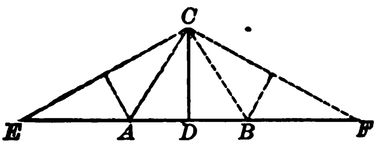
EXERCISES. SET XCV. PROBLEMS CALLING FOR ANALYSIS

Construct a triangle, having given:

1316. a, b, m_b. **1317.** a, b, h_b. **1318.** a, h_a, m_a.
1319. $a, m_a, \sphericalangle B$. **1320.** $m_a, h_a, \sphericalangle B$. **1321.** b, c, h_a.
1322. a, h_a, h_c. **1323.** $h_a, \sphericalangle B, \sphericalangle C$. **1324.** $h_a, h_c, \sphericalangle C$.
1325. $a, b, \sphericalangle A + \sphericalangle B$.
1326. The base, the altitude, and an angle at the base.
1327. The base, the altitude, and the angle at the vertex.

Construct an isosceles triangle, having given:

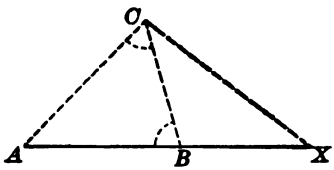
1328. The base and the angle at the vertex.
1329. The base and the radius of the circumscribed circle.



1330. The base and the radius of the inscribed circle.

1331. The perimeter and the altitude.

Let ABC be the required $\triangle$, EF the given perimeter. The altitude CD passes through the mid-point of EF, and the $\triangle$s EAC, BFC are isosceles.



1332. Construct a triangle, having given two angles and the sum of two sides.

Can the third $\sphericalangle$ be found? Assume the problem solved. If $AX = AB + BC$, what kind of triangle is $\triangle BXC$? What does $\sphericalangle CBA$ equal? Is $\sphericalangle X$ known? How can C be fixed?

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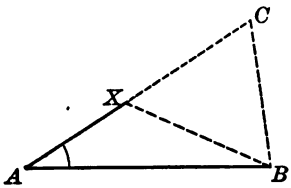
1333. Construct a triangle, having given a side, an adjacent angle, and the difference of the other sides.



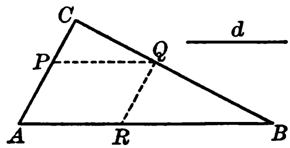
If AB, $\measuredangle A$, and $AC - BC$ are known, what points are determined? Then can XB be drawn? What kind of triangle is $\triangle XBC$? How can C be located?

1334. Construct an isosceles triangle, having given the sum of the base and an arm, and a base angle.

1335. Construct a triangle, having given the base, the sum of the other two sides, and the angle included by them.

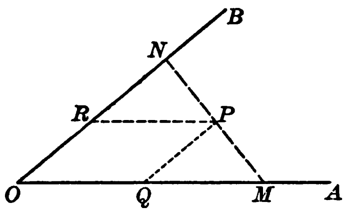
1336. Construct a triangle given the mid-points of its sides.

1337. Draw between two sides of a triangle a sect parallel to the third side, and equal to a given sect.



If $PQ = d$, what does AR equal? How will you reverse the reasoning?

1338. Draw through a given point P between the sides of an angle AOB a sect terminated by the sides of the angle and bisected at P.



If $PM = PN$, and $PR \parallel AO$, what can you say about OR and RN? Can you now reverse this? Similarly, if $PQ \parallel BO$, is $OQ = QM$?

1339. Given two perpendiculars, AB and CD, intersecting in O, and a line intersecting these perpendiculars in E and F; construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF. (Two solutions.)

1340. Draw from a given point P in the base AB of a triangle ABC a line to AC produced, so that it may be bisected by BC.

Construct a rectangle, having given:

1341. One side and the diagonal.

1342. One side and the angle formed by the diagonals.

1343. The perimeter and the diagonal.

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Construct a parallelogram, having given:

1344. Two independent sides and one altitude

1345. Two independent sides and an angle.

1346. One side and the two diagonals.

1347. One side, one angle, and one diagonal.

1348. The diagonals and the angle formed by the diagonals

Construct a rhombus, having given:

1349. The two diagonals.

1350. The perimeter and one diagonal.

1351. One angle and a diagonal.

1352. The altitude and the base.

1353. The altitude and one angle.

1354. Construct a square, having given the diagonal.

Construct a trapezoid, having given:

1355. The four sides.

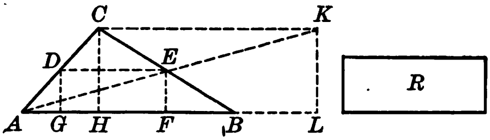
1356. The bases, another side, and one base angle.

1357. The bases and the diagonals.

1358. One base, the diagonals, and the angle formed by the diagonals.

1359. Inscribe a square in a given triangle.

1360. In a given triangle inscribe a rectangle similar to a given rectangle.



Hint: Let ABC be the given triangle and R the given rectangle. On the altitude CH construct a rectangle CL similar to the given rectangle. The line AK will determine a point E which will be one of the vertices of the required rectangle. Why?

1361 Inscribe a square in a semicircle.

1362. Divide a given triangle into two equal parts by a line parallel to one of the sides.

1363. Bisect a triangle by a line parallel to a given line.

1364. Bisect a triangle by a line drawn perpendicular to the base.

Construct a circle which shall be tangent to a given:

1365. Line, and to a given circle at a given point.

1366. Circle, and to a given line at a given point.

1367. Transform a triangle ABC so that $\sphericalangle A$ is not altered, and the side opposite the angle A becomes parallel to a given line $\overline{MN}$.

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CHAPTER VIII

SUMMARIES AND APPLICATIONS

A. SYLLABUS OF THEOREMS

1. Vertical angles are equal.

2. Two sides and the included angle determine a triangle.

3. Two angles and the included side determine a triangle.

4. The bisector of the vertex angle of an isosceles triangle divides it into two congruent triangles.

Cor. 1. The angles opposite the equal sides of an isosceles triangle are equal.

Cor. 2. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

Cor. 3. An equilateral triangle is equiangular.

Cor. 4. The bisectors of the angles of an equilateral triangle bisect the opposite sides and are perpendicular to them.

Cor. 5. The bisectors of the angles of an equilateral triangle are equal.

5. A triangle is determined by its sides.

5*a*. Only one perpendicular can be drawn through a given point to a given line.

5*b*. Two sects drawn from a point in a perpendicular to a given line, cutting off on the line equal sects from the foot of the perpendicular, are equal and make equal angles with the perpendicular.

5*c*. The sum of two sects drawn from any point within a triangle to the ends of one of its sides is less than the sum of its remaining sides.

5*d*. Of two sects drawn from a point in a perpendicular to a given line and cutting off unequal sects from the foot of the perpendicular, the more remote is the greater, and conversely.

Cor. 1. All possible obliques from a point to a line are equal in pairs, and each pair cuts off equal sects from the foot of the perpendicular from that point to the line.

6. The perpendicular is the shortest sect from a point to a line.

6*a*. The shortest sect from a point to a line is perpendicular to it.

7. The hypotenuse and an adjacent angle determine a right triangle.

8. The hypotenuse and another side determine a right triangle.

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9. Lines perpendicular to the same line are parallel.

10. A line perpendicular to one of a series of parallels is perpendicular to the others

11. If when lines are cut by a transversal the alternate-interior angles are equal, the lines thus cut are parallel.

Cor. 1. If the alternate-exterior angles or corresponding angles are equal when lines are cut by a transversal, the lines thus cut are parallel.

Cor. 2. If either the consecutive-interior angles or the consecutive-exterior angles are supplementary when lines are cut by a transversal, the lines thus cut are parallel.

12. Parallels cut by a transversal form equal alternate-interior angles.

Cor. 1. Parallels cut by a transversal form equal corresponding angles and equal alternate-exterior angles.

Cor. 2. Parallels cut by a transversal form supplementary consecutive-interior angles and supplementary consecutive exterior angles.

12*a*. Two angles whose sides are parallel each to each or perpendicular each to each, are either equal or supplementary.

13. The sum of the angles of a triangle is a straight angle.

Cor. 1. A triangle can have but one right or one obtuse angle.

Cor. 2. Triangles having two angles mutually equal are mutually equiangular.

Cor. 3. A triangle is determined by a side and any two homologous angles.

Cor. 4. An exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.

14. The sum of the angles of a polygon is equal to a straight angle taken as many times less two as the polygon has sides.

Cor. 1. Each angle of an equiangular polygon of n sides equals the $\frac{n-2}{n}$th part of a straight angle.

Cor. 2. The sum of the exterior angles of a polygon is two straight angles.

Cor. 3. Each exterior angle of an equiangular polygon of n sides is equal to the $\frac{2}{n}$th part of a straight angle.

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15. If two angles of a triangle are equal, the sides opposite them are equal.

Cor. 1. Equiangular triangles are equilateral.

16. Either diagonal of a parallelogram bisects it.

Cor. 1. The parallel sides of a parallelogram are equal, and the opposite angles are equal.

Cor. 2. Parallels are everywhere equidistant.

17. A quadrilateral whose opposite sides are equal is a parallelogram.

18. A quadrilateral having a pair of sides both equal and parallel is a parallelogram.

19. A parallelogram is determined by two adjacent sides and an angle; or parallelograms are congruent if two adjacent sides and an angle are equal each to each.

20. The diagonals of a parallelogram bisect each other.

21. A quadrilateral whose diagonals bisect each other is a parallelogram.

21*a*. The difference between any two sides of a triangle is less than the third side.

21*b*. If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.

21*c*. If two angles in a triangle are unequal, the sides opposite them are unequal in the same order.

21*d*. If two triangles have two sides equal each to each, but the included angles unequal, their third sides are unequal in the same order as those angles.

21*e*. If two triangles have two sides equal each to each, but the third sides unequal, the angles opposite those sides are unequal in the same order.

21*f*. If one acute angle of a right triangle is double the other the hypotenuse is double the shorter leg, and conversely.

22. Rectangles having a dimension of one equal to that of another compare as their remaining dimensions.

23. Any two rectangles compare as the products of their dimensions.

24. The area of a rectangle is equal to the product of its base and altitude.

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25. The area of a parallelogram is equal to the product of its base and altitude.

Cor. 1. Any two parallelograms compare as the products of their bases and altitudes.

Cor. 2. Parallelograms having one dimension equal compare as their remaining dimensions.

Cor. 3. Parallelograms having equal bases and equal altitudes are equal.

26. The area of a triangle is equal to half the product of its base and altitude.

Cor. 1. Any two triangles compare as the products of their bases and altitudes.

Cor. 2. Triangles having one dimension equal compare as their remaining dimensions.

Cor. 3. Triangles having equal bases and equal altitudes are equal.

26*a*. The square on the hypotenuse of a right triangle equals the sum of the squares on the two legs.

26*b*. The areas of two triangles that have an angle of one equal to an angle of the other, are to each other as the products of the sides including those angles.

27. The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.

28. Any proportion may be transformed by alternation, *i.e.*, the first term is to the third as the second is to the fourth.

29. In any proportion the terms may be combined by addition (usually called composition); *i.e.*, the ratio of the sum of the first and second terms to the second term (or first term) equals the ratio of the sum of the third and fourth terms to the fourth term (or the third term).

N.B.—"Addition" and "sum" are used in the algebraic sense.

30. In a series of equal ratios, the ratio of the sum of any number of antecedents to the sum of their consequents equals the ratio of any antecedent to its consequent.

31. A line parallel to one side of a triangle divides the other sides proportionally.

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Cor. 1. One side of a triangle is to either of the sects cut off by a line parallel to a second side as the third side is to its homologous sect.

Cor. 2. Parallels cut off proportional sects on all transversals.

Cor. 3. Parallels which intercept equal sects on one transversal do so on all transversals.

Cor. 4. A line which bisects one side of a triangle, and is parallel to the second, bisects the third.

Cor. 5. A sect which bisects two sides of a triangle is parallel to the third side and equal to half of it.

Cor. 6. The line (usually called median) joining the mid-points of the non-parallel sides of a trapezoid is parallel to the bases and equal to one-half their sum.

Cor. 7. The area of a trapezoid equals the product of its median and altitude.

32. A line dividing two sides of a triangle proportionally is parallel to the third side.

Cor. 1. A line dividing two sides of a triangle so that these sides bear the same ratio to a pair of homologous sects, is parallel to the third side.

32*a*. The bisector of an angle of a triangle divides the opposite side into sects which are proportional to the adjacent sides.

32*b*. The bisector of an exterior angle of a triangle divides the opposite side externally into sects which are proportional to the adjacent sides.

Cor. 1. The bisectors of an adjacent interior and exterior angle of a triangle divide the opposite side harmonically.

33. The homologous angles of similar triangles are equal and their homologous sides have a constant ratio.

34. Triangles are similar when two angles of one are equal each to each to two angles of another.

34*a*. Triangles which have their sides parallel or perpendicular each to each are similar.

35. Triangles which have two sides of one proportional to two sides of another, and the included angles equal, are similar.

36. If the ratio of the sides of one triangle to those of another is constant, the triangles are similar.

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36*a*. The homologous angles of similar polygons are equal, and their homologous sides have a constant ratio.

Cor. 1. The homologous diagonals drawn from a single vertex of similar polygons divide the polygons into triangles similar each to each.

36*b*. Polygons are similar if their homologous angles are equal and homologous sides proportional.

Cor. 1. If homologous diagonals drawn from a single vertex of polygons divide them into triangles similar each to each, the polygons are similar.

37. The perimeters of similar triangles are proportional to any two homologous sides, or any two homologous altitudes.

Cor. 1. Homologous altitudes of similar triangles have the same ratio as homologous sides.

Cor. 2. The perimeters of similar polygons have the same ratio as any pair of homologous sides or diagonals.

38. The areas of similar triangles compare as the squares of any two homologous sides.

Cor. 1. The areas of similar polygons compare as the squares of any two homologous sides or diagonals.

Cor. 2. Homologous sides or diagonals of similar polygons have the same ratio as the square roots of the areas.

38*a*. If two parallels are cut by concurrent transversals, the ratio of homologous sects of the parallels is constant.

38*b*. If the ratio of homologous sects of two parallels cut by three or more transversals is constant, the transversals are either parallel or concurrent.

39. The altitude upon the hypotenuse of a right triangle divides it into triangles similar to each other and to the original.

Cor. 1. Each side of a right triangle is a mean proportional between the hypotenuse and its projection upon the hypotenuse.

Cor. 2. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

Cor. 3. The altitude upon the hypotenuse of a right triangle is a mean proportional between the sects it cuts off on the hypotenuse.

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39*a*. In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of those sides by the projection of the other upon it.

39*b*. In an obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of those sides by the projection of the other upon it.

39*c*. I. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side, increased by twice the square of the median to it.

II. The difference of the squares of two sides of a triangle is equal to twice the product of the third side and the projection of the median upon it.

Cor. 1. If m_a represents the length of the median upon side a of the triangle whose sides are a, b, c, then $m_a \equiv \frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$.

39*d*. If h_a represents the altitude upon side a of a triangle whose sides are a, b, c, and s stands for its semiperimeter, *i.e.*, $s \equiv \frac{a+b+c}{2}$, then $h_a \equiv \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$.

Cor. 1. If A stands for the area of a triangle whose sides are a, b, c, and whose semiperimeter is s, then $A \equiv \sqrt{s(s-a)(s-b)(s-c)}$.

39*e*. If similar polygons are constructed on the sides of a right triangle, as homologous sides, the polygon on the hypotenuse is equal to the sum of the polygons on the other two sides.

40. The locus of points equidistant from the ends of a sect is the perpendicular bisector of the sect.

Cor. 1. Two points equidistant from the ends of a sect fix its perpendicular bisector.

41. The locus of points equidistant from the sides of an angle is the bisector of the angle.

Cor. 1. The locus of a point equidistant from two intersecting lines is a pair of lines bisecting the angles.

41*a*. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides of the triangle.

41*b*. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.

41*c*. The altitudes of a triangle are concurrent.

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41*d*. The medians of a triangle are concurrent in a point of trisection of each.

42. Three points not in a straight line fix a circle.

43. In equal circles, equal central angles intercept equal arcs, and conversely.

43*a*. In equal circles, the greater of two central angles intercepts the greater arc, and conversely.

44. In equal circles, equal arcs are subtended by equal chords, and conversely.

44*a*. In equal circles, unequal arcs are subtended by chords unequal in the same order, and conversely.

45. A diameter perpendicular to a chord bisects it and its subtended arcs.

Cor. 1. A radius which bisects a chord is perpendicular to it.

Cor. 2. The perpendicular bisector of a chord passes through the center of the circle.

46. In equal circles, equal chords are equidistant from the center, and conversely.

46*a*. In equal circles, the distances of unequal chords from the center are unequal in the opposite order.

47. A line perpendicular to a radius at its outer extremity is tangent to the circle.

Cor. 1. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

Cor. 2. The perpendicular to a tangent at the point of contact passes through the center of the circle.

Cor. 3. A radius perpendicular to a tangent passes through the point of contact.

Cor. 4. Only one tangent can be drawn to a circle at a given point on the circle.

48. Sects of tangents from the same point to a circle are equal.

49. The line of centers of two tangent circles passes through their point of contact.

49*a*. The line of centers of two intersecting circles is the perpendicular bisector of their common chord.

50. In equal circles central angles have the same ratio as their intercepted arcs.

Cor. 1. A central angle is measured by its intercepted arc.

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51. **Parallels intercept equal arcs on a circle.**

52. An inscribed angle, or one formed by a tangent and a chord is measured by one-half its intercepted arc.

52*a*. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.

53. An angle whose vertex is inside the circle is measured by half the sum of the arcs intercepted by it and its vertical.

54. An angle whose vertex is outside the circle is measured by half the difference of its intercepted arcs.

54*a*. The opposite angles of a quadrilateral inscribed in a circle are supplementary.

Cor. 1. A quadrilateral is inscriptible if its opposite angles are supplementary.

55. A tangent is the mean proportional between any secant and its external sect, when drawn from the same point to a circle.

Cor. 1. The product of a secant and its external sect from a fixed point outside a circle is constant.

55*a*. If chords intersect inside a circle, the product of their sects is constant.

55*b*. The square of the bisector of an angle of a triangle is equal to the product of the sides of this angle, diminished by the product of the sects made by the bisector on the third side.

55*c*. In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle and the altitude on the third side.

Cor. 1. If R denote the radius of the circle circumscribed about a triangle whose sides are a, b, c, and semiperimeter s,

then $$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

56. A circle may be circumscribed about, and inscribed within any regular polygon.

Cor. 1. An equilateral polygon inscribed in a circle is regular.

Cor. 2. An equiangular polygon circumscribed about a circle is regular.

Cor. 3. The area of a regular polygon is equal to half the product of its apothem and perimeter.

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57. If a circle is divided into any number of equal arcs, the chords joining the successive points of division form a regular inscribed polygon; and the tangents drawn at the points of division form a regular circumscribed polygon.

Cor. 1. Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.

Cor. 2 Lines drawn from each vertex of a regular inscribed polygon to the mid-points of the adjacent arcs subtended by its sides, form a regular inscribed polygon of double the number of sides.

Cor. 3. Tangents at the mid-points of the arcs between consecutive points of contact of the sides of a regular circumscribed polygon form a regular circumscribed polygon of double the number of sides.

Cor. 4. The perimeter of a regular inscribed polygon is less than that of a regular inscribed polygon of double the number of sides, and the perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.

Cor. 5. Tangents to a circle at the mid-points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon, of which the sides are parallel to the sides of the original and the vertices lie on the prolongations of the radii of the inscribed polygon.

58. A regular polygon the number of whose sides is $3 \cdot 2^n$ may be inscribed in a circle.

59. If i_n represent the side of a regular inscribed polygon of n sides, and i_{2n} the side of one of $2n$ sides, and r the radius of the circle, $i_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - i_n^2}}$.

60. If i_n represent the side of a regular inscribed polygon of n sides, c_n that of a regular circumscribed polygon of n sides, and r the radius of the circle, $c_n = \dfrac{2ri_n}{\sqrt{4r^2 - i_n^2}}$.

61. The perimeters of regular polygons of the same number of sides compare as their radii and also as their apothems.

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62. Circumferences have the same ratio as their radii.

Cor. 1. The ratio of any circle to its diameter is constant.

Cor. 2. In any circle $c = 2\pi r$.

63. The value of π is approximately 3.14159.

64. The area of a circle is equal to one-half the product of its radius and its circumference.

Cor. 1. The area of a circle is equal to π times the square of its radius.

Cor. 2. The areas of circles compare as the squares of their radii.

Cor. 3. The area of a sector is equal to half the product of its radius and its arc.

Cor. 4. Similar sectors and similar segments compare as the squares of their radii.

B. SYLLABUS OF CONSTRUCTIONS

Problem 1. Draw a perpendicular to a given line from a given point (*a*) outside the line, (*b*) on the line.

Problem 2. Bisect a given (*a*) sect, (*b*) angle, (*c*) arc.

Problem 3. Reproduce a given angle.

Problem 4. Draw a line through a given point, and parallel to a given line.

Problem 5. Construct a triangle given any three independent parts; (*a*) two angles and the included side, (*b*) two sides and the included angle, (*c*) three sides, (*d*) the hypotenuse and a leg of a right triangle.

Problem 6. Divide a sect into n equal parts.

Problem 7. Find a common measure of two commensurable sects.

Problem 8. Pass a circle through three non-collinear points.

Cor. 1. Circumscribe a circle about a triangle.

Problem 9. Divide a given sect into parts proportional to n given sects.

Problem 10. Divide a sect harmonically in the ratio of two given sects.

Problem 11. Find a fourth proportional to three given sects.

Cor. 1. Find a third proportional to two given sects.

Problem 12. Upon a given sect as homologous to a designated

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side of a given polygon construct another similar to the original polygon.

Problem 13. Construct a square equal to the sum of two or more given squares.

Cor. 1. Construct a square equal to the difference of two given squares.

Cor. 2. Construct a polygon similar to two given similar polygons and equal to (*a*) their sum, and (*b*) their difference.

Problem 14. Inscribe in a circle regular polygons the number of whose sides is (*a*) $3 \cdot 2^n$, (*b*) $4 \cdot 2^n$.

Problem 15. Construct a triangle, given two sides and the angle opposite one of them.

Problem 16. Through a given point draw a tangent to a given circle (*a*) when the point is on the circle, (*b*) when the point is outside the circle.

Problem 17. Construct a common (*a*) external and (*b*) internal tangent to two circles.

Problem 18. Inscribe a circle in a triangle.

Cor. 1. Find the centers of the escribed circles of a triangle.

Problem 19. Construct upon a given sect the segment of a circle capable of containing a given angle.

Problem 20. Construct a mean proportional between two given sects.

Problem 21. Divide a given sect into mean and extreme ratio.

Problem 22. Inscribe in a circle a regular decagon.

Cor. 1. Inscribe in a circle regular polygons the number of whose sides is (*a*) $5 \cdot 2^n$, (*b*) $15 \cdot 2^n$.

Problem 23. Transform a polygon into a triangle.

Problem 24. Construct the square equal to a given (*a*) parallelogram, (*b*) triangle, (*c*) polygon.

Problem 25. Construct a parallelogram equal to a given square, having given (*a*) the sum of base and altitude equal to a given sect, (*b*) the difference of base and altitude equal to a given sect.

Problem 26. Construct a polygon similar to one and equal to another of two given polygons.

Problem 27. Construct a square which shall have a given ratio to a given square.

Cor. 1. Construct a polygon equal to any part of a given polygon and similar to it.

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C. SUMMARY OF FORMULAS

I. SUM OF THE ANGLES OF A POLYGON.

a. Interior. $S = n - 2$. (S stands for sum of angles in st. ∡s.)

Equiangular. $E = \frac{n-2}{n}$. (E stands for each angle of an equiangular n-gon, in st. ∡s.)

b. Exterior. $S = 2$.

Equiangular. $E = \frac{2}{n}$.

II. AREAS.

a. Parallelogram. $A = bh$.

b. Triangle $A = \frac{1}{2}bh$.

c. Trapezoid. 1. $A = \frac{1}{2}h(b + b_1)$.

2. $A = hm$.

III. METRIC RELATIONS IN TRIANGLES.

a. Right triangle. $c^2 = a^2 + b^2$. (∡C = rt. ∡.)

b. Acute triangle. $c^2 = a^2 + b^2 - 2ap_{b_a}$. (∡$C$ < rt. ∡.)

c. Obtuse triangle. $c^2 = a^2 + b^2 + 2ap_{ba}$. (∡$C$ > rt. ∡.)

d. In any triangle.

1. $a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2m_c^2$.
2. $a^2 - b^2 = 2cp_{m_c}$.
3. $m_c = \frac{1}{2}\sqrt{2(a^2+b^2) - c^2}$.
4. $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.
5. $ab = h_cD$. (D stands for diameter of circumscribed circle.)
6. $t_c^2 = ab - pq$. (p and q are sects of c made by t.)
7. $\frac{p}{q} = \frac{a}{b}$.
8. $A = \sqrt{s(s-a)(s-b)(s-c)}$.

IV. MENSURATION OF THE CIRCLE.

a. $R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$.

b. $i_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - i_n^2}}$.

c. $c_n = \frac{2ri_n}{\sqrt{4r^2 - i_n^2}}$.

d. $C = 2\pi r$ or $C = \pi D$.

e. $\pi = 3.14159 \ldots$

f. $A = \pi r^2$ or $A = \frac{Cr}{2}$.

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V. RATIO AND PROPORTION.

Given: $\frac{a}{b}=\frac{c}{d}$, then,

a. Product of means.... $ad=bc$.

b. Alternation. $\frac{a}{c}=\frac{b}{d}$.

c. Inversion. $\frac{b}{a}=\frac{d}{c}$.

d. Addition (Composition) $\frac{a+b}{b}=\frac{c+d}{d}$ or $\frac{a+b}{a}=\frac{c+d}{c}$.

(Division) $\frac{a-b}{b}=\frac{c-d}{d}$ or $\frac{a-b}{a}=\frac{c-d}{c}$.

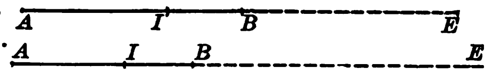
(Both) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

e. Series of equal ratios. Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, then,

$$\frac{a+c+e+\ldots}{b+d+f+\ldots}=\frac{a}{b}=\ldots.$$

VI. DIVISION OF SECTS.

a. Harmonic. $\frac{AI}{BI}=\frac{AE}{BE}$.



b. Extreme and mean ratio.

$$\frac{AB}{AI}=\frac{AI}{IB} \text{ and } \frac{AB}{BE}=\frac{BE}{AE}$$

VII. SIMILAR FIGURES.

a. $\frac{P}{p}=\frac{S}{s}$. (P and p stand for perimeter, S and s for homologous sides.)

b. $\frac{A}{a}=\frac{S^2}{s^2}$.

c. Circle. 1. $\frac{A}{a}=\frac{R^2}{r^2}$.

2. $\frac{C}{c}=\frac{R}{r}=\frac{D}{d}$.

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VIII. MEASUREMENT OF ANGLES.

a. $C \overset{\circ}{=} a$. (*C* stands for number of degrees in central angle.)

b. $I \overset{\circ}{=} \frac{a}{2}$. (*I* stands for number of degrees in inscribed angle or one formed by tangent and chord.)

c. $W \overset{\circ}{=} \frac{a+v}{2}$. (*W* stands for number of degrees in angle with vertex inside circle.)

d. $O \overset{\circ}{=} \frac{l-s}{2}$. (*O* stands for number of degrees in angle with vertex outside circle.)

(*a* stands for number of degrees in intercepted arc.)

(*v* stands for number of degrees in arc intercepted by vertical angle.)

(*l* stands for number of degrees in larger intercepted arc.)

(*s* stands for number of degrees in smaller intercepted arc.)

IX. TANGENT AND SECANT.

a. $T^2 = S \cdot E$. (*T* stands for length of sect of tangent from point to circle.)
(*S* stands for length of secant from same point to circle.)
(*E* stands for length of external sect of the secant.)

b. $pq = ab$. (*pq* stand for sects of a chord made by the intersection of a chord whose sects are *a* and *b*.)

D. SUMMARY OF METHODS OF PROOF

I. TRIANGLES CONGRUENT.

Show that they have

(*a*) two sides and included angle respectively equal,
(*b*) a side and any two angles respectively equal,
(*c*) three sides respectively equal,

or, that they are parts of

(*d*) an isosceles triangle formed by the bisector of the vertex angle,
(*e*) a parallelogram formed by the diagonal,

or, that they are right triangles, and have

(*f*) the hypotenuse and adjacent angle respectively equal,
(*g*) the hypotenuse and adjacent side respectively equal.

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EXERCISES. SET XCVI. CONGRUENCE OF TRIANGLES

1368. If a perpendicular is erected at any point on the bisector of an angle and produced to cut the sides of the angle, two congruent triangles are formed.

1369. In triangle ABC, BD (D in AC), the bisector of angle B intersects AE (E in BC), a perpendicular to BD, in F. Prove that triangles ABF and BFE are congruent.

1370. Triangles are congruent if two sides and the median to one of these sides are equal respectively to two sides and the homologous median in the other. (Is this always true or is there any exception?)

1371. Isosceles triangles are congruent if the base and a leg of one are equal respectively to the base and a leg of the other.

1372. The three sects joining the mid-points of the sides of a triangle divide the figure into four congruent triangles.

1373. If equal distances are laid off from the same vertex on the legs of an isosceles triangle, and these points of division are joined with the opposite vertices, congruent triangles are formed.

1374. If two angles of a triangle are equal, the bisector of the third angle divides the figure into two congruent triangles.

II. SECTS EQUAL.

Show that they are

(*a*) homologous parts of congruent polygons,
(*b*) sects from a point in a perpendicular cutting off equal distances from its foot,
(*c*) sects of a line cut off from the foot of a perpendicular to it by equal sects from the same point in the perpendicular,
(*d*) bisectors of the angles of an equilateral triangle,
(*e*) sects of the base of an isosceles triangle formed by the bisector of the vertex angle,
(*f*) sides opposite equal angles of a triangle,
(*g*) parallel sides of a parallelogram,
(*h*) sects of one diagonal of a parallelogram made by the other,
(*i*) sects on a transversal cut off by parallels which cut off equal sects on some other transversal,

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(*j*) chords in equal circles subtending equal arcs,
(*k*) chords in equal circles equally distant from the center,
(*l*) sects of tangents to a circle from the same point.

EXERCISES. SET XCVII. EQUALITY OF SECTS

1375. The bisectors of the base angles of an isosceles triangle are equal.

1376. The sects joining the mid-points of the legs of an isosceles triangle with the mid-point of the base are equal.

1377. The medians to the legs of an isosceles triangle are equal.

1378. If the base of an isosceles triangle is trisected, the sects joining the points of division with the vertex are equal.

1379. If from any point in the circumference of a circle two chords are drawn making equal angles with the radius to the point, these chords are equal.

1380. A circumscribed parallelogram is equilateral.

III. ANGLES EQUAL.

Show that they are
(*a*) straight or right angles,
(*b*) supplements or complements of equal angles,
(*c*) vertical angles,
(*d*) homologous parts of congruent polygons,
(*e*) angles opposite the equal sides of a triangle,
(*f*) alternate-interior, alternate-exterior, or corresponding angles of parallels,
(*g*) opposite angles of a parallelogram,
(*h*) homologous angles of similar polygons,
(*i*) measured by equal arcs,
or, that their sides are respectively
(*j*) perpendicular or parallel.

EXERCISES. SET XCVIII. EQUALITY OF ANGLES

1381. The tangent at the vertex of an inscribed equilateral triangle forms equal angles with the adjacent sides.

1382. If in the base AC of an isosceles triangle ABC, two points D and E are taken so that $AE=AB$ and $CD=BC$, prove that $\measuredangle DBE=\measuredangle A$.

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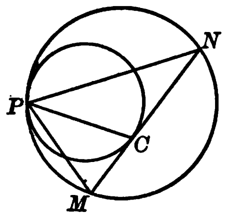
1383. If triangles are similar to the same triangle, they are similar to each other.

1384. If through the vertices of an isosceles triangle lines are drawn parallel to the opposite sides, they form an isosceles triangle.

1385. If the bisector of an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.

1386. Two circles intersect at the points A and B. Through A a variable secant is drawn, cutting the circles at C and D. Prove that the angle DBC is constant.



1387. Two circles touch each other internally at P. MN is a chord of the larger, tangent to the smaller at C. Prove $\measuredangle MPC = \measuredangle CPN$.

IV. LINES PARALLEL.

Show that they are

(*a*) perpendicular or parallel to the same line,

(*b*) opposite sides of a quadrilateral which can be shown to be a parallelogram,

(*c*) one side of a triangle and a sect cutting the other two sides proportionally,

(*d*) side of a regular inscribed polygon and the side of a polygon the sides of which are tangent to the circle at the midpoints of the arcs subtended by the inscribed polygon,

or, that they have

(*e*) alternate-interior, alternate-exterior, or corresponding angles equal,

(*f*) consecutive-interior or consecutive-exterior angles supplementary.

EXERCISES. SET XCIX. PARALLELISM OF LINES

1388. If two sides of a triangle are produced their own lengths through the common vertex, a line joining their ends is parallel to the third side of the triangle.

1389. The line joining the feet of the perpendiculars dropped from the extremities of the base of an isosceles triangle to the opposite sides is parallel to the base.

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1390. The tangents drawn through the extremities of a diameter are parallel.

1391. If from one pair of opposite vertices of a parallelogram lines are drawn bisecting the opposite sides respectively, the lines are parallel.

1392. The bisectors of a pair of opposite angles of a parallelogram are parallel.

1393. If two circles intersect and a sect be drawn through each point of intersection terminated by the circumferences, the chords which join the extremities of these sects are parallel.

1394. If through the point of contact of two tangent circles two secants are drawn, the chords joining the points where the secants cut the circles are parallel. Discuss both cases.

1395. If a straight line be drawn through the point of contact of two tangent circles so as to form chords, the radii drawn to the other extremities of these chords are parallel. (What two cases?)

1396. If a straight line be drawn through the point of contact of two tangent circles so as to form chords, the tangents drawn at the other extremities of the chords are parallel. (What two cases?)

V. LINES PERPENDICULAR.

Show that

(*a*) two adjacent angles formed by them are equal,

(*b*) one is the base and the other the bisector of the vertex angle of an isosceles triangle,

(*c*) one of them is perpendicular to a line parallel to the other,

(*d*) one is a tangent to a circle and the other a radius drawn to the point of contact,

(*e*) one is a radius bisecting the other which is a chord of the same circle,

(*f*) one is the shortest sect from a point to the other.

EXERCISES. SET C. PERPENDICULARITY OF LINES

1397. Every parallelogram inscribed in a circle is a rectangle.

1398. The bisectors of two interior angles on the same side of a transversal to two parallels are perpendicular to each other.

1399. If either leg of an isosceles triangle be produced through the vertex by its own length, and the extremity joined to the extremity of the base, the joining line is perpendicular to the base.

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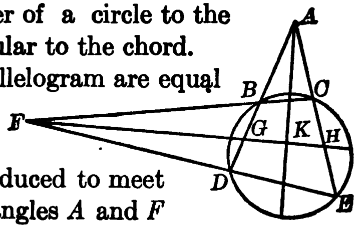
1400. Two circles are tangent externally at A, and a common external tangent touches them in B and C. Show that angle BAC is a right angle.

1401. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other two angles are perpendicular.

1402. The line joining the center of a circle to the mid-point of a chord is perpendicular to the chord.

1403. If the diagonals of a parallelogram are equal the figure is a rectangle.



1404. If the opposite sides of an inscribed quadrilateral be produced to meet in A and F, the bisectors of the angles A and F meet at right angles.

Hint: Prove $\angle BGK = \angle CHK$

VI. SECTS UNEQUAL.

Show that they are

(*a*) sects from a point in a perpendicular to a line cutting off unequal sects from its foot,
(*b*) sects on a line cut off by unequal sects drawn from a point in the perpendicular to that line,
(*c*) sides of a triangle opposite unequal angles,
(*d*) third sides of two triangles which have the other two sides respectively equal but the included angles unequal,
(*e*) chords of equal circles subtending unequal arcs,
(*f*) chords of equal circles unequally distant from the centers.

EXERCISES. SET CI. INEQUALITY OF SECTS

1405. A, B, C, and D are points taken in succession on a semicircumference and arc AC is greater than arc BD. Prove that chord AB is greater than chord CD.

1406. Two chords drawn from a point in the circumference are unequal if they make unequal angles with the radius drawn from that point. Which of the chords is the greater?

1407. Two chords drawn through an interior point are unequal if they make unequal angles with the radius drawn through that point. Which is the greater one?

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1408. If two unequal chords be produced to meet, the secants thus formed are unequal.

1409. In the equilateral triangle *ABC*, side *BC* is produced to *D*, and *DA* is drawn. Prove that $BD > AD$.

1410. If the bisector of the right angle *A* of a right isosceles triangle *BAC* cuts *BC* in *D* and is produced to *K* so that $DK = AD$, then $KC < AK$; also than *BC*.

VII. ANGLES UNEQUAL.

Show that they are

(*a*) one exterior and one non-adjacent interior angle of a triangle,

(*b*) opposite the unequal sides of a triangle,

(*c*) the angles opposite the third sides of two triangles which have two sides respectively equal but the third sides unequal,

(*d*) measured by unequal arcs.

EXERCISES. SET CII. INEQUALITY OF ANGLES

1411. If *A* is the vertex of an isosceles triangle *ABC* and the leg *AC* is produced to point *D* and *DB* drawn, prove that $\angle ABD > \angle ADB$.

1412. If the side *BC* of the square *ABCD* is produced to *P* and *P* is joined with *A*, prove that $\angle APB < \angle BAP$.

1413. The angle formed by two tangents is equal to twice the angle between the chord of contact and the radius drawn to a point of contact.

1414. If the diagonals of quadrilateral *ABCD* bisect each other at *P*, and side *BC* is longer than side *AB*, prove that $\angle BPC > \angle BPA$.

1415. In the quadrilateral *MNRS*, *MS* is the longest and *NR* the shortest side. Prove that $\angle MNR > \angle MSR$; also that $\angle NRS > \angle NMS$.

1416. If in parallelogram *ABCD*, side $BC > AB$, and the diagonals intersect in *P*, prove that $\angle BPC > \angle BPA$.

VIII. TRIANGLES SIMILAR.

Show that they have

(*a*) a center of similitude,

(*b*) two angles respectively equal,

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(c) two pairs of sides proportional and the included angles equal,

or, that

(d) their sides bear a constant ratio,

(e) they are triangles formed by homologous diagonals of similar polygons,

(f) they are triangles formed by the altitude upon the hypotenuse of a right triangle.

EXERCISES. SET CIII. SIMILARITY OF TRIANGLES

1417. If through the point of contact of two tangent circles three secants are drawn, cutting the circumferences in A, B, C, and A_1, B_1, C_1, respectively, then triangles ABC and $A_1B_1C_1$ are similar.

1418. If from the point P outside a circle two secants are drawn to meet the circumference in B and C, and D and E, respectively, the triangles PBE and PCD are similar.

1419. If the bisector AD of $\measuredangle A$ in the inscribed triangle ABC is produced to meet the circumference in E, then triangles ABD and AEC are similar.

1420. If two chords, AB and CD, intersect in E, the triangles AEC and BED are similar.

1421. If the altitudes AD and BE of triangle ABC intersect at F, triangles AFE and BFD are similar.

1422. AD and BE are two altitudes of triangle ABC. Prove that triangles CDE and CBA are similar.

IX. SECTS PROPORTIONAL.

Show that they are

(a) homologous parts of similar polygons,

(b) sects of sides of a triangle cut off by line parallel to third side,

(c) sects of transversals cut off by parallels,

(d) two sides of a triangle and the sects of the third side made by the bisector of the opposite (1) interior and (2) exterior angle,

(e) the altitude upon the hypotenuse and the sects of the hypotenuse made by it, or a leg, hypotenuse, and the projection of the leg upon the hypotenuse.

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EXERCISES. SET CIV. PROPORTIONALITY OF SECTS

1423. Triangles are similar if an angle of one is equal to an angle of another and the altitudes drawn from the vertices of the other angles are proportional.

1424. If two circles are tangent externally, and through the point of contact a secant is drawn, the chords formed are proportional to the radii.

1425. If C is the mid-point of the arc AB, and a chord CD meets the chord AB in E, then $\frac{CE}{CA}=\frac{CA}{CD}$.

1426. The diagonals of any trapezoid divide each other in the same ratio.

X. PRODUCTS OF SECTS EQUAL.

Show that the

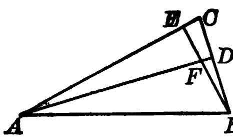
(*a*) factors are sides of similar polygons,

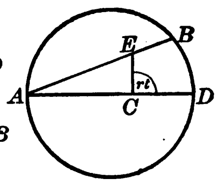
(*b*) sects are sects of chords intersecting inside a circle,

(*c*) factors of one product are the tangent and the factors of the other are the entire secant from the same point and its external sect.

EXERCISES. SET CV. EQUALITY OF PRODUCTS OF SECTS

1427. If from any point E in the chord AB the perpendicular EC be drawn upon the diameter AD, then $\overline{AC}\times\overline{AD}\equiv\overline{AB}\times\overline{AE}$.





1428. If in the triangle ABC the altitudes AD and BE meet in F, then $\overline{BD}\times\overline{DC}\equiv\overline{DF}\times\overline{AD}$.

1429. In the same diagram, $\overline{BD}\times\overline{AC}\equiv\overline{BF}\times\overline{AD}$.

1430. If a chord be bisected by another, either sect of the first is a mean proportional between the sects of the other.

The preceding summary is incomplete. The idea contained in it may be developed by asking the pupil to make similar lists of methods of establishing such geometric relations as the following: inequality of arcs, conditions under which a quadrilateral is a parallelogram, sums of sects unequal, differences of products unequal, lines concurrent, points collinear, points concyclic.

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EXERCISES. SET CVI. MISCELLANEOUS

1431. Draw, through a given point, a secant from which two equal circles shall cut off equal chords.

1432. In a right isosceles triangle the hypotenuse of which is 10 in., find the length of the projection of either arm upon the hypotenuse.

1433. Find the projection of one side of an equilateral triangle upon another if each side is 6 in.

1434. If two sides of a triangle are 10 and 12, and their included angle is 120°, what is the value of the third side?

1435. If two sides of a triangle are 12 and 16, and their included angle is 45°, find the third side.

1436. Assuming the diameter of the earth to be 8000 mi., how far can you see from the top of a mountain a mile high?

1437. Write the formula involving the median to b, to c.

1438. If the sides of a triangle ABC are 5, 7, and 8, find the lengths of the three medians.

1439. If the sides of a triangle are 12, 16, and 20, find the median to side 20. How does it compare in length with the side to which it is drawn? Why?

1440. In triangle ABC, $a=16$, $b=22$, and $m_c=17$. Find c.

1441. In a right triangle, right-angled at C, $m_c=8\frac{1}{2}$; what is c? Find one pair of values for a and b that will satisfy the conditions of the problem.

1442. If the sides of a triangle are 7, 8, and 10, is the angle opposite 10 obtuse, right, or acute? Why?

1443. Draw the projections of the shortest side of a triangle upon each of the other sides (a) in an acute triangle (b) in a right triangle, (c) in an obtuse triangle. Draw the projections of the longest side in each case.

1444. Two sides of a triangle are 8 and 12 in. and their included angle is 60°. Find the projection of the shorter upon the longer.

1445. In Ex. 1444 find the projection of the shorter side upon the longer if the included angle is 30°; 45°.

1446. Write the formula for the projection of a upon b.

1447. In triangle ABC, $a=15$, $b=20$, $c=25$; find the projection of b upon c. Is angle A acute, right, or obtuse?

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1448. If the altitude upon the hypotenuse of a right triangle divides the hypotenuse in extreme and mean ratio, the smaller arm is equal to the non-adjacent sect of the hypotenuse.

1449. If two circles are tangent externally, their common external tangent is a mean proportional between their diameters.

1450. The sides of a triangle are 13, 17, 19. Find the lengths of the sects into which the angle bisectors divide the opposite sides.

1451. The angles of a triangle are 30°, 60°, 90°. Find the lengths of the sects into which the angle bisectors divide the opposite sides, if the hypotenuse is 10.

1452. Find the sum of (*a*) the acute angles of a starred pentagon, (*b*) all the interior angles.

1453. The difference of the squares of two sides of a triangle is equal to the difference of the squares of the sects made by the altitude upon the third side.

1454. The perpendicular erected at the mid-point of the base of an isosceles triangle passes through the vertex and bisects the angle at the vertex.

Construct a right triangle, having given:

1455. The hypotenuse and one side.

1456. One side and the altitude upon the hypotenuse.

1457. The median and the altitude upon the hypotenuse.

1458. The hypotenuse and the altitude upon the hypotenuse.

1459. The radius of the inscribed circle and one side.

1460. The radius of the inscribed circle and an acute angle.

1461. The hypotenuse and the difference between the arms.

1462. The hypotenuse and the sum of the arms.

1463. The sides of a triangle are 6, 7, and 8 ft. Find the areas of the two parts into which the triangle is divided by the bisector of the angle included by 6 and 7.

1464. The square constructed upon the sum of two sects is equal to the sum of the squares constructed upon these two sects, increased by twice the rectangle of the sects.

1465. The square constructed upon the difference of two sects is equal to the sum of the squares constructed upon these sects, diminished by twice their rectangle.

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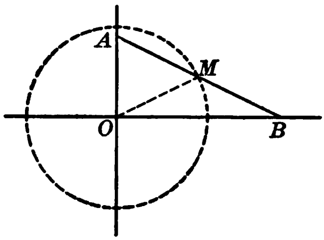
1466. The difference between the squares constructed upon two sects is equal to the rectangle of their sum and their difference.

1467. A straight rod moves so that its ends constantly touch two fixed rods perpendicular to each other. Find the locus of its mid-point.



1468. If a quadrilateral has each side tangent to a circle, the sum of one pair of opposite sides equals the sum of the other pair.

1469. Analysis of the regular inscribed hexagon—prove that:

(*a*) Three of the diagonals are diameters.

(*b*) The perimeter contains three pairs of parallel sides.

(*c*) Any diagonal which is a diameter divides the hexagon into two isosceles trapezoids.

(*d*) Radii drawn to the alternate vertices divide the hexagon into three congruent rhombuses.

(*e*) The diagonals joining the alternate vertices form an equilateral triangle of which the area equals one-half that of the hexagon.

(*f*) The diagonals joining the corresponding extremities of a pair of parallel sides of the hexagon form with these sides a rectangle.

1470. Inscribe in a given circle a regular polygon similar to a given regular polygon.

1471. Construct the following angles: 60°, 30°, 72°, 18°, 24°, 42°.

1472. Construct $\triangle ABC$ given $\measuredangle B$, b, and m_b.

1473. Construct a triangle, having given the base, the altitude, and the angle at the vertex.

1474. (*a*) Construct a square that shall be to a given triangle as 5 is to 4.

(*b*) Construct a square that shall be to a given triangle as m is to n, when m and n are two given sects.

1475. The diagonals of a regular pentagon divide each other into extreme and mean ratio.

1476. If the sect l is divided internally in extreme and mean ratio, and if s is its greater sect, what is the value of s in terms of l?

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1477. A sect 10 in. long is divided internally in extreme and mean ratio. Find the lengths of its sects.

1478. A sect 8 in. long is divided externally in extreme and mean ratio. Find the length of its longer sect.

1479. Experience has shown that a book, photograph, or other rectangular object is most pleasing to the eye when its length and width are obtained by dividing the semiperimeter into extreme and mean ratio. Find to the nearest integer the width of such a book where its length is 8 in.

1480. If the bisector of an inscribed angle be produced until it meets the circumference, and through this point of intersection a chord be drawn parallel to one side of the angle, it is equal to the other side.

1481. If from the extremities of a diameter perpendiculars be drawn upon any chord (produced if necessary), the feet of the perpendicular are equidistant from the center.

1482. Find the locus of points, the distances of which from two intersecting lines L and L_1 are as $\frac{m}{n}$.

The locus consists of two straight lines. Draw parallels to L and L_1, such that their distances from L and L_1 respectively shall be as $\frac{m}{n}$; these parallels will intersect in points belonging to the required locus. Special case: $\measuredangle LL_1 = 60°$, $m = 2$, $n = 1$.

1483. Between the sides of a given angle a series of parallels are drawn; find the locus of points which divide these parallels in the ratio $\frac{m}{n}$. Special case: $m = 4$, $n = 1$.

1484. Construct a triangle equal to the sum of two given triangles.

1485. Construct a triangle equal to the difference of two given parallelograms.

1486. Construct a regular pentagon, given one of the diagonals.

1487. Prove that the following solution of the problem to divide a given circle into any number of equal parts (say 3), by drawing concentric circles, is correct.

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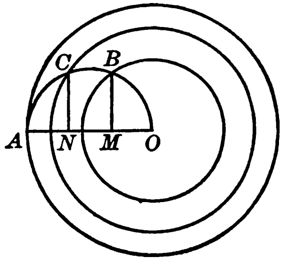
Trisect the radius at M and N. Draw semicircle on AO as diameter. Erect perpendiculars MB and NC meeting semicircle at B and C. With centers at O and radii OB and OC, draw circles.



1488. Transform a rectangle into a parallelogram, having given a diagonal.

1489. Transform a given triangle into a right triangle containing a given acute angle.

1490. The sides of a triangle are 8, 15, and 17. Find the radius of the inscribed circle.

1491. The sum of the squares on the sects of two perpendicular chords is equal to the square of the diameter of the circle.

If AB, CD are the chords, draw the diameter BE, and draw AC, ED, BD. Prove that $AC = ED$.

1492. Calculate the lengths of the common external and internal tangents to two circles whose radii are 16 and 12 units respectively, and whose line of centers is 40.

1493. Describe a circle whose circumference is equal to the difference of two circumferences of given radii.

1494. Construct a circle equal to three-fifths of a given circle.

1495. Construct a circle equal to three times a given circle.

1496. Construct a semicircle equal to a given circle.

1497. Construct a circle equal to the area bounded by two concentric circumferences.

Required to circumscribe about a given circle the following regular polygons:

1498. Triangle. **1499.** Quadrilateral. **1500.** Hexagon.

1501. Octagon. **1502.** Pentagon. **1503.** Decagon.

1504. Construct a square equal to a given (*a*) parallelogram, (*b*) triangle, (*c*) polygon.

1505. Construct a triangle similar to a given triangle and equal to another given triangle.

1506. Construct a polygon similar to a given polygon and having a given ratio to it.

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1507. Construct $x \equiv \frac{ab}{\sqrt{a^2+b^2}}$.

1508. Construct $x \equiv \sqrt{a^2+b^2-ab}$.

1509. Construct the roots of $x^2+ax+b \equiv O$.

1510. Given the sects a, b, c, construct sect x if:

(a) $x \equiv \sqrt{3ab}$. (b) $x \equiv \sqrt{a^2-b^2}$. (c) $x \equiv \sqrt{a^2-bc}$.

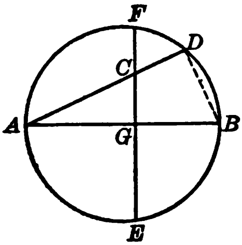
1511. Transform a given triangle into an equilateral triangle.

1512. Transform a given parallelogram into an equilateral triangle.

1513. Construct an equilateral triangle equal to one-half a given square.

1514. Transform a triangle into a right isosceles triangle.

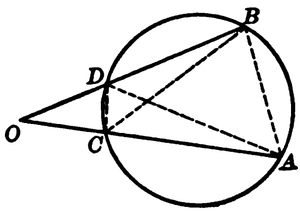
1515. Divide a given sect into two sects such that one is to the given sect as $\sqrt{2}$ is to $\sqrt{5}$.



1516. In a circle a line EF is drawn perpendicular to a diameter AB, and meeting it in G. Through A any chord AD is drawn, meeting EF in C. Prove that the product $\overline{AD} \times \overline{AC}$ is constant, whatever the direction of AD.

Draw BD and compare the ⊿ ACG, ADB. Is the theorem also true when G lies outside the circle?

1517. If two sects OA, OB, drawn through a point O are divided in C, D, respectively, so that $\overline{OA} \times \overline{OC} \equiv \overline{OB} \times \overline{OD}$, a circle can be described through the points A, B, C, D.



Show that ⊿ DAO, CBO are similar, and the ∡s DAC and CBD equal. Therefore, if a segment is described upon CD capable of containing the ∡DAO, the arc of this segment will pass through B.

1518. Divide a quadrilateral into four equal parts by lines drawn from a point in one of its sides.

1519. Draw a common secant to two given circles exterior to each other, such that the intercepted chords shall have the given lengths a and b.

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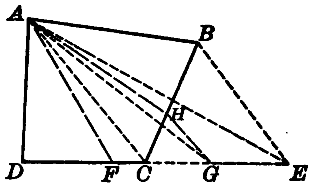
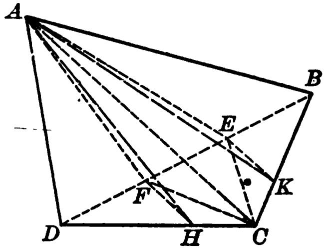
1520. Divide quadrilateral $ABCD$ into three equal parts by straight lines passing through A.



(*a*) Transform $ABCD$ into $\triangle ADE$. Divide $\triangle ADE$ into the three equal parts ADF, AFG, and AGE.

As the last two parts do not lie entirely in the given quadrilateral draw $GH \parallel CA$.



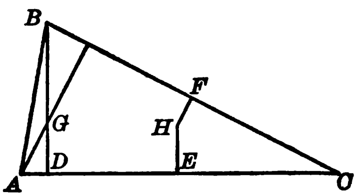
Then $AFG = AFCH$, and AF and AH are the required lines.

Or (*b*) Trisect DB. Draw AF, AE, CF, and CE.

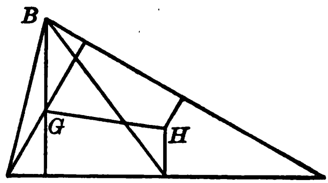
Then the broken lines AFC and AEC divide the figure into three equal parts. To transform these parts so as to fulfill the conditions, draw FH and EK parallel to AC. AH and AK are the required lines.

1521. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

1522. If the altitude BD of $\triangle ABC$ is intersected by another altitude in G, and EH and HF are perpendicular bisectors of $\overline{AC}$ and $\overline{CB}$, prove $\overline{BG} = 2\overline{HE}$.



1523. The line joining the point of intersection of the altitudes of a triangle and the point of intersection of the three perpendicular bisectors, cuts off one-third of the corresponding median.



1524. The points of intersection of the altitudes, the medians, and the perpendicular bisectors of a triangle lie in a straight line.

1525. If, through the points of intersection of two circumferences, parallels be drawn terminated by the circumferences, they are equal.

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1526. If from any point in the circumference of a circle chords be drawn to the vertices of an inscribed equilateral triangle, the longest chord equals the sum of the smaller chords.

1527. Triangles are similar if two sides and the radius of the circumscribed circle of one are proportional to the homologous parts of another.

1528. Construct a square that shall be to a given triangle as 5 is to 4.

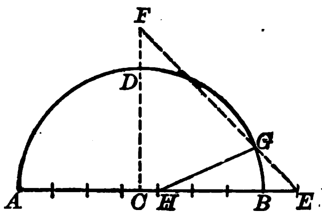
1529. Construct a square that shall be to a given triangle as m is to n, when m and n are two given sects.

1530. If in a circle a regular decagon and a regular pentagon be inscribed, the side of the decagon increased by the radius is equal to twice the apothem of the pentagon.

1531. If from a point O, OA, OB, OC, and OD are drawn so that the $\measuredangle AOB$ is equal to the $\measuredangle BOC$, and the $\measuredangle BOD$ equal to a right angle, any line intersecting OA, OB, OC, and OD is divided harmonically.

1532. Inscribe in a given circle a regular polygon of n sides, n being any whole number.

The following construction is found in most cases to be sufficiently exact for practical purposes:



Divide the diameter AB into n equal parts (in the figure $n=7$). Draw the radius $CD \perp AB$, produce CB to E, and CD to F, making BE and DF each equal to one of the parts of the diameter; draw EF, cutting the circle for the first time in G. Then the line GH joining G and the *third* point of division of AB, counting from B will be very nearly equal to one side of the inscribed polygon of n sides.

For $n=3$ and $n=4$, this construction is impossible; for $n=5$ it is useless, on account of its inaccuracy; but for $n>5$ it gives a very close approximation to the exact value of the side required.

Inscribe a hexagon, a heptagon, an octagon, a nonagon, and a decagon.

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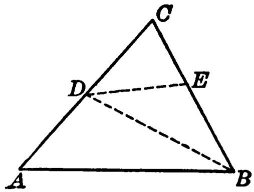
1533. Draw a line meeting the sides CA, CB, of $\triangle ABC$ in D, E, respectively, so that:

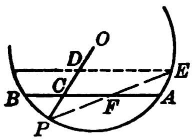


(*a*) $DE \parallel AB$, $\overline{BE}=\overline{ED}$. N.B. Bisect $\angle B$.

(*b*) $DE \parallel AB$, $\overline{DE}=\overline{AD}+\overline{BE}$.

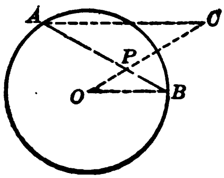
(*c*) $\overline{DE}=\overline{CD}=\overline{BE}$. (See figure.)

Analysis: Suppose the problem solved, and draw BD. The $\triangle$s BDE and DCE are isosceles, whence $\angle DBE=\angle BDE=\frac{1}{2}\angle DEC=\frac{1}{2}\angle DCE$, which is known. This determines the point D, and E is easily found, since $\overline{DE}=\overline{DC}$.



Examine this problem for the special cases when $\angle ACE=90°$, and when $\angle ACE=120°$.

1534. Draw through a given point P in the arc subtended by a chord AB a chord which shall be bisected by AB.

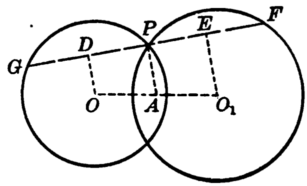


On radius OP take CD equal to CP. Draw $DE \parallel BA$.

1535. Through a given point P inside a given circle draw a chord AB so that the ratio $\frac{AP}{BP}=\frac{m}{n}$.

Draw OPC so that $\frac{OP}{PC}=\frac{n}{m}$. Draw CA equal to the fourth proportional to n, m, and the radius of the circle.

1536. Draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be in the given ratio m to n.



1537. Draw a common tangent to two given circles:

(*a*) Non-intersecting circles
(1) unequal, (2) equal.

(*b*) Intersecting
(1) unequal, (2) equal.

(*c*) Tangent (internally and externally)
(1) unequal, (2) equal.

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1538. Find the shortest path from P to P_1 which shall touch a given line, if P and P_1 are two given points on the same side of the line.

1539. Given points P, P_1, P_2; through P draw a line which shall be equidistant from P_1 and P_2.

1540. Given P, P_1, P_2; through P draw a line so that the distances from P to the feet of the perpendiculars dropped from P_1, P_2 to the line, shall be equal.

1541. Find the direction in which a billiard ball must be shot from a given point on the table, so as to strike another ball at a given point after first striking one side of the table. (The angle at which the ball is reflected from a side is equal to the angle at which it meets the side, that is, $\measuredangle ECB = \measuredangle ACF$.)

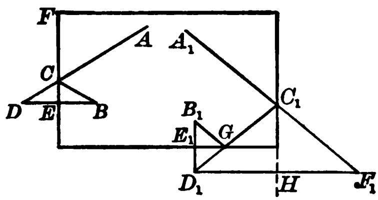


FIG. 1.

Suggestion: Construct $BE \perp$ that side of the table which the ball is to strike, and make $ED = BE$.

1542. The same as the preceding problem, except that the cue ball is to strike two sides of the table before striking the other ball. (Fig. 1.)

Suggestion: $B_1E_1 = E_1D_1$, $D_1H = HF_1$.

1543. Solve Ex. 1542 if the cue ball is to strike three sides before striking the other ball, also if it is to strike all four sides. (Fig. 2.)

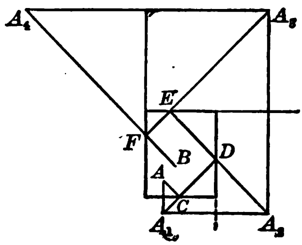
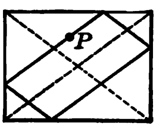


FIG. 2.

1544. A billiard ball is placed at a point P on a billiard table. In what direction must it be shot to return to the same point after hitting all four sides?



Suggestion: (*a*) Show that the opposite sides of the quadrilateral along which the ball travels are parallel. (*b*) If the ball is started parallel to a diagonal of the table, show that it will return to the starting point.

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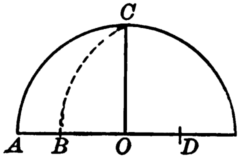
1545. Show that in the preceding problem the length of the path traveled by the ball is equal to the sum of the diagonals of the table.



1546. Show that in the diagram $BC = i_5$ and $OB = i_{10}$ and $OC = i_6$ where $OC \perp AD$, $\overline{OD} = \frac{1}{2}\overline{OC}$, and $BD = DC$.

d1547. Describe a circle the ratio of whose area to that of a given circle shall be equal to the given ratio m to n.

d1548. In an inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides. (Ptolemy.)

d1549. Find a point such that the perpendiculars from it to the sides of a given triangle shall be in the ratio p to q to r.

d1550. Find a point within a triangle such that the sects joining the point with the vertices shall form three triangles, having the ratio 3 to 4 to 5.

d1551. Given a circle and its center; find the side of an inscribed square by means of the compasses alone. ("Napoleon's Problem.")

d1552. Bisect a trapezoid by a line parallel to the bases.

d1553. The feet of the perpendiculars dropped upon the sides of a triangle from any point in the circumference of the circumscribed circle are collinear. ("Simpson's Line.")

d1554. The points A, B, C, D, are collinear. Find the locus of a point P from which the sects $\overline{AB}$ and $\overline{CD}$ subtend the same angle.

d1555. Transform a given triangle into one containing two given angles.

d1556. Transform a given triangle into an isosceles triangle, having a given vertex angle.

Hint: Construct a $\triangle$ similar to a given isosceles $\triangle$ and equal to a given $\triangle$, or transform a $\triangle$ into one containing two given $\angle$s.

d1557. Construct an equilateral triangle that shall be to a given rectangle as 4 is to 5.

d1558. In a given triangle, ABC, inscribe a parallelogram similar to a given parallelogram, so that one side lies in AB, and the other two vertices lie in BC and AC respectively.

d1559. Divide a pentagon into four equal parts by lines drawn through one of its vertices.

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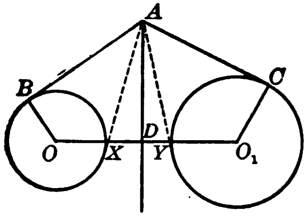
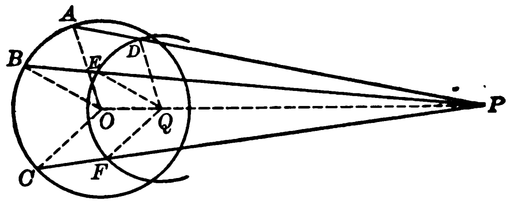
d1560. Right triangles are similar if the hypotenuse and an arm of one triangle are proportional to the hypotenuse and an arm of another.



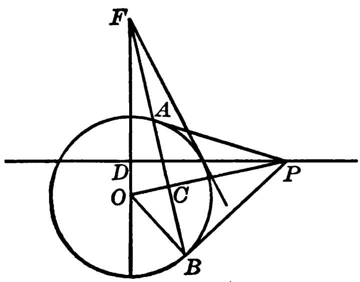
d1561. If from a point A two equal tangents, AB and AC, are drawn to two circles, O and O_1, and AD is perpendicular to OO_1, then $\overline{OD}^2 - \overline{O_1D}^2 = \overline{OB}^2 - \overline{O_1C}^2$.

d1562. Conversely, if, in the same diagram, D is taken so that $\overline{OD}^2 - \overline{O_1D}^2 = \overline{OB}^2 - \overline{O_1C}^2$, then the tangents drawn from any point in the perpendicular, AD, to the circles are equal.



d1563. Divide a trapezoid into two similar trapezoids by a line parallel to the bases.

d1564. Through P secants are drawn to a circle O; find the locus of points which divide the entire secants in the ratio $\frac{m}{n}$.



d1565. Through a fixed point F a secant is drawn to a given circle, and through its intersections A, B with the circumference tangents are drawn intersecting in a point P. If the secant revolves about F, find the locus of P.

d1566. Find the locus of the vertex of a triangle, having given the base and the ratio of the other two sides.

CONSTRUCTIONS LEADING TO THE PROBLEM OF APOLLONIUS

(200 B.C.)

d1567. Construct a circle which shall pass through two given points and be tangent to a given line.

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d1568. Construct a circle which shall pass through two given points and be tangent to a given circle.

d1569. Construct a circle which shall pass through a given point and be tangent to two given lines.

d1570. Construct a circle which shall pass through a given point and be tangent to a given line and to a given circle.

d1571. Construct a circle which shall pass through a given point and be tangent to two given circles.

d1572. Construct a circle which shall be tangent to three given lines.

d1573. Construct a circle which shall be tangent to two given lines and to a given circle.

d1574. Construct a circle which shall be tangent to a given line and to two given circles.

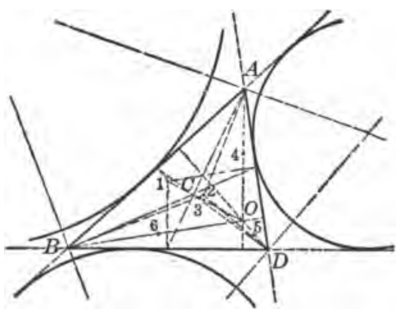
d1575. Construct a circle which shall be tangent to three given circles. ("Problem of Apollonius.")

THE TRIANGLE AND NINE OF ITS CIRCLES

The three escribed circles are represented by arcs; the other six by centers. Prove the construction of the nine circles. Find three others.

Circles.	Centers.	Radii.
Circumscribed,	1	$\overline{1A}$
Inscribed,	2	Perp. from 2
Nine points,	3	$\overline{34}$
Pedal (3),	4, 5, 6	$\overline{40}, \overline{50}, \overline{60}$

C is the *centroid;* and *O*, the *orthocenter;* $\overline{13} = \overline{30}$.



d1576. The circumscribed circle bisects the straight lines joining the center of the inscribed circle with the centers of the escribed circles.

d1577. Each vertex of the triangle is collinear with centers of two of the escribed circles.

d1578. The center of the inscribed circle is collinear with the center of any escribed circle and the opposite vertex.

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d1579. Each center of the inscribed or the escribed circles is the orthocenter of the triangle having the other three centers as its vertices.

d1580. The four circles, each of which passes through three of the centers of the escribed and inscribed circles, are equal.

d1581. The three circles, the circumference of each of which passes through the extremities of any side of a triangle and the orthocenter, equal one another.

Nineteen circles in all have been mentioned.

THE NINE-POINTS CIRCLE

The **orthocenter** *is the point at which the altitudes of a triangle meet.*

The **centroid** *of any triangle is the point at which the medians of the triangle meet.*

1582. The mid-points of the sides of a triangle are concyclic with the feet of the perpendicular from the opposite vertices, and with the mid-points of the sects joining the orthocenter with the vertices. (Nine-points circle.)

1583. The center of the nine-points circle is the mid-point of the sect joining the orthocenter and the center of the circumscribed circle.

1584. The diameter of the nine-points circle is equal to the radius of the circumscribed circle.

1585. The orthocenter and the centroid are collinear with the centers of the nine-points and the circumscribed circles.

1586. The nine-points circle is tangent to the inscribed and escribed circles of a triangle.

Give proofs different from those suggested in the text for the following theorems:

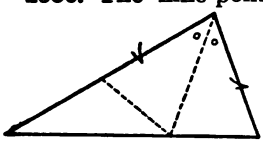

Fig. 1

1587. Theorem 21*b*. Suggestion: (Fig. 1).

1588. Theorem 21*d*. Suggestion: (Figs. 2 and 3).

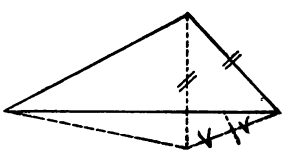

Fig. 2

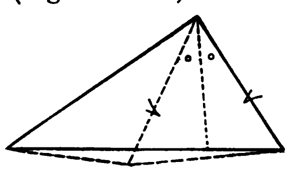

Fig. 3

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1589. Theorem 21*c*. Prove by the method of exclusion.

1590. Theorem 21*f*. Suggestion: (Fig. A).

1591. Theorem 26*a*. For suggestions see Heath's Mathematical Monographs, Numbers 1 and 2.

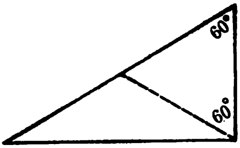

FIG. A

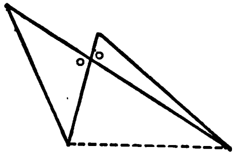

FIG. B

1592. Theorem 26*b*. Suggestion: (Fig. B).

1593. Theorem 40. Prove by means of direct and opposite.

1594. Theorem 41. Prove as in Ex. 1593.

1595. Theorem 53. Suggestion: (Fig. 1).

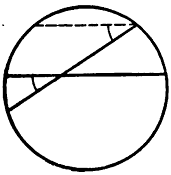

FIG. 1

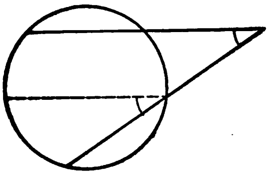

FIG. 2

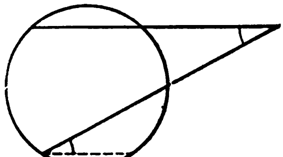

FIG. 3

1596. Theorem 54. Suggestions: (Figs. 2 and 3).

1597. Theorem 55, Cor. 1. Suggestion: (Fig. 4).

1598. Use the following analysis to find another solution for Problem 21.

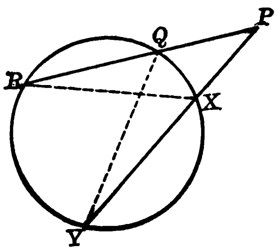

FIG. 4

Analysis: Suppose the construction completed.

Then $\frac{s}{a} = \frac{a}{s-a}$

or $a^2 = s(s-a)$

or $a^2 = s^2 - sa$

or $a^2 + as = s^2$

Completing the square, $a^2 + as + \left(\frac{s}{2}\right)^2 = s^2 + \left(\frac{s}{2}\right)^2$

or $\left(a + \frac{s}{2}\right)^2 = s^2 + \left(\frac{s}{2}\right)^2$

which suggests that $\left(a + \frac{s}{2}\right)$ is the hypotenuse in which the legs are s and $\frac{s}{2}$ respectively.

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CHAPTER IX

COLLEGE ENTRANCE EXAMINATIONS

THE UNIVERSITY OF CHICAGO

EXAMINATION FOR ADMISSION, JUNE, 1908

MATHEMATICS (2)—PLANE GEOMETRY

(TIME ALLOWED—ONE HOUR AND THIRTY MINUTES.)

[In writing, use only one side of the paper, put your name in full at the top of each sheet, and number your work according to the numbers on the printed paper.]

[When required, give all reasons in full, and work out proofs and problems in detail.]

State:

(*a*) At what school you studied this subject.

(*b*) How many weeks.

(*c*) How many recitations per week.

(*d*) What textbook you used.

I. Given two circles of unequal radii and lying exterior to each other; make the construction by which a straight line may be drawn tangent to both circles and shall cross the line joining their centers.

II. Prove that the sum of the exterior angles of any convex polygon, made by producing each of its sides in consecutive order, is equal to four right angles.

III. In any triangle PQR perpendiculars are let fall to the opposite sides from the vertices P and Q. Show that the lines joining the feet of these perpendiculars to the middle point of the side PQ are equal. [Draw two figures, in one of which the angle at Q shall be obtuse, in the other acute.]

IV. Two fields are of similar shape, one having five times the area of the other. (*a*) If they are both circles and the radius of the first is 25 rods, find the radius of the second. (*b*) If they are both equilateral triangles, and the side of the first is 25 rods, find the side of the other.

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Examination for Admission, September, 1916

MATHEMATICS (2)—PLANE GEOMETRY

(TIME ALLOWED—ONE HOUR AND FIFTEEN MINUTES.)

[In writing use only one side of the paper, put your name in full at the top of each sheet, and number your work according to the numbers on the printed paper.]

[When required, give all reasons in full, and work out proofs and problems in detail.]

1. Prove that a straight line parallel to one side of a triangle divides the other two sides proportionally.

2. Find the locus of all points such that the two lines joining each to two fixed points always make a given angle with each other. What does the locus become when the given angle is a right angle?

3. Prove that in a triangle with a given fixed base the median from the vertex opposite this base is greater than, equal to, or less than half the base according as the vertical angle is less than, equal to, or greater than a right angle.

4. Construct a circle passing through a given point and tangent to two given intersecting straight lines.

5. Two tangents to a circle intersect in a point which is 50 inches from the center of the circle. The area of the four-sided figure formed by the two tangents and the two radii drawn to the two points of contact is 625 square inches. Find the length of the tangents and the radius of the circle.

6. Show how to construct geometrically a square that shall contain the same area as a given rectangle whose base is b and whose altitude is a. Prove the result.

HARVARD UNIVERSITY

JUNE, 1894

(In solving problems use for π the approximate value $3\frac{1}{7}$.)

1. Prove that any quadrilateral the opposite sides of which are equal, is a parallelogram.

A certain parallelogram inscribed in a circumference has two

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sides 20 feet in length and two sides 15 feet in length; what are the lengths of the diagonals?

2. Prove that if one acute angle of a triangle is double another, the triangle can be divided into two isosceles triangles by a straight line drawn through the vertex of the third angle.

Upon a given base is constructed a triangle one of the base angles of which is double the other. The bisector of the larger base angle meets the opposite side at the point P. Find the locus of P.

3. Show how to find a mean proportional between two given straight lines, but do not prove that your construction is correct.

Prove that if from a point, O, in the base, BC, of a triangle, ABC, straight lines be drawn parallel to the sides, AB, AC, respectively, so as to meet AC in M and AB in N, the area of the triangle AMN is a mean proportional between the areas of the triangles BNO and CMO.

4. Assuming that the areas of two parallelograms which have an angle and a side common and two other sides unequal, but commensurable, are to each other as the unequal sides, prove that the same proportion holds good when these sides have no common measure.

5. Every cross-section of the train house of a railway station has the form of a pointed arch made of two circular arcs the centres of which are on the ground. The radius of each arc is equal to the width of the building (210 feet); find the distance across the building measured over the roof, and show that the area of the cross-section is 3675 $(4\pi - 3\sqrt{3})$ square feet.

SEPTEMBER, 1894

One question may be omitted.

(In solving problems use for π the approximate value $3\frac{1}{7}$.)

1. Prove that any quadrilateral the diagonals of which bisect each other is a parallelogram.

The diagonals of a parallelogram circumscribed about a circumference are 60 inches and 80 inches long respectively. How long are the sides?

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2. Prove that the difference of the angles at the base of a triangle is double the angle between a perpendicular to the base and the bisector of the vertical angle.

The sum of the base angles of each of a number of triangles constructed on a given base 10 inches long is 150°. What is the locus of the vertices of these triangles?

3. Show how to find a fourth proportional to three given lines, but do not prove that your construction is correct.

One circle touches another internally at O, and a chord AB of the larger circle touches the smaller one at C. Prove that AO makes with the common tangent to the circles an angle equal to ABO and that CO bisects the angle AOB. State without proof some relation that exists between the lines AO, CB, BO, and AC.

4. Assuming that the areas of two rectangles which have equal altitudes are to each other as their bases when the latter are commensurable, show that the same proportionality exists when the bases have no common measure.

5. A kite-shaped racing track is formed by a circular arc and two tangents at its extremities. The tangents meet at an angle of 60°. The riders are to go round the track, one on a line close to the inner edge, the other on a line everywhere $5\frac{1}{4}$ feet outside the first line. Show that the second rider is handicapped by about 22 feet.

JUNE, 1895

One question may be omitted.

(In solving problems use for π the approximate value $3\frac{1}{7}$.)

1. Prove that if two straight lines are so cut by a third that corresponding alternate-interior angles are equal, the two lines are parallel to each other.

2. Prove that an angle formed by two chords intersecting within a circumference is measured by one-half the sum of the arcs intercepted between its sides and between the sides of its vertical angle.

Two chords which intersect within a certain circumference divide the latter into parts the lengths of which, taken in order, are as 1, 1, 2, and 5; what angles do the chords make with each other?

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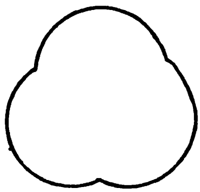
3. Through the point of contact of two circles which touch each other externally, any straight line is drawn terminated by the circumferences; show that the tangents at its extremities are parallel to each other.

What is the locus of the point of contact of tangents drawn from a fixed point to the different members of a system of concentric circumferences?

4. Prove that, if from a point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and the part without the circle.

Show (without proving that your construction is correct) how you would draw a tangent to a circumference from a point without it.

5. Prove that the area of any regular polygon of an even number of sides ($2n$) inscribed in a circle is a mean proportional between the areas of the inscribed and the circumscribed polygons of half the number of sides. If n be indefinitely increased, what limit or limits do these three areas approach?



6. The perimeter of a certain church window is made up of three equal semi-circumferences, the centres of which form the vertices of an equilateral triangle which has sides $3\frac{1}{2}$ feet long. Find the area of the window and the length of its perimeter.

SEPTEMBER, 1895

One question may be omitted.

(*In solving problems use for* π *the approximate value* $3\frac{1}{7}$.)

1. Prove that every point in the bisector of an angle is equally distant from the sides of the angle. State the converse of this proposition. Is this converse true?

2. Prove that an angle formed by two secants intersecting without a circumference is measured by half the difference of the arcs which the sides of the angle intercept.

A certain pair of secant lines which intersect without a circle divide the circumference into parts the lengths of which, taken in

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order, are to one another as 1, 2, 3, and 4. What angles do the lines make with each other?

3. Two given circles touch each other externally at the point P, where they have the common tangent PC. They are also touched by the line AB in the points A and B respectively. Show that the circle described on AB as diameter has its centre on PC, and touches at P the straight line which joins the centres of the two given circles.

4. Show how to describe upon a given straight line a segment which shall contain a given angle.

A and B are two fixed points on the circumference of a circle, and PQ is any diameter. What is the locus of the intersection of PA and QB?

5. C is any point on the straight portion, AB, of the boundary of a semicircle. CD, drawn at right angles to AB, meets the circumference at D. DO is drawn to the centre, O, of the circle, and the perpendicular dropped from C upon OD meets OD at E Show that DC is a mean proportional to AO and DE.

State the fundamental theorem in the method of limits as used in Plane Geometry.

6. A horse is tethered to a hook on the inner side of a fence which bounds a circular grass plot. His tether is so long that he can just reach the centre of the plot. The area of so much of the plot as he can graze over is $\frac{98}{3}(4\pi - 3\sqrt{3})$ square rods; find the length of the tether and the circumference of the plot.

JUNE, 1896

One question may be omitted.

(In solving problems use for π the approximate value $3\frac{1}{7}$.)

1. Prove that if two oblique lines drawn from a point to a straight line meet this line at unequal distances from the foot of the perpendicular dropped upon it from the given point, the more remote is the longer.

2. Prove that the distances of the point of intersection of any two tangents to a circle from their points of contact are equal.

A straight line drawn through the centre of a certain circle and

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through an external point, P, cuts the circumference at points distant 8 and 18 inches respectively from P. What is the length of a tangent drawn from P to the circumference?

3. Given an arc of a circle, the chord subtended by the arc, and the tangent to the arc at one extremity, show that the perpendiculars dropped from the middle point of the arc on the tangent and chord, respectively, are equal.

One extremity of the base of a triangle is given and the centre of the circumscribed circle. What is the locus of the middle point of the base?

4. Prove that in any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other upon that side.

Show very briefly how to construct a triangle having given the base, the projections of the other sides on the base, and the projection of the base on one of these sides.

5. Show that the areas of similar triangles are to one another as the areas of their inscribed circles.

The area of a certain triangle the altitude of which is $\sqrt{2}$, is bisected by a line drawn parallel to the base. What is the distance of this line from the vertex?

6. Two flower beds have equal perimeters. One of the beds is circular and the other has the form of a regular hexagon. The circular bed is closely surrounded by a walk 7 feet wide bounded by a circumference concentric with the bed. The area of the walk is to that of the bed as 7 to 9. Find the diameter of the circular bed and the area of the hexagonal bed.

SEPTEMBER, 1896

One question may be omitted

(*In solving problems use for π the approximate value* $3\frac{1}{7}$.)

1. Prove that, if one of two convex broken lines which have the same extremities envelops the other, the first is the longer.

2. Prove that, when two circumferences intersect each other, the line which joins their centres bisects at right angles their common chord.

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The centres of two circles of radii 8 inches and 6 inches respectively are 10 inches apart. Show that the common chord is 9.6 inches long.

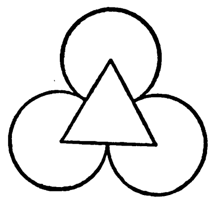
3. Show that, if two parallel tangents to a circle are intercepted by a third tangent, the part of the third tangent between the other two subtends a right angle at the centre of the circle.

State briefly how you might find a fourth proportional to three given straight lines.

4. Prove that in any obtuse-angled triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of these sides and the projection of the other upon that side.

What is the locus of the vertices of all the triangles so constructed on a given base that the radii of their circumscribed circles are all equal to a given line?

5. The line passing through the centres of two circles which touch each other externally at A, meets a common tangent, which touches the circles at B and C respectively, at the point S. Show that SA is a mean proportional between SB and SC.



6. The perimeter of a certain church window is made up of three equal circular arcs the centres of which are the vertices of an equilateral triangle. Each of the arcs subtends an angle of 300° at its own centre. Find the area of the window, assuming the length of the perimeter to be 110 feet.

JUNE, 1906

(One Hour and a Half)

The University Provides a Syllabus

1. Prove that the sum of the three angles of any triangle is equal to two right angles.

On a line AE choose a point B, and construct an isosceles triangle ABC with AB as base, the base angles being less than 45°. With B as vertex construct an isosceles triangle BCD whose base CD lies in AC produced. Show that the angle DBE is three times the angle A.

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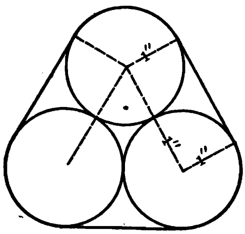
2. Prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the sides of the angle.

The hypotenuse of a right triangle is 10 inches long and one of the acute angles is 30°. Compute the lengths of the segments into which the short side is divided by the bisector of the opposite angle.

3. A chord BC of a given circle is drawn, and a point A moves on the longer arc BC. Draw the triangle ABC, and find the locus of the centre of a circle inscribed in this triangle.



4. Three equal circular plates are so placed that each touches the other two, and a string is tied tightly around them. If the length of the string is 10 feet, find the radius of the circles correct to three significant figures.

5. Let O be the centre of a circle and P any point outside. With P as centre and radius PO draw the arc of a circle, cutting the given circle at A and B. With A and B as centres and AB as radius draw arcs intersecting in Q. Prove:

(*a*) that the points O, Q, P are in a straight line; and

(*b*) that $OP \times OQ = (OA)^2$.

Suggestion.—Join A and B with O, P, and Q, and join Q with O and P.

SEPTEMBER, 1907

(ONE HOUR AND A HALF)

The University Provides a Syllabus

1. Prove that in an isosceles triangle the angles opposite the equal sides are equal.

On BC, the longest side of a triangle ABC, points B' and C' are taken so that $BB' = BA$ and $CC' = CA$. Show that the angle $B'AC'$ equals half the sum of the angles B and C.

2. Prove that the area of a triangle is one-half the product of the base and the altitude.

Show that if a point move about within a regular polygon, the sum of the perpendiculars let fall upon the sides (or the sides produced) will be constant.

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3. A rod 8 feet long is free to move within a rectangle 8 feet long and 6 feet wide. Describe accurately the boundary of the region within which the middle point of the rod will always be found.

4. A circle is described upon one side of an equilateral triangle as diameter. Compute the area of the part of the triangle which lies outside the circle, correct to one per cent. of its value.

5. Two circles intersect at right angles. The radius of one of them is of length a, and its centre is the point O. Show that if any line be drawn through O cutting the second circle in the points P and P', then

$$OP \times OP' = a^2.$$

1908

(One Hour and a Half)

The University Provides a Syllabus

1. Prove that if in a quadrilateral a pair of opposite sides be equal and parallel, the figure is a parallelogram.

In a certain quadrilateral, one diagonal, and a line connecting the middle points of a pair of opposite sides bisect each other. Prove that the quadrilateral is a parallelogram.

2. Prove that in any circle equal chords subtend equal arcs.

Show that the bisector of an angle of a triangle meets the perpendicular bisector of the opposite side on the circumference of the circumscribed circle.

3. The radii of two circles are 1 inch, and $\sqrt{3}$ inches respectively, and the distance between their centres is 2 inches. Compute their common area to three significant figures.

4. Determine a point P without a given circle so that the sum of the lengths of the tangents from P to the circle shall be equal to the distance from P to the farthest point of the circle.

5. The image of a point in a mirror is, apparently, as far behind the mirror as the point itself is in front. If a mirror revolve about a vertical axis, what will be the locus of the apparent image of a fixed point one foot from the axis?

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1909

(Two Hours)

The University Provides a Syllabus

1. Show that if a triangle be equilateral, it is also equiangular. Is this theorem true in the case of a quadrilateral? Give your reason.

2. Prove that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

Deduce from this a proof that of two oblique lines drawn from the same point of a perpendicular and cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.

3. Prove that if the products of the segments into which the diagonals of a quadrilateral divide one another are equal, a circle may be circumscribed about the quadrilateral.

4. A square 10 inches on a side is changed into a regular octagon by cutting off the corners. Find the area of this octagon.

5. A wheel 40 inches in diameter has a flat place 5 inches long on the rim. Describe carefully the locus of the centre as the wheel rolls along the level.

1910

(Two Hours)

The University Provides a Syllabus

1. Prove that if the sides of one angle be perpendicular respectively to those of another, the angles are either equal or supplementary.

2. Show how to inscribe a circle in a given triangle.

How many circles can be drawn to touch three given lines? Are there any positions of these lines for which the number is less?

3. Define "incommensurable magnitudes." Give a careful proof of some theorem where such magnitudes occur.

4. $ABCD$ are the vertices in order of a quadrilateral which is circumscribed to a circle whose centre is O. Prove that $\angle AOB$ and $\angle COD$ are supplementary.

5. Two radii of a circle OA and OB make a right angle. A second circle is described upon AB as diameter. Prove that the area of the crescent-shaped region outside of the first circle, but inside of the second, is equal to that of the triangle AOB. (Hippocrates, fifth century B.C.)

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1915

(Two Hours)

The University Provides a Syllabus

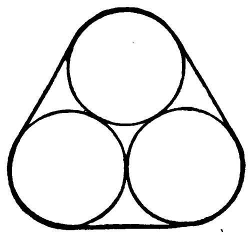
1. Two straight lines are cut by a third, and the alternate interior angles are equal. Prove that the two straight lines are parallel.

Prove that the bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base.

2. Prove that an angle formed by a tangent to a circle and a chord through the point of contact is measured by one-half of the intercepted arc.

Tangents are drawn at the extremities of a chord of a circle, and the perpendicular bisector of the chord is drawn. The point in which this last line cuts the minor arc is connected with one end of the chord. Show that this connecting line bisects the angle between the chord and the tangent.

3. Two unequal circles are tangent to each other externally, and their common tangent at the point of contact meets one of the other common tangents at the point P. Lines are drawn from P to the centres of the two circles. Prove that these two lines are perpendicular.



4. Three equal circular plates of radius r are so placed that each is tangent to the other two. Find the length of the shortest string that can be tied around the three circles, and the area enclosed by this string.

5. Two circles ABC and ADE touch internally at A, and through A straight lines, ABD and ACE, are drawn to cut the circles. Prove that $AB \cdot DE = AD \cdot BC$.

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CHAPTER X

SUGGESTIONS

The aim of this chapter is to suggest appropriate material for reading and discussion outside of class, and in mathematics clubs. To this end certain books and topics are mentioned, but it is understood that these lists are intended to be suggestive rather than comprehensive, and include only subject matter adapted to pupils at the stage of development represented by this text.

After topics in the second list, parenthetical reference is frequently made to books from the first list, in which at least something about the topic may be found, and in the third list definite references for a few topics are given as a guide to the beginner. The practice of seeking such references without help is particularly valuable as a training for college work; hence, this list includes only a few topics, and is to serve for direction at the outset only.

A. LIST OF TOPICS SUITABLE FOR STUDENTS' DISCUSSION

GENERAL

Mathematical Games. (Ball.)
    Card Tricks.
    Problems on a Chess Board.
History and Elementary Idea of Calculus. (White.)
Some Applications of Mathematics to Astronomy. (Ball.)
Parcel Post Problems. (School Science and Mathematics.)
Mathematics of Common Things.
    Optical Illusions. (Smith's Teaching of Geometry.)
Navigation. (Richards.)
Instruments.
    Historic. (School Science and Mathematics.)
    Astrolabe, squadra, carpenter's level, baculus mensorus, sundial, etc.
Pantograph. (Phillips and Fisher's Elements of Geometry.)
    Map-making.
Planimeter. (Scientific American.)
Symmetry. (White; Dobbs, Symmetry, Chapt. VI.)
    In nature.
    Design.
Maxima and Minima. (Texts on Plane Geometry.)

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Linkages. (White.)
Fourth Dimension. (Flatland.)
Angles of a General Polygon. (School Science and Mathematics.)
Study of Rupert's "Famous Problems in Geometry." (Heath's Monographs.)
Fallacies. (Ball.)
Odd or Purely Mechanical Constructions. (Becker.)

ARITHMETIC

History of. (Ball, Fink, Cajori, Brooks.)
- Primitive Numeration.
- Systems of Notation.
- Problems in Number Systems Other Than Our Own.
- Special Consideration of Duodecimal System.
- Kinds of Number.
- Fundamental Operations.
  - Russian Peasant Multiplication.

Calculation. (Langley's Computation. Longmans, Green & Co.)
- Short Cuts. (Jones; White.)
- Calculating Machines.
- Slide Rule. (The Teaching of Mathematics, by Schultze, Macmillan.)
- Approximation. (White.)
- Repeating Decimals. (White.)

Peculiarities and Possibilities of Our Electoral System.
Number Curiosities.
- 9 and its properties. (White.)
- Tricks. (Dudeney; Jones; White.)

Alligation. (The Mathematics Teacher.)

ALGEBRAIC

History of. (Ball, Fink, Cajori, etc.)
- Stages—Rhetorical, Syncopated and Symbolic.
- Symbols.

Fallacies. (White; Jones.)
Choice and Chance. (Text on higher algebra.)

GEOMETRIC

History. (Smith's Teaching of Geometry.)
- More detailed study of some period other than is given in text.

Famous Problems. (Heath's Monographs.)
- Trisection of an Angle.
  - Instruments.
- Squaring of the Circle.
- Pythagorean Proposition.
  - Various proofs and applications.

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B. TOPICS WITH DEFINITE REFERENCES

GEOMETRIC FALLACIES

Right angle is obtuse. Ball's Recreations, p. 40.
Part of a sect equals the sect. Ball's Recreations, p. 41.
Every triangle is isosceles. Ball's Recreations, p. 42.
Miscellaneous fallacies. Ball's Recreations, pp. 43–46.
Hypotenuse equals sum of legs of triangle. Canterbury Puzzles, p. 26.
1 = 0. Wrinkles, p. 93.
Two perpendiculars from a point to a line. Wrinkles, p. 95.

NUMBER CURIOSITIES

Mystic Properties of Numbers.
  Repeating products. White, p. 11 *et seq.*
  Nine. White, p. 25.
Fallacies. Ball, p. 23 *et seq.*
Magic Squares. See Andrews's book and his references.

PYTHAGOREAN PROPOSITION

Various proofs.
  Rupert's "Famous Geometrical Theorems and Problems."
Applications (curious).
  Fly Problem. Jones p. 2, No. 10.
  Historical Problems. See Beman and Smith's Academic Algebra, p. 153.

C. LIST OF BOOKS SUITABLE FOR STUDENTS' READING.

HISTORY

ALLMAN: Greek History from Thales to Euclid. Longmans.
BALL: History of Mathematics. Macmillan.
  Primer of the History of Mathematics. Macmillan.
BROOKS: Philosophy of Arithmetic. Normal Publishing Co.
CAJORI: History of Mathematics. Macmillan.
  History of Elementary Mathematics. Macmillan.
CONANT: Number Concept. Macmillan.
FINE: Number Systems of Algebra. Heath.
FINK: Brief History of Mathematics. Open Court Pub. Co.
FRANKLAND: The Story of Euclid. Wessel and Co.
GOW: History of Greek Mathematics. Cambridge Press.
KLEIN: Famous Problems of Elementary Geometry. Ginn.
HEATH: Diophantus of Alexandria. Cambridge Press.
MANNING: Non-Euclidean Geometry. Ginn.
MILLER, G. A.: Historical Introduction to Mathematical Literature. Macmillan.

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SMITH, D. E.: Rara Arithmetica. Ginn.
Teaching of Geometry. Ginn.
Teaching of Elementary Mathematics. Macmillan.
SMITH AND KARPINSKI: The Hindu-Arabic Numerals. Ginn.
History of Japanese Mathematics. Open Court Pub. Co.
BOYER: Histoire des Mathématiques. Paris.
CANTOR: Vorlesungen Über Geschichte der Mathematik. Leipzig.
Portraits of Mathematicians by Prof. D. E. SMITH.
Portfolios. Open Court Pub. Co.
Lantern Slides by Prof. D. E. SMITH, of Teachers' College, Columbia University.

RECREATIONS

ABBOTT: Flatland. Little, Brown.
ANDREWS: Magic Squares and Cubes. Open Court Pub. Co.
ANONYMOUS: Flatland. Boston.
BALL: Mathematical Recreations. Macmillan.
CAVENDISH: Recreations with Magic Squares. London.
DUDENEY: The Canterbury Puzzles. Dutton & Co.
HARPSON: Paradoxes of Nature and Science. Dutton & Co.
HATTON: Recreations in Mathematics. London.
HILL: Geometry and Faith. Lee and Shepard.
JONES, S. I.: Mathematical Wrinkles. Author, Gunther, Texas.
KEMPE: How to Draw a Straight Line. Macmillan.
LATOON: On Common and Perfect Magic Squares. Cambridge.
DE MORGAN: A Budget of Paradoxes. London.
MANNING: Fourth Dimension Simply Explained. Munn.
PERRY: Spinning Tops. London.
SCHOFIELD: Another World. Swan, Sonnenschein; London.
SCHUBERT: Mathematical Recreations. Open Court Pub. Co.
WHITE: Scrap Book of Elementary Mathematics. Open Court Pub. Co.
AHRENS: Mathematische Unterhaltungen und Spiele. Leipzig.
LUCAS: Recreations Mathématiques. Paris.
L'Arithmétique amusante. Paris.
MAUPIN: Opinions et curiosités touchant la mathématique. Paris.
REBIÈRE: Mathématiques et Mathématiciens, Penseés et Curiosités. Paris.

PRACTICAL

BRECKENRIDGE, MERSEREAU AND MOORE: Shop Problems in Mathematics. Ginn.
CALVIN, F. H.: Shop Calculations. McGraw Hill.
CASTLE: Manual of Practical Mathematics. Macmillan.
COBB: Applied Mathematics. Ginn.
COX: Manual of Slide Rule. Keuffel and Esser.
DOOLEY: Vocational Mathematics. Heath.
HINDS, NOBLE AND ELDREDGE: How to Become Quick at Figures.

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LANGLEY: Computation. Longmans.
MARSH: Vocational Mathematics. Wiley and Sons.
MURRAY: Practical Mathematics. Longmans.
RICHARDS: Navigation and Nautical Astronomy. American Book Co.
SAXELBY: Practical Mathematics. Longmans.

GENERAL

CARUS: Foundations of Mathematics. Open Court Pub. Co.
CLIFFORD: Common Sense of the Exact Sciences. Appleton.
FRANKLAND: Theories of Parallelism. Cambridge Press.
HENRICI: Congruent Figures. Longmans.
LAGRANGE: Lectures on Elementary Mathematics. Open Court Pub. Co.
ROW: Geometric Paper Folding. Open Court Pub. Co.
SYKES: Source Book of Problems for Geometry. Allyn and Bacon.
VON H. BECKER: Geometrisches Zeichnen. Sammlung Göschen.

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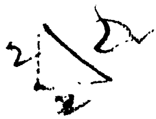

$2 = 4+4$

$\sqrt{8}$

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