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PLANE AND SOLID

GEOMETRY

BY

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PREFACE

It is generally conceded that geometry is the most interesting of all the mathematical sciences, yet many students have failed to find either pleasure or profit in studying it. The most serious hindrance to the proper understanding of the subject has been the failure on the part of the student to grasp the geometrical concept which he has been endeavoring to establish by a process of reasoning. Many attempts have been made by thorough teachers to remedy the difficulty, but there is a very general agreement that the most successful method has been by exercises in "Inventional" Geometry. Students who have been fortunate enough to have the subject presented in that way have usually understood it, and, better still, they have enjoyed it.

While Inventional Geometry has been full of interest to the student, it has often failed to develop that knowledge of the science which is necessary to thorough mastery, because it has not been progressive, and, what is more to be deplored, it has failed to give that acquaintance with the forms of rigid deductive reasoning which is one of the chief objects sought in the study of the science. The student has often been led by this objective method of study to rely upon his visual recognition of the relations of lines and angles in a drawing rather than upon the demonstration based upon definitions, axioms, and propositions that have been proved.

In this book the effort is made to introduce the student to geometry through the employment of inventional steps, but the somewhat fragmentary and unsatisfactory result of such teaching is supplemented by demonstrations, in consecutive order, of the fundamental propositions of the science. The desirability of training students to form proper inferences from the study of accurately drawn figures has been recognized by the author; such a method awakens keen interest in the subject and develops right habits of investigation, but there is necessity also for the accuracy of statement and the logical training of the older methods to assure the pleasure and profit that belong with both.

Every theorem has been introduced by questions designed to lead the student to discover the geometrical concept clearly and fully before a demonstration is attempted. They are not intended to lead to a demonstration, but rather to a correct and definite idea of what is to be proved.

Many of the exercises at the foot of the page require the student to infer the truth involved in the relations given. The interrogative form is employed for the purpose of compelling the student to obtain the ideas for himself, and the answers he must give to the questions furnish an admirable training in accuracy of expression.

A great abundance of undemonstrated theorems and of unsolved problems is supplied, and teachers will find them quite numerous enough for the needs of any class. The demonstration of original theorems and the solution of original problems are of so great consequence in developing the power to reason that every teacher should insist upon such work.

Much aid in originating demonstrations may be obtained from the *Summaries* which follow each of the first six books. These summaries are not collections of propositions that have been demonstrated, but are rather groups of the truths established in the book to which they are appended. If the student makes himself thoroughly acquainted with them, much of the difficulty experienced in demonstrating original theorems, in solving problems, and in determining loci will be removed.

A very small proportion of those who study elementary geometry expect to become mathematicians in any broad sense of the term, and so geometry must serve to give them almost the only training they will get in formal and logical argument in secondary schools and in colleges. For this reason mathematical elegance in demonstrations and in solutions has often been sacrificed in the interest of clear and simple steps, even though such a plan has required some expansion of the text. Elegant demonstrations are appreciated by mathematicians, but training in formal deductive reasoning is of more consequence to most students.

The author is indebted to many authors, both American and foreign, who have preceded him. Their efforts to present the subject in the best way have aided him very greatly in preparing this work. He has selected large numbers of supplementary theorems and problems from several European authors of renown, and yet he is unable to give credit to any author in particular, because they all seem to have selected their exercises from some common source of supply.

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SUGGESTIONS TO TEACHERS

1. Thorough teaching and frequent reviews, especially at the beginning of plane and of solid geometry, will be rewarded by intelligent progress and deep interest on the part of the students.

2. Before the assignment of any lesson, the teacher should require the students to draw the figures and answer the questions which are introductory to the propositions that are to be proved at the next lesson.

After the questions have been answered, require the students to express their inferences in the form of a theorem.

3. While the students are answering the introductory questions or stating the inferences suggested by the exercises at the bottom of the page, the inquiry, "How do you know that this is true?" will often lead to a demonstration.

4. The section numbers are convenient in written demonstrations, but in oral proofs the reason for each step should be given fully and accurately and all why's should be answered.

5. Students may sometimes be allowed to express definitions, axioms, theorems, etc., in their own language, but as a general rule their expressions are inaccurate and faulty. The teacher should in such instances call attention to the errors and require concise and accurate statements. It will then be discovered that they approximate very closely those given in the book.

6. The practice of requiring the students to outline, in a general way, the steps they are to take in establishing the truth of a proposition will develop much logical power and cause them to look at the argument rather than at its details.

The following are suggestive *outlines of steps:*

Prop. XXXVI., page 61.

1. Draw the diagonal AC.
2. Prove $\triangle ABC$ and ADC equal.
3. Prove $AB \parallel DC$ and $AD \parallel BC$.

   Then, $ABCD$ is a parallelogram.

Prop. XII., page 185.

1. Make the required construction, drawing *CE*, *BJ*, and *CK*.
2. Prove $\triangle$ *AEC* and *AJB* equal.
3. Prove $AEKL \approx 2 \triangle AEC$, and $ACHJ \approx 2 \triangle AJB$.
4. Prove $AEKL \approx ACHJ$.
5. Similarly $BDKL \approx BCGF$.
   Then, $ABDE \approx BCGF + ACHJ$.

7. Demonstrations should never be memorized. If suggestion 6 is observed carefully, students will not be likely to commit to memory the words of the book.

8. Encourage students to prove propositions in their own way, even though the proofs be less elegant than those which are given. Elegant methods will be acquired by practice.

9. Written demonstrations should be required frequently. They serve a double purpose, viz.: they train the eye and develop accuracy in reasoning.

All written work should be done neatly, and all figures should be drawn as accurately as possible.

10. The undemonstrated theorems and unsolved problems are probably more numerous than most classes can prove or solve in the time allotted to the subject, consequently teachers are expected to make selections from the lists given. The exercises are carefully graded so that the more difficult ones come at the end of each list. These may be omitted at the first reading and reserved for a final review.

It is suggested that the exercises in the interrogative form at the foot of the page in Books I and II, except the numerical ones, be employed at first only for the purpose of developing correct geometrical concepts and accuracy in expressing the truth inferred. In review the proofs of the inferences may be required.

11. Particular attention should be given to the *Summary* at the end of each book. The students should be required to state all the conditions under which the facts given in black-faced type have been shown to be true. They will thus have at immediate command all the facts which can be employed in the demonstration they are attempting.

If the demonstration of the inferences and theorems found at the bottom of the page is required, the students should be referred to the summary. They should understand, however, that they can use no truth given in the summary whose section number indicates that it was established subsequently to the point in the text where the proposition or exercise is found.

The method of using the summaries is illustrated upon page 78.

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GEOMETRY

PRELIMINARY DEFINITIONS

1. Every material object occupies a limited portion of space and is called a **Physical Solid** or **Body.**

2. The portion of space occupied by a physical solid is identical in form and in extent with that solid, and is called a **Geometrical Solid.**

In this work only *geometrical solids* are considered, and for brevity they are called simply *solids.*

3. Any limited portion of space is called a **Solid.**

A solid has three dimensions, *length, breadth,* and *thickness.*

The drawing in the margin is represented as having three dimensions.

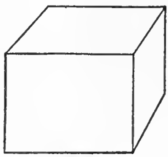


FIG. 1.

4. The limit of a solid, or the boundary which separates it from all surrounding space, is called a **Surface.**

A surface has only two dimensions, *length* and *breadth.*

A page of a book is a *surface*, but a leaf of a book is a *solid.*

5. The limit or boundary of a surface is called a **Line.**

A line has only *one* dimension, *length.* It has neither breadth nor thickness.

The edges of a cube are *lines.*

6. The limits, or extremities, of a line are called **Points.**

A *point* has *position* only. It has neither length, breadth, nor thickness.

The dots and lines made by a pencil or crayon are not geometrical points and lines, but are convenient representations of them.

7. Lines, surfaces, and solids are called **Geometrical Magnitudes,** or simply **Magnitudes.**

8. A line may be conceived of as generated by a point in motion. Hence a line may be considered as independent of a surface, and it may be of unlimited extent.

A surface may be conceived of as generated by a line in motion. Hence a surface may be considered as independent of a solid, and it may be of unlimited extent.

A solid may be conceived of as generated by a surface in motion. Hence a solid may be considered as independent of a material object.

LINES AND SURFACES

9. 1. Select two points upon your paper and draw several lines connecting them.

a. Which is the shortest line you have drawn? If this line is not the shortest that can be drawn between the points, what kind of a line is the shortest?

b. What other kinds of lines have you drawn besides a straight line?

2. When a carpenter places a straightedge upon a board and moves it about over the surface, what is he endeavoring to determine regarding the surface?

3. If the straightedge does not touch every point of the surface of the board to which it is applied, what has been discovered about the surface?

4. How does he know whether or not the surface is an even or a *plane* surface?

5. If any two points on the surface of a ball or sphere are joined by a straight line, where does the line pass?

6. How much of the surface of a perfect sphere is a plane surface?

10. A line which has the same direction throughout its whole extent is called a **Straight Line.**

A straight line is also called a *Right Line*, or simply a *Line*.

In this work the term "line" means a *straight line* unless otherwise specified.

11. A line no part of which is straight is called a **Curved Line.**

Consequently, a curved line changes its direction at every point.

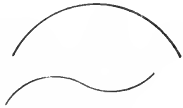


Fig. 2.

12. A line made up of several straight lines which have different directions is called a **Broken Line.**

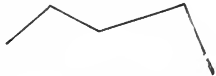


Fig. 3.

13. A line made up of straight and curved lines is called a **Mixed Line.**

Any portion of a line may be called a *segment* of that line.

Fig. 4.

14. A surface such that a straight line joining any two of its points lies wholly in the surface is called a **Plane Surface, or a Plane.**

15. A surface, no part of which is plane, is called a **Curved Surface.**

16. Any combination of points, lines, surfaces, or solids is called a **Geometrical Figure.**

A *geometrical figure* is ideal, but it can be represented to the eye by drawings or objects.

17. A figure formed by points and lines in the same plane is called a **Plane Figure.**

18. A figure formed by straight or right lines only is called a **Rectilinear Figure.**

19. The science which treats of points, lines, surfaces, and solids, and of the properties, construction, and measurement of geometrical figures, is called **Geometry.**

20. That portion of geometry which treats of plane figures is called **Plane Geometry.**

21. That portion of geometry which treats of figures whose points and lines do not all lie in the same plane is called **Solid Geometry.**

ANGLES

22. 1. From any point draw two straight lines in different directions. Draw two straight lines from each of several other points, and thus form several *angles.*

2. How does the angle at the corner of this page compare in size with the angle at the corner of the room? Show your answer to be true by an actual test.

How is the size of any angle affected by the length of the lines which form its *sides?*

3. Form several angles at the same point; that is, several angles having a *common vertex.*

4. How many of them have a common vertex and one common side between them and are, at the same time, on opposite sides of the common side; that is, how many angles are *adjacent angles?*

5. Draw a straight line meeting another straight line so as to form two equal adjacent angles; that is, two *right angles.*

6. Draw from a point or vertex two straight lines in opposite directions; that is, form a *straight angle.* How does a straight angle compare in size with a right angle?

7. Draw several angles, some greater and some less, than a right angle.

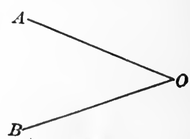
8. Draw a right angle and divide it into two parts, or into two *complementary angles.*

9. Draw a straight angle and divide it into two parts, or into two *supplementary angles.*

10. Draw two straight lines crossing or intersecting each other, thus forming two pairs of opposite or *vertical angles.*

23. The difference in direction of two lines which meet is called a **Plane Angle,** or simply an **Angle.**

The lines are called the *sides* of the angle, and the point where they meet is called its *vertex.*



The lines OA and OB are the *sides* of the *angle* formed at the point O, and O is the *vertex* of the angle.

FIG. 5.

The size of an angle does not depend upon the length of its sides, but upon the divergence of the sides or upon the opening between them. Compare Figs. 5 and 6.

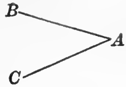


FIG. 6.

24. When there is but one angle at a point, it may be designated by the single letter at the vertex, or by three letters.

In Fig. 6 the angle may be called the angle A, or the angle BAC, or the angle CAB.

When several angles have a common vertex, it is customary to use three letters in designating each, placing the letter at the vertex between the other two.

An angle is sometimes designated by a figure or small letter placed in the opening of the angle.

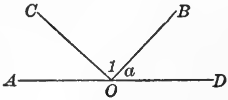


FIG. 7.

The angles formed by the lines meeting at O may be designated by AOC, the figure 1, and the small letter a.

25. Angles which have a common vertex and a common side, and which are upon opposite sides of the common side, are called **Adjacent Angles.**

In Fig. 7 angles COA and COB are *adjacent angles*, having a common vertex O, and a common side CO and lying upon opposite sides of the common side. Also COB and BOD are *adjacent angles.*

26. When one straight line meets another straight line so as to form two equal adjacent angles, each of the angles is called a **Right Angle**; and each line is said to be *perpendicular* to the other.

The sides of a right angle are therefore perpendicular to each other, and lines perpendicular to each other form right angles with each other.

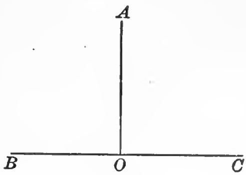


FIG. 8.

27. An angle whose sides extend in opposite directions from the vertex, thus forming one straight line, is called a **Straight Angle.**

If the sides OA and OB, Fig. 9, extend in opposite directions from the vertex O, the angle AOB is a *straight angle.*

FIG. 9.

A straight angle is equal to two right angles.

28. An angle less than a right angle is called an **Acute Angle.**

FIG. 10.

29. An angle greater than a right angle and less than a straight angle is called an **Obtuse Angle.**

FIG. 11.

30. An angle greater than a straight angle and less than two straight angles is called a **Reflex Angle.**

Acute, obtuse, and reflex angles are called *oblique angles* in distinction from right angles and straight angles.

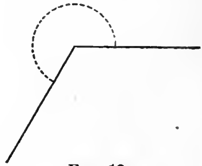


FIG. 12.

31. When two angles are together equal to a right angle, they are called **Complementary Angles,** and each is said to be the **Complement** of the other.

If the angle COE is a right angle, the angles COD and DOE are *complementary angles;* the angle COD is the *complement* of the angle DOE; and the angle DOE is the *complement* of the angle COD.

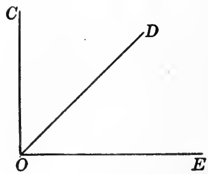


FIG. 13.

32. When two angles are together equal to two right angles, they are called **Supplementary Angles,** and each is said to be the **Supplement** of the other.

If the angles AOD and DOB are together equal to two right angles, the angles AOD and DOB are *supplementary angles;* the angle AOD is the *supplement* of the angle DOB, and the angle DOB is the *supplement* of the angle AOD.

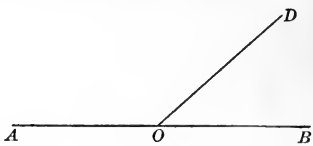


FIG. 14.

33. When two lines intersect, the opposite angles are called **Vertical Angles.**

The angles AOC and DOB, and the angles AOD and COB are *vertical angles.*

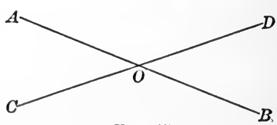


FIG. 15.

34. A line or a plane which divides any geometrical magnitude into two equal parts is called the **Bisector** of that magnitude.

MEASUREMENT OF ANGLES

35. To measure a magnitude is to find how many times it contains a certain other magnitude assumed as a unit of measure.

The unit of measure for angles is sometimes a *right angle,* but very often it is a *degree.*

Suppose the line OB, having one of its extremities fixed at O, moves from a position coincident with OA to the position indicated by OB. By this motion the angle AOB has been generated.

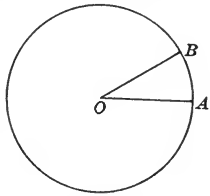


FIG. 16.

When the rotating line OB has passed one half the distance from OA around to OA, the lines extend in opposite directions from O, and a *straight angle* has been generated; and since a straight angle is equal to two right angles (§ 27), when the line has passed one fourth of the distance around to OA, a *right angle* has been generated, and the lines OB and OA are perpendicular to each other (§ 26). When the line has rotated entirely around from OA to OA, it has generated *two straight angles,* or *four right angles.* Consequently: *The total angular magnitude about a point in a plane is equal to four right angles.*

Inasmuch as it is frequently convenient to employ a smaller unit of angular measure than a right angle, the entire angular magnitude about a point has been divided into 360 equal parts, called *degrees;* a degree into 60 equal parts, called *minutes;* a minute into 60 equal parts, called *seconds.*

Degrees, minutes, and seconds are indicated in connection with numbers by the respective symbols °, ′, ″.

25 degrees, 18 minutes, 34 seconds is written 25° 18′ 34″.

A right angle is an angle of 90°.

EQUALITY OF GEOMETRICAL MAGNITUDES

36 Geometrical magnitudes which coincide exactly when one is placed upon or applied to the other are equal. Since, however, geometrical magnitudes are ideal they are not actually taken up and placed the one upon the other, but this is conceived to be done.

This method of establishing equality is called the *Method of Superposition.*

If one straight line is conceived to be placed upon another straight line so that the extremities of both coincide, the lines are equal.

If an angle is conceived to be placed upon another angle so that their vertices coincide and their sides take the same directions, respectively, the angles are equal.

If any figure is conceived to be placed upon any other figure so that they coincide exactly throughout their whole extent, they are equal.

Figures that are superposable are sometimes called *congruent.*

EXERCISES

37. Draw as accurately as possible the figures which are suggested; study them carefully; infer the answers to the questions; state your inference or conclusion in as accurate form as possible; give the reason for your conclusion when you can.

The student is asked to represent by a drawing any figure that may be required so that it may simply appear to the eye to be accurate. Geometrical methods of construction are given at suitable points in the book, but they cannot be insisted upon at this stage.

1. Draw two straight lines intersecting in as many points as possible. In how many points do they intersect?

Inference: *Two straight lines can intersect in only one point.*

2. Draw a straight line; draw another meeting it. How does the sum of the adjacent angles thus formed compare with two right angles?

Inference: *When one straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.*

3. Draw a straight line; from any point in it draw several lines extending in different directions. How does the sum of the consecutive angles formed on one side of the given line compare with a right angle? With a straight angle?

4. Draw a straight line; also another meeting it so as to form two adjacent angles, one of which is an acute angle. What kind of an angle is the other?

5. Draw two intersecting lines. How many angles are formed? How do the opposite or vertical angles compare in size?

6. Draw two lines intersecting so as to form a right angle. How does each of the other angles formed compare with a right angle? How do right angles compare in size? How do straight angles compare in size?

7. Draw two equal angles. How do their complements compare? How do their supplements compare?

8. Draw a straight line; select any point in that line and draw as many perpendiculars as possible to the line at that point. How many such perpendiculars can be drawn on one side of the line?

DEMONSTRATION OR PROOF

38. The inferences which the student has just made are probably correct, but they must be proved to be true before they can be relied upon with certainty unless their truth is self-evident.

Many truths have been inferred, and used as the basis of important enterprises before they have been logically demonstrated.

Carpenters believe that their squares are true if a line from the 12-inch mark on one side to the 16-inch mark on the other is 20 inches long; but they may not be capable of giving satisfactory reasons for their convictions.

Many valuable facts of geometry may be inferred by observation of figures and objects, but the value of the study to a student consists not so much in the knowledge acquired as in the development of the logical faculty by the rigid course of reasoning required to prove the truth or falsity of the inference.

Much attention must therefore be given to the *demonstration* or *proof* of inferences from known data, and of statements even though they may seem to be true.

39. A course of reasoning which establishes the truth or falsity of a statement is called a **Demonstration,** or **Proof.**

40. A statement of something to be considered or done is called a **Proposition.**

"All men are mortal" and "It is required to bisect an angle" are *propositions.*

41. A proposition so elementary that its truth is self-evident is called an **Axiom.**

An axiom is a self-evident truth to those only who understand the terms employed in expressing it.

Axioms may be illustrated, but they do not require proof.

Axioms have often a general application. Some, however, apply only to geometrical magnitudes and relations.

"A whole is equal to the sum of all its parts" is a *general axiom.* It can be employed in demonstrating propositions in arithmetic and algebra as well as in geometry. "A straight line is the shortest distance between two points" is a *geometrical axiom.* It can be used only in proving propositions which express some geometrical truth.

42. A proposition which requires demonstration or proof is called a **Theorem.**

"In any proportion the product of the extremes is equal to the product of the means" is an algebraic *theorem.*

43. A theorem whose truth may be easily deduced from a preceding theorem is often attached to it, and called a **Corollary.**

The arithmetical theorem, "A number is divisible by 3 when the sum of its digits is divisible by 3" may be readily deduced from the theorem, "A number is divisible by 9 when the sum of its digits is divisible by 9," and may be attached to it as a *corollary.*

44. A proposition requiring something to be done is called a **Problem.**

"Construct an angle equal to a given angle" is a geometrical *problem.*

45. A problem so simple that its solution is admitted to be possible is called a **Postulate.**

"A straight line may be drawn from one point to another" is a *postulate.* Postulates are numerous. Some of those employed in geometry may be found in § 50.

46. A remark made upon one or more propositions, and showing, in a general way, their extension or limitations, their connection, or their use is called a **Scholium.**

Thus, after the processes of dividing a common fraction by a common fraction, and a decimal by a decimal, have been taught, a remark showing that precisely the same principles are involved in each process is a *scholium.*

47. The enunciation of a theorem may be separated into the following parts:

1. The things given, or granted, called the **Data** (singular *datum*).

2. A statement of what is to be proved, called the **Conclusion.**

The term *Hypothesis* may be used instead of the term *data.*

A supposition made in the course of a demonstration is also called an *Hypothesis.*

48. *In proofs, or demonstrations, only definitions, axioms, and propositions which have been proved can be employed to establish the truth of the proposition.*

AXIOMS

49. 1. *Things which are equal to the same thing are equal to each other.*

2. *If equals are added to equals, the sums are equal.*
3. *If equals are taken from equals, the remainders are equal.*
4. *If equals are added to unequals, the sums are unequal.*
5. *If equals are taken from unequals, the remainders are unequal.*
6. *Things which are doubles of equal things are equal.*
7. *Things which are halves of equal things are equal.*
8. *The whole is greater than any of its parts.*
9. *The whole is equal to the sum of all its parts.*
10. *A straight line is the shortest distance between two points.*
11. *If two straight lines coincide in two points, they will coincide throughout their whole extent, and form one and the same straight line.*
12. *Between the same two points but one straight line can be drawn.*

POSTULATES

50. 1. *A straight line may be produced indefinitely.*

2. *A straight line may be drawn from any point to any other point.*

3. *On the greater of two straight lines a part can be laid off equal to the less.*

4. *A figure can be moved unaltered to a new position.*

SYMBOLS

$+$	plus, or increased by.	$\triangle$	triangle.
$-$	minus, or diminished by.	$\triangle\!\!\!s$	triangles.
$\times$	multiplied by.	$\square$	parallelogram.
$\cdot$	multiplied by.	$\square\!\!\!s$	parallelograms.
$\div$	divided by.	$\odot$	circle.
$=$	equals, or is (or are) equal to.	Ⓢ	circles.
$\Leftrightarrow$	is (or are) equivalent to.	$\parallel$	parallel, or is (or are) parallel to.
$>$	is (or are) greater than.	$\parallel$s	parallels.
$<$	is (or are) less than.	$\perp$	perpendicular, or is (or are) perpendicular to.
$\therefore$	therefore, or hence.	$\perp$s	perpendiculars.
$\angle$	angle.	$''$	inch or inches.
$\angle$s	angles.		

ABBREVIATIONS

Adj.	adjacent.	N.	note.
Alt.	alternate.	Opp.	opposite.
Ax.	axiom.	Post.	postulate.
Circum.	circumference.	Prob.	problem.
Comp.	complement.	Pt.	point.
Const.	construction.	Rect.	rectangle.
Cor.	corollary.	Rt.	right.
Def.	definition.	Sch.	scholium.
Ex.	exercise.	Sect.	sector.
Ext.	exterior.	Seg.	segment.
Fig.	figure.	Sim.	similar.
Int.	interior.	St.	straight.
Lat. Surf.	lateral surface.	Sup.	supplement.

The letters Q.E.D. are placed at the end of a *proof;* they are the initial letters of the Latin words *quod erat demonstrandum*, meaning *which was to be proved.*

The letters Q.E.F. are placed at the end of a solution of a *problem* for *quod erat faciendum*, meaning *which was to be done.*

PLANE GEOMETRY

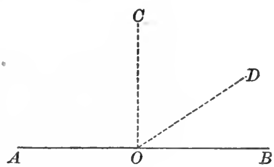
BOOK I

LINES AND RECTILINEAR FIGURES

Proposition I

51. Draw a straight line and as many perpendiculars as possible to the line at one point. How many can be drawn? (§ 37)

Theorem. *At any point in a straight line one perpendicular to the line can be drawn, and only one.*



Data: Any straight line, as AB, and any point in that line, as O.

To prove that a perpendicular to AB can be drawn at the point O, and that only one can be drawn.

Proof. Suppose a line DO to rotate about the point O as a pivot, from the position BO to AO.

As DO rotates from the position BO toward the position AO, the angle DOB will, at first, be smaller than the angle DOA.

As DO continues to rotate, the angle DOB will increase continuously, and will eventually become larger than angle DOA.

Therefore, since angle DOB is at first smaller than angle DOA, and afterwards larger than angle DOA, there must be one position of DO, as, for example, CO, in which the two angles are equal.

By § 26, CO is then perpendicular to AB.

Since there is but one position in which the line DO makes equal angles with the line AB, there can be but *one* perpendicular.

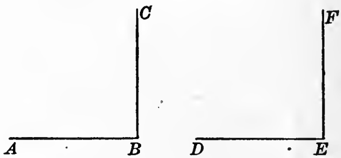
Therefore, *at any point in a straight line one perpendicular to the line can be drawn, and only one.* Q.E.D.

Proposition II

52. 1. Draw two lines intersecting so as to form a right angle. How does each of the other angles formed compare in size with a right angle? How do right angles compare in size? How do straight angles compare? (§ 37)

2. Draw two equal angles and their complements. How do their complements compare in size? How do their supplements compare? (§ 37)

Theorem. *All right angles are equal.*



Data: Any right angles, as *ABC* and *DEF*.

To prove angles *ABC* and *DEF* equal.

Proof. Suppose that ∠ *DEF* is placed upon ∠ *ABC* in such a way that the point *E* falls upon the point *B* and the line *ED* takes the direction of the line *BA*.

Since by § 26, *BC* is perpendicular to *BA* and *EF* is perpendicular to *ED* and on the same side of the line,

line *EF* must take the same direction as line *BC*,

for otherwise there would be two perpendiculars to *BA* at the point *B* and by § 51, this is impossible.

Consequently, the line *EF* falls upon the line *BC*,

and ∠ *DEF* coincides with ∠ *ABC*.

Hence, § 36, ∠s *ABC* and *DEF* are equal.

Therefore, *all right angles are equal.* **Q.E.D.**

53. Cor. I. *All straight angles are equal.*

54. Cor. II. *The complements of equal angles are equal, also the supplements of equal angles are equal.*

Ex. **1.** Find the complement of an angle of 15°; 27°; 35°; 49°.

Ex. **2.** Find the supplement of an angle of 38°; 96°; 114°.

Ex. **3.** The complement of an angle is 63°. What is the angle?

Ex. **4.** The supplement of an angle is 103°. What is the angle?

Ex. **5.** Find the complement of the supplement of an angle of 165°; 140°; 122°; 113°; 108°; 99°.

Ex. **6.** Find the supplement of the complement of an angle of 48°; 84°; 27°; 16°; 31°; 54°; 39°.

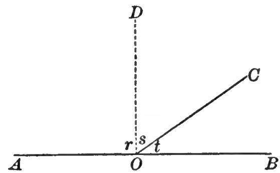
Proposition III

55. 1. Draw a straight line, and another meeting it. How does the sum of the adjacent angles thus formed compare with a right angle? With a straight angle? (§ 37)

2. Draw a straight line, and from any point in it draw several lines extending in different directions. How does the sum of the consecutive angles formed on one side of the line compare with a right angle? With a straight angle? How does the sum of the consecutive angles on both sides of the line compare with a right angle? With a straight angle? (§ 37)

Theorem. *If one straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.*

Data: Any straight line, as AB, and any other straight line, as CO, meeting it in the point O.



To prove the sum of the adjacent angles, AOC and t, equal to two right angles.

Proof. When CO is perpendicular to AB,

by § 26, each of the $\angle$s AOC and t is a rt. $\angle$,

and their sum is two rt. $\angle$s.

When CO is not perpendicular to AB,

draw DO perpendicular to AB at the point O.

Then, by § 26, $\angle$s r and DOB are rt. $\angle$s,

and $\angle r + \angle DOB = 2$ rt. $\angle$s;

by Ax. 9, $\angle DOB = \angle s + \angle t$,

$\therefore$ by substitution,

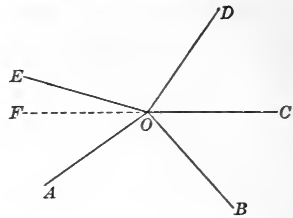
$$\angle r + \angle s + \angle t = 2 \text{ rt. } \angle\text{s}.$$

By Ax. 9, $\angle AOC = \angle r + \angle s$,

and by substitution, $\angle AOC + \angle t = 2$ rt. $\angle$s.

Therefore, *if one straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.* Q.E.D.

56. Cor. I. *The sum of all the consecutive angles which have a common vertex in a line, and which lie on one side of it, is equal to two right angles, or a straight angle.*



57. Cor. II. *The sum of all the consecutive angles that can be formed about a point is equal to four right angles, or two straight angles.*

Ex. **7**. One line meets another, making two angles with it. One angle contains 87°. How many degrees are there in the other?

Ex. **8**. Four of the five consecutive angles about a point contain 17°, 36°, 89°, and 110° respectively. How many degrees are there in the fifth angle?

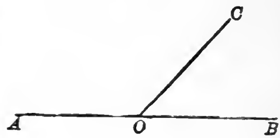
Ex. **9**. If two lines meet a third line at the same point, making with the third line angles of 27° and 63° respectively, what is the angle between the two lines?

Proposition IV

58. Construct two angles which are adjacent such that their sum is equal to two right angles. What kind of a line do their exterior sides form?

Theorem. *If the sum of two adjacent angles is equal to two right angles, their exterior sides form one straight line.*

Data: Any two adjacent angles, as AOC and COB, whose sum is equal to two right angles.



To prove that the exterior sides, AO and OB, form one straight line.

Proof.

From data, $\angle AOC + \angle COB = 2$ rt. $\angle$s;

$\therefore$ by § 27, $\angle AOC + \angle COB =$ a st. $\angle$.

But by Ax. 9, $\angle AOC + \angle COB = \angle AOB$;

$\therefore$ by Ax. 1, $\angle AOB =$ a st. $\angle$.

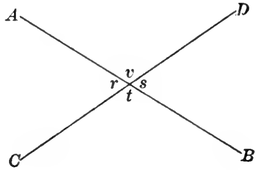
Hence, by § 27, AO and OB, the sides of $\angle AOB$ extending in opposite directions from the point O, form one straight line.

Therefore, etc. Q.E.D.

Proposition V

59. Draw two intersecting lines. How many angles are formed? How do the opposite or vertical angles compare in size?

Theorem. *If two straight lines intersect, the vertical angles are equal.*



Data: Any two intersecting straight lines, as AB and CD.

To prove the vertical angles, as v and t, equal.

Proof.

By § 55, $\angle r + \angle v = 2$ rt. $\angle$s,

and $\angle r + \angle t = 2$ rt. $\angle$s;

hence, by Ax. 1, $\angle r + \angle v = \angle r + \angle t$.

Subtracting $\angle r$ from both sides of this equality,

by Ax. 3, $\angle v = \angle t$.

In like manner it may be proved that $\angle r = \angle s$.

Therefore, etc. Q.E.D.

Ex. **10.** The complement of an angle is 43°. What is the supplement of the angle?

Ex. **11.** The supplement of an angle is 125°. What is the complement of the angle?

Ex. **12.** How many degrees are there in the supplement of the complement of an angle of 60°? Of 43° 25′ 50″?

Ex. **13.** How many degrees are there in the complement of the supplement of an angle of 159°? Of 133° 15′ 25″?

Ex. **14.** How many degrees are there in the angle formed by the bisectors of two supplementary adjacent angles?

Ex. **15.** If a line drawn through the vertex of two vertical angles bisects one angle, how does it divide the other? (§ 37)

Ex. **16.** If one of the vertical angles formed by the intersection of two straight lines is 37°, what is the value of each of the other angles?

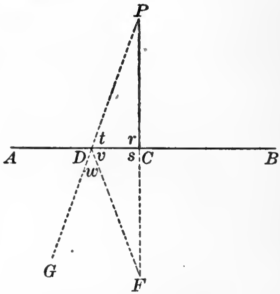
Ex. **17.** Lines are drawn to bisect the two pairs of vertical angles formed by two intersecting straight lines. What is the direction of these bisectors with reference to each other? (§ 37)

Proposition VI

60. 1. Draw a straight line; from a point not in the line draw as many perpendiculars to the line as possible. How many can be drawn?

2. How does the perpendicular compare in length with any other line drawn from the point to the given line?

Theorem. *From a point without a straight line only one perpendicular can be drawn to the line.*



Data: Any straight line, as AB, any point without it, as P, and the perpendicular PC drawn from the point P to the line AB.

To prove that PC is the only perpendicular that can be drawn from the point P to the line AB.

Proof. Prolong PC to F, making $CF = PC$; from P draw any other line to AB, as PD; and draw DF.

Then, PCF is a straight line.

§ 26, $\angle$s r and s are rt. $\angle$s,

and, § 52, $\angle r = \angle s$.

Revolve the figure PCD about AB as an axis and apply it to the figure FCD.

DC being in the axis remains fixed,

and since $\angle r = \angle s$,

CP will take the direction of CF.

Since, const., $CP = CF$,

P and F will coincide;

∴, Ax. 11, DP and DF will coincide,

and, § 36, $\angle t = \angle v$.

Revolve PCD back to its original position and prolong PD to G.

Then, § 56, $\angle t + \angle v + \angle w = 2$ rt. $\angle$s, and since $\angle t = \angle v$,

$$2\angle t + \angle w = 2 \text{ rt. } \angle s;$$

$\therefore$ $\angle t + \frac{1}{2}\angle w = 1 \text{ rt. } \angle;$

that is, $\angle t$ is less than a rt. $\angle$.

$\therefore$ *PD* is not perpendicular to *AB*.

But since *PD* represents any line from *P* to *AB* other than *PC*, *PC* is the only perpendicular that can be drawn to *AB* from *P*.

Therefore, etc. Q.E.D.

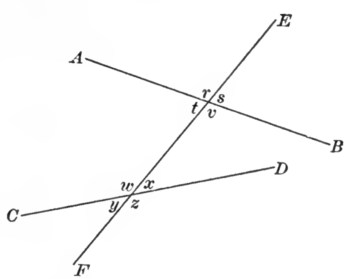
61. Cor. *A perpendicular is the shortest line that can be drawn from a point to a line.*

1. Since *PCF* is a straight line, is *PDF* a straight line? Ax. 12.
2. Which line, then, is the shorter, *PCF* or *PDF*? Ax. 10.
3. What part of *PCF* is *PC*? Of *PDF* is *PD*?
4. Then, how do *PC* and *PD* compare in length?
5. Since *PD* represents any line from *P* to *AB* other than the perpendicular *PC*, what is the shortest line that can be drawn from a point to a line?

62. The distance from a point to a line is always understood to be the *perpendicular* or shortest distance.

PARALLEL LINES

63. Lines which lie in the same plane, and which cannot meet however far they may be extended, are called **Parallel Lines.**



64. A straight line which crosses or cuts two or more straight lines is called a **Transversal.**

EF is a *transversal* of *AB* and *CD*.

Eight angles are formed by the transversal *EF* with the lines *AB* and *CD*.

65. The angles above *AB* and those below *CD*, or those *without* the two lines cut by the transversal, are called **Exterior Angles.**

Angles *r*, *s*, *y*, and *z* are *exterior angles.*

66. The angles between, or *within* the two lines cut by the transversal, are called **Interior Angles.**

Angles t, v, w, and x are *interior angles.*

67. Non-adjacent angles without the two lines, and on opposite sides of the transversal, are called **Alternate Exterior Angles.**

Angles r and z, or s and y, are *alternate exterior angles.*

68. Non-adjacent angles within the two lines, and on opposite sides of the transversal, are called **Alternate Interior Angles.**

Angles t and x, or v and w, are *alternate interior angles.*

69. Non-adjacent angles, which lie one without and one within the two lines, and on the *same side* of the transversal, are called **Corresponding Angles.**

Angles r and w, s and x, t and y, or v and z, are *corresponding angles.*

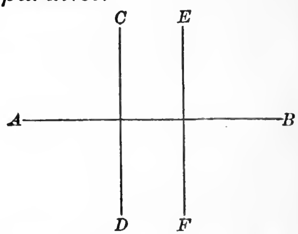
Corresponding angles are also called *Exterior Interior Angles.*

70. Ax. 13. *Through a given point but one straight line can be drawn parallel to a given straight line.*

Proposition VII

71. Draw a straight line; also two other lines each perpendicular to the first line. In what direction do the perpendiculars extend with reference to each other?

Theorem. *If two straight lines are perpendicular to the same straight line, they are parallel.*



Data: Any straight line, as AB, and any two straight lines each perpendicular to AB, as CD and EF.

To prove CD and EF parallel.

Proof. Since, by data, both CD and EF are perpendicular to AB, they cannot meet, for, if they should meet, there would then be two perpendiculars from the same point to the line AB, which is impossible. § 60.

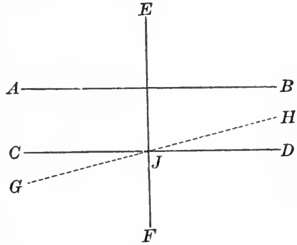
Hence, § 63, $CD \parallel EF$.

Therefore, etc. Q.E.D.

Proposition VIII

72. Draw two parallel lines; also a transversal perpendicular to one of them. What is the direction of the transversal with reference to the other parallel line?

Theorem. *If a straight line is perpendicular to one of two parallel straight lines, it is perpendicular to the other.*



Data: Any two parallel straight lines, as AB and CD, and any straight line perpendicular to AB, as EF, cutting CD at the point J.

To prove EF perpendicular to CD.

Proof. If EF is not perpendicular to CD at the point J, it will be perpendicular to some other line drawn through that point.

Suppose GH is that line.

Then, hyp., $EF \perp GH$.

But, data, $EF \perp AB$,

then, § 71, $GH \parallel AB$.

But, data, $CD \parallel AB$,

then, § 70, GH and CD passing through J cannot both be parallel to AB.

Hence, the hypothesis that EF is not perpendicular to CD is antenable.

Consequently, $EF \perp CD$.

Therefore, etc., Q.E.D.

Ex. **18.** Two lines are drawn each parallel to AB, and another line making an angle of 90° with AB. What is the direction of this line with reference to each of the other two lines?

Ex. **19.** State and illustrate the differences between a plumb line, a perpendicular line, and a vertical line.

Ex. **20.** Two parallel lines are cut by a third line making one interior angle 35°. What is the value of the adjacent interior angle?

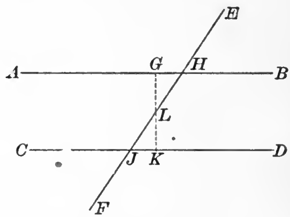
Proposition IX

73. 1. Draw two parallel lines; also a transversal. How many pairs of vertical angles are formed? How many pairs of supplementary adjacent angles? How many sizes of angles are formed? How many angles of each size? When may they all be of the same size?

2. Name a pair of angles whose sum is equal to two right angles. Name seven other pairs. Name a set of four angles whose sum is equal to four right angles. Name three other sets.

3. Name the pairs of alternate interior angles. How do the angles of any pair compare in size?

Theorem. *If two parallel straight lines are cut by a transversal, the alternate interior angles are equal.*



Data: Any two parallel straight lines, as AB and CD, cut by a transversal, as EF, in the points H and J.

To prove the alternate interior angles, as AHJ and DJH, equal.

Proof. Through L, the middle point of HJ, draw $GK \perp CD$.

Then, § 72, $GK \perp AB$.

Revolve the figure JLK about the point L and apply it to the figure HLG, so that LJ coincides with LH.

Then, since, § 59, $\angle JLK = \angle HLG$,

LK takes the direction of LG.

Const., $JK \perp GK$ and $HG \perp GK$,

and since the point J falls upon the point H,

§ 60, JK must fall upon HG.

Since LJ coincides with LH,

and JK takes the same direction as HG,

§ 36, $\angle GHL = \angle KJL$.

Therefore, etc. Q.E.D.

74. If two theorems are related in such a way that the data and conclusion of one become the conclusion and data, respectively, of the other, the one is said to be the *converse* of the other.

Thus, the converse of the theorem just proved is, "Two straight lines cut by a transversal are parallel, if the alternate interior angles are equal."

Converse propositions cannot be assumed to be true. They *may* be true, but their truth must be established by proof.

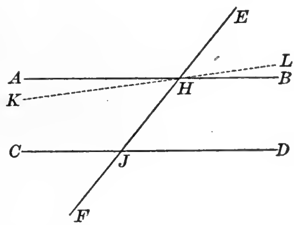
Thus, the truth of the proposition, "The product of two even numbers is an even number," can be established readily, but its converse, "An even number is the product of two even numbers," is evidently false.

Proposition X

75. Draw two lines; also a transversal. In what direction do the lines extend with reference to each other, if the alternate interior angles are equal?

Theorem. *Two straight lines cut by a transversal are parallel, if the alternate interior angles are equal.* (Converse of Prop. IX.)

Data: Two straight lines, as *AB* and *CD*, such that when cut by any transversal, as *EF*, the alternate interior angles, as *AHF* and *EJD*, are equal.



To prove *AB* and *CD* parallel.

Proof. If *AB* is not parallel to *CD*, then some other line, as *KL*, drawn through the point *H* is parallel to *CD*.

Then, hyp. and § 73, $\angle KHF = \angle EJD$;

but, data, $\angle AHF = \angle EJD$;

hence, Ax. 1, $\angle KHF = \angle AHF$,

which is absurd, since a part cannot be equal to the whole.

Hence, the hypothesis, that some other line, as *KL*, drawn through the point *H* is parallel to *CD*, is untenable.

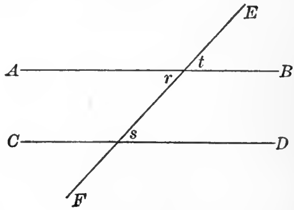
Consequently, $AB \parallel CD$.

Therefore, etc. Q.E.D.

Proposition XI

76. Draw two parallel lines; also a transversal. Name the pairs of corresponding angles. How do the angles of any pair compare in size?

Theorem. *If two parallel straight lines are cut by a transversal, the corresponding angles are equal.*



Data: Any two parallel straight lines, as AB and CD, cut by any transversal, as EF.

To prove the corresponding angles, as t and s, equal.

Proof. § 59, $\angle t = \angle r$,

§ 73, $\angle s = \angle r$;

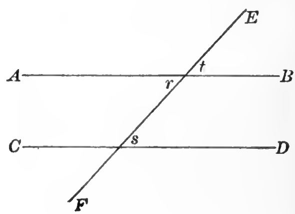
hence, Ax. 1, $\angle t = \angle s$.

Therefore, etc. **Q.E.D.**

Proposition XII

77. Draw two lines; also a transversal. In what direction do the lines extend with reference to each other, if the corresponding angles are equal?

Theorem. *Two straight lines cut by a transversal are parallel, if the corresponding angles are equal.* (Converse of Prop. XI.)



Data: Two straight lines, as AB and CD, such that when cut by any transversal, as EF, the corresponding angles, as t and s, are equal.

To prove AB and CD parallel.

Proof. § 59, $\angle r = \angle t$,

data, $\angle s = \angle t$;

then, Ax. 1, $\angle r = \angle s$.

Hence, § 75, $AB \parallel CD$.

Therefore, etc. **Q.E.D.**

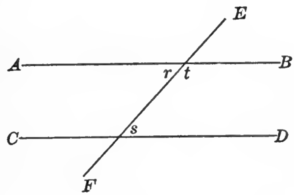
Proposition XIII

78. Draw two parallel lines; also a transversal. How does the sum of the two interior angles on the same side of the transversal compare with a right angle?

Theorem. *If two parallel straight lines are cut by a transversal, the sum of the two interior angles on the same side of the transversal is equal to two right angles.*

Data: Any two parallel straight lines, as AB and CD, cut by a transversal, as EF.

To prove the sum of the two interior angles on the same side of the transversal, as t and s, equal to two right angles.



Proof. § 73, $\angle r = \angle s.$

Adding $\angle t$ to each member of this equation,

Ax. 2, $\angle r + \angle t = \angle s + \angle t.$

But, § 55, $\angle r + \angle t = 2$ rt. $\angle$s;

$\therefore$ Ax. 1, $\angle s + \angle t = 2$ rt. $\angle$s.

Therefore, etc. Q.E.D.

Ex. **21.** If two parallel lines are cut by a transversal, what is the sum of the two exterior angles on the same side of the transversal?

Ex. **22.** The straight lines AB and CD are cut by EF in G and H respectively; angle $EHD = 38°$. What must be the value of the angle EGB in order that AB and CD may be parallel?

Ex. **23.** A transversal cutting two parallel lines makes an interior angle of 50°. What is the value of the other interior angle on the same side of the transversal?

Ex. **24.** Two parallel lines are cut by a third line making one interior angle 35°. What is the value of each of the other interior angles? How many degrees are there in the sum of the interior angles upon the same side of the transversal?

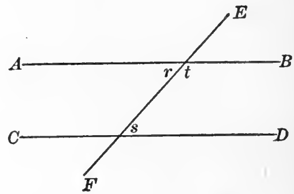
Ex. **25.** How do lines bisecting any two alternate interior angles, formed by two parallel lines cut by a transversal, lie with reference to each other?

Ex. **26.** The straight lines AB and CD are cut by EF in G and H respectively; angle $EHD = 40°$. What must be the value of the angle AGF, if AB and CD are parallel?

Proposition XIV

79. Draw two lines; also a transversal. In what direction do the lines extend with reference to each other, if the sum of the two interior angles on the same side of the transversal is equal to two right angles?

Theorem. *Two straight lines cut by a transversal are parallel, if the sum of the two interior angles on the same side of the transversal is equal to two right angles.* (Converse of Prop. XIII.)



Data: Two straight lines, as AB and CD, such that when cut by any transversal, as EF, the sum of the two interior angles on the same side of the transversal, as t and s, is equal to two right angles.

To prove AB and CD parallel.

Proof. § 55, $\angle r + \angle t = 2$ rt. $\angle$s;

data, $\angle t + \angle s = 2$ rt. $\angle$s;

$\therefore$ Ax. 1, $\angle r + \angle t = \angle t + \angle s$.

Taking $\angle t$ from each member of this equation,

Ax. 3, $\angle r = \angle s$.

Hence, § 75, $AB \parallel CD$.

Therefore, etc. Q.E.D.

Ex. **27**. AB and CD are two lines cut in G and H, respectively, by EF; $\angle BGF = 123°$, and $\angle GHD = 62°$. Are the lines AB and CD parallel?

Ex. **28**. If two lines are cut by a transversal and the sum of the two exterior angles on the same side of the transversal is equal to 180°, are the lines parallel?

Ex. **29**. Two parallel lines are cut by a transversal so that one exterior angle is 105°. How many degrees are there in the sum of each pair of alternate interior angles?

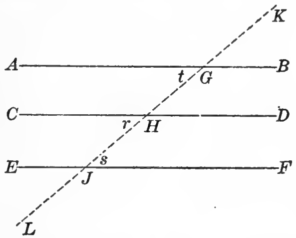
Ex. **30**. The bisectors of two adjacent angles are perpendicular to each other. What is the relation of the given angles to each other?

Ex. **31**. Two lines are cut by a transversal. In what direction do they extend with reference to each other, if the alternate exterior angles are equal?

Proposition XV

80. Draw a straight line; also two other lines each parallel to the given line. In what direction do these two lines extend with reference to each other?

Theorem. *Straight lines which are parallel to the same straight line are parallel to each other.*



Data: Any straight lines, as AB and CD, each parallel to another straight line, as EF.

To prove AB parallel to CD.

Proof. Draw any transversal, as KL, cutting the lines AB, CD, and EF.

Since, data,	$CD \parallel EF$,
§ 73,	$\angle r = \angle s$.
Since, data,	$AB \parallel EF$,
§ 73,	$\angle t = \angle s$.
Then, Ax. 1,	$\angle r = \angle t$.
Hence, § 77,	$AB \parallel CD$.

Therefore, etc. Q.E.D.

Ex. **32**. The straight lines AB, CD, and EF are cut in G, H, and J respectively, by KL; angle $KGB = 37°$; angle $KHC = 149°$; angle $FJL = 143°$. Are the lines AB and CD parallel? AB and EF? CD and EF?

Ex. **33**. Can two intersecting straight lines both be parallel to the same straight line?

Ex. **34**. How many degrees are there in the angle formed by the bisectors of two complementary adjacent angles?

Ex. **35**. If the line BD bisects the angle ABC, and EF is drawn through B perpendicular to BD, how do the angles CBE and ABF compare in size?

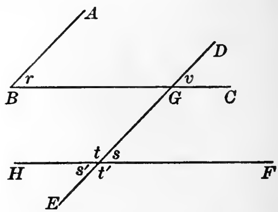
Ex. **36**. If a straight line is perpendicular to the bisector of an angle at the vertex, how does it divide the supplementary adjacent angle formed by producing one side of the given angle through the vertex?

Proposition XVI

81. 1. Construct two angles whose corresponding sides are parallel. How do the angles compare in size, if *both* corresponding pairs of sides extend in the *same* direction from their vertices? If *both* pairs extend in *opposite* directions from their vertices?

2. Discover whether it is possible for the angles to have their sides parallel and yet not be equal.

Theorem. *Angles whose corresponding sides are parallel are either equal or supplementary.*



Data: AB parallel to DE, and BC parallel to HF, forming the angles r, s, t, s', and t'.

To prove 1. $\angle r = \angle s$, or $\angle s'$.

2. $\angle r$ and $\angle t$ or $\angle t'$ supplementary.

Proof. 1. Produce BC and ED, if necessary, to intersect as at G.

§ 76, $\angle r = \angle v$, and $\angle s = \angle v$; $\therefore$ Ax. 1, $\angle r = \angle s$.

§ 59, $\angle s = \angle s'$; $\therefore$ Ax. 1, $\angle r = \angle s'$.

2. § 55, $\angle s + \angle t = 2$ rt. $\angle$s;

but $\angle r = \angle s$;

$\therefore$ $\angle r + \angle t = 2$ rt. $\angle$s.

Hence, § 32, $\angle r$ and $\angle t$ are supplementary;

also, since, § 59, $\angle t = \angle t'$, $\angle r$ and $\angle t'$ are supplementary.

Therefore, etc. Q.E.D.

82. Scholium. The angles are *equal*, if *both* corresponding pairs of sides extend in the *same* or in *opposite* directions from their vertices; they are *supplementary*, if *one* pair extends in the *same* and the *other* in *opposite* directions.

Ex. **37**. If two straight lines are perpendicular each to one of two parallel straight lines, in what direction do they extend with reference to each other?

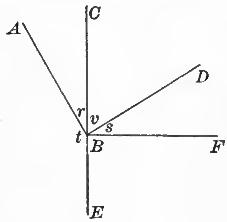
Ex. **38**. How do lines bisecting any two corresponding angles, formed by parallel lines, cut by a transversal, lie with reference to each other?

Proposition XVII

83. 1. Construct two angles whose corresponding sides are perpendicular to each other. How do the angles compare in size, if *both* are *acute?* If *both* are *obtuse?*

2. Discover whether it is possible for the angles to have their sides perpendicular and yet not be equal.

Theorem. ***Angles whose corresponding sides are perpendicular to each other are either equal or supplementary.***



Data: AB perpendicular to DB, and CE perpendicular to FB, forming the angles r, s, and t.

To prove 1. $\angle r = \angle s$.

2. $\angle t$ and $\angle s$ supplementary.

Proof. 1. § 26, $\angle s\ ABD$ and CBF are rt. $\angle s$;

$\therefore$ § 52, $\angle ABD = \angle CBF$,

and, Ax. 9, $\angle r + \angle v = \angle v + \angle s$.

Taking $\angle v$ from each member of this equation,

Ax. 3, $\angle r = \angle s$.

2. §§ 55, 32, $\angle t$ and $\angle r$ are supplementary;

but $\angle r = \angle s$;

hence, $\angle t$ and $\angle s$ are supplementary.

Therefore, etc. Q.E.D.

84. Sch. The angles are *equal*, if *both* are *acute* or if *both* are *obtuse;* they are *supplementary*, if *one* is *acute* and the other *obtuse.*

Ex. **39.** The bisectors of two adjacent angles form an angle of 45°. What is the relation of the given angles to each other?

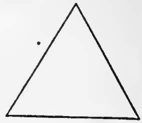
Ex. **40.** Two angles are supplementary, and the greater is five times the less. How many degrees are there in each angle?

Ex. **41.** Two angles are complementary, and the greater is five times the less. How many degrees are there in each angle?

Ex. **42.** Two parallel straight lines are cut by a transversal so that one of the two interior angles on one side of the transversal is eleven times the other. How many degrees are there in each of the exterior angles?

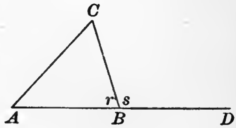
TRIANGLES

85. A portion of a plane bounded by three straight lines is called a **Plane Triangle,** or simply a **Triangle.**



The straight lines which bound a triangle are called its *sides*, their sum is called its *perimeter*, and the vertices of the angles of a triangle are called the *vertices of the triangle.*

86. The angle formed by any side of a triangle and the prolongation of another side is called an **Exterior Angle of the triangle.**



Angle s is an *exterior angle.*

87. An angle formed within a triangle by any two of its sides is called an **Interior Angle of the triangle.**

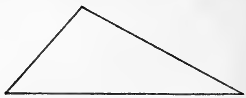
Whenever the angles of a triangle or other enclosed figure are mentioned, the *interior angles* are referred to unless otherwise specified.

Angles A, C, and r are *interior angles.*

88. The interior angles which are not adjacent to the exterior angles are called **Opposite Interior Angles.**

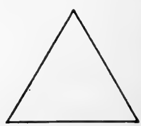
Angles A and C are *opposite interior angles* when s is the exterior angle.

89. A triangle whose three sides are unequal is called a **Scalene Triangle.**



90. A triangle two of whose sides are equal is called an **Isosceles Triangle.**

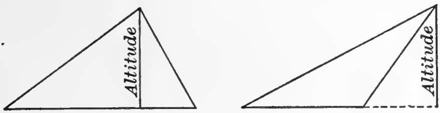
91. A triangle whose three sides are equal is called an **Equilateral Triangle.**



92. The side upon which a triangle is assumed to stand is called the **Base of the triangle.**

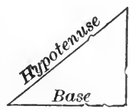
See figure accompanying § 95.

93. The angle opposite the base of a triangle is called the **Vertical Angle,** and its vertex is called the **Vertex of the triangle.**



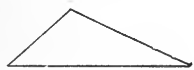
94. The perpendicular distance from the vertex of a triangle to its base, or its base produced, is called the **Altitude of the triangle.**

Since any side of a triangle may be considered as its base, it is evident that a triangle may have three altitudes and that they will be unequal, if the sides of the triangle are unequal. If the triangle is equilateral, then all three altitudes will be equal; if the triangle is isosceles, only two of the altitudes will be equal.

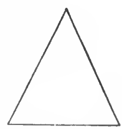


95. A triangle, one of whose angles is a right angle, is called a **Right Triangle.**

In a right triangle, the side opposite the right angle is called the *hypotenuse.*



96. A triangle, one of whose angles is an obtuse angle, is called an **Obtuse Triangle.**

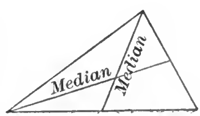


97. A triangle, each of whose angles is an acute angle, is called an **Acute Triangle.**

Obtuse triangles and acute triangles are called *oblique triangles.*

98. A triangle whose three angles are equal is called an **Equiangular Triangle.**

See figure accompanying § 91.



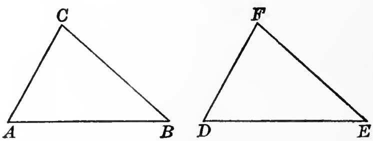
99. A line drawn from any vertex of a triangle to the middle of the opposite side is called a **Median, or Median Line of the triangle.**

Proposition XVIII

100. 1. Make two triangles such that two sides of one, and the angle formed by them, shall be equal to the corresponding parts of the other. How do the triangles compare? How do the third sides compare? How do the angles of one compare with the corresponding angles of the other?

2. Under what conditions are two triangles equal?

Theorem. *Two triangles are equal, if two sides and the included angle of one are equal to two sides and the included angle of the other, each to each.*



Data: Any two triangles, as ABC and DEF, in which $AB = DE$, $AC = DF$, and angle $A =$ angle D.

To prove triangles ABC and DEF equal.

Proof. Place $\triangle ABC$ upon $\triangle DEF$, AB coinciding with DE.

Data, $\angle A = \angle D$,

hence, AC will take the direction of DF;

and since $AC = DF$,

the point C will fall upon the point F.

Since the point B falls upon E and the point C upon F,

BC will coincide with EF.

Then, $\triangle$s ABC and DEF coincide in all their parts.

Hence, § 36, $\triangle ABC = \triangle DEF$.

Therefore, etc. Q.E.D.

Prove Prop. XVIII

(1) when $AB = DE$, $BC = EF$, and angle $B =$ angle E.

(2) when $AC = DF$, $BC = EF$, and angle $C =$ angle F.

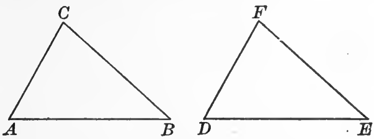
101. Sch. Every triangle has six parts or elements; namely, three sides and three angles. Two equal triangles may be made to coincide in all their parts. Therefore, each part of one is equal to the corresponding part of the other.

Proposition XIX

102. 1. Make two triangles such that a side of one, and the angles formed at its extremities, shall be equal to the corresponding parts of the other. How do the triangles compare? What parts are equal?

2. Under what conditions are two triangles equal?

Theorem. *Two triangles are equal, if a side and two adjacent angles of one are equal to a side and two adjacent angles of the other, each to each.*



Data: Any two triangles, as ABC and DEF, in which $AB = DE$, angle $A =$ angle D, and angle $B =$ angle E.

To prove triangles ABC and DEF equal.

Proof. Place $\triangle ABC$ upon $\triangle DEF$, AB coinciding with DE.

Data, $\angle A = \angle D$;

hence, AC will take the direction of DF,

and the point C will fall upon DF, or upon DF produced.

Also, data, $\angle B = \angle E$;

hence, BC will take the direction of EF,

and the point C will fall upon EF, or upon EF produced.

Since the point C falls upon each of the lines DF and EF, it must fall upon their point of intersection, F.

Then, $\triangle$ ABC and DEF coincide in all their parts.

Hence, § 36, $\triangle ABC = \triangle DEF$.

Therefore, etc. Q.E.D.

Prove Prop. XIX

(1) when angle $C =$ angle F, angle $B =$ angle E, and $BC = EF$.

(2) when angle $A =$ angle D, angle $C =$ angle F, and $AC = DF$.

Ex. **43.** Are two triangles equal, if the three angles of one are equal to the three angles of the other, each to each?

Ex. **44.** Can two triangles, having two sides and an angle of one respectively equal to two sides and an angle of the other, be unequal?

Proposition XX

103. 1. Draw a straight line and a perpendicular to that line at its middle point; select any point in the perpendicular and from that point draw straight lines to the extremities of the given line. How do these lines compare in length? How do the angles made by these lines with the perpendicular compare in size? How do the angles made by these lines with the given line compare?

2. Select any point *not* in the perpendicular and from that point draw straight lines to the extremities of the given line. How do they compare in length?

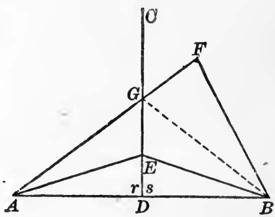
3. Draw a straight line and find a point equidistant from its extremities; find another point equidistant from its extremities; connect these points by a line and if necessary extend it until it intersects the given line. At what point does it intersect the given line? What kind of angles does it make with the given line?

4. What line contains every point that is equidistant from the extremities of a straight line?

Theorem. ***If a perpendicular is drawn to a straight line at its middle point,***

1. Any point in the perpendicular is equidistant from the extremities of the line.

2. Any point not in the perpendicular is unequally distant from the extremities of the line.



Data: Any straight line, as AB; a perpendicular to it at its middle point, as CD; any point in CD, as E; and any point not in CD, as F.

To prove

1. E equidistant from A and B.
2. F unequally distant from A and B.

Proof. 1. Draw AE, BE, AF, and BF.

Data and § 26, $\angle r$ and $\angle s$ are rt. $\angle$s,

and, § 52, $\angle r \quad \angle s$.

In $\triangle$s ADE and BDE, $AD = BD$,

ED is common,

and $\angle r = \angle s$;

$\therefore$ § 100, $\triangle ADE = \triangle BDE$, and, § 101, $AE = BE.$

That is, E is equidistant from A and B.

2. From the point G, where AF cuts CD, draw GB.

Ax. 10, $BF < BG + GF$;

but $BG = AG$; Why?

$\therefore$ substituting AG for its equal BG,

$$BF < AG + GF,$$

or $$BF < AF.$$

That is, F is unequally distant from A and B.

Therefore, etc. Q.E.D.

104. Cor. I. *Every point that is equidistant from the extremities of a straight line lies in the perpendicular at the middle point of that line.*

105. Cor. II. *If a perpendicular is erected at the middle point of a straight line, the lines joining the extremities of this line with any point in the perpendicular make equal angles with the line and also with the perpendicular.* § 101

106. Cor. III. *Two points each equidistant from the extremities of a straight line determine the perpendicular at the middle point of that line.*

Ex. **45.** How does the distance between two parallel lines at a given point compare with the distance between them at any other point?

Ex. **46.** Can two angles which are not adjacent have a common vertex and a common side?

Ex. **47.** If in an equilateral triangle a line is drawn from the vertex to the middle point of the base, how do the triangles thus formed compare in size?

Ex. **48.** If two lines bisect each other, in what direction do the lines joining their opposite extremities extend with reference to each other?

Ex. **49.** If two sides of a triangle are equal, and a line is drawn bisecting their included angle and intersecting the third side, how do the segments of the third side compare in length?

Ex. **50.** Perpendiculars are erected at the extremities of a line and terminate in any bisector of the line that is not perpendicular to the line. How do the perpendiculars compare in length?

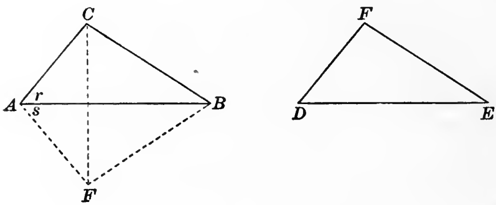
Ex. **51.** If through the middle point of a straight line terminating in two parallel lines, a second straight line is drawn also terminating in the parallels, how do the parts of the second line compare in length?

Proposition XXI

107. 1. Make two triangles such that the sides of one shall be equal to the corresponding sides of the other. How do the triangles compare? How do the corresponding angles compare?

2. Under what conditions are two triangles equal?

Theorem. *Two triangles are equal, if the three sides of one are equal to the three sides of the other, each to each.*



Data: Any two triangles, as ABC and DEF, in which $AB = DE$, $AC = DF$, and $BC = EF$.

To prove triangles ABC and DEF equal.

Proof. Place $\triangle DEF$ in the position ABF so that the equal sides, DE and AB, coincide, and the vertex F falls opposite C. Draw CF.

Data, $AF = AC$, and $BF = BC$;

$\therefore$ A and B are each equidistant from F and C;

hence, § 106, $AB \perp CF$ at its middle point,

and, § 105, $\angle r = \angle s$.

In $\triangle$s ABC and ABF, $AC = AF$,

AB is common,

and $\angle r = \angle s$;

$\therefore$ § 100, $\triangle ABC = \triangle ABF$.

That is, $\triangle ABC = \triangle DEF$.

Therefore, etc. Q.E.D

Prove by placing the triangle so that

(1) DF will coincide with AC.

(2) EF will coincide with BC.

108. Sch. It is evident that in equal triangles the parts which are similarly situated are equal; that is, the angles included by the equal sides are equal; the angles opposite the equal sides are equal, the sides included between equal angles are equal, and the sides opposite the equal angles are equal.

109. In equal figures, the parts which are similarly situated are called **Homologous** parts.

Ex. **52.** Draw two parallel lines intersecting two parallel lines, and draw a line joining two opposite points of intersection. How do the triangles thus formed compare?

Ex. **53.** Perpendiculars are drawn from the extremities of a line to any line that bisects it and is not perpendicular to it. How do the perpendiculars compare in length?

Ex. **54.** The line BD is the bisector of the angle ABC whose sides are equal. Lines are drawn from any point of BD, as E, to A and C. How do AE and CE compare in length?

Ex. **55.** In a triangle ABC angle A equals angle B; a line parallel to AB intersects AC in D and BC in E. How do the angles ADE and BED compare?

Ex. **56.** If D is the middle point of the side BC of the triangle ABC, and BE and CF are perpendiculars from B and C to AD, or AD produced, how do BE and CF compare in length?

Proposition XXII

110. 1. Cut out a paper triangle ABC. Cut off the corners and place the vertices A, B, and C together. To how many right angles is the sum of the three angles equal?

2. If one angle of a triangle is a right angle, how does the sum of the other two angles compare with a right angle?

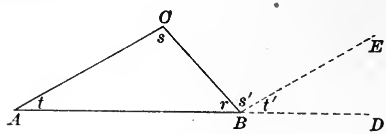
3. What is the greatest number of obtuse angles that a triangle may have? The greatest number of right angles?

4. If there are two triangles such that the sum of two angles of one is equal to the sum of two angles of the other, how do the third angles compare in size?

5. If there are two right triangles such that a side and an acute angle of one are equal to the corresponding parts of the other, how do the triangles compare?

6. Extend one side of a triangle through a vertex; through the same vertex draw a line parallel to the opposite side of the triangle. Since the figure thus formed contains two parallel lines and a transversal, what angles of the figure are equal? How does the exterior angle of the triangle compare with the sum of the two opposite interior angles?

Theorem. *The sum of the angles of a triangle is equal to two right angles.*



Datum: Any triangle, as ABC.

To prove $\angle r + \angle s + \angle t =$ two right angles.

Proof. Produce AB to D and draw $BE \parallel AC$.

§ 56, $\angle r + \angle s' + \angle t' = 2$ rt. $\angle$s;

but, § 73, $\angle s' = \angle s$,

and, § 76, $\angle t' = \angle t$;

$\therefore$ substituting $\angle s$ and $\angle t$ for $\angle s'$ and $\angle t'$ in the first equation,

$$\angle r + \angle s + \angle t = 2 \text{ rt. } \angle\text{s}.$$

Therefore, etc. Q.E.D.

111. Cor. I. *In a right triangle the sum of the two acute angles is equal to a right angle.*

112. Cor. II. *A triangle cannot have more than one right angle, nor more than one obtuse angle.*

113. Cor. III. *If two angles of one triangle are equal to two angles of another, the third angles are equal.*

114. Cor. IV. *Two right triangles are equal, if a side and an acute angle of one are equal to a side and an acute angle of the other, each to each.*

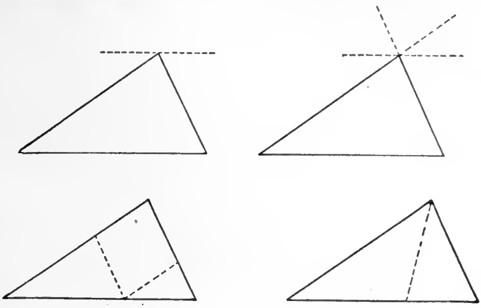
115. Cor. V. *Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

1. What interior angle is equal to $\angle t'$? Why?
2. What interior angle is equal to $\angle s'$? Why?
3. To what, then, is the whole exterior angle equal?

Ex. **57**. May a triangle be formed whose angles are 93°, 40°, and 61° respectively? 98°, 24°, and 58°? 57°, 49°, and 74°?

Ex. **58**. Two angles of a triangle are together equal to 76°. What is the value of the third angle?

Ex. **59.** Show by each of the following figures that the sum of the three angles of a triangle is equal to two right angles, assuming that the construction lines are drawn as they appear to be drawn.

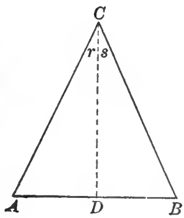


Proposition XXIII

116. 1. Draw an isosceles triangle. How do the angles opposite the equal sides compare in size?

2. How do the angles of an equilateral triangle compare?

Theorem. In an isosceles triangle the angles opposite the equal sides are equal.



Data: Any isosceles triangle, as ABC, in which $AC = BC$.

To prove angle A = angle B.

Proof. Draw CD bisecting $\angle C$.

Then, in $\triangle$s ADC and BDC,

data, $AC = BC$,

CD is common,

and, const., $\angle r = \angle s$;

$\therefore$ § 100, $\triangle ADC = \triangle BDC$,

and, § 108, $\angle A = \angle B$.

Therefore, etc. Q.E.D.

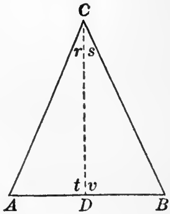
117. Cor. *An equilateral triangle is also equiangular.*

Proposition XXIV

118. 1. Draw a triangle such that two of its angles are equal. How do the sides opposite these angles compare in length? What kind of a triangle is it?

2. How do the sides of an equiangular triangle compare in length?

Theorem. *If two angles of a triangle are equal, the sides opposite the equal angles are equal and the triangle is isosceles.*



Data: Any triangle, as ABC, having

angle A = angle B.

To prove $AC = BC$, and triangle ABC isosceles.

Proof. Draw CD bisecting $\angle C$.

Then, in $\triangle$ ADC and BDC,

data,	$\angle A = \angle B$,
const.,	$\angle r = \angle s$;
$\therefore$ § 113,	$\angle t = \angle v$;
and, since	CD is common,
§ 102,	$\triangle ADC = \triangle BDC$,
and, § 108,	$AC = BC$.
Hence, § 90,	$\triangle ABC$ is isosceles.

Therefore, etc. Q.E.D.

119. Cor. *An equiangular triangle is also equilateral.*

Ex. **60.** If equal distances from the vertices of an equilateral triangle are laid off on its sides in the same order, what kind of a triangle do the lines joining these points form?

Ex. **61.** From the extremities of the base of an isosceles triangle perpendiculars are drawn to the opposite sides; the points where the perpendiculars meet the opposite sides are joined by a straight line. What is the direction of this line with reference to the base?

Ex. **62.** The base of an isosceles triangle is 6 inches and the opposite angle is 60°. How many degrees are there in each of the base angles? What is the length of each of the other two sides?

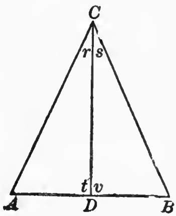
Proposition XXV

120. 1. Draw an isosceles triangle and a line bisecting its vertical angle. How does this line divide the base? What kind of angles does it form with the base?

2. Draw a line perpendicular to the base of an isosceles triangle at its middle point. How does it divide the triangle? How does it divide the vertical angle?

3. Draw a line from the vertex of an isosceles triangle and perpendicular to the base. How does this line divide the base of the triangle? How does it divide the vertical angle?

Theorem. ***The bisector of the vertical angle of an isosceles triangle is perpendicular to the base at its middle point.***



Data: Any isosceles triangle, as ABC, in which $AC = BC$, and CD bisects the angle C.

To prove CD perpendicular to AB at its middle point.

Proof. **Data,** $AC = BC$,

$\angle r = \angle s$,

and, § **116**, $\angle A = \angle B$;

∴ § 102, $\triangle ADC = \triangle BDC$,

and, § 108, $AD = BD$;

that is, D is the middle point of AB;

also, $\angle t = \angle v$;

hence, § 26, $CD \perp AB$.

Therefore, etc. Q.E.D.

121. **Cor. I.** *A perpendicular which bisects the base of an isosceles triangle bisects the vertical angle.*

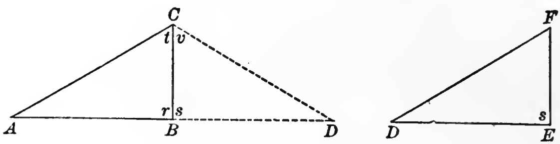
122. **Cor. II.** *A line perpendicular to the base of an isosceles triangle and passing through the vertex bisects both the base and the vertical angle of the triangle.*

Ex. **63.** How do the lines joining the extremities of the bases of two opposite, or vertical, isosceles triangles compare in length?

Proposition XXVI

123. Draw two right triangles such that the hypotenuse and a side of one shall be equal to the corresponding parts of the other. How do the triangles compare?

Theorem. *Two right triangles are equal, if the hypotenuse and a side of one are equal to the hypotenuse and a side of the other, each to each.*



Data: Any two right triangles, as ABC and DEF, in which the hypotenuse $AC =$ the hypotenuse DF, $BC = EF$, and angles r and s are the right angles.

To prove triangles ABC and DEF equal.

Proof. Place $\triangle DEF$ in the position DBC so that the equal sides EF and BC coincide, and the vertex D falls opposite A.

Data, $\angle r$ and $\angle s$ are rt. $\angle$s;

then, $\angle r + \angle s = 2$ rt. $\angle$s,

and, § 58, AB and BD form one straight line.

Data, $AC = DC$;

$\therefore$ § 90, $\triangle ADC$ is isosceles.

Hence, in $\triangle$s ABC and DBC,

$AC = DC$,

§ 116, $\angle A = \angle D$,

and, § 113, $\angle t = \angle v$;

$\therefore$ § 102, $\triangle ABC = \triangle DBC$.

That is, $\triangle ABC = \triangle DEF$.

Therefore, etc. **Q.E.D.**

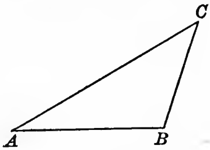
Ex. **64**. The median line from the vertex to the base of a certain triangle is equal to one half the base. What kind of an angle is the vertical angle?

Proposition XXVII

124. 1. Draw any triangle. How does any side compare in length with the sum of the other two sides?

2. How does the sum of any two sides compare with the third side?

Theorem. *Any side of a triangle is less than the sum of the other two sides.*



Data: Any triangle, as ABC, and any side, as AC.

To prove AC less than $AB + BC$.

Proof. By Ax. 10, the straight line AC, which is a side of the triangle, is the shortest distance between the points A and C.

Hence, AC is less than the broken line ABC which joins the points A and C.

That is, AC is less than $AB + BC$.

Therefore, etc. **Q.E.D.**

125. Cor. *The sum of any two sides of a triangle is greater than the third side.*

Ex. **65.** May a triangle be formed with lines 4, 2, and 3 inches long? With lines 6, 1, and 2 inches long? 5, 2, and 3 inches long?

Ex. **66.** If a line is drawn joining the middle points of the equal sides of an isosceles triangle, what kind of a triangle is formed?

Ex. **67.** If the bisectors of the base angles of an isosceles triangle are produced to the opposite sides, how do they compare in length?

Ex. **68.** The sum of the two angles at the base of an isosceles triangle is 64°. What is the value of each angle of the triangle?

Ex. **69.** If the straight line which joins the vertex of a triangle with the middle point of the base is perpendicular to the base, what kind of a triangle is it?

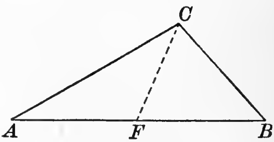
Ex. **70.** The perpendicular distance between two parallel lines is 20 inches, and a line is drawn across the parallels making an angle of 45° with the perpendicular at its upper extremity. What distance does this line cut off from the foot of the perpendicular?

Ex. **71.** If from a point within a right angle perpendiculars are drawn to the sides containing the right angle and each perpendicular is produced its own length, what kind of a line will join the extremities of the produced lines and the vertex of the right angle?

Proposition XXVIII

126. Draw a scalene triangle. Where is the smaller angle situated with reference to the shorter side? Where is the greater angle situated with reference to the greater side?

Theorem. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



Data: Any triangle, as ABC, in which AB is greater than BC.

To prove angle ACB, opposite AB, is greater than angle A, opposite BC.

Proof. On BA take BF equal to BC, and draw CF.

Ax. 8, $\angle ACB$ is greater than $\angle BCF$;

but, § 116, $\angle BCF = \angle BFC$;

$\therefore$ $\angle ACB$ is greater than $\angle BFC$.

But since, § 115, $\angle BFC = \angle A + \angle ACF$,

$\angle BFC$ is greater than $\angle A$.

Then, $\angle ACB$, the angle opposite AB, is greater than $\angle A$, the angle opposite BC.

In like manner, if AB is greater than AC, $\angle ACB$ may be proved greater than $\angle B$.

Therefore, etc. Q.E.D.

Prove $\angle ACB$ greater than $\angle B$ when AB is greater than AC.

Ex. **72.** How do the angles of a scalene triangle compare?

Ex. **73.** What is the value of the angle formed by the bisectors of the acute angles of a right triangle?

Ex. **74.** How many degrees are there in an angle of an equilateral triangle?

Ex. **75.** How many degrees are there in each of the equal angles of an isosceles triangle, the angle at the vertex being $35° 50'$?

Ex. **76.** In an isosceles triangle one base angle is 35°. What is the value of the vertical angle?

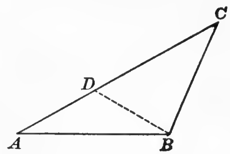
Ex. **77.** AD is the bisector of a base angle of the isosceles triangle ABC, the bisector meeting the side BC in D; the vertical angle C is 28°. How many degrees are there in angle ADC?

Proposition XXIX

127. 1. Draw any triangle. Where is the shorter side situated with reference to the smaller angle? Where is the greater side situated with reference to the greater angle?

2. Which side of a right triangle is the greatest?

Theorem. *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.* (Converse of Prop. XXVIII)



Data: Any triangle, as ABC, in which angle ABC is greater than angle A.

To prove AC, opposite angle ABC, greater than BC, opposite angle A.

Proof. Draw BD so that $\angle ABD = \angle BAD$.

Then, § 118, $AD = BD$.

In $\triangle BCD$, § 125, $BD + DC > BC$.

Substituting AD for its equal BD,

$$AD + DC > BC,$$

or

$$AC > BC.$$

That is, AC, opposite angle ABC, is greater than BC, opposite angle A.

In like manner, if angle ABC is greater than angle C, AC may be proved greater than AB.

Therefore, etc. Q.E.D.

Prove that AC is greater than AB when $\angle ABC$ is greater than $\angle C$.

128. Cor. *The hypotenuse is the greatest side of a right triangle.*

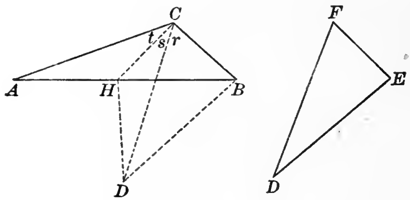
Ex. **78**. The angles A, B, and C of the triangle ABC are 40°, 60°, and 80° respectively. How do AC and AB compare in length? AB and BC? AC and BC?

Ex. **79**. CD bisects the base of an isosceles triangle ABC, a base angle of which is 55°. How many degrees are there in angle ADC? In angle CDB? In angle ACD? In angle DCB?

Proposition XXX

129. Construct two triangles having two sides of one equal to two sides of the other, and the angles included between the sides unequal. How do the third sides compare in length? Which triangle has the greater third side?

Theorem. *If two sides of one triangle are equal to two sides of another, each to each, and the included angles are unequal, the remaining sides are unequal, and the greater side is in the triangle which has the greater included angle.*



Data: Any two triangles, as ABC and DEF, in which $AC = DF$, $BC = EF$, and angle ACB is greater than angle F.

To prove AB greater than DE.

Proof. Of the two sides DF and EF, suppose that EF is the side which is not greater. Place $\triangle DEF$ in the position DBC so that the equal sides, EF and BC, coincide.

Draw CH bisecting $\angle ACD$ and draw DH.

In $\triangle$s AHC and DHC, $AC = DC$,

CH is common,

and, const., $\angle t = \angle s$;

$\therefore$ § 100, $\triangle AHC = \triangle DHC$,

and, § 108, $AH = DH$.

In $\triangle DHB$, § 125, $DH + HB > DB$.

Substituting AH for its equal DH,

$$AH + HB > DB;$$

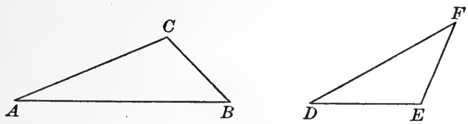
that is, $AB > DB$ or DE.

Therefore, etc. **Q.E.D.**

Proposition XXXI

130. Construct two triangles that have two sides of one equal respectively to two sides of the other, but the third sides unequal. How do the angles opposite the third sides compare in size?

Theorem. *If two sides of one triangle are equal to two sides of another, each to each, and the third sides are unequal, the angles opposite the third sides are unequal, and the greater angle is in the triangle which has the greater third side.* (Converse of Prop. XXX.)



Data: Any two triangles, as ABC and DEF, in which $AC = DF$, $BC = EF$, and AB is greater than DE.

To prove angle C greater than angle F.

Proof. Since $AC = DF$,

and $BC = EF$,

if $\angle C = \angle F$,

then, § 100, $\triangle ABC = \triangle DEF$,

and, § 108, $AB = DE$,

which is contrary to data.

If $\angle C$ is less than $\angle F$,

then, $\angle F$ is greater than $\angle C$,

and, § 129, $DE > AB$,

which is also contrary to data.

Therefore, both hypotheses, namely, that $\angle C = \angle F$, and that $\angle C$ is less than $\angle F$, are untenable.

Consequently, $\angle C$ is greater than $\angle F$.

Therefore, etc. Q.E.D.

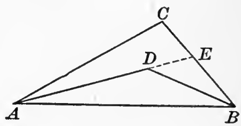
Ex. **80.** If one angle of a triangle is equal to the sum of the other two, what is the value of that angle? What kind of a triangle is the triangle?

Ex. **81.** AD is perpendicular to BC, one of the equal sides of the isosceles triangle ABC whose vertical angle is 30°. How many degrees are there in each of the angles CAD, DAB, and ABC?

Proposition XXXII

131. Choose some point within any triangle and from it draw lines to the extremities of one side. How does the sum of these lines compare with the sum of the other two sides of the triangle?

Theorem. *The sum of two lines drawn from a point within a triangle to the extremities of one side is less than the sum of the other two sides.*



Data: Any triangle, as ABC; any point within it, as D; and the two lines, AD and BD, drawn from D to the extremities of AB.

To prove $AD + BD$ less than $AC + BC$.

Proof. Produce AD to meet BC in E.

In $\triangle AEC$, § 124, $\quad AE < AC + CE.$

Adding BE to both members of this inequality,

Ax. 4, $\quad AE + BE < AC + CE + BE,$

or $\quad AE + BE < AC + BC.$

In $\triangle DBE$, $\quad BD < DE + BE.$ $\quad$ Why?

Adding AD to both members of this inequality,

Ax. 4, $\quad AD + BD < AD + DE + BE,$

or $\quad AD + BD < AE + BE.$

It has been shown that

$$AE + BE < AC + BC;$$

hence, $\quad AD + BD < AC + BC.$

Therefore, etc. $\quad$ Q.E.D.

Proposition XXXIII

132. 1. Draw a straight line and a perpendicular to it; select a point in the perpendicular, and from that point draw two oblique lines meeting the given line at equal distances from the foot of the perpendicular. How do the oblique lines compare in length?

2. Draw oblique lines from that point to points unequally distant from the foot of the perpendicular. How do they compare in length? Which is the greater?

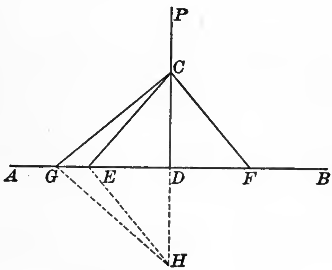
3. Draw two unequal lines from that point to the given line. Which one meets the line at the greater distance from the foot of the perpendicular?

4. How many equal straight lines can be drawn from a point to a straight line?

Theorem. *If from a point in a perpendicular to a given straight line, oblique lines are drawn to the given line,*

1. The oblique lines which meet the given line at equal distances from the foot of the perpendicular are equal.

2. Of oblique lines which meet the given line at unequal distances from the foot of the perpendicular the more remote is the greater.



Data: Any straight line, as AB; any perpendicular to AB, as PD; and any point in PD, as C, from which oblique lines, as CE, CF, and CG, are drawn meeting AB so that $DE = DF$, and DG is greater than DE.

To prove 1. $CE = CF$.

2. CG greater than CE.

Proof. 1. Data, $CD \perp EF$ at its middle point.

Then, § 103, $CE = CF$.

2. Produce CD to H, making $DH = CD$; draw EH and GH.

Then, data and const., $AB \perp CH$ at its middle point.

$\therefore$ § 103, $EH = CE$, and $GH = CG$;

hence, Ax. 2, $CE + EH = 2\,CE$, and $CG + GH = 2\,CG$.

But, § 131, $CE + EH < CG + GH$;

$\therefore$ $2\,CE < 2\,CG$, or $CE < CG$;

that is, $CG > CE$.

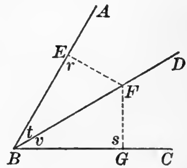
Therefore, etc. Q.E.D.

133. Cor. *Only two equal straight lines can be drawn from a point to a straight line; and of two unequal lines the greater cuts off the greater distance from the foot of a perpendicular drawn to the line from the given point.*

Proposition XXXIV

134. Bisect any angle; from any point in the bisector draw lines perpendicular to the sides of the angle. How do the perpendiculars compare in length? How do the distances of the point from the sides of the angle compare?

Theorem. *Every point in the bisector of an angle is equidistant from the sides of the angle.*



Data: Any angle, as ABC, and any point in its bisector BD, as F.

To prove F equidistant from AB and CB.

Proof. Draw the perpendiculars FE and FG representing the distances of the point F from AB and CB respectively.

§ 26, $\angle r$ and $\angle s$ are rt. $\angle$s.

Then, in the rt. $\triangle$s BFE and BFG,

BF is common,

and, data, $\angle t = \angle v$;

$\therefore$ § 114, $\triangle BFE = \triangle BFG$,

and, § 108, $FE = FG$;

that is, F is equidistant from AB and CB.

Therefore, etc. Q.E.D.

Ex. **82.** The perpendicular let fall from the vertex to the base of a triangle divides the vertical angle into two angles. How does the difference of these angles compare with the difference of the base angles of the triangle?

Ex. **83.** ABC is a triangle. Angle $A = 60°$, angle $B = 40°$. The bisector of angle A is produced until it cuts the side BC. How many degrees are there in each angle thus formed?

Ex. **84.** A perpendicular is let fall from one end of the base of an isosceles triangle upon the opposite side. How does the angle formed by the perpendicular and the base compare with the vertical angle?

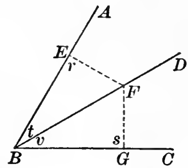
Ex. **85.** If an angle of a triangle is equal to half the sum of the other two, what is the value of that angle?

Ex. **86.** How does the sum of the lines from a point within a triangle to the vertices of the triangle compare with the sum of the sides of the triangle? With half that sum?

Proposition XXXV

135. Within an angle select any number of points that are each equidistant from its sides. Will the lines joining these points form a straight line? How will it divide the angle?

Theorem. *Every point within an angle and equidistant from its sides lies in the bisector of the angle.* (Converse of Prop. XXXIV.)



Data: Any angle, as ABC, and any point within the angle equidistant from AB and CB, as F.

To prove F is in the bisector of the angle ABC.

Proof. Through the point F draw BD; also draw the perpendiculars FE and FG representing the distances of the point F from AB and CB respectively.

Then, § 26, $\angle r$ and $\angle s$ are rt. $\angle$s.

In the rt. $\triangle$s BEF and BGF,

BF is common,

and, data, $FE = FG$;

$\therefore$ § 123, $\triangle BEF = \triangle BGF$,

and, § 108, $\angle t = \angle v$;

that is, BD is the bisector of $\angle ABC$.

Hence, F is in the bisector of $\angle ABC$.

Therefore, etc. Q.E.D.

Ex. **87.** ABC is an isosceles triangle having a vertical angle of 30°. From each extremity of the base perpendiculars are drawn to the opposite sides. What angles are formed at the intersection of these perpendiculars?

Ex. **88.** The exterior angle at the vertex of an isosceles triangle is 110°. How many degrees are there in each angle of the triangle?

Ex. **89.** The exterior angle at the base of an isosceles triangle is 110°. How many degrees are there in each angle of the triangle?

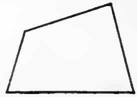
Ex. **90.** The angle C at the vertex of the isosceles triangle ABC is one fourth of the exterior angle at C. How many degrees are there in angle A? In the exterior angle at B?

Ex. **91.** How does the angle formed by the bisectors of the base angles of an isosceles triangle compare with an exterior angle at the base?

QUADRILATERALS

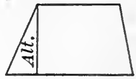
136. A portion of a plane bounded by four straight lines is called a **Quadrilateral.**

137. A quadrilateral which has no two sides parallel is called a **Trapezium.**



138. A quadrilateral which has only two sides parallel is called a **Trapezoid.**

The parallel sides of a trapezoid are called its *bases.*



139. A trapezoid whose non-parallel sides are equal is called an **Isosceles Trapezoid.**

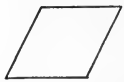
140. A quadrilateral whose opposite sides are parallel is called a **Parallelogram.**

141. A parallelogram whose angles are right angles is called a **Rectangle.**

142. A parallelogram whose angles are oblique angles is called a **Rhomboid.**

143. An equilateral rectangle is called a **Square.**

144. An equilateral rhomboid is called a **Rhombus.**



145. The straight lines which join the vertices of the opposite angles of a quadrilateral are called **Diagonals.**

146. The side upon which a figure is assumed to stand is called the **Base.**

The side upon which a trapezoid or a parallelogram is assumed to stand is called its *lower base,* and the side opposite is called its *upper base.*

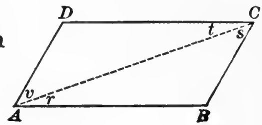
147. The perpendicular distance between the bases of a trapezoid or of a parallelogram is called its **Altitude.**

Proposition XXXVI

148. 1. Draw a quadrilateral whose opposite sides are equal. What kind of a quadrilateral is it?

2. How do the opposite angles of a parallelogram compare in size?

Theorem. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Data: Any quadrilateral, as $ABCD$, in which $AB = DC$ and $AD = BC$.

To prove $ABCD$ a parallelogram.

Proof. Draw AC.

Then, in the $\triangle s$ ABC and ADC,

data,	$AB = DC,\ BC = AD,$
and	AC is common;
$\therefore$ § 107,	$\triangle ABC = \triangle ADC,$
§ 108,	$\angle r = \angle t$, and $\angle s = \angle v$;
$\therefore$ § 75,	$AB \parallel DC$, and $AD \parallel BC$.
Hence, § 140,	$ABCD$ is a parallelogram.

Therefore, etc. Q.E.D.

149. Cor. *The opposite angles of a parallelogram are equal.*

$$\angle r = \angle t \text{ and } \angle v = \angle s;$$

$$\therefore \angle r + \angle v = \angle t + \angle s.$$

Ex. **92.** If lines are drawn joining in succession the middle points of the sides of a square, what figure will be formed?

Ex. **93.** To how many right angles is the sum of the angles of a parallelogram equal? To what is the sum of any two angles of a parallelogram, which are not opposite, equal?

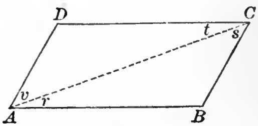
Ex. **94.** If medians are drawn from two vertices of a triangle and each is produced its own length, what kind of a line will join the extremities of the produced medians and the other vertex of the triangle?

Proposition XXXVII

150. 1. Draw a quadrilateral having two of its sides equal and parallel to each other. What kind of a quadrilateral is it?

2. Draw two parallel lines and two parallel transversals. How do the segments of the transversals between the parallel lines compare in length?

Theorem. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Data: Any quadrilateral, as $ABCD$, in which two of the sides, as AB and DC, are equal and parallel.

To prove $ABCD$ a parallelogram.

Proof. Draw AC.

In the $\triangle s$ ABC and ADC,

data,	$AB = DC$,
	AC is common,
and, § 73,	$\angle r = \angle t$;
∴ § 100,	$\triangle ABC = \triangle ADC$,
and	$\angle s = \angle v$.
Hence, § 75,	$BC \parallel AD$,
and since, data,	$AB \parallel DC$,
§ 140,	$ABCD$ is a parallelogram.

Therefore, etc. Q.E.D.

151. Cor. *Parallel lines intercepted between parallel lines are equal, and parallel lines are everywhere equally distant.*

Ex. **95.** If the sides of a parallelogram are bisected and these middle points joined in succession, what figure is formed by the connecting lines?

Ex. **96.** If the four interior angles formed by a transversal crossing two parallel lines are bisected and the bisectors produced until they meet, what figure will be formed?

Ex. **97.** If a line is drawn through the vertices of two isosceles triangles on the same base, how does it divide the base?

Ex. **98.** If two equal straight lines are drawn from a point to a line, how do the angles formed with the given line compare?

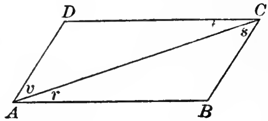
Ex. **99.** If lines are drawn from the vertex of an isosceles triangle to points in the base equally distant from its extremities, how do they compare in length?

Proposition XXXVIII

152. 1. Draw a parallelogram and either diagonal. How do the triangles thus formed compare in size?

2. How do the opposite sides of the parallelogram compare in length?

Theorem. *The diagonal of a parallelogram divides the figure into two equal triangles.*



Data: Any parallelogram, as $ABCD$, and one of its diagonals, as AC.

To prove triangles ABC and ADC equal.

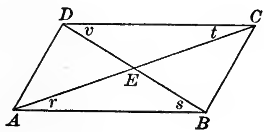
Proof. To be given by the student.

153. Cor. *The opposite sides of a parallelogram are equal.*

Proposition XXXIX

154. Draw a parallelogram and its diagonals. How do the segments of each diagonal compare in length?

Theorem. *The diagonals of a parallelogram bisect each other.*



Data: Any parallelogram, as $ABCD$, and its diagonals, AC and BD, intersecting at E.

To prove $AE = CE$ and $BE = DE$.

Proof. In the $\triangle$s ABE and CDE,

§ 153, $AB = DC$,

§ 73, $\angle r = \angle t$,

and $\angle s = \angle v$;

$\therefore$ § 102, $\triangle ABE = \triangle CDE$,

§ 108, $AE = CE$, and $BE = DE$.

Therefore, etc. Q.E.D.

Ex. **100.** If the diagonals of a quadrilateral are equal and bisect each other, what kind of a figure is the quadrilateral?

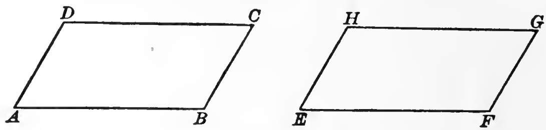
Ex. **101.** From a figure representing a parallelogram and its diagonals, select four pairs of equal triangles.

Proposition XL

155. 1. Draw two parallelograms such that two sides of one, and the angle between them, shall be equal to the corresponding parts of the other. How do the parallelograms compare?

2. How do two rectangles compare, if the base and altitude of one are equal to the corresponding parts of the other?

Theorem. *Two parallelograms are equal, if two sides and the included angle of one are equal to two sides and the included angle of the other, each to each.*



Data: Any two parallelograms, as $ABCD$ and $EFGH$, in which $AB = EF$, $AD = EH$, and angle A = angle E.

To prove parallelograms $ABCD$ and $EFGH$ equal.

Proof. Place $\square EFGH$ upon $\square ABCD$ so that EF coincides with its equal AB and $\angle E$ with its equal $\angle A$.

Then, EH coincides with its equal AD,

§ 70, HG takes the direction of DC,

and G falls upon DC, or upon DC produced.

Also, FG takes the direction of BC,

and G falls upon BC, or upon BC produced.

Since G falls upon both DC and BC, it must fall upon their point of intersection C, which is the only point common to DC and BC.

Hence, § 36, $\square ABCD = \square EFGH$.

Therefore, etc. Q.E.D.

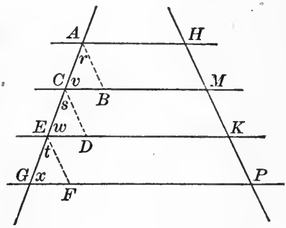
156. Cor. *Two rectangles are equal, if the base and altitude of one are equal to the base and altitude of the other, each to each.*

Ex. 102. From any point in the base of an isosceles triangle lines are drawn parallel to the equal sides and produced until they meet the sides of the triangle. How does the sum of these two lines compare with one of the equal sides of the triangle?

Proposition XLI

157. Draw three or more parallel lines intercepting equal parts on a transversal; draw any other transversal. How do the parts which the parallels intercept on the second transversal compare in length?

Theorem. *If three or more parallel lines intercept equal parts on any transversal, they intercept equal parts on every transversal.*



Data: Any parallel lines, as AH, CM, EK, and GP, intercepting the equal parts AC, CE, and EG on the transversal AG, and the parts HM, MK, and KP on any other transversal, as HP.

To prove $HM = MK = KP$.

Proof. Draw AB, CD, and EF each parallel to HP.

Then, § 140, $ABMH$, $CDKM$, and $EFPK$ are parallelograms,

and, § 153, $HM = AB$, $MK = CD$, and $KP = EF$.

Now, in △ ABC, CDE, and EFG,

§ 80, $AB \parallel CD \parallel EF$;

hence, § 76, $\angle r = \angle s = \angle t$,

$\angle v = \angle w = \angle x$,

and, data, $AC = CE = EG$;

∴ § 102, $\triangle ABC = \triangle CDE = \triangle EFG$,

and $AB = CD = EF$.

But since $HM = AB$, $MK = CD$, and $KP = EF$,

Ax. 1, $HM = MK = KP$.

Therefore, etc. Q.E.D.

Ex. **103.** In the triangle ABC angle A is double angle B and the exterior angle at C is 105°. How many degrees are there in angles A and B respectively?

Ex. **104.** If one angle of a parallelogram is a right angle, what is the value of each of the other angles?

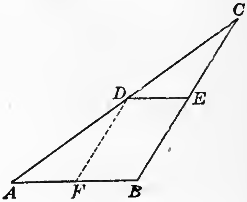
Ex. **105.** One angle of a parallelogram is three times its supplement. What is the value of each angle of the parallelogram?

Proposition XLII

158. 1. Draw a triangle and a line parallel to the base, bisecting one of the sides. How does it divide the other side? How does the part of this line intercepted by the sides of the triangle compare in length with the base of the triangle?

2. Draw a triangle and a line connecting the middle points of two of its sides. What is the direction of this line with reference to the third side of the triangle?

Theorem. *If a straight line drawn parallel to the base of a triangle bisects one of its sides, it bisects the other side, and is equal to one half of the base.*



Data: Any triangle, as ABC, and a straight line DE drawn parallel to AB bisecting AC at D.

To prove 1. $BE = EC$.

2. $DE = \frac{1}{2} AB$.

Proof. 1. Draw $FD \parallel BC$.

Then, in $\triangle$s DEC and AFD,

data,	$DC = AD$,
§ 76,	$\angle DCE = \angle ADF$,
and	$\angle CDE = \angle DAF$;
∴ § 102,	$\triangle DEC = \triangle AFD$,
and	$EC = FD$.
But, § 151,	$BE = FD$;
∴ Ax. 1,	$BE = EC$.
2. § 108,	$AF = DE$,
and, § 151,	$FB = DE$;
∴	$AF = FB = \frac{1}{2} AB$;
but	$FB = DE$; Why?
hence,	$DE = \frac{1}{2} AB$.

Therefore, etc. Q.E.D.

159. Cor. *The line joining the middle points of two sides of a triangle is parallel to the third side.*

For if the line is not parallel to the third side, suppose a line drawn through D, the middle point of AC, parallel to AB. By § 158, it will pass through E, the middle point of BC, and we shall have two straight lines drawn between the same two points, which by Ax. 12 is impossible. Consequently, the line joining the middle points of two sides of a triangle is parallel to the third side.

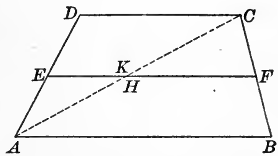
Proposition XLIII

160. Draw a trapezoid and a line connecting the middle points of the non-parallel sides. What is the direction of this line with reference to the bases of the trapezoid? How does it compare in length with the sum of the bases?

Theorem. *The line which joins the middle points of the non-parallel sides of a trapezoid is parallel to the bases and is equal to one half their sum.*

Data: Any trapezoid, as $ABCD$, and the line EF joining the middle points of the non-parallel sides AD and BC.

To prove EF parallel to AB and DC and equal to one half $AB + DC$.



Proof. Draw AC intersecting EF at K, and from H, the middle point of AC, draw HE and HF.

Data, $AE = ED$,
and, const., $AH = HC$;
∴ § 159, $HE \parallel DC$,
and, § 158, $HE = \frac{1}{2} DC$.
In like manner, $HF \parallel AB$ and $HF = \frac{1}{2} AB$.
Then, § 80, $HF \parallel DC$;
but $HE \parallel DC$;
∴ § 70, EHF is a straight line parallel to AB and DC.
But, data, EKF is a straight line,
then, EHF and EKF coincide,
and the point H coincides with the point K.
Now, HE, or $KE = \frac{1}{2} DC$,
and HF, or $KF = \frac{1}{2} AB$;
hence, Ax. 2, $EF = \frac{1}{2}(AB + DC)$.
Therefore, etc. Q.E.D.

POLYGONS

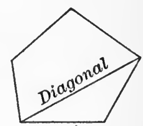
161. A portion of a plane bounded by any number of straight lines is called a **Polygon.**

The sum of the straight lines which bound a polygon is called its *perimeter.*

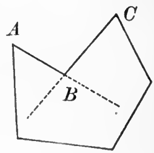
The term *polygon* is usually applied to figures of more than four sides.

162. A polygon of three sides is called a *trigon* or *triangle;* one of four sides, a *tetragon* or *quadrilateral;* one of five sides, a *pentagon;* one of six sides, a *hexagon;* one of seven sides, a *heptagon;* one of eight sides, an *octagon;* one of ten sides, a *decagon;* one of twelve sides, a *dodecagon;* one of fifteen sides, a *pentadecagon.*

163. A polygon such that none of its sides, if produced, extend within it is called a **Convex Polygon.**



164. A polygon such that two or more of its sides, if produced, extend within it is called a **Concave Polygon.**



The reflex angle ABC is called a *re-entrant angle.*

Unless otherwise stated, polygons considered hereafter will be understood to be convex.

165. A straight line joining the vertices of two non-adjacent angles of a polygon is called a **Diagonal of the Polygon.**

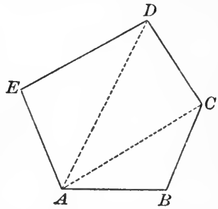
Proposition XLIV

166. 1. Draw convex polygons, each having a different number of sides, and from any vertex of each draw its diagonals. How does the number of triangles into which each polygon is divided compare with the number of sides of the polygon?

To how many right angles is the sum of the angles of a triangle equal? To how many times two right angles is the sum of the interior angles of a polygon equal?

2. Produce the sides of any polygon in succession. To how many right angles is the sum of all the exterior and interior angles equal? To how many right angles is the sum of the exterior angles of a polygon equal?

Theorem. *The sum of the angles of any convex polygon is equal to twice as many right angles as the polygon has sides less two.*



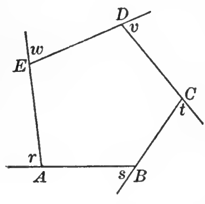
Data: A convex polygon of any number (n) of sides, as *ABCDE*.

To prove the sum of the angles, A, B, C, D, and E equal to twice as many right angles as the polygon has sides less two.

Proof. From any vertex, as A, draw the diagonals, AC and AD. The number of triangles thus formed is two less than the number of sides of the polygon, or $(n-2)$ triangles.

By § 110, the sum of the angles of each triangle is equal to two right angles, therefore, the sum of the angles of all the triangles; that is, the sum of the angles of the polygon is equal to $(n-2)$ 2 rt. ∡s.

Therefore, etc. Q.E.D.



167. Cor. *The sum of the exterior angles of any convex polygon formed by producing the sides of the polygon in succession is equal to four right angles.*

Ex. **106.** If from the extremities of the shorter base of an isosceles trapezoid lines are drawn parallel to the equal sides, two triangles are formed. How do they compare?

Ex. **107.** If in a parallelogram any two points in a diagonal equally distant from its extremities are joined to the vertices of the opposite angles, what kind of a figure is thus formed?

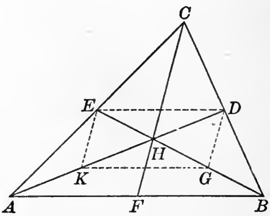
Ex. **108.** How many degrees are there in each angle of an equiangular polygon of five sides?

Ex. **109.** How many sides has a polygon the sum of whose interior angles is double the sum of its exterior angles?

Proposition XLV

168. Draw any triangle and its three medians. Do the medians intersect in a point? Measure the distance from this point to each vertex. How do these distances compare with the medians of which they are a part?

Theorem. *The medians of a triangle pass through a point which is two thirds of the distance from each vertex to the middle of the opposite side.*



Data: Any triangle, as ABC, and its medians, AD, BE, and CF.

To prove that AD, BE, and CF pass through a point, which is two thirds of the distance from A, B, and C to the middle of the opposite sides respectively.

Proof. Since two of the medians will intersect, if sufficiently produced, it needs to be shown only that the third median passes through the point of intersection, to prove that the three pass through the same point.

Let any two of the medians, as AD and BE, intersect at H.

Draw KG, joining K and G, the middle points of AH and BH respectively; also draw KE, ED, and GD.

Then, § 159, $KG \parallel AB$, and $ED \parallel AB$;

∴ § 80, $KG \parallel ED$;

also, § 158, $KG = \frac{1}{2} AB$, and $ED = \frac{1}{2} AB$;

∴ $KG = ED$;

and, § 150, $KGDE$ is a parallelogram.

Hence, § 154, $KH = HD$, and $GH = HE$.

But since, const., $AK = KH$, and $BG = GH$,

$AH = \frac{2}{3} AD$, and $BH = \frac{2}{3} BE$.

Then, since the medians from any two vertices intersect in a point which is two thirds of the distance from each vertex to the middle of the opposite side, the median from C intersects AD at H.

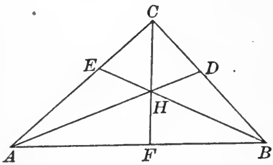
That is, CF passes through H, and $CH = \frac{2}{3} CF$.

Therefore, etc. Q.E.D.

Proposition XLVI

169. Draw any triangle and lines bisecting its angles. Do these lines intersect in a point? How do the distances of the point from the sides of the triangle compare?

Theorem. *The bisectors of the three angles of a triangle pass through a point which is equidistant from the sides of the triangle.*



Data: Any triangle, as ABC, and the lines AD, BE, and CF, bisecting the angles A, B, and C respectively.

To prove that AD, BE, and CF pass through a point which is equidistant from AB, BC, and AC.

Proof. Since two of the bisectors will intersect, if sufficiently produced, it needs to be shown only that the third bisector passes through the point of intersection to prove that the three pass through the same point.

Let any two of the bisectors, as AD and BE, intersect in H.

Then, § 134, H is equidistant from AB and AC, and also from AB and BC.

Hence, H is equidistant from AC and BC;

∴ § 135, H lies in the bisector of angle C.

That is, CF passes through the point H, which is equidistant from AB, BC, and AC.

Therefore, etc. Q.E.D.

Ex. **110.** How does the angle formed by the diagonals of a square compare with a right angle?

Ex. **111.** How does the angle formed by the diagonals of a rhombus compare with a right angle? How do the diagonals divide each other?

Ex. **112.** How do the diagonals of a rectangle compare in length?

Ex. **113.** If from any point in the bisector of an angle straight lines are drawn parallel to the sides of the angle and are produced to meet the sides, what figure is thus formed?

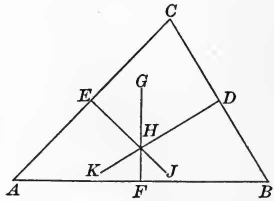
Ex. **114.** The difference between two angles of a parallelogram which have a common side is 60°. What is the value of each angle of the parallelogram?

Ex. **115.** If the middle points of any two opposite sides of a quadrilateral are joined to each of the middle points of the diagonals, what kind of a figure will the four joining lines form?

Proposition XLVII

170. Draw any triangle and lines perpendicular to its sides, bisecting them. Do these lines intersect in a point? How do the distances of the point from the vertices of the triangle compare?

Theorem. *The perpendicular bisectors of the sides of a triangle pass through a point which is equidistant from the vertices of the triangle.*



Data: Any triangle, as ABC, and FG, DK, and EJ, the perpendicular bisectors of AB, BC, and AC respectively.

To prove that FG, DK, and EJ pass through a point which is equidistant from A, B, and C.

Proof. Since two of the perpendiculars will intersect, if sufficiently produced (why?), it needs to be shown only that the third perpendicular passes through the point of intersection, to prove that the three pass through the same point.

Let any two of the perpendiculars, as FG and DK, intersect at H.

Then, § 103, H is equidistant from A and B, and also from B and C.

Hence, H is equidistant from A and C;

∴ § 104, H lies in the perpendicular bisector of AC.

That is, EJ passes through the point H, which is equidistant from A, B, and C.

Therefore, etc. Q.E.D.

Ex. **116**. The middle points of the sides of an equilateral triangle are joined. What kind of triangles are formed?

Ex. **117**. How do the lines drawn from the middle points of the equal sides of an isosceles triangle to the opposite extremities of the base compare in length?

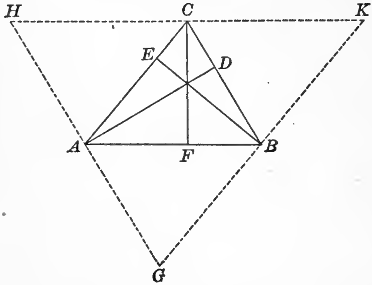
Ex. **118**. The parallel sides of a trapezoid are 35 and 55 feet respectively. What is the length of the line joining the middle points of the non-parallel sides?

Ex. **119**. If from the extremities of the shorter base of an isosceles trapezoid perpendiculars are drawn to the longer base, two triangles are formed. How do they compare?

Proposition XLVIII

171. Draw any triangle and lines from the vertices perpendicular to the opposite sides. Do these lines intersect in a point?

Theorem. *The perpendiculars from the vertices of a triangle to the opposite sides pass through the same point.*



Data: Any triangle, as ABC, and the lines AD, BE, and CF drawn from the vertices A, B, and C respectively, perpendicular to the opposite sides.

To prove that AD, BE, and CF pass through the same point.

Proof. Through the vertices A, B, and C draw GH, GK, and HK parallel to BC, AC, and AB respectively, and intersecting in G, H, and K.

§ 153, $AG = BC$,

and $AH = BC$;

$\therefore$ Ax. 1, $AG = AH$.

§ 72, $AD \perp GH$;

$\therefore$ AD is the perpendicular bisector of GH.

In like manner,

BE is the perpendicular bisector of GK,

and CF is the perpendicular bisector of HK.

Hence, § 170, AD, BE, and CF pass through the same point.

Therefore, etc. Q.E.D.

Ex. **120.** How many sides has a polygon the sum of whose exterior angles is double the sum of its interior angles?

Ex. **121.** How many sides has a polygon the sum of whose interior angles is equal to the sum of its exterior angles?

Ex. **122.** The perimeter of an isosceles triangle is 176 feet, and the base is $1\frac{1}{5}$ times one of the equal sides. What is the length of each side of the triangle?

Ex. **123.** How many sides has an equiangular polygon which can be divided into equilateral triangles by lines drawn from a point within to the vertices of the polygon?

SUMMARY

172. Truths established in Book I.

1. Two lines are equal,

a. If they can be made to coincide. § 36

b. If they are sides of an equilateral triangle. § 91

c. If they represent the distances from the extremities of a straight line to any point in the perpendicular erected at its middle point. § 103

d. If they are homologous sides of equal triangles. § 108

e. If they are sides of a triangle opposite equal angles. § 118

f. If they are sides of an equiangular triangle. § 119

g. If they are drawn from any point in a perpendicular to a line and cut off equal distances on that line from the foot of the perpendicular. § 132

h. If they represent the distances of any point in the bisector of an angle from its sides. § 134

i. If they are the non-parallel sides of an isosceles trapezoid. § 139

j. If they are the sides of a square. § 143

k. If they are the sides of a rhombus. § 144

l. If they are parallel and are intercepted between parallel lines. § 151

m. If they are opposite sides of a parallelogram. § 153

n. If they are parts intercepted on one transversal by parallel lines which intercept equal parts on another transversal. § 157

o. If one is half a side of a triangle and the other is drawn parallel to it and bisecting one of the other sides. § 158

p. If one joins the middle points of the non-parallel sides of a trapezoid and the other is equal to half the sum of the parallel sides. § 160

2. Two lines are parallel,

a. If both are perpendicular to the same line. § 71

b. If when cut by a transversal the alternate interior angles are equal. § 75

c. If when cut by a transversal the corresponding angles are equal. § 77

d. If when cut by a transversal the sum of the two interior angles on the same side of the transversal is equal to two right angles. § 79

e. If both are parallel to a third line. § 80

f. If they are the bases of a trapezoid. § 138

g. If they are opposite sides of a parallelogram. § 140

h. If one is a side of a triangle and the other joins the middle points of the other two sides. § 159

i. If one is either base of a trapezoid and the other joins the middle points of the non-parallel sides. § 160

3. Two lines are perpendicular to each other,

a. If they form equal adjacent angles with each other. § 26

b. If one is perpendicular to a line which is parallel to the other. § 72 *c.* If any two or more points in one are each equidistant from the extremities of the other. §§ 106, 104

d. If one is the base of an isosceles triangle and the other is the bisector of the vertical angle. § 120

4. Two lines form one and the same straight line,

a. If they are the sides of a straight angle. § 27

b. If they are the exterior sides of adjacent supplementary angles. § 58

5. Two lines are unequal,

a. If one is a perpendicular from a point to a straight line and the other is any other line from that point to the straight line. § 61

b. If they represent the distances from the extremities of a straight line to any point without the perpendicular erected at its middle point. § 103

c. If they are sides of a triangle and lie opposite unequal angles. § 127

d. If they are the third sides of two triangles whose other sides are equal, each to each, and include unequal angles. § 129

e. If they are drawn from any point in a perpendicular to a line and cut off unequal distances on that line from the foot of the perpendicular. § 132

f. If they are distances cut off on a line from the foot of a perpendicular to it by unequal lines from any point in the perpendicular. § 133

g. If one is any side of a triangle and the other is equal to the sum of the other two sides. §§ 124, 125

h. If one is equal to the sum of two lines from a point within a triangle to the extremities of one side, and the other is equal to the sum of the other two sides. § 131

6. A line is bisected,

a. If it is the base of an isosceles triangle, by the bisector of the vertical angle. § 120

b. If it is the base of an isosceles triangle, by a perpendicular from the vertex. § 122

c. If it is either diagonal of a parallelogram, by the other diagonal. § 154

d. If it is the side of a triangle, by a straight line drawn parallel to the base and bisecting the other side. § 158

7. Lines pass through the same point,

a. If they are the medians of a triangle. § 168

b. If they are the bisectors of the three angles of a triangle. § 169

c. If they are the perpendicular bisectors of the sides of a triangle. § 170

d. If they are perpendiculars from the vertices of a triangle to the opposite sides. § 171

8. A perpendicular, and only one, can be drawn to a straight line,

a. At a point in the line. § 51

b. From a point without the line. § 60

9. Two angles are equal,

a. If they can be made to coincide. § 36

b. If they are right angles. § 52

c. If they are straight angles. § 53

d. If they are complements of equal angles. § 54

e. If they are supplements of equal angles. § 54

f. If they are vertical angles. § 59

g. If they are alternate interior angles formed by a transversal and parallel lines. § 73

h. If they are corresponding angles formed by a transversal and parallel lines. § 76

i. If their sides are parallel and both pairs extend in the same or in opposite directions from their vertices. § 81

j. If their sides are perpendicular to each other and both angles are acute or both are obtuse. § 83

k. If they are angles of an equiangular triangle. § 98

l. If they are formed adjacent to a straight line by lines joining the extremities of that line with any point in its perpendicular bisector. § 105

m. If they are formed by the perpendicular bisector of a straight line and lines from any point in it to the extremities of the straight line. § 105

n. If they are homologous angles of equal triangles. § 108

o. If they are the third angles of two triangles whose other angles are equal, each to each. § 113

p. If they are opposite the equal sides of an isosceles triangle. § 116

q. If they are angles of an equilateral triangle. § 117

r. If they are the opposite angles of a parallelogram. § 149

10. Two angles are supplementary,

a. If their sum is equal to two right angles. § 32

b. If their corresponding sides are parallel and one pair extends in the same direction and the other in opposite directions from their vertices. § 81

c. If their corresponding sides are perpendicular and one angle is acute and the other obtuse. § 83

11. Two angles are unequal,

a. If they are angles of a triangle and lie opposite unequal sides. § 126

b. If they are the angles opposite unequal sides of two triangles whose other two sides are equal, each to each. § 130

12. An angle is bisected,

a. If it is the vertical angle of an isosceles triangle, by the perpendicular bisector of the base. § 121

b. If it is the vertical angle of an isosceles triangle, by a line from the vertex perpendicular to the base. § 122

c. By a line every point of which is equidistant from the sides of the angle. § 155

13. An angle is equal to the sum of two angles,

a. If it is an exterior angle of a triangle, and the two angles are the opposite interior angles. § 115

14. The sum of angles is equal to a right angle,

a. If they are complements of each other. § 31

b. If they are the acute angles of a right triangle. § 111

15. The sum of angles is equal to two right angles,

a. If they are supplements of each other. § 32

b. If they are adjacent angles formed by one straight line meeting another. § 55

c. If they are all the consecutive angles which have a common vertex in a line and lie on the same side of the line. § 56

d. If they are the interior angles formed by a transversal and parallel lines and lie on the same side of the transversal. § 78

e. If they are the angles of a triangle. § 110

16. The sum of angles is equal to four right angles,

a. If they are all the consecutive angles that can be formed about a point. § 57

b. If they are the exterior angles of any convex polygon formed by producing the sides in succession. § 167

17. The sum of angles is equal to $(n-2)$ 2 rt. $\angle$s,

a. If they are the angles of any convex polygon. § 166

18. Two triangles are equal,

a. If two sides and the included angle of one are equal to the corresponding parts of the other. § 100

b. If a side and two adjacent angles of one are equal to the corresponding parts of the other. § 102

c. If the three sides of one are equal to the three sides of the other. § 107

d. If they are right triangles, and a side and an acute angle of one are equal to the corresponding parts of the other. § 114

e. If they are right triangles, and the hypotenuse and a side of one are equal to the corresponding parts of the other. § 122

f. If they are formed by dividing a parallelogram by one of its diagonals. § 152

19. Two parallelograms are equal,

a. If they can be made to coincide. § 36

b. If two sides and the included angle of one are equal to the corresponding parts of the other. § 155

20. A quadrilateral is a parallelogram,

a. If its opposite sides are parallel. § 140

b. If its opposite sides are equal. § 148

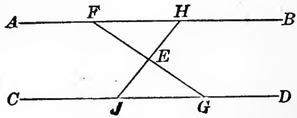
c. If two of its sides are equal and parallel. § 150

SUPPLEMENTARY EXERCISES

Ex. **124.** *If through a point halfway between two parallel lines two transversals are drawn, they intercept equal parts on the parallel lines.*

Suggestions for Demonstration. 1. What are the data of the proposition?

2. What lines and point in the figure in the margin represent the data of the proposition?



3. What parts of the figure are to be proved equal?

4. How may two lines be proved equal? Summary, § 172, 1.

5. Since FH and JG, which are to be proved equal, are parts of triangles, what propositions might we expect to employ in the proof?

6. In what ways may two triangles be proved equal? Summary, § 172, 18.

7. What facts in the data suggest aid in determining the equality of angles? *Ans.* Parallel lines.

8. What homologous angles in the two triangles are equal?

9. What other homologous elements of the two triangles must also be equal before the triangles can be proved to be equal?

10. By careful examination of the given figure discover whether any two homologous sides can be proved equal.

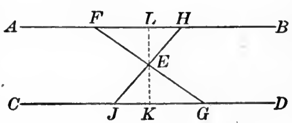
11. Since the homologous sides cannot be proved equal from the given figure, if they can be proved equal at all, what expedient must be resorted to?

Ans. Construction lines must be drawn which will enable us to prove a side of one of the triangles equal to an homologous side of the other.

12. What fact in the data has not yet been considered which might suggest aid in drawing the construction lines?

13. What kind of a line measures the distance between two parallel lines? If such a line be drawn through the given point, how is it divided at the given point? Then, what line may aid in the proof?

14. Drawing the figure as in the margin, with LK perpendicular to the parallel lines, and passing through the point E, which is halfway between the parallel lines, discover how the triangles FEL and GEK compare; also, how FE and GE compare.



15. Since the homologous angles of the original triangles have been discovered to be equal, and since the equality of two homologous sides, FE and GE, has also been shown, how do the original triangles compare? How do the sides FH and JG compare?

Write out the demonstration.

General Suggestions. I. *Study the theorem carefully to discover the data.*

II. *Construct a figure, or figures, to correspond with the data.*

III. *Discover what parts of the figure correspond to the conclusion given in the theorem.*

IV. *Study the theorem and the figure to discover as many truths as possible regarding the lines, angles, or other parts.*

V. *Keeping in mind the truths just discovered and the facts to be proved, consult the Summary and find which truth will best aid in establishing the proposition.*

VI. *If none of the truths in the Summary seem to be directly applicable to the demonstration sought, draw construction lines which may aid in applying some one of the truths.*

Ex. **125**. A straight line cutting the sides of an isosceles triangle and parallel to the base makes equal angles with the sides.

Ex. **126**. If the base of a triangle is divided into two parts by a perpendicular from the vertex, each part of the base is less than the adjacent side of the triangle.

Ex. **127**. Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.

Ex. **128**. The perpendiculars to the diagonal of a parallelogram from the opposite vertices are equal.

Ex. **129**. If one side of a quadrilateral is extended in both directions, the sum of the exterior angles formed is equal to the sum of the two interior angles opposite the side produced.

Ex. **130**. If in an isosceles triangle perpendiculars are drawn from the middle point of the base to the equal sides, the perpendiculars are equal.

Ex. **131**. A straight line drawn from any point in the bisector of an angle to either side and parallel to the other side makes, with the bisector and the side it meets, an isosceles triangle.

Ex. **132**. The difference between two sides of a triangle is less than the third side.

Ex. **133**. Any straight line through the middle point of a diagonal of a parallelogram, and terminated by the opposite sides, is bisected at that point.

Ex. **134**. If either of the equal sides of an isosceles triangle is produced through the vertex, the line bisecting the exterior angle thus formed is parallel to the base of the triangle.

Ex. **135**. If the bisector of one of the angles of a triangle meets the opposite side, the lines from the point of meeting parallel to the other sides and terminated by them are equal.

Ex. **136.** If each of the angles at the base of an isosceles triangle is one fourth the vertical angle, every line perpendicular to the base forms an equilateral triangle with the other two sides, produced when necessary.

Ex. **137.** If the straight line bisecting an exterior angle of a triangle is parallel to a side, the triangle is isosceles.

Ex. **138.** If the non-parallel sides of a trapezoid are equal, the base angles are equal, and the diagonals are equal.

SUGGESTION. Through one extremity of the shorter parallel side draw a line parallel to the opposite non-parallel side.

Ex. **139.** If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are also equal.

SUGGESTION. Produce the non-parallel sides.

Ex. **140.** If the upper base of an isosceles trapezoid equals the sum of the non-parallel sides, lines drawn from the middle point of the upper base to the extremities of the lower divide the figure into three isosceles triangles.

Ex. **141.** The opposite angles of an isosceles trapezoid are supplements of each other.

Ex. **142.** The segments of the diagonals of an isosceles trapezoid form with the upper and lower bases two isosceles triangles.

Ex. **143.** The triangle formed by joining the middle points of the sides of an isosceles triangle is isosceles.

Ex. **144.** If the two angles at the base of an isosceles triangle are bisected, the line which joins the intersection of the bisectors with the vertex of the triangle bisects the vertical angle.

SUGGESTION. Refer to § 172, 9, n.

Ex. **145.** $ABCD$ is a parallelogram; E and F are the middle points of AD and BC respectively. Show that BE and FD trisect the diagonal AC.

SUGGESTION. Refer to § 172, 6, d.

Ex. **146.** The exterior angle of an equiangular hexagon is equal to the interior angle of an equiangular triangle.

Ex. **147.** If one diagonal of a quadrilateral bisects two of the angles, it is perpendicular to the other diagonal.

Ex. **148.** If one triangle has two sides and a median to one of them equal respectively to the corresponding parts of another triangle, the triangles are equal.

Ex. **149.** The diagonals of a rhombus are unequal.

Ex. **150.** If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

SUGGESTION. From the vertex of $\angle B$ which is equal to the sum of the other two angles, draw BD to meet AC at D, making $\angle ABD = \angle BAD$.

Ex. **151.** If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Ex. **152.** The bisectors of two adjacent angles of a parallelogram intersect each other at right angles.

Ex. **153.** If the bisectors of the equal angles of an isosceles triangle are produced until they meet, they form with the base an isosceles triangle.

Ex. **154.** The diagonals of a rhombus bisect the opposite angles.

Ex. **155.** If two equal straight lines bisect each other at right angles, the lines joining their extremities form a square.

Ex. **156.** If the base of any triangle is produced in both directions, the sum of the two exterior angles diminished by the vertical angle is equal to two right angles.

Ex. **157.** In a quadrilateral, if two opposite sides which are not parallel are produced to meet, the perimeter of the greater triangle thus formed is greater than the perimeter of the quadrilateral.

Ex. **158.** If from any point in the base of an isosceles triangle lines parallel to the sides are drawn, a parallelogram is formed whose perimeter is equal to the sum of the equal sides of the triangle.

Ex. **159.** $ABCD$ is a quadrilateral having angle ABC equal to angle ADC; AB and DC produced meet in E; AD and BC in F. Show that angle AED equals angle AFB.

Ex. **160.** ABC is an isosceles triangle having AC equal to BC, and AC is produced through C its own length to D. Then, ABD is a right angle.

Ex. **161.** ABC is a triangle, and through D, the intersection of the bisectors of the angles B and C, EDF is drawn parallel to BC, meeting AB in E and AC in F. Then $EF = EB + FC$.

Suggestion. Prove $ED = EB$ and $FD = FC$.

Ex. **162.** $ABCD$ is an isosceles trapezoid and AC and BD its diagonals intersecting at O. Prove that the following pairs of triangles are equal: ABC and ABD; ADC and BDC; AOD and BOC.

Ex. **163.** In any right triangle the line drawn from the vertex of the right angle to the middle of the hypotenuse is equal to one half the hypotenuse.

Suggestion. From the middle point of the hypotenuse draw a line parallel to one of the other sides.

Ex. **164.** If through each of the vertices of a triangle a line is drawn parallel to the opposite side, a new triangle is formed equal to four times the given triangle.

Ex. **165.** Two equal lines, AB and CD, intersect at E, and the triangles CAE and BDE are equal. Show that CB is parallel to AD.

Ex. **166.** ABC and ABD are two equilateral triangles on opposite sides of the same base; BE and BF are the bisectors respectively of the angles ABC and ABD, meeting AC and AD in E and F respectively. Then, the triangle BEF is equilateral.

Suggestion. Refer to § 172, 1, f.

Ex. **167.** ABC is any triangle, and on AB and AC equilateral triangles ADB and AEC are constructed externally. Show that CD equals BE.

Suggestion. Refer to § 172, 18. a

Ex. **168.** If through the extremities of each diagonal of a quadrilateral lines parallel to the other diagonal are drawn, a parallelogram double the given quadrilateral will be formed.

Ex. **169.** $ABCD$ is a parallelogram; E and F are points on AC, such that $AE = FC$; G and H are points on BD, such that $BG = HD$. Then, $EGFH$ is a parallelogram.

Ex. **170.** The lines joining the middle points of the sides of a rhombus taken in order form a rectangle.

Ex. **171.** The bisector of the vertical angle of a triangle and the bisectors of the exterior angles at the base formed by producing the sides about the vertical angle meet in a point which is equidistant from the base and the sides produced.

Suggestion. Use the method of proof employed in Prop. XLVI.

Ex. **172.** If in a right triangle one of the acute angles is twice the other, the hypotenuse is equal to twice the side opposite the smaller acute angle.

Suggestion. From the vertex of the right angle draw a line to the hypotenuse, making with one side an angle equal to the acute angle adjacent to that side.

Ex. **173.** A parallelogram is bisected by any straight line passing through the middle point of one of its diagonals.

Ex. **174.** If two quadrilaterals have three sides and the two included angles of one equal, each to each, to three sides and the two included angles of the other, the quadrilaterals are equal.

Suggestion. Draw homologous diagonals.

Ex. **175.** If two quadrilaterals have three angles and the two included sides of one equal, each to each, to three angles and the two included sides of the other, the quadrilaterals are equal.

Ex. **176.** The bisectors of the exterior angles of a rectangle form a square.

Ex. **177.** The bisectors of the interior angles of a parallelogram form a rectangle.

Ex. **178.** The bisectors of the exterior angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary.

Ex. **179.** The bisectors of the interior angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary.

Ex. **180.** The straight line drawn from any vertex of a triangle to the middle point of the opposite side is less than half the sum of the other two sides.

Suggestion. Draw lines from the middle point of the side opposite the given vertex, parallel to each of the other two sides.

Ex. **181.** The lines which join the middle points of the sides of a quadrilateral successively form a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral.

BOOK II

CIRCLES

173. A plane figure bounded by a curved line, every point of which is equally distant from a point within, is called a **Circle**; the point within is called the **Center**; and the bounding line is called the **Circumference.**

174. Circles which have the same center are called **Concentric Circles.**

175. A straight line from the center to the circumference of a circle is called a **Radius of the circle.**

OB, Fig. 1, is a *radius* of the circle *AEDC.*

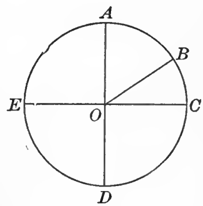


FIG. 1.

176. A straight line which passes through the center of a circle, and whose extremities are in the circumference, is called a **Diameter of the circle.**

AD and *EC*, Fig. 1, are *diameters* of the circle *AEDC.*

177. Any part of a circumference is called an **Arc.**

The curved lines between *A* and *B*, and between *A* and *E*, Fig. 1, are *arcs* of the circumference *AEDC.*

178. An arc which is one half a circumference is called a **Semicircumference.**

179. A straight line which joins any two points in a circumference is called a **Chord of the circle.**

A chord which joins the extremities of an arc is said to *subtend* that arc, and an arc is said to *be subtended* by its chord.

Every chord of a circle subtends two arcs.

The chord *AB*, Fig. 2, *subtends* the arc *ADB*, and also the arc *ACB*.

When a chord and its subtended arc are mentioned, the less arc is meant unless otherwise specified.

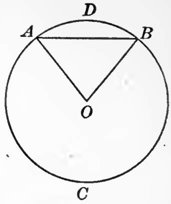


FIG. 2.

180. The part of a circle included between an arc and its chord is called a **Segment of the circle.**

ADB, Fig. 2, is a *segment* of the circle *ADBC*.

181. A segment which is one half a circle is called a **Semicircle.**

182. The part of a circle included between an arc and the radii drawn to its extremities is called a **Sector of the circle.**

OADB, Fig. 2, is a *sector* of the circle *ADBC*.

183. An arc which is one fourth of a circumference, or a sector which is one fourth of a circle, is called a **Quadrant.**

184. A straight line which touches a circle and does not cut the circumference if produced is called a **Tangent of the circle.**

The *circle* is then said to be *tangent* to the line.

The point where a tangent touches a circle is called the *point of tangency*, or the *point of contact.*

AB, Fig. 3, is a *tangent* of the circle; and *P* is the *point of tangency.*

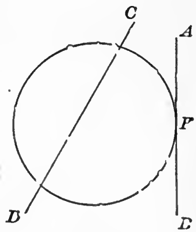


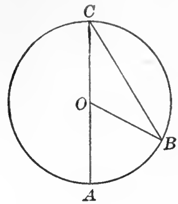
FIG. 3.

185. Two circles are said to be *tangent to each other*, if they are both tangent to the same straight line at the same point.

They are tangent *internally* or *externally* according as one circle lies within or without the other.

186. A straight line which cuts a circumference in two points is called a **Secant of the circle.**

CD, Fig. 3, is a *secant* of the circle.

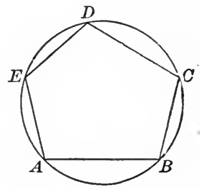


187. An angle whose vertex is at the center of a circle, and whose sides are radii of the circle, is called a **Central Angle.**

Angle AOB is a *central angle.*

188. An angle whose vertex is in the circumference of a circle, and whose sides are chords, is called an **Inscribed Angle.**

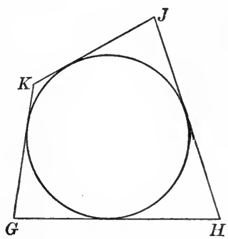
Angle ACB is an *inscribed angle.*



An angle whose vertex is in the arc of a segment, and whose sides pass through the extremities of the arc, is said to be *inscribed in the segment.*

189. A *polygon* is said to be *inscribed in a circle,* if the vertices of its angles are in the circumference.

The *circle* is then said to be *circumscribed* about the polygon.



190. A *polygon* is said to be *circumscribed about a circle,* if each side is tangent to the circle.

The *circle* is then said to be *inscribed* in the polygon.

191. Ax. 14. ***All radii of the same circle, or of equal circles, are equal.***

15. *All diameters of the same circle, or of equal circles, are equal.*

16. *Two circles are equal, if their radii or diameters are equal.*

17. *A tangent has only one point in common with a circle.*

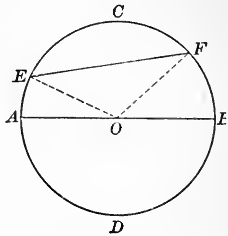
Proposition I

192. 1. Draw a circle and one of its diameters; draw several chords of the circle. How does the diameter compare in length with any other chord? How does the diameter divide the circle? The circumference?

2. How do two arcs of the same circle, or of equal circles, compare, if their extremities can be made to coincide?

Theorem. A diameter of a circle is greater than any other chord, and bisects the circle and its circumference.

Data: A circle whose center is O; any diameter, as AB; and any other chord, as EF.



To prove 1. AB greater than EF.

2. AB bisects the circle and its circumference.

Proof. 1. Draw the radii OE and OF.

§ 125, $OE + OF > EF$;

but, Ax. 14, $OE = AO$, and $OF = OB$;

∴ $AO + OB > EF$, or $AB > EF$.

2. § 176, AB passes through the center O.

Revolve the segment ACB upon AB as an axis until it comes into the plane of the segment ADB.

Then, the arc ACB must coincide with the arc ADB; for, if the arcs do not coincide, some points in the two arcs are unequally distant from the center.

But, § 173, every point in the arcs is equally distant from O.

Hence, the arcs ACB and ADB coincide and are equal.

That is, AB bisects the circle and its circumference.

Therefore, etc. Q.E.D.

193. Cor. *In the same circle, or in equal circles, arcs whose extremities can be made to coincide are equal.*

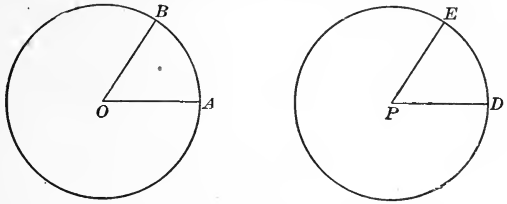
Proposition II

194. 1. Draw a circle and divide a part of its circumference into a number of equal arcs; from the points of division draw lines to the center. How do the central angles thus formed compare in size?

2. In a circle construct two equal central angles. How do the arcs which subtend them compare in size?

3. Draw a circle and take two unequal arcs; from their extremities draw lines to the center. How do the angles at the center compare in size? Which arc subtends the larger angle? Which angle at the center is subtended by the smaller arc?

Theorem. *In the same circle, or in equal circles, equal arcs subtend equal central angles;* **conversely,** *the sides of equal central angles intercept equal arcs.*



Data: The equal circles whose centers are O and P, and any equal arcs, as AB and DE.

To prove angle O = angle P.

Proof. Place the circle whose center is O upon the equal circle whose center is P so that arc AB coincides with the equal arc DE, and the point O with the point P,

then, OA coincides with PD, and OB with PE.

Hence, § 36, $\angle O = \angle P$. Q.E.D.

Conversely: Data: Any equal central angles in these circles, as $\angle O$ and $\angle P$.

To prove arc AB = arc DE.

Proof. Place the circle whose center is O upon the circle whose center is P so that the point O coincides with the point P, and OA takes the direction of PD.

Data, $\angle O = \angle P$;

∴ OB takes the direction of PE;

and, since, Ax. 14, $OA = PD$, and $OB = PE$;

A coincides with D, and B with E.

Hence, § 193, arc AB = arc DE.

Therefore, etc. Q.E.D.

195. Cor. *In the same circle, or in equal circles, the greater of two arcs subtends the greater central angle;* conversely, *the sides of the greater of two central angles intercept the greater arc.*

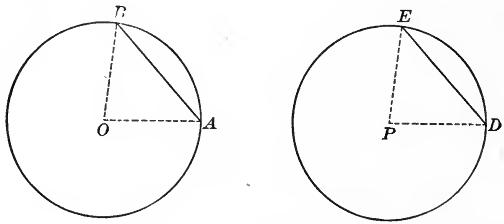
Proposition III

196. 1. Draw two equal circles and two equal chords, one in each circle. How do the subtended arcs compare in length?

2. Draw two chords, one in each of two equal circles, subtending equal arcs. How do the chords compare in length?

3. Draw two equal circles and two unequal chords, one in each circle. Which chord subtends the larger arc? Which arc is subtended by the less chord?

Theorem. *In the same circle, or in equal circles, equal chords subtend equal arcs; conversely, equal arcs have equal chords.*



Data: The equal circles whose centers are O and P, and any equal chords, as AB and DE.

To prove arc $AB =$ arc DE.

Proof. Draw the radii OA, OB, PD, and PE.

In the $\triangle$ OAB and PDE, $AB = DE$,

Ax. 14, $OA = PD$, and $OB = PE$;

$\therefore$ $\triangle OAB = \triangle PDE$, Why?

and $\angle O = \angle P$.

Hence, § 194, arc $AB =$ arc DE. Q.E.D.

Conversely: Data: Any two equal arcs in these circles, as AB and DE, and the chords subtending them.

To prove chord $AB =$ chord DE.

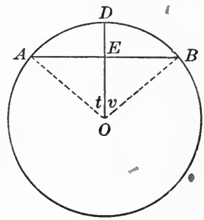
Proof. By the student.

197. Cor. *In the same circle, or in equal circles, the greater of two chords subtends the greater arc;* conversely, *the greater of two arcs has the greater chord.*

Proposition IV

198. Draw a circle and a chord of the circle; draw a radius perpendicular to the chord. How does this radius divide the chord? How does it divide the arc subtended by the chord?

Theorem. *A radius which is perpendicular to a chord bisects the chord and its subtended arc.*



Data: A circle whose center is O; any chord, as AB; and a radius, as OD, perpendicular to AB at E.

To prove $AE = BE$, and arc AD = arc BD.

Proof. Draw the radii OA and OB.

In the rt. ⊿s AEO and BEO,

	$OA = OB$,	Why?
and	OE is common;	
∴ § 123,	$\triangle AEO = \triangle BEO$,	
and	$AE = BE$.	Why?
Also,	$\angle t = \angle v$;	
hence, § 194,	arc AD = arc BD.	
Therefore, etc.		Q.E.D.

Ex. **182.** If two circumferences intersect, how does the distance between their centers compare with the difference of their radii?

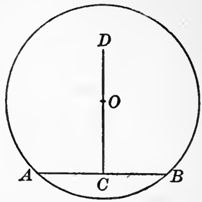
Proposition V

199. 1. Draw a chord of any circle and a perpendicular to that chord at its middle point. Determine whether the perpendicular passes through the center of the circle.

2. Draw a line through the center of a circle perpendicular to a chord. How does it divide the chord? How does it divide the subtended arc?

3. Draw a circle and two chords which are not parallel. Erect perpendiculars at their middle points and produce these perpendiculars until they intersect. At what point in the circle do the perpendicular bisectors of the chords meet?

Theorem. *A line perpendicular to a chord at its middle point passes through the center of the circle.*



Data: A circle whose center is O; any chord, as AB; and CD a perpendicular to AB at its middle point.

To prove that CD passes through O.

Proof. § 173, A and B are equally distant from O,
data, CD is the perpendicular bisector of AB;
hence, § 104, O must lie in CD;
that is, CD passes through O.

Therefore, etc. Q.E.D.

200. Cor. I. *A line passing through the center of a circle and perpendicular to a chord bisects the chord and its subtended arc.*

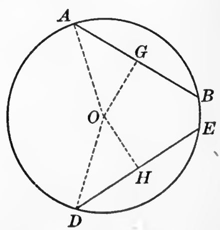
201. Cor. II. *The point of intersection of the perpendicular bisectors of two non-parallel chords is the center of the circle.*

Proposition VI

202. 1. Draw a circle and two equal chords. How do their distances from the center compare?

2. Draw a circle and two chords equally distant from the center. How do the chords compare in length?

Theorem. *In the same circle, or in equal circles, equal chords are equally distant from the center;* ***conversely,*** *chords equally distant from the center are equal.*



Data: Any two equal chords, as AB and DE, in the circle whose center is O.

To prove AB and DE equally distant from O.

Proof. Draw the radii OA and OD; also draw the perpendiculars OG and OH representing the distances from O to AB and DE, respectively.

Then, § 200, $AG = \frac{1}{2}AB$, and $DH = \frac{1}{2}DE$;

∴ Ax. 7, $AG = DH$.

In the rt. $\triangle$ AOG and DOH,

Ax. 14, $OA = OD$,

$AG = DH$;

∴ § 123, $\triangle AOG = \triangle DOH$,

and $OG = OH$; Why?

that is, AB and DE are equally distant from O. Q.E.D.

Conversely: Data: Any two chords, as AB and DE, equally distant from O, the center of the circle.

To prove $AB = DE$.

Proof. In the rt. $\triangle$ AOG and DOH,

data, $OG = OH$,

Ax. 14, $OA = OD$;

∴ § 123, $\triangle AOG = \triangle DOH$,

and $AG = DH$. Why?

But, § 200, $AG = \frac{1}{2}AB$, and $DH = \frac{1}{2}DE$;

hence, Ax. 6, $AB = DE$.

Therefore, etc. Q.E.D

Ex. **183.** In how many points can a straight line cut a circumference?

Ex. **184.** How many centers may a circle have?

Ex. **185.** If a straight line bisects a chord and its subtended arc, what is its direction with reference to the chord?

Ex. **186.** Do the perpendicular bisectors of the sides of an inscribed quadrilateral meet in a common point?

Ex. **187.** If a diameter of a circle bisects a chord, how does it divide the subtended arc? In what direction does it extend with reference to the chord?

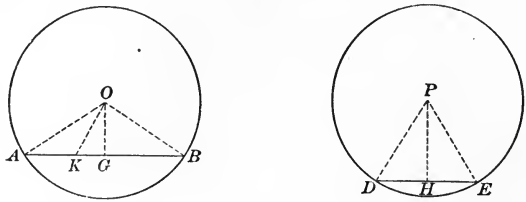
Ex. **188.** If a diameter of a circle bisects an arc, how does it divide the chord of the arc? What is its direction with reference to the chord?

Ex. **189.** If the distance from the center of a circle to a straight line is less than the radius, will the line cut the circumference? If the distance is greater than the radius, will the line cut the circumference?

Proposition VII

203. Draw two unequal chords in the same circle, or in equal circles. Which chord is nearer the center, the longer or the shorter one?

Theorem. *In the same circle, or in equal circles, the less of two unequal chords is at the greater distance from the center.*



Data: In the equal circles whose centers are O and P, any two chords, as AB and DE, of which DE is the less.

To prove DE at a greater distance from P than AB is from O.

Proof. Draw the radii OA, OB, PD, and PE; also draw the perpendiculars OG and PH representing the distances from O to AB, and from P to DE, respectively.

Data, $AB > DE$,

§ 200, $AG = \frac{1}{2} AB$, and $DH = \frac{1}{2} DE$;

$\therefore$ $AG > DH$.

Then, take AK equal to DH, and draw OK.

In the isosceles $\triangle$ ABO and DEP,

§ 130, $\angle AOB$ is greater than $\angle DPE$;

$\therefore$ $\angle PDE$ is greater than $\angle OAB$. Why?

Then, in $\triangle$ DHP and AKO, $DP = AO$, Why?

const., $DH = AK$,

and $\angle PDH$ is greater than $\angle OAK$;

$\therefore$ $PH > OK$; Why?

but $OK > OG$; Why?

$\therefore$ $PH > OG$;

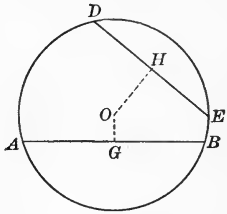
that is, DE is at a greater distance from P than AB is from O.

Therefore, etc. Q.E.D

Proposition VIII

204. In the same circle, or in equal circles, draw two chords unequally distant from the center. Which one is the shorter?

Theorem. *In the same circle, or in equal circles, of two chords unequally distant from the center, the one at the greater distance is the less.* (Converse of Prop. VII.)



Data: In the circle whose center is O, any two chords, as AB and DE, of which DE is at the greater distance from O.

To prove DE less than AB.

Proof. Draw the perpendiculars OG and OH representing the distances from O to AB and DE, respectively.

Now, if	$DE = AB$,
then, § 202,	$OH = OG$, which is contrary to data.
If	$DE > AB$,
then,	$AB < DE$,
and, § 203,	$OG > OH$, which is also contrary to data.

Then, since DE is neither equal to, nor greater than AB,

DE is less than AB.

Therefore, etc. Q.E.D.

Ex. **190.** Where does the line drawn through the middle points of two parallel chords in a circle pass with reference to the center?

Ex. **191.** If two circles are concentric, how do any two chords of the greater, which are tangent to the less, compare in length?

Ex. **192.** If an isosceles triangle is constructed on any chord of a circle as its base, where does the vertex lie with reference to the diameter that is perpendicular to the chord, or to that diameter produced?

Ex. **193.** If two chords of a circle cut each other and make equal angles with the straight line which joins their point of intersection with the center, how do the chords compare in length?

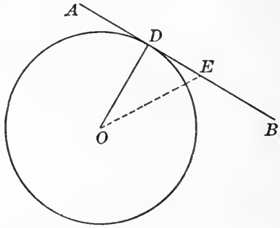
Ex. **194.** If from any point within a circle two equal straight lines are drawn to the circumference, where will the bisector of the angle thus formed pass with reference to the center of the circle?

Proposition IX

205. 1. Draw a circle and one of its radii; also a line perpendicular to the radius at its extremity. Is this line a tangent or a secant?

2. Draw a tangent to a circle and a radius to the point of contact. What kind of an angle is formed by these lines?

Theorem: *A line perpendicular to a radius at its extremity is tangent to the circle;* ***conversely,*** *a tangent is perpendicular to the radius drawn to the point of contact.*



Data: A circle whose center is O; any radius, as OD; and a line AB perpendicular to OD at D.

To prove AB tangent to the circle.

Proof. From O draw any other line to AB, as OE.

Then, $OD < OE$. Why?

Since every point in the circumference is at a distance equal to OD from the center, and E is at a greater distance, E is without the circumference.

Therefore, every point of AB, except D, is without the circumference.

Hence, § 184, AB is tangent to the circle at D. Q.E.D.

Conversely: Data: Any tangent to this circle, as AB, and the radius drawn to the point of contact, as OD.

To prove AB perpendicular to OD.

Proof. Ax. 17, every point of AB, except D, is without the circumference.

$\therefore$ OD is the shortest line that can be drawn between O and AB.

Hence, § 61, $OD \perp AB$.

Therefore, etc. Q.E.D.

Ex. **195.** If in a circle a chord is perpendicular to a radius at any point, how does it compare in length with any other chord which can be drawn through that point?

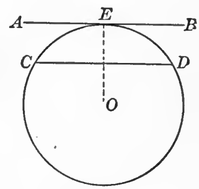
Ex. **196.** If tangents are drawn through the extremities of a diameter, what is their direction with reference to each other?

Proposition X

206. 1. Draw a circle, a tangent to it, and a chord parallel to the tangent. How do the arcs intercepted between the point of tangency and the extremities of the chord compare?

2. Draw a circle and two parallel secants or chords. How do the intercepted arcs compare?

Theorem. *Parallel lines intercept equal arcs on a circumference.*



Data: A circle whose center is O, and any two parallel lines, as AB and CD, intercepting arcs on the circumference.

To prove that the arcs intercepted by AB and CD are equal.

Proof. Case I. *When AB is a tangent and CD is a chord.*

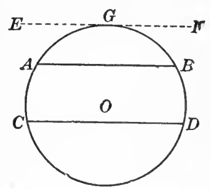
Draw to the point of tangency the radius OE.

Then, § 205, $OE \perp AB$;

∴ § 72, $OE \perp CD$;

hence, § 198, arc CE = arc DE.

Case II. *When both AB and CD are chords.*



Draw $EF \parallel AB$, and tangent to the circumference at G.

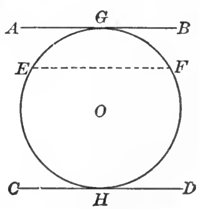
Then, § 80, $EF \parallel CD$,

Case I, arc CG = arc DG,

and arc AG = arc BG;

∴ Ax. 3, arc CA = arc DB.

Case III. *When AB and CD are tangents as at G and H respectively.*



Draw the chord $EF \parallel AB$.

Then, § 80, $EF \parallel CD$,

Case I, arc EH = arc FH,

and arc EG = arc FG;

∴ Ax. 2, arc HEG = arc HFG.

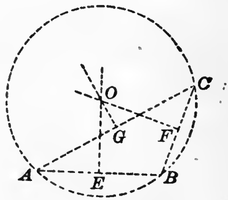
Therefore, etc. Q.E.D.

Proposition XI

207. Select three points not in the same straight line. How many circumferences can be passed through them?

Theorem. *Through three points not in the same straight line one circumference can be drawn, and only one.*

Data: Any three points not in the same straight line, as A, B, and C.



To prove that one circumference can be drawn through A, B, and C, and only one.

Proof. Draw AB, BC, and AC; and at their middle points, E, F, and G, respectively, erect perpendiculars.

§ 170, these perpendiculars meet in a point, as O, which is equidistant from A, B, and C;

∴ § 173, a circumference described from O as a center, and with a radius equal to the distance OA, passes through the points A, B, and C; and, since the perpendiculars intersect in but *one* point, there can be but *one* center, and consequently but *one* circumference passing through the points A, B, and C. Q.E.D.

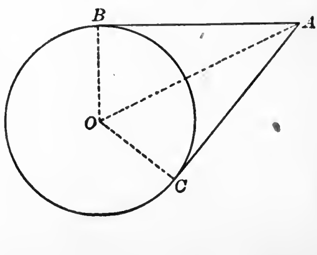
208. Cor. I. *Circles circumscribing equal triangles are equal.*

Cor. II. *Two circumferences can intersect in only two points.* If two circumferences have three points in common, they coincide and form one circumference.

Proposition XII

209. Draw a circle and from a point outside draw two tangents. How do the tangents compare in length?

Theorem. *The tangents drawn to a circle from a point without are equal.*



Data: A circle whose center is O; any point without it, as A; and AB and AC the tangents to the circle at the points B and C, respectively.

To prove $AB = AC$.

Proof. Draw OA, OB, and OC.

§ 205, $\angle s$ B and C are rt. $\angle s$.

Then, in the rt. $\triangle s$ OAB and OAC,

$OB = OC$, Why?

and OA is common;

$\therefore$ $\triangle OAB = \triangle OAC$, Why?

and $AB = AC$.

Therefore, etc. Q.E.D.

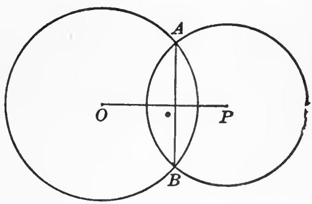
210. The line which joins the centers of two circles is called their *line of centers.*

211. A common tangent to two circles which cuts their line of centers is called a *common interior tangent;* one which does not cut their line of centers is called a *common exterior tangent.*

Proposition XIII

212. Draw two intersecting circles and a chord that is common to both. What kind of an angle does a line joining their centers make with this common chord?

Theorem. *If the circumferences of two circles intersect, the line of centers is perpendicular to the common chord at its middle point.*



Data: Two circles whose centers are O and P, whose circumferences intersect at A and B; AB the common chord; and OP the line of centers.

To prove OP perpendicular to AB at its middle point.

Proof. § 173, P is equidistant from A and B,

and also, O is equidistant from A and B;

$\therefore$ § 104, both P and O lie in the perpendicular bisector of AB.

Hence, Ax. 11, OP coincides with this perpendicular bisector; that is, $OP \perp AB$ at its middle point.

Therefore, etc. Q.E.D

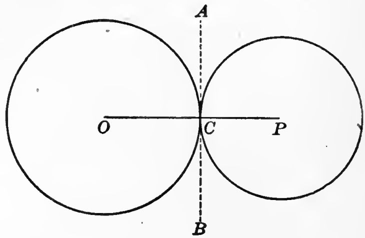
Proposition XIV

213. Draw two circles tangent to each other, and a line joining their centers. Through what point will this line pass?

Theorem. *If two circles are tangent to each other, their line of centers passes through the point of contact.*

Data: The two tangent circles whose centers are O and P; OP their line of centers; and C their point of contact.

To prove that OP passes through C.



Proof. At C draw the common tangent AB.

Ax. 17,	C lies in both circumferences;	
∴ § 205,	if radius PC is drawn, $PC \perp AB$,	
also,	if radius OC is drawn, $OC \perp AB$;	
∴	$\angle OCA + \angle PCA = 2$ rt. $\angle$s;	**Why?**
hence, § 58,	OCP is a straight line;	
that is,	OP passes through C.	

Therefore, etc. **Q.E.D.**

MEASUREMENT

214. The theorems thus far presented and proved have usually established only the equality or inequality of two magnitudes, but it is sometimes desirable to measure accurately the magnitudes that are given.

A magnitude is measured when we find how many times it contains another magnitude of the same kind, called the *unit of measure.*

The number which expresses how many times a magnitude contains a unit of measure is called its *numerical measure.*

215. The relation of two magnitudes which is determined by finding how many times one contains the other, or what part one is of the other, is called their **Ratio**.

The ratio of two magnitudes is the ratio of their numerical measures. It may be either an integer or a fraction.

The *ratio* of a line 12 ft. long to one 4 ft. long is 3; that is, the 12 ft. line is three times the 4 ft. line. Also the ratio of an angle of 10° to an angle of 60° is $\frac{1}{6}$; that is, an angle of 10° is one sixth of an angle of 60°.

216. Two magnitudes of the same kind which contain a common unit of measure an *integral* number of times are called **Commensurable Magnitudes.**

217. Two magnitudes of the same kind which have no common unit of measure are called **Incommensurable Magnitudes.**

218. The ratio of incommensurable magnitudes is called an *incommensurable ratio*; that is, it cannot be expressed exactly by numbers; and yet we can approximate to the exact numerical value as nearly as we please.

Thus, if the side of a square is 1 ft. in length, the diagonal is $\sqrt{2}$ ft. in length, as will be shown later, and the ratio of the diagonal to the side of the square is $\frac{\sqrt{2}}{1}$.

Now, no integer or mixed number can be found which is exactly equal to $\sqrt{2}$, but by expressing the square root of 2 in a decimal form, the ratio can be determined within a fraction as small as we please.

Thus, $\sqrt{2} = 1.414213+$; that is, $\sqrt{2}$ lies between 1.414213 and 1.414214; therefore, the ratio of the diagonal of a square to its side lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$.

That is, if the one-millionth part of a foot be assumed approximately as the common unit of measure, the side of the square will be equal to 1,000,000, and the diagonal will be equal to between 1,414,213 and 1,414,214 such units.

By continuing the process of finding the square root we can approximate as closely to the actual ratio as we please, or until the fraction contains an error so small that it may be disregarded, though it cannot be eliminated.

It is evident, therefore, that the ratio of two magnitudes of the same kind, even when they are incommensurable, may be obtained to any required degree of precision.

To generalize: suppose Q to be divided into n equal parts, and that P contains m of these parts with a remainder less than one of the parts; then, $\frac{P}{Q} = \frac{m}{n}$ within less than $\frac{1}{n}$.

Since n may be taken as large as we please, $\frac{1}{n}$ may be made less than any assigned measure of precision, and, consequently, the value of $\frac{m}{n}$ may be regarded as the approximate value of the ratio $\frac{P}{Q}$ within any assigned degree of precision.

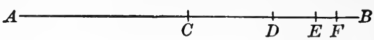
THEORY OF LIMITS

219. A magnitude which remains unchanged throughout the same discussion is called a **Constant.**

220. A magnitude which, under the conditions imposed upon it, may have an indefinite number of different successive dimensions is called a **Variable.**

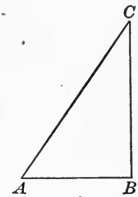
221. When a variable increases or decreases so that the difference between it and a constant may be made as small as we please, but cannot be made absolutely equal to zero, the constant is called the **Limit** of the variable, and the variable is said to *approach* this limit.

Suppose, for example, that a point moves from A to B under the condition that it shall move one half the distance during the first interval of time; one half the remaining distance the second interval; one half the distance still remaining the next interval; and so on. The distance from A to the moving point is an *increasing variable,* which approaches the distance AB as a *limit,* though it cannot actually reach it. Also, the distance from B to the moving point is a *decreasing variable,* which approaches zero as a *limit,* though it cannot actually reach it.



Again, the sum of the descending series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, etc., approaches 2 as a limit, but never quite reaches it. The sum of the first two terms is $1\frac{1}{2}$, of the first three terms $1\frac{3}{4}$, of the first four terms $1\frac{7}{8}$, etc. It is evident that the sum approaches 2, and that, by taking terms enough, it may be made to differ from 2 by as small a quantity as we please, but it cannot be actually equal to 2. That is, the sum of the series approaches the limit 2 as the number of terms is increased.

For further illustration, if, in the right triangle *ABC*, the vertex *C* indefinitely approaches the vertex *B*, the angle *A* diminishes and indefinitely approaches zero as its limit, and if it should actually become zero, the triangle would vanish and become the straight line *AB*. Again, if the vertex *C* indefinitely moves away from the vertex *B*, the angle *A* increases and indefinitely approaches a right angle as its limit, and if it should actually become a right angle the triangle would vanish and *AC* and *BC* would be two parallel lines, each perpendicular to *AB*. Hence, the value of angle *A* lies between the limits zero and a right angle, but it can never actually reach either limit so long as the triangle exists.

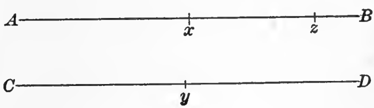


Proposition XV

222. What is a variable? What is the limit of a variable? If two variables are always equal, how do their limits compare?

Theorem. *If, while approaching their respective limits, two variables are always equal, their limits are equal.*

Data: Two equal variables, as *Ax* and *Cy*, which approach the limits *AB* and *CD*, respectively.



To prove *AB* and *CD* equal.

Proof. Suppose that *AB* is greater than *CD*; then some part of *AB*, as *Az*, is equal to *CD*.

Then, the variable *Ax* may have values between *Az* and *AB*; that is, the variable *Ax* may have values greater than *CD*, while the variable *Cy* cannot have values greater than *CD*.

This is contrary to the condition that the variables are equal.

Hence, *AB* cannot be greater than *CD*.

In like manner it can be proved that *AB* cannot be less than *CD*.

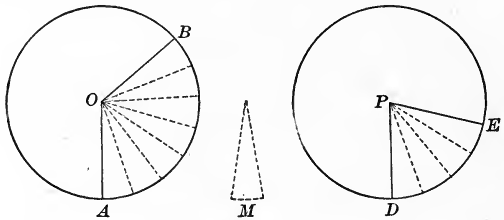
Consequently, *AB* = *CD*.

Therefore, etc. Q.E.D.

Proposition XVI

223. In the same circle, or in equal circles, draw two central angles such that the arc intercepted by the sides of the first shall be three times the arc intercepted by the sides of the second. How does the first angle compare in size with the second? How does the ratio of the central angles compare with the ratio of the arcs intercepted by their sides?

Theorem. *In the same circle, or in equal circles, two central angles have the same ratio as the arcs intercepted by their sides.*



Data: In the equal circles whose centers are O and P, any central angles, as AOB and DPE, whose sides intercept the arcs AB and DE respectively.

To prove $\quad$ ratio $\dfrac{\angle AOB}{\angle DPE} =$ ratio $\dfrac{\text{arc } AB}{\text{arc } DE}$.

Proof. **Case I.** *When the arcs are commensurable.*

Suppose that M is the common unit of measure for the two arcs, and that it is contained in arc AB 7 times and in arc DE 4 times.

Then, $\quad$ ratio $\dfrac{\text{arc } AB}{\text{arc } DE} =$ ratio $\dfrac{7}{4}$.

Divide the arcs AB and DE into parts each equal to the common measure M, and to each point of division draw a radius.

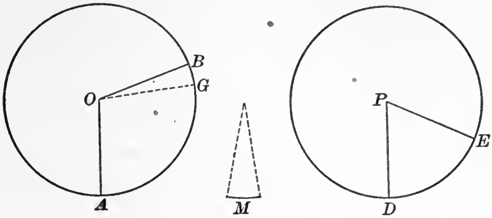
By § 194, each of these central angles formed by any two adjacent radii is equal to every other central angle so formed.

The number of equal parts into which the angles AOB and DPE are divided by these radii is equal to the number of times M is contained in the arcs AB and DE respectively.

$\therefore$ $\quad$ ratio $\dfrac{\angle AOB}{\angle DPE} =$ ratio $\dfrac{7}{4}$.

Hence, Ax. 1, $\quad$ ratio $\dfrac{\angle AOB}{\angle DPE} =$ ratio $\dfrac{\text{arc } AB}{\text{arc } DE}$.

Case II. *When the arcs are incommensurable.*



Since arcs AB and DE are incommensurable, take these commensurable arcs AG and DE, and suppose that GB is less than M.

Then, Case I, ratio $\dfrac{\angle AOG}{\angle DPE} = $ ratio $\dfrac{\text{arc } AG}{\text{arc } DE}$.

If M is indefinitely diminished, angle GOB and arc GB decrease, and the ratios $\dfrac{\angle AOG}{\angle DPE}$ and $\dfrac{\text{arc } AG}{\text{arc } DE}$ remain equal, and indefinitely approach the limiting ratios $\dfrac{\angle AOB}{\angle DPE}$ and $\dfrac{\text{arc } AB}{\text{arc } DE}$ respectively.

Hence, § 222, ratio $\dfrac{\angle AOB}{\angle DPE} = $ ratio $\dfrac{\text{arc } AB}{\text{arc } DE}$. Q.E.D.

224. It was stated in § 35 that the total angular magnitude about a point in a plane is divided into *degrees, minutes,* and *seconds.* In the same way the circumference of a circle is divided into 360 equal arcs, called *arc degrees,* or simply *degrees;* each arc degree is divided into 60 equal parts, called *minutes;* and each arc minute into 60 equal parts, called *seconds.*

Thus it will be seen that the angle degree and the arc degree, the angle minute and the arc minute, the angle second and the arc second correspond, each to each. The sides of a central angle of one degree therefore intercept an arc of one degree, the sides of a central angle of ten degrees intercept an arc of ten degrees, and, in general, the sides of a central angle of any number of degrees intercept an arc of an equal number of degrees.

Therefore, *a central angle is measured by its intercepted arc;* that is, *the central angle contains between its sides the same proportion of the total angular magnitude about a point in a plane, that the arc intercepted by its sides is of the whole circumference.*

Proposition XVII

225. 1. Draw a circle and an inscribed angle one of whose sides is a diameter; draw a radius to the extremity of the other side. How does the inscribed angle compare in size with the central angle subtended by the same arc? Since the central angle is measured by the arc which subtends it, by what part of the arc is the inscribed angle measured?

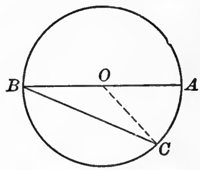
2. Draw other inscribed angles no one of whose sides is a diameter and draw a diameter from the vertex of each. By what part of its arc is each inscribed angle measured?

3. If inscribed angles are subtended by the same arc or chord or are inscribed in the same segment, how do they compare in size?

4. How many degrees are there in a semicircumference? What, then, will be the size of all angles inscribed in a semicircle?

5. How does an angle inscribed in a segment greater than a semicircle compare with a right angle? How, if in one less than a semicircle?

Theorem. *An inscribed angle is measured by one half the arc intercepted by its sides.*



Data: A circle whose center is O, and any inscribed angle, as ABC.

To prove angle ABC measured by $\frac{1}{2}$ arc AC.

Proof. Case I. *When one side of the angle is a diameter of the circle.*

When AB is a diameter.

Draw the radius OC.

Then, $OB = OC$, Why?

§ 90, $\triangle BOC$ is isosceles,

and $\angle B = \angle C$. Why?

§ 115, $\angle AOC = \angle B + \angle C = 2\angle B$;

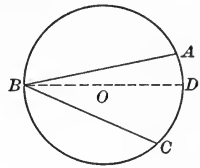
or $\angle B = \frac{1}{2}\angle AOC$.

But, § 224, $\angle AOC$ is measured by arc AC;

hence, $\angle B$ is measured by $\frac{1}{2}$ arc AC.

Case II. *When the diameter from the vertex of the angle lies between the sides.*

Draw the diameter BD.



Case I., $\angle ABD$ is measured by $\frac{1}{2}$ arc AD,

and $\angle CBD$ is measured by $\frac{1}{2}$ arc CD;

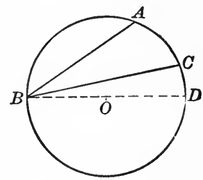
but, Ax. 9, $\angle ABD + \angle CBD = \angle ABC$,

and arc AD + arc CD = arc AC;

hence, $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

Case III. *When both sides of the angle are on the same side of the diameter from the vertex.*

Draw the diameter BD.



Case I., $\angle ABD$ is measured by $\frac{1}{2}$ arc AD,

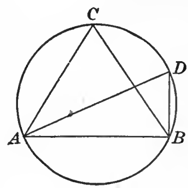
and $\angle CBD$ is measured by $\frac{1}{2}$ arc CD;

but $\angle ABD - \angle CBD = \angle ABC$,

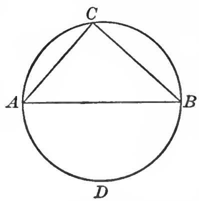
and arc AD − arc CD = arc AC;

hence, $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

Therefore, etc. Q.E.D.



226. Cor. I. *Angles inscribed in the same segment of a circle, or in equal segments of the same circle, or of equal circles are equal.*



227. Cor. II. *An angle inscribed in a semicircle is a right angle.*

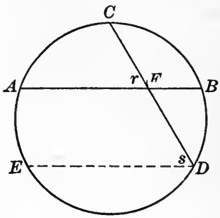
228. Cor. III. *An angle inscribed in a segment greater than a semicircle is less than a right angle.*

229. Cor. IV. *An angle inscribed in a segment less than a semicircle is greater than a right angle.*

Proposition XVIII

230. Draw a circle and two intersecting chords; construct an inscribed angle equal to one of the vertical angles thus formed, by drawing from an extremity of one chord a chord parallel to the other. How does the arc subtending the inscribed angle compare with the sum of the arcs intercepted by the sides of the vertical angles? What, then, is the measure of the angle formed by the two intersecting chords?

Theorem. An angle formed by two intersecting chords is measured by one half the sum of the arc intercepted by its sides and the arc intercepted by the sides of its vertical angle.



Data: Any two intersecting chords, as AB and CD, forming the angle r.

To prove angle r measured by

$$\tfrac{1}{2}(\text{arc } AC + \text{arc } BD).$$

Proof. Draw $DE \parallel AB$.

Then, § 76, $\angle s = \angle r$.

§ 225, $\angle s$ is measured by $\frac{1}{2}$ arc CAE;

but arc $CAE =$ arc $AC +$ arc AE,

and, § 206, arc $AE =$ arc BD;

$\therefore$ arc $CAE =$ arc $AC +$ arc BD;

hence, $\angle s$ is measured by $\frac{1}{2}(\text{arc } AC + \text{arc } BD)$.

Consequently, $\angle r$ is measured by $\frac{1}{2}(\text{arc } AC + \text{arc } BD)$.

Therefore, etc. Q.E.D.

Ex. **197**. Prove by Prop. XVII that the sum of the angles of a triangle is equal to two right angles.

Ex. **198**. The opposite angles of an inscribed quadrilateral are supplementary.

Ex. **199**. If two chords of a circle intersect at right angles, to what is the sum of any pair of opposite arcs equal?

Ex. **200**. If one of the equal sides of an isosceles triangle is the diameter of a circle, the circumference bisects the base.

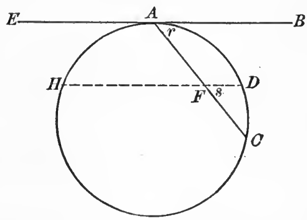
Ex. **201**. If one side of an angle of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite angle of the quadrilateral.

Proposition XIX

231. 1. At any point in the circumference of a circle form an angle by a tangent and a chord; construct vertical angles equal to this by drawing through the given chord a chord parallel to the tangent. How does the sum of the arcs subtending these vertical angles compare with the arc intercepted by the sides of the given angle? What, then, is the measure of the given angle?

2. How does an angle between a tangent and a diameter compare with a right angle? What is its arc measure?

Theorem. *An angle formed by a tangent and a chord is measured by one half the intercepted arc.*



Data: Any tangent, as EB, and any chord, as AC, forming with EB the angle r.

To prove angle r measured by $\frac{1}{2}$ arc AC.

Proof. Draw any chord, as HD, parallel to EB and cutting AC in F.

Then, § 76, $\angle s = \angle r$.

§ 230, $\angle s$ is measured by $\frac{1}{2}(\text{arc } AH + \text{arc } DC)$;

but, § 206, $\text{arc } AH = \text{arc } AD$;

hence, $\angle s$ is measured by $\frac{1}{2}(\text{arc } AD + \text{arc } DC)$, or $\frac{1}{2}$ arc AC.

Consequently, $\angle r$ is measured by $\frac{1}{2}$ arc AC.

Therefore, etc. Q.E.D.

232. Cor. *A right angle is measured by one half a semicircumference.*

Ex. **202.** The angle between a tangent to a circle and a chord drawn from the point of contact is half the angle at the center subtended by that chord.

Ex. **203.** A line which is tangent to the inner of two concentric circles, and is a chord of the outer circle, is bisected at the point of tangency.

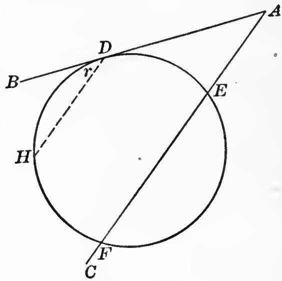
Ex. **204.** The diagonals of a rectangle inscribed in a circle are diameters of the circle.

Ex. **205.** The bisector of the vertical angle of an inscribed isosceles triangle passes through the center of the circle.

Proposition XX

233. At any point without the circumference of a circle form an angle between a tangent and a secant; construct an angle equal to this by drawing from the point of tangency a chord parallel to the secant. How does the arc intercepted by the sides of this angle compare with the difference of the arcs intercepted by the sides of the given angle? What, then, is the measure of the given angle?

Theorem. *An angle formed by a tangent and a secant which meet without a circumference is measured by one half the difference of the intercepted arcs.*



Data: Any tangent, as AB, and any secant, as AC, meeting AB without the circumference and forming with it the angle A.

To prove angle A measured by $\frac{1}{2}$(arc DF − arc DE).

Proof. From D, the point of tangency, draw $DH \parallel AC$.

Then, § 76, $\angle r = \angle A$.

§ 231, $\angle r$ is measured by $\frac{1}{2}$ arc DH;

but arc DH = arc DF − arc HF,

and, § 206, arc HF = arc DE;

∴ arc DH = arc DF − arc DE;

hence, $\angle r$ is measured by $\frac{1}{2}$(arc DF − arc DE).

Consequently, $\angle A$ is measured by $\frac{1}{2}$(arc DF − arc DE).

Therefore, etc. Q.E.D.

Ex. **206**. Two chords perpendicular to a third chord at its extremities are equal.

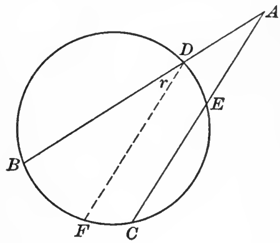
Ex. **207**. If a quadrilateral is inscribed in a circle and its diagonals are drawn, the diagonals will divide the angles of the quadrilateral so that there will be four pairs of equal angles.

Ex. **208**. If from the center of a circle a perpendicular is drawn to either side of an inscribed triangle, and a radius is drawn to one end of this side, the angle between the radius and the perpendicular is equal to the opposite angle of the triangle.

Proposition XXI

234. At any point without the circumference of a circle form an angle between two secants; construct an inscribed angle equal to this by drawing from the point of intersection of either secant and the circumference a chord parallel to the other secant. How does the arc subtending the inscribed angle compare with the difference of the arcs intercepted by the secants? What, then, is the measure of the given angle?

Theorem. *An angle formed by two secants which meet without a circumference is measured by one half the difference of the intercepted arcs.*



Data: Any two secants, as AB and AC, meeting without the circumference and forming the angle A.

To prove angle A measured by

$\frac{1}{2}(\text{arc } BC - \text{arc } DE)$.

Proof. Draw the chord $DF \parallel AC$.

Then, § 76, $\angle r = \angle A$.

§ 225, $\angle r$ is measured by $\frac{1}{2}$ arc BF;

but arc BF = arc BC − arc FC,

and, § 206, arc FC = arc DE;

$\therefore$ arc BF = arc BC − arc DE;

hence, $\angle r$ is measured by $\frac{1}{2}(\text{arc } BC - \text{arc } DE)$.

Consequently, $\angle A$ is measured by $\frac{1}{2}(\text{arc } BC - \text{arc } DE)$.

Therefore, etc. Q.E.D.

Ex. **209**. The tangent at the vertex of an inscribed equilateral triangle forms equal angles with the adjacent sides.

Ex. **210**. The angle between two tangents from the same point is 24° 15′. How many degrees are there in each of the intercepted arcs?

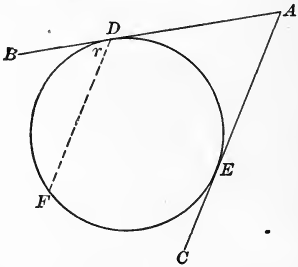
Ex. **211**. One angle of an inscribed triangle is 38°, and one of its sides subtends an arc of 124°. What are the other angles of the triangle?

Ex. **212**. AB, a chord of a circle, is the base of an isosceles triangle whose vertex C is without the circle, and whose equal sides intersect the circumference at D and E. Prove that CD is equal to CE.

Proposition XXII

235. At any point without the circumference of a circle form an angle between two tangents; construct an angle equal to this by drawing from the point of contact of either tangent a chord parallel to the other. How does the arc intercepted by the sides of this angle compare with the difference of the arcs intercepted by the tangents? What, then, is the measure of the given angle?

Theorem. An angle formed by two tangents is measured by one half the difference of the intercepted arcs.



Data: Any two tangents, as AB and AC, forming the angle A.

To prove angle A measured by $\frac{1}{2}$ (arc DFE − arc DE).

Proof. From D, the point of tangency, draw the chord $DF \parallel AC$.

Then, § 76, $\angle r = \angle A$.

§ 231, $\angle r$ is measured by $\frac{1}{2}$ arc DF;

but arc DF = arc DFE − arc FE,

and, § 206, arc FE = arc DE;

∴ arc DF = arc DFE − arc DE;

hence, $\angle r$ is measured by $\frac{1}{2}$ (arc DFE − arc DE).

Consequently, $\angle A$ is measured by $\frac{1}{2}$ (arc DFE − arc DE).

Therefore, etc. Q.E.D.

Ex. **213.** In any quadrilateral circumscribing a circle any pair of opposite sides is equal to half the perimeter of the quadrilateral.

Ex. **214.** If three circles touch each other externally, and the three common tangents are drawn, these tangents meet in a point equidistant from the points of contact of the circles.

Ex. **215.** If two triangles are inscribed in a circle, and two sides of one are parallel, each to each, to two sides of the other, the third sides are equal.

Ex. 216. Every parallelogram inscribed in a circle is a rectangle.

Ex. 217. Two sides of an inscribed triangle subtend $\frac{1}{5}$ and $\frac{1}{6}$ of the circumference, respectively. What are the angles of the triangle?

Ex. 218. If a circle is circumscribed about the triangle ABC, and a line is drawn bisecting angle A and meeting the circumference in D, angle DCB is equal to one half angle BAC.

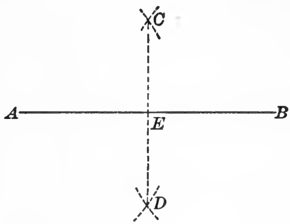
Ex. 219. AB is an arc of 65°, DC an arc of 75° in a circle whose center is O. AC is a diameter. How many degrees are there in each angle of the triangles AOD and BOC?

Ex. 220. If an equilateral triangle is inscribed in a circle, and a diameter is drawn from one vertex, the triangle formed by joining the other extremity of the diameter and the center of the circle with one of the other vertices of the inscribed triangle is also equilateral.

Ex. 221. If tangents are drawn to a circle from a point without, the line joining that point with the center of the circle bisects (1) the angle formed by the tangents; (2) the angle formed by the radii drawn to the points of tangency; and (3) the arc intercepted by these radii.

Proposition XXIII

236. ***Problem.**** *To bisect a straight line.*



Datum: Any line, as AB.

Required to bisect AB.

Solution. From A and B as centers, with equal radii each greater than one half AB, describe arcs intersecting at C and D.

Draw CD intersecting AB at E.

Then, CD bisects AB at E. Q.E.F.

Proof. Const., C and D are each equidistant from A and B.

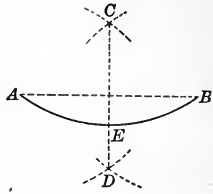
Hence, § 106, CD is the perpendicular bisector of AB.

* 1. The student is urged to attempt to solve each problem before he studies the solution given in the book. He will very likely discover for himself the same method of solution or perhaps another one equally good.

2. The only implements used in solving problems in plane geometry are the *straightedge* and *compasses*.

Proposition XXIV

237. *Problem.* *To bisect an arc of a circle.*



Datum: Any arc of a circle, as AB.

Required to bisect arc AB.

Solution. Draw the chord AB.

From A and B as centers, with equal radii each greater than one half chord AB, describe arcs intersecting at C and D.

Draw CD intersecting arc AB at E.

Then, CD bisects arc AB at E. Q.E.F.

Proof. Const., C and D are each equidistant from A and B;

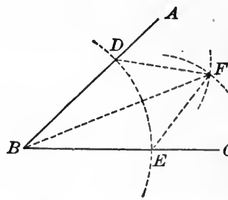
∴ § 106, CD is perpendicular to the chord AB at its middle point.

and, § 199, CD passes through the center of the circle.

Hence, § 200, CD bisects the arc AB at E.

Proposition XXV

238. *Problem.* *To bisect an angle.*



Datum: Any angle, as ABC.

Required to bisect the angle ABC.

Solution. From B as a center with any radius, as BE, describe an arc intersecting AB in D and CB in E.

From D and E as centers, with equal radii each greater than one half the distance DE, describe arcs intersecting at F, and draw BF.

Then, BF bisects the angle ABC. Q.E.F

Proof. Draw DF and EF.

Then, in ⚠ BDF and BEF,

const., $BD = BE$,

$DF = EF$,

and BF is common;

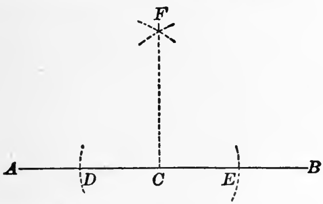
$\therefore$ § 107, $\triangle BDF = \triangle BEF$,

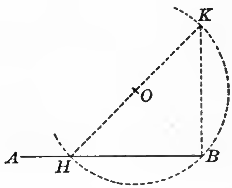
and $\angle DBF = \angle EBF$; Why?

that is, BF bisects the angle ABC.

Proposition XXVI

239. Problem. *At a given point in a line to erect a perpendicular to the line.*





Case I. *When the given point is between the extremities of the line.*

Data: Any line, as AB, and any point in AB, as C.

Required to erect a perpendicular to AB at C.

Solution. From C as a center, with any radius, describe arcs intersecting AB at D and E.

From D and E as centers, with equal radii, describe arcs intersecting at F. Draw CF.

Then, CF is the perpendicular required. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 106.

Case II. *When the given point is at the extremity of the line.*

Data: Any line, as AB, and the point at either extremity, as B.

Required to erect a perpendicular to AB at B.

Solution. From O, any point without AB, as a center, with OB as a radius, describe an arc intersecting AB in H.

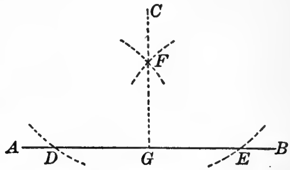
From H, draw a line through O intersecting this arc in K, and draw KB.

Then, KB is the perpendicular required. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 227.

Proposition XXVII

240. *Problem.* *To draw a perpendicular to a line from a point without the line.*



Data: Any line, as AB, and any point without the line, as C.

Required to draw a perpendicular from C to AB.

Solution. From C, as a center, with a radius greater than the distance from C to AB, describe an arc intersecting AB at D and E.

From D and E, as centers, with equal radii each greater than one half of DE, describe arcs intersecting at F.

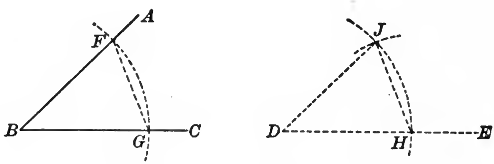
Draw CF and produce it to meet AB as at G.

Then, CG is the perpendicular required. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 106.

Proposition XXVIII

241. *Problem.* *To construct an angle equal to a given angle.*



Datum: Any angle, as ABC.

Required to construct an angle equal to ABC.

Solution. Draw any line, as DE.

From B as a center, with any radius, describe an arc intersecting BA and BC in F and G respectively.

From D as a center, with the same radius, describe an arc intersecting DE in H.

From H as a center, with a radius equal to the distance GF, describe a second arc intersecting the first at J, and draw DJ.

Then, $\angle JDH$ is the angle required. Q.E.F.

Proof. Draw GF and HJ.

In $\triangle$s BGF and DHJ,

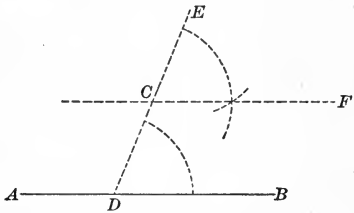
const.,	$BG = DH,\ BF = DJ$, and $GF = HJ$;
∴ § 107,	$\triangle BGF = \triangle DHJ$,
and	$\angle B = \angle D$.

Proposition XXIX

242. *Problem.* *Through a given point to draw a line parallel to a given line.*

Data: Any line, as AB, and any point not in AB, as C.

Required to draw a line through C parallel to AB.



Solution. Through C draw any line meeting AB, as ED.

Construct $\angle ECF$ equal to $\angle EDB$.

Then, CF is parallel to AB. Q.E.F.

Proof. Const.,	$\angle ECF = \angle EDB$;
∴ § 77,	$CF \parallel AB$.

Ex. **222**. Two angles of a triangle being given to construct the third.

Ex. **223**. Two secants cut each other without a circle; the intercepted arcs are 72° and 48°. What is the angle between the secants?

Ex. **224**. A tangent and a secant cut each other without a circle; the intercepted arcs are 94° and 32°. What is the angle between the tangent and the secant?

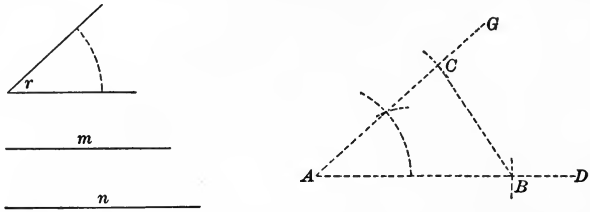
Ex. **225**. Two chords of a circle intersect and two opposite intercepted arcs are 88° and 26°. What are the angles between the chords?

Ex. **226**. A tangent of a circle and a chord from the point of contact intercept an arc of 110°. What is the angle between the tangent and the chord?

Ex. **227**. If the radii of two intersecting circles are 4^{dm} and 9^{dm}, respectively; what is the greatest and the least possible distance between their centers?

Proposition XXX

243. Problem. *To construct a triangle when two sides and the included angle are given.*



Data: Two sides of a triangle, as m and n, and the included angle, as r.

Required to construct the triangle.

Solution. Draw any line, as AD, and on it measure AB equal to n.

Construct the $\angle A$ equal to $\angle r$, and on AG measure AC equal to m.

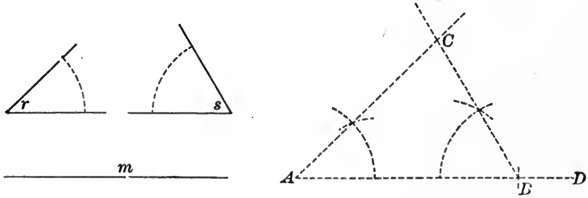
Draw CB.

Then, $\triangle ABC$ is the $\triangle$ required. Q.E.F.

Proof. By the student.

Proposition XXXI

244. Problem. *To construct a triangle when a side and two adjacent angles are given.*



Data: A side of a triangle, as m, and the angles, as r and s, adjacent to it.

Required to construct the triangle.

Solution. Draw any line, as AD, and on it measure AB equal to m.

Construct $\angle A$ equal to $\angle r$, and $\angle B$ equal to $\angle s$.

Produce the sides of these two angles until they intersect, as at C.

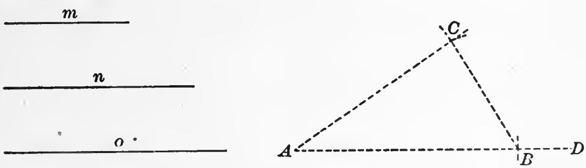
Then, $\triangle ABC$ is the $\triangle$ required. Q.E.F.

Proof. By the student.

245. Sch. The problem is impossible when the sum of the given angles equals or exceeds two right angles. Why?

Proposition XXXII

246. ***Problem.*** *To construct a triangle when the three sides are given.*



Data: The three sides of a triangle, as m, n, and o.

Required to construct the triangle.

Solution. Draw any line, as AD, and on it measure AB equal to o.

From A as a center, with a radius equal to n, describe an arc.

From B as a center, with a radius equal to m, describe a second arc intersecting the first in C. Draw AC and BC.

Then, $\triangle ABC$ is the $\triangle$ required. Q.E.F.

Proof. By the student.

247. Sch. The problem is impossible when any one side is equal to or greater than the sum of the other two sides. Why?

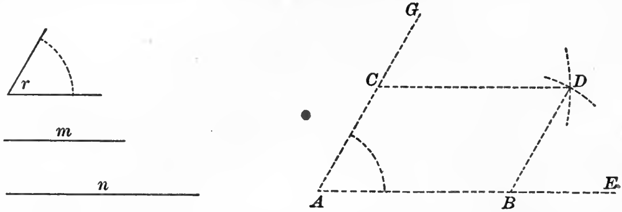
Ex. **228.** Construct an equilateral triangle.

Ex. **229.** Prove that the radius of a circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.

Ex. **230.** In an inscribed trapezoid how do the non-parallel sides compare? How do the diagonals compare?

Proposition XXXIII

248. Problem. *To construct a parallelogram when two sides and the included angle are given.*



Data: Two sides of a parallelogram, as m and n, and the included angle, as r.

Required to construct the parallelogram.

Solution. Draw any line, as AE, and on it measure AB equal to n.

Construct the $\angle A$ equal to $\angle r$, and on AG measure AC equal to m.

From C as a center, with a radius equal to n, describe an arc.

From B as a center, with a radius equal to m, describe a second arc intersecting the first in D.

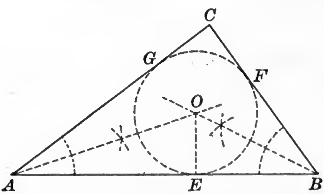
Draw CD and BD.

Then, $ABDC$ is the parallelogram required. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 148.

Proposition XXXIV

249. Problem. *To inscribe a circle in a given triangle.*



Datum: Any triangle, as ABC.

Required to inscribe a circle in $\triangle ABC$.

Solution. Bisect $\angle s$ A and B, produce the bisectors to intersect at O, and draw $OE \perp AB$.

From O as a center, with OE as a radius, describe the circle EFG.

Then, EFG is the circle required. Q.E.F.

Proof. Const., O lies in the bisectors of $\angle$s A and B;

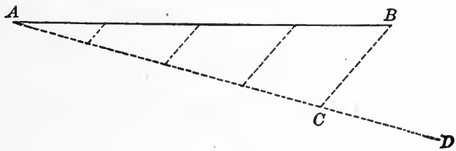
$\therefore$ § 134, O is equidistant from AB, AC, and BC.

Hence, a circle described from O as a center, with a radius equal to OE, touches AB, AC, and BC.

That is, § 190, the circle EFG is inscribed in $\triangle ABC$.

Proposition XXXV

250. *Problem.* *To divide a straight line into equal parts.*



Datum: Any straight line, as AB.

Required to divide AB into equal parts.

Solution. From A draw a line of indefinite length, as AD, making any convenient angle with AB.

On AD measure off in succession equal distances corresponding in number with the parts into which AB is to be divided.

From the last point thus found on AD, as C, draw CB, and from each point of division on AC draw lines $\parallel$ CB and meeting AB.

These lines divide AB into equal parts. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 157.

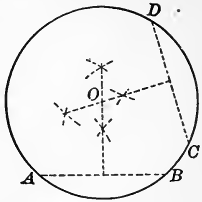
Ex. **231.** If the sides of a central angle of 35° intercept an arc of 75^{cm}, what will be the length of an arc intercepted by the sides of a central angle of 80° in the same circle?

Ex. **232.** AB and CD are diameters of the circle whose center is O; BD is an arc of 116°. How many degrees are there in each angle of the triangles AOC and DOB?

Ex. **233.** If a circle is circumscribed about a triangle ABC, and perpendiculars are drawn from the vertices to the opposite sides and produced to meet the circumference in the points D, E, and F, the arcs EF, FD, and DE are bisected at the vertices.

Proposition XXXVI

251. *Problem.* *To find the center of a circle.*



Datum: Any circle, as *ABD*.

Required to find the center of *ABD*.

Solution. Draw any two non-parallel chords, as *AB* and *CD*.

Draw the perpendicular bisectors of *AB* and *CD*, and produce them until they intersect, as in *O*.

Then, *O* is the center of the circle.

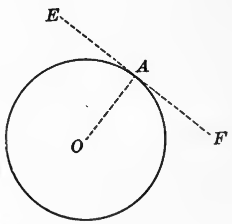
Proof. By the student. SUGGESTION. Refer to § 201.

Ex. **234.** To circumscribe a circle about a given triangle.

Ex. **235.** *AB* is a chord of a circle and *AC* is a tangent at *A*; a secant parallel to *AB*, as *EFD*, cuts *AC* in *E* and the circumference in *F* and *D*; the lines *AF*, *AD*, and *BD* are drawn. Prove that the triangles *AEF* and *ADB* are mutually equiangular.

Proposition XXXVII

252. *Problem.* *Through a given point to draw a tangent to a given circle.*



Data: A circle whose center is *O*, and any point, as *A*.

Required to draw through *A* a tangent to the circle.

Solution. Case I. *When A is on the circumference.*

Draw the radius *OA*.

At *A* draw $EF \perp OA$.

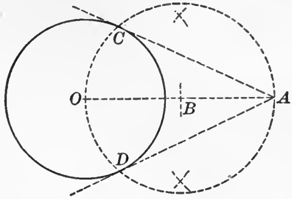
Then, *EF* is the tangent required.

Proof. By the student.

Case II. *When A is without the circumference.*

Draw OA.

From B, the middle point of OA, as a center, with BO as a radius, describe a circle intersecting the given circle at C and D.



Draw AC and AD.

Then, AC or AD is the tangent required. Q.E.F.

Proof. By the student.

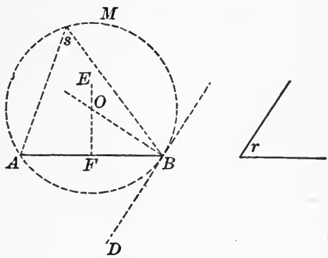
SUGGESTION. Draw the radii OC and OD, and refer to § 227 and § 205.

Proposition XXXVIII

253. *Problem.* *To describe upon a given straight line a segment of a circle which shall contain a given angle.*

Data: Any straight line, as AB, and any angle, as r.

Required to describe a segment of a circle upon AB which shall contain $\angle r$.



Solution. Construct $\angle ABD$ equal to $\angle r$.

Draw $FE \perp AB$ at its middle point.

Erect a perpendicular to DB at B, and produce it to intersect FE at O.

From O as a center, with a radius equal to OB, describe a circle.

Then, AMB is the segment required. Q.E.F.

Proof. Inscribe any angle in segment AMB, as $\angle s$.

Const., $\angle ABD = \angle r$,

§ 231, $\angle ABD$ is measured by $\frac{1}{2}$ arc AB;

but, § 225, $\angle s$ is measured by $\frac{1}{2}$ arc AB,

$\therefore$ $\angle s = \angle ABD = \angle r$.

Hence, AMB is the segment required.

SUMMARY

254. Truths established in Book II.

1. Two lines are equal,

a. If they are radii of the same circle, or of equal circles. Ax. 14

b. If they are diameters of the same circle, or of equal circles. Ax. 15

c. If they are chords which subtend equal arcs in the same circle, or in equal circles. § 196

d. If they represent the distances of equal chords in the same circle, or in equal circles, from the center. § 202

e. If they are chords equally distant from the center of the same circle, or of equal circles. § 202

f. If they are tangents drawn to a circle from a point without. § 209

g. If they are the limits of two variable lines which constantly remain equal and indefinitely approach their respective limits. § 222

2. Two lines are perpendicular to each other,

a. If one is a tangent to a circle and the other is a radius drawn to the point of contact. § 205

b. If one is the common chord of two intersecting circles and the other is their line of centers. § 212

3. Two lines are unequal,

a. If one is a diameter of a circle and the other is any other chord of that circle. § 192

b. If they are chords of the same circle, or of equal circles, subtending unequal arcs. § 197

c. If they represent the distances of unequal chords in the same circle, or in equal circles, from the center. § 203

d. If they are chords of the same circle, or of equal circles, unequally distant from the center. § 204

4. A line is bisected,

a. If it is a chord of a circle, by a radius perpendicular to it. § 198

b. If it is a chord of a circle, by a line perpendicular to it and passing through the center. § 200

c. If it is the common chord of two intersecting circles, by their line of centers. § 212

5. A line passes through a point,

a. If it is the perpendicular bisector of a chord and the point is the center of the circle. § 199

b. If it is the line of centers of two tangent circles, and the point is their point of contact. § 213

6. Two angles are equal,

a. If they are central angles subtended by equal arcs in the same circle, or in equal circles. § 194

b. If they are inscribed in the same segment of a circle, or in equal segments of the same circle, or of equal circles. § 226

7. Two angles are unequal,

a. If they are central angles subtended by unequal arcs in the same circle, or in equal circles. § 195

8. An angle is measured,

a. If it is a central angle, by the intercepted arc. § 224

b. If it is an inscribed angle, by one half the intercepted arc. § 225

c. If it is between a tangent and a chord, by one half the intercepted arc. § 231

d. If it is a right angle, by one half a semicircumference. § 232

e. If it is between two intersecting chords, by one half the sum of the intercepted arcs. § 230

f. If it is between a tangent and a secant, by one half the difference of the intercepted arcs. § 233

g. If it is between two secants intersecting without the circle, by one half the difference of the intercepted arcs. § 234

9. Two arcs are equal,

a. If they are arcs of the same circle, or of equal circles and their extremities can be made to coincide. § 193

b. If they subtend equal central angles in the same circle, or in equal circles. § 194

c. If they are subtended by equal chords in the same circle, or in equal circles. § 196

d. If they are intercepted on a circumference by parallel lines. § 206

10. Two arcs are unequal,

a. If they subtend unequal central angles in the same circle, or in equal circles. § 195

b. If they are subtended by unequal chords in the same circle, or in equal circles. § 197

11. An arc is bisected,

a. By the radius perpendicular to the chord that subtends the arc. § 198

b. By a line through the center perpendicular to the chord. § 200

12. Two circles are equal,

a. If their radii or diameters are equal. Ax. 16

b. If they circumscribe equal triangles. § 208

13. A line is tangent to a circle,

a. If it is perpendicular to a radius at its extremity. § 205

SUPPLEMENTARY EXERCISES

Ex. **236**. ABC is an inscribed isosceles triangle; the vertical angle C is three times each base angle. How many degrees are there in each of the arcs AB, AC, and BC?

Ex. **237**. If a hexagon is circumscribed about a circle, the sums of its alternate sides are equal.

Ex. **238**. Two radii of a circle at right angles to each other are intersected, when produced, by a line tangent to the circle. Prove that the tangents drawn to the circle from the points of intersection are parallel to each other.

Ex. **239**. Two circles are tangent to each other externally and each is tangent to a third circle internally. Show that the perimeter of the triangle formed by joining the three centers is equal to the diameter of the exterior circle.

Ex. **240**. From two points, A and B, in a diameter of a circle, each the same distance from the center, two parallel lines AE and BF are drawn toward the same semicircumference, meeting it in E and F. Show that EF is perpendicular to AE and BF.

Ex. **241**. Two circles are tangent externally at A. BC is a tangent to the two circles at B and C. Prove that the circumference of the circle described on BC as a diameter passes through A.

Ex. **242**. OA is a radius of a circle whose center is O; B is a point on a radius perpendicular to OA; through B the chord AC is drawn; at C a tangent is drawn meeting OB produced in D. Prove that CBD is an isosceles triangle.

Suggestion. Draw a tangent at A.

Ex. **243**. Through a given point P without a circle whose center is O two lines PAB and PCD are drawn, making equal angles with OP and intersecting the circumference in A and B, C and D, respectively. Prove that AB equals CD, and that AP equals CP.

Suggestion. Draw OM and ON perpendicular to AB and CD, respectively.

Ex. **244**. If the angles at the base of a circumscribed trapezoid are equal, each non-parallel side is equal to half the sum of the parallel sides.

Ex. **245**. If a circle is inscribed in a right triangle, the sum of the diameter and the hypotenuse is equal to the sum of the other two sides of the triangle.

Ex. **246**. Any parallelogram which can be circumscribed about a circle is equilateral.

Ex. **247**. AB and CD are perpendicular diameters of a circle; E is any point on the arc ACB. Then, D is equidistant from AE and BE.

Ex. **248.** If two equal chords of a circle intersect, their corresponding segments are equal.

Ex. **249.** If the arc cut off by the base of an inscribed triangle is bisected and from the point of bisection a radius is drawn and also a line to the opposite vertex, the angle between these lines is equal to half the difference of the angles at the base of the triangle.

Ex. **250.** The two lines which join the opposite extremities of two parallel chords intersect at a point in that diameter which is perpendicular to the chords.

Ex. **251.** If a tangent is drawn to a circle at the extremity of a chord, the middle point of the subtended arc is equidistant from the chord and the tangent.

Ex. **252.** A line is drawn touching two tangent circles. Prove that the chords, that join the points of contact with the points in which the line through the centers meets the circumferences, are parallel in pairs.

Ex. **253.** Two circles intersect at the points A and B; through A a secant is drawn intersecting one circumference in C and the other in D; through B a secant is drawn intersecting the circumference CAB in E and the other circumference in F. Prove that the chords CE and DF are parallel.

Suggestion. Refer to Ex. 198 and 201.

Ex. **254.** The length of the straight line joining the middle points of the non-parallel sides of a circumscribed trapezoid is equal to one fourth the perimeter of the trapezoid.

Ex. **255.** A quadrilateral is inscribed in a circle, and two opposite angles are bisected by lines meeting the circumference in A and B. Prove that AB is a diameter.

Ex. **256.** The centers of the four circles circumscribed about the four triangles formed by the sides and diagonals of a quadrilateral lie on the vertices of a parallelogram.

Ex. **257.** If an equilateral triangle is inscribed in a circle, the distance of each side from the center is equal to half the radius of the circle.

Ex. **258.** The vertical angle of an oblique triangle inscribed in a circle is greater or less than a right angle by the angle contained by the base and the diameter drawn from the extremity of the base.

Ex. **259.** If from the extremities of any diameter of a given circle perpendiculars to any secant that is not parallel to this diameter are drawn, the less perpendicular is equal to that segment of the greater which is contained between the circumference and the secant.

Ex. **260.** Two circles are tangent internally at A, and a chord BC of the larger circle is tangent to the smaller at D. Prove that AD bisects the angle CAB.

Suggestion. Draw AT, the common tangent of the circles.

Ex. **261**. The tangents at the four vertices of an inscribed rectangle form a rhombus.

Ex. **262**. If a line is drawn through the point of contact of two circles which are tangent internally, intersecting the circle whose center is A at C, and the circle whose center is B at D, AC and BD are parallel.

Ex. **263**. If lines are drawn from the center of a circle to the vertices of any circumscribed quadrilateral, each angle at the center is the supplement of the central angle that is not adjacent to it.

Ex. **264**. Three circles are tangent to each other externally at the points A, B, and C. From A lines are drawn through B and C meeting the circumference which passes through B and C at the points D and E. Prove that DE is a diameter.

Ex. **265**. If an angle between a diagonal and one side of a quadrilateral is equal to the angle between the other diagonal and the opposite side, the same will be true of the three other pairs of angles corresponding to the same description, and the four vertices of the quadrilateral lie on a circumference.

Ex. **266**. Let the diameter AB of a circle be produced to C, making BC equal to the radius; through B draw a tangent, and from C draw a second tangent to the circle at D, intersecting the first at E; AD and BE produced meet at F. Prove that DEF is an equilateral triangle.

Suggestion. Draw a line from the center to E.

Ex. **267**. If from any point without a circle tangents are drawn, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through one of them.

Ex. **268**. The lines, which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, meet at the same point.

Ex. **269**. AB and AC are tangents at B and C respectively, to a circle whose center is O; from D, any point on the circumference, a tangent is drawn, meeting AB in E and AC in F. Prove that angle EOF is equal to one half angle BOC.

Ex. **270**. A circle whose center is O is tangent to the sides of an angle ABC at A and C; through any point in the arc AC, as D, a tangent is drawn, meeting AB in E, and CB in F. Prove (1) that the perimeter of the triangle BEF is constant for all positions of D in AC; (2) that the angle EOF is constant.

Ex. **271**. If AE and BD are drawn perpendicular to the sides BC and AC, respectively, of the triangle ABC, and DE is drawn, the angles AED and ABD are equal.

Suggestion. Describe a circle passing through A, D, and B.

Ex. **272**. The perimeter of an inscribed equilateral triangle is equal to one half the perimeter of the circumscribed equilateral triangle.

Ex. **273**. In the circumscribed quadrilateral $ABCD$, the angles A, B, and C are 110°, 95°, and 80°, respectively, and the sides AB, BC, CD, and DA touch the circumference at the points E, F, G, and H respectively. Find the number of degrees in each angle of the quadrilateral $EFGH$.

Ex. **274**. If an inscribed isosceles triangle has each of its base angles double the vertical angle, and its vertices the points of contact of three tangents, these tangents form an isosceles triangle each of whose base angles is one third its vertical angle.

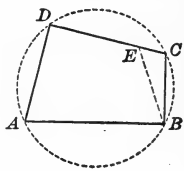
Ex. **275**. ABC is a triangle inscribed in a circle; the bisectors of the angles A, B, and C meet in D; AD produced meets the circumference in E. Prove that DE equals CE.

Suggestion. Produce CD to meet the circumference at F, and draw AF.

Ex. **276**. If the diagonals of a quadrilateral inscribed in a circle intersect at right angles, the perpendicular from their intersection upon any side, if produced, bisects the opposite side.

Ex. **277**. If the opposite angles of a quadrilateral are supplementary, the quadrilateral may be inscribed in a circle.

Suggestion. § 207. A circumference may be passed through A, B, and D, and if it does not pass through C, it will cut DC, or DC produced, as at E. Draw EB.



Then, from data and the supposition that the circumference passes through E, it may be shown by measurement of inscribed angles, that $\angle DCB = \angle DEB$.

But, § 115, $\angle DEB = \angle CBE + \angle DCB$;

$\therefore \quad \angle DCB = \angle CBE + \angle DCB$,

which is absurd, and the supposition that the circumference passes through E and not through C is untenable.

Hence, the circumference through A, B, and D passes through C.

Ex. **278**. The four lines bisecting the angles of any quadrilateral form a quadrilateral which may be inscribed in a circle.

Suggestion. Refer to Ex. 179 and 277.

Ex. **279**. The line, drawn from the center of the square described upon the hypotenuse of a right triangle to the vertex of the right angle, bisects the right angle.

Ex. **280**. AB and CD are two chords of a circle intersecting at E; through A a line is drawn to meet a line tangent at C so that the angle AFC equals the angle BEC. Then, EF is parallel to BC.

Ex. **281**. ABC is a triangle; AD and BE are the perpendiculars from A and B upon BC and AC respectively; DF and EG are the perpendiculars from D and E upon AC and BC respectively. Then, FG is parallel to AB.

PROBLEMS

255. Problems are valuable for developing the ingenuity of the student in discovering the auxiliary lines necessary for solution or demonstration, and for fixing clearly in mind the knowledge previously acquired. They do not, however, involve any new fundamental fact or principle of geometry, and may therefore be omitted without impairing the logical development of the science.

No definite rules can be given for solving problems, but close attention to the following suggestions, and a thorough study of the Summary will be of great assistance in developing a proper and logical method of procedure.

I. *Study carefully the data of the problem to discover every fact that is given, and notice also what is required.*

II. *From the facts given deduce all possible conclusions, and try to relate them to what is required.*

III. *If the principles upon which the solution is based are not readily discovered from the data, try to get some clew to the solution by studying the Summary and by drawing lines perpendicular or parallel to other lines; by forming triangles; by joining given points; by describing circles; etc.*

IV. *The solution is often readily discovered by drawing a figure representing the problem solved, and then from a study of the relations of the known and unknown parts of the figure discover the facts which bear upon the solution.*

Ex. **282**. Construct the complement of a given angle.

Ex. **283**. Construct the supplement of a given angle.

Ex. **284**. At a given point in a given straight line, to construct an angle of 45°.

Ex. **285**. Divide an angle into four equal parts.

Ex. **286**. At a given point in a given straight line, to construct an angle of 60°.

Ex. **287**. Trisect a right angle.

Ex. **288**. Construct a square, having given one side.

Ex. **289**. Construct an isosceles triangle, having given the base and the perpendicular from the vertex to the base.

Ex. **290**. Construct an equilateral triangle, having given the perimeter.

Ex. 291. Construct an isosceles triangle, having given the perimeter and base.

Ex. 292. Construct a rectangle, having given two adjacent sides.

Ex. 293. Construct a rectangle, having given the shorter side and the difference of two sides.

Ex. 294. Construct a rectangle, having given the longer side and the difference of two sides.

Ex. 295. Construct a rectangle, having given the sum and difference of two adjacent sides.

Ex. 296. Construct a rhombus, having given one of its angles and the length of its side.

Ex. 297. Construct an isosceles triangle, having given the base and one of the two equal angles at the base.

Ex. 298. Construct an isosceles right triangle, having given its hypotenuse.

Ex. 299. Construct a rhomboid, having given the perimeter, one side, and one angle.

Ex. 300. Construct a right triangle, having given the hypotenuse and one side.

Ex. 301. Construct a right triangle, having given the hypotenuse and one acute angle.

Ex. 302. Construct an isosceles trapezoid, having given two sides and the included angle.

Ex. 303. Construct a trapezoid, having given two adjacent sides, the included angle, and the angle at the other extremity of the given parallel base.

Ex. 304. From a given point without a given line, to draw a line making a given angle with the given line.

Ex. 305. Construct a square, having given a diagonal.

Ex. 306. Construct a rectangle, having given one side and the angle included between it and a diagonal.

Ex. 307. Construct a rectangle, having given a diagonal and an angle between it and a side.

Ex. 308. Construct a rhombus, having given its perimeter and one diagonal.

Ex. 309. Construct a rhomboid, having given two diagonals and an angle between them.

Ex. 310. Construct a rectangle, having given one diagonal and the angle included between the two diagonals.

Ex. 311. Construct a rectangle, having given the perimeter and one side.

Ex. **312**. Construct a trapezoid, having given two sides, the included angle, and the difference between the two parallel sides.

Ex. **313**. Construct a trapezium, having given three consecutive sides and the two included angles.

Ex. **314**. Construct a trapezium, having given two adjacent sides and the three angles adjacent to these sides.

Ex. **315**. Construct an isosceles triangle, having given the base and the vertical angle.

Ex. **316**. Construct an equilateral triangle, having given its altitude.

Ex. **317**. Construct a triangle, having given two sides and the angle opposite one.

Ex. **318**. Construct a triangle, having given its base, the median to the base, and the angle included between them.

Ex. **319**. Construct a right triangle, having given its hypotenuse and having one of its acute angles double the other.

Ex. **320**. Construct a trapezoid, having given the sum and difference of the parallel sides, and the sum and difference of the angles at the base.

Ex. **321**. Construct a rhombus, having given its diagonals.

Ex. **322**. Construct a rhomboid, having given two adjacent sides and an angle not included by them.

Ex. **323**. Construct a rhomboid, having given one side and the angles included between it and the diagonals.

Ex. **324**. Construct an isosceles trapezoid, having given the bases and the altitude.

Ex. **325**. Construct an isosceles trapezoid, having given the altitude and the sum and difference of the parallel sides.

Ex. **326**. Construct a triangle, having given two angles and a side opposite one.

Ex. **327**. Draw a line which shall pass through a given point and make equal angles with two given intersecting lines.

Ex. **328**. Construct a right triangle, having given one side and the angle opposite.

Ex. **329**. Construct an isosceles trapezoid, having given the bases and a diagonal.

Ex. **330**. Construct an isosceles trapezoid, having given the bases and one angle.

Ex. **331**. From two given points, draw two equal straight lines which shall meet in the same point of a line given in position.

Ex. **332**. ABC is an isosceles triangle. Draw a straight line parallel to the base AB and meeting the equal sides in E and F, so that BF, EF, and EA are all equal.

Ex. **333**. Given two straight lines which cannot be produced to their intersection, to draw a third line which would pass through their intersection and bisect their contained angle.

Ex. **334**. Construct a triangle, having given the altitude and the angles at the base.

Ex. **335**. Given the middle point of a chord in a given circle, to draw the chord.

Ex. **336**. Construct a circle to pass through two given points and have its center on a given straight line. When is this impossible?

Ex. **337**. Draw a tangent to a circle parallel to a given straight line.

Ex. **338**. Draw a tangent to a circle perpendicular to a given straight line.

Ex. **339**. Draw a straight line tangent to a given circle and making with a given line a given angle.

Ex. **340**. Construct a circle of given radius to pass through two given points. When is this impossible?

Ex. **341**. Construct a circle tangent to two intersecting lines with its center at a given distance from their intersection. How many such circles can be drawn?

Ex. **342**. From a given point as a center, to describe a circle tangent to a given circle. How many solutions may there be?

Ex. **343**. Construct a circle of given radius tangent to a given circle at a given point. How many solutions may there be?

Ex. **344**. Draw a common tangent to two given circles. How many solutions may there be?

Ex. **345**. In a given circle, to inscribe a triangle equiangular to a given triangle.

Ex. **346**. About a given circle, to circumscribe a triangle equiangular to a given triangle.

Ex. **347**. Construct a triangle, having given the vertical angle, one of the sides containing it, and the altitude.

Ex. **348**. Construct a triangle, having given the base, the vertical angle, and one other side.

Suggestion. On the given base construct a segment that will contain an angle equal to the given angle.

Ex. **349**. Construct a triangle, having given the base, the vertical angle, and the foot of the perpendicular from the vertex to the base.

Ex. **350**. Construct a triangle whose vertex is on a given straight line, and having given its base and vertical angle.

Ex. **351**. Construct a triangle, having given the base, the vertical angle, and the altitude.

Ex. **352**. Describe a circle with a given center to intersect a given circle at the extremities of a diameter. Is this ever impossible?

Ex. **353**. Construct a circle to pass through a given point and be tangent to a given circle at a given point. When is this impossible?

Ex. **354**. Construct a circle to pass through a given point and touch a given straight line at a given point.

Ex. **355**. Construct a circle to touch three given straight lines.

Ex. **356**. Within an equilateral triangle, to describe three circles each tangent to the other two and to two sides of the triangle.

Ex. **357**. Construct a circle of given radius to touch two given straight lines.

Ex. **358**. Construct a circle of given radius, having its center on a given straight line and touching another given straight line. How many solutions may there be?

Ex. **359**. Construct a right triangle, having given the radius of the inscribed circle and one of the sides containing the right angle.

Ex. **360**. Construct a triangle, having given the base, the vertical angle, and the length of the median to the base.

Ex. **361**. Construct a triangle, having given the three middle points of its sides.

Ex. **362**. Construct a circle of given radius to pass through a given point and touch a given straight line.

Ex. **363**. From the vertices of a triangle as centers, to describe three circles which shall be tangent to each other.

Ex. **364**. Construct a triangle, having given the base, the altitude, and the radius of the circumscribed circle.

Ex. **365**. Three given straight lines meet at a point; draw another straight line so that the two portions of it intercepted between the given lines are equal. How many solutions may there be?

Suggestion. Form a parallelogram.

Ex. **366**. Through a given point, between two intersecting straight lines, to draw a line terminated by the given lines and bisected at the given point.

Ex. **367**. Construct a circle to intercept equal chords of given length on three given straight lines.

Ex. **368**. Construct a triangle, having given one angle, the opposite side, and the sum of the other two sides.

The Locus of a Point

256. When a point equidistant from the extremities of a straight line is to be found, the middle point of the line meets the conditions. But there are other points which also fulfill the required conditions, for every point in the perpendicular to the given line at its middle point is equidistant from the extremities of the line.

Such a perpendicular is called the *locus* of the points which are equidistant from the extremities of the line.

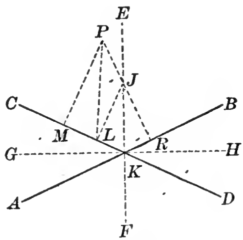
The *line, or system of lines, containing every point which satisfies certain given conditions, and no other points, is called the **Locus** of those points.

A locus may also be described as a line, or the lines, traced by a point which moves in accordance with given conditions.

To prove that a certain line, or system of lines, is the required locus, it must be shown:

1. *That every point in the lines satisfies the given conditions.*
2. *That any point not in the lines cannot satisfy the given conditions.*

Ex. **369**. *Find the locus of a point which is equidistant from two intersecting straight lines.*



Data: Any two straight lines, as AB and CD, intersecting at the point K.

Required to find the locus of a point equidistant from AB and CD.

Solution. A point equidistant from two intersecting straight lines suggests a point in the bisector of an angle.

Draw EF bisecting $\angle s$ CKB and AKD, and also GH bisecting $\angle s$ AKC and BKD.

Then, § 134, every point in EF is equidistant from AB and CD, and every point in GH is equidistant from AB and CD.

If all other points are unequally distant from AB and CD, then EF and GH is the required locus.

From P any point without the bisectors draw $PM \perp CD$, and $PR \perp AB$, intersecting EF in J. From J draw $JL \perp CD$, and also draw PL.

Then, § 61, $PL > PM$, and, § 125, $PJ + JL > PL$;

$\therefore$ $PJ + JL > PM$.

But $JL = JR$; Why?

$PJ + JR > PM$, or $PR > PM$.

That is, the point P is unequally distant from AB and CD.

Hence EF and GH is the required locus.

Ex. **370**. Find the locus of a point at a given distance from a given point.

Ex. **371**. Find the locus of a point equidistant from two parallel straight lines.

* In this and the next four paragraphs the *lines* mentioned may be either straight or curved.

Ex. **372**. Find the locus of a point at a given distance from a given straight line.

Ex. **373**. Find a point which is equidistant from three given points not in the same straight line.

Ex. **374**. Find the locus of a point equidistant from the circumferences of two concentric circles.

Ex. **375**. Find a point in a given straight line which is equidistant from two given points.

Ex. **376**. Find the locus of the center of a circle tangent to each of two parallel lines.

Ex. **377**. Find the locus of the center of a circle which touches a given line at a given point.

Ex. **378**. Find the locus of the center of a circle of given radius that passes through a given point.

Ex. **379**. Find the locus of the center of a circle which is tangent to a given circle at a given point.

Ex. **380**. Find the locus of the center of a circle of given radius and tangent to a given circle.

Ex. **381**. Find the locus of the center of a circle passing through two given points.

Ex. **382**. Find the locus of the center of a circle of given radius and tangent to a given line.

Ex. **383**. Find the locus of the center of a circle tangent to each of two intersecting lines.

Ex. **384**. Find the locus of the middle points of a system of parallel chords drawn in a circle.

Ex. **385**. Find the locus of the middle points of equal chords of a given circle.

Ex. **386**. Find the locus of the extremities of tangents of fixed length drawn to a given circle.

Ex. **387**. Find the locus of the middle points of straight lines drawn from a given point to meet a given straight line.

Ex. **388**. Find the locus of the vertex of a right triangle on a given base as hypotenuse.

Ex. **389**. Find the locus of the middle points of all chords of a circle drawn from a fixed point in the circumference.

Ex. **390**. Find the locus of the middle point of a straight line moving between the sides of a right angle.

Ex. **391**. Find the locus of the points of contact of tangents from a fixed point to a system of concentric circles.

Ex. **392**. Find the locus of the middle points of secants drawn from a given point to a given circle.

BOOK III

RATIO AND PROPORTION

257. **1.** How is a magnitude measured?

2. What is the numerical measure of a magnitude?

3. What is the common measure of two or more magnitudes?

4. What is meant by the ratio of two magnitudes?

5. How may the ratio of two magnitudes be determined?

6. Since the ratio of two magnitudes is the ratio of their numerical measures, what is the relation of two magnitudes whose numerical measures are 8 and 16 respectively? 5 and 10? 12 and 36? 15 and 45?

7. How does 8 compare with 2? What is the relation of 3 to 9? Of 12 to 4? Of 18 to 3? Of 20 to 40? Of 25 to 75? Of 35 to 70?

8. What is the ratio of 1 ft. to 1 yd.? 3 in. to 1 ft.? 2^{cm} to 1^{dm}? 5^{dm} to 2^{m}? 2 sq. ft. to 2 sq. yd.? 3 cu. ft. to 1 cu. yd.?

258. The quantities compared are called the **Terms of the ratio.**

A ratio is denoted by a colon placed between the terms.

The ratio of 2 to 5 is expressed 2 : 5.

259. The first term of a ratio is called the **Antecedent of the ratio.** The second term of a ratio is called the **Consequent of the ratio.**

260. The antecedent and consequent together form a **Couplet.**

261. Since the ratio of two quantities may be expressed by a fraction, as $\frac{a}{b}$, it follows that:

The changes which may be made upon the terms of a fraction without altering its value may be made upon the terms of a ratio without altering the ratio.

262. 1. What two numbers have the same relation to each other as 3 has to 6? 2 to 8? 5 to 15? 8 to 4?

2. What numbers have the same relation to each other that 4 in. has to 2 ft.? 5 ft. to 2 yd.? 5^{cm} to 1^{m}? 3^{dm} to 8^{cm}?

3. What number has the same relation to 6 that 2 has to 4?

4. What number has the same relation to 12 that 3 has to 9?

5. What number has the same ratio to 8 that 5 has to 15?

263. An equality of ratios is called a **Proportion.**

The sign of equality is written between the equal ratios.

$a : b = c : d$ is a *proportion*, and is read: the ratio of a to b is equal to the ratio of c to d, or a is to b as c is to d.

The double colon, : :, is frequently used instead of the sign of equality.

264. The antecedents of the ratios which form a proportion are called the **Antecedents of the proportion,** and the consequents of those ratios are called the **Consequents of the proportion.**

In the proportion $a : b = c : d$, a and c are the *antecedents*, and b and d are the *consequents* of the proportion.

265. The first and fourth terms of a proportion are called the **Extremes** and the second and third terms are called the **Means of the proportion.**

In the proportion $a : b = c : d$, a and d are the *extremes*, and b and c are the *means.*

266. A quantity which serves as both means of a proportion is called a **Mean Proportional.**

In the proportion $a : b = b : c$, b is a *mean proportional.*

267. Since a proportion is an equality of ratios, and the ratio of one quantity to another is found by dividing the antecedent by the consequent, it follows that:

A proportion may be expressed as an equation in which both members are fractions.

The proportion $a : b = c : d$ may be written $\frac{a}{b} = \frac{c}{d}$.

Such an expression is to be read as the ordinary form of a proportion is read.

268. **Since a proportion** may be regarded as an equation in which both members are fractions, it follows that:

1. *The changes which may be made upon the members of an equation without destroying the equality may be made upon the couplets of a proportion without destroying the equality of the ratios.*

2. *The changes which may be made upon the terms of a fraction without altering the value of the fraction may be made upon the terms of each ratio of a proportion without destroying the proportion.*

Proposition I

269. 1. **Form several** proportions, as $3:5=9:15$, and discover how the product of the extremes compares with the product of the means in each.

2. If the means in any proportion are the same, how may the means be found from the product of the extremes?

3. Form a proportion whose consequents are equal. How do the antecedents compare?

4. Form a proportion in which either antecedent is equal to its consequent. How does the other antecedent compare with its consequent?

Theorem. In any proportion, the product of the extremes is equal to the product of the means.

Data: $a:b=c:d.$

To prove $ad=bc.$

Proof. From data, § 267, $\frac{a}{b}=\frac{c}{d}.$

Multiplying each member of this equation by bd,

$$ad=bc.$$

Therefore, etc. Q.E.D.

270. Cor. I. *A mean proportional between two quantities is equal to the square root of their product.*

If $a:b=b:c$, find the value of b.

271. Cor. II. *If in any proportion any antecedent is equal to its consequent, the other antecedent is equal to its consequent.*

272. Cor. III. *If the consequents of any proportion are equal, the antecedents are equal, and conversely.*

For, if $a : b = c : b$,

$$\frac{a}{b} = \frac{c}{b},$$

and multiplying by b, $a = c$.

Proposition II

273. 1. If the product of the extremes of a proportion is 48, what may the extremes be? If 72? If 30? If 36? If $6a^2$? If $12ab$? If abc?

2. If the product of the means is 48, what may the means be? If 96? If 108? If $6bcd$? If $abcd$? If a^2b^2? If abc?

3. Form a proportion the product of whose extremes or means is 60; 72; 84; 80; 64; 144; x^2y^2, xyz; $xyzv$.

Theorem. *If the product of two quantities is equal to the product of two others, either two may be made the extremes of a proportion of which the other two are the means.*

Data: $ad = bc$.

To prove $a : b = c : d$.

Proof. Data, $ad = bc$.

Dividing each member of this equation by bd,

$$\frac{a}{b} = \frac{c}{d};$$

that is, $a : b = c : d$.

Therefore, etc. Q.E.D.

Ex. **393.** If the vertical angle of an isosceles triangle is 30°, what is its ratio to each of the base angles?

Ex. **394.** If the exterior angle at the base of an isosceles triangle is 100°, what is its ratio to each angle of the triangle?

Ex. **395.** If one of the acute angles of a right triangle is 40°, what is its ratio to the other acute angle? To the right angle?

Ex. **396.** The interior angles on the same side of a transversal cutting two parallel lines are to each other as 3 to 2. How many degrees are there in each angle?

Ex. **397.** The vertical angle of an isosceles triangle has the same ratio to a right angle that an angle of 40° has to an angle of an equilateral triangle. How many degrees are there in each angle of the isosceles triangle?

Proposition III

274. 1. Form a proportion and transpose the means. How do the resulting ratios compare?

2. Transpose the extremes. How do the resulting ratios compare?

3. Transform similarly and investigate other proportions.

Theorem. *In any proportion, the first term is to the third as the second is to the fourth; that is, the terms are in proportion by* ***alternation.***

Data: $a : b = c : d.$

To prove $a : c = b : d.$

Proof. From data, § 267, $\frac{a}{b} = \frac{c}{d}.$

Multiplying each member of this equation by $\frac{b}{c}$,

$$\frac{a}{c} = \frac{b}{d};$$

that is, $a : c = b : d.$

Therefore, etc. **Q.E.D.**

Proposition IV

275. 1. Form a proportion. If the antecedent of each ratio becomes the consequent, and the consequent the antecedent, how do the resulting ratios compare?

2. Transform similarly and investigate other proportions.

Theorem. *In any proportion, the ratio of the second term to the first is equal to the ratio of the fourth term to the third; that is, the terms are in proportion by* ***inversion.***

Data: $a : b = c : d.$

To prove $b : a = d : c.$

Proof. From data, § 269, $bc = ad.$

Dividing each member of this equation by ac,

$$\frac{b}{a} = \frac{d}{c};$$

that is, $b : a = d : c.$

Therefore, etc. **Q.E.D.**

Proposition V

276. 1. Form a proportion. How does the ratio of the sum of the first two terms to either term compare with the ratio of the sum of the last two terms to the corresponding term?

2. Transform similarly and investigate other proportions.

Theorem. *In any proportion, the ratio of the sum of the first two terms to either term is equal to the ratio of the sum of the last two terms to the corresponding term; that is, the terms are in proportion by* ***composition.***

Data: $a : b = c : d.$

To prove $a + b : b = c + d : d$, and $a + b : a = c + d : c.$

Proof. § 267, $\frac{a}{b} = \frac{c}{d}.$

Adding 1 to each member of this equation,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d};$$

that is, $$a + b : b = c + d : d.$$

In like manner it may be shown that $a + b : a = c + d : c.$

Therefore, etc. Q.E.D.

Proposition VI

277. 1. Form a proportion. How does the ratio of the difference of the first two terms to either term compare with ratio of the difference of the last two terms to the corresponding term?

2. Transform similarly and investigate other proportions.

Theorem. *In any proportion, the ratio of the difference between the first two terms to either term is equal to the ratio of the difference between the last two terms to the corresponding term; that is, the terms are in proportion by* ***division.***

Data: $a : b = c : d.$

To prove $a - b : b = c - d : d$, and $a - b : a = c - d : c.$

Proof. § 267, $\frac{a}{b} = \frac{c}{d}.$

Subtracting 1 from each member of this equation,

$$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d};$$

that is, $$a - b : b = c - d : d.$$

In like manner it may be shown that $a - b : a = c - d : c.$

Therefore, etc. Q.E.D.

Proposition VII

278. 1. Form a proportion. How does the ratio of the sum of the first two terms to their difference compare with the ratio of the sum of the last two terms to their difference?

Theorem. *In any proportion, the ratio of the sum of the first two terms to their difference is equal to the ratio of the sum of the last two terms to their difference; that is, the terms are in proportion by* ***composition*** *and* ***division.***

Data: $a : b = c : d.$

To prove $a + b : a - b = c + d : c - d.$

Proof. §§ 276, 267, $$\frac{a+b}{b} = \frac{c+d}{d}, \qquad (1)$$

and, §§ 277, 267, $$\frac{a-b}{b} = \frac{c-d}{d}. \qquad (2)$$

Dividing (1) by (2),

$$\frac{a+b}{a-b} = \frac{c+d}{c-d};$$

that is, $$a + b : a - b = c + d : c - d.$$

Therefore, etc. Q.E.D.

Proposition VIII

279. 1. Form a series of equal ratios, as $2:3=4:6=8:12=10:15$. How does the ratio of the sum of the antecedents to the sum of the consequents compare with the ratio of any antecedent to its consequent?

2. Transform similarly and investigate other series.

Theorem. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

Data: Any series of equal ratios, as $a:b=c:d=e:f=g:h$.

To prove $a+c+e+g:b+d+f+h=a:b$, or $c:d$, etc.

Proof. Denoting each ratio by r, $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=r.$ (1)

From (1), $a=br,\ c=dr,\ e=fr,$ and $g=hr.$ (2)

Adding equations (2),

$$a+c+e+g=(b+d+f+h)r. \quad (3)$$

Dividing (3) by $(b+d+f+h)$,

$$\frac{a+c+e+g}{b+d+f+h}=r.$$

Substituting the value of r in (1),

$$\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}, \text{ or } \frac{c}{d}, \text{ etc.};$$

that is, $a+c+e+g:b+d+f+h=a:b$, or $c:d$, etc.

Therefore, etc. Q.E.D.

Ex. **398.** If $a:b=c:d$, prove that $a:a+b=c:c+d$.

Ex. **399.** If $a:b=b:c$, prove that $a+b:b+c=a:b$.

Ex. **400.** AD bisects angle A at the base of the isosceles triangle ABC, and meets the side BC in D. If angle C is 68°, what is its ratio to angle ADB?

Ex. **401.** The sum of the angles of a polygon expressed in right angles is to the number of its sides as 4 is to 3. How many sides has the polygon?

Ex. **402.** If the angle formed by two secants intersecting without a circle is 30° and the smaller of the intercepted arcs is 20°, what is the ratio of the smaller arc to the larger?

Ex. **403.** If the angle formed by two intersecting chords of a circle is 40° and one of the intercepted arcs is 30°, what is the ratio of that arc to the opposite intercepted arc?

Proposition IX

280. Form two or more proportions in which the corresponding consequents are equal, as $2:3=6:9$ and $5:3=15:9$. How does the ratio of the sum of the antecedents of the first couplets to their common consequent compare with the ratio of the sum of the antecedents of the second couplets to their common consequent?

Theorem. *When two or more proportions have the same quantity as the consequents of the first couplets and another quantity as the consequents of the second couplets, the sum of the antecedents of the first couplets is to their common consequent as the sum of the antecedents of the second couplets is to their common consequent.*

Data: $a:b=c:d,$ (1)

$e:b=f:d,$ (2)

and $g:b=h:d.$ (3)

To prove $a+e+g:b=c+f+h:d.$

Proof. From (1) $\frac{a}{b}=\frac{c}{d};$ (4)

from (2) $\frac{e}{b}=\frac{f}{d};$ (5)

from (3) $\frac{g}{b}=\frac{h}{d};$ (6)

$\therefore$ (4) + (5) + (6), $\frac{a+e+g}{b}=\frac{c+f+h}{d};$

that is, $a+e+g:b=c+f+h:d.$

Therefore, etc. Q.E.D.

281. Cor. *When two or more proportions have the same quantity as the antecedents of the first couplets, and another quantity as the antecedents of the second couplets, the common antecedent of the first couplets is to the sum of their consequents as the common antecedent of the second couplets is to the sum of their consequents.*

Ex. **404.** If $a:b=c:d$, prove that $2a+b:b=2c+d:d$.

Ex. **405.** If $a:b=c:d$, prove that $a:3a+b=c:3c+d$.

Ex. **406.** If $a:b=b:c$, prove that $2a-b:a=2b-c:b$.

Ex. **407.** If $a:b=b:c$, prove that $a+3b:b=b+3c:c$.

Proposition X

282. 1. Form a proportion; multiply or divide the terms of either ratio by any number. How do the resulting ratios compare?

2. Transform similarly and investigate other proportions.

Theorem. *If in a proportion the terms of either couplet are multiplied by any quantity, the resulting ratios form a proportion.*

Data: $a : b = c : d.$

To prove $ma : mb = c : d.$

Proof. $$\frac{a}{b} = \frac{c}{d}.$$

Multiplying both terms of the first fraction by m,

$$\frac{ma}{mb} = \frac{c}{d};$$

that is, $ma : mb = c : d.$

Therefore, etc. Q.E.D.

283. Cor. *If in a proportion the terms of either couplet are divided by any quantity, the resulting ratios form a proportion.*

Proposition XI

284. 1. Form a proportion; multiply or divide the antecedents or the consequents by any number. How do the resulting ratios compare?

Theorem. *If in any proportion the antecedents or the consequents are multiplied by the same quantity, the resulting ratios are in proportion.*

Data: $a : b = c : d.$

To prove $ma : b = mc : d,$

and $a : nb = c : nd.$

Proof. $$\frac{a}{b} = \frac{c}{d}. \qquad (1)$$

Multiplying (1) by m, $$\frac{ma}{b} = \frac{mc}{d};$$

that is, $ma : b = mc : d.$

Dividing (1) by n, $\dfrac{a}{nb} = \dfrac{c}{nd}$;

that is, $a : nb = c : nd$.

Therefore, etc. Q.E.D.

285. Cor. *If in any proportion the antecedents or the consequents are divided by the same quantity, the resulting ratios are in proportion.*

Proposition XII

286. 1. Form several proportions. Multiply together their corresponding terms, and discover whether the resulting quantities form a proportion.

2. If there is an equal antecedent and consequent in the same couplet, or in corresponding couplets, cancel them from the products of the corresponding terms. Do the resulting quantities form a proportion?

Theorem. *The products of the corresponding terms of any number of proportions are in proportion.*

Data: $a : b = c : d$, $e : f = g : h$, and $k : l = m : o$.

To prove $aek : bfl = cgm : dho$.

Proof. $\dfrac{a}{b} = \dfrac{c}{d}$, $\dfrac{e}{f} = \dfrac{g}{h}$, and $\dfrac{k}{l} = \dfrac{m}{o}$.

Multiplying these equations together,

$$\frac{aek}{bfl} = \frac{cgm}{dho};$$

that is, $aek : bfl = cgm : dho$.

Therefore, etc. Q.E.D.

287. Cor. *In finding the proportion formed by the products of the corresponding terms of any number of proportions, an equal antecedent and consequent in the same couplet, or in corresponding couplets, may be dropped.*

For, if $a : b = c : d$,

and $b : e = f : c$,

§ 286, $ab : be = cf : dc$.

Dividing the terms of the first couplet by b and the terms of the second by c, § 283, $a : e = f : d$.

Proposition XIII

288. 1. Form a proportion; raise the terms of both ratios to the same power. How do the resulting ratios compare?

2. Extract the same root of the terms of both ratios in a proportion, as $4:9 = 16:36$. How do the resulting ratios compare?

3. Transform similarly and investigate other proportions.

Theorem. *In any proportion, like powers or like roots of the terms are in proportion.*

Data: $a : b = c : d.$

To prove $a^n : b^n = c^n : d^n$, and $a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

Proof.
$$\frac{a}{b} = \frac{c}{d}. \qquad (1)$$

Raising both fractions in (1) to the nth power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n};$$

that is,
$$a^n : b^n = c^n : d^n.$$

Extracting the nth root of both fractions in (1),

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}};$$

that is,
$$a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

Therefore, etc. Q.E.D.

Ex. **408.** Make the changes that may be made upon the following proportion without destroying the equality of the ratios: $16 : 36 = 4 : 9$.

Ex. **409.** If $a : b = c : d$, prove that $ma : nb = mc : nd$.

Ex. **410.** If $a : b = c : d$, prove that $a + 4b : b = c + 4d : d$.

Ex. **411.** If $a : b = b : c$, prove that $a^2 + ab : b^2 + bc = a : c$.

Ex. **412.** If $a : b = b : c$, prove that $a : c = (a + b)^2 : (b + c)^2$.

Ex. **413.** If $a : b = m : n$, and $b : c = n : o$, prove that $a : c = m : o$.

Ex. **414.** If $a : b = c : d$, prove that
$$ma - nb : ma + nb = mc - nd : mc + nd.$$

Ex. **415.** If $a : b = c : d$, prove that
$$3a + 4b : 4a - 5b = 3c + 4d : 4c - 5d.$$

BOOK IV

PROPORTIONAL LINES AND SIMILAR FIGURES

Proposition I

289. 1. Draw a line parallel to the base of a triangle through the middle point of one side and cutting the other side. How do the segments of the other side compare in length?

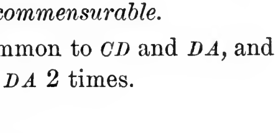
2. Draw a line parallel to the base one fourth, one sixth, or any part of the distance from the extremity of the base to the vertex. How do the segments of the other side compare?

3. How does the ratio of one of these sides to either of its segments compare with the ratio of the other to its corresponding segment?

Theorem. *A line which is parallel to one side of a triangle and meets the other two sides divides those sides proportionally.*

Data: Any triangle, as ABC, and any line parallel to AB, as DE, meeting AC and BC in D and E, respectively.

To prove $CD : DA = CE : EB$.



Proof. **Case I.** *When CD and DA are commensurable.*

Suppose that M is a unit of measure common to CD and DA, and that M is contained in CD 3 times and in DA 2 times.

Then, hyp., $$CD : DA = 3 : 2.$$

Divide CD and DA into parts each equal to the common measure M, and from each point of division draw lines parallel to AB.

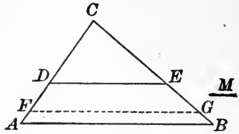
§ 157, these lines divide CE into 3 and EB into 2 equal parts;

$\therefore$ $$CE : EB = 3 : 2,$$

and, Ax. 1, $$CD : DA = CE : EB.$$

Case II. *When CD and DA are incommensurable.*

Since CD and DA are incommensurable, suppose that CD and DF are commensurable and that $FA < M$.



Draw $FG \parallel DE$.

Case I, $CD : DF = CE : EG$.

If M is indefinitely diminished, the ratios $CD : DF$ and $CE : EG$ remain equal, and indefinitely approach their limiting ratios $CD : DA$ and $CE : EB$, respectively.

Hence, § 222, $CD : DA = CE : EB$.

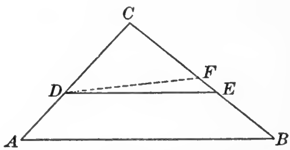
Therefore, etc. Q.E.D.

290. Cor. *A line which is parallel to one side of a triangle and meets the other two sides divides them so that one side is to either of its segments as the other side is to its corresponding segment.*

Proposition II

291. Draw a line dividing each of two sides of a triangle into halves, or into other proportional parts. What is the direction of this line with reference to the third side?

Theorem. *A line which divides two sides of a triangle proportionally is parallel to the third side.* (Converse of Prop. I.)



Data: Any triangle, as ABC, and the line DE dividing AC and BC so that $CA : CD = CB : CE$.

To prove $DE \parallel AB$.

Proof. If DE is not parallel to AB, some other line drawn through D will be parallel to AB.

Suppose that DF is that line.

Then, § 290, $CA : CD = CB : CF$;

but, data, $CA : CD = CB : CE$;

$\therefore$ $CB : CF = CB : CE$;

hence, § 272, $CF = CE$.

But this is impossible unless F coincides with E;
that is, Ax. 11, unless DF coincides with DE.

Therefore, the hypothesis, that some line other than DE drawn through D is parallel to AB, is untenable.

Hence, $DE \parallel AB$.

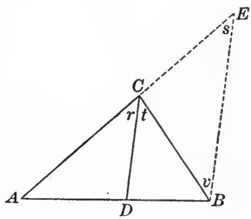
Therefore, etc. Q.E.D.

Proposition III

292. Draw a triangle whose sides are 6″, 5″, and 3″, or any other dimensions; bisect any one of its angles and produce the bisector to meet the opposite side.

How does the ratio of the segments of this side made by the bisector compare with the ratio of the sides of the triangle adjacent to these segments?

Theorem. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.*



Data: Any triangle, as ABC, and CD the bisector of one of its angles, ACB.

To prove $AD : DB = AC : CB$.

Proof. From B draw a line parallel to CD and meeting AC produced in E.

Then, § 289, $AD : DB = AC : CE$,

also, $\angle r = \angle s$, Why?

and $\angle t = \angle v$; Why?

but, data, $\angle r = \angle t$;

$\therefore$ $\angle s = \angle v$, Why?

and, § 118, $CB = CE$.

Substituting CB in the proportion for its equal CE,

$$AD : DB = AC : CB.$$

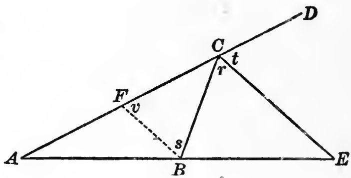
Therefore, etc. Q.E.D.

Proposition IV

293. Draw a triangle whose sides are 6″, 5″, and 3″, or any other dimensions; bisect an exterior angle at any vertex and produce the bisector to meet the opposite side produced. How does the ratio of the distances from the point of meeting to each extremity of the opposite side compare with the ratio of the other sides of the triangle?

Theorem. The bisector of an exterior angle of a triangle meets the opposite side produced at a point the distances of which from the extremities of this side are proportional to the other two sides.

Data: A triangle, as ABC; an exterior angle, as BCD; and its bisector CE meeting AB produced in E.



To prove $AE : BE = AC : BC$.

Proof. Draw $BF \parallel CE$.

Then, § 290, $AE : BE = AC : FC$,

also, $\angle r = \angle s$, Why?

and $\angle t = \angle v$; Why?

but, data, $\angle r = \angle t$;

$\therefore$ $\angle s = \angle v$, Why?

and, § 118, $FC = BC$.

Substituting BC in the proportion for its equal FC,

$$AE : BE = AC : BC.$$

Therefore, etc. Q.E.D.

294. Sch. This proposition is not true, if the triangle is equilateral. Why?

Ex. **416.** The base of a triangle is 10 ft. and the other sides 8 ft. and 12 ft. Find the segments of the base made by the bisector of the vertical angle.

Ex. **417**. The sides AC and BC of the triangle ABC are 5 ft. and 8 ft. respectively. If a line drawn parallel to the base divides AC into segments of 2 ft. and 3 ft., what are the segments into which it divides BC?

295. If a straight line is divided at a point between its extremities, it is said to be *divided internally.* The line is equal to the sum of the internal segments.

If a straight line is produced and divided at a point on the part produced, it is said to be *divided externally.* The line is equal to the difference between the external segments.

Fig. 1.

In Fig. 1, AB is *divided internally* at C, and

$$AB = AC + CB.$$

Fig. 2.

In Fig. 2, AB is produced and *divided externally* at E, and

$$AB = BE - AE.$$

296. If a line is divided at a given point so that one segment is a mean proportional between the whole line and the other segment, it is said to be divided in *extreme and mean ratio.*

In Fig. 1, if $\qquad AB : AC = AC : CB,$

AB is divided *internally* in *extreme and mean ratio* at the point C.

In Fig. 2, if $\qquad AB : AE = AE : BE,$

AB is divided *externally* in *extreme and mean ratio* at the point E.

297. If a line is divided internally and externally into segments which have the same ratio, it is said to be divided *harmonically.*

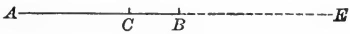


Fig. 3.

In Fig. 3, if $\qquad AC : CB = 6 : 3,$

and if $\qquad AE : BE = 6 : 3,$

$$AC : CB = AE : BE,$$

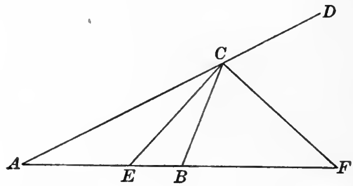
and AB is divided *harmonically* at the points C and E.

Ex. **418.** A line drawn parallel to the base of triangle ABC divides AC into segments of 3^{dm} and 8^{dm} respectively, and the segment of BC, corresponding to 3^{dm}, is 5^{dm}. What is the length of BC?

Proposition V

298. Draw a triangle whose sides are 6″, 5″, and 3″, or any other dimensions; bisect an interior and an exterior angle at one vertex and produce the bisectors to meet the opposite side and the opposite side produced respectively. How do the ratios of the internal and external segments of the opposite side compare?

Theorem. The bisectors of an interior and of an exterior angle at one vertex of a triangle divide the opposite side harmonically.



Data: Any triangle, as ABC, and the bisectors CE and CF of the interior and an exterior angle at C, respectively.

To prove $AE : EB = AF : BF$.

Proof. § 292, $AE : EB = AC : BC$,

§ 293, $AF : BF = AC : BC$;

hence, $AE : EB = AF : BF$. Why?

Therefore, etc. Q.E.D.

299. Polygons whose homologous angles are equal, and whose homologous sides are proportional, are called **Similar Polygons.**

In similar polygons, points, lines, and angles that are similarly situated are called *homologous* points, lines, and angles.

Equal polygons are similar, but similar polygons are not necessarily equal.

Thus, all equilateral triangles are similar, but not all equilateral triangles are equal.

Ex. **419**. The sides AB, BC, and AC of the triangle ABC are respectively 8 in., 5 in., and 10 in. The bisector of the exterior angle at C meets AB produced in E. What is the length of BE?

Ex. **420**. In triangle ABC, AC is 15^m and BC is 5^m. The bisector of the exterior angle at C meets AB produced in E. If AE is 21^m, what is the length of the side AB?

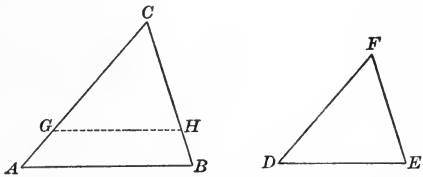
Ex. **421**. The bisectors of an interior and exterior angle at C of the triangle ABC meet the opposite side and the opposite side produced in E and F, respectively. If AB is 14 in. and EB is 4 in., what are the internal and external segments of AB?

Proposition VI

300. Draw a triangle; draw another whose angles are equal, each to each, to the angles of the first and one of whose sides is double, or any other number of times, the homologous side of the first. How do the ratios of any two pairs of homologous sides compare? What kind of triangles are they? Why?

Theorem. *Two triangles are similar, if the angles of one are equal to the angles of the other, each to each.*

Data: Any two triangles, as ABC and DEF, in which angle A = angle D, angle B = angle E, and angle C = angle F.



To prove $\triangle ABC$ and DEF similar.

Proof. In the greater triangle, ABC, measure CG equal to DF, CH equal to EF, and draw GH.

Then, § 100, $\triangle GHC = \triangle DEF$,

and $\angle CGH = \angle D = \angle A$; Why?

$\therefore$ $GH \parallel AB$. Why?

Hence, § 290, $AC : GC = BC : HC$,

or, substituting DF for its equal GC, and EF for its equal HC,

$$AC : DF = BC : EF.$$

In like manner, $AB : DE = BC : EF$.

Since, in the $\triangle ABC$ and DEF the homologous sides are proportional, and, from data, the homologous angles are equal,

§ 299, $\triangle ABC$ and DEF are similar.

Therefore, etc. Q.E.D.

301. Cor. I. *Two triangles are similar, if two angles of one are equal to two angles of the other, each to each.*

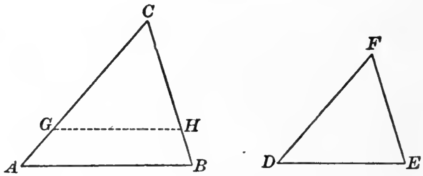
302. Cor. II. *Two right triangles are similar, if an acute angle of one is equal to an acute angle of the other.*

Ex. **422.** The sides of a triangle are 5, 7, and 9. If the side of a similar triangle homologous to 7 is 8, what are the other sides of the triangle?

Proposition VII

303. Draw any two triangles such that the sides of one are proportional to the sides of the other; for example, draw one triangle whose sides are 3″, 4″, and 5″, and another whose sides are 6″, 8″, and 10″. How do the homologous angles compare in size? What kind of triangles are they? Why?

Theorem. *Two triangles are similar, if the sides of one are proportional to the sides of the other, each to each.*



Data: Any two triangles, as ABC and DEF, such that

$$AC : DF = BC : EF$$
$$= AB : DE.$$

To prove $\triangle s$ ABC and DEF similar.

Proof. In the greater triangle, ABC, measure CG equal to DF, CH equal to EF, and draw GH.

Data, $AC : DF = BC : EF$;

∴ $AC : GC = BC : HC$.

Then, § 291, $GH \parallel AB$;

hence, $\angle A = \angle CGH$, $\angle B = \angle CHG$, Why?

and, § 301, $\triangle s$ ABC and GHC are similar;

∴ § 299, $AC : GC = AB : GH$;

that is, $AC : DF = AB : GH$.

But, data, $AC : DF = AB : DE$;

∴ $AB : GH = AB : DE$;

whence, § 272, $GH = DE$,

and, § 107, $\triangle GHC = \triangle DEF$.

But $\triangle s$ ABC and GHC are similar;

hence, $\triangle s$ ABC and DEF are similar. Q.E.D.

304. Sch. In § 299 the characteristics of similar polygons were defined as:

1. Their homologous angles are equal.
2. Their homologous sides are proportional.

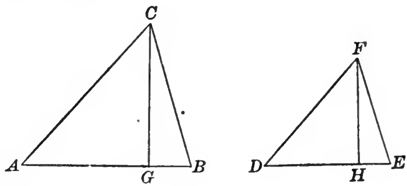
From § 300 and § 303 it is seen that in the case of *triangles,* either condition involves the other; that is, if the homologous angles of two triangles are equal, the homologous sides are proportional, and conversely; hence, triangles are similar, if their homologous angles are equal or if their homologous sides are proportional. In the case of polygons of *more than three sides* either condition may exist without involving the other.

Thus, a square and a rhombus may have their sides all equal and, consequently, proportional, but the angles of the square are right angles, and those of the rhombus are oblique; therefore, the figures are not similar. Also a square and a rectangle have their angles all equal, but their sides may not be proportional; consequently, the figures are not similar.

Proposition VIII

305. Draw two similar triangles whose sides are 3″, 4″, and 5″, and 6″, 8″, and 10″ respectively, or any two similar triangles; draw lines representing their altitudes. How does the ratio of their altitudes compare with the ratio of any two homologous sides?

Theorem. *The altitudes of similar triangles are to each other as any two homologous sides.*



Data: Any two similar triangles, as ABC and DEF, and their altitudes, as CG and FH, respectively.

To prove $CG : FH = AC : DF = BC : EF = AB : DE$.

Proof. Data, $\triangle ABC$ and DEF are similar;

$\therefore$ § 299, $\angle A = \angle D$,

§ 94, $\angle s\ AGC$ and DHF are rt. $\angle s$;

$\therefore$ § 302, rt. $\triangle s\ AGC$ and DHF are similar,

and, § 299, $CG : FH = AC : DF$.

In like manner it may be shown that

$$CG : FH = BC : EF.$$

But, § 299, $BC : EF = AB : DE$;

hence, $CG : FH = AC : DF = BC : EF = AB : DE$. Why?

Therefore, etc. Q.E.D.

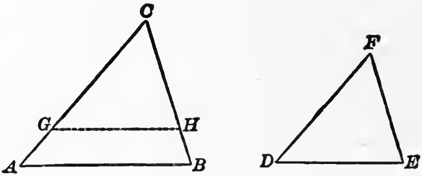
Proposition IX

306. Draw two triangles such that an angle of one is equal to an angle of the other, and the including sides in the first triangle are 3″ and 5″, and in the second 6″ and 10″. How do the homologous angles compare? How do the ratios of any two pairs of homologous sides compare? What name is given to triangles that have such relations to each other?

Theorem. *Two triangles are similar, if an angle of one is equal to an angle of the other, and the sides about these angles are in proportion.*

Data: Any two triangles, as *ABC* and *DEF*, in which angle C = angle F, and $AC : BC = DF : EF$.

To prove $\triangle$s *ABC* and *DEF* similar.



Proof. In the greater triangle, *ABC*, measure *CG* equal to *DF*, *CH* equal to *EF*, and draw *GH*.

Then, since, data, $\angle C = \angle F$,

§ 100, $\triangle GHC = \triangle DEF$.

Data, $AC : BC = DF : EF$;

$\therefore$ $AC : BC = GC : HC$. Why?

Hence, § 291, $GH \parallel AB$,

$\therefore$ $\angle A = \angle CGH$, and $\angle B = \angle CHG$; Why?

hence, § 301, $\triangle$s *ABC* and *GHC* are similar;

that is, $\triangle$s *ABC* and *DEF* are similar.

Therefore, etc. Q.E.D.

Ex. **423.** The sides of a triangle are 8^{dm}, 10^{dm}, and 12^{dm} in length respectively. If a line 9^{dm} long, parallel to the longest side, terminates in the other two, what are the segments into which it divides them?

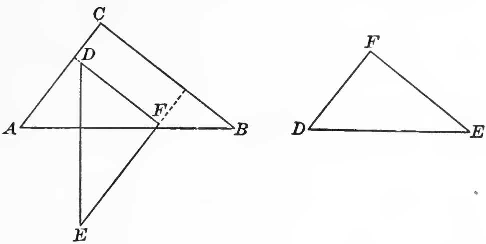
Ex. **424.** If the bisector of an interior angle of a triangle divides the side opposite the angle into two segments which are 6 ft. and 8 ft. respectively, and if the side of the triangle adjacent to the 8 ft. segment is 20 ft., what is the length of the other side of the triangle?

Proposition X

307. Draw two triangles whose sides are parallel, each to each, or perpendicular, each to each. What may be inferred regarding the relative size of the homologous angles? Then, what kind of triangles are they?

Theorem. *Two triangles are similar, if their sides are parallel, each to each, or are perpendicular, each to each.*

Data: Any two triangles, as ABC and DEF, in which AB, AC, and BC are parallel or perpendicular respectively to DE, DF, and EF.



To prove $\triangle s$ ABC and DEF similar.

Proof. By §§ 81, 83, angles which have their sides either parallel or perpendicular are either equal or supplementary.

1. Suppose that each of the angles of one triangle is supplementary to the corresponding angles of the other; that is, suppose $\angle A + \angle D = 2$ rt. $\angle s$; $\angle B + \angle E = 2$ rt. $\angle s$; $\angle C + \angle F = 2$ rt. $\angle s$.

Then, the sum of the interior angles of the two triangles is equal to 6 rt. $\angle s$, which is impossible.

2. Suppose that one angle of one triangle is equal to the corresponding angle of the other, and that the other two angles of the triangles are supplementary, each to each; that is, suppose $\angle A = \angle D$; $\angle B + \angle E = 2$ rt. $\angle s$; $\angle C + \angle F = 2$ rt. $\angle s$.

Then, the sum of the angles of the two triangles exceeds 4 rt. $\angle s$, which is impossible.

3. Suppose that two angles of one triangle are equal to the corresponding angles of the other, each to each; then the third angles must be equal;

that is, suppose $\angle A = \angle D$; $\angle B = \angle E$,

then, $\angle C = \angle F$;

that is, $\triangle s$ ABC and DEF are mutually equiangular.

Hence, § 300, $\triangle s$ ABC and DEF are similar.

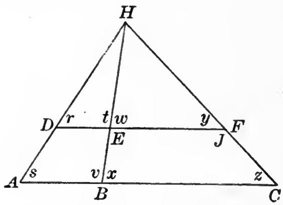
Therefore, etc. Q.E.D.

Proposition XI

308. 1. Draw three or more lines which meet in a point and two parallel lines cutting them. Discover whether any pairs of triangles thus formed are similar. Are the pairs of bases proportional?

2. If three non-parallel lines intersect two parallel lines, making the intercepted segments 4″ and 6″ on one side of the middle line and 8″ and 12″ on the other side, will the non-parallel lines meet in the same point if produced?

Theorem. *Lines which meet in a point intercept proportional segments upon two parallel lines;* ***conversely,*** *non-parallel lines which intercept proportional segments upon two parallel lines meet in a point.*



Data: Any lines, as AH, BH, and CH, which meet at a point, as H, and intercept the segments AB, BC, DE, and EF upon two parallel lines, AC and DF.

To prove $AB : DE = BC : EF$.

Proof. $\angle r = \angle s$, $\angle t = \angle v$, $\angle w = \angle x$, and $\angle y = \angle z$; Why?

∴ § 301, $\triangle$ ABH and DEH are similar,

and $\triangle$ BCH and EFH are similar.

Then, § 299, $AB : DE = BH : EH$,

and $BC : EF = BH : EH$;

∴ $AB : DE = BC : EF$. Q.E.D.

Conversely: **Data**: Non-parallel lines, as AD, BE, and CF, intersecting parallel lines AC and DF, so that $AB : DE = BC : EF$.

To prove that AD, BE, and CF, if produced, meet in a point.

Proof. Produce AD and BE to meet in H and draw CH.

Suppose that J is the point in which DF intersects CH.

Then, $AB : DE = BC : EJ$, Why?

but, since, data, $AB : DE = BC : EF$,

this is impossible, unless $EJ = EF$,

and J and F coincide.

Then, CF passes through H.

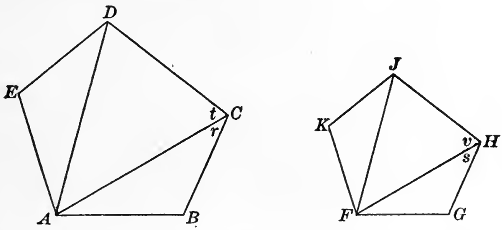
Consequently, AD, BE, and CF meet in a point.

Therefore, etc. Q.E.D.

Proposition XII

309. Draw two polygons such that they may be divided into the same number of triangles, similar, each to each, and similarly situated. How do the homologous angles of these polygons compare in size? How do the ratios of any two pairs of homologous sides compare? What kind of polygons are they? Why?

Theorem. *Two polygons are similar, if each is composed of the same number of triangles which are similar, each to each, and similarly placed.*



Data: Any two polygons, as $ABCDE$ and $FGHJK$, composed of triangles ABC, ACD, and ADE; and FGH, FHJ, and FJK, respectively, which are similar, each to each, and are similarly placed.

To prove $ABCDE$ and $FGHJK$ similar.

Proof. $\angle B = \angle G$. Why?

Also, $\angle r = \angle s$,

and $\angle t = \angle v$;

$\therefore$ $\angle BCD = \angle GHJ$. Why?

In like manner it may be shown that $\angle CDE = \angle HJK$, etc.

Hence, the homologous angles of the polygons are equal.

Again, § 299, $AB : FG = BC : GH = AC : FH = CD : HJ =$ etc.,

or $AB : FG = BC : GH = CD : HJ =$ etc.;

that is, the homologous sides of the polygons are proportional.

Hence, § 299, $ABCDE$ and $FGHJK$ are similar.

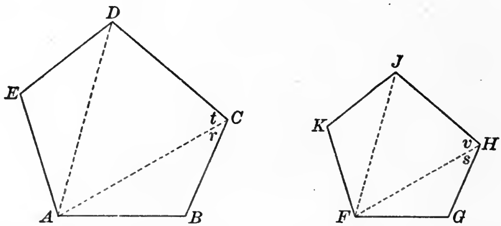
Therefore, etc. Q.E.D.

Ex. **425**. If a stick 3 ft. long, in a vertical position, casts a shadow 1 ft. $7\frac{1}{2}$ in. long, how high is a church steeple which at the same time casts a shadow 78 ft. in length?

Proposition XIII

310. Draw two similar polygons and from two homologous vertices draw diagonals dividing the polygons into triangles. How many triangles are there in each polygon? How do the homologous angles of the corresponding triangles compare in size? How do the ratios of their sides compare? Then, what kind of triangles are they?

Theorem. *If two polygons are similar, they may be divided by diagonals into the same number of triangles which are similar, each to each, and similarly placed.* (Converse of Prop. XII.)



Data: Any two similar polygons, as $ABCDE$ and $FGHJK$.

To prove that the polygons $ABCDE$ and $FGHJK$ may be divided by diagonals into the same number of triangles which are similar, each to each, and are similarly placed.

Proof. From any two homologous vertices, as A and F, draw the diagonals AC, AD, FH, and FJ.

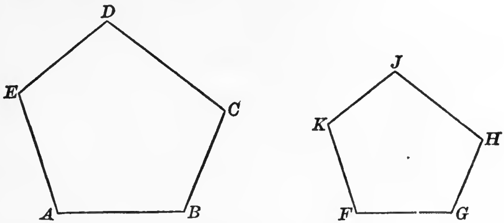
In $\triangle$ ABC and FGH,	$\angle B = \angle G$,	Why?
and	$AB : FG = BC : GH$;	
$\therefore$ § 306,	$\triangle$ ABC and FGH are similar,	
and	$\angle r = \angle s$;	Why?
but	$\angle BCD = \angle GHJ$;	Why?
$\therefore$	$\angle t = \angle v$;	Why?
	$BC : GH = AC : FH$,	Why?
and	$BC : GH = CD : HJ$;	Why?
$\therefore$	$AC : FH = CD : HJ$,	Why?
and, § 306,	$\triangle$ ACD and FHJ are similar.	
In like manner,	$\triangle$ ADE and FJK are similar.	

Therefore, etc. **Q.E.D.**

Proposition XIV

311. Draw two similar polygons; measure the sides of each. How does the ratio of the perimeters, or the sums of the homologous sides, compare with the ratio of any two homologous sides?

Theorem. *The perimeters of similar polygons are to each other as any two homologous sides.*



Data: Any two similar polygons, as $ABCDE$ and $FGHJK$. Denote their perimeters by P and Q respectively.

To prove $P : Q = AB : FG =$ etc.

Proof. $AB : FG = BC : GH = CD : HJ =$ etc.; Why?

$\therefore$ § 279, $AB + BC +$ etc. $: FG + GH +$ etc. $= AB : FG =$ etc.;

that is, $P : Q = AB : FG =$ etc.

Therefore, etc. Q.E.D.

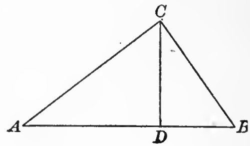
Proposition XV

312. 1. Draw a right triangle whose sides are 3″, 4″, and 5″, or any other right triangle; from the vertex of the right angle draw a perpendicular to the hypotenuse. How do the angles of each of the triangles thus formed compare in size with the homologous angles of the original triangle? How does the ratio of the longer segment of the hypotenuse to the perpendicular compare with the ratio of the perpendicular to the shorter segment?

2. How does the ratio of the hypotenuse to either side about the right angle compare with the ratio of the same side to the segment of the hypotenuse adjacent to it?

3. Draw a circle and its diameter; from any point in the circumference draw a perpendicular to the diameter. How does the ratio of the longer segment of the diameter to the perpendicular compare with the ratio of the perpendicular to the shorter segment?

Theorem. *If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the perpendicular is a mean proportional between the segments of the hypotenuse.*



Data: Any right triangle, as ABC, and the perpendicular CD, from the vertex of the right angle C, upon AB.

To prove $AD : CD = CD : BD$.

Proof. In the rt. $\triangle$ ABC and ACD,

$\angle A$ is common;

$\therefore$ § 302, $\triangle$ ABC and ACD are similar.

In like manner it may be shown that $\triangle$ ABC and CBD are similar;

hence, $\triangle$ ACD and CBD are similar. Why?

Now, AC, AD, and CD are the sides of $\triangle ACD$ homologous respectively to BC, CD, and BD of $\triangle CBD$;

hence, § 299, $AD : CD = CD : BD$.

Therefore, etc. Q.E.D.

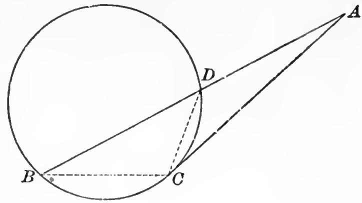
313. Cor. I. *Each side about the right angle is a mean proportional between the hypotenuse and the adjacent segment.*

314. Cor. II. *The perpendicular to a diameter from any point in the circumference of a circle is a mean proportional between the segments of the diameter.*

Proposition XVI

315. From a point without a circumference draw a tangent and a secant; from the point of tangency draw chords to the points at which the secant intersects the circumference. What angles of the figure are equal? What two triangles are similar? Then, how does the ratio of the secant to the tangent compare with the ratio of the tangent to the external segment of the secant?

Theorem. *If from a point without a circle a secant and a tangent are drawn, the tangent is a mean proportional between the whole secant and the external segment.*



Data: Any circle, as *BCD*; any point without, as *A*; any secant from *A*, as *ADB*; and the tangent from *A*, as *AC*.

To prove

$AB : AC = AC : AD.$

Proof. Draw *BC* and *DC*.

In $\triangle$ *ABC* and *ADC*, $\angle A$ is common,

§ 225, $\angle B$ is measured by $\frac{1}{2}$ arc *DC*,

and, § 231, $\angle ACD$ is measured by $\frac{1}{2}$ arc *DC*;

$\therefore$ $\angle B = \angle ACD.$ Why?

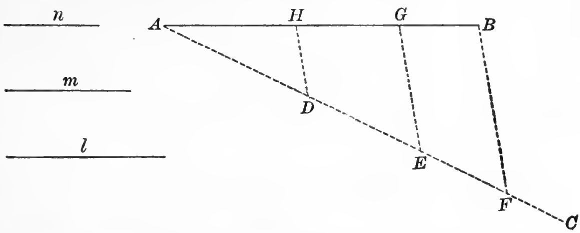
Hence, § 301, $\triangle$ *ABC* and *ADC* are similar,

and, § 299, $AB : AC = AC : AD.$

Therefore, etc. Q.E.D.

Proposition XVII

316. *Problem.* *To divide a straight line into parts proportional to any number of given lines.*



Data: Any straight line, as *AB*; also the lines *l*, *m*, and *n*.

Required to divide *AB* into parts proportional to *l*, *m*, and *n*.

Solution. From one extremity of *AB*, as *A*, draw a line, as *AC*, making with *AB* any convenient angle.

On *AC* measure *AD*, *DE*, and *EF* equal to *l*, *m*, and *n* respectively. Draw *FB*.

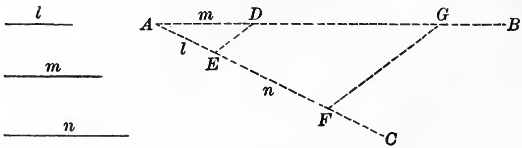
Through *D* and *E* draw lines parallel to *FB*, meeting *AB* in *H* and *G* respectively.

Then, *AH*, *HG*, and *GB* are the parts required. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 289.

Proposition XVIII

317. Problem. *To find a fourth* proportional to three given lines.*



Data: Any three lines, as l, m, and n.

Required a fourth proportional to l, m, and n.

Solution. Draw any two lines, as AB and AC, forming any convenient angle at A.

On AB take $AD = m$.

On AC take $AE = l$, and $EF = n$.

Draw ED.

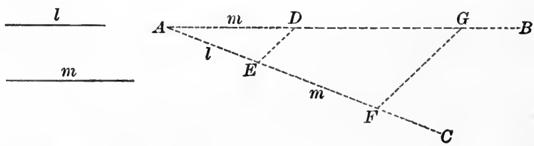
From F draw a line parallel to ED meeting AB in G.

Then, DG is the proportional required. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 289.

Proposition XIX

318. Problem. *To find a third† proportional to two given lines.*



Data: Any two lines, as l and m.

Required a third proportional to l and m.

Solution. Draw any two lines, as AB and AC, forming any convenient angle at A.

On AB take $AD = m$.

On AC take $AE = l$, and $EF = m$.

* When $a : b = c : d$, d is termed the *fourth proportional* to a, b, and c.

† When $a : b = b : c$, c is termed the *third proportional* to a and b.

Draw ED.

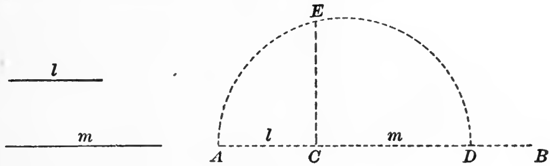
From F draw a line parallel to ED meeting AB in G.

Then, DG is the proportional required. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 289.

Proposition XX

319. ***Problem.*** *To find a mean proportional between two given lines.*



Data: Any two lines, as l and m.

Required a mean proportional between l and m.

Solution. Draw any line, as AB.

On AB take $AC = l$, and $CD = m$.

On AD as a diameter describe a semicircumference.

At C erect a perpendicular to AD meeting the semicircumference in E.

Then, CE is the required proportional. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 314.

Ex. **426**. Three lines are 10^{cm}, 12^{cm}, and 16^{cm}. Construct their fourth proportional.

Ex. **427**. Two lines are 11^{cm} and 9^{cm}. Construct their third proportional.

Ex. **428**. Two lines are 6^{cm} and 2^{cm}. Construct their mean proportional.

Ex. **429**. Tangents are drawn through a point 6^{m} from the circumference of a circle whose radius is 9^{m}. Find the length of the tangents.

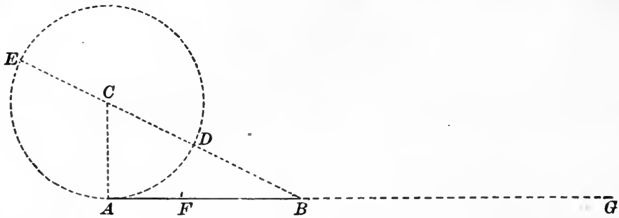
Ex. **430**. If one side of a polygon is 2 ft. 6 in. long, what is the length of the corresponding side of a similar polygon, if their perimeters are respectively 15 ft. and 25 ft.?

Ex. **431**. The shortest distance from a given point to the circumference of a given circle is 2 ft. The length of a tangent from the same point to the circumference is 3 ft. Find the diameter of the circle.

Ex. **432**. Five straight lines passing through the same point intercept segments on one of two parallel lines, of 12^{dm}, 20^{dm}, 28^{dm}, and 36^{dm}. The segment of the other parallel corresponding to the 20^{dm} segment is 15^{dm}. Find the other segments.

Proposition XXI

320. *Problem.* *To divide a line in extreme and mean ratio.*



Datum: Any line, as AB.

Required to divide AB in extreme and mean ratio.

Solution. At one extremity of AB, as A, draw AC perpendicular to AB and equal to $\frac{1}{2}AB$.

With C as a center and AC as a radius describe a circumference.

Through C draw BE cutting the circumference in D and meeting it in E. On AB take $BF = BD$ and on AB produced take $BG = BE$.

Then, $\quad AB : BF = BF : AF,$

and $\quad AG : BG = BG : AB;$

that is, § 296, AB is divided at F internally, and at G externally, in extreme and mean ratio. Q.E.F.

Proof. § 315, $\quad BE : AB = AB : BD;$

∴ § 277, $\quad BE - AB : AB = AB - BD : BD, \quad (1)$

and, § 276, $\quad AB + BE : BE = BD + AB : AB. \quad (2)$

Const., $\quad DE = 2\,AC = AB;$

hence, $\quad BE - AB = BE - DE = BD = BF.$

Substituting in (1) for $BE - AB$ its equal BF, for $AB - BD$ its equal AF, and for BD its equal BF,

$$BF : AB = AF : BF,$$

or, § 275, $\quad AB : BF = BF : AF.$

Const., $\quad AB + BE = AG$, and $BD + AB = BE.$

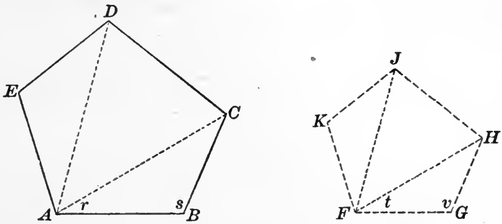
Substituting in (2) for $AB + BE$ its equal AG, and for $BD + AB$ its equal BE, $\quad AG : BE = BE : AB.$

Since, const., $\quad BE = BG,$

$$AG : BG = BG : AB.$$

Proposition XXII

321. ***Problem.*** *Upon a given line to construct a polygon similar to a given polygon.*



Data: Any polygon, as $ABCDE$, and any line, as FG.

Required to construct on FG a polygon similar to $ABCDE$.

Solution. Draw AC and AD.

At F and G construct $\angle t$ and $\angle v$ equal respectively to $\angle r$ and $\angle s$.

Produce the sides from F and G until they meet at H.

In like manner on FH construct $\triangle FHJ$, and on FJ, $\triangle FJK$, similar respectively to ⚠ ACD and ADE.

Then, $FGHJK$ is the required polygon. Q.E.F.

Proof. By the student. SUGGESTION. Refer to §§ 301, 309.

SUMMARY

322. Truths established in Book IV.

1. Two lines are parallel,

a. If one divides two sides of a triangle proportionally and the other is the third side. § 291

2. Lines are in proportion,

a. If they are segments of two sides of a triangle made by a line parallel to the third side. § 289

b. If they are two sides of a triangle and their corresponding segments made by a line parallel to the third side. § 290

c. If they are two sides of a triangle and the segments of the third side made by the bisector of the angle opposite that side. § 292

d. If they are two sides of a triangle and the external segments of the third side made by the bisector of the exterior angle at the vertex opposite that side. § 293 *e.* If they are the internal and external segments of a side of a triangle made by the bisectors of an interior and exterior angle at the vertex opposite that side. § 298

f. If they are the altitudes and homologous sides of similar triangles. § 305

g. If they are segments of parallel lines made by lines which meet in a point. § 308

h. If they are homologous sides of similar polygons. § 299

i. If they are perimeters of similar polygons and any two homologous sides. § 311

3. A line is a mean proportional between two other lines,

a. If it is the perpendicular to the hypotenuse of a right triangle from the vertex of the right angle, and the other lines are the segments of the hypotenuse. § 312

b. If it is either side about the right angle of a right triangle, and the other lines are the hypotenuse and the segment of it adjacent to that side made by the perpendicular from the vertex of the right angle. § 313

c. If it is the perpendicular to the diameter of a circle from any point in the circumference, and the other lines are the segments of the diameter. § 314

d. If it is a tangent to a circle from any point without, and the other lines are a secant from the same point and its external segment. § 315

4. Lines pass through the same point,

a. If they are non-parallel lines that intercept proportional segments upon two parallel lines. § 308

5. Two angles are equal,

a. If they are homologous angles of similar polygons. § 299

6. Two triangles are similar,

a. If the angles of one are respectively equal to the angles of the other. § 300

b. If two angles of the one are respectively equal to two angles of the other. § 301

c. If they are right triangles and an acute angle of one is equal to an acute angle of the other. § 302

d. If the sides of one are proportional respectively to the sides of the other. § 303

e. If an angle of one is equal to an angle of the other and the including sides are in proportion. § 306

f. If their sides are parallel, each to each. § 307 *g.* If their sides are perpendicular, each to each. § 307

h. If they are the corresponding triangles of similar polygons divided by homologous diagonals. § 310

7. Two polygons are similar,

a. If they have their homologous angles equal and their homologous sides proportional. § 299

b. If each is composed of the same number of triangles similar each to each and similarly placed. § 309

SUPPLEMENTARY EXERCISES

Ex. **433.** Construct a triangle whose sides are 6, 8, and 10; then construct a similar triangle whose side homologous to 8 is 5.

Ex. **434.** Divide a line 10^{cm} long internally in extreme and mean ratio.

Ex. **435.** The median from the vertex of a triangle bisects every line drawn parallel to the base and terminated by the sides, or the sides produced.

Ex. **436.** Two circles intersect at A and B, and at A tangents are drawn, one to each circle, to meet the circumference of the other in C and D respectively; BC, BD, and AB are drawn. Prove that BD is a third proportional to BC and AB.

Ex. **437.** The diameter AB of a circle whose center is O is divided at any point C, and CD is drawn perpendicular to AB, meeting the circumference in D; OD is drawn, and CE perpendicular to OD. Prove that DE is a third proportional to AO and DC.

Ex. **438.** In the triangle ABC, AD is the median to BC; the angles ADC and ADB are bisected by DE and DF, meeting AC and AB in E and F respectively. Then, FE is parallel to BC.

Ex. **439.** A secant from a given point without a circle is 1 ft. 6 in. long, and its external segment is 8 in. long. Find the length of a tangent to the circle from the same point.

Ex. **440.** The radius of a circle is 6 in. What is the length of the tangents drawn from a point 12 in. from the center?

Ex. **441.** If the tangent to a circle from a given point is 2^{m} and the radius of the circle is 15^{dm}, find the distance from the point to the circumference.

Ex. **442.** If from the vertex D of the parallelogram $ABCD$ a straight line is drawn cutting AB at E and CB produced at F, prove that CF is a fourth proportional to AE, AD, and AB.

Ex. **443.** If the segments of the hypotenuse of a right triangle made by the perpendicular from the vertex of the right angle are 6 in. and 4 ft., find the length of the perpendicular and the length of each of the sides about the right angle.

Ex. **444.** Find the length of the longest and of the shortest chord that can be drawn through a point $7\frac{1}{2}$ in. from the center of a circle whose radius is $19\frac{1}{2}$ in.

Ex. **445.** If the greater segment of a line divided internally in extreme and mean ratio is 36 in., what is the length of the line?

Ex. **446.** The shorter segment of a line divided externally in extreme and mean ratio is 240^{dm}. Find the length of the greater segment in meters.

Ex. **447.** Find the shorter segment of a line 12^{dm} long when it is divided internally in extreme and mean ratio. When it is divided externally in extreme and mean ratio.

Ex. **448.** The tangents to two intersecting circles drawn from any point in their common chord produced are equal.

Ex. **449.** If the common chord of two intersecting circles is produced, it will bisect their common tangents.

Ex. **450.** ABC is a straight line, ABD and BCE are triangles on the same side of it, having angle ABD equal to angle CBE and $AB : BC = BE : BD$. If AE and CD intersect in F, triangle AFC is isosceles.

Ex. **451.** If in the triangle ABC, CE and BD are drawn perpendicular to the sides AB and AC respectively, these sides are reciprocally proportional to the perpendiculars upon them; that is, $AB : AC = BD : CE$.

Ex. **452.** $ABCD$ is a parallelogram. If through O, any point in the diagonal AC, EF and GH are drawn, terminating in AB and DC, and in AD and BC respectively, EH is parallel to GF.

Ex. **453.** Lines are drawn from a point P to the vertices of the triangle ABC; through D, any point in PA, a line is drawn parallel to AB, meeting PB at E, and through E a line parallel to BC, meeting PC at F. If FD is drawn, triangle DEF is similar to triangle ABC.

Ex. **454.** If two lines are tangent to a circle at the extremities of a diameter, and from the points of contact secants are drawn terminated respectively by the opposite tangent and intersecting the circumference at the same point, the diameter is a mean proportional between the tangents.

Ex. **455.** AB and AC are secants of a circle from the common point A, cutting the circumference in D and E respectively. Then, the secants are reciprocally proportional to their external segments; that is, $AB : AC = AE : AD$.

SUGGESTION. Draw CD and BE, and refer to § 322, 6, *b*.

Ex. **456**. AB and CD are two chords of a circle intersecting at E. Prove that $AE : DE = CE : BE$.

Ex. **457**. Two secants intersect without a circle. The segments of one are 4 ft. and 20 ft., and the external segment of the second is 16 ft. Find the length of the second secant.

Ex. **458**. From a point without a circle two secants are drawn, whose external segments are respectively 7^{dm} and 9^{dm}, the internal segment of the latter being 13^{dm}. What is the length of the first secant?

Ex. **459**. The segments of a chord intersected by another chord are 7 in. and 9 in., and one segment of the latter is 3 in. What is the other segment?

Ex. **460**. Two secants from the same point without a circle are 24^{dm} and 32^{dm} long. If the external segment of the less is 5^{dm}, what is the external segment of the greater?

Ex. **461**. Through a point 7^{m} from the circumference of a circle a secant 28^{m} long is drawn. If the internal segment of this secant is 17^{m}, what is the radius of the circle?

Ex. **462**. If from any point in the diameter of a circle produced a tangent is drawn and a perpendicular from the point of contact is let fall on the diameter, the distances from the point without the circle to the foot of the perpendicular, the center of the circle, and the extremities of the diameter are in proportion.

SUGGESTION. Draw the radius to the point of contact.

Ex. **463**. If the sides of a triangle are respectively 1.5^{Dm}, $.12^{Hm}$, and 10^{m} long, what are the segments into which each side is divided by the bisector of the opposite angle?

Ex. **464**. If an angle of one triangle is equal to an angle of another, and the perpendiculars from the vertices of the remaining angles to the sides opposite are proportional, the triangles are similar.

SUGGESTION. Refer to § 322, 6, *c* and *e*.

Ex. **465**. If two circles are respectively 6 in. and 3 in. in diameter and their centers are 10 in. apart, find the distance from the center of the smaller one to the point of intersection of their common exterior tangent with their line of centers produced.

Ex. **466**. Two intersecting chords of a circle are 38 ft. and 34 ft. respectively; the segments of the first are 8 ft. and 30 ft. Find the segments of the second.

Ex. **467**. What is the length of a chord joining the points of contact of the tangents drawn from a point 13 in. from the center of a circle whose radius is 5 in.?

Ex. **468**. Chords AB and CD of a circle are produced in the direction of B and D respectively to meet in the point E, and through E the line EF is drawn parallel to AD to meet CB produced in F. Prove that EF is a mean proportional between FB and FC.

Ex. **469.** AB is a diameter of a circle, and through A any straight line is drawn to cut the circumference in C and the tangent at B in D. Prove that AC is a third proportional to AD and AB.

Ex. **470.** From any point in the base of a triangle straight lines are drawn parallel to the sides. Prove that the intersection of the diagonals of every parallelogram so formed lies in a line parallel to the base of the triangle.

Ex. **471.** If E is the middle point of one of the parallel sides DC of the trapezoid $ABCD$, and AE and BE produced meet BC and AD produced in F and G respectively, prove that GF is parallel to AB.

Ex. **472.** If a line tangent to two circles cuts their line of centers, the segments of the latter are to each other as the diameters of the circles.

Ex. **473.** The bisector of the vertical angle C of the inscribed triangle ABC cuts the base at D and meets the circumference in E. Prove that $AC : CD = CE : BC$.

Ex. **474.** Through any point A of the circumference of a circle a tangent is drawn, and from A two chords, AB and AC; the chord FG parallel to the tangent cuts AB and AC in D and E respectively. Prove $AB : AE = AC : AD$.

Ex. **475.** The greatest distance of a chord 8 ft. in length from its arc is 4 in. Find the diameter of the circle.

Ex. **476.** If two circles are tangent externally, their common exterior tangent is a mean proportional between the diameters of the circles.

Suggestion. Draw radii to the points of contact, draw the common interior tangent to intersect the common exterior tangent, and connect the point of intersection with the centers.

Ex. **477.** The perpendicular from any point of a circumference upon a chord is a mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.

Suggestion. Draw lines from the given point to the extremities of the chord, and refer to § 322, 6, *c*.

Ex. **478.** From a point A tangents AB and AC are drawn to a circle whose center is O, and BD is drawn perpendicular to CO produced. Prove that BD is a fourth proportional to AC, CD, and CO.

Suggestion. Draw AO and BC.

Ex. **479.** From a point E in the common base of two triangles ABC and ABD, straight lines are drawn parallel to AC and AD, meeting BC and BD at F and G respectively. Prove that FG is parallel to CD.

Ex. **480.** If tangents to a circle are drawn at the extremities of a diameter, the radius is a mean proportional between the segments of any third tangent intercepted between them and divided at its point of tangency.

Suggestion. Draw lines to form a right triangle, having the third tangent for its hypotenuse and a vertex at the center.

BOOK V

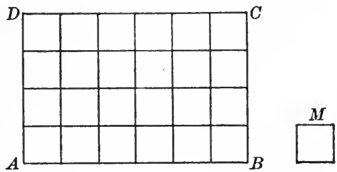
AREA AND EQUIVALENCE

323. The amount of surface in a plane figure is called its **Area.**

A surface is *measured* by finding how many times it contains some given square which is taken as a unit of measure.

The ordinary units of measure for surfaces are the *square inch,* the *square foot,* the *square centimeter,* the *square decimeter,* etc.

Suppose that the square M is the unit of measure, and that $ABCD$ is the rectangle to be measured.



By applying M to $ABCD$ it is evident that the rectangle may be divided into as many rows of squares, each equal to M, as the side of M is contained times in the altitude of $ABCD$; that in each row there are as many squares as the side of M is contained times in the base of $ABCD$; and therefore, that the product of the numerical measures of the base and altitude of $ABCD$ is equal to the number of times that M is contained in $ABCD$.

In this case the side of M is contained 4 times in AD and 6 times in AB; consequently, M is contained 24 times in $ABCD$; that is, the rectangle contains 24 square units.

Therefore, if the side of a square is a common measure of the base and altitude of a rectangle, the product of the numerical measures of the base and altitude expresses the number of times that the rectangle contains the square, and is the numerical measure of the surface, or the area of the rectangle.

324. For the sake of brevity, the *product of the base and altitude* is used instead of the *product of the numerical measures of the base and altitude.*

The product of two lines is, strictly speaking, an absurdity, but since the expression is used to denote the area of a rectangle it follows, that the *geometrical concept* of the *product* of two lines is the *rectangle* formed by them.

Thus, $AB \times CD$ implies a product, which is a *numerical* result, but it must be interpreted *geometrically* to mean rect. $AB \cdot CD$.

For similar reasons, if AB represents a line, $\overline{AB}^2$ must be interpreted to mean *geometrically* the square described upon the line AB, and *conversely*, the square described upon a line may be indicated by the square of the line.

325. It has been stated in § 36 that equal figures may be made to coincide, consequently such figures have equal areas.

Figures, however, which cannot be made to coincide may have equal areas, and they are called *equivalent figures*.

All equal figures are equivalent, but not all equivalent figures are equal.

If a square and a triangle each contains one square foot of surface, they are *equivalent;* but since they cannot be made to coincide, they are not equal.

The symbol of equivalence is $\backsimeq$.

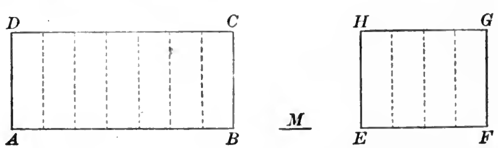
326. Since *equivalent* means *equal* in *area*, or in *volume* as will be shown, then, § 222 may be extended to apply to *equivalent* magnitudes; consequently, *if, while approaching their respective limits, two variables are always equivalent, their limits are equivalent.*

Proposition I

327. 1. If a rectangle is 3″ long and another of the same altitude is 6″ long, how do they compare in area? How, then, do rectangles having equal altitudes compare in area?

2. How do rectangles that have equal bases, but different altitudes, compare in area?

Theorem. *Rectangles which have equal altitudes are to each other as their bases.*



Data: Any two rectangles, as $ABCD$ and $EFGH$, whose altitudes, AD and EH, are equal.

To prove $\quad ABCD : EFGH = AB : EF.$

Proof. Case I. *When AB and EF are commensurable.*

Suppose that M is a common unit of measure for AB and EF.

Apply M to each base, and suppose that it is contained in AB 7 times and in EF 4 times.

Then, $\quad AB : EF = 7 : 4.$

Divide AB into 7 equal parts and EF into 4 equal parts, and at each point of division erect a perpendicular.

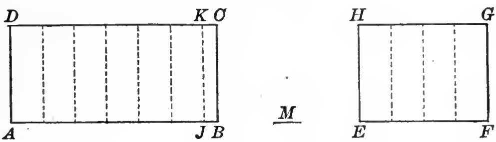
$ABCD$ is thus divided into 7 rectangles, and $EFGH$ into 4 rectangles.

Since, § 156, these rectangles are all equal,

$$ABCD : EFGH = 7 : 4.$$

$\therefore \quad ABCD : EFGH = AB : EF.$ Why?

Case II. *When AB and EF are incommensurable.*



Since AB and EF are incommensurable, suppose that M is a common unit of measure for AJ and EF, and that JB is less than M. Draw $JK \parallel AD$.

Then, Case I, $\quad AJKD : EFGH = AJ : EF;$

and $JBCK$ is less than any one of the rectangles whose base is equal to M.

Now, if M is indefinitely diminished, the ratios $AJKD : EFGH$ and $AJ : EF$ remain equal and indefinitely approach the limiting ratios $ABCD : EFGH$ and $AB : EF$ respectively.

Hence, § 222, $\quad ABCD : EFGH = AB : EF.$

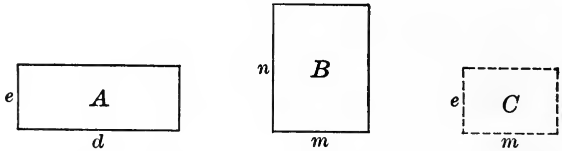
Therefore, etc. Q.E.D.

328. Cor. *Rectangles which have equal bases are to each other as their altitudes.*

Proposition II

329. Draw two rectangles whose bases are respectively 5″ and 3″ and altitudes 2″ and 4″, or any other dimensions; divide them into squares having a side of 1″. How many square inches are there in the first rectangle? In the second? How does the ratio of the areas of the two rectangles compare with the ratio of the products of their bases by their altitudes?

Theorem. *Rectangles are to each other as the products of their bases by their altitudes.*



Data: Any two rectangles, as A and B, of which d and m are the bases, and e and n the altitudes, respectively.

To prove $\quad A : B = d \times e : m \times n.$

Proof. Construct a rectangle C, having the base m and the altitude e.

Then, § 327, $\quad A : C = d : m,$

and, § 328, $\quad C : B = e : n;$

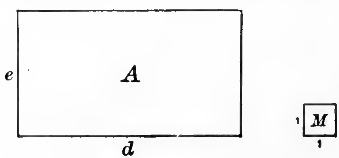
hence, § 287, $\quad A : B = d \times e : m \times n.$

Therefore, etc. Q.E.D.

Proposition III

330. How many square inches of surface are there in a rectangle that is 5″ long and 1″ wide? 5″ long and 2″ wide? 5″ long and 6″ wide? 8″ long and 7″ wide? How may the amount of surface, or the area of any rectangle, be found?

Theorem. *The area of a rectangle is equal to the product of its base by its altitude.*



Data: Any rectangle, as A, whose base is d and altitude e.

To prove area of $A = d \times e$.

Proof. Assume that the unit of measure is a square M, whose side is the linear unit.

§ 329, $A : M = d \times e : 1 \times 1$, or $\frac{A}{M} = \frac{d \times e}{1 \times 1} = d \times e$.

But, § 323, the surface of A is measured by the number of times it contains the unit of measure M;

$\therefore$ $\frac{A}{M} =$ area of A.

But $\frac{A}{M} = d \times e$.

Hence, area of $A = d \times e$.

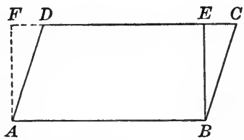
Therefore, etc. Q.E.D.

Proposition IV

331. 1. Draw an oblique parallelogram, and on the same base a rectangle having an equal altitude. How do the triangles thus formed at the ends of this figure compare? How does the area of the parallelogram compare with the area of the rectangle?

2. What ratio do two rectangles have to each other? (§ 329) What, then, is the ratio of two parallelograms to each other?

Theorem. *A parallelogram is equivalent to the rectangle which has the same base and altitude.*



Data: Any parallelogram, as $ABCD$, whose base is AB and altitude BE.

To prove $ABCD$ equivalent to the rectangle whose base is AB and altitude BE.

Proof. Draw $AF \parallel BE$ and meeting CD produced in F.

Const., $ABEF$ is a rectangle which has the same base and altitude as $ABCD$.

In rt. $\triangle$s BCE and ADF, $BE = AF$, Why?

and $BC = AD$; Why?

$\therefore$ $\triangle BCE = \triangle ADF$. Why?

Hence, $\triangle BCE + ABED \approx \triangle ADF + ABED$;

that is, $ABCD \approx ABEF$.

Therefore, etc. Q.E.D.

332. Cor. I. *The area of a parallelogram is equal to the product of its base by its altitude.*

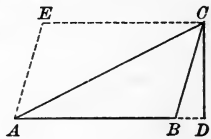
333. Cor. II. *Parallelograms are to each other as the products of their bases by their altitudes;* consequently, *parallelograms which have equal altitudes are to each other as their bases, parallelograms which have equal bases are to each other as their altitudes, and parallelograms which have equal bases and equal altitudes are equivalent.*

Proposition V

334. 1. Draw any triangle, and through two of its vertices draw lines parallel to the opposite sides, producing them until they meet. What part of the parallelogram thus formed is the original triangle? How does this parallelogram compare with a rectangle having the same base and altitude? What part of such a rectangle is the triangle?

2. What ratio do two rectangles have to each other? (§ 329) What, then, is the ratio of two triangles to each other?

Theorem. ***A triangle is equivalent to one half the rectangle which has the same base and altitude.***



Data: Any triangle, as ABC, whose base is AB and altitude CD.

To prove $\triangle ABC \approx \frac{1}{2}$ rect. $AB \cdot CD$.

Proof. Draw $AE \parallel BC$ and $CE \parallel BA$.

Then, $ABCE$ is a parallelogram, AC is its diagonal,

and, § 152, $\triangle ABC = \triangle AEC$,

or $\triangle ABC \approx \frac{1}{2} ABCE$.

But, § 331, $ABCE \approx$ rect. $AB \cdot CD$.

Hence, $\triangle ABC \approx \frac{1}{2}$ rect. $AB \cdot CD$. Q.E.D.

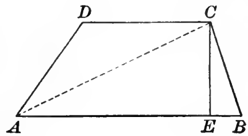
335. Cor. I. *The area of a triangle is equal to one half the product of its base by its altitude.*

336. Cor. II. *Triangles are to each other as the products of their bases by their altitudes;* consequently, *triangles which have equal altitudes are to each other as their bases, triangles which have equal bases are to each other as their altitudes, and triangles which have equal bases and equal altitudes are equivalent.*

Proposition VI

337. Draw a trapezoid and one of its diagonals. How does the area of the trapezoid compare with the combined areas of the triangles thus formed? Since both triangles have the same altitude, how does the area of the trapezoid compare with the area of the rectangle which has the same altitude and a base equal to the sum of the parallel sides of the trapezoid?

Theorem. *A trapezoid is equivalent to one half the rectangle which has the same altitude and a base equal to the sum of its parallel sides.*



Data: Any trapezoid, as $ABCD$, whose altitude is CE and whose parallel sides are AB and CD.

To prove $ABCD \backsimeq \frac{1}{2}$ rect. $CE \cdot (AB + CD)$.

Proof. Draw the diagonal AC.

Then, $ABCD \backsimeq \triangle ABC + \triangle ADC$.

§ 334, $\triangle ABC \backsimeq \frac{1}{2}$ rect. $CE \cdot AB$,

and $\triangle ADC \backsimeq \frac{1}{2}$ rect. $CE \cdot CD$;

hence, $\triangle ABC + \triangle ADC \backsimeq \frac{1}{2}$ rect. $CE \cdot AB + \frac{1}{2}$ rect. $CE \cdot CD$;

that is, $ABCD \backsimeq \frac{1}{2}$ rect. $CE \cdot (AB + CD)$.

Therefore, etc. **Q.E.D.**

338. Cor. *The area of a trapezoid is equal to one half the product of its altitude by the sum of its parallel sides.*

339. Sch. It will be observed that the corollaries § 332, § 335, and § 338 are *arithmetical rules* for computing areas.

Such rules are readily formed from the theorems to which they are attached by employing the terms *product* and *equal* instead of *rectangle* and *equivalent*.

Ex. **481.** Triangles on the same base and having their vertices in the same line which is parallel to the base are equivalent.

Ex. **482.** The parallel sides of a trapezoid are 12^{dm} and 8^{dm}, and their distance apart is 5^{dm}. What is the area of the trapezoid?

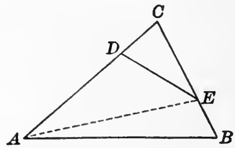
Ex. **483.** The area of a trapezoid is 52 sq. in., and the sum of the two parallel sides is 13 in. What is the distance between the parallel sides?

Ex. **484.** The area of a triangle is 36 sq. ft. If its base is 9 ft., what is its altitude?

Proposition VII

340. Draw a triangle one of whose sides is 5″, base 6″, and altitude 3″, or other dimensions; draw another triangle having an equal side and altitude but any base whatever, as 10″, and the angle between the base and given side equal to the corresponding angle of the first triangle. How does the ratio of the areas of these triangles compare with the ratio of the products of the sides that include their equal angles?

Theorem. *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Data: Any two triangles, as ABC and DEC, having the common angle C.

To prove

$\triangle ABC : \triangle DEC = AC \times BC : DC \times EC.$

Proof. Draw AE.

Since BC and EC may be regarded as the bases of the $\triangle$s ABC and AEC, their bases are in the same straight line; and since they have their common vertex at A, they have the same altitude.

$\therefore$ § 336, $\triangle ABC : \triangle AEC = BC : EC.$

In like manner, $\triangle AEC : \triangle DEC = AC : DC.$

Hence, § 287, $\triangle ABC : \triangle DEC = AC \times BC : DC \times EC.$

Therefore, etc. Q.E.D.

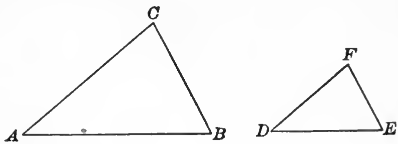
341. Cor. *If the products of the sides including the equal angles are equal, the triangles are equivalent.*

Ex. **485**. Two triangles have an angle in each equal, the including sides in one being 8 ft. and 12 ft., and in the other 6 ft. and 20 ft. The area of the smaller triangle is 27 sq. ft. Find the area of the larger triangle.

Proposition VIII

342. Find the area of a triangle whose base is 14″, another side 8″, and the altitude 6″, or other dimensions; also the area of a similar triangle whose base is 7″, or any other convenient length. How does the ratio of the areas of the two triangles compare with the ratio of the areas of the squares described upon their bases? With the ratios of the squares upon other homologous sides or lines?

Theorem. *Similar triangles are to each other as the squares upon their homologous sides.*



Data: Any two similar triangles, as ABC and DEF.

To prove $\triangle ABC : \triangle DEF = \overline{AB}^2 : \overline{DE}^2 =$ etc.

Proof. § 299, $\angle A = \angle D$;

$\therefore$ § 340, $\triangle ABC : \triangle DEF = AB \times AC : DE \times DF.$ (1)

But, § 299, $AC : DF = AB : DE.$ (2)

Multiplying the antecedents by AB and the consequents by DE,
§ 284, $AB \times AC : DE \times DF = \overline{AB}^2 : \overline{DE}^2.$

Substituting in (1),

$$\triangle ABC : \triangle DEF = \overline{AB}^2 : \overline{DE}^2.$$

In like manner the same may be proved for any homologous sides.

Therefore, etc. Q.E.D.

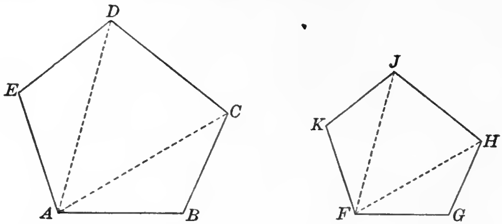
343. Cor. *Similar triangles are to each other as the squares upon any of their homologous lines.*

Ex. **486.** The homologous sides of two similar triangular fields are in the ratio of 5 : 3. How many times the area of the second field is the area of the first?

Proposition IX

344. Divide two similar polygons into triangles by diagonals drawn from a pair of homologous vertices. Since the homologous triangles are similar, to what is the ratio of their areas equal? (§ 342) Write all the ratios of the areas of each pair of triangles. Discover from the ratios whether every pair can be shown to have the same ratio. How, then, does the ratio of the sum of the triangles of one polygon — that is, its area — to the sum of the triangles of the other compare with the ratio of any two corresponding triangles? (§ 279) Then, how does the ratio of the polygons compare with the ratio of the squares described upon their homologous sides?

Theorem. *Similar polygons are to each other as the squares upon their homologous sides.*



Data: Any two similar polygons, as $ABCDE$ and $FGHJK$.

To prove $\quad ABCDE : FGHJK = \overline{AB}^2 : \overline{FG}^2 =$ etc.

Proof. Draw the homologous diagonals AC, AD, FH, and FJ.

Then, § 310, the corresponding triangles thus formed are similar,

and, § 299, $AB : FG = BC : GH = CD : HJ = DE : JK =$ etc.;

$\therefore$ § 288, $\overline{AB}^2 : \overline{FG}^2 = \overline{BC}^2 : \overline{GH}^2 = \overline{CD}^2 : \overline{HJ}^2 = \overline{DE}^2 : \overline{JK}^2 =$ etc.,

also, § 342, $\quad \triangle ABC : \triangle FGH = \overline{AB}^2 : \overline{FG}^2 =$ etc.,

$\triangle ACD : \triangle FHJ = \overline{CD}^2 : \overline{HJ}^2 =$ etc.,

and $\quad \triangle ADE : \triangle FJK = \overline{DE}^2 : \overline{JK}^2 =$ etc.

Since the ratios of these $\triangle$s are all equal, by § 279,

$\triangle ABC + \triangle ACD + \triangle ADE : \triangle FGH + \triangle FHJ + \triangle FJK = \triangle ABC : \triangle FGH$;

that is, $\quad ABCDE : FGHJK = \triangle ABC : \triangle FGH$.

But $\quad \triangle ABC : \triangle FGH = \overline{AB}^2 : \overline{FG}^2 =$ etc.

Hence, $\quad ABCDE : FGHJK = \overline{AB}^2 : \overline{FG}^2 =$ etc.

Therefore, etc. Q.E.D.

345. Cor. I. *Similar polygons are to each other as the squares upon any of their homologous lines.*

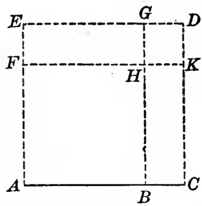
346. Cor. II. *The homologous sides of any similar polygons are to each other as the square roots of the areas of those polygons.*

Ex. **487.** In two similar polygons two homologous sides are 15 ft. and 25 ft. The area of the smaller polygon is 450 sq. ft. Find the area of the larger one.

Proposition X

347. Draw two lines respectively 3″ and 5″ long, or any other lengths; construct a square on each and a square on their sum; also construct the rectangle of these lines. How does the area of the square on their sum compare with the combined areas of the other squares and double the area of the rectangle?

Theorem. *The square upon the sum of two lines is equivalent to the sum of the squares upon the lines plus twice the rectangle formed by the lines.*



Data: Any two lines, as AB and BC, and their sum AC.

To prove $\overline{AC}^2 \Leftrightarrow \overline{AB}^2 + \overline{BC}^2 + 2 \text{ rect. } AB \cdot BC$.

Proof. On AC construct the square $ACDE$; draw $BG \parallel CD$; take DK equal to BC; and draw $FK \parallel AC$, cutting BG in H.

Since the sides of $HKDG$ are respectively parallel to the sides of the square $ACDE$, its angles are rt. $\angle$s.

§ 151, $GD = HK = BC$,

and $GH = DK$;

but, const., $DK = BC$;

$\therefore$ $GD = HK = GH = DK = BC$, Why?

and, § 143, $HKDG$ is a square whose side equals BC;

that is, $HKDG = \overline{BC}^2$.

Similarly, $ABHF = \overline{AB}^2$.

Since the sides of $BCHK$ are respectively parallel to the sides of the square $ACDE$, its angles are rt. $\angle$s;

and since, $HB = AB$,

$BCKH = \text{rect. } AB \cdot BC$.

Similarly, $FHGE = \text{rect. } AB \cdot BC$.

But $ACDE \Leftrightarrow ABHF + HKDG + BCKH + FHGE$;

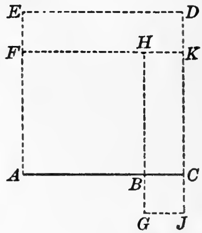
that is, $\overline{AC}^2 \Leftrightarrow \overline{AB}^2 + \overline{BC}^2 + 2 \text{ rect. } AB \cdot BC$.

Therefore, etc. Q.E.D.

Proposition XI

348. Draw two lines respectively 3″ and 5″ long, or any other lengths; construct a square on each, and a square on their difference; also construct the rectangle of these lines. How does the area of the square on their difference compare with the combined areas of the other squares minus double the area of the rectangle?

Theorem. *The square upon the difference of two lines is equivalent to the sum of the squares upon the lines minus twice the rectangle formed by the lines.*



Data: Any two lines, as AC and BC, and their difference AB.

To prove $\overline{AB}^2 \Leftrightarrow \overline{AC}^2 + \overline{BC}^2 - 2$ rect. $AC \cdot BC$.

Proof. On AC construct the square $ACDE$, on BC the square $BGJC$, and on AB the square $ABHF$. Produce FH to meet CD in K.

Then, $\angle GBC$ is a rt. $\angle$; Why?

$\therefore$ $\angle ABG$ is a rt. $\angle$; Why?

but $\angle ABH$ is a rt. $\angle$;

$\therefore$ GBH is a straight line. Why?

Similarly, JCK is a straight line.

Now, $\angle$s G and J are rt. $\angle$s, Why?

also $\angle$s GHK and HKJ are rt. $\angle$s; Why?

$\therefore$ $HGJK$ is a rectangle.

$HB = AB$, and $BG = BC$; Why?

hence, $HG = AC$,

also $GJ = BC$; Why?

$\therefore$ $HGJK =$ rect. $AC \cdot BC$.

Similarly, $FKDE =$ rect. $AC \cdot BC$.

Const., $ABHF = \overline{AB}^2$, $ACDE = \overline{AC}^2$, and $BGJC = \overline{BC}^2$.

But $ABHF \Leftrightarrow ACDE + BGJC - (HGJK + FKDE)$;

that is, $\overline{AB}^2 \Leftrightarrow \overline{AC}^2 + \overline{BC}^2 - 2$ rect. $AC \cdot BC$.

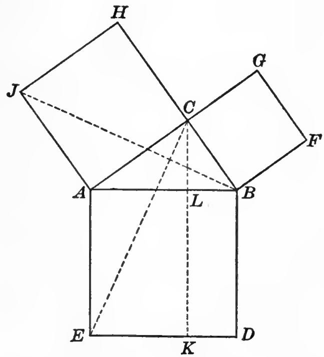
Therefore, etc. Q.E.D.

Proposition XII

349. Draw a right triangle whose sides are 3″, 4″, and 5″, or any other right triangle; construct a square on each side and find the area of each square. How does the square on the hypotenuse compare in area with the sum of the squares on the other sides?

Theorem. *The square upon the hypotenuse of a right triangle is equivalent to the sum of the squares upon the other two sides.*

FIRST METHOD



Data: Any right triangle, as *ABC*; the square on the hypotenuse, as *ABDE*; and the squares on the other two sides, as *BCGF* and *ACHJ* respectively.

To prove *ABDE* ⇔ *BCGF* + *ACHJ*.

Proof. From *C* draw *CK* ∥ *BD*, cutting *AB* in *L* and meeting *ED* in *K*, and draw *CE* and *BJ*.

$\angle$s *ACB*, *ACH*, and *BCG* are rt. $\angle$s;

∴ § 58, *ACG* and *BCH* are straight lines.

In $\triangle$s *AEC* and *AJB*, $AE = AB$, $AC = AJ$, Why?

and $\angle EAC = \angle JAB$; Why?

∴ § 100, $\triangle AEC = \triangle AJB$;

but, § 334, *AEKL* ⇔ 2 $\triangle$ *AEC*,

and *ACHJ* ⇔ 2 $\triangle$ *AJB*;

∴ *AEKL* ⇔ *ACHJ*.

In like manner, *BDKL* ⇔ *BCGF*.

But *ABDE* ⇔ *AEKL* + *BDKL*.

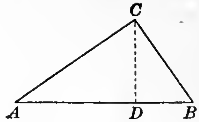
Hence, *ABDE* ⇔ *BCGF* + *ACHJ*.

Therefore, etc. Q.E.D.

Prove that *BDKL* ⇔ *BCGF*.

Second Method

Proof. Draw the perpendicular CD.



Then, § 313, $AB : AC = AC : AD,$

or $\overline{AC}^2 = AB \times AD,$

and $AB : BC = BC : DB,$

or $\overline{BC}^2 = AB \times DB.$

Ax. 2, $\overline{AC}^2 + \overline{BC}^2 = AB(AD + DB) = AB \times AB = \overline{AB}^2,$

or $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$

Therefore, etc. Q.E.D.

350. Cor. I. *Either side of a right triangle is equal to the square root of the difference between the squares of the hypotenuse and the other side.*

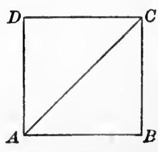
The following is an easy method of determining integral numbers which are measures of the sides of right triangles:

Write in a column the squares of the numbers of the scale as far as desired; subtract each square from all of the others following it. When the remainder is a perfect square its square root is the measure of one side of a right triangle, the square root of the minuend of this subtraction is the measure of the hypotenuse, and the square root of the subtrahend is the measure of the other side. By taking equimultiples of these numbers the measures of the sides of similar right triangles may be found.

The following sets of numbers are some of the integral measures of the sides of right triangles:

3	4	5
5	12	13
7	24	25
8	15	17
9	40	41

351. Cor. II. *The ratio of the diagonal of a square to a side is* $\sqrt{2}$.



For, in the square $ABCD$, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2;$

but $\overline{BC}^2 = \overline{AB}^2;$ hence, $\overline{AC}^2 = 2\,\overline{AB}^2.$

Dividing by $\overline{AB}^2$, $\dfrac{\overline{AC}^2}{\overline{AB}^2} = 2;$ whence, $\dfrac{AC}{AB} = \sqrt{2}.$

In the figure on page 185:

Ex. **488.** Prove CE perpendicular to JB.

Ex. **489.** If the lines AH and BG are drawn, prove that they are parallel.

Ex. **490.** Prove that the sum of the perpendiculars from F and J to AB produced is equal to AB.

Ex. **491.** Prove that J, C, and F are in the same straight line.

Ex. **492.** If the lines EJ and DF are drawn, prove that the sum of the angles AEJ, AJE, BDF, and BFD is equal to one right angle.

Ex. **493.** If EM and DN are drawn perpendicular respectively to JA and FB produced, prove that the triangles AEM and BDN are each equal to triangle ABC.

Ex. **494.** If the lines JH, KC, and FG are produced, prove that they meet in a common point.

Ex. **495.** If D is the middle point of the side BC of the right triangle ABC, and DE is drawn perpendicular to the hypotenuse AB, prove that $\overline{AC}^2 \approx \overline{AE}^2 - \overline{BE}^2$.

Ex. **496.** The area of a rectangle is $26.40^{\text{sq dm}}$ and its altitude is 4.8^{dm}. Find the length of its diagonal.

Ex. **497.** The perpendicular distance between two parallel lines is 20 in. and a line is drawn across them at an angle of 45°. What is the length of the part intercepted between the parallel lines?

Ex. **498.** Find the area of a right isosceles triangle, if the hypotenuse is 100 rd. in length.

Ex. **499.** The diameter of a circle is 12^{cm} and a chord of the circle is 10^{cm}. What is the length of a perpendicular from the center to this chord?

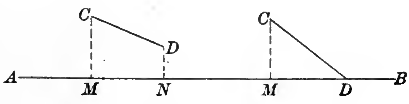
Ex. **500.** Two parallel chords in a circle are each 8 ft. in length, and the distance between them is 6 ft. Find the radius of the circle.

Ex. **501.** Two sides of a triangle are 13^{dm} and 15^{dm} and the altitude on the third side is 12^{dm}. Find the third side and also the area of the triangle.

Ex. **502.** Two parallel lines are 12 ft. apart, and from a point on one of them two lines, one 20 ft. and the other 13 ft. long, are drawn to the other parallel. What is the area of the triangle thus formed?

352. When from the extremities of a given straight line perpendiculars are let fall upon an indefinite straight line, the portion of the indefinite line between the perpendiculars is called the *projection of the given line.*

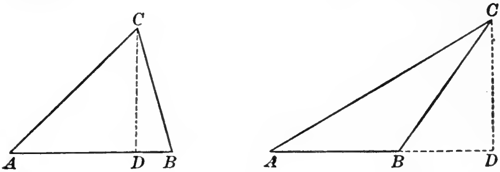
MN is the *projection* of the line CD upon the line AB. If the point D is in the line AB, then MD is the projection of CD.



Proposition XIII

353. Draw a triangle whose sides are 2″, 3″, and 4″, or any other oblique triangle; construct a square on the side opposite an acute angle; construct squares on the other two sides and also the rectangle of one of those sides and the projection of the other upon that side; find the area of each figure constructed. How does the area of the first square compare with the combined area of the otner squares less twice the area of the rectangle?

Theorem. *In any oblique triangle the square upon the side opposite an acute angle is equivalent to the sum of the squares upon the other two sides minus twice the rectangle formed by one of those sides and the projection of the other upon that side.*



Data: Any oblique triangle, as ABC, in which A is an acute angle and AD the projection of AC on AB, or AB produced.

To prove $\overline{BC}^2 \Leftrightarrow \overline{AB}^2 + \overline{AC}^2 - 2$ rect. $AB \cdot AD$.

Proof. When CD lies within $\triangle ABC$,

$$BD = AB - AD;$$

when CD lies without $\triangle ABC$,

$$BD = AD - AB;$$

and in either case,

§ 348, $\overline{BD}^2 \Leftrightarrow \overline{AB}^2 + \overline{AD}^2 - 2$ rect. $AB \cdot AD$.

Adding $\overline{CD}^2$ to both members of this equation,

$$\overline{BD}^2 + \overline{CD}^2 \Leftrightarrow \overline{AB}^2 + \overline{AD}^2 + \overline{CD}^2 - 2 \text{ rect. } AB \cdot AD.$$

But, § 349, $\overline{BD}^2 + \overline{CD}^2 \Leftrightarrow \overline{BC}^2$,

and $\overline{AD}^2 + \overline{CD}^2 \Leftrightarrow \overline{AC}^2$.

Hence, substituting $\overline{BC}^2$ and $\overline{AC}^2$ for their equivalents,

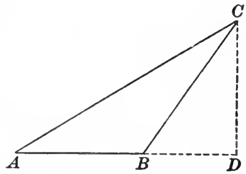
$$\overline{BC}^2 \Leftrightarrow \overline{AB}^2 + \overline{AC}^2 - 2 \text{ rect. } AB \cdot AD. \qquad \text{Q.E.D.}$$

Proposition XIV

354. Draw a triangle whose sides are 2″, 3″, and 4″, or any other obtuse triangle; construct a square on the side opposite the obtuse angle; construct squares on the other two sides and also the rectangle of one of those sides and the projection of the other upon that side produced; find the area of each figure constructed. How does the area of the first square compare with the combined area of the other squares and twice the area of the rectangle?

Theorem. *In any obtuse triangle the square upon the side opposite the obtuse angle is equivalent to the sum of the squares upon the other two sides plus twice the rectangle formed by one of those sides and the projection of the other upon that side.*

Data: Any obtuse triangle, as ABC, in which B is the obtuse angle and BD the projection of BC upon AB produced.



To prove

$$\overline{AC}^2 \Leftrightarrow \overline{AB}^2 + \overline{BC}^2 + 2 \text{ rect. } AB \cdot BD.$$

Proof. $$AD = AB + BD;$$

then, § 347, $$\overline{AD}^2 \Leftrightarrow \overline{AB}^2 + \overline{BD}^2 + 2 \text{ rect. } AB \cdot BD.$$

Adding $\overline{CD}^2$ to both members of this equation,

$$\overline{AD}^2 + \overline{CD}^2 \Leftrightarrow \overline{AB}^2 + \overline{BD}^2 + \overline{CD}^2 + 2 \text{ rect. } AB \cdot BD.$$

But, § 349, $$\overline{AD}^2 + \overline{CD}^2 \Leftrightarrow \overline{AC}^2,$$

and $$\overline{BD}^2 + \overline{CD}^2 \Leftrightarrow \overline{BC}^2.$$

Hence, substituting $\overline{AC}^2$ and $\overline{BC}^2$ for their equivalents,

$$\overline{AC}^2 \Leftrightarrow \overline{AB}^2 + \overline{BC}^2 + 2 \text{ rect. } AB \cdot BD.$$

Therefore, etc. Q.E.D.

Ex. **503.** The diagonals of a rhombus are 30 in. and 16 in. What is the length of the sides?

Ex. **504.** A square lawn with the walk around it contains $\frac{1}{16}$ of an acre. If the walk contains $\frac{15}{64}$ of the entire area, what is the width of the walk?

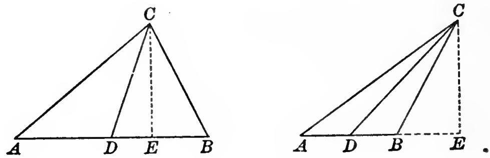
Ex. **505.** Find the area of a field in the form of a trapezoid whose bases are 45 rd. and 27 rd., and each of whose non-parallel sides is 15 rd.

Proposition XV

355. 1. Draw a triangle whose sides are 2″, 3″, and 4″, or any other oblique triangle; construct the squares on any two sides; construct a square on one half of the third side and also a square on the median to that side; find the area of each of these squares. How does the combined area of the first two compare with double the combined area of the other two?

2. Construct and find the area of the rectangle of the third side and the projection of the median upon that side. How does the difference in the area of the first two squares compare with double the area of this rectangle?

Theorem. *In any oblique triangle the sum of the squares upon any two sides is equivalent to twice the square upon one half the third side, plus twice the square upon the median to that side.*



Data: Any oblique triangle, as ABC, and the median CD, making with AB the obtuse angle ADC and the acute angle BDC.

To prove $\overline{AC}^2 + \overline{BC}^2 \approx 2\,\overline{AD}^2 + 2\overline{CD}^2$.

Proof. Draw $CE \perp AB$, or AB produced.

Then, in the $\triangle ADC$ and DBC respectively,

§ 354, $\overline{AC}^2 \approx \overline{AD}^2 + \overline{CD}^2 + 2$ rect. $AD \cdot DE$, (1)

and, § 353, $\overline{BC}^2 \approx \overline{DB}^2 + \overline{CD}^2 - 2$ rect. $DB \cdot DE$. (2)

But, data, $AD = DB$.

Substituting for DB in the second equation its equal AD and adding (1) and (2), $\overline{AC}^2 + \overline{BC}^2 \approx 2\,\overline{AD}^2 + 2\,\overline{CD}^2$.

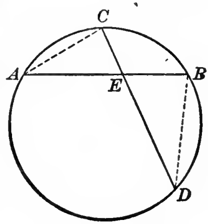
Therefore, etc. Q.E.D.

356. Cor. *In any oblique triangle the difference of the squares upon any two sides is equivalent to twice the rectangle formed by the third side and the projection of the median upon that side.*

Proposition XVI

357. Draw a circle and two intersecting chords; draw two chords, which do not meet, to connect the extremities of the given chords, thus forming two triangles. What angles of the figure are equal? Are the triangles equal, equivalent, or similar? How does the ratio of the longer segments of the given chords (sides of the similar triangles) compare with the ratio of their shorter segments? How does the rectangle formed by the segments of one chord compare with the rectangle formed by the segments of the other?

Theorem. *If two chords of a circle intersect, the rectangle formed by the segments of one chord is equivalent to the rectangle formed by the segments of the other.*



Data: Any two chords of a circle, as AB and CD, intersecting, as at E.

To prove rect. $AE \cdot BE \Leftrightarrow$ rect. $DE \cdot CE$.

Proof. Draw AC and BD.

Then, in $\triangle AEC$ and DEB,

§ 225, $\angle A = \angle D$, each being measured by $\frac{1}{2}$ arc CB,

and $\angle C = \angle B$, each being measured by $\frac{1}{2}$ arc AD;

∴, § 301, $\triangle AEC$ and DEB are similar.

Hence, $AE : DE = CE : BE$, Why?

and, § 269, $AE \times BE = DE \times CE$;

that is, § 324, rect. $AE \cdot BE \Leftrightarrow$ rect. $DE \cdot CE$.

Therefore, etc. Q.E.D.

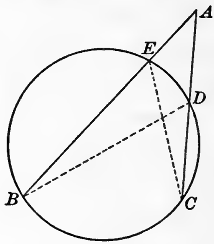
358. Cor. *If a chord passes through a fixed point, the area of the rectangle formed by its segments is constant in whatever direction the chord is drawn.*

Ex. **506.** A ladder 25^m long, with its foot in the street, will reach on one side to a window 20^m high, and on the other to a window 15^m high. What is the distance between the windows?

Proposition XVII

359. From a point without a circle draw two secants; draw two intersecting chords to connect the points of intersection of the secants and the circumference; select two triangles each of which has a secant for one of its sides. What angles of these triangles are equal? Are the triangles equal, equivalent, or similar? How does the ratio of the secants (sides of the similar triangles) compare with the ratio of their external segments? How does the rectangle formed by one secant and its external segment compare with the rectangle formed by the other and its external segments?

Theorem. *If from a point without a circle two secants are drawn, the rectangle formed by one secant and its external segment is equivalent to the rectangle formed by the other secant and its external segment.*



Data: Any point without a circle, as A, and any two secants from A, as AB and AC, cutting the circumference in E and D respectively.

To prove rect. $AB \cdot AE \approx$ rect. $AC \cdot AD$.

Proof. Draw BD and CE.

Then, in $\triangle$ ABD and ACE,

$\angle A$ is common,

and $\angle B = \angle C$; Why?

$\therefore$ $\triangle$ ABD and ACE are similar. Why?

Hence, $AB : AC = AD : AE$, Why?

and rect. $AB \cdot AE \approx$ rect. $AC \cdot AD$.

Therefore, etc. Q.E.D.

Ex. **507**. Find the area of a triangle each of whose sides is 12 ft.

Ex. **508**. The side of a rhombus is 29^{cm} and one of its diagonals is 40^{cm}. What is the length of the other diagonal?

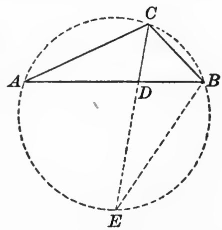
Ex. **509**. The area of a rhombus is 1176 sq. in. and one of its diagonals is 42 in. What are its sides and the other diagonal?

Ex. **510**. The radius of a circle is 8^{dm} and a tangent to the circle is 15^{dm}. What is the length of a secant drawn from the same point as the tangent, if the secant is 5^{dm} from the center?

Proposition XVIII

360. Draw a triangle and its circumscribing circle; bisect the vertical angle and produce the bisector to meet the circumference; connect this point of meeting with the point of intersection of the base and shortest side. What angles of the figure are equal? What triangles are similar? From the ratios of the sides of similar triangles and of the segments of intersecting chords discover how the rectangle formed by the sides of the given triangle compares with the rectangle formed by the segments of the base plus the square upon the bisector of the vertical angle?

Theorem. *If the bisector of the vertical angle of a triangle intersects the base, the rectangle formed by the two sides is equivalent to the rectangle formed by the segments of the base plus the square upon the bisector.*



Data: Any triangle, as ABC, and the bisector of its vertical angle, as CD, intersecting the base in D.

To prove

rect. $AC \cdot BC \approx$ rect. $AD \cdot BD + \overline{CD}^2$.

Proof. Circumscribe a circle about $\triangle ABC$, produce CD to meet the circumference in E, and draw EB.

Then, in $\triangle$ ADC and EBC,

data, $\angle ACD = \angle ECB$,

and $\angle A = \angle E$; Why?

$\therefore$ $\triangle$ ADC and EBC are similar. Why?

Hence, $AC : EC = CD : BC$,

and rect. $AC \cdot BC \approx$ rect. $EC \cdot CD$,

or rect. $AC \cdot BC \approx$ rect. $(DE + CD) \cdot CD \approx$ rect. $DE \cdot CD + \overline{CD}^2$.

But, § 357, rect. $DE \cdot CD \approx$ rect. $AD \cdot BD$.

Hence, rect. $AC \cdot BC \approx$ rect. $AD \cdot BD + \overline{CD}^2$.

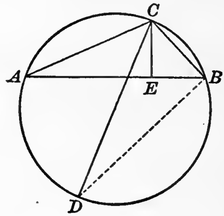
Therefore, etc. Q.E.D.

Ex. **511.** A pole standing on level ground was broken 75 ft. from the top and fell so that the end struck 60 ft. from the foot. Find the length of the pole.

Proposition XIX

361. Draw a triangle, a line representing its altitude, the circumscribing circle, and a diameter from the vertex; connect the other extremity of this diameter with the point of intersection of the base and shortest side. What right triangles are similar? How does the ratio of their longest sides compare with the ratio of their shortest sides? How does the rectangle formed by the sides of the given triangle compare with the rectangle formed by its altitude and the diameter of the circumscribing circle?

Theorem. *The rectangle formed by any two sides of a triangle is equivalent to the rectangle formed by the altitude upon the third side and the diameter of the circumscribing circle.*



Data: Any triangle, as ABC; a diameter of the circumscribing circle, as CD; and the altitude upon AB, as CE.

To prove rect. $AC \cdot BC \Leftrightarrow$ rect. $CD \cdot CE$.

Proof. Draw DB.

Data, § 94, $\angle AEC$ is a rt. $\angle$,

§ 227, $\angle DBC$ is a rt. $\angle$.

Then, in rt. $\triangle$ AEC and DBC,

$\angle A = \angle D$; Why?

$\therefore$ § 302, $\triangle$ AEC and DBC are similar.

Hence, $AC : CD = CE : BC$,

and rect. $AC \cdot BC \Leftrightarrow$ rect. $CD \cdot CE$.

Therefore, etc. Q.E.D.

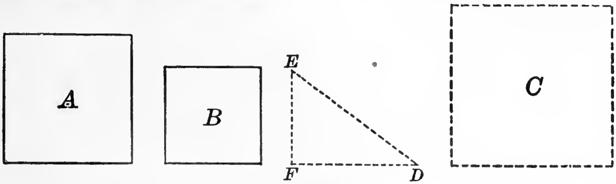
Ex. **512.** Upon the diagonal of a rectangle 28^m by 21^m a triangle equivalent to the rectangle is constructed. What is the altitude of the triangle?

Ex. **513.** The base and altitude of a right triangle are 6 ft. and 8 ft. respectively. What is the length of the perpendicular drawn from the vertex of the right angle to the hypotenuse?

Ex. **514.** The parallel sides of a trapezoid are 12 in. and 16 in. and the non-parallel sides are 10 in. What is the area of the triangle formed by joining the middle point of the shorter base with the extremities of the longer?

Proposition XX

362. Problem. *To construct a square equivalent to the sum of two given squares.*



Data: Any two squares, as A and B.

Required to construct a square equivalent to $A + B$.

Solution. Draw FD equal to a side of A. At one extremity, as at F, draw $FE \perp FD$ and equal to a side of B. Draw ED.

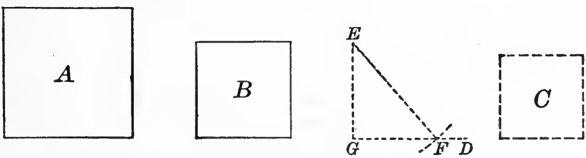
Construct a square C, having each of its sides equal to ED.

Then, C is the required square. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 349.

Proposition XXI

363. Problem. *To construct a square equivalent to the difference of two given squares.*



Data: Any two squares, as A and B.

Required to construct a square equivalent to $A - B$.

Solution. Draw an indefinite line, as GD.

At G, erect a perpendicular to GD, as GE, equal to a side of B.

With E as a center and a radius equal to a side of A, describe an arc intersecting GD at F. Draw EF.

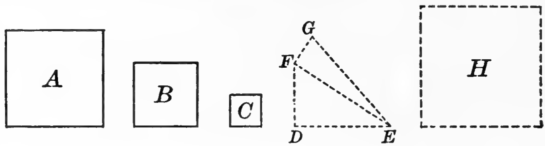
Construct a square C, having each of its sides equal to GF.

Then, C is the required square. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 349.

Proposition XXII

364. Problem. *To construct a square equivalent to the sum of any number of given squares.*



Data: Any squares, as A, B, and C.

Required to construct a square equivalent to $A + B + C$.

Solution. Draw DE equal to a side of A.

At D erect a perpendicular to DE, as DF, equal to a side of B. Draw FE.

At F erect a perpendicular to FE, as FG, equal to a side of C. Draw GE.

Construct a square H, having its sides each equal to GE.

Then, H is the required square. Q.E.D.

Proof. By the student. SUGGESTION. Refer to § 349.

Ex. **515.** Divide a triangle into two equivalent triangles by a line drawn through any vertex.

Ex. **516.** Construct a triangle equivalent to a given triangle and having the same base.

Ex. **517.** Construct an isosceles triangle equivalent to a given triangle and having the same base.

Ex. **518.** Construct a right triangle equivalent to a given triangle.

Ex. **519.** Construct a triangle equivalent to a given triangle, having the same base and an angle at the base equal to a given angle.

Ex. **520.** Construct a triangle similar to a given triangle and four times the given triangle.

Ex. **521.** Divide a parallelogram into two equivalent parts by a line through any point in its perimeter.

Ex. **522.** Divide a rectangle into four equivalent parts by lines through any vertex.

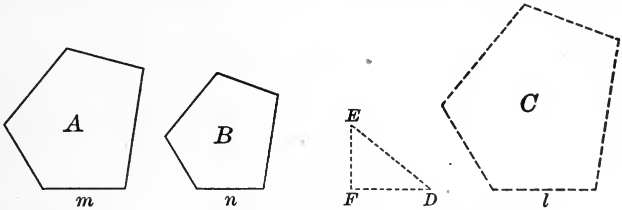
Ex. **523.** Construct a square equivalent to a triangle whose base is 18^{cm} and altitude 4^{cm}.

Ex. **524.** Construct a square equivalent to a rectangle whose dimensions are 16^{cm} and 4^{cm}.

Ex. **525.** Construct a square equivalent to the difference between two squares whose areas are $25^{sq\ cm}$ and $16^{sq\ cm}$.

Proposition XXIII

365. ***Problem.*** *To construct a polygon similar to two given similar polygons and equivalent to their sum.*



Data: Any two similar polygons, as A and B.

Required to construct a polygon similar to A and B and equivalent to $A+B$.

Solution. Draw a line, as FD, equal to m, a side of A.

At one extremity, as F, erect a perpendicular FE, equal to n, the homologous side of B. Draw ED.

Taking l, a line equal to ED, as homologous to m and n, construct a polygon C, similar to A and B.

Then, C is the required polygon. Q.E.F.

Proof. $\overline{FD}^2+\overline{FE}^2=\overline{ED}^2$; Why?

$\therefore$ $m^2+n^2=l^2$.

Now, § 344, $A:C=m^2:l^2$,

and $B:C=n^2:l^2$;

$\therefore$ § 280, $A+B:C=m^2+n^2:l^2$.

But $m^2+n^2=l^2$.

Hence, § 271, $A+B \approx C$.

Ex. **526.** Construct a right triangle equivalent to a given square.

Ex. **527.** Construct a right triangle equivalent to a given rectangle.

Ex. **528.** Construct a right triangle equivalent to a given parallelogram.

Ex. **529.** Construct an isosceles triangle equivalent to a given square.

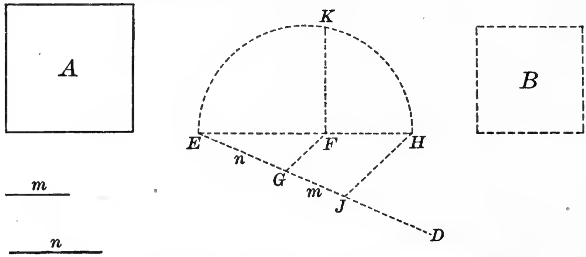
Ex. **530.** Construct a square equivalent to the sum of two squares whose sides are 5 in. and 10 in.

Ex. **531.** Construct a square equivalent to the difference of two squares whose sides are 15^{cm} and 17^{cm}.

Ex. **532.** Construct a polygon similar to two given similar polygons, and equivalent to their difference.

Proposition XXIV

366. *Problem.* *To construct a square having a given ratio to a given square.*



Data: Any square, as A, and any ratio, as $m:n$.

Required to construct a square B, such that $B:A=m:n$.

Solution. Draw EF equal to a side of A, and draw ED making any acute angle with EF.

On ED take EG equal to n, and GJ equal to m.

Draw GF; also draw $JH \parallel GF$, meeting EF produced at H.

On EH describe a semicircumference, and at F erect FK perpendicular to EH and meeting the semicircumference in K.

On a line equal to FK as a side, construct the square B.

Then, $B:A=m:n$. Q.E.F.

Proof. § 314, $EF:FK=FK:FH$;

hence, $\overline{FK}^2 = EF \times FH.$ Why?

Now, $\overline{FK}^2 : \overline{EF}^2 = \overline{FK}^2 : \overline{EF}^2$;

$\therefore$ $\overline{FK}^2 : \overline{EF}^2 = EF \times FH : \overline{EF}^2$,

or, dividing each term of the second ratio by EF,

§ 283, $\overline{FK}^2 : \overline{EF}^2 = FH : EF.$

Const., $FH:EF=GJ:EG=m:n$;

$\therefore$ $\overline{FK}^2 : \overline{EF}^2 = m:n$;

that is, $B:A=m:n$.

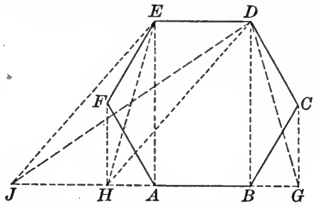
Ex. **533**. Construct an isosceles triangle equivalent to a given rectangle.

Ex. **534**. Construct an isosceles triangle equivalent to a given parallelogram.

Ex. **535**. Construct a square which shall be equivalent to a right isosceles triangle, having given the perpendicular from the vertex of the right angle upon the hypotenuse.

Proposition XXV

367. *Problem.* *To construct a triangle equivalent to a given polygon.*



Datum: Any polygon, as *ABCDEF.*

Required to construct a triangle equivalent to *ABCDEF.*

Solution. Draw *DB,* and from *C* to *AB* produced draw $CG \parallel DB$; also draw *DG.*

Draw *EA,* and from *F* to *BA* produced draw $FH \parallel EA$; also draw *EH.*

Draw *DH,* and from *E* to *BA* produced draw $EJ \parallel DH$; also draw *DJ.*

Then, *JGD* is the required triangle. Q.E.F.

Proof. In the polygons *AGDEF* and *ABCDEF,*

ABDEF is common,

and, § 336, $\triangle DBG \approx \triangle DBC$;

$\therefore$ $AGDEF \approx ABCDEF$.

In the polygons *HGDE* and *AGDEF,*

AGDE is common,

and $\triangle EAH \approx \triangle EAF$; Why?

$\therefore$ $HGDE \approx AGDEF$.

In the polygons *JGD* and *HGDE,*

HGD is common,

and $\triangle HDJ \approx \triangle HDE$. Why?

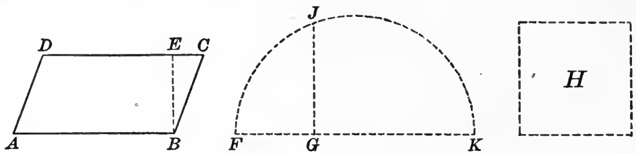
Hence, $\triangle JGD \approx HGDE \approx AGDEF \approx ABCDEF$.

Ex. **536.** Construct a parallelogram equivalent to a given parallelogram and having an angle equal to a given angle.

Ex. **537.** Bisect a given parallelogram (1) by a line passing through a given point within; (2) by a line perpendicular to a side; (3) by a line parallel to a side.

Proposition XXVI

368. Problem. *To construct a square equivalent to a given parallelogram.*



Datum: Any parallelogram, as *ABCD*.

Required to construct a square equivalent to *ABCD*.

Solution. Draw the altitude *BE*; also draw *FG* equal to *BE*.

Produce *FG* to *K*, making *GK* equal to *AB*. On *FK* as a diameter describe a semicircumference, draw *GJ* meeting it in *J* and $\perp$ *FK*.

On a line equal to *GJ* as a side, construct the square *H*.

Then, *H* is the required square. Q.E.F.

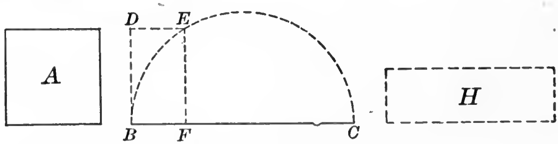
Proof. By the student. Suggestion. Refer to § 314.

369. Sch. A square may be constructed equivalent to a given triangle by taking for its side a mean proportional between the base and one half the altitude of the triangle.

To construct a square equivalent to any given polygon, first reduce the polygon to an equivalent triangle, and then construct a square equivalent to this triangle.

Proposition XXVII

370. Problem. *To construct a rectangle equivalent to a given square, and having the sum of its base and altitude equal to a given line.*



Data: Any square, as *A*, and the line *BC*.

Required to construct a rectangle equivalent to *A*, and having the sum of its base and altitude equal to *BC*.

Solution. Upon BC as a diameter describe a semicircumference.

At one extremity of BC, as B, erect a perpendicular to BC, as DB, equal to a side of A. Draw $DE \parallel BC$ meeting the semicircumference in E. Draw $EF \parallel DB$ meeting BC in F.

With base CF and altitude BF construct rectangle H.

Then, H is the required rectangle. Q.E.F.

Proof. $DB = EF$; Why?

$\therefore$ $\overline{DB}^2 = \overline{EF}^2 \Leftrightarrow A$.

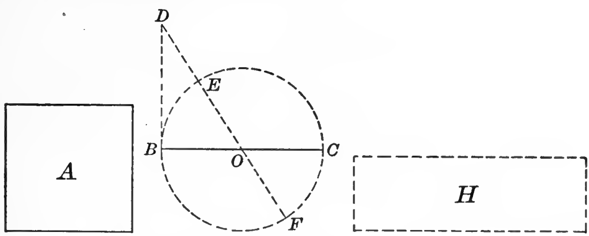
But, § 314, $BF : EF = EF : CF$;

$\therefore$ $BF \times CF = \overline{EF}^2$;

that is, $H \Leftrightarrow A$.

Proposition XXVIII

371. ***Problem.*** *To construct a rectangle equivalent to a given square, and having the difference of its base and altitude equal to a given line.*



Data: Any square, as A, and the line BC.

Required to construct a rectangle equivalent to A, and having the difference of its base and altitude equal to BC.

Solution. On BC as a diameter describe a circumference.

At one extremity of BC, as B, erect a perpendicular to BC, as BD, equal to a side of A.

Through O, the center of the circle, draw DF intersecting the circumference in E and meeting it in F.

Then, $FD - ED = EF$, or BC.

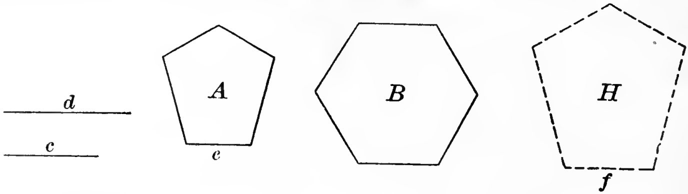
With base FD and altitude ED construct rectangle H.

Then, H is the required rectangle. Q.E.F.

Proof. By the student. SUGGESTION. Refer to §§ 315, 269.

Proposition XXIX

372. *Problem.* *To construct a polygon similar to a given polygon and equivalent to any other given polygon.*



Data: Any two polygons, as A and B.

Required to construct a polygon similar to A and equivalent to B.

Solution. Find c, the side of a square equivalent to A, and d, the side of a square equivalent to B, and let e be a side of A.

Find a fourth proportional to c, d, and e, as f.

Upon f homologous to e construct H similar to A.

Then, H is the required polygon. Q.E.F.

Proof. Const., H is similar to A.

Also $c : d = e : f$;

$\therefore$ $c^2 : d^2 = e^2 : f^2$. Why?

But, const., $A \approx c^2$, and $B \approx d^2$;

$\therefore$ $A : B = e^2 : f^2$.

But, § 344, $A : H = e^2 : f^2$;

$\therefore$ $A : H = A : B$.

Hence, § 272, $H \approx B$.

SUMMARY

373. Truths established in Book V.

1. A rectangle is equivalent,

a. If it is the rectangle formed by the segments of one of two intersecting chords, to the rectangle formed by the segments of the other. § 357

b. If it is formed by a secant and its external segment, to a rectangle formed by another secant from the same point and its external segment. § 359

c. If it is formed by the two sides of a triangle, to the rectangle formed by the segments of the base, made by the bisector of the vertical angle, plus the square upon the bisector. § 360 *d.* If it is formed by two sides of a triangle, to the rectangle formed by the altitude upon the third side and the diameter of the circumscribing circle. § 361

2. Rectangles are in proportion,

a. If they have equal altitudes, to their bases. § 327

b. If they have equal bases, to their altitudes. § 328

c. To the products of their bases by their altitudes. § 329

3. A parallelogram is equivalent,

a. To the rectangle which has the same base and altitude. § 331

b. To another parallelogram which has an equal base and an equal altitude. § 333

4. Parallelograms are in proportion,

a. If they have equal altitudes, to their bases. § 333

b. If they have equal bases, to their altitudes. § 333

c. To the products of their bases by their altitudes. § 333

5. A triangle is equivalent,

a. To one half the rectangle which has the same base and altitude. § 334

b. To another triangle which has an equal base and an equal altitude. § 336

c. To another triangle which has an angle equal to an angle of the first, and the products of the sides, including the equal angles, equal. § 341

6. Triangles are in proportion,

a. If they have equal altitudes, to their bases. § 336

b. If they have equal bases, to their altitudes. § 336

c. To the products of their bases by their altitudes. § 336

d. If they have an angle of one equal to an angle of the other, to the products of the sides including the equal angles. § 340

e. If they are similar triangles, to the squares upon their homologous sides. § 342

f. If they are similar triangles, to the squares upon any of their homologous lines. § 343

7. A trapezoid is equivalent,

a. To one half the rectangle which has the same altitude and a base equal to the sum of the parallel sides. § 337

8. The square upon a line is equivalent,

a. If the line is the sum of two lines, to the sum of the squares upon the lines plus twice the rectangle formed by them. § 347

b. If the line is the difference of two lines, to the sum of the squares upon the lines minus twice the rectangle formed by them. § 348

c. If the line is the hypotenuse of a right triangle, to the sum of the squares upon the other two sides. § 349 *d.* If the line is the side of an oblique triangle, opposite an acute angle, to the sum of the squares upon the other two sides minus twice the rectangle formed by one of those sides and the projection of the other upon that side. § 353

e. If the line is the side opposite an obtuse angle of a triangle, to the sum of the squares upon the other two sides plus twice the rectangle formed by one of those sides and the projection of the other upon that side. § 354

9. The sum of two squares is equivalent,

a. If they are the squares upon any two sides of an oblique triangle, to twice the square upon one half the third side plus twice the square upon the median to that side. § 355

10. The difference of two squares is equivalent,

a. If they are the squares upon any two sides of an oblique triangle, to twice the rectangle formed by the third side and the projection of the median upon that side. § 356

11. Similar polygons are in proportion,

a. To the squares upon their homologous sides. § 344

b. To the squares upon any of their homologous lines. § 345

12. The area of a figure is equal,

a. If it is a rectangle, to the product of its base by its altitude. § 330

b. If it is a parallelogram, to the product of its base by its altitude. § 332

c. If it is a triangle, to half the product of its base by its altitude. § 335

d. If it is a trapezoid, to half the product of its altitude by the sum of its parallel sides. § 338

SUPPLEMENTARY EXERCISES

Ex. **538**. The straight line joining the middle points of the parallel sides of a trapezoid bisects the trapezoid.

Ex. **539**. The lines joining the middle point of either diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equivalent parts.

Ex. **540**. Two triangles are equivalent, if they have two sides of one respectively equal to two sides of the other, and if the included angles are supplementary.

Ex. **541**. O is any point on the diagonal AC of the parallelogram $ABCD$. If the lines DO and BO are drawn, prove that the triangles AOB and AOD are equivalent.

Ex. **542**. A rhombus and a rectangle have equal bases and equal areas. One side of the rhombus is 15^{m} and the altitude of the rectangle is 12^{m}. What are their perimeters?

Ex. 543. **The area of a rhombus is equal to one half the product of its diagonals.**

Ex. **544.** The diagonals of a rhombus are 64 rd. and 37 rd. What is the area of the rhombus?

Ex. **545.** The base of a triangle is 75^m, and its altitude is 60^m. Find the perimeter of an equivalent rhombus, if its altitude is 45^m.

Ex. **546.** Find the area of a rhombus, if the sum of its diagonals is 12 in. and their ratio is 3 : 5.

Ex. **547.** A man travels 25 miles east from a certain town, and another man travels 36 miles north from the same town. How far apart are the men?

Ex. **548.** The shortest side of a triangle acute-angled at the base is 45 ft. long, and the segments of the base made by a perpendicular from the vertex are 27 ft. and 77 ft. How long is the other side?

Ex. 549. The sides of a triangle are 25^m and 17^m, and the lesser segment of the base made by a perpendicular from the vertex is 8^m. What is the length of the base?

Ex. **550.** In a right triangle the base is 3^{dm}, and the difference between the hypotenuse and perpendicular is 1^{dm}. What are the hypotenuse and perpendicular?

Ex. **551.** In a right triangle the hypotenuse is 5^{dm}, and the difference between the base and perpendicular is 1^{dm}. Find the base and perpendicular.

Ex. **552.** The sides of a right triangle are in the ratio of 3, 4, and 5, and the perpendicular upon the hypotenuse from the vertex of the right angle is 20 yd. What is the area of the triangle?

Ex. **553.** If in any triangle a perpendicular is drawn from the vertex to the base, the difference of the squares upon the sides is equivalent to the difference of the squares upon the segments of the base.

Ex. **554.** In a right triangle the square on either side containing the right angle is equivalent to the rectangle contained by the sum and the difference of the other sides.

Ex. **555.** If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares upon one pair of opposite sides is equivalent to the sum of the squares upon the other pair.

Ex. **556.** The altitude of an equilateral triangle is 60 in. How long are its sides?

Ex. **557.** Through D and E, the middle points of the sides AC and BC of the triangle ABC, any two parallel straight lines are drawn meeting AB or AB produced in the points F and G. Prove that the parallelogram $DFGE$ is equivalent to half the triangle ABC.

Ex. **558.** The four triangles into which a parallelogram is divided by its diagonals are equivalent.

Ex. **559.** The diagonals of a trapezoid divide it into four triangles, two of which are similar, while the other two are equivalent.

Ex. **560.** In any trapezoid the triangle, having for its base one of the non-parallel sides and for its vertex the middle point of the opposite side, is equivalent to one half of the trapezoid.

Ex. **561.** The triangle, formed by drawing a line from any vertex of a parallelogram to the middle point of one of the opposite sides, is equivalent to one fourth of the parallelogram.

Ex. **562.** A triangle is equivalent to one half the rectangle of its perimeter and the radius of the inscribed circle.

SUGGESTION. Draw radii to the points of contact and lines from the vertices of the triangle to the center of the circle.

Ex. **563.** If the perimeter of a quadrilateral circumscribed about a circle is 400 ft. and the radius of the circle is 25 ft., what is the area of the quadrilateral?

Ex. **564.** The area of a triangle is 875 sq. yd. Find its base and altitude, if they are in the ratio of 14 to 5.

Ex. **565.** The homologous sides of two similar fields are in the ratio of 3 to 5, and the sum of their areas is 416^{Ha}. What is the area of each field?

Ex. **566.** A board 12 ft. long is 10 in. wide at one end and 6 in. at the other. What length must be cut from the narrower end to contain a square foot?

Ex. **567.** The side of one equilateral triangle is equal to the altitude of another. What is the ratio of their areas?

Ex. **568.** The perimeter of an isosceles triangle whose base is its shortest side is 100^{dm}; the difference between the base and an adjacent side is 23^{dm}. What is the altitude of the triangle? What is its area?

Ex. **569.** Two chords on opposite sides of the center of a circle are parallel; one is 16 ft. long and the other is 12 ft. If the distance between them is 14 ft., what is the diameter of the circle?

Ex. **570.** If from the vertex of an acute angle of a right triangle a straight line is drawn bisecting the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square upon half the line bisected.

Ex. **571.** In the right triangle ABC, $\overline{BC}^2 = 3\,\overline{AC}^2$. If CD is drawn from the vertex of the right angle to the middle point of AB, angle ACD equals 60°.

Ex. **572.** If ACB and ADB are two angles inscribed in a semicircle, and AE and BF are drawn perpendicular to CD produced, prove that

$$\overline{CE}^2 + \overline{CF}^2 = \overline{DE}^2 + \overline{DF}^2.$$

Ex. **573**. If lines are drawn perpendicular to the diagonals of a square at their extremities, a second square is formed equivalent to twice the original square.

Ex. **574**. The square upon the base of an isosceles triangle whose vertical angle is a right angle is equivalent to four times the triangle.

Ex. **575**. The sum of the squares on the lines joining any point in the circumference of a circle with the vertices of an inscribed square is equivalent to twice the square of the diameter of the circle.

Ex. **576**. If AF and BE are the medians drawn from the extremities of the hypotenuse of the right $\triangle ABC$, prove that $4\,\overline{AF}^2 + 4\,\overline{BE}^2 = 5\,\overline{AB}^2$.

Ex. **577**. If perpendiculars PF, PD, and PE are drawn from any point P to the sides AB, BC, and AC of a triangle, prove that

$$\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2.$$

Ex. **578**. If any point P within the rectangle $ABCD$ is joined to the vertices, prove that $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.

Ex. **579**. If CD and AE are the perpendiculars from the vertices C and A of the acute triangle ABC to the opposite sides, prove that

$$\overline{AC}^2 = BC \times CE + AB \times AD.$$

SUGGESTION. Refer to § 373, 8, *d*.

Ex. **580**. If AC and BC are the equal sides of an isosceles triangle, and AD is drawn perpendicular to BC, prove that $\overline{AB}^2 = 2\,BC \times BD$.

Ex. **581**. The sum of the squares on the diagonals of a parallelogram is equivalent to the sum of the squares on its four sides.

Ex. **582**. Three times the sum of the squares on the sides of a triangle is equivalent to four times the sum of the squares on the medians of the triangle.

Ex. **583**. Two sides of an oblique triangle are 137 and 111 respectively, and the difference of the segments of the third side made by a perpendicular from the opposite vertex is 52. What is the third side?

Ex. **584**. The chord of an arc is 80 in.; the chord of half the arc is 41 in. What is the diameter of the circle?

Ex. **585**. From a point without a circle two tangents are drawn which with the chord of contact form an equilateral triangle whose side is 18 in. Find the diameter of the circle.

Ex. **586**. If the center of each of two equal circles is on the circumference of the other, the square on the common chord is equivalent to three times the square on the radius.

Ex. **587**. A tangent and a secant meet without a circle, forming an angle of 45°; the tangent is 2 ft. long and the diameter of the circle is 4 ft. Find the length of the secant.

PROBLEMS OF CONSTRUCTION

Ex. **588**. Divide a given parallelogram into two equivalent parts by a line drawn parallel to a given line.

Ex. **589**. Divide a given triangle into two parts, whose ratio is 3 : 4, by a line drawn from one vertex.

Ex. **590**. Divide a given parallelogram into two parts, whose ratio is 2 : 3, by a line parallel to a side.

Ex. **591**. Construct a parallelogram equivalent to the sum of two given parallelograms of equal altitude.

Ex. **592**. Construct a parallelogram equivalent to the difference of two given parallelograms of equal bases.

Ex. **593**. Construct a square equivalent to five times a given square.

Ex. **594**. Transform a given trapezoid into an equivalent isosceles trapezoid.

Ex. **595**. Construct a rhombus equivalent to a given parallelogram, and having one side of the parallelogram for a diagonal.

Ex. **596**. Construct a square equivalent to a given rectangle.

Ex. **597**. Construct a square equivalent to four sevenths of a given square.

Ex. **598**. Construct a rectangle equivalent to a given square, and having a given side.

Ex. **599**. Construct a rectangle equivalent to a given rectangle, and having a given side.

Ex. **600**. Construct a square equivalent to a given rhombus.

Ex. **601**. Construct a parallelogram having a given altitude, and equivalent to a given parallelogram.

Ex. **602**. Construct a rectangle having a given altitude and equivalent to a given parallelogram.

Ex. **603**. Construct a rhombus having a given side, and equivalent to a given parallelogram.

Ex. **604**. Construct a rhombus having a given altitude and equivalent to a given parallelogram.

Ex. **605**. Transform a triangle into an equivalent parallelogram whose base shall be the base of the triangle and one of whose base angles shall be equal to a base angle of the triangle.

Ex. **606**. Construct a triangle having a given angle, and equivalent to a given parallelogram.

Ex. **607**. Construct a triangle equivalent to a given trapezium.

Ex. **608**. Construct a parallelogram equivalent to a given trapezium.

Ex. **609**. Construct an isosceles triangle on a given base and equivalent to a given trapezium.

Ex. **610**. Construct a right triangle equivalent to a given triangle, having given one of the sides about the right angle.

Ex. **611**. Construct a right triangle equivalent to a given triangle, having given the hypotenuse.

Ex. **612**. Construct a triangle equivalent to a given triangle and having its base and altitude equal.

Ex. **613**. Construct an equilateral triangle equivalent to a given triangle.

Ex. **614**. Construct an equilateral triangle equivalent to a given square.

Ex. **615**. Construct a rectangle having a given diagonal and equivalent to a given rectangle.

Ex. **616**. Construct a rectangle having a given diagonal and equivalent to a given square.

Ex. **617**. Construct a square equivalent to a given trapezoid.

Ex. **618**. Construct a triangle equivalent to a given trapezoid.

Ex. **619**. Construct a parallelogram equivalent to a given trapezoid and having for its base the longer base of the trapezoid.

Ex. **620**. Construct a triangle equivalent to a given triangle and similar to another given triangle.

Ex. **621**. Construct a parallelogram equivalent to the sum of two given parallelograms.

Ex. **622**. The area of a square is 16. Construct a square that shall be to it in the ratio of 5 to 3.

Ex. **623**. Construct a hexagon similar to a given hexagon, having its ratio to the given hexagon as 5 is to 3.

Ex. **624**. Construct a square equivalent to two thirds of a given hexagon.

Ex. **625**. Construct a square equivalent to the sum of a given pentagon and a given parallelogram.

Ex. **626**. Divide a given triangle into two equivalent parts by a line perpendicular to one side.

Ex. **627**. Divide a given triangle into two equivalent parts by a line parallel to one side.

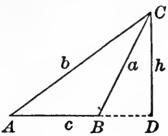
Ex. **628**. Bisect a given trapezoid by a line parallel to the bases.

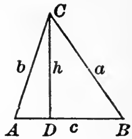
Ex. **629**. Bisect a given quadrilateral by a line drawn from one of the vertices.

Ex. **630**. Bisect a given quadrilateral by a line drawn from any point in its perimeter.

ALGEBRAIC SOLUTIONS

Ex. **631**. *Given the sides of any triangle, to compute the altitude.*





Solution. In $\triangle ABC$, suppose that angle A is acute.

Then, § 353, $$a^2 = b^2 + c^2 - 2c \times AD;$$

$\therefore$ $$AD = \frac{b^2 + c^2 - a^2}{2c}.$$

In $\triangle ADC$, $$h^2 = b^2 - \overline{AD}^2.$$

Substituting for $\overline{AD}^2$ its value,

$$h^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2 = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4c^2}$$

$$= \frac{\{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}}{4c^2}$$

$$= \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4c^2}.$$

Let $$a + b + c = 2s,$$

then, $$b + c - a = 2(s - a),$$

$$a + b - c = 2(s - c),$$

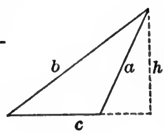
and $$a - b + c = 2(s - b).$$

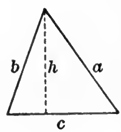
Hence, $$h^2 = \frac{2s \times 2(s - a) \times 2(s - b) \times 2(s - c)}{4c^2};$$

$\therefore$ $$h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

Ex. **632**. *Given the sides of any triangle, to compute its area.*

Denote the area by A.

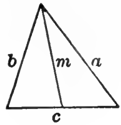




Solution. Ex. 631, $h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$,

then, $$A = \frac{c}{2} \times \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$$

$$= \sqrt{s(s - a)(s - b)(s - c)}.$$

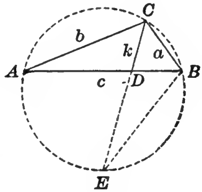


Ex. 633. *Given the sides of a triangle, to compute the medians.*

Solution. § 355, $$a^2 + b^2 = 2\,m^2 + 2\left(\frac{c}{2}\right)^2.$$

Hence, $$4\,m^2 = 2(a^2 + b^2) - c^2,$$

and $$m = \tfrac{1}{2}\sqrt{2(a^2 + b^2) - c^2}.$$



Ex. 634. *Given the sides of a triangle, to compute the bisectors of the angles.*

Solution. Circumscribe a circle about $\triangle ABC$; produce CD to meet the circumference in E; and draw BE.

§ 360, $$ab = AD \times BD + k^2;$$

$\therefore$ $$k^2 = ab - AD \times BD.$$

§ 292, $$AD : BD = b : a;$$

$\therefore$ $$AD + BD : AD = a + b : b,$$

and $$AD + BD : a + b = AD : b = BD : a;$$

that is, $$\frac{c}{a+b} = \frac{AD}{b} = \frac{BD}{a},$$

whence, $$AD = \frac{bc}{a+b}, \text{ and } BD = \frac{ac}{a+b};$$

hence, $$k^2 = ab - \frac{abc^2}{(a+b)^2}$$

$$= ab\left(1 - \frac{c^2}{(a+b)^2}\right)$$

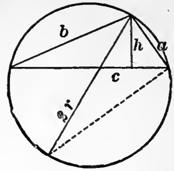
$$= \frac{ab\{(a+b)^2 - c^2\}}{(a+b)^2}$$

$$= \frac{ab(a+b+c)(a+b-c)}{(a+b)^2}$$

$$= \frac{ab \times 2\,s \times 2(s-c)}{(a+b)^2}.$$

Hence, $$k = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

Ex. **635**. *Given the sides of a triangle and the radius of the circumscribing circle, to compute the area of the triangle.*



Solution. Denote the radius of the circle by r.

Then, § 361, $$ab = 2\,rh;$$

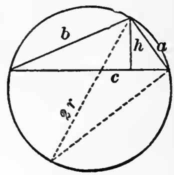
$$\therefore \quad abc = 2\,rch.$$

But $$ch = 2\,A;$$

$$\therefore \quad abc = 4\,Ar.$$

Hence, $$A = \frac{abc}{4\,r}.$$

Ex. **636**. *Given the sides of a triangle, to compute the radius of the circumscribed circle.*



Solution. § 361, $$ab = 2\,rh.$$

But, Ex. 631, $$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)};$$

$$\therefore \quad ab = \frac{4\,r}{c}\sqrt{s(s-a)(s-b)(s-c)},$$

whence, $$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Ex. **637**. The three sides of a triangle are 58 ft., 51 ft., and 41 ft. in length. What is the area of the triangle?

Ex. **638**. Find the altitude on each of the sides of a triangle whose sides are respectively 7 in., 9 in., and 11 in.

Ex. **639**. If the sides of a triangle are respectively 4^m, 6^m, and 8^m long, what are its three medians?

Ex. **640**. What is the area of a triangle, if the radius of the circumscribing circle is 6.196^m and the sides of the triangle are respectively 9^m, 6^m, and 12^m in length?

Ex. **641**. The sides of a triangle are respectively 12 in., 11 in., and 9 in. in length. Find the radius of the circumscribing circle.

Ex. **642**. The sides of a triangle are respectively 30^{dm}, 50^{dm}, and 70^{dm}. Find the lengths of the three angle bisectors.

Ex. **643**. If two sides and one of the diagonals of a parallelogram are respectively 12 in., 15 in., and 18 in., what is length of the other diagonal? What is the area of the parallelogram?

BOOK VI

REGULAR POLYGONS AND MEASUREMENT OF THE CIRCLE

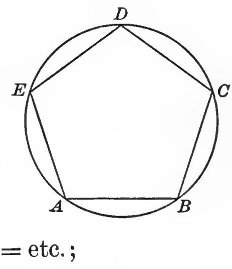
374. A polygon which is equilateral and equiangular is called a **Regular Polygon.**

An equilateral triangle and a square are *regular polygons.*

Proposition I

375. Draw a circle and inscribe in it any equilateral polygon. How do the arcs subtended by the sides of the polygon compare? How do the arcs intercepted by the sides of the angles of the polygon compare? How do the angles themselves compare? What may any equilateral polygon that is inscribed in a circle be called?

Theorem. *Any equilateral polygon inscribed in a circle is a regular polygon.*



Data: Any equilateral polygon, as $ABCDE$, inscribed in a circle.

To prove $ABCDE$ a regular polygon.

Proof. § 196, arc $AB =$ arc $BC =$ arc $CD =$ etc.;

$\therefore$ arc $BCDE =$ arc $CDEA =$ arc $DEAB =$ etc.,

and $\angle A = \angle B = \angle C =$ etc.; Why?

that is, $ABCDE$ is equiangular.

But, data, $ABCDE$ is equilateral;

hence, § 374, $ABCDE$ is a regular polygon.

Therefore, etc. Q.E.D.

Proposition II

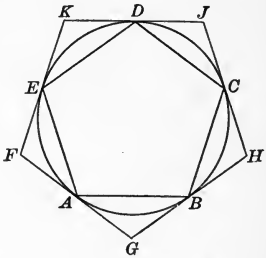
376. 1. Divide the circumference of a circle into any number of equal arcs; draw the chords of these arcs in succession. What kind of an inscribed polygon is thus formed?

2. Draw tangents to the circle at the extremities of the chords and produce them until they intersect. How do they compare in length? How do the angles formed by each pair of tangents compare in size? What kind of a circumscribed polygon has been formed?

Theorem. *If the circumference of a circle is divided into any number of equal arcs,*

1. The chords joining the extremities of the arcs in succession form a regular inscribed polygon.

2. The tangents drawn at the extremities of the arcs form a regular circumscribed polygon.



Data: Any circumference divided into equal arcs at A, B, C, etc.; the chords AB, BC, CD, etc.; and the tangents GBH, HCJ, etc.

To prove

$ABCDE$ and $FGHJK$ regular polygons.

Proof. 1. $AB = BC = CD =$ etc., Why?

and $\angle A = \angle B = \angle C =$ etc.; Why?

$\therefore$ § 374, $ABCDE$ is a regular polygon.

2. $\angle BAG = \angle ABG = \angle CBH =$ etc.; Why?

$\therefore$ $\triangle ABG$, BCH, CDJ, etc., are equal isosceles $\triangle$. Why?

Hence, $\angle G = \angle H = \angle J =$ etc.,

and $GB = BH = HC =$ etc.; Why?

$\therefore$ tangents GBH, HCJ, etc., are equal.

Hence, $FGHJK$ is a regular polygon.

Therefore, etc. Q.E.D.

377. The radius of a circle inscribed in a regular polygon is called the *apothem of the polygon.*

378. The radius of a circle circumscribed about a regular polygon is called the *radius of the polygon.*

379. The common center of the circles inscribed in and circumscribed about a regular polygon is called the *center of the polygon.*

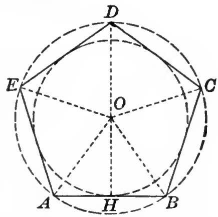
380. The angle between the radii drawn to the extremities of any side of a regular polygon is called the *angle at the center of the polygon.*

Proposition III

381. 1. Draw any regular polygon; pass a circumference through three of its vertices. Does it pass through the other vertices? Why?

2. With the same center and a radius equal to the apothem describe a circle. How many sides of the polygon does this circle touch? Why?

Theorem. ***A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.***



Datum: Any regular polygon, as $ABCDE$.

To prove 1. That a circle may be circumscribed about $ABCDE$.

2. That a circle may be inscribed in $ABCDE$.

Proof. 1. Describe a circle passing through A, B, and C, and from the center O, draw OA, OB, OC, OD, and OE.

In the $\triangle BCO$,	$OB = OC$;	Why?
$\therefore$	$\angle OBC = \angle OCB$.	Why?
But, § 374,	$\angle ABC = \angle BCD$;	
$\therefore$ Ax. 3,	$\angle OBA = \angle OCD$.	
Also	$AB = CD$, and $OB = OC$;	
hence, § 100,	$\triangle ABO = \triangle CDO$,	
and	$OA = OD$.	

Consequently, the circle passing through A, B, and C passes through D.

In like manner it may be shown that this circle passes through E.

Therefore, a circle described with the center O and a radius equal to OA will be circumscribed about the polygon.

2. Since the sides *AB, BC, CD,* etc., are equal chords of the circumscribed circle, they are equally distant from the center and the perpendiculars drawn from the center to the chords are all equal.

Hence, a circle described with the center *O* and a radius equal to one of these perpendiculars, as *OH,* will be inscribed in the polygon, for each of the sides of the polygon will be perpendicular to a radius at its extremity and tangent to the circle. § 205

Therefore, etc. Q.E.D.

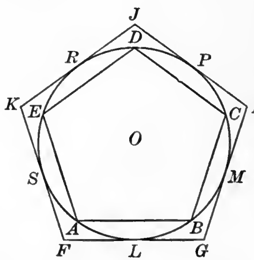
382. Cor. I. *The radius drawn to the vertex of any interior angle of a regular polygon bisects the angle.*

383. Cor. II. *Each angle at the center of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

Proposition IV

384. Draw a circle and a regular inscribed polygon; at the middle points of the arcs subtended by its sides draw tangents and produce them until they intersect. How do they compare in length? How do the angles formed by each pair of tangents compare in size? What kind of a circumscribed polygon has been formed?

Theorem. *Tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon.*



Data: A circle whose center is *O*; any regular inscribed polygon, as *ABCDE*; and the tangents *FG, GH, HJ,* etc., at the middle points *L, M, P,* etc., of the arcs *AB, BC, CD,* etc.

To prove *FGHJK* a regular circumscribed polygon.

Proof. arc $AB =$ arc $BC =$ arc $CD =$ etc., Why?

arc $AL =$ arc $LB =$ arc $BM =$ etc.; Why?

arc $LM =$ arc $MP =$ etc.

But the sides of $FGHJK$ are tangents at the extremities of these arcs.

Hence, § 376, $FGHJK$ is a regular circumscribed polygon.

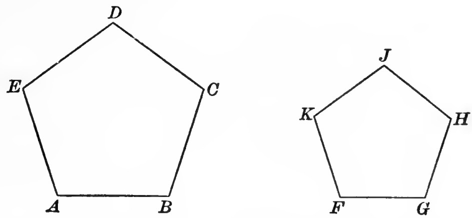
Therefore, etc. Q.E.D.

385. Cor. *Regular inscribed and circumscribed polygons of the same number of sides may be so placed that their sides are parallel and their vertices will then lie upon the radii (prolonged) of the inscribed polygon.*

Proposition V

386. Draw two regular polygons of the same number of sides. How do the homologous angles compare in size? How do the ratios of any two pairs of homologous sides compare? What name is given to polygons that have such relations to each other?

Theorem. ***Regular polygons which have the same number of sides are similar.***



Data: Any two regular polygons, as $ABCDE$ and $FGHJK$, which have the same number of sides.

To prove $ABCDE$ and $FGHJK$ similar.

Proof. By § 166, the sum of the angles of each polygon is equal to twice as many right angles as the polygon has sides less two.

Since, § 374, each polygon is equiangular, and since each contains the same number of angles;

all the angles of both polygons are equal.

§ 374, $AB = BC = CD =$ etc., and $FG = GH = HJ =$ etc.;

$\therefore$ $AB : FG = BC : GH =$ etc.

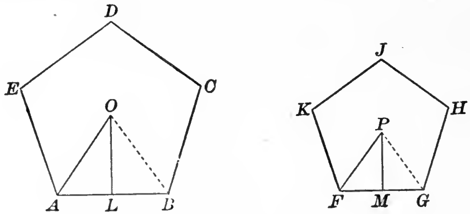
Hence, § 299, $ABCDE$ and $FGHJK$ are similar.

Therefore, etc. Q.E.D.

Proposition VI

387. Draw two regular polygons of the same number of sides; draw radii to the extremities of a pair of homologous sides. What kind of triangles are formed? How does the ratio of their bases compare with the ratio of the radii? With the ratio of the apothems? Since the polygons are similar, how does the ratio of their perimeters compare with the ratio of any two homologous sides? With the ratio of their radii? Of their apothems?

Theorem. *The perimeters of regular polygons of the same number of sides are to each other as their radii and also as their apothems.*



Data: Any two regular polygons of the same number of sides, as $ABCDE$ and $FGHJK$, whose radii are OA and PF, and apothems OL and PM, respectively.

Denote their perimeters by Q and S respectively.

To prove $Q : S = OA : PF = OL : PM.$

Proof. Draw OB and PG.

In the $\triangle$ AOB and FPG,

§ 383, $\angle O \doteq \angle P,$

and $OA : PF = OB : PG;$ Why?

$\therefore$ § 306, $\triangle$ AOB and FPG are similar.

Hence, $AB : FG = OA : PF,$

and $AB : FG = OL : PM.$ Why?

But, §§ 386, 311, $Q : S = AB : FG.$

Hence, $Q : S = OA : PF = OL : PM.$

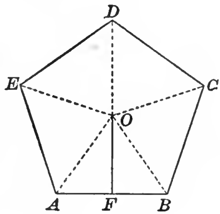
Therefore, etc. Q.E.D.

Ex. **644**. The apothem of a regular pentagon is 41^{cm} and a side is 6^{dm}. Find the perimeter of a regular pentagon whose apothem is 82^{cm}.

Proposition VII

388. Draw a regular polygon and draw the radii to the vertices of its angles. How does each triangle thus formed compare with the rectangle of its base and altitude? How does the sum of the bases of the triangles compare with the perimeter of the polygon? Since the triangles are of equal altitude, how does the polygon compare with the rectangle of its perimeter and apothem?

Theorem. *A regular polygon is equivalent to one half the rectangle formed by its perimeter and apothem.*



Data: Any regular polygon, as $ABCDE$, and its apothem OF.

Denote its perimeter by P.

To prove $ABCDE \Leftrightarrow \frac{1}{2}$ rect. $P \cdot OF$.

Proof. Draw the radii OA, OB, OC, OD, and OE.

These radii divide the polygon into triangles whose altitudes are each equal to the apothem and the sum of whose bases is equal to the perimeter.

Now, § 334, $\triangle ABO \Leftrightarrow \frac{1}{2}$ rect. $AB \cdot OF$,

$\triangle BCO \Leftrightarrow \frac{1}{2}$ rect. $BC \cdot OF$, etc.

Hence, $\triangle ABO + \triangle BCO +$ etc. $\Leftrightarrow \frac{1}{2}$ rect. $(AB + BC +$ etc.$) \cdot OF$;

that is, $ABCDE \Leftrightarrow \frac{1}{2}$ rect. $P \cdot OF$.

Therefore, etc. Q.E.D.

389. Cor. I. *The area of a regular polygon is equal to one half the product of its perimeter by its apothem.*

390. Cor. II. *Regular polygons of the same number of sides are to each other as the squares upon their radii and also as the squares upon their apothems.* §§ 386, 345

Ex. **645.** The sides of a regular circumscribed polygon are bisected at the points of tangency.

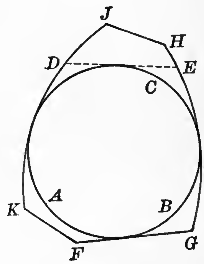
Ex. **646.** The angle at the center of a regular polygon is the supplement of the angle of the polygon.

Proposition VIII

391. Draw a circle and circumscribe a polygon about it. How does the circumference of the circle compare in length with the perimeter of the polygon? How does the circumference compare with any enveloping line?

Theorem. *The circumference of a circle is less than the perimeter of a circumscribed polygon or any enveloping line.*

Data: Any circumference, as ABC, and any enveloping line, as $FGHJK$.



To prove $ABC < FGHJK$.

Proof. Of all the lines enveloping the area ABC there must be a least line.

Draw DE tangent to ABC, and cutting $FGHJK$ in D and E.

Then, Ax. 10, $ED < EHJD$;

$\therefore$ $FGEDK < FGHJK$;

hence, $FGHJK$ is not the least enveloping line.

Similarly, it may be shown that no other line than ABC can be the least line enveloping the area ABC.

Hence, $ABC < FGHJK$.

Therefore, etc. Q.E.D.

Ex. **647.** Find the angle and the angle at the center of a regular dodecagon.

Ex. **648.** If the radius of a regular inscribed hexagon is r, prove that its side $= r$, and its apothem $= \frac{1}{2} r\sqrt{3}$.

Ex. **649.** If the radius of an inscribed equilateral triangle is r, prove that its side $= r\sqrt{3}$, and its apothem $= \frac{1}{2} r$.

Ex. **650.** If the radius of an inscribed square is r, prove that its side $= r\sqrt{2}$, and its apothem $= \frac{1}{2} r\sqrt{2}$.

Ex. **651.** The radius of a circle being r, find the area of an inscribed equilateral triangle.

Ex. **652.** The radius of a circle being r, find the area of an inscribed square.

Ex. **653.** Find the area of a regular hexagon whose side is 10 ft.

Proposition IX

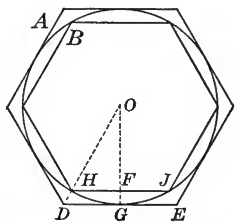
392. Draw a circle and inscribe in it a regular polygon; circumscribe about it a similar polygon whose sides are parallel to the sides of the inscribed polygon. If the number of sides of each polygon is increased indefinitely, what line will their perimeters approach as a limit? What will their areas approach as a limit?

Theorem. *If a regular polygon is circumscribed about a circle and a similar polygon is inscribed in the circle, and if the number of their sides is indefinitely increased,*

1. Their perimeters approach the circumference as a common limit.

2. The polygons approach the circle as a common limit.

Data: Any two regular polygons of the same number of sides, as A and B, respectively circumscribed about and inscribed in a circle whose center is O.



Denote the circle by S, its circumference by C, and the perimeters of the polygons by P and Q respectively.

To prove that, if the number of sides of the polygons is indefinitely increased,

1. P and Q approach C as their common limit.
2. A and B approach S as their common limit.

Proof. 1. Place the polygons so that their sides are parallel; draw a line from O to G, the point of contact of the side DE of the circumscribed polygon and draw OD which by § 385 will pass through the extremity H of the side HJ of the inscribed polygon.

Since the polygons have the same number of sides,

§ 387, $$P : Q = OD : OG;$$

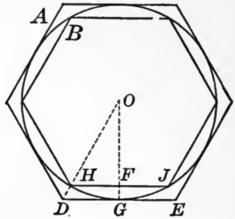
∴ § 277, $$P - Q : Q = OD - OG : OG,$$

and, § 269, $$OG \times (P - Q) = Q \times (OD - OG);$$

whence, $$P - Q = \frac{Q}{OG} \times (OD - OG). \quad (1)$$

Now, Ax. 10, $OD < OG + DG$, or $OD - OG < DG$. (2)

But, if the number of sides of each polygon is increased indefinitely, the two polygons continuing to have the same number of sides, the length of each side decreases indefinitely and approaches the limit 0; therefore DG, which is half the side DE, approaches the limit 0, and in (2), $OD - OG$ approaches the limit 0.



Hence, from (1), $P - Q$ approaches the limit 0.

Since, § 391, P is always greater than C,
and, Ax. 10, Q is always less than C,
the difference between C and either P or Q is less than $P - Q$; therefore, the difference approaches the limit 0.

Consequently, P and Q approach C as their common limit.

2. Since the polygons are regular and similar,

§ 390, $$A : B = \overline{OD}^2 : \overline{OG}^2;$$

$\therefore$ $$A - B : B = \overline{OD}^2 - \overline{OG}^2 : \overline{OG}^2.$$ Why?

Now $$\overline{OD}^2 - \overline{OG}^2 = \overline{DG}^2;$$ Why?

$\therefore$ $$A - B : B = \overline{DG}^2 : \overline{OG}^2,$$

whence, $$A - B = \frac{B}{\overline{OG}^2} \times \overline{DG}^2. \qquad (3)$$

But DG, which is half of the side DE, approaches the limit 0.

Hence, from (3), $A - B$ approaches the limit 0.

Since, Ax. 8, A is always greater than S,
and B is always less than S,
the difference between S and either A or B is less than $A - B$; therefore, the difference approaches the limit 0.

Consequently, A and B approach S as their common limit.

Therefore, etc. Q.E.D.

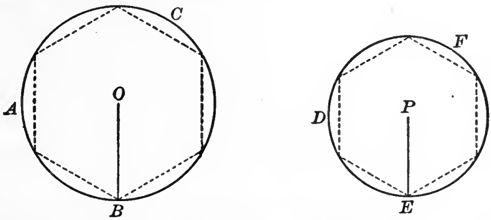
393. Sch. It is evident that this is a special case of the theorem, which may be proved, *that the perimeters of polygons inscribed in and circumscribed about a closed convex curve,* when the number of their sides is indefinitely increased, approach the curve as a limit, and the polygons approach the figure bounded by the curve as a limit.*

* A convex curve is a curve which a straight line can cut in only two points.

Proposition X

394. Draw two circles and inscribe in them regular polygons of the same number of sides. How does the ratio of their perimeters compare with the ratio of their radii? How does the ratio of the circumferences compare with the ratio of *their* radii? Of their diameters?

Theorem. *Circumferences are to each other as their radii.*



Data: Any two circumferences, as ABC and DEF, whose radii are OB and PE, respectively.

To prove $ABC : DEF = OB : PE.$

Proof. In ABC and DEF inscribe regular polygons of the same number of sides, and denote their perimeters by Q and S, respectively.

Then, § 387, $Q : S = OB : PE.$

If the number of sides of the polygons is indefinitely increased, the polygons still remaining regular and similar, § 392, Q and S approach ABC and DEF, respectively, as their limits.

Hence, $ABC : DEF = OB : PE.$ Q.E.D.

395. Cor. *The ratio of the circumference of a circle to its diameter is constant.*

The ratio of the circumference of a circle to its diameter is represented by the Greek letter π whose approximate value, as shown in Ex. 698, is 3.1416.

If the circumference of a circle is denoted by C, its diameter by D, and its radius by R,

$$\pi = \frac{C}{D},$$

$$\therefore\ C = \pi D, \text{ or } 2\,\pi R;$$

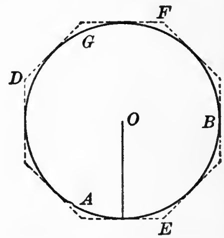
that is, *the circumference of a circle is equal to π times the diameter or $2\,\pi$ times the radius.*

396. *Similar arcs, similar sectors,* and *similar segments* are those which correspond to equal central angles.

Proposition XI

397. Draw a circle and circumscribe about it any regular polygon. How does the apothem of the polygon compare with the radius of the circle? If the number of sides of the polygon is indefinitely increased, how does the limit of its perimeter compare with the circumference of the circle? How, then, does the polygon at its limit compare with the circle? Since the polygon is equivalent to one half the rectangle of its perimeter and apothem, how does the circle compare with the rectangle of its circumference and radius?

Theorem. *A circle is equivalent to one half the rectangle formed by its circumference and radius.*



Data: Any circle, as ABG, whose center is O.

Denote its circumference by C and its radius by R.

To prove $ABG \approx \frac{1}{2}$ rect. $C \cdot R$.

Proof. Circumscribe about the circle any regular polygon, as DEF, and denote its perimeter by P.

Then, the apothem of the polygon is equal to R,

and, § 388, $DEF \approx \frac{1}{2}$ rect. $P \cdot R$.

Now, if the number of sides of the polygon is indefinitely increased,

§ 392, P indefinitely approaches C as its limit,

and DEF indefinitely approaches ABG as its limit.

But, however great the number of sides,

$DEF \approx \frac{1}{2}$ rect. $P \cdot R$.

Hence, § 326, $ABG \approx \frac{1}{2}$ rect. $C \cdot R$.

Therefore, etc. Q.E.D.

Arithmetical Rule: To be framed by the student. § 339

398. Cor. I. *The area of a circle is equal to π times the square of its radius.*

§ 395, $$C = 2\pi R;$$

$$\therefore \text{ Area} = \tfrac{1}{2}(2\pi R \times R), \text{ or } \pi R^2.$$

399. Cor. II. *The areas of circles are to each other as the squares of their radii.*

400. Cor. III. *The area of a sector is equal to one half the product of its arc by its radius.*

401. Cor. IV. *Similar sectors are to each other as the squares of their radii.*

Ex. **654.** If the circumferences of two circles are 314.16^{cm} and 157.08^{cm} respectively, and the radius of the first is 50^{cm}, what is the radius of the second? What is the area of the first?

Ex. **655.** Find the circumference and area of a circle whose radius is 2.5^{dm}.

Ex. **656.** What is the ratio of the radii of two circles, if the area of one circle is twice that of the other?

Ex. **657.** What is the area of a sector whose arc is $\frac{1}{6}$ of the circumference, if the radius of the circle is 18^{dm}?

Ex. **658.** If the radius of a circle is 63 in., how long is the arc of a sector whose angle is 45°?

Ex. **659.** Calling the equatorial radius of the earth 3962.8 miles, what is the length of a degree on the equator?

Ex. **660.** Find the radius of a circle whose area is $6^{sq\,m}$.

Ex. **661.** Find the circumference of a circle whose area is 100 sq. in.

Ex. **662.** What is the area of a circle whose circumference is 100 ft.?

Ex. **663.** What is the area of a circle circumscribed about a square whose side is a?

Ex. **664.** The diagonals drawn from a vertex of a regular pentagon to the opposite vertices trisect the angle at that vertex.

Ex. **665.** If the chord of an arc is 72^{dm} and the chord of half that arc is 36.9^{dm}, what is the diameter of the circle?

Ex. **666.** The chord of an arc is 24 in. and its altitude is 9 in. What is the diameter of the circle?

Ex. **667.** The chord of half an arc is 12^{m} and the radius of the circle is 18^{m}. What is the altitude of the arc?

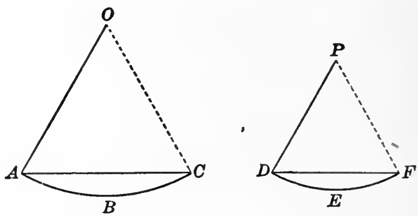
Ex. **668.** The altitude of an equilateral triangle is equal to one and a half times the radius of the circumscribed circle.

Ex. **669.** Find the central angle subtended by an arc whose length is equal to the radius of the circle.

Proposition XII

402. Draw two similar segments and their radii. What kind of sectors and what kind of triangles are thus formed? How does the difference in the area of each sector and the corresponding triangle compare with the area of the corresponding segment? Since the ratio of the sectors and also the ratio of the triangles equals the ratio of the squares of the radii, how does the ratio of the segments compare with the ratio of the same squares?

Theorem. *Similar segments are to each other as the squares upon their radii.*



Data: Any two similar segments, as ABC and DEF, whose radii are AO and DP respectively.

To prove

seg. ABC : seg. $DEF = \overline{AO}^2 : \overline{DP}^2$.

Proof. Draw OC and PF.

In the $\triangle$s ACO and DFP, $\angle O = \angle P$, Why?

and $AO : DP = CO : FP$; Why?

$\therefore$ $\triangle$s ACO and DFP are similar; Why?

hence, § 342, $\triangle ACO : \triangle DFP = \overline{AO}^2 : \overline{DP}^2$.

But, § 401, sect. $ABCO$: sect. $DEFP = \overline{AO}^2 : \overline{DP}^2$;

$\therefore$ sect. $ABCO$: sect. $DEFP = \triangle ACO : \triangle DFP$,

and sect. $ABCO : \triangle ACO =$ sect. $DEFP : \triangle DFP$. Why?

Hence, § 277,

sect. $ABCO - \triangle ACO : \triangle ACO =$ sect. $DEFP - \triangle DFP : \triangle DFP$;

that is, seg. $ABC : \triangle ACO =$ seg. $DEF : \triangle DFP$,

or seg. ABC : seg. $DEF = \triangle ACO : \triangle DFP$. Why?

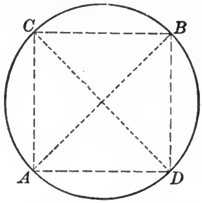
But $\triangle ACO : \triangle DFP = \overline{AO}^2 : \overline{DP}^2$.

Hence, seg. ABC : seg. $DEF = \overline{AO}^2 : \overline{DP}^2$. Why?

Therefore, etc. Q.E.D.

Proposition XIII

403. *Problem.* *To inscribe a square in a circle.*



Datum: Any circle.

Required to inscribe a square in the circle.

Solution. Draw any two diameters at right angles to each other, as *AB* and *CD*.

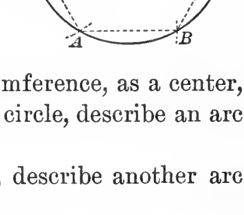
Draw *AD*, *DB*, *BC*, and *CA*.

Then, *ADBC* is the required square. Q.E.F.

Proof. By the student.

Proposition XIV

404. *Problem.* *To inscribe a regular hexagon in a circle.*



Datum: Any circle.

Required to inscribe a regular hexagon in the circle.

Solution. From *A*, any point in the circumference, as a center, and with a radius equal to the radius of the circle, describe an arc intersecting the circumference as at *B*.

From *B* as a center with the same radius, describe another arc intersecting the circumference as at *C*.

In like manner determine the points *D*, *E*, and *F*.

Draw chords connecting these points in succession.

Then, *ABCDEF* is the required hexagon. Q.E.F

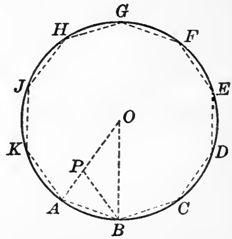
Proof. By the student.

Ex. **670.** To circumscribe an equilateral triangle about a circle.

Ex. **671.** To circumscribe a square about a circle.

Proposition XV

405. *Problem.* *To inscribe a regular decagon in a circle.*



Datum: Any circle.

Required to inscribe a regular decagon in the circle.

Solution. Draw any radius, as OA, and divide it in extreme and mean ratio as at P; that is, so that $AO : PO = PO : AP$.

From A as a center and with PO as a radius, describe an arc intersecting the circumference as at B. Draw AB.

From B as a center with the same radius, or AB, describe an arc intersecting the circumference as at C.

In like manner determine the points D, E, F, G, H, J, and K.

Draw chords connecting these points in succession.

Then, $ABCD\text{–}K$ is the required decagon. Q.E.F.

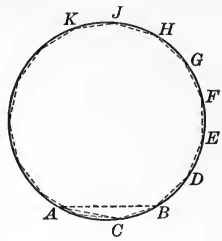
Proof. Draw BP and BO.

Const.,	$AO : PO = PO : AP$,	
and	$AB = PO$;	
∴ in ⨞ ABO and ABP,	$AO : AB = AB : AP$,	
and	$\angle A$ is common;	
∴	⨞ ABP and ABO are similar.	Why?
Now,	$\triangle ABO$ is isosceles;	
∴	$\triangle ABP$ is isosceles,	
and	$AB = BP = PO$;	Why?
hence,	$\triangle BPO$ is isosceles.	
But	$\angle APB = \angle O + \angle PBO = 2 \angle O$.	Why?
Then,	$\angle A + \angle ABP + \angle APB = 5 \angle O$,	
and	$\angle O = \frac{1}{5}$ of 2 rt. ∠s, or $\frac{1}{10}$ of 4 rt. ∠s;	
∴	arc AB is $\frac{1}{10}$ of the circumference,	

and the chord AB, which subtends the arc AB, is a side of the regular inscribed decagon $ABCD\text{–}K$.

Proposition XVI

406. *Problem.* *To inscribe a regular pentadecagon in a circle.*



Datum: Any circle.

Required to inscribe a regular pentadecagon in the circle.

Solution. Draw the chord AB equal to a side of the regular inscribed hexagon, and from A draw the chord AC equal to a side of the regular inscribed decagon. Draw CB.

From B as a center with CB as a radius, describe an arc intersecting the circumference as at D.

In like manner determine the points E, F, G, H, etc.

Draw chords connecting these points in succession.

Then, $CBDEF$ etc., is the required pentadecagon. Q.E.F.

Proof. Arc $AB = \frac{1}{6}$ of the circum.,

and arc $AC = \frac{1}{10}$ of the circum.;

$\therefore$ arc $BC =$ arc $AB -$ arc $AC = \frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$ of the circum.,

and the chord CB, which subtends the arc CB, is a side of the regular inscribed pentadecagon $CBDEF$ etc.

407. Cor. I. *By joining the alternate vertices of any regular inscribed polygon of an even number of sides, a regular polygon of half the number of sides is inscribed.*

408. Cor. II. *By joining to the vertices of any regular inscribed polygon the middle points of the arcs subtended by its sides, a regular polygon of double the number of sides is inscribed.*

Ex. **672.** To inscribe a regular octagon in a circle.
Ex. **673.** To inscribe a regular dodecagon in a circle.
Ex. **674.** To circumscribe a regular hexagon about a circle.
Ex. **675.** To circumscribe a regular octagon about a circle.
Ex. **676.** To inscribe a regular hexagon in an equilateral triangle.

Ex. **677.** To divide an angle of an equilateral triangle into five equal parts.

Ex. **678.** The segment of a circle is equal to $\frac{5}{3}$ of a similar segment. What is the ratio of their radii?

Ex. **679.** How many degrees are there in an arc 18 in. long on a circumference whose radius is 5 ft.?

Ex. **680.** The radii of two similar segments are as $3:5$. What is the ratio of their areas?

Ex. **681.** In a circle 3 ft. in diameter an equilateral triangle is inscribed. What is the area of a segment without the triangle?

Ex. **682.** Two chords drawn from the same point in a circumference to the extremities of a diameter of a circle are 6 in. and 8 in. respectively. What is the area of the circle?

MAXIMA AND MINIMA

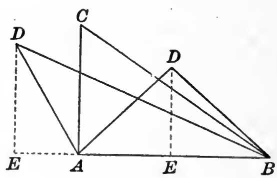
409. Of any number of magnitudes of the same kind the greatest is called the **Maximum,** and the least is called the **Minimum.**

Of all chords of any circle the diameter is the *maximum;* and of all lines from any point to a given line the perpendicular is the *minimum.*

410. Figures which have equal perimeters are called **Isoperimetric.**

Proposition XVII

411. ***Theorem.*** *Of all triangles having two given sides, that in which these sides are perpendicular to each other is the maximum.*



Data: Any two triangles, as ABC and ABD, such that $AC = AD$, AB is common, and AC and AB are perpendicular to each other.

To prove $\triangle ABC$ the maximum.

Proof. Draw $ED \perp AB$.

Area of $\triangle ABC = \frac{1}{2} AC \times AB$,

and area of $\triangle ABD = \frac{1}{2} ED \times AB$. Why?

But $AD > ED$, and $AC = AD$; Why?

$\therefore$ $AC > ED$,

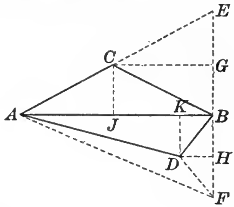
and $\frac{1}{2} AC \times AB > \frac{1}{2} ED \times AB$. Why?

Hence, $\triangle ABC$ is the maximum.

Therefore, etc. Q.E.D.

Proposition XVIII

412. *Theorem.* *Of all isoperimetric triangles which have the same base, the isosceles triangle is the maximum.*



Data: Any two isoperimetric triangles upon the same base AB, as ABC and ABD, of which ABC is isosceles.

To prove $\triangle ABC$ the maximum.

Proof. Produce AC to E, making CE equal to AC. Draw EB. Since C is equidistant from A, B, and E, $\angle ABE$ may be inscribed in a semicircumference;

$\therefore$ $\angle ABE$ is a right angle.

Draw DF equal to DB meeting EB produced in F; CG and DH parallel to AB; CJ and DK perpendicular to AB; and draw AF.

Then, $AE = AC + BC = AD + BD = AD + FD$. Why?

But $AD + FD > AF$; Why?

$\therefore$ $AE > AF$;

hence, § 133, $EB > FB$.

But $GB = \frac{1}{2} EB$, and $HB = \frac{1}{2} FB$; Why?

$\therefore$ $GB > HB$.

Also, $GB = CJ$, and $HB = DK$, the altitudes of $\triangle ABC$ and ABD, respectively;

$\therefore$ $CJ > DK$.

Now, area of $\triangle ABC = \frac{1}{2} AB \times CJ$,

and area of $\triangle ABD = \frac{1}{2} AB \times DK$; Why?

$\therefore$ area of $\triangle ABC >$ area of $\triangle ABD$;

that is, $\triangle ABC$ is the maximum.

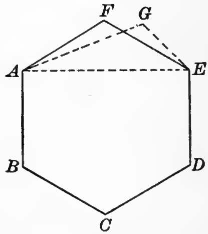
Therefore, etc. Q.E.D.

413. Cor. *Of all isoperimetric triangles, the equilateral triangle is the maximum.*

Ex. 683. Of all equivalent parallelograms having equal bases, the rectangle has the least perimeter.

Proposition XIX

414. Theorem. *Of isoperimetric polygons which have the same number of sides, the maximum is equilateral.*



Data: The maximum of isoperimetric polygons of a given number of sides, as *ABCDEF*.

To prove *ABCDEF* equilateral.

Proof. Draw *AE*.

Then, △ *AEF* must be the maximum of all the △s that can be formed upon *AE* with a perimeter equal to that of △ *AEF*, for if not, a greater △, as *AEG*, could be substituted for △ *AEF* without changing the perimeter of *ABCDEF*.

But it would be impossible to enlarge *ABCDEF*, for, data, it is the maximum.

Hence, § 412, △ *AEF* is isosceles,

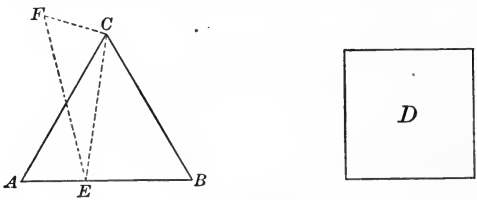
and $AF = EF$.

Similarly any two consecutive sides may be shown equal.

Hence, *ABCDEF* is equilateral. Q.E.D.

Proposition XX

415. Theorem. *Of isoperimetric regular polygons, that which has the greatest number of sides is the maximum.*



Data: Any two isoperimetric regular polygons, as *ABC* and *D*, of which *D* has one side more than *ABC*.

To prove *D* the maximum.

Proof. To *E*, any point in *AB*, draw *CE* and construct the △ *CEF* equal to the △ *ACE*.

Then, $EBCF \approx ABC$,

and $EBCF$ and D are isoperimetric.

But, § 414, $D > EBCF$;

$\therefore$ $D > ABC$;

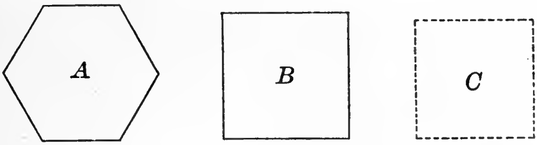
that is, D is the maximum.

Therefore, etc. Q.E.D.

416. Cor. *The area of a circle is greater than the area of any isoperimetric polygon.*

Proposition XXI

417. *Theorem.* *Of regular polygons which have equal areas, that which has the greatest number of sides has the least perimeter.*



Data: Any regular polygons which have equal areas, as A and B, of which A has a greater number of sides than B.

To prove the perimeter of A less than the perimeter of B.

Proof. Construct the regular polygon C, having its perimeter equal to that of A and having the same number of sides as B.

Then, § 415, $A > C$.

But, data, $A \approx B$;

$\therefore$ $B > C$,

and, § 346, the perimeter of B is greater than that of C.

But the perimeter of C is equal to that of A;

$\therefore$ the perimeter of B is greater than that of A;

that is, the perimeter of A is less than the perimeter of B.

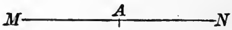
Therefore, etc. Q.E.D.

418. Cor. *The circumference of a circle is less than the perimeter of any polygon which has an equal area.*

Ex. **684.** Of all rectangles of a given area, the square has the least perimeter.

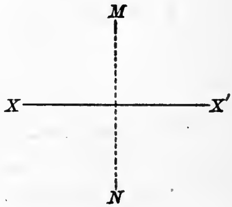
SYMMETRY

419. If a point bisects the straight line joining two other points, the two points are said to be *symmetrical with respect to a point,* and this point is called the *center of symmetry.*



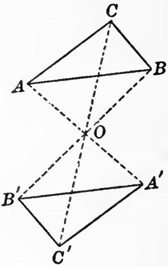
M and N are *symmetrical* with respect to the *center* A, if A bisects the straight line MN.

420. If a straight line is the perpendicular bisector of the straight line joining two points, the points are said to be *symmetrical with respect to a straight line,* and this line is called the *axis of symmetry.*



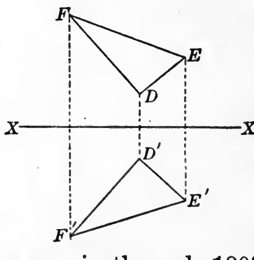
M and N are *symmetrical* with respect to the *axis* XX', if XX' is the perpendicular bisector of the straight line MN.

421. If every point of one figure has a corresponding symmetrical point in another, the two figures are said to be *symmetrical with respect to a center* or *an axis.*



If every point in the figure ABC has a symmetrical point in $A'B'C'$ with respect to O as a center, then, the figures ABC and $A'B'C'$ are *symmetrical* with respect to the *center* O.

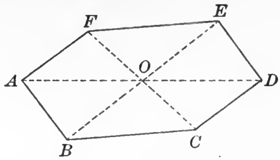
If every point in the figure DEF has a symmetrical point in $D'E'F'$ with respect to XX' as an axis. then, the figures DEF and $D'E'F'$ are *symmetrical* with respect to the *axis* XX'.



Two plane figures that are symmetrical with respect to an axis can be applied one to the other by revolving either one about the axis; consequently they are equal, and if two figures can be made to coincide by revolving one of them about an axis through 180°, they are symmetrical with respect to the axis.

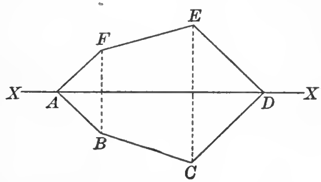
422. If a point bisects every straight line drawn through it and terminated in the boundary of a figure, the figure is said to be *symmetrical with respect to a point.*

If O bisects every straight line drawn through it and terminated by the boundary of $ABCDEF$, then, $ABCDEF$ is *symmetrical* with respect to the *point* O.



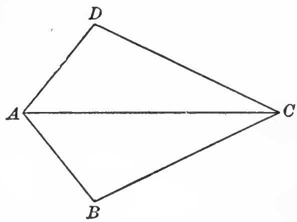
423. If a straight line divides a plane figure into two parts which are symmetrical with respect to the line, the figure is said to be *symmetrical with respect to a straight line.*

If the parts $ABCD$ and $AFED$ are symmetrical with respect to XX', then, the figure $ABCDEF$ is *symmetrical* with respect to the *straight line* XX'.



Proposition XXII

424. *Theorem.* *A quadrilateral which has two adjacent sides equal and the other two sides equal, is symmetrical with respect to the diagonal joining the vertices of the angles formed by the equal sides.*



Data: A quadrilateral, as $ABCD$, having $AB = AD$, $CB = CD$, and the diagonal AC.

To prove $ABCD$ symmetrical with respect to AC.

Proof. In the $\triangle$s ABC and ADC,

data, $AB = AD$, $CB = CD$,

and AC is common;

$\therefore$ $\triangle ABC = \triangle ADC$, Why?

$\angle BAC = \angle DAC$, and $\angle BCA = \angle DCA$. Why?

Hence, if ADC is turned on AC as an axis, it may be made to coincide with ABC.

$\therefore$ § 421, ADC and ABC are symmetrical with respect to AC;

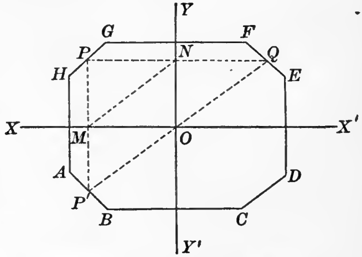
that is, § 423, $ABCD$ is symmetrical with respect to AC.

Therefore, etc. Q.E.D.

Proposition XXIII

425. Theorem. *If a figure is symmetrical with respect to two axes perpendicular to each other, it is symmetrical with respect to their intersection as a center.*

Data: A figure, as $ABCD-H$, symmetrical with respect to the two perpendicular axes, XX' and YY', which intersect at O.



To prove $ABCD-H$ symmetrical with respect to O as a center.

Proof. From any point in the perimeter, as P, draw $PMP' \perp XX'$ and $PNQ \perp YY'$.

Draw MN, $P'O$, and OQ.

Now, § 420, $PM = P'M$,

and $PM = ON$; Why?

$\therefore$ $P'M = ON$,

and, § 71, $P'M \parallel ON$;

consequently, § 150, $MP'ON$ is a parallelogram;

$\therefore$ $P'O$ is equal and parallel to MN.

Similarly, OQ is equal and parallel to MN.

Hence, points P', O, Q are in the same straight line $P'OQ$, which is bisected at O. Why?

But since P is any point in the perimeter, $P'OQ$ is any straight line drawn through O.

Hence, § 422, $ABCD-H$ is symmetrical with respect to O as a center.

Therefore, etc. Q.E.D.

Ex. **685**. A segment of a circle is symmetrical with respect to the perpendicular bisector of its chord as an axis.

Ex. **686**. A circle is symmetrical with respect to its center or with respect to any diameter as an axis.

Ex. **687**. A parallelogram is symmetrical with respect to the point of intersection of its diagonals as a center.

SUMMARY

426. Truths established in Book VI.

1. Two lines are equal,

a. If they are sides of a regular polygon. § 374

2. Lines are in proportion,

a. If they are the perimeters of regular polygons of the same number of sides, and their radii. § 387

b. If they are the perimeters of regular polygons of the same number of sides, and their apothems. § 387

c. If they are circumferences and their radii. § 394

3. Two angles are equal,

a. If they are angles of a regular polygon. § 374

4. An angle is bisected,

a. If it is an interior angle of a regular polygon, by the radius drawn to its vertex. § 382

5. A polygon is regular,

a. If it is equilateral and equiangular. § 374

b. If it is equilateral and inscribed in a circle. § 375

c. If it is formed by chords joining the extremities of arcs which are equal divisions of the circumference of a circle. § 376

d. If it is formed by tangents drawn at the extremities of arcs which are equal divisions of the circumference of a circle. § 376

e. If it is formed by tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon. § 384

6. Polygons are similar,

a. If they are regular and have the same number of sides. § 386

7. A regular polygon is equivalent,

a. To half the rectangle formed by its perimeter and apothem. § 388

8. A circle is equivalent,

a. To half the rectangle formed by its circumference and radius. § 397

9. A circumference is the limit,

a. Of the perimeter of a regular inscribed polygon when the number of its sides is indefinitely increased. § 392

b. Of the perimeter of a regular circumscribed polygon when the number of its sides is indefinitely increased. § 392

10. A circle is the limit,

a. Of a regular inscribed polygon when the number of its sides is indefinitely increased. § 392

b. Of a regular circumscribed polygon when the number of its sides is indefinitely increased. § 392

11. Figures are in proportion,

a. If they are regular polygons of the same number of sides, to the squares upon their radii. § 390

b. If they are regular polygons of the same number of sides, to the squares upon their apothems. § 390

c. If they are circles, to the squares of their radii. § 399

d. If they are similar sectors, to the squares of their radii. § 401

e. If they are similar segments, to the squares of their radii. § 402

12. The area of a figure is equal,

a. If it is a regular polygon, to one half the product of its perimeter by its apothem. § 389

b. If it is a circle, to one half the product of its circumference by its radius. § 397

c. If it is a circle, to π times the square of its radius. § 398

d. If it is a sector, to one half the product of its arc by its radius. § 400

SUPPLEMENTARY EXERCISES

Ex. **688**. If the perimeter of each of the figures, equilateral triangle, square, and circle is 396 ft., what is the area of each figure?

Ex. **689**. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

Ex. **690**. If an equilateral triangle is inscribed in a circle, any side will cut off one fourth of the diameter from the opposite vertex.

Ex. **691**. The square inscribed in a circle is equivalent to one half the square circumscribed about that circle.

Ex. **692**. A circle is inscribed in a square whose side is 4 in. How much of the area of the square is without the circle?

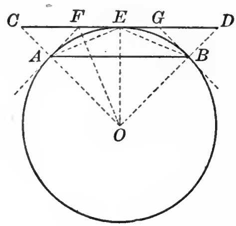
Ex. **693**. What is the width of the ring between the circumferences of two concentric circles whose circumferences are 48 ft. and 36 ft. respectively?

Ex. **694**. Of all squares that can be inscribed in a given square, the minimum has its vertices at the middle points of the sides.

Ex. **695**. Every equiangular polygon circumscribed about a circle is regular.

Ex. **696**. In any regular polygon of an even number of sides, the lines joining opposite vertices are diameters of the circumscribing circle.

Ex. **697**. *Given the side of a regular inscribed polygon and the side of a similar circumscribed polygon, to compute the perimeters of the regular inscribed and circumscribed polygons of double the number of sides.*



Data: AB, the side of a regular inscribed polygon, and CD, the side of a similar circumscribed polygon, tangent to the arc AB at its middle point E.

Denote the perimeters of these polygons by P and Q respectively, and the number of sides in each by n; denote the perimeters of the inscribed and circumscribed polygons which have $2n$ sides by S and T respectively.

Required to compute the value of S and of T.

Solution. Through A and B draw tangents to meet CD in F and G respectively; also draw AE and BE.

Then, § 376, AE and FG are sides of the polygons whose perimeters are S and T respectively.

$$AB = \frac{P}{n},\ CD = \frac{Q}{n},\ AE = \frac{S}{2n},\ \text{and}\ FG = \frac{T}{2n}.$$

Draw the radii CO, FO, EO, and DO.

Since, § 385, A lies in CO,

by Ex. 221, FO bisects $\angle AOE$, or $\angle COE$;

$\therefore$ § 292, $EF : CF = EO : CO$;

but $P : Q = EO : CO$;

$\therefore$ $P : Q = EF : CF$,

and $P + Q : P = EF + CF : EF = CE : EF$.

But $CE = \frac{1}{2}CD = \frac{Q}{2n}$, and $EF = \frac{1}{2}FG = \frac{T}{4n}$;

$\therefore$ substituting, $P + Q : P = \frac{Q}{2n} : \frac{T}{4n}$,

or $P + Q : P = 2Q : T$;

whence, $T = \frac{2Q \times P}{Q + P}$.

Again, in the isosceles $\triangle ABE$ and AEF,

$$\angle ABE = \angle AEF;$$

$\therefore$ § 301, $\triangle ABE$ and AEF are similar,

and $AE : AB = EF : AE$;

hence, $\overline{AE}^2 = AB \times EF$,

and substituting for AE, AB, and EF their values,

$$\frac{S^2}{4n^2} = \frac{P}{n} \times \frac{T}{4n};$$

whence, $S^2 = P \times T$,

and $S = \sqrt{P \times T}$.

Ex. **698**. *To compute the approximate ratio of a circumference to its diameter.*

Solution. If the diameter of a circle is 1, the side of a circumscribed square is 1, and its perimeter is 4; the side of an inscribed square is $\frac{1}{2}\sqrt{2}$, and its perimeter is $2\sqrt{2}$, or 2.82843.

Thus, $Q = 4$, and $P = 2\sqrt{2}$ for computing the octagon.

Substituting these values in the formulæ, $T = \frac{2Q \times P}{Q + P}$, $S = \sqrt{P \times T}$ (Ex. 697), and solving, the results tabulated below are found, showing the perimeters to five decimal places.

No. of Sides	Computation of T	T	Computation of S	S
	$T = \frac{2Q \times P}{Q + P}$		$S = \sqrt{P \times T}$	
8	$\frac{2 \times 4 \times 2.82843}{4 + 2.82843}$	3.31371	$\sqrt{2.82843 \times 3.31371}$	3.06147
16	$\frac{2 \times 3.31371 \times 3.06147}{3.31371 + 3.06147}$	3.18260	$\sqrt{3.06147 \times 3.18260}$	3.12145
32	$\frac{2 \times 3.18260 \times 3.12145}{3.18260 + 3.12145}$	3.15172	$\sqrt{3.12145 \times 3.15172}$	3.13655
64	$\frac{2 \times 3.15172 \times 3.13655}{3.15172 + 3.13655}$	3.14412	$\sqrt{3.13655 \times 3.14412}$	3.14033
128	$\frac{2 \times 3.14412 \times 3.14033}{3.14412 + 3.14033}$	3.14222	$\sqrt{3.14033 \times 3.14222}$	3.14128
256	$\frac{2 \times 3.14222 \times 3.14128}{3.14222 + 3.14128}$	3.14175	$\sqrt{3.14128 \times 3.14175}$	3.14151
512	$\frac{2 \times 3.14175 \times 3.14151}{3.14175 + 3.14151}$	3.14163	$\sqrt{3.14151 \times 3.14163}$	3.14157
1024	$\frac{2 \times 3.14163 \times 3.14157}{3.14163 + 3.14157}$	3.14159	$\sqrt{3.14157 \times 3.14159}$	3.14159

The results of the last two computations show that the circumference of a circle whose diameter is 1 is approximately 3.1416; that is, the ratio of the diameter of a circle to its circumference is equal to the ratio of 1 to 3.1416, approximately.

Ex. **699**. The sides of an inscribed rectangle are 30^{cm} and 40^{cm}. What is the area of the part of the circle without the rectangle?

Ex. **700**. What is the area of a figure bounded by four semicircumferences described on the sides of a three foot square?

Ex. **701**. A square piece of land and a circular piece of land each contain one acre. Which perimeter is the greater, and how much?

Ex. 702. The area of an inscribed equilateral triangle is one half the area of a regular hexagon inscribed in the same circle.

Ex. **703.** Of all triangles that have the same vertical angle and whose bases pass through a given point, the minimum is the one whose base is bisected at that point.

Ex. **704.** An arc of a circle whose radius is 6 ft. subtends a central angle of 20°; an equal arc of another circle subtends a central angle of 30°. What is the radius of the second circle?

Ex. **705.** Two tangents make with each other an angle of 60°, and the radius of the circle is 7 in. What are the lengths of the arcs between the points of contact?

Ex. **706.** If the apothem of a regular hexagon is 10^{m}, what is the area of the ring between the circumferences of its inscribed and circumscribed circles?

Ex. **707.** If a circle 18^{cm} in diameter is divided into three equivalent parts by two concentric circumferences, what are their radii?

Ex. **708.** The square upon the side of a regular inscribed pentagon is equivalent to the sum of the squares upon the radius of the circle and the side of a regular inscribed decagon.

Ex. **709.** The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of the similar circumscribed polygon.

Ex. **710.** If the radius of a regular inscribed octagon is r, prove that its side $= r\sqrt{2-\sqrt{2}}$, and its apothem $= \frac{r}{2}\sqrt{2+\sqrt{2}}$.

Ex. **711.** If the radius of a regular inscribed decagon is r, prove that its side $= \frac{r}{2}(\sqrt{5}-1)$, and its apothem $= \frac{r}{4}\sqrt{10+2\sqrt{5}}$.

Ex. **712.** If the radius of a regular inscribed dodecagon is r, prove that its side $= r\sqrt{2-\sqrt{3}}$, and its apothem $= \frac{r}{2}\sqrt{2+\sqrt{3}}$.

Ex. **713.** If the radius of a regular inscribed pentagon is r, prove that its side $= \frac{r}{2}\sqrt{10-2\sqrt{5}}$, and its apothem $= \frac{r}{4}\sqrt{6+2\sqrt{5}}$.

Ex. **714.** The square upon a side of an inscribed equilateral triangle is equivalent to three times the square upon the side of a regular inscribed hexagon.

Ex. **715.** The area of an inscribed square is $16^{sq\,m}$. Find the length of a side of a regular inscribed octagon.

Ex. **716.** If the radius of a circle is r, prove that a side of a regular circumscribed hexagon is $\frac{2r}{3}\sqrt{3}$.

Ex. **717.** The area of a regular inscribed dodecagon is equal to three times the square of the radius.

Ex. **718**. Find the side of a regular hexagon circumscribed about a circle whose diameter is 1.

Ex. **719**. The apothem of an inscribed regular hexagon is equal to one half the side of the inscribed equilateral triangle.

Ex. **720**. The area of a ring bounded by two concentric circumferences is equal to the area of a circle whose diameter is a chord of the outer circumference and is tangent to the inner circumference.

Ex. **721**. If the radius of a circle is r, find the area of a segment whose chord is one side of a regular inscribed hexagon.

Ex. **722**. Three equal circles with a radius of 12 ft. are drawn tangent to each other. What is the area between them?

Ex. **723**. The area of an inscribed regular hexagon is equal to three fourths that of a regular hexagon circumscribed about the same circle.

Ex. **724**. The altitude of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base of the first triangle.

Ex. **725**. If the radius of a circle is r and the side of a regular inscribed polygon is a, prove that the side of a similar circumscribed polygon is $\frac{2ar}{\sqrt{4r^2-a^2}}$.

Ex. **726**. If the alternate vertices of a regular hexagon are joined by straight lines, another regular hexagon is formed which is one third as large as the original hexagon.

Ex. **727**. The diagonals of a regular pentagon divide each other in extreme and mean ratio.

PROBLEMS OF CONSTRUCTION

Ex. **728**. Construct x, if $x = \sqrt{ab}$.

Ex. **729**. Inscribe a circle in a given sector.

Ex. **730**. In a given circle describe three equal circles tangent to each other and to the given circle.

Ex. **731**. Divide a circle into two segments such that an angle inscribed in one shall be three times an angle inscribed in the other.

Ex. **732**. Construct a circumference equal to the sum of two given circumferences.

Ex. **733**. Inscribe a square in a given quadrant.

Ex. **734**. Inscribe a square in a given segment of a circle.

Ex. **735**. Through a given point draw a line so that it shall divide a given circumference into two parts having the ratio 3 : 7.

Ex. **736**. Construct a circle equivalent to twice a given circle.

Ex. **737**. Construct a circle equivalent to three times a given circle.

SOLID GEOMETRY

BOOK VII

PLANES AND SOLID ANGLES

427. A *plane* is a surface such that a straight line joining any two of its points lies wholly in the surface. § 14.

A plane is considered to be indefinite in extent, but in a diagram it is usually represented by a quadrilateral segment.

428. The student will be aided in obtaining correct concepts of the truths presented in the geometry of planes by using pieces of cardboard or paper to represent planes, and drawing such lines upon them as are required. Pins may be used to represent the lines which are perpendicular or oblique to the planes.

429. 1. By using cardboard to represent a plane and the point of a pin or pencil to represent a point in space, discover in how many directions the plane may be passed through the point.

2. By using a card as before and the points of a pair of dividers to represent *two* fixed points in space, discover whether the number of directions that the plane may take is greater or less than when it was passed through *one* fixed point.

3. Suppose a plane is passed through *three* fixed points not in the same straight line, how many directions may it take? How many points, then, determine the position of a plane?

4. Since two of the points must be in a straight line, what else besides three points determine the position of a plane?

5. Since a straight line through the other point may intersect the straight line joining the two points, what else will determine the position of a plane?

6. Since a straight line may join two of the points and a straight line parallel to that may be drawn through the other point, how else may the position of a plane be determined?

In what ways, then, may the position of a plane be determined?

430. A plane is determined by certain points or lines, when it is the only plane which contains those points or lines.

A plane is determined by

1. *Three points not in the same straight line.*
2. *A straight line and a point without that line.*
3. *Two intersecting straight lines.*
4. *Two parallel straight lines.*

431. The point at which a line meets a plane is called the **Foot of the line.**

432. A straight line that is perpendicular to every straight line in a plane drawn through its foot is *perpendicular to the plane.*

In this case the plane is *perpendicular to the line.*

433. A straight line that is not perpendicular to every line in a plane drawn through its foot is *oblique to the plane.*

434. A straight line and a plane which cannot meet, however far they may be produced, are *parallel* to each other.

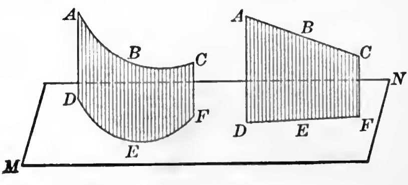
435. Two planes which cannot meet, however far they may be produced, are *parallel* to each other.

436. The locus of the points common to two non-parallel planes is the **Intersection of the planes.**

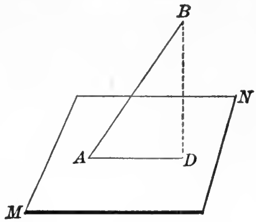
437. The foot of the perpendicular, let fall from a point to a plane, is called the **Projection of the point** on the plane.

438. The locus of the projections on a plane of all points in a line is called the **Projection of the line.**

The point D is the *projection* of the point A upon the plane MN, and DEF is the *projection* of the line ABC on the plane MN.



439. The *angle* which a straight line makes *with a plane* is the acute angle between the line and its projection on the plane, and is called the *inclination of the line* to the plane.



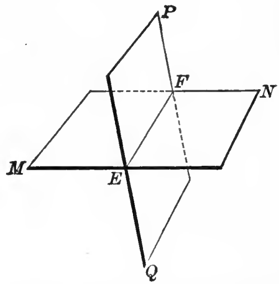
MN is a plane; *AB* a straight line meeting *MN*; and *AD* the projection of *AB* on *MN*. Then, angle *BAD* is the *angle* which *AB* makes with the plane *MN*.

440. The *distance* from a point to a plane is understood to be the *perpendicular* distance from that point to the plane.

Proposition I

441. Place two planes * so that they intersect. What kind of a line is the line of their intersection?

Theorem. *The intersection of two planes is a straight line.*



Data: Any two intersecting planes, as *MN* and *PQ*.

To prove the intersection of *MN* and *PQ* a straight line.

Proof. Suppose that *E* and *F* are any two of the points in which *MN* and *PQ* intersect. Draw the straight line *EF*.

Since *E* and *F* are points in the plane *MN*, § 427, the straight line joining them must lie in *MN*; and since they are also points in *PQ*, the straight line joining them must lie in *PQ*.

Hence, *EF* is common to *MN* and *PQ*;

and since, § 430, only one plane can contain a line and a point without that line, no point without *EF* can be common to *MN* and *PQ*;

∴ § 436, *EF* is the intersection of *MN* and *PQ*.

But, const., *EF* is a straight line;

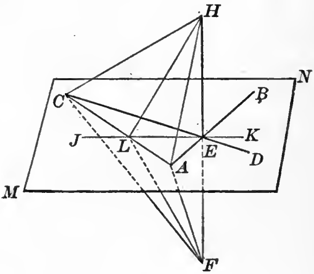
hence, the intersection of *MN* and *PQ* is a straight line. Q.E.D.

* The student may represent planes and lines as suggested in § 428.

Proposition II

442. In a plane draw two intersecting straight lines. If a third straight line is perpendicular to each of these at their point of intersection, what is its direction with reference to the plane?

Theorem. *If a straight line is perpendicular to each of two other straight lines at their point of intersection, it is perpendicular to the plane of the two lines.*



Data: Any two straight lines, as *AB* and *CD*, intersecting at *E*; *MN*, the plane of these lines; and *HE*, a perpendicular to *AB* and *CD* at *E*.

To prove *HE* perpendicular to *MN*.

Proof. Through *E*, in the plane *MN*, draw any other straight line, as *JK*; also draw *AC* intersecting *JK* in *L*.

Produce *HE* through *MN* to *F*, making $EF = HE$, and draw *HA*, *HL*, *HC*, *FA*, *FL*, and *FC*.

In $\triangle$s *ACH* and *ACF*, *AC* is common,

§ 103, $HA = FA$, and $HC = FC$;

∴ § 107, $\triangle ACH = \triangle ACF$,

and $\angle HAC = \angle FAC$;

∴ § 100, $\triangle ALH = \triangle ALF$,

and $HL = FL$;

∴ § 106, $LE \perp HE$;

that is, $HE \perp JK$.

Consequently, *HE* is perpendicular to every straight line drawn in *MN* through *E*.

Hence, § 432, *HE* is perpendicular to *MN*.

Therefore, etc. Q.E.D.

443. Cor. *A straight line, which is perpendicular to a plane at any point, is perpendicular to every straight line which can be drawn in that plane through that point.*

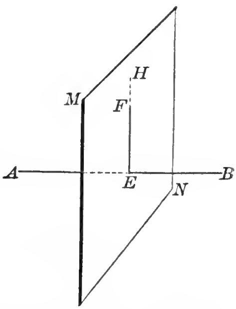
Proposition III

444. 1. At any point in a given straight line erect two perpendiculars to the line, and through them pass a plane. What is the direction of the plane with reference to the given line? Can any perpendiculars be drawn to the given line, at this point, which do not lie in this plane?

2. Can any other plane be passed through this point perpendicular to the given line?

3. Through a point without a straight line pass as many planes as possible perpendicular to the line. How many such planes can be passed through the point?

Theorem. *Every perpendicular to a straight line at a given point lies in a plane which is perpendicular to the line at that point.*



Data: Any straight line, as AB; and a plane, as MN, perpendicular to AB at E; also any line, as EF, perpendicular to AB at E.

To prove that EF lies in MN.

Proof. Suppose that the plane of AB and EF intersects MN in the line EH.

Then, § 443, $AB \perp EH$.

Since, § 51, in the plane of AB and EF only one perpendicular can be drawn to AB at E, EF and EH coincide, and EF lies in MN.

Hence, every perpendicular to AB at E lies in the plane MN.

Therefore, etc. Q.E.D.

445. Cor. I. *At a given point in a straight line one plane perpendicular to the line can be passed, and only one.*

446. Cor. II. *Through a given point without a straight line one plane perpendicular to the line can be passed, and only one.*

Ex. **738.** What is the locus of the perpendiculars to a given straight line at a given point in the line?

Proposition IV

447. 1. Erect a perpendicular to a plane; connect a point in the perpendicular with points in the plane which are equally distant from the foot of the perpendicular. How do these oblique lines compare in length?

2. Connect the same point in the perpendicular with points in the plane unequally distant from the foot of the perpendicular. How do these oblique lines compare in length?

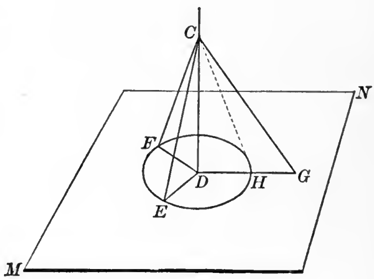
3. Represent a perpendicular and several other lines from a point to a plane. Which line is the shortest?

Theorem. *If from a point in a perpendicular to a plane oblique lines are drawn to the plane,*

1. Those lines which meet the plane at equal distances from the foot of the perpendicular are equal.

2. Of two lines which meet the plane at unequal distances from the foot of the perpendicular, that which meets it at the greater distance is the greater.

Data: A perpendicular to the plane MN, as CD, and the oblique lines CE, CF, and CG, which meet MN so that $DE = DF$, and $DG > DE$.



To prove 1. $CE = CF$.

2. $CG > CE$.

Proof. 1. Data, $CD \perp MN$;

$\therefore$ $CD \perp DE$, and $CD \perp DF$. Why?

In rt. $\triangle$ EDC and FDC, $DE = DF$,

CD is common,

and, § 52, $\angle EDC = \angle FDC$;

$\therefore$ § 100, $\triangle EDC = \triangle FDC$,

and $CE = CF$.

2. On DG take $DH = DE$, and draw CH.

Then, $CH = CE$. Why?

Hence, § 132, $CG > CH$, or CE. Q.E.D.

448. Cor. I. *A perpendicular is the shortest line that can be drawn from a point to a plane.*

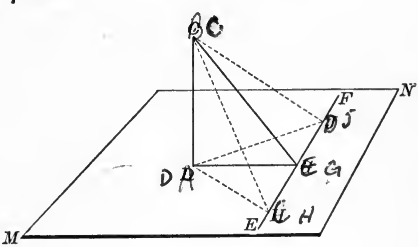
449. Cor. II. *Equal oblique lines from a point in a perpendicular to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

450. Cor. III. *The locus of a point in space equidistant from all points in the circumference of a circle is a straight line passing through the center and perpendicular to the plane of the circle.*

Proposition V

451. Erect a perpendicular to a plane; from the foot of the perpendicular draw a straight line at right angles to any other straight line of the plane; join the point of intersection of these two lines with any point in the perpendicular. What is the direction of this joining line with reference to the line in the plane that does not pass through the foot of the perpendicular?

Theorem. *If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any straight line in the plane, the line drawn from the point of meeting to any point in the perpendicular is perpendicular to the line of the plane.*



Data: A perpendicular to the plane *MN*, as *CD*; any straight line in *MN*, as *EF*; *DG* perpendicular to *EF*; and *CG* joining any point in *CD* with *G*.

To prove *CG* perpendicular to *EF*.

Proof. From *C* and *D* draw lines to *H* and *J*, two points in *EF* equally distant from *G*.

Then, § 103, $HD = JD$;

∴ § 447, $HC = JC$;

hence, § 106, $CG \perp HJ$;

that is, *CG* is perpendicular to *EF*.

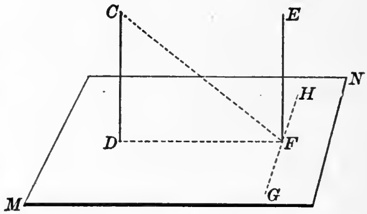
Therefore, etc. Q.E.D

452. Cor. *The locus of a point in space equidistant from the extremities of a straight line is the plane perpendicular to the line at its middle point.*

Proposition VI

453. At any two points in a plane erect perpendiculars to the plane. What is the direction of the perpendiculars with reference to each other?

Theorem. *Two straight lines perpendicular to the same plane are parallel.*



Data: Any two straight lines perpendicular to plane MN, as CD and EF.

To prove CD and EF parallel.

Proof. Draw CF and DF, and through F draw $GH \perp DF$.

Then, § 443, $EF \perp GH$,

const., $DF \perp GH$,

and, § 451, $CF \perp GH$;

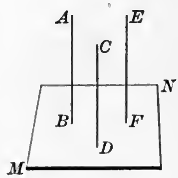
∴ § 444, EF, DF, and CF lie in the same plane.

Hence, CD and EF lie in the same plane.

But, § 443, $CD \perp DF$, and $EF \perp DF$;

hence, § 71, CD and EF are parallel.

Therefore, etc. Q.E.D.



454. Cor. I. *If one of two parallel straight lines is perpendicular to a plane, the other is also perpendicular to the plane.*

455. Cor. II. *Two straight lines that are parallel to a third straight line in another plane are parallel to each other.*

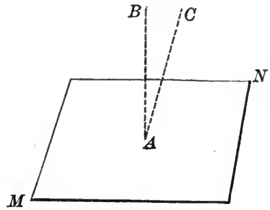
Ex. **739.** The length of a perpendicular from a given point to a plane is 5^{dm}. What is the diameter of the circle which is the locus of the foot of an oblique line drawn from the same point to the plane, if the oblique line is 13^{dm} long?

Proposition VII

456. 1. At any point in a plane erect as many perpendiculars to the plane as possible. How many can be erected?

2. Choose a point above or below the plane, and from that point draw as many perpendiculars to the plane as possible. How many such perpendiculars can be drawn?

Theorem. *From a given point only one perpendicular to a given plane can be drawn.*



Data: Any plane, as MN, and any point, as A.

To prove that from A only one perpendicular to MN can be drawn.

Proof. **Case I.** *When the given point is in the given plane.*

Draw $AB \perp MN$, and from A draw any other line, as AC.

If AC is perpendicular to MN,

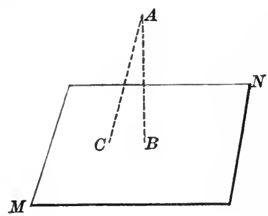
§ 453, AB and AC are parallel to every line that is perpendicular to MN;

but, § 70, this is impossible;

$\therefore$ AC is not perpendicular to MN.

Hence, only one perpendicular to MN can be drawn from A.

Case II. *When the given point is without the given plane.*



Draw $AB \perp MN$, and from A draw any other line to MN, as AC.

If AC is perpendicular to MN, § 453, AB and AC are parallel to every line that is perpendicular to MN;

but, § 70, this is impossible;

$\therefore$ AC is not perpendicular to MN.

Hence, only one perpendicular to MN can be drawn from A.

Therefore, etc. Q.E.D.

Proposition VIII

457. 1. Represent two parallel straight lines, only one of which is in a given plane. What is the direction of the other line with reference to the plane?

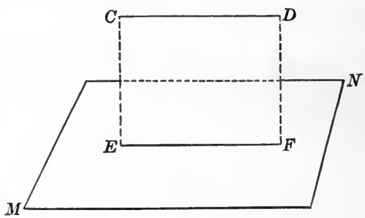
2. Represent a plane and a straight line parallel to it; pass any plane through the line so that it intersects the given plane. What is the direction of the intersection with reference to the given line?

3. Represent two intersecting planes and a straight line parallel to their intersection. What is the direction of this line with reference to each of the planes?

4. Represent any two straight lines in space; through one of them pass any plane. Can this plane be turned on the line as an axis into a position parallel to the other given line?

5. Represent any two straight lines in space and a point without them; represent a line passing through this point and parallel to one of the given lines; represent a second line through the same point and parallel to the other given line; represent the plane of these lines that intersect at the given point. What is the direction of this plane with reference to each of the given lines?

Theorem. *If a straight line without a plane is parallel to any straight line in the plane, it is parallel to the plane.*



Data: Any straight line in plane *MN*, as *EF*, and any straight line without *MN* and parallel to *EF*, as *CD*.

To prove *CD* parallel to *MN*.

Proof. Through *CD* and *EF* pass the plane *ED*.

Now, if *CD* is not parallel to *MN*, it must meet *MN* in the inter section of *MN* and *ED*, that is, in *EF*.

But, data, *CD* cannot meet *EF*;

hence, *CD* cannot meet *MN*;

that is, *CD* is parallel to *MN*. Q.E.D.

458. Cor. I. *If a straight line is parallel to a plane, the intersection of the plane with any plane passed through the line is parallel to the line.*

459. Cor. II. *A straight line parallel to the intersection of two planes is parallel to each of the planes.*

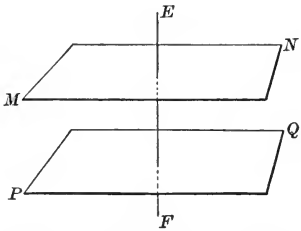
460. Cor. III. *Through any given straight line a plane may be passed parallel to any other given straight line not intersecting the first; if the lines are not parallel, only one such plane can be passed.*

461. Cor. IV. *Through a given point a plane may be passed parallel to any two given straight lines in space; and if the lines are not parallel, only one such plane can be passed.*

Proposition IX

462. Represent two planes each perpendicular to the same straight line. What is the direction of the planes with reference to each other?

Theorem. *Two planes perpendicular to the same straight line are parallel.*



Data: Any two planes perpendicular to EF, as MN and PQ.

To prove MN and PQ parallel.

Proof. If MN and PQ are not parallel, they will meet, and thus there will be two planes passing through the same point and perpendicular to the same line EF.

But, § 446, this is impossible.

Hence, MN and PQ cannot meet;

that is, MN and PQ are parallel.

Therefore, etc. Q.E.D.

Ex. **740.** If a straight line and a plane are perpendicular to another straight line, they are parallel.

Ex. **741.** If a line is equal to its projection on a plane, it is parallel to the plane.

Ex. **742.** If a line makes equal angles with three intersecting lines in the same plane, it is perpendicular to that plane.

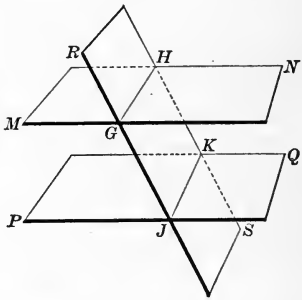
Ex. **743.** If a plane bisects a straight line at right angles, any point in the plane is equidistant from the extremities of the line.

Proposition X

463. 1. Represent two parallel planes each intersected by a third plane. In what direction, with reference to each other, do the lines of intersection extend?

2. Represent two parallel straight lines included between two parallel planes. How do the lines compare in length?

Theorem. *The intersections of two parallel planes by a third plane are parallel.*



Data: Any two parallel planes, as *MN* and *PQ*, intersected by a third plane, as *RS*, in *GH* and *JK*, respectively.

To prove *GH* and *JK* parallel.

Proof. § 435, *MN* and *PQ* cannot meet;

∴ *GH* and *JK*, which are lines lying in *MN* and *PQ* respectively, cannot meet.

But *GH* and *JK* lie in the same plane *RS*;

hence, *GH* and *JK* are parallel.

Therefore, etc. Q.E.D.

464. Cor. I. *Parallel straight lines included between parallel planes are equal.*

465. Cor. II. *Two parallel planes are everywhere equally distant.*

Ex. **744.** Draw a perpendicular to a given plane from any point without the plane.

Ex. **745.** Erect a perpendicular to a given plane at a given point in the plane.

Ex. **746.** A line parallel to two intersecting planes is parallel to their intersection.

Ex. **747.** If two lines are parallel, the intersections of any planes passing through them are parallel.

Ex. **748.** If a plane is passed through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

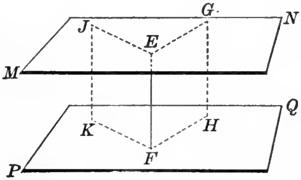
Proposition XI

466. 1. Represent a straight line perpendicular to one of two parallel planes. What is the direction of the line with reference to the other plane?

2. Pass as many planes as possible through a point and parallel to a given plane. How many such planes can be passed?

3. Represent two intersecting straight lines each parallel to a given plane. What is the direction of the plane of these lines with reference to the given plane?

Theorem. *A straight line perpendicular to one of two parallel planes is perpendicular to the other.*



Data: Any two parallel planes, as *MN* and *PQ*, and any straight line perpendicular to *MN*, as *EF*.

To prove *EF* perpendicular to *PQ*.

Proof. Through *EF* pass any two planes, as *EH* and *EK*, intersecting *MN* in *EG* and *EJ*, and *PQ* in *FH* and *FK*, respectively.

Then, § 463, $EG \parallel FH$, and $EJ \parallel FK$;

and, § 443, $EF \perp EG$ and EJ;

$\therefore$ $EF \perp FH$ and FK. Why?

Hence, § 442, *EF* is perpendicular to *PQ*.

Therefore, etc. Q.E.D.

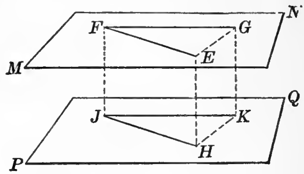
467. Cor. I. *Through a given point one plane, and only one, can be passed parallel to a given plane.*

468. Cor. II. *If two intersecting straight lines are each parallel to a given plane, the plane of those lines is parallel to the given plane.*

Proposition XII

469. Draw an angle in one plane, and in another, an angle whose sides are respectively parallel to the sides of the first angle and extending in the same direction. How do the angles compare in size? In what direction do their planes extend with reference to each other?

Theorem. *If two angles, not in the same plane, have their sides respectively parallel and extending in the same direction, they are equal and their planes are parallel.*



Data: Any two angles, as F and J, in the planes MN and PQ respectively, having the sides FE and FG parallel to and extending in the same direction with JH and JK respectively.

To prove $\angle F = \angle J$, and $MN \parallel PQ$.

Proof. 1. Take $FE = JH$, and $FG = JK$; and draw FJ, EH, GK, EG, and HK.

Then, § 150, $FEHJ$ and $FGKJ$ are parallelograms;

$EH = FJ = GK$, and $EH \parallel FJ \parallel GK$; Why?

∴ $EGKH$ is a parallelogram, and $EG = HK$. Why?

Hence, $\triangle EFG = \triangle HJK$, Why?

and $\angle F = \angle J$.

2. § 457, $FE \parallel PQ$, and $FG \parallel PQ$;

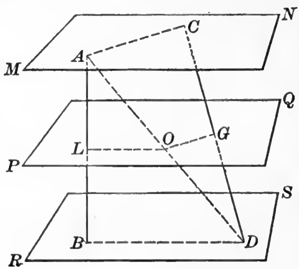
hence, § 468, $MN \parallel PQ$.

Therefore, etc. Q.E.D.

Proposition XIII

470. Represent two straight lines intersected by three parallel planes. If one line is divided into segments which are in the ratio of 2 : 3, what is the ratio of the segments of the other line? If one line is divided into segments which are in any ratio whatever, how does the ratio of the segments of the other line compare with this ratio?

Theorem. *If two straight lines are intersected by three parallel planes, their corresponding segments are proportional.*



Data: Any two lines, as AB and CD, intersected by any three parallel planes, as MN, PQ, and RS, in the points A, L, B, and C, G, D, respectively.

To prove $AL : LB = CG : GD$.

Proof. Draw AD intersecting PQ in O; and draw LO, OG, AC, and BD.

Then, §§ 430, 463, $LO \parallel BD$, and $OG \parallel AC$;

∴ § 289, $AL : LB = AO : OD$,

and $CG : GD = AO : OD$;

hence, $AL : LB = CG : GD$.

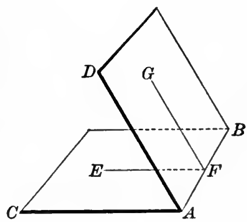
Therefore, etc. Q.E.D.

DIHEDRAL ANGLES

471. The opening between two intersecting planes is called a **Dihedral Angle,** or simply a **Dihedral.**

The intersection of the planes is called the *edge* of the dihedral angle, and the two planes are called its *faces*.

472. A dihedral angle may be designated by letters at four points, two in its edge and one in each face, the two letters at the edge being written between the other two.



When but one dihedral angle is formed at the same edge it is designated simply by two letters at this edge.

AB is the *edge*, and *BC* and *BD* are the *faces* of the *dihedral angle AB*, or *C–AB–D*.

473. The angle formed at any point in the edge of a dihedral angle by two perpendiculars to the edge, one in each face, is called the **Plane Angle of the dihedral angle.**

EF and *GF* in *BC* and *BD* respectively, both perpendicular to *AB* at *F*, form the *plane angle EFG* of the *dihedral angle C–AB–D*.

The plane angle is of the same size at whatever point in the edge it is constructed. (§§ 71, 469).

The size of a dihedral angle does not depend upon the extent of its faces, but upon their difference in direction.

474. Two dihedral angles which can be made to coincide are *equal.*

475. Dihedral angles are *adjacent, right, acute, obtuse, complementary, supplementary,* or *vertical,* according as their plane angles conform to the definitions of those terms given in plane geometry.

Two dihedral angles are *adjacent*, if they have a common edge, and a common face between them; they are *right*, if they are formed by two perpendicular intersecting planes; they are *vertical*, if the faces of one are prolongations of the faces of the other.

EXERCISES

476. 1. Represent a plane meeting another plane. How does the sum of the two dihedral angles thus formed compare with two right dihedral angles?

2. Represent two adjacent dihedral angles whose sum is equal to two right dihedral angles. How do their exterior faces lie?

3. Represent two intersecting planes. How do the vertical dihedral angles compare in size?

4. Represent two parallel planes intersected by a third plane. How do the alternate interior dihedral angles compare in size? How do the corresponding dihedral angles compare? To how many right dihedral angles is the sum of the two interior dihedral angles on the same side of the intersecting plane equal?

5. Represent two planes intersected by a third plane. In what direction do the two planes extend with reference to each other, if the alternate interior dihedral angles are equal? If the corresponding dihedral angles are equal? If the sum of the interior dihedral angles on the same side of the intersecting plane is equal to two right dihedral angles?

6. Represent two dihedral angles whose corresponding faces are parallel. How do these dihedrals compare in size, if both corresponding pairs of faces extend in the *same* direction from their edges? If both extend in *opposite* directions?

Discover whether it is possible for the dihedrals to have their faces parallel and yet not be equal.

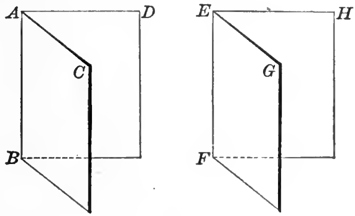
7. Represent two dihedral angles whose corresponding faces are perpendicular to each other. How do the dihedrals compare in size, if *both* are *acute?* If *both* are *obtuse?*

Discover whether it is possible for the dihedrals to have their faces perpendicular and yet not be equal.

Proposition XIV

477. Represent two dihedral angles whose plane angles are equal. How do the dihedral angles compare in size?

Theorem. *Two dihedral angles are equal, if their plane angles are equal.*



Data: Any two dihedral angles, as AB and EF, whose plane angles CAD and GEH are equal.

To prove dihedral $\angle AB =$ dihedral $\angle EF$.

Proof. Suppose that dihedral $\angle AB$ is applied to dihedral $\angle EF$ in such a way that the plane $\angle CAD$ coincides with the equal plane $\angle GEH$.

Then, point A coincides with point E,

and, § 430, plane CAD coincides with plane GEH;

$\therefore$ § 456, AB, the perpendicular to plane CAD, coincides with EF, the perpendicular to plane GEH.

Since AB coincides with EF, and AC with EG,

§ 430, plane BC coincides with plane FG.

In like manner it may be proved that

plane BD coincides with plane FH.

Hence, § 474, dihedral $\angle AB =$ dihedral $\angle EF$.

Therefore, etc. Q.E.D.

Ex. **749.** If two planes intersect each other, the vertical dihedral angles are equal.

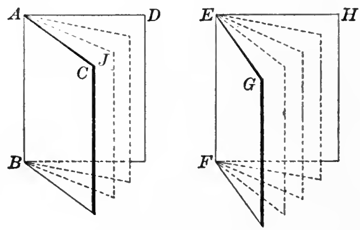
Ex. **750.** If a plane intersects two parallel planes, the alternate interior dihedral angles are equal.

Ex. **751.** The line ABC pierces three parallel planes in A, B, and C, respectively, and the line DEF pierces the same planes in D, E, and F, respectively. If AB is 6 in., BC 8 in., and DF 12 in., what is the length of DE and of EF?

Proposition XV

478. Represent two dihedral angles whose plane angles are in the ratio of 3 to 4, or any other ratio. What is the ratio of the dihedral angles?

Theorem. *Dihedral angles are to each other as their plane angles.*



Data: Any two dihedral angles, as $C\text{–}AB\text{–}D$ and $G\text{–}EF\text{–}H$, whose plane angles are CAD and GEH respectively.

To prove $C\text{–}AB\text{–}D : G\text{–}EF\text{–}H = \angle\, CAD : \angle\, GEH.$

Proof. Suppose that $\angle s$ CAD and GEH have a common unit of measure, as $\angle\, CAJ$, which is contained in $\angle\, CAD$ three times and in $\angle\, GEH$ four times.

Then, $\angle\, CAD : \angle\, GEH = 3 : 4.$

Divide the plane angles CAD and GEH into parts each equal to $\angle\, CAJ$, and through the several lines of division and the edges AB and EF pass planes..

By § 477, these planes divide $C\text{–}AB\text{–}D$ into three and $G\text{–}EF\text{–}H$ into four equal parts;

$\therefore$ $C\text{–}AB\text{–}D : G\text{–}EF\text{–}H = 3 : 4.$

Hence, $C\text{–}AB\text{–}D : G\text{–}EF\text{–}H = \angle\, CAD : \angle\, GEH.$

By the method of limits, exemplified in § 223, the same may be proved when the dihedral angles are incommensurable.

Therefore, etc. Q.E.D.

479. Sch. It is evident that the plane angle of a dihedral may be taken as the measure of the dihedral.

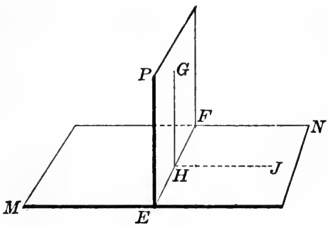
Ex. **752.** If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.

Proposition XVI

480. 1. Represent two planes that are perpendicular to each other; in one of them draw a straight line perpendicular to their intersection. What is the direction of the line with reference to the other plane?

2. Represent two planes perpendicular to each other and a straight line perpendicular to one of them at any point of their intersection. How does this line lie with reference to the other plane?

Theorem. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Data: Any two planes perpendicular to each other, as MN and PF, intersecting in EF, and any line in PF perpendicular to EF, as GH.

To prove
GH perpendicular to MN.

Proof. In MN draw $HJ \perp EF$.

Then, § 473, the angle GHJ is the plane angle of the right dihedral angle $P\text{–}EF\text{–}N$

∴ § 475, $\angle GHJ$ is a rt. $\angle$.

Data, $\angle GHE$ is a rt. $\angle$;

hence, $GH \perp HJ$ and EF at their point of intersection;

and, § 442, GH is perpendicular to MN.

Therefore, etc. Q.E.D.

481. Cor. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection lies in the other.*

Ex. **753.** Find the locus of all points equidistant from two parallel planes.

Ex. **754.** Parallel lines which pierce the same plane make equal angles with it.

Ex. **755.** If two planes intersect each other, the sum of the two adjacent dihedral angles on the same side of either plane is equal to two right dihedral angles.

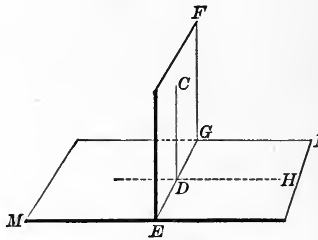
Ex. **756.** If a plane intersects two parallel planes, the interior dihedral angles on the same side of the intersecting plane are supplementary.

Proposition XVII

482. 1. Represent a straight line perpendicular to a plane. What is the direction, with reference to this plane, of every plane passing through the perpendicular?

2. Represent a dihedral angle and a plane perpendicular to its edge. What is the direction of this plane with reference to each of the faces of the dihedral?

Theorem. *If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to that plane.*



Data: Any straight line perpendicular to plane *MN*, as *CD*, and any plane passing through *CD*, as *EF*.

To prove

EF perpendicular to *MN*.

Proof. In *MN* draw $DH \perp EG$, the intersection of *EF* and *MN*.

§ 443, $CD \perp EG$;

∴ § 473, $\angle CDH$ is the plane angle of dihedral $\angle F\text{–}EG\text{–}N$.

But, § 443, $\angle CDH$ is a rt. $\angle$;

hence, $F\text{–}EG\text{–}N$ is a right dihedral angle;

that is, *EF* is perpendicular to *MN*.

Therefore, etc. Q.E.D.

483. Cor. *A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.*

Ex. **757.** Find the locus of all points in space equidistant from two given points.

Ex. **758.** Find the locus of all points at a given distance from a given plane.

Ex. **759.** A line and its projection on a plane determine a second plane perpendicular to the first.

Ex. **760.** If a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

Ex. **761.** *D* is any point in the perpendicular *AF* from *A* to the side *BC* of the triangle *ABC*. If *DE* is perpendicular to the plane *ABC*, and *GH* passing through *E* is parallel to *BC*, then, *AE* is perpendicular to *GH*.

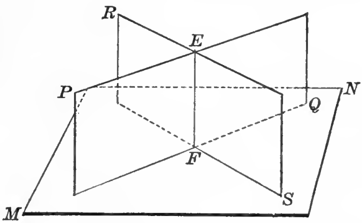
Proposition XVIII

484. 1. Represent two intersecting planes each perpendicular to a third plane. What is the direction of their intersection with reference to the third plane?

2. What is the direction of the third plane with reference to the intersection of the other two planes?

3. Represent two planes perpendicular to each other and another plane perpendicular to each of them. What is the direction of the intersection of any two of these planes with reference to the third plane? What is the direction of each intersection with reference to the other two intersections?

Theorem. *If two intersecting planes are each perpendicular to a third plane, their intersection is perpendicular to the third plane.*



Data: Any two planes, as *PQ* and *RS*, intersecting in *EF*, and perpendicular to a third plane, as *MN*.

To prove

EF perpendicular to *MN*.

Proof. At *F*, the point common to the three planes, erect a perpendicular to the plane *MN*.

By § 481, this perpendicular lies in both *PQ* and *RS*, and hence must coincide with their intersection *EF*.

Consequently, *EF* is perpendicular to *MN*.

Therefore, etc. Q.E.D.

485. Cor. I. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*

486. Cor. II. *If a plane is perpendicular to two planes which are perpendicular to each other, the intersection of any two of these planes is perpendicular to the third plane, and each of the three intersections is perpendicular to the other two.*

Ex. **762.** If from a point within a dihedral angle perpendiculars are drawn to its faces, the angle contained by these perpendiculars is equal to the plane angle of the adjacent dihedral angle formed by producing one of the planes.

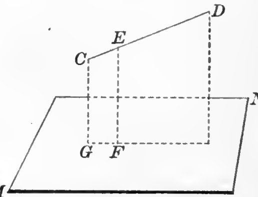
Proposition XIX

487. Represent a straight line oblique to a plane. How many planes can be passed through that line perpendicular to the given plane?

Theorem. *Through any straight line not perpendicular to a plane, one plane perpendicular to that plane can be passed, and only one.*

Data: Any plane, as MN, and any straight line not perpendicular to MN, as CD.

To prove that through CD one plane perpendicular to MN can be passed, and only one.



Proof. From any point of CD, as E, draw $EF \perp MN$.

Through CD and EF pass a plane, as GD, intersecting MN.

Then, § 482, GD is perpendicular to MN.

Now, if any other plane perpendicular to MN could be passed through CD,

§ 484, CD would be perpendicular to MN.

But, data, CD is not perpendicular to MN;

hence, only one plane perpendicular to MN can be passed through CD.

Therefore, etc. Q.E.D.

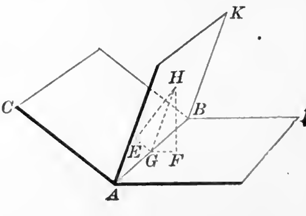
Proposition XX

488. Represent two intersecting planes forming a dihedral angle, and a third plane bisecting this angle; select any point in the bisecting plane. How do the distances of the point from the faces of the angle compare?

Theorem. *Every point in the plane which bisects a dihedral angle is equidistant from the faces of the angle.*

Data: Any dihedral angle, as $C\text{–}AB\text{–}D$; the plane bisecting it, as AK; and any point in AK, as H.

To prove H equidistant from the faces CB and AD.



Proof. Draw HE and HF, the perpendiculars from H to CB and AD respectively, and through them pass a plane intersecting the planes CB, AD, and AK in the lines EG, FG, and HG respectively.

Then, § 482, plane $EGFH \perp CB$ and AD;

$\therefore$ § 485, $EGFH \perp AB$.

Hence, § 443, EG, FG, and HG are perpendicular to AB.

Data, $C\text{–}AB\text{–}K = D\text{–}AB\text{–}K$;

hence, § 479, $\angle EGH = \angle FGH$.

In rt. $\triangle$ EGH and FGH, GH is common,

and $\angle EGH = \angle FGH$;

$\therefore$ $\triangle EGH = \triangle FGH$, Why?

and $HE = HF$;

that is, § 440, H is equidistant from the faces CB and AD.

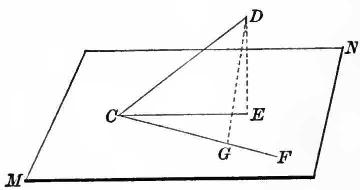
Therefore, etc. Q.E.D.

Proposition XXI

489. Represent a line oblique to a given plane and represent its projection upon that plane. How does the acute angle formed by the line with its projection compare in size with the angle which the line makes with any other line of the plane?

Theorem. *The acute angle formed by a line and its projection upon a plane is the least angle which the line makes with any line of the plane.*

Data: Any plane, as MN; any line, as CD, meeting MN in C; the projection of CD upon MN, as CE; and any other line drawn in MN through C, as CF.



To prove

$\angle DCE$ less than $\angle DCF$.

Proof. Take $CG = CE$, and draw DG and DE.

In $\triangle$ CED and CGD, CD is common,

$CE = CG$, and $ED < GD$; Why?

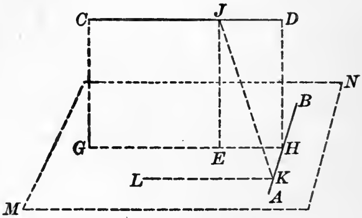
hence, § 130, $\angle DCE$ is less than $\angle DCF$.

Therefore, etc. Q.E.D.

Proposition XXII

490. Represent two straight lines in different planes and a common perpendicular to them. How many such perpendiculars can there be?

Theorem. *Between two straight lines, not in the same plane, one common perpendicular can be drawn, and only one.*



Data: Any two straight lines not in the same plane, as *AB* and *CD*.

To prove that one common perpendicular can be drawn to *AB* and *CD*, and only one.

Proof. 1. Through any point, as *K*, in *AB* draw *LK* ∥ *CD*, and let *MN* be the plane of *LK* and *AB*; through *CD* pass a plane, as *GD*, perpendicular to *MN*, intersecting *MN* in *GH* and *AB* in *H*.

Then, § 457, *CD* ∥ *MN* and, § 458, *GH* ∥ *CD*.

In the plane *GD* draw *HD* ⊥ *GH*.

Then, *HD* ⊥ *CD*, Why?

and, § 480, *HD* ⊥ *MN*;

∴ § 443, *HD* ⊥ *AB*.

2. Now, suppose that any other line, as *JK*, is perpendicular to *AB* and *CD*.

In *MN* draw *KL* ∥ *CD*, and in *GD* draw *JE* ⊥ *GH*.

Then, hyp. and § 72, *JK* ⊥ *KL*;

hence, § 442, *JK* ⊥ *MN*.

But, § 480, *JE* ⊥ *MN*.

Hence there are two perpendiculars from *J* to *MN*;

but, § 456, this is impossible.

Hence, *JK* is not perpendicular to *AB* and *CD*, and *HD* is the only perpendicular common to those lines.

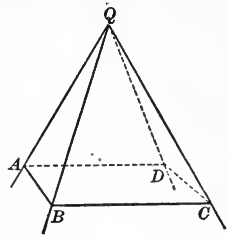
Therefore, etc. Q.E.D.

Ex. **763**. The shortest line that can be drawn between two straight lines not in the same plane is their common perpendicular.

POLYHEDRAL ANGLES

491. The angle formed by three or more planes which meet at but one point is called a **Polyhedral Angle,** or **Polyhedral.**

The point in which the planes meet is called the *vertex;* the intersections of the planes are called the *edges;* the portions of the planes included between the edges are called the *faces;* and the angles formed by the edges are called the *face angles of the polyhedral angle.*



In the *polyhedral angle* $Q\text{-}ABCD$, Q is the *vertex;* QAB, QBC, etc., are the *faces;* QA, QB, etc., are the *edges;* and angles AQB, BQC, etc., are the *face angles.*

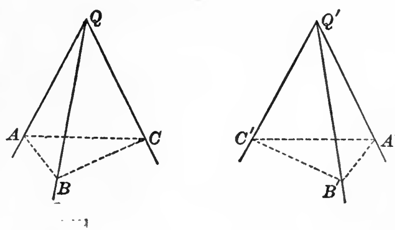
492. The faces of a polyhedral angle are of indefinite extent, but for convenience they are represented as limited by an intersecting plane called the **Base.**

$ABCD$ is the *base* of $Q\text{-}ABCD$.

493. A polyhedral angle whose base is a convex polygon is called a **Convex Polyhedral Angle.**

494. Polyhedral angles which have their face angles and their dihedral angles equal, each to each, and arranged in the *same* order, are *equal,* for they can be made to coincide. Polyhedral angles which have their face angles and their dihedral angles equal, each to each, and arranged in *reverse* order, are *symmetrical,* but not generally equal.

The trihedral angles $Q\text{-}ABC$ and $Q'\text{-}A'B'C'$ are *symmetrical,* when the face angles AQB, BQC, CQA are equal respectively to the face angles $A'Q'B'$, $B'Q'C'$, $C'Q'A'$, and the dihedral angles QA, QB, QC are equal respectively to the dihedral angles $Q'A'$, $Q'B'$, $Q'C'$.



Two symmetrical polyhedral angles cannot, generally, be made to coincide.

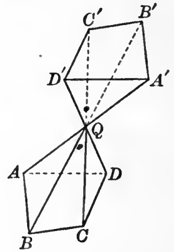
495. If the edges of one of two polyhedral angles are prolongations of the edges of the other through their common vertex the angles are called **Vertical Polyhedral Angles.**

496. A polyhedral angle having three faces is called a *trihedral angle;* one having four faces is called a *tetrahedral angle;* etc.

Proposition XXIII

497. Represent two vertical polyhedral angles. How do the face angles of one compare with the face angles of the other? How do the dihedral angles of one compare with the dihedral angles of the other? Are they arranged in the same or in a reverse order in the two polyhedrals? What name is given to such polyhedral angles?

Theorem. *Two vertical polyhedral angles are symmetrical.*



Data: Any two vertical polyhedral angles, as $Q\text{–}ABCD$ and $Q\text{–}A'B'C'D'$.

To prove $Q\text{–}ABCD$ and $Q\text{–}A'B'C'D'$ symmetrical.

Proof. § 59, face $\angle AQB =$ face $\angle A'QB'$,

and face $\angle BQC$, etc. $=$ face $\angle B'QC'$, etc., respectively.

§§ 473, 477, dihedral $\angle QA =$ dihedral $\angle QA'$,

and dihedral $\angle QB$, etc. $=$ dihedral $\angle QB'$, etc., respectively.

But the face and dihedral angles of $Q\text{–}ABCD$ are arranged in an order which is the reverse of the equal face and dihedral angles of $Q\text{–}A'B'C'D'$.

Hence, § 494, $Q\text{–}ABCD$ and $Q\text{–}A'B'C'D'$ are symmetrical.

Therefore, etc. Q.E.D.

Ex. **764.** A plane can be passed perpendicular to only one edge and only two faces of a polyhedral angle.

Ex. **765.** Every point within a dihedral and equidistant from its faces lies in the plane which bisects that dihedral.

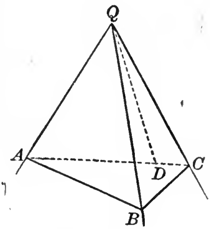
Ex. **766.** The sides of an isosceles triangle are equally inclined to any plane through its base.

Proposition XXIV

498. Represent a trihedral angle. How does the sum of any two of its face angles compare with the third face angle?

Theorem. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*

The theorem requires proof only when the third angle is greater than each of the others.



Data: Any trihedral angle, as $Q\text{–}ABC$, having one face angle, as AQC, greater than either of the other face angles.

To prove

$\angle AQB + \angle BQC$ greater than $\angle AQC$.

Proof. In the face AQC draw QD, making $\angle AQD = \angle AQB$; through any point, as D, of QD draw ADC in the plane AQC; take $QB = QD$, and through line AC and point B pass a plane.

Then, $\triangle AQB = \triangle AQD$, and $AB = AD$. Why?

In $\triangle ABC$, $AB + BC > AD + DC$; Why?

but $AB = AD$;

$\therefore$ Ax. 5, $BC > DC$.

In $\triangle$s BQC and DQC, $QB = QD$,

QC is common,

and $BC > DC$;

$\therefore$ $\angle BQC$ is greater than $\angle DQC$. Why?

Const., $\angle AQB = \angle AQD$;

$\therefore$ Ax. 4, $\angle AQB + \angle BQC$ is greater than $\angle AQD + \angle DQC$,

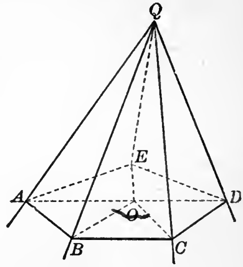
or $\angle AQB + \angle BQC$ is greater than $\angle AQC$.

Therefore, etc. Q.E.D.

Proposition XXV

499. Represent any convex polyhedral angle; around some point in a plane as a common vertex construct in succession angles equal to the face angles of this polyhedral. How does their sum compare with four right angles?

Theorem. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Datum: Any convex polyhedral angle, as Q.

To prove that the sum of the face angles of Q is less than four right angles.

Proof. Pass a plane intersecting the edges of Q in A, B, C, etc.

Then, $ABCDE$ is a convex polygon.

From O, any point within $ABCDE$, draw OA, OB, OC, etc.

The number of triangles whose common vertex is Q equals the number whose common vertex is O.

Hence, the sum of the angles of the triangles whose vertex is Q equals the sum of the angles of the triangles whose vertex is O.

But in the trihedral angles whose vertices are A, B, C, etc.,

§ 498, $\angle QBA + \angle QBC$ is greater than $\angle ABC$, or $\angle ABO + \angle CBO$,

and $\angle QCB + \angle QCD$ is greater than $\angle BCD$, or $\angle BCO + \angle DCO$.

Hence, reasoning in a similar manner regarding the other base angles of the triangles, the sum of the base angles of all the triangles whose vertex is Q is greater than the sum of the base angles of the triangles whose vertex is O.

Therefore, the sum of the face angles at Q is less than the sum of the angles at O. Why?

But the sum of the angles at O equals four right angles.

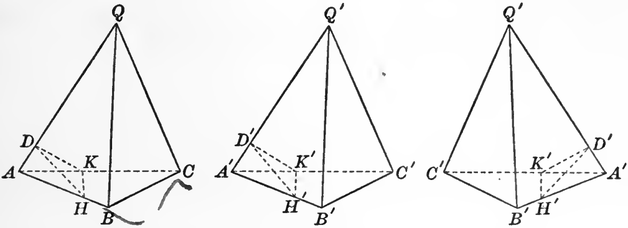
Hence, the sum of the face angles of Q is less than four right angles.

Therefore, etc. Q.E.D.

Proposition XXVI

500. Represent two trihedral angles having the three face angles of one equal respectively to the three face angles of the other. How do the trihedrals compare? Can there be two trihedrals which fulfill the same conditions and yet not be equal? What name is given to such trihedrals?

Theorem. *Two trihedral angles are either equal or symmetrical, if the three face angles of one are equal to the three face angles of the other, each to each.*



Data: Any two trihedral angles, as Q and Q', having the face angles AQB, BQC, AQC equal to the face angles $A'Q'B'$, $B'Q'C'$, $A'Q'C'$, each to each.

To prove Q either equal or symmetrical to Q'.

Proof. On the edges of Q and Q' take the equal distances QA, QB, QC, $Q'A'$, $Q'B'$, $Q'C'$, and draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

Then, $\triangle$s QAB, QBC, QAC are equal to $\triangle$s $Q'A'B'$, $Q'B'C'$, $Q'A'C'$, each to each. Why?

Hence, $\triangle ABC = \triangle A'B'C'$, and $\angle BAC = \angle B'A'C'$. Why?

On the edge QA take AD and on $Q'A'$ take $A'D' = AD$. At D and D' construct the plane $\angle$s HDK and $H'D'K'$ of the dihedrals QA and $Q'A'$ respectively, the sides meeting AB, AC, $A'B'$, and $A'C'$ as at H, K, H' and K' respectively, inasmuch as $\angle$s QAB, QAC, etc., are acute, being angles of isosceles $\triangle$s QAB, etc. Draw HK and $H'K'$.

Then, const., $AD = A'D'$,

and $\angle DAH = \angle D'A'H'$; Why?

$\therefore$ rt. $\triangle ADH =$ rt. $\triangle A'D'H'$, and $AH = A'H'$; Why?

similarly, $AK = A'K'$,

and, since $\angle BAC = \angle B'A'C'$;

$\therefore$ $\triangle AHK = \triangle A'H'K'$, and $HK = H'K'$;

but $DH = D'H'$, and $DK = D'K'$;

$\therefore$ $\triangle HDK = \triangle H'D'K'$, and $\angle HDK = \angle H'D'K'$.

Hence, § 477, dihedral $\angle QA =$ dihedral $\angle Q'A'$.

In like manner it may be shown that the dihedral angles QB and QC are equal to the dihedral angles $Q'B'$ and $Q'C'$ respectively.

Hence, § 494, if the equal angles are arranged in the same order, as in the first two figures, the two trihedral angles are equal; but if they are arranged in the reverse order, as in the first and third figures, the two trihedral angles are symmetrical.

Therefore, etc. Q.E.D.

501. Cor. *If two trihedral angles have three face angles of the one equal to three face angles of the other, then the dihedral angles of the one are respectively equal to the dihedral angles of the other.*

SUPPLEMENTARY EXERCISES

Ex. **767**. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

Ex. **768**. If a straight line intersects two parallel planes it makes equal angles with them.

Ex. **769**. If a line is parallel to each of two planes, the intersections which any plane passing through it makes with the planes are parallel.

Ex. **770**. The projections of parallel straight lines on any plane are either parallel or coincident.

Ex. **771**. Find the locus of points which are equidistant from three given points not in the same straight line.

Ex. **772**. From any point within the dihedral angle A–BC–D, EF and EG are drawn perpendicular to the faces AC and BD, respectively, and GH perpendicular to AC at H. Prove that FH is perpendicular to BC.

Ex. **773**. If a plane is passed through the middle point of the common perpendicular to two straight lines in space, and parallel to both lines, it bisects every straight line drawn from any point in one line to any point in the other line.

Ex. **774**. If the intersections of several planes are parallel, the perpendiculars drawn to them from any point lie in one plane.

Ex. **775**. If two face angles of a trihedral are equal, the dihedral angles opposite them are also equal.

Ex. **776**. A trihedral angle, having two of its dihedral angles equal, may be made to coincide with its symmetrical trihedral angle.

Ex. **777**. In any trihedral the three planes bisecting the three dihedrals intersect in the same straight line.

Ex. **778**. In any trihedral the planes which bisect the three face angles, and are perpendicular to those faces, respectively, intersect in the same straight line.

BOOK VIII

POLYHEDRONS

502. A solid bounded by planes is called a **Polyhedron.**

The intersections of the planes which bound a polyhedron are called its *edges;* the intersections of the edges are called its *vertices;* and the portions of the planes included by its edges are called its *faces.*

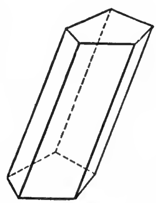
The line joining any two vertices of a polyhedron, not in the same face, is called a *diagonal of the polyhedron.*

503. A polyhedron having four faces is called a *tetrahedron;* one having six faces is called a *hexahedron;* one having eight faces is called an *octahedron;* one having twelve faces is called a *dodecahedron;* one having twenty faces is called an *icosahedron.*

504. If the section made by any plane cutting a polyhedron is a convex polygon, the solid is called a **Convex Polyhedron.**

Only convex polyhedrons are considered in this work.

PRISMS

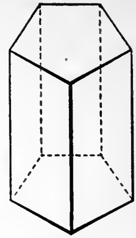


505. A polyhedron two of whose faces are equal polygons, which lie in parallel planes and have their homologous sides parallel, and whose other faces are parallelograms, is called a **Prism.**

The two equal and parallel faces of the prism are called its *bases;* the other faces are called *lateral faces;* the intersections of the lateral faces are called *lateral edges;* the sum of the lateral faces is called the *lateral,* or *convex* surface; and the sum of the areas of the lateral faces is called the *lateral area* of the prism.

The lateral edges of a prism are parallel and equal. § 153.

The perpendicular distance between the bases of a prism is its *altitude.*



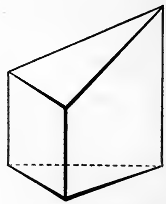
506. A prism is called *triangular, quadrangular, hexagonal,* etc., according as its bases are triangles, quadrilaterals, hexagons, etc.

507. A prism whose lateral edges are perpendicular to its bases is called a **Right Prism.**

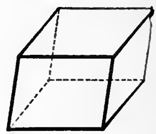
508. A prism whose lateral edges are not perpendicular to its bases is called an **Oblique Prism.**

509. A right prism whose bases are regular polygons is called a **Regular Prism.**

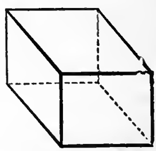
510. A section of a prism made by a plane perpendicular to its lateral edges is called a **Right Section.**



511. The part of a prism included between one base and a section made by a plane oblique to that base, and cutting all the lateral edges, is called a **Truncated Prism.**



512. A prism whose bases are parallelograms is called a **Parallelopiped.**



513. A parallelopiped whose lateral edges are perpendicular to its bases is called a **Right Parallelopiped.**

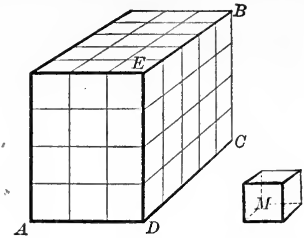
514. A parallelopiped all six of whose faces are rectangles is called a **Rectangular Parallelopiped.**

515. A parallelopiped whose six faces are all squares is called a **Cube.**

516. The quantity of space inclosed by the surfaces which bound a solid is called the **Volume of the solid.**

A solid is *measured* by finding how many times it contains some other solid adopted as the unit of measure.

The units of measure for volume are the *cubic inch,* the *cubic foot,* the *cubic yard,* the *cubic centimeter,* the *cubic decimeter,* etc.



Suppose that the cube M is the unit of measure and that AB is the rectangular parallelopiped to be measured.

Apply an edge of M to each edge of AB and at the points of division pass planes respectively perpendicular to those edges. These planes divide AB into cubes, each equal to the unit M.

It is evident that there will be as many layers of these cubes as the edge of M is contained times in the altitude of AB, that each layer will contain as many rows of cubes as the edge of M is contained times in the width of AB, and that each row will contain as many cubes as the edge of M is contained times in the length of AB; and, therefore, that the product of the numerical measures of the three dimensions of AB is equal to the number of times that M is contained in AB.

In this case the edge of M is contained 4 times in DE, 3 times in DA, and 5 times in DC; consequently, there are 5 cubes in each row, 3 rows in each layer, and 4 layers in the parallelopiped; that is, M is contained in AB $5 \times 3 \times 4 = 60$ times, or the rectangular parallelopiped contains 60 cubic units.

Therefore, if the edge of M is a common unit of measure of the three dimensions of a rectangular parallelopiped, the product of the numerical measures of the three dimensions expresses the number of times that the rectangular parallelopiped contains the cube, and is the numerical measure of the volume of the rectangular parallelopiped.

517. For the sake of brevity, the *product of the three dimensions* is used instead of the *product of the numerical measures of the three dimensions.*

The product of three lines is, strictly speaking, an absurdity, but since the expression is used to denote the volume of a rectangular parallelopiped, it follows that the *geometrical* concept of the product of three lines is the *rectangular parallelopiped* whose edges they are.

Thus, $DC \times DA \times DE$ implies a product, which is a numerical result, but it must be interpreted geometrically to mean the rectangular parallelopiped whose edges are DC, DA, and DE.

For similar reasons, the cube of a line must be interpreted *geometrically* to mean the cube constructed upon the line as an edge, and *conversely*, the cube constructed upon a line may be indicated by the cube of the line.

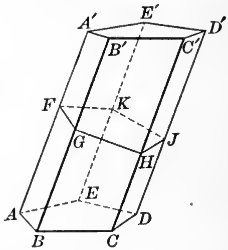
518. Solids which have the same form are *similar;* those which have the same volume are *equivalent;* and those which have the same form and volume are *equal.*

Proposition I

519. 1. Form * a prism. Since the faces are parallelograms, how does each face compare with a rectangle having the same base and altitude? Considering a lateral edge as the base of each, how does the sum of the altitudes compare with the perimeter of the right section? Since the lateral edges are equal, how does the lateral surface of a prism compare with the rectangle of its lateral edge and the perimeter of a right section?

2. To what rectangle is the lateral surface of a right prism equivalent?

Theorem. *The lateral surface of a prism is equivalent to the rectangle formed by a lateral edge and the perimeter of a right section.*



Data: Any prism, as AD', of which AA' is a lateral edge, and $FGHJK$ any right section.

To prove lateral surface of

$AD' \approxeq$ rect $AA' \cdot (FG + GH + \text{etc.})$.

Proof. § 505, AB', BC', CD', etc., are parallelograms,

data, $FGHJK$ is a right section;

∴ § 443, $FG \perp AA'$, $GH \perp BB'$, $HJ \perp CC'$, etc.

Now the lateral surface of $AD' \approxeq AB' + BC' +$ etc.;

* Objective representations of the solids referred to in this and the following books will aid the student very greatly in acquiring the correct geometrical concepts. Solids made from wood or glass may be procured, but it will be far better for the student to form them for himself from some plastic material, like clay or putty. He can then cut them readily in any desired manner with a thin-bladed knife.

but, § 331, $AB' \approx \text{rect. } AA' \cdot FG,$

$BC' \approx \text{rect. } BB' \cdot GH, \text{ etc.};$

$\therefore \quad AB' + BC' + \text{etc.} \approx \text{rect. } AA' \cdot FG + \text{rect. } BB' \cdot GH + \text{etc.},$

or lat. surf. of $AD' \approx \text{rect. } AA' \cdot FG + \text{rect. } BB' \cdot GH + \text{etc.}$

But, § 505, $AA' = BB' = CC' = \text{etc.}$

Hence, lat. surf. of $AD' \approx \text{rect. } AA' \cdot (FG + GH + \text{etc.})$. Q.E.D.

520. Cor. *The lateral surface of a right prism is equivalent to the rectangle formed by its altitude and the perimeter of its base.*

Arithmetical Rules: To be framed by the student. § 339

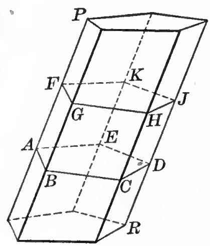
Proposition II

521. 1. Form a prism; cut it by parallel planes. What figures are the sections made by these planes? How do they compare?

2. How does any section of a prism parallel to the base compare with the base?

3. How do all right sections of the same prism compare?

Theorem. *The sections of a prism made by parallel planes are equal polygons.*



Data: Any prism, as PR, cut by any parallel planes, as AD and FJ, making the sections $ABCDE$ and $FGHJK$.

To prove $ABCDE = FGHJK$.

Proof. § 463, $AB \parallel FG$, $BC \parallel GH$, etc.;

$\therefore$ § 469, $\angle ABC = \angle FGH$, $\angle BCD = \angle GHJ$, etc.

Also, § 151, $AB = FG$, $BC = GH$, etc.

Then, $ABCDE$ and $FGHJK$ are mutually equiangular and equilateral, and one can be applied to the other so that they will exactly coincide.

Hence, § 36, $ABCDE = FGHJK$. Q.E.D.

522. Cor. I. *Any section of a prism parallel to the base is equal to the base.*

523. Cor. II. *All right sections of the same prism are equal.*

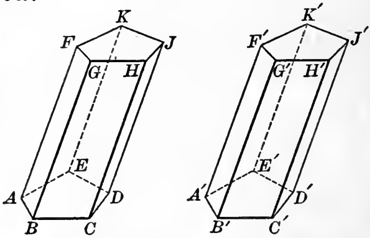
Proposition III

524. 1. Form two prisms such that three faces including a trihedral angle of one are equal to the corresponding faces of the other and similarly placed in each prism. How do the prisms compare?

2. Form two truncated prisms such that three faces including a trihedral angle of one are equal to the corresponding faces of the other and similarly placed in each prism. How do the prisms compare?

3. Form two right prisms having equal bases and equal altitudes. How do they compare?

Theorem. *Two prisms are equal, if three faces including a trihedral angle of one are equal to three faces including a trihedral angle of the other, each to each, and these faces are similarly placed.*



Data: Any two prisms, as AJ and $A'J'$, having the faces AG, AD, and AK equal to the faces $A'G'$, $A'D'$, and $A'K'$, each to each, and similarly placed.

To prove $AJ = A'J'$.

Proof. Data, the face angles BAE, BAF, and EAF are equal to the face angles $B'A'E'$, $B'A'F'$, and $E'A'F'$, respectively;

$\therefore$ § 500, trihedral angle $A\text{–}BEF =$ trihedral angle $A'\text{–}B'E'F'$.

Apply prism $A'J'$ to AJ so that the faces of trihedral $\angle A'$ shall coincide with the equal faces of the trihedral $\angle A$.

Then, the points C' and D' fall upon C and D, respectively, and, § 505, $C'H'$ and $D'J'$ take the direction of CH and DJ, respectively.

Since the points F', G', K' coincide with F, G, K, respectively, § 430, the planes of the upper bases must coincide.

Then, H' coincides with H, and J' with J.

Hence, the prisms AJ and $A'J'$ coincide in all their parts;

that is, $AJ = A'J'$. Q.E.D.

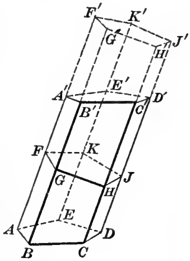
525. Cor. I. *Two truncated prisms are equal, if three faces including a trihedral angle of one are equal to three faces including a trihedral angle of the other, each to each, and these faces are similarly placed.*

526. Cor. II. *Two right prisms are equal, if they have equal bases and equal altitudes.*

Proposition IV

527. Form any oblique prism; on a base equal to a right section of the oblique prism form a right prism whose altitude is equal to a lateral edge of the oblique prism. How do these prisms compare in volume?

Theorem. *An oblique prism is equivalent to a right prism which has its base equal to a right section of the oblique prism, and its altitude equal to a lateral edge of the oblique prism.*



Data: Any oblique prism, as AD'; a right section of it, as $FGHJK$; and a lateral edge, as AA'.

To prove AD' equivalent to a right prism whose base is $FGHJK$ and altitude equal to AA'.

Proof. Produce AA' to F' making $FF' = AA'$, and at F' pass a plane perpendicular to FF' cutting all the faces of AD' produced and forming the right section $F'G'H'J'K'$ parallel to $FGHJK$.

Then, § 521, section $F'G'H'J'K'$ = section $FGHJK$,

and FJ' is a right prism whose base is $FGHJK$ and altitude equal to AA'.

In the truncated prisms AJ and $A'J'$,

§ 505, the bases AD and $A'D'$ are equal.

Const., $AA' = FF'$, and $BB' = GG'$;

∴ Ax. 3, $AF = A'F'$, and $BG = B'G'$.

AB and FG are equal and parallel to $A'B'$ and $F'G'$, respectively; and $\angle s\ FAB$, ABG, etc., of the face AG are equal respectively to $\angle s\ F'A'B'$, $A'B'G'$, etc., of the face $A'G'$; Why?

∴ AG and $A'G'$ are mutually equiangular and equilateral, and one can be applied to the other so that they will exactly coincide.

Hence, § 36, $AG = A'G'$.

In like manner AK may be proved equal to $A'K'$.

Hence, § 525, prism AJ = prism $A'J'$.

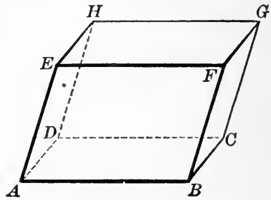
Adding to each the prism FD',

then, $AD' \Leftrightarrow FJ'$. Q.E.D.

Proposition V

528. Form any parallelopiped. How do the opposite faces compare? In what direction do they extend with reference to each other?

Theorem. *The opposite faces of a parallelopiped are equal and parallel.*



Data: Any parallelopiped, as AG, and any opposite faces of AG, as AF and DG.

To prove AF and DG equal and parallel.

Proof. $AB \parallel DC$, and $BF \parallel CG$; Why?

$\therefore$ § 469, $\angle ABF = \angle DCG$.

Also, $AB = DC$, and $BF = CG$; Why?

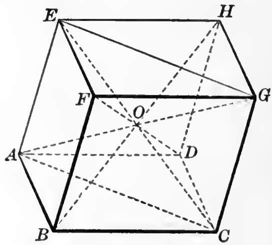
$\therefore$ $AF = DG$, Why?

and, § 469, $AF \parallel DG$. Q.E.D.

Proposition VI

529. 1. Form any parallelopiped; pass planes through any three pairs of diagonally opposite edges. What plane figures are formed by these edges and the intersections of these planes with the faces of the parallelopiped? How do the diagonals of these parallelograms correspond with the diagonals of the parallelopiped? How do the segments of each diagonal compare in length?

Theorem. *The diagonals of a parallelopiped bisect each other.*



Data: Any parallelopiped, as AG, whose diagonals are AG, BH, CE, and DF.

To prove that AG, BH, CE, and DF bisect each other.

Proof. Through the opposite edges AE and CG pass a plane.

§ 505, AE and CG are equal and parallel;

$\therefore$ $ACGE$ is a parallelogram; Why?

and, § 154, diagonals AG and CE bisect each other at O.

In like manner, AG, BH, and AG, DF also bisect each other at O.

Hence, AG, BH, CE, and DF bisect each other. Q.E.D.

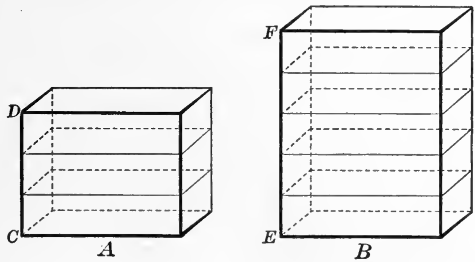
530. Cor. *The diagonals of a rectangular parallelopiped are equal.*

Proposition VII

531. 1. Form two rectangular parallelopipeds whose bases are equal and whose altitudes are in the ratio of $2:3$, or any other ratio. How does the ratio of their volumes compare with the ratio of their altitudes?

2. How does the ratio of two rectangular parallelopipeds having two dimensions in common compare with the ratio of their third dimensions?

Theorem. *Rectangular parallelopipeds which have equal bases are to each other as their altitudes.*



Data: Any two rectangular parallelopipeds, as A and B, whose bases are equal and whose altitudes are CD and EF respectively.

To prove $A:B = CD:EF$.

Proof. Suppose the altitudes CD and EF have a common measure which is contained in CD 3 times and in EF 5 times.

Then, $CD:EF = 3:5$.

Divide CD into three and EF into five equal parts by applying this common measure to them, and through the several points of division pass planes perpendicular to these lines.

§ 462, these planes are parallel to each other and to the bases of A and B;

$\therefore$ §§ 523, 526, A is divided by these parallel planes into three, and B into five equal rectangular parallelopipeds;

$\therefore$ $A:B = 3:5$.

Hence, $A:B = CD:EF$.

By the method of limits exemplified in § 327 the same may be proved when the altitudes are incommensurable.

Therefore, etc. Q.E.D.

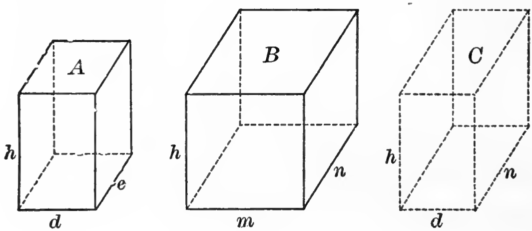
532. Cor. *Rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.*

Proposition VIII

533. 1. Form two rectangular parallelopipeds whose altitudes are equal and the areas of whose bases are in the ratio of 2 : 3, or any other ratio. How does the ratio of their volumes compare with the ratio of their bases?

2. If two rectangular parallelopipeds have one dimension in common, how does the ratio of their volumes compare with the ratio of the products of their other two dimensions?

Theorem. *Rectangular parallelopipeds which have equal altitudes are to each other as their bases.*



Data: Any two rectangular parallelopipeds, as A and B, which have a common altitude, as h, and the dimensions of whose bases are d, e, and m, n, respectively.

To prove $\quad A : B = d \times e : m \times n.$

Proof. Construct a third rectangular parallelopiped C, having the altitude h, and the dimensions of its base d and n.

Then, § 532, $\quad A : C = e : n,$

and $\quad C : B = d : m;$

hence, § 287, $\quad A : B = d \times e : m \times n.$

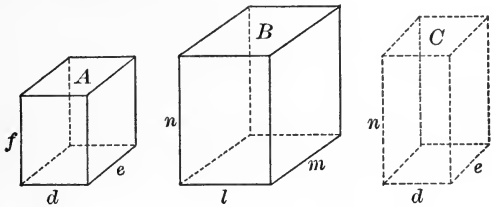
Therefore, etc. **Q.E.D.**

534. Cor. *Rectangular parallelopipeds which have one dimension in common are to each other as the products of their other two dimensions.*

Proposition IX

535. Form any two rectangular parallelopipeds, and also a third one whose base is equal to the base of the first and whose altitude is equal to that of the second. By comparing each of the first two with the third, discover how the ratio of their volumes compares with the ratio of the products of their three dimensions.

Theorem. *Rectangular parallelopipeds are to each other as the products of their three dimensions.*



Data: Any two rectangular parallelopipeds, as A and B, whose dimensions are d, e, f, and l, m, n, respectively.

To prove $A : B = d \times e \times f : l \times m \times n$.

Proof. Construct a third rectangular parallelopiped C, having the dimensions d, e, and n.

Then, § 532, $A : C = f : n$,

and, § 534, $C : B = d \times e : l \times m$;

hence, § 287, $A : B = d \times e \times f : l \times m \times n$.

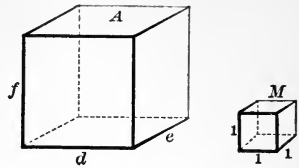
Therefore, etc. Q.E.D.

Proposition X

536. Form any rectangular parallelopiped, and a cube, whose edge is some linear unit, as a unit of measure. Since the ratio of these solids is equal to the ratio of the products of their three dimensions, find the measure of the volume of the parallelopiped in terms of its three dimensions.

Theorem. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*

Data: Any rectangular parallelopiped, as A, whose dimensions are d, e, and f.



To prove

volume of $A = d \times e \times f$.

Proof. Assume that the unit of volume is a cube, M, whose edge is the linear unit.

Then, § 535, $\quad A : M = d \times e \times f : 1 \times 1 \times 1,$

or
$$\frac{A}{M} = \frac{d \times e \times f}{1 \times 1 \times 1} = d \times e \times f.$$

But, § 516, the volume of A is measured by the number of times it contains the unit of measure, M;

$$\therefore \quad \frac{A}{M} = \text{volume of } A.$$

But
$$\frac{A}{M} = d \times e \times f.$$

Hence, volume ot $A = d \times e \times f$.

Therefore, etc. Q.E.D.

537. Cor. *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

Proposition XI

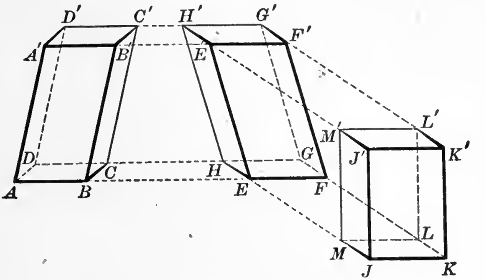
538. 1. Form any oblique parallelopiped. How does it compare in volume with a rectangular parallelopiped having an equivalent base and the same altitude?

2. What, then, is the measure of the volume of any parallelopiped in terms of its base and altitude?

Theorem. *Any parallelopiped is equivalent to a rectangular parallelopiped having the same altitude and an equivalent base.*

Data: Any parallelopiped, as AC', whose base is $ABCD$.

To prove AC' equivalent to a rectangular parallelopiped having the same altitude and a base equivalent to $ABCD$.



Proof. On AB produced take EF equal to AB, and through E and F pass planes $\perp EF$, as EH' and FG'. Produce the faces AC, $A'C'$, AB', and DC' to intersect the planes EH' and FG', forming the right parallelopiped EG'.

Then, § 527, $\qquad AC' \approx EG'$.

On HE produced take MJ equal to HE, and through M and J pass planes $\perp MJ$, as ML' and JK'. Produce the faces EG, $E'G'$, EH', and FG' to intersect the planes ML' and JK', forming the right parallelopiped JL'.

Then, § 527, $\qquad EG' \approx JL'$;

$\therefore \qquad AC' \approx JL'$.

Const., $\qquad EF = AB$, and $AF \parallel DG$;

$\therefore$ § 333, $\qquad EFGH \approx ABCD$.

Also, const., $\qquad MJ = HE$, and $HJ \parallel GK$;

$\therefore \qquad JKLM \approx EFGH \approx ABCD$. $\qquad$ Why?

Const., plane $A'G'J' \parallel$ plane AGJ, and hence the three solids have the same altitude.

Const., § 482, faces EH' and FG' are perpendicular to AGJ;

hence, $\qquad$ faces JM' and KL' are perpendicular to AGJ.

Also, const., § 482, faces ML' and JK' are perpendicular to AGJ;

$\therefore \qquad$ the faces of JL' are rectangles;

hence, § 514, $\quad JL'$ is a rectangular parallelopiped.

But $\qquad AC' \approx JL'$, and $JKLM \approx ABCD$.

Hence, AC' is equivalent to a rectangular parallelopiped having the same altitude and a base equivalent to $ABCD$.

Therefore, etc. Q.E.D.

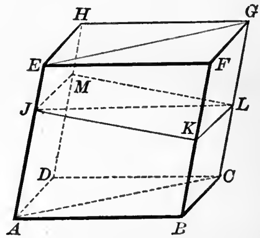
539. Cor. *The volume of any parallelopiped is equal to the product of its base by its altitude.*

Proposition XII

540. Form a parallelopiped; divide it into two triangular prisms by a plane passing through two diagonally opposite edges. How do these prisms compare in volume? What, then, is the volume of a triangular prism in terms of its base and altitude?

Form any prism; divide it into triangular prisms by planes through a lateral edge. What is the volume of each triangular prism? What, then, is the volume of any prism?

Theorem. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Data: Any parallelopiped, as AG, and a plane, as $ACGE$, passing through two diagonally opposite edges, as AE and CG.

To prove prism $ABC\text{–}F \approx$ prism $ADC\text{–}H$.

Proof. Through the parallelopiped AG pass a plane forming the right section $JKLM$ and intersecting $ACGE$ in JL.

§ 528, $AF \parallel DG$, and $AH \parallel BG$;

hence, § 463, $JK \parallel ML$, and $JM \parallel KL$;

∴ $JKLM$ is a parallelogram, and JL is its diagonal.

Hence, $\triangle JKL = \triangle JML$. Why?

Now, § 527, prism $ABC\text{–}F$ is equivalent to a right prism whose base is JKL, and whose altitude is AE;

also prism $ADC\text{–}H$ is equivalent to a right prism whose base is JML, and whose altitude is AE.

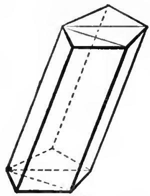
But, § 526, these two right prisms are equal;

hence, $ABC\text{–}F \approx ADC\text{–}H$.

Therefore, etc. Q.E.D.

541. Cor. I. *A triangular prism is equivalent to one half of a parallelopiped having the same altitude and a base twice as great.*

542. Cor. II. *The volume of a triangular prism is equal to the product of its base by its altitude.*

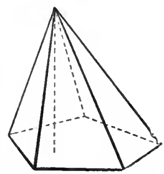


543. Cor. III. *The volume of any prism is equal to the product of its base by its altitude.*

544. Cor. IV. *Prisms are to each other as the products of their bases by their altitudes;* consequently, *prisms which have equivalent bases are to each other as their altitudes; prisms which have equal altitudes are to each other as their bases; and prisms which have equivalent bases and equal altitudes are equivalent.*

PYRAMIDS

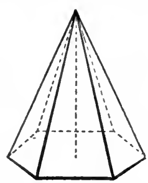
545. A polyhedron whose base is a polygon, and whose lateral faces are triangles which have a common vertex, is called a **Pyramid.**



The common vertex of the triangles which form the lateral faces of a pyramid is called the *vertex* of the pyramid; the lines in which the lateral faces intersect are called the *lateral edges;* the sum of the lateral faces is called the *lateral,* or *convex surface;* and the sum of the areas of the lateral faces is called the *lateral area* of the pyramid.

The perpendicular distance from the vertex of a pyramid to the plane of its base is its *altitude.*

546. A pyramid is called *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.



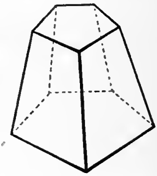
547. A pyramid whose base is a regular polygon, and whose vertex lies in the perpendicular to the base erected at its center, is called a **Regular Pyramid.**

Since a regular polygon may be inscribed in a circle, it is evident, from § 450, that the vertex of a regular pyramid is equidistant from the vertices of the polygon forming its base; and hence the lateral edges of a regular pyramid are equal, and its lateral faces are equal isosceles triangles.

548. The perpendicular distance from the vertex of a regular pyramid to the base of any one of its lateral faces is called the **Slant Height of the pyramid.**

The slant height is therefore the altitude of the equal isosceles triangles which form the lateral faces of the pyramid.

549. The part of a pyramid included between its base and a plane which cuts all its lateral edges is called a **Truncated Pyramid.**



550. A truncated pyramid whose bases are parallel is called a **Frustum of a pyramid.**

The perpendicular distance between the bases of a frustum is its *altitude.*

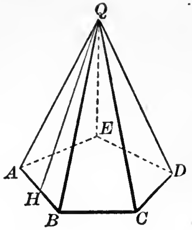
The lateral faces of the frustum of a regular pyramid are equal isosceles trapezoids, and the common altitude of these trapezoids is the *slant height of the frustum.*

Proposition XIII

551. 1. Form a regular pyramid. Since the lateral faces are triangles, how does each compare with the rectangle having the same base and altitude? How does the altitude of each compare with the slant height of the pyramid? How does the sum of the bases of the lateral faces compare with the perimeter of the base of the pyramid? How, then, does the lateral surface of a regular pyramid compare with the rectangle of the perimeter of its base and its slant height?

2. Form the frustum of a regular pyramid. What plane figures are its faces? To what, then, is the surface of each equivalent? How does the sum of the upper bases of the faces compare with the perimeter of the upper base of the frustum? The sum of the lower bases of the faces with the perimeter of the lower base of the frustum? How, then, does the lateral surface of the frustum of a regular pyramid compare with the rectangle of its slant height and the sum of the perimeters of its bases?

Theorem. *The lateral surface of a regular pyramid is equivalent to one half the rectangle formed by the perimeter of its base and its slant height.*



Data: Any regular pyramid, as $Q\text{–}ABCDE$, whose slant height is QH.

To prove lateral surface of
$Q\text{–}ABCDE \approx \frac{1}{2}$ rect. $(AB + BC + \text{etc.}) \cdot QH$.

Proof. § 545, lat. surf. of $Q\text{–}ABCDE \approx \triangle\, QAB + \triangle\, QBC +$ etc., and, since, § 548, the altitudes of these triangles $= QH$,

$\triangle\, QAB \approx \frac{1}{2}$ rect. $AB \cdot QH$, Why?

$\triangle\, QBC \approx \frac{1}{2}$ rect. $BC \cdot QH$, etc.;

$\therefore\ \triangle\, QAB + \triangle\, QBC +$ etc. $\approx \frac{1}{2}$ rect. $AB \cdot QH + \frac{1}{2}$ rect. $BC \cdot QH +$ etc.,

or lat. surf. of $Q\text{–}ABCDE \approx \frac{1}{2}$ rect. $(AB + BC + \text{etc.}) \cdot QH$.

Therefore, etc. Q.E.D.

552. Cor. *The lateral surface of a frustum of a regular pyramid is equivalent to one half the rectangle formed by its slant height and the sum of the perimeters of its bases.*

Arithmetical Rules: To be framed by the student. § 339

Ex. **779**. The perimeter of the base of a regular pyramid is 14 ft. and its slant height is 6 ft. What is its lateral area?

Proposition XIV

553. 1. Form a pyramid; cut it by a plane parallel to its base. How do the ratios of the segments of the lateral edges compare with each other, and with the ratio of the segments of the altitude?

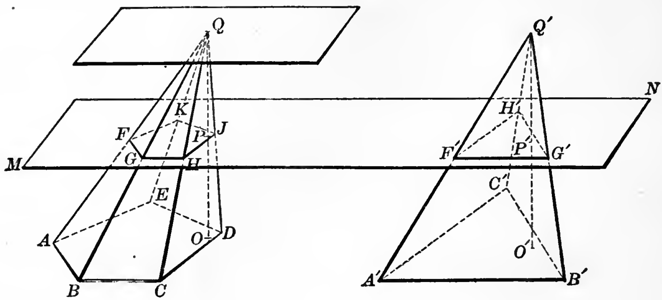
2. Is the section equal, equivalent, or similar to the base?

Theorem. *If a pyramid is cut by a plane parallel to its base,*

1. The lateral edges and the altitude are divided proportionally.

2. The section is a polygon similar to the base.

Data: Any pyramid, as Q–$ABCDE$, whose altitude is QO, and any plane parallel to the base, as MN, which cuts the pyramid in the section $FGHJK$, and the altitude in P.



To prove 1. $QF : QA = QG : QB = QP : QO =$ etc.

2. $FGHJK$ and $ABCDE$ similar.

Proof. 1. Through Q pass a plane parallel to $ABCDE$.

Then, the lateral edges and the altitude are intersected by three parallel planes;

∴ § 470, $QF : QA = QG : QB = QP : QO$, etc.

2. § 463, $FG \parallel AB$, $GH \parallel BC$, $HJ \parallel CD$, etc.;

∴ § 469, $\angle FGH = \angle ABC$, $\angle GHJ = \angle BCD$, etc.

Also, § 306, $\triangle$s QFG, QGH, etc., are similar to $\triangle$s QAB, QBC, etc., each to each;

∴ $FG : AB = QG : QB$, and $GH : BC = QG : QB$;

hence, $FG : AB = GH : BC$.

In like manner, $GH : BC = HJ : CD$, etc.

Hence, $FGHJK$ and $ABCDE$ are mutually equiangular and have their homologous sides proportional;

that is, § 299, $FGHJK$ and $ABCDE$ are similar.

Therefore, etc. Q.E.D.

554. Cor. I. *Parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

For, § 344, $FGHJK : ABCDE = \overline{FG}^2 : \overline{AB}^2$;

but, § 553, $FG : AB = QG : QB = QP : QO$;

hence, $FGHJK : ABCDE = \overline{QP}^2 : \overline{QO}^2$.

555. Cor. II. *If two pyramids have equal altitudes, sections parallel to their bases and equally distant from their vertices are to each other as the bases.*

For, § 554, $FGHJK : ABCDE = \overline{QP}^2 : \overline{QO}^2$,

and $F'G'H' : A'B'C' = \overline{Q'P'}^2 : \overline{Q'O'}^2$.

But $QP = Q'P'$, and $QO = Q'O'$;

$\therefore$ $FGHJK : ABCDE = F'G'H' : A'B'C'$,

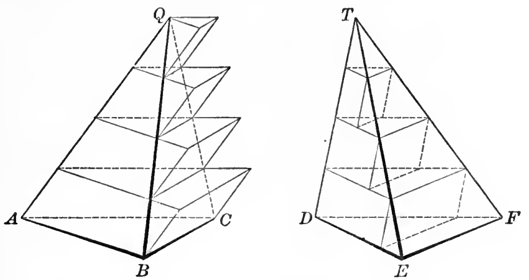
or $FGHJK : F'G'H' = ABCDE : A'B'C'$. Why?

556. Cor. III. *If two pyramids have equal altitudes and equivalent bases, sections parallel to their bases and equally distant from their vertices are equivalent.*

Proposition XV

557. Form two triangular pyramids which have equivalent bases and equal altitudes. How do they compare in volume?

Theorem. *Triangular pyramids which have equivalent bases and equal altitudes are equivalent.*



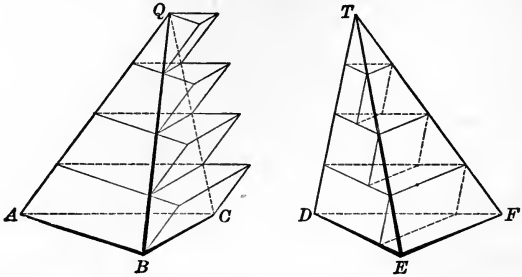
Data: Any two triangular pyramids, as Q–ABC and T–DEF, with equivalent bases ABC and DEF, and equal altitudes.

To prove $Q\text{–}ABC \Leftrightarrow T\text{–}DEF$.

Proof. Divide the equal altitudes into equal parts, each n units long, and through the points of division pass planes parallel to ABC and DEF respectively.

By § 556, the corresponding sections of the two pyramids, formed by these planes, are equivalent.

If the pyramids are not equivalent, suppose *Q–ABC* is the greater. On its base, and on each section as a lower base, construct a prism with lateral edges parallel to *AQ* and altitude equal to *n*.



On each section of *T–DEF*, as an upper base, construct a prism with lateral edges parallel to *DT* and altitude equal to *n*.

Then, § 544, each prism in *T–DEF* is equivalent to the prism next above it in *Q–ABC*; consequently, the difference between the two sets of prisms is the lowest prism of the first set.

Now, if *n* is decreased indefinitely, the lowest prism is decreased indefinitely, and the difference between the two sets of prisms may be made less than any assigned volume, however small.

But the sum of all the prisms of the first set is greater than *Q–ABC* and the sum of all the prisms of the second set is less than *T–DEF*; therefore, the difference between *Q–ABC* and *T–DEF* is less than the difference between the two sets of prisms, and consequently, less than any assigned volume, however small.

Hence, *Q–ABC* ≎ *T–DEF*.

Therefore, etc. Q.E.D.

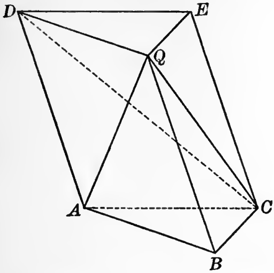
Proposition XVI

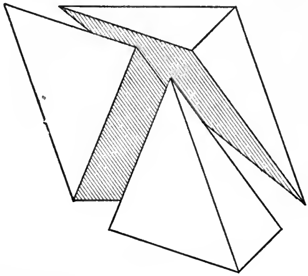
558. 1. Form any triangular prism and divide it into three triangular pyramids. How do the pyramids compare with each other? To what part, then, of the prism is the pyramid, which has the same base and altitude, equivalent?

2. Form any pyramid; divide it into triangular pyramids by planes through a lateral edge. What is the volume of each triangular pyramid? What, then, is the volume of any pyramid?

Theorem. *A triangular pyramid is equivalent to one third of a triangular prism which has the same base and altitude.*

Data: Any triangular pyramid, as $Q\text{–}ABC$, and a triangular prism, as $DQE\text{–}ABC$, which has the same base and altitude.





To prove $Q\text{–}ABC \approx \frac{1}{3}\, DQE\text{–}ABC.$

Proof. Through QC and QD pass a plane intersecting the parallelogram $ACED$ in CD. Then, $DQE\text{–}ABC$ is composed of the three triangular pyramids $Q\text{–}ABC$, $Q\text{–}ACD$, and $Q\text{–}CDE$.

$\triangle ACD = \triangle CDE$; Why?

that is, $Q\text{–}ACD$ and $Q\text{–}CDE$ have equal bases, and the same altitude;

$\therefore$ § 557, $Q\text{–}ACD \approx Q\text{–}CDE.$

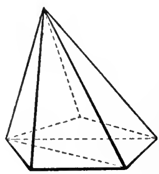
Regarding C as the vertex and DQE as the base of $Q\text{–}CDE$, § 505, $Q\text{–}ABC$ and $C\text{–}DQE$ have equal bases and equal altitudes;

$\therefore$ § 557, $Q\text{–}ABC \approx C\text{–}DQE$, or $Q\text{–}CDE$;

consequently, $Q\text{–}ABC \approx Q\text{–}ACD \approx Q\text{–}CDE.$

Hence, $Q\text{–}ABC \approx \frac{1}{3}\, DQE\text{–}ABC.$ Q.E.D.

559. Cor. I. *The volume of a triangular pyramid is equal to one third the product of its base by its altitude.*



560. Cor. II. *The volume of any pyramid is equal to one third the product of its base by its altitude.*

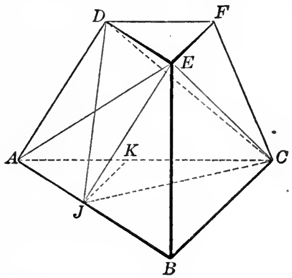
561. Cor. III. *Pyramids are to each other as the products of their bases by their altitudes;* consequently, *pyramids which have equivalent bases are to each other as their altitudes; pyramids which have equal altitudes are to each other as their bases; and pyramids which have equivalent bases and equal altitudes are equivalent.*

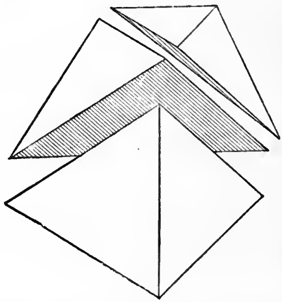
Proposition XVII

562. Form the frustum of a triangular pyramid; divide it into three triangular pyramids one of which shall have for its base the lower base of the frustum, another the upper base, and both the altitude of the frustum for their altitude. It will be shown that the third pyramid is equivalent to a pyramid whose altitude is the altitude of the frustum and whose base is a mean proportional between the bases of the frustum.

To the sum of what triangular pyramids, then, is the frustum of a triangular pyramid equivalent?

Theorem. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids of the same altitude as the frustum, and whose bases are those of the frustum and a mean proportional between them.*





Data: A frustum of any triangular pyramid, as *ABC–DEF*, whose bases are *ABC* and *DEF*.

Denote its altitude by *H*.

To prove *ABC–DEF* equivalent to the sum of three pyramids whose common altitude is *H* and whose bases are *ABC*, *DEF*, and a mean proportional between them.

Proof. Through the points *A*, *E*, *C*, and *D*, *E*, *C* pass planes, thus dividing the frustum into the three pyramids *E–ABC*, *C–DEF*, and *E–ADC*.

Then, *E–ABC* and *C–DEF* have the bases *ABC* and *DEF*, respectively, and the common altitude *H*.

It remains to prove *E–ADC* equivalent to a pyramid whose altitude is *H* and whose base is a mean proportional between *ABC* and *DEF*.

In the face DB, draw $EJ \parallel DA$, and pass the plane EJC; also draw JD.

§ 457, $EJ \parallel$ plane $ACFD$;

hence, E and J are equally distant from plane $ACFD$;

$\therefore$ $E\text{–}ADC \approx J\text{–}ADC$. Why?

Now, D may be regarded as the vertex, and AJC as the base of $J\text{–}ADC$.

Then, $E\text{–}ADC$ is equivalent to $D\text{–}AJC$, whose altitude is H.

Draw $JK \parallel EF$.

Then, § 469, $\angle AJK = \angle DEF$, $\angle JAK = \angle EDF$,

and, § 151, $AJ = DE$;

$\therefore$ $\triangle AJK = \triangle DEF$, Why?

and $AK = DF$. Why?

Now the altitudes of $\triangle$s ABC and AJC on AB are equal,

and the altitudes of $\triangle$s AJC and AJK on AC are equal;

$\therefore$ § 336, $\triangle ABC : \triangle AJC = AB : AJ = AB : DE$,

and $\triangle AJC : \triangle AJK = AC : AK = AC : DF$.

But, § 553, $\triangle$s ABC and DEF are similar;

hence, $AB : DE = AC : DF$; Why?

$\therefore$ $\triangle ABC : \triangle AJC = \triangle AJC : \triangle AJK$.

But $\triangle AJK = \triangle DEF$;

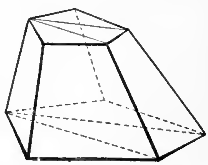
$\therefore$ $\triangle ABC : \triangle AJC = \triangle AJC : \triangle DEF$;

that is, $\triangle AJC$ is a mean proportional between $\triangle$s ABC and DEF.

Hence, $ABC\text{–}DEF$ is equivalent to the sum of three pyramids, whose common altitude is H and whose bases are ABC, DEF, and a mean proportional between them.

Therefore, etc. Q.E.D

563. Cor. I. *A frustum of any pyramid is equivalent to the sum of three pyramids of the same altitude as the frustum and whose bases are those of the frustum and a mean proportional between them.*

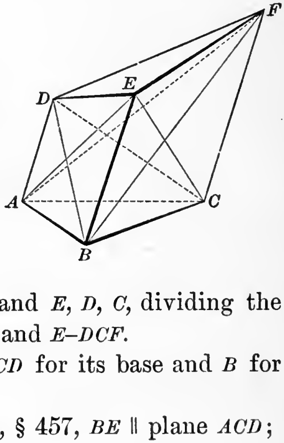


564. Cor. II. *The volume of a frustum of any pyramid is equal to one third the product of its altitude by the sum of its bases and a mean proportional between them.*

Proposition XVIII

565. Form a truncated triangular prism and through one of its upper vertices pass planes dividing it into three triangular pyramids. Since any face of a pyramid may be considered as its base, discover whether each one of these pyramids is equivalent to some one of the three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.

Theorem. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the three vertices of the inclined section.*



Data: Any truncated triangular prism, as $ABC–DEF$, and the three pyramids $E–ABC$, $D–ABC$, and $F–ABC$.

To prove $ABC–DEF \approx E–ABC + D–ABC + F–ABC$.

Proof. Pass planes through E, A, C, and E, D, C, dividing the prism into the pyramids $E–ABC$, $E–ACD$, and $E–DCF$.

$D–ABC$ may be regarded as having ACD for its base and B for its vertex.

Now, § 505, $AD \parallel BE \parallel CF$; and hence, § 457, $BE \parallel$ plane ACD;

$\therefore$ $B–ACD$ and $E–ACD$ have equal altitudes;

hence, § 557, $D–ABC \approx B–ACD \approx E–ACD$.

$F–ABC$ may be regarded as having ACF for its base and B for its vertex;

but $\triangle ACF \approx \triangle DCF$, and $BE \parallel$ plane DCF; Why?

hence, $F–ABC \approx B–ACF \approx E–DCF$; Why?

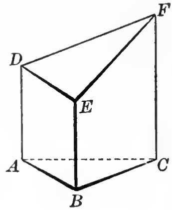
$\therefore$ $E–ABC + D–ABC + F–ABC \approx E–ABC + E–ACD + E–DCF$.

But $ABC–DEF \approx E–ABC + E–ACD + E–DCF$;

hence, $ABC–DEF \approx E–ABC + D–ABC + F–ABC$.

Therefore, etc. Q.E.D.

566. Cor. I. *The volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges.*



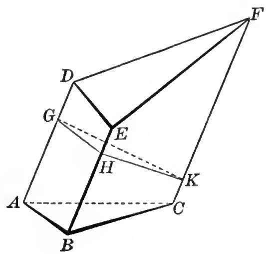
1. What is the direction of the lateral edges *AD, BE, CF* with reference to the base *ABC*?

2. How, then, do *AD, BE,* and *CF* compare with the altitudes of the three pyramids whose sum is equivalent to *ABC–DEF*?

3. To what is the volume of each of these pyramids equal?

4. To what, then, is the volume of *ABC–DEF* equal?

567. Cor. II. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*



1. If *GHK* is a right section, to what is the volume of *GHK–DEF* equal?

2. To what is the volume of *GHK–ABC* equal?

3. To what, then, is the volume of *ABC–DEF* equal?

Ex. **780.** What is the lateral area of a right prism whose altitude is 12 in. and the perimeter of whose base is 20 in.?

Ex. **781.** Find the ratio of two rectangular parallelopipeds, if their altitudes are each 7^m and their bases 3^m by 4^m and 7^m by 9^m, respectively.

Ex. **782.** Find the ratio of two rectangular parallelopipeds, if their dimensions are 2^{dm}, 4^{dm}, 3^{dm}, and 6^{dm}, 7^{dm}, 8^{dm}, respectively.

Ex. **783.** What is the volume of a rectangular parallelopiped whose edges are 20.5^m, 12.75^m, and 8.6^m, respectively?

Ex. **784.** The altitude of a regular hexagonal prism is 12 ft., and each side of its base is 10 ft. What is its volume?

Ex. **785.** What is the volume of a pyramid whose altitude is 18^{dm} and whose base is a rectangle 10^{dm} by 6^{dm}?

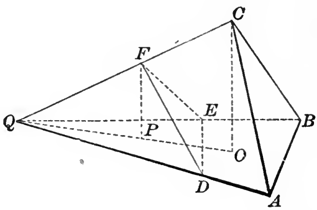
Ex. **786.** What is the volume of a truncated right triangular prism, if each side of its base is 3 ft. and its edges are 3 ft., 4 ft., and 6 ft., respectively?

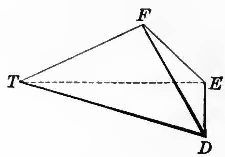
Ex. **787.** What is the lateral area of the frustum of a square pyramid whose slant height is 13^m, each side of the lower base being 3.5^m, and of the upper base 2^m?

Proposition XIX

568. Form two tetrahedrons which have a trihedral angle of one equal to a trihedral angle of the other. Considering homologous faces of these trihedrals as bases of the tetrahedrons, how does the ratio of their volumes compare with the ratio of the products of their bases by their altitudes? (§ 561) How does the ratio of their bases compare with the product of the ratios of the homologous edges which include the basal face angles of the equal trihedrals? (§ 340) How does the ratio of the altitudes of the tetrahedrons compare with the ratio of the third edges of the equal trihedrals? (§ 299) From these equal ratios discover how the ratio of the volumes of the tetrahedrons compares with the ratio of the products of the three edges of the equal trihedral angles.

Theorem. *Tetrahedrons which have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the edges of the equal trihedral angles.*





Data: Any two tetrahedrons, as $Q\text{–}ABC$ and $T\text{–}DEF$, which have the trihedral angles Q and T equal.

To prove $Q\text{–}ABC : T\text{–}DEF = QA \times QB \times QC : TD \times TE \times TF.$

Proof. Apply $T\text{–}DEF$ to $Q\text{–}ABC$ so that the equal trihedral angles T and Q coincide.

Draw CO and FP perpendicular to the plane QAB.

Then, their plane intersects QAB in QPO.

Now, CO and FP are the altitudes of the triangular pyramids $C\text{–}QAB$ and $F\text{–}QDE$;

$\therefore$ § 561, $\quad C\text{–}QAB : F\text{–}QDE = QAB \times CO : QDE \times FP.$ (1)

But, § 340, $\quad QAB : QDE = QA \times QB : QD \times QE.$

Substituting in (1),

$$C\text{–}QAB : F\text{–}QDE = QA \times QB \times CO : QD \times QE \times FP. \quad (2)$$

Now, rt. $\triangle$ QOC and QPF are similar; Why?

$\therefore$ § 299, $CO : FP = QC : QF.$

Substituting in (2),

$$C\text{–}QAB : F\text{–}QDE = QA \times QB \times QC : QD \times QE \times QF;$$

that is, $$Q\text{–}ABC : T\text{–}DEF = QA \times QB \times QC : TD \times TE \times TF.$$

Therefore, etc. Q.E.D.

SIMILAR AND REGULAR POLYHEDRONS

569. Polyhedrons which have their corresponding polyhedral angles equal, and have the same number of faces similar each to each, and similarly placed, are called **Similar Polyhedrons**.

Faces, edges, angles, etc., which are similarly placed in similar polyhedrons are called *homologous faces, edges, angles,* etc.

570. Since the homologous sides of similar polygons are proportional, the homologous edges of similar polyhedrons are proportional.

571. Since similar polygons are proportional to the squares upon any of their homologous lines, the homologous faces of similar polyhedrons are proportional to the squares upon any of their homologous edges.

572. From § 571 it is evident that the entire surfaces of similar polyhedrons are proportional to the squares upon any of their homologous edges.

Proposition XX

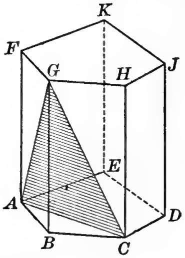
573. 1. Form two similar polyhedrons, and if possible divide them into the same number of tetrahedrons, similar each to each.

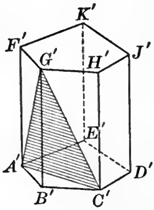
2. How does the ratio of any two homologous lines in two similar polyhedrons compare with the ratio of any two homologous edges?

Theorem. *Similar polyhedrons may be divided into the same number of tetrahedrons, similar each to each, and similarly placed.*

Data: Any two similar polyhedrons, as AJ and $A'J'$.

To prove that AJ and $A'J'$ may be divided into the same number of tetrahedrons, similar each to each, and similarly placed.





Proof. Select any trihedral angle in AJ, as B, and through the extremities of its edges, as A, G, C, pass a plane. Also through the homologous points, A', G', C', pass a plane.

Then, § 310, in the tetrahedrons $B\text{–}AGC$ and $B'\text{–}A'G'C'$, the faces BAC, BAG, BGC are similar to the faces $B'A'C'$, $B'A'G'$, $B'G'C'$, each to each.

Hence, $\quad AG : A'G' = BG : B'G' = CG : C'G'$,

and $\quad AG : A'G' = AB : A'B' = AC : A'C'$;

$\therefore \quad AG : A'G' = CG : C'G' = AC : A'C'$.

Hence, face ACG is similar to face $A'C'G'$. Why?

$\therefore$ the homologous faces of these tetrahedrons are similar.

Also, § 500, the homologous trihedral angles of these tetrahedrons are equal.

Hence, § 569, tetrahedrons $B\text{–}AGC$ and $B'\text{–}A'G'C'$ are similar.

Now, if these similar tetrahedrons are removed from the similar polyhedrons of which they are a part, the polyhedrons which remain will continue to be similar; for the faces and polyhedral angles of the original polyhedrons will be similarly modified.

By continuing to remove similar tetrahedrons from them, the original polyhedrons may be reduced to similar tetrahedrons, and they will then have been divided into the same number of tetrahedrons, similar each to each, and similarly placed.

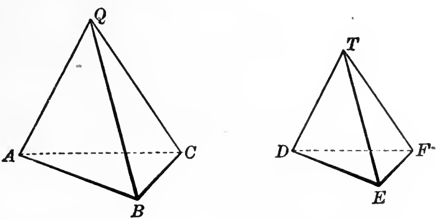
Therefore, etc. Q.E.D.

574. Cor. *Homologous lines in similar polyhedrons are proportional to their homologous edges.*

Proposition XXI

575. Form two similar tetrahedrons. How do the trihedral angles at the vertices compare? Then, how does the ratio of the tetrahedrons compare with the product of the ratios of the homologous edges of the corresponding trihedral angles? (§ 568) How do the ratios of these edges compare with each other? Then, how does the ratio of the tetrahedrons compare with the ratio of the cubes of any two homologous edges?

Theorem. *Similar tetrahedrons are to each other as the cubes of their homologous edges.*



Data: Any two similar tetrahedrons, as $Q\text{–}ABC$ and $T\text{–}DEF$.

To prove $\quad Q\text{–}ABC : T\text{–}DEF = \overline{QA}^3 : \overline{TD}^3 = \text{etc.}$

Proof. § 569, trihedral $\angle Q =$ trihedral $\angle T$;

$\therefore$ § 568, $Q\text{–}ABC : T\text{–}DEF = QA \times QB \times QC : TD \times TE \times TF$,

or
$$\frac{Q\text{–}ABC}{T\text{–}DEF} = \frac{QA \times QB \times QC}{TD \times TE \times TF} = \frac{QA}{TD} \times \frac{QB}{TE} \times \frac{QC}{TF}.$$

But $\quad QA : TD = QB : TE = QC : TF$, or $\dfrac{QA}{TD} = \dfrac{QB}{TE} = \dfrac{QC}{TF}$.

Hence,
$$\frac{Q\text{–}ABC}{T\text{–}DEF} = \frac{QA}{TD} \times \frac{QA}{TD} \times \frac{QA}{TD} = \frac{\overline{QA}^3}{\overline{TD}^3};$$

that is,
$$Q\text{–}ABC : T\text{–}DEF = \overline{QA}^3 : \overline{TD}^3.$$

In like manner, the same may be proved for any two homologous edges.

Therefore, etc. Q.E.D.

576. Cor. *Similar polyhedrons are to each other as the cubes of their homologous edges.*

1. Into what may two similar polyhedrons be divided? § 573
2. How do the ratios of these portions compare with the ratios of the cubes of their homologous edges? § 575
3. How do these ratios compare with the ratios of the cubes of any two homologous edges of the polyhedrons? § 574
4. How, then, does the ratio of the sums of these portions compare with the ratio of the cubes of any two homologous edges of the polyhedrons?

577. A polyhedron whose faces are equal regular polygons, and whose polyhedral angles are equal, is called a **Regular Polyhedron.**

578. 1. What is the least number of faces that a convex polyhedral angle may have? How does the sum of the face angles of any convex polyhedral angle compare with 360°? Since each angle of an equilateral triangle is 60°, may a convex polyhedral angle be formed by combining three equilateral triangles? Four? Five? Six? Why? Then, how many regular convex polyhedrons are possible with equilateral triangles for faces?

2. How many degrees are there in the angle of a square? May a convex polyhedral angle be formed by combining three squares? By combining four? Why? Then, how many regular convex polyhedrons are possible with squares for faces?

3. Since each angle of a regular pentagon is 108°, may a convex polyhedral angle be formed by combining three regular pentagons? By combining four? Why? Then, how many regular convex polyhedrons are possible with regular pentagons for faces?

4. Since each angle of a regular hexagon is 120°, may a convex polyhedral angle be formed by combining three regular hexagons? Why? By combining three regular heptagons? Why? What is the limit of the number of sides of a regular polygon that may be used in forming a convex polyhedral angle, and therefore in forming a regular convex polyhedron?

What, then, is the greatest number of regular convex polyhedrons possible?

579. There are only five regular convex polyhedrons possible, called from the number of their faces the *tetrahedron*, the *hexahedron*, the *octahedron*, the *dodecahedron*, and the *icosahedron*.

The tetrahedron, octahedron, and icosahedron are bounded by equilateral triangles; the hexahedron by squares; and the dodecahedron by pentagons.

580. The point within a regular polyhedron that is equidistant from all the faces of the polyhedron is called the *center* of the polyhedron.

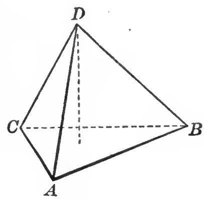
The center is also equidistant from the vertices of all the polyhedral angles of the polyhedron.

Therefore, *a sphere may be inscribed in, and a sphere may be circumscribed about, any regular polyhedron.* §§ 640, 641

Proposition XXII

581. Problem. *Upon a given edge to construct the regular polyhedrons.*

Datum: An edge, as AB.



Required to construct the regular polyhedrons on AB.

Solutions. 1. *The regular tetrahedron.*

Upon AB construct an equilateral triangle, as ABC.

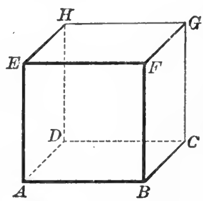
At the center of $\triangle ABC$ erect a perpendicular, and take a point D in this perpendicular such that $DA = AB$.

Draw lines from D to the vertices of $\triangle ABC$.

Then, the polyhedron $D\text{–}ABC$ is a regular tetrahedron. Q.E.F.

Proof. By the student. SUGGESTION. Refer to §§ 450, 500.

2. *The regular hexahedron.*



Upon AB construct the square $ABCD$, and upon its sides construct the squares AF, BG, CH, and DE perpendicular to $ABCD$.

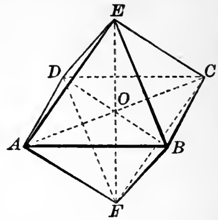
Then, the polyhedron AG is a regular hexahedron. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 500.

3. *The regular octahedron.*

Upon AB construct the square $ABCD$, and through its center O pass a line perpendicular to its plane.

On this perpendicular take the points E and F such that AE and AF are each equal to AB.



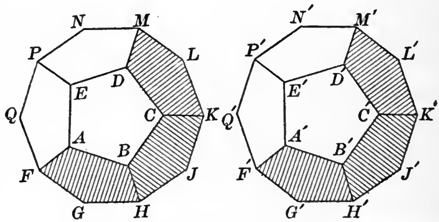
Draw lines from E and F to the vertices of $ABCD$.

Then, the polyhedron $E\text{–}ABCD\text{–}F$ is a regular octahedron. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 450.

4. *The regular dodecahedron.*

Upon AB construct a regular pentagon $ABCDE$, and to each side of it apply an equal pentagon so inclined to the plane of $ABCDE$ as to form trihedral angles at A, B, C, D, E.



Then, a convex surface $FHKMP$, composed of six regular pentagons, has been constructed.

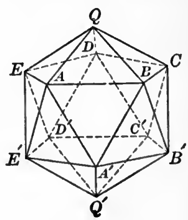
Construct a convex surface $F'H'K'M'P'$ equal to $FHKMP$, and apply one to the other so as to form a single convex surface.

The surface thus formed is that of a regular dodecahedron. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 500.

5. *The regular icosahedron.*

Upon AB construct a regular pentagon $ABCDE$; at its center erect a perpendicular; and take a point Q in this perpendicular such that $QA = AB$.



Draw lines from Q to the vertices of the pentagon forming a regular pentagonal pyramid $Q\text{–}ABCDE$.

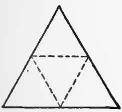
Complete the pentahedral angles at A, B, C, D, E by adding to each three equilateral triangles, each equal to $\triangle QAB$.

Construct a regular pentagonal pyramid $Q'\text{-}A'B'C'D'E'$ equal to $Q\text{-}ABCDE$, and join it to the convex surface already formed so as to form a single convex surface.

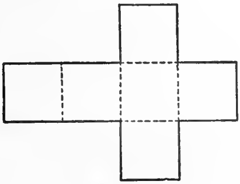
The surface thus formed is that of a regular icosahedron. Q.E.F.

Proof. By the student. SUGGESTION. Refer to § 450.

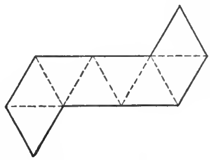
582. Sch. The five regular polyhedrons may be made from cardboard in patterns as given below, by cutting half through along the dotted lines, folding, and pasting strips of paper along the edges.



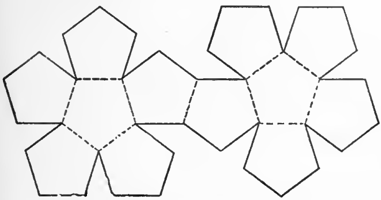
TETRAHEDRON



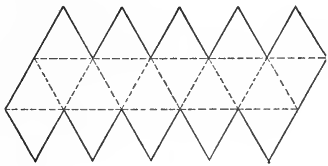
HEXAHEDRON



OCTAHEDRON



DODECAHEDRON



ICOSAHEDRON

583. **FORMULÆ**

Notation

B = base.

b = upper base.

P = perimeter of base.

P' = perimeter of upper base.

P'' = perimeter of right section.

H = altitude.

L = slant height.

E = lateral edge.

$\left.\begin{matrix} d \\ e \\ f \end{matrix}\right\}$ = dimensions of a parallelopiped.

A = lateral area.

V = volume.

Prism.

$A = E \times P''$ . . . . § 519

$A = H \times P$ (Right Prism) . . . . § 520

$V = B \times H$ . . . . §§ 542, 543

Rectangular Parallelopiped.

$V = d \times e \times f$ . . . . § 536

$V = B \times H$ (True also for any parallelopiped) §§ 537, 539

Pyramid.

$A = \frac{1}{2} P \times L$ (Regular Pyramid) . . . . § 551

$V = \frac{1}{3} B \times H$. . . . . §§ 559, 560

Frustum of a Pyramid.

$A = \frac{1}{2} L(P + P')$ (Regular Pyramid) . . . . . . § 552

$V = \frac{1}{3} H(B + b + \sqrt{B \times b})$ . . . . § 564

SUPPLEMENTARY EXERCISES

Ex. **788**. The edge of a cube is $5\frac{1}{2}$ in. What are its volume and surface?

Ex. **789**. What are the entire area and volume of a right prism 4.5 ft. in altitude, if the bases are equilateral triangles 13 in. on a side?

Ex. **790**. What is the total area of a regular triangular pyramid whose slant height is 15^{dm} and each side of whose base is 9^{dm}?

Ex. **791**. What is the volume of a triangular pyramid whose altitude is 11 ft. and the sides of whose base are 3 ft., 4 ft., and 5 ft.?

Ex. **792**. What is the edge of a cube whose volume equals that of a rectangular parallelopiped whose edges are 9 in., 12 in., and 16 in.?

Ex. **793**. The altitude of a prism is 6^{dm} and the area of its base is $2.5^{\text{sq dm}}$. What is the altitude of a prism of the same volume, if the area of its base is $3.75^{\text{sq dm}}$?

Ex. **794**. The homologous edges of two similar tetrahedrons are as 4 : 5. What is the ratio of their surfaces? What is the ratio of their volumes?

Ex. **795**. What is the altitude of a pyramid whose volume is $36^{\text{cu m}}$ and the sides of whose triangular base are 6^{m}, 5^{m}, and 4^{m}?

Ex. **796**. The area of the upper base of the frustum of a pyramid is 48 sq. ft. and that of the lower base is 72 sq. ft. If the altitude of the frustum is 60 ft., what is its volume?

Ex. **797**. What is the altitude of the frustum of a regular hexagonal pyramid, if its volume is $16^{\text{cu m}}$ and the sides of its bases are respectively 1.5^{m} and 2.5^{m}?

Ex. **798**. A pyramid 20 ft. high has 100 sq. ft. in its base; a section parallel to the base has an area of 55 sq. ft. How far is the section from the base?

Ex. **799**. What is the volume of an oblique truncated triangular prism whose edges are 5^{m}, 7^{m}, and 9^{m}, and the area of whose right section is $16^{\text{sq m}}$?

Ex. **800**. What is the edge of a cube whose entire area is $1^{\text{sq m}}$?

Ex. **801**. The base of a pyramid contains 121 sq. ft.; a section parallel to the base and 3 ft. from the vertex contains 49 sq. ft. What is the altitude of the pyramid?

Ex. **802**. What is the lateral area of a regular hexagonal pyramid whose base is inscribed in a circle whose diameter is 15 ft., the altitude of the pyramid being 8 ft.? What is the volume of the pyramid?

Ex. **803**. Any lateral edge of a right prism is equal to the altitude.

Ex. **804**. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three edges.

Ex. **805**. If the edges of a tetrahedron are all equal, the sum of the angles at any corner is equal to two right angles.

Ex. **806**. The section of a triangular pyramid made by a plane parallel to two opposite edges is a parallelogram.

Ex. **807**. The lateral faces of right prisms are rectangles.

Ex. **808**. The section of a prism made by a plane parallel to a lateral edge is a parallelogram.

Ex. **809**. The diagonal of a cube is equal to the product of its edge and $\sqrt{3}$.

Ex. **810**. The volume of a regular prism is equal to the product of its lateral area and one half the apothem of the base.

Ex. **811**. Any straight line passing through the center* of a parallelopiped and terminated by two faces is bisected at the center.

Ex. **812**. If any two non-parallel diagonal planes of a prism are perpendicular to the base, the prism is a right prism.

Ex. **813**. The base of a pyramid is 16 sq. ft. and its altitude is 7 ft. What is the area of a section parallel to the base, if it is 2 ft. 6 in. from the apex?

Ex. **814**. The edges of a rectangular parallelopiped are 3 in., 4 in., and 6 in. What is the area of its diagonal planes and the length of its diagonal line?

* The *center* of a parallelopiped is the intersection of its diagonals.

Ex. **815**. A portion of a railway embankment is 18 ft. by 380 ft. at the top, and 40 ft. by 380 ft. at the bottom. If its height is 12 ft., how many cubic yards, or loads, of earth does it contain?

Ex. **816**. If the four diagonals of a four-sided prism pass through a common point, the prism is a parallelopiped.

Ex. **817**. If a pyramid is cut by a plane parallel to its base, the pyramid cut off is similar to the given pyramid.

Ex. **818**. The lateral area of a right prism is less than the lateral area of any oblique prism having the same base and altitude.

Ex. **819**. If a section of a pyramid made by a plane parallel to the base bisects the altitude, the area of the section is one fourth the area of the base, and the pyramid cut off is one eighth of the original pyramid.

Ex. **820**. The volume of a right triangular prism is equal to one half the product of any lateral face by its distance from the opposite edge.

Ex. **821**. If the diagonals of three unequal faces of a rectangular parallelopiped are given, compute the edges.

Ex. **822**. What is the lateral area of a regular pyramid whose slant height is 10 ft., the base being a pentagon inscribed in a circle whose radius is 6 ft.? What is the volume?

Ex. **823**. The volume of a rectangular parallelopiped is $336^{\text{cu m}}$, its total area is $320^{\text{sq m}}$, and its altitude is 4^{m}. What are the dimensions of its base?

Ex. **824**. A pyramid weighs 30^{Kg}, and its altitude is 12^{dm}. A plane parallel to the base cuts off a frustum which weighs 15^{Kg}. What is the altitude of the frustum?

Ex. **825**. Each side of the base of a regular triangular pyramid is 3 in., and its altitude is 8 in. What are its lateral edge and lateral area?

Ex. **826**. The volume of a regular tetrahedron is equal to the product of the cube of its edge and $\frac{1}{12}\sqrt{2}$.

Ex. **827**. The volume of a regular octahedron is equal to the product of the cube of its edge and $\frac{1}{3}\sqrt{2}$.

Ex. **828**. Any plane passing through the center of a parallelopiped divides it into two equal solids.

Ex. **829**. The lateral area of a regular pyramid is greater than its base.

Ex. **830**. The lateral edge of the frustum of a regular triangular pyramid is $4\frac{1}{2}$ ft., a side of one base is 5 ft., and of the other 4 ft. What is the volume?

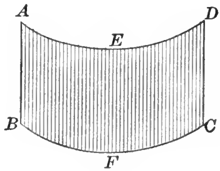
Ex. **831**. The sum of the perpendiculars from any point within a regular tetrahedron to each of its four faces is equal to its altitude.

Ex. **832**. In a regular tetrahedron an altitude is equal to three times the perpendicular from its foot on a face.

BOOK IX

CYLINDERS AND CONES

584. A surface, generated by a moving straight line which always remains parallel to its original position and continually touches a given curved line, is called a **Cylindrical Surface.**



The moving straight line is called the *generatrix,* and the given curved line is called the *directrix.*

The generatrix in any position is called an *element of the surface.*

585. A solid bounded by a cylindrical surface and two parallel planes which cut all its elements is called a **Cylinder.**

The plane surfaces are called the *bases* and the cylindrical surface is called the *lateral surface of the cylinder.*

All elements of a cylinder are *equal.* § 464.

The perpendicular distance between its bases is the *altitude of the cylinder.*

586. A section of a cylinder made by a plane perpendicular to its elements is called a **Right Section.**

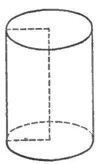
587. A cylinder whose elements are perpendicular to its base is called a **Right Cylinder.**

588. A cylinder whose elements are not perpendicular to its base is called an **Oblique Cylinder.**

589. A cylinder whose bases are circles is a **Circular Cylinder.**

The straight line joining the centers of the bases of a circular cylinder is called the *axis of the cylinder.*

590. A right circular cylinder is called a **Cylinder of Revolution,** because it may be generated by the revolution of a rectangle about one of its sides.



Cylinders of revolution generated by similar rectangles revolving about homologous sides are *similar.*

591. A plane which contains an element of a cylinder and does not cut the surface is a **Tangent Plane to the cylinder.**

The element is called the *element of contact.*

592. Any straight line that lies in a tangent plane and cuts the element of contact is a **Tangent Line to the cylinder.**

593. When the bases of a prism are inscribed in the bases of a cylinder and its lateral edges are elements of the cylinder, the prism is said to be *inscribed in the cylinder.*

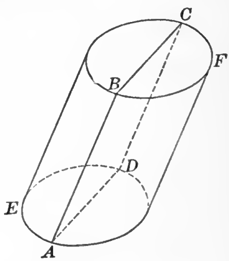
594. When the bases of a prism are circumscribed about the bases of a cylinder and its lateral edges are parallel to the elements of the cylinder, the prism is said to be *circumscribed about the cylinder.*

Proposition I

595. 1. Form a cylinder and cut it by any plane through an element of its surface (§ 519 N.). What plane figure is the section made by the cutting plane?

2. If the cylinder is a right cylinder, what plane figure does such a plane make?

Theorem. *Any section of a cylinder made by a plane passing through an element is a parallelogram.*



Data: Any section of the cylinder EF, as $ABCD$, made by a plane passing through AB, an element of the surface.

To prove $ABCD$ a parallelogram.

Proof. The plane passing through the element AB cuts the circumference of the base in a second point, as D. Draw $DC \parallel AB$.

Then, § 63, DC is in the plane BAD;

and, § 584, DC is an element of the cylinder.

Hence, DC, being common to the plane and the lateral surface of the cylinder, is their intersection.

Also, § 463, $AD \parallel BC$;

hence, § 140, $ABCD$ is a parallelogram.

Therefore, etc. Q.E.D

596. Cor. *Any section of a right cylinder made by a plane passing through an element is a rectangle.*

Proposition II

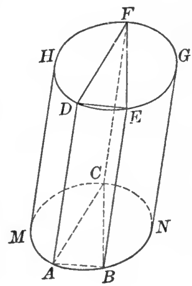
597. 1. Form a cylinder. How do its bases compare?

2. Cut the cylinder by parallel planes which cut all its elements. How do the sections thus made compare with each other?

3. How does a section made by a plane parallel to the base compare with the base?

Theorem. *The bases of a cylinder are equal.*

Data: Any cylinder, as MG, whose bases are HG and MN.



To prove $HG = MN$.

Proof. Take any three points in the perimeter of the upper base, as D, E, F, and from them draw the elements of the surface DA, EB, FC, respectively.

Draw AB, BC, AC, DE, EF, and DF.

§§ 585, 584, AD, BE, and CF are equal and parallel;

∴ § 150, AE, AF, and BF are parallelograms;

and $AB = DE$, $AC = DF$, $BC = EF$; Why?

hence, $\triangle ABC = \triangle DEF$. Why?

Apply the upper base to the lower base so that DE shall fall upon AB.

Then, F will fall upon C.

But F is any point in the perimeter of the upper base, therefore, every point in the perimeter of the upper base will fall upon the perimeter of the lower base.

Hence, § 36, $HG = MN$.

Therefore, etc. Q.E.D.

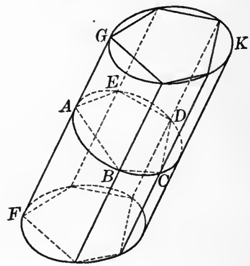
598. Cor. I. *The sections of a cylinder made by parallel planes cutting all its elements are equal.*

599. Cor. II. *The axis of a circular cylinder passes through the centers of all the sections parallel to the bases.*

Proposition III

600. To what is the lateral surface of any prism equivalent? (§ 519) If the number of its lateral faces is indefinitely increased, what solid does the prism approach as its limit? How, then, does the lateral surface of any cylinder compare with the rectangle formed by an element and the perimeter of a right section?

Theorem. *The lateral surface of a cylinder is equivalent to the rectangle formed by an element of the surface and the perimeter of a right section.*



Data: Any cylinder, as FK; any right section of it, as $ABCDE$; and any element of its surface, as FG.

Denote the lateral surface of FK by S, and the perimeter of its right section by P.

To prove $S \approx$ rect. $FG \cdot P$.

Proof. Inscribe in the cylinder a prism; denote its lateral surface by S' and the perimeter of its right section by P'.

Then, § 593, each lateral edge is an element of the cylinder, and, § 585, the elements are all equal;

$\therefore$ § 519, $S' \approx$ rect. $FG \cdot P'$.

Now, if the number of lateral faces of the inscribed prism is indefinitely increased,

§ 393, P' approaches P as its limit;

$\therefore$ S' approaches S as its limit.

But, however great the number of faces,

$S' \approx$ rect. $FG \cdot P'$.

Hence, § 326, $S \approx$ rect. $FG \cdot P$.

Therefore, etc. Q.E.D.

601. Cor. *The lateral surface of a cylinder of revolution is equivalent to the rectangle formed by its altitude and the circumference of its base.*

Arithmetical Rules: To be framed by the student. § 339

602. Formulæ: Let A denote the lateral area, T the total area H the altitude, and R the radius of the base of a cylinder of revolution.

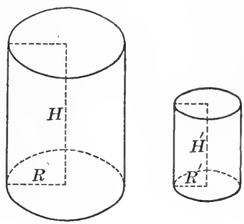
Then, § 395, $A = 2\pi R \times H,$

and, § 398, $T = 2\pi R \times H + 2\pi R^2 = 2\pi R(H + R).$

Proposition IV

603. Compute the areas of any two similar cylinders of revolution, as those whose altitudes are 4″ and 2″ and whose radii are 2″ and 1″, respectively. How does the ratio of their lateral areas, or of their total areas, compare with the ratio of the squares of their altitudes, or with the ratio of the squares of their radii?

Theorem. *The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of their radii.*



Data: Any two similar cylinders of revolution, whose altitudes are H and H', and radii R and R', respectively.

Denote their lateral areas by A and A', and their total areas by T and T', respectively.

To prove 1. $A : A' = H^2 : H'^2 = R^2 : R'^2.$

2. $T : T' = H^2 : H'^2 = R^2 : R'^2.$

Proof. 1. Since the generating rectangles are similar,

§§ 299, 279, $$\frac{R}{R'} = \frac{H}{H'} = \frac{R+H}{R'+H'};$$

∴ § 602, $$\frac{A}{A'} = \frac{2\pi RH}{2\pi R'H'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{H^2}{H'^2} = \frac{R^2}{R'^2},$$

or $$A : A' = H^2 : H'^2 = R^2 : R'^2.$$

2. § 602, $$\frac{T}{T'} = \frac{2\pi R(R+H)}{2\pi R'(R'+H')} = \frac{R}{R'}\left(\frac{R+H}{R'+H'}\right) = \frac{R^2}{R'^2} = \frac{H^2}{H'^2},$$

or $$T : T' = H^2 : H'^2 = R^2 : R'^2.$$

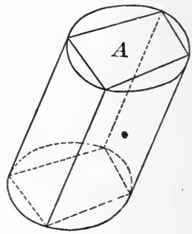
Therefore, etc. Q.E.D.

604. Cor. *The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of any of their like dimensions.*

Proposition V

605. To what is the volume of any prism equal? (§ 543) If the number of its lateral faces is indefinitely increased, what solid does the prism approach as its limit? To what, then, is the volume of any cylinder equal?

Theorem. *The volume of any cylinder is equal to the product of its base by its altitude.*



Data: Any cylinder, as A, whose base is B and altitude H.

Denote its volume by V.

To prove $V = B \times H$.

Proof. Inscribe in the cylinder a prism, and denote its volume by V' and its base by B'.

Then, the altitude of the prism is H,

and, § 543, $V' = B' \times H$.

Now, if the number of lateral faces of the inscribed prism is indefinitely increased,

§ 393, B' approaches B as its limit;

∴ V' approaches V as its limit.

But, however great the number of faces,

$$V' = B' \times H.$$

Hence, § 222, $V = B \times H$.

Therefore, etc. Q.E.D.

606. Formula: Let R denote the radius of the base of a cylinder of revolution.

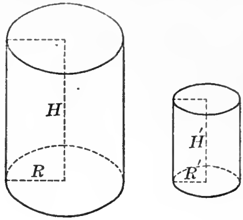
Then, § 398, $B = \pi R^2$;

∴ $V = \pi R^2 \times H$.

Proposition VI

607. Compute the volumes of any two similar cylinders of revolution, as those whose altitudes are 4″ and 2″ and whose radii are 2″ and 1″ respectively. How does the ratio of their volumes compare with the ratio of the cubes of their altitudes, or with the ratio of the cubes of their radii?

Theorem. *The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of their radii.*



Data: Any two similar cylinders of revolution, whose altitudes are H and H', and radii R and R' respectively.

Denote their volumes by V and V' respectively.

To prove $V : V' = H^3 : H'^3 = R^3 : R'^3.$

Proof. Since the generating rectangles are similar,

§ 299, $$\frac{R}{R'} = \frac{H}{H'};$$

∴ § 606, $$\frac{V}{V'} = \frac{\pi R^2 H}{\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{H^3}{H'^3} = \frac{R^3}{R'^3},$$

or $$V : V' = H^3 : H'^3 = R^3 : R'^3.$$

Therefore, etc. Q.E.D.

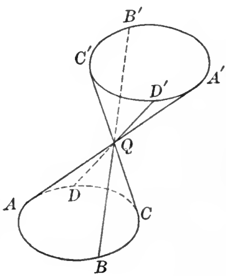
608. Cor. *The volumes of similar cylinders of revolution are to each other as the cubes of any of their like dimensions.*

CONES

609. A surface, generated by a moving straight line which passes through a fixed point and continually touches a given curved line, is called a **Conical Surface.**

The moving straight line is called the *generatrix*, the fixed point the *vertex*, and the given curved line the *directrix*.

The generatrix in any position is called an *element of the surface.*



If the generatrix extends on both sides of the vertex, the whole surface consists of two portions which are called the *lower* and *upper nappes* respectively.

Q–$ABCD$ and Q–$A'B'C'D'$ are the *lower* and *upper nappes* respectively of a *conical surface* of which AA' is the *generatrix*, Q the *vertex*, $ABCD$ the *directrix*, and AA', BB', CC', DD', etc., are *elements.*

610. A solid bounded by a conical surface and a plane which cuts all its elements is called a **Cone.**

The plane surface is called the *base* and the conical surface is called the *lateral surface of the cone.*

The perpendicular distance from its vertex to the plane of its base is the *altitude of the cone.*

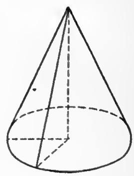
611. A cone whose base is a circle is called a **Circular Cone.**

The straight line joining the vertex and the center of the base of a cone is called the *axis of the cone.*

612. A cone whose axis is perpendicular to its base is called a **Right Cone.**

613. A cone whose axis is not perpendicular to its base is called an **Oblique Cone.**

614. A right circular cone is called a **Cone of Revolution,** because it may be generated by the revolution of a right triangle about one of its perpendicular sides.



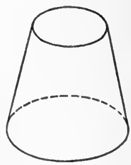
All the elements of a cone of revolution are *equal,* and any one of them is called the *slant height of the cone.*

Cones of revolution, which are generated by similar right triangles revolving about homologous perpendicular sides, are *similar.*

615. A plane which contains an element of a cone and does not cut the surface is a **Tangent Plane to the cone.**

The element is called the *element of contact.*

616. Any straight line that lies in a tangent plane and cuts the element of contact is a **Tangent Line to the cone.**



617. The portion of a cone included between its base and a section parallel to the base and cutting all the elements is called a **Frustum of a cone.**

The base of the cone is called the *lower base* of the frustum, and the parallel section the *upper base.*

The perpendicular distance between its bases is the *altitude of the frustum.*

The portion of an element of a cone of revolution included between the parallel bases of a frustum is the *slant height of the frustum.*

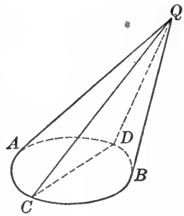
618. When the base of a pyramid is inscribed in the base of a cone and its lateral edges are elements of the cone, the pyramid is said to be *inscribed in the cone.*

619. When the base of a pyramid is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone, the pyramid is said to be *circumscribed about the cone.*

Proposition VII

620. Form a cone and cut it by any plane through its vertex. What plane figure is the section made by the cutting plane?

Theorem. *Any section of a cone made by a plane passing through its vertex is a triangle.*



Data: Any cone, as $Q\text{–}ABD$, and any section of it, as CQD, made by a plane passing through the vertex Q.

To prove CQD a triangle.

Proof. Draw the straight lines QC and QD.

Then, § 609, QC and QD are elements of the cone,

and, § 427, since QC and QD each have two points in common with the plane CQD, they lie in that plane;

∴ QC and QD are the intersections of the cutting plane and the lateral surface.

Also, § 441, CD is a straight line.

Hence, § 85, CQD is a triangle.

Therefore, etc. Q.E.D.

Ex. **833**. What is the lateral area of a cylinder of revolution whose altitude is 18 ft. and the diameter of whose base is 6 ft.?

Ex. **834**. What is the volume of a cylinder of revolution whose altitude is 7 ft. and the circumference of whose base is 5 ft.?

Ex. **835**. How many cubic feet are there in a cylindrical log 14 ft. long and 2.5 ft. in diameter?

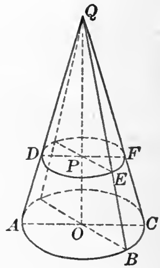
Ex. **836**. The altitude of a cylinder of revolution is 16^{dm} and the diameter of its base is 11^{dm}. What is its total area? What is its volume?

Proposition VIII

621. 1. Form a circular cone and cut it by any plane parallel to its base. What plane figure is the section made by the cutting plane?

2. In what points will the axis of the cone pierce all the sections that are parallel to the base?

Theorem. *Any section of a circular cone made by a plane parallel to the base is a circle.*



Data: Any circular cone, as $Q\text{–}ABC$ and any section of it, as DEF, made by a plane parallel to the base.

To prove DEF a circle.

Proof. Draw QO, the axis of the cone piercing DEF in P.

Through QO and any elements, QA, QB, etc., pass planes cutting the base in the radii OA, OB, etc., and the parallel section in the straight lines PD, PE, etc.

Data, planes DEF and ABC are parallel;

∴ § 463, $PD \parallel OA$, and $PE \parallel OB$.

Consequently, $\triangle$ QPD and QOA are mutually equiangular and therefore similar; likewise $\triangle$ QPE and QOB are similar.

Hence, § 299, $PD : OA = QP : QO$,

and $PE : OB = QP : QO$;

∴ $PD : OA = PE : OB$.

But $OA = OB$; Why?

∴ § 272, $PD = PE$;

that is, all the straight lines drawn from the point P to the perimeter of the section DEF are equal.

Hence, § 173, DEF is a circle.

Therefore, etc. Q.E.D.

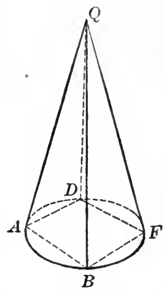
622. Cor. *The axis of a circular cone passes through the centers of all the sections that are parallel to the base.*

Ex. **837**. The total area of a cylinder of revolution is $659^{\text{sq dm}}$ and its altitude is 15^{dm}. What is the diameter of its base?

Proposition IX

623. To what is the lateral surface of any regular pyramid equivalent? If the number of its lateral faces is indefinitely increased, what solid does the pyramid approach as its limit? How, then, does the lateral surface of a cone of revolution compare with the rectangle formed by the circumference of its base and its slant height?

Theorem. *The lateral surface of a cone of revolution is equivalent to one half the rectangle formed by the circumference of its base and its slant height.*



Data: Any cone of revolution, as $Q\text{–}ABFD$, whose slant height is L, and the circumference of whose base is C.

Denote its lateral surface by S.

To prove $S \approx \frac{1}{2}$ rect. $C \cdot L$.

Proof. Inscribe in the cone a regular pyramid of any number of faces and denote its lateral surface by S', its slant height by L', and the perimeter of its base by P.

Then, § 618, each lateral edge of the pyramid is an element of the cone;

and, § 551, $S' \approx \frac{1}{2}$ rect. $P \cdot L'$.

Now, if the number of lateral faces of the inscribed pyramid is indefinitely increased,

§ 392, P approaches C as its limit;

∴ L' approaches L as its limit,

and S' approaches S as its limit.

But, however great the number of faces,

$$S' \approx \tfrac{1}{2} \text{ rect. } P \cdot L'.$$

Hence, § 326, $S \approx \frac{1}{2}$ rect. $C \cdot L$.

Therefore, etc. Q.E.D.

Arithmetical Rule: To be framed by the student.

624. Formulæ: Let R denote the radius of the base of a cone of revolution, A its lateral area, and T its total area.

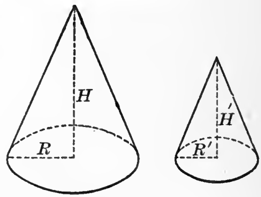
Then, § 395, $A = \frac{1}{2}(2\,\pi R \times L) = \pi RL$,

and, § 398, $T = \pi RL + \pi R^2 = \pi R(L + R)$.

Proposition X

625. Compute the areas of any two similar cones of revolution, as those whose altitudes are 8″ and 4″, slant heights 10″ and 5″, and the radii of whose bases are 6″ and 3″, respectively. How does the ratio of their lateral areas, or of their total areas, compare with the ratio of the squares of their altitudes, or with the ratio of the squares of the radii of their bases?

Theorem. The lateral areas, or the total areas, of similar cones of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases.



Data: Any two similar cones of revolution, whose altitudes are H and H', slant heights L and L', and the radii of whose bases are R and R', respectively.

Denote their lateral areas by A and A', and their total areas by T and T', respectively.

To prove 1. $A : A' = H^2 : H'^2 = R^2 : R'^2$.

2. $T : T' = H^2 : H'^2 = R^2 : R'^2$.

Proof. 1. Since the generating triangles are similar,

§§ 299, 279, $$\frac{H}{H'} = \frac{L}{L'} = \frac{R}{R'} = \frac{L+R}{L'+R'};$$

∴ § 624, $$\frac{A}{A'} = \frac{\pi RL}{\pi R'L'} = \frac{R}{R'} \times \frac{L}{L'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2},$$

or $$A : A' = H^2 : H'^2 = R^2 : R'^2.$$

2. § 624, $$\frac{T}{T'} = \frac{\pi R(L+R)}{\pi R'(L'+R')} = \frac{R}{R'}\left(\frac{L+R}{L'+R'}\right) = \frac{R^2}{R'^2} = \frac{H^2}{H'^2},$$

or $$T : T' = H^2 : H'^2 = R^2 : R'^2.$$

Therefore, etc. Q.E.D.

626. Cor. *The lateral areas, or the total areas, of similar cones of revolution are to each other as the squares of their like dimensions.*

Ex. **838.** What is the lateral area of a cone of revolution whose slant height is 13 ft. and the diameter of whose base is 5 ft.?

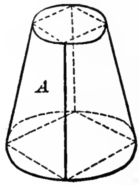
Ex. **839** What is the ratio of the lateral surface of a right circular cylinder to that of a right circular cone having the same base and altitude, if the altitude is 5 times the radius of the base?

Proposition XI

627. 1. To what is the lateral surface of a frustum of any regular pyramid equivalent? If the number of its lateral faces is indefinitely increased, what solid does the frustum of the pyramid approach as its limit? How, then, does the lateral surface of the frustum of any cone of revolution compare with the rectangle formed by its slant height and the sum of the circumferences of its bases?

2. How does the circumference of a section equidistant from the bases compare with half the sum of the circumferences of the bases?

Theorem. *The lateral surface of a frustum of a cone of revolution is equivalent to one half the rectangle formed by its slant height and the sum of the circumferences of its bases.*



Data: Any frustum of a cone of revolution, as A, whose slant height is L, and the circumferences of whose lower and upper bases are C and C', respectively.

Denote the lateral surface of the frustum by S.

To prove $S \Leftrightarrow \frac{1}{2}$ rect. $L \cdot (C + C')$.

Proof. Inscribe in the frustum of the cone the frustum of a regular pyramid, and denote its lateral surface by S', its slant height by L', and the perimeters of its lower and upper bases by P and P', respectively.

Then, § 552, $S' \Leftrightarrow \frac{1}{2}$ rect. $L' \cdot (P + P')$.

Now, if the number of lateral faces of the inscribed frustum is indefinitely increased,

§ 392, P and P' approach C and C', respectively, as their limits;

∴ L' approaches L as its limit,

and S' approaches S as its limit.

But, however great the number of faces,

$S' \Leftrightarrow \frac{1}{2}$ rect. $L' \cdot (P + P')$.

Hence, § 326, $S \Leftrightarrow \frac{1}{2}$ rect. $L \cdot (C + C')$.

Therefore, etc. Q.E.D.

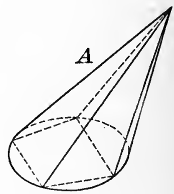
628. Cor. *The lateral surface of a frustum of a cone of revolution is equivalent to the rectangle formed by its slant height and the circumference of a section equidistant from its bases.*

Arithmetical Rules: To be formed by the student.

Proposition XII

629. To what is the volume of any pyramid equal? If the number of its lateral faces is indefinitely increased, what solid does the pyramid approach as its limit? To what, then, is the volume of any cone equal?

Theorem. *The volume of any cone is equal to one third the product of its base by its altitude.*



Data: Any cone, as A, whose base is B and altitude H.

Denote its volume by V.

To prove $V = \frac{1}{3} B \times H$.

Proof. Inscribe in the cone a pyramid, and denote its volume by V' and its base by B'.

Then, the altitude of the pyramid is H,

and, § 560, $V' = \frac{1}{3} B' \times H$.

Now, if the number of lateral faces of the inscribed pyramid is indefinitely increased,

§ 393, B' approaches B as its limit;

∴ V' approaches V as its limit.

But, however great the number of faces,

$$V' = \frac{1}{3} B' \times H.$$

Hence, § 222, $V = \frac{1}{3} B \times H$.

Therefore, etc. Q.E.D.

630. Formula: Let R denote the radius of a cone of revolution

Then, § 398, $B = \pi R^2$;

∴ $V = \frac{1}{3} \pi R^2 \times H$.

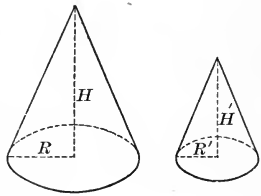
Ex. **840.** The slant height of a right circular cone is 21 ft. and its altitude is 15 ft. What is its total area?

Ex. **841.** The slant height of a right circular cone is 6^m and the radius of its base is 5^m. What is its lateral area? What is its volume?

Proposition XIII

631. Compute the volumes of any two similar cones of revolution, as those whose altitudes are 8″ and 4″, and the radii of whose bases are 6″ and 3″ respectively. How does the ratio of their volumes compare with the ratio of the cubes of their altitudes, or with the ratio of the cubes of the radii of their bases?

Theorem. *The volumes of similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Data: Any two similar cones of revolution, whose altitudes are H and H', and the radii of whose bases are R and R', respectively.

Denote their volumes by V and V', respectively.

To prove $V : V' = H^3 : H'^3 = R^3 : R'^3$.

Proof. Since the generating triangles are similar,

§ 299, $$\frac{H}{H'} = \frac{R}{R'};$$

∴ § 630, $$\frac{V}{V'} = \frac{\frac{1}{3}\pi R^2 H}{\frac{1}{3}\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3},$$

or $$V : V' = H^3 : H'^3 = R^3 : R'^3.$$

Therefore, etc. Q.E.D.

632. Cor. *The volumes of similar cones of revolution are to each other as the cubes of any of their like dimensions.*

Ex. **842.** What is the volume of a cone whose altitude is 13 ft. and the circumference of whose base is 9 ft.?

Ex. **843.** What is the total area of the frustum of a cone of revolution whose slant height is 17^{dm} and the radii of whose bases are 5^{dm} and 3^{dm}?

Ex. **844.** How far from the base must a cone of revolution whose altitude is 15 in. be cut by a plane parallel to the base so that the volume of the frustum shall be one half that of the entire cone?

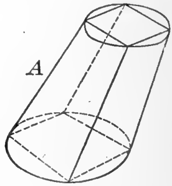
Ex. **845.** At what distances from the vertex must a cone of revolution be cut by planes parallel to the base to divide it into three equivalent solids?

Ex. **846.** A plane parallel to the base of a cone of revolution cuts the altitude at a point $\frac{2}{3}$ of the distance from the vertex. What is the ratio of the volume of the cone cut off to that of the original cone?

Proposition XIV

633. To what is the volume of a frustum of any pyramid equal? If the number of its faces is indefinitely increased, what solid does the frustum of the pyramid approach as its limit? To what, then, is the volume of the frustum of any cone equal?

Theorem. *The volume of a frustum of any cone is equal to one third the product of its altitude by the sum of its bases and a mean proportional between them.*



Data: Any frustum of any cone, as A, whose altitude is H, and whose lower and upper bases are B and b respectively.

Denote the volume of the frustum by V.

To prove $V = \frac{1}{3} H(B + b + \sqrt{B \times b})$.

Proof. Inscribe in the frustum of the cone the frustum of a pyramid, and denote its volume by V' and its lower and upper bases by B' and b', respectively.

Then, the altitude of the frustum of the pyramid is H,

and, § 564, $V' = \frac{1}{3} H(B' + b' + \sqrt{B' \times b'})$.

Now, if the number of lateral faces of the inscribed frustum is indefinitely increased,

§ 393, B' and b' approach B and b respectively as their limits;

∴ V' approaches V as its limit.

But, however great the number of faces,

$$V' = \frac{1}{3} H(B' + b' + \sqrt{B' \times b'}).$$

Hence, § 222, $V = \frac{1}{3} H(B + b + \sqrt{B \times b})$.

Therefore, etc. Q.E.D.

634 Formula: Let R and R' denote the radii of the bases of a frustum of a cone of revolution.

Then, § 398, $B = \pi R^2$, $b = \pi R'^2$,

and $\sqrt{B \times b} = \pi RR'$;

∴ $V = \frac{1}{3} \pi H(R^2 + R'^2 + RR')$.

Ex. **847.** What is the volume of the frustum of a cone whose altitude is 21 ft. and the circumferences of whose bases are 17 ft. and 13 ft. respectively?

635. FORMULÆ

Notation

B = base.	R = radius of base.
b = upper base.	R' = radius of upper base.
C = circumference of base.	H = altitude.
C' = circumference of upper base.	L = slant height.
C'' = circumference of mid-section.	A = lateral area.
	T = total area.
	V = volume.

Cylinder of Revolution.

$A = C \times H.$

$A = 2\pi RH.$

$T = 2\pi R(H + R).$

$V = B \times H.$ (True also for any cylinder.)

$V = \pi R^2 H.$

Cone of Revolution.

$A = \frac{1}{2} C \times L.$

$A = \pi RL.$

$T = \pi R(L + R).$

$V = \frac{1}{3} B \times H.$ (True also for any cone.)

$V = \frac{1}{3}\pi R^2 H.$

Frustum of a Cone of Revolution.

$A = \frac{1}{2} L(C + C').$

$A = L \times C''.$

$V = \frac{1}{3} H(B + b + \sqrt{B \times b}).$ (True also for any frustum.)

$V = \frac{1}{3}\pi H(R^2 + R'^2 + RR').$

SUPPLEMENTARY EXERCISES

Ex. **848**. What is the lateral area, total area, and volume of a cylinder of revolution whose diameter is 8 in. and altitude 12 in.?

Ex. **849**. What is the lateral area, total area, and volume of a cone of revolution whose base is 10^{cm} in diameter and whose altitude is 12^{cm}?

Ex. **850**. What is the lateral area, total area, and volume of a frustum of a cone of revolution, the radii of whose bases are 6 in. and 4 in., respectively, and whose altitude is 9 in.?

Ex. **851**. On a cylindrical surface only one straight line can be drawn through a given point.

Ex. **852**. The intersection of two planes tangent to a cylinder is parallel to an element.

Ex. **853**. How many square yards of canvas are required for a conical tent 18 ft. high and 10 ft. in diameter?

Ex. **854**. How many cubic feet are there in a piece of round timber 40 ft. long, whose ends are respectively 3 ft. and 1 ft. in diameter?

Ex. **855**. A cylindrical vessel is 40^{dm} long and 20^{dm} in diameter. What is the weight, in grams, of the water that it will hold?

Ex. **856**. What is the weight, in grams, of a piece of lead, if when put under water in a cylindrical tank 24^{cm} in diameter it causes the level of the water to rise 8^{cm}, the specific gravity of lead being 11.4?

Ex. **857**. A cylindrical cistern is 6.4^{m} deep and 5.2^{m} in diameter. How long will it take to fill it, if 2^{Hl} flow into it per minute?

Ex. **858**. A plane through a tangent to the base of a circular cylinder and the element drawn to the point of contact, is tangent to the cylinder.

Ex. **859**. A plane through a tangent to the base of a circular cone and the element drawn to the point of contact, is tangent to the cone.

Ex. **860**. The volumes of two similar cylinders of revolution are as 27 : 64. If the diameter of the first is 3 ft., what is the diameter of the second?

Ex. **861**. A cylindrical vessel holds 1728 grams of water. What are the dimensions of the vessel, if the diameter is one third of the altitude?

Ex. **862**. Show that any lateral face of a pyramid circumscribed about a circular cone is tangent to the cone.

Ex. **863**. What is the height of a cylinder 4.8^{dm} in diameter, if it is equivalent to a cone of revolution 5.6^{dm} in diameter and 6.4^{dm} high?

Ex. **864**. A cylindrical vessel is 12^{cm} in diameter and 20^{cm} high. How many grams of mercury would it hold, the specific gravity of mercury being 13.6?

Ex. **865**. The volumes of two similar cones of revolution are to each other as 512 : 729. What is the ratio of their lateral areas?

Ex. **866**. How many centigrams of alcohol will a cylindrical bottle hold, if the bottle is 8^{cm} in diameter and 24^{cm} high, the specific gravity of alcohol being .79?

Ex. **867**. The specific gravity of marble is 2.8. What is the weight, in kilograms, of a conical piece of marble, if the radius of its base is 20^{cm} and its height 50^{cm}?

Ex. **868**. The slant height of a cone of revolution is 3^{m}. How far from the vertex must the elements be cut by a plane parallel to the base in order that the lateral surface may be divided into two equivalent parts?

Ex. **869**. If the altitude of a cylinder of revolution is equal to the diameter of its base, the volume is equal to the product of its total area by one third of its radius.

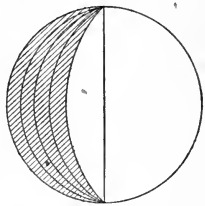
BOOK X

SPHERES

636. A solid bounded by a surface, every point of which is equally distant from a point within, is called a **Sphere.***

The point within is called the *center*.

A sphere may be generated by the revolution of a semicircle about its diameter as an axis.



637. A straight line drawn from the center to any point of the surface of a sphere is called a *radius*.

A straight line which passes through the center of a sphere, and whose extremities are in the surface, is called a *diameter*.

638. A line or plane which has one, and only one, point in common with the surface of a sphere is *tangent to the sphere*.

The sphere is then said to be *tangent* to the line or plane.

639. Two spheres whose surfaces have one, and only one, point in common are *tangent* to each other.

640. When all the faces of a polyhedron are tangent to a sphere, the sphere is said to be *inscribed in the polyhedron*.

641. When all the vertices of a polyhedron lie in the surface of a sphere, the sphere is said to be *circumscribed about the polyhedron*.

* In teaching *Spherical Geometry*, the class-room should be furnished with a spherical blackboard, on which the student should draw the diagrams required. It is also advised that each student be provided with some sort of a blackened or slated sphere for use in the preparation of lessons in cases where figures are to be drawn on its surface. A hemispherical cup to fit the sphere will enable him to draw great circles on the sphere.

642. Ax. 18. *All radii of the same sphere, or of equal spheres, are equal.*

19. *All diameters of the same sphere, or of equal spheres, are equal.*

20. *Two spheres are equal, if their radii or diameters are equal.*

Proposition I

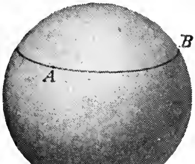
643. 1. Form a sphere and cut it by any plane (§ 519 N.). What plane figure is the section thus formed?

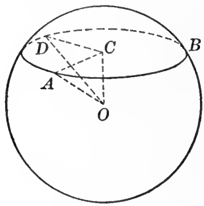
2. If a line joins the center of the sphere with the center of a circle of the sphere, what is its direction with reference to the plane of the circle?

3. Cut a sphere by planes which are equally distant from its center. How do the sections thus formed compare?

4. If the cutting planes are unequally distant from the center, which circle is the larger?

Theorem. *Any section of a sphere made by a plane is a circle.*





Data: A sphere; its center O; and any section, as ABD.

To prove ABD a circle.

Proof. Draw OC perpendicular to the plane ABD; draw the radii OA and OD to any two points in the perimeter of the section; and draw CA and CD.

Since O is a point in the perpendicular OC,

and, Ax. 18, $OA = OD$,

§ 449, $CA = CD$.

But A and D are any two points in the perimeter of section ABD.

Hence, § 173, ABD is a circle whose center is C. Q.E.D.

644. Cor. I. *The line joining the center of a sphere to the center of a circle of the sphere is perpendicular to the plane of the circle.*

645. Cor. II. *Circles of a sphere made by planes equally distant from the center are equal.*

646. Cor. III. *Of two circles of a sphere made by planes unequally distant from the center, the nearer is the larger.*

647. A section of a sphere made by a plane which passes through the center is called a **Great Circle of the sphere.**

648. A section of a sphere made by a plane which does not pass through the center is called a **Small Circle of the sphere.**

649. The diameter, which is perpendicular to the plane of a circle of a sphere, is called the **Axis of the circle.**

650. The ends of the axis of a circle of a sphere are called the **Poles of the circle.**

651. 1. Form a sphere and cut it by any plane, thus forming a circle of the sphere. Through what point of the circle does its axis pass?

2. Form a sphere and cut it by two parallel planes, thus forming two parallel circles. How are their axes situated with reference to each other? How, then, are the poles of one of these circles situated with reference to the poles of the other?

3. By passing planes form any two great circles of the same, or of equal spheres. How do they compare with each other?

4. Form a sphere and divide it into two parts by a great circle. How do these parts compare with each other?

5. By passing planes form any two great circles of a sphere. Since their intersection passes through the center and is a diameter of each circle, how do two great circles divide each other?

6. Form two great circles of a sphere by passing two planes through it perpendicular to each other. Where do these circles pass with reference to each other's poles? If two great circles pass through each other's poles, what is the direction of their planes with reference to each other?

7. Form a sphere and pass a plane through its center and any two points on its surface. What kind of a circle is the section thus formed? What kind of an arc, then, may be drawn through any two points on the surface of a sphere?

8. Form a sphere and pass a plane through any three points on its surface. What plane figure is the section thus formed? How many planes may be passed through the three points? How many circles, then, may be drawn through any three points on the surface of a sphere?

652. *The axis of a circle passes through the center of that circle.*

653. *Parallel circles have the same axis and the same poles.*

654. *Great circles of the same sphere, or of equal spheres, are equal.*

655. *Any great circle of a sphere bisects the sphere.*

656. *Two great circles of the same sphere bisect each other.*

657. *Two great circles whose planes are perpendicular pass through each other's poles;* and conversely.

658. *Through two given points on the surface of a sphere an arc of a great circle may be drawn.*

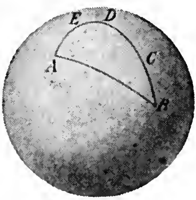
659. *Through three given points on the surface of a sphere one circle may be drawn, and only one.*

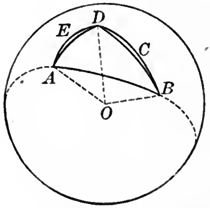
Proposition II

660. Form a sphere and select any two points on its surface; through these points and the center of the sphere pass a plane. What kind of a circle is this section? Then, what kind of an arc joins the given points?

Join them by any other line on the surface. Which line represents the shortest distance between the given points?

Theorem. *The shortest distance on the surface of a sphere between any two points on that surface is the arc, not greater than a semicircumference, of the great circle which joins them.*





Data: Any two points on the surface of a sphere, as A and B, joined by the arc of a great circle, as AB, not greater than a semicircumference; also any other line on the surface joining A and B, as $AECB$.

To prove AB less than $AECB$.

Proof. Take any point in $AECB$, as D, and pass arcs of great circles through A and D, and B and D. Draw OA, OB, and OD from O, the center of the sphere.

Then, ∠s AOB, AOD, and BOD are the face ∠s of the trihedral ∠ whose vertex is at O;

∴ § 498, $\angle AOD + \angle BOD$ is greater than $\angle AOB$.

But, § 224, arcs AD, BD, and AB are the measures of ∠s AOD, BOD, and AOB respectively;

∴ $\text{arc } AD + \text{arc } BD > \text{arc } AB$.

In like manner, joining any point in AED with A and D, and any point in DCB with D and B by arcs of great circles, the sum of these arcs will be greater than arc AD + arc BD, and therefore greater than arc AB.

If this process is indefinitely repeated, the points common to $AECB$ and the path from A to B on the great circle arcs will approach as near each other as we please, and the sum of these arcs will continually increase and approach $AECB$ as a limit.

But the sum of the great circle arcs is always greater than AB.

Therefore, AB is less than $AECB$. Q.E.D.

661. By the *distance* between two points on the surface of a sphere is meant the *shortest distance;* that is, the *arc of a great circle* joining them.

662. The distance from the nearer pole of a circle to any point in its circumference is called the **Polar Distance of the circle.**

Proposition III

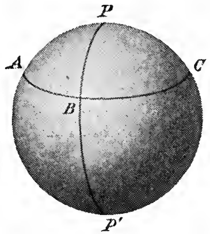
663. 1. Form a sphere and cut it by any plane; pass planes through the axis of the circle thus formed and any points in its circumference. What kind of arcs, then, connect the pole of the circle and the points of its circumference? How do these arcs compare? Then, how do the distances from the pole of a circle of a sphere to all the points in its circumference compare?

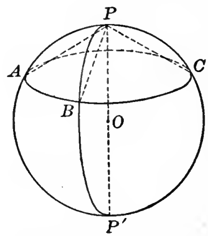
2. If the circle is a great circle, what part of a circumference is its polar distance?

3. By passing planes form two equal circles of the same or of equal spheres. How do their polar distances compare?

4. Select a point on the surface of a sphere which is at a quadrant's distance from each of two other points. Where is this point situated with reference to a pole of a great circle that passes through the other two points?

Theorem. *All points in the circumference of a circle of a sphere are equally distant from a pole of the circle.*





Data: Any circle of a sphere, as ABC, and its poles, P and P'.

To prove all points in the circumference of ABC equally distant from P and also from P'.

Proof. Draw great circle arcs from P to any points in the circumference of ABC, as A, B, and C.

§§ 649, 652, $PP' \perp ABC$ at its center;

∴ § 450, chords PA, PB, and PC are equal;

hence, § 196, arcs PA, PB, and PC are equal.

In like manner, arcs $P'A$, $P'B$, and $P'C$ may be proved equal.

But A, B, and C are any points in the circumference of ABC.

Hence, § 661, all points in the circumference of ABC are equally distant from P and also from P'.

Therefore, etc. Q.E.D.

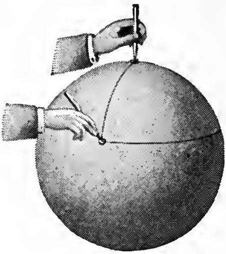
664. Cor. I. *The polar distance of a great circle is a quadrant.**

665. Cor. II. *The polar distances of equal circles on the same, or on equal spheres, are equal.*

* In *Spherical Geometry* the term *quadrant* generally means the *quadrant of a great circle.*

666. Cor. III. *A point, which is at the distance of a quadrant from each of two other points on the surface of a sphere, is a pole of the great circle passing through those points.*

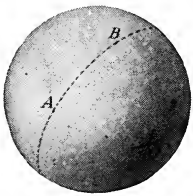
667. Sch. I. By using the facts demonstrated in § 663 and in § 664 we may draw the circumferences of small and great circles on the surface of a material sphere.



To draw the circumference of a circle, take a cord equal to its polar distance, and, placing one end of it at the pole, cause a pencil held at the other to trace the circumference, as in the figure.

To describe the circumference of a great circle, a quadrant must be used as the arc.

668. Sch. II. By means of § 666 we are enabled to pass the circumference of a great circle through any two points, as A and B, on the surface of a material sphere in the following manner:



From each of the given points, as poles, and with a quadrant arc, draw arcs to intersect, as at O. The circumference described from this intersection with a quadrant arc will be the circumference required.

Proposition IV

669. 1. If a plane is perpendicular to a radius of a sphere at its extremity, how many points do the sphere and the plane have in common? What name is given to such a plane?

2. How many points do a straight line and a sphere have in common, if the line is perpendicular to a radius of the sphere at its extremity? What name is given to such a line?

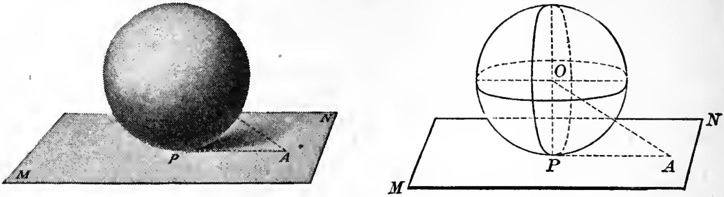
3. What is the direction of every plane or line, that is tangent to a sphere, with reference to the radius drawn to the point of contact?

4. If a straight line is tangent to any circle of a sphere, how does it lie with reference to a plane tangent to the sphere at the point of contact?

5. If a plane is tangent to a sphere, what is the relation to the sphere of any line drawn in that plane and through the point of contact?

6. If two straight lines are tangent to a sphere at the same point, what is the relation of the plane of those lines to the sphere?

Theorem. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



Data: Any sphere, and a plane, as MN, perpendicular to a radius, as OP, at its extremity P.

To prove MN tangent to the sphere.

Proof. Take any point except P in MN, as A, and draw OA.

Then, § 448, $OP < OA$;

$\therefore$ point A is without the sphere.

But A is any point in MN except P;

$\therefore$ every point in MN except P is without the sphere.

Hence, § 638, MN is tangent to the sphere at P. Q.E.D.

670. Cor. I. *Any straight line perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*

671. Cor. II. *Any plane or line tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

672. Cor. III. *A straight line tangent to any circle of a sphere lies in the plane tangent to the sphere at the point of contact.*

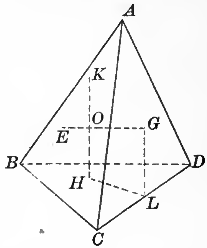
673. Cor. IV. *Any straight line drawn in a tangent plane and through the point of contact is tangent to the sphere at that point.*

674. Cor. V. *Any two straight lines tangent to a sphere at the same point determine the tangent plane at that point.*

Proposition V

675. Select any four points not in the same plane; form a tetrahedron of which these points are the vertices; and at the centers of the circles circumscribed about any two of its faces erect perpendiculars. How do the distances from any point in either perpendicular to the vertices of the face to which it is perpendicular compare? If these perpendiculars intersect, how, then, do the distances from their intersection to the four given points compare? Is there any other point that is equidistant from the four given points? What surface, then, may be passed through these four points? How many such surfaces may be passed through them?

Theorem. *Through any four points not in the same plane one spherical surface may be passed, and only one.*



Data: Any four points not in the same plane, as A, B, C, D.

To prove that one spherical surface may be passed through A, B, C, D, and only one.

Proof. Suppose H and G to be the centers of circles circumscribed about the triangles BCD and ACD, respectively.

Draw $HK \perp$ plane BCD, and $GE \perp$ plane ACD.

§ 450, every point in HK is equidistant from points B, C, D, and every point in GE is equidistant from points A, C, D.

From H and G draw lines to L, the middle point of CD.

Then, § 106, $HL \perp CD$, and $GL \perp CD$;

∴ § 444, the plane through HL and GL is perpendicular to CD, and, § 483, this plane is perpendicular to planes BCD and ACD.

Const., $GE \perp$ plane ACD at G;

∴ § 481, GE lies in the plane HLG.

In like manner it may be shown that HK lies in plane HLG.

Hence, the perpendiculars HK and GE lie in the same plane, and, being perpendicular to planes which are not parallel, they must intersect at some point, as at O.

Since O is in the perpendiculars HK and GE, it is equidistant from B, C, D, and from A, C, D.

Hence, O is equidistant from A, B, C, D, and the surface of the sphere, whose center is O and radius OB, will pass through the points A, B, C, D.

Now, § 450, the center of any sphere whose surface passes through the four points A, B, C, D must be in the perpendiculars HK and GE.

Hence, O, the intersection of HK and GE, must be the center of the *only* sphere whose surface can pass through A, B, C, D.

Therefore, etc. Q.E.D.

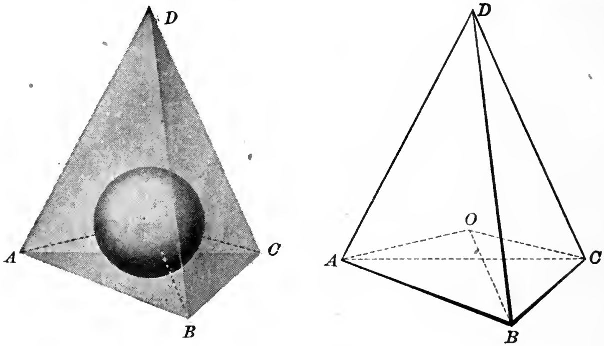
676. Cor. I. *A sphere may be circumscribed about any tetrahedron.*

677. Cor. II. *The four perpendiculars to the faces of a tetrahedron through their centers meet at the same point.*

Proposition VI

678. Form any tetrahedron and pass planes bisecting any three of its dihedral angles which have one face in common. How do the distances from the point of intersection of these bisecting planes to the faces of the tetrahedron compare? What figure, then, may be inscribed in any tetrahedron?

Theorem. *A sphere may be inscribed in any tetrahedron.*



Datum: Any tetrahedron, as $D\text{–}ABC$.

To prove that a sphere may be inscribed in $D\text{–}ABC$.

Proof. Bisect any three of the dihedral angles which have one face common, as AB, BC, AC, by the planes OAB, OBC, OAC, respectively.

By § 488, every point in the plane OAB is equidistant from the faces ABC and ABD.

Also every point in plane OBC is equidistant from the faces ABC and BCD; and every point in the plane OAC is equidistant from the faces ABC and ACD.

Therefore, point O, the intersection of these three planes, is equidistant from the four faces of the tetrahedron.

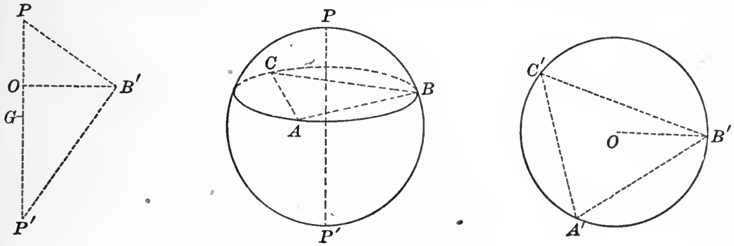
Hence, § 638, a sphere with O as a center and with a radius equal to the distance from O to any face will be tangent to each face, and, § 640, it will be inscribed in the tetrahedron.

Therefore, etc. Q.E.D.

679. Cor. *The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.*

Proposition VII

680. *Problem.* *To find the radius of a material sphere.*



Datum: Any sphere, as APP'.

Required to find the radius of APP'.

Solution. From any point P as a pole describe any circumference on the surface. § 667. From any three points in this circumference, as A, B, C, measure the chord distances AB, BC, AC. (Use compasses with curved branches.)

Construct the $\triangle A'B'C'$ having its sides equal respectively to AB, BC, AC, and circumscribe about it a circle.

With the radius OB' as a side, and PB' equal to the chord of the arc joining P and B, as hypotenuse, construct the rt. $\triangle PB'O$.

Draw a line from B' perpendicular to $B'P$, and produce it to meet PO produced in P'.

Then, PP' thus determined is equal to the diameter of the sphere, and its half, PG, is the required radius. Q.E.F.

Proof. By the student.

Ex. **870**. Equal straight lines whose extremities are in the surface of a sphere are equally distant from the center of the sphere.

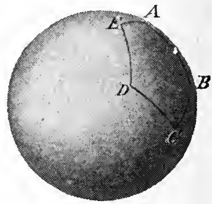
Ex. **871**. The six planes which bisect at right angles the six edges of a tetrahedron all intersect at the same point.

SPHERICAL ANGLES AND POLYGONS

681. The *angle between two intersecting curves* is the angle contained by the two tangents to the curves at their intersection.

682. The angle between two intersecting arcs of great circles is called a **Spherical Angle.**

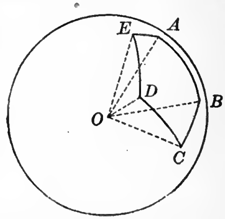
683. A portion of the surface of a sphere bounded by three or more arcs of *great circles* is called a **Spherical Polygon.**



The bounding arcs are the *sides* of the polygon; the angles which they form are the *angles* of the polygon; and the points of intersection are the *vertices* of the polygon.

An arc of a great circle joining any two non-adjacent vertices of a spherical polygon is a *diagonal.*

684. The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the center of the sphere, and whose face angles are measured by the sides of the polygon.



O–ABCDE is a polyhedral angle whose vertex *O* is the center of the sphere and whose face angles *EOD*, *DOC*, etc., are measured by the sides *ED*, *DC*, etc., of the spherical polygon *ABCDE*.

685. A spherical polygon whose corresponding polyhedral angle is *convex* is called a **Convex Spherical Polygon.**

Spherical polygons will be regarded as *convex* unless otherwise specified.

686. A spherical polygon of three sides is called a **Spherical Triangle.**

A spherical triangle is *right, oblique, equilateral, isosceles,* etc., under the same conditions that plane triangles are right, oblique, etc.

687. The sides of a spherical polygon, being arcs, are usually measured in *degrees, minutes,* and *seconds.*

688. Any two points on the surface of a sphere may be joined by two arcs of a great circle, one of which will usually be greater and the other usually less than a semicircumference.

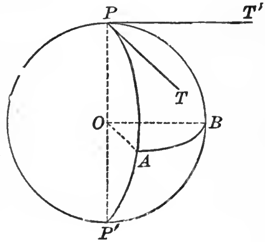
Unless otherwise stated the less arc is always meant.

Proposition VIII

689. **1.** Form a sphere and construct on it a spherical angle (§ 667); pass planes through the sides of the angle and the center of the sphere; on the part thus cut out draw a great circle arc with the vertex of the angle as a pole, and draw the radii to the extremities of this arc. How does the angle between the radii compare with the angle contained by the tangents to the sides of the spherical angle at their intersection? What arc measures the angle between the radii? Then, what arc measures the given spherical angle?

2. How does a spherical angle compare with the dihedral angle formed by the planes of its sides?

Theorem. *A spherical angle is measured by the arc of a great circle described from its vertex as a pole and included by its sides, produced if necessary.*



Data: Any spherical angle, as APB, and the arc of a great circle, as AB, described from the vertex P, and included between the sides AP and BP.

To prove $\angle APB$ measured by arc AB.

Proof. Draw PT and PT' tangent to AP and BP, respectively at P; also draw the radii OA, OB, and the diameter PP'.

Const., § 205, $PT \perp PP'$ in plane PAP',

and since, data, PA is a quadrant,

$OA \perp PP'$ in plane PAP'; Why?

∴ § 71, $PT \parallel OA$,

and similarly, $PT' \parallel OB$;

∴ § 469, $\angle TPT' = \angle AOB$.

But, § 224, $\angle AOB$ is measured by arc AB;

∴ $\angle TPT'$ is measured by arc AB;

that is, § 681, $\angle APB$ is measured by arc AB.

Therefore, etc. Q.E.D.

690. Cor. I. *A spherical angle has the same measure as the dihedral angle formed by the planes of its sides.*

691. Cor. II. *If two sides of a spherical triangle are quadrants, the third side measures the angle opposite.*

692. Cor. III. *If each side of a spherical triangle is a quadrant, each angle is a right angle.*

693. Cor. IV. *If two arcs of great circles cut each other, their vertical angles are equal.*

694. Cor. V. *The angles of a spherical polygon are equal to the dihedral angles between the planes of the sides of the polygon.*

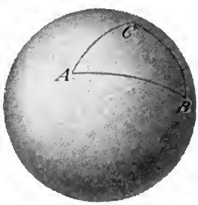
695. Sch. Since, §§ 684, 694, the sides and angles of a spherical polygon have respectively the same measures as the face and dihedral angles of the corresponding polyhedral angle, *we may, from any property of polyhedral angles, infer an analogous property of spherical polygons;* and conversely.

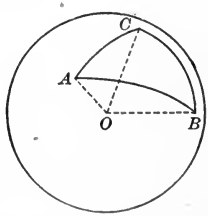
Proposition IX

696. 1. Form a sphere and draw on it a spherical triangle; pass planes through the sides of the triangle and the center of the sphere, and thus cut out the corresponding trihedral angle. How does the sum of any two face angles of the trihedral compare with the third face angle? How, then, does the sum of any two sides of the spherical triangle compare with the third side?

2. How does any side compare with the difference of the other two sides?

Theorem. *The sum of any two sides of a spherical triangle is greater than the third side.*





Data: Any spherical triangle, as ABC, on the sphere whose center is O.

To prove the sum of any two sides, as $AC + BC$, greater than the third side AB.

Proof. In the corresponding trihedral angle $O\text{–}ABC$,

§ 498, $\angle AOC + \angle BOC$ is greater than $\angle AOB$;

$\therefore$ § 695, $AC + BC$ is greater than AB.

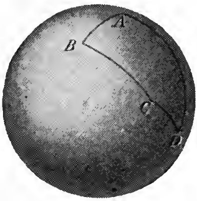
Therefore, etc. Q.E.D.

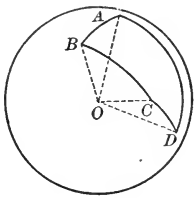
697. Cor. I. *Any side of a spherical triangle is greater than the difference of the other two sides.*

Proposition X

698. Form a sphere and draw on it a spherical polygon; pass planes through the sides of the polygon and the center of the sphere, and thus cut out the corresponding polyhedral angle. How does the sum of the face angles of the polyhedral compare with four right angles? How, then, does the sum of the sides of the spherical polygon compare with the circumference of a great circle?

Theorem. *The sum of the sides of a spherical polygon is less than the circumference of a great circle.*





Data: Any spherical polygon, as $ABCD$, on the sphere whose center is O.

To prove $AB + BC + CD + DA <$ the circumference of a great circle.

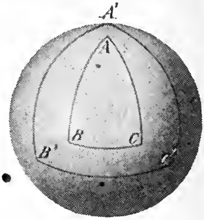
Proof. In the corresponding polyhedral angle $O\text{–}ABCD$,

§ 499, $\angle AOB + \angle BOC + \angle COD + \angle DOA < 4$ rt. $\angle$s;

$\therefore$ § 695, $AB + BC + CD + DA <$ the circum. of a great circle.

Therefore, etc. Q.E.D.

699. If from the vertices of a spherical triangle as poles arcs of great circles are described, these arcs form by their intersections a second triangle which is called the **Polar Triangle** of the first.



If A, B, and C are the poles of the great circle arcs $B'C'$, $A'C'$, and $A'B'$, respectively, then, $A'B'C'$ is the *polar triangle* of ABC.

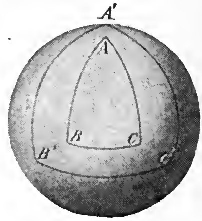
If from A, B, and C as poles entire great circles instead of arcs are described, these circles will divide the surface of the sphere into eight spherical triangles.

Of these eight triangles, that one is the polar of ABC whose vertex A' corresponding to A lies on the same side of BC as the vertex A; and in the same way the other corresponding vertices may be determined.

Proposition XI

700. On the surface of a sphere draw a spherical triangle; draw also its polar triangle. Test the first triangle to see if it is the polar triangle of the second.

Theorem. *If one of two spherical triangles is the polar triangle of the other, then the other is the polar triangle of that one.*



Data: Any spherical triangle, as ABC, and its polar triangle, $A'B'C'$.

To prove ABC the polar triangle of $A'B'C'$.

Proof. § 699, A is the pole of arc $B'C'$;

∴ § 664, B' is at a quadrant's distance from A.

Also, C is the pole of arc $A'B'$;

∴ B' is at a quadrant's distance from C.

Hence, § 666, B' is the pole of arc AC.

In like manner it may be shown that,

A' is the pole of arc BC, and C' the pole of arc AB.

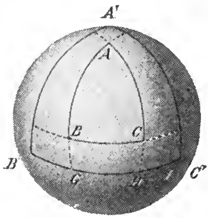
Hence, § 699, ABC is the polar triangle of $A'B'C'$.

Therefore, etc. **Q.E.D.**

Proposition XII

701. On the surface of a sphere, draw a spherical triangle and its polar triangle; select an angle of one of these and extend its sides, if necessary, to meet the opposite side of the other triangle. What part of a circumference is the distance from each point of meeting to the farthest extremity of that opposite side? Then, how does the *arc* intercepted on that side by the sides of the given angle compare with two quadrants less the *whole* side, or with 180° less the whole side? How, then, does the measure of the given angle compare with the supplement of the opposite side of the polar triangle?

Theorem. *In two polar triangles any angle of the one is measured by the supplement of the opposite side of the other.*



Data: Any two polar triangles, as ABC and $A'B'C'$, and any angle of either triangle, as A.

To prove $\angle A$ measured by $180° - B'C'$.

Proof. Produce the arcs AB and AC until they meet $B'C'$ at the points G and H, respectively.

Since, data, A is the pole of arc GH,

§ 689, $\angle A$ is measured by GH.

Since, data, B' and C' are the poles of arcs AH and AG, respectively,

arcs $B'H$ and $C'G$ are quadrants.

Hence, $B'H + C'G =$ a semicircumference;

that is, $B'C' + GH =$ a semicircumference $= 180°$;

$\therefore$ $GH = 180° - B'C'$.

But $\angle A$ is measured by GH.

Hence, $\angle A$ is measured by $180° - B'C'$.

Therefore, etc. Q.E.D.

702. Two polar triangles are also called **Supplemental Triangles.**

For, if we denote the number of angle degrees in each angle by the letter at the vertex, and the number of arc degrees in each opposite side by the corresponding small letter, we have the following relations:

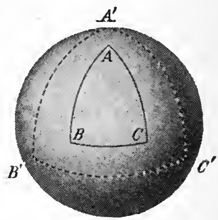
$$\angle A = 180° - a', \ \angle B = 180° - b', \ \angle C = 180° - c',$$
$$\angle A' = 180° - a, \ \angle B' = 180° - b, \ \angle C' = 180° - c.$$

Proposition XIII

703. On the surface of a sphere draw a spherical triangle; draw also its polar triangle. What is the measure of each angle of the given triangle? What, then, is the sum of the measures of its three angles? Since the sum of the arcs which form the sides of the polar triangle is greater than an arc of 0° and (§ 698) less than an arc of 360°, what is the greatest number and also the least number of degrees that there can be in the sum of the angles of the given triangle or any spherical triangle? Express your conclusions in terms of right angles.

Theorem. *The sum of the angles of a spherical triangle is greater than two and less than six right angles.*

Data: Any spherical triangle, as ABC, whose angles are A, B, and C.



To prove 1. $\angle A + \angle B + \angle C > 2$ rt. $\angle$s.

2. $\angle A + \angle B + \angle C < 6$ rt. $\angle$s.

Proof. 1. Construct $A'B'C'$, the polar triangle of ABC, and denote the number of degrees in $B'C'$, $A'C'$, and $A'B'$ by a', b', and c', respectively.

Then, § 701, $\angle A = 180° - a'$, $\angle B = 180° - b'$, and $\angle C = 180° - c'$;

$\therefore$ Ax. 2, $\angle A + \angle B + \angle C = 540° - (a' + b' + c')$.

But, § 698, $B'C' + A'C' + A'B' <$ the circum. of a great circle;

that is, $a' + b' + c' < 360°$;

$\therefore$ $\angle A + \angle B + \angle C > 180°$, or 2 rt. $\angle$s.

2. $a' + b' + c' > 0°$;

hence, $\angle A + \angle B + \angle C < 540°$, or 6 rt. $\angle$s.

Therefore, etc. Q.E.D.

704. Cor. *A spherical triangle may have two, or even three, right angles; or it may have two, or even three, obtuse angles.*

Ex. **872**. The sides of a spherical triangle are 65°, 86°, and 98°. What are the angles of its polar triangle?

Ex. **873**. The angles of a spherical triangle are 53°, 77°, and 92°. What are the sides of its polar triangle?

Ex. **874**. The angles of a spherical triangle are 65°, 80°, and 110°. What are the sides of its polar triangle?

705. A spherical triangle having two right angles is said to be *birectangular;* and one having three right angles is said to be *trirectangular.*

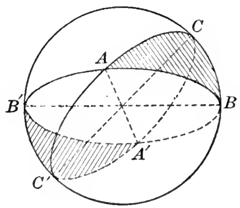
706. The excess of the sum of the angles of a spherical triangle over two right angles is called the **Spherical Excess of the triangle.**

707. The excess of the sum of the angles of a spherical polygon over two right angles, taken as many times as the polygon has sides less two, is called the **Spherical Excess of the polygon.**

If a polygon has n sides, its spherical excess is equal to the sum of the spherical excesses of the $n-2$ spherical triangles into which the polygon may be divided by diagonals from any vertex.

708. Spherical triangles in which the sides and angles of the one are equal respectively to the sides and angles of the other, but arranged in the reverse order, are called **Symmetrical Spherical Triangles.**

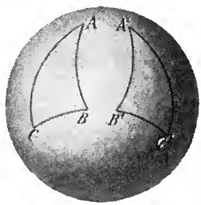
Two spherical triangles are *symmetrical,* when the vertices of one are at the ends of the diameters from the vertices of the other.



Triangles ABC and $A'B'C'$ are *symmetrical spherical triangles.*

Symmetrical spherical triangles are mutually equilateral and equiangular, and the equal sides are opposite the equal angles, yet they cannot generally be made to coincide.

To make the symmetrical triangles ABC and $A'B'C'$ coincide, any arc BC must be made to coincide with its equal $B'C'$. This can be done in only two ways—with B either on B' or on C'. When superposed with B on C', unless the triangles are isosceles, angles B and C' are unequal and the triangles will not coincide; with B on B', A and A' fall on opposite sides of $B'C'$ and the triangles will not coincide.

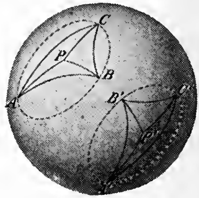


Symmetrical spherical triangles which are isosceles can be made to coincide.

Proposition XIV

709. On the surface of a sphere draw two symmetrical triangles. How do they compare in area? Are the triangles equal or equivalent?

Theorem. *Two symmetrical spherical triangles are equivalent.*



Data: Any two symmetrical spherical triangles, as ABC and $A'B'C'$.

To prove $\triangle ABC \Leftrightarrow \triangle A'B'C'$.

Proof. Case I. *When they are isosceles.*

If isosceles, they may be made to coincide;

$\therefore$ area $ABC =$ area $A'B'C'$.

Case II. *When the triangles are not isosceles.*

Suppose P and P' to be the poles of the small circles passing through the points A, B, C, and A', B', C', respectively.

Data, arcs AB, AC, $BC =$ arcs $A'B'$, $A'C'$, $B'C'$, respectively;

$\therefore$ § 196, chords of arcs AB, AC, $BC =$ chords of arcs $A'B'$, $A'C'$, $B'C'$, respectively; hence, § 107, the plane triangles formed by these chords are equal;

$\therefore$ § 208, the small circles through A, B, C, and A', B', C' are equal.

Draw the great circle arcs PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

Then, § 665, these arcs are equal.

Now, §§ 695, 501, the angles of $\triangle PAB$ are equal to the angles of $\triangle P'A'B'$, respectively, and the equal parts of the triangles are in reverse order;

$\therefore$ §§ 708, 686, $\triangle$s PAB and $P'A'B'$ are symmetrical and isosceles, and, Case I, area $PAB =$ area $P'A'B'$.

In like manner, area $PBC =$ area $P'B'C'$, and area $PAC =$ area $P'A'C'$;

$\therefore$ area $PAB + PBC + PAC =$ area $P'A'B' + P'B'C' + P'A'C'$,

or area $ABC =$ area $A'B'C'$; that is, $\triangle ABC \Leftrightarrow \triangle A'B'C'$.

If the pole P should be without $\triangle ABC$, then P' would be without $\triangle A'B'C'$, and each triangle would be equivalent to the sum of two isosceles triangles diminished by the third; consequently, the result would be the same as before.

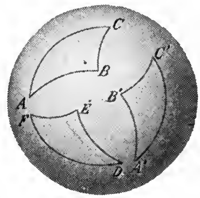
Therefore, etc. Q.E.D.

Proposition XV

710. 1. On the same sphere, or on equal spheres, draw two spherical triangles having two sides and the included angle of one equal to the corresponding parts of the other, and arranged in the same order. Can the triangles be made to coincide? Then, how do they compare?

2. Draw two spherical triangles as before, but with the given equal parts arranged in the reverse order; draw another triangle symmetrical to one of these. How does it compare with the other? Then, are the given triangles equal or equivalent?

Theorem. *Two triangles on the same, or on equal spheres, having two sides and the included angle of one equal to two sides and the included angle of the other, each to each, are either equal or equivalent.*



Data: Two spherical triangles, as ABC and DEF, in which $AB = DE$, $AC = DF$, and angle A = angle D.

Case I. *When the given equal parts of the two triangles are arranged in the same order.*

To prove $\triangle ABC = \triangle DEF$.

Proof. The $\triangle ABC$ can be applied to the $\triangle DEF$, as in the corresponding case of plane triangles, and they will coincide.

Hence, § 36, $\triangle ABC = \triangle DEF$.

Case II. *When the given equal parts of the two triangles are arranged in reverse order*, as in triangles ABC and $A'B'C'$ in which $AB = A'B'$, $AC = A'C'$, and $\angle A = \angle A'$.

To prove $\triangle ABC \approx \triangle A'B'C'$.

Proof. Suppose the $\triangle DEF$ to be symmetrical with respect to the $\triangle A'B'C'$.

Then, § 708, the sides and angles of $\triangle DEF$ are equal respectively to those of $\triangle A'B'C'$;

$\therefore$ in the $\triangle$s ABC and DEF, $\angle A = \angle D$, $AB = DE$, and $AC = DF$, and the equal parts are arranged in the same order;

$\therefore$ Case I, $\triangle ABC = \triangle DEF$.

But, § 709, $\triangle A'B'C' \approx \triangle DEF$.

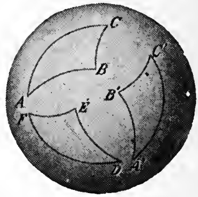
Hence, $\triangle ABC \approx \triangle A'B'C'$. Q.E.D.

Proposition XVI

711. 1. On the same sphere, or on equal spheres, draw two spherical triangles having a side and two adjacent angles of one equal to the corresponding parts of the other, and arranged in the same order. Can these triangles be applied to each other so that they will coincide? Then, how do they compare?

2. Draw two spherical triangles as before, but with the given equal parts arranged in the reverse order; draw another triangle symmetrical to one of these. How does it compare with the other? Then, are the given triangles equal or equivalent?

Theorem. *Two triangles on the same sphere, or on equal spheres, having a side and two adjacent angles of one equal to a side and two adjacent angles of the other, each to each, are either equal or equivalent.*



Proof. One of the triangles may be applied to the other, or to its symmetrical triangle, as in the corresponding case of plane triangles.

Therefore, etc. Q.E.D.

Proposition XVII

712. On the same sphere, or on equal spheres, draw two mutually equilateral spherical triangles. How do the angles of one compare with the angles of the other? If the equal parts are arranged in the same order in each, how do the triangles compare? If the equal parts are in reverse order, are the triangles equal or equivalent?

Theorem. *Two mutually equilateral triangles on the same sphere, or on equal spheres, are mutually equiangular, and are either equal or equivalent.*

Proof. By §§ 695, 501, the triangles are mutually equiangular;

$\therefore$ they are equal or symmetrical. Why?

If they are symmetrical,

then, § 709, they are equivalent.

Hence, they are either equal or equivalent.

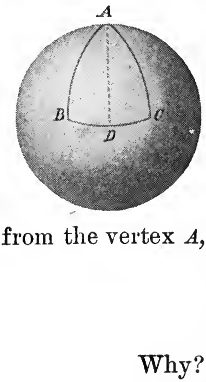
Therefore, etc. Q.E.D.

Proposition XVIII

713. 1. On the surface of a sphere, draw an isosceles spherical triangle; draw the arc of a great circle from its vertex to the middle of the opposite side. In the two triangles thus formed, how do the sides of one compare with the sides of the other? Then, how do the angles of one compare with the angles of the other? In the original triangle, how do the angles opposite the equal sides compare with each other?

2. How does the great circle arc from the vertex to the middle of the base of an isosceles spherical triangle divide the vertical angle? What is its direction with reference to the base? Into what kind of triangles does it divide the given triangle?

Theorem. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*



Data: An isosceles spherical triangle, as ABC, in which $AB = AC$.

To prove $\angle B = \angle C$.

Proof. Draw the arc of a great circle, as AD, from the vertex A, bisecting the side BC.

Then, in $\triangle ABD$ and ACD,

AD is common,

$AB = AC$, $DB = DC$; Why?

that is, the triangles are mutually equilateral;

$\therefore$ § 712, $\triangle ABD$ and ACD are mutually equiangular.

Hence, $\angle B = \angle C$.

Therefore, etc. Q.E.D.

714. Cor. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.*

Ex. **875.** If the sides of a spherical triangle are 50°, 75°, and 110°, what are the angles of its polar triangle?

Ex. **876.** If the sides of a spherical triangle are 54°, 89°, and 103°, what is the spherical excess of its polar triangle?

Proposition XIX

715. On the surface of a sphere, draw two mutually equiangular triangles; draw also their polars. How do the sides of their polars compare, each to each? Then, how do the angles of the polars compare, each to each? How, then, do the sides of the given triangles compare, each to each? If the equal parts in the given triangles are arranged in the same order in each, how do the triangles compare? If the equal parts are arranged in reverse order, are the triangles equal or equivalent?

Theorem. *Two mutually equiangular triangles on the same sphere, or on equal spheres, are mutually equilateral, and are either equal or equivalent.*

Data: Two spherical triangles, as A and B, that are mutually equiangular.

To prove $\triangle$s A and B mutually equilateral, and either equal or equivalent.

Proof. Suppose $\triangle A'$ to be the polar of $\triangle A$, and $\triangle B'$ the polar of $\triangle B$.

Data, $\triangle$s A and B are mutually equiangular;

∴ § 701, their polar $\triangle$s, A' and B', are mutually equilateral;

hence, § 712, $\triangle$s A' and B' are mutually equiangular;

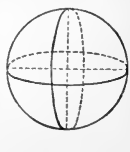
∴ § 701, $\triangle$s A and B are mutually equilateral.

Hence, § 712, $\triangle$s A and B are either equal or equivalent.

Therefore, etc. Q.E.D.

716. Cor. I. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.*

717. Cor. II. *If three planes are passed through the center of a sphere, each perpendicular to the other two, they divide the surface of the sphere into eight equal trirectangular triangles.* § 695

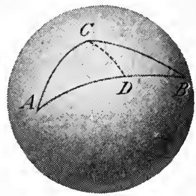


Proposition XX

718. 1. On the surface of a sphere draw a spherical triangle, two of whose angles are unequal. How do the sides opposite these angles compare? Which one is the greater?

2. Draw a spherical triangle, two of whose sides are unequal. How do the angles opposite these sides compare? Which one is the greater?

Theorem. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle;* ***conversely,*** *if two sides are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



Data: A spherical triangle, as ABC, in which the angle ACB is greater than the angle ABC.

To prove $AB > AC$.

Proof. Draw CD, the arc of a great circle, making $\angle BCD = \angle B$.

Then, § 716, $DB = CD$.

Now, § 696, $AD + CD > AC$;

$\therefore$ $AD + DB > AC$,

or $AB > AC$.

Conversely: Data: A spherical triangle, as ABC, in which the side AB is greater than the side AC.

To prove $\angle ACB$ greater than $\angle B$.

Proof. If $\angle ACB = \angle B$,

then, § 716, $AB = AC$,

which is contrary to data.

If $\angle ACB$ is less than $\angle B$,

then, $\angle B$ is greater than $\angle ACB$,

and $AC > AB$,

which is also contrary to data.

Therefore, both hypotheses, namely, that $\angle ACB = \angle B$ and that $\angle ACB$ is less than $\angle B$, are untenable.

Consequently, $\angle ACB$ is greater than $\angle B$.

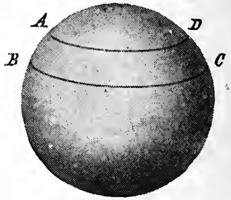
Therefore, etc. Q.E.D.

SPHERICAL MEASUREMENTS

719. The portion of the surface of a sphere included between two parallel planes is called a **Zone**.

The perpendicular distance between the planes is the *altitude of the zone*, and the circumferences of the sections made by the planes are called the *bases of the zone*.

If one of the parallel planes is tangent to the sphere, the zone is called a *zone of one base*.

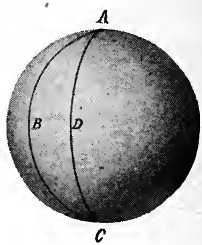


ABCD is a *zone* of the sphere.

720. The portion of the surface of a sphere bounded by two semicircumferences of great circles is called a **Lune**.

The angle between the semicircumferences which form its boundaries is called the *angle of the lune*.

ABCD is a *lune* of which *BAD* is the *angle*.



721. Lunes on the same sphere, or on equal spheres, having equal angles may be made to coincide, and are *equal*.

722. A convenient *unit of measure* for the surfaces of spherical figures is the *spherical degree*, which is equal to $\frac{1}{360}$ of the surface of a hemisphere.

Like the unit of arcs, it is not a unit of fixed magnitude, but depends upon the size of the sphere upon which the figure is drawn.

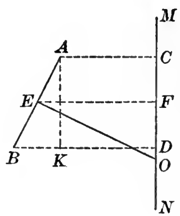
It may be conceived of as a birectangular spherical triangle whose third angle is an angle of one degree.

The distinction between the three different uses of the term degree should be kept clearly in mind; an *angular degree* is a *difference of direction* between two lines, and it is the 360th part of the total angular magnitude about a point in a plane (§ 35); an *arc degree* is a *line*, which is the 360th part of the circumference of a circle (§ 224); a *spherical degree* is a *surface*, which is the 360th part of the surface of a hemisphere, or the 720th part of the surface of a sphere.

Proposition XXI

723. Represent an axis and a line oblique to it, but not meeting it; draw lines from the extremities and middle point of this line perpendicular to the axis; from the nearer extremity draw a line parallel to the axis; also a line perpendicular to the given line at its middle point and terminating in the axis. If the given line revolves about the axis, what kind of a surface will it generate? To what is this surface equivalent? (§ 628) By means of the proportion of lines from similar right triangles, express the surface in terms of the projection of the given line on the axis and the circumference of a circle whose radius is the perpendicular from the middle point of the given line. Would this result hold true, if the line should meet the axis or be parallel to it?

Theorem. *The surface generated by a straight line revolving about an axis in its plane is equivalent to the rectangle formed by the projection of the line on the axis and the circumference whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.*



Data: Any line, as AB, revolving about an axis, as MN; its projection upon MN, as CD; and EO perpendicular to AB at its middle point and terminating in the axis.

To prove surface $AB \Leftrightarrow$ rect. $CD \cdot 2\,\pi EO$.

Proof. Draw $EF \perp MN$ and $AK \parallel MN$.

If AB neither meets nor is parallel to MN it generates the lateral surface of a frustum of a cone of revolution whose slant height is AB and axis CD;

∴ § 628, surface $AB \Leftrightarrow$ rect. $AB \cdot 2\,\pi EF$.

§ 307, $\triangle$s ABK and EOF are similar,

and $AB : AK = EO : EF$;

∴ rect. $AB \cdot EF \Leftrightarrow$ rect. $AK \cdot EO$.

But, § 151, $AK = CD$;

hence, rect. $AB \cdot EF \Leftrightarrow$ rect. $CD \cdot EO$,

and rect. $AB \cdot 2\,\pi EF \Leftrightarrow$ rect. $CD \cdot 2\,\pi EO$;

that is, surface $AB \Leftrightarrow$ rect. $CD \cdot 2\,\pi EO$.

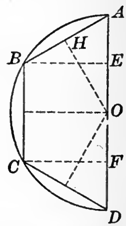
If AB meets axis MN, or is parallel to it, a conical or a cylindri cai surface is generated, and the truth of the theorem follows.

Therefore, etc. Q.E.D.

Proposition XXII

724. Draw a semicircumference and inscribe in it a regular semipolygon. How does the sum of the projections of the sides of the polygon on the diameter of the semicircle compare with the diameter? How do the perpendiculars to the sides of the polygon at their middle points compare in length, if they terminate in the diameter? If the figure is revolved about the diameter as an axis, to what is the surface generated by the perimeter of the semipolygon equivalent? How does the perimeter of the semipolygon at its limit compare with the semicircumference, if the number of its sides is indefinitely increased? What is the limit of the perpendicular to the middle point of a side of the semipolygon? How, then, does the surface of a sphere compare with the rectangle formed by its diameter and the circumference of a great circle?

Theorem. *The surface of a sphere is equivalent to the rectangle formed by its diameter and the circumference of a great circle.*



Data: A sphere, whose center is O, generated by the revolution of the semicircle $ABCD$ about the diameter AD.

Denote the surface of the sphere by S, and its radius by R.

To prove $S \approx$ rect. $AD \cdot 2\pi R$.

Proof. Inscribe in the semicircle half of a regular polygon of an *even* number of sides, as $ABCD$, and let S' denote the surface generated by its sides.

Draw BE and $CF \perp AD$, and the perpendiculars from O to the chords AB, BC, and CD.

§§ 202, 200, these perpendiculars are equal, and bisect the chords.

Then, § 723, surface $AB \approx$ rect. $AE \cdot 2\pi OH$,

surface $BC \approx$ rect. $EF \cdot 2\pi OH$,

and surface $CD \approx$ rect. $FD \cdot 2\pi OH$.

But the sum of the projections AE, EF, and FD equals the diameter AD;

$\therefore$ $S' \approx$ rect. $AD \cdot 2\pi OH$.

Now, if the number of sides of the inscribed semipolygon is indefinitely increased,

§ 392, the semiperimeter will approach the semicircumference as its limit;

$\therefore$ OH will approach R as its limit,

and S' will approach S as its limit.

But, however great the number of sides of the semipolygon,

$$S' \approx \text{rect. } AD \cdot 2\,\pi OH.$$

Hence, § 326, $S \approx \text{rect. } AD \cdot 2\,\pi R.$

Therefore, etc. Q.E.D.

725. Cor. I. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

726. Cor. II. § 725, area $= AD \times 2\,\pi R = 2\,R \times 2\,\pi R = 4\,\pi R^2$;

that is, *the area of the surface of a sphere is equal to the area of four great circles.*

727. Cor. III. Let R and R' denote the radii, D and D' the diameters, and A and A' the areas of the surfaces of two spheres.

Then, § 726, $A = 4\,\pi R^2$, and $A' = 4\,\pi R'^2$;

$\therefore$ $A : A' = 4\,\pi R^2 : 4\,\pi R'^2 = R^2 : R'^2 = D^2 : D'^2$;

that is, *the areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

728. Cor. IV. Area of a zone, as $BC = EF \times 2\,\pi R$;

that is, *the area of a zone is equal to the product of its altitude by the circumference of a great circle.*

729. Cor. V. *Zones on the same sphere, or on equal spheres, are to each other as their altitudes.*

730. Cor. VI. § 728, area of a zone of one base, as

$$AB = AE \times 2\,\pi R = \pi AE \times AD.$$

Draw BD. Then, § 313, $AE \times AD = \overline{AB}^2$;

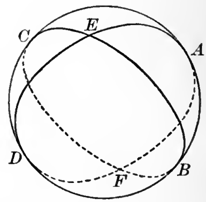
$\therefore$ area of zone $AB = \pi \overline{AB}^2$;

that is, *the area of a zone of one base is equal to the area of a circle whose radius is the chord of the generating arc.*

Proposition XXIII

731. Divide the surface of a sphere into hemispheres by a great circle; on one of the hemispheres form two opposite triangles by drawing two great circle arcs to intersect; complete the circumferences of which these arcs are parts. By comparing one of these opposite triangles with a triangle on the other hemisphere that completes a lune of which the other of the given triangles is a part, discover how the sum of the given triangles compares with a lune whose angle is the angle between the given arcs.

Theorem. *If two arcs of great circles intersect on the surface of a hemisphere, the sum of the two opposite triangles thus formed is equivalent to a lune whose angle is the angle between the given arcs.*



Data: Opposite ⚠s, as AEB and DEC, formed by two great circle arcs, as AED and BEC, on the hemisphere $E\text{–}ABDC$.

To prove $\triangle AEB + \triangle DEC \approx$ lune $AEBF$.

Proof. Produce arcs AED and BEC around the sphere intersecting as at F.

§ 656, arc $DE =$ arc AF (each being the supplement of arc AE),

arc $CE =$ arc BF (each being the supplement of arc BE),

and, § 693, $\angle DEC = \angle AEB = \angle AFB$;

∴ § 710, $\triangle DEC \approx \triangle AFB$.

Adding $\triangle AEB$ to each side of this expression of equivalence,

$$\triangle AEB + \triangle DEC \approx \triangle AEB + \triangle AFB.$$

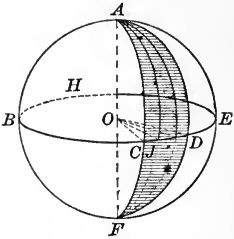
Hence, $\triangle AEB + \triangle DEC \approx$ lune $AEBF$. Q.E.D.

Proposition XXIV

732. On the surface of a sphere draw a lune whose angle is to four right angles as $3:12$; from the vertex of its angle as a pole describe the circumference of a great circle. What is the ratio of the arc included between the sides of the lune to the whole circumference? Divide the circumference into 12 equal parts and through the points of division and the poles pass great circle arcs. Into how many equal lunes do these arcs divide the surface of the sphere? The given lune? How, then, does the ratio of the given lune to the surface of the sphere compare with the ratio of the angle of the lune to four right angles?

Theorem. *A lune is to the surface of a sphere as the angle of the lune is to four right angles.*

Data: A lune, as $ACFD$, whose angle is CAD, on the sphere whose center is O.



Denote the lune by L and the surface of the sphere by S.

To prove $L : S = \angle CAD : 4$ rt. $\angle$s.

Proof. With A as a pole, describe the circumference of a great circle $BCEH$.

Then, § 689, arc CD measures $\angle CAD$, and circumference $BCEH$ measures 4 rt. $\angle$s.

Suppose arc CD and $BCEH$ have a common unit of measure, as CJ, contained in CD m times and in $BCEH$ n times.

Then, $$CD : BCEH = m : n,$$

or $$\angle CAD : 4 \text{ rt. } \angle\text{s} = m : n.$$

Beginning at C, divide $BCEH$ into parts, each equal to the unit of measure CJ, and through the points of division and the poles, A and F, of this circumference pass great circles.

By §§ 689, 721, these circles divide the whole surface of the sphere into n equal lunes of which the given lune contains m.

Then, $$L : S = m : n.$$

Hence, $$L : S = \angle CAD : 4 \text{ rt. } \angle\text{s}.$$

By the method of limits exemplified in § 223, the same may be proved, when arc CD and $BCEH$ are incommensurable.

Therefore, etc. Q.E.D.

733. Cor. I. Let A denote the degrees in the angle of a lune.

Then, $$L : S = A : 360°.$$

Since, § 722, S contains 720 spherical degrees,

$$L : 720 = A : 360;$$

whence, $$L = 2A;$$

that is, *the numerical measure of a lune expressed in spherical degrees is twice the numerical measure of its angle expressed in angular degrees.*

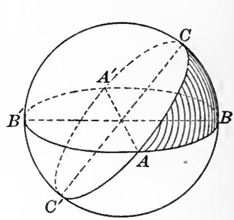
734. Cor. II. *Lunes on the same sphere, or on equal spheres, are to each other as their angles.*

Proposition XXV

735. On a sphere draw a spherical triangle and complete the great circles whose arcs are its sides. How many triangles having a common vertex with the given triangle occupy the surface of a hemisphere? Since the given triangle plus any one of the others is equivalent to a lune whose angle is equal to one of the angles of the given triangle, or to twice as many spherical degrees as that angle contains angular degrees, how does three times the given triangle plus the other three compare with twice the number of spherical degrees that there are angular degrees in the angles of the given triangle? How many spherical degrees are there in the four triangles occupying the surface of the hemisphere? Then, discover how the number of spherical degrees in the given triangle compares with the sum of the angular degrees in its angles less 180°, that is, with its spherical excess.

Theorem. *A spherical triangle is equivalent to as many spherical degrees as there are angular degrees in its spherical excess.*

Data: A spherical triangle, as ABC, whose spherical excess is E degrees.



To prove $\triangle ABC \Leftrightarrow E$ spherical degrees.

Proof. Complete the great circles whose arcs are sides of $\triangle ABC$.

These circles divide the surface of the sphere into eight spherical triangles, any four of which having a common vertex, as A, form the surface of a hemisphere, whose measure is 360 spherical degrees.

§ 731, $\triangle ABC + \triangle AB'C' \Leftrightarrow$ a lune whose angle equals angle A;

$\therefore$ § 733, $\triangle ABC + \triangle AB'C' \Leftrightarrow 2\,A$ spherical degrees. (1)

In like manner, $\triangle ABC + \triangle AB'C \Leftrightarrow 2\,B$ spherical degrees, (2)

and $\triangle ABC + \triangle ABC' \Leftrightarrow 2\,C$ spherical degrees. (3)

Adding (1), (2), and (3),

$3\,\triangle ABC + \triangle AB'C' + \triangle AB'C + \triangle ABC' \Leftrightarrow 2(A + B + C)$ sph. deg.

But $\triangle ABC + \triangle AB'C' + \triangle AB'C + \triangle ABC' = 360$ spherical degrees;

$\therefore 2\,\triangle ABC + 360$ spherical degrees $\Leftrightarrow 2(A + B + C)$ spherical degrees;

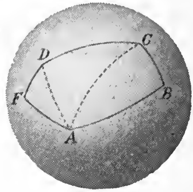
hence, $\triangle ABC \Leftrightarrow (A + B + C - 180)$ spherical degrees;

that is, § 706, $\triangle ABC \Leftrightarrow E$ spherical degrees. Q.E.D.

Proposition XXVI

736. Draw any spherical polygon and divide it into spherical triangles by diagonals from any vertex. To how many spherical degrees is each triangle equivalent? How does the number of spherical degrees in the sum of the triangles compare with the number of angular degrees in the sum of the spherical excesses of the triangles? To how many spherical degrees, then, is any spherical polygon equivalent?

Theorem. Any spherical polygon is equivalent to as many spherical degrees as there are angular degrees in its spherical excess.



Data: Any spherical polygon, as $ABCDF$, whose spherical excess is E degrees.

To prove $ABCDF \approx E$ spherical degrees.

Proof. Divide the polygon into spherical triangles by diagonals from any vertex, as A.

By § 735, each triangle is equivalent to as many spherical degrees as there are angular degrees in its spherical excess.

Hence, the polygon is equivalent to as many spherical degrees as there are angular degrees in the sum of the spherical excesses of the triangles; that is, § 707, in the spherical excess of the polygon.

Hence, $ABCDF \approx E$ spherical degrees.

Therefore, etc. Q.E.D.

737. Cor. *The area of any spherical polygon is to the area of the surface of the sphere as the number which expresses its spherical excess is to 720.*

Ex. **877**. The angle of a lune is 40°. What part of the surface of the sphere is the lune?

Ex. **878**. What is the area of a spherical triangle whose angles are 85°, 120°, and 110°, if the radius of the sphere is 10^{dm}?

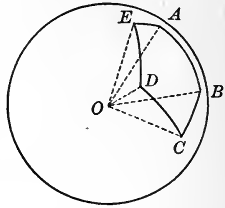
Ex. **879**. The area of the surface of a sphere is 160 sq. in.; the angles of a spherical triangle on this sphere are 93°, 117°, and 132°. What is the area of the triangle?

Ex. **880**. Two spherical triangles on the same sphere, or on equal spheres, are equivalent, if the perimeters of their polar triangles are equal.

738. A solid bounded by a spherical polygon and the planes of its sides is called a **Spherical Pyramid**.

The center of the sphere is the *vertex* of the pyramid, and the spherical polygon is its *base*.

O–ABCDE is a *spherical pyramid* whose *vertex* is *O* and *base ABCDE*.



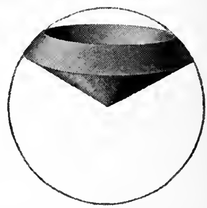
739. The portion of a sphere contained between two parallel planes is called a **Spherical Segment**.

The sections made by the parallel planes are the *bases* of the spherical segment; the perpendicular distance between its bases is the *altitude of the segment*.

If one of the parallel planes is tangent to the sphere, the segment is called a *segment of one base*.

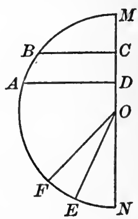
740. The portion of a sphere bounded by a lune and the planes of its sides is called a **Spherical Wedge**, or **Ungula**.

741. The portion of a sphere generated by the revolution of a circular sector about a diameter of the circle is called a **Spherical Sector**.



The zone generated by the arc of the circular sector is the *base of the spherical sector*.

742. *MNEFAB* is a semicircle, *AD* and *BC* are lines from the semicircumference perpendicular to the diameter *MN*, and *OE* and *OF* are radii. Then, if the semicircle is revolved about *MN* as an axis, it generates a *sphere*.



The arc *AB* generates a *zone* whose *altitude* is *DC*, and whose *bases* are the circumferences generated by the points *A* and *B*.

The figure *ABCD* generates a *spherical segment* whose *altitude* is *DC* and whose *bases* are the circles generated by *AD* and *BC*.

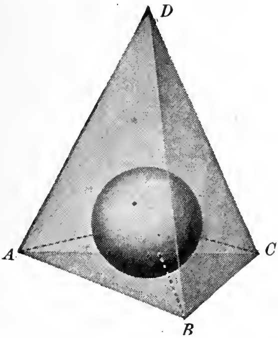
The arc *BM* generates a *zone of one base*, and the figure *BCM* a *spherical segment of one base*.

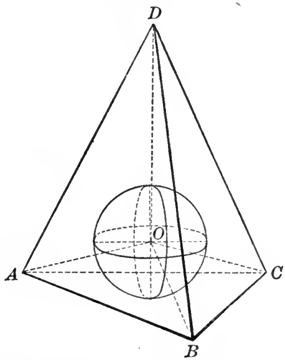
The circular sector *OEF* generates a *spherical sector* whose bounding surfaces are its *base*, the zone generated by the arc *EF*, and the conical surfaces generated by the radii *OE* and *OF*.

Proposition XXVII

743. Represent a polyhedron circumscribed about a sphere. If pyramids are formed having the faces of the polyhedron as bases and the center of the sphere as a common vertex, how will the altitudes of these pyramids compare with each other and with the radius of the sphere? What is the volume of each pyramid? What, then, is the volume of the sum of the pyramids? If the number of faces of the polyhedron is indefinitely increased, how will its volume compare with the volume of the sphere? To what, then, is the volume of a sphere equal?

Theorem. *The volume of a sphere is equal to the product of its surface by one third of its radius.*





Data: A sphere whose center is O, surface S, and radius R. Denote its volume by V.

To prove $V = S \times \frac{1}{3} R$.

Proof. Circumscribe about the sphere any polyhedron, as $D\text{–}ABC$, and denote its surface by S' and its volume by V'.

Form pyramids, as $O\text{–}ABC$, etc., having the faces of the polyhedron as bases and the center of the sphere as a common vertex.

Then these pyramids will have a common altitude equal to R, and, § 560, the volume of each pyramid = its base $\times \frac{1}{3} R$.

$\therefore$ $V' = S' \times \frac{1}{3} R$.

Now, if the number of pyramids is indefinitely increased by passing planes tangent to the sphere at the points where the edges of the pyramids cut the surface of the sphere,

S' approaches S as its limit; $\therefore$ V' approaches V as its limit.

But, however great the number of pyramids,

$$V' = S' \times \tfrac{1}{3} R.$$

Hence, § 222, $V = S \times \tfrac{1}{3} R.$ Q.E.D.

744. Formulæ: $V = S \times \tfrac{1}{3} R = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi D^3.$

745. Cor. I. Let R, R' denote the radii, D, D' the diameters, and V, V' the volumes, respectively, of two spheres.

Then, § 744, $V = \tfrac{4}{3}\pi R^3$, and $V' = \tfrac{4}{3}\pi R'^3$;

$\therefore$ $V : V' = \tfrac{4}{3}\pi R^3 : \tfrac{4}{3}\pi R'^3 = R^3 : R'^3 = D^3 : D'^3$;

that is, *the volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

746. Cor. II. *The volume of a spherical pyramid is equal to the product of its base by one third of the radius of the sphere.*

747. Cor. III. *The volume of a spherical sector is equal to the product of the zone which forms its base by one third of the radius of the sphere.*

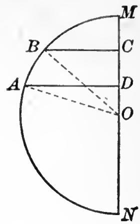
748. Formula: Let R denote the radius of a sphere, C the circumference of a great circle, V the volume of a spherical sector, and H and Z the altitude and area, respectively, of the corresponding zone.

Then, since $C = 2\pi R$, and $Z = 2\pi RH$, $V = \tfrac{2}{3}\pi R^2 H.$

Proposition XXVIII

749. Draw a semicircle; to the extremities of any arc of the semicircumference draw radii, and from their extremities draw lines perpendicular to the diameter. If this figure is revolved about the diameter as an axis, what kind of a solid is generated by the part included between the perpendiculars and the given arc? Between the radii and the given arc? Between each radius and the perpendicular from its extremity? Find an expression for the volume of the spherical segment in terms of the spherical sector and the two cones.

Theorem. *The volume of a spherical segment is equal to one half the product of its altitude by the sum of its bases, plus the volume of a sphere of which that altitude is a diameter.*



Data: A spherical segment, as that generated by the revolution of $ABCD$ about MN as an axis.

Denote the volume of the segment by V; its altitude CD, by H; the radii of its bases AD and BC, by r and r', respectively; and the radius of the semicircle by R.

To prove $V = \frac{1}{2}H(\pi r^2 + \pi r'^2) + \frac{1}{6}\pi H^3.$

Proof. Draw the radii OA and OB.

The volume of the segment generated by $ABCD$ equals the volume of the spherical sector generated by OAB plus the volume of the cone generated by OBC minus the volume of the cone generated by OAD;

$\therefore$ §§ 748, 629, $V = \frac{2}{3}\pi R^2H + \frac{1}{3}\pi r'^2 OC - \frac{1}{3}\pi r^2 OD.$

But $H = OC - OD,\ R^2 - r'^2 = \overline{OC}^2$, and $R^2 - r^2 = \overline{OD}^2$.

Then, $V = \frac{1}{3}\pi\{2\,R^2(OC - OD) + (R^2 - \overline{OC}^2)OC - (R^2 - \overline{OD}^2)OD\}$

$= \frac{1}{3}\pi\{2\,R^2(OC - OD) + R^2(OC - OD) - (\overline{OC}^3 - \overline{OD}^3)\}$

$= \frac{1}{3}\pi H\{3\,R^2 - (\overline{OC}^2 + OC \times OD + \overline{OD}^2)\}.$

But, § 348, $(OC - OD)^2 = \overline{OC}^2 - 2\,OC \times OD + \overline{OD}^2 = H^2$;

$$\therefore \quad \overline{OC}^2 + OC \times OD + \overline{OD}^2 = \frac{3}{2}(\overline{OC}^2 + \overline{OD}^2) - \frac{H^2}{2}$$

$$= 3\,R^2 - \frac{3}{2}(r^2 + r'^2) - \frac{H^2}{2}.$$

Hence, $$V = \frac{1}{3}\pi H\left\{\frac{3}{2}(r^2 + r'^2) + \frac{H^2}{2}\right\}$$

$$= \frac{1}{2}H(\pi r^2 + \pi r'^2) + \frac{1}{6}\pi H^3.$$

Therefore, etc. Q.E.D.

750. Cor. Let the segment be a segment of one base, as that generated by $MBAD$.

Then, the radius $r' = 0,$

and $V = \frac{1}{2}\pi r^2 H + \frac{1}{6}\pi H^3;$

that is, *the volume of a spherical segment of one base is equal to one half the volume of the cylinder having the same base and the same altitude plus the volume of a sphere of which that altitude is the diameter.*

751. FORMULÆ

Notation

B = base.	H = altitude.
D = diameter.	S = surface.
R = radius.	A = area (or area of surface).
r = radius of lower base.	V = volume.
r' = radius of upper base:	

Sphere.

$A = 4\,\pi R^2$ . . . . § 726

$V = S \times \frac{1}{3} R$ . . . . § 743

$V = \frac{4}{3}\,\pi R^3$ . . . . § 744

$V = \frac{1}{6}\,\pi D^3$ . . . . § 744

Zone.

$A = 2\,\pi RH$ . . . . § 728

Spherical Pyramid.

$V = B \times \frac{1}{3} R$ . . . . § 746

Spherical Sector.

$V = B \times \frac{1}{3} R$ . . . . § 747

$V = \frac{2}{3}\,\pi R^2 H$ . . . . § 748

Spherical Segment.

$V = \frac{1}{2} H(\pi r^2 + \pi r'^2) + \frac{1}{6}\,\pi H^3$ . . . . § 749

SUPPLEMENTARY EXERCISES

Ex. **881**. What is the volume of a sphere whose radius is 9 in.?

Ex. **882**. The circumference of a great circle of a sphere is 36 ft. What is the area of the surface of the sphere?

Ex. **883**. The diameter of a sphere is 13^{dm}. How many cubic decimeters does it contain?

Ex. **884**. The volume of a sphere is $1870^{cu\ m}$. What is its radius?

Ex. **885**. The area of the surface of a sphere is 69 sq. ft. What is its diameter?

Ex. **886**. The edge of a cube is 16^{cm}. What is the volume of the circumscribed sphere?

Ex. **887**. If the sides of a spherical triangle are 75°, 93°, and 110°, what is the spherical excess of its polar triangle?

Ex. **888**. The angles of a spherical triangle are 98°, 110°, and 160°. What is the area of a symmetrical triangle on the same sphere, the radius being 12^{m}?

Ex. **889**. Find the volume of a triangular spherical pyramid, the angles of the base being 58°, 116°, and 145°, and the diameter of the sphere being 20 in.

Ex. **890**. What is the area of a lune whose angle is 60° on the surface of a sphere whose radius is 6 in.?

Ex. **891**. What is the area of a zone whose altitude is 3^{dm} on the surface of a sphere whose radius is 8^{dm}?

Ex. **892**. What is the volume of a spherical sector whose altitude is 3.5^{m}, if the radius of the sphere is 11^{m}?

Ex. **893**. What is the volume of a spherical wedge whose angle is 72°, the volume of the sphere being 1728 cu. in.?

Ex. **894**. What is the area of a zone of one base, if the chord of its generating arc is 13^{dm}?

Ex. **895**. The area of a zone of a sphere 20^{dm} in diameter is $150^{sq\ dm}$ What is the altitude of the zone?

Ex. **896**. The angles of the base of a triangular spherical pyramid are 90°, 121°, and 135°. What is the volume of the pyramid, the volume of the sphere being 194 cu. in.?

Ex. **897**. Spherical polygons are to each other as their spherical excesses.

Ex. **898**. The base of a spherical pyramid is a trirectangular triangle. What part of the sphere is the pyramid?

Ex. **899**. The surface of a sphere is equivalent to the lateral surface of the circumscribing cylinder.

Ex. **900**. What is the spherical excess of a triangle whose area is 261.8 sq. in., if the radius of the sphere is 10 in.?

Ex. **901**. A lune and a trirectangular spherical triangle on the same sphere are to each other as the angle of the lune is to an angle of 45°.

Ex. **902**. Trirectangular triangles on equal spheres are equal.

Ex. **903**. The diameters of two spheres are 12 in. and 14 in. respectively. What is the ratio of their surfaces? What is the ratio of their volumes?

Ex. **904**. The areas of the surfaces of two spheres are as 144 to 24. What is the ratio of their diameters? What is the ratio of their volumes?

Ex. **905**. The diameters of the sun and earth are in the ratio of 109 : 1. What is the ratio of their volumes?

Ex. **906**. How many quarts of water will a hemispherical kettle hold, if its inside diameter is 12 in.?

Ex. **907**. If lines are drawn from any point in the surface of a sphere to the ends of a diameter, they are perpendicular to each other.

Ex. **908**. What is the circumference of a small circle of a sphere whose diameter is 9^{dm}, if the circle is at a distance of 3^{dm} from the center?

Ex. **909**. The dihedral angles of a spherical pyramid are 40°, 80°, and 120°, and its edge is 9 ft. What is the volume of the pyramid?

Ex. **910**. The dihedrals of a trihedral angle whose vertex is at the center of a sphere are 75°, 90°, and 130°. What is the volume of the part of the sphere included by the faces of the trihedral, the radius of the sphere being 8^{m}?

Ex. **911**. What is the radius of a sphere which is equivalent to the sum of two spheres whose radii are respectively 4 in. and 7 in.?

Ex. **912**. How many cubic decimeters does a segment of a single base contain, if it is cut from a sphere 12^{dm} in diameter, the altitude of the segment being 4^{dm}?

Ex. **913**. In a sphere whose diameter is 20 ft., what is the volume of a segment, the bases of which are on the same side of the center, one 3 ft. and the other 5 ft. from it?

Ex. **914**. Find the area of the surface of a sphere inscribed in a cube whose surface is 726 sq. ft.

Ex. **915**. A trirectangular triangle and a lune on the same sphere are in the ratio of 14 : 9. What is the angle of the lune, and what part of the surface of the sphere is the lune?

Ex. **916**. Find the area of a spherical quadrilateral whose angles are 120°, 130°, 140°, and 150°, the volume of the sphere being 1000 cu. ft.

Ex. **917**. The base of a cone of revolution is the great circle of a sphere, and its altitude is the radius of the sphere. What is the ratio of the surface of the sphere to the lateral surface of the cone?

Ex. **918**. The base of a cone is equal to a great circle of a sphere, and its altitude is equal to the diameter of the sphere. What is the ratio of their volumes?

Ex. **919**. Find the altitude of a zone whose area is equal to that of a great circle of the sphere, the radius of the sphere being 8^{dm}.

Ex. **920**. How many spherical bullets $\frac{1}{2}$ in. in diameter can be molded from a spherical piece of lead $\frac{1}{2}$ ft. in diameter?

Ex. **921**. A cannon ball put into a cylindrical tub 2 ft. in diameter causes the water in the tub to rise 2 in. What is the diameter of the cannon ball?

Ex. **922**. A section parallel to the base of a hemisphere bisects its altitude. What is the ratio of the volumes of the spherical segments thus formed?

Ex. **923**. The volume of a sphere is to that of the circumscribed cube as π is to 6.

Ex. **924**. The volume of a sphere is to that of the inscribed cube as π is to $\frac{2}{\sqrt{3}}$.

Ex. **925**. The surface of a sphere is to the total surface of the circumscribing cylinder as 2 is to 3.

Ex. **926**. The volume of a sphere is to the volume of a circumscribing cylinder as 2 is to 3.

Ex. **927**. A sphere is cut by five parallel planes at the distance of 2^{dm}, 3^{dm}, 4^{dm}, and 5^{dm} from each other, respectively. What are the relative areas of the zones included between the planes?

Ex. **928**. The sides opposite the equal angles of a birectangular spherical triangle are quadrants.

Ex. **929**. The slant height of a cone of revolution is equal to the diameter of its base. What is the ratio of its total area to that of the inscribed sphere?

Ex. **930**. The smallest circle whose plane passes through a given point within a sphere is that one whose plane is perpendicular to the radius through the given point.

Ex. **931**. The intersection of the surfaces of two spheres is the circumference of a circle whose plane is at right angles to the line joining the centers of the spheres, and whose center is on that line.

Ex. **932**. What is the area of the circle of intersection of two spheres whose radii are respectively 5^{dm} and 8^{dm}, if their centers are 10^{dm} apart?

Ex. **933**. What is the weight of an iron ball, the area of whose surface is $2^{sq\,m}$, the specific gravity of iron being 7.5?

Ex. **934**. If the exterior diameter of a spherical shell is 12 in., what should be the thickness of its wall in order that it may contain 696.9 cu. in.?

Ex. **935**. What is the weight of a hollow iron shell whose wall is 2 in. thick, if it will hold $31\frac{1}{4}$ pounds of water, the specific gravity of iron being 7.5?

Ex. **936**. An equilateral triangle revolves about its altitude. Compare the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.

Ex. **937**. From a sphere whose surface is 69 sq. ft. a segment of one base is cut, which has an altitude of 3 ft. What is the convex surface of the segment?

Ex. **938**. What is the radius of a sphere inscribed in a regular tetrahedron whose entire area is 16 sq. ft.?

Ex. **939**. What is the area of the surface of a sphere inscribed in a regular tetrahedron whose edge is 6 in.?

Ex. **940**. How much of the surface of the earth could a man see, if he were at the distance of a diameter above it?

Ex. **941**. How far from the surface of the earth must a man be in order that he may see one fifth of it?

Ex. **942**. All arcs of great circles drawn through the pole of a given great circle are perpendicular to its circumference.

Ex. **943**. The sum of the squares of three chords perpendicular to each other at any point in the surface of a sphere is equal to the square of the diameter.

Ex. **944**. If a zone of one base is a mean proportional between the remaining surface of the sphere and its total surface, how far is the base of the zone from the center of the sphere?

Ex. **945**. If any number of lines in space meet in a point, the feet of the perpendiculars drawn to these lines from another point lie in the surface of a sphere.

PROBLEMS OF CONSTRUCTION

Ex. **946**. Bisect an arc of a great circle.

Ex. **947**. Bisect a spherical angle.

Ex. **948**. At a given point in a given arc of a great circle construct a spherical angle equal to a given spherical angle.

Ex. **949**. Construct a spherical triangle, the poles of the respective sides being given.

Ex. **950**. Construct a spherical triangle, having given two sides and the included angle.

Ex. **951**. Construct a spherical triangle, having given a side and two adjacent angles.

Ex. **952**. Construct a spherical triangle, having given the three sides.

Ex. **953**. Construct a spherical triangle, having given the three angles.

Ex. **954**. Draw an arc of a great circle perpendicular to a given spherical arc from a point without.

Ex. **955**. Draw an arc of a great circle perpendicular to a given spherical arc at a point in it.

Ex. **956**. Pass a plane tangent to a sphere at a given point on the surface of the sphere.

Ex. **957**. Pass a plane tangent to a sphere through a given straight line without the sphere.

Ex. **958**. Cut a given sphere by a plane passing through a given straight line so that the section shall have a given radius.

Ex. **959**. Through a given point on a sphere draw a great circle tangent to a given small circle.

Ex. **960**. Through a given point on a sphere draw a great circle tangent to two equal small circles whose planes are parallel.

Ex. **961**. Describe a circle to pass through three given points on the surface of a sphere.

Ex. **962**. Circumscribe a circle about a given spherical triangle.

EXERCISES FOR REVIEW

Ex. **1**. The perpendicular erected at the middle point of one side of a triangle meets the longer of the other two sides.

Ex. **2**. Of the bisectors of two unequal angles of a triangle, produced to the point of intersection, the bisector of the smaller angle is the longer.

Ex. **3**. The straight lines which join the middle points of the opposite sides of any quadrilateral bisect each other.

Ex. **4**. If a line is drawn from the middle point of one base of a trapezoid to pass through the intersection of the diagonals, it will bisect the other base.

Ex. **5**. If the opposite sides of a pentagon are produced to intersect, the um of the angles at the vertices of the triangles thus formed is equal to two right angles.

Ex. **6**. The sum of the four lines drawn from the vertices of any quadrilateral to any point except the intersection of the diagonals is greater than the sum of the diagonals.

Ex. **7**. If the internal bisector of one base angle of a triangle and the external bisector of the other base angle are produced until they meet, the angle included between them is equal to half the vertical angle of the triangle.

Ex. **8**. The angle contained by the bisectors of two exterior angles of any triangle is equal to half the sum of the two adjacent interior angles.

Ex. **9**. If each of two angles of a quadrilateral is a right angle, the bisectors of the other angles are either perpendicular or parallel to each other.

Ex. **10**. If the side CB of the triangle ABC is greater than the side CA, and CA is produced to D and CB to E, making AD and BE equal, AE is greater than DB.

Ex. **11**. In the triangle ABC a straight line AD is drawn perpendicular to the straight line BD which bisects angle B. Prove that a straight line through D parallel to BC bisects AC.

Ex. **12**. If one side of a triangle is greater than the other, any line from the vertex of the included angle to the base is less than the longer side.

Ex. **13**. Lines drawn from one vertex of a parallelogram to the middle points of the opposite sides trisect a diagonal.

Ex. **14**. No two straight lines drawn from two vertices of a triangle and terminated by the opposite sides can bisect each other.

Ex. **15**. The base of a triangle whose sides are unequal is divided into two parts by a straight line bisecting the vertical angle. Prove that the greater part is adjacent to the greater side.

Ex. **16**. If two exterior angles of a triangle are bisected, and from the point of intersection of the bisectors a straight line is drawn to the vertex of the third angle, this line bisects that angle.

Ex. **17**. ABC is a triangle; D is the middle point of BC, and E of AD; BE produced meets AC in F. Prove that AC is trisected in F.

Ex. **18**. In the triangle ABC the sides AB, BC, and CA are trisected at the consecutive points D and E, F and G, and H and K respectively. Prove that the lines EF, GH, and KD, when produced, form a triangle equal to ABC.

Ex. **19**. If one of the equal sides of an isosceles triangle is produced below the base to a certain length, if an equal length is cut off above the base from the other equal side, and if the two points are joined by a straight line, this line is bisected by the base.

Ex. **20**. ABC is a triangle, and BE and CF are drawn perpendicular to AG, a line through A; D is the middle point of BC. Show that FD equals ED.

Ex. **21**. The angle contained by the bisectors of the base angles of any triangle is equal to the vertical angle of the triangle plus half the sum of the base angles.

Ex. **22**. The bisectors of two angles of an equilateral triangle intersect, and from their point of intersection lines are drawn parallel to any two sides. Prove that these lines trisect the third side.

Ex. **23**. The opposite sides of a regular hexagon are parallel.

Ex. **24**. If in a quadrilateral the diagonals are equal and two sides are parallel, the other sides are equal.

Ex. **25**. If from any point in the base of an isosceles triangle perpendiculars are drawn to the equal sides, their sum is equal to the perpendicular drawn from either extremity of the base to the opposite side.

Ex. **26**. The sum of the perpendiculars from any point within an equilateral triangle to its sides is equal to the altitude.

Ex. **27**. If from the vertex of any triangle two lines are drawn, one of which bisects the angle at the vertex and the other is perpendicular to the base, the angle between these lines is half the difference of the angles at the base of the triangle.

Ex. **28**. In any triangle, the sides of the vertical angle being unequal, the median drawn from the vertical angle lies between the bisector of that angle and the longer side.

Ex. **29**. In any triangle, the sides of the vertical angle being unequal, the bisector of that angle lies between the median and the perpendicular drawn from the vertex to the base.

Ex. **30**. Lines are drawn through the extremities of the base of an isosceles triangle, making angles with it, on the side opposite the vertex, each equal to one third of a base angle of the triangle, and meeting the sides produced. Prove that the three triangles thus formed are isosceles.

Ex. **31**. If two circumferences are tangent internally and the radius of the larger is the diameter of the smaller, any chord of the larger drawn from the point of contact is bisected by the circumference of the smaller.

Ex. **32**. If perpendiculars are drawn to any chord at its extremities and produced to intersect a diameter of the circle, the points of intersection are equally distant from the center.

Ex. **33**. If perpendiculars are drawn from the ends of a diameter of a circle upon any secant, their feet are equally distant from the points in which the secant intersects the circumference.

Ex. **34**. Given an arc of a circumference, the chord subtended by it, and the tangent at one extremity. Prove that the perpendiculars dropped from the middle point of the arc upon the tangent and chord, respectively, are equal.

Ex. **35**. The bisectors of the angles contained by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.

Ex. **36**. If two opposite sides of an inscribed quadrilateral are equal, the other two sides are parallel.

Ex. **37**. In a given square, inscribe an equilateral triangle having its vertex in the middle of a side of the square.

Ex. **38**. Find, in a side of a triangle, a point from which straight lines, drawn parallel to the other sides of the triangle and terminated by them, are equal.

Ex. **39**. Construct a triangle, having given the base, one of the angles at the base, and the sum of the other two sides.

Ex. **40**. Construct a triangle, having given the base, one of the angles at the base, and the difference of the other two sides.

Ex. **41**. Construct a triangle, having given the perpendicular from the vertex to the base, and the difference between each side and the adjacent segment of the base.

Ex. **42**. If two circles are each tangent to two parallel lines and a transversal crossing them, the line of centers is equal to the segment of the transversal intercepted between the parallels.

Ex. **43**. If through the point of contact of two circles which are tangent to each other externally any straight line is drawn terminated by the circumferences, the tangents at its extremities are parallel to each other.

Ex. **44**. If two circles are tangent to each other externally and parallel diameters are drawn, the straight line joining the opposite extremities of these diameters will pass through the point of contact.

Ex. **45**. Construct the three escribed * circles of a given triangle.

* A circle tangent to one side of a triangle and to the other two sides produced is called an *escribed circle.*

Ex. **46**. Construct an isosceles right triangle, having given the sum of the hypotenuse and one side.

Ex. **47**. Construct a right triangle, having given the hypotenuse and the sum of the sides.

Ex. **48**. Construct a right triangle, having given the hypotenuse and the difference of the sides.

Ex. **49**. A and B are two fixed points on the circumference of a circle and CD is any diameter. What is the locus of the intersection of CA and DB?

Ex. **50**. Construct a triangle, having given a median and the two angles into which the angle is divided by that median.

Ex. **51**. Construct a triangle, having given the base, the difference between the sides, and the difference between the angles at the base.

Ex. **52**. Construct an isosceles triangle, having given the perimeter and altitude.

Ex. **53**. The circles described on two sides of a triangle as diameters intersect on the third side, or the third side produced.

Ex. **54**. ABC is a triangle having AC equal to BC; D is any point in AB. Prove that the circles circumscribed about triangles ADC and DBC are equal.

Ex. **55**. Construct a triangle, having given two sides and the median to the third side.

Ex. **56**. Construct a triangle, having given its perimeter, and having its angles equal to the angles of a given triangle.

Ex. **57**. Construct a triangle, having given one side and the medians to the other sides.

Ex. **58**. Construct a circle of given radius to touch a given circle and a given straight line. How many such circles may there be?

Ex. **59**. Construct a circle of given radius which shall be tangent to two given circles. How many solutions may there be?

Ex. **60**. If an equilateral triangle is inscribed in a circle and from any point in the circumference lines are drawn to the vertices, the longest of these lines is equal to the sum of the other two.

Ex. **61**. If two circles intersect each other, two parallel lines passing through the points of intersection and terminated by the exterior arcs are equal.

Ex. **62**. An isosceles triangle has its vertical angle equal to an exterior angle of an equilateral triangle. Prove that the radius of the circumscribed circle is equal to one of the equal sides of the isosceles triangle.

Ex. **63**. If a chord of a circle is extended by a length equal to the radius, and from the extremity a secant is drawn through the center of the circle, the length of the greater included arc is three times the length of the less.

Ex. 64. Construct three circles having equal diameters and being tangent to each other.

Ex. 65. Construct two circles of given radii to touch each other and a given straight line on the same side of it.

Ex. 66. Construct a triangle, having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.

Ex. 67. Construct a circle to touch a given circle and also to touch a given straight line at a given point.

Ex. 68. Construct a circle to touch a given straight line, and to touch a given circle at a given point.

Ex. 69. Construct a circle to touch a given circle, have its center in a given line, and pass through a given point in that line.

Ex. 70. If $a : b = c : d$, prove that

$$a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2.$$

Ex. 71. Given three lines a, b, and c. Construct $x = \frac{bc}{a}$.

Ex. 72. Construct x, having given $\frac{4}{x} = \frac{x}{9}$.

Ex. 73. The diagonals of a trapezoid divide each other into segments which are proportional.

Ex. 74. If one side of a right triangle is double the other, the perpendicular from the vertex upon the hypotenuse divides the hypotenuse into parts which are in the ratio of 1 to 4.

Ex. 75. $ABCD$ is an inscribed quadrilateral. The sides AB and DC are produced to meet at E. Prove triangles ACE and BDE similar.

Ex. 76. If AB is a chord of a circle, and CD is any chord drawn from the middle point C of the arc AB cutting the chord AB at E, prove that the chord AC is a mean proportional between CE and CD.

Ex. 77. If perpendiculars are drawn from two vertices of a triangle to the opposite sides, the triangle cut off by the line joining the feet of the perpendiculars is similar to the original triangle.

Ex. 78. AB is the hypotenuse of the right triangle ABC. If perpendiculars are drawn to AB at A and B, meeting BC produced at E, and AC produced at D, the triangles ACE and BCD are similar.

Ex. 79. If two circles intersect in the points A and B, and a secant through B cuts the circumferences in C and D respectively, the straight lines AC and AD are in the same ratio as the diameters of the circles.

Ex. 80. Inscribe a square in a given right isosceles triangle.

Ex. 81. From the obtuse angle of a triangle draw a line to the base, which shall be a mean proportional between the external segments into which it divides the base.

Ex. **82**. Through a given point draw a straight line, so that the parts of it intercepted between that point and perpendiculars drawn to it from two other given points may have a given ratio.

Ex. **83**. Show that the diagonals of a trapezoid, one of whose bases is double the other, cut each other at a point of trisection.

Ex. **84**. A tangent to a circle at the point A intersects two parallel tangents whose points of contact are D and E, in B and C respectively. BE and CD intersect at F. Prove that the line AF is parallel to the tangents BD and CE.

Ex. **85**. The angle C of the triangle ABC is bisected by CD, which cuts the base AB at D; O is the middle point of AB. Prove that OD has the same ratio to OA that the difference of the sides has to their sum.

Ex. **86**. A and B are two points on the circumference of a circle of which O is the center; tangents at A and B meet at E; and from A the line AD is drawn perpendicular to OB. Prove $BE : BO = BD : AD$.

Ex. **87**. AB is a diameter of a circle, CD is a chord at right angles to it, and E is any point in CD; AE and BE are drawn, and produced to cut the circumference in F and G respectively. Prove that $CFDG$ has any two of its adjacent sides in the same ratio as the remaining two.

Ex. **88**. Two circles whose centers are O and P intersect in A, and the tangent to each at A meets the circumference of the other in C and B respectively. Prove that $AB : AC = OA : PA$.

Ex. **89**. O is the center of the circle inscribed in the triangle ABC; AO meets BC in D. Prove $AO : DO = AB + AC : BC$.

Ex. **90**. ABC is an isosceles triangle; the perpendicular to AC at C meets the base AB, or the base produced, at E; D is the middle point of AE. Prove that AC is a mean proportional between AB and AD.

Ex. **91**. AB and CD are two parallel straight lines; E is the middle point of CD; AC and BE meet in F, and AE and BD meet in G. Prove that FG is parallel to AB.

Ex. **92**. If two circles are tangent to each other, either internally or externally, any two straight lines drawn through the point of contact will be cut proportionally by the circumferences.

Ex. **93**. From one of the points of intersection of two intersecting circles a diameter of each circle is drawn. Prove (1) that the line joining the extremities of these diameters passes through the other point of intersection; and (2) that this line is parallel to the line of centers of the circles.

Ex. **94**. If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse, and circles are inscribed in the triangles thus formed, the diameters are proportional to the sides of the given right angle.

Ex. **95**. The distance from the center of a circle to a chord 8^{dm} long is 4^{dm}. What is the distance from the center to a chord 5^{dm} long?

Ex. 96. If a chord 18^{dm} long is bisected by another chord 22^{dm} long, what are the segments of the latter?

Ex. **97.** If two intersecting chords divide the circumference of a circle into parts whose lengths taken in order are as 1, 1, 2, and 5, what angles do the chords make with each other?

Ex. **98.** The square on the hypotenuse of a right isosceles triangle is equivalent to four times the triangle.

Ex. **99.** If the sides of a triangular field are respectively 11^{Hm}, 9^{Hm}, and 8^{Hm} long, how many hektares are there in the area of the field?

Ex. **100.** The sides of a triangle are respectively 39, 42, and 45 inches in length. What is the radius of the inscribed circle?

Ex. **101.** The sides of a triangle are respectively 5 ft., 5 ft., and 6 ft. What is the diameter of the circumscribed circle?

Ex. **102.** A triangular field has its sides respectively 16 rd., 24 rd., and 36 rd. long. What is the length of a line from the middle of the longest side to the opposite corner? What is the area of the field?

Ex. **103.** If a chord 10^{cm} long is 5^{cm} distant from the center of a circle, what is the radius of the circle, and the distance from the end of the chord to the end of the radius that is perpendicular to the chord?

Ex. **104.** How many square meters are there in the area of the quadrilateral $ABCD$, if $AB = 6^m$, $BC = 11^m$, $CD = 4^m$, $AD = 13^m$, and the diagonal $AC = 15^m$?

Ex. **105.** If two equivalent triangles have a common base, and lie on opposite sides of it, the base, or the base produced, will bisect the line joining their vertices.

Ex. **106.** ABC is a given triangle. Construct an equivalent triangle, having its vertex at a given point in BC, and its base in the same straight line as AB.

Ex. **107.** Through the vertex A of the parallelogram $ABCD$ draw a line meeting the side CB produced in F, and the side CD produced in E. Prove that the rectangle of the produced parts of the sides is equivalent to the rectangle of the sides.

Ex. **108.** ABC is a right triangle having its right angle at B. At A and C perpendiculars to AC are erected to meet CB and AB produced in E and F respectively, and EF is drawn. Prove that the triangles BEF and ABC are equivalent.

Ex. **109.** The square upon the altitude of an equilateral triangle is equivalent to three times the square upon half of one of the sides of the triangle.

Ex. **110.** If from a point D in the base AB of the triangle ABC straight lines are drawn parallel to the sides AC and BC respectively, so as to meet BC in F and AC in E, triangle EFC is a mean proportional between triangles ADE and DBF.

Ex. **111**. Two sides of a triangle are 70^m and 65^m, and the difference of the segments of the other side made by a perpendicular from the opposite vertex is 9^m. What is the length of the other side?

Ex. **112**. The sum of two sides of a triangle is 128 ft., and a perpendicular from the vertex opposite the other side divides that side into segments of 60 ft. and 28 ft. What are the sides of the triangle?

Ex. **113**. Two sides of a triangle are in proportion to each other as 6 is to 5, and the adjacent segments of the other side made by a perpendicular from the opposite vertex are 36 ft. and 14 ft. respectively. What are the sides?

Ex. **114**. The difference of the two sides of an oblique triangle, obtuse-angled at the base, is 9^m, and the segments of the base produced made by a perpendicular from the opposite vertex are 30^m and 9^m. What are the sides?

Ex. **115**. A flag pole 140 ft. long, standing on an eminence 30 ft. high, broke so that the top struck the level ground at a distance from the base of the pole equal to the length of the part standing. What was the length of the part broken off?

Ex. **116**. If from the extremities of the base of any triangle lines are drawn bisecting the other two sides, these lines intersect within the triangle and form another triangle on the same base equivalent to one third of the original triangle.

Ex. **117**. Upon the sides of a right triangle, as homologous sides, three similar polygons of any number of sides are constructed. Prove that the polygon upon the hypotenuse is equivalent to the sum of the polygons upon the other two sides.

Ex. **118**. $ABCD$ is a rectangle, and BD is its diagonal; a circle whose center is O is inscribed in the triangle DBC; EO and FO are drawn perpendicular to AD and AB respectively. Then, the rectangle $AFOE$ is equivalent to one half the rectangle $ABCD$.

Ex. **119**. If squares are described upon the three sides of a right triangle, and the extremities of the adjacent sides of any two squares are joined, the triangle so formed is equivalent to the given triangle.

Ex. **120**. Inscribe a circle in a given rhombus.

Ex. **121**. A segment whose arc is 60° is cut off from a circle whose radius is 15 ft. What is the area of the segment?

Ex. **122**. If the bisectors of all the angles of a polygon meet in a point, a circle may be inscribed in the polygon.

Ex. **123**. If the area of a certain circle is $154^{sq\,m}$, how many degrees are there in an angle at the center, if it is subtended by an arc of the circumference 5.5^m long?

Ex. **124**. Prove that an equiangular polygon inscribed in a circle is regular, if the number of its sides is odd.

Ex. **125.** Two parallel chords in a circle are the sides of regular inscribed polygons, one a hexagon and the other a dodecagon. If the radius of the circle is 11 in., how far apart are the chords?

Ex. **126.** What is the length of the side of a square equivalent to a circle in which a chord of 30^{dm} has an arc whose height is 5^{dm}?

Ex. **127.** If a 4-inch pipe will fill a cistern in 2 hr. 30 min., how long will it take a 2-inch pipe to fill it?

Ex. **128.** If an equilateral triangle and a regular decagon each has a perimeter of 6^{m}, what is the difference in area between them?

Ex. **129.** If an equilateral triangle is inscribed in a circle, the line joining the middle points of the arcs cut off by two of its sides will be trisected by those sides.

Ex. **130.** If the area between three equal circles, each tangent to the other two, is $40^{sq\,m}$, what are the radii of the circles?

Ex. **131.** Construct a circle equal to three fourths of a given circle.

Ex. **132.** If a circle is circumscribed about a right triangle, and on each of its sides as a diameter a semicircle is described exterior to the triangle, the sum of the areas of the crescents thus formed is equal to the area of the triangle.

Ex. **133.** The area of an inscribed regular octagon is equal to that of a rectangle whose sides are respectively equal to the sides of the inscribed and circumscribed squares.

Ex. **134.** If the radius of a circle is r, prove that the area of a regular inscribed octagon is $2r^2\sqrt{2}$.

Ex. **135.** The area of a circle is a mean proportional between the areas of any two similar polygons, one of which is circumscribed about the circle and the other is isoperimetric with the circle. (*Galileo's Theorem.*)

Ex. **136.** Prove that the sum of the perpendiculars drawn to the sides of a regular polygon from any point within is equal to the apothem of the polygon multiplied by the number of its sides.

Ex. **137.** If upon the sides of a regular hexagon squares are constructed outwardly, the exterior vertices of these squares are the vertices of a regular dodecagon.

Ex. **138.** A horse is tethered to a hook on the inner side of a fence which bounds a circular grass plot. His tether is so long that he can just reach the center of the plot. The area of so much of the plot as he can graze over is $\frac{98}{3}(4\pi - 3\sqrt{3})$ sq. rd.; find the length of the tether and the circumference of the plot. (*Harvard.*)

Ex. **139.** If equal straight lines are drawn from a given point to a given plane they make equal angles with the plane.

Ex. **140.** Two planes which are each perpendicular to a third plane are parallel, if their intersections with the third plane are parallel.

Ex. **141**. The line joining the extremities of two equal lines which are perpendicular to a plane, on the same side of it, is parallel to the plane.

Ex. **142**. Through a given line in a given plane pass a plane to make a given angle with the given plane.

Ex. **143**. Through a given line parallel to a given plane pass a plane to make a given angle with the given plane.

Ex. **144**. Through the edge of a given dihedral angle pass a plane to bisect that angle.

Ex. **145**. Find the locus of the points in space which are equidistant from two parallel lines.

Ex. **146**. If a straight line is perpendicular to a plane, its projection on any other plane is perpendicular to the intersection of the two planes.

Ex. **147**. If two planes are perpendicular, a straight line drawn from any point of one plane perpendicular to the other will lie in the first plane.

Ex. **148**. The projection of a straight line upon a plane is a straight line.

Ex. **149**. Find the locus of the points in space which are equidistant from three given planes.

Ex. **150**. Find the locus of the points in space equidistant from two intersecting straight lines.

Ex. **151**. Two trihedrals are equal or symmetrical, if two face angles, and the dihedral between their faces, in one are equal, each to each, to the corresponding parts in the other.

Ex. **152**. Find the locus of the points in space which are equidistant from three given straight lines in the same plane.

Ex. **153**. Find the locus of the points in a given plane which are equidistant from two given points without the plane.

Ex. **154**. The angles AOB and AOC in different planes are equal. Prove that the plane bisecting the dihedral angle between their planes is perpendicular to the plane BOC.

Ex. **155**. The planes through any two pairs of lines that pass through a point intersect in a line which passes through the same point.

Ex. **156**. In a given plane find a point equidistant from three given points without the plane.

Ex. **157**. Through a given point in space, draw a straight line which shall cut two given straight lines not in the same plane.

Ex. **158**. Find the locus of the points in space which are equidistant from two given planes and also equidistant from two given points.

Ex. **159**. Two planes are perpendicular respectively to two non-parallel lines which are not in the same plane. Prove that their intersection is perpendicular to any plane that is parallel to both lines.

Ex. **160**. From the vertex of a trihedral angle a line is drawn within the angle. Prove that the sum of the angles which this line makes with the edges is less than the sum, but greater than half the sum, of the face angles.

Ex. **161**. Two trihedrals are equal or symmetrical, if two dihedrals and the included face angle of one are equal, each to each, to the corresponding parts of the other.

Ex. **162**. A plane parallel to two sides of a quadrilateral in space (that is, a quadrilateral whose sides do not all lie in the same plane) divides the other two sides proportionally.

Ex. **163**. In any trihedral, the three planes, passed through the edges perpendicular to the opposite faces respectively, intersect in the same straight line.

Ex. **164**. In any trihedral, the three planes, passed through the edges and the bisectors of the opposite face angles respectively, intersect in the same straight line.

Ex. **165**. What is the edge of a cubical vessel that holds one half ton of water?

Ex. **166**. Represent the base edge of a regular four-sided pyramid by e, its altitude by h, and its total surface by T. Compute the base edge in terms of h and T.

Ex. **167**. What is the difference in volume between the frustum of a pyramid and a prism, each 12^{dm} high, if the bases of the frustum are squares whose sides are 10^{dm} and 8^{dm} respectively, and the base of the prism is a section of the frustum parallel to its bases and midway between them?

Ex. **168**. What is the volume of a regular tetrahedron whose edge is 10^{dm}?

Ex. **169**. The altitude of a regular hexagonal pyramid is 13 in., and its slant height is 16 in. What is its lateral edge?

Ex. **170**. The lateral faces of a regular quadrangular pyramid are equilateral triangles, and its altitude is 9^{m}. What is the area of the base?

Ex. **171**. The altitude of a frustum of a regular quadrangular pyramid is 10^{cm}, and the sides of its bases are respectively 16^{cm} and 6^{cm}. What is the lateral area of the frustum?

Ex. **172**. If the altitude of a pyramid is h, at what distance from the vertex will it be cut by a plane parallel to the base and dividing the pyramid into two parts which are in the ratio of $3:4$?

Ex. **173**. At what distances from the vertex will a lateral edge of a pyramid be cut by two planes parallel to the base, if they divide the pyramid into three equivalent parts, the length of the edge being m?

Ex. **174**. If the base edge of a regular square pyramid is m, and its total surface is T, what is its volume?

Ex. **175**. The perimeter of the base of a regular quadrangular pyramid is p, and the area of a section through two diagonally opposite edges is A. What is the lateral area of the pyramid?

Ex. **176**. If two tetrahedrons have the faces including a trihedral of one similar to the faces including a trihedral of the other, each to each, and similarly placed, the tetrahedrons are similar.

Ex. **177**. If two tetrahedrons have a dihedral angle of one equal to a dihedral angle of the other, and the faces including these dihedrals similar, each to each, and similarly placed, the tetrahedrons are similar.

Ex. **178**. The perpendicular from the middle point of the diagonal of a rectangular parallelopiped upon a lateral edge bisects the edge, and is equal to one half the projection of the diagonal upon the base.

Ex. **179**. In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces. (*Euler's Theorem.*)

Ex. **180**. The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices.

Ex. **181**. If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the base.

Ex. **182**. If a plane is tangent to a circular cylinder, its intersection with the plane of the base is tangent to the base.

Ex. **183**. What are the dimensions of a cylindrical measure whose altitude is half its diameter, if it holds a half bushel?

Ex. **184**. Find the weight of the water that will be contained in a vertical pipe 40 ft. high and 1 ft. in diameter. Also find the pressure per square inch on the base of the pipe.

Ex. **185**. A rectangle revolves successively about two adjacent sides whose lengths are m and n respectively. Compare the volumes of the cylinders thus generated.

Ex. **186**. A right triangle revolves successively about the perpendicular sides whose lengths are m and n respectively. Compare the volumes of the cones thus generated.

Ex. **187**. If the sides including the right angle of a right triangle are m and n, what is the area of the surface generated by revolving the triangle about its hypotenuse as an axis?

Ex. **188** Find the altitude of a cylinder of revolution of radius r, if the cylinder is equivalent to a rectangular parallelopiped whose dimensions are l, m, and n.

Ex. **189**. Find the altitude of a cone of revolution of radius r, equivalent to a rectangular parallelopiped whose dimensions are l, m, and n.

Ex. **190**. The altitudes of two equivalent cylinders of revolution are in the ratio $a : b$. If the radius of one is r, what is the radius of the other?

Ex. **191**. Find the altitude of a regular quadrangular prism whose base edge is m, the prism being equivalent to a cylinder of revolution whose altitude is h and radius r.

Ex. **192.** How must the dimensions of a cylinder of revolution be increased in order to form a similar cylinder whose total surface shall be n times that of the original cylinder?

Ex. **193.** How must the dimensions of a cylinder of revolution be increased in order to form a similar cylinder whose volume shall be n times that of the original cylinder?

Ex. **194.** What is the radius of the base of a circular cone whose altitude is h, the longest and the shortest elements being l and l' respectively?

Ex. **195.** What is the slant height of a frustum of a cone of revolution whose lateral surface is S and whose lower and upper bases are B and b respectively?

Ex. **196.** A cone of revolution whose radius is r and altitude h is divided into two equivalent parts by a plane parallel to the base. What is the total area of the frustum thus formed?

Ex. **197.** The volume of a cylinder of revolution is equal to the area of its generating rectangle multiplied by the circumference generated by the point of intersection of the diagonals of the rectangle.

Ex. **198.** On each base of the frustum of a cone of revolution there is a cone whose vertex is in the center of the other base. If the radii of the lower and upper bases are r and r' respectively, what is the radius of the circle of intersection of the two cones?

Ex. **199.** A stone bridge 20 ft. wide has a circular arch of 140 ft. span at the water level. The crown of the arch is $140(1-\frac{1}{2}\sqrt{3})$ ft. above the surface of the water. How many square feet of surface must be gone over in cleaning so much of the under side of the arch as is above water? (*Harvard.*)

Ex. **200.** What part of the whole surface of a sphere is a spherical triangle whose angles are $57°57'$, $75°27'$, and $100°36'$?

Ex. **201.** What is the volume of a right cone whose altitude is 15 ft., inscribed in a sphere whose radius is 10 ft.?

Ex. **202.** How far from the base of a hemisphere must a plane be passed to divide the surface into two equivalent zones?

Ex. **203.** The volume of a spherical segment of one base is V and its altitude is h. What is the radius of the sphere?

Ex. **204.** Find an expression for the volume of a cube inscribed in a sphere whose radius is r.

Ex. **205.** Two equal circles intersect in a diameter. If a plane is passed perpendicular to that diameter, prove that the four points in which it intersects the circumferences lie in the circumference of a circle.

Ex. **206.** The square on the diameter of a sphere and the square on the edge of an inscribed cube are to each other as 3 is to 1.

Ex. **207.** Find an expression for the altitude of a zone of a sphere whose radius is r, the area of the zone being equal to that of a great circle of the sphere.

Ex. **208**. Find an expression for the altitude of a zone whose area is A on a sphere whose volume is V.

Ex. **209**. Assuming the atmosphere to extend to a height of 50 miles above the earth's surface and the earth to be a sphere whose radius is approximately 4000 miles, what is the volume of the atmosphere?

Ex. **210**. Assuming the earth to be a sphere whose radius is approximately 4000 miles, how far at sea is a lighthouse visible, if it is 80 ft. high?

Ex. **211**. A swimmer, whose eye is at the surface of the water, can just see the top of a buoy a mile distant. If the buoy is 8 in. out of the water, what is the radius of the earth?

Ex. **212**. How high above the surface of the earth must a man be in order that he may see $\frac{1}{n}$ of it?

Ex. **213**. What is the area of the zone illuminated by a taper h decimeters from the surface of a sphere whose radius is r decimeters?

Ex. **214**. In a cube whose edge is 1 ft. there are inscribed a cylinder, a cone, a sphere, and a square pyramid. What is the volume of each of these solids?

Ex. **215**. A cylindrical boiler with hemispherical ends has a total length of 12 ft. and its circumference is 10 ft. What is its surface? What weight of water is required to fill it?

Ex. **216**. Find the diameter of a sphere which is circumscribed about a regular square pyramid whose base is 4 in. square and altitude 8 in.

Ex. **217**. A sphere 8 in. in diameter has a 3-inch hole bored through its center. What is the remaining volume?

Ex. **218**. What is the volume of the portion of a sphere lying outside of an inscribed cylinder of revolution whose altitude is h and radius r?

Ex. **219**. Inscribe a circle in a given spherical triangle.

Ex. **220**. Find the locus of the centers of the sections of a given sphere made by planes passing through a given straight line.

Ex. **221**. Find the locus of the centers of the sections of a given sphere made by planes passing through a given point without the sphere.

Ex. **222**. Having given the radius, construct a spherical surface to pass through three given points.

Ex. **223**. Having given the radius, construct a spherical surface to pass through two given points and be tangent to a given plane or to a given sphere.

Ex. **224**. Having given the radius, construct a spherical surface to pass through a given point and be tangent to two given planes.

Ex. **225**. Having given the radius, construct a spherical surface to be tangent to three given planes.

METRIC TABLES

Measures of Length

10 **Millimeters** (mm)	= 1	**Centimeter** (cm)
10 Centimeters	= 1	Decimeter (dm)
10 Decimeters	= 1	**Meter** (m)
10 Meters	= 1	Dekameter (Dm)
10 Dekameters	= 1	Hektometer (Hm)
10 Hektometers	= 1	**Kilometer** (Km)

Measures of Surface

100 Sq. Millimeters (sq mm)	= 1	**Sq. Centimeter** (sq cm)
100 Sq. Centimeters	= 1	Sq. Decimeter (sq dm)
100 Sq. Decimeters	= 1	**Sq. Meter** (sq m)
100 Sq. Meters	= 1	Sq. Dekameter (sq Dm)
100 Sq. Dekameters	= 1	Sq. Hektometer (sq Hm)
100 Sq. Hektometers	= 1	**Sq. Kilometer** (sq Km)

A square hektometer is also called a **hektare** (Ha).

Measures of Volume

1000 Cu. Millimeters (cu mm)	= 1	**Cu. Centimeter** (cu cm)
1000 Cu. Centimeters	= 1	Cu. Decimeter (cu dm)
1000 Cu. Decimeters	= 1	**Cu. Meter** (cu m)

Measures of Capacity

10 Milliliters (ml)	= 1	**Centiliter** (cl)
10 Centiliters	= 1	Deciliter (dl)
10 Deciliters	= 1	**Liter** (l)
10 Liters	= 1	Dekaliter (Dl)
10 Dekaliters	= 1	Hektoliter (Hl)
10 Hektoliters	= 1	Kiloliter (Kl)

The liter contains a volume equal to a cube whose edge is a **decimeter**.

MEASURES OF WEIGHT

10 **Milligrams** (mg)	= 1 **Centigram** (cg)
10 Centigrams	= 1 Decigram (dg)
10 Decigrams	= 1 **Gram** (g)
10 Grams	= 1 Dekagram (Dg)
10 Dekagrams	= 1 Hektogram (Hg)
10 Hektograms	= 1 **Kilogram** (Kg)

The weight of a **gram** is the weight of a **cubic centimeter of distilled water** t its greatest density.

METRIC EQUIVALENTS

1 **Meter**	= **39.37** in. = 1.0936 yd.	1 Yard	= .9144^{m}
1 Kilometer	= .62138 Mile	1 Mile	= 1.6093^{Km}
1 Hektare	= 2.471 Acres	1 Acre	= .4047^{Ha}
1 Liter	= .908 qt. dry; 1.0567 qt. liquid	1 qt. dry	= 1.101^{l}
		1 qt. liq.	= .9463^{l}
1 Gram	= 15.432 Grains	1 Grain	= .0648^{g}
1 Kilogram	= 2.2046 lb.	1 Pound	= .4536^{Kg}

APPROXIMATE METRIC EQUIVALENTS

1 Decimeter	= 4 in.	1 Liter	= $\frac{9}{10}$ qt. dry; $\frac{17}{16}$ qt. liquid
1 **Meter**	= 40 in.		
1 Kilometer	= $\frac{5}{8}$ Mile	1 Hektoliter	= $2\frac{5}{6}$ bu.
1 Hektare	= $2\frac{1}{2}$ Acres	1 Kilogram	= $2\frac{1}{5}$ lb.

NOTES.—1. The **specific gravity** of a substance (solid or liquid) is the ratio between the weight of any volume of the substance and the weight of a like volume of distilled water at its greatest density; consequently, since a **cubic centimeter** of distilled water at its greatest density weighs one **gram**, the weight of any substance may be found if its specific gravity and volume are known.

2. A cubic foot of water weighs $62\frac{1}{2}$ lb., or 1000 oz.
3. A bushel contains 2150.42 cu. in.
4. A gallon contains 231 cu. in.