Plane and Solid Geometry

INDUCTIVE METHOD

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Florian Cajori
March, 1901

PLANE AND SOLID

GEOMETRY

INDUCTIVE METHOD

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PRESS OF
HUDSON-KIMBERLY PUBLISHING CO.,
KANSAS CITY, MO.

PREFACE.

The purpose of this elementary treatment of Plane Geometry is to give to the pupils of the Kansas City Manual Training High School a course in geometric reasoning based principally on the Inductive method of presentation.

In the beginning, by many questions and suggestions carefully arranged, the pupil is led to grasp some of the fundamental ideas of Geometry, and in the same manner the first propositions are established.

Instead of giving the formal proposition at the beginning of a demonstration and then the proof, the pupil reaches the general truth as a *result*. This result or proposition is to be neatly written and numbered on the blank pages at the back of the book. With the lessons prepared, partly by answering the questions, and under the direction of skillful teachers, it is hoped that the majority of the pupils will be able to make logical deductions from data given and to become self-reliant. It is not the *amount* of knowledge possessed, but rather the *method* by which the knowledge is gained, which is the *important* thing. What *power* results from the investigation of the truths in Geometry is the *all-important* question.

No boy or girl will learn to ride a bicycle by memorizing the most carefully prepared directions. Actual struggle and persevering efforts with thoughtful direction bring skill to the learner.

No amount of memorizing of reasoning *processes* will make a pupil proficient in reasoning. It will tend to keep him from using his originative and reasoning powers.

It is supposed that the pupils who take this course have had, at least, one term in Inventional or Constructional Geometry; if the class has not had this preliminary work, the teacher should spend some time in introducing the subject concretely, familiarizing the pupils with dividers and rule; to pupils thus prepared the construction of most plane figures and the grasping of simple geometric ideas should present no serious difficulties. The energies of the pupil can be directed to the processes of reasoning. In this school especially do we try to teach by *doing*, guided by skilled directors. What the actual experimental work in the laboratory, shop, or cooking-room is to the theory of the science taught, so is the original solution or demonstration to the principles and theorems in Geometry.

For this reason, numerous graded exercises are given along with the propositions. Many of these exercises are intended to make the pupil feel that Geometry is a science which has to do with common affairs. An early introduction of the properties of the triangle is easily made to the pupil by the method of superposition; and the utility of this method is of great value in acquiring other geometric truths.

Accuracy, neatness in demonstration, and the giving of authority are insisted on in all the work to be done. Independent solutions are encouraged.

Nearly every English Elementary Geometry has been made use of in securing suggestions and hints on demonstrations, but the method of reaching the general truths of Elementary Geometry is unique and is believed to be on the laboratory method of teaching. It must not be forgotten that those who take this course should have finished a book like Spencer's Inventional Geometry, or Nichols' Constructional Geometry, or Hornbrook's Concrete Geometry, or to have had a good preliminary introduction to Demonstrative Geometry. In most instances *the* question whose answer is the proposition required is printed in pica, thus helping the pupil to keep the main question in mind.

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SUGGESTIONS TO TEACHERS.

It is of the greatest importance that the pupil clearly understand the status of his work in Constructional Geometry. Of the three books mentioned in the Preface, Nichols' is the only one which attempts pure demonstrations. But even these cannot be accepted (see pages 32, 41, 48, 54, etc.), since the problems, (1) To construct an angle equal to a given angle; (2) .To bisect a given angle, etc., have not yet been *proved to be geometrically true*. In Proposition I. he does not *know* that he has two triangles in which two respective sides and included angles are equal *because he constructed them equal*, but because they are *so given in the hypothesis*.

In beginning the *formal* demonstration of Proposition I. the teacher should clearly outline the *method of proof* and by numerous exercises and illustrations see that this is fully understood by the pupils. Let the pupils understand that in Proposition I. and others the questions asked about the figures are *no part of the demonstration*—they are asked simply to *lead the pupil to discover* that proof and the steps in it which he is required to give in logical order.

ORDER OF PROOF.

I. General Enunciation, which should be clearly shown to consist of two parts, the *hypothesis* (or supposition), and the *conclusion*. Thus in Proposition I. we have: *Hypothesis.*—If two sides and the included angle of any triangle are equal, respectively, to two sides and the included angle of another triangle, *conclusion*, the triangles are equal.

II. The Particular Enunciation, which refers to the particular figure or figures which fulfill all the given conditions of the General Enunciation. Thus in Proposition I.:

Given: The triangles A B C and D E F, in which A B, A C, and $\angle$ B A C, respectively; equal D E, D F, and $\angle$ E D F.

To prove: The triangles A B C and D E F equal.

III. *The Construction*, which consists of the drawing of *aid* lines, superposition of figures, etc. Here the authority for the work should be shown to rest upon the geometric postulates which are discussed in the text. In *problems* the Construction is given a prominent place. At all times the work should be done with the greatest care and accuracy.

IV. The Demonstration, which is shown to rest solidly upon definitions, axioms, and previously proved theorems and problems previously constructed and proved.

It is of vital importance that the pupil fully understand that the truth set to be proved in the Particular Enunciation is not established until the very *best authority has been given*, and then the pupil should be led to see clearly just how the conclusion of the General Enunciation follows the proof of the Particular Enunciation. The pupil must not be permitted to conclude that his work in Constructional Geometry is useless because he can not use it as authority for his present work. The *definitions* and axioms there given *are authority* in the present work. (The fundamental notions there developed will be of great value in the work in hand.) Truths which are there proved and are shown to depend solely upon the definition for authority or which followed from the application of axioms *are in full force* here. But he must understand once for all that the further truths which he discovered and carefully *tested* with instruments must be now formally established by the strictest of logical reasoning. He has studied *Practical Geometry*.

He can be shown that the fundamental notions there developed will be of the greatest value in the present *science of reasoning*, which deals with the truths there discovered and practically used.

When the study of Concrete Geometry, as recommended by the Committee of Ten, has been widely introduced into the grammar grades of our schools, this work can be begun in the beginning of the High School.

TABLE OF CONTENTS.

PLANE GEOMETRY.

BOOK I. LINES AND RECTILINEAR FIGURES, PROBLEMS.

BOOK II. CIRCLES, PROBLEMS.

Book III. Proportion, Similar Figures.

Book IV. Areas.

Book V. Measurement of the Circle.

SOLID GEOMETRY.

Book VI. Planes and Polyedral Angles.

Book VII. Polyedrons.

BOOK VIII. THE THREE ROUND BODIES.

SYMBOLS AND ABBREVIATIONS.

$\angle$ = angle.
$\angle$s = angles.
$\triangle$ = triangle.
$\triangle$s = triangles.
▭ = rectangle.
▭s = rectangles.
$\odot$ = circle.
$\odot$s = circles.
⌢ = arc.
⌢s = arcs.
$\perp$ = perpendicular.
$\perp$s = perpendiculars.
$\parallel$ = parallel.
$\parallel$s = parallels.
▱ = parallelogram.
▱s = parallelograms.
$>$ = is, or are, greater than.
$<$ = is, or are, less than.
$\therefore$ = therefore.
$\because$ = since, or because.
$\doteq$ = approaches as a limit.
$'$ = foot or feet.
$''$ = inch or inches.
Isos. = isosceles.
Ax. = axiom.

Cor. = corollary.
Iden. = identical.
Rt. = right.
Ex. = exercise.
Auth. = give authority = why?
Hyp. = hypothesis.
St. = straight.
Pt. = point.
Pts. = points.
P. P. = previous proposition.
Prop. = proposition.
Def. = definition.
Sug. = suggestion.
Sup. = supplementary.
Adj. = adjacent.
Ext. = exterior.
Int. = interior.
Alt. = alternate.
Ext.-int. = exterior-interior.
Alt.-int. = alternate-interior.
Q. E. D. = *quod erat demonstrandum*—which was to be proved.
Q. E. F. = *quod erat faciendum*—which was to be done.

INTRODUCTION.

What dimensions has a solid?

What are the boundaries of a solid? Give examples.

What are the dimensions of a surface? Give examples.

Give the boundaries of a surface. Illustrate.

What dimension has a line? What are its boundaries? Can we apply the word "dimension" to a point?

1. Think of a square lying horizontally. Raise it vertically. What solid is described? What surfaces do the sides of the square describe?

2. Think of a circle lying horizontally. Raise it vertically. What solid is described? What surface is described by the circumference? Revolve a rectangle about one of its sides as an axis. What solid is generated? What surfaces?

3. Imagine a semicircle revolved about its diameter. What solid is formed by this revolution? What surface is generated by the semi-circumference? What does the revolution of the diameter generate?

4. As you fill a vessel with water, what is the solid traced by the surface of the water?

If a point move through space, what will it describe?

If a line move keeping parallel to its original position, what will it generate?

If a plane move at right angles to its original position, what will it generate?

Revolve a line about one end as a center. What surface is described by the line? What is described by the other end?

Revolve a right triangle about each side in order. Describe each solid and each surface formed.

Revolve an obtuse triangle about each side in order. Describe the solids and the surfaces formed.

Can you imagine a hollow glass cube? Can you picture other hollow glass figures? Give examples.

Can you imagine the cube, were the glass cube shattered? Can you see a cube with your eyes closed? Do you see the upper surface? the lower surface? the upper front edge? the lower front edge? the other edges? the upper right front corner? the other corners?

Think of a cube bisected. What kind of surfaces bound the parts?

In how many ways can you think of bisecting a cylinder? How many ways of bisecting a sphere?

Think of other solids; conceive them bisected. What new solids and plane figures are thus formed?

Write short, clear definitions of solid, surface, line, and point.

If you can *think* of a cube apart from the material, of its sides, of its edges, and its corners, you have a geometrical *concept* of a cube. If you can think of a surface apart from the solid which it bounds, you have a geometrical *concept* of a surface. If you can think of a line apart from the surface which it bounds, you have a geometrical *concept* of a line.

If you can think of a point apart from the extremities of a line or the intersection of two or more lines, you have a *notion* of the geometrical point.

Can you conceive of a cylinder, its boundaries, its surfaces? Can you form geometrical concepts of other solids?

POSTULATES.

Can any two points in the same plane be joined by a straight line? Can you think of any two points not in the same plane? Can you think of any three points not in the same plane?

Is it self-evident that any straight line may be produced to any length in either direction?

May a circle be drawn with any point as a center and with any finite straight line as a radius?

Can a figure be moved unaltered to a new position?

Is it possible to think of two equal geometric cubes being so placed that they will coincide?

Can you state five postulates?

AXIOMS.

What is an axiom?

Give conclusion in the following examples and state the axiom applicable.

1. Tom and John are each the same age as I; therefore—

2. I have the same amount of money as Brown or Smith; therefore—

3. $A = B$ and $C = B$; therefore—

4. Brown has as much money as Smith, and Jones as Robinson; ∴ Brown and Jones together have—

$A = B$ and $C = D$; $\therefore A + C = ?$

5. Two armies are equal in number, each loses 500 men in battle; consequently—

6. Brown and Smith are each double the height of the dwarf Jones; ∴ —

7. M is 9 times N, R is also 9 times N; ∴ —

8. A is $\frac{1}{2}$ of B, C is $\frac{1}{2}$ of B; ∴ —

9. X is $\frac{6}{9}$ of Y, Z is $\frac{2}{3}$ of Y; ∴ —

10. My whole hand is larger than my thumb. State axiom.

11. One-third of an apple is less than the whole of it. State axiom.

12. I am older than you. In 5 years I shall still be older. State axiom.

13. Smith has less money than Jones, each spends $5. Draw conclusion and state axiom.

14. If, when a sheet of paper is placed on another, their edges exactly coincide— State conclusion and axiom.

15. If, when one line is placed on another, their extremities coincide and every point in the first line coincides with a corresponding point in the second line, the lines are equal. State axiom.

16. Is it possible for the extremities of two straight lines to coincide when the other parts of the lines do not coincide?

17. Can you state the axiom which your answer suggests?

18. How does the carpenter get a straight line between two points without using a straight-edge? What is the shortest distance between any two points? Do you think your answer self-evident?

19. How many points are necessary to determine the direction of a straight line? How many straight lines can be drawn between the same two points?

20. In how many points can two straight lines intersect? Why? Can you give an axiom for your answer?

DEFINITIONS.

It is of vital importance that the pupil shall be able to give *clear*, *exact* definitions to all terms used in Geometry. Of course, he must fully understand, and be prepared to illustrate any definition given. The questions previously given were designed to lead the pupil so far as possible to formulate his own definitions. But many of the terms in Geometry are difficult to define, and the pupil can compare his definitions with those here given.

Upon these definitions and upon the axioms and postulates rest the demonstrations of the truths of Geometry. But do not mistake the mere learning of these truths to be the *object* of the study. *It is the ability to reason which we acquire—their demonstration.*

1.

A solid is a limited portion of space. Its dimensions are *length*, *breadth*, and *thickness.*

The pupil can conceive space to be divided into a multitude of forms or shapes. Each form pictured is a solid. The geometrical solid contains no *matter;* it is the limited portion of space *conceived by the mind.*

2.

A surface is the boundary of a solid. It divides space into parts and can be conceived without the solid, so the definition is often given: *A surface is that which has length and breadth without thickness.*

(1) A *plane surface* or *plane* is a surface in which if *any* two points are joined by a straight line, every point in the line will lie in the surface.

(2) A surface, no part of which is plane, is called a *curved surface.*

3.

A line is the boundary of a surface. We can conceive the line without the surface and define it to be *that which has length, but neither breadth nor thickness.*

(1) A *straight* line has the same direction throughout its entire length.

(2) A *curved* line is a line no part of which is straight Hereafter the term *line* will be understood to mean *straight line*, and *curve* to mean *curved line*.

(3) *Parallel* lines are everywhere equidistant, or lines which will never meet, no matter how far they are produced.

4.

A point is the extremity of a line. It is also defined as *that which has no dimension, but position only*.

We may further explain points, lines, and surfaces as follows:

(1) The simplest concept that can be formed is of a point which has no magnitude.

(2) A line is described by a moving point. When the point does *not change its direction, a straight line is described;* when the point *constantly changes its direction, a curved line is described.*

(3) A surface may be described by a moving line. [How may a line move and not form a surface?]

(4) When a surface is moved in a certain manner, a solid is generated [How may a plane surface be moved without generating a solid?]

5.

Solids, surfaces, and lines are called *geometrical magnitudes.*

6.

Geometry is the science which treats of *geometrical concepts.*

7.

A *geometrical figure* is any combination of points, lines, and surfaces.

8.

A *plane figure* is one in which all the points and lines lie in the same plane.

9.

Plane Geometry treats of *plane figures.*

10.

Solid Geometry treats of figures all of which are not in the same plane.

11.

A plane angle is the difference in direction of two lines which meet or might meet.

Thus A B C, or C B A, is an *angle* with *sides*, B C and B A, and *vertex* at B. The size of the angle depends upon

the *amount of divergence* of the lines, and not upon their *length.*

(1) When the lines point in exactly opposite directions, the angle is called a *straight angle.* If two lines be drawn from B, as in the figure, two angles are formed, each less than four right angles. Of these angles the smaller is always understood if "the angle at B" is mentioned, unless it is otherwise stated. The two angles at B having the same sides are called *conjugate* angles.

Suppose the line B A continues to revolve about the vertex B until it passes a straight angle and comes around to the line B C, or forms two straight angles, a *perigon* is formed. Angles are usually measured in degrees, minutes, and seconds. A perigon contains 360 degrees. A *right* angle is one-half of a straight angle.

(2) The lines which form a right angle are said to be perpendicular to each other.

(3) A straight angle equals two right angles. An *acute angle* is less than a right angle. An *obtuse angle* is greater than a right angle and less than a straight angle.

(4) If an angle is greater than a straight angle, and less than a perigon, it is said to be *reflex*.

(5) *Oblique angles* are either acute or obtuse; and *oblique lines* are those which are not perpendicular to each other.

(6) The point where the lines which form the sides of an angle meet is called the vertex.

(7) When two angles have the same vertex and a common side, they are called *adjacent angles.*

(*a*) Draw two adjacent angles, both of which are acute. Can their sum be a straight angle? an obtuse angle? an acute angle? a right angle?

(*b*) Draw two right angles which are adjacent. What is their sum?

(*c*) Draw two right angles which have the same vertex, but *are not adjacent.*

(*d*) Draw an obtuse angle and a right angle which are adjacent; compare their sum with a straight angle.

(8) When the sum of two angles is a right angle, each is called the *complement* of the other, and they are called *complementary angles.* Draw two *complementary* angles (1) that are also *adjacent* angles; (2) that are *not adjacent* angles.

(9) When the sum of two angles is a straight angle, each is said to be the *supplement* of the other, and the angles are called *supplementary angles;* or supplementary angles are angles whose sum is a straight angle.

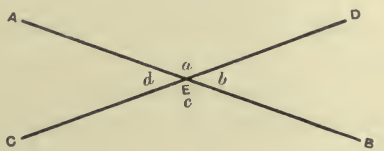
(*a*) Draw two *supplementary adjacent* angles.

(*b*) Draw two supplementary angles which are *not adjacent*, but yet have the *same vertex*.

(*c*) Can you draw two supplementary angles, (1) when both are *acute?* (2) when both are *obtuse?* (3) when one is a *right* angle and the other *acute?* (4) when one is *obtuse* and the other *acute?*

(*d*) What are the conditions under which two angles may be supplementary?

(10) When two lines intersect, the opposite angles are said to be *vertical angles*.



Lines A B and C D intersect at E, forming angles a, b, c, d.

(*a*) Name the pairs of vertical angles? Do they *appear* to be equal? to be complementary? to be supplementary?

(*b*) Select four pairs of supplementary angles. Tell *why* they are *supplementary angles*.

(*c*) Name two *straight* angles.

12.

A proposition is a statement of something to be considered or to be done.

(1) A *theorem* is a proposition stating a geometrical truth.

(2) A *problem* is a proposition requiring something to be done.

(3) An *axiom* is a theorem so elementary that no proof is required.

It is *self-evident* to those who understand the terms used in expressing it.

(4) A *postulate* is a problem so simple that its construction is admitted as possible to be done.

(5) A *corollary* is a theorem whose truth is easily deduced from a preceding proposition.

(6) A *scholum* is a remark upon a preceding proposition.

13.

A *theorem* may be divided into two parts:

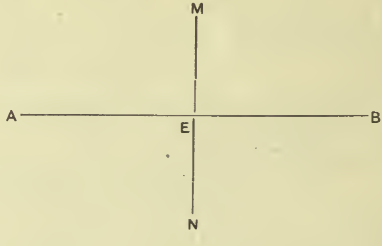
(1) The *hypothesis*, which gives the *data*, or the facts admitted to be true.

(2) The *conclusion*, or that which we *wish to prove must follow* from the facts admitted by the hypothesis; *e. g.*:

If two triangles have two sides and the included angle of the one equal, respectively, to two sides and the included angle of the other, *the triangles are equal.*

What is *known*, or *admitted to be true*, about the two triangles above mentioned? What are we required to *prove* is *true*?

Or, if M N is a perpendicular intersecting A B at E, *then M N is the only line that can be drawn perpendicular to A B at E.*



What is granted to be true? What are we required to prove?

AXIOMS.

14.

(1) Things equal to the same thing, or equal things, are equal to each other.

(2) If the same operation be performed on equals, the results will be equal.

(3) The whole is greater than any of its parts.

(4) The whole is equal to the sum of all its parts.

(5) If equals are added to unequals, the sums will be unequal in the same sense.

(6) If equals are subtracted from unequals, the remainders will be unequal in the same sense.

(7) Equals may be substituted for equals.

(8) Magnitudes whose boundaries coincide are equal.

(9) Two points determine but one straight line.

(10) Two straight lines can intersect in but one point.

(11) A straight line is the shortest distance between two points.

(12) Through the same point but one line can be drawn parallel to a given line.

POSTULATES.

15.

Let it be granted:

(1) That a straight line can be drawn joining any two given points.

(2) That a straight line can be produced to any extent in either direction.

(3) That a circle can be drawn with any point as the center and any finite straight line the radius.

(4) That on the greater of any two lines can be cut off a line equal to the less.

(5) That a figure can be moved unaltered to a new position.

(6) That two equal magnitudes can be made to coincide.

SUGGESTIONS TO PUPILS.

The first step in the solution of a geometrical problem is to study it very carefully to understand the meaning of the language. The second step is the careful construction of a figure which shall afford a clear conception of what we have to do or prove. By constructing your figure carefully, relations between lines, angles, etc., are often suggested which might otherwise escape attention if the figure were carelessly constructed. *Neatness* is conducive to accuracy, while *carelessness* tends to inaccuracy. If your figure suggests certain relations, you are now ready to satisfy yourself whether they are real or apparent.

Your success and progress in solving geometrical problems will depend on your *habit* of watching for new properties that present themselves in various ways. The construction of a figure should be such that it shall not exhibit *apparent* relations not involved in the problem illustrated. That is, lines should not seem equal, or to be at right angles when they are not necessarily so. Triangles should not seem to be isosceles or right-angled unless the conditions of the problem require it. It is better to make a triangle whose angles are about 75°, 45°, 60°, for illustrating in general.

If quadrilaterals are spoken of in a problem, use the trapezium, and not the parallelogram. It is a good plan to draw the figure of the problem in heavy lines, and those used as helping lines more lightly, or in dotted or broken lines. Always *letter* every point of the figure which may be referred to as you proceed with your discussion. Keep the same figure as long as possible. Drawing new figures may distract the attention from a course of reasoning. After having exhausted the properties of the given figure, auxiliary lines may be drawn and resulting properties noted. The most useful auxiliary lines are obtained by—

(1) Joining two given points.

(2) Drawing a line through a given point parallel to a given line.

(3) Drawing a line perpendicular to a given line at a given point within the line, or from a given point without the line.

(4) Producing a line its own length, or the length of another given line.

In preparing your lessons, write the statement of the proposition very carefully, *as you have worked it out*, to bring to class. After it has been corrected, then write it in your book.

PLANE GEOMETRY.

16.

The *demonstration* or *proof* of a theorem must be based upon definitions, axioms, postulates, and previously proved theorems. One of the simplest methods of proof of the equality of two figures is to show that when one is superposed upon the other their boundaries coincide, and the figures are consequently equal, by Axiom 8.

17.

Thus—suppose we wish to prove that

All straight angles are equal.

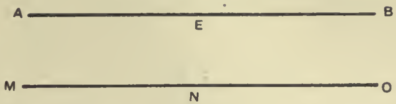
What is a straight angle? *We know what the definition says, nothing more. The pupil must here review the definition until there is no doubt in his mind about what it states, and he can fully illustrate it.*

(1) What is an *angle* in general?

(2) What is a *straight* angle?

After the definition has been *mastered*, let him draw two straight angles and attempt to prove them equal by showing that their sides must coincide when one is placed upon the other. Write the authority for each step in brackets after each statement. Compare your proof with that given below and see if yours fails in any essential point.

Theorem. All straight angles are equal.



Given: A E B and M N O, any two straight angles.

Required: To prove that angle A E B equals angle M N O.

Proof:

(1) Superpose ∠ A E B upon ∠ M N O so that point *E* will fall upon point *N* and side E B will take the direction and coincide with N O. [§15, Post. 5—Any figure can be moved unaltered to a new position.]

(2) Side E A will take the direction of side N M. [§11, 1—A straight angle is an angle whose sides point in exactly opposite directions.]

(3) ∴ ∠ A E B = ∠ M N O. [§14, 8—Magnitudes which can be made to coincide are equal.] But ∠ A E B and ∠ M N O were given *any* two straight angles; ∴ we conclude that *all* straight angles are equal.

18.

Cor. All right angles are equal.

(The proof follows directly from the definition of a right angle.)

The sections and exercises are numbered consecutively throughout the entire book.

N. B.—*The proof of every section and exercise in the book is required of the pupil.* When considered too difficult for the average pupil, partial proofs and suggestions are given to assist him. But he *should never refer to these hints* unless he has *first exhausted* his own resources to discover a *proof of his own.*

TRIANGLES.

19.

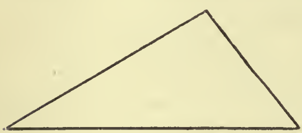
Def. A *triangle* is a portion of a plane bounded by three straight lines. Each triangle has three *sides* and three *angles*. The *vertices* of the angles of the triangle are called the *vertices of the triangle*. The sum of all the sides is called the *perimeter*.

20.

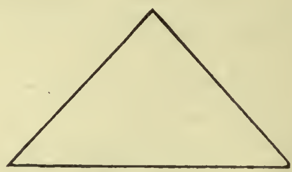
Triangles are classified in two ways; viz., (1) with respect to *sides;* (2) with respect to *angles*.

21.

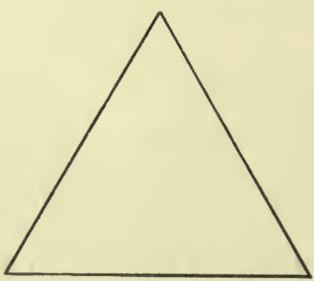
The *Scalene triangle* has no sides equal; the *Isosceles triangle* has two sides equal; and the *Equilateral triangle* has all sides equal.



Scalene.



Isosceles.



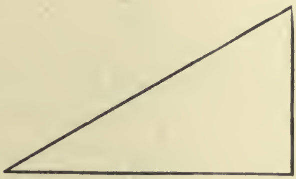
Equilateral.

22.

A triangle is called *Acute* when all angles are *acute; Obtuse* when one angle is *obtuse;* and *Right* when one angle is a *right* angle.

Acute.

Obtuse.



Right.

(1) Draw an *obtuse-isosceles* triangle.

(2) Draw a *right-isosceles* triangle.

(3) Is an *equilateral* triangle *acute?*

23.

In the scalene triangle A B C, produce A B to E; then angle C B E is called an *exterior angle.*

(1) Form the exterior angle by producing (*a*) A C, (*b*) B C, (*d*) B A, (*c*) C A, (*e*) C B. *Redraw* the triangle each time. Define an *exterior angle.*

(2) Draw a triangle and then draw the exterior angle, which shall equal the adjacent interior angle. Classify the triangle.

(3) Draw a triangle in which one exterior angle is acute. Classify the triangle.

(4) Which classes of triangles always have *all* exterior angles obtuse angles?

24.

The *base* of a triangle is the side upon which it is assumed to stand. *Any* side may be considered the base.

25.

The angle opposite the base is called the *vertical* angle.

Which is the vertical angle of triangle A B C when A B is assumed the base? when B C? when A C?

26.

The vertex of the *vertical angle* is called the *vertex of the triangle.*

27.

The *altitude* of a triangle is the perpendicular distance from the vertex of the triangle to its base, or its base produced.

Can you draw the altitude of triangle A B C when C is the vertex? A? B? [Show the three drawings. Make C an obtuse $\angle$, and have the $\triangle$ scalene.]

Show the three altitudes of a *right* triangle. [How many are drawn?]

28.

In a right triangle the side opposite the right angle is called the *hypotenuse.*

Does the altitude of a right triangle ever fall without the base? Which side is assumed the base when the altitude falls within the base?

29.

Obtuse triangles and acute triangles are called *oblique* triangles.

30.

The three lines drawn from the vertices of the triangle to the middle points of the opposite sides are called the *medians* of the triangle.

Draw triangle A B C and draw its three medians.

31.

The three lines bisecting the angles of the triangle are called the *bisectors* of the angles of the triangle.

In what classes of triangles do the *bisectors* of the angles and *medians* appear to be the same lines? Observe closely these triangles and state what you discover. What do you observe about the intersection of the three *medians* of any triangle? the three *bisectors* of the angles?

BOOK I.

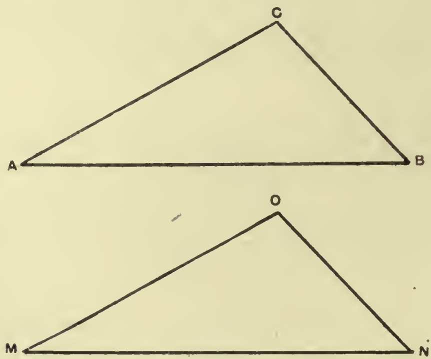
32.

PROPOSITION I.

When are two angles equal? two triangles? any two magnitudes?

Draw a horizontal line and with compasses cut off equal parts.

Construct a triangle, A B C, making length of A B, 8, of A C, 6, and of B C, 4, of the equal parts.



Then construct a triangle, M N O, making M N equal to A B, angle at M equal to angle at A, and side M O = A C. Will the remaining angles and side of △ M N O be equal to corresponding angles and side of △ A B C? You may test the accuracy of your answer with compasses, but Prop. I. will state a general truth about all triangles, and the *proof* depends upon a course of reasoning, wherein the only authority we may give are definitions, axioms, and postulates.

Make: (1) M N = A B; (2) ∠ M = ∠ A; (3) M O = A C. Then draw N O.

Clearly fix in mind the parts of the triangles which *are known* to be *equal.* Can you place one upon the other in such a way that they must coincide? By what authority can you do this? If you place △ M N O upon △ A B C, upon what point of △ A B C will you place point M? What direction will you let M N take? Where must point N fall? What axiom proves your answer? Will you fold △ M N O above A B or below? If above, will M O take the direction of A C? By what axiom? Will O fall on C? Why? Must the line N O *wholly* coincide with B C? Give authority.

Are the △s then equal? [Auth.]

Again review the sides and angles that were *known* to be equal.

Does N O = B C? [Auth.]

Does ∠ N = ∠ B? [Auth.]

Does ∠ O = ∠ C? [Auth.]

Now let us formally prove the general truth, or *theorem*, that the drawing and reasoning lead us to declare. The theorem makes a statement, not about *these particular triangles*, but about *any two triangles* in which the given *conditions are known to exist.* The reasoning does *not depend upon the mechanical accuracy of the drawing.* We state that M N = A B, M O = A C and ∠ M = ∠ A, *not because they were so constructed* (in fact, their *geometrical* construction depends upon *problems* not yet studied), but because they are *so given as the conditions upon which we base our demonstration.*

The order of arrangements should be as given below:

I. *General enunciation* of the *theorem.*

(1) Hypothesis.

(2) Conclusion.

II. *Particular enunciation* of the figure drawn.

(1) Hypothesis.

(2) Conclusion.

III. *Proof.*

Note carefully the steps in the following demonstration:

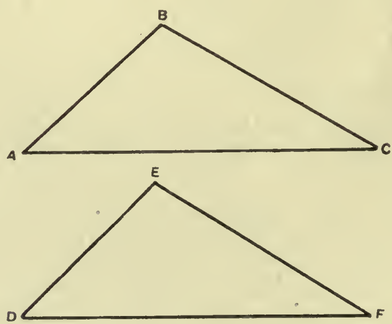
DEMONSTRATION OF PROP. I. (MODEL FOR PUPILS.)

I. Theorem. If any two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other, the triangles are equal.

II. Given: △ A B C and △ D E F, in which
(1) A B = D E,
(2) A C = D F, and
(3) ∠ B A C = ∠ E D F.

Required: *To prove* that △ A B C = △ D E F.

III. Proof:



(1) Superpose △ A B C upon △ D E F so that point A shall fall upon point D, and A C shall take the direction of D F, and fold △ A B C above the line D F. [§ 15, 5, Postulate—A figure can be moved unaltered to a new position.]

(2) A C = D F; [Hypothesis.]

∴ C will fall on F. [§ 15, 6 (Post. 6)—Equal magnitudes can be made to coincide.]

(3) ∠ A = ∠ D; [Hyp.]

∴ A B will take the direction of D E. [§ 15, 6.]

(4) A B = D E; [Hyp.]

∴ B will fall upon E. [§ 15, 6.]

(5) C falls upon F, and B falls upon E; [P. P.]

∴ B C coincides with E F. [§ 14, 9—Two points determine but one straight line.]

(6) A C coincides with D F, [P. P.]
A B coincides with D E, [P. P.] and
B C coincides with E F; [P. P.]

∴ △ A B C = △ D E F. [§ 14, 8—Magnitudes whose boundaries coincide are equal] Q. E. D.

Let the pupil carefully prepare written proof of Prop I.; use the above figure, but assume ∠s B and E and the including sides respectively equal.

33.

CIRCLES—Definitions.

(1) A *circle* is a plane figure bounded by a curved line every point of which is equidistant from a point within, called the *center*.

(2) The curved line which bounds the circle is called the *circumference*.

(3) Any part of the circumference is called an *arc*.

(4) A straight line which joins the ends of an arc is called a *chord*.

(5) A *radius* is a straight line which joins the center to any point in the circumference.

(6) The longest possible chord passes through the center, and is called the *diameter*.

(7) *Theorem.* Circles which have equal radii, or equal diameters are equal; and *conversely*, if circles are equal, they have equal radii and equal diameters.

Pupil will give the proof.

[*Hint.*—Use method of superposition.]

Definitions—Homologous Parts of Equal Figures.

34.

A *polygon* is a plane figure bounded by straight lines.

35.

A *quadrilateral* is a polygon having four sides.

36.

A *parallelogram* is a quadrilateral having its opposite sides parallel.

37.

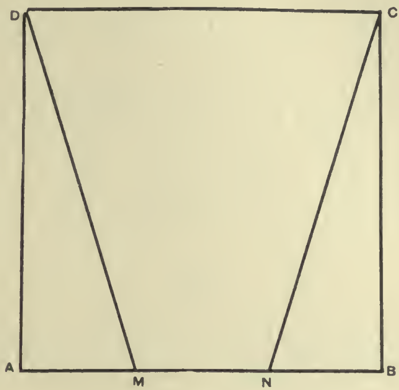
A *rectangle* is a parallelogram having right angles.

38.

A *square* is a rectangle having equal sides.

Review Prop. I. (Quote it.)

Hereafter we can prove that two triangles are equal by showing that two sides and the included angle of the one are equal respectively to two sides and the included angle of the other, and then quoting Prop. I.; *e. g.*, if we have *given:*



(1) The square A B C D,

(2) A M = B N,

we can prove △ A D M = △ B C N;

(1) ∵ A D = B C, [?]

(2) ∠ A = ∠ B, [?] and

(3) A M = B N; [Hyp.]

(4) ∴ △ A D M = △ B C N. [Prop. I.]

It is *not necessary* to superpose again and show that the triangles must coincide. That would be re-proving Prop. I., which is a *previously proved proposition* (note abbreviation "P. P.") and can be cited as authority just as we cite axioms, definitions, and postulates.

So we have proved the above triangles equal, using only four steps, but we do *not know* which angles correspond and are equal; we must not say angle at D equals angle at C because it *appears* to be true. The reasoning does not at all depend upon the *appearance* of figures. It is possible for the sides to be so nearly equal that we could not tell whether ∠ at D equaled angle at C or at N; furthermore the drawing might be inaccurate and not agree with the conditions [hypothesis], and the appearance would greatly mislead us. We look for the equal angle by first looking for the sides which are *given equal.* A M is given equal to B N, and the *angles opposite these equal sides are equal;* $\therefore$ in $\triangle$s A D M and B C N $\angle$ at D = $\angle$ at C.

Also A D was proved equal to B C; hence the angles opposite these equal sides are equal; $\therefore$ $\angle$ M = $\angle$ N.

In the same way we can prove D M = C N, being opposite the equal angles A and D respectively.

We have the definition given below, in § 39.

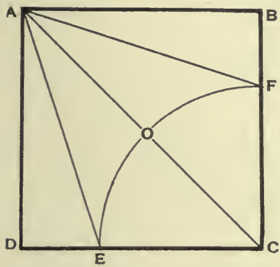
39.

Def. In equal figures, *equal sides* are opposite the angles which are known to be equal, and *equal angles* are opposite the sides which are known to be equal. Or, in equal figures, the *homologous parts* are equal. *Homologous* sides are opposite equal angles and homologous angles are opposite equal sides.

TRIANGLES—EXERCISES.

The exercises in this book are given for two reasons: (1) To give the pupil facility in making deductions from data given. (2) To afford abundant application of this previously proved proposition. Let the pupil draw just so much of the figure as is required in each exercise. In Ex. 2 the pupil who has mastered Prop. I. and who fully understands the discussion of *homologous parts of equal figures* will be able to prove many triangles, angles and sides equal. An exercise upon a *previously proved* proposition is called a *rider.* Let the pupil see how many *riders* he can make by answering the questions asked in Ex. 2.

Exercises.



1. Given a square and one of its diagonals, what can you prove?

2. Given the square A B C D and the arc E O F with radius C E and center C, E and F being points in the lines C D and C B, respectively, and O being in the diagonal A C. Draw F A and E A. (1) How many lines, $\angle$s, and $\triangle$s can you prove equal? (2) By joining fixed points, how many pairs of $\triangle$s can you prove equal?

40.

Proposition II.

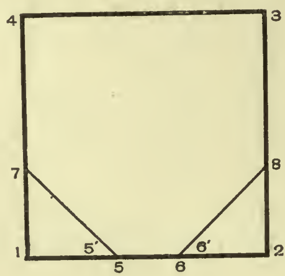
Draw 2 oblique $\triangle$s, A B C and D E F, making A B $=$ D E, $\angle$ D $=$ $\angle$ A, and $\angle$ E $=$ $\angle$ B.

Can you prove these $\triangle$s equal? Use method of superposition. Write Prop. II.

[*Hint.*—Superpose $\triangle$D E F upon $\triangle$A B C so that D will fall upon A and D E will take the direction of A B. Must E fall upon B? Can you now show that F *must* fall upon C?

Must D F take the direction of A C? Why *must* F fall upon A C or A C produced? Must E F take the direction of B C? *Must* F also fall upon B C or B C produced? Must F then fall upon two lines or those lines produced? What point is *common* to A C and B C? How many points in common is it possible for two lines to have?]

Exercises.



3. Given the square 1 2 3 4, 1 5 = 6 2, and ∠ 5′ = ∠ 6′. What conclusions can you draw?

4. Draw diagonals. Using what you have just proved, what new △s can you *prove* equal? What new lines and angles can you prove equal?

5. Join 5 and 4, 6 and 3. What new △s can you prove equal? What new lines and angles can you prove equal?

6. Draw an isosceles △, A B C, with A B and A C the equal sides. Lay off B D on B A, and lay off C K = B D on C A. Draw C D and B K. (*a*) Why is A D = A K? (*b*) What ∠ in △ A B K is equal to an ∠ in △ A C D? Draw separately the pairs of △s which appear equal, placing them in similar positions. (*c*) In △s A B K and A C D, what pairs of sides are equal? (*d*) Why is △ A B K equal to △ A C D? (*e*) What homologous or corresponding parts are equal as a result? (*f*) Can you prove other pairs of △s equal?

41.

Proposition III.

What have you learned about the base angles of any isosceles △?

Now we wish to prove this truth by Demonstrative Geometry.

Draw any isosceles △, A B C, making A B and A C the equal sides. Produce A B and A C. Measure off on A B produced B E, and on A C produced C D equal to B E. Join points E and C. What 2 new △s have been formed? Join B and D. What other 2 new △s have been formed which appear to be respectively similar and equal to the first 2 △s?

Can you prove the pair of larger △s equal?

[*Hint.*—What ∠ is common to both △s? How does A E compare with A D? Why? If you are still unable to prove the △s equal, redraw them, placing them in similar positions.]

Note carefully the sides and ∠s in the pair of smaller △s which are common to the larger △s. Now prove the pair of smaller △s equal in all their parts.

What have you proved about ∠ A B D and ∠ A C E?

What have you proved about ∠ D B C and ∠ E C B?

What axiom can you apply to show that ∠ A B C equals ∠ A C B? Write a general statement and call it Prop. III.

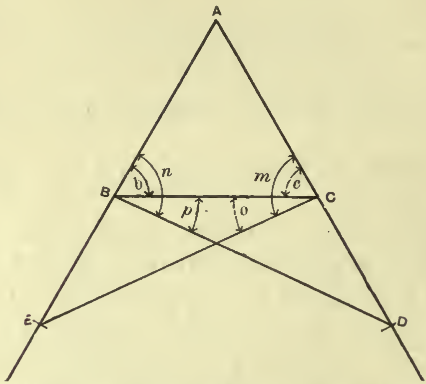
Let the pupil carefully prepare written proof of Prop. III. and then compare with the demonstration given below.

MODEL DEMONSTRATION.

Theorem. If a triangle is isosceles, the angles opposite the equal sides are equal.

Given: The isosceles △ A B C, in which A B = A C.
Required: To prove that ∠ C = ∠ B.

Proof: Produce the equal sides A B and A C, and upon these sides produced lay off the equal lines B E and C D.



Join E to C and D to B, forming △ E C A and △ D B A, also △ B C E and △ D B C.

I. Prove △ E C A = △ D B A, and consequently ∠ *m* = ∠ *n*.

(1) A B = A C, [Hyp.]

(2) B E = C D; [Construction.]

(3) ∴ A B + B E = A C + C D, [§ 14, 2—If the same operation be performed on equals the results will be equal.]

(4) A E = A D. [§ 14, 7—Equals may be substituted for equals.]

(5) A C = A B, [Hyp.]

(6) ∠ A is common to both triangles;

(7) ∴ △ E C A = △ D B A, [§ 32—If two triangles have two sides and the included angle, etc.] and ∠ *m* = ∠ *n*. [§ 39—Being homologous angles of equal △s opposite the equal sides, A E and A D.]

II. Prove △ B C E = △ D B C, and consequently ∠ *p* = ∠ *o*.

(1) B E = C D, [So constructed.]

(2) E C = D B, [§ 39—Being homologous sides of the equal triangles A C E and A B D, opposite the common angle at A.]

(3) ∠ E = ∠ D; [§ 39—Opposite the equal sides A D and A E, respectively.]

(4) ∴ △ B E C = △ B D C, [§ 32.] and ∠ *p* = ∠ *o*. [§ 39—Being opposite the equal sides D C and B E.]

III. (1) $\angle m = \angle n$, [P. P.]
(2) $\angle o = \angle p$; [P. P.]
(3) $\therefore m - o = n - p$, [§ 14, 2.] or $\angle C = \angle B$. [§ 14, 7.] Q. E. D.

EXERCISES.

7. Can you prove that an equilateral $\triangle$ is equiangular?

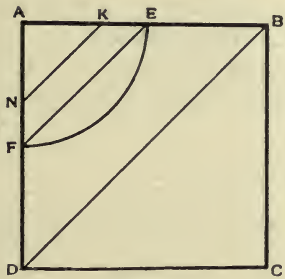


FIG. 1.

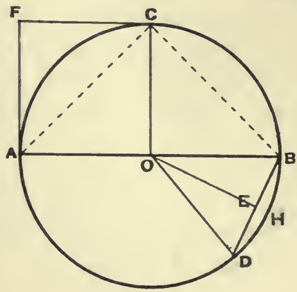


FIG. 2.

8. Given the square A B C D. Draw B D, and make D N = B K. Draw arc E F with center A and radius A E.

Prove (1) $\angle$ A B D = $\angle$ A D B; (2) $\angle$ A K N = $\angle$ A N K; (3) $\angle$ A E F = $\angle$ A F E.

9. In the second figure, A B is diameter of $\odot$; A O C F is a square erected on radius A O; B D is a chord, and E its middle point. Prove, (1) A C = C B; (2) $\angle$ O A C = $\angle$ O C A; (3) $\angle$ A C B = sum of the $\angle$s C A B and C B A; (4) $\angle$ O B D = $\angle$ O D B; (5) The sum of the $\angle$s O C B and O D B = $\angle$ C B D.

10. Construct A B C an equilateral $\triangle$, and A B D an isosceles $\triangle$, on the same base, A B. Prove the $\angle$ C A D = $\angle$ C B D, whether the $\triangle$s are on the same side or on opposite sides of A B. (1) Join D and C. Produce D C, if necessary, until A B is cut. Is A B bisected? Prove. Make further deductions.

11. If x and y are the middle points of the equal sides A B and A C, respectively, of the isosceles $\triangle$ A B C, prove in two ways that C x = B y. Make deductions. E and F are points in the base B C, and B E = C F. Prove A E = A F.

42.

Proposition IV.

Draw two acute-angled $\triangle$s, A B C and D E F, so that A C = D F, A B = D E, B C = E F. Place $\triangle$ D E F on $\triangle$ A B C so that E F falls on B C and vertex D falls opposite to vertex A. Join A and D. How many isosceles $\triangle$s are formed? Can you prove $\angle$ D equal to $\angle$ A? What follows?

Draw again, making $\angle$s at E and B obtuse $\angle$s. What follows? Write a general statement and call it Prop. IV.

43.

Definition.—A right bisector of a line meets it at right angles and bisects it.

44.

Proposition V.

Draw any straight line, A B, and fix two points, P and Q, equidistant from the ends of A. B.

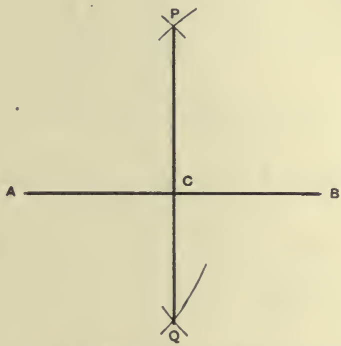
Do these two points determine the right bisector of A B?

Draw P Q and *if necessary* produce it to meet A B at C. Can you prove the angles at C are right angles? What is a straight angle? a right angle? Can you prove A B is divided into two equal parts? Can you state this proposition?

If you fail to state Prop. V., or to prove it, do not get discouraged (the proof is difficult for the beginner), but carefully study the "hints" given below. *Master each step before reading the next*, and just so soon as you discover the proof, finish it *without reading further* until after you have carefully prepared your own proof *in full.* Ability to originate a *single step* in a demonstration will give you power with which to attack future demonstrations. Struggle, repeatedly and with determination, for this *power to originate.* It is the *highest order of mental achievement.*

[*Hint.*—Let A B be the given line and let P and Q be two points equidistant from the ends.]

To prove that P Q, or P Q produced, is the rt. bisector of A B.



Let P Q intersect A B at C. Think of the definition of a rt. bisector. We must prove what lines are equal to prove that A B is bisected? We have learned to prove lines equal in what ways? If B C is superposed on C A, can we prove that the lines *must* coincide and are consequently equal? If not, can we prove B C and C A *similar sides of equal figures*, and are consequently equal? Draw two lines which will form two triangles of which B C and C A are sides, respectively. Do you *know* any sides and angles of these triangles equal? Are you able to prove the triangles equal? (Think of the previous propositions you have proved, and just what is necessary to prove the triangles equal by using Prop. I., or Prop. II., or Prop. IV.) If you do not *know sufficient* sides and angles of these triangles equal to prove the triangles equal by either of the previously proved propositions, you must form *other triangles* and try to prove *them* equal.

(Note well just what *was lacking* to prove the above triangles equal—was it not the *angles at* P, or at Q?)

If you need the angles at P equal in order to prove

△ A C P = △ B C P, by joining other points you will have two *new triangles* which contain these angles at P. Now if you can prove the new triangles, A P Q and B P Q, equal (redraw figure on new page if necessary), you can then prove angles at P equal, and then the triangles A C P and B C P equal, and finally the sides B C and C A equal.

[The pupil must understand each step in the order presented. Review until you clearly see just *why each step is necessary*.]

Then B C = C A, [P. P.] and A B = B C + C A; [Ax. ?]

∴ A B is bisected by P Q, and P Q is the bisector of A B.

But it is required to prove that points P and Q determine the *right* bisector; hence it is necessary to prove angles A C P and B C P *right* angles. What *are* right angles? (See the definition.)

What is the sum of two adjacent angles at C?

Can you prove them equal?

Carefully prove Prop. V. Compare with proof given to Prop. III. if necessary.

Exercises.

12. Letter the intersection of A D and B C, in § 42, O. What pairs of equal △s in the figure? Give the equal homologous or corresponding parts resulting. What angles at O are equal?

13. If 2 oblique lines are drawn from the same point in a perpendicular cutting off equal distances from the foot of the ⊥, how are these two lines related?

45.

Proposition VI.

Problem. To bisect a given straight line.

[What proposition treats of the bisection of a line? What is necessary to bisect it? How can you find the necessary points?]

Given: A B a straight line.

Required: To bisect A B.

Construction: [Let the pupil construct the figure.]

(1) With centers A and B and any radius greater than one-half of A B, describe arcs which intersect at *C* and *D*.

(2) Draw C D, cutting A B at E.

Then A B is bisected at E.

Proof:

(1) A C = B C, [?]

(2) Also A D = B D; [?]

(3) ∴ A B is bisected by C D at E. [?]

Q. E. F.

46.

Proposition VII.

Problem. To bisect a given angle.

[Draw an angle and bisect it. The construetion has been learned in Constructional Geometry, but it is now required to *prove* that the angle has been bisected. You are required to prove angles equal; review the propositions in which this is done. By joining fixed points, construct two triangles which contain the angles. Prove these triangles equal. N. B.—There are *two* pairs of triangles which may be drawn.]

Given: The angle A B C.

Required: To bisect A B C.

Construction: [Let the pupil construct the figure.]

(1) With center B and any radius less than B A or B C,

describe an arc cutting B A and B C at D and E respectively.

(2) With centers D and E and any radius greater than one-half of D E, describe arcs which intersect at F, remote from B.

(3) Draw B F, forming angles A B F and C B F. Thus A B C is bisected by B F.

Proof: Left to the pupil.

47.

Proposition VIII.

Problem. To draw a perpendicular to a given line at a given point in it.

Given: (1) The line A B; (2) P, a point in A B.

Required: To draw a perpendicular to A B at P.

Construction: Left to the pupil.

Proof: Left to the pupil.

If unable to construct and prove the above problem, consult the "hint" below.

[*Hint.*—What proposition enables you to draw a perpendicular to a given line? But at which point in the line does it enable you to draw the perpendicular? Is P that point? If *not*, cut off of A B a line of which P *is* the middle point. Be careful to fix no unnecessary points.]

48.

Proposition IX.

Problem. To erect a perpendicular to a line from a given point without that line.

The hypothesis, construction, and proof are left for the pupil.

If unable to make the construction, consult the "hint" below.

[*Hint.*—Why do you *not* wish to draw the perpendicular to the mid-point of the given line? But it is *only by having* *two points equidistant from the ends of some line* that you can draw a perpendicular. How then can you cut off a *part of the given line* of which P is equidistant from the ends? After cutting off *this part*, you can easily fix another point equidistant from its ends; and thus you have a line and two points equidistant from its ends; and P is one of these two points.]

EXERCISE.

14. Divide a line into four equal parts; into sixteen equal parts.

49.

PROPOSITION X.

Problem. At a given point in a given straight line constrnct an angle equal to a given angle.

Left to the pupil.

EXERCISES.

15. Construct an isosceles triangle: (1) When base and one side are given; (2) when the angle at the vertex and one side are given; (3) when an angle at the base and the base are given.

16. If a line is drawn from the vertex of an isosceles △ bisecting the base, prove that it is ⊥ to the base.

17. Construct a △ when 2 sides and the included ∠ are given.

18. Construct a △ equal to a given △. Show three ways.

19. Let A B C be an isosceles △, A B = A C. Let bisector of ∠ B meet A C in x and the bisector of ∠ C meet A B in y. Show that B x = C y.

20. If, in Fig 2, Ex. 7, a point in the circumference, H, is equidistant from B and D, and H is joined to O, B, and D, what △s are equal? What lines must coincide?

21. Draw any two intersecting ⊙s; join the centers and join each center to points where circumferences intersect. Can you prove the equalityof the △s formed? Can you form other equal △s by joining the points where circumferences intersect?

[*Suggestion.*—Draw ⊙s in all possible positions. Seek for exceptions to your answers.]

50.

PROPOSITION XI.

How many perpendiculars may be erected to a given line at a given point in that line? (The lines are assumed to be in the *same plane*.)

The answer at first seems self-evident, and it appears hardly necessary to *prove* that there can be but *one*.

All the proofs given hitherto have been *direct* proofs, but here we use an *indirect* method of proof. *We suppose that what we wish to prove true is false* and then show that *our supposition leads to an absurd conclusion.* This is one of the *indirect* methods of proof, and is called "*reductio ad absurdum*," a reduction to an absurdity.

[State the proposition.]

Given: (1) the line A B; (2) P, a point in A B; (3) C P E, a ⊥ to A B at P. [Pupil draw the figure.]

Required: To prove that C P E is the only ⊥ that can be drawn to A B through P.

Proof: Draw any other line through P, F P D, and suppose, *if possible*, that F P D is another ⊥ to A B at P.

[Let the pupil draw the lines, making F P D a dotted line and let C and F be *above* the line A B, and place F at the right of C.]

(1) ∵ ∠ C P B is a rt. ∠; [?]

(2) also, ∠ F P B is a rt. ∠; [?]

(3) ∴ ∠ C P B = ∠ F P B. [?]

But this is *absurd* for ∠ C P B *is greater* than ∠ F P B; [Ax. ?]

∵ the supposition that F P D was another ⊥ led us to an *absurd conclusion* (No. 3, where a *part is proved equal to the whole*), we must reject the supposition and conclude that F P D is *not* a ⊥ to A B at P; and *since F P D is any other line than C P E* (*the true* ⊥), we conclude that there is *no other* ⊥ and that C P E is the only ⊥.

[The pupil should carefully review each step of the proof until he understands the full force of the reasoning.]

51.

Proposition XII.

Draw two adjacent angles. How many lines are used? Draw, using three lines, two lines.

If two lines meet so as to form two adjacent angles, what is their sum compared with a right angle?

State the proposition and give the proof.

[*Hint.*—The proof is not difficult, but be careful not to use any statement which is not given in previous proposition, or axiom, or definition.]

52.

Cor. I. What is the sum of all the angles that can be formed on the same side of a given straight line at a given point in that line?

State and prove.

53.

Cor. II. What is the sum of all the angles that can be formed in the same plane around a given point?

State the corollary.

Given: (1) Point P; (2) angles a, b, c, d, etc., formed around P.

Required: To prove the sum of angles a, b, c, d, etc., is four right angles.

Proof: (The pupil should fix point P and draw any number of angles, a, b, c, d, etc., so that any side produced will *not coincide* with any other side. Then produce any side, and use Cor. 1.)

Questions.

1. Are all straight angles equal?

2. What is the complement of an angle? the supplement?

3. If an angle is double its complement, what fraction is it of a right angle? of a straight angle?

4. If an angle is three times its supplement, what fraction is it of a straight angle? of a right angle?

5. What are supplementary adjacent angles? How large is an angle whose supplement is three times its complement?

6. What is the angle between the bisectors of two supplementary angles? What is the angle between the bisectors of two complementary adjacent angles?

7. What is the hypothesis of a theorem? the conclusion? What is the hypothesis of Prop. III? the conclusion. If the hypothesis and conclusion of Prop. III were interchanged, how would it be stated? Do you think it is true?

54.

Definition.—The converse of a theorem is the theorem when the hypothesis and conclusion are interchanged.

55.

Proposition XIII.

If two adjacent angles are supplementary, how are their exterior sides related?

State and prove Prop. XIII.

[*Hint.*—What is a straight angle?]

Is Prop. XIII the *converse* of any proposition?

56.

Proposition XIV.

Given the isosceles △ A B C with A B and A C the equal sides. Join A with the middle point of the base B C. Prove the 2 △s formed equal.

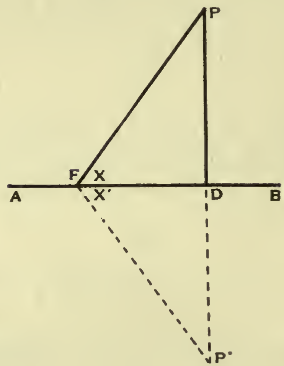
How does the median meet the base? Why? How does it divide the triangle A B C? How does it divide the angle at the vertex?

Write a formal statement of these three truths and call it Prop. XIV.

Erect a ⊥ at the mid-point of the base of any isosceles △. Through what point must it pass?

57.

Proposition XV.



Let P be a point without the straight line A B, and P D be a ⊥ to it. Suppose it is possible to draw another ⊥ from P and let P F be that ⊥. Can you think of any axiom which this supposition violates?

Produce P D to P′ making D P = D P′. Join F and P′. Can you prove F P′ equal to F P?

If P F is ⊥ to A B, how large is ∠ x? What is the value of ∠ x + ∠ x'? What kind of a line is P F P′? But what axiom is violated if our supposition is true?

Can P F be ⊥ to A B?

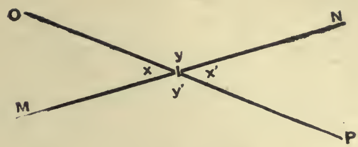
Can any other line than P D be drawn from P ⊥ to A B? Why?

What then must we say of the *supposition* which led to this absurd conclusion?

Write a formal statement of the truth discovered and call it Prop. XV.

58.

PROPOSITION XVI.



Let the two straight lines O P and M N intersect at I, forming the vertical angles x and x', y and y'.

What is the value of $\angle y + \angle x$? Quote P. P.

What is the value of $\angle y + \angle x'$? Quote P. P.

Can you prove that $\angle y = \angle y'$?

Prove in a similar manner that $\angle x = \angle x'$.

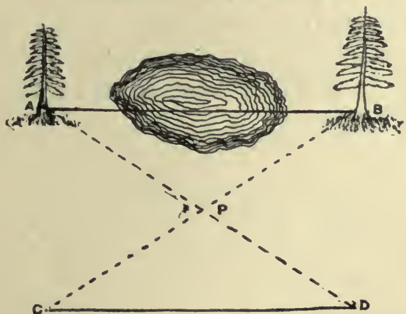
Write Prop. XVI.

EXERCISES.

22. If $\angle y$ and $\angle y'$ in §58 are bisected, prove what is true of the bisectors.

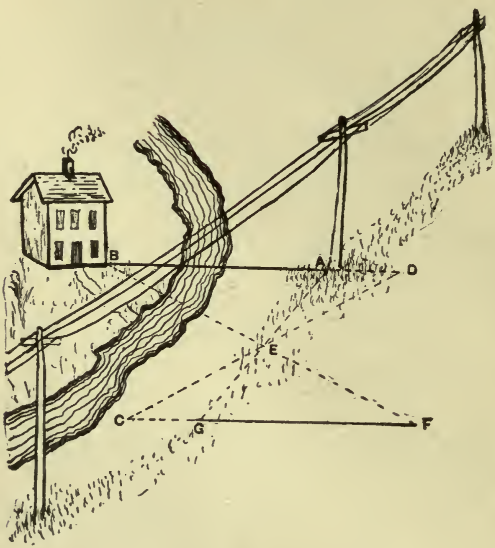
23. If $\angle x$ and $\angle y$ are bisected, prove what is true of the bisectors. What can you say of the bisector of $\angle x$ produced through $\angle x'$?

SURVEYOR'S PROBLEM.



Suppose a surveyor wishes to know the distance between A and B, with a lake between. Suppose that the distance

A B is not over 300 feet and that the surrounding land is level; tell how to measure the distance required by the hint in diagram, and prove your work.



It is required to find the distance from A to B, with a river between. Supposing the surrounding land to be level and that the surveyor has a transit which measures angles and which enables him to run straight lines; can you tell from the suggestion in the diagram how to estimate the distance required?

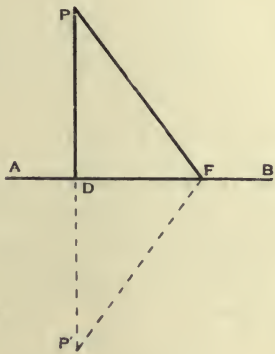
59.

PROPOSITION XVII.

Given P a point outside of the line A B.

What is the shortest line that can be drawn from P to A B?

If P D is the shortest line, how should it be drawn?



[Try to state and prove Prop. XVII. Use the above figure.]

(If you fail, consult the "hint" given below.)

[*Hint.—Given:* P D a ⊥ to A B at D.]

Required: To prove P D the shortest line from P to A B.

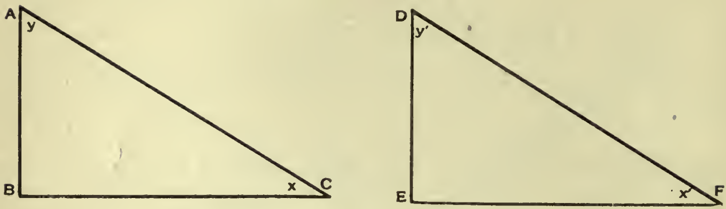
Proof: Draw any other line, P F. Produce P D its own length to P′. Draw P′ F.

P P′ < P F P′. [Auth.]

P D < P F. [Auth. *Pupil must prove P′ F = P F, etc.*]

60.

Proposition XVIII.



Given the right $\triangle$s A B C and D E F having the hypotenuse A C of the first $\triangle$ equal to the hypotenuse D F of the second $\triangle$, and $\angle x = \angle x'$.

Can you prove the $\triangle$s equal?

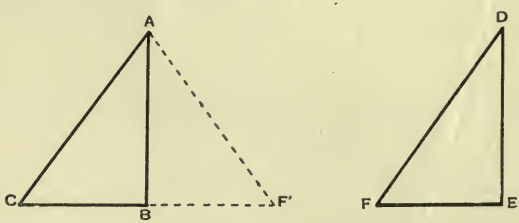
If not, superpose $\triangle$ D E F upon $\triangle$ A B C so that D shall fall on A and D F take the direction of A C. Where will F fall? Why? What direction will F E take? Why? Will E fall on B? Why? If not, how many $\perp$s would be drawn from point A to the same line?

Suppose $\angle y = \angle y'$ and that we know nothing of angles x and x'; prove the $\triangle$s equal.

Write the formal statement of the truth proved and call it Prop. XVIII.

61.

Proposition XIX.



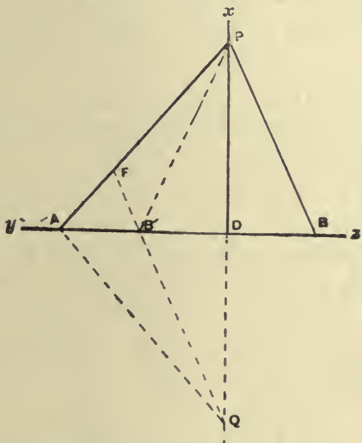
Given the right $\triangle$s A B C and D E F with A C = D F and A B = D E.

Can you prove the $\triangle$s equal?

Place $\triangle$ D E F on $\triangle$ A B C so that D will fall on A and let D E take the direction of A B and let F fall on F′. Must E fall on B? Why? Will C B and F E form one straight line? Why? Will C A F′ form a $\triangle$? Why? Prove $\angle$ F $=$ $\angle$ C. Write Prop. XIX.

62.

PROPOSITION XX.



Given: (1) Any line $y\ z$; (2) x D $\perp$ to $y\ z$ at D; (3) P any point in the $\perp$ x D; (4) P B and P A lines cutting off of $y\ z$ the unequal distances D B and D A, D B being less than D A.

Can you prove PB less than PA?

Produce P D and make D Q $=$ D P. Make D B′ $=$ D B. Join Q to B′ and to A. Produce Q B′ to F. Prove B′Q$=$B′P, and A Q $=$ A P. How does the broken line P B′ Q compare with P D Q? Prove P F B′ $>$ P B′. Prove P F B′ Q $>$ P B′ Q. Can you prove Q A P $>$ Q F P?

How does Q A P compare with Q B′ P?

How does A P compare with B′ P?

How does A P compare with B P?

Write the general truth proved and call it Prop. XX.

Exercises.

Given: Construct the following triangles:

24. Two angles and the included side.

25. Two sides and the angle opposite one of them. Discuss, showing under what conditions the construction is possible.

26. Three sides. Discuss.

27. Hypotenuse and one adjacent angle of a rt. △. Discuss.

28. Two legs of a rt. △.

29. One side and an adjacent acute angle of a rt. △.

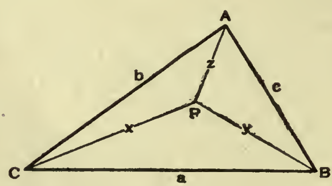
30. Hypotenuse and one leg of a rt. △. Discuss.

31. Prove the side of a △ is greater than the difference of the other two sides.

32. How are the altitudes to the equal sides of an isosceles triangle related? Prove your answer.

63.

Proposition XXI.



Given △ A B C and P any point within the △. Compare $x + y$ with a. Compare $x + z$ with b. Compare $y + z$ with c.

Can you now prove that $x + y + z > \frac{1}{2}(a+b+c)$?

Write the general truth and call it Prop. XXI.

PARALLELS.

64.

PROPOSITION XXII.

How many straight lines can be drawn through a given point parallel to a given straight line? Is your answer self-evident?

[See § 14, 12.]

If two straight lines in the same plane are ⊥ to to the same line, can you prove these two lines ||?

If they are not ||, can you show that any P. P. is violated?

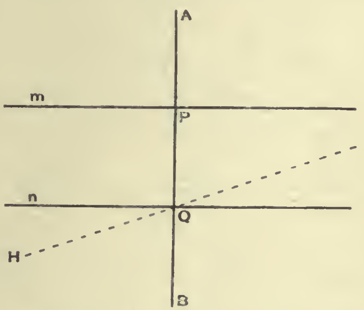
Write the formal statement of the truth proved and call it Prop. XXII.

65.

PROPOSITION XXIII.

In how many ways can you draw a line through a given point parallel to a given line?

Show them and try to prove them.



Given *m* and *n* 2 || lines, and A B ⊥ to *m* at P.

Is A B ⊥ to *n*?

If we *suppose* that *n* is not ⊥ to A B at Q, draw H I through Q ⊥ to A B How is H I related to *m?* But how is *n* related to *m?* What then is true of the line H I and *n?* Write the proposition and call it Prop. XXIII.

Exercise.

33. Draw from any point a ⊥ to 2 || lines.

66.

Proposition XXIV.

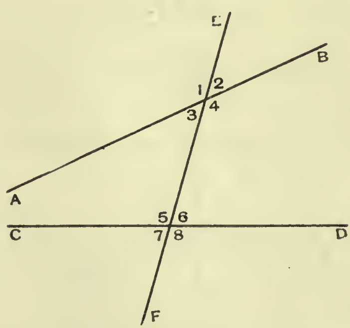
Given E F and C D || to A B.

Can you prove E F and C D ∥ to each other?

[*Hint.*—Draw a ⊥ to A B.]

Write the general statement and call it Prop. XXIV.

67.



Given any 2 lines, A B and C D, cut by a third line, E F How many angles are formed? If there were three lines cut by E F, how many ∠s would be formed?

E F is called a *transversal* or *secant* line.

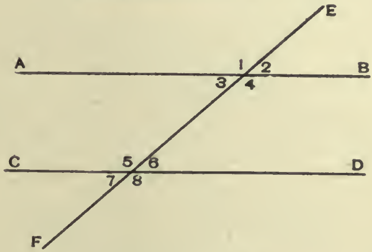
Write carefully a definition of a transversal.

Name the vertical ∠s. Which ∠s lie between the lines cut? These are called interior ∠s.

Which are the *exterior* ∠s? Which are the interior ∠s on the same side of the transversal? Which are the *alternate interior* ∠s? Which are the *alternate exterior* ∠s? ∠ 2 and ∠ 6 are called *corresponding* ∠s. Name the other corresponding ∠s.

68.

Proposition XXV.



Given A B and C D ||, cut by E F. Bisect that part of the line E F included between A B and C D. From the mid-point draw a ⊥ to A B. Extend the line until it reaches C D. How is this line related to C D? How are the 2 △s related?

How is ∠ 3 related to ∠ 6? How is ∠ 4 related to ∠ 5? What kind of ∠s are 3 and 6? ∠s 4 and 5? Can you write a general statement?

Call it Prop. XXV.

69.

Cor. I. How are ∠ 2 and ∠ 6 related under § 68? What are the other pairs of corresponding ∠s? Are they equal? Can you state the general truth?

70.

Cor. II. Compare ∠ 1 and ∠ 8, also ∠ 2 and ∠ 7.

71.

Cor. III. What is the sum of ∠5 and ∠6? of ∠4 and ∠6? of ∠3 and ∠5?

72.

PROPOSITION XXVI.

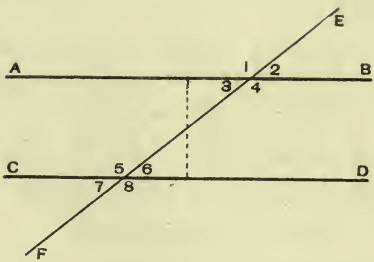
Problem. To draw through a given point a line parallel to a given line. [Use § 64.]

EXERCISE.

34. Draw through the vertex of a △ a line parallel to the base.

73.

PROPOSITION XXVII.



Given 2 lines, A B and C D, not known to be ||, cut by the third line, E F, so that ∠4 = ∠5. What other ∠s are equal? What pairs of ∠s = 2 right ∠s?

Note carefully how many deductions may be made when ∠4 = ∠5.

Construct another figure like the one above. Erect a ⊥ to A B from the mid-point of that part of the transversal included between the lines A B and C D. Produce this ⊥ to C D. How are the 2 △s related?

Can you now show how A B and C D are related?

Write a general statement of the truth proved and call it Prop. XXVII.

74.

Cor. I. State and prove the converse of § 69.

75.

Cor. II. State and prove the converse of § 70.

76.

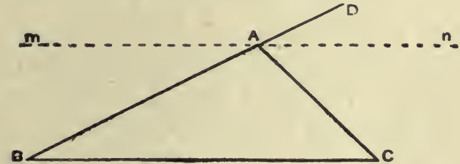
Cor. III. State and prove the converse of § 71.

Construct Prop. XXVI., using § 73, etc.

TRIANGLES.

77.

PROPOSITION XXVIII.



Given the △ A B C. Produce any side B A forming the exterior ∠ D A C. Draw *m n* || to B C.

Can you prove the exterior angle equal to the sum of the angles B and C?

Does this prove a general proposition? State it. Call it Prop. XXVIII.

[Let the pupil prove when A B is produced.]

78.

Proposition XXIX.

What is the value of the sum of the $\angle$s of any $\triangle$?

Prove your answer. See figure in Prop. XXVIII.

Write Prop. XXIX.

Exercises.

35. Why can a $\triangle$ not have 2 right $\angle$s? What is the sum of the 2 acute $\angle$s of a right $\triangle$? What is the relation of the two acute $\angle$s of a right $\triangle$?

36. If 2 $\angle$s of a $\triangle$ are equal respectively to 2 $\angle$s of another $\triangle$, how do the third $\angle$s compare?

37. If in 2 right $\triangle$s an acute $\angle$ of one equals an acute $\angle$ of the other, how are the other acute $\angle$s related?

79.

Proposition XXX.

Quote the propositions that prove 2 right $\triangle$s equal to each other.

Construct a right scalene $\triangle$ A B C, having the right $\angle$ at B. Construct another right $\triangle$ A′ B′ C′, having the right $\angle$ at B′ and the side A′ B′ equal to side A B and $\angle$ at A′ = $\angle$ at C.

Can you prove the triangles equal?

Is it possible to draw a right $\triangle$ A′ B′ C′ equal to the right $\triangle$ A B C when only one side and one acute $\angle$ of the first equals a side and an homologous acute $\angle$ of the second?

Can you discover a new proposition about the equality of 2 right $\triangle$s?

Write a clear statement of the truth discovered and call it Prop. XXX.

80.

PROPOSITION XXXI.

Given the △ A B C with ∠ A = ∠ B.

Is the △ isosceles?

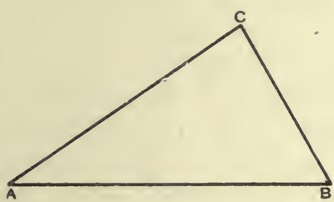
[*Hint.*—Drop a ⊥ from C to the base A B.]
Write Prop. XXXI.

EXERCISE

38. Prove that an equiangular △ is equilateral.

81.

PROPOSITION XXXII.



Given △ A B C, in which ∠ B > ∠ A.

Compare the sides opposite the unequal ∠s. Are they equal. If not, which is *greater?*

Prove your answer.

[*Hint.*—At B and above A B construct an ∠ equal to ∠ A having A B for one side. Must the side fall within B C? Why? Letter point of intersection with A C, D. Does A D = B D? Does A D C = B D C? Is B D C > B C? Give reason for each step.]

5—

82.

Proposition XXXIII.

Given $\triangle$ A B C, in which A C > B C.

Compare the $\angle$s opposite the unequal sides.

(1) Either $\angle$ B = $\angle$ A,

(2) or $\angle$ B < $\angle$ A,

(3) or $\angle$ B > $\angle$ A.

Show that (1) and (2) are impossible; hence (3) must be true.

This proof is *based on the doctrine of exclusion.*

83.

Proposition XXXIV.

Draw two $\triangle$s, A B C and D E F, making sides A B and A C respectively equal to sides D E and D F, but the $\angle$ at A greater than the $\angle$ at D. Now place $\triangle$ D E F upon $\triangle$ A B C so that D falls upon A and D E takes the direction of A B. Where must E fall? Why? Where must D F fall? Why? Does the third side of the second $\triangle$ *appear* to be as long as the third side of the first $\triangle$? Our problem requires us to prove which is the longer. If we bisect $\angle$ C A F and letter the new point on B C, G, and join F to G, we have two new $\triangle$s, G A C and G A F. Are they equal? Why? Does G F = G C? Why?

Can you now prove which is greater, B C or E F? Write Prop. XXXIV.

84.

Proposition XXXV.

What is the converse of Prop. XXXIV.?

Try to prove this by supposing—(1) that the included $\angle$s are equal; (2) that the included $\angle$ in the first $\triangle$ is less than the included $\angle$ in the second $\triangle$. Write Prop. XXXV. in the notes.

PARALLELOGRAMS.

What is a quadrilateral?

What are its diagonals? What is meant by the angles of a quadrilateral?

Make a quadrilateral the angles of which shall each be less than a straight angle.

Can you make a quadrilateral having an angle greater than a straight angle?

How does this figure differ from the first?

What name is given to a quadrilateral whose opposite sides are parallel? if only two sides are parallel? if no two sides are parallel?

What are the bases of a parallelogram? of a trapezoid?

Note.—Remember that all rectangles are parallelograms, but all parallelograms are not rectangles. A rectangle should not be used in proving properties about parallelograms. Why?

85.

A *diagonal* of a polygon is a line joining the vertics of two angles not adjacent.

86.

A *trapezium* is a quadrilateral having no sides parallel.

The "kite" trapezium has two pairs of equal sides and each angle less than a straight angle. The "arrow" trapezium has two pairs of equal sides and one of its angles is reflex or greater than a straight angle. Draw trapeziums, illustrating each class.

87.

A *trapezoid* is a quadrilateral having only two sides parallel.

1. The non-parallel sides are called the *legs* of the trapezoid.

2. The parallel sides are called the *bases*.

3. Draw a trapezoid in which the *legs* are equal. It is called an *isosceles* trapezoid. Draw a trapezoid containing a rt. ∠.

88.

A *rhomboid* is a parallelogram having adjacent sides unequal and angles oblique.

89.

A *rhombus* is a parallelogram having adjacent sides equal and angles oblique.

90.

Definition.—An *intercept* is a line or a part of a line intercepted between two other lines.

91.

PROPOSITION XXXVI.

Review the definition of a parallelogram. Remember that in any figure which is given a parallelogram you must assume no relations of the sides and angles which are *not warranted in the definition.*

Given: The parallelogram A B C D and its diagonal A C.

Are the triangles into which the parallelogram is divided by the diagonal, equal?

[Prove when diagonal B D is drawn.]

92.

Cor. I. Prove that the opposite sides of a ▱ are equal.

93.

Cor. II. Prove that the opposite angles of a ▱ are equal.

Question: State the converse of Prop. XXXVI. Is it always true?

94.

Cor. III. Prove that parallel intercepts between parallel lines are equal.

EXERCISES.

39. *Given:* Parallelogram A B C D, in which ∠ B is a rt. ∠. Make deduction and prove.

40. *Given:* Parallelogram A B C D, in which ∠ D is a rt. ∠ and side A B = side B C. Make deduction and prove it.

41. Draw the bisectors of two opposite angles of a parallelogram. *Prove that these bisectors are parallel.*

(If you fail, see "hint" below.)

[*Hint.*—Produce bisectors until sides of the parallelogram are cut and prove the new figure a parallelogram.]

95.

Proposition XXXVII.

Draw a parallelogram and both diagonals.

Compare the parts of each diagonal.

Make deduction. State and prove Prop. XXXVII.

Exercises.

42. Given the rhomboid A B C D and the diagonals A B and B D intersecting at E; also A B > B C.

(1) Compare the four △s. State and prove which are equal.

(2) Compare ∠s C E B and B E A. Prove deductions.

(3) Why *must* the diagonals of a rhomboid be unequal?

43. Draw a rhombus and both diagonals.

(1) Compare the ∠s at intersection.

(2) Compare the four triangles.

(3) Why must the diagonals be unequal?

Prove all deductions made.

44. Draw a rectangle and both diagonals.

(1) Compare the diagonals.

(2) Compare the four triangles in order.

(3) Make a deduction which is true of each of the four triangles.

(4) Compare the rectangle with the rhomboid and state points of *difference* and *similarity*.

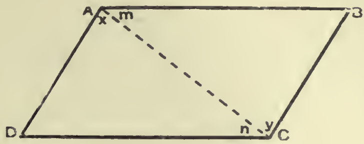
45. Draw a square and its diagonals.

(1) Make and prove deductions.

(2) Make careful comparison of square and rhombus.

96.

Proposition XXXVIII.



Given: 1. The *quadrilateral* A B C D and the diagonal A C, forming ∠s x, m, n and y.

2. Also (1) A B = D C, and (2) A D = B C.

Is the quadrilateral a parallelogram?

State and prove Prop. XXXVIII.

(If you fail, see "hint" below.)

[*Hint.*—Prove △s equal; consequently ∠m = ∠n and A B parallel to D C. Then prove B C parallel to A D, and tell why the figure is a parallelogram.]

97.

Proposition XXXIX.

Given the quadrilateral A B C D, in which A D = B C and A D is parallel to B C.

Can you prove the figure a parallelogram?

(If you fail, see "hint" below.)

[*Hint.*—Draw either diagonal and use method given in the "hint" to §96.]

98.

Proposition XL.

Draw two parallelograms A B C D and A′ B′ C′ D′, in which any two adjacent sides and the included angle of the one equal respectively two adjacent sides and the included angle of the other.

Are the parallelograms equal?

[N. B.—*Equal* figures can be made to coincide in all parts. Figures may be *equal in area*, or equivalent, which are not *equal*.]

State and prove Prop. XL.

(If you fail, consult "hint" below.)

[*Hint.*—Superpose A′ B′ C′ D′ on A B C D so that the equal sides and the included $\angle$s will coincide. Then show that the opposite sides and the remaining vertex must coincide. See § 14, 12, and § 14, 10,]

Exercises.

46. If the diagonals of a quadrilateral bisect each other, what is the figure? Distinguish in the above when the diagonals are (1) equal, (2) unequal.

47. If the diagonals of a quadrilateral are (1) equal, (2) unequal, and bisect each other at right angles, what deductions can you make?

Questions.

1. If the diagonals of a quadrilateral are equal, can you make and prove any deduction?

2. If the opposite angles of a quadrilateral are equal, what deduction can you make and prove?

EXERCISES.

48. By drawing the diagonals A C and A′ C′ in § 98 can you prove Prop. XL. in another way?

49. Given the △ A B C, in which ∠ A plus twice ∠ B minus three times ∠ C equals 110°, and ∠ A minus twice ∠ B plus ∠ C equals 90°; find ∠s A, B, C.

50. Given two rectangles having the base and altitude of one respectively equal to the base and altitude of the other; show how they compare in area.

51. What is the value of the sum of all the angles of a parallelogram?

99.

PROPOSITION XLI.

Problem. To construct a parallelogram when two adjacent sides and the included angle are given.

EXERCISES.

52. Construct a rectangle when a side is given and the adjacent side is 2¾ times the given side.

53. Construct a square when a diagonal is given. [Show two ways.]

54. Construct a rhombus when the diagonals are given.

55. Construct a rhombus when one diagonal is given and the other is 3⅛ times it.

56. Construct a "kite" trapezium when the diagonals are given. Discuss the possible lengths and position of the diagonals.

57. Construct an *isosceles* trapezoid when a diagonal and distance between the bases are given. *Discuss.*

POLYGONS.

What is a polygon? its perimeter? Compare the "arrow" trapezium with the "kite" trapezium. What is a *convex* polygon? a *re-entrant angled* or *concave* polygon? What is a *regular* polygon? the *angle* of a regular polygon? the *angle at the center?* What is an exterior angle of a polygon?

What are the diagonals of a polygon? How are polygons classified?

100.

Polygons are classified according to the number of sides. A polygon of three sides is a *triangle*, or trigon; of four sides is a *quadrilateral;* of five sides is a *pentagon;* of six sides is a *hexagon;* of seven sides is a *heptagon;* of eight sides is an octagon; of nine sides is a nonagon; of ten sides is a decagon; of eleven sides is an undecagon; of twelve sides is a dodecagon.

101.

The sum of the sides of a polygon is called the *perimeter*.

102.

A *convex polygon* is one in which no side produced will enter the polygon.

103.

A *concave polygon* is one in which at least two sides when produced enter the polygon. The angle whose sides produced enter the polygon is called a *re-entrant angle*.

104.

A regular polygon is both equilateral and equiangular. When the term *polygon* is used, a *convex* polygon is meant.

105.

PROPOSITION XLII.

Into how many triangles may we divide any polygon by joining any vertex to all other vertices when there are four sides? five sides? six sides? n sides?

What is the sum of all the interior angles of a polygon of four sides? five sides? six sides? n sides?

The sum of the interior angles of a polygon is equal to as many straight angles as —

Finish the above statement and prove it. Call it Prop. XLII.

106.

Cor. I. If the sides of a polygon are produced in order, the exterior angles thus formed are equal to two straight angles.

[*Hint.*—What is the sum of the interior angles? of the interior and exterior angles taken together?]

107.

Cor. II. If a polygon is equiangular, each interior angle is equal to as many straight angles —

Finish and prove.

[*Hint.*—What is the sum of the $\angle$s of any triangle? Then what is the size of each angle of an equiangular triangle? What is each angle of an equiangular quadrilateral? etc.]

Write the fraction which represents the number of st. $\angle$s in each $\angle$ of a polygon of n sides.

108.

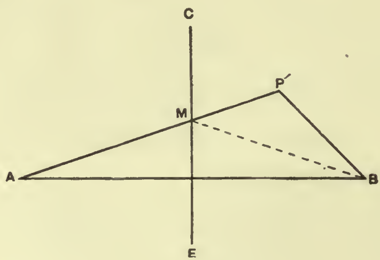
PROPOSITION XLIII.

Let C E be the right bisector to the line A B; *i. e.*, a $\perp$ at the mid-point of A B. Take any point P in the right bisector and join it with the extremities of A B.

(1) Compare P A and P B.

Take any point P′ outside the right bisector and join it with the extremities of the line A B.

(2) What can you prove concerning P′ A and P′ B?



(If unable to prove the second part of the above, see "hint" below.)

[*Hint.*—Let P′ be without and to the right of the right bisector C E.

Draw P′ B and P′ A;

To Prove P′ B $<$ P′ A.

Since P′ is to the right of the right bisector, P′ A will intersect it; letter the point of intersection M.

Draw M B.

M A $=$ M B, [?]

And M B $+$ M P′ $>$ P′ B, [?]

And M A $+$ M P′ $>$ P′ B, [?]

Or A P′ $>$ P′ B. [?]

Let the pupil take P′ to the left of the right bisector and prove P′ B and P′ A are unequal.

109.

Distance from a point to a line is measured by a perpendicular drawn from the point to the line. The distance between parallel lines is a perpendicular from any point in one line to the other line.

110.

Proposition XLIV.

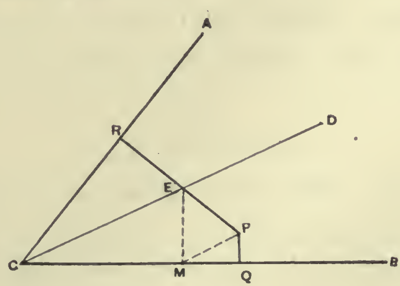
I. Given any angle A C B and its bisector D C; also P any point in the bisector.

Compare the distances from P to the sides of the sides of the angle, C B and C A

[*Hint.*—How do you measure distance from a point to a straight line? Draw the required lines and compare the △s formed.]

II. Given any angle A C B and its bisector D C; also P any point *without* the bisector.

Compare the distances from P to the sides of the angle, C A and C B.



Let P be below the bisector C D. Draw ⊥s, P Q and P R, to the sides of the angle, C B and C A respectively.

Then prove P Q < P R.

Since P is below C D, P R will intersect C D; letter point of intersection E.

Draw E M $\perp$ to C B;

E R = E M. [?]

Also draw M P;

: M P > P Q. [?]

But E M + E P > M P; [?]

$\therefore$ E M + E P > P Q, [?]

And E R + E P > P Q, [?]

Or P R > P Q. [?]

Let the pupil take any point above the bisector C D and prove that the distances to the sides of the $\angle$ are unequal.

Write Prop. XLIV., the proof of which includes both of the above proofs.

111.

State and prove the converse of Prop. XLIII.

112.

State and prove the converse of Prop. XLIV.

113.

Proposition XLV.

Theorem. The three right bisectors of the sides of any triangle meet in a common point, which is equidistant from the vertices of the triangle.

114.

Proposition XLVI.

Theorem. The three bisectors of the angles of any triangle intersect in a common point which is equidistant from the three sides of the triangle.

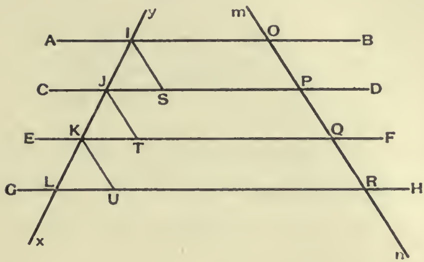
115.

When three or more lines intersect in a common point they are said to be *concurrent*.

TRANSVERSALS.

116.

PROPOSITION XLVII.



Let A B, C D, etc., be || lines cut by the transversal *x y* so that the parts intercepted, I J, J K, etc., are equal. Now draw any other transversal *m n* Do the parts intercepted by the ||s on *m n appear* to be equal?

Through points I, J, etc., draw lines || to *m n*. Does I S = O P? Why? Make further deductions. Can you prove I S = J T = K U?

[*Hint.*—Are △s I S J, J T K, etc , equal? Why?]

What then can you prove about parts intercepted on *m n?* Do you discover any general truth?

Carefully retrace the steps by which you reach the conclusion and write Prop. XLVII.

117.

PROPOSITION XLVIII.

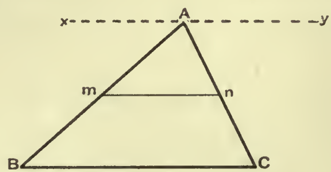
Problem. To divide a line into n equal parts.

(See "hint" if necessary.)

[*Hint.*—Let *n* equal 5 or some known number. Draw an indefinite line making an angle with the given line at either end. Lay off, beginning at the vertex, 5 equal parts on the indefinite line and join the remote end of the last part with the other end of the given line. Parallel to this line draw through the points of division other lines cutting the given line. Prove that the parts intercepted by the parallels are the required equal parts.]

118.

PROPOSITION XLIX.



Suppose A B C any △. Through m, the middle point of any side, A B, draw a line || to the side B C, cutting A C at n. Through A draw $x\,y$ || to B C. How many || lines have you? Why?

(1) Does A n appear to be equal to n C? Can you prove your supposition?

(2) What part of B C does $m\,n$ appear to equal?

Do you see any way to prove your answer? State Prop. XLIX. There are *two* parts.

(See "hint" below if you *fail.*)

Let the pupil use either of the other sides for the base and prove the above proposition.

[*Hint.*—Draw through n a line parallel to A B.]

119.

Proposition L.

(1) If the middle points of two sides of a triangle are joined, will the line joining these points *be parallel to the third side?*

[*Hint.*—Through one of the mid-points draw a line || to the third side. Prove the lines must coincide.]

(2) What part of the base does the line joining the mid-points of the sides of a triangle equal?

Write the two truths as Prop. L.

120.

The line joining the mid-points of the legs of a trapezoid is called the *median* of the trapezoid.

121.

Draw any trapezoid, and through the mid-point of either leg draw a line parallel to either base.

Cor. I. (1) How does this parallel meet the other leg? (2) Compare the length of the parallel to the sum of the bases.

Write the two truths discovered as Cor. I. to Prop. L.

122.

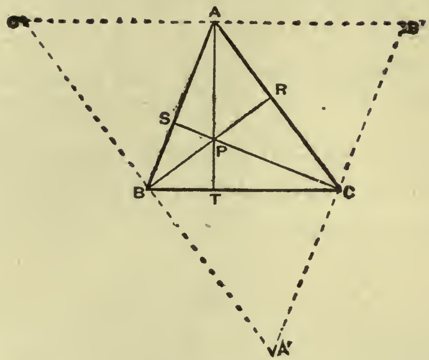
Cor. II. Draw any trapezoid and its median.

What relations does the median bear to the two bases?

Prove your answer. Write the two truths discovered as Cor. II., Prop. L.

123.

PROPOSITION LI.



I. Given the acute △ A B C with the altitudes B R, C S, A T. Through the vertices C, A, B draw lines A′ B′, B′ C′, C′ A′, respectively, || to the sides A B, B C, C A.

In the figure A B C B′, how are A B′ and B C related? Why? How are A B and B′ C? Why?

What figure is A B C B′? Prove in similar manner A C′ = B C. How are A B′ and A C′ related? [Ax.]

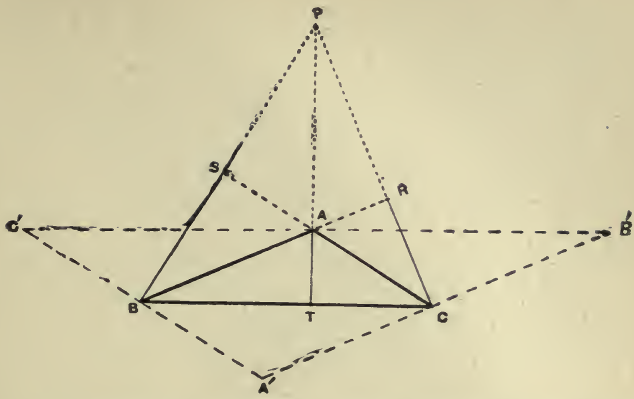
Prove C′ B and B A′ each equal to A C. Compare B C′ and B A′.

Prove B′ C and A′ C equal to each other.

What are A T, C S, and B R to the △ A′ B′ C′?

By P. P. what have we proven concerning these three lines? But what are these three lines of the △ A B C?

What then can you prove of the altitudes of any acute triangle?



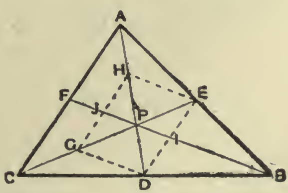
II. Construct an oblique △ with two of the altitudes falling without the △. Prove that the altitudes are concurrent.

EXERCISE.

58. The ⊥s and the ⊥s produced from two opposite vertices of a rhomboid to the sides opposite, or to those sides produced, form with the sides of the rhomboid and the sides produced, two rectangles and two rhomboids.

124.

PROPOSITION LII.



Given the oblique △ A B C and the medians A D, B F, and C E.

Let P be the point of intersection of any two medians, C E and A D, and take H the middle of A P, and G the middle of C P. Draw H G, G D, D E, E H.

In the △ A B C, how is E D related to A C? How does E D compare in length with A C? Why? In △ A P C, how does H G compare with A C? Compare H G and E D. What is the figure H E D G? Why? Compare H P, P D, and G P, P E. Compare D P with A D and E P with E C. How do the two medians A D and C E cut each other?

Since A D and C E are any two medians, how will B F cut C E or A D?

What two truths have we discovered about the three medians of any triangle?

State these truths formally and number the Prop. LII.

Exercises.

59. Join the middle points of the sides of a △. What new figures do you get? Prove how they are related to the original △.

60. How does the median drawn to the hypotenuse of a rt. △ compare with the hypotenuse?

[*Hint.*—Draw the rt. △ and the required median. Draw through mid-point of the hypotenuse a line parallel to the base. How will this parallel meet the altitude of the rt. △? Compare the parts of the altitude. Do you not now see the relation of the median to the hypotenuse?]

61. One angle of an isosceles △ is 60°. What can you prove about the △?

LOCI.

What have you learned about every point in the line bisecting a given $\angle$? How then could you find a line which contains all the points equidistant from the sides of a given $\angle$?

Can you find a ball when it is known to be on the ground just 20 yards from the foot of a given tree? Can you find a line which contains all the points which are a given distance from a given point?

Can you find two lines which contain all the points which are a given distance from a given line?

Can you find a line which contains all the points equidistant from the extremities of a given line?

126.

Definition.—The place, line, or system of lines which contains all the points, and only those points, which satisfy a given condition, is called the *locus of the point.*

Exercises.

What is the locus of the point:

62. Equally distant from two given points?

63. Equally distant from two given straight lines—(*a*) When the lines intersect? (*b*) When the lines are parallel?

64. Equally distant from the extremities of a given st. line?

65. Equidistant from three given points, A, B, C, which are not in the same st. line?

66. A, B, C are three towns on a st. road. B is two miles north of A and four miles south of C. Find a point equidis-

tant from A and C and four miles from B. Is there only *one?*

67. What is the locus of the vertex of an isosceles △ having a given base?

Each locus requires a *geometrical proof; e. g.*, in Ex. 62—to prove the answer, "The locus of a point equidistant from two given points is the right bisector of a line joining the given points," fix the points, draw the line joining them, and the right bisector. Then select *any* point in the locus (the right bisector) and prove that it is equidistant from the given points. Then select any point without the locus and prove that it is not equidistant from the points.

In general we may say that when we prove a theorem concerning the locus of points, it is necessary to prove two things:

(1) That all points in the locus satisfy the given conditions.

(2) That any point not in the locus does not satisfy the given conditions.

Can you tell why both of the above proofs are necessary?

BOOK II.

CIRCLES.

Define (1) a circle, (2) a circumference, (3) an arc, (4) a chord, (5) a radius, (6) a diameter.

Are all radii and all diameters of the same or equal circles equal?

127.

A *segment* of a circle is a part cut off by a chord. It is bounded by an arc and the chord cutting it off.

128.

A *semi-circle* is one-half of a circle. A *semi-circumference* is one-half of the circumference. A semi-circle is bounded by a diameter and a semi-circumference.

129.

A *sector* is a part of a circle bounded by an arc and two radii.

130.

A tangent to a circle is a line which touches it, but will not enter the circle, no matter how far the tangent is produced.

The *point of tangency* is the point where the line touches the circle.

Two *circles* are *tangent* when their circumferences touch but do not cut each other. They may be tangent *internally* when one is wholly *within* the other; or tangent *externally* when one is wholly *without* the other.

131.

A *secant* is a line which cuts the circumference in two points.

132.

A *central angle* is an angle formed by two radii.

133.

(1) An *inscribed angle* is an angle whose vertex is a point in the circumference, and whose sides are chords.

(2) An *inscribed polygon* has its vertices in the circumference of the circle.

134.

Proposition I.

Which chord divides the circle into two equal parts?

Prove your answer. State Prop. I.

135.

Proposition II.

Draw a circle and any chord not the diameter.
Draw a radius perpendicular to the chord.

Compare the parts of the chord.

State and prove Prop. II.

136.

Cor. I. Draw a circle and a chord. Draw the right bisector of th chord.

Will it pass through the center of the circle?

State and prove Cor. I., Prop II.

[*Hint.*—If you fail to prove the above, consult § 111.]

Exercise.

68. Construct a chord when the circle and the mid-point of the chord are given.

(See "hint" below *if you fail.*)

[*Hint.*—Draw the radius through the given point; then draw a ⊥ to the radius at that point.]

137.

Proposition III.

(1) Draw in equal circles two equal central angles.

Do the arcs which subtend the equal angles *appear* to be equal?

[How did you prove that circles having equal radii or equal diameters are equal? How then can you prove arcs equal?]

(*Subtend* means *to be below*, or *under*, or *to be opposite; e.g.*, *arcs* are said to *subtend central* angles, and chords *subtend arcs*.)

(2) *Again*, in equal circles take equal arcs.

Do the central angles *appear* equal?

State the truths discovered as one proposition, the second part being the converse of the first. Prove Prop. III.

138.

Cor. I. Draw a circle and a chord. Draw the right bisector of the chord.

Does it pass through the center? [Auth.]
Does it bisect the central angle?

Prove your answer.

Does it bisect the subtended arc?

Prove it.
State the truths discovered as one corollary.

Exercise.

69. (1) Bisect a given arc. (2) Bisect a given angle, not as in § 46.

139.

Proposition IV.

Given two equal circles and two equal arcs.

Compare the chords which subtend the arcs.

Again, given two equal circles and two equal chords.

Compare the arcs subtended.

State and prove Prop. IV.

Exercise.

70. Construct and prove § 49 again, using a method you were unable to use when first given.

140.

Proposition V.

Problem. To find the center of a given circle.

[*Hint.*—See § 135.]

Exercise.

71. Find the center of a circle when only an arc is given. [Can you find the center when only a chord is given?]

141.

Proposition VI.

Problem. To construct an arc equal to a given arc.

142.

Proposition VII.

How do you find the distance from a chord to the center of the circle?

Draw two equal circles and two equal chords.

Compare the distances of these chords to the centers of the circles.

State and prove the converse of the above.

Write both conclusions as Prop. VII.

143.

Proposition VIII.

Given in equal circles unequal arcs, each being less than a semi-circumference.

Compare the central angles.

State and prove the converse.

144.

PROPOSITION IX.

Given in equal circles unequal arcs, each being less than a semi-circumference.

Compare the chords subtending the arcs.

[Which has the greater central angle? Compare the triangles formed.]

State and prove the converse.

[Use doctrine of exclusion.]

145.

PROPOSITION X.

How many circumferences can be made to pass through any three points which are not in the same straight line?

[*Hint.*—To discover a solution, suppose the problem solved. Draw a ⊙ and consider any three points on the circumference the given points. Study the figure. How could you replace the circumference were it erased? Could you find the center of the ⊙? What proposition, corollary, or exercise declares that certain lines pass through the center? Have you the data necessary to draw the lines which must pass through the center?]

146.

Cor. I. In how many points can two circles intersect?

Why?

EXERCISES.

72. A circle cannot have two centers.

73. If two circles have a common center, they must coincide.

74. How many equal lines can be drawn from a point to a line?

75. If from any point in a circle two equal lines are drawn to the circumference, the bisector of the angle formed must pass through what point?

76. The right bisectors of the sides of an inscribed polygon must pass through what common point?

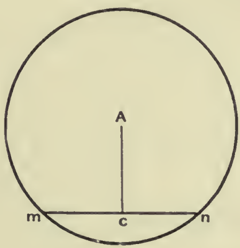
77. A radius perpendicular to a side of an inscribed equilateral triangle isbisected by the side.

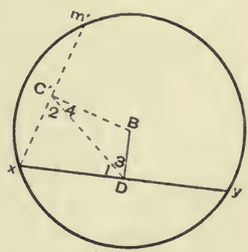
147.

Proposition XI.

Given the equal circles, A and B and the unequal chords $m\,n$ and $x\,y$; $m\,n < x\,y$.

Which chord is nearer the center?





(If unable to answer and prove, study the figure. If still unable, see the "hint.")

[*Hint.*—Draw ⊥s A C and B D. Where are C and D? Which is longer, M C or X D? Why? Construct X M = M N. Draw ⊥ B C′. Compare B C′ with B D. Which is greater, ∠ 1 or ∠ 2? Then which is greater, ∠ 3 or ∠ 4? Can you now tell which is greater, B D or B C?]

State and prove the converse of the above. Write the above theorem and its converse as Prop. XI.

Exercises.

78. Through a given point within a ⊙ construct the shortest possible chord.

79. Given two chords which intersect and which make equal ∠s with the radius drawn through the point of intersection. Do the chords appear to be equal? Can you prove them equal or unequal?

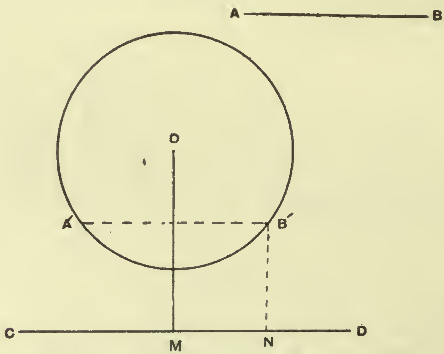
[*Hint.*—Draw radii ⊥ to the chords.]

80. In a given ⊙ construct a chord ∥ to a given line and equal to another given line. Use lines of different lengths and in different positions. When is this impossible?

[*Hint.*—Draw a ⊥ from center to the line to which the chord is to be parallel. Imagine the required chord drawn, and study how to fix the extremity of the chord.]

If you fail to construct the chord, study the following suggestions:

Given the ⊙, O, and A B, the chord, and C D the line to which the chord is to be parallel.



Imagine *A′ B′* the required chord.

It is parallel to ——. [?]

It is ⊥ to ——. [?]

It is bisected by ——. [?]

Note that we wish to *discover how to fix point B′*.

Imagine ⊥ B′ N dropped. This ⊥ cuts off of M D a line equal to ½ of A B.

But we *know* ½ of A B, hence we can retrace the steps taken; and we—

(1) Bisect A B.

(2) Measure M N = ½ A B.

(3) Erect ⊥ to C D at N, which fixes B′.

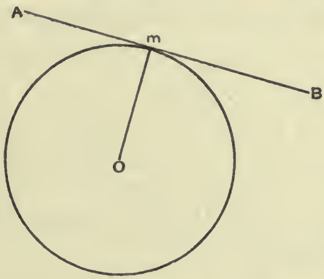
(4) Draw B′ A′ ⊥ to O M, thus giving the required chord.

81. Given two equal intersecting chords. Compare their parts.

148.

Proposition XII.

What is a tangent? the point of tangency?



O M is any radius of circle O. If A B is a line ⊥ to O M at its extremity,—

Is A B a tangent to circle O?

Prove your answer. State the truth discovered. State and prove the converse. State both as *one* proposition (Prop. XII.).

(If you can not prove Prop. XII., see the "hint" given.)

[*Hint.*—By the *definition* a tangent touches but does not enter the ⊙. A B *does* touch the ⊙, since *M* is the end of the radius. Then it is only necessary for us to prove that A B will not enter the ⊙.

We must show that every point in A B or A B produced, except M, *is without the* ⊙.

How does O M meet A B? Is it then the shortest distance from O to A B? Then if *any* other point in A B than M, say P, is joined to O, the distance is greater than the radius O M, and its end, which is in A B, is without the circle. But P is *any* other pt. in A B than *M*; ∴ *every* point but *M* is without the ⊙; hence A B is a tangent by definition.

Note the theorem:

If a line is perpendicular to a radius at its extremity, it is a tangent to the circle.

Converse:

A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

To prove the converse:

Draw a ⊙ and a tangent. Draw a radius to point of tangency.

Is the shortest distance from a point to a line a perpendicular? Prove it. Can you prove this radius which is drawn to the point of tangency to be the shortest distance and consequently a perpendicular?

When you join *any* other point in the tangent than the point of tangency to the center, why is the line longer than the radius?

149.

Cor. I. A perpendicular from the center of a circle to a tangent passes through the—

150.

PROPOSITION XIII.

Problem. To draw a tangent to a circle at a given point in the circumference.

Exercises.

82. If tangents are drawn to the extremities of a diameter, they are ——. Fill blank and prove.

83. If one chord of one circle is equal to one chord of another, are the circles necessarily equal? If not, what other conditions are necessary?

84. What is the locus of the center of a circle whose circumference (1) shall pass through two given points?

(2) Which shall be tangent to two intersecting lines?

151.

A *circle* is *inscribed* in a polygon when each side of the polygon is tangent to the circle. The polygon is said to be *circumscribed* about the circle.

152.

Two circles are *concentric* when they have the same center.

Exercise.

85. Prove that the inscribed circle and circumscribed circle of an equilateral triangle are concentric.

153.

Proposition XIV.

I. Draw two parallel chords in a circle.

Compare the arcs intercepted by the chords.

[*Hint.*—Draw radius perpendicular to one of the chords and make deductions.]

II. Draw a tangent and a chord parallel to it.

Compare the intercepted arcs.

III. Draw two parallel tangents.

Compare the intercepted arcs.

State Prop. XIV., which includes the three truths discovered.

154.

Cor. Through what point does the line pass which joins the points of tangency of two parallel tangents, or the mid-point of a chord and the point of tangency of a parallel tangent, or the mid-points of two parallel chords?

What then is this line (when produced if necessary)? State corollary.

[Remember that *every problem, corollary, excrcise, theorem, and locus requires geometrical proof.*]

155.

PROPOSITION XV.

Draw two circles which intersect—(1) equal circles; (2) unequal circles. Draw the *line of centers* (line joining the two centers). Draw the *common chord.*

(1) How does the line of centers meet the common chord?

(2) Compare the segments of the common chord.

Prove your answers. State Prop. XV.

156.

PROPOSITION XVI.

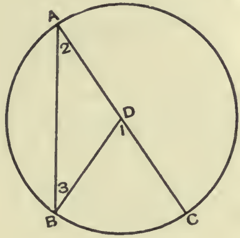
Draw two circles which are tangent externally.

Does the line of centers pass through the point of tangency?

[*Hint.*—Draw the common tangent and erect ⊥ to it at point of contact. When produced in both directions, through what points must the perpendicular pass?]

157.

Proposition XVII.



What is an inscribed angle? Draw an inscribed angle and the central angle that intercepts the same arc.

Compare the inscribed with the central angle.

(1) Let the drawing show one side of the inscribed angle a diameter. Make deduction and prove this case.

(2) Draw an inscribed ∠ wholly to the right or to the left of the center. [Use the first case in proving this.]

(3) Let the center be within the inscribed angle. State Prop. XVII.

(If you fail in the above after studying the figure, making drawings of your own, see the "hint" given below.)

[*Hint.*—See Figure. Compare ∠ 2 with ∠ 3. Compare the sums of ∠ 2 and ∠ 3 with ∠ 1. When you attempt Case 2, draw a diameter through the vertex of the inscribed ∠. Note that the inscribed ∠ is a *part* of an inscribed ∠ which has *a diameter for a side* and that there are *two* inscribed ∠s having the diameter for a side.]

158.

Cor. I. Draw a semicircle and inscribe several angles in it. [The vertex must lie in the semi-circumference and the sides of the inscribed angle must terminate in the ends of the arc.]

How large is each angle inscribed in the semi-circle?

159.

Cor. II. How large is an angle inscribed in a segment greater than a semi-circle when compared with a rt. ∠?

160.

Cor. III. How large is an angle inscribed in a segment less than a semi-circle when compared with a rt. ∠?

161.

What can you say of all angles inscribed in the same segment?

Exercises.

86. What is the sum of the opposite angles of an inscribed quadrilateral?

87. *Problem.—Erect a perpendicular to a given line at a given point in it.* [Use a different method from that used in § 47.]

[*Hint.*—See § 158.] [With center without the line, draw circumference passing through the required point.]

88. *Problem.—Erect a perpendicular to a line from a given point without the line.* [Use a different method than that in § 48.]

[*Hint.*—Draw any oblique line through point to the line and use it as a diameter.]

89. Construct a right triangle when hypotenuse and one adjacent ∠ are given. [Use § 158.]

90. Construct a right triangle when hypotenuse and one leg are given.

91. Construct a right triangle when the hypotenuse and the altitude from the right angle to the hypotenuse are given.

92. From a point without a circle draw two tangents.

[*Hint.*—Join point to the center and describe a circle upon that line as a diameter.]

Prove that—

(1) The tangents are equal in length.

(2) The line drawn from the point without the circle to the center bisects the angle formed by the tangents and also bisects the central angle formed by drawing radii to the points of tangency.

93. What is the locus of the vertex of a right triangle when the hypotenuse is given?

94. Two chords are perpendicular to a third chord at its extremities. Compare the two chords.

162.

Proposition XVIII.

Draw any two chords which intersect within the circle.

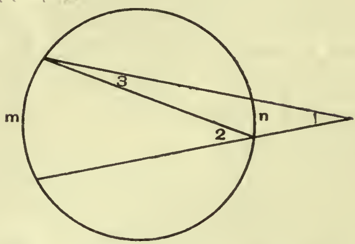
How large is each vertical $\angle$ when compared with the central $\angle$s which intercept the arcs which subtend the vertical $\angle$s?

Write your answer as Prop. XVIII. [Let the chords intersect in all possible positions and draw the central $\angle$s and compare.]

[*Hint.*—Show that the required angle is an exterior angle of a triangle whose opposite interior angles are measured by one-half the sum of the arcs which subtend the central angles.]

163.

PROPOSITION XIX.



Draw two secants intersecting without the circle.

Compare the angle of the secants with central angles subtended by the intercepted arcs.

[*Hint.*—Compare ∠ 1 with ∠ 2 and ∠ 3. But how large is ∠ 2? how large is ∠ 3?]

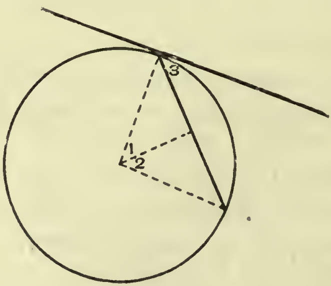
Write Prop. XIX.

164.

PROPOSITION XX.

Draw a tangent to a ⊙ and any chord to the point of tangency.

Compare the ∠ made by the chord and tangent with the central ∠ which is subtended by the intercepted arc.



Compare ∠ 3 and ∠ 1 in the figure. Write Prop. XX.

165.

Proposition XXI.

Draw any circle and a tangent. Draw any secant intersecting the tangent.

Compare the angle formed with the central angles subtended by the intercepted arcs.

166.

Proposition XXII.

From a point without a circle two tangents are drawn.

Compare the angle formed with central angles subtended by the intercepted arcs.

Exercises.

94. Given two tangent circles. [Draw in different positions.] Draw the *common tangent* at the point of tangency.

95. When are two ⊙s tangent? externally? internally?

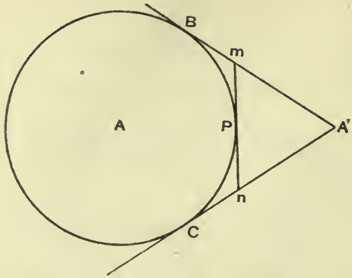
Problem.—At a given point in the circumference of a given ⊙ draw a ⊙ with a given radius tangent to the given ⊙.

96. What is the locus of the center of a circle tangent to a given circle at a given point in the circumference?

97. Given a ⊙ with a quadrilateral circumscribed about it. Compare the sum of two opposite sides with the sum of the other two sides.

[*Hint.*—Draw the figure. How many tangents or parts of tangents have you? Which of the parts are equal?]

98. What is the locus of the center of a ⊙ tangent to two given lines? [Suppose the lines were ||?] Give two cases.

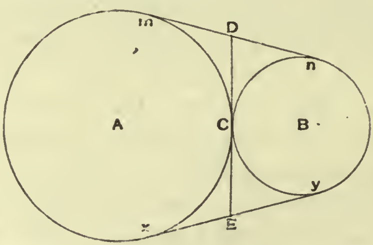


99. Given ⊙ A, and A′ B and A′ C two tangents from any point A′ without the ⊙; *m n* is any tangent within A′ C and A′ B. What is the sum of the sides of the △ A′ *m n?* If P moves on the arc B C, will the perimeter of the △ increase or decrease?

100. What is the locus of the center of a ⊙ tangent to a given line at a given point?

101. What is the locus of the center of a ⊙ of given radius always tangent to a given ⊙? Discuss all possible positions of the tangent circle and the length of its radius.

102. If two ⊙s are tangent to a line at T, and if from any point (A) in the line a tangent (A K, A L) is drawn to each ⊙, can you prove the lengths equal?



103. (1) Can you prove that E D bisects *m n?*
(2) Can you prove *m n* = *x y?*
(3) Can you prove D E = *m·n* = *x y?*
(4) Can you prove *m* C *n* = right ∠?
(5) Can you prove *x* C *y* = right ∠?

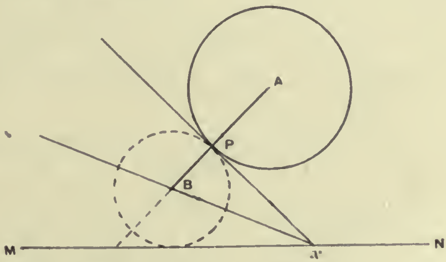
104. Construct a ⊙ passing through a given point and also tangent to a given line at a given point. [What is the locus of the centers of circumferences passing through two given points? What is the locus of the centers of ⊙s tangent to a given line at a given point? Will these loci intersect?]

105. Construct a ⊙ tangent to a given line at a given point and also tangent to another given line. [May these lines intersect? What is the locus of the center of a ⊙ tangent to two || lines? to two intersecting lines?] Show both solutions.

(1) When the lines are ||.

(2) When they intersect, or when produced intersect.

106. Construct a ⊙ tangent to a given ⊙ at a given point and also tangent to a given line. [Construct a tangent to the given ⊙ at the given point.]



Assume the problem solved for the *purpose of discovering a solution*, and assume that the given circle is A, the given point is P, *and* the given line is *m n*, and that the required ⊙ is B. With completed figure before us it will be easier to discover a solution whereby we may obtain the center B. Let us try to find two intersecting loci which will fix point B for us. Do you see two given points which joined and the line produced *must* pass through B? Why *must* it? *State the locus required.* [Ex. 96]

Is it possible to draw another line besides *m n* to which ⊙ B must be tangent? [Ex. 94.]

Will this common tangent intersect *m n*? What is the locus of the center of a circle tangent to intersecting lines?

Must these loci intersect? If the common tangent and *m n* do not intersect, where must point P lie? What is the locus of the center of a circle tangent to two parallel lines?

The above discussion is called the *analysis* of the problem. Having discovered the solution, we now give the *direct* construction as follows:

Given: (1) Line *m n*.

(2) Circle A.

(3) Point P in circumference of ⊙ A.

Required: To draw a ⊙ tangent to ⊙ A at P and to the line *m n*.

Construction: (1) Draw A P and produce it.

(2) Draw tangent to ⊙ A at P and produce it until *m n* is intersected (if possible).

(3) Letter point of intersection of tangent and *m n*, *x*, and then bisect ∠ *m x* P and produce the bisector until A P produced is intersected. Letter point of intersection B. Then B is the center of the required ⊙.

Proof: (1) A P produced is the locus of the center of a ⊙ tangent to ⊙ A at P. [Ex. 96.]

(2) P *x* is ⊥ to A B. [§ 148.]

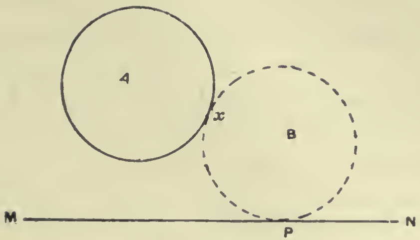
(3) The bisector of ∠ *m x* P is the locus of the center of a ⊙ tangent to *m n* and *x* P. [Ex. 84 and Ex. 98.]

∴ ⊙ B is the required ⊙.

[Pupil will give solution when common tangent and *m n* are ||; also when A P and *m n* are ||; also when A P produced (beyond P) *will not cut m n*.]

107. Construct a ⊙ tangent to a given line at a given point and also tangent to a given ⊙.

What have we given? (1) A ⊙ to which the required ⊙ is somewhere to be tangent. (2) A *line* and the *point* of tangency to that line.



Again suppose the problem solved—for what purpose?

Given: (1) *m n* the given line and A the given ⊙.

(2) P the point of tangency of the required ⊙ to that line.

(3) And B the required ⊙ (supposed to be).

Again try to find two loci which intersect and thus fix *B*. [See Ex. 106.]

Can you get the locus of the center of a ⊙ tangent to another ⊙?

Can you get the locus of the center of a ⊙ tangent to a line at a given point?

Can you get another locus to intersect the locus found? Review the locii you have learned. See Ex. 62, etc.

Can you apply Ex. 62? (1) Is B equidis ant from the two fixed points which were given? If not, is it possible, with data given, to fix a point which you can use with P or with A?

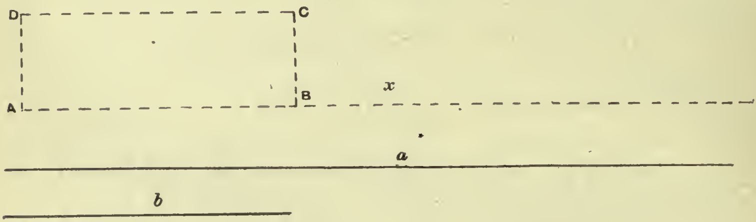
[What straight line is given? What straight lines can you draw? Try to fix a point by measuring from fixed points on given lines or lines which you can draw.]

Ex. 62 *can* be used here. But if it could not, you should try another locus, etc., etc., till you get a solution.

108. Construct a rectangle when the perimeter and the diagonal are given. This is a difficult problem and it is of much importance that the pupil fully weigh each question in the order given, and frequently review the ground gone over to hold in mind *just what has been discovered*, that it may be readily used in further investigation.

Note carefully the *data given* (*what we know*). Draw any rectangle—suppose it to be the required rectangle and study it carefully. What data are necessary to draw it? Have we sufficient *data* given? If not, how can we by using data given get the necessary lines and points to draw it?

Given: Perimeter, a. Diagonal, b. } Data.



How many corners of the required rectangle do data enable us to fix?

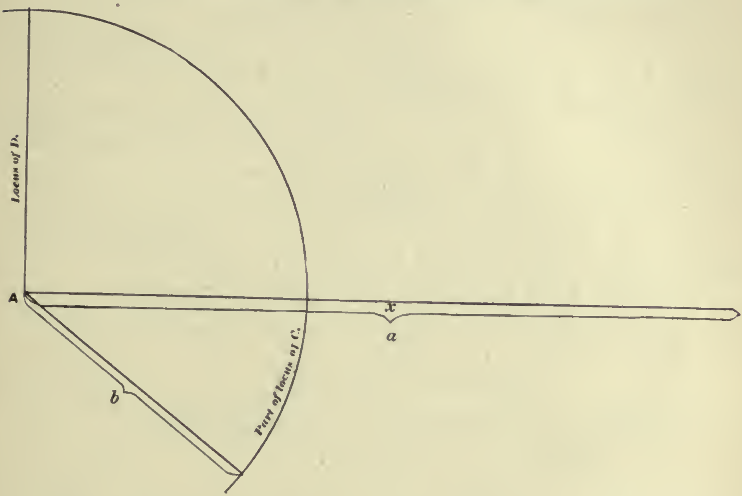
If we assume A as fixed, can we get B, or the opposite corner, C? Can we get the locus of either of these corners?

What is the locus of C? Is it a fixed distance from A? Do you *know* that distance?

Let us try to discover another locus which will enable us to fix C.

[But *why* do we want to fix C? How could you draw the rectangle if it *were* fixed?]

Let us now construct the rectangle so far as possible.



But we must have *another locus* of C, or of D.

We have *a* and also *b*; ∴ we can get a part of them should we so desire. What use could we make of ½ the diagonal, or of ½ the perimeter? Study the supposed rectangle. You can see there another locus of C. What is it? [Circumference of a ⊙ having B the center and B C the radius.] You no doubt think that we have neither point B, nor radius, B C, but there is a *point* in that circumference which we *can* fix. Produce A B till it is the length of the known perimeter. At what point does the above circumference cut A B produced? Can you not fix that point? Letter it x. Note the ∠ C B x. What kind of △ would it form the sides of? How could you draw at x the third side of the △, *or a line that must contain C*, giving a second locus of C?

109. Construct a ⊙ with a given radius:
   (1) Tangent to two given lines.
   (2) Tangent to a given line and to a given ⊙.
   [Find intersection of loci.]
   (3) Tangent to two given ⊙s.
   [Find intersection of loci.]

167.

PROPOSITION XXIII.

Problem. To circumscribe a ⊙ about a given △.

168.

Cor. I. Pass the circumference of a ⊙ through three given points. When is this impossible?

Cor. II. Find the center when an arc is given.

169.

PROPOSITION XXIV.

Problem. To inscribe a ⊙ in a given △.

170.

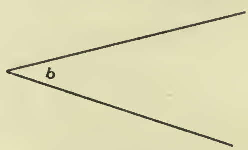
Cor. I.—To draw a ⊙ tangent to three given lines. When is this impossible?

171.

PROPOSITION XXV.

Problem. To construct on a given chord a segment which will contain a given inscribed ∠.

Given chord.



Given ∠.

[Use any convenient point on either side of ∠ *b* as a center and with *a* for radius cut the other side of ∠ *b*. Pass a circumference through the three points fixed. Pupil should note the segment and give the geometrical proof. Prove that any ∠ inscribed in this segment equals ∠ *b*.]

EXERCISES.

109. To construct a △ when the base, altitude, and ∠ of vertex is given. [Use the above problem.]

110. To construct a △ when two sides and the ∠ opposite one side are given. Show solution when the ∠ is (1) acute, (2) right, (3) obtuse.

111. (1) To construct a △ when the base, the vertical ∠, and the median from base are given. When is this impossible?

(2) What is the locus of the vertex of a △ when the base and the vertical ∠ are given?

112. *Problem. To draw a common tangent to two given circles.*

How many common tangents may be drawn to these ⊙s? How many common tangents cross the line of centers?

[Join O to O′. With O for center, draw a ⊙ with radius R—*r*. From O′ draw a tangent to this ⊙. Draw its radius to the point of tangency and continue to the circumference of large ⊙. At this point draw a tangent to large ⊙ O. Can you prove this line also a tangent to ⊙ O′? Try to invent a way to draw the tangents which cross the line of centers.]

Exercises.

113. Can you find the center of a ⊙ without bisecting any straight line?

114. Inscribe a rectangle in a ⊙ using diameters only to fix points.

115. Draw a ⊙ and take any point in it. What is the longest possible line that can be drawn from this point to a point in the circumference?

116. Draw a ⊙ and fix any point without it. Draw the shortest possible line from the point fixed to the circumference of the circle.

117. What is the locus of the mid-point of a chord of a given length in a given circle.

118. Can a tangent of any kind be drawn to a ⊙ from a point within it?

119. Can a tangent be drawn to a point in the circumference when the center is not known?

120. Describe a circle on one of the sides of an isosceles △ and show how the circumference cuts the base.

121. Find the locus of the mid-point of a ladder as the foot of the ladder is pulled away from a vertical wall.

122. Through a given point within an angle draw a line intercepted between the sides and bisected at the given point. [See Ex. 107 and Ex. 108 for method; and, if you *fail*, see § 118.]

123. Construct an equilateral △ when the altitude is given.

124. Trisect a given rt. ∠.

125. A circle is wholly without or within another circle, according as their central distance is greater than the sum, or less than the difference, of their radii.

[From Ex. 125 can you show how the central distance is related to the radii if two ⊙s intersect?]

BOOK III.

MEASUREMENT.

172.

Plane Geometry deals with lines, angles, surfaces, and areas of surfaces. In comparing these magnitudes hitherto it has been deemed sufficient to prove their equality, or to prove that one is greater.

[When it is known that two $\angle$s of a $\triangle$ are unequal, what follows? If in the same $\odot$ or in equal $\odot$s chords are unequally distant from the center, what relation do these chords have? Can you think of other propositions wherein lines have been proved unequal? Think of propositions in which $\angle$s have been proved unequal, etc., etc]

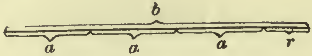
But it now becomes necessary for us to find the *exact* relation of the size of two or more magnitudes.

Case I.—If in comparing the lengths of two lines, *a* and *b*, we find that *b* contains, *or* measures, *a* an *exact* number of times, say three times, what is the relation of *b* to *a*? of *a* to *b*?

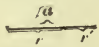
Case II.—If in comparing the lines *a* and *b* we find that one is *not* an *exact* measure of the other, but that a third line, *c*, is exactly contained in *a* four times and in *b* exactly nine times, what is the relation of *a* to *b*? of *b* to *a*? What is their *common measure?* Can you tell the *common measure* in Case I.? When is one line a common measure of another? Suppose *a* and *b* are given and *c* is required. Can you find *c*?

[How did you find the highest common factor of two or more quantities in Arithmetic and Algebra when you were unable to factor the quantities?]

Given the lines a, , and b, ———————————.
It is required to find their common measure. Apply the smaller, a, to the larger, b.



We find a is contained three times in b with a remainder, r. Apply r to a.

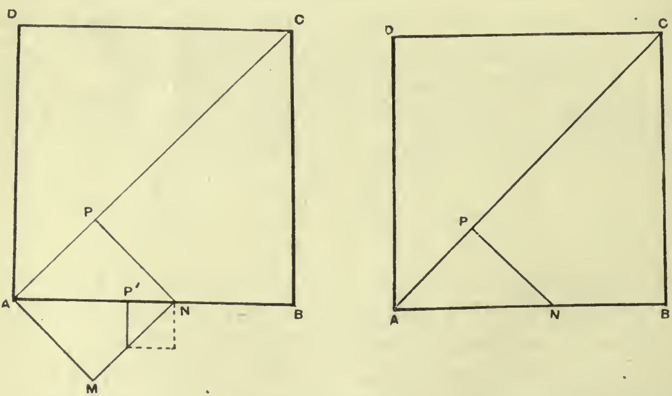


[Why is r less than a?]

We find r is contained in a once with a remainder, r'. Apply r' to r.

We find r' is contained in r twice and *there is no remainder*. What must be added to r to produce a? a is then how many times r'? b is also how many times r'? What then is the *common measure* of a and b? What is the relation of b to a? of a to b? What is the relation of r to a? to b? to r'?

Case III.—Let us try to find a common measure of the diagonal and a side of a square.



Let A B C D be the required square. Draw the diagonal C A. Apply C B to C A (C B < C A; [Auth.] C A is not twice C B; [Auth.] ∴ C B is contained in C A *once with a* *remainder.* Lay off on C A, C P equal to C B; then the remainder is A P. Erect a $\perp$ to C A at P, which cuts A B at N. then B N = N P; [Auth.] also N P = A P, and A P N is a rt. isos. $\triangle$, and A P N M is a square. [Give proof of each statement.] When, then, we apply the remainder, A P, to a side, C B, or to its equal, A B, A P is contained *twice* with the remainder N P′. Show that N P′ (second remainder) is contained in A P (first remainder) twice with a remainder, and consequently, show there will always be a remainder and that there is *no common measure of the diagonal and a side of a square.*

Such lines are said to be *incommensurable.*

Do you know of other lines that are incommensurable?

While it is impossible to get the *exact* relation of two incommensurable magnitudes, we may approximate that relation to any required degree of accuracy. (1) Thus, when C B is divided into 10 equal parts, [Review § 117.] A C contains more than 14 of those parts and less than 15. (2) If C B is divided into 100 parts, A C will contain more than 141 and less than 142 of them, etc.

In the first instance A C is more than $\frac{14}{10}$ and less than $\frac{15}{10}$ times B C.

Tell the relation of A C to B C in the second instance. Tell the relation of B C to A C in each instance.

Extract the square root of 2 true to six decimal places and further approximate the relation of A C to B C. Can you in each instance find two lines, one a little less than A C and one a little more than A C, to which B C does bear an *exact* relation? Find the difference of these two lines in each instance. Do these lines get closer and closer to A C, and does their difference grow less and less? What is the difference when you use all of the six decimal places required above?

RATIO.

173.

What is the relation of line *a* to line *b* when *a* is 2 ft. in length and *b* is 9 ft.? when *a* is 3 yds. and *b* is 2 rods? When *a* is $2\frac{1}{2}$ inches and *b* is $2\frac{1}{2}$ ft.?

The fraction or quotient which expresses the relation of two magnitudes is called the *ratio* of those magnitudes. Is it possible to find the ratio of two magnitudes of different kinds? What is the ratio of 9 ft. to 10 yds.? of 40 rods to 2 miles? of $\frac{2}{3}$ of an inch to $2\frac{1}{3}$ ft.? Can you find the ratio of 2 inches to 2 sq. ft.? of $3\frac{1}{2}$ sq. yds. to 50 sq. ft.?

Define ratio.

The ratio of a quantity *a* to a like quantity *b* is expressed $a : b$, or $\frac{a}{b}$, and is read in each case, *the ratio of a to b*. *a* is the *first term* of the ratio, or the *antecedent*, and *b* is the *second term*, or *consequent*.

When the diagonal of a square is 1 foot, write the fraction that expresses the ratio of the diagonal to the side. Is the fraction a *rational* fraction? [See Algebra for definition of *rational*.] Write the rational fraction that is less than one-millionth *greater*, and also the rational fraction that is less than one-millionth *less* than the required ratio when the *side* of the square is 1 foot.

When two incommensurable quantities of the same kind are given, is it always possible to find the ratio within any required degree of accuracy?

When the side of a square is 1 yard, find two fractions each of which approximates the ratio of the diagonal to the side within one ten-thousandth; within one ten-millionth.

Let A and B be two incommensurable quantities. Let A be divided into any number of equal parts, say *n* equal parts, and let P be each part. Then A = ? Then B must contain some number of those parts, say m parts, with a remainder less than P. Then B is greater than mP, [Why?] and less than $(m+1)$P. [Why?]

If it is required to approximate the ratio of B to A, we have

$$\frac{B}{A} > \frac{Pm}{Pn} \text{ and } < \frac{P(m+1)}{Pn}; \text{ or } \frac{B}{A} > \frac{m}{n} \text{ and } < \frac{m+1}{n}.$$

How much greater than $\frac{m}{n}$ is $\frac{m+1}{n}$?

But A was divided into n equal parts; so if we make n a very great number, the difference between the ratios $\frac{m}{n}$ and $\frac{m+1}{n}$, which is always $\frac{1}{n}$, will become *very small*. What, then, can you say of the ratio of two incommensurable numbers?

[*Hint.*—The pupil *must* thoughtfully read each question *in the order asked. Master it.*]

Exercises.

126. What is the ratio of 1 lb. to 9 ozs.? to 32 ozs.? to $2\frac{1}{2}$ lbs.? to $3\frac{5}{9}$ lbs.? to $5\frac{1}{2}$ ozs.?

127. What is the ratio of 90° to $2\frac{3}{10}$ st. $\angle$s?

128. If a base $\angle$ of an isosceles $\triangle$ is 40°, what is its ratio to the vertical $\angle$?

129. If the vertical $\angle$ of an isos. $\triangle$ is 42°, what is its ratio to each base $\angle$?

130. If the exterior angle at the base of an isos. $\triangle$ is 110° degrees, what is its ratio to each of the interior $\angle$s?

PROPORTION.—PRELIMINARY.

174.

What is the ratio of 2 feet to 3 inches? of 13⅓ yards to 5 feet? What can you say of these two ratios? What number has the same ratio to 10 feet that 160 rods has to one mile? What two numbers have the same ratio to each other that 3 a has to 7 a? that 5 feet has to 4 rods? When two pairs of quantities have equal ratios, they are said to be *in proportion.* Thus if it is known that the ratio of A to B equals the ratio of x to y, the four quantities are *in proportion*, and may be written $\frac{A}{B} = \frac{x}{y}$, or A : B $= x : y$, and each is read *the ratio of* A *to* B *equals the ratio of* x *to* y; or each may be read A *is to* B *as* x *is to* y. Which term is the antecedent in any ratio? the consequent? What are the antecedents in the above proportion? the consequents?

175.

(1) The *extremes* in a proportion are the first and fourth terms; (2) the *means* are the second and third terms; (3) when three numbers are in proportion, as $a : b = b : c$; b is called a *mean proportional* and c is called a *third proportional.*

Can there be a ratio between two unlike magnitudes? Can there be a proportion when the four quantities are not all the same kind?

[Give examples of youı answers.]

176.

In ratio we may measure one *magnitude* by another magnitude; *e. g.*, the ratio of line a to line b equals $\frac{4}{9}$ when the lines a and b contain a common measure, c, 4 and 9 times, respectively, and we write $\frac{a}{b} = \frac{4}{9}$. We call $\frac{4}{9}$ the *numerical ratio* of a to b.

177.

Suppose we wish to give the geometrical proof of the above, which stated is—

The ratio of two magnitudes is equal to the ratio of their numerical measures when referred to the same unit.

We have $a = 4\,c$; [Hyp.]

Also $b = 9\,c$; [?]

$\therefore\ \frac{a}{b} = \frac{4\,c}{9\,c}$. [?]

But $\frac{4\,c}{9\,c} = \frac{4}{9}$; [?]

$\therefore\ \frac{a}{b} = \frac{4}{9}$. [?]

[*Hint.*—To make this proof *general* substitute m for 4 and n for 9.]

178.

Theorem. Prove that if four magnitudes are in proportion, their numerical measures are in proportion and conversely.

[*Hint.*—Let the four magnitudes be A, B, C, and D, and their numerical measures be a, b, c, and d, respectively.

Then $\frac{A}{B} = \frac{a}{b}$, [?]

And $\frac{C}{D} = \frac{c}{d}$, [?]

Also $\frac{A}{B} = \frac{C}{D}$, [?] etc.]

[Pupil will finish the above proof.]

PROPORTION.—Propositions.

Preliminary Questions.

1. Observe that when four *numbers* are in proportion, we have two equal fractions. Select two equal fractions and make all the deductions you can.

(1) Are the results equal when the fractions are inverted? When the denominators are interchanged? When the denominator of the first is interchanged with the numerator of the second? etc., etc.

(2) Are their powers equal? their roots?

(3) Does the sum of the numerators divided by the sum of the denominators equal each of the fractions? Is this true when there are three or more equal fractions?

(4) Is the sum of the terms of the first divided by either term equal to the sum of the terms of the second divided by the corresponding term? etc., etc.

(5) May the numerators or the denominators be multiplied or divided by the same number and the results remain equal?

2. (1) May four magnitudes be in proportion when they are like magnitudes? *e. g.*, four rectangles? four solids?

(2) May four magnitudes be in proportion when the magnitudes are not *all* of the same kind?

[*Hint.*—Carefully review the definition of a proportion.]

179.

Proposition I.

Select four *numbers* which are in proportion.

Does the product of the extremes equal the product of the means?

Let the numbers be unknown to you; *e. g.*, a, b, c, and d, and let $a : b = c : d$. What is the question you wish to answer about these four numbers? What other way to write the proportion? By a single operation you can obtain the desired proof. Try to discover it. What axiom do you use?

Select four *magnitudes* which are in proportion. Is it possible to multiply the extremes together? Does the above truth apply to magnitudes when they are all of the same kind? when the pairs of magnitudes are of different kinds? Does it *ever* apply to magnitudes?

[N. B.—The above is no part of the *formal proof*, which it is desired the pupil shall originate for himself. These questions are asked to lead him to *discover the truth and its proof*.]

The proof of Prop. I. is given below to serve as an example of proofs in Proportion.

I. If four *numbers* are in proportion, the *product* of the extremes equals the *product* of the means.

Given: $a : b = c : d$.

To prove: $ad = bc$.

Proof: $\frac{a}{b} = \frac{c}{d}$; [Hyp.]

$\therefore ad = bc$. [*Ax.* If the same operation be performed upon equals, the results will be equal.]

[What was done to each fraction?]

II. If four *lines* are in proportion, the *rectangle* of the extremes equals the rectangle of the means.

[*Hint.*—Let the magnitudes be represented by capitals and their numerical measures represented by small letters.]

Given: A : B = C : D.

Then $a : b = c : d$. [*Prop.* When four magnitudes are in the proportion, their numerical measures are in proportion.]

Or $\frac{a}{b} = \frac{c}{d}$;

$\therefore ad = bc$. [?]

But ad represents the product of the numerical measures of the lines A and D, or the rectangle formed by A and D;

also bc represents the product of the numerical measures of B and C.

$\therefore$ $AD = BC$. [Which means the rectangle formed by A and D equals the rectangle formed by B and C.]

180.

Proposition II.

What is the converse of Prop. I.? Write the converse to each part and give the formal proof.

181.

Proposition III.

I. Write two or more proportions.

Are the products of the corresponding terms in proportion?

Does your answer apply to lines as well as to numbers? Write Prop. III. and its proof.

182.

Proposition IV.

When four *numbers* are in proportion, are *like powers or like roots of these numbers in proportion?*

Write Prop. IV. and its proof.

Can this proposition be applied to lines or other geometric magnitudes?

183.

Proposition V.

If four *like* quantities are in proportion, *are they in proportion by alternation?* that is, *is* the *first to the* third as the *second* is to the *fourth?*

Is this true when the pairs of quantities are *unlike*?

Write and prove Prop. V.

[*Hint.*—Let us attempt the proof when the quantities are like.

Given: 1. A, B, C, and D, the like quantities.

2. a, b, c, and d, their numerical measures.

3. Also, $\frac{A}{B} = \frac{C}{D}$.

To prove: $\frac{A}{C} = \frac{B}{D}$.

When we can change what is given by the hypothesis *directly* into the required statement, we have what is called the *direct proof.* But it is often difficult to see the steps necessary to do this, and we will now outline a method whereby we are often enabled to *discover* the necessary steps.

1st. Fix *firmly* in mind what is *given;* this we *know* to be true.

2d. Clearly fix in mind what is *required to be proved.*

3d. *Suppose that which we wish to prove is really proved*, and then make deductions based upon the truth of this supposition. After each deduction, try to find a *proof* for the statement made; if you are unable to discover a proof, make further deductions and again try to prove each in turn. When finally able to prove the last deduction made, prove the one immediately preceding the last by using the last deduction and continue to retrace the steps taken, proving each in reverse order until the supposed truth is finally established, based upon a chain of evidence from the last to the first; *e. g.*, in Prop. V. *we know*

$\frac{A}{B} = \frac{C}{D}$, and *we wish to prove* that

$\frac{A}{C} = \frac{B}{D}$.

Now apply the method outlined.

(1) $\frac{A}{C} = \frac{B}{D}$. [Supposed to be true.]

(2) Then $\frac{a}{c} = \frac{b}{d}$. [First deduction.] [When four magnitudes are in proportion ....]

(3) And $a\,d = b\,c$. [?] [Second deduction.]

(4) $\frac{ad}{bd} = \frac{bc}{bd}$, [?]

Or, $\frac{a}{b} = \frac{c}{d}$; [Third deduction.]

(5) $\therefore \frac{A}{B} = \frac{C}{D}$. [?] [Fourth deduction.]

But *we know* that this fourth deduction is true *by hypothesis:* consequently we have *discovered* the steps whereby we may give the *direct* proof, which is:

1. $\frac{A}{B} = \frac{C}{D}$. [Hyp.] [*Fourth* deduction.]

2. Then $\frac{a}{b} = \frac{c}{d}$. [?] [*Third* deduction.]

3. $a\,d = b\,c$. [?] [*Second* deduction.]

4. $\frac{a}{c} = \frac{b}{d}$. [?] [*First* deduction.]

5. $\therefore \frac{A}{C} = \frac{B}{D}$. [?] [The required equality.]

184.

Proposition VI.

Are four *numbers* which are in proportion in proportion by *inversion;* that is, is the *second* to the *first* as the *fourth* is to the *third?*

Are four magnitudes which are in proportion in proportion by *inversion?*

(1) When the magnitudes are *like?*

(2) When the pairs of magnitudes are *unlike?*

Do you conclude that *any* four quantities which are in proportion are in proportion by inversion?

Try to prove it is true.

Let A, B, C, and D be the four magnitudes, and a, b, c, and d be their numerical measures.

Then we have:

Given: $\frac{A}{B} = \frac{C}{D}$.

To prove: $\frac{B}{A} = \frac{D}{C}$.

Proof: $\frac{A}{B} = \frac{C}{D}$. [?]

Then $\frac{a}{b} = \frac{c}{d}$. [?]

Let the pupil finish the proof and write carefully Prop. VI.

185.

Proposition VII

By composition is meant the sum of the first and second is to either the first or to the second as the sum of the third and fourth is to either the third or to the fourth.

Ask a comprehensive set of questions with respect to *composition* about four quantities which are in proportion [see questions under Prop. VI.] and give their answers and carefully write the formal statement of Prop. VII. and give its proof.

186.

Proposition VIII.

By *division* is meant the difference of the first and the second is to either the first or the second as the difference of the third and fourth is to the third or to the fourth.

Treat division in a similar manner to that of composition in Prop. VII.

187.

Proposition IX.

When four *numbers* are in proportion, are they in proportion by *composition* and *division;* that is, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference?

Is this true when four *like* magnitudes are in proportion? when the pairs are *unlike*?

Carefully study all possible kinds of proportions with reference to *composition* and *division* and then state and prove Prop. IX.

Exercises.

131. If $a : b = c : d$, prove that $a : a + b = c : c + d$ Is this true when a and b are unlike c and d?

132. If $a : b = c : d$, prove that $a - b : c - d = b : d$. May a, b, c, and d represent any quantities whatever? State clearly the exceptions if there are any.

188.

Proposition X.

1. Write a series of equal ratios, (1) using all numbers, (2) using lines, (3) using other *like* quantities, (4) using unlike quantities, *if possible.*

In each *case* the quantities used are said to be in *continued proportion.*

Carefully examine each continued proportion and answer the following question about it: Is the sum of all the antecedents to the sum of all the consequents as any antecedent in that continued proportion is to its consequent?

If your continued proportions comprehend all possible cases, you should now be able to state Prop. X. Try to prove your statement.

[*Hint.*—Let A, B, C, D, E, F, etc., be any like quantities and A : B = C : D = E : F, etc. Then $a : b = c : d = e : f$, etc. [Auth.]

Or, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$, etc.

To prove: A + C + E + ... : B + D + F + ... = A : B = C : D = E : F .....

(1) If $\frac{a}{b} = r$,

Then $\frac{c}{d} = r$, [?]

And $\frac{e}{f} = r$, [?] etc.

(2) $\therefore a = br$ [?]
$c = dr$, [?]
$e = fr$, [?] etc.

(3) And $a + c + e + \ldots\ldots = (b + d + f + \ldots\ldots) r$, [?]

And $\frac{a + c + e + \ldots\ldots}{b + d + f + \ldots\ldots} = r$. [?]

Pupil will finish the proof.]

189.

PROPOSITION XI.

1. Write a proportion (1) using numbers, (2) using lines or other like magnitudes, (3) using magnitudes having the pairs unlike.

2. (1) Will the numbers or magnitudes be in proportion when the antecedents are multiplied by the same number? (2) When the consequents are multiplied by the same number? (3) *When the antecedents are multiplied by one number and the consequents by another number?*

State and prove Prop. XI.

190.

PROPOSITION XII.

Is the value of a ratio changed when both terms are multiplied or divided by the same number?

State and prove Prop. XII.

191.

Cor. If A : B = P : Q when A and B are like quantities and P and Q are also like quantities,

(1) Is $M\,A : M\,B = N\,P : N\,Q$?

(2) Is $\frac{1}{M}\,A : \frac{1}{M}\,B = \frac{1}{N}\,P : \frac{1}{N}\,Q$?

State and prove the corollary.

EXERCISES.

133. If $a : b = b : x$, find x.

134. If $a : x = x : c$, find x.

135. If $a : b = c : x$, find x.

136. If $a : b = c : d$, and
$a : b = e : f$.
Show that $c : d = e : f$.

137. If A, B, C, D, E, F, etc , are like quantities, and $A : B = C : D = E : F = \ldots$, and m, n, o, p, etc., are numbers,

Then $\dfrac{A\,m + C\,n + E\,o + \ldots}{B\,m + D\,n + F\,o + \ldots} = \dfrac{A}{B} = \dfrac{C}{D} = \dfrac{E}{F} = \ldots$

192.

PROPOSITION XIII.

Given *any* two proportions in which three terms of one are respectively equal to three terms of the other; *can you prove that the fourth terms are equal?*

[The proof should comprehend all possible proportions.]

LIMITS.—PRELIMINARY QUESTIONS.

193.

[The pupil should carefully review *measurement* under Proportion.]

1. If m and n are two points a given distance apart and A move from m directly toward n and travel one-half of the distance from m to n the first day, and then one-half the remaining distance the second day, and *then* one-half the distance *still remaining* the third day, etc., will A ever reach n?

2. Draw a diagram showing A at the end of each of the three days; also show the position of A at the end of the fourth day.

3. What can you say of the distance from m to A at the end of each succeeding day? What is the *limit* of that distance, or the distance which m A can *never reach?*

4. What can you say of the distance from n to A from day to day? Has n A a *limit* which it can never reach?

5. Would you call m A and n A *variable quantities* or

9—

variables? Can you distinguish between them and again tell the *limit* of each?

6. Does the distance which A travels also *vary* from day to day? Name three *variables* in the preceding discussion. Which two of the variables are of the *same kind?* What is the *limit of each?*

7. If A travel a uniform distance, say c, each day, would you call c a *variable?* Would A ever reach n? How many days are required for A to reach n?

8. Define a *variable;* a *constant.* Do you think $m\,n$ is a *constant?*

[*Hint.*—Does its value remain *fixed* during this entire discussion?]

What else have you called $m\,n$ in this discussion?

9. As another illustration, consider the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

What kind of variable is the *last term* of the series?

[(1) Think of the series having two terms, (2) three terms, (3) four terms, etc.]

What does it *approach as a limit?*

What is the sum of the terms of the series when three terms are taken?

What is the sum of the terms of the series when four terms are taken?

What is the sum of the terms of the series when five terms are taken?

What is the sum of the terms of the series when six terms are taken?

What is the sum of the terms of the series when seven terms are taken?

What other *variable* do you discover? What is its *limit?*

10. Given the line $m\,n$; bisect it at A; bisect A n at A′; bisect A′ n at A″, etc. Let x represent the distance from m to any of the points A A′, A″, etc. What do you call x? Why?

Does the value of $m\ n$ change in this discussion? Call $m\ n$, a. What is a? What does x *approach as a limit?*

(The sign $\doteq$ means *approach as a limit.*)

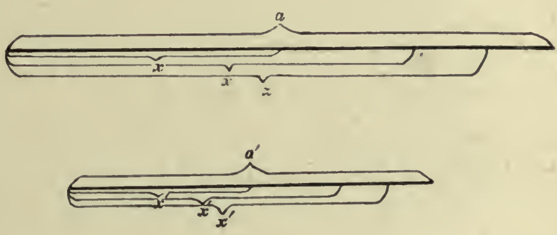
We write $x \doteq a$. *Read the statement.*

Is $a - x$ a variable? What does *it* approach as a limit? Make the statement using the above symbol.

194.

1. *Theorem.* We now wish to discuss two variables, each approaching its respective limit. *If the variables have a constant ratio, what can we say of their limits?*

2.



3. Let a and a' be two constants, and x and x' two variables approaching the respective constants as limits.

4. Also let $\dfrac{x}{x'} =$ the constant, r.

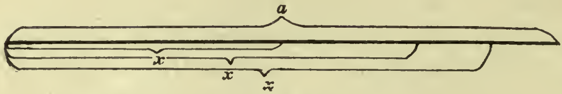
5. [*Note* the variable $x \doteq a$.]

6. [Also, *note* the variable $x' \doteq a'$.]

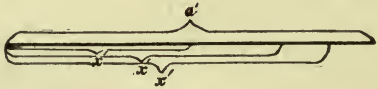
7. *Get a clear notion of the ratio,* $\dfrac{x}{x'}$. The *ratio is constant*, although x and x' are *variables.*

8. Those who fail to get a clear conception of the mean-

ing of the above should study the *concrete illustration* given below:



a = a constant (line 6 in. long).



a' = a constant (line 4 in. long).

For the *purpose of illustration* we take x, x', a, a', *known* lines. In the *theorem* x, x', a, a' are *unknown* quantities; it is *only known* that $\frac{x}{x'} = r$, and that r is a *constant;* and we *wish to prove* that $\frac{a}{a'} = \frac{x}{x'}$, or $\frac{a}{a'} = r$

Note variables x and x'.

(1) If we *first* take $x = 1\frac{1}{2}$ inches and $x' = 1$ inch, we have $\frac{x}{x'} = \frac{1\frac{1}{2}}{1} = \frac{3}{2} = r$.

(2) If we next take $x = 3$ inches and $x' = 2$ inches, we have $\frac{x}{x'} = \frac{3}{2} = r$.

(3) If we next take $x = 4\frac{1}{2}$ inches and $x' = 3$ inches, we have $\frac{x}{x'} = \frac{4\frac{1}{2}}{3} = \frac{3}{2} = r$.

Note $\frac{x}{x'}$ or r in *each case* is $\frac{3}{2}$.

(4) No matter how closely $x \doteq a$, and $x' \doteq a'$, $\frac{x}{x'} = r$ $(\frac{3}{2})$.

In this illustration we easily see that $\frac{x}{x'} = \frac{a}{a'}$, because a and a' are known, being 6 inches and 4 inches, respectively.

9. In order to grasp the full force of this proof it is *absolutely necessary* for the pupil—

(1) To get a *clear idea* of the meaning of *each term* used,

(2) And to hold that idea firmly in mind during the entire discussion,

(3) And to understand fully the authority for each statement made,

(4) And to keep constantly in mind the steps whereby each statement is established.

To prove: $\frac{a}{a'} = \frac{x}{x'} = r.$

Proof: $\frac{a}{a'} =, >, \text{or} < r.$ [?]

I. Prove $\frac{a}{a'}$ is not *greater* than r. Suppose, *if possible*, that—

(1) $\frac{a}{a'} > r;$

(2) Then $\frac{a}{a' + b} = r.$

[When the numerator is constant, the value of a fraction is decreased by increasing the denominator. Then let b represent the required increase.]

(3) Also, $a = (a' + b)\,r.$ [?]

(4) But $x = r\,x';$ [?]

(5) And $a - x = (a' - x' + b)\,r.$ [?]

(6) Now $a - x \doteq 0;$ [?]

$\therefore$ $a - x$ may be made as small as we choose.

(7) And $(a' - x')\,r + b\,r = (a' - x' + b)\,r.$ [?]

(8) $b\,r =$ a constant positive quantity. [?]

(9) $(a' - x')\,r =$ an indefinitely small but positive quantity. [?]

(10) Then $b\,r > a - x.$ [?]

(11) But $(a' - x' + b)\,r > b\,r;$ [?]

(12) $\therefore (a' - x' + b)\, r > a - x$. [?]

But, *above* (in No. 5) $a - x = (a' - x' + b)\, r$, *which is absurd.*

[N. B. (5) is reached by *supposing* $\frac{a}{a'} > r$, but (12) is based upon (6), (7), (8), (9), (10), and (11), all of which are *proved* by Geometry.]

(13) Hence $\frac{a}{a'}$ is not greater than r. [?]

II. Prove $\frac{a}{a'}$ is not less than r.

(1) Suppose, *if possible*, that $\frac{a}{a'} < r$.

(2) Then $\frac{a}{a' - b} = r$, [?]

(3) And $a = (a' - b)\, r$. [?]

(4) But $x = x'\, r$; [?]

(5) $\therefore a - x = (a' - x' - b)\, r$. [?]

(6) But $a - x$ is positive, [?]

(7) And $(a' - x' - b)\, r$ is negative; [?]

8) $\therefore$ No. 5 *is absurd*, [?]

(9) And $\frac{a}{a'}$ is *not* less than r. [?]

[The final reasoning is left to the pupil.]

195.

Cor. I. If, while approaching their respective limits, two variables are always equal, are their limits equal?

Give the proof of your answer.

196.

Cor. II. If, while approaching their respective limits, two variables have a constant ratio, and one of them is always greater than the other, what do you conclude about their respective limits?

Prove your answer.

PROPORTIONAL LINES.

1. When a line drawn parallel to the base of a △ bisects one side, how will it meet the other side?

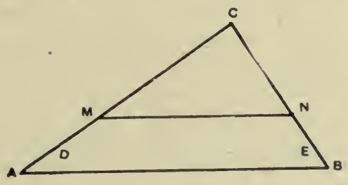
2. When a line is drawn parallel to one of the parallel sides of a trapezoid and bisects one of the non-parallel sides, how does it meet the other non-parallel side?

3. When parallel lines cut off equal parts on a given transversal, how will they meet any other transversal?

4. Draw a scalene triangle, A B C, and a line $m\ n$ ∥ to A B cutting off $\frac{1}{6}$ of A C; what part of B C is cut off?

197.

PROPOSITION XIV.



Let A C B be *any* △ and M *any* point in A C, and M N ∥ to A B. *Compare the ratio of the parts of A C (A M : M C) with the ratio of the parts of B C (B N : N C).*

Case I. Suppose that C M and M A are *commensurable.* [Review § 172.] Apply a common measure, k, to C M and M A.

Then A M $= p\,k$, and M C $= q\,k$. [?]

Through points of division of A M and M C draw lines parallel to A B.

Then parts intercepted on B C are equal, [?] and B N = $p\,l$ and N C = $q\,l$. [?]

[What is l?]

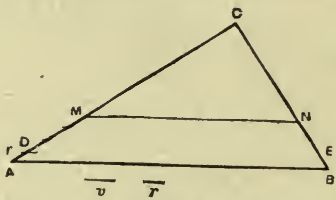
Then $\frac{\text{A M}}{\text{M C}} = ?$ [?]

And $\frac{\text{B N}}{\text{N C}} = ?$ [?]

$\therefore \frac{\text{A M}}{\text{M C}} = \frac{\text{B N}}{\text{N C}}$.

Case II. Suppose A M and M C are *incommensuralle.*

Divide C M into any number, say s, equal parts. Call each part v. Apply v to M A; it is contained in M A a given number of times, say t, with a remainder, r, which is less than v. Letter the point at the end of the last measurement D, and through D draw a line || to A B, intersecting C B at E.



(1) Then C M : M D = C N : N E. [?]

Now v may be made very small, consequently r will approach 0 as a limit. But no matter how small v becomes, (1), above, holds true.

What does M D approach as a limit?

What does N E approach as a limit?

What does the ratio C M : M D approach as a limit?

What does the ratio C N : N E approach as a limit?

Why then does C M : M A = C N : N B?

Write and carefully prove Prop. XIV.

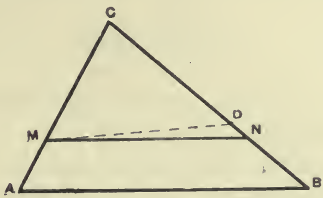
198.

Cor. Prove that a line drawn || to the base through any point in the side of the △ divides the other side so that a side is to either segment as the other side is to its corresponding segment.

199.

Proposition XV.

What is the converse of Prop. XIV.?



Given △ A C B and points M and N so that $\frac{CM}{MA} = \frac{CN}{NB}$.

Can you prove M N ∥ to A B?

If not, draw M D ∥ to A B.

What proportions follow? Compare with proportion given by Hyp. and show absurdity.

[*Hint.*—Try to show that C D = C N. Consult § 192 if you fail.]

State and prove Prop. XV.

200.

Proposition XVI.

1. Draw a scalene △ having sides 4, 5, and 6. Bisect each ∠ in turn, and each time compare the ratio of the sides of the △ including the angle bisected with the ratio of the segments of the opposite side when, if necessary, the bisector is produced to cut it.

2. Draw other scalene △s and make similar comparisons.

[*Hint.*—Much care must be taken in the above drawings in order to reach any definite conclusion by the comparisons.]

Try to prove that the principle you have discovered is geometrically true. If *you fail*, consult the *hints* given below.

[*Hints.*—Produce either side of the ∠ bisected and through the extremity of the other side draw a line parallel to the bisector. Produce these new lines until they meet, forming a new △ wholly exterior to the original △. Then prove this new △ is isos., and then use Prop. XIV.]

Exercises.

138. The sides of a △ are 5 and 9 feet. A line drawn ∥ to the base divides the one into segments of $1\frac{1}{2}$ and $3\frac{1}{2}$ feet; calculate the segments of the other side.

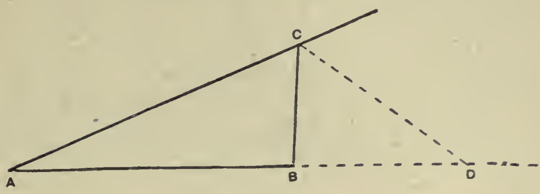
139. The sides of a △ are 6 and 4 feet, and the base is 10 feet. (1) Calculate the segments of the base made by the bisector of the vertical angle. (2) Made by the bisector of each of the other ∠s when the side opposite is considered the base of the △.

140. A line drawn ∥ to one of the parallel sides of a trapezoid cuts the non-parallel sides proportionally, and each non-parallel side is to each of its segments as the other non-parallel side is to its corresponding segment.

141. When three or more parallel lines are cut by two transversals, the segments of either transversal are proportional to corresponding segments of the other transversal, and either transversal included by the parallels is to each of its segments as the other transversal included by the parallels is to its corresponding segment: and if these transversals meet when produced, any part between the point of intersection and a parallel is to any of its intercepts as the corresponding part of the other transversal is to its corresponding intercept.

201.

Proposition XVII.



Draw any scalene △ whose sides are known to you, A B C. Produce either side, A C. Bisect the exterior ∠ and produce the bisector until it meets the opposite side, A B produced, at D.

Compare the ratio of A D to B D with that of A C to C B.

Do not jump to conclusions, but draw different shaped △s and produce each side in order and bisect exterior angle and make comparisons.

What truth do you discover?

Try to prove it by Geometry. If *you fail*, try again, using all you know about methods for discovering original demonstrations.

The hint below is given for those who finally *fail after repeated trials*.

[*Hint.*—Through B draw line || to D C and study the segments of the sides of △ A D C, etc., etc.]

202.

Definitions.

A line may be divided into segments in two ways: (1) when the point of division is *in the line*, (2) when the point of division is in *the* line *produced*. The line is said to be divided *internally* in the first cases, and *externally* in the second. In each case the segments are the distances from the point of division to the extremities of the line. In Prop. XVII. A B is said to be divided *externally* at *D*. What are the segments of A B when divided *externally* at D?

Draw a line and divide it (1) internally, (2) externally. Tell the segments in each division.

EXERCISES.

142. State and prove the converse of Prop. XVI.

143. State and prove the converse of Prop. XVII.

203.

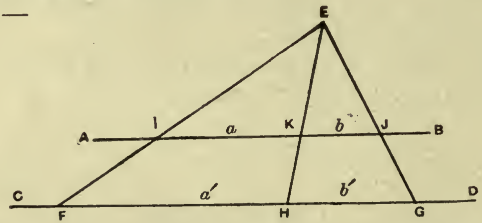
PROPOSITION XVIII.

1. Draw two parallels 5 in. and 3 in. in length. Join their extremities and produce the lines until they meet. Through this point draw a transversal which cuts off one-half of the longer parallel. How is the shorter parallel cut? Draw a transversal which cuts off one-fourth the shorter parallel. How does it cut the longer parallel?

2. Draw any two parallels and draw three or more transversals which meet at a common point. Compare the intercepts on one parallel with the intercepts on the other parallel. *Are the intercepts proportional?*

State and prove Prop. XVIII.

[*Hint.*—



Prove $a' : a = b' : b$. Draw through I and J lines parallel to E H and study $\triangle$s E F H and E G H. Also study the quadrilaterals formed.]

204.

Cor. In the above figure prove $a' : a = \text{F E} : \text{I E} = \text{H E} : \text{K E}$, etc.

SIMILAR FIGURES.

205.

When are two figures equivalent? equal? May a square *equal* a triangle? a rectangle?

(1) *Equal figures* are not only equal in area, but they are *similar in form;* as we have learned, they can be made to *coincide in all their parts.*

(2) *Equivalent figures are equal in area,* but they are not necessarily similar in form.

(3) *Similar polygons* are mutually equiangular, and their homologous sides are proportional?

Draw any △ and assume any side as the base; then draw a line which is parallel to the base and which cuts off $\frac{1}{4}$ of one side. How does it cut the other side? Note the two triangles. Are they equiangular? [?] Are their sides proportional? [?] Are they *similar?*

4. *Homologous parts*, sides, angles, etc., are those which are similarly situated.

5. *Similar arcs* subtend equal central angles; *similar sectors and segments* are those whose arcs subtend equal central angles.

Are all circles similar?

206.

PROPOSITION XIX.

(1) Draw any triangle, assume any side the base, and fix any point in either of the sides. Let the △ be A B C, and A B the base, and P the point fixed in A C. Now draw a line through P which cuts B C at D and makes the ∠ C P D = ∠ C A B. Is P D parallel to the base? Can you prove the triangles similar?

(2) Given the triangles A B C and D E F in which angles A, B, and C are respectively equal to angles D, E, and F. *Can you prove the triangles similar?*

Can you prove two triangles similar *which are mutually equiangular?*

[See (2) above.]

What *two* properties have similar polygons? [See § 205, 3.]

[*Hint.*—Superpose.]

If *you fail*, after using the above exercises and hints as helps, see the further hint given below.

[*Hint.*—Superpose, placing an ∠ upon its equal ∠ and similar side upon similar side. Show that sides are parallel; ∴ sides are proportional.]

207.

Cor. I. Given any two triangles in which two angles of one equal two angles of the other. Prove the triangles similar.

208.

Cor. II. Given two rt. triangles in which an acute ∠ of one equals an acute angle of the other. Prove the triangles similar.

209.

Cor. III. Given two triangles similar to a third triangle. Make deduction and prove it.

Exercises.

144. Given two unequal lines, *a* and *b*. Construct two squares having these lines respectively for sides. Can you prove these squares similar? Make general conclusion.

145. Given two equilateral triangles having unequal sides Can you prove them similar?

146. Given two isosceles △s having vertical angles equal. Can you prove them similar?

147. Given two regular hexagons. Prove that they are similar.

148. Given two regular polygons, each having n sides. Prove that they are similar.

210.

PROPOSITION XX.

You have proved in Prop. XIX. that if two triangles are mutually equiangular, the corresponding sides are proportional, and consequently the triangles are similar.

Now can you prove that two triangles are similar if the *sides of one are proportional to the sides of the other*, *each to each?*

This proof is difficult, so *do not get discouraged* if you fail. But make a stubborn fight before you give it up. Review all the suggestions given to pupils for attacking original demonstrations.

If, having exhausted all your powers, you fail, consult the "hint," prove the Prop. yourself without further reading just as soon as you discover the proof, and compare your proof with the rest of the "hint." Possibly, *part* of your proof will be *original*. Try to see just *what effort you failed to put forth* in your trying to discover the demonstration. This may give success to future efforts.

[*Hint.*—Given the two triangles A B C and D E F, in which A B : D E = B C : E F = A C : D F. (Pupil may draw the △s having sides 4″, 5″, and 6″, and 6″, 7½″, and 9″, respectively. Make the sides of A B C longer than those of D E F.) On C A measure off C G = F D and on C B measure off C H = F E. Draw G H. (Now try to prove △s C G H and C A B similar. Then try to prove △s C G H and D E F equal.)

(1) A C : D F = B C : E F. [?]
(2) Then A C : G C = B C : H C. [?]
(3) $\triangle$s A B C and G H C are similar. [?]
(4) A C : G C = A B : G H. [?]
(5) A C : D F = A B : D E. [?]
(6) A C : G C = A B : D E. [?]
(Compare (4) and (6) and make deductions.)
(7) $\therefore$ G H = D E. [?]
(8) And $\triangle$s G C H and D E F are equal. [?]
(9) Also $\triangle$s A B C and D E F are similar, Q. E. D.]

We have seen that if triangles are equiangular, their sides must be proportional, and conversely. Would you form the same conclusion about equiangular quadrilaterals and other equiangular polygons?

(1) Compare a square with a rectangle, a rhombus with a rhomboid. (Draw figures.)

(2) Also compare a square with a rhombus, a rectangle with a rhomboid. (Their sides may be equal or proportional.)

(3) What conclusion do you reach?

211.

Proposition XXI.

Draw two $\triangle$s A B C and D E F, making A B = 10″, A C = 8″, B C = 6″, and D E = 5″, D F = 4″, E F = 3′. Are the $\triangle$s similar? Why?

Draw the altitudes upon A B and D E. Compare them. Do they have the same ratio as A B and D E? as A C and D F? as B C and E F?

Draw the other altitudes and compare their ratio with that of any two homologous sides.

Draw other similar triangles and make further comparisons of the ratio of similar altitudes to the ratio of homologous sides.

What deduction can you make of the *altitudes of all similar triangles?*

Write Prop. XXI. and its formal proof.

What is the converse of Prop. XXI? Can you prove it?

EXERCISES.

149. In a city are two rectangular lots, one is 50 feet by 150 feet, and the other is 100 feet by 250 feet. Are they similar? Why?

150. Draw any scalene triangle. Can you draw an isosceles triangle similar to the triangle drawn? Prove your answer.

212.

PROPOSITION XXII.

Problem. To divide a line into parts proportional to two or more given lines.

There are two equal fractions whose numerators are 3 and 4, respectively, and the sum of whose denominators is 49. Required the denominators.

(1) Solve by Algebra.

(2) Solve by Geometry.

[*Hint.*—Use lines to represent the numbers.]

[*Hint.*—If *you fail* on (2), draw any scalene △ and fix a point in one side. Note how the point divides the side. Now, how can you divide the other side into parts proportional to these two?]

There are two equal fractions, the first is $\frac{2}{5}$ and the numerator of the second is 3. Required the denominator of the second.

(1) Solve by Algebra.

(2) Solve by Geometry.

213.

Proposition XXIII.

Problem. To find a fourth proportional to three given lines.

214.

Proposition XXIV.

(1) Draw any acute ∠, a, and any line, b. Then draw two ∠s, each equal to ∠ a, but the sides of one $3b$ and $4b$, while the sides of the other are $6b$ and $8b$. Join the extremities of these sides, forming two △s. Compare the remaining ∠s of these △s. Compare the remaining sides of these △s. What deduction can you make respecting the two triangles?

(2) Draw any obtuse ∠, c, and any two unequal lines, d and e. Then draw two △s each having an ∠ = ∠ c and the sides including ∠ c of one $2d$ and $3e$ and the sides including ∠ c of the other $6d$ and $9e$.

Make deductions respecting these △s.

(3) What deduction *in general* about △s that have an ∠ *of one equal to an* ∠ *of the other and the sides including the equal angles proportional?*

State and prove Prop. XXIV.

Exercises.

151. Two sides of a △ are 8 inches and 10 inches, and the base is 12 inches. If a line 9 inches long, parallel to the base, terminates in the sides, what are lengths of the segments of the sides?

152. The sides of a triangle are 5, 7, and 9, and the shortest side of a similar △ is 14. Find the remaining sides.

153. The base of a triangle is 10″ and the altitude is 6″. In a similar triangle, the base is unknown and the altitude is 15″. What is the base?

154. Given the scalene acute-angled △ having the sides a, b, c, and the altitude upon a, m; the altitude upon b, n; and the altitude upon c, o. Prove that

(1) $a : b = n : m$.

(2) $a : c = o : m$.

(3) $b : c = o : n$.

Write general statement of the altitudes of a triangle. [If you fail in proof, see § 208.]

215.

Proposition XXV.

(1) Draw two ∠s having sides respectively parallel and extending the same direction from the vertex. What deduction can you make respecting these angles? Prove it.

[*Hint.*—Produce a side of one until the non-corresponding side, or that side produced, is cut.]

(2) Draw two ∠s having sides respectively parallel and extending respectively in opposite directions from the vertex.

[Note.—Such lines are said to be *anti*-parallel. In (1) they are said to be sym-parallel.]

Can you make the same deduction about these ∠s that you made in (1)?

(3) Draw two ∠s, a side of one sym-parallel to a side of the other. but the other side of the first anti-parallel to the other side of the second.

Make deduction and prove it.

Make a general statement respecting the three deductions made and proved.

Use the fewest possible words necesary to make the statement *clear* and *comprehensive.*

216.

Cor. I. (1) Can you draw two unequal △s having all pairs of corresponding sides parallel and extending the same direction from the vertex, or sym-parallel? If so, what deduction can you make of these △s?

(2) Can you draw two unequal △s having all pairs of corresponding sides anti-parallel? If so, what deduction can you make?

(3) Can you draw two unequal △s having from the vertex of each angle one pair of sides sym-parallel while the other pair are anti-parallel?

Prove your answer.

What general conclusion do you reach about two △s whose corresponding sides are parallel?

[Note that parallel lines may be either sym-parallel or anti-parallel.]

State and carefully prove Cor. I., Prop. XXV.

217.

Proposition XXVI.

(1) Draw two acute ∠s having the sides of one perpendicular to the sides of the other. Make deduction and prove it.

(2) Draw two obtuse ∠s having the sides of one perpendicular to the sides of the other. Make deduction and prove.

(3) Draw an acute angle and an obtuse angle having the sides of one perpendicular to the sides of the other. Compare the angles and prove deduction made.

(4) What general deduction can you make about any two ∠s which have their sides respectively perpendicular?

State and prove Prop. XXVI.

218.

Cor. I. (1) Draw two acute-angled triangles having the sides of one perpendicular to the sides of the other. Compare the angles included by the sides which are respectively perpendicular to each other.

(2) Draw two obtuse-angled triangles having sides respectively perpendicular. Make deductions and prove them.

Make further drawings in which triangles have their sides respectively perpendicular.

State Cor. I., Prop. XXVI., and prove it.

Be sure that your proof comprehends all possible conditions.

219.

PROPOSITION XXVII.

Draw a rt. △ and consider the hypotenuse the base. Draw the altitude and compare —

(1) The original rt. △ with each △ formed by the altitude;

(2) The rt. △s formed by the altitude;

(3) The altitude with the segments of the hypotenuse;

(4) Each leg with the whole hypotenuse and the adjacent segment.

State Prop. XXVII., which should comprehend the four deductions made.

220.

Cor. I. Prove that the square of the altitude upon the hypotenuse of a rt. △ equals the product of the segments of the base.

221.

Cor. II. When the altitude of a rt. △ upon the hypotenuse is drawn, the square of either leg is equal . . . .

[Pupil will complete the sentence and prove.]

222.

Cor. III. The squares of the legs of a rt. △ are proportional to the adjacent segments made by the altitude drawn to the hypotenuse.

[*Hint.*—Use § 221.]

223.

Cor. IV. Draw a semicircle and fix any point in the circumference. Draw a perpendicular from the fixed point to the diameter. (1) Compare the perpendicular with the segments of the diameter. (2) Draw chords from the fixed point to the ends of the diameter. Compare each chord with the diameter and the adjacent segment.

State Cor IV., Prop XXVII.

224.

PROPOSITION XXVIII.

Problem. To construct a mean proportional to two given straight lines.

EXERCISES.

155. Construct a square equivalent to a given rectangle (assuming that the area of a rectangle equals the product of the base by the altitude).

156. Construct a square equivalent to a given rt. △.

157. Given the side of a square and one side of an equivalent rectangle. Construct the square and the rectangle.

Consult the hint if *you fail*. It is not difficult. Do not give up easily,

[*Hint.*—I. Can you find a *third proportional* to two given lines? or, II. Construct Ex. 156 again and try to determine how you could replace one segment of the hypotenuse if it and the semicircumference were erased. If you still fail, construct the entire circle in the above; produce the altitude below the diameter until the lower semicircumference is cut. How many points on the circumference do you now have? How many of these points would be erased if the required segment of the diameter (required side of the rectangle) were erased? How many points on the circumference are required to determine the center?]

225.

PROPOSITION XXIX.

Carefully draw the figure and review Section 221. What does the square of each leg equal? What does the sum of the squares equal?

Compare the sum of the squares with the square of the hypotenuse.

State Prop. XXIX.

226.

Cor. I. What does the square of either leg equal when compared with the square of the hypotenuse and the square of the other leg?

EXERCISES.

158. Construct a square having twice the area of a given square.

159. Construct a square having the area (1) of three unequal given squares, (2) of three equal given squares.

160. Given 3 unequal lines, a, b, and c. Construct a square equivalent to $2a^2 + b^2 - c^2$. Is this ever impossible?

161. Draw an indefinite straight line and cut off equal distances. Using these parts, construct (1) $\sqrt{2}$, (2) $\sqrt{3}$, (3) $\sqrt{5}$ (two ways), (4) $\sqrt{6}$, (5) $\sqrt{8}$ (two ways), (6) $\sqrt{29}$ (two ways), (7) $\sqrt{24}$ (two ways).

[*Hint.*—Make a list of perfect squares. Find sums, difference, etc.]

162. What is the length of the tangent to a circle whose diameter is 14, from a point whose distance from the center is 25.

163. Lay off equal parts on an indefinite line and construct $\sqrt{3}$. Calculate the altitude of an equilateral $\triangle$ whose base is $4\sqrt{3}$.

Construct the $\triangle$ and compare its altitude with result obtained by calculation.

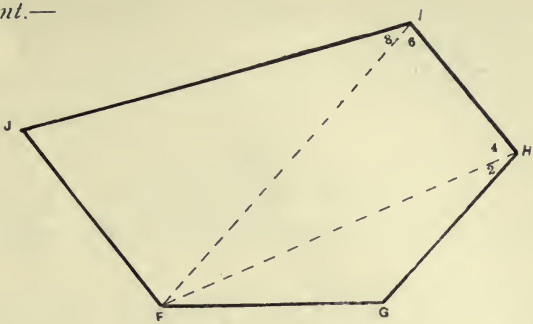
227.

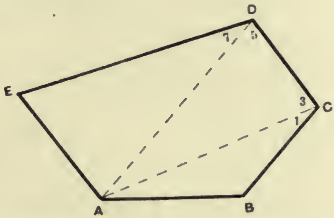
Proposition XXX.

Given two polygons which are composed of the same number of triangles which are similar each to each and are similarly placed. Compare the polygons.

What are similar polygons? Make deduction and prove it.

[*Hint.*—





Given the polygons A B — E and F G — J, composed of the similar △s, which are similarly placed.

(1) A B C and F G H.

(2) A C D and F H I.

(3) A D E and F I J.

To prove the polygons similar.

Proof:

$\angle 1 = \angle 2$; [?]

$\angle 3 = \angle 4$; [?]

$\therefore \angle C = \angle H$. [?]

In like manner we can prove other corresponding $\angle$ s of the polygons equal.

$\frac{A\ C}{F\ H} = \frac{C\ B}{H\ G}$; [?]

$\frac{A\ C}{F\ H} = \frac{C\ D}{H\ I}$; [?]

$\therefore \frac{C\ B}{H\ G} = \frac{C\ D}{H\ I}$. [?]

In like manner the remaining corresponding sides may be proved proportional.

$\therefore$ the polygons are similar. [?]]

228.

Proposition XXXI.

State and prove the converse of Prop. XXX.

See "hint," if you fail.

[*Hint.*—Given two similar polygons. Can you prove them composed of the same number of $\triangle$s, similar each to each and similarly placed?

(Redraw the figure of Prop. XXX.)

(1) $\angle E = \angle J$; [?]

(2) $\frac{AE}{FJ} = \frac{ED}{JI}$; [?]

(3) $\therefore$ $\triangle$s A E D and F J I are similar. [?]

(4) $\angle D = \angle I$. [?]

(5) $\angle 7 = \angle 8$. [?]

(6) $\angle 5 = \angle 6$. [?]

(7) $\frac{ED}{JI} = \frac{DA}{IF}$. [?]

(8) $\frac{ED}{JI} = \frac{DC}{IH}$. [?]

Pupil will finish the proof.]

229.

Proposition XXXII.

Given two similar polygons. Compare the ratio of their perimeters with the ratio of any pair of homologous sides.

[*Hint.*—Draw similar polygons having sides, a, b, c, d, etc., and a', b', c', d', etc. Then we are required to prove that

$$\frac{a + b + c + d + \ldots.}{a' + b' + c' + d' + \ldots.} = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \ldots.]$$

EXERCISE.

164. The perimeters of two similar polygons are 119 and 68; if side of the first is 21, what is the homologous side of the second?

230.

Definition. (1) The *projection* of a point upon a straight line is the foot of the perpendicular drawn from the point to the line.

(2) The projection of a finite straight line upon a line is that portion of the second line included between the projections of the ends of the finite line.

EXERCISES.

165. I. Draw a scalene △ and assume the longest side the base.
    - (1) What is the projection of the vertex upon the base?
    - (2) Show the projection of each side upon the base.

  II. Draw a rt. △ and assume either leg the base.
    - (1) What is the projection of the hypotenuse upon the base?
    - (2) What is the projection of the vertex?
    - (3) What is the projection of the other leg?

  III. Draw any obtuse scalene △ and assume the shortest of the two sides including the obtuse ∠ the base.
    - (1) What is the projection of the vertex of the △ upon the base, or the base produced?
    - (2) What is the projection of the side opposite the obtuse ∠?
    - (3) What is the projection of the other side including the obtuse ∠?

Compare these projections with those when each of the other sides of the △ is assumed the base.

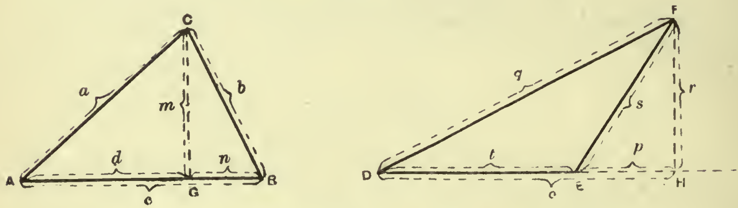
166\. (1) Can the projection of a line ever be longer than that line?

(2) When does the projection of a line equal that line?

(3) When is the projection of a line shorter than the line? When is it a point.

231.

Proposition XXXIII.



1\. In the acute-angled $\triangle$ A B C what is the projection of B C? of A C? of C?

2\. It is desired to find the relation the square of a side opposite an acute $\angle$ (in the $\triangle$ A B C *all* the $\angle$s are acute) to the squares of the other sides and the rectangle formed by one of them and the projection of the other side upon it.

If we take $\angle$ A, we wish to find the relation of b^2 to a^2, c^2, and rectangle formed by c and d.

(1) $b^2 = m^2 + n^2$; [?]

(2) $m^2 = a^2 - d^2$; [?]

(3) $n^2 = (c - d)^2 = c^2 + d^2 - 2\,c\,d$; [?]

$\therefore$ $b^2 = ?$ [?]

State the deduction proved.

Let the pupil find the value of a^2 [opposite $\angle$ B] in terms b, c, and n; also the value of c^2.

3\. In the obtuse-angled $\triangle$ D E F find the value of s^2, or of t^2.

(1) *Of* s^2 in terms of q^2, t^2, and *ot*. $s^2 = r^2 + p^2$. [?]
Then find the value of r^2 and p^2 and substitute.

(2) Find the value of t^2.

Make general deduction about the square of the side opposite an acute angle in any $\triangle$.

232.

Proposition XXXIV.

What does the square of any side opposite an obtuse $\angle$ in any $\triangle$ equal?

[See $\triangle$ D E F, § 231.]
[What is the projection of s upon t (produced)?]
Discuss all possible cases and form general conclusion.

233.

Proposition XXXV.

Draw any $\triangle$ and assume any side the base and draw the median from the vertex to the base.

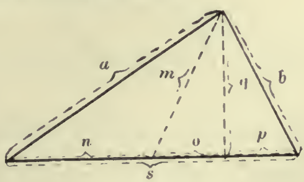
1. Compare the sum of the squares of the other two sides of the $\triangle$ with the square of the median and the square of half the base.

(What does the sum of the squares of the sides equal in terms of median and half the base?)

2. Compare the difference of the squares of the other two sides with the rectangle formed by the base and the projection of the median upon the base.

State the general conclusion comprehending the results of both of the above comparisons.

[*Hint.*—



(1) $a^2 = (n + o)^2 + q^2$, [?]
(2) $b^2 = p^2 + q^2$, or
(2) $b^2 = (n - o)^2 + q^2$; [?]
(3) $\therefore a^2 + b^2 = 2\,n^2 + 2\,m^2$. [?] Or,
(4) $a^2 - b^2 = 4\,n\,o = 2\,s\,o$. [?]]

234.

PROPOSITION XXXVI.

Fix any point in a circle and through it draw any two chords.

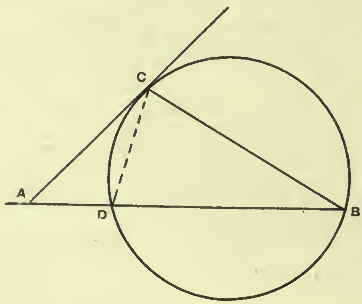
Compare the product of the segments of one chord with the product of the segments of the other. What conclusion do you reach?

Write and prove Prop. XXXVI.

[*Hint.*—Form two $\triangle$s by joining the ends of the chords. Compare the $\triangle$s.]

235.

PROPOSITION XXXVII.



Given: A secant, A B, and a tangent, A C, drawn from a point without the circle.

Compare the square of the tangent with the product of the secant and its external segment.

State and prove Prop. XXXVII.

[*Hint.*—Note the $\triangle$s, which are similar? What proportion can you form using A B, A D, and A C?]

236.

Cor. I. The tangent is a mean proportional between....

Finish and prove.

237.

Cor. II. Draw another secant and compare the products of each whole secant and its external segment.

Exercises.

167. Construct a fourth proportional to 3 given lines, using § 237.

168. The length of the common chord of two intersecting circles is 16, and their radii are 10 and 17. What is the distance between their centers?

169. C and D are the middle points of a chord, A B, and and its subtended arc. If A D = 13 and C D = 12, what is the diameter of the circle?

170. In a scalene △ two sides are 8″ and 5″ and the projection of the median drawn to the third side upon that side is 3″.

Required the third side.

171. The three sides of a △ are x, y, and z. If $x = 8$, $y = 6$, and the projection of y upon $x = 1\frac{7}{12}$; calculate z, z being opposite an acute ∠.

172. The radius of a circle is 8″, and a tangent to the circle is 15″. What is the length of a secant drawn from the same point as the tangent, (1) if the secant passes through the center, (2) if the distance from the center to the secant is 5″?

173. A side of a rhombus is 10 and one diagonal is 52. Find the other diagonal.

238.

PROPOSITION XXXVIII.

Draw any scalene triangle and bisect the vertical angle; produce the bisector to meet the base.

Compare the rectangle formed by the two sides of the △ with the rectangle made by the segments of the base plus the square of the bisector.

[*Hint.*—Circumscribe a circle about the △. Produce the bisector until it becomes a chord of the circle. Join the end found to that vertex of the △ made by the base and the shortest side. Make deductions. What equal ∠s do you find? What similar △s? What proportions involving sides of similar △s? involving intersecting chords?]

EXERCISES.

174. The base of a △ is 15″, and the sides are 7″ and 10″, and the length of the bisector of the vertical ∠ is 4″. Find the segments of the base.

175. The base of a △ is 20″, and the sides are 12″ and 14″. Calculate the bisector of the vertical angle when the shortest segment is 6″.

239.

PROPOSITION XXXIX.

Draw any △ and its circumscribing circle. Draw the altitude to the longest side as base and draw the diameter through the vertex of the △. Also join the other end of the diameter to the other end of the shortest side.

Compare the rectangle of the sides of the △ with the rectangle of the diameter and the altitude.

[*Hint.*—How many rt. △s in the figure? Can you discover any similar rt. △s?]

Draw the altitude to a shorter side and prove that the above deduction is true.

Exercises.

176. The sides of a △ are 12, 18, and 24 units, respectively. Find the diameter of the circumscribed circle.

Test the accuracy of your answer by a drawing.

177. Given the sides of the following △s. Is the greatest angle of each acute, right, or obtuse? Give full reason for each answer.

(1) 3, 5, and 6.
(2) 3, 4, and 5.
(3) 8, 9, and 12.
(4) 16, 17, and 25.
(5) 10, 11, and 16.

178. (1) Can you show that the diagonals of a trapezium divide it into four △s which are proportional?

(2) Draw three equal circles, tangent to each other, and join their centers. Compare this △ with the one formed by joining the points of contact.

179. A carpenter wishes to brace a corner post; the piece of timber he proposes to use is 7 feet long. If he sets the brace $4\frac{1}{2}$ feet from the foot of the corner post, how high will it reach?

180. What is the longest line that can be drawn in a room 20 feet by 30 feet by 10 feet?

181. A stair-builder wishes to cut a diagonal brace for a flight whose horses are 3 feet apart. If the horizontal distance between the extremities of the stairs is 8 feet, and the vertical distance 7 feet, how long must the brace be?

182. Produce two equal chords till they meet. What can you prove concerning the secants and their external segments?

183. What is the length of a rafter which projects over the outside of the plate 15 inches, if the comb of the house is 8 feet above the level of the plate and the house is 24 feet wide?

184. If two secants drawn from the same point without

the circle are equally distant from the center, how are they related?

185. Given any two intersecting circles and their common chord produced. From any point on this chord produced draw a tangent to each circle and compare them.

Supplementary Exercises.

186. If one leg of a right triangle is double the other, show how the perpendicular from the vertex of the right angle to the hypotenuse divides it.

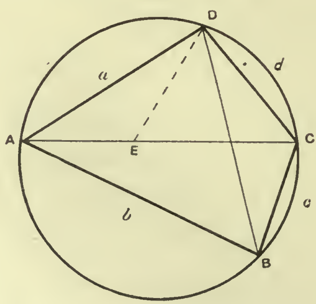
187. T and W are the mid points of a chord R S and its subtended arc respectively. If R W $= 9$, and T W $= 3$, what is the diameter of the circle?

188. Two secants are drawn to a circle from an outside point. If their external segments are 10 and 6, while the internal segment of the first is 5, what is the internal segment of the latter?

189. The sides of a $\triangle$ are $x\,y = 4$, $x\,z = 6$, $y\,z = 8$. Find the length of the bisector of $\angle\, x$.

240.

Proposition XL.



Given: A B C D any inscribed trapezium with A C and B D the diagonals.

To compare the product of the diagonals with the sum of the products of the opposite sides.

Sug. 1. Draw D E so that ∠ A D E = ∠ B D C.
Can you prove (1) △s E C D and A B D similar?
(2) D C : D B = E C : A B?
I. A B · D C = D B · E C?
How much of the required result has been found?
Sug. 2. Compare △s A D E and B D C.
Can you prove II. A D · B C = D B · A E?
What results if I. and II. are added and factored?
Generalize the truth reached.

241.

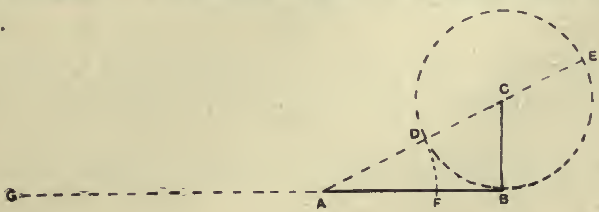
Definition: A straight line is said to be divided by a given point in *extreme and mean ratio* when one of the segments is a mean proportional between the whole line and the other segment.

The straight line A B is divided *internally* at C in extreme and mean ratio when A B : A C = A C : B C. It is divided *externally* at D in mean and extreme ratio when A B : A D = A D : B D.

242.

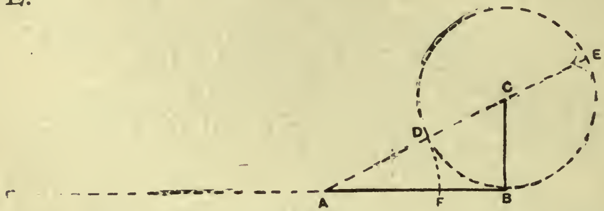
PROPOSITION XLI.

Problem. To divide a line in extreme and mean ratio.



Given: Straight line A B.
To Divide: A B in extreme and mean ratio.
Construction: Let B C be a perpendicular to A B at B,

and made equal to ½ A B. With C B as a radius, describe the circle C, and draw the line A C cutting the circumference at D and E. Make A F = A D, and produce B A making A G = A E.



Sug. What is the relation between A B and A E?

Can you show—

(1) A E : A B = A B : A F?

(2) A E − A B : A B = A B − A F : A F?

(3) A F : A B = F B : A F?

(4) A B : A F = A F : F B?

What can you say of (4)?

From (1), by composition (§ 185), show—

(1)′ A E + A B : A E : : A B + A F : A B.

(2)′ B G : A E = A E : A B.

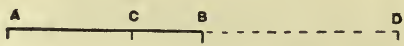
(3)′ B G : A G = A G : A B, or A B : A G = A G : B G.

What can you say of (3)′?

(Study this proposition carefully; it will be needed in future work.)

243.

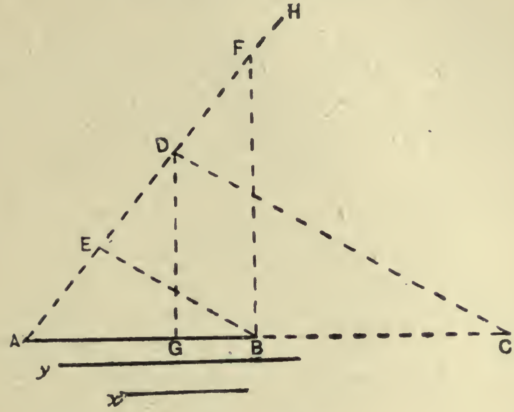
Definition: If a straight line is divided internally and externally in the *same ratio*, it is said to be divided *harmonically*.



If line A B has C and D located so that A C : C B = A D : B D, it is divided harmonically.

244.

PROPOSITION XLII.



Given: The straight line A B.

To divide it harmonically in a given ratio, as in the ratio of lines x and y.

Construction. Draw A H making any convenient angle with A B.

Measure off A D $= y$, and D E $=$ D F $= x$.

Join B E, B F, and through D draw D G $\parallel$ F B and D C $\parallel$ E B meeting A B produced in C.

Can you prove (1) A G : G B $=$ A D : D F $= y : x$?

(2) A C : B C $=$ A D : E D $= y : x$?

What conclusion can you draw? Is A B divided harmonically in ratio of y and x?

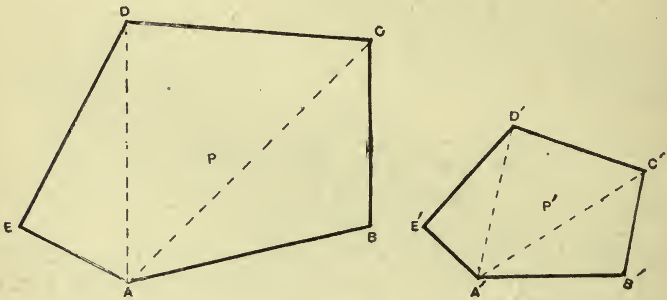
EXERCISE.

190. Can you prove that G C is divided harmonically at the points A and B?

245.

Proposition XLIII.

Problem. Upon a given line, homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.



Given: The line A′ B′ and the polygon P.

To Construct—On A′ B′, homologous to A B, a polygon, P′, similar to P.

Sug. Draw the diagonals A C and A D. Under what conditions will △ A′ B′ C′ be similar to △ A B C? §205.

When will P and P′ be similar? §205.

Supplementary Exercises.

191. To inscribe in a given circle a △ similar to a given △.

192. To circumscribe about a given circle a △ similar to a given △.

BOOK IV.

AREAS.

Quadrilaterals.

(1) What is meant by a unit of length? a unit of surface? a unit of volume? Give examples of each.

(2) What is meant by a line 5 yards long? a surface containing 25 square feet? a volume of 125 cubic feet? Give other examples.

(3) Illustrate the difference between equal and equivalent figures.

(4) What is meant by the base of a polygon? Illustrate.

(5) What is meant by the altitude of a polygon? Illustrate.

246.

Proposition I.

Draw 2 parallelograms on equal bases and having equal altitudes. Write the steps to show how they are related.

247.

Cor. Using the same figures, can you compare 2 △s having equal bases and equal altitudes?

Exercises.

193. What is the path of the vertex of a △ of constant area on a fixed base?

194. Draw a line from the vertex of a △ to the middle of the opposite side. How does it divide the △? Prove.

Draw lines from vertex to points which are distant $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ of the base from either extremity of it. Compare these △s.

195. Draw a parallelogram, A B C D. Join B D. Take any point, P, on B D and join with A and C. Compare △s D P C and D P A.

196. Draw 2 △s, A B C and A′ B′ C′, in which B C = B′ C′ and A C = A′ C′ and ∠ C = the supplement of ∠ C′. Can you prove the △s equivalent?

197. Construct △ A B C, then on the same base construct a rectangle having twice the area of the △. Prove.

248.

Proposition II.

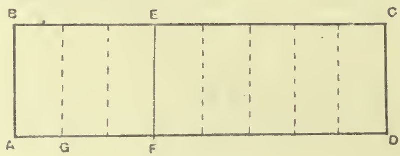


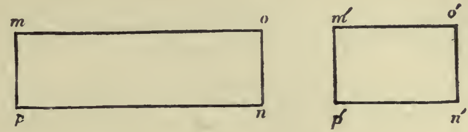
Fig. 1.

A G = the common linear unit of measurement.

Given: The two rectangles B D and B F with equal altitudes, but unequal bases. *Compare the rectangles.*

Case I. Let the base A D contain the unit A G 8 times, and the base A F contain the unit 3 times. Write an equation expressing the relation of A F to A D. Call it (1). At the points of division erect ⊥s to B C. Into what is B D divided?

B F? Express the relation between these two rectangles by an equation. Call it (2). Compare (1) and (2).



Given: The rectangle $m\,n$ and $m'\,n'$ with equal altitudes? Suppose $p\,n = 1$ dm. and $p'\,n' = \frac{1}{2}$ dm., using the mm. as the unit, what is the ratio of these bases? Erect ⊥s at the extremities of each mm. Compare the areas of $m\,n$ and $m'\,n'$. Are the bases commensurable?

In Fig. 1, could the relation between B D and B F be expressed if we used $\frac{1}{2}$ A G as the common unit? Does the relation between B D and B F change if the unit is halved? What effect does it have on the small rectangles when we decrease the size of the common unit? Could we express the relation between B D and B F if we took $\frac{1}{4}$ A G as a common unit? How small a fractional part may we take and still express the true relation?

Suppose we think of the common unit as 1 millionth of A G, how many rectangles may be formed in B F? in B D? in E D?

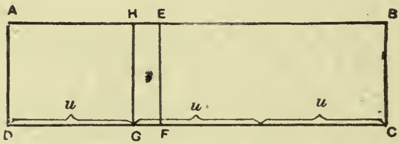
[*Review on Principles of Limits.*—(*Review* § 195.)

Let A M and A′ M′ be two *equal* variables which constantly $\doteq$* A B and A′ B′ respectively. Let us compare A B and A′ B′. If possible, *suppose* A B > A′ B′, measure off on A B a distance A C = A′ B′. What is the *limit* A M approaches?

(* $\doteq$ means approach as a limit.)

May the limit of A M pass A C? What is the *limit* A′ M′ approaches? Can A′ M′ ever reach A′ B′ or A C? What is the relation of A M and A′ M′? Do you see any absurdity in supposing A B $>$ A′ B′? Now *suppose* that A B $<$ A′ B′ and let A′ B′ be measured off on A B produced and let A D $=$ A′ B′. What will A M $\doteq$? A′ M′ $\doteq$? Can A′ M′ become greater than A B? Can A M become *equal* to A B? How are the variables A M and A′ M′ related? What absurdity by our last supposition? Now if A B *cannot* be greater than A′ B′ and it *cannot* be less than A′ B′, what condition must exist? Write a general statement of what we have proved and *memorize* it.]

Case II.



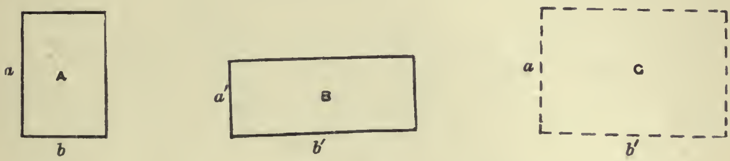
Let D C and D F in the 2 ▭s A C and A F be incommensurable, but having same altitude. Suppose a unit of length, u, is contained an exact number of times in D C, say 3 times, and in D G once, with remainder. What do we say of D C and D G? Erect a $\perp$ at G forming the ▭ A G. How are A G and A C related? Now imagine the unit of comparison to decrease. How does it affect D G? G F? What does D G $\doteq$? What does A G $\doteq$? Now do you see that $\frac{AG}{AC}=\frac{DG}{DC}$? Do you see that each member of the equation is a variable? Does each approach a limit? Are they always equal? What have we learned about two equal variables as they approach limits? Draw conclusion. Call it Prop. II.

249.

Which side of a rectangle may be considered the base? Can A D be considered the base of each of the rectangles, A F and A C? Can you then state a corollary from the preceding proposition?

250.

Proposition III.



Given the rectangles A and B with altitudes a and a' and bases b and b'. Let C be a third rectangle with altitude a and base b'. Compare A and C in the fractional form. Call the equation (1). Compare C and B in the same manner as in (1), call new equation (2). Multiply (1) by (2).

What does the new equation show?

Write a general statement and call it Prop. III.

Exercises.

198. Find the ratio of a rectangular field 72 rods by 49 rods to a board 18 inches by 14 inches.

199. A rectangular court-yard is 18½ yards by 15½ yards. Find the relation of its surface to a rectangular paving-stone 31 inches by 18 inches.

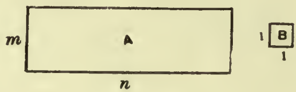
200. On a certain map the scale is 1 inch to 10 miles. How many acres are represented on this map by a square whose perimeter is 1 inch?

201. A square and rectangle have the same perimeter, 100 yards; the leugth of the rectangle is 4 times its breadth. Compare the areas.

251.

PROPOSITION IV.

What is meant by finding the surface of a plane figure? the numerical measure?



Given: The rectangle A and the square B. Let B be the unit of surface. How are these two figures related? Express this relation in a fractional form; clear of fractions. What is B?

What then does the area of A equal? What does this prove about the area of any rectangle?

Write the general truth and call it Prop. IV.

252.

As a corollary to this proposition, show how to find the area of a square. Suppose the sides of the rectangle are multiples of the linear unit, can you show any easy way to illustrate the area of the rectangle?

[Remark: Rectangle and triangle are often used for their areas. The *product of two lines* means the product of their *numerical measure*. The *unit of surface* means a square, each side of which is a *unit of length*. The measurement of a surface is the *number of times* it contains the unit of surface. It equals the *product of the numerical measures* of the base and altitude.]

253.

PROPOSITION V.

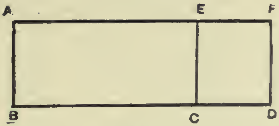


Fig. 1.

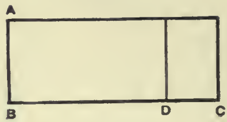


Fig. 2.

Given rectangles A C, A D in Fig. 1, and A D, A C in Fig 2. In Fig. 1 write an expression for B D. The sum of what rectangles = A D? The product of what lines = A C? E D? The product of what lines = A D? Place these three products into an equation. Can you interpret this equation in a general way? What does B D = in Fig. 2? What does it = in Fig. 1? What does A B · B D = in Fig. 1? in Fig. 2? Write the equivalent products for A B · B D in Fig. 1, and just under this equation write the equivalent products of A B · B D in Fig. 2. If we let a, b, c be the *numerical* measure of A B, B C, C D respectively, then by substitution in the last two equations we get $a\,(b \pm c) = a\,b \pm a\,c$.

254.

Cor. I. Let M and N each represent a line. $(M + N)^2$ equivalent to rectangle $(M + N)\,(M + N)$ equivalent to rectangle $M\,(M + N) + N\,(M + N)$ equivalent to M^2 + rectangle M N + rectangle M N + N^2 equivalent to $M^2 + N^2 + 2$ rectangle M N. Let m and n be the numeral measure of M and N. What algebraic expression represents the areas? Let M − N be the difference of two lines; work out the value of $(M-N)^2$. Write a general statement for $(M \pm N)^2$ in which M and N represent lines.

255.

Cor. II. In same general way write out the value of $(M + N)\,(M - N)$.

Construct figures to illustrate each of the above corollaries.

May some algebraic expressions have a geometrical interpretation? Illustrate.

EXERCISES.

202. Upon the diagonal of a rectangle 24 feet × 10 feet a △ equivalent to the rectangle is constructed. Find its altitude.

203. The area of a rectangle is 3456 square inches and the base is 2 yards. Find perimeter in feet.

204. A rectangle 18 × 6. Find the side of an equivalent square.

256.

PROPOSITION VI.

Compare the rectangle and rhomboid having the same base, of any length, and equal altitudes in each of the following cases:

Fig. 1.

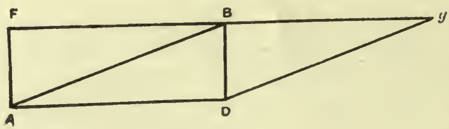


Fig. 2.

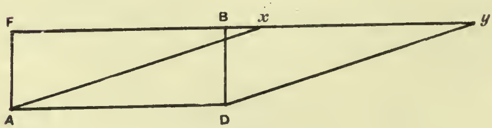


Fig. 3.

(1). Suppose X to fall between F and B.
(2). Suppose X to fall on B.
(3). Suppose X to fall to the right of B.

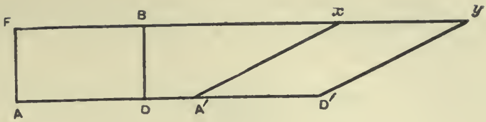


Fig. 4.

In Fig. 4, F y and A D′ are parallel, and A B is any rectangle, and A D = A′ D′.

Compare the rhomboid A′ y with the rectangle A B.

Write a statement of what has been proved in the four cases above. Call it Prop. VI.

257.

Cor. I. Show that two parallelograms having the same base and equal altitudes are equivalent.

258.

Cor. II. Show that two parallelograms having equal bases and equal altitudes are equivalent.

259.

Cor. III. Show that two parallelograms having equal altitudes are to each other as their bases; two parallelograms having equal bases are to each other as their altitudes; and any two parallelograms are to each other as the products of their bases by their altitudes.

260.

Cor. IV. Show how to find the area of any parallelogram.

261.

Cor. V. Can you show how to find the area of any triangle?

262.

Cor. VI. Can you show that triangles with equal bases and equal altitudes are equivalent?

263.

Cor. VII. Can you show that triangles with equal altitudes are to each other as their bases; and those with equal bases are to each other as their altitudes?

264.

Cor. VIII. Can you show that any two triangles are to each other as the products of their bases and altitudes.

EXERCISES.

205. The altitude and base of a △ being 35 and 12, respectively, what is its area?

206. The area of a △ is 221 square feet; its base is 5⅔ yards. What is its altitude in inches?

207. The bases of two parallelograms are 15 cm. and 16 cm. respectively; and their altitudes are 8 cm. and 10 cm. respectively. What is the ratio of their areas?

208. Two △s of equal areas have their bases 26 mm. and 36 mm. respectively. What is the ratio of their altitudes?

209. Draw any straight line through the point of intersection of the diagonals of a parallelogram terminating in a pair of opposite sides and show how the parallelogram is divided.

210. If E is the middle point of C D, one of the non-parallel sides of the trapezoid A B C D, prove that the triangle A B E [draw A E and B E] is equivalent to one-half the trapezoid.

211. If E and F are the middle points of the sides A B and A C of a triangle, and D is any point in B C, show how the quadrilateral A E D F is related to the △ A B C.

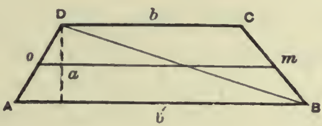
212. Join the middle points of the adjacent sides of any quadrilateral. What is the new figure? How is it related to the quadrilateral?

213. If two equivalent △s have a common base and have their vertices on opposite sides of the base, the line joining their vertices is bisected by the base (produced if necessary).

214. Construct a parallelogram, A B C D, and draw the diagonal A C. Take any point, P, on A C and join it with B and D. Compare the areas of A B P and A D P, of B P C and D P C.

265.

Proposition VII.



What is the figure A B C D if A B is ∥ D C? What is the altitude? *What is its area?*

[*Hint.*—What 2△s compose A B C D? A B D = ? B C D = ? Therefore A B C D = ?]

Generalize this equation and call it Prop. VII.

266.

Cor. 1. Show that one-half the sum of the parallel bases equals the median of the trapezoid?

Exercises.

215. Altitude of a trapezoid is 5, bases 8 and 10, find area.

216. Construct an irregular pentagon, and, having compasses and rule, show how to compute area.

217. The area of a rhombus = one-half the product of its diagonals. Prove.

218. Rough boards are usually narrower at one end than at the other, for which reason the lumber merchant usually measures their width in the middle. Can you explain the principle involved in such measurement?

219. A carpenter wishes a trapezoidal board whose non-parallel sides must be equal. He lays off equal angles with one of the bases and saws out his board.

[Let the sides of the ∠s which coincide with the base of the trapezoid extend in opposite directions from the vertices.]

Prove that his method is right.

220. Suppose the trapezoidal board mentioned in Ex. 219 simply required that the base angles made by the non-parallel sides should be equal. What could the carpenter discover concerning the non-parallel sides?

221. (1) Can you give two methods of finding the area of any polygon?

(2) Show that equiangular △s are similar.

267.

Proposition VIII.

Areas of Polygons—Continued.

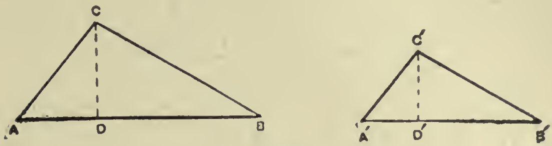


Fig.

Given: The similar △s A B C and A′ B′ C′.

To Show—That corresponding altitudes are to each other as any two homologous sides.

Sug. If C D and C′ D′ are homologous altitudes, can you show that A C D is similar to A′ C′ D′?

[When are triangles similar?]

Can you show that C D : C′ D′ = A C : A′ C′ = A B : A′ B′ = B C : B′ C′?

Write the general truth as Prop. VIII.

Scholium: The ratio of any two homologous sides of similar polygons is called the *ratio of similitude.*

222. Can you show that any two similar △s are to each other as the squares of any two homologous lines, or are in the ratio of similitude of the triangles?

223. In Fig., § 267, if A B = 10, A′ B′ = 6 and area △ A′ B′ C′ = 36, what is the area of A B C?

268.

PROPOSITION IX.

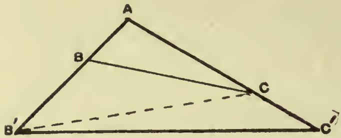


Fig.

Given: △s A B C and A B′ C′ any 2 △s having an angle of one equal to an angle of the other.

To Compare—A B C and A B′ C′.

Sug. Compare A B C and A B′ C. Compare A B′ C and A B′ C′. Express these two comparisons in fractional forms. Multiply the two equations together and simplify the result. Express the result in a general statement. This is Prop. IX.

269.

Cor. I. If 2 parallelograms have an angle of the one equal to an angle in the other, how are they related?

224. Given the perimeter of a triangle and the radius of the inscribed circle to find its area.

270.

Cor. II. If the products of the sides about the equal angles are equal, what can you say of the triangles?

271.

Proposition X.

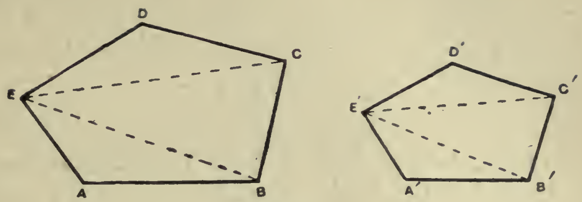


Fig.

Given: A—C and A′—C′, similar polygons, and let S and S′ represent the areas respectively. Draw the diagonals E B, E C and E′ B′ and E′ C′. (See Ex. under Prop. VIII.) Can you form an equation between △s A B E and A′ B′ E′ on the sides A B and A′ B′? Similarly compare E B C, E′ B′ C′ and E C D and E′ C′ D′. What can you say of the homologous sides of the polygons? A B : A′ B′ = B C : B′ C′ = C D : C′ D′, etc. But in a proportion like powers are in proportion, § 182; hence $\overline{AB}^2 : \overline{A'B'}^2 = \overline{BC}^2 : \overline{B'C'}^2 = \overline{CD}^2 : \overline{C'D'}^2$. Now substitute these values in your first equations. By proportion, §188, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent. Can you write an equation so that the sum of the △s in the first figure shall be to the sum of the △s in the second as any △ in first is to any corresponding △ in second? Now substitute for the ratio of the △s its value above. Substitute for the sums of the △s in A — C and A′ — C′, S and S′ in your last equation. Generalize and call the statement Prop. X.

272.

Cor. I. Can you show that similar polygons are to each other as the squares of any homologous lines?

273.

Cor. II. Prove that the homologous sides of any two similar polygons are as the square roots of the areas of those polygons.

274.

Proposition XI.

In the rt. △ A B C, let A B be the hypotenuse. Erect square A B E F on A B. Upon the legs A C and B C construct squares A C G H, B C K L, respectively, and join H B, F C and draw C D ⊥ to A B, and produce it to cut E F at *m*. Can you prove that △ B A H is equal to △ C A F? Can you show how △ A B H is related to the square A G? How is △ A C F related to the rectangle A M? Compare the square and the rectangle. By what axiom do you make the comparison? Join A and L, C and E. Can you show that the square C L is equivalent to the rectangle D E? Add squares C L and A G. How does the result compare with that obtained in § 225? Read the short biography of Pythagoras in the notes.

275.

Cor. I. If the legs of a rt. △ be given, write an equation which will indicate how to find the hypotenuse.

276.

Cor. II. Draw any square, A B C D. Can you show that $\overline{AC}^2 = 2\,\overline{AB}^2$? Also, that $\frac{AC}{AB} = \sqrt{2}$?

What does this equation mean? [§ § 172, 173.]

Exercises.

225. Length of rectangle 60, altitude 5. Find diagonal.

226. In the Fig. of Prop. XI. [§ 274], show how C F and B H are related.

227. Prove that lines B K and A G are ∥. [§ 274]

228. Show that the sum of the ⊥s from H and L to AB produced is equal to A B. [§ 274]

229. Compare △ K C G, △ B E L, and △ C G K with △ A B C. [§ 274]

230. Prove that C, H, and L are in the same straight line. [§ 274]

231. Construct a square equivalent to the sum of any number of squares.

232. Construct a line whose length is $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$ (in two ways), $\sqrt{11}$ (in three ways).

233. If similar polygons are constructed on sides of a right △, show that the polygon on the hypotenuse is equivalent to the sum of those on the other two sides.

234. In Fig., § 271, if A B : A′ B′ = $x : y$, what will be the ratio of the square described on E B to that described on E′ B′?

235. In the same figure, if $s : s' = x : y$, what is the ratio of any two homologous lines, as E C and E′ C′?

236. Find the area of a right isosceles △, if the hypotenuse is 50 rods in length.

237. Two parallel chords are each 10 feet in length and the distance between them is 8 feet. What is the radius of the circle?

238. A rectangular table contains 26.4 square feet; its width is 2.2 feet. Find the length of its diagonal.

239. If one angle of a right triangle is 30°, how does the hypotenuse compare with the side opposite the angle of 30°?

240. Construct an equilateral △ on each side of a right triangle. Show that the equilateral △ on the hypotenuse is equivalent to the sum of those on the other two sides.

241. One angle of a right triangle is 60°. Construct an equilateral triangle on the hypotenuse and compare its area with the rectangle whose sides are the two legs of the right △.

242. If two triangles have two sides of one equal respect-

ively to two sides of the other and the included angles supplementary, how are the areas related?

243. The side of an equilateral triangle is 20 decimeters. What is its altitude?

244. Prove how a trapezoid is divided by a line joining the mid-points of the parallel sides.

245. (1) On a given line, $m n$, the hypotenuse, construct a rt. △ equivalent to a given △. When is this impossible?

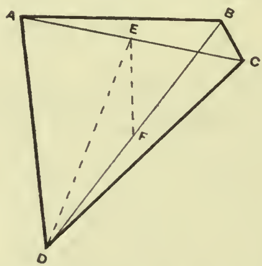
(2) Prove that 3 times the square of the side of an equilateral △ equals 4 times the square of the altitude.

246. If the acute ∠ B of the rt. △ A B C is double the ∠ A, prove that $\overline{AC}^2 = 3\,\overline{BC}^2$.

247. If the ∠ A of the △ A B C above is 30°, prove that area of A B C $= \frac{1}{4}$ A B × A C.

277.

Proposition XII.



Given: Any quadrilateral, A B C D, and the diagonals A C, B D. Call the centers of these diagonals E and F. Join them.

Can you show that the sum of the squares of the 4 sides is equivalent to the sum of the squares of the diagonals plus 4 times the square of the line joining the middle points E and F?

[*Hint.*—Join B E and D E. By §233 write an equation for $\overline{AB}^2 + \overline{BC}^2$, call it equation (1); also write an equation for $\overline{AD}^2 + \overline{DC}^2$. Call this (2). Add (1) and (2). In $\triangle$ B E D what does $\overline{BE}^2 + \overline{DE}^2$ equal? $2\,\overline{BE}^2 + 2\,\overline{DE}^2 = ?$ Can you substitute this in equation (3)? Show that $4\,\overline{AE}^2 = 2\,(\overline{AE})^2$ $= \overline{AC}^2$. Similarly find the value of $4\,\overline{BF}^2$ and substitute.]

Write the complete statement and call it Prop. XII.

278.

What corollary can you state concerning any parallelogram?

Let A = area of an equilateral $\triangle$ and a one side. Find area in terms of its side.

279.

Proposition XIII.

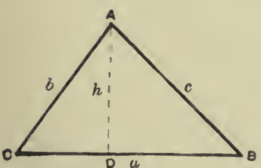


Fig. 1.

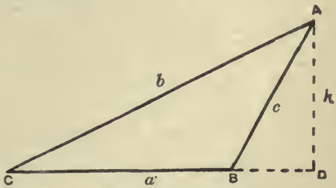


Fig. 2.

Given: The sides of any triangle, as a, b, c.

To Find—I. An expression for the altitude in terms of the sides.

II. The area in terms of the sides.

Sug. Let h be the altitude.

By § 231 can you show that $c^2 = a^2 + b^2 - 2\,a \cdot \text{C D}$?

What is the value of C D?

By substituting and factoring,

$$h^2 = b^2 - \overline{CD}^2. \quad \text{Why?}$$

$$h^2 = b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2 =$$

(1) $$\frac{4a^2 b^2 - (a^2 + b^2 - c^2)^2}{4a^2} =$$

(2) $$\frac{(2ab + a^2 + b^2 - c^2)\,(2ab - a^2 - b^2 + c^2)}{4a^2} =$$

(3) $$\frac{\left[(a + b)^2 - c^2\right]\left[c^2 - \overline{(a - b)}^2\right]}{4a^2} =$$

(4) $$\frac{(a + b + c)\,(a + b - c)\,(c + a - b)\,(c - a + b)}{4a^2} =$$

[To shorten the work, let $a + b + c = 2\,S$;

Then $a + b - c = 2\,S - 2\,c = 2\,(S - c)$,

$c + a - b = 2\,S - 2\,b = 2\,(S - b)$,

$c - a + b = 2\,S - 2\,a = 2\,(S - a)$.

Now substitute these values in (4), and]:

(5) $$\frac{2\,S \cdot 2\,(S - a) \cdot 2\,(S - b) \cdot 2\,(S - c)}{4a^2};$$

$$\therefore \quad h = \frac{2}{a}\sqrt{S\,(S - a)\,(S - b)\,(S - c)}.$$

Write expression II.

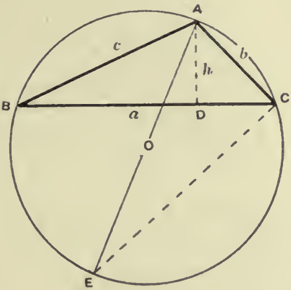
Call the Prop. XIII.

Note.—The solution given is algebraic.

248. If the sides of a triangle are 10 dm., 17 dm., 21 dm., what is the area? What is each altitude?

280.

PROPOSITION XIV.



Given: Any triangle, A B C, with sides a, b, c, and altitude h. Circumscribe a circle about the triangle and draw a diameter, A E, through the vertex A. Join E with the other extremity of the shortest side.

To Find—The area of the triangle in terms of the radius. Call the area A.

Sug. $b\,c = ?$ [§ 239] Instead of A E, we may use $2\,r$, r being the radius of the circumscribed circle. $h\,a = ?$

$\therefore a\,b\,c = ?$ A = ?

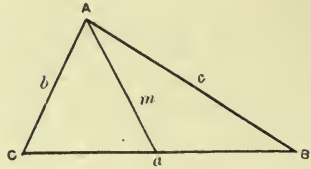
Call this Prop. XIV. Write it carefully.

249. Find the diameter of the circumscribed circle in Ex. 248.

250. By using the value of h in § 279 and using § 280, find the value of the radius of the circumscribing circle about a triangle in terms of the sides when they are given.

281.

Proposition XV.



Given: Any triangle, A B C, with the median, m, from vertex A and the sides a, b, c.

To Find—An expression for m in terms of the sides.

Sug. By § 233 $b^2 + c^2 = 2m^2 + 2\left(\frac{a}{2}\right)^2$.

Find m. Draw the median from the vertices C and B and compare the three expressions for the medians.

282.

Proposition XVI.

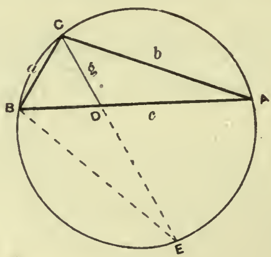


Fig.

Given: Any triangle, A B C, with the sides a, b, c and the bisector of the angle at C, C D = g.

To Find—An expression for the bisector, g, in terms of the sides.

Sug. Circumscribe a circle about the triangle.

From § 238 (1) $a\,b = \text{A D} \cdot \text{D B} + g^2$;

$\therefore$ (2) $g^2 = a\,b - \text{A D} \cdot \text{D B}$.

(3) $\text{A D} : \text{D B} = b : a$. Why?

(4) $\text{A D} + \text{B D} : \text{A D} = b + a : b$. Why?

(5) $\text{A D} + \text{B D} : b + a = \text{A D} : b$. Why?

$= \text{D B} : a$. Why?

(6) $c : b + a = \text{A D} : b = \text{D B} : a$.

(7) What is the value of A D? Of D B?

Call $a + b + c$, 2s, then $a + b - c = 2s - 2c$.

(8) Show that $g^2 = \dfrac{a\,b - 2s \cdot 2\,(s - e)}{(a + b)^2}$

$$g = \frac{2}{a + b}\sqrt{a\,b\,s\,(s-c)}$$

Write the proposition.

251. Find an expression for the bisector of $\angle$ B in § 282.

Supplementary Exercises.

252. If the sides of an isosceles triangle are denoted by a, a, and b respectively, prove that its area $= \dfrac{b}{4}\sqrt{4a^2 - b^2}$.

253. The area of an isosceles right triangle is equal to one-fourth the area of the square described upon the base.

254. Three times the square of the side of an equilateral triangle is equal to four times the square of the altitude.

255. If the acute angle B of the right triangle A B C is double the angle A, prove that $\overline{\text{A C}}^2 = 3\,\overline{\text{B C}}^2$.

256. If the angle A of the triangle A B C is 30°, prove that area $\text{A B C} = \frac{1}{4}\,\text{A B} \times \text{A C}$.

257. If E is any point in the side B C of the parallelogram A B C D, the triangle A E D is equivalent to one-half the parallelogram.

258. If E is any point within the parallelogram A B C D, the triangles A B E and C D E are together equivalent to one-half the parallelogram.

259. If A D is the perpendicular from A to the side B C of the triangle A B C, prove that $\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2$.

260. If D is the intersection of the perpendiculars from the vertices of the triangle A B C to the opposite sides, prove that $\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2$.

261. The area of a rhombus is 24 and its side is 5. Find the lengths of its diagonals.

262. If one diagonal of a quadrilateral bisects the other, it divides the quadrilateral into two equivalent triangles.

263. Any straight line drawn through the point of intersection of the diagonals of a parallelogram, terminating in a pair of opposite sides, divides the parallelogram into two equivalent quadrilaterals.

264. The sum of the squares of the lines joining any point in the circumference of a circle with the vertices of an inscribed squa e is equal to twice the square of the diameter of the circle. (§ 204.)

265. If D is the middle point of the side B C of the triangle A B C right-angled at C, prove that $\overline{AB}^2 - \overline{AD}^2 = 3\,\overline{CD}^2$.

266. If D is the middle point of the hypotenuse A B of the right triangle A B C, prove that

$$\overline{CD}^2 = \tfrac{1}{8}(\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2).$$

267. If a line is drawn from the vertex C of an isosceles triangle meeting the base A B produced at D, prove that

$$\overline{CD}^2 - \overline{CB}^2 \quad AD \times BD.$$

268. If A D is the perpendicular from one of the extremities of the base A C to the opposite side in the isosceles triangle A B C, prove that

$$3\,\overline{AD}^2 + 2\,\overline{BD}^2 + \overline{CD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2.$$

269. One of the equal sides of an isosceles triangle is 18 dkm. and the base is 30 m. Find the area of the triangle.

270. One of the sides of an equilateral triangle is 20 m, Find its area.

271. The sides of a △ are 20 dm., 30 dm., 40 dm. Find the length of the bisectors of the angle.

272. If the sides of a triangle are 9 m., 11 m., 12 m., what is the diameter of the circumscribed circle?

273. The two sides of a parallelogram are a, and b, and one diagonal is e. What is the length of the other diagonal?

274. The hypotenuse of a right isosceles triangle is a. Find its area.

275. The diagonals of a rhombus are to each other as 4 : 7 and their sum is 16. Find the area.

276. What is the area of a triangle two of whose sides are 8 dm. and 12 dm., and the angle included between them is 30°?

277. Two sides of a triangle are a and b, and the included angle is 30°. What is the area?

278. In the last example, suppose the included angle, were 150°, what would be the area?

279. If the two sides of a triangle are a and b, and the included angle is 45° or 135°, can you show that the area is $\frac{1}{4} a\, b \sqrt{2}$?

280. In the last example, if the included angle is 60° or 120°, the area is $\frac{1}{4} a\, b \sqrt{3}$. Prove.

281. The area of a trapezoid is 46 ares, and its bases are 97 m. and 133 m. Can you find its altitude?

Problems in Construction.

282. Construct a ▭, A B C D, with D E the altitude and A B the base of the ▭. Find a mean proportional between the base and the altitude of A B C D. Construct a square equivalent to the ▭.

283. Construct a square equivalent to a given △.

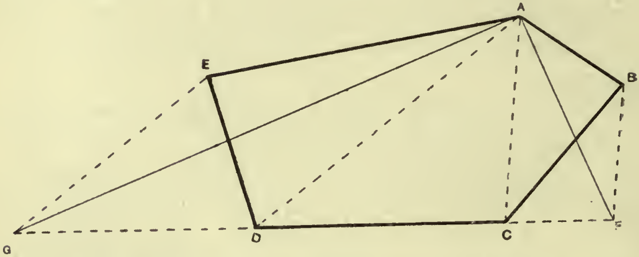
284. Find a fourth proportional to these lines $\underline{\quad m \quad}$ $\underline{\quad n \quad}$, $\underline{\quad o \quad}$.

285. Given rectangle A B C D and the line E F. Construct a rectangle on E F having the area of rectangle A B C D.

283.

Proposition XVII.

Problem. To construct a triangle equivalent to a given polygon.



Let A B C D E be any polygon. Let B F be || diagonal A C, and E G || diagonal A D. Compare △ A B C and △ A F C, also △ A E D and △ A G D. Compare △ A F G with polygon A B C D E.

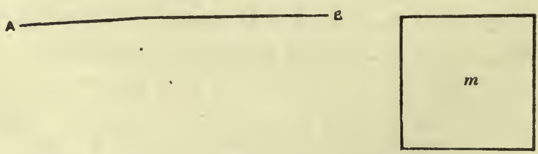
Exercises.

286. Can you construct a square equivalent to a given polygon?

287. Construct a rectangle upon a given line, A B, equivalent to a given polygon.

284.

Proposition XVIII.



Given the square *m* and the line A B.

Can you construct a rectangle equivalent to m, having the sum of its base and altitude equal to A B?

[*Hint.*—(1) Assume the problem solved and then study the rectangle. Distinguish the *known* from the *required.* Review method of discovering solutions to originals.]

If you fail, see the "hint" given below.

[*Hint.*—(2) *How do you find a mean proportional to two given lines?* In answering this question, try to see what you have *given* which can be used as data. Of what two lines is A B the sum? What is the mean proportion of these lines? (*Ans.* Side of *m.*) Can you construct a rt. △ of which A B is the hypotenuse? Construct a semicircle on A B. Draw a line parallel to A B (distance equal to a side of *m*). In how many points will this parallel, produced if necessary, cut the semi-circumference? Draw a perpendicular to diameter from either of these points.]

State Prop. XVIII.

Discuss this problem: When is it impossible? etc.

Exercises.

288. The sum of two numbers is 13; their product is 36. Required the numbers.

(1) Solve by Algebra.

(2) Solve by Geometry.

289. Given △ A B C and *m n* the base of an isosceles △ equivalent to A B C. To construct the isosceles △. Is this ever impossible?

290. Given the rectangle A B C D, the ∠ *a*, and the line *m n*. To construct a △ equivalent to A B C D having base equal to *m n*, and adjacent angle equal to ∠ A.

291. Upon a given line draw an isosceles △ equivalent to the sum of a given △ and a given quadrilateral.

285.

Proposition XIX.—Problem.

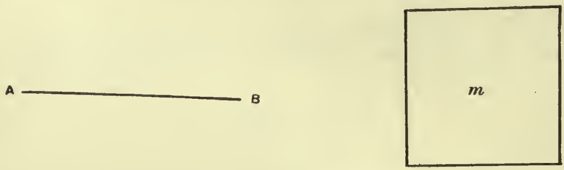
Given the square m and the line A B.

Construct a rectangle equivalent to m, the difference of whose base and altitude equals A B.

[*Hint.*—What proposition for finding the mean proportional of two given lines did you use in § 284?

Think of another proposition in which you have the mean proportional of two given lines.

Note that in § 284 you have given the *sum* of two lines, while in § 285 you have given the —. *Try* hard to finish without consulting the figure given below or reading further.



Construct a circle having diameter A B.

Construct a tangent, one end terminating at the circumference, having a length equal to a side of m. Through the extremity without the circumference draw a secant passing through the center. State the proposition you now have. Find what the square m equals.]

State Prop. XIX.

Exercise.

292. The difference of two numbers is 10 and their product is 56. Required the numbers.

(1) Solve by Algebra.

(2) Solve by Geometry.

[*Hint.*—Take any unit of length. Construct the line, $\sqrt{56}$.]

286.

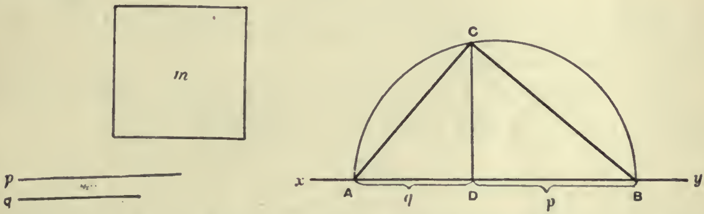
PROPOSITION XX.– PROBLEM.

Given the square m and the unequal lines p and q.

It is desired to construct a square that shall have the same ratio to m as that of p to q.

Try to solve without consulting "hint." It may be quite difficult, but try repeatedly by using all you have learned about methods of discovering the solution of original prolems.

[*Hint.—Study* § 222. Note that here we have the *squares* of lines proportional to *lines*. Try to apply this in the above problem.]



In the above figure p and q are proportional to what squares? But does the side of one of them equal the side of m? How can we get other squares having the ratio of $\overline{AC}^2$ to $\overline{CB}^2$? How can we get two squares, one of which equals m?

If still unable to solve, cut off C A, C E a distance = a side of m. Then through E draw a line parallel to A B, cutting C B and F. Then C F is the side of the required square. Discuss the above problem. Is it ever impossible? Solve when $p \angle q$; when $p = q$.

EXERCISE.

293. If area of m = 100, $p = 7$, and $q = 5$, find the area of the required square. Test the accuracy of your answer by a drawing.

287.

PROPOSITION XXI.—PROBLEM.

Given any polygon, A B C D E, and the unequal lines m and n.

Construct a polygon similar to A B C D E and having the ratio to it of m to n.

If you fail, see "hint" below.

[*Hint.*—How do similar polygons vary? Can you construct a square whose ratio to the square of a side of A B C D E shall equal $m : n$?]

Discuss the problem.

EXERCISES.

294. If A B = 12 $m = 15$, and $n = 10$, find A′ B′ side of the required polygon.

295. Construct a △ equivalent to a given polygon having given the base and median to the base of the △.

288.

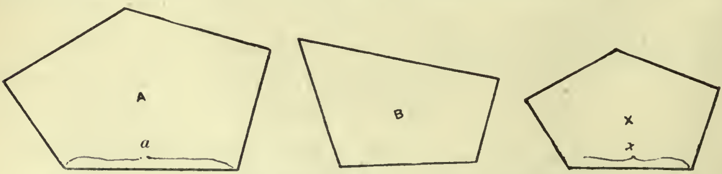
PROPOSITION XXII.—PROBLEM.

Given two dissimilar polygons, A and B.

It is required to construct a polygon similar to A *and equivalent to* B.

[*Hint.*—(1) Assume the problem solved.

Then we have given dissimilar polygons A and B, and the *required* polygon X, similar to A and equivalent to B.



Let a = a side of A, and x = a side of X, the required polygon.

Review the propositions about similar polygons. Form proportions.

You wish to discover a method to find the length of what line? (With it known you can construct X.)]

[*Hint.*—(2) $A : X = a^2 : x^2$. [?]

How many unknown terms in the above proportion?

Put for one of the *unknown* terms a *known* term.

Are all the magnitudes in the above proportion of the same kind?

Get another proportion from this, in which the magnitudes are not areas. What kind of magnitude is A?

What does the $\sqrt{A}$ *mean?* Can you construct it?]

296. Construct a rhombus equivalent to a given rhomboid

(1) Having one diagonal equal to the short diagonal of the rhomboid;

(2) Having one diagonal equal to the long diagonal of the rhomboid.

297. How does the radius of an inscribed circle in an equilateral △ compare with the radius of the circumscribed circle?

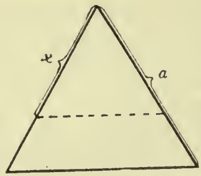
298. Construct a △ similar to one of two given dissimilar △s and equivalent to their difference. Discuss.

299. Construct an isosceles △ equivalent to a given △, having given one of the equal sides. Discuss.

300. Draw a line parallel to a side of an equilateral △ which will divide it into two equivalent parts.

[*Hint.*—(1) Suppose the problem solved, and then compare the area of the △ cut off with that of the original △. How do these △s vary? Form the proportion, etc.]

If you still fail, consult figure and further "hint."

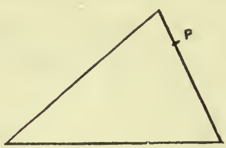


[*Hint.*—(2) Let T = large △ and one of its sides = a,
T′ = small △ cut off, and side = x.
$T : T' = a^2 : x^2$. [?] But T = 2 T; [?]
$\therefore 2 : 1 = a^2 : x^2$; (What is the unit of measure here?)
And $\sqrt{2} : 1 = a : x$. (What is the unit of measure here?)
Construct the $\sqrt{2}$ where the unit is given. Construct the converse.

301. Draw a line parallel to the base of a given △ dividing it into two equivalent parts.

302. Draw a line through a given point in the side of a △ dividing it into two equivalent parts.

[*Hint.*—Draw a scalene △ and solve, fixing the point in different positions in the sides.



When P is joined to opposite vertex, is the △ divided into equivalent △s? Why not? Where must P be in order that the △s shall be equivalent? Fix this point. Call it *m*, join *P* and *m* to opposite vertex. The line from *m* cuts off *half.* How much does the line through *P lack* of cutting off half? How then can you draw a line through *P* which will cut off in addition to △ already cut off a △ equivalent to △ lacking?]

303. Draw a line through any point in the side of a parallelogram dividing it into two equivalent parts. When will the parts be △s? trapezoids?

MOULDING OF POLYGONS.

§283 may be stated as follows: Mould a polygon into an equivalent △. Note the figure carefully. △ A C F is equivalent to △ A C B. Why? In △ A C B consider A C the base and let the vertex B be moved parallel to the base A C. Suppose we start to move B toward F, but stop at intermediate points, x, y, z, etc. Why are △s B A C, X A C, y A C, z A C ....and F A C equivalent? Are any of these △s isosceles? right? equilateral?

304. I. Redraw △ B A C (§283) and mould it into (1) an isosceles △; (2) a right △; (3) a different isosceles △ from (1). What is this base here? (4) Still a different isosceles △ from (1) or 3. What is now the base?

II. Now redraw △ B A C and take A B the base. What is the vertex? Mould B A C into the following △s, using A B the base: (1) isosceles; (2) right. Can you mould it into other isosceles △s while A B is considered the base? Why?

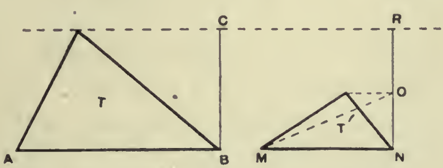
III. Answer questions as asked in II. using B C the base.

IV. Discuss the possibilities of moulding a scalene △ into different shaped equivalent isosceles △s.

305. Divide any quadrilateral into two equivalent parts by drawing a line through any given point in any one of its sides.

[*Hint.*—Mould the quadrilateral into a △.] Discuss.

306. Draw two scalene △s having unequal altitudes. Draw an isosceles △ equivalent to their sum.



[*Hint.*—(1) Mould each △ into a right △. Raise the altitude of T′ until its altitude equals that of T. T is equivalent to A B C. [?] T′ is equivalent to M N O. [?] To solve: (2) Draw M R and then draw through O a line parallel to M R, cutting M N at S. Draw S R. △ S R N is equivalent to △ M N O, which is equivalent to T′. Why?

(3) Now add the △s by adding their bases and mould into the required isosceles △.

307. Draw three scalene △s of unequal altitudes, and (1) Mould into a rt. △ equivalent to their sum; (2) Draw a rectangle equivalent to their sum; (3) Draw a square equivalent to their sum.

308. (1) Draw a quadrilateral and mould it into an equivalent △. Now mould the △ into an equivalent quadrilateral. Discuss the data necessary to mould the △ into the original quadrilateral.

(2) Draw a square $\frac{4}{7}$ as large as a given square.

309. Draw an irregular pentagon and then draw a square having $\frac{3}{5}$ the area of the pentagon.

310. Mould a scalene △ (1) into a "kite" trapezium; (2) into an "arrow" trapezium.

311. Construct a △ similar to a given △ and having twice the area.

312. Draw a circle having twice the area of a given circle.

313. Draw a square having $\frac{3}{4}$ the area of a given square.

314. Draw a circle having half the area of a given circle.

315. Draw an irregular polygon; then draw a similar polygon having $3/8$ the area.

316. Draw a "kite" trapezium and mould it into an equivalent △.

317. Draw an irregular polygon having two re-entrant ∠s. Mould it into an equivalent square.

318. Divide a given line into $2\frac{1}{2}$ equal parts.

319. Draw two similar but unequal rectangles. Then draw a rectangle similar and equivalent to their sum.

320. Divide a given line into 3 equal parts, not using §117.

321. Draw a line through the vertex of a given △ so that the △ will be divided into two △s which shall have the ratio 2 : 3.

322. Construct a right isosceles △ equivalent to a given square.

323. Find the locus of the middle points of all the chords in a given circle which can be drawn through a given point (1) in the circumference, (2) within the circle, (3) without the circle.

324. Given the line A B and the angle *a*. Construct largest possible △ that shall have A B for the base, an ∠ *a* for the vertical angle.

325. From a point without a circle draw a secant so that the intercepted chord shall subtend $\frac{1}{4}$ of the circumference.

326. From a point without a circle draw a secant so that the internal segment and the external segment shall be equal. Discuss Ex. 326.

327. Construct a triangle having given—

(1) The base, altitude, and the median to the base. Discuss.

(2) The angles and one median. Discuss.

(3) One side, an ∠ adjacent to that side, and the sum or the difference of the other two sides. Discuss.

(4) The perimeter, one ∠, and the altitude drawn from the vertex of this ∠. Discuss.

(5) The radius of the circumscribed circle, and the angles. Discuss.

BOOK V.

REGULAR POLYGONS.

MEASUREMENT OF THE CIRCLE.

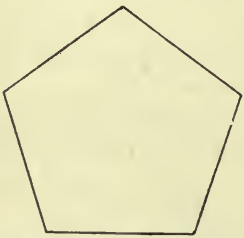
328.

Recall your definition of a *regular polygon*. See § 104.

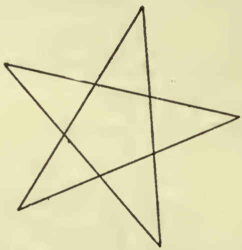
The term *regular polygon* means a convex polygon unless otherwise stated.

What would you call an equilateral triangle? a square?

329.



Regular Convex Pentagon.



Regular Concave or Cross Polygon.

The subject of the regularity of polygons may be looked at from the standpoint of *symmetry*. By *symmetry* in Geomtry we mean that if a figure be turned half way round on a point as a pivot, each part of the figure will occupy the same space previously occupied by another part. If the figure, on being turned half way round, occupies its original position, it is said to have *two-fold symmetry*.

If an equilateral triangle be revolved about its center one-third of 360°, what will be its second position? Suppose it is turned two-thirds of 360° about the center, what will be its third position?

Discuss revolving a square about its center.

Is a right triangle symmetrical with regard to its center? an isosceles triangle?

If a figure be turned one-third of a revolution and it occupies its original position, the figure is said to have three-fold symmetry.

What is four-fold symmetry? five-fold symmetry?

What is the symmetry with regard to a point illustrated by the following figures? Make figures illustrating other forms of symmetry.

For examples, observe wall paper and other decorations.

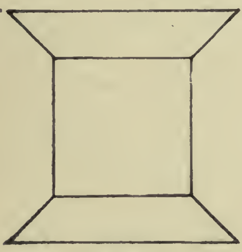


Fig. 1

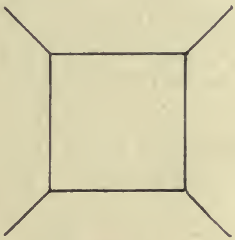


Fig. 2.

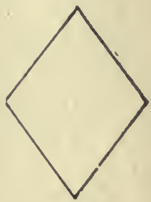


Fig. 3.

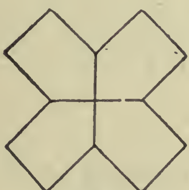


Fig. 4.

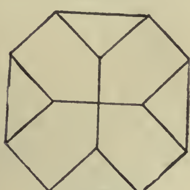


Fig. 5.

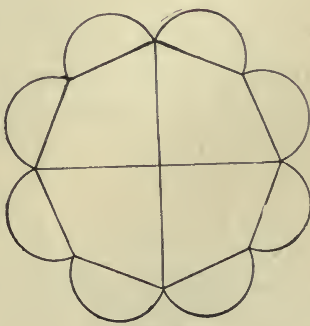


Fig. 6.

330.

From the standpoint of symmetry, polygons that are symmetrical are regular. Thus, a triangle is regular if it has three-fold symmetry. A heptagon is regular if it has seven-fold symmetry.

A polygon of n sides is regular if it has n-fold symmetry.

331.

By means of revolution show that a symmetrical octagon has (1) its sides equal, (2) its angles equal; (3) that a circle may be circumscribed about a regular polygon having the same center as the polygon; (4) that with the center of the polygon for center a circle may be inscribed within it.

From the special case just given, can you prove a general truth? State it.

Notice that we prove by symmetry (1) and (2), which are given as a definition in §104.

332.

Into how many isosceles triangles may a regular triangle be divided by joining the center to the vertices? a regular quadrilateral? a regular pentagon? a regular hexagon?

Proposition I.

Can you show that a regular polygon, P, of n sides, may be divided into n isosceles triangles?

Write Prop. I.

333.

Cor. I. What do the bisectors of any two $\angle$s of a regular polygon determine by their intersection?

334.

Cor. II. A point is equidistant from all the vertices of a regular polygon. How is it related to the sides of the polygon? Proof.

335.

Cor. III. Show that a ⊙ may be circumscribed about or inscribed within a regular polygon and both ⊙s have the same center.

The center in Cor. III. is called the *center of a regular polygon* and the radius of the circumscribing circle is called the *radius of a regular polygon;* the radius of the inscribed ⊙ is called the *apothem* of a regular polygon. Draw figure and fix these terms in mind. What is meant by the ∠ at the center of a regular polygon?

336.

Cor. IV. If a regular polygon have n sides, show that each ∠ at the center = 4 rt. ∠s ÷ n.

337.

PROPOSITION II.

Given: A B C D E F, an equilateral polygon inscribed in a circle. Can you prove it to be regular?

338.

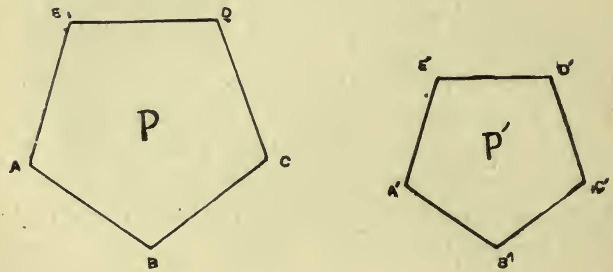
Cor. I. If a circumference be divided into n equal arcs, ($n > 2$), what will the chords of these arcs form?

339.

Cor. II. If the arcs subtended by a regular polygon of n sides be bisected, what will the chords of these arcs form?

340.

Proposition III.



Suppose A B C D E and A′ B′ C′ D′ E′ are regular polygons of the *same number of sides.* (1) Compare the sum of all the ∠s in P and P′. (2) Compare ∠ A and ∠ A′, ∠ B and ∠ B′. Why does ∠ A = ∠ A′? ∠ B = ∠ B′? Compare A B, B C, C D, etc. Compare A′B′, B′C′, C′D′, etc.

Can you show that P and P′ are *similar?*

Call this Prop. III.

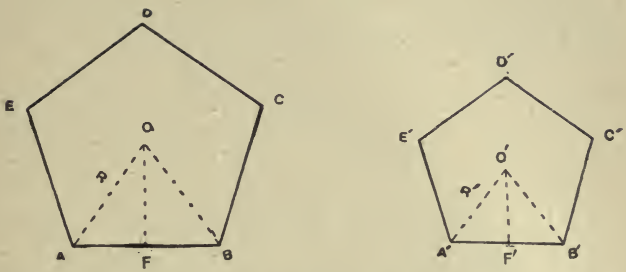
341.

Proposition IV.

Prove that regular polygons of the same number of sides may be divided into the same number of similar △s, similarly placed.

342.

Proposition V.



In A—D and A′—D′ we have two regular polygons of the same number of sides. Let P and P′ be the perimeters and S and S′ be the areas of the figures. Let O and O′ be the centers. (How find them?) Join O A, O B, O′ A′, O′ B′, and draw the ⊥s O F and O′ F′ to A B and A′ B′ respectively. Call O A, R; O′ A′, R′; O F, r; O′ F′, r'. Compare ∠s A O B and A′ O′ B′. Can you compare the ratio of O A and O′ A′ with ratio of O B and O′ B′? What conclusion concerning the △s? Compare the ratio of A B and A′ B′ with the ratio of O F and O′ F′? Substitute for O B, O′ B′, O F, O′ F their values in the last comparison.

Can you now show that $\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}$?

Show that $\frac{S}{S'} = \frac{r^2}{(r')^2} = \frac{R^2}{(R')^2}$.

What are R, R′, r, r' ?

Can you state the proposition and prove?

Call this Prop. V.

343.

Proposition VI.

Show how to find the area of any regular polygon. Word the conclusion Prop. VI.

344.

Proposition VII.

Show how to inscribe a square in a given ⊙. Call this Prop. VII.

345.

Proposition VIII.

Can you prove how to inscribe a regular hexagon in a circle?

Exercises.

328. In how many ways can you construct an equilateral triangle?

329. If $r =$ radius of the ⊙, and s the side of the inscribed equilateral △, show that $s = r\sqrt{3}$.

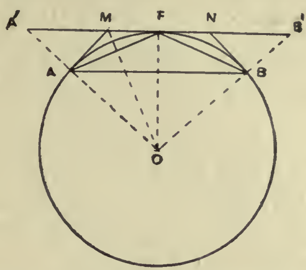
330. The distance from the center of an inscribed equilateral △ to a side is $\frac{r}{2}$.

331. Inscribe an equilateral △ and a regular hexagon in the same ⊙. Compare their areas.

332. Circumscribe an equilateral △ about the ⊙ in Exercise 331. Compare the hexagon with the 2 △s. State your conclusion in neat form.

346.

PROPOSITION IX.—PROBLEM.



In this figure let A′ B′ be one side of a regular circumscribed polygon of n sides. Call its perimeter P, and let O be the center of the polygon and circle. Draw O A′, O B′ cutting the arc at A and B. Join A and B. Prove A B ∥ to A′ B′. Can you show that A B is the side of a regular inscribed polygon of n sides? Call its perimeter p.

Join A F, B F and draw tangents at A and B. How often is A B used in the regular inscribed polygon? Show that A F is the side of a regular inscribed polygon. How many sides? Can you show that M N is the side of a regular circumscribed polygon? How many sides to the polygon of which M N is a side? Call the perimeter whose side is M N, P′, and the regular polygon whose side is A F, p'. Show that $A'\,B' = \frac{P}{n}$. Write the values of A B, M N, A F. Join O M, O F. Prove that M O bisects ∠ A′ O F. Show that $\frac{A'\,M}{M\,F} = \frac{O\,A'}{O\,F}$. What are O A′ and O F of the polygons? How are the radii of regular polygons of the same number of sides related? (§ 342.) What axiom shows this, $\frac{P}{p} = \frac{A'\,M}{M\,F}$?

Rewrite this equation by composition. What is the sum of A′ M and M F? How is M F related to M N?

Do you see that $\frac{P + p}{p} = \frac{\frac{1}{2} A' B'}{\frac{1}{2} M N}$? Substitute the values of $A' B'$ and $M N$. Can you now show (1) $P' = \frac{2 P \cdot p}{P + p}$?

What is P'? Express this equation in words.

Can you prove $\triangle$s A B F and A M F similar? Show that $\overline{AF}^2 = AB \cdot \frac{1}{2} MN$. Substitute the values of A F, A B, M N, and show that (2) $p' = \sqrt{p \cdot P'}$.

By means of (1) and (2) the perimeter of regular inscribed and circumscribed polygons of double the number of sides may be found, for any values of P and p.

Call the above Prop. IX. Write a statement of it.

Exercises.

333. The side of an inscribed sq. $= r\sqrt{2}$

334. If s denotes the side, $R =$ radius, $r =$ apothem, and $S =$ area, and the radius of the $\odot = 1$, prove that:

(*a*) In a regular inscribed octagon $s = \sqrt{2 - \sqrt{2}}$, $r = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $S = 2\sqrt{2}$.

(*b*) In a regular circumscribed octagon $s = 2\sqrt{2} - 2$, $R = \sqrt{4 - 2\sqrt{2}}$, $S = 8\sqrt{2} - 8$.

(*c*) In a regular inscribed dodecagon $s = \sqrt{(2 - \sqrt{3})}$, $r = \frac{1}{2}\sqrt{2 + \sqrt{3}}$, $S = 3$.

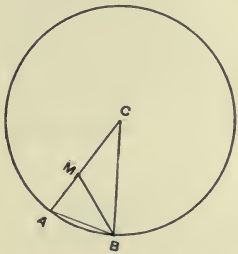
(*d*) In a regular circumscribed dodecagon $s = 4 - 2\sqrt{3}$, $R = \sqrt{8 - 4\sqrt{3}}$, $S = 24 - 12\sqrt{3}$.

335. In a given sector whose $\angle$ at the center $= 90°$ inscribe a square. Prove the area $= \frac{R^2}{2}$.

347.

Proposition X.

Problem. To inscribe a regular decagon in a circle.



Sug. 1. Divide the radius C A into mean and extreme ratio and draw the chord A B equal to C M. Join M with B, C with B. When is a line divided into mean and extreme ratio? Write proportion.

Sug. 2. In place of C M put its equal A B in the proportion above. Can you compare △s C A B and M A B through the last proportion? Quote proof.

Sug. 3. Compare B A and B M, B M and C M. Compare ∠ A B M with ∠s A C B and M B C, ∠ A B C with ∠ A C B, ∠ B A C with ∠ A C B. How many times the angle A C B is the sum of the angles of the triangle A C B?

What is the sum of the ∠s in the △?

Show that ∠ A C B = 36°.

A B is the chord of what angle at the center? How many times may it be applied to the circumference?

Give the method of inscribing a regular decagon in a circle.

348.

Cor. I. How inscribe a regular pentagon?

349.

Cor. II. At a given point in the circumference make a chord equal to the side of the inscribed decagon; from the same point in the same direction along the circumference make a chord equal to the side of a hexagon. What part of a circumference is the difference between the extremities of these 2 chords? The chord of the difference of these arcs is the side of what regular inscribed polygon?

5. Name the series of regular polygons that we have learned to inscribe. Read a biography of the great mathematician Gauss.

EXERCISES.

336. In the figure under § 347 let $x = C M$. Can you show that $A B = \frac{R(\sqrt{5} - 1)}{2}$? Word this equation.

337. When the side of a regular decagon is 10 feet, calculate the radius of the decagon and test the accuracy of your answer by drawing to scale of .1 of an inch to the foot.

338. Prove that the angle at the center of a regular polygon is the supplement of the angle of the polygon.

350.

PROPOSITION XI.—PROBLEM.

Draw a regular inscribed polygon. Bisect the arcs subtended by the equal chords, and at these points of bisection draw tangents.

Prove that these tangents form a regular circumscribed polygon.

Call this Prop. XI.

351.

Cor. By joining certain points in the last figure, show that you can make a regular *inscribed* polygon of twice the number of sides of the original; and, by joining certain other points, a regular *circumscribed* polygon of double the number of sides of the original circumscribed polygon.

352.

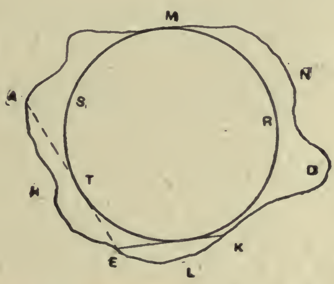
What regular circumscribed polygons may be constructed?

EXERCISE.

339. Prove the diagonals of a regular pentagon are equal.

353.

PROPOSITION XII.



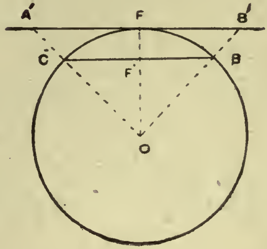
Let the irregular curve M N D K....envelop the circumference M R S. Draw A E tangent at T. Compare A E with A H E. Tangent E K with E L K. Compare A E L K D M with A H E L K D M. What is the shortest enveloping line of the surface within M R T S? How does the circumference compare with any enveloping line? State Prop. XII.

354.

Cor. How does the circumference of a ⊙ compare with the perimeter of any regular circumscribed polygon? with any regular inscribed polygon?

355.

PROPOSITION XIII.



I. Let P and p be the perimeters of the regular circumscribed and inscribed polygons of the *same number of sides*, and S and s the areas, and C the circumference of the ⊙. Let A′ B′ be a side of the polygon whose perimeter is P. Draw O A′, O B′, C B and O F ⊥ A′ B′. What is C B? [§346.] Prove that (1) $\frac{P}{p}=\frac{OA'}{OF}$. By division what does (1) equal? Show that $P-p=\frac{p}{OF}(OA'-OF)$.

In △ O A′ F show that O A′—O F < A′ F. What does A′ F $\doteq$ as you continue to increase the number of sides? What will O A′ – O F $\doteq$? What will C F′ $\doteq$? What will P – C and C – $p \doteq$? What do P and $p \doteq$? Express this in the form of a proposition.

II. Let us find the limit of the variables S and s.

$\frac{S}{s}=\frac{\overline{OA'}^2}{\overline{OF}^2}$. Why? Prove that $\frac{S-s}{s}=\frac{\overline{A'F}^2}{\overline{OF}^2}$ and that

$S-s=\frac{s}{\overline{OF}^2}\cdot\overline{A'F}^2$. Increase the number of sides indefinitely.

What does $\overline{A'F}^2 \doteq$? What does $S-s \doteq$? How does the area of the ⊙ compare with S? with s? Call area ⊙ A. What does $S-A \doteq$? $A-s \doteq$? What are the variables?

What then do S and $s \doteq$? Make a general statement of this proposition, including both parts.

356.

Cor. What limit has the radius of the circumscribed polygon and the apothem of the inscribed polygon?

Exercises.

340. Can you prove that the diagonals of a regular pentagon cut each other in extreme and mean ratio?

341. In the regular pentagon A B C D E let A D and B E cut each other at P; prove that A P : A E : : A E : A D.

342. Construct a regular pentagon equivalent to the sum of two given regular pentagons.

343. Let d = side of an inscribed regular decagon, p = that of an inscribed regular pentagon, r = radius of the ⊙. Prove that $p^2 = d^2 + r^2$.

344. Given the side of a regular pentagon to construct the pentagon.

345. Given the side of a regular hexagon to construct the hexagon.

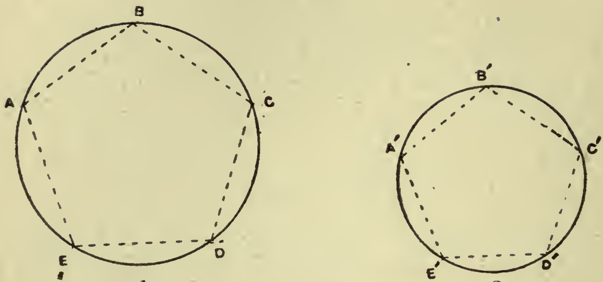
346. Construct the regular hexagon A B C D E F. Draw the diagonals D F, E A, F B. What new figure is found? Compare with the original figure.

357.

Definition: In unequal circles, similar arcs, sectors or segments are those which have equal angles at the center.

358.

Proposition XIV.



1 2

In 1 and 2 let R, r, and C, c, and S, s denote respectively the radii, circumferences, and surfaces of the ⊙s. Inscribe the 2 regular polygons of the same number of sides, letting P, p and A, a denote respectively the perimeters and areas. Compare the perimeters with their radii? Write the relation. Compare the areas with their radii. Increase the number of sides indefinitely, keeping them the same in number. Does the ratio written above ever change? What does P $\doteq$? What does $p \doteq$? What does A $\doteq$? $a \doteq$? If 2 variables are in a constant ratio, what can be said of their limits? Prove your answer.

Show that $\frac{C}{c} = \frac{R}{r}$ and that $\frac{S}{s} = \frac{R}{r^2}$ Express this clearly.

359.

Cor. I. Prove $\frac{C}{c} = \frac{D}{d}$ and $\frac{S}{s} = \frac{D^2}{d^2}$. Express this clearly.

360.

Cor. II. (1) Write the proportion of Cor. I. by alternation.

Do you see that this proportion may be interpreted as showing that the ratio of the circumference of a ⊙ to its diameter is a constant quantity? This constant $\frac{C}{D}$ is denoted by the Greek letter π. It is the initial letter of the Greek word for circumference (*periphereia*). It is proved by methods in higher mathematics that Π is incommensurable?

(2) Prove that $C = \pi D = 2 \pi R$.

(3) Show how similar *sectors* are related? (How are two ⊙s related?

EXERCISES.

347. Find in terms of the radius and diameter of the circle the perimeter of a regular inscribed hexagon.

348. Find in terms of the radius and diameter the perimeter of a regular circumscribed hexagon.

349. Solve as in Exercises 347 for the perimeter of a regular inscribed dodecagon.

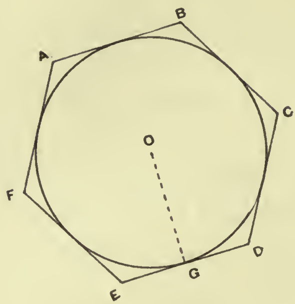
350. Solve as in Exercises 348 for the perimeter of a regular circumscribed dodecagon.

351. Show that the area of a circle is four times the area of a circle described on the radius as a diameter.

352. How does the square inscribed in a semicircle compare with the area of the circle?

361.

Proposition XV.



Let R, C, and A denote the radius, circumference, and area of the ⊙. Construct a polygon of n sides. Call its perimeter P, and apothem R, and area S. Write an expression for the area of the polygon. As the number of sides are indefinitely increased, what does S $\doteq$? what does P $\doteq$? what does $\frac{1}{2}$ P R $\doteq$? Now if S $\doteq$ A or $\frac{1}{2}$ P R $\doteq$ A and $\frac{1}{2}$ P R $\doteq$ $\frac{1}{2}$ C R, then how are A and $\frac{1}{2}$ C R related? Review each step of your proposition. Make it clear. State the theorem.

362.

Cor. I. Substitute the value of C in terms of R and show that $A = \pi R^2$.

363.

Cor. II. Show that the area of a sector $= \frac{1}{2}$ of the product of the arc by its radius. Write the steps of your proof.

364.

Taking the formulæ in §346, $P' = \frac{2\,P \times p}{P + p}$ and $p' = \sqrt{p \times P'}$, and calling the diameter of the circle 1, can you show how we may approximate the ratio of the diameter of a circle to its circumference?

From the above formulæ the following table has been computed:

No. Sides.	Perimeter of Circumscribed Polygon.	Perimeter of Inscribed Polygon.
4	4.00000	2.82813
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14152
512	3.14163	3 14157

Notice that by the last result the approximate value of the ratio of the circumference to the diameter is 3.1416, correct to the 4th decimal place. That is, $\pi = 3.1416$.

Exercises.

353. An oyster can is 4 inches in diameter and 8 inches high. How many square inches of tin are required to make it?

354. Find the length of an arc of 180° in a circle of radius 4.

355. A circus ring contains 40 square rods. Find its radius and circumference. Call π, $3\frac{1}{7}$.

356.. The apothem of a hexagonal paving-stone is 18 cm. Find the area of its circumscribing $\odot$.

357. How many degrees in an arc whose length = the length of the radius of the circle? This arc is called a *radian* and is one of the units for measuring circles.

358. A cow is tethered with a chain 15 m. long; the stake is driven 10 m. from a straight fence. Over how much ground can the cow graze?

359. A railroad fence meets a farmer's fence at an angle of 50°; the farmer tethers a cow between the fences at the corner post; the chain is 30 m. long. Over how much ground can the cow graze?

360. Construct a rt. △, circumscribe a circle about the △, and on each side, about the rt. ∠ as a diameter, describe a semicircle exterior to the △. Compare the sum of the crescents with the area of the △.

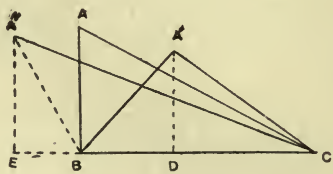
Maxima and Minima of Plane Figures.

365.

When thinking of qualities of the same kind, that which is greatest is called *maximum* and that which is least is called *minimum*. What is the maximum chord in a ⊙? What is the minimum line from a point to a line?

366.

Proposition I.



Two figures are *isoperimetric* when they have equal perimeters.

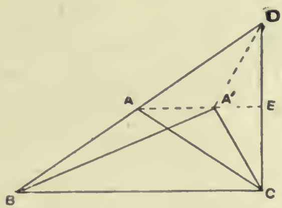
367.

In this figure let the 2 △s A B C and A′ B C have two sides of one = two sides of the other; *i. e.*, A B = A′ B and B C = B C. △ A B C is a right △ having right ∠ at B. Drop the ⊥ A′ D. Compare A′ B and A′ D. Express this relation. How does A B compare with A′ D? Suppose we draw any other line = A B, as A″ B. Join A″ to C. Draw ⊥ A″ E. Compare A″ E with A B. Compare the △s A′ B C and A″ B C with the right △.

Of all △s having 2 sides equal, which is the maximum?

368.

Proposition II.



Let A B C be an isosceles △ and A′ B C be an equivalent △ on same base. Compare their perimeters.

[*Hint.*—Produce B A to D, making A D = A B = A C Join D and C and A A′. Produce A A′ to E. Compare altitudes of △s A B C and A′ B C. Compare A A′ E and B C, E D with E C, A E with D C, A′ D with A′ C.]

Can you show that the perimeter of △ A B C is less than the perimeter of △ A′ B C?

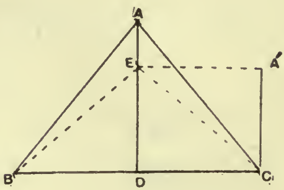
Generalize this theorem.

369.

Cor. Of all equivalent △s, which has the least perimeter?

370.

PROPOSITION III.



Construct △ A B C with A B = A C. Draw A D ⊥ B C and make A′ B + A′ C = A B + A C. Draw A′ E ∥ B C meeting A D, or A D produced, in E. Join E B and E C. Compare △s E B C and A′ B C. Compare A′ B + A′ C with E B + E C, A B + A C with E B + E C, A B with E B, A D with E D.

[Use Doctrine of Exclusion. Why is D E not equal to D A? Why not greater?]

Which is the maximum △?

Write a general statement of the truth proved.

371.

Cor. Of all isoperimetric △s, which has the greatest area?

Exercises.

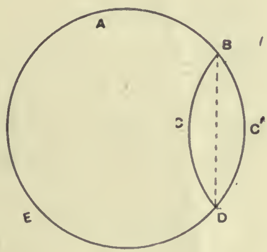
361. The perimeter of a maximum △ is n meters. Find its area.

362. Show what is the greatest △ that can be inscribed in a ⊙.

363 If the diagonals of a parallelogram are given, when is its area a maximum? When, if ever, may the maximum parallelogram be a square?

372.

Proposition IV.

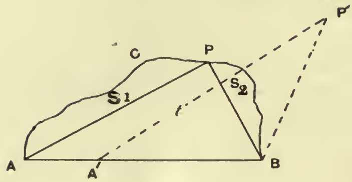


Let A B C D E be a plane figure bounded by the convex arc B E D and the concave arc B C D with the staight line B D joining the ends of the concave arc. Show that A B C D E cannot be the maximum of isoperimetric figures.

Sug. Revolve B C D on the axis B D till it comes into the plane of the original figure. Compare the two perimeters. Can you draw any conclusion as to the *form* of a closed figure of given perimeter if it is to have a maximum area?

373.

Proposition V.



In this figure let the curve A C B have a given length. Join the ends A and B with the line A B. Suppose the figure to be a *maximum*.

Required to find its form.

Take any point P on the line and join P and A, P and B. Call the segments cut off by P A and P B, S_1 and S_2, and the $\triangle$ formed, *t*. *Suppose* the $\triangle$ is not a maximum, what do you know of $\angle$ A P B? What must $\angle$ P be that the $\triangle$ may be a maximum?

Imagine S_1 and S_2 hinged at P as a pair of compasses with unequal legs. Imagine $\angle$ P $>$ a right $\angle$, and suppose the point A slipped along A B to A′ till $\angle$ A′ P′ B is a right $\angle$. What has been done to the area of the $\triangle$? Have you changed the area of S_1 and S_2? Have you changed the area of $S_1 + t + S_2$, or the figure A B C? Is this possible if A B C is a maximum?

In what kind of a curve is the $\triangle$ located?

Why?

Write a generalization of your conclusion. Call it Prop. V.

374.

PROPOSITION VI.

Given the convex figure A C B D. Let A and B be 2 points that bisect the *perimeter*. Can you show that if the figure is a maximum, the straight line A B bisects the area?

Sug. *Suppose* $S_2 > S_1$ and then revolve S_2 on A B into the plane of the paper. What would follow?

If S_1 and S_2 have maximum areas, what must their forms be? Why?

What is A B of the maximum figure A C B D? Generalize.

375.

PROPOSITION VII.

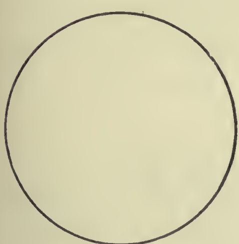


Fig. 1.

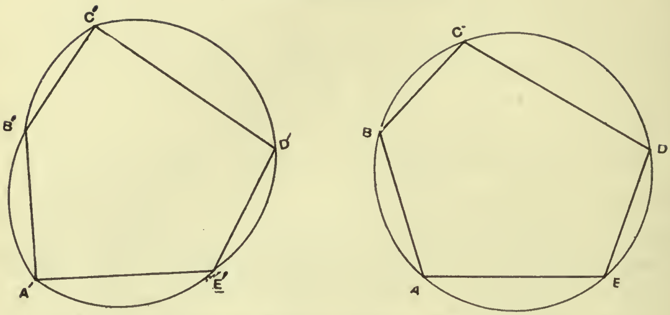
Fig. 2.

Fig 3.

Suppose the ⊙ Figure 1 and the plane Figure 2 to have equal areas and that the perimeter of Figure 2 equals the perimeter of circle Figure 3. Compare the areas of Figure 2 and Figure 3, of Figure 1 and Figure 3; the circumferences of Figure 1 and Figure 3; the circumference of Figure 1 with the perimeter of Figure 2. Generalize.

376.

Proposition VIII.



Given the polygon A B — E inscribed within the circle A B — E. Let A′ B′ — E′ be another polygon having the sides A′ B′ = A B, B′ C′ = B C, etc. Construct segments A′ B′, B′ C′, etc., = to the corresponding segments on sides A B, B C, etc.

Compare the perimeter of the circle and the curvilinear figure. Compare their areas. Can you now show that the plane figure A B — E > A′ B′ — E′? Write theorem.

377.

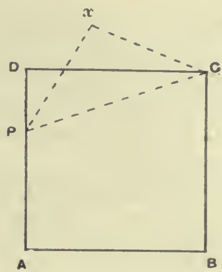
Proposition IX.

Let A B C D E be the maximum polygon having a given perimeter and n sides. Join A C. *Suppose*, if possible, A B and B C *unequal* and let A B′ C be an isos. △, A B′ = C B′ and isoperimetric with A B C. Compare area A B′ C D E .... with the original figure. Can A B′ C D E .... be greater than A B C D E ....? What conclusion can you draw concerning A B and B C?

Can you show that A B = A E = E D = D C? Is this polygon inscriptible? Give reason for answer. Is it regular? Write theorem.

378.

Proposition X.



Let A B C D be a square. Take any point P on side A D and construct △ P x C isoperimetric with P D C and having the side P x = C x.

(1) Compare △s P D C, P x C. (2) Compare area of square with pentagon A B C x P. (3) *Suppose* pentagon were made regular and isoperimetric with A B C x P, how would area compare with irregular pentagon and square? (4) Can you show that a regular hexagon > a regular isoperimetric pentagon? (5) How far may this reasoning extend? Generalize the truth reached.

Exercises.

364. Considering only the relation of space enclosed to amount of wall, what would be the most economical form for the ground plan of a house?

365. Why is the most economical form for piping that with a circular cross-section?

366. Of all △s in a given circle, what is the shape of the one of maximum area? Prove your work.

367. A cross-section of a bee's cell is a regular hexagon. Show that this is the best form for securing the greatest capacity with a given amount of wax (perimeter).

368. Find a point in a given straight line such that the tangents drawn from it to a given circle contain the maximum angle.

369. A straight ruler, 1 foot long, slips between the 2 edges of a rectangle. Find the position of the ruler when the $\triangle$ thus formed is a maximum. What is its area?

370. Of all $\triangle$s of a given base and area, show which $\triangle$ has the greatest vertical $\angle$.

371. Of 3 similar figures constructed on the 3 sides of a rt. $\triangle$, the figure constructed on the hypotenuse is equivalent to the sum of the other 2 figures.

372. Of all parallelograms of a given base and area, which has the least perimeter? Prove.

373. Given a square and a rectangle of the same area. Compare their perimeters.

374. What is the largest rectangle whose dimensions are the two segments of a line?

375. From a given point without a circle, draw a secant whose outer segment is a minimum. What about the inner segment?

376. Side of a regular hexagon $= 1$; it is required to find the sides of a rectangle that shall exactly enclose it, and to find the area of the hexagon and the area of the rectangle, and the ratio between them.

SELECTED EXAMINATION PAPERS IN PLANE GEOMETRY SET FOR ADMISSION TO SOME OF THE LEADING COLLEGES OF THE UNITED STATES.

Massachusetts Institute of Technology, September, 1898.

Every reason must be stated in full.

1. If two straight lines are cut by a third so as to make the alternate interior angles equal, the two lines are parallel.

2. If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.

3. The diagonals of a rhombus bisect each other at right angles.

4. In the same circle or equal circles, equal chords are equally distant from the center.

5. The angle between a secant and a tangent is measured by one-half the difference of the intercepted arcs.

6. Any two rectangles are to each other as the products of their bases by their altitudes.

7. The area of a circle is equal to one-half the product of its circumference and radius.

8. A regular hexagon, A B C D E F, is inscribed in a circle whose radius is 4. Find the length of the diagonal A C.

Cornell, 1898.

1. Through a given point, P, without the line, one and only one perpendicular can be drawn to a given straight line, A B.

2. Show how to bisect a given angle.

3. Given a triangle; find the center of the circumscribed

circle, the center of the inscribed circle, and the median center.

4. Construct a pentagon similar to a given pentagon when the sum of the sides is given.

5. If a square and a rhombus have equal perimeters, and the altitude of the rhombus is four-fifths its side, compare the areas of the two figures.

6. Find the area of a trapezoid of which the bases are a and b, and the other sides are equal to c.

7. Compare the areas of an inscribed and circumscribed hexagon about a given circle.

Harvard, June, 1896.

One question may be omitted.

[In solving problems, use for π the approximate value $3\frac{1}{7}$.]

1. Prove that if two oblique lines drawn from a point to a straight line meet this line at unequal distances from the foot of the perpendicular dropped upon it from the given point, the more remote is the longer.

2. Prove that the distances of the point of intersection of any two tangents to a circle from their points of contact are equal.

A straight line drawn through the center of a certain circle and through an external point, P, cuts the circumference at points distant 8 and 18 inches respectively from P. What is the length of tangent drawn from P to the circumference?

3. Given an arc of a circle, the chord subtended by the arc, and the tangent to the arc at one extremity, show that the perpendiculars dropped from the middle point of the arc on the tangent and chord, respectively, are equal.

One extremity of the base of a triangle is given and the center of the circumscribed circle. What is the locus of the middle point of the base?

4. Prove that in any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other upon that side.

Show very briefly how to construct a triangle having given the base, the projections of the other sides on the base, and the projection of the base on one of these sides.

5. Show that the areas of similar triangles are to one another as the areas of their inscribed circles.

The area of a certain triangle the altitude of which is $\sqrt{2}$, is bisected by a line drawn parallel to the base. What is the distance of this line from the vertex?

6. Two flower-beds have equal perimeters. One of the beds is circular and the other has the form of a regular hexagon. The circular bed is closely surrounded by a walk 7 feet wide bounded by a circumference concentric with the bed. The area of the walk is to that of the bed as 7 to 9. Find the diameter of the circular bed and the area of the hexagonal bed.

Yale, June, 1896.

GEOMETRY (A).

TIME, ONE HOUR.

1. The sum of the three angles of a triangle is equal to two right angles.

2. Construct a circle having its center in a given line and passing through two given points.

3. The bisector of the angle of a triangle divides the opposite side into segments which are proportioned to the two other sides.

4. If two angles of a quadrilateral are bisected by one of its diagonals, the quadrilateral is divided into two equal

triangles and the two diagonals of the quadrilateral are perpendicular to each other.

5. The circumferences of two circles are to each other as their radii (Use the method of limits.)

Yale, June, 1896.

GEOMETRY (B).

TIME ALLOWED, FORTY-FIVE MINUTES.

1. A tree casts a shadow 90 feet long, when a vertical rod 6 feet high casts a shadow 4 feet long. How high is the tree?

2. The distance from the center of a circle to a chord 10 inches long is 12 inches. Find the distance from the center to a chord 24 inches long.

3. The diameter of a circular grass plot is 28 feet. Find the diameter of a grass plot just twice as large. (Use logarithms.)

4. Find the area of a triangle whose sides are $a = 12.342$ metres $b = 31.456$ metres $c = 24.756$ metres, using the formula

Area $= \sqrt{s(s-a)\,(s-b)\,(s-c)}$ where $s = \dfrac{a+b+c}{2}$. (Use logarithms.)

Princeton, June, 1896.

State what text-book you have read and how much of it.

1. Prove that the sum of the three angles of a triangle is equal to two right angles; and that the sum of all the interior angles of a polygon of n sides is equal to $(n-2)$ times two right angles.

2. Show that the portions of any straight line intercepted between the circumferences of two concentric circles are equal.

3. Define similar polygons and show that two triangles whose sides are respectively parallel or perpendicular are similar polygons according to the definition.

4. Prove that, if from a point without a circle a secant and a tangent are drawn, the tangent is a mean proportional between the whole secant and its external segment.

5. Prove what the area of a triangle is equal to; also of a trapezoid; also of a regular polygon. Define each of the figures named.

6. Explain how to construct a triangle equivalent to a given polygon.

7. Prove that of all isoperimetric polygons of the same number of sides, the maximum is equilateral.

Johns Hopkins University, October, 1896.

1. Prove that the bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.

2. Show that through three points not lying in the same straight line one circle, and only one, can be made to pass.

3. The bases of a trapezoid are 16 feet and 10 feet respectively; each leg is 5 feet. Find the area of the trapezoid. Also find the area of a similar trapezoid, if each of its legs is 3 feet.

4. Define regular polygon. Prove that every equiangular polygon circumscribed about a circle is a regular polygon.

5. Prove that the opposite angles of a quadrilateral inscribed in a circle are supplements of each other.

6. Construct a square, having given its diagonal.

7. Prove that the area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle.

8. What is the area of a ring between two concentric circumferences whose lengths are 10 feet and 20 feet respectively?

Sheffield Scientific School, June, 1896.

[NOTE.—State at the head of your paper what text-book you have studied on the subject and to what extent]

1. Two angles whose sides are parallel each to each are either equal or supplementary. When will they be equal, and when supplementary?

2. An angle formed by two chords intersecting within the circumference of a circle is measured by one-half the sum of the intercepted arcs.

3. A triangle having a base 8 inches is cut by a line parallel to the base and 6 inches from it. If the base of the smaller triangle thus formed is 5 inches, find the area of the larger triangle.

4. Construct a parallelogram equivalent to a given square, having given the sum of its base and altitude. Give proof.

5. What are regular polygons A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.

The University of Chicago, September, 1896.

TIME ALLOWED, ONE HOUR AND FIFTEEN MINUTES.

[When required, give all reasons in full and work out proofs and problems in detail]

1. Show that if on a diagonal of a parallelogram two points be taken equally distant from the extremities, and these points be joined to the opposite vertices of the parallelogram, the four-sided figure thus formed will be a parallelogram.

2. State and prove the converse of the following theorem.

In the same circle, equal chords are equally distant from the center.

3. Given a circle, a point, and two straight lines meeting in the point and terminating in the circumference of the

circle. State what four lines or segments form a proportion and in what order they must be taken:

(1) When the point is outside the circle, and

(*a*) both lines are secants,

(*b*) one line is a secant, and the other a tangent,

(*c*) both lines are tangents.

(2) When the point is within the circle, and the two lines are chords.

Prove in full (1) (*a*). Show that (1) (*c*) is a limiting case of (1) (*a*).

4. To a given circle draw a tangent that shall be perpendicular to a given line.

5. Show how to construct a triangle, having given the base, the angle at the opposite vertex, and the median from that vertex to the base. Discuss the cases depending upon the length of the given median.

Wellesley College, June, 1895.

1. An angle formed by two tangents is how measured? Prove.

2. The diagonals of a rhombus bisect each other at right angles.

3. (*a*) If a line bisects an angle of a triangle and also bisects the opposite side, the triangle is isosceles.

(*b*) State and demonstrate the general case for the ratio of the segments of the side opposite to a bisected angle.

4. With a given line as a chord, construct a circle so that this chord shall subtend a given inscribed angle.

5. (*a*) On a circle of 4 feet radius, how long is an arc included between two radii forming an angle of 20°? Prove, deriving the formula employed.

(*b*) Find the area of the regular circumscribed hexagon of a circle whose radius is 1.

6. Two similar triangles are to each other as the squares of their homologous sides.

BOOK VI.

SOLID GEOMETRY.

Preliminary Discussion.

378.

1. (1) Construct three plane figures; two not plane. (2) Why call the first *plane* figures? (3) What is a plane? What are its limits? How illustrate or represent a plane? (4) How test a plane surface? a curved surface?

2. What is your idea of space?

3. Explain the terms *finite* and *infinite*. (*Definition:* A finite (see § 1) portion of space regarded as separated from the rest is called a *solid*.) Draw distinction between a physical solid and a geometrical solid.

4. What does a surface do to space? What does a closed surface separate? What separates one portion of space from another?

5. Suppose one definite portion of surface is separated from the rest, what kind of a line do we find?

6. What separates one part of a line from another?

7. What can you assert of a point? Can we compare two points? How may a point be determined?

8. Explain the difference between a square and a cube; a circle and a sphere; a triangle and a triangular prism.

9. What is the difference between plane figures and solid figures?

10. What is meant by the position of a plane? Illustrate with cardboard.

11. How many planes may be made to pass through a

given point? two points? a straight line? three points? Use pins to illustrate lines and points.

12. Does one line determine a plane? Illustrate with a card.

13. Can you recall your definition of a *postulate?* State a postulate about changing the position of a figure. Suppose one point of a figure to be fixed, how does it affect the figure? Suppose a second point fixed? a third?

14. What is the locus of a moving point? Is a point a part of a line? (How many points make a line one foot long?) What determines a straight line?

15. How many points determine a curved line? How many straight lines are determined by two points?

16. Think of two points on *any* surface. How many lines may be passed through these two points? Can there be between two points on *any* surface *one* line passing along the surface shorter than all the others? Such a line is called a *direct* or *geodesic* line.

17. What must we have in order to locate one point on a surface relative to another? (A *surface* is a magnitude of *two* dimensions.)

18. How many items must be considered to locate two points relatively in space? Illustrate. Define a plane surface; a curved surface.

379.

1. When is a straight line || to a plane?
2. When are planes parallel?

380.

PROPOSITION I.

(1) Can you think of an easy way to pass a plane through a line and a point? Illustrate. How many planes may be passed through a given line and a given point? Can you show that this plane is fixed?

(2) Think of three points, *m*, *n*, *o*, not in the same straight line. What is one way to pass a plane through these three points? How many planes may be passed through these three points?

(3) Draw two ‖ lines, A B and C D. Pass a plane, *m n*, through A B. Can you pass the same plane through C D? Will these two lines fix the position of *m n*? Why? Illustrate with straight wires and a card.

(4) Draw two lines, A B and C D, that intersect. Pass a plane through A B. Can you pass the same plane through C D? Is the plane fixed? How many planes can you pass through these two lines?

Make a summary of the conditions which will locate a plane.

Call it Prop. I.

Preliminary Discussion.

381.

The beginner in Solid Geometry will find his ideas growing clearer if he will use pieces of card board to represent planes and straight wires to represent lines and points. To illustrate: Take two postal cards and cut each half in two, and then fit the cards together. This will represent to the mind the intersection of two planes which appears to be a straight line. A darning-needle or hat-pin put through these cards will represent a line piercing two planes. An ordinary room will illustrate some of the problems in Solid Geometry. The pupil will understand that these are simply *aids* to the imagination. When the learner is able to see planes intersecting, lines making angles, solids cut by planes, solids cut by other solids, without the aid of physical illustrations, he is making a good start in Solid Geometry.

Exercise.

377. Name some problems in Solid Geometry which the carpenter must solve; some the plumber must solve; some the brickmason must solve.

382.

In how many points may a straight line intersect a plane? Prove?

Sug. If we suppose the line to meet the plane in two points, what definition is violated?

383.

Proposition II.

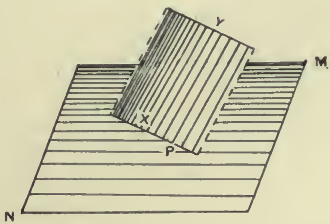
Suppose two planes meet, what can you prove of their intersection?

Sug. Take any two points in the line of intersection and join them. What is the line? Where does it lie? Illustrate using a pair of scissors and two pieces of cardboard. *Draw* figures also. Write a neat proof.

Generalize these truths, calling the statement Prop. II.

384.

Proposition III.



In the figure let y be a line || the line x in the plane M N.

Required to find how y is related to the plane M N.

Sug. Pass a plane through x and y.

Suppose y could meet M N (in P, say). In what plane would it meet M N? in what line? Any violation of conditions? Write a careful proof. Generalize.

Call this Prop. III.

385.

Proposition IV.

In the figure above let y be any line || to M N. Pass any plane through y intersecting M N.

Write a full proof showing how the intersection of the two planes is related to y.

Call the generalization Prop. IV.

386.

Cor. Suppose that a line, c, is || to y, in § 385, and passes through a point, P, in the plane M N.

Prove where the line c lies.

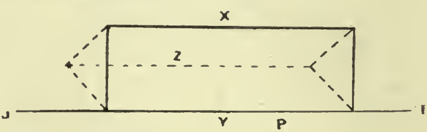
Sug. Can you prove it coincident with x? Write a statement of this corollary.

387.

Proposition V.

Can you show that if two lines are || to a third line they are || to each other

Use cards to illustrate.



In the figure let x and y be || z.

Call some point on y, P. Pass planes through x z ; y z ; x and P. Call the line intersecting y z and x P, i j. Can you show that $i\,j$ coincides with y? Write your proof and state theorem.

Call it Prop. V.

Can you prove this theorem by projecting one line?

Z || plane x P. [?]

Z || y [?]

y lies in plane x P. [?]

x || line J P I. [?]

y lies in plane x P and passes through P;

∴ J P I and y must coincide. [?] Q. E. D.

Exercises.

378. Can you show that any theorem in Plane Geometry in regard to a △ is true also in Solid Geometry?

Illustrate your exercises with good drawings.

379. If a plane is passed through each of 2 || lines, the intersection of the planes is || to each of the lines.

380. If 3 planes meet in 3 lines, the intersections either meet in a point or they are ||.

381. If each of the two intersecting lines is || to a plane, the plane of these lines is || to the first plane.

382. Parallel lines included between || planes are equal.

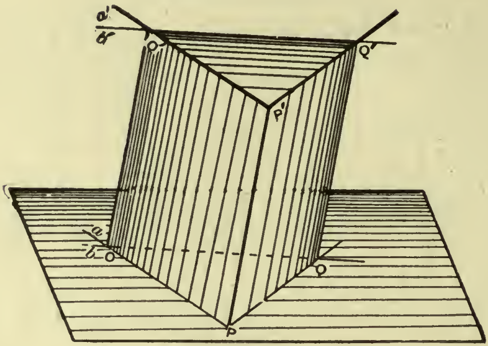
383. Construct through a given point a plane || to a given plane.

384. Construct through a given line a plane || to a given line.

385. Construct through a given point a plane || to each 2 lines. Is this ever impossible? Discuss.

388.

PROPOSITION VI.



Stand a book on the desk partly open and suppose the pages to represent parallelograms. Compare the angle made by the two top edges with the angle made by the two corresponding lower edges. Can you illustrate the same truth in the room?

In the figure suppose *a* and *b* two intersecting lines and *a*′ and *b*′ two intersecting lines respectively || to *a* and *b*.

Compare their ∠s.

Call the intersections O and O′. Join these points. From any points on *a* and *b*, as P and Q, erect ||s to O O′.

(1) Can you pass a plane through *a*, *a*′, and O O′?

(2) Will the parallel from P meet *a*′? Why? Letter the point, P′.

(3) Will parallel through Q intersect *b*′? Letter intersection Q?

(4) Compare O P and O′ P′; O Q and O′ Q′.

(5) Join P Q and P′ Q′. How are P Q and P′ Q′ related?

(6) Compare ∠ P O Q with ∠ P′ O′ Q′.

(7) Show that ∠ P O Q is supplementary to two angles formed by *a*′ and *b*′.

Draw conclusion, and call it Prop. VI.

389.

Prove that if each of two intersecting lines is $\parallel$ to a plane, the plane of these lines is $\parallel$ to the first plane.

Use *reductio ad absurdum*. *Suppose* the plane of the intersecting lines meets the given plane in line x. Then each of the intersecting lines meets the plane. (Pupil finish.)

Call this Prop. VII.

Exercises.

386. If a line cuts one of two $\parallel$ lines, must it cut the other? Are the corresponding $\angle$s equal.

387. Prove that $\parallel$ lines included between $\parallel$ planes are equal.

388. If 2 $\parallel$ lines intersect a plane, compare the angles formed.

389. If a straight line intersects 2 $\parallel$ planes, compare the angles formed.

390. If a line is $\parallel$ to each of 2 intersecting planes, how is it related to their intersection?

Definitions.

390.

The distance from a point to a plane is the *perpendicular distance*.

391.

The point where a $\perp$ meets a plane is called the *foot of the perpendicular*.

392.

A line is said to be $\perp$, or *normal to a plane*, when it is $\perp$ to every line in that plane which passes through its foot. When is a line oblique to a plane?

393.

Lines or points which lie in the same plane are *coplanar*

394.

(1) Three or more points which lie in the same *line* are said to *collinear*.

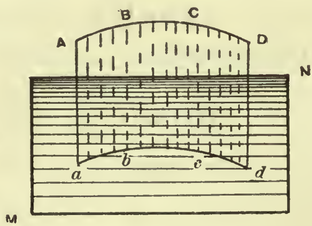
(2) A line is || to a plane if it cannot meet it however far produced. The plane is said to be || to the line.

395.

The projection of a point on a *plane* is the foot of the ⊥ from the point to the plane. Illustrate with your pencil and desk.

396.

The projection of a *line* on a plane is a straight line joining the projections of the extremities of the line on the plane. In the figure, *a*, *b*, *c*, *d* are projections of the points A, B, C, D and line *a d* in the projection of line A D on the plane M N.



397.

The *smaller angle* formed by a line and its projection is called the *inclination of the line to the plane.*

398.

The angle which a line makes with a plane is the angle which it makes with its projection. Illustrate with pencil and desk.

399.

The plane which a line makes with its projection is called the *projecting plane.*

400.

Where are all the *common* points of 2 planes?

EXERCISES.

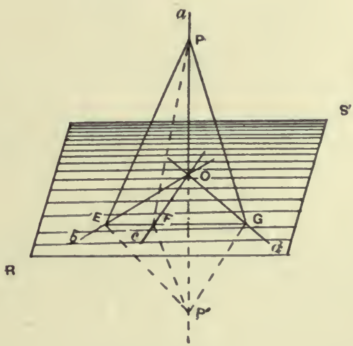
391. How many planes are determined by 6 points, 3 being collinear?

392. How many planes in general are determined by 4 points in space, no 3 being collinear?

393. A point, P, is in three planes, P, Q, R. Is it necessarily fixed?

401.

PROPOSITION VII.



In the figure suppose line $a \perp$ to both b and d. How is a related to any other line, as c, lying in the plane of b and d and passing through their intersection?

Sug. I. On line a make $OP = OP'$; draw any line cutting b, c, d in E, F, G, and join points with P and P'.

Sug. II. Compare (1) △s E G P′ and E G P; (2) △s E F P and E F P′; (3) and △s P O F and P′ O F.

How then is c related to a?

State the general truth, Prop. VII.

402.

Cor. I. If a line is ⊥ to each of 2 intersecting lines, show from the figure above how it is related to their plane.

403.

Cor. II. Prove what line of all those drawn from a given point to a plane is the shortest. Which lines are equal?

404.

Cor. III. What is the locus of points equally distant from 2 points?

405.

Cor. IV. What is the locus of straight lines which cut a given straight line at a given point at right angles?

Note.—The proof that a is ⊥ to every straight line that meets it in the plane R S is due the great French mathematician Legendre. (See his biography.)

EXERCISES.

394. Through a given point in a plane, how many ⊥s may be erected to the plane?

395. Through a given point in a line, how many ⊥ planes can be drawn to that line?

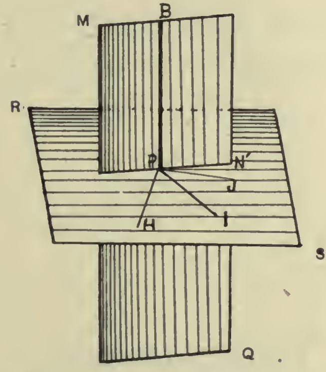
396. *Suppose* the hand of a clock to be ⊥ to its moving axle. Show what kind of a figure it describes in revolving.

397. How many planes are determined by 5 concurrent lines, no 3 of which are coplanar? By x lines?

398. If a line cuts one of 2 $\parallel$ lines, must it cut the other? If it does, are the corresponding $\angle$s equal?

406.

Proposition VIII.—Theorem.



In the figure suppose B P $\perp$ to the three lines P H, P I, P J at their point of concurrence. Can you show how these three lines are situated?

Sug. 1. Let R S be the plane determined by P H and P I, and M Q the plane determined by B P and P J.

Sug. 2. *Suppose* P J is not in R S and let P N′ be the intersection of the two planes. What does this supposition involve? What can you now state concerning the three lines which are $\perp$ to B P at their point of concurrence?

Write the general truth and call it Prop. VIII.

407.

Cor. I. Show that lines $\perp$ to the same line at the same point are coplanar.

408.

Cor. II. Through a given point in a plane how many ⊥s can be erected?

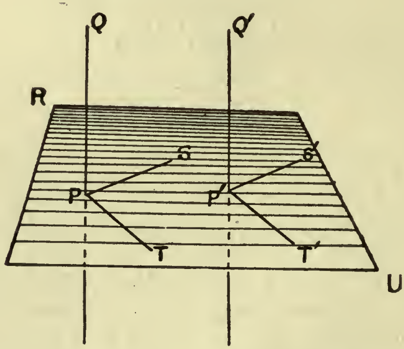
[*Hint.—Suppose* two perpendicular lines can be erected. Pass a plane through those two lines. What follows?]

409.

Cor. III. Can you show from the figure above that only one plane can be passed through a given point in a line ⊥ to that line?

410.

Proposition IX.



Given: The line P Q ⊥ to plane R U and P′ Q′ || P Q.

How is P′ Q′ related to R U?

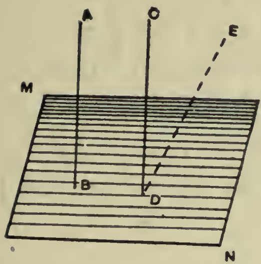
Sug. Draw P S, P T in the plane through P, and through P′ draw P′ S′, P′ T′ || P S and P T respectively.

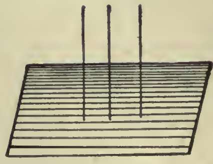
Compare ∠ Q′ P′ S′ with ∠ Q P S, ∠ Q′ P′ T′ with Q P T. How then is Q′ P′ related to the plane R U?

Write the general truth, and call it Prop. IX.

411.

PROPOSITION X.





Suppose A B and C D ⊥ to plane M N. how are these two lines related?

Sug. Suppose, if possible, a third line, D E, to be erected || A B.

What do you know of E D? Is there any previous proposition violated?

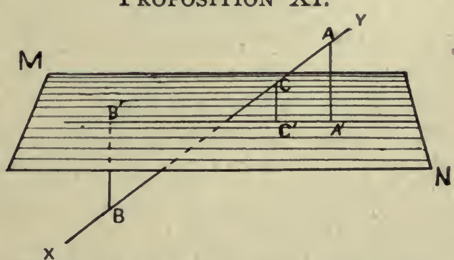
Suppose 3 lines ⊥ to the same plane. Show how the lines are related? Compare with Prop. V.

412.

Cor. How many ⊥s from a point to a plane?

413.

PROPOSITION XI.



Given: The points A and B in the line *x y*, which pierces plane M N. Project these points in the plane M N.

Join the projections of A and B. Call the line A′ B′. Does the projection of any other point in the line xy, C lie in the same line with A′ B′?

Sug. What can you say of A A′? B B′? C C′? Pass a plane through B B′, A A′. Is the point C in this plane? C C′? Does A′ C′ B′ lie in the straight line A′ B′?

Does the line determined by the projections of any two points of a given line give its projection? Generalize.

This will be Prop. XI.

Can you prove this directly from the definition of the projection of a line?

414.

Cor. Suppose a line 20 dm. long pierces a plane. Does it meet its projection? Where? Compare the lengths of the projections if the line were (1) projected on the plane pierced; (2) projected on a plane o which it is parallel.

EXERCISES.

399. Are lines which make equal angles with a given line always ||? Illustrate your answer.

400. A line equals its projection; construct a figure to compare the positions of the two lines.

401. Can you show when the projection of a line is half the length of the line? When is it zero?

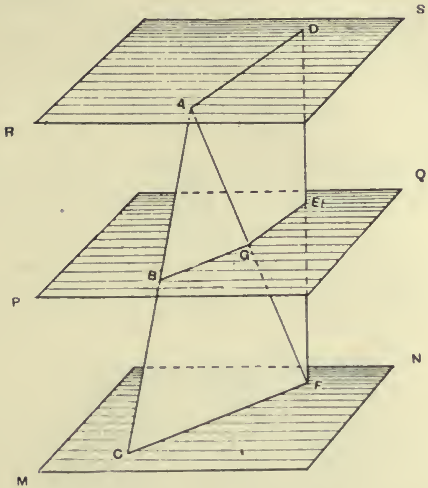
402. Two || lines have their projections in the same plane. What can you show of their projections?

403. If the projections of two lines are || or coincident, prove whether or not the lines are ||.

404. If the projection of a line 24 inches long is 10 inches, find the projection on the same plane of a parallel line 32 inches long.

415.

Proposition XII.



In the figure let A C, D F be any two lines cut by three ∥ planes, R S, P Q, M N, in the points A, B, C, D, E, F.

Compare the segments of the lines.

Sug. Join A, F. Pass a plane through A C and A F, and F A and F D. (Why can we do this?) Let C F and B G be the intersections made by the first plane with the others and A D and G E the intersections made by the second plane. Can you now show that $\frac{AB}{BC} = \frac{DE}{EF}$?

Write Prop. XII.

416.

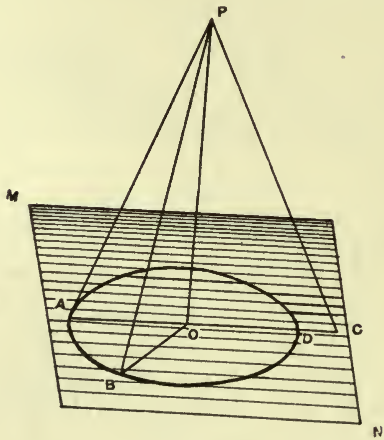
Cor. I. Suppose two lines are cut by any number of ∥ planes. What can you say or prove of the corresponding segments?

417.

Cor. II. If x straight lines be cut by three ∥ planes, prove how the corresponding segments are related.

418.

Proposition XIII.



Given: P, a point without the plane M N, and P O, a perpendicular to it.

1. Compare P O with any other line drawn from P to the plane.

In how many ways can you make the comparison? Generalize.

2. *Given:* P O ⊥ M N and ∠ B = ∠ A (angles of inclination), to compare P B and P A. State the converse and prove it.

Generalize.

3. If the projections of two lines from the same point to the same plane are equal, how are the lines related? State and prove the converse.

Generalize each.

4. Given the unequal oblique ∠s P B O > P C O, to compare P B and P C. State and prove the converse.

Generalize each.

5. Suppose the projections of two oblique lines drawn from a point, P, to the plane M N to be unequal, compare the lines projected.

Write a general statement comprehending the five statements above. Begin in this manner: Of all lines that can be drawn from a point to a plane 1. ——, 2. ——, 3. ——, 4. ——, 5. ——.

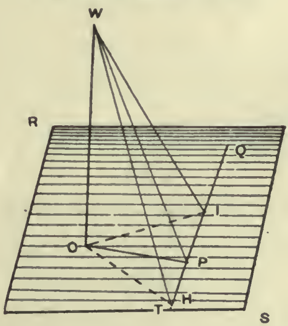
Exercises.

405. Parallel line segments are proportional to their projections on a plane.

406. Can you project two lines on three different planes at the same time.

419.

Proposition XIV.



Let W P be any line intersecting the plane R S at point P, and let O P be its projection on R S. Let Q T be a line in R S ⊥ O P.

Show how Q T is related to the given line W P.

Sug. Measure off on Q T, P H = P I. Join O and H, O and I, W and H, W and I. Compare △s P O I and P O H, I W and H W; also △s I P W and H P W. How is W P related to Q T? Generalize the truth reached?

EXERCISES.

407. If a line is || to each of two intersecting planes, how is it related to their intersection? Prove.

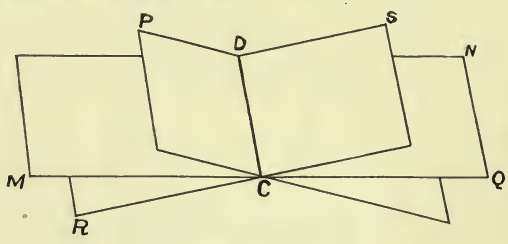
408 Can you construct a plane containing a given line and || to another line?

409. If two || lines intersect the same plane, show that they are equally inclined to it.

DIEDRAL ANGLES.

420.

Definition: When any number of planes pass through the same line, they are said to form a *pencil of planes*, and any two of the planes form a *diedral angle*.



421.

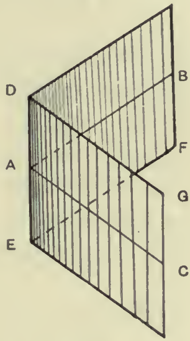
The planes are the faces of the diedral angle, and their intersection the edge.

422.

A diedral angle may be designated by two letters on its edge, but if several diedral angles have a common edge, then four letters are necessary, one in each face and two on the edge, thus: S D C P, N D C M.

423.

Definition: The *plane angle* of a diedral ∠ is the angle formed by two straight lines, one in each plane, drawn perpendicular to the edge at any given point.



Thus if B A and C A in the faces D F and E G, respectively, are each ⊥ D E, they form the plane ∠ of the diedral D E.

424.

By using your cardboard and by drawings, illustrate *vertical* diedral angles, *adjacent* diedral angles, *right* diedral angles. Write a definition of each.

[*Note.*—The faces of the diedral angle are indefinite in extent, but for convenience in study we take a limited portion of the bounding planes. The pupil should take pains in learning to draw the figures in Solid Geometry.]

425.

What plane angle will be formed if a plane be passed perpendicularly to the edge of a diedral angle and intersecting its sides?

Through a given line in a plane, how many perpendicular planes can be passed to the given plane? Why?

Exercises and Questions.

410. How many diedral ∠s are formed by two intersecting planes.

Can you show by illustration how diedral ∠s may vary?

411. How many diedral ∠s has a cube? a square pyramid? a triangular pyramid?

412. Make a drawing showing two planes meeting. What is the sum of the diedral angles formed?

413. Represent two complementary diedral angles.

414. Draw two parallel planes cut by an oblique plane. Name the equal angles and pairs whose sums equal two right diedral angles. Compare with § 68.

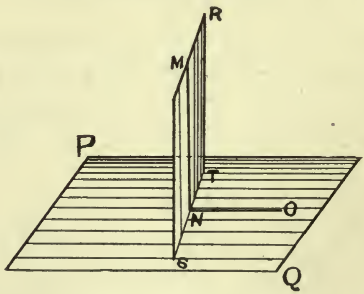
415. State for diedral angles what § 73 does for plane angles.

416. Draw two diedral angles whose corresponding sides are parallel. In how many ways can you draw this? Compare the diedral angles

417. Draw two diedral angles whose corresponding faces are perpendicular to each other. Discuss the diedral angles formed.

426.

Proposition XV.



Suppose R S ⊥ P Q and line M N ⊥ S T the intersection of the two planes.

How is M N related to P Q?

Sug. At N draw N O ⊥ S T in plane P Q. What is the angle M N O? Why? What two lines determine P Q? Write Prop. XV.

427.

Cor. I. From figure above show that a ⊥ to either plane at any point of S T lies in the other plane.

EXERCISE.

418. A line is ⊥ to a plane. Can you show that every plane passed through the line is ⊥ to the plane.

428.

Cor. II. Suppose two planes ⊥ to each other. Can you show that a ⊥ from any point of the one plane to the other must lie in the first plane?

429.

Cor. III. Through a point without a line prove how many ⊥ planes can be passed ⊥ to the line.

430.

PROPOSITION XVI.

Can you show that vertical diedral ∠s are equal?

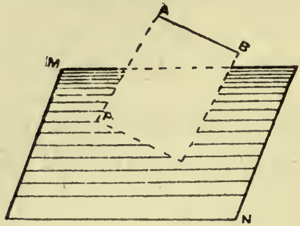
431.

PROPOSITION XVII.

If a plane is ⊥ the edge of a diedral ∠, how is it related to each face of the diedral ∠?

432.

Proposition XVIII.



Given: A B, any line not ⊥ to the plane M N.

How many planes can be passed through A B ⊥ M N?

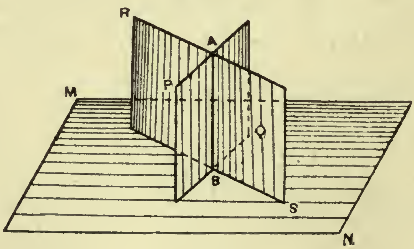
[*Hint.*—Project the point A to the plane M N. Pass a plane through A B and A P.]

Draw conclusion and call this Prop. XVIII.

Exercise.

419. Take the different kinds of plane △s and pass planes through the sides (including the sides) ⊥ to the planes of the △s. Prove what kinds of diedral ∠s are formed.

433.



Suppose M N ⊥ to the intersecting planes R S and P Q.

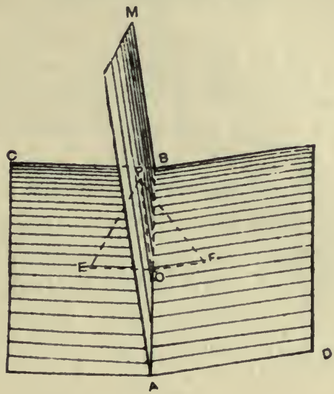
How is it related to the line of intersection?

Sug. Erect a ⊥ to the plane M N at B. Show how it is related to R S and P Q. Draw conclusion.

Write Prop. XIX.

434.

Proposition XX.



In the figure, let A M be the bisector of the diedral ∠ C A B D. From any point, P, in A M, draw P E ⊥ A C and P F ⊥ B D. Pass a plane through P E and P F cutting A M in O P, A C in E O, and B D in O F.

(1) How is plane P E O F related to A C? to B D? to A B?

(2) What are the plane angles which measure the two equal diedral angles? Prove it.

(3) Compare P E, P F.

How is any point in the plane which bisects the diedral ∠ located with regard to the faces of the angle?

Exercises.

420. What is the locus of the foot of an oblique line 1 m. long drawn from a point 8 dm. above a plane?

421. What is the locus of all points in space equidistant from two parallel planes? from two intersecting planes? from a given point? from two given points?

422. What is the locus of all points in space any given distance from a given plane?

423. In the proposition, do P, E, O, F lie in the same circumference, or are they *concyclic*.

424. If two adjacent diedral ∠s are supplementary, show how their exterior faces are related.

425. Pass a plane through one of the diagonals of a parallelogram. Draw perpendiculars from the extremities of the other diagonal to the plane and compare them. What is the limit of the length of these ⊥s?

426. If two lines are perpendicular to each other, are their projections always perpendicular to each other?

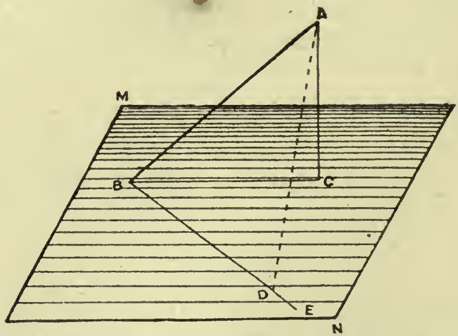
427. Two parallel planes are intersected by two other parallel planes. How do the four lines of intersection compare?

428. If O F = F P, Prop. XX., how many degrees in ∠ M A B D? ∠ C B A D?

435.

PROPOSITION XXI.

Let A B be any straight line and B C its projection on the plane M N.



How does the acute angle formed by the line and its projection compare with the angle formed by the line and any other line in the plane?

Sug. Draw any other line through B, as B E. Measure off B D = B C.

Compare A C and A D. Auth.
Compare ∠ A B C and ∠ A B D (§ 84).
Write the proposition, and call it Prop. XXI.

436.

Cor. With what line in the plane M N does the line A B make the greatest angle?

EXERCISE.

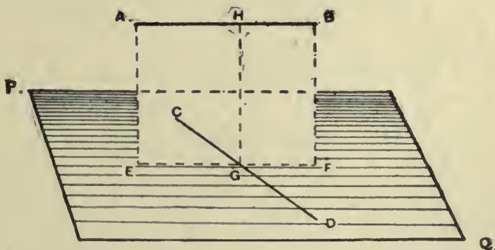
429. The projections of three lines on the same plane are parallel and of equal length. Can you draw any definite conclusions concerning the three lines?

437.

PROPOSITION XXII.

Can you illustrate from the room that two straight lines in different planes may have a perpendicular between them?

Suppose A B and C D two straight lines not in the same plane.



(1) Can we find a perpendicular between the lines? (2) How many perpendiculars are there?

Sug. Let P Q be a plane passed through C D and ∥ to A B. Project A B on P Q. How does E F compare with A B? Is E F ∥ to C D? Why? Call their intersection G Does A B and its projection determine a plane? At G erect ⊥ G H in the projecting plane.

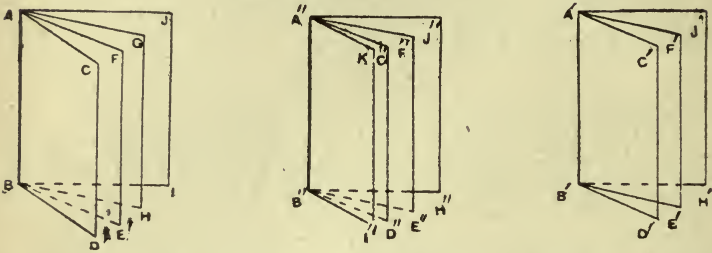
Pupil complete proof. Can you prove (2)? [§ 57.]

438.

Cor. What is the shortest distance between two lines not in the same plane?

439.

Proposition XXIII.



Given: The diedral ∠s J A B D, J′ A′ B′ D′ and their plane ∠s C A J and C′ A′ J′.

To show that the diedral ∠s are to each other as their plane ∠s.

(1) Suppose ∠ J A C and J′ A′ C′ are commensurable and let ∠ G A J be the common unit. Express the relation of ∠ C A J and ∠ C′ A′ J′.

Can you pass planes through A G and A B? A F and A B? A′ F′ and A′ B′? How are the diedral ∠s formed related? Can you now show that the diedral ∠s are to each other as their plane ∠s when the plane ∠s are commensurable?

(2) Suppose the unit plane ∠ G A J is contained twice with a remainder in J″ A″ B″ K. If we let the unit ∠ continually decrease, what does ∠ J″ A″ C″ ≐ ? Considering the diedral ∠s formed from these plane ∠s and decreasing the unit diedral ∠ indefinitely, what does J″ A″ B″ C″ ≐ ?

What is the ratio $\frac{\angle \text{ J A C}}{\angle \text{ J}'' \text{ A}'' \text{ C}''}$? $\frac{\angle \text{ J A B C}}{\angle \text{ J}'' \text{ A}'' \text{ B}'' \text{ C}''}$?

But $\frac{\text{J A C}}{\text{J}'' \text{ A}'' \text{ C}''} \doteq \ldots .$ and $\frac{\text{diedral } \angle \text{ J A B C}}{\text{diedral } \angle \text{ J}'' \text{ A}'' \text{ B}'' \text{ C}''} \doteq \ldots .$

∴ . . . .

Write Prop. XXIII.

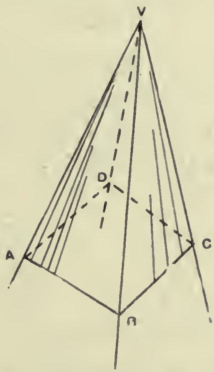
POLYEDRAL ANGLES.

DEFINITIONS.

440.

When three or more planes meet in a point, they form a polyedral angle or polyedral.

[Thus the planes V A B, V B C, V C D, V D A, meeting in the point V, form a polyedral angle. The point at which the planes meet



is called the *vertex* of the polyedral; the intersection of the planes are the *edges;* the planes are called the *faces;* and the angles A V B, B V C, etc., are called the *face angles* of the polyedral.]

Note.—As in other problems, the planes are of indefinite extent, but to show the relation of the edges in a figure it is clearer to have a plane cutting the edges.

441.

A polyedral angle bounded by three faces is called a *triedral angle;* if bounded by four faces, it is called a *tetraedral angle.*

442.

In the figure in § 440, A B C D is called a *section.* How is it formed? If the *section* is a convex polygon, the polyedral is a *convex polyedral.* What is a *concave* polyedral?

443.

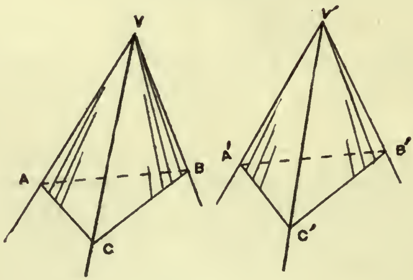
The *parts* of any polyedral angle are its face angles and its diedral angles. Illustrate with a pyramid.

444.

The *magnitude* of a polyedral angle depends entirely upon the divergence of its faces.

445.

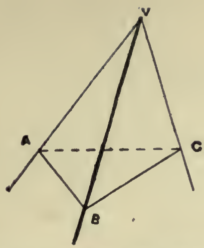
Two polyedral angles which have the face angles and diedral angles of one respectively equal to the homologous face angles and diedral angles of the other and arranged in the *same order* are said to be equal.

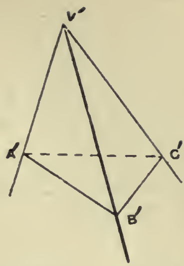


The face angles A V C, C V B, B V A are equal, respectively, to A′ V′ C′, C′ V′ B′, B′ V′ A′, and the diedral angles V A, V C, V B, to V′ A′, V′ C′, V′ B′; hence we may apply one to the other and they will coincide in all their parts; hence the polyedrals are *equal*.

446.

Polyedral angles which have their face angles and their diedral angles equal, each to each, and arranged in *reverse* order, are said to be *symmetrical*.





The triedral angles V — A B C and V′ — A′ B′ C′ are *symmetrical* if the face angles A V B, B V C, C V A are equal, respectively, to the face angles A′ V′ B′, B′ V′ C′, C′ V′ A′, and the diedral angles V A, V B, V C equal, respectively, the diedral angles V′ A′, V′ B′, V′ C′.

Observe the position of the faces.

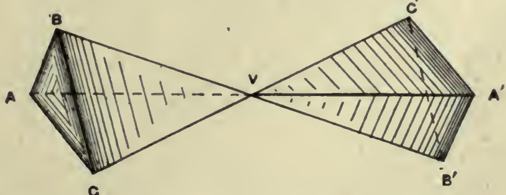
[Note.—The two hands or the two feet or the two sides of the face illustrate *symmetrical* solids. Are the right shoe and the left shoe equal? What should be said about the right glove and the left glove?]

447.

Two polyedrals are *vertical* when the edges of one are the prolongations of the edges of the other.

448.

PROPOSITION XXIV.



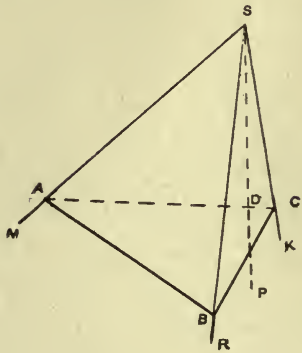
Can you prove vertical polyedrals symmetrical?

[*Hint.*—What are the *parts* of a polyedral? Are the corresponding vertical parts equal? What is the order of the equal parts?]

Write the proposition, and call it Prop. XXIV.

449.

PROPOSITION XXV.



Let S — M R K S be a triedral angle.

To show that the two face ∠s M S R + R S K are greater than ∠ M S K.

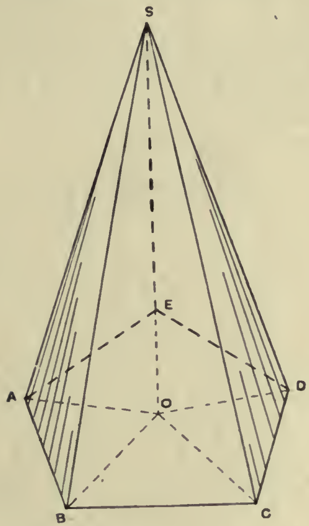
Sug. In M S K draw S P so that ∠ M S P = M S R. On lines S R and S P make S D = S B. Pass a plane through D and B cutting the solid ∠ in A B, B C, A C. Compare A B, A D; also A B + B C with A D + D C. Can you finish proof?

Write Prop. XXV.

QUESTION.—Would Prop. XXV. need proof if the three face angles were equal?

450.

Proposition XXVI.



Given: The polyedral ∠ S — A B C D E to prove that the sum of the plane ∠s formed by the edges is less than 4 rt. ∠s.

Sug. Intersect the faces of the polyedral ∠ by a plane forming the plane figure A B C D E. From any point in the polygon, as O, draw lines to the vertices, as O B, O C, etc. How many △s having vertices at O? How many △s with vertices at S? What do you observe in the two sets of △s? What planes contain or bound the triedral ∠ B? triedral ∠ C? etc. What did we learn in Prop. XXV.? Can you show:

That ∠ S B A + ∠ S B C > ∠ A B O + C B O?

That ∠ S C B + ∠ S C D > ∠ B C O + D C O?

That ∠ S D C + ∠ S D E > ∠ C D O + E D O? etc.

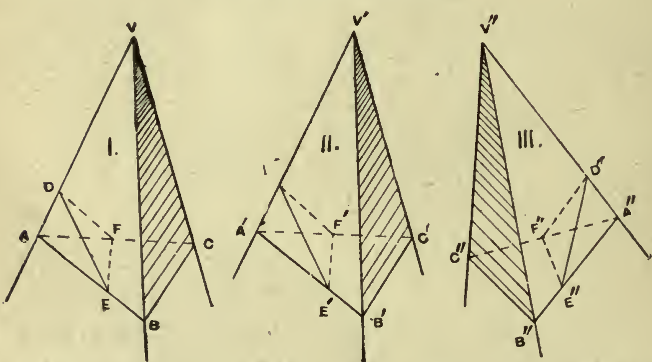
How does the sum of the ∠s at the base of the △s whose vertices are at S compare with the sum of the ∠s at the base of the △s whose vertices are at O?

How many right ∠s are there in each of the two sets of △s? Now from these equal sums take the sum of the angles at the bases of the two sets of △s. How do the remainders compare? What is the sum of all the ∠s at O? What then follows about the sum of all the angles at S?

Generalize the truth discovered, and call it Prop. XXVI.

451

PROPOSITION XXVII.



Given: The two triedral ∠s V—A B C, V'—A' B' C' with ∠ A V B = ∠ A' V' B'; ∠ B V C = ∠ B' V' C'; ∠ C V D = ∠ C' V' D'.

Can you prove the triedrals *equal* or symmetrical?

Sug. I. Compare the diedral ∠s V A and V' A', V B and V' B', V C and V' C'.

Measure off equal distances from the vertices on each diedral ∠, as V A = V B = V C = V' A', etc. and compare the △s A B C and A' B' C'.

Sug. II. Measure off on V A and V' A' a distance A D = A' D' and draw D E and D' E' respectively ⊥ to A V and A' V' in the faces A V B and A' V' B'; also draw D F and D' F' respectively ⊥ to A V and A' V' in the faces A V C and A' V' C'.

Join the points of intersections E and F, E′ and F′. Can you show that D E = D′ E′? D F = D′ F′? E F = E′ F′? Compare ∠ E D F and ∠ E′ D′ F′. Compare diedral ∠s V A and V′ A′. Generalize the truth discovered. Do you think I. and II. can be made to coincide? Suppose in III. V″ A″ = V A in I., and V″ B″ = V B, and V″ C″ = V C. Can you make them coincide? What do we say of such solids as I. and III.?

452.

Cor. Suppose two symmetrical triedral ∠s to be isosceles what follows?

453.

A problem of construction in Solid Geometry is considered solved when it is reduced to one of the following elementary constructions.

(1) A straight line can be drawn through any given point ⊥ to any given plane.

(2) A plane can be passed through any three given points.

(3) That the intersection of a plane with any other plane or any line can be determined.

In Solid Geometry the constructions cannot be made with ruler and compasses only.

Exercises.

430. Given two straight lines x and y. Through a given point in space determine a line that shall cut the two given lines.

431. A plane intersects 2 ∥ planes. Show (1) that the alternate interior diedral ∠s are equal; (2) That the corresponding diedral ∠s are equal: (3) That the sum of the interior diedral ∠s on the same side equals two rt. ∠s.

432. Given a plane to $\perp$ a line at its middle point. Show how any point in the plane is related to the extremities of the line.

433. Can you show that the 3 planes which bisect the diedral $\angle$s of a triedral $\angle$ meet in the same straight line?

434. Find the locus of all points, any one of which is equally distant from two given planes.

435. Two $\parallel$ lines intersected by 3 $\parallel$ planes are 12 inches and 20 inches in length, If the segments of the former are 9 and 3, what are the segments of the latter.

436. Suppose a polyedral $\angle$ formed by three equilateral $\angle$s. What is the sum of the face angles at the vertex? If formed by four? by five?

437. Can you pass a plane through a point $\parallel$ to two given straight lines?

BOOK VII.

POLYEDRONS.

DEFINITIONS.

454.

A polyedron is a geometric solid bounded by planes.

The intersections of the planes bounding the polyedron are called the *edges;* the intersections of the edges are called *vertices:* the portions of the planes included by the edges are called the *faces.* Any face may be thought of as the base.

455.

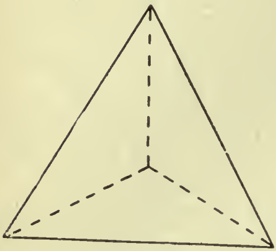
Polyedrons are classified as to the number of faces required to bound them.

[*Question:* What is the fewest number of faces necessary to form a polyedron ?]

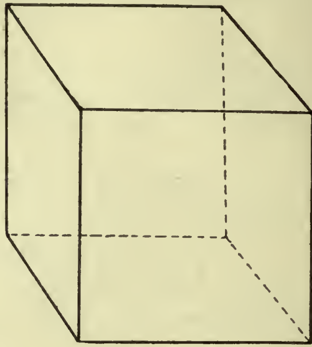
Note.—The pupil will find that a few cents invested in putty or moulding clay will be well spent, since with a thin-bladed knife many concrete illustrations of solids can be readily made.

456.

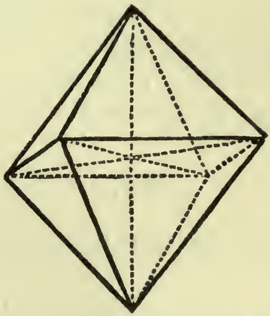
A polyedron of four faces is called *tetraedron,* one of five faces a *pentaedron,* one of six faces a *hexaedron,* one of eight faces an *octaedron* one of ten faces a *decaedron*, one of twelve faces a *dodecaedron*, one of twenty faces an *icosaedron*, etc.



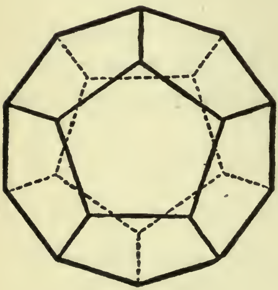
Tetraedron.



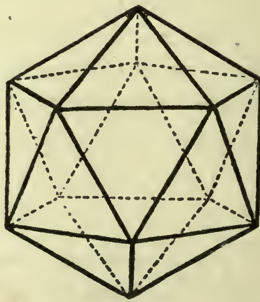
Hexaedron.



Octaedron.



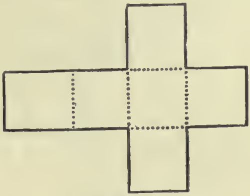
Dodecaedron.



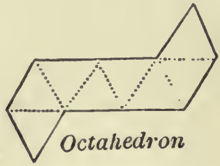
Icosaedron.

Scholium. Having drawn on cardboard the diagrams below, cut through the heavy lines and half through the dotted lines. By folding these figures the regular polyedrons can be formed.

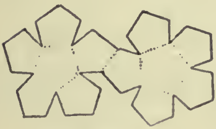
Tetrahedron



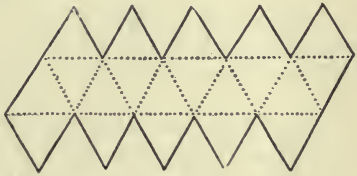
Hexahedron.



Octahedron



Dodecahedron



Icosahedron

457.

A polyedron is *convex* when any plane section is a convex polygon.

In the convex polyedron no face will enter the polyedron when produced.

Polyedrons will be considered convex in this book unless otherwise stated.

458.

A straight line joining any two vertices not in the same face is called a *diagonal.*

459.

The *volume* of a solid is the number expressing its ratio to another solid arbitrarily taken as the unit of volume.

The edge of the unit is a linear unit.

If a cubic cm. is contained in a given solid 50 times, its volume is 50 cubic cm.

460.

Two volumes are said to be *equivalent* when their volumes are equal.

PRISMS.

461.

A *prism* is a polyedron two of whose faces, the bases, are equal polygons having their corresponding sides parallel, and the remaining faces are parallelograms formed by planes passing through the corresponding sides of the bases.

The parallelograms are called the *lateral faces.*

Question: What form the *basal edges?* the *lateral edges?*

462.

The lateral edges of a prism are parallel and equal. Can you prove it?

463.

A *right section* of a prism is a section formed by a plane passing at right angles to the lateral edges.

464.

The *altitude* of a prism is the perpendicular distance between its bases.

465.

Prisms are *triangular*, *quadrangular*, etc., according as their bases are triangles, quadrilaterals, etc.

466.

A *right prism* is one whose lateral edges are perpendicular to its faces.

467.

An *oblique prism* is one whose lateral edges are oblique to its faces.

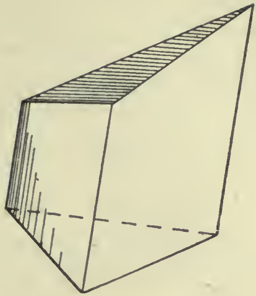
468.

A *regular prism* is a right prism whose bases are regular polygons.

469.

The lateral faces of a prism form a *prismatic surface.* The faces may extend beyond the bases.

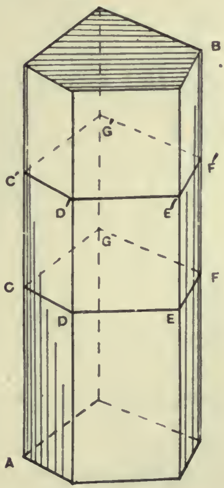
470.



A *truncated prism* is a portion of a prism included between a base and a plane not parallel to the bases which cuts all the iateral edges.

471.

Proposition I.



Given: The prism A B cut by the parallel planes C F and C′F′.

Compare the polygon C D E F G and C′ D′ E′ F′ G′.

Sug. Show how C D and C′ D′, D E and D′ E′, etc., are related, and also how ∠ C D E and ∠ C′ D′ E′, ∠ D E F and ∠ D′ E′ F′, etc., are related.

Can you now show the relation between the polygons?

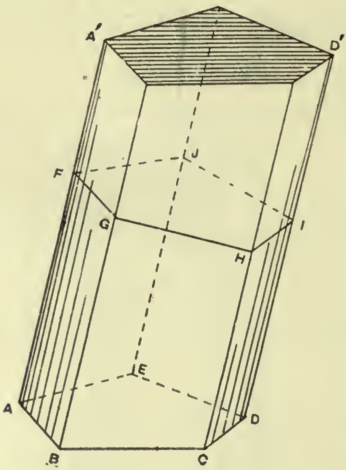
472.

Cor. (1) Prove what the section is when the prism is cut by a plane ∥ to the base.

(2) How do right sections compare?

473.

PROPOSITION II.



Suppose A D′ represent any oblique prism and F G H I J a right section.

Can you find an expression for its lateral surface?

Sug. What are the faces? How do the edges compare? How does the area of any face compare with a rectangle having the same base and an equal altitude? May we consider a lateral edge as base of the parallelograms?

If, for example, we take A A′ as the base of the parallelogram A′ B, what is its altitude?

How does the sum of the altitudes of the faces compare with the perimeter of a right section.

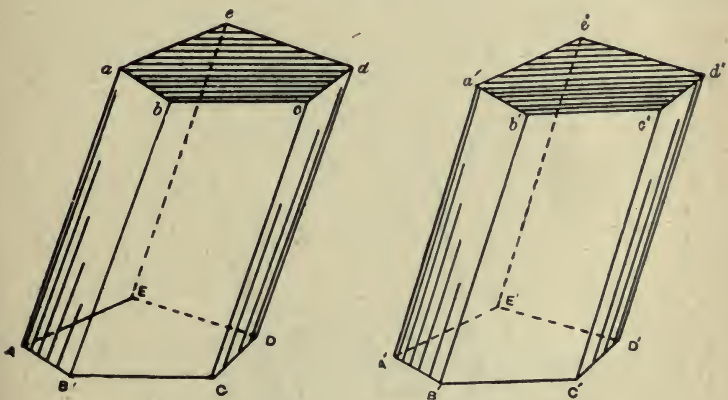
Complete proof and write the generalization. Call the statement Prop. II.

474.

Cor. How find the area of a right prism?

475.

Proposition III.



Given: In the prisms A *d*, A′ *d*′ the faces B E, B *c*, B *a* respectively equal to B′ E′, B′ *c*′, B′ *a*′, and similarly arranged

Compare the volumes of the prisms.

Sug. Compare any two corresponding triedral ∠s, as B and B′. Can you make A D coincide with A′ D′? A *b* coincide with A′ *b*′? B *c* coincide with B′ *c*′? *a b* with *a*′ *b*′? *b c* with *b*′ *c*′? Show that *a d* coincides with *a*′ *d*′.

Finish the proof and write the generalization. Call this Prop. III.

476.

Cor. I. When are two right prisms equal?

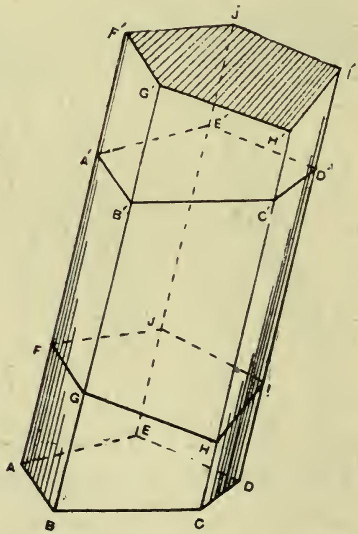
477.

Cor. II. Suppose the figures above are truncated prisms and the conditions the same. Compare the solids.

478.

PROPOSITION IV.

Given. The *oblique* prism A D′ and F G H I J a right section.



Can you find a right prism, having for its base a right section of the oblique prism and an altitude equal to a lateral edge of the oblique prism, which shall be equivalent to the given oblique prism?

Sug. Extend the lateral edges A A′, B B′, etc., making F F′ = A A′. At F′ pass a plane ⊥ F F′. How is this plane related to F I? What is F I′? A′ I′? A I? Compare B B′ with G G′.

How do B G and B′ G′ campare? A B and A′ B′? F G and F′ G′? What is A G? A′ G′? Compare them.

In a similar manner compare B H and B′ H′.

Compare ∠ A B C with ∠ A′ B′ C′ and base B E with base B′ E′. Compare the triedral angle A B G C with the triedral A′ B′ G′ C′.

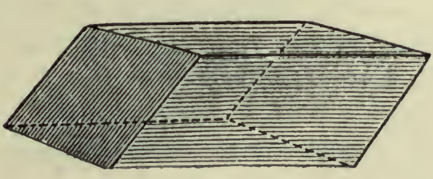
Compare the truncated prisms A I and A′ I′. Can you now complete the proof to the answer of the original question?

Write the proposition, and call it Prop. V.

PARALLELOPIPEDS.

Definitions.

479.



A *parallelopiped* is a prism whose bases are parallelograms.

480.

A *right parallelopiped* is one whose lateral edges are perpendicular to the bases.

481.

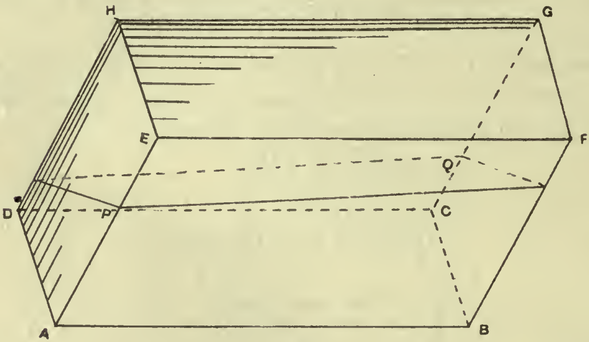
A *rectangular parallelopiped* is a right parallelopiped having all its faces rectangles. How would you define a cube? (See Fig. § 485.)

Exercise.

438. Draw a right parallelopiped whose bases are, (1) trapeziums, (2) trapezoids, (3) rhomboids, (4) rhombuses.

482.

Proposition V.



Given any parallelopiped, A G, and consider A H and B G the bases. Compare the opposite faces.

[*Hint.*—Compare E F, H G, D C, A B; also B C, G F, E H, A D. Compare ∠ F E H and ∠ B A D, ∠ E H G and ∠ A D C. Compare faces A C and E G.]

What can you say of the opposite faces of a parallelopiped?

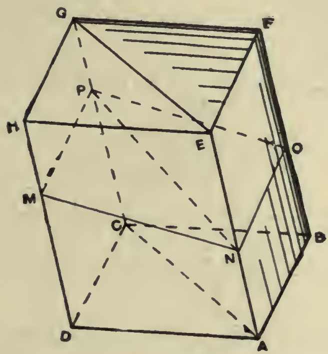
Write the general statement, and call it Prop. V.

Exercise.

439. Let P Q be a section formed by passing a plane through the parallelopiped A G, cutting only 2 pairs of opposite sides. Prove what the section is.

483.

PROPOSITION VI.



Given any parallelopiped, D F, required to pass a plane, through two diagonally opposite edges.

What solids are formed? Compare them.

[*Hint*.—Let A C G E be the required plane. Pass a plane M N O P, ⊥ to the edges of the solid, cutting the plane A G in N P. What is M N O P? P N?]

Compare the oblique prisms B — F G E with another prism whose base is N O P with altitude A E.

Complete the demonstration and write the general truth. Call it Prop VI.

484.

PROPOSITION VII.

Any triangular prism......

EXERCISES.

440. Find the entire surface of a right triangular prism the sides of whose base are 8, 12, and 16 inches and altitude 30 inches.

441. In how many ways can a polyedral ∠ be formed with equilateral △s? with squares? with regular pentagons? with a regular hexagon? with regular heptagons?

442. What is the entire surface of a regular hexagonal prism, each side of the base being 10 centimeters and the altitude 20 centimeters.

443. The four diagonals of a parallelopiped bisect each other.

444. The diagonals of a rectangular parallelopiped are equal.

445. The diagonals of an oblique parallelopiped are unequal.

446. How many sets of || lines in a parallelopiped?

447. In any parallelopiped the sum of the squares on the 4 diagonals equals the sum of the squares on the 12 edges.

448. Given a rectangular parallelopiped to show that if the diagonals of all the faces are drawn and the points of intersection of the diagonals of the opposite faces are connected, these connecting lines are concurrent at the middle point of each.

449. If a plane cuts one edge of a prismatic space, it cuts all the edges. What follows if it cuts one edge perpendicularly?

450. What should be the edge of a cube so that its entire surface shall be a square foot?

451. A rectangular parallelopiped is x, by y, by z. What is the longest straight line that can be drawn in the solid?

452. Show what every section of a prism is when a plane cuts it parallel to the base.

453. Construct a pentagonal prism and show what is the sum of the plane angles of the lateral diedral angles.

454. In the last exercise how many *face angles?* diedral angles? triedral angles? If the base have r sides instead of five, how many face angles? etc.?

455. Given a cube with an 8-inch edge. Join the middle point of each face with the other middle points. What are the plane figures formed? Can you compare the sum of their

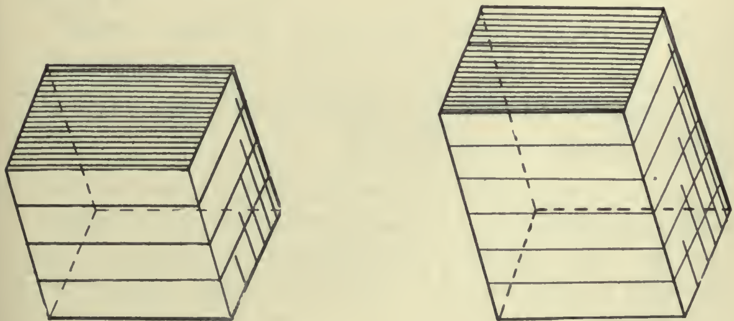
surfaces with the surface of the original cube? What is the solid figure formed?

456. Given a rectangular parallelopiped whose base is 6 by 8 and whose altitude is 12. Join the middle points of each face to the middle points of the others. What are the plane figures formed? Can you compare the sum of their surfaces with the surface of the parallelopiped? What is the solid figure formed?

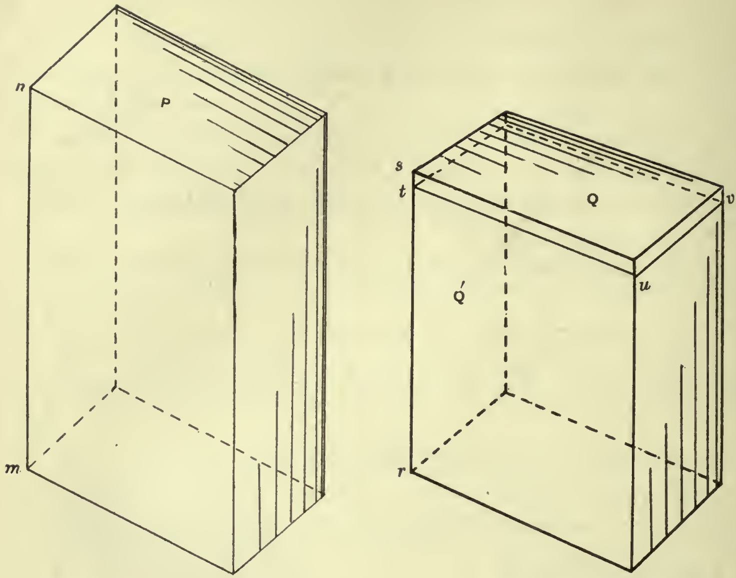
Note.—The last two exercises illustrate two forms of crystals sometimes seen in the mineral kingdom.

485.

Proposition VIII.



Can you show that two rectangular parallelopipeds having equal bases are to each other as their altitudes when the altitudes are commensurable?



Compare the two parallelopipeds when the altitudes are incommensurable.

Sug. Let P and Q be the solids and suppose $m\,n$ and $r\,t$ to be commensurable, $t\,s$ being less than the unit of measure. Call the lower parallelopiped formed by passing a plane through t || to the base, Q′. How are P and Q′ related? Write the relation. Is it always true for any unit parallelopiped? Now suppose the linear unit on $m\,n$ to be indefinitely decreased, how will it affect $r\,t$? Q′? Write $\frac{m\,n}{r\,t} \doteq ?\ \frac{P}{Q'} \doteq ?$

Therefore . . . . . . Generalize.

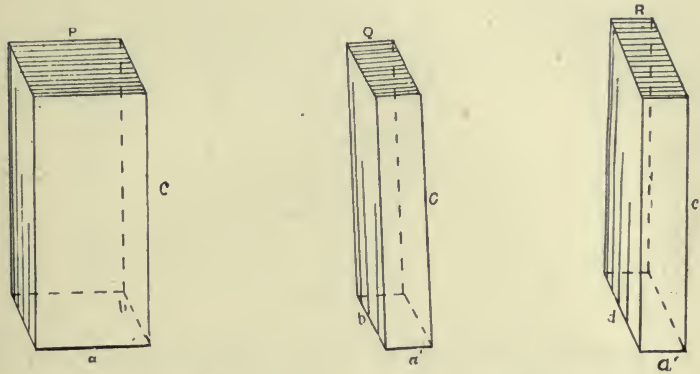
Call this Prop. VIII.

EXERCISE.

457. Explain the statement that if two rectangular parallelopipeds have two dimensions in common, they are to each other as their third dimensions.

486.

Proposition IX.



Given: The two rectangular parallelopipeds P and Q having two dimensions, b and c, equal. Write an expression to show the relation of P and Q. Call this equation (1). Next let Q and R be two rectangular parallelopipeds having two dimensions, a and c, equal. Write a second equation showing their relation. Multiply (1) by (2) and simplify.

What is the meaning of the resulting equation? Can you write the proposition deduced?

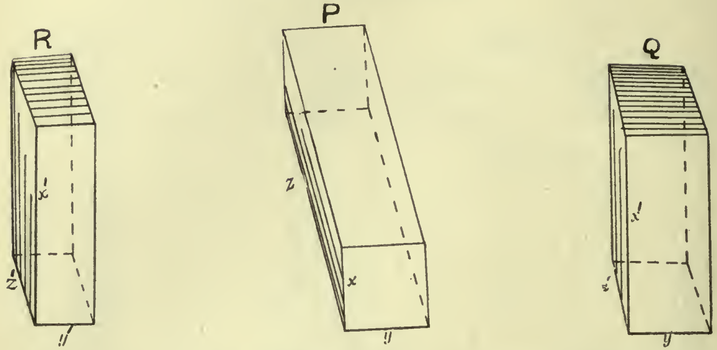
Call it Prop. IX.

Exercise.

458. If P in §486 have edges 3, 4, 5 cm., what is the area of a diagonal plane?

487.

PROPOSITION X.



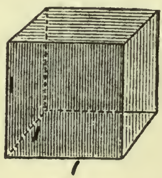
Compare the rectangular parallelopipeds P and R, whose edges are x, y, z and x', y', z'.

Sug. Construct a third parallelopiped, Q, with edges x', z', y. Write an equation comparing P and Q. Call it (1). In the same manner compare Q and R. Call this equation (2). Multiply (1) by (2) and simplify. Explain the meaning of this third equation. Generalize.

Call this Prop. X.

488.

Cor. Suppose that P were a cube with edges 1, 1, 1 Write the equation expressing the relation of R to P.



If we call P a *unit of volume*, what does your equation show as to the number of units of volume in R?

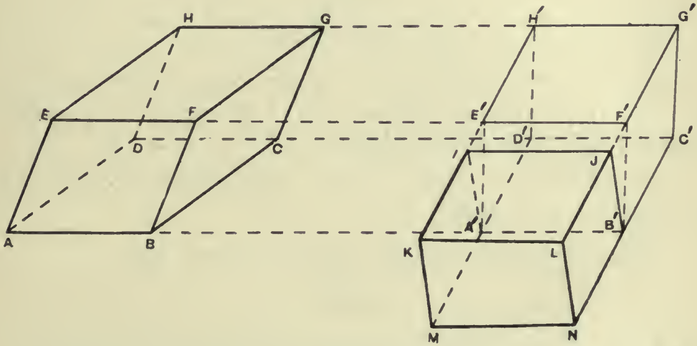
What expresses the volume of a rectangular parallelopiped?

EXERCISE.

459. When the edges of the rectangular parallelopiped are *multiples* of the linear unit, construct a figure showing how to compute the volume of the solid.

489.

PROPOSITION XI.



Given any oblique parallelopiped, A G; *i. e.*, all angles oblique.

Produce the edges A B, E F, D C, H G.

Make E′ F′ = E F. Pass planes E′ D′, F′ C′ ⊥ to E′ F′.

Sug. 1. By considering E′ D′ and F′ C′ bases, what kind of a solid have we? [§ 478.]

How does the edge E′ F′ compare with E F?

Compare E C and E′ C′. [§ 478.]

Sug. 2. Now consider D′ C′ G′ H′ as the base of A′ G′. Produce D′ A′, making A′ M = D′ A′. Produce H′ E′, G′ F′, C′ B′, making B′ N = C′ B′. Make right section A′ B′ J I and M N L K. What kind of solid is A′ L? Compare A′ L with A′ G′. [Prop. V., § 478.]

How does it compare with A G?

Can you generalize the result?

Call this Prop. XI.

490.

Cor. I. How do find the volume of any parallelopiped?

491.

Cor. II. Construct a figure and show how to find the volume of any triangular prism.

492.

PROPOSITION XII.

Given a prism whose base is a polygon of n sides. Into how many triangular prisms may it be divided by passing planes through the lateral edges?

Construct a figure and show how to find the volume of any prism?

Write the general truth, and call it Prop. XII.

493.

Cor. Can you show that any two prisms are to each other as the products of their bases and altitudes?

Compare two prisms having equivalent bases. Compare two prisms having = altitudes.

In figure Prop. XI. may we compare A G and A′ G′ as prisms? A G and A′ L? What statement can you make concerning these prisms? of any two prisms having equivalent bases and equal altitudes?

EXERCISES.

460. The dimensions of the base of a rectangular parallelopiped are 3 and 4 centimeters and the entire surface is 52 square centimeters. Find the volume.

461. The volume of a rectangular parallelopiped is 60 cubic centimeters, the entire surface 94 square centimeters, and the altitude is 3 centimeters. Find the dimensions of the base.

462. Find the lateral surface of a regular triangular prism, each side of whose base is 5 centimeters, and whose altitude is 10 centimeters. Suppose the base were 6, 5, 5 centimeters, what would be the entire surface?

463. Find convex surface and volume of a regular hexagonal prism each side of whose base is 1 dm. and whose altitude is 10 dm.

464. Two triangular prisms P and Q, have the same altitude; P has for its base a right isosceles triangle; Q has for its base an equilateral triangle of side equal to the hypotenuse of the base of P. What is the ratio of the volumes of P and Q?

465. Find the ratio of the convex surfaces of P and Q, in Ex. 464.

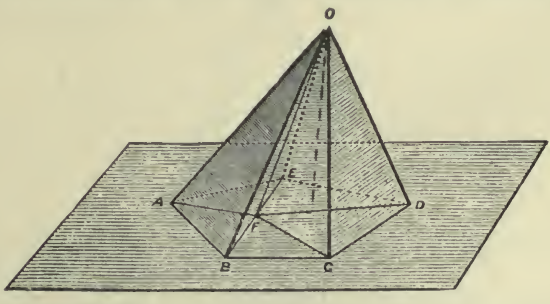
466. A rectangular parallelopiped is 4, 6, and 9 centimeters. What is the edge of an equivalent cube?

PYRAMIDS.

494.

A *pyramid* is a polyedron bounded by a polygon, and a series of triangles which meet in a common point and whose bases are the sides of the polygon.

Thus the polygon A B C D E is called the *base* of the pyra-

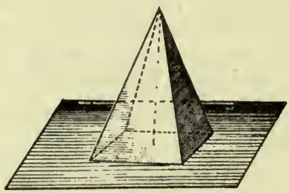


mid. The common vertex of the triangular faces is called the *vertex* of the pyramid. The edges passing through the vertex are called the *lateral* edges. The perpendicular from the vertex to the base is called the *altitude.*

495.

A pyramid is *triangular*, *quadrangular*, *pentagonal*, etc., according as its base is a triangle, a quadrilateral, a pentagon, etc.

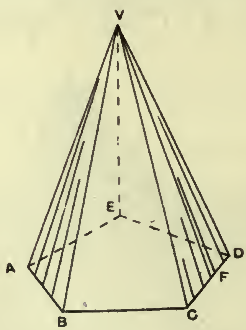
496.



A *regular pyramid* has for its base a regular polygon, and its vertex lies in the perpendicular erected at the center of the polygon.

497.

PROPOSITION XIII.



Can you show how the lateral edges of a regular pyramid are related?

498.

Cor. I. What kind of triangles are the lateral faces of a regular pyramid?

499.

Cor. II. How do the altitudes of the lateral faces drawn from the common vertex, V, of a regular pyramid compare?

500.

The altitude of any of the lateral faces of a regular pyramid drawn from the common vertex is called the *slant height.*

501.

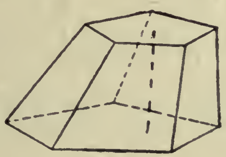
The *lateral surface* of a pyramid is the sum of the areas of its lateral *faces.*

502.

A *truncated pyramid* is the portion of a pyramid included between its base and a plane cutting all the lateral edges. [§ 470.]

503.

A *frustum* of a pyramid is a truncated pyramid whose bases are parallel.



The *altitude* of a frustum is the perpendicular distance between the planes of its bases.

The *slant height* of a frustrum of a regular pyramid is the altitude of any lateral face.

Exercises.

467. What are the faces of the frustum of a regular pyramid? Can you prove that they are equal?

468. Can you find an expression for the lateral surface of the frustum of a pyramid?

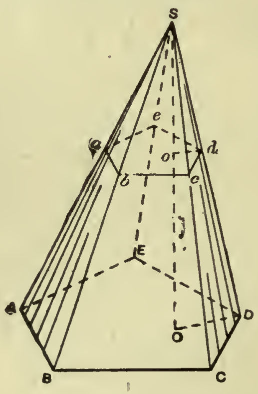
469. Can you find an expression for the lateral area of any regular pyramid?

470. Can you draw a figure illustrating geometrically the formula $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + xy^3$?

471. A rectangular drain-tile has the dimensions 60 cm., 12 cm., 10 cm. The mold is 65 cm. long and proportionally wide and deep. Supposing there were no orifice, what per cent does the tile decrease in baking?

504.

Proposition XIV.



Given: Any oblique pyramid, as S — A B C D E, cut by a plane || to its base intersecting the edges in $a\ b\ c\ d\ e$ and the altitude, S O, in o.

1. Can you show how the edges and altitudes are divided?

Sug. How are $a\ b$ and A B related? Why? $b\ c$ and B C? etc.

2. Compare the section of the pyramid with the base.

Sug. Compare ∠ *a b c* and ∠ A B C, ∠ *b c d* and ∠ B C D, etc.

What can you say of the two polygons?

Can you prove the homologous sides proportional?

What results?

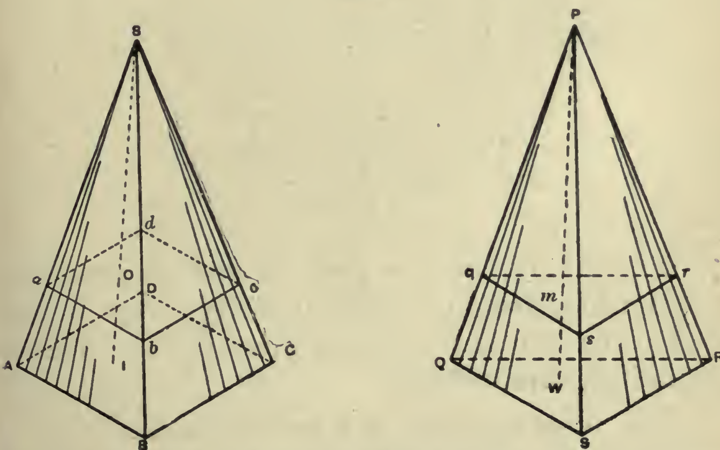
Put the answers to 1 and 2 into a general statement, and call it Prop. XIV.

505.

Cor. I. Can you show that any section of a pyramid ∥ to the base is to the base as the square of its distance from the vertex is to the square of the altitude of the pyramid?

Sug. Compare the two polygons with two homologous sides, the segments of an edge, etc.

506.



Cor. II. (1) Suppose S — A B C D and P — Q R S to be two pyramids of equal altitudes. Let each pyramid be cut by a plane || to its base in *a b c d* and *q r s*, respectively, and let each plane be equally distant from the vertex.

Compare the ratio of the sections with the ratio of the bases.

Altitude S I = altitude P *w*, and altituc S = altitude P *m*.

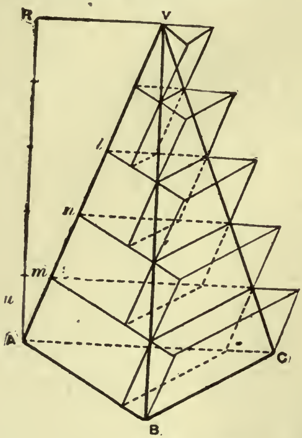
Sug. Use Cor. I.

(2) Suppose the base of S — A B C D equivalent to base of P — Q R S, what deduction follows from (1).

Write Cor. II.

507.

PROPOSITION XIV.



If in any pyramid we inscribe and circumscribe a series of prisms of equal altitudes, what is the *limit* of the sum of each series when the altitude is indefinitely diminished?

Sug. Given pyramid V — A B C with altitude A R.

Suppose the altitude divided into any number of equal parts, letting *u* be a unit of measure. Pass planes through these points of division parallel to the base, cutting the pyramid at L, *m*, and *n*. What will the sections be? Upon A B C and the sections at L, *m*, and *n* as *lower* bases, construct prisms having altitudes equal to *u* and the lateral edges parallel to A V. These are the *circumscribed* prisms. Now with the sections at L, *m*, and *n* as *upper bases* and altitudes equal to *u* and lateral edges parallel to A V, construct prisms. These are the *inscribed* prisms.

How does the sum of two inscribed prisms in any pyramid compare with the sum of 5 inscribed prisms? with 10? with 100?

How does the sum of 3 circumscribed prisms of any pyramid compare with the sum of 5 circumscribed prisms? with 10? with 100? (It is very important that the pupil master these questions before proceeding.) If the altitude of each series of prisms be indefinitely decreased, how does it affect the sum of each series?

What is the *limit* of the sum of the series of inscribed prisms? of the sum of the circumscribed series?

Write a general statement, and call it Prop. XIV.

Exercises.

472. Each face of a triangular pyramid is an equilateral triangle whose side is 4 cm. Find the surface.

473. The side of the base of a square pyramid is 4 m., the altitude is 10 m.; how many square meters of tin is required to cover it?

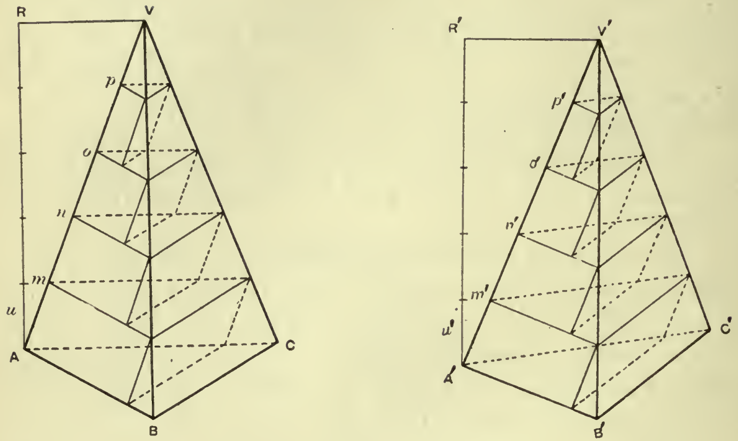
474. The diagonal of the face of a cube is $a\sqrt{2}$. Find the volume.

475. Can you cut a cube with a plane so that the section shall be a regular hexagon?

476. Can you show that the lateral area of any pyramid is greater than the area of its base?

508.

Proposition XV.



Given any two pyramids, as V — A B C, V′ — A′ B′ C′, having equivalent bases in the same plane and equal altitudes, A R = A′ R′.

Compare the volumes of the pyramids.

Sug. 1. Divide A R into any number of equal parts, letting u be one of the equal parts. Pass planes through the points of division parallel to the bases. Compare corresponding sections.

Sug. 2. Inscribe a series of prisms in each pyramid having the sections as *upper* bases and common altitude u. Compare any two corresponding prisms. Compare the total sum of the prisms, P, in V — A B C with the total sum of the prisms, P′, in V′ — A′ B′ C′. Now suppose u be indefinitely decreased, how does it affect the prisms? What does P $\doteq$ P′ $\doteq$?

Complete the work, and write Prop. XV.

509.

Cor. I. Suppose a pyramid has a parallelogram for its base and a plane be passed through two opposite edges, cutting the base. What follows?

510.

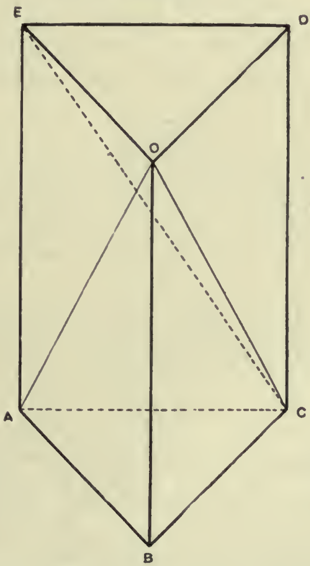
Cor. II. Given a pyramid having a parallelogram for its base which is double the area of the base of a triangular pyramid of the same altitude. How do they compare?

EXERCISE.

477. How may the volume of any polyedron be found?

511.

PROPOSITION XVI.



Given: The triangular prism O — A B C.

Prove that it may be divided into three equivalent pyramids.

Sug. 1. Pass a plane through O and A C.

How is the prism divided?

Prove that the quadrangular pyramid may be divided into two equal pyramids.

Sug. 2. Compare C – D E O with O – D E C, with O – A B C.

Finish the proof and write the general statement. Call this Prop. XVI.

512.

Cor. I. Compare the volume of a triangular prism with that of a pyramid having the *same* base and altitude.

513.

Cor. II. Compare a triangular pyramid with a prism having an *equivalent* base and altitude.

514.

Cor. III. How find the volume of a triangular pyramid?

515.

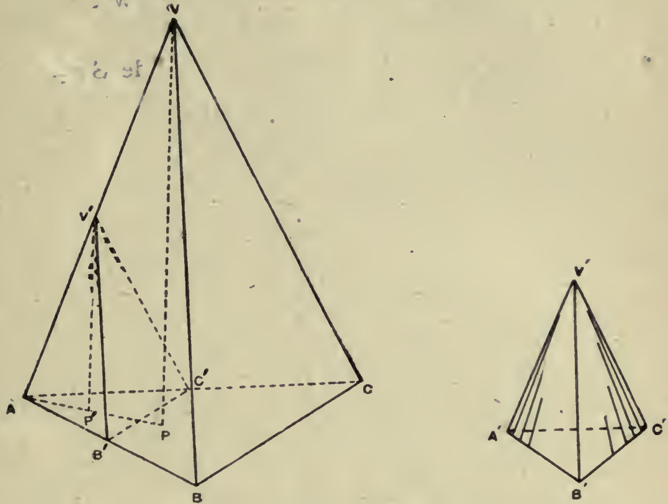
Cor. IV. Show that any pyramid may be divided into triangular pyramids. How many?

516.

Cor. V. Show how pyramids of equivalent bases are related. Suppose two pyramids have equal altitudes, show how they are related. Suppose two pyramids have equivalent bases and equal altitudes, show how they are related.

517.

Proposition XVII.



Given: Any two tetraedrons, as V — A B C and V′ — A′ B′ C′, having the triedral ∠ A equal to the triedral ∠ A′.

Prove that the tetraedrons are to each other as the products of the edges of the corresponding triedral ∠s.

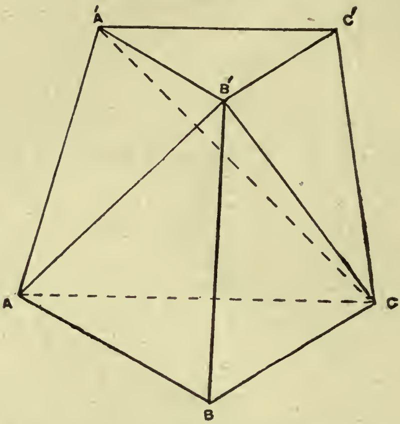
Sug. Apply the smaller tetraedron to the larger, so that ∠ A′ coincides with A. Drop ⊥s V P, V′ P′ to the bases A B C and A′ B′ C′. (Are the two bases in the same plane?) Can you pass a plane through V P and V′ P′? Will it pass through V V′? through A? What is its line of intersection with A B C? How are two triangular prisms related? two triangular pyramids? two tetraedrons? Call the tetraedrons T and T′. (1) T : T′ : : A B C · — : A′ B′ C′ · —. (2) A B C : A′ B′ C′ : : — : —. How are V P and V′ P′ related? Substitute and finish the proof.

Write the proposition, and call it Prop. XVII.

Question. If V — A B C and V′ — A′ B′ C′ are similar tetraedrons, can you show that they are to each other as the cubes of their homologous edges?

518.

Proposition XVIII.



Construct a figure and show how to find the volume of any pyramid. (§ 511.)

Write the proposition, and call it Prop. XVIII.

Given: The frustum of any *triangular* pyramid.

Can you divide the frustum into three pyramids, one having for its base the lower base of the frustum, another having for its base the upper base of the frustum, and the third having a base equivalent to a mean between the upper and lower bases of the frustum, and each pyramid to have the altitude of the frustum of the pyramid?

Sug. 1. Pass a plane through B′, A C and B′, A′, C. Consider B′ — A B C and B′ — A A′ C as having a common vertex, C.

Write an equation showing the relation between C — A B B′ and C — A B′ A′. Call this (1).

What do you notice about the altitudes of the bases of these pyramids? Write an equation showing the relation between the two triangles. Call it (2).

Substitute in (1). New equation (3). How then is B′ — A B C and B′ — A A′ C related?

Sug. 2. Write an equation showing the relation between B′ — A A′ C and B′ — A′ C C′. Call this (4). Compare $\triangle$s A A′ C and A C C′. Call this (5). Substitute (5) in (4). New equation (6).

How does A′ B′ C′ compare with A B C? [§508.]

Sug. 3. Can you show that A B : A′ B′ : : A C : A′ C′?

Form a new proportion from this by substituting pyramids. What is B′ — A A′ C from this proportion? Call the altitude of the frustum *h*, area of lower base B, area of upper base *b*. What expresses the volume of B′ — A B C? Of C — A′ B′ C′? or B′ — A′ C′ C? Express B′ — A′ A C in terms of B, *b*, *h*.

Write a generalization of the truth developed, and call it Prop. XVIII.

519.

Cor. (1) Let V = volume of frustum. [§518.]

Prove $V = \frac{h}{3}(B + b + \sqrt{B \times b}).$

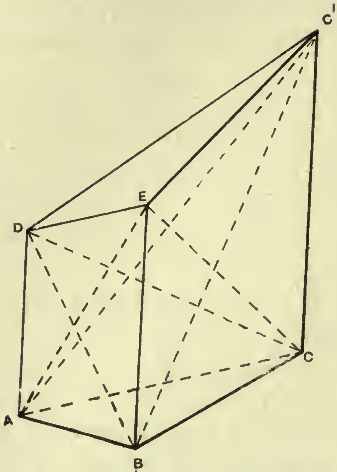
(2) From (1) can you show that

$V = \frac{h}{3}(B + b + \sqrt{B \times b})$ is true for the frustum of *any* pyramid?

Sug. Construct a triangular frustum in the same plane, having a lower base equivalent to the lower base of the given frustum and of equal altitude.

520.

Proposition XIX.



Given: A B C′ — D E C, any truncated triangular prism.

Can you show that the truncated prism is equivalent to three pyramids having a common base, A B C, with the vertices at D, E, C′?

Sug. 1. Pass a plane through E, A, C, and D, B, C, and C′, A, B. Can you take E as a vertex and name three pyramids that compose the truncated prism? Compare E — A D C and B — A D C. Compare B — A D C and D — A B C. What then is the equivalent of E — A D C?

Sug. 2. Compare △ D C C′ and A C C′. Compare E — A C C′ and B — A C C′? Also B — A C C′ with C′ — A B C. What is the equivalent of E — A C C′? ∴ A B C — D E C′ is equivalent to —, —, —, —.

Write the generalization of the truth developed, and call it Prop. XIX.

521.

Cor. I. Show that the volume of any truncated right triangular prism equals the product of its base by ⅓ of the sum of its lateral edges.

522.

Cor. II. Show that the volume of any truncated triangular prism equals the product of a right section by $\frac{1}{3}$ the sum of its lateral edges.

Sug. After passing a plane at right angles to the edges, compute each part separately.

SIMILAR POLYEDRONS

523.

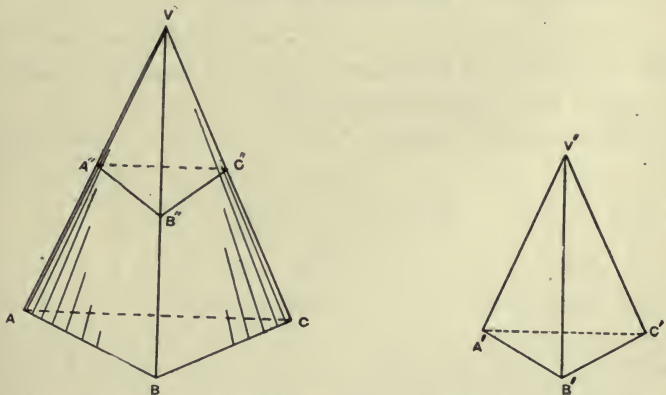
DEFINITION.

Similar polyedrons are those polyedrons having the same number of faces, similar each to each and similarly placed, and their homologous polyedral angles equal.

The faces, lines, and angles of similar polyedrons which are similarly placed are called *homologous faces, lines, and angles.*

524.

PROPOSITION XX.



Given: V — A B C and V′ — A′ B′ C′, similar polyedrons.

Compare their homologous edges.

Sug. Can V′ — A′ B′ C′ be made to coincide with any part of V — A B C?

Generalize. Call it Prop. XX.

525.

Cor. I. Can you show that any two homologous faces of two similar polyedrons are proportional to the squares of any two homologous edges?

526.

Cor. II. Can you show that the entire surfaces of any two similar polyedrons are proportional to the squares of any two homologous edges.

527.

Cor. III. How do the homologous diedral angles of similar polyedrons compare?

528.

PROPOSITION XXI.

If a tetraedron is cut by a plane parallel to one of the faces, show that the tetraedron cut off is similar to the first.

Sug. Use figure in Prop. XX.

Call this truth Prop. XXI.

529.

PROPOSITION XXII.

In figure for Prop. XX. let V — A B C and V′ — A′ B′ C′ be two tetraedrons with the diedral angles V A and V′ A′ equal, and the faces V A B and V′ A′ B′ similar and V A C similar to V′ A′ C′, and these pairs of faces similarly placed.

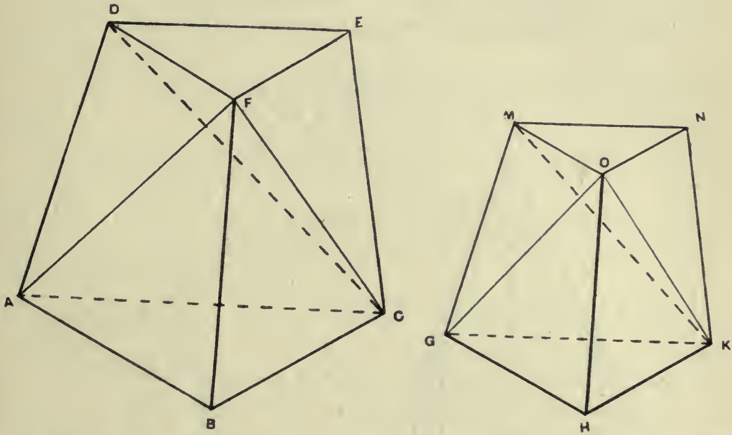
Can you prove the tetraedrons similar?

Sug. *Suppose* V′ — A′ B′ C′ on V — A B C.

Write the general proposition, and call it Prop. XXII.

530.

PROPOSITION XXIII.



Given: The two similar polyedrons B — D F E and H — N O M, in which triedral ∠ H = triedral ∠ B, etc.

To Prove—That they may be decomposed into the same number of tetraedrons similar each to each and similarly placed.

Sug. Draw the diagonals A F, C F, D C, G O, K O, K N. Can you show that the tetraedron F — A B C is the corresponding tetraedron to O — G H K? Prove them similar. Complete the proof.

Write the general truth, and number it Prop. XXIII.

531.

PROPOSITION XXIV.

Given: F — A B C and O — G H K, two polyedrons composed of the same number of tetraedrons similar each to each and similarly placed. (See figure under Prop. XXIII.)

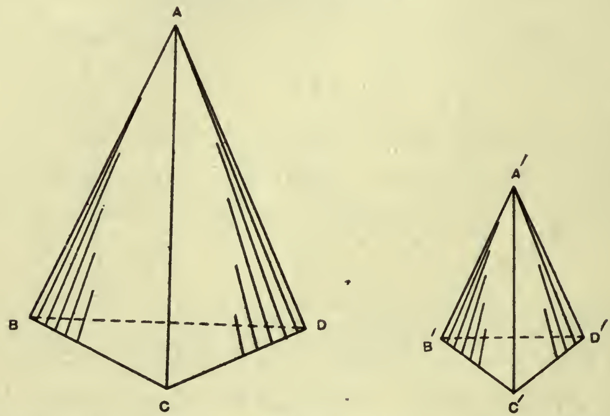
To Prove—F — A B C similar to O — G H K.

Sug. Can you show how A B F D and G H O N are related, etc.? Can you show how the polyedral ∠ at A is related to the polyedral ∠ at G? Complete proof.

Draw conclusion, and call this Prop. XXIV.

532.

PROPOSITION XXV.



Given: A — B C D and A′ — B′ C′ D′, two similar tetraedrons.

To Prove—That they are to each other as the cubes of their homologous edges.

Sug. See § 517.

Write the proposition, and call it Prop. XXV.

533.

Cor. Show that any two similar polyedrons are to each other as the cubes of their homologous edges.

EXERCISES.

478. In Ex. 455, compare the volume of the solid formed with the original solid.

479. In Ex. 455, compare the volume of the solid formed with the original solid.

480. A farmer wishes to build a cubical bin that will hold 100 bushels of wheat. What will be an inside edge in inches?

481. What is the edge of a cube whose entire surface is 1 square foot?

482. What is the entire surface of a common building brick?

483. What is the edge of a cube that will contain a gallon, dry measure?

484. The base of a pyramid is 12 square feet and its altitude is 6 feet. What is the area of a section parallel to the base and 2 feet from it?

485. Prove the diagonals of a rectangular parallelopiped equal.

486. The volume of any triangular prism equals one half the product of any lateral face by its distance from the opposite edge. Prove.

487. Show that the four diagonals of a parallelopiped bisect each other. (The point of intersection is called the center of the parallelopiped.)

488. Can you prove that a straight line passing through the center of a parallelopiped and terminated by two faces is bisected at the center?

489. Can you show that the middle points of the edges of a regular tetraedron are the vertices of a regular octaedron.

Is the altitude of a regular tetraedron equal to the sum of the perpendiculars to the faces from any point within the figure?

490. Find the volume of a regular triangular pyramid whose basal edge is 4 feet and whose altitude is $\sqrt{3}$ feet.

491. Find the lateral edge, lateral area, and volume of a frustum of a regular triangular pyramid the sides of whose bases are $10\sqrt{3}$ and $2\sqrt{3}$ and whose altitude is 10.

492. If the homologous edges of two similar polyedrons are 2 and 3, what is the ratio of their entire surfaces and of their volumes?

493. Can you show that the volume of a regular tetraedron equals the cube of an edge multiplied by $\frac{1}{12}\sqrt{2}$?

494. Can you show that the volume of a regular octaedron equals the cube of an edge multiplied by $\frac{1}{3}\sqrt{2}$?

495. A side of a cube is the base of a pyramid whose vertex is at the center of the cube. Compare the volumes of the cube and pyramid.

496. If a pyramid is cut by a plane parallel to its base, the pyramid cut off is similar to the first, and the two pyramids are to each other as the cubes of any two homologous edges. Prove.

REGULAR POLYEDRONS.

Definition.

A *regular polyedron* is a polyedron whose faces are equal regular polygons and whose diedral angles are all equal.

[See §456 for figures.]

Questions.

534.

1. What is the fewest number of faces necessary to form a polyedron? Name the solid.

2. How many faces meet at each vertex of the tetraedon.

3. What is the fewest number of faces required to form a convex polyedral angle? What is it called?

4. Illustrate how the sum of the face angles of any polyedral angle compares with 4 right angles.

5. What is the angle of an equilateral triangle? How many equilateral △s are required to form a solid angle? What is the sum of the face angles forming the solid angle? Could

a convex polyedral angle be formed with four equivalent $\triangle$s? Illustrate. With five? Illustrate. With six? Illustrate. How many regular solid figures may be made using equilateral $\triangle$s?

6. What is the limit of the number of squares that can be used in forming regular convex polyedrons? Illustrate. How many convex polyedrons can be formed with squares? Name the solid.

7. How many regular pentagons may be used to form a solid angle? Illustrate. How many convex polyedrons can be formed with regular pentagons? Name the solid.

8. What is the limit of the number of regular polyedrons formed of regular hexagons? Why?

9. What is the limit of the number of sides of a regular polygon that can be used in forming regular polyedrons.

10. What is the greatest number of regular convex polyedrons? Name those that are bounded by triangles; by squares; by pentagons. These five regular polyedrons are called the Platonic Bodies.

535.

The point within a regular polyedron equally distant from the sides is called the *center* of the polyedron.

How is the center located from the vertices? Prove it.

What is the locus of all points equally distant from a given point?

Do you see how a regular polyedron may be related to the surface of a sphere?

[See § 456 for diagram of the Platonic Bodies.]

536.

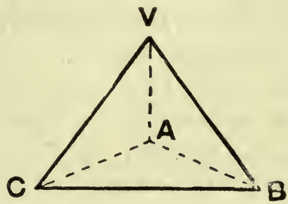
Proposition XXVI.

Given the line A__________B.

Problem: With an edge equal to A B, can you construct a regular tetraedron?

(1) How many faces are required for the solid?

(2) Where is the center of a regular tetraedron? Let V — A B C be the tetraedron required.

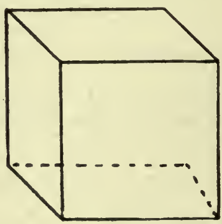


(3) Can you find the center of the base A B C? How do you erect a perpendicular to a plane at a given point?

(4) In what perpendicular to the base does the vertex of the tetraedron lie? How can you find the required point?

537.

Proposition XXVII.



Problem: Construct a regular hexaedron whose edge equals a given line?

538.

Proposition XXVIII.

Problem: Construct a regular octaedron whose edge equals a given line.

539.

Proposition XXIX.

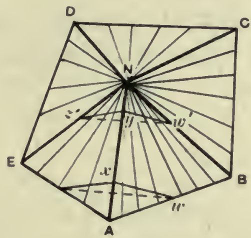


Fig. 1.

Problem: Construct a regular icosaedron.

[*Hint.* 1. With A B as a side, construct the regular pentagon A B C D E. Find the center and erect a ⊥. Where on this ⊥ must a point be taken to form a regular pentagonal pyramid? Let the point be N.

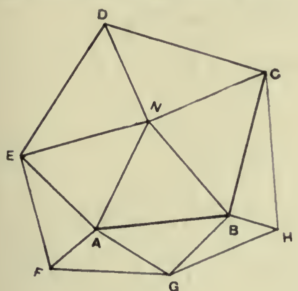


Fig. 2.

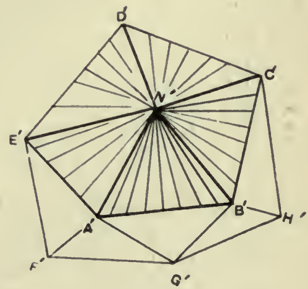


Fig. 3.

2. With A and B as vertices, form pentaedral angles. Study Fig. 2. How many pentaedral ∠s in Fig. 2? The

parts of how many more? How many faces at G? at H? at C? etc? How many faces used in Fig. 2?]

To construct (2): Does N B = A B? Why? Does N E = N B?

In the plane determined by N B and N E form a regular pentagon having N a vertex. *Will N B and N E be sides of that pentagon?*

Where is A, with reference to the plane of N B and N E? Where is it, with reference to the points B, N, and E? With center A and radius A B, describe a ⊙ in the plane of N B and N E. Will B N and N E be chords of that circle?

It is necessary to prove that ∠ B N E is the ∠ of a regular pentagon. Compare it with ∠ B A E. In what plane is ∠ B A E? How large is it?

Can you prove the two ∠s equal? If so, you can construct a second pentaedron whose vertex is A.

Sug. Take two points, x and y on a N, making x A $= y$ N. Through x pass a plane perpendicular to A N, cutting the plane of A B C D E in $z\ w$, z being in A E and w in A B. Through y pass a plane perpendicular to A N, cutting plane of B N E in $z'\ w'$, z' being in N E, and w' being in B N. What ∠s are we trying to prove equal? In which △s are these ∠s? Can you prove these △s =? Try to prove the 3 sides of one = to 3 sides of the other by proving—

(1) △ z A x = △ z' N y.

(2) △ A $x\ w$ = △ N $y\ w'$.

(3) △ $z\ x\ w = z'\ y\ w'$.

(1) (*a*) A x = N y, [?]

(*b*) ∠ z A x = ∠ z' N y, [?]

(*c*) ∠ A $x\ z$ = N $y\ z'$; [Prove each is a right ∠].

∴ △ z A x = △ z' N y.

(2) Use method similar to that used in (1).

(3) ∠ $z\ x\ w$ = ∠ $z'\ y\ w'$, [?]

∴ △ etc., etc.

∴ △ z A w = △ z' N w' (?)

and ∠ E N B = ∠ E A B,
and ∠ E N B is an ∠ of a pentagon,
and N E and N B are sides of a regular pentagon,
and A is the vertex of a second pentaedral ∠.
Thus we can construct a regular pentaedral at B.

3. Construct a convex surface like Fig. 2. Call it Fig. 3.

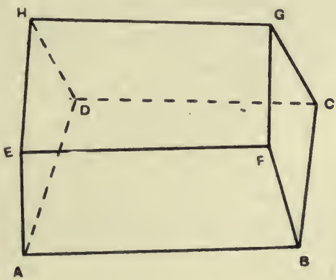
Show that the convex suface of Fig. 2 can be made to combine with that of Fig. 3 so as to form the required icosaedron.

Note.—The pupil will have his ideas cleared concerning the Platonic Bodies by making them out of cardboard. Some of these forms of regular polyedrons are studied in chemistry, botany, geology, and crystallography.

540.

PROPOSITION XXX.—EULER'S THEOREM.

Can it be proven that the number of edges increased by two in any polyedron equals the number of vertices increased by the number of faces?



Let A B C D be one face of any polyedron. Let E = number of edges; V = number of vertices; F = number of faces. Does E + 2 = V + F?

Sug. 1. In face A B C D there are 4 edges and 4 vertices; *i. e.*, E, = V.

Now add the face B C G F by placing its edge B C to the edge B C of the first face. How many vertices in common? edges? How do the edges and vertices compare thus far? Is this true (1) $E = V + 1$ for the two surfaces joined?

Sug. 2. Now add face A B F E.

How many edges are in common with the first two faces? How many vertices in common with the first two faces?

Now the total number of edges is how many more than the total number of vertices?

Is the following equation true for the three faces joined? (2) $E = V + 2$.

Sug. 3. Add the face A D H E.

How many edges are in common with the first three?

How many vertices in common with the first three.

The total number of edges is how many more than the total number of vertices?

Can you show that (3) $E = V + 3$ when four faces are joined?

Sug. 4. Continuing the process till every face but one has been annexed, what effect is produced on the right member of the equation as each additional face is added? At any stage of the process of adding faces, $E = V +$ a number less by one than the number of faces. Study (1), (2), (3).

Sug. 5. In the incomplete solid lacking *one* face the number of faces is $F - 1$.

$\therefore E = V + (F - 1) - 1$, or $E = V + F - 2$. (Pupil will finish this proof.)

Remark.—The pupil should test this law with the Platonic Solids. Find out what you can about Euler, the mathematician.

541.

Proposition XXXI.

Considering the faces of a polyedron as separate polygons, how does the number of edges compare with the number of edges in the polyedron?

Use the notation of Prop. XXX. and call the sum of all the angles of the faces of the polyedron S, and let $R =$ a right angle.

Find an expression for S in terms of R and V.

Sug. Form an exterior angle at the vertex of each polygon.

What is the sum of the interior and the exterior angles at each vertex?

Since the whole number of edges of the polygons which form the face of the polyedron $= 2E$ (first quarter), then the sum of all the interior and exterior angles of the faces is $2R \cdot 2E = 4R \cdot E$.

What is the sum of the exterior angles of any polygon or face?

Then the sum of all the exterior angles is $F \cdot 4R$;

$\therefore$ the sum of all the interior angles of the sum of the face angles is (1) $4R \cdot E - 4R \cdot F = 4R(E - F)$, or $S = 4R(E - F)$.

But from Prop. XXX. $E + 2 = V + F$, or, transposing, $E - F = V - 2$. Substituting this value of $E - F$ in (1), we have $S = 4R(V - 2)$.

Write the interpretation of the last equation, and call it Prop. XXXI.

BOOK VIII.

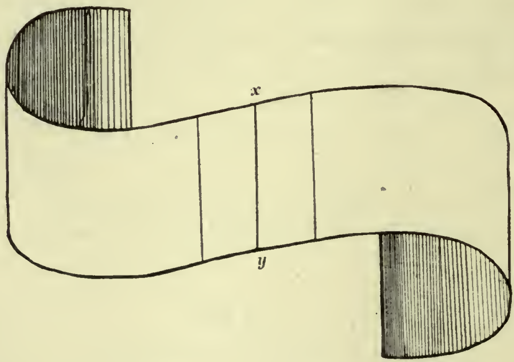
THE CYLINDER.

DEFINITIONS.

542.

A cylindrical surface is a curved surface formed by a moving straight line which constantly touches a given curve and is at all times parallel to a fixed straight line not in the plane. The moving straight line is called the *generatrix*. The given curved line is called the *directrix*.

543.



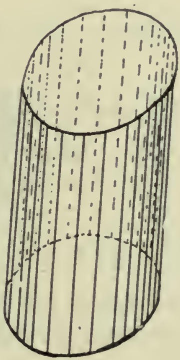
Any straight line in the cylindrical surface is called an *element* of the surface, as $x\ y$ above.

There may be any number of cylindrical surfaces.

544.

A *cylinder* is a solid bounded by a cylindrical surface and two parallel planes. The plane surfaces are called the bases and the curved surface the *lateral* or *convex* surface of the cylinder.

(How do the elements compare in length?)



545.

A line joining the centers of the bases is called the ***axis*** of the cylinder.

546.

A *right section* of a cylinder is the intersection of the cylinder and a plane perpendicular to an element.

547.

The *altitude* of a cylinder is the perpendicular distance between its bases.

548.

An *oblique cylinder* is one whose bases form oblique angles with the elements.

549.

A *right cylinder* is one whose elements are perpendicular to its bases.

550.

A *circular cylinder* is one whose bases are circles.

551.

In how many ways may a right circular cylinder be generated?

552.

A right circular cylinder generated by the revolution of a rectangle about one of its sides as an axis is called a *cylinder of revolution.*

553.

Similar cylinders of revolution are generated by the revolution of similar rectangles about homologous sides.

554.

An *axial section* is formed by passing a plane through the axis.

555.

A *plane is tangent* to a cylinder when it passes through an element of the cylinder, but does not cut the surface.

556.

A *tangent line* to a cylinder is a line which touches the cylindrical surface at a point, but does not intersect the surface.

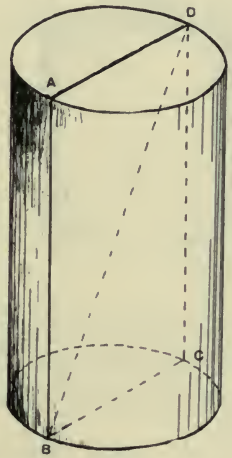
557.

Remark: The generatrix is supposed to be indefinite in extent; hence the surface generated is also of indefinite extent,

558.

PROPOSITION I.

Can you show that any section of a cylinder made by passing a plane through an element is a parallelogram?



Let the plane A C contain an element A B. Is A B C D a parallelogram?

Sug. Is point D common to the plane and the surface of the cylinder? Draw a line through D || to A B. Where will it lie?

Pupil complete proof and write the proposition, calling it Prop. I.

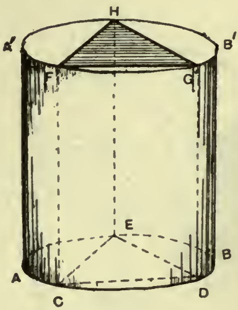
559.

Cor. Every section of a right cylinder made by passing a plane through an element is a ——. Proof.

560.

Proposition II.

Can you show that the bases of a cylinder are equal?



Sug. 1. Suppose A B′ to be any cylinder and F, G, any two points in the perimeter of the upper base. Pass the plane F D containing the line F G and the element F C. Compare F G and C D

Sug. 2. Take H as any other point in the perimeter of the upper base and H E an element through H. Pass planes through H E, F C and H E, G D. Compare F H, C E and G H, D E.

Sug. 3. Compare △s. Superpose the upper base on the lower base so F G will fall on C D and △ F G H wil fall on △ C D E. What follows?

Make the generalization. Write it and call it Prop. II.

561.

Cor. I. In the cylinder A B let C D and E F be ∥ sections made by ∥ planes cutting all the elements. Compare the sections.

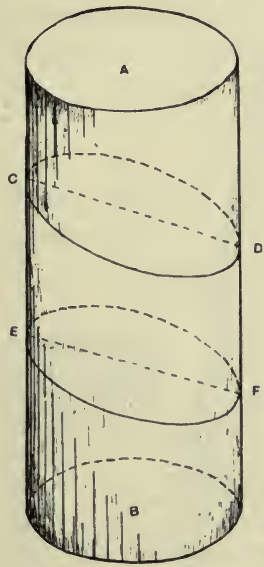


Fig. 1.

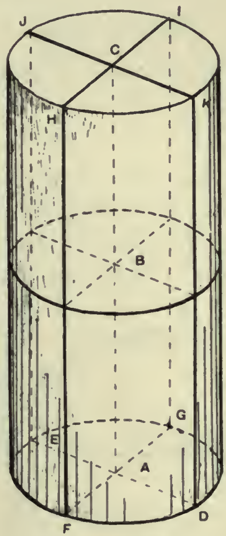


Fig 2.

562.

Cor. II. Compare the section of a cylinder made by a plane ∥ to the base with the base.

563.

Cor. III. Can you show that the axis of a circular cylinder passes through the centers of all the sections ∥ to the base ?

Sug. Draw any two diameters of one of the bases, as D E, F G. Through these diameters and elements of the cylinder pass planes cutting the upper base in H I and J K. Are H I and J K diameters?

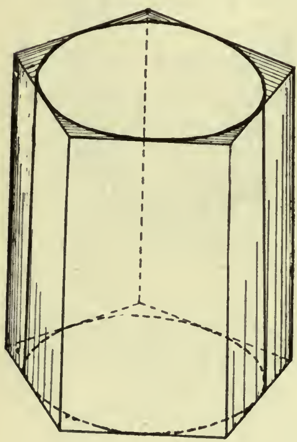
Finish proof.

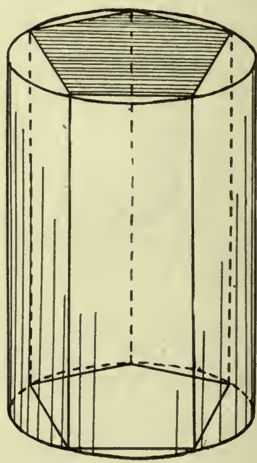
564.

Cor. IV. Suppose A C to be the axis of a circular cylinder. Can you show that it is equal and parallel to the elements of the cylinder?

565.

A cylinder is said to be *inscribed in a prism* when each lateral face is tangent to the cylinder and the bases of the prism circumscribe the bases of the cylinder. The prism is also said to be *circumscribed about the cylinder.*





566.

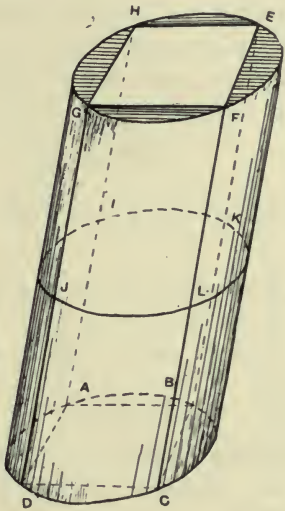
A cylinder is *circumscribed about a prism* when each edge of the prism is an element of the cylinder and the bases of the prism are inscribed in the bases of the cylinder. The prism is said to be *inscribed in the cylinder.*

567.

PROPOSITION III.

Let H C be any cylinder and I L a right section.

How do we compute the convex surface ?



Sug. 1. How find the lateral surface of any prism ? If the lateral faces of the inscribed prism H C are indefinitely increased, to what solid does the prism approach as a limit? How, then, does the convex surface of any cylinder compare with the rectangle formed by any element and the perimeter of a right section ?

Sug. 2. Let the number of sides of H C be increased by bisecting the arcs subtended by the sides, and joining the points of bisection.

Sug 3. What is the limit of the lateral area of the prism as the sides are indefinitely increased ? What does the perimeter of the polygon approach as a limit ?

Sug. 4. As the lateral area increases, what is the expression for its surface ?

Sug. 5. Call convex surface of cylinder S and circumference of I L, P, and let $S' =$ surface of the prism H C and $P' =$ the perimeter of rt. section I L.

(1) S' is equivalent to $G D \cdot P'$ Why? But $S' \doteq$? $G D \cdot P \doteq$?

Is (1) always true for any number of sides?

Are both sides of (1) variables?

Pupil will finish the proof and write the general statement, calling it Prop. III.

568.

Cor. How find the lateral surface of a cylinder of revolution?

Two cylinders are said to be *similar* when the ratio of the axis to the radius of the base of the one equals the ratio of the axis to the radius of the base of the other.

Exercise.

497. If $S =$ lateral surface of a cylinder, $h =$ altitude, $r =$ radius of the base, and $A =$ total or entire surface, find an expression for A in terms of h, r, and π.

569.

Proposition IV.

What are similar rectangles?

What are similar cylinders? Illustrate.

Compare the lateral areas of two similar cylinders whose altitudes are 6 inches and 2 inches respectively, and whose radii are 3 inches and 1 inch respectively.

How does their ratio compare with the ratio of the squares of the radii? with the ratio of the squares of the altitudes?

Can you show that the lateral areas of similar cylinders of revolution are to each other as the squares of their radii, or as the squares of their altitudes?

Sug. 1. Let S, h, r and S′ h', r' represent the lateral surface, altitude, and radius of each of the similar cylinders (1) and (2).

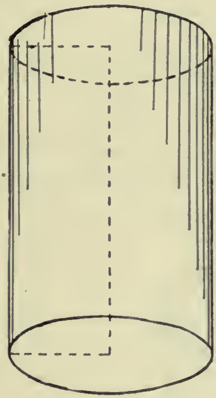


Fig. (1.)

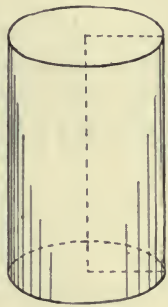


Fig. (2.)

Sug. 2. Can you prove that $\frac{r}{r'} = \frac{h}{h'}$?

That $\frac{r+h}{r'+h'} = \frac{r}{r'} = \frac{h}{h'}$?

Convex surface of a cylinder of revolution $S = 2\,\pi\,r\,h$. Why? [§ 359.]

$$\therefore \frac{S}{S'} = \frac{2\,\pi\,r \cdot h}{2\,\pi\,r' \cdot h'} = \frac{r \cdot h}{r' \cdot h'} = \frac{r}{r} \cdot \frac{h}{h'}.$$

What does $\frac{r}{r'} = $?

$$\therefore \frac{S}{S'} = \frac{r^2}{r'^2} \text{ or } \frac{h^2}{h'^2}.$$

Write the generalization, and call it Prop. IV.

570.

Cor. By Ex. 497 can you show that the *entire* surface of Fig. (1) is to entire surface of Fig. (2) as $\frac{r^2}{r'^2}$ or as $\frac{h^2}{h'^2}$?

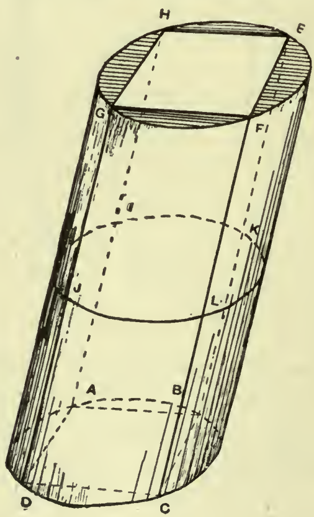
571.

PROPOSITION V.

Can you prove that the volume of *any* cylinder equals the product of its base by its altitude?

Sug. 1. How do you find the volume of any prism?

If the lateral faces of the prism be indefinitely increased, to what does the prism approach as its limit?



Sug. 2. Let G C be any cylinder and inscribe a prism within the cylinder. Now let the number of sides increase indefinitely.

Sug. 3. For convenience call volume of the cylinder

v, altitude h, base b, and volume of prism v', altitude h', base b'.

$v' = ?$ For any number of sides?

$v' \doteq ?$ $b' \doteq ?$

Draw and state conclusion, and call it Prop. V.

572.

Cor. I. Call r the radius of a cylinder of revolution, and show that $v = \pi r^2 h$.

573.

Cor. II. Can you show that the volumes of any two similar cylinders of revolution are to each other as the cubes of their radii, or as the cubes of their altitudes?

Sug. 1. (1) $\frac{r}{r'} = \frac{h}{h'}$. Why?

$v = \pi r^2 \cdot h;$

$v' = \pi r'^2 \cdot h';$ $\therefore \frac{v}{v'} = ?$

Complete proof.

574.

Cor. III. Can you show that similar cylinders of revolution are to each other as the cubes of any like dimensions?

THE CONE.

DEFINITIONS.

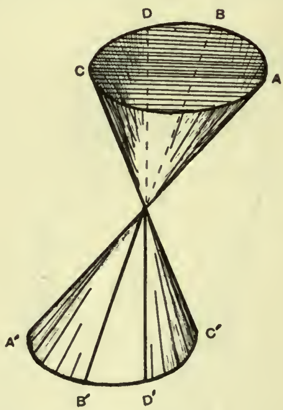
575.

A conical surface is the surface generated by a moving straight line called the *generatrix*, passing through a fixed point and constantly touching a fixed curve.

576.

The fixed point is called the *vertex*, and the fixed curve is called the *directrix*. Let the figure represent a conical surface.

577.



The generatrix in any position, as A A′ C C′, is called an *element* of the surface.

578.

When the generatrix is of indefinite length, the surface is composed of two portions, one above the vertex, the other below the vertex. These parts are called the *upper* and *lower nappes*, respectively.

579.

A *cone* is a solid bounded by a conical surface and a plane. The plane is called the *base* of the cone, the conical surface is called the *lateral, or convex surface.*

580.

The *altitude* of a cone is the perpendicular distance from the vertex to the base, or the base produced. The *axis* of a cone is the line joining its vertex to the center of its base.

581.

A *circular* cone is one whose base is a circle.

582.

If the axis is perpendicular to the base, the cone is called a *right cone.*

583.

If the axis is oblique to the base, it is called an *oblique* cone.

584.

A *right circular* cone is called a *cone of revolution*, since it may be generated by revolving a right triangle about one of its legs as an axis.

What does the hypotenuse generate? the leg not used for the axis? What is the hypotenuse in any position?

The element of a right cone is the *slant height* of the cone.

All the elements of a *cone of revolution* are equal. Why?

Has an oblique cone a slant height?

585.

Similar cones of revolution are cones generated by similar right triangles revolving about homologous legs.

586.

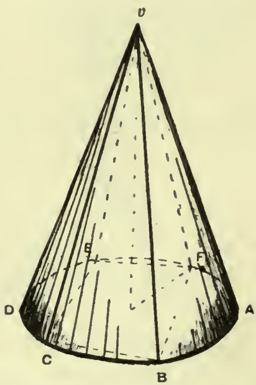
A *tangent line* to a cone is a line, not an element, which touches the cone, but does not cut it, no matter how far it is produced.

587.

A *tangent plane* contains an element of the cone, but does not, when produced, cut the surface.

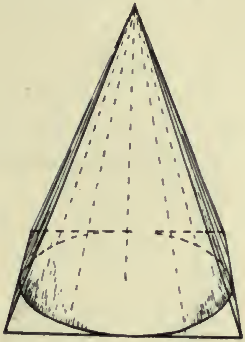
The element of contact is the element contained by the tangent plane.

588.



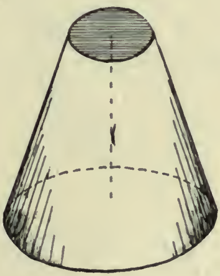
A pyramid is *inscribed* in a cone when its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

589.



A pyramid is *circumscribed* about a cone when its base circumscribes the base of the cone and its vertex coincides with the vertex of the cone.

590.



The *frustum of a cone* is that part of the cone included between the base and a section parallel to the base. The base of the cone is called the *lower base* of the frustum, and the parallel section is called the *upper base of the frustum.*

591.

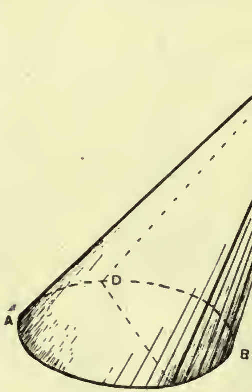
The *altitude* of the frustum is the perpendicular distance between the bases.

592.

The *slant height of a frustum* of a cone of revolution is that part of the slaut height of the cone which is included between the bases.

593.

PROPOSITION VI.



Can you prove what section is formed by passing a plane through any part of the cone and its vertex?

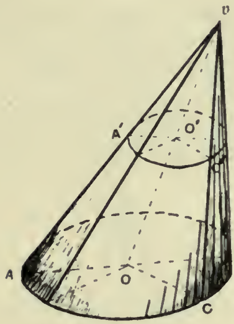
What is V C D? Prove it.

Write the generalization, and call it Prop. VI.

What is the section when the cone is a cone of revolution?

594.

PROPOSITION VII.



Can you show what any section of a circular cone is when formed by a plane parallel to the base?

Sug. 1. Suppose V O the axis of the cone and O′ the point where the axis pierces the section.

Sug. 2. Through axis V O pass any plane intersecting the base in A O and the section in A′ O′. In the same manner through V O pass any other plane intersecting the base in O C and the section in O′ C′.

Sug. 3. Compare O′ A′ and O A, also O′ C′ and O C.

Proof:

(1) V O′ : V O :: O′ A′ :: O A. [?]

(2) V O′ : V O :: O′ C′ : O C. Why?

(3) O′ A′ : O A :: O′ C′ : O C. Why?

Sug. 4. What is the base of the cone?

Now O A = O C. Why?

∴ O′ A′ = O′ C′. Why?

But A′ and C′ are *any* two points in the perimeter of the section; therefore—

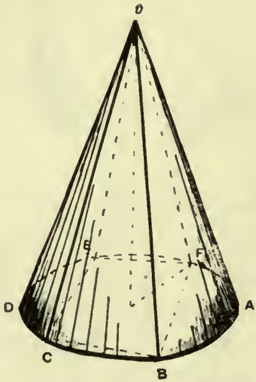
595.

Cor. How does the axis of a circular cone pierce sections parallel to the base? Proof.

596.

PROPOSITION VII.

Can you find an expression for the convex surface of a cone of revolution?



Sug. Inscribe a regular pyramid within the cone V — A B C D.

How find the lateral surface of the pyramid?

Complete the demonstration.

See method in Prop. III.

597.

Cor. Let S = convex surface of a right cone, r = radius of the base, and s = slant height. Show that $S = \frac{1}{2}(2\pi r \cdot s) = \pi r \cdot s$. If A = total area, show that $A = \pi r(s + r)$. These are convenient formulæ in arithmetic.

EXERCISES.

498. The lateral areas of two similar cones of revolution are to each other as the squares of their slant height, or the

squares of their altitudes, or the squares of the radii of their bases.

[*Hint.*—See Prop. IV.]

499. The altitudes of two similar cones of revolution are 8 inches and 4 inches, slant heights 10 inches and 5 inches, and the radii of their bases 6 inches and 3 inches. Compare total areas with the ratio of the squares of their altitudes.

500. If a cone is circumscribed about a pyramid, prove that each lateral edge of the pyramid is an element of the cone.

598.

PROPOSITION VIII.

How do you find the lateral surface of the frustum of a right pyramid?

As the number of sides of the frustum of the pyramid is indefinitely increased, to what solid does the frustum of the pyramid approach?

Can you show that the convex surface of a frustum of a cone of revolution equals one-half the product of its slant height by the sum of the circumferences of its bases?

Sug. See method of Prop. III.

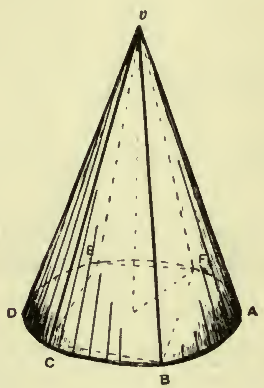
Write the proposition. Can you make an arithmetical formula for this proposition?

599.

Proposition IX.

How do you compute the volume of a pyramid? To what does the pyramid approach as its limit if the sides be increased indefinitely?

Can you prove that the volume of a cone equals one-third of the product of its base by its altitude?



Sug. Inscribe a pyramid within the cone. Pupil complete demonstration.

600.

Cor. Show that if $v =$ volume of *circular* cone, the formula $v = \frac{1}{3} \pi v^2 h$ is true if $r =$ radius of base, and $h =$ altitude of the cone.

601.

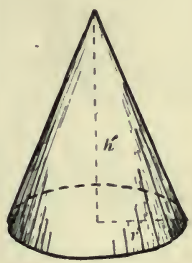
PROPOSITION X.

What are similar cones of revolution?

Compute the volumes of two cones whose altitudes are 6 inches and 3 inches and whose diameters are 8 inches and 4 inches.

Compare their volumes with the cubes of the altitudes; with the cubes of the radii of their bases; with the cubes of the diameters of their bases.

(1)



(2)

Given (1) and (2), two similar cones of revolution with h, h' the respective altitudes and r, r' the respective radii.

Can you show that the volumes of these cones are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases, or as the cubes of the diameters of their bases?

Sug. Let $V =$ volume of (1) and $v =$ volume of (2).

$V = ?\quad v = ?\quad \frac{V}{v} = ?\quad \frac{h}{h'} = ?\quad \therefore$

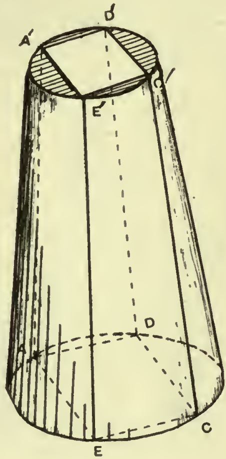
Write the conclusion, and call it Prop. X.

602.

Proposition XI.

How do you find the volume of the frustum of any pyramid?

As the number of sides is indefinitely increased, to what does the frustum of the pyramid approach as a limit?



Let A E C D — E′ be an inscribed frustum of a pyramid. Its volume = V′. Call h the altitude, B′ the area of the lower base, b' the area of the upper base, and $\sqrt{B' b'}$ the area of the mean base. [P. P.] Call V the volume of the cone.

Sug. 1. § 519, $V' = \frac{1}{3} h \left(B' + b' + \sqrt{B' \cdot b'} \right)$.

Increase the lateral faces of the frustum of the pyramid indefinitely. To what does $B' \doteq$? $b' \doteq$? $V' \doteq$? Is the above equation always true for any number of sides? Apply law of limits and complete the demonstration.

Exercise.

501. If r and r' denote the radii of the base, can you prove that $v = \frac{1}{3} \pi h \left(r^2 + (r')^2 + r r' \right)$.

EXERCISES.

502. If B = base, b = upper base, C = circumference of base, c = circumference of upper base, c' = circumference of mid-section, r = radius of base, r' = radius of upper base, h = altitude, d = diameter, s = lateral area, S = total area, and v = volume.

Write formulas for a cylinder of revolution, a cone of revolution, and the frustum of a cone of revolution.

503. Find s, S, and v by the formula above when $d = 8$ dm. and $h = 10$ dm.

504. The dimensions of the frustum of a cone of revolution are $r = 6$, $r' = 4$, and $h = 9$. Find s, S, and v.

505. How many square yards of tin will be required to cover a conical tower whose base is 12 feet in diameter and whose altitude is 20 feet?

506. A ship's mast is 40 feet long, 12 inches in diameter at the lower base, and 5 inches at the upper base. Find its volume.

507. A cylindrical cistern is 5 m. deep and $2\frac{1}{2}$ m. in diameter. If 2 Hl. flow into it per minute, how long will the cistern be in filling?

508. Can you prove that the lateral surface of a pyramid circumscribed about a cone is tanget to the cone?

509. In a cylinder of revolution $d = h$, can you show that $v = \frac{1}{3} r\, S$?

510. Two right circular cylinders have the diameter and the altitude of each equal. If the volume of one is $\frac{1}{64}$ of that of the other, what is the relation of their altitudes?

THE SPHERE.

Definitions.

603.

A *sphere* is a solid bounded by a surface all points of which are equally distant from a point within called the *center*.

604.

The *radius* of a sphere is the distance from the center to any point on the surface.

605.

The *diameter* of a sphere is any straight line passing through the center terminated by the surface of the sphere. It follows that all radii are equal, that any diameter is twice the radius, and that all diameters are equal.

606.

A line is *tangent* to a sphere when it has only one point in common with the sphere.

A *plane is tangent* to a sphere when it has only one point in common with the sphere.

Two *spheres are tangent* when they have only one point in common.

607.

Two spheres are *concentric* when they have a common center.

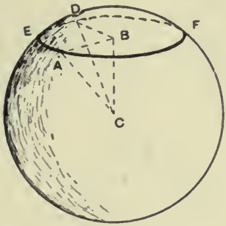
608.

A polyedron is said to be inscribed in a sphere when its vertices lie in the surface of the sphere. The sphere may be said to be circumscribed about the polyedron.

A polyedron is *circumscribed about a sphere* when its faces are tangent to the surface of the sphere. In this case the sphere is inscribed within the polyedron.

609.

PROPOSITION I.



How are points on the surface of a sphere located with regard to the center?

Let E C F represent a sphere and E, D, F, A be points in the intersection made by cutting the sphere with a plane.

What is the section E A F D?

Sug. 1. Let A and D be any two points in the intersection, and join them with the center of the sphere.

Erect a $\perp$ to the plane from the center, C, meeting the plane at B.

Sug. 2 Compare $\triangle$s A C B and D C B, A B and B D.

Draw conclusion and write it, calling it Prop. I.

610.

Cor. I. Can you show how the line drawn from the center of the sphere to the center of a circle of the sphere is related to the plane of the circle?

611.

Cor. II. Two circles are equally distant from the center. Compare them.

612.

Cor. III. Of two circles unequally distant from the center of a sphere, prove which is the larger.

613.

Cor. IV. Compare circles which contain the center of the sphere.

614.

A *great circle of a sphere* is one that contains the center of the sphere.

A *small circle of a sphere* is one that does not contain the the center of the sphere.

615.

The *poles of a circle of a sphere* are the extremities of the diameter which is perpendicular to the plane of the circle.

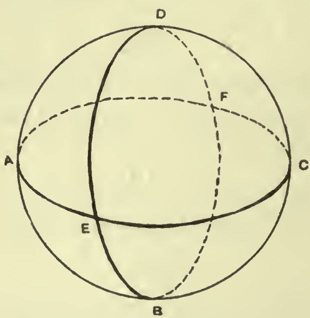
This diameter is often called the axis of the sphere.

616.

What section is formed if a plane passes through the center of a sphere? Prove.

617.

PROPOSITION II.



Let A B C D be any sphere cut by the two intersecting great circles A E C F and B E D F. E F is the line of intersection of the two circles.

Compare the parts of each great circle?

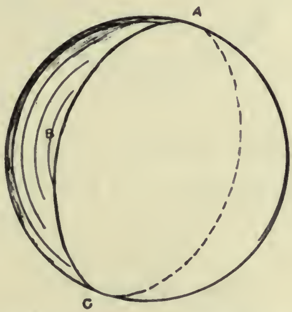
Sug. 1. Where is the center of the sphere with regard to the centers of the two circles?

Sug. 2. Does the line of intersection pass through the center of the sphere? Through the center of the circles? What does the line E F do to each circle?

Write conclusion, and call it Prop. II.

618.

Proposition III.



How many points determine a plane?

How does a plane cut a sphere?

How many points on the surface of a sphere determine a circle?

Sug. Can you pass a plane through points A, B, C?

619.

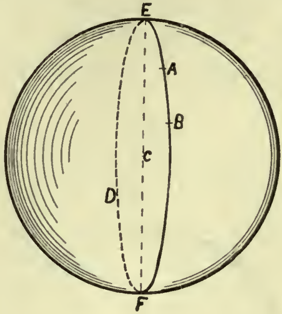
Cor. Is there any circle which may be determined by two points on the surface?

Exercise.

511. Two great circles have their planes perpendicular to each other. How are they related to each other's poles?

620.

When the *distance* between two points on the surface of a sphere is spoken of, the shorter of the two arcs is meant.

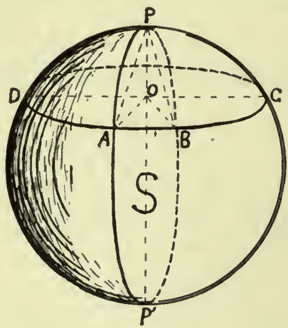


Thus, in the great circle D E B F the *distance* between A and B is the shorter arc.

621.

Proposition IV.

Let S represent any sphere, P and P′, the poles of the circle A B C D.



Can you show that all points of the circumference of A B C D are equally distant from P and P′?

Sug. 1. Take any two points, as A and B, in the circumference of A B C D, and pass the arcs of great circles P A and P B. Let the axis P P′, pierce the plane A B C D at O and draw O A, O B, O C, O D.

Sug. 2. Compare O A, O B; P A, P B; P′ A, P′ B.

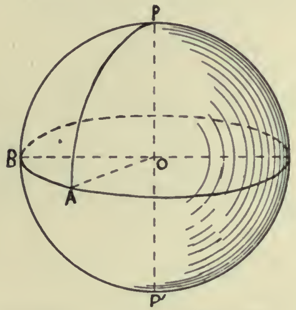
Draw conclusion and write it. Call this theorem Prop. IV.

622.

The *polar distance* of a circle of a sphere is the distance from the nearer of its poles to the circumference of the circle. Thus in § 621 P A and P B are polar distances.

623.

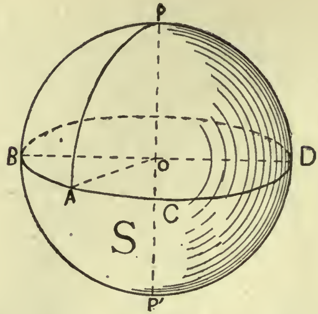
The *polar distance* of a great circle is a quardrant. Show this from the figure.



624.

PROPOSITION V.

Let S be any sphere, and P be at a quadrant's distance from two points, A and B, on a great circle, A B C D.



To show that P is the pole of the arc A B.

Sug. Join the center of the sphere, O, with A, B, P.

Compare ∠s P O A and P O B.

How is P O related to the plane A B C D? What is P of arc A B.

Write a general statement, and call it Prop. V.

Remark: Discuss the question when the two points are at the extremity of a diameter, as B and D.

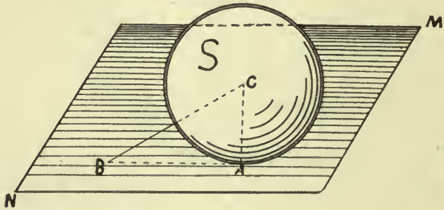
625.

Scholium: An arc of a great or small circle may be drawn between two given points on the surface of sphere by placing one foot of the compasses at the pole.

The distance from the point to the pole must be known.

626.

PROPOSITION VI.



Given the plane M N perpendicular to the radius of the sphere, S, at its extremity, A.

Can you show how the plane is related to the sphere?

Sug. Compare C A and C B, B being any point in the plane except A.

Complete the proof. Write Prop. VI.

627.

Cor. I. Write and prove the converse of Prop. VI.

628.

Cor. II. Prove that a straight line tangent to any circle of a sphere lies in the plane tangent to the sphere at the point of contact.

629.

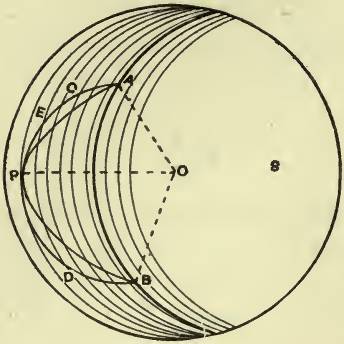
Cor. III. Show that any straight line in a tangent plane at the point of contact is tangent to the sphere.

630.

Cor. IV. Prove that any two straight lines tangent to a sphere at the same point determine the tangent plane.

631.

Proposition VII.



Given the sphere s. Choose any two points as, A, B, less than a circumference apart on a great circle and with the center pass a plane. What kind of a section is formed? What kind of an arc is A B?

Draw any other line on the surface of the sphere joining these two points and compare its length with A B.

Sug. 1. Let this line between A and B be A E D B. Take any point P on A E and D B pass arcs of great circles, through A and P, B and P. Draw O A, O B, O P. Pass planes through these lines.

Sug. 2. What kind of an angle is O — A P B? Compare the face angle A O B with the sum of the face angles A O P and B O P. Compare the arc A B with the sum of the arcs A P and B P.

Sug. 3. Take any other point in A E P D B, as C, and connect it with arcs of great circles with A and P. Connect D with P and B in the same manner. Compare the sum of the great circle arcs A C, P C, P D, D B with the sum of great circle arcs A P and P B. Also with A B.

Sug. 4. Suppose we continue to take points on the line A E P D B and proceed as above. Do you see that the sum of the arcs of the great circles is a variable? What does this sum $\doteq$?

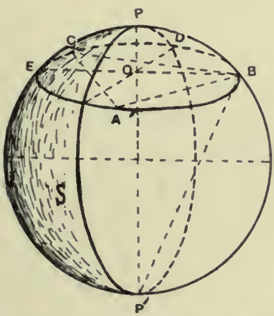
Sug. 5. As the number of points on A E P D B is increased, how does the sum of the arcs compare with the sum just preceding?

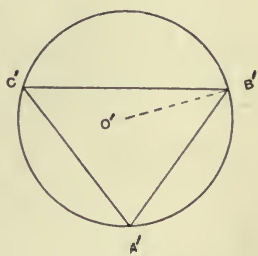
Sug. 6. How does A E P B D compare with the arc A B? Write the conclusion, and call it Prop. VII.

632.

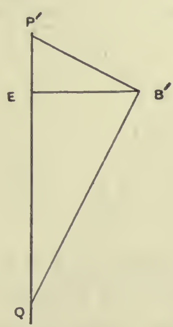
Proposition VIII.

Problem: Can you find the radius of a physical sphere?





Given: Any sphere, S.



Sug. 1. Take any point, P, on the surface as a pole, and with any opening of the compasses describe a circle.

Take any three points on the circumference, as A, B, C, and measure the distances A B, B C, C A.

Sug. 2. Construct a triangle, A′ B′ C′, with sides A B, B C, C A, and circumscribe a circle about it. Call the radius O′ B′.

With O′ B′ as one side of a right triangle and P B the hypotenuse, construct the triangle P′ B′ E.

If O is the intersection of the diameter of the sphere and the diameter of the circle A B C, compare $\triangle$s P O B and P′ B′ E.

Sug. 3. At B′ draw B′ Q $\perp$ P′ B′ and compare $\triangle$ P′ B′ Q with P B P′.

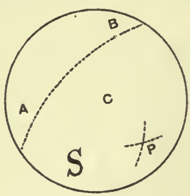
Can you measure P′ Q? Can you find the radius of the sphere?

633.

Cor. I. Can you find the chord of a quadrant of a sphere when the radius of the sphere is known?

634.

Cor. II. From Cor. I. can you describe a circumference of a great circle through any two points on the surface of a sphere whose radius $= r$?

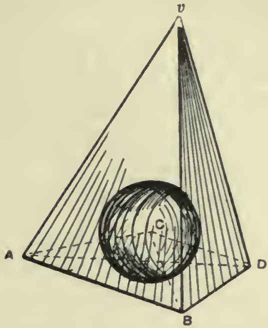


Sug. Find the chord of a quadrant from Cor. I. Take any two points, A, B, on the sphere.

Pupil demonstrate.

635.

PROPOSITION IX.



Bisect three diedral angles at the base of a triangular pyramid with planes.

How does any point in either of the bisecting planes relate to the sides of the diedral angle bisected? How will the three bisecting planes meet?

Given: Any tetraedron, v — A B D.

Can you inscribe a sphere within it?

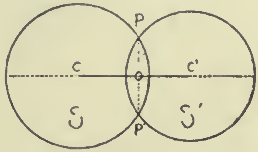
Write the theorem, and call it Prop. IX.

636.

Cor. Can you prove how the six bisecting planes of the diedral angles of a tetraedron meet?

637.

PROPOSITION X.



Imagine part of a hollow glass sphere to receive in part another smaller sphere.

Can you picture the section made by the surfaces coming in contact?

What is the section of the intersection of the the surfaces of two spheres? How is the section related to the line of centers?

Sug. 1. Let S and S′ represent two intersecting spheres whose centers are C and C′. Pass a plane through C and C′. What are the circles formed?

Sug. 2. Letter points of intersection of the circles P and P′ and draw the chord P P′.

Extend the line C C′ to meet the circumferences.

Sug. 3. How is C C′ related to P P′? Why?

Sug. 4. Imagine the lower half of the figure composed of the intersecting circles to revolve about the axis C C′ produced. What will the two semicircles generate? the line O P′? the point P′?

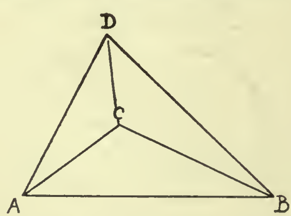
Give a complete demonstration, and write Prop. X.

EXERCISES.

512. Three equal lines have their extremities in the surface of a sphere. How are these lines related to the center of the sphere?

513. If a cone of revolution roll upon a plane with its vertex fixed, what kind of a surface is generated by the surface of the cone?

Can you prove that one, and only one, surface of a sphere may be passed through any four points not in the same plane?



Sug. 1. Suppose A, B, C, D to be the four points not in he same plane.

What is the locus of all points equally distant from A and B, B and C? What is the intersection of these two loci?

Sug. 2. What is the locus of all points equally distant from B, D?

Note the intersection of the loci.

The pupil will work out the demonstration.

SPHERICAL ANGLES AND POLYGONS.

DEFINITIONS.

638.

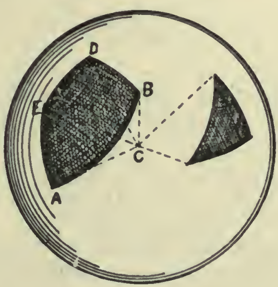
The angle of two intersecting curves is the angle of the two tangents to the curves at their point of intersection. This definition applies to all curved surfaces.

639.

A spherical angle is the angle included between two arcs of great circles of a sphere. The arcs are the sides of the angle and their intersection the vertex.

640.

A portion of the surface of a sphere bounded by three or more arcs of *great circles* is called a *spherical polygon.*



The *sides* of the spherical polygon are the bounding arcs;

the *angles* of the polygon are the angles of the intersecting arcs; the *vertices* of the polygon are the points of intersection of the arcs.

Thus, A B D E is a spherical polygon. Its sides may be expressed in degrees.

641.

The *diagonal* of any spherical polygon is the arc of a great circle joining any two vertices not adjacent.

642.

The planes of the sides of a spherical polygon form a *polyedral angle* whose vertex is at the center of the sphere.

Thus, C — A B D E is a polyedral angle with its vertex at C.

643.

A *convex spherical polygon* is a spherical polygon whose corresponding polyedral angle is convex.

644.

A *spherical triangle* is a spherical polygon of three sides.

645.

A spherical triangle may be right, acute, equilateral, isosceles, etc., under the same restrictions as plane triangles.

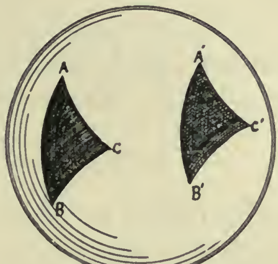
646.

Any two points on the surface of a sphere may be joined by two arcs of a great circle; one will usually be greater than a semicircumference, the other less.

The smaller arc is always meant unless otherwise stated.

647.

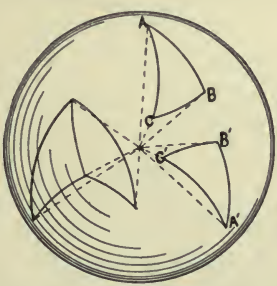
Two spherical polygons are *equal* when one may be applied to the other so that they will coincide in all their parts. Thus,



the spherical triangles A B C and A′ B′ C′ are equal if A B, B C, C A are equal respectively to A′ B′, B′ C′, C′ A′, and the angles A, B, C respectively equal the angles A′, B′, C′.

648.

Two spherical polygons are symmetrical when the sides and angles of one are equal respectively to the sides and angles of the other when taken *in reverse order*. Study the figures.



Can you make these figures coincide?

[The question of equality and symmetry of spherical triangles may be cleared by using the rind of an orange.

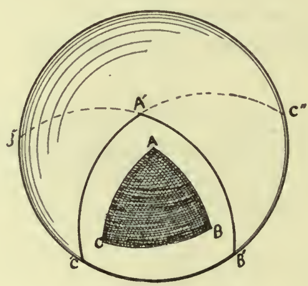
Let the pupil draw a sphere and show that the vertices of one symmetrical triangle are at the ends of the diameters from the vertices of the other.]

649.

What is meant by *vertical* spherical polyedral angles? Draw a figure to illustrate. How are the corresponding polygons of two vertical polyedral angles related to each other?

650.

A *polar triangle* is formed by taking the vertices of a given spherical triangle as poles, and then describing three intersecting arcs of great circles. Thus, in the figure.



A is the pole of B′ C′;
B is the pole of A′ C′;
C is the pole of A′ B′.

651.

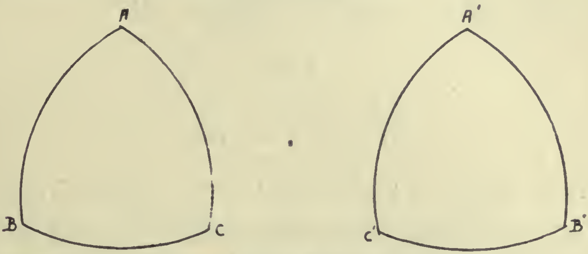
The great circles B′ B″, C′ C″, B′ C′ form eight spherical triangles, four of which are on the opposite side of the sphere. That one of the eight is a polar △ which has A′ homologous to A on the same side of B C, C′ and C on the same side of A B, B′ and B on the same side of A C.

652.

Proposition XI.

(1) In the two given $\triangle$s let A B C and A′ B′ C′ be *symmetrical isosceles* spherical $\triangle$s; *i. e.*, A B = A′ B′, A C = A′ C′, B C = B′ C′, and $\angle$ A = $\angle$ A′, $\angle$ B = $\angle$ B′, and $\angle$ C = $\angle$ C′.

Can they be made to coincide?



[*Hint.*—Compare A B and A′ C′, A C and A′ B′. [Auth.] Compare $\angle$ A with $\angle$ A′.]

Can you superpose $\triangle$ A B C upon $\triangle$ A′ B′ C?

Will the two $\triangle$s coincide? Why?

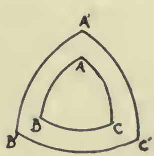
Give a complete demonstration.

(2) When the two triangles are not isosceles.

Can you show how they are related?

653.

Proposition XII.



1

In the figure let A B C be any spherical triangle and A′ B′ C′ its polar triangle. Draw on the surface of a sphere.

Can you prove that A B C is the polar triangle of A′ B′ C′ ?

Sug. 1. A is the pole of what arc?

How many degrees from A to B′?

Sug. 2. C is the pole of what arc?

How far from C to B′? B′ is the pole of what arc?

Pupil complete proof.

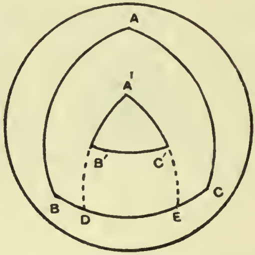
Write the theorem, and call it Prop. XII.

654.

Proposition XIII.

In the figure let A B C and A′ B′ C′ be polar triangles.

Can you show that each angle of one is measured by the supplement of the side opposite it in the other?



Sug. 1. Select any angle as A′, and extend its sides, if necessary, to meet the opposite side of the other triangle at D and E. What part of a circumference is the distance from each point, D and E, to the extremities of the opposite sides? *i. e.*, B is the pole of what arc? E is the pole of what arc?

Sug. 2. D C + B E = ?

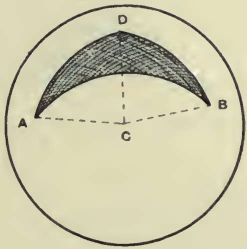
B C + D E = ?

∴ D E = ?

Complete the demonstration, and write Prop. XIII.

655.

PROPOSITION XIV.



In the given sphere form any spherical triangle, as A B D, and pass planes through the sides and the center of the sphere. What is the figure formed? Compare a face angle of any triedral angle with the sum of the other two. [§ 449.]

Can you prove that the sum of any two sides of a spherical triangle is greater than the third side?

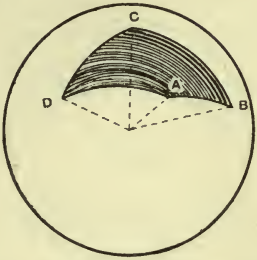
656.

Cor. Can you prove that any side of a spherical triangle is greater than the difference of the other two sides?

657.

PROPOSITION XV.

What is the limit of the sum of all the face angles about a polyedral angle? [§ 534]



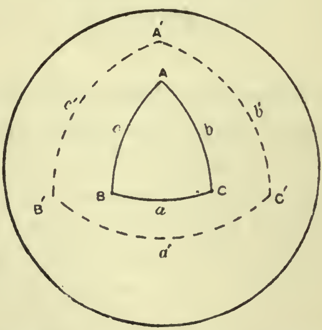
In the figure let A B C D be any polygon on the surface of the sphere. Pass planes through the sides and the center, C, of the sphere. What is formed by these planes?

Can you prove that the sum of the sides of any spherical polygon is less than the circumference of a great circle?

Give demonstration, and write Prop. XV.

658.

PROPOSITION XVI.



In the figure let A B C be any spherical triangle and A′ B′ C′ its polar triangle. How do you measure each angle?

Write an expression for the measurement of $\angle$ A + $\angle$ B + $\angle$ C. What is the limit of $a' + b' + c'$? [Prop. XV.]

Can you now demonstrate that the sum of the angles of any spherical triangle is greater than two and less than six right angles.

659.

Cor. How many right angles may a spherical triangle have? how many obtuse angles?

660.

A *birectangular* spherical triangle is one which has two right angles.

A *trirectangular spherical triangle* is one which has three right angles.

661.

PROPOSITION XVII.

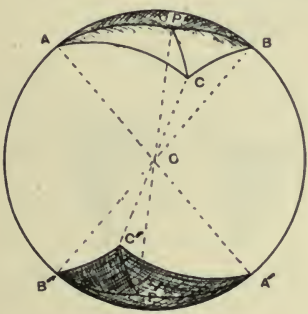


Fig. 1.

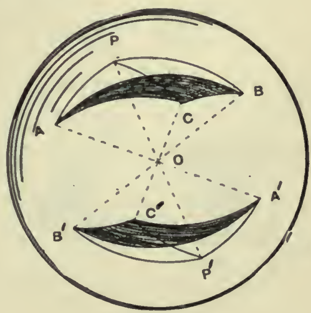


Fig. 2.

Let A B C and A′ B′ C′ be two symmetrical spherical triangles having their homologous vertices diametrically opposite to each other.

Are the triangles equivalent?

Sug. 1. Let P be the pole of a small circle passing through the points A, B, C and let P O P′ be a diameter. Draw the great circle arcs P A, P B, P C, P′ A′, P′ B′, P′ C′.

Sug. 2. Compare P A, P B and P C.

Also compare P A and P′ A′; P B, P′ B′; P C, P′ C′.

Also compare P′ A′, P′ B′, and P′ C′; then compare △s P A C and P′ A′ C′. (1) Are they symmetrical? Are they isosceles? Draw conclusion.

Sug. 3. In the same manner compare (1) △s P A B and P′ A′ B′. (2) △s P B C and P′ B′ C′.

Sug. 4. Now, by referring to the three pairs of △s above, compare △s A B C and A′ B′ C′.

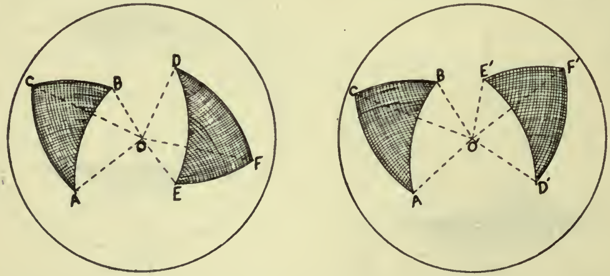
Sug. 5. But suppose P, the pole of the small ⊙ passing through the points A, B, C, shall fall without the △ A B C.

Can you show that each of the two △s A B C and A′ B′ C′ is equivalent to the sum of two isosceles △s diminished by the third? (See second figure)

Draw general conclusion, and write Prop. XVII.

662.

Proposition XVIII.



Given: On the same or equal sphere two triangles in which two sides and the included angle of one equal respectively two sides and the included angle of the other.

Make deductions and prove.

Sug. 1. In Fig. 1 let △s A B C and D E F on the same sphere have $\angle$ A and the sides A B, A C of the one respectively equal to $\angle$ D and sides D E and D F of the other. *Can the triangles be made to coincide?*

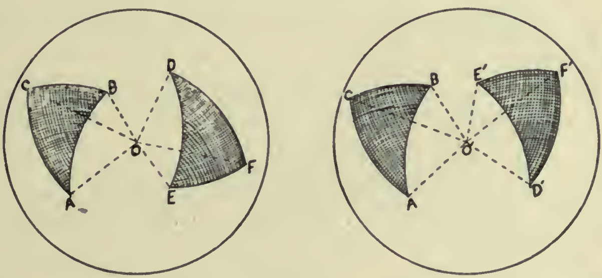
Sug. 2. In Fig. 2, on a sphere equal to that in Fig. 1, redraw $\triangle$ A B C and draw $\triangle$ D′ E′ F′ symmetrical to $\triangle$ D E F of Fig. 1.

Compare $\triangle$s D E F and D′ E′ F′. [§ 662.]

Pupil complete demonstration, and write Prop. XVIII.

663.

Proposition XIX.



Given: Two triangles on the same sphere, or equal spheres, having two angles and the included side of one equal respectively to two angles and the included side of the other.

(1) Can you prove the triangles equal?

(2) Can you prove the triangles symmetrical?

Study the figures and give proof.

664.

Proposition XX.

Suppose on the same or equal spheres the three sides of one triangle equal the three sides of another respectively.

Compare the triangles.

In the figure let △ A B C and △ A′ B′ C′ have the three sides of one respectively equal to the three sides of the other.

Sug. 1. Connect the vertices of each △ with the centers of the spheres.

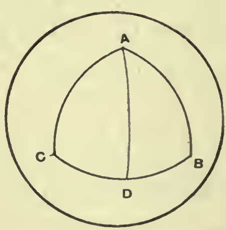
Sug. 2. Compare the face angles of the triedral ∠ at O with the face angles of the triedral ∠ at O′. Compare the diedral ∠s of the two solid angles.

Sug. 3. Can you now compare △ A B C and △ A′ B′ C′, (1) when sides are arranged in the same order, (2) when sides are in reverse order?

Write Prop. XX.

665.

Proposition XXI.



Given: The isosceles spherical △ A B C with A B = A C.

Can you prove ∠ B = ∠ C?

Sug. Pass the arc of a great ⊙ through A and D, the mid-point of B C.

Prove the △s equal or equivalent and consequently equiangular. Give proof in full.

Write Prop. XXI.

666.

Proposition XXII.

Given: The △s A B C, A′ B′ C′ on the same or equal spheres, with the angles of the one respectively equal to the angles of the other.

How do the △s compare?

Sug. 1. Construct the polar △s T and T′ for the given △s. Compare the sides of T and T′. (What Prop.?)

Compare the angles of T and T′. (What Prop.?)

Sug. 2. Compare the sides of the △s A B C and A′ B′ C′. Therefore by Prop. — — — —

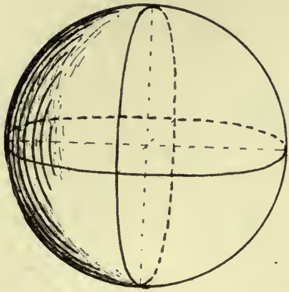
Write Prop. XXII.

667.

Cor. I. If two angles of a spherical triangle are equal, can you show that the sides opposite these angles are equal, and the triangle isosceles?

In figure, § 666, what measures ∠ C? ∠ B? Then what sides are equal? what angles? etc.

668.

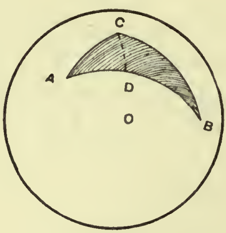


Cor. II. In the figure suppose three planes each perpendicular to the other two and passing through the center of the sphere. Can you show how many equal trirectangular triangles are formed?

669.

PROPOSITION XXIII.

I. *Given:* A B C any spherical triangle in which two angles are unequal; ∠ C > ∠ A.



How do the sides opposite these angles compare? Which is the greater?

Sug. 1. Draw C D arc of a great ⊙ so that ∠ A C D shall equal ∠ C A D.

Compare A D and C D.

Compare D B + C D with B C. [§ 655.]

Compare D B + A D with B C.

∴ ——— — — —

Pupil give complete demonstration.

II. Suppose the sides A B and B C are unequal, A B > B C.

How do the angles opposite these sides compare? Which is the greater?

Use method of exclusion. Write Prop. XXIII.

Exercises.

514. The arc of a great circle drawn from the vertex of an isosceles spherical △ to the middle point of the base is ⊥ to the base and bisects the vertical angle.

515. Suppose one circle of a sphere passes through the poles of another circle of a sphere, how are the two circles related?

516. Compare the volume formed by revolving a rectangle about its shorter side with that formed by revolving it about its longer side.

517. If the sides of a spherical △ are respectively 63°, 115°, and 84°, how many degrees in each angle of the polar △?

518. If the angles of a spherical △ are 100°, 90°, and 75°, how many degrees are there in each side of its polar △?

519. What are the maximum and minimum limits of the sum of the angles of a spherical pentagon?

520. At a given point in a given arc of a great circle, to construct a spherical angle equal to given spherical angle.

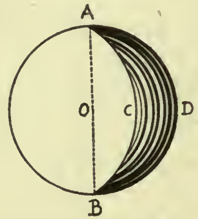
521. The radius of a small circle on a sphere is less than the radius of the sphere.

SPHERICAL MEASUREMENTS.

DEFINITIONS.

670.

A *lune* is a portion of the surface of a sphere included between two semicircumferences of great circles; as A B C D.



The *angle* of the lune is the angle formed by its bounding arcs; as ∠ C A D.

671.

Lunes on the same or on equal spheres, having equal angles, may be made to coincide.

672.

A *spherical wedge or ungula* is the solid bounded by the lune and the planes of its sides; as A O B C D.

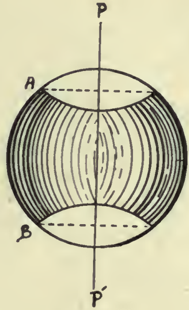
The lune A B C D is called the *base* of the wedge and the diameter A B is the *edge*.

673.

A *zone* is a portion of the surface of a sphere included between two parallel planes. The circumferences of the sections are called the *bases* of the zone. The distance between the bases is the *altitude* of the zone.

674.

A *zone of one base* is a zone one of whose bases is tangent to the sphere.



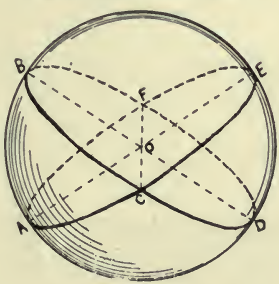
If the circle P A B P′ be revolved about the diameter P P′, the arc A B will describe a zone and points A and B will describe the circumferences of the bases of the zone.

675.

The surface of a sphere has eight trirectangular triangles, and it is convenient to divide each into 90 equal parts, called *spherical degrees.* The surface of every sphere has how many spherical degrees? What kind of a triangle is a spherical degree?

676.

Proposition XXIV.



In the figure let A C E and B C D be any two arcs of great circles intersecting at point C on the hemisphere A D E B forming the two vertical spherical triangles A C B and D C E.

Can you show that the sum of the two △s mentioned is equivalent to a lune whose angle equals ∠ A C B?

Sug. 1. Complete the circles of arcs A C E and B C D. What is C D F E? Why? C A F B?

Sug. 2. Compare arcs C B and D F, A C and E F, A B and E D.

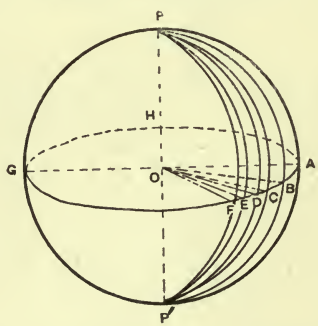
Sug. 3. Compare △s D E F and A C B. [§ 664.]

Now compare the sum of A C B and D C E with C D E F.

Give complete demonstration, and write Prop. XXIV.

677.

Proposition XXV.



In the figure let P and P′ be the poles of a sphere and draw the arc of a great ⊙ A F. Let P A P′ F be a lune.

(1) Compare the area of the lune with the area of the sphere when the angle of the lune and four right angles are commensurable.

Sug. 1. On the arc A F measure off equal distances, each of which is the common unit between arc A F and the circumference A H G F. From O, the center of the sphere, draw lines to the points of division on the arc A F.

Sug. 2. Suppose the unit contained in A O F *a* times and in four right angles *b* times; then $\frac{\text{A O F}}{4 \text{ rt. } \angle\text{s}} = ?$

Now pass planes through P P′ and the points of division, A, B, C, D, E, F. How is the surface of the lune divided? Compare the number of lunes with the number of ∠s at O.

Then the $\frac{\text{surface of the lune}}{\text{surface of the sphere}} = ?$

Complete the demonstration.

(2) Compare the area of the lune with the area of the sphere when ∠ A O F and four right angles are not commensurable.

Sug. Use method of limits.

678.

Cor. I. Can you show that a lune contains twice as many spherical degrees as its angle contains angular degrees?

Sug. How many spherical degrees in the su face of a sphere?

What is the sum of all the angular degrees of all the lunes converging at P in the figure? Let L = lune, S = surface of sphere, A = angle of lune.

Can you prove that

(1) L : S :: A : 360°?

(2) The number of spherical degrees of L : 720° : : A : 360°.

(3) L = 2 A?

679.

Cor. II. Can you prove that lunes on the same or on equal spheres are to each other at their angles?

680.

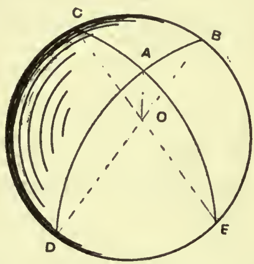
The spherical excess of a triangle is the excess of the sum of the angles of a spherical triangle over two right angles.

The spherical excess of a polygon is the excess of the sum of the angles of the spherical polygon over two right angles taken as many times as the polygon has sides less two.

Thus, if a polygon has n sides, it has $n - 2$ spherical triangles. Its spherical excess equals the sum of the spherical excesses of its triangles.

681.

Proposition XXVI.



In the figure let A D E be any spherical triangle. Complete the great circle of which D E is an arc and produce the sides E A and D A till they meet the great circle D E B C.

Compare the number of *spherical* degrees in triangle D A E with the number of angular degrees.

Sug. 1. What are D E B, D A B, and D C B? [Auth.]

(1) △ D A E + △ B A C equivalent? Its angle?

(2) △ D A E + △ A B E equivalent? Its angle?

(3) △ D A E + △ D A C equivalent? Its angle?

Sug. 2. How many *spherical* degrees in a lune whose angle is A? in a lune whose angle is B? in a lune whose angle is C?

Sug. 3. In (1), (2), (3), how many times extra have we used the △ D A E in taking the surface of the hemisphere?

How many spherical degrees in a hemisphere?

Sug. 4. 2 △ D A E + 360° equivalent?

△ D A E + 180° equivalent ∠ A + ∠ B + ∠ C. Why?

∴ △ D A E is equivalent to (∠ A + ∠ B + ∠ C — 180°) spherical degrees.

If we call the spherical excess, E, then △ D A E contains E spherical degrees.

Go over these hints carefully again and give a complete demonstration.

Write Prop. XXVI.

Scholium. If a spherical triangle contains 120 spherical degrees, or c spherical degrees, it means that the surface of the triangle is $\frac{120}{360}$, or $\frac{c}{360}$ of the surface of a hemisphere.

How would you express the relation of a spherical triangle which has r spherical degrees to the surface of a sphere?

EXERCISES.

522. How many spherical degrees in a spherical triangle whose angles are 180°, 140°, 120°?

523. Compare the surface of a spherical triangle whose angles 120°, 150°, 180° with that of a sphere.

524. Each angle of a spherical triangle is 90°. Compare the triangle with the surface of a sphere.

525. The angle of a lune is 60°. What part of the surface of a sphere is it?

526. The sides of a spherical △ are respectively 75°, 120°, and 90°. How many degrees in each angle of its polar triangle?

527. If the angles of a spherical △ are respectively 100°, 90°, 75°, how many degrees are there in each side of its polar triangle?

528. The sum of the angles of a spherical pentagon is greater than six, and less than ten, right angles.

529. The sides opposite the equal angles of a birectangular triangle are quadrants.

530 If the radii of the bases of the frustum of a cone are r and r' and the slant height is a, and the altitude is h, can you show that $S = \pi a (r + r')$?

531. The volume of the frustum of a right cone is V. Find the difference between the volumes of the circumscribed and the inscribed regular hexagonal frustums.

532. A cannon ball 4 inches in diameter weighs 8 lbs. What is the cost of one of the same material whose diameter is 6 inches, if the metal is worth three cents per pound?

533. The height of the frustum of a cone is $\frac{2}{5}$ of the height of the entire cone. Compare the volume of the frustum and the entire cone.

534. What are the dimensions of a right cylinder $\frac{15}{16}$ as large as a similar cylinder whose height is 20 feet and whose diameter is 10 feet?

535. The total surface of a sphere is 16 square feet. What is the number of square feet in a lune of the same sphere whose angle is 30°?

536. If the surface of a lune is 2 square feet and the surface of the sphere is 18 square feet, what is the angle of the lune?

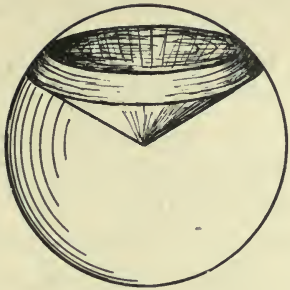
682.

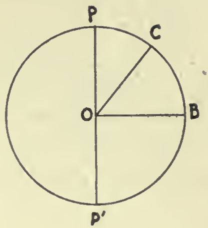
A *spherical segment* is the portion of a sphere included between two parallel planes.

683.

A *spherical sector* is the portion of a sphere generated by the revolution of a circular sector about a diameter.

I. Thus, let P P′ be the diameter of any circle, and P O C a circular sector. When the semicircle P B P′ is revolved, the *circular* sector P O C will generate a *spherical* sector, whose base is described by the arc P C, and whose conical surface is described by the radius O C.



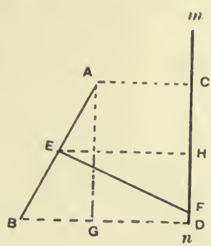


The arc P C describes a zone which is called the *base of the spherical sector.*

II. What will be the circular sector C O B describe? How many surfaces bound it? What do you call the surfaces described by O C and O B? C B? What is the base of the spherical sector described by C O B?

684.

Proposition XXVII.



Let $m\ n$ be an axis and A B any line in the same plane making any angle with the axis, but not meeting it.

Project the extremities and mid-point of A B upon the axis, thus fixing points C, D, and H.

If the line A B be revolved about the axis $m\ n$, what surface will be generated?

How do you compute the surface?

From A draw a ⊥ to B D meeting it at G, and at E erect a ⊥ to A B meeting the axis *m n* in F. Compare △s A B G and H E F.

$$\therefore \quad \frac{AB}{AG} = \frac{EF}{EH}, \text{ and } AB \cdot EH = EF \cdot AG = CD \cdot EF.$$

What expresses the convex surface described by A B?

∴ area of the surface described by A B = C D · 2 π E F.

Complete the demonstration and state Prop. XXVI.

685.

Cor. I. If A B is parallel to the axis *m n*, what is generated? How is the lateral surface computed?

686.

Cor. II. Suppose the point A to lie in the axis *m n*,

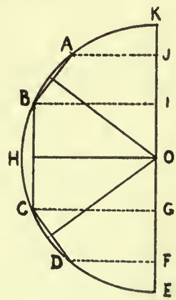
What is generated by the revolution of A B about the axis *m n?*

Compute the lateral surface.

Sug. Use the method as when A B did not meet *m n*. A C = O.

687.

PROPOSITION XXVIII.



Given: K B D E a semicircle, and A D an arc. Revolve this arc about the axis K E. What is generated?

Suppose the arc to be divided into any number of equal parts, as A B, B C, C D. Draw the chords of these arcs. Project the extremities of these chords on the axis K E. Call K O, the radius, R.

Compare the chords A B, B C, C D.

At the mid-points of these chords erect ⊥s terminating in the axis, K E.

Where will these ⊥s meet?

Compare the length of the ⊥s.

Can you show that the area of a zone equals its altitude by the circumference of a great circle?

Sug. 1. What is the area generated by the chord A B? by the chord B C? C D?

Sug. 2. Can you express the area generated by the sum of the chords in one equation?

Sug. 3. Suppose the arcs are bisected and chords drawn as before, what will represent the area generated?

Sug. 4. Let the arcs be bisected indefinitely; what does the broken line A B C D $\doteq$?

Sug. 5. What does the sum of surfaces described by the chords $\doteq$?

What does 2π L O $\doteq$?

Review and give a complete demonstration. Write Prop. XXVIII.

Scholium. If we let S represent the surface of the zone and h the altitude and r the radius of sphere on which the zone lies, Prop. XXVIII may be expressed in the formula $S = 2 \pi r h$.

688.

Proposition XXIX.

Recall the definition of a zone. Can you think of a zone whose altitude equals the diameter? Make a drawing to illustrate.

How do we find the area of the surface of a zone.

Can you show how to find the area of the surface of a sphere?

Can you prove this by the method used in Prop. XXIX?

689.

Cor. I. Can you express the area of the surface of a sphere in terms of the radius?

Sug. Instead of using π D as the circumference, use the equivalent of D.

690.

Cor. II. Can you show from Cor. I. that the area of the surface of a sphere equals the area of four great circles?

691.

Cor. III. Can you show that the area of the surface of a sphere equals the area of a circle whose radius is the diameter of the sphere?

692.

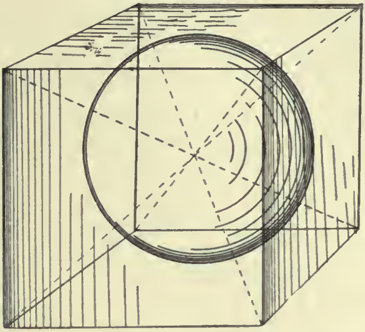
Cor. IV. Show that the areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.

693.

Cor. V. Can you show that the area of a spherical degree equals $\frac{4 \pi r^2}{720}$?

694.

Proposition XXX.



Let the figure represent a cube circumscribed about a sphere whose radius is r. Join the vertices of the cube with the center of the sphere. Pass a plane through each edge and the two lines which join its ends to the center. What solid figures are formed with vertices at the center of the sphere? What are their bases? their altitudes? What is their sum? How does it compare with the volume of the sphere? How do you find the volume of each pyramid? Now at points where the edges of the pyramids pierce the surface of the sphere draw tangent planes.

What will these planes do to the cube? Then how will the new circumscribing solid compare with the cube?

What will the edges made by these tangent planes form?

Pass planes through these edges and the center of the sphere as before. What will now be formed? How find the volume of the sum of all these pyramids?

Now call the volume of circumscribing solid V′ and its surface S′, and the volume of the sphere V and its surface S.

Now let the number of pyramids be indefinitely increased by passing tangent planes to the sphere at the points where the edges of the pyramids pierce the surface of the sphere.

What does $S' \doteq$? What does $S' \times \frac{1}{3} r \doteq$?

What does $V' \doteq$? Is $V' = S' \times \frac{1}{3} r$ true for any number of faces?

Can you draw the conclusion?

Write a statement expressing the volume of a sphere, and call it Prop. XXX.

Note.—Let the pupil inscribe a sphere in a pyramid and give a demonstration in full for finding the volume of a sphere. Try it again by starting with the sphere inscribed in an icosaedron.

695.

Show that $V = \frac{4}{3} \pi r^3$ or $\frac{1}{6} \pi D^3$.

Sug. Express S in terms of r.

696.

Cor. II. Denote the volumes of two spheres by v and v', and their radii by r and r', and their diameters by d and d'.

Show that $v : v' = r^3 : r'^3 = d^3 : d'^3$.

697.

Cor. III. Can you prove that the volume of a spherical pyramid equals the product of its base by one-third of the radius of the sphere?

698.

Cor. IV. Can you prove that the volume of a spherical sector equals the product of the zone which forms its base by one-third of the radius of the sphere?

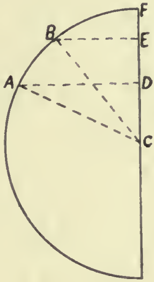
EXERCISE.

537. If r = radius of a sphere, C = circumference of a great circle, v = volume of a spherical sector, h and S = the altitude and area, respectively, of the corresponding zone, show that $v = \frac{2}{3} \pi r^2 h$.

699.

PROPOSITION XXXI.

Draw a semicircle. Take any arc, A B, and to the extremities draw radii A C, B C, and draw the ⊥s to the diameter, B E and A D.



Let this figure be revolved about the diameter F G.

What does C B E generate? A C B? A C D?

If we add the volume generated by C B E to that generated by A C B, and then deduct the volume generated by A C D, what solid will remain?

Can you show how to find the volume of a spherical segment?

Sug. 1. Call the radius of the sphere r, radius of upper base of segment r', radius of lower base r'', altitude of segment D E, h, the volume of the segment v.

Sug. 2. Find an expression for the cone generated by C B E.

What represents the volume of the sector generated by A C B?

Find an expression for the cone generated by A D C.

Sug. 3. Find the value of $\overline{\mathrm{CE}}^2$ in terms of r and r', $h = \mathrm{EC} - \mathrm{DC}$.

Find the value of $\overline{\mathrm{DC}}^2$ in terms of r and r''.

Can you show that

(1) $v = \frac{1}{3}\pi\,[2\,r^2(\mathrm{CE} - \mathrm{CD}) + (r^2 - \overline{\mathrm{CE}}^2)\,\mathrm{CE} - (r^2 - \overline{\mathrm{CD}}^2)\,\mathrm{CD}]$?

(2) $v = \frac{1}{3}\pi\,[2\,r^2(\mathrm{CE} - \mathrm{CD}) + r^2(\mathrm{CE} - \mathrm{CD}) - (\overline{\mathrm{CE}}^3 - \overline{\mathrm{CD}}^3)]$?

(3) $v = \frac{1}{3}\pi h\,[3\,r^2 - (\overline{\mathrm{CE}}^2 + \mathrm{CE}\cdot\mathrm{CD} + \overline{\mathrm{CD}}^2)]$?

(4) $(\mathrm{CE} - \mathrm{CD})^2 = \overline{\mathrm{CE}}^2 = 2\,\mathrm{CE}\cdot\mathrm{CD} + \overline{\mathrm{CD}}^2 = h^2$?

(5) $3\,\overline{\mathrm{CE}}^2 + 3\,\overline{\mathrm{CD}}^2 = h^2 + 2\,\overline{\mathrm{CE}}^2 + 2\,\mathrm{CE}\cdot\mathrm{CD} + 2\,\overline{\mathrm{CD}}^2$?

(6) $\overline{\mathrm{CE}}^2 + \mathrm{CE}\cdot\mathrm{CD} + \overline{\mathrm{CD}}^2 = \frac{3}{2}(\overline{\mathrm{CE}}^2 + \overline{\mathrm{CD}}^2) - \frac{h}{2}$?

$$= 3\,r^2 - \tfrac{3}{2}(r'^2 + r''^2) - \frac{h^2}{2}?$$

(7) $\therefore$ from (3) $v = \frac{1}{3}\pi h\,[\frac{3}{2}(r'^2 + r''^2) + \frac{h^2}{2}]$.

(8) $v = \frac{1}{2}h(\pi r'^2 + \pi r''^2) + \frac{1}{6}\pi h^3$.

Review these steps carefully until you thoroughly understand this proposition.

Write Prop. XXXI.

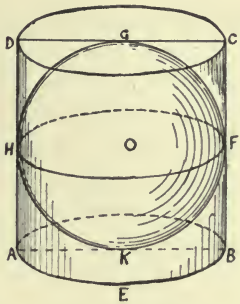
700.

Cor. If the segment be of one base, as that generated by F B E, show that

$v = \frac{1}{2}\pi r''^2 h + \frac{1}{6}\pi h^3)$.

701.

PROPOSITION XXXII.



Given: The cylinder A E B C D with the inscribed sphere H K F G. Call A K or O K, r, the radius of the sphere and the radius of the base of the cylinder.

Can you prove (1) that the surface of the cylinder : to surface of the sphere : : 3 : 2, and (2) that the volume of the cylinder : the volume of the sphere : : 3 : 2?

Sug. 1. Express the factors in terms of π and A D or in terms of π and r.

Note.—The celebrated geometer Archimedes discovered this interesting theorem. Read his biography.

702.

Cor. Suppose a cone to have the same base and altitude as the cylinder circumscribing the sphere.

Can you show that the cylinder : sphere : cone :: 3 : 2 : 1?

Sug. Express volume of cone in terms of π and r.

EXERCISES.

538. What is the convex surface of the largest cylinder that can be made from a cube whose edge is 14 feet?

539. A cube of steel weighs 9 pounds. What will be the largest bicycle cone that can be turned from it? 1 cubic inch of steel weighs 4.53 + ounces.

540. If the edge of a regular tetraedron is 4, can you show that the radius of the inscribed and circumscribed spheres equals $\frac{1}{3}\sqrt{6}$ and $\sqrt{6}$? Compare the volume of a cube inscribed in a sphere with that circumscribed about the sphere.

541. Let an equilateral triangle revolve about an altitude. Compare the convex surface of the cone generated with the surface of the sphere generated by the inscribed circle.

542. In Ex. 541 compare the volumes generated.

543. Given a cone the radius of whose base equals the radius of a sphere, and whose altitude equals the diameter of the sphere. Can you prove the volume to each other as 1 : 2?

544. On the same sphere, or on equal spheres, zones of equal altitudes are equal in area.

545. How many square feet in a spherical triangle whose angles are 200°, 156°, 95°, the radius of the sphere being 15 inches?

546. How many sheets of tin 20 inches by 28 inches are required to cover a globe 32 inches in diameter.

547. Compare the volume of the moon with that of the earth, assuming the diameter of the moon to be 2,000 miles and that of the earth 8,000 miles.

548. What is the cost of cementing the bottom and curved surface of a cylindrical cistern 10 feet deep and 8 feet in diameter, at 20 cents per square yard?

549. What is the ratio of the surface of a sphere to the *entire* surface of its hemisphere?

550. From Ex. 547 compare the amount of light reflected to a given point in space equally distant from both the earth and moon.

551. Prove how the areas of two zones on the same or equal spheres are related.

552. Let r' and r'' be the radii of two spheres. How are they related?

553. The altitude of two zones on a given sphere are 3 inches and 8 inches. What is the ratio of their surfaces?

554. What is the polar of a trirectangular triangle?

555. A spherical triangle is to the surface of a sphere as the spherical excess is to eight right angles.

556. All triangles on the same or equal spheres having equal angle-sums are equivalent.

557. What is the volume of a spherical segment of one base, whose altitude is 6 cm. and the radius of whose sphere is 20 cm.

558. Compare the surface of a sphere of diameter d with the convex surface of a circumscribed cylinder.

559. A circular sector has its central angle 30° and radius 12 dm. If this sector is revolved about a diameter perpendicular to one of its radii, find the volume generated.

FORMULAS FROM PREVIOUS PROPOSITIONS.

b = base.
r = radius.
c = circumference.
r' = radius upper base.
r'' = radius lower base.
s = slant height.
d = diameter
h = altitude.
a = apothem.
S = surface.
p = perimeter.
A = area.
a = arc.
v = volume.
E = spherical excess.
T = trirectangular triangle.

Polygons:

Rectangle, $A = b\,h$. § 251.
Parallelogram, $A = b\,h$. § 260.
Triangle, $A = \frac{1}{2}\,b\,h$. § 261.
Trapezoid, $A = \frac{1}{2}\,(b + b')\,h$. § 265.
Regular polygons, $A = \frac{1}{2}\,a\,p$. § 343.
Circle, $C = 2\,\pi\,r = \pi\,d$. § 360.
$A = \frac{1}{2}\,c\,r = \pi\,r^2 = \frac{1}{4}\,\pi\,d^2$. § 362.

Polyedrons.

Prism, $v = b\,h$. § 492.
Lateral, $A = p\,h$. § 473.

Parallelopiped, $v = b\,h$. § 488.
Pyramid, $v = \frac{1}{3}\,b\,h$. § 515.
Lateral, $A = \frac{1}{2}\,p\,s$. § 501.
Frustum of pyramid, $v = \frac{1}{3}\,h\,(b + b' + \sqrt{b\,b'})$. § 519.
Lateral area, $A = \frac{1}{2}\,s\,(p + p')$. Ex. 468.
Cylinder, $v = \pi\,r^2\,h$. § 571.
Lateral area, $A = 2\,\pi\,r\,h$. § 567.
Entire area, $A = 2\,\pi\,r\,(r + h)$. Ex. 497.
Right circular cone, $v = \frac{1}{3}\,\pi\,r^2\,h$. § 600.
Lateral area, $A = \pi\,r\,s$. § 596.
Frustum, $v = \frac{1}{3}\,\pi\,h\,(r^2 + r'^2 + r\,r')$ § 602.
Lateral area, $A = \pi\,s\,(r + r')$. § 602.
Sphere, $A = c\,d = 4\,\pi\,r^2 = \pi\,d^2$. §§ 687, 689.
$v = \frac{1}{3}\,r\,A = \frac{4}{3}\,\pi\,r^3 = \frac{1}{6}\,\pi\,d^3$. § 694.
Lune, $A = 2\,a\,T$. § 678.
Spherical triangle, $A = E \cdot T$.
Spherical polygon, $A = E \cdot T$.
Zone, $A = 2\,\pi\,r\,h$.
Spherical sector, $v = \frac{2}{3}\,\pi\,r^2\,h$. § 697.
Spherical segment, $v = \frac{1}{2}\,\pi\,h\,(r' + r'') + \frac{1}{6}\,\pi\,h^3$.

USEFUL NUMERICAL RESULTS.

$\sqrt{2} = 1.41421$.	$\sqrt[3]{2} = 1.25992$.
$\sqrt{3} = 1.73205$.	$\sqrt[3]{3} = 1.44224$.
$\sqrt{5} = 2.23606$.	$\sqrt[3]{4} = 1.5874$.
$\sqrt{6} = 2.4495$.	$\sqrt[3]{5} = 1.70998$.
$\sqrt{7} = 2.64575$.	$\sqrt[3]{7} = 1.91293$.
$\sqrt{10} = 3.1623$.	$\sqrt[3]{9} = 2.0801$.
$\sqrt{\frac{1}{2}} = 0.7071$.	$\sqrt[3]{10} = 2.1544$.

METRIC MEASURES.

Linear Measure.

10 millimeters, marked mm.,	are	1 centimeter,	marked	cm.
10 centimeters	"	1 decimeter,	"	dm.
10 decimeters	"	1 meter,	"	m.
10 meters	"	1 dekameter,	"	Dm.
10 dekameters	"	1 hektometer,	"	Hm.
10 hektometers	"	1 kilometer,	"	Km.
10 kilometers	"	1 myriameter,	"	Mm.

Rem.—The measures chiefly used are the meter and kilometer. The meter, like the yard, is used in measuring cloth and short distances; the kilometer is used in measuring long distances.

1 m. = 39.37 in.
1 km. = .6214 mi.

Square Measure.

100 sq. decimeters = $\left\{\begin{array}{l}\text{1 sq. meter.}\\ \text{1 centare (ca.).}\end{array}\right.$
100 centares = 1 are (a.).
100 ares = 1 hektare (Ha.).
1 sq. m. = 1.196 sq. rds.
1 are = 3.954 sq. yds.
1 Ha. = 2.471 acres.

Volume.

1000 cu. mm. = 1 cu. cm.
1000 cu. cm. = 1 cu. dm.
1000 cu. dm. = 1 cu. m. = 1 stere.
1 cu. cm. = .061 cu. in.
1 cu. m. = 1.308 cu. yd.
1 stere = .2759 cord.

Capacity Table.

10 centiliters, marked cl.,	are	1 deciliter,	marked	dl.
10 deciliters	"	1 liter,	"	l.
10 liters	"	1 dekaliter,	"	Dl.
10 dekaliters	"	1 hektoliter,	"	Hl.

1 liter = 1 cu. dm. = 1.0567 qt.

MEASURES OF WEIGHT.

The gram is the unit of weight; it is legal at 15.432 grains.

Table.

10 milligrams, marked mg.,	are	1 centigram,	marked	cg.
10 centigrams	"	1 decigram,	"	dg.
10 decigrams	"	1 gram,	"	g.
10 grams	"	1 dekagram,	"	Dg.
10 dekagrams	"	1 hektogram,	"	Hg.
10 hektograms	"	1 kilogram,	"	Kg.
10 kilograms	"	1 myriagram,	"	Mg.
10 myriagrams	"	1 quintal,	"	Q.
10 quintals, or 1000 kilograms,	"	1 metric ton,	"	M. T.

1 gram = 15.432 grains.
1 Kg. = 2.2046 lbs.
1 tonneau = 1.1023 tons.

ENGLISH MEASURES.

DRY MEASURE.

Dry measure is used in measuring grain, vegetables, fruit, coal, etc.

Table.

2 pints (pt.)	make	1 quart,	marked	qt.
8 quarts	"	1 peck,	"	pk.
4 pecks	"	1 bushel,	"	bu.

Rem.—The *standard unit* of dry measure is the; bushel it is a cylindrical measure $18\frac{1}{2}$ inches in diameter, 8 inches deep, and contains $2150\frac{2}{5}$ cubic inches.

1 bu. = .3524 Hl.
1 dry gallon = 4.404 liters.

LIQUID MEASURE.

Liquid measure is used for measuring all liquids.

Table.

4 gills (gi.)	make	1 pint,	marked	pt.
2 pints	"	1 quart,	"	qt.
4 quarts	"	1 gallon,	"	gal.

Rem.—The *standard unit* of liquid measure is the *gallon*, which contains 231 cubic inches, = 3.785 liters.

Comparative Table of Measures of Capacity.

	Liquid Measure.	Dry Measure.
1 gallon	= 231 cu. in.	$268\frac{4}{5}$ cu. in.
1 quart	= $57\frac{3}{4}$ cu. in.	$67\frac{1}{5}$ cu. in.
1 pint	= $28\frac{7}{8}$ cu. in.	$33\frac{3}{5}$ cu. in.

Measures of Weight.

Troy weight is used in weighing gold, silver and jewels.

24 grains (gr.)	make	1 pennyweight,	marked	pwt.
20 pennyweights	"	1 ounce,	"	oz.
12 ounces	"	1 pound,	"	lb.

The *standard unit* of all weight in the United States is the *Troy pound*, containing 5760 grains.

Avoirdupois Weight.

16 ounces (oz.)	= 1 pound,	marked	lb.
100 pounds	= 1 hundredweight,	"	cwt.
20 hundredweight	= 1 ton,	"	T.

1 lb. = 7000 grains = .4536 kilo.

1 T. = .9071 tonneau.

Long Measure.

Long measure is used it measuring distances, or length, in any direction.

Table.

12 inches (in.)	make	1 foot,	marked	ft.
3 feet	"	1 yard,	"	yd.
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet,	"	1 rod,	"	rd.
320 rods	"	1 mile	"	mi.

Rem.—The standard unit of length is the *yard*. The standard yard for the United States is preserved at Washington. A copy of this standard is kept at each State capital.

1 yard = .9144 m.

1 mile = 1.6093 Km.

Square Measure.

144	square inches	make	1 square foot,	marked	sq. ft.
9	square feet	"	1 square yard,	"	sq. yd.
$30\frac{1}{4}$	square yards	"	1 square rod,	"	sq. rd.
160	square rods	"	1 acre,	"	A.
640	acres	"	1 square mile,	"	sq. mi.

1 square yard = .8361 sq. m.
1 A. = .4047 hectare.

Cubic Measure.

1728 cubic inches (cu in.)	make	1 cubic foot,	marked	cu. ft.
27 cubic feet	"	1 cubic yard,	"	cu. yd.
128 cubic feet ($8 \times 4 \times 4$), 8 ft. long, 4 ft. wide, and 4 ft. high, make		1 cord,	"	C.

Rem.—A cord foot is 1 foot in length of the pile which makes a cord. It is 4 feet wide, 4 feet high, and 1 foot long; hence it contains 16 cubic feet, and 8 cord feet make 1 cord.

1 cu. yd. = .7646 cu. m.
1 C = 3.625 steres.

Circular Measure.

60 seconds (″)	make	1 minute,	marked	′.
60 minutes	"	1 degree,	"	°.
360 degrees	"	1 circle.		

Rem.—The circumference is also divided into *quadrants* of 90 degrees each, and into *signs* of 30 degrees each.

Note.—1. A degree at the equator, also the average degree of latitude, is equal to 69.16 statute miles.

2. Minutes on the earth's surface are called geographic miles.

A BRIEF ACCOUNT OF GEOMETRY AMONG THE BABYLONIANS, EGYPTIANS, AND GRECIANS.

Nearly all of the ancient nations leave some traces of a knowledge of the simpler notions of geometry. It is said that besides the knowledge of the division of the circumference into six equal parts by the radius, the Babylonians knew something of the properties of the triangle and quadrangle. Like the Hebrews (I. Kings vii. 23), they took $\pi = 3$. There are evidences that the Babylonians knew something of astronomy.

Herodotus and other ancient historians state that geometry had its origin in Egypt. The measurement of land and the building of the pyramids are referred to by some of the ancient historians as evidences of their knowing some practical geometry.

Before 1700 B. C., there was a mathematical manual, called the *Papyrus*, containing problems in arithmetic and geometry. It was written by *Ahmes*. The Egyptians knew how to compute the area of an isosceles triangle and also that of an isosceles trapezoid. Their value of π was 3.1604.

It is probable that the Egyptians knew how to construct a right triangle with the lines whose ratios are 3 : 4 : 5. Later geometricans proved that some of the rules used by he Egyptians in their constructions were not absolutely correct. They did not have a system of geometry based on a few axioms and postulates.

About 700 B. C., an active commercial intercourse sprang up between Greece and Egypt, and as a result an interchange of ideas arose. Thales, Pythagoras, Plato, and many other philosophers visited Egypt to study the learning of that nation.

Many things in Greek culture originally came from the land of the pyramids. The Grecians radically changed Egyptian geometry and the subject began to take more of the form of a science.

Thales is supposed to have created the geometry of lines in an abstract character. He formulated many of the truths which the Egyptians were acquainted with. Thus, Eudemus ascribes to Thales the theorems on the equality of vertical angles, equality of the angles at the base of an isosceles triangle, the bisection of a circle by any diameter, and the equality of any two triangles having a side and any two adjacent angles equal respectively.

Thales was one of the earliest geometers to apply theoretical geometry to practical uses. It is said he measured the distances of ships from shore by geometry As a result of his study of geometry he *calculated eclipses.*

Among other Grecians who added to the science of geometry are Pythagoras, Hippocrates, and Anaxagoras. For about 400 years, 650 B. C. to 250 B. C., the science of geometry had many of its principles worked out. It is called the golden age of geometry. The great body of the propositions in plane geometry is about as it was formulated by Euclid. Many improvements in methods have been made in more recent times, such as the study of loci, collinearity, concurrence, and other subjects in what is called Modern Geometry. These subjects are studied in colleges and universities.

Brief Biographies.

Note.—Nearly all the persons mentioned in this table have their names connected in some way with important principles in geometry. The numbers refer to the particular section in which the person mentioned has made some discovery or improvement. *b* stands for born, *d* for died, and *c* (circa) about.

Ahmes, *c* 1700 B. C., an Egyptian priest, was one of the earliest writers on mathematics. His work is called "Directions for Knowing All Dark Things." It is a collection of problems in arithmetic and geometry. He gives the answers to the problems, but usually does not show the processes used to obtain them.

In one arithmetical problem he states that the sum of

$$\frac{1}{24}, \frac{1}{58}, \frac{1}{174}, \text{ and } \frac{1}{232} \text{ is } \frac{2}{29}.$$

He had some idea of algebra and always used for the unknown quantity the symbol meaning a heap. He represented addition by a pair of legs walking forward, and subtraction by a pair of legs walking backward, or by a flight of arrows. Equality he represented by the sign ⁄∟.

In the part treating of geometry he gives the contents of barns; but since he does not give the shape of the Egyptian barn, we cannot verify his results. He finds the areas of rectilinear figures and of a circular field of diameter 12. The value of π he approximates closely 3.1416.

[See "A Short History of Mathematics," by W. W. R. Ball, pp. 3 to 8.]

Anaxagoras, *c* 450 B. C., tried to find the side of a square which should equal that of a given circle. Anaxagoras belonged to the Pythagorean school of philosophy.

Archimedes, *c* 290—215 B. C. Born at Syracuse, educated at Alexandria, where the famous Euclid had attended fifty years before. Archimedes was one of the earliest writers on measuring the circle. He proves that the area of a circle equals that of a right triangle having the circumference for the length of its base, and the radius for its altitude. He assumes that there is a straight line equal in length to the circumference. He showed that this line exceeds three times the diameter by a part which is less than $\frac{1}{7}$ but more than $\frac{10}{71}$ of the diameter. To quote Ball's "History of Mathematics":

"In the old and mediæval world Archimedes was unanimously reckoned as the first of mathematicians; and in the modern world there is no one but Newton who can be compared with him."

His mechanical ingenuity was astonishing. He invented the Archimedian screw, used to pump water out of the hold of a ship and to drain the fields inundated by the Nile. He invented engines of war to fight the Romans. A story is told of his burning-glass, which consisted of concave mirrors, a hexagon surrounded by 24-sided polygons; this he used to set the Roman ships on fire.

Euclid wrote systematic treatises, while Archimedes wrote brilliant essays addressed to famous mathematicians of his day. On some of these he played practical jokes by misstating the results, "to deceive those vain geometricians who say they have found everything, but never give their proofs."

On Plane Geometry Archimedes wrote three works, viz.: (1) ' The Measure of *the Circle*"; (2) "The Quadrature of the Parabola"; (3) "*Spirals*."

On Solid Geometry he wrote: (1) "The Sphere and the Cylinder"; (2) "The Conoids and Spheroids." He wrote a treatise on the thirteen *semi-regular polyedrons*, solids bounded by regular but dissimilar *polygons*.

He wrote a treatise on Arithmetic in which he calculates the number of grains of sand required to fill the universe is less than 10^{63}.

He wrote a treatise on Mechanics; another on Levers, in which he declared he could move the whole earth if he had a fixed fulcrum.

He wrote a work on *floating bodies*, and while bathing discovered a method to prove that the crown of Hiero was not pure gold.

When Syracuse was taken, he was killed by a Roman soldier. It is said that he was contemplating a geometrical figure drawn on the floor in the sand when the soldier entered and was ordered off the figure by Archimedes, who was afraid he would spoil it. The Roman general, Marcellus, did not desire the death of Archimedes, and had given orders to spare his house and person.

The Romans built a splendid monument over his grave, upon which, according to his wish, was engraved a sphere inscribed in a cylinder in memory of his discovery that the inscribed sphere is two-thirds of the cylinder, and that the area of the surface of the sphere equals the area of four great circles. [§§ 123, 701.]

Descartes (1596—1650) was a French philosopher as well as a great mathematician. Descartes was among the first to apply algebra to geometry. His great work was founding the science of Analytic Geometry.

Euclid, *c* 300 B. C. Euclid's greatest activity was in the reign of the first Ptolemy, 306—283 B. C. Euclid was a student of the Platonian philosophy, and an eminent writer on geometry. He gave to the world the first scientific text-book, "Elements," in thirteen books, which is still a standard text-book in many English schools. Ptolemy once asked Euclid if geometry could not be mastered by an easier process than by studying his "Elements." The answer was, "There is no royal road to geometry." § 225 is the 47th Prop. in Euclid's "Elements," and is often referred to as the 47th of Euclid.

Euler, *b* 1707, *d* 1783, one of the greatest of modern mathematicians. He was a Swiss. He solved in three days mathematical problems which eminent mathematicians had said would require months. His intense study caused him to go blind and he dictated his "*Elements of Algebra*" to his servant, who was quite ignorant of mathematics. This work is still considered one of the best of its class. Besides making many discoveries in geometry, he did much work in higher mathematics and astronomy. [§§ 277, 540.]

Hero, *c* 190 B. C. He was a practical surveyor of Alexandria and made many mechanical inventions, among them being an instrument resembling a modern theodolite. This mathematician wrote a commentary on Euclid's "Elements."

Among other formulas, he developed the one which expresses the area of a triangle in terms of its sides.

Johannes Kepler, *b* 1571, *d* 1630, was a great student of science and many publications are from his pen. He has enriched pure mathematics as well as astronomy.

He was one of the first great mathematicians to make extensive use of logarithms. He conceived the circle to be composed of an infinite number of triangles with their common vertices at the center and their bases in the circumference, and the sphere to consist of an infinite number of pyramids. He made a study of the ellipse, parabola, and hyperbola; he gave to the world his laws concerning the movements of heavenly bodies.

Legendre (le zhondr), *b* 1752, *d* 1833, one of the most eminent of modern mathematicians. Besides many valuable additions to higher mathematics, he gave to the world one of the most celebrated works on geometry. It is called the "Eléments de Géometrie." [§ .05.]

Sir Isaac Newton (1642—1727) was a great English philosopher and mathematician. Newton was a student of geometry, but his works are on higher mathematics, and applications of algebra and geometry to the solution of astronomical problems. He discovered the *Binomial Theorem.*

Blaise Pascal, *b* 1623, *d* 1662, lived in Paris, where his father taught him. The father would not trust his son's education to others. Blaise Pascal's genius in geometry showed itself when he was only twelve years old. The father tried to keep mathematical work from his son till he had learned Latin and Greek, but with charcoal and paving tiles the boy studied the methods of drawing the circle and the equilateral triangle.

He discovered for himself the sum of the three angles of a triangle. Pascal's genius was so great for geometry that at sixteen he wrote a treatise on conics which had not been equaled since the time of Archimedes. Pascal was a great student in other subjects.

Plato, who lived about 400 B. C., was the founder of a school of philosophy. Plato studied mathematics and gave the *analytic* method of attacking a proposition in geometry. The Platonic Bodies are so named because of the study given them in Plato's school. [§ 456]

Pythagoras, *c* 580—500 B. C. To Pythagoras is attributed the important theorem that the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides. He probably learned the special truth when the sides are 3, 4, 5, respectively, from the Egyptians, and then developed the general truth. The Pythagorean school of mathematics taught that the plane about a point is completely filled by six equilateral triangles, four squares, or three regular hexagons. Pythagoras called the circle the most beautiful of all plane figures and the sphere the most beautiful of all solid figures. The star-shaped pentagram was used as a sign of recognition by the Pythagoreans, and was called by them Health.

INDEX.

References are to Pages.