PLANE AND SOLID GEOMETRY

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PLANE AND SOLID GEOMETRY.

W. P. 3

FOR THOSE WHOSE PRIVILEGE

IT MAY BE TO ACQUIRE A KNOWLEDGE OF

GEOMETRY

THIS VOLUME HAS BEEN WRITTEN

AND TO THE BOYS AND GIRLS WHO LEARN THE ANCIENT SCIENCE

FROM THESE PAGES, AND WHO ESTEEM THE POWER

OF CORRECT REASONING THE MORE

BECAUSE OF THE LOGIC OF

PURE GEOMETRY

THIS VOLUME IS DEDICATED

PREFACE

THE motives actuating the author in the preparation of this text in Geometry have been :

(*a*) *To present a book that has been written for the pupil.*

The object sought in the study of Geometry is not solely to train the mind to accept only those statements as truth for which convincing reasons can be provided, but to cultivate a foresight that will appreciate both the purpose in making a statement and the process of reasoning by which the ultimate truth is established. Thus, the study of this formal science should develop in the pupil the ability to pursue argument coherently, and to establish one truth by the aid of other known truths, in logical order.

The more mature members of a class do not require that the reason for every declaration be given in full in the text; still, to omit it altogether, wrongs those pupils who do not know and cannot perceive the correct reason. But to ask for the reason and to print the paragraph reference meets the requirements of the various degrees of intellectual capacity and maturity in every class. The pupil who knows and knows that he knows need not consult the paragraph cited; the pupil who does not know may learn for himself the correct reason by the reference. It is obvious that the greater progress an individual makes in assimilating the subject and in entering into its spirit, the less need there will be for the printed reference.

(*b*) *To stimulate the mental activity of the pupil.*

To compel a young student to supply his own demonstrations, in other words, to think and reason for himself, frequently proves unprofitable as well as unpleasant, and engenders in the learner a distaste for a study he has the right to admire and to delight in. The short-sighted youth absorbs his Geometry by memorizing, only to find that his memory has been an enemy, and while he himself is becoming more and more confused, his thoughtful companion is making greater and greater progress. The earlier he discovers his error the better, and the plan of this text gives him an opportunity to reëstablish himself with his class. It is not calculated to produce accomplished geometricians at the completion of the first book, but to aid the learner in his progress throughout the volume, wherever experience has shown that he is likely to require assistance. It is calculated, under good instruction, to develop a clear conception of the geometric idea, and to produce at the end of the course a rational individual and a friend of this particular science.

(*c*) *To bring the pupil to the theorems and their demonstrations — the real subject-matter of Geometry — as early in the study as possible.*

(*d*) *To explain rather than formally demonstrate the simple fundamental truths.*

(*e*) *To apply each theorem in the demonstration of other theorems as promptly as possible.*

(*f*) *To present a text that will be clear, consistent, teachable, and sound.*

The experienced teacher will observe:

(*a*) *The economy of arrangement.*

Many of the smaller figures are placed at the side of the page rather than at the center. The individual numbers of theorems are omitted.

(*b*) *The superior character of the diagrams.*

(*c*) *The omission of the words "since" and "for."*

The advance statement is made and the reason asked for and usually cited. The inquiring mind fails to understand the force of preceding and following some statements with the same reason.

(*d*) *Originals that are carefully classified, graded, and placed after the natural subdivisions of the subject-matter.*

(*e*) *The independence of these originals.*

Every exercise can be solved or demonstrated without the use of any other exercise. Only the truths in the numbered paragraphs are necessary in working originals.

(*f*) *The setting of every theorem, corollary, and problem of the text proper in fullface type.*

(*g*) *The consistent use of such terms as "vertical angles," "vertex-angle," "adjacent angles," "angles adjoining a side," and others.*

(*h*) *The full treatment of measurement and the illustrations of the terms employed.*

(*i*) *The summaries that precede earlier collections of original exercises.*

(*j*) *The emphasis given to the discussion of original constructions.*

As in all subjects that are new to a class, the successful teacher will be content with short lessons at the beginning, and will progress slowly until the class is thoroughly familiar with the language and the general method and purpose of the new science.

The author sincerely desires to extend his thanks to those friends who, by suggestion and encouragement, have inspired him in the preparation of these pages.

EDWARD R. ROBBINS.

THE WILLIAM PENN CHARTER SCHOOL,
PHILADELPHIA.

CONTENTS

PLANE GEOMETRY

INTRODUCTION

1. Geometry is a science which treats of the measurement of magnitudes.

2. A **definition** is a statement explaining the significance of a word or a phrase.

Every definition should be clear, simple, descriptive, and correct; that is, it should contain the essential qualities or exclude all others, or both.

3. A **point** is that which has position but not magnitude.

4. A **line** is that which has length but no other magnitude.

5. A **straight line** is a line which is determined (fixed in position) by *any* two of its points. That is, two lines that coincide entirely, if they coincide at *any* two points, are straight lines.

6. A **rectilinear figure** is a figure containing straight lines and no others.

7. A **surface** is that which has length and breadth but no other magnitude.

8. A **plane** is a surface in which if any two points are taken, the straight line connecting them lies wholly in that surface.

9. Plane Geometry is a science which treats of the properties of magnitudes in a plane.

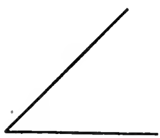
10. A **solid** is that which has length, breadth, and thickness. A solid is that which occupies space.

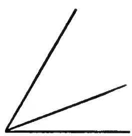
11. Boundaries. The boundaries (or boundary) of a solid are surfaces. The boundaries (or boundary) of a surface are lines. The boundaries of a line are points. These boundaries can be no part of the things they limit. A surface is no part of a solid; a line is no part of a surface; a point is no part of a line.

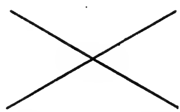
12. Motion. If a point moves, its path is a line. Hence, if a point moves, it generates (describes or traces) a line; if a line moves (except upon itself), it generates a surface; if a surface moves (except upon itself), it generates a solid.

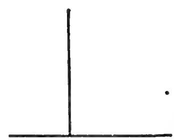
NOTE. Unless otherwise specified the word "line" hereafter means *straight line.*

ANGLES


ANGLE


ADJACENT ANGLES


VERTICAL ANGLES


RIGHT ANGLES PERPENDICULAR

13. A **plane angle** is the amount of divergence of two straight lines that meet. The lines are called the **sides** of the angle. The **vertex** of an angle is the point at which the lines meet.

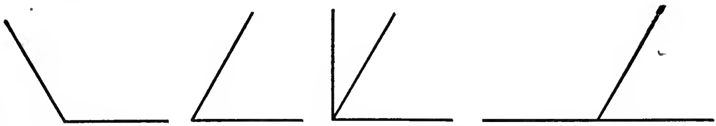
14. Adjacent angles are two angles that have the same vertex and a common side *between them.*

15. Vertical angles are two angles that have the same vertex, the sides of one being prolongations of the sides of the other.

16. If one straight line meets another and makes the adjacent angles equal, the angles are **right angles.**

17. One line is **perpendicular** to another if they meet at right angles. Either line is perpendicular to the other. The point at which the lines meet is the **foot** of the perpendicular. **Oblique lines** are lines that meet but are not perpendicular.

18. A **straight angle** is an angle whose sides lie in the same straight line, but extend in opposite directions from the vertex.



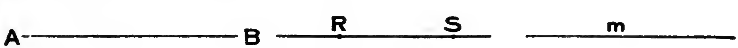
OBTUSE ANGLE ACUTE ANGLE COMPLEMENTARY ANGLES SUPPLEMENTARY ANGLES

19. An **obtuse angle** is an angle that is greater than a right angle. An **acute angle** is an angle that is less than a right angle. An **oblique angle** is any angle that is not a right angle.

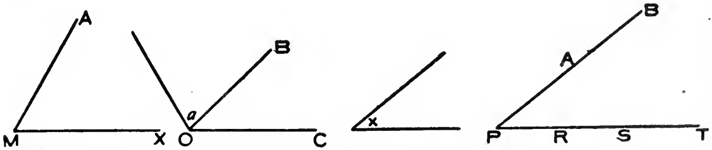
20. Two angles are **complementary** if their sum is equal to *one* right angle. Two angles are **supplementary** if their sum is equal to *two* right angles. Thus the complement of an angle is the difference between one right angle and the given angle. The supplement of an angle is the difference between two right angles and the given angle.

21. A **degree** is one ninetieth of a right angle. The degree is the familiar unit used in measuring angles. It is evident that there are 90° in a right angle; 180° in two right angles, or a straight angle; 360° in four right angles.

22. Notation. A point is usually denoted by a capital letter, placed near it. A line is denoted by two capital letters, placed one at each end, or one at each of two of its points. Its length is sometimes represented advantageously by a small letter written near it. Thus, the line AB; the line RS; the line m.



An angle is usually denoted by three capital letters, placed one at the vertex and one on each side. If only one angle is at a vertex, the capital letter at the vertex is sufficient to designate the angle. Sometimes it is advantageous to name an angle by a small letter placed within the angle. The word "angle" is usually denoted by the symbol "$\angle$" in geometrical processes.

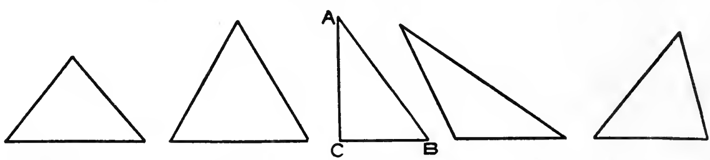


$\angle AMX$ OR $\angle XMA$ OR $\angle M$. $\angle a$ AND $\angle BOC$, NOT $\angle O$ $\angle x$

It is important that in naming an angle by the use of three letters, the vertex-letter should be placed between the others. The size of an angle does not depend upon the length of the sides, but only on the amount of their divergence. Thus, $\angle x = \angle P$ and $\angle P$ is the same as $\angle APR$ or $\angle APS$ or $\angle BPS$, etc. An angle is said to be **included** by its sides. An angle is **bisected** by a line drawn through the vertex and dividing the angle into two equal angles.

TRIANGLES

23. A **triangle** is a portion of a plane bounded by three straight lines. These lines are the **sides**. The **vertices** of a triangle are the three points at which the sides intersect. The **angles** of a triangle are the three angles at the three vertices. Each side of a triangle has two angles adjoining it. The symbol for triangle is "$\triangle$".



ISOSCELES $\triangle$ EQUILATERAL $\triangle$ EQUIANGULAR $\triangle$ RIGHT $\triangle$ OBTUSE $\triangle$ ACUTE $\triangle$ SCALENE $\triangle$s

The **base** of a triangle is the side on which the figure appears to stand. The **vertex** of a triangle is the vertex opposite the base. The **vertex-angle** is the angle opposite the base.

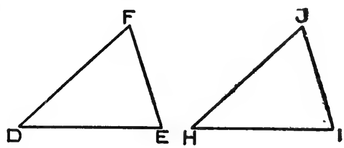
24. Kinds of triangles:

A **scalene** triangle is a triangle no two **sides** of which are equal.
An **isosceles** triangle is a triangle two **sides** of which are equal.
An **equilateral** triangle is a triangle all **sides** of which are equal.
A **right** triangle is a triangle **one angle** of which is a right angle.
An **obtuse** triangle is a triangle **one angle** of which is an obtuse angle.
An **acute** triangle is a triangle **all angles** of which are acute angles.
An **equiangular** triangle is a triangle **all angles** of which are equal.

25. The **hypotenuse** of a right triangle is the side opposite the right angle. The sides forming the right angle are called the **legs.** In an isosceles triangle the equal sides are sometimes called the **legs,** and the other side, the **base.**

26. Homologous Parts. If two triangles have the three angles of one equal respectively to the three angles of the other, the pairs of equal angles are **homologous. Homologous sides** in two triangles are opposite the homologous angles.

27. Homologous parts of equal figures are equal.



If the triangles *DEF* and *HIJ* are equal in all respects, $\angle D$ is homologous to, and $= \angle H$, hence *EF* is homologous to, and $=$ *IJ*. And $\angle E$ is homologous to, and $= \angle I$, hence, *DF* is homologous to, and $=$ *HJ*, and so on.

SUPERPOSITION. SYMBOLS

28. Equality and coincidence. Two geometrical figures are **equal** if they can be made to coincide in all respects. Angles coincide, and are equal, if their vertices are the same point and the sides of one angle are identical with the sides of the other. **Superposition** is the process of placing one figure upon another. This method of showing the equality of two geometrical figures is employed only in establishing fundamental principles.

29. Symbols. The usual symbols and abbreviations employed in geometry are the following:

$+$	plus.	Ⓢ	circles.	ax.	axiom.
$-$	minus.	$\angle$	angle.	hyp.	hypothesis.
$=$	equals, *or* is (*or* are) equal to.	$\angle$s	angles.	comp.	complementary.
≎	is (*or* are) equivalent to.	rt. $\angle$	right angle.	supp.	supplementary.
$>$	is (*or* are) greater than.	rt. $\angle$s	right angles.	const.	construction.
$<$	is (*or* are) less than.	$\triangle$	triangle.	cor.	corollary.
$\therefore$	hence, therefore, consequently.	$\triangle$s	triangles.	st.	straight.
$\perp$	perpendicular.	rt. $\triangle$s	right triangles.	def.	definition.
$\perp$s	perpendiculars.	$\parallel$	parallel.	alt.	alternate.
$\odot$	circle.	$\parallel$s	parallels.	int.	interior.
		▱	parallelogram.	ext.	exterior.
		▱s	parallelograms.		

AXIOM, POSTULATE, AND THEOREM

30. An **axiom** is a truth assumed to be self-evident. **It is** a truth which is received and assented to immediately.

31. Axioms.

1. Magnitudes that are equal to the same thing, or to equals, are equal to each other.

2. If equals are added to, or subtracted from, equals, the results are equal.

3. If equals are multiplied by, or divided by, equals, the results are equal.

[Doubles of equals are equal; halves of equals are equal.]

4. The whole is equal to the sum of all of its parts.

5. The whole is greater than any of its parts.

6. A magnitude may be displaced by its equal in any process. [Briefly called "substitution."]

7. If equals are added to, or subtracted from, unequals, the results are unequal in the same sense.

8. If unequals are added to unequals in the same sense, the results are unequal in that sense.

9. If unequals are subtracted from equals, the results are unequal in the opposite sense.

10. Doubles or halves of unequals are unequal in the same sense.

11. If the first of three magnitudes is greater than the second, and the second is greater than the third, the first is greater than the third.

12. A straight line is the shortest line that can be drawn between two points.

13. A geometrical figure may be moved from one position to another without any change in form or magnitude.

32. A **postulate** is something required to be done, the possibility of which is admitted as evident.

33. POSTULATES.

1. It is possible to draw a straight line from any point to any other point.

2. It is possible to extend (prolong or produce) a straight line indefinitely, or to terminate it at any point.

34. A geometric **proof** or **demonstration** is a logical course of reasoning by which a truth becomes evident.

35. A **theorem** is a statement that requires proof.

In the case of the preliminary theorems which follow, the proof is very simple; but as these theorems are not self-evident they cannot be classified with the axioms.

A **corollary** is a truth immediately evident, or readily established, from some other truth or truths.

EXERCISE 1. Draw an $\angle ABC$. In $\angle ABC$ draw line BD.
What does $\angle ABD + \angle DBC = ?$
What does $\angle ABC - \angle ABD = ?$

Ex. 2. In a rt. $\angle ABC$ draw line BD.
If $\angle ABD = 25°$, how many degrees are there in $\angle DBC$?
How many degrees are there in the complement of an angle of $38°$? How many degrees are there in the supplement?

Ex. 3. Draw a straight line AB and take a point X on it.
What line does $AX + BX = ?$
What line does $AB - BX = ?$

Ex. 4. Draw a straight line AB and prolong it to X so that $BX = AB$. Prolong it so that $AX = AB$.

BOOK I

ANGLES, LINES, RECTILINEAR FIGURES

PRELIMINARY THEOREMS

36. A right angle is equal to half a straight angle.

Because of the definition of a right angle. (See 16.)

37. A straight angle is equal to two right angles. (See 36.)

38. Two straight lines can intersect in only one point.

Because they would coincide entirely if they had two common points. (See 5.)

39. Only one straight line can be drawn between two points. (See 5.)

40. A definite (limited or finite) **straight line can have only one midpoint.**

Because the halves of a line are equal.

41. All straight angles are equal.

Because they can be made to coincide. (See 28 and Ax. 13.)

42. All right angles are equal.

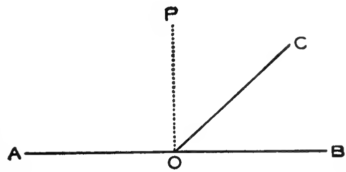
They are halves of straight angles (36), and hence equal (Ax. 3).

43. Only one perpendicular to a line can be drawn from a point in the line.

Because the right angles would not be equal if there were two perpendiculars; and all right angles are equal. (See 42.)

44. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

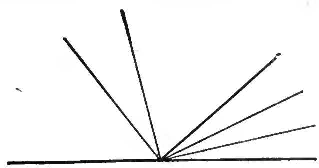
Because they together = two rt. ∠s. (See 20.)



45. If two adjacent angles are supplementary, their exterior sides are in the same straight line.

Because their sum is two rt. ∠s (20); or a straight ∠ (37). Hence the exterior sides are in the same straight line (18).

46. The sum of all the angles on one side of a straight line at a point equals two right angles.
(See Ax. 4 and 37.)



47. The sum of all the angles about a point in a plane is equal to four right angles. (See 46.)

48. Angles that have the same complement are equal. Or, **complements of the same angle, or of equal angles, are equal.**

Because equal angles subtracted from equal right angles leave equals. (See Ax. 2.)

49. Angles that have the same supplement are equal. Or, **supplements of the same angle, or of equal angles, are equal.** (See Ax. 2.)

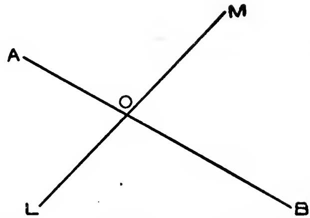
50. If two angles are equal and supplementary, they are right angles.

Because each is half a straight ∠; hence each is a rt. ∠. (See 36.)

NOTE. A single number, given as a reference, always signifies the truth stated in that paragraph and is usually the statement in **full face type** only. In reciting or writing the demonstrations the pupil should quote the correct reason for each statement, and *not* give the number of its paragraph. [Consult model demonstrations on page 24.]

THEOREMS AND DEMONSTRATIONS

51. Theorem. Vertical angles are equal.



Given: ∠s *AOM* and *BOL*, a pair of vertical angles.

To Prove: $\angle AOM = \angle BOL$.

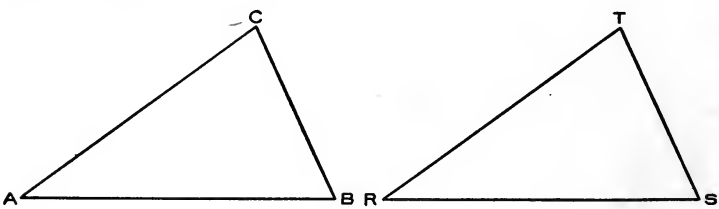
Proof: $\angle AOM$ is the supplement of $\angle MOB$. (Why?) (See 44.)

$\angle BOL$ is the supplement of $\angle MOB$. (Why?) (See 44.)

$\therefore \angle AOM = \angle BOL$. (Why?) (See 49.)

∠s *AOL* and *BOM* are a pair of vertical angles. These may be proved equal in precisely the same manner. If $\angle AOL = 80°$, how many degrees are there in the other ∠s?

52. Theorem. Two triangles are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.



Given: △s *ABC* and *RST*; $AB = RS$; $AC = RT$; $\angle A = \angle R$.

To Prove: $\triangle ABC = \triangle RST$.

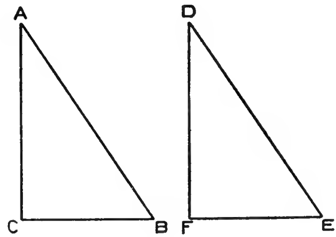
Proof: Place the $\triangle ABC$ upon the $\triangle RST$ so that $\angle A$ coincides with its equal, $\angle R$; then AB will fall upon RS and point B upon S. (It is given that $AB = RS$.) AC will fall upon RT and point C upon T. (It is given that $AC = RT$.)

$\therefore BC$ will coincide with ST. (Why?) (See 39.)

Hence, the triangles coincide in every respect and are equal (28).

53. Theorem. Two right triangles are equal if the two legs of one are equal respectively to the two legs of the other.

Given: Rt. ⟁ ABC and DEF; $AC = DF$; $CB = FE$.

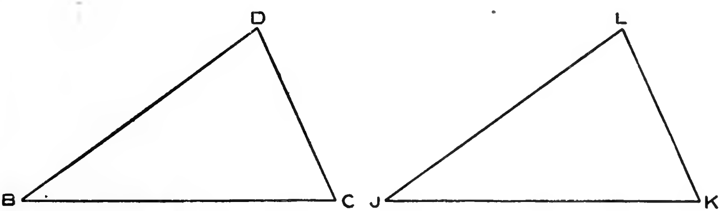


To Prove: $\triangle ABC = \triangle DEF$.

Proof: In the ⟁ ABC and DEF, $AC = DF$ (Given); $CB = FE$ (Given); $\angle C = \angle F$. (Why?) (See 42.)

∴ the ⟁ are equal. (Why?) (Theorem of 52.)

54. Theorem. Two triangles are equal if a side and the two angles adjoining it in the one are equal respectively to a side and the two angles adjoining it in the other.



Given: ⟁ BCD and JKL; $BC = JK$; $\angle B = \angle J$; $\angle C = \angle K$.

To Prove: $\triangle BCD = \triangle JKL$.

Proof: Place $\triangle$ BCD upon $\triangle$ JKL so that $\angle$ B coincides with its equal, $\angle J$, BC falling on JK.

Point C will fall on K. (It is given that $BC = JK$.)

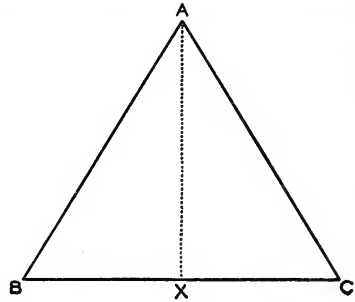
BD will fall on JL. (Because $\angle B$ is given $= \angle J$.)

CD will fall on KL. (Because $\angle C$ is given $= \angle K$.)

Then point D which falls on both the lines JL and KL will fall at their intersection, L. (Why?) (See 38.)

∴ the ⟁ are =. (Why?) (See 28.)

55. THEOREM. The angles opposite the equal sides of an isosceles triangle are equal.



Given: $\triangle ABC$, $AB = AC$.

To Prove: $\angle B = \angle C$.

Proof: Suppose AX is drawn dividing $\angle BAC$ into two equal angles, and meeting BC at X. In $\triangle$s BAX and CAX, $AX = AX$ (Identical); $AB = AC$ (Given); $\angle BAX = \angle CAX$. (Because AX made them =.) $\therefore \triangle ABX = \triangle ACX$. (Why?) (52.) $\therefore \angle B = \angle C$. (Why?) (See 27.)

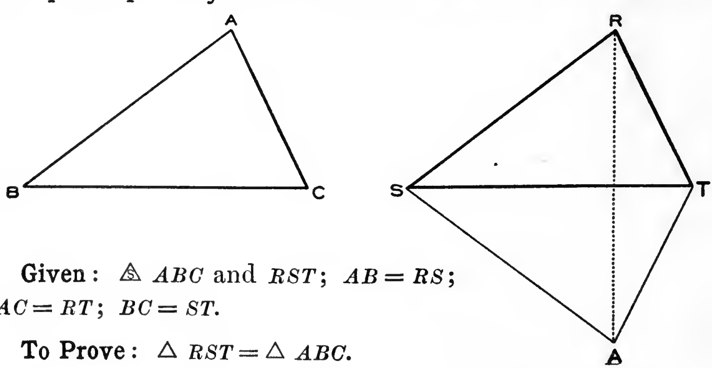
56. THEOREM. An equilateral triangle is equiangular. (See 55.)

57. THEOREM. The line bisecting the vertex-angle of an isosceles triangle is perpendicular to the base, and bisects the base.

Prove $\triangle$s ABX and ACX equal as in 55. Then, $\angle AXB = \angle AXC$. (Why?) (27.) $\therefore \angle$s AXB and AXC are rt. $\angle$s (16).

$\therefore AX$ is $\perp$ to BC. (Why?) (17.) And, also, $BX = CX$. (Why?) (27.)

58. THEOREM. Two triangles are equal, if the three sides of one are equal respectively to the three sides of the other.



Given: $\triangle$s ABC and RST; $AB = RS$; $AC = RT$; $BC = ST$.

To Prove: $\triangle RST = \triangle ABC$.

Proof: Place $\triangle\ ABC$ in the position of $\triangle\ AST$ so that the longest equal sides (BC and ST) coincide and A is opposite ST from R. Draw RA. $RS = AS$ (Given). $\therefore$ ASR is an isosceles $\triangle$. (Def. 24.)

$\therefore \angle\ SRA = \angle\ SAR$. (Why?) (55.) Likewise $TR = AT$ (?) and
$\angle\ TRA = \angle\ TAR$. (Why?) Adding these equals we obtain
$\angle\ SRT = \angle\ SAT$ (Ax. 2). $\therefore \triangle\ RST = \triangle\ AST$ (52).
That is, $\triangle\ RST = \triangle\ ABC$. (Substitution, Ax. 6.)

59. Elements of a theorem. Every theorem contains two parts, the one is assumed to be true and the other results from this assumption. The one part contains the given conditions, the other part states the resulting truth.

The assumed part of a theorem is called the **hypothesis.**

The part whose truth is to be proved is the **conclusion.**

Usually the hypothesis is a clause introduced by the word "if." When this conjunction is omitted, the subject of the sentence is known and its qualities, described in the qualifying words, constitute the "given conditions." Thus, in the theorem of 58, the assumed part follows the word "if," and the truth to be proved is: "Two triangles are equal."

60. Elements of a demonstration. All correct demonstrations should consist of certain distinct parts, namely:

1. Full statement of the **given conditions** as applied to a particular figure.

2. Full statement of the truth which it is required **to prove.**

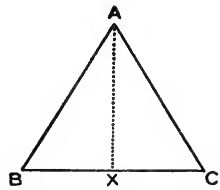
3. The **Proof.** This consists in a series of successive statements, for each of which a valid reason should be quoted. (The drawing of auxiliary lines is sometimes essential, but this part is accomplished by *imperatives* for which no reasons are necessary.)

4. The conclusion declared to be true.

The letters "Q.E.D." are often annexed at the end of a demonstration and stand for "*quod erat demonstrandum,*" which means, "which was to be proved."

MODEL DEMONSTRATIONS

The angles opposite the equal sides of an isosceles triangle are equal.



Given: $\triangle ABC$; $AB = AC$.

To Prove: $\angle B = \angle C$.

Proof: Suppose AX is drawn bisecting $\angle BAC$ and meeting BC at X.

In the $\triangle$s BAX and CAX

$AX = AX$ (Identical).

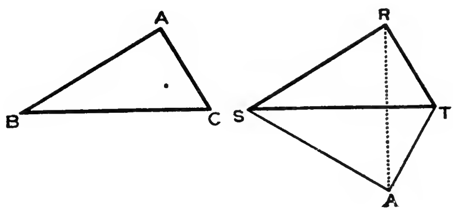
$AB = AC$ (Hypothesis).

$\angle BAX = \angle CAX$ (Construction).

$\therefore \triangle ABX = \triangle ACX$. (Two $\triangle$s are $=$ if two sides and the included $\angle$ of one are $=$ respectively to two sides and the included $\angle$ of the other.)

Hence, $\angle B = \angle C$. (Homologous parts of equal figures are equal.) Q.E.D.

Two triangles are equal if the three sides of one are equal respectively to the three sides of the other.



Given: $\triangle$s ABC and RST; $AB = RS$; $AC = RT$; $BC = ST$.

To Prove: $\triangle RST = \triangle ABC$.

Proof: Place $\triangle ABC$ in the position of $\triangle AST$ so that the longest equal sides (BC and ST) coincide, and A is opposite ST from R. Draw RA.

$RS = AS$ (Hypothesis).

$\triangle ASR$ is isosceles. (An isosceles $\triangle$ is a $\triangle$ two sides of which are equal.)

$\therefore \angle SRA = \angle SAR \ldots (1) \ldots$ (The $\angle$s opp. the $=$ sides of an isos. $\triangle$ are $=$.)

Again, $TR = AT$ (Hypothesis).

$\triangle TRA$ is isosceles. (Same reason as before.)

$\angle TRA = \angle TAR \ldots (2) \ldots$ (Same reason as for (1).) Adding equations (1) and (2).

$\angle SRT = \angle SAT$. (If $=$'s are added to $=$'s the results are $=$.)

Consequently, the $\triangle RST = \triangle AST$. (Two $\triangle$s are $=$ if two sides and the included $\angle$ of one are $=$ respectively to two sides and the included $\angle$ of the other.)

That is, $\triangle RST = \triangle ABC$. (Substitution; $\triangle ABC$ is the same as $\triangle AST$.) Q.E.D.

The preceding form of demonstration will serve to illustrate an excellent scheme of writing the proofs. It will be observed that the *statements* are at the left of the page and their *reasons* at the right. This arrangement will be found of great value in the saving of time, both for the pupil who writes the proofs and for the teacher who reads them.

61. The **converse of a theorem** is the theorem obtained by interchanging the hypothesis and conclusion of the original theorem. Consult 44 and 45; 79, 80, and others.

Every theorem which has a simple hypothesis and a simple conclusion has a converse, but only a few of these converses are actually true theorems.

For example: Direct theorem: "Vertical angles are equal."

Converse theorem: "If angles are equal, they are vertical." This statement cannot be universally true.

The theorem of 120 is the converse of that of 55.

62. Auxiliary lines. Often it is impossible to give a simple demonstration without drawing a line (or lines) not described in the hypothesis. Such lines are usually dotted for no other reason than to aid the learner in distinguishing the lines mentioned in the hypothesis and conclusion from lines whose use is confined to the proof. Hence, lines mentioned in the hypothesis and conclusion should never be dotted. (The figure used in 57 should contain no dotted line.)

63. Converse of definitions. The converse of a definition is true. It is often advantageous to quote the converse of a definition, as a reason, instead of the definition itself.

64. Homologous parts. Triangles are proved equal in order that their homologous sides, or homologous angles, may be proved equal. This is a very common method of proving lines equal and angles equal.

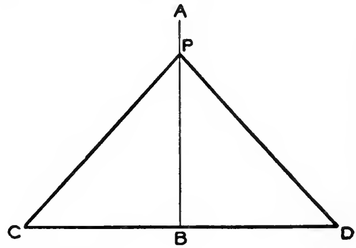
65. The **distance** from one point to another is the length of the straight line joining the two points.

66. Theorem. **If lines be drawn from any point in a perpendicular erected at the midpoint of a straight line to the ends of the line,**

I. They will be equal.

II. They will make equal angles with the perpendicular.

III. They will make equal angles with the line.



Given: $AB \perp$ to CD at its midpoint, B; P any point in AB; PC and PD.

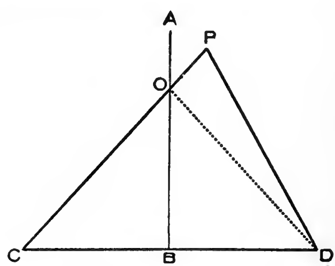
To Prove: I. $PC = PD$;
II. $\angle CPB = \angle DPB$; and
III. $\angle C = \angle D$.

Proof: In rt. $\triangle$ PBC and PBD, $BC = BD$ (Hyp.); $BP = BP$ (Iden.).
$\therefore \triangle PBC = \triangle PBD$. (Why?) (53.)

$\therefore$ I. $PC = PD$ (Why?) (27;) II. $\angle CPB = \angle DPB$ (Why?);
III. $\angle C = \angle D$ (Why?). Q.E.D.

67. Theorem. **Any point in the perpendicular bisector of a line is equally distant from the extremities of the line.** (See 66, I.)

68. Theorem. **Any point not in the perpendicular bisector of a line is not equally distant from the extremities of the line.**



Given: $AB \perp$ bisector of CD; P any point not in AB; PC and PD.

To Prove: PC not $= PD$.

Proof: Either PC or PD will cut AB.

Suppose PC cuts AB at O. Draw OD.

$DO + OP > PD$. (Why?) (Ax. 12.) But $CO = OD$ (67).
$\therefore CO + OP > PD$. (Substitution; Ax. 6.)
That is, $PC > PD$, or PC is not $= PD$. Q.E.D.

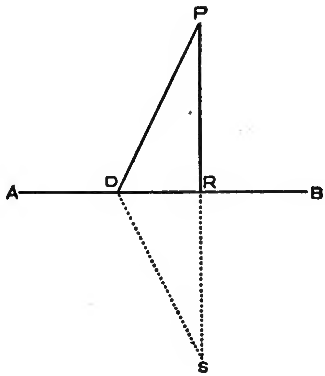
69. Theorem. If a point is equally distant from the extremities of a line, it is in the perpendicular bisector of the line. (See 67 and 68.)

70. Theorem. Two points each equally distant from the extremities of a line determine the perpendicular bisector of the line.

Each point is in the ⊥ bisector (69); two points determine a line (5).

71. Theorem. Only one perpendicular can be drawn to a line from an external point.

Given: *PR* ⊥ to *AB* from *P*; *PD* any other line from *P* to *AB*.



To Prove: *PD* cannot be ⊥ to *AB*; that is, *PR* is the only ⊥ to *AB* from *P*.

Proof: Extend *PR* to *S*, making *RS* = *PR*; draw *DS*.

In rt. △ *PDR* and *SDR*, *PR* = *RS* (Const.).

DR = *DR* (Iden.). ∴ △ *PDR* = △ *SDR*. (Why?) (53.)

∴ ∠ *PDR* = ∠ *SDR* (27). That is, ∠ *PDR* = half of ∠ *PDS*.

Now *PRS* is a straight line (Const.).

∴ *PDS* is not a straight line (39).

∴ ∠ *PDS* is not a straight angle (18).

∴ ∠ *PDR*, the half of ∠ *PDS*, is not a right angle (36).

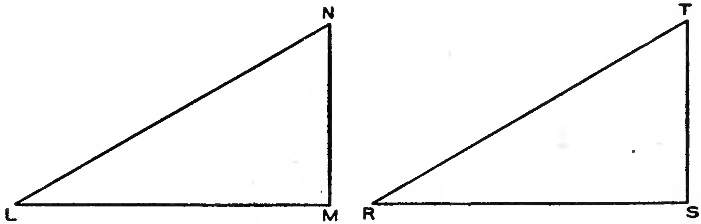
∴ *PD* is not ⊥ (17). ∴ *PR* is the only ⊥. Q.E.D.

Ex. 1. Through how many degrees does the minute hand of a clock move in 15 min.? in 20 min.? Through how many degrees does the hour hand move in one hour? in 45 minutes? in 10 minutes?

Ex. 2. How many degrees are there in the angle between the hands of a clock at 9 o'clock? at 10 o'clock? at 12:30? at 2:15? at 3:45?

Ex. 3. Theorem. If two lines be drawn bisecting each other, and their ends be joined in order, the opposite pairs of triangles will be equal. [Use 51 and 52.]

72. Theorem. **Two right triangles are equal if the hypotenuse and an adjoining angle of one are equal respectively to the hypotenuse and an adjoining angle of the other.**



Given: Rt. ⊿ *LMN* and *RST*; $LN = RT$; and $\angle L = \angle R$.

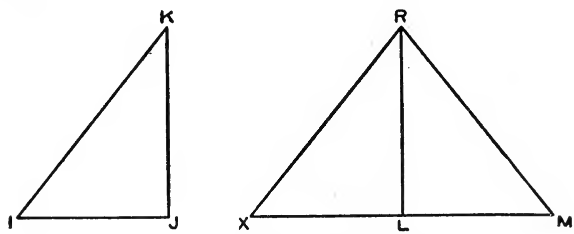
To Prove: $\triangle LMN = \triangle RST$.

Proof: Superpose $\triangle LMN$ upon $\triangle RST$ so that $\angle L$ coincides with its equal, $\angle R$, *LM* falling along *RS*. Then *LN* will fall on *RT* and point *N* will fall exactly on *T* ($LN = RT$ by Hyp.).

Now *NM* and *TS* will both be $\perp$ to *RS* from *T* (Rt. ⊿ by Hyp.). $\therefore$ *NM* will coincide with *TS* (71).

$\therefore \triangle LMN = \triangle RST$ (28). Q.E.D.

73. Theorem. **Two right triangles are equal if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.**



Given: Rt. ⊿ *IJK* and *LMR*; $KI = RM$; $KJ = RL$.

To Prove: $\triangle IJK = \triangle LMR$.

Proof: Place $\triangle IJK$ in the position of $\triangle XLR$ so that the equal sides, *KJ* and *RL*, coincide and *I* is at *X*, opposite *RL* from *M*.

Now, $\triangle$ *RLM* and *RLX* are supplementary. (Why ?) (20.)
$\therefore XLM$ is a str. line (45).

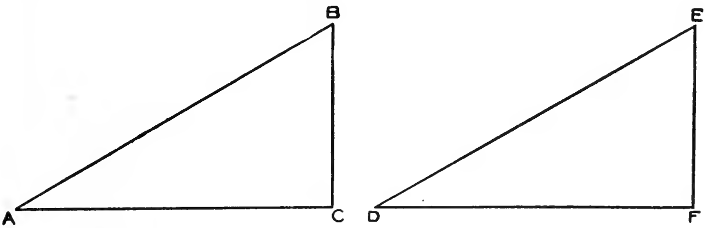
Also, $\triangle RMX$ is isosceles. (Why ?) ($RX = RM$ by Hyp.)
$\therefore \angle X = \angle M$. (Why ?) (55.)

$\therefore \triangle XLR = \triangle MLR$. (Why ?) (72.) That is, $\triangle IJK = \triangle LMR$. (Ax. 6.) Q.E.D.

Cor. The perpendicular from the vertex of an isosceles triangle to the base bisects the base.

Proof: $\triangle XLR = \triangle MLR$. (Why ?) $\therefore XL = ML$. (Why ?)

74. Theorem. Two right triangles are equal if a leg and the adjoining acute angle of one are equal respectively to a leg and the adjoining acute angle of the other.



Given: Rt. $\triangle$ *ABC* and *DEF*; $AC = DF$; $\angle A = \angle D$.

To Prove: $\triangle ABC = \triangle DEF$.

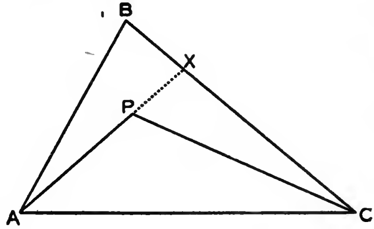
Proof: In the $\triangle$ *ABC* and *DEF*, $AC = DF$. (Why ?) (Hyp.)
Also $\angle A = \angle D$ (Why ?) and $\angle C = \angle F$. (Why ?) (42.)
$\therefore \triangle ABC = \triangle DEF$. (Why ?) (54.) Q.E.D.

Ex. 1. How many pairs of equal parts must two triangles have, in order that they may be proved equal? How many pairs is it necessary to mention in the case of two right triangles?

Ex. 2. Theorem. If a perpendicular be erected at any point in the bisector of an angle, two equal right triangles will be formed. [Use 74.]

Ex. 3. Through the midpoint of a line *AB* any oblique line is drawn:
I. The lines $\perp$ to it from *A* and *B* are equal. [Use 72.]
II. The lines $\perp$ to *AB* at *A* and *B*, terminated by the oblique line, are equal. [Use 74.]

75. Theorem. The sum of two sides of a triangle is greater than the sum of two lines drawn to the extremities of the third side, from any point within the triangle.



Given: P, any point in $\triangle ABC$; lines PA and PC.

To Prove: $AB + BC > AP + PC$.

Proof: Extend AP to meet BC at X.

$$AB + BX > AP + PX. \text{ (Why?) (Ax. 12.)}$$
$$CX + PX > PC. \text{ (Why?) (Ax. 12.) Add:}$$

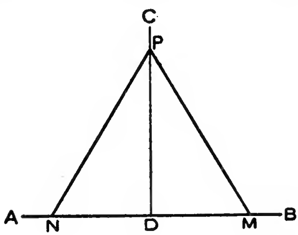
$$AB + \overline{BX + CX} + PX > AP + PC + PX \text{ (Ax. 8).}$$
Subtract $PX = PX$.
$$\therefore AB + BC > AP + PC \text{ (Ax. 7).}$$ Q.E.D.

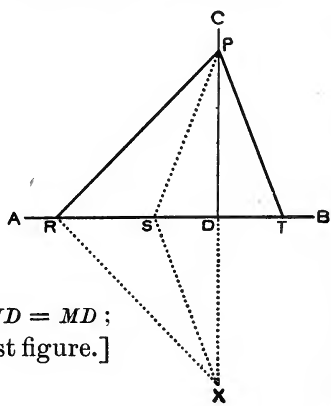
76. Theorem. If from any point in a perpendicular to a line two oblique lines be drawn,

I. Oblique lines cutting off equal distances from the foot of the perpendicular will be equal.

II. Equal oblique lines will cut off equal distances (converse).

III. Oblique lines cutting off unequal distances will be unequal, and that one which cuts off the greater distance will be the greater.





I. **Given**: $CD \perp$ to AB; $ND = MD$; oblique lines PN and PM. [First figure.]

To Prove: $PN = PM$.

Proof: PD is $\perp$ bisector of NM (Hyp.). $\therefore PN = PM$ (67).

II. **Given:** $CD \perp$ to AB; $PN = PM$. [First figure.]

To Prove: $ND = MD$.

Proof: In the rt. $\triangle$ PND and PMD, $PD = PD$ (Iden.); and $PN = PM$ (Hyp.). $\therefore \triangle PND = \triangle PMD$. (Why?) (73.)
$\therefore ND = MD$. (Why?) (27.) Q.E.D.

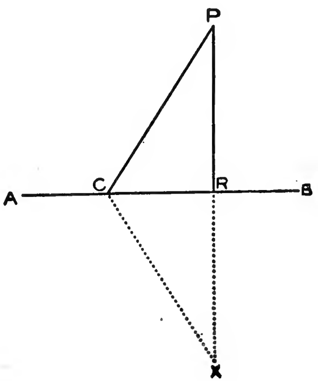
III. **Given:** $CD \perp$ to AB; oblique lines PR, PT; also $RD > DT$. [Second figure.]

To Prove: $PR > PT$.

Proof: Because RD is $> DT$, we may mark DS (on RD) $= DT$. Draw PS. Extend PD to X, making $DX = PD$; draw RX and SX. AD is $\perp$ to PX at its midpoint (Const.). $\therefore PR = RX$ and $PS = SX$ (66).
Now $PR + RX > PS + SX$ (75).
Hence $PR + PR > PS + PS$ (Ax. 6).
That is, $2\,PR > 2\,PS$. $\therefore PR > PS$ (Ax. 10).
But $PS = PT$ (76, I). $\therefore PR > PT$ (Ax. 6). Q.E.D.

Cor. Therefore, from an external point it is not possible to draw three equal lines to a given straight line.

77. Theorem. The perpendicular is the shortest line that can be drawn from a point to a straight line.



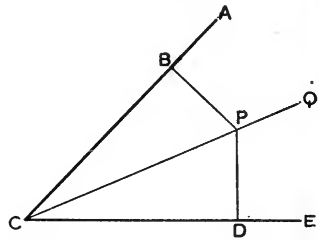
Given: $PR \perp$ to AB; PC not $\perp$.

To Prove: $PR < PC$.

Proof: Extend PR to X, making $RX = PR$. Draw CX. $PR + RX < PC + CX$ (Ax. 12).
But AR is $\perp$ to PX at its midpoint (Const.).
$\therefore PC = CX$ (66).
$\therefore PR + PR < PC + PC$ (Ax. 6).
That is, $2PR < 2PC$.
$\therefore PR < PC$ (Ax. 10). Q.E.D.

78. The **distance** from a point to a line is the length of the perpendicular from the point to the line. Thus "distance from a line" involves the perpendicular. If the perpendiculars from a point to two lines are equal, the point is said to be **equally distant** from the lines.

79. THEOREM. **Every point in the bisector of an angle is equally distant from the sides of the angle.**



Given: $\angle ACE$; bisector CQ; point P in CQ; distances PB and PD.

To Prove: $PB = PD$.

Proof: $\triangle PBC$ and PDC are rt. $\triangle$ (78).

In rt. $\triangle PBC$ and PDC, $PC = PC$ (Iden.); $\angle PCB = \angle PCD$ (Hyp.). $\therefore \triangle PBC = \triangle PDC$ (?) (72). $\therefore PB = PD$ (?). Q.E.D.

80. THEOREM. **Every point equally distant from the sides of an angle is in the bisector of the angle.**

Given: $\angle ACE$; P a point, such that $PB = PD$ (distances); CQ a line from vertex of the angle, and containing P.

To Prove: $\angle ACQ = \angle ECQ$.

Proof: $\triangle PBC$ and PDC are right $\triangle$ (?). In rt. $\triangle PBC$ and PDC, $PC = PC$ (?) ; $PB = PD$ (?).

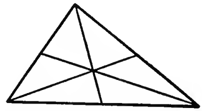
$\therefore \triangle PBC = \triangle PDC$ (?) (73). $\therefore \angle ACQ = \angle ECQ$ (?). Q.E.D.

81. THEOREM. **Any point not in the bisector of an angle is not equally distant from the sides of the angle.** [Because if it were equally distant, it would be in the bisector (80).]

82. THEOREM. **The vertex of an angle and a point equally distant from its sides determine the bisector of the angle.** (See 80 and 5.)

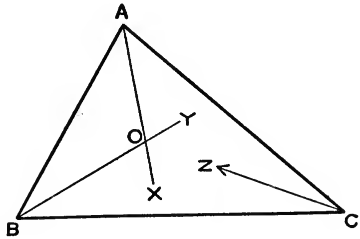
Ex. Describe the path of a moving point which shall be equally distant from two intersecting lines.

83. The **altitude** of a triangle is the perpendicular from any vertex to the opposite side (prolonged if necessary). A triangle has three altitudes. The **bisector** of an angle of a triangle is the line dividing any angle into equal angles. A triangle has three bisectors of its angles. The **median** of a triangle is the line drawn from any vertex to the midpoint of the opposite side. A triangle has three medians.



THE THREE MEDIANS

84. Theorem. The bisectors of the angles of a triangle meet in a point which is equally distant from the sides.



Given: $\triangle ABC$, AX bisecting $\angle A$, BY and CZ the other bisectors.

To Prove: AX, BY, CZ meet in a point equally distant from AB, AC, and BC.

Proof: Suppose that AX and BY intersect at O.

O in AX is equally distant from AB and AC (?) (79).

O in BY is equally distant from AB and BC (?).

$\therefore$ point O is equally distant from AC and BC (Ax. 1).*

$\therefore$ O is in bisector CZ (?) (80).

That is, all three bisectors meet at O, and O is equally distant from the three sides. Q.E.D.

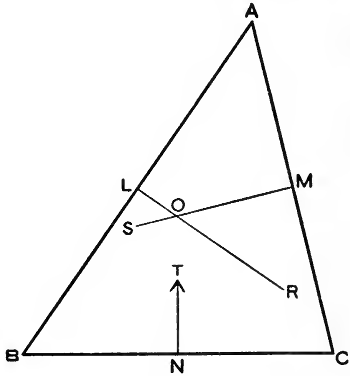
* The $\perp$ distances from O to the three sides are the three equals.

Ex. 1. Draw the three altitudes of an acute triangle; of an obtuse triangle.

Ex. 2. Prove that in an equilateral triangle:

(*a*) An altitude is also a median. [Use 73.]
(*b*) A median is also an altitude. [Use 58, 27, 16, 17.]
(*c*) An altitude is also the bisector of an angle of the triangle.
(*d*) The bisector of an angle is also an altitude. [Use 52.]
(*e*) The bisector of an angle is also a median.

85. Theorem. The three perpendicular bisectors of the sides of a triangle meet in a point which is equally distant from the vertices.



Given: $\triangle ABC$; LR, MS, NT, the three $\perp$ bisectors.

To Prove: LR, MS, NT meet at a point equally distant from A and B and C.

Proof: Suppose that LR and MS intersect at O.

O in LR is equally distant from A and B. (?) (67.)

O in MS is equally distant from A and C. (?)

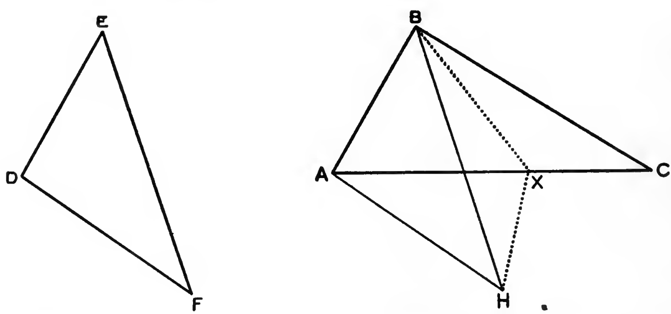
$\therefore$ point O is equally distant from B and C (Ax. 1).*

Hence O is in $\perp$ bisector NT (?) (69).

That is, all three $\perp$ bisectors meet at O, and O is equally distant from A and B and C. Q.E.D.

* The three lines from O to the vertices are the three equals.

86. Theorem. If two triangles have two sides of one equal to two sides of the other, but the included angle in the first greater than the included angle in the second, the third side of the first is greater than the third side of the second.



Given: $\triangle\!\!\triangle$ ABC, DEF; $AB = DE$; $BC = EF$; $\angle ABC > \angle E$.

To Prove: $AC > DF$.

Proof: Place the $\triangle DEF$ upon $\triangle ABC$ so that line DE coincides with its equal AB, $\triangle DEF$ taking the position of $\triangle ABH$. There will remain an angle, HBC. ($\angle ABC$ is $> \angle E$.)

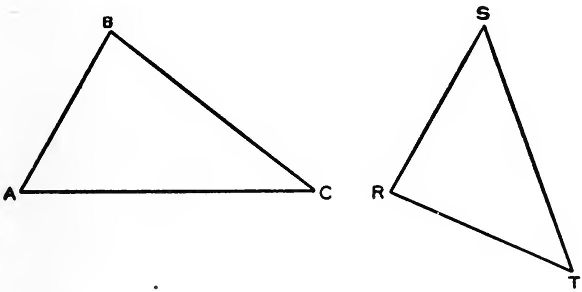
Suppose BX to be the bisector of $\angle HBC$, meeting AC at X. Draw XH.

In $\triangle$s HBX and CBX, $BX = BX$ (?); $BH = BC$ (?); $\angle HBX = \angle CBX$ (?) (Const.). $\therefore \triangle HBX = \triangle CBX$ (?) (52). $\therefore HX = XC$ (?).

Now, $AX + XH > AH$ (?). $\therefore AX + XC > AH$ (Ax. 6).

That is, $AC > DF$. Q.E.D.

87. Theorem. If two triangles have two sides of one equal to two sides of the other but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second. [Converse.]



Given: $\triangle$s ABC and RST; $AB = RS$; $BC = ST$; $AC > RT$.

To Prove: $\angle B > \angle S$.

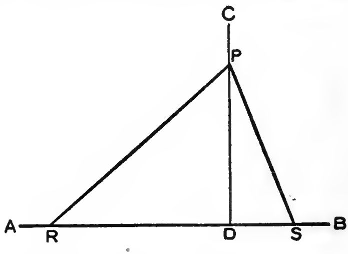
Proof: It is evident that $\angle B < \angle S$, or $\angle B = \angle S$, or $\angle B > \angle S$.

1. If $\angle B < \angle S$, $AC < RT$ (86).
   But $AC > RT$ (Hyp.). $\therefore \angle B$ is *not* $< \angle S$.
2. If $\angle B = \angle S$, the $\triangle$s are $=$ (52), and AC is $= RT$ (27).
   But $AC > RT$ (Hyp.). $\therefore \angle B$ is *not* $= \angle S$.
3. $\therefore$ the only possibility is that $\angle B > \angle S$. Q.E.D.

88. The preceding method of demonstration is termed the **method of exclusion.** It consists in making all possible suppositions, leaving the probable one last, and then proving all these suppositions impossible, except the last, which must necessarily be true.

89. The method of proving the individual steps is called *reductio ad absurdum* (reduction to an absurd or impossible conclusion). This method consists in assuming as false the truth to be proved and then showing that this assumption leads to a conclusion altogether contrary to known truth or the given hypothesis. (Examine 87.) This is sometimes called the *indirect method.* The theorems of 93 and 94 are demonstrated by a single use of this method.

90. Theorem. **If two unequal oblique lines be drawn from any point in a perpendicular to a line, they will cut off unequal distances from the foot of the perpendicular, and the longer oblique line will cut off the greater distance.** [Converse of 76, III.]



Given: $CD \perp$ to AB; PR and PS oblique lines; $PR > PS$.

To Prove: $DR > DS$.

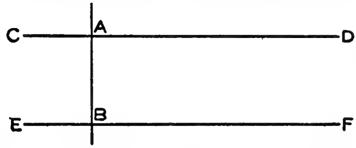
Proof: It is evident that $DR < DS$, or $DR = DS$, or $DR > DS$.

If $DR < DS$, $PR < PS$ (76, III). But $PR > PS$ (Hyp.).
$\therefore DR$ is *not* $< DS$.
If $DR = DS$, $PR = PS$ (76, I). But $PR > PS$ (Hyp.).
$\therefore DR$ is *not* $= DS$.
Therefore the only possibility is that $DR > DS$. Q.E.D.

91. Parallel lines are straight lines that lie in the same plane and that never meet, however far extended in either direction.

92. Axiom. **Only one line can be drawn through a point parallel to a given line.**

93. Theorem. Two lines in the same plane and perpendicular to the same line are parallel.



Given: CD and EF in same plane and both $\perp$ to AB.

To Prove: CD and EF $\parallel$.

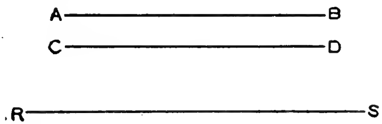
Proof: If CD and EF were not $\parallel$, they would meet if sufficiently prolonged. Then there would be two lines from the point of meeting $\perp$ to AB. (By Hyp. they are $\perp$ to AB.)

But this is impossible (?) (71).

$\therefore$ CD and EF do not meet, and are parallel (91). Q.E.D.

94. Theorem. Two lines in the same plane and parallel to the same line are parallel.

Given: $AB \parallel$ to RS and $CD \parallel$ to RS and in the same plane.



To Prove: $AB \parallel$ to CD.

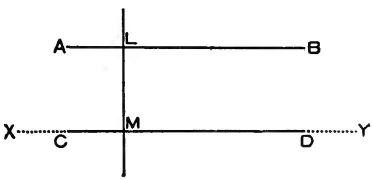
Proof: If AB and CD were not $\parallel$, they would meet if sufficiently prolonged.

Then there would be two lines through the point of meeting $\parallel$ to the line RS. (By Hyp. they are $\parallel$ to RS).

But this is impossible (92).

$\therefore$ AB and CD do not meet, and are $\parallel$ (?) (91). Q.E.D.

95. Theorem. If a line is perpendicular to one of two parallels, it is perpendicular to the other also.



Given: $LM \perp$ to AB and $AB \parallel$ to CD.

To Prove: $LM \perp$ to CD.

Proof: Suppose XY is drawn through $M \perp$ to LM. AB is $\parallel$ to XY (?) (93).

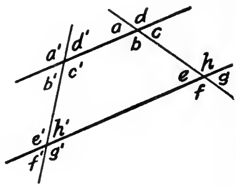
But AB is $\parallel$ to CD (Hyp.).

Now CD and XY both contain M (Const.).

$\therefore$ CD and XY coincide (92).

But LM is $\perp$ to XY (?). That is, LM is $\perp$ to CD. Q.E.D.

96. If one line cuts other lines, it is called a **transversal.** Angles are formed at the several intersections, and these receive the following names:



Interior $\angle$s are between the lines [b, c, e, h].

Exterior $\angle$s are without the lines [a, d, f, g].

Alternate $\angle$s are on opposite sides of the transversal [b and h; c and e; a and g; etc.].

Alternate-Interior $\angle$s are b and h; and c and e.

Alternate-Exterior $\angle$s are a and g; d and f.

Corresponding $\angle$s are a and e; d and h; b and f; c and g.

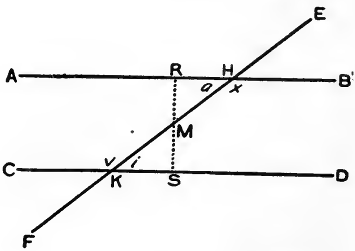
Adjoining-Interior $\angle$s are c and h; b and e.

Adjacent-Interior $\angle$s are b and c; e and h.

The left-hand transversal makes eight angles similarly related. The "primes" are used only to designate angles which are different from those in the right-hand part of the figure.

97. Theorem. **If a transversal intersects two parallels, the alternate interior angles are equal.**

Given: $AB \parallel$ to CD; transversal EF cutting the $\parallel$s at H and K.



To Prove: $\angle a = \angle i$ and $\angle x = \angle v$.

Proof: Suppose through M, the midpoint of HK, RS is drawn $\perp$ to AB. Then RS is $\perp$ to CD (95). In rt. $\triangle$s RMH and KMS, $HM = KM$ (Const.); $\angle RMH = \angle KMS$ (?) (51). $\therefore \triangle RMH = \triangle KMS$ (?) (72).

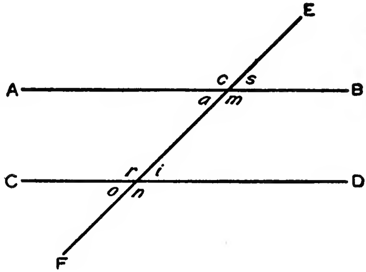
$\therefore \angle a = \angle i$ (?).

$$\left.\begin{array}{l}\angle x \text{ is the supp. of } \angle a \text{ (?) (44);} \\ \angle v \text{ is the supp. of } \angle i \text{ (?).}\end{array}\right\} \therefore \angle x = \angle v \text{ (?) (49).}$$

Q.E.D.

Exercise. If $\angle a = 70°$, in the figure of 97, how many degrees are there in $\angle x$? in $\angle i$? $\angle v$? $\angle AHE$? $\angle EHB$? $\angle CKF$? $\angle DKF$?

98. THEOREM. If a transversal intersects two parallels, the corresponding angles are equal.



Given: *AB* ∥ to *CD*; transversal *EF* cutting the ∥s and forming the 8 $\angle$s.

To Prove: $\angle s = \angle i$; $\angle c = \angle r$; $\angle o = \angle a$; $\angle n = \angle m$.

Proof: $\angle s = \angle a$ (51); $\angle a = \angle i$ (97). $\therefore \angle s = \angle i$ (?).
$\angle c = \angle m$ (?); $\angle m = \angle r$ (?). $\therefore \angle c = \angle r$ (?). Etc. Q.E.D.

99. THEOREM. If a transversal intersects two parallels, the alternate-exterior angles are equal.

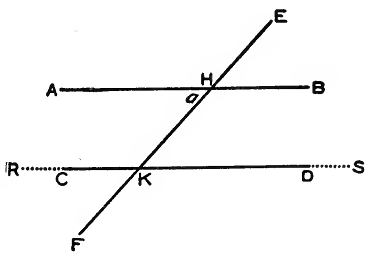
Given: (?). **To Prove:** (?). **Proof:** $\angle c = \angle r$ (?); and $\angle r = \angle n$ (?). $\therefore \angle c = \angle n$ (?). Etc. Q.E.D.

100. THEOREM. If a transversal intersects two parallels, the sum of the adjoining-interior angles equals two right angles.

Given: (?). **To Prove:** $\angle a + \angle r = 2$ rt. $\angle$s. Etc.

Proof: $\angle a + \angle m = 2$ rt. $\angle$s (?) (46). But $\angle m = \angle r$ (?). $\therefore \angle a + \angle r = 2$ rt. $\angle$s (Ax. 6). Etc. Q.E.D.

101. THEOREM. If a transversal intersects two lines and the alternate-interior angles are equal, the lines are parallel. [Converse of 97.]

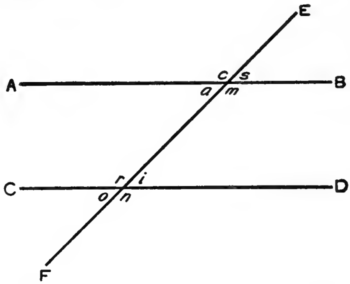


Given: *AB* and *CD* two lines; transversal *EF* cutting them at *H* and *K* respectively; $\angle a = \angle HKD$.

To Prove: *CD* ∥ to *AB*.

Proof: Through *K*, suppose *RS* is drawn ∥ to *AB*. Then $\angle a = \angle HKS$ (?) (97). But $\angle a = \angle HKD$ (Hyp.). $\therefore \angle HKS = \angle HKD$ (?). $\therefore$ *KD* and *KS* coincide; that is, *CD* and *RS* are the same line. $\therefore$ *CD* is ∥ to *AB*. (Because it coincides with *RS* which is ∥ to *AB*). Q.E.D.

102. THEOREM. If a transversal intersects two lines and the corresponding angles are equal, the lines are parallel.



Given: AB and CD cut by transversal EF; $\angle c = \angle r$.

To Prove: $AB \parallel$ to CD.

Proof: $\angle c = \angle m$ (?); $\angle c = \angle r$ (?). $\therefore \angle m = \angle r$ (?). $\therefore AB$ is $\parallel$ to CD (101). Q.E.D.

If $\angle a$ were given $= \angle o$ or $\angle s = \angle i$, etc., the proof that AB is $\parallel$ to CD would be the same.

103. THEOREM. If a transversal intersects two lines and the alternate-exterior angles are equal, the lines are parallel.

Given: AB and CD cut by EF, and $\angle c = \angle n$.

To Prove: $AB \parallel$ to CD.

Proof: $\angle c = \angle m$ (?); $\angle c = \angle n$ (?). $\therefore \angle m = \angle n$ (?). $\therefore AB$ is $\parallel$ to CD (?). Etc. Q.E.D.

104. THEOREM. If a transversal intersects two lines and the sum of the adjoining-interior angles equals two right angles, the lines are parallel.

Given: AB and CD cut by EF and $\angle a + \angle r = 2$ rt. $\angle$s.

To Prove: $AB \parallel$ to CD.

Proof: $\angle a + \angle c = 2$ rt. $\angle$s (46); $\angle a + \angle r = 2$ rt. $\angle$s (?). $\therefore \angle a + \angle c = \angle a + \angle r$ (?). Hence $\angle c = \angle r$ (Ax. 2). $\therefore AB$ is $\parallel$ to CD (?) (102). Q.E.D.

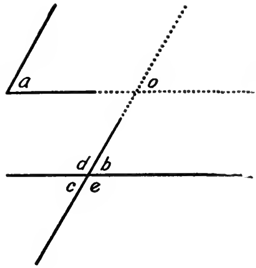
Another proof: $\left\{\begin{array}{l}\angle a \text{ is supp. of } \angle c\ (?);\\ \angle a \text{ is supp. of } \angle r\ (?).\end{array}\right\} \therefore \angle c = \angle r$ (?). Etc.

105. THEOREM. If two angles have their sides parallel each to each, the angles are equal or supplementary.

I. **Given:** $\angle a$ and $\angle b$ with their sides $\parallel$ each to each and extending in the *same directions* from their vertices.

To Prove: $\angle a = \angle b$.

Proof: If the non-parallel lines do not intersect, produce



them till they meet, forming $\angle o$. Now, $\angle a = \angle o$. (?) (98). $\angle o = \angle b$ (?).

$\therefore \angle a = \angle b$ (?). Q.E.D.

II. **Given:** $\angle a$ and $\angle c$ with their sides ∥ each to each and extending in *opposite directions* from their vertices.

To Prove: $\angle a = \angle c$.

Proof: $\angle a = \angle b$ (Proved in I); $\angle b = \angle c$ (?).

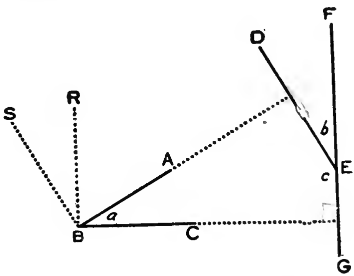
Hence $\angle a = \angle c$ (?). Q.E.D.

III. **Given:** $\angle a$ and $\angle d$ with their sides ∥ each to each, but one pair extending in the *same direction*, the other pair extending in *opposite directions* from their vertices.

To Prove: $\angle a$ and $\angle d$ supplementary.

Proof: $\angle b$ and $\angle d$ are supplementary (44). $\angle a = \angle b$ (I). $\therefore \angle a$ and $\angle d$ are supp. (Ax. 6.). Etc. Q.E.D.

106. Theorem. **If two angles have their sides perpendicular each to each, the angles are equal or supplementary.**



I. **Given:** ∠s a and b with sides ⊥ each to each.

To Prove: $\angle a = \angle b$.

Proof: At B suppose BR is drawn ⊥ to BC and BS ⊥ to AB. BR is ∥ to FE and BS is ∥ to DE (?) (93).

$\therefore \angle RBS = \angle b$ (?) (105).

$\left.\begin{array}{l}\text{Now, } \angle a \text{ is the comp. of } \angle ABR \text{ (20),} \\ \text{and } \angle RBS \text{ is the comp. of } \angle ABR \text{ (?).}\end{array}\right\} \begin{array}{l}\therefore \angle a = \angle RBS \text{ (?).} \\ \text{(48.)}\end{array}$

$\therefore \angle a = \angle b$. (Ax. 1.) Q.E.D.

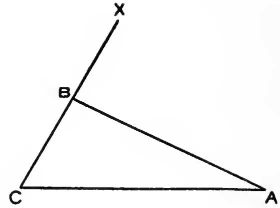
II. **Given:** ∠s a and c with sides ⊥ each to each.

To Prove: $\angle a$ and $\angle c$ supplementary.

Proof: $\angle b$ and $\angle c$ are supp. (?); $\angle a = \angle b$ (I).

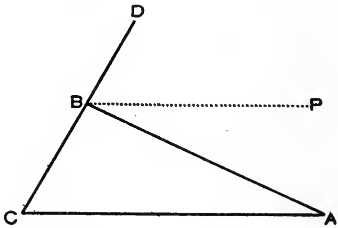
$\therefore \angle a$ and $\angle c$ are supp. (Ax. 6). Q.E.D.

107. An **exterior angle** of a triangle is an angle formed outside the triangle, between one side of the triangle and another side prolonged. [$\angle ABX$.]



The angles within the triangle, at the other vertices are the **opposite interior** angles. [$\angle A$ and $\angle C$.]

108. THEOREM. **An exterior angle of a triangle is equal to the sum of the opposite interior angles.**



Given: $\triangle ABC$; exterior $\angle ABD$.

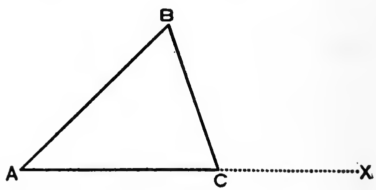
To Prove: $\angle ABD = \angle A + \angle C$.

Proof: Suppose BP to be drawn through $B \parallel$ to AC.

$\angle ABD = \angle ABP + \angle PBD$ (Ax. 4).
$\angle ABP = \angle A$ (?)(97);
$\angle PBD = \angle C$ (?) (98).
$\therefore \angle ABD = \angle A + \angle C$ (Ax. 6). Q.E.D.

109. THEOREM. **An exterior angle of a triangle is greater than either of the opposite interior angles.** (See Ax. 5.)

110. THEOREM. **The sum of the angles of any triangle is two right angles; that is, 180°.**



Given: $\triangle ABC$.

To Prove: $\angle A + \angle B + \angle ACB = 2$ rt. $\angle s = 180°$.

Proof: Prolong AC to X, making the ext. $\angle BCX$.

$\angle BCX + \angle ACB = 2$ rt. $\angle s$ (?) (46).
But $\angle BCX = \angle A + \angle B$ (?) (108).
$\therefore \angle A + \angle B + \angle ACB = 2$ rt. $\angle s = 180°$ (Ax. 6). Q.E.D.

111. Cor. **The sum of any two angles of a triangle is less than two right angles.** (See Ax. 5.)

112. Cor. **A triangle cannot have more than one right angle oı more than one obtuse angle.**

113. Cor. **Two angles of every triangle are acute.** (See 112.)

114. Theorem. **The acute angles of a right triangle are complementary.**

Proof: Their sum = 1 rt. ∠ (110 and Ax. 2). Hence they are complementary. (See 20.)

115. Cor. **Each angle of an equiangular triangle is 60°.**

116. Theorem. **If two right triangles have an acute angle of one equal to an acute angle of the other, the remaining acute angles are equal.** (See 114 and 48.)

117. Theorem. **If two triangles have two angles of the one equal to two angles of the other, the third angle of the first is equal to the third angle of the second.** (See 110 and Ax. 2.)

118. Theorem. **Two triangles are equal if a side and any two angles of the one are equal respectively to a homologous side and the two homologous angles of the other.**

Proof: The third ∠ of one △ = third ∠ of other △ (117).
∴ the $\triangle$s are = (54).

119. Theorem. **Two right triangles are equal if a leg and the opposite acute angle of one are equal respectively to a leg and the opposite acute angle of the other.** (See 118.)

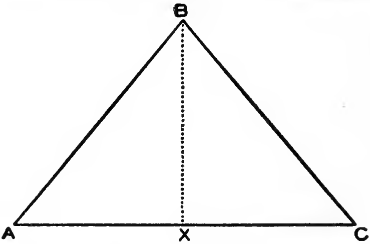
Ex. 1. In the figure of 107, if $\angle A = 40°$ and $\angle C = 70°$, how many degrees are there in $\angle ABX$? in $\angle ABC$?

Ex. 2. If each of the equal angles of an isosceles triangle is 50°, how many degrees are there in the third angle?

Ex. 3. If one of the acute angles of a right triangle is 25°, how many degrees are there in the other?

Ex. 4. State and prove the converse of 114.

120. Theorem. **If two angles of a triangle are equal, the triangle is isosceles.** [Converse of 55.]



Given: $\triangle ABC$; $\angle A = \angle C$.

To Prove: $AB = BC$.

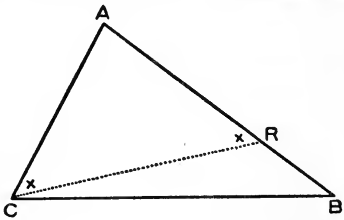
Proof: Suppose BX drawn $\perp$ to AC.

In rt. $\triangle$ ABX and CBX, $BX = BX$ (?); $\angle A = \angle C$ (?).

$\therefore \triangle ABX = \triangle CBX$ (?) (119). $\therefore AB = BC$ (?). Q.E.D.

121. Theorem. **An equiangular triangle is equilateral.**

122. Theorem. **If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.**



Given: $\triangle ABC$; $AB > AC$.

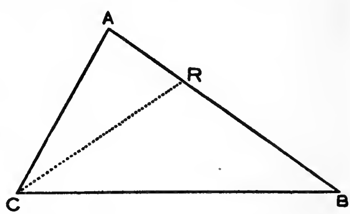
To Prove: $\angle ACB > \angle B$.

Proof: On AB take $AR = AC$. [We may, because $AB > AC$.] Draw CR and let $\angle ARC = x$.

$\angle ARC$ is an ext. $\angle$ of $\triangle CBR$ (?). $\therefore \angle x > \angle B$ (109).

Also, $\angle ACR = \angle ARC = \angle x$ (?) (55). Again, $\angle ACB > \angle x$ (?) (Ax. 5). $\therefore \angle ACB > \angle B$ (Ax. 11). Q.E.D.

123. Theorem. **If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.**



Given: $\triangle ABC$; $\angle ACB > \angle B$.

To Prove: $AB > AC$.

Proof: In $\angle ACB$, suppose $\angle BCR$ constructed $= \angle B$.

Then, $CR = BR$ (?) (120).

Also $AR + CR > AC$ (?).

$\therefore AR + BR > AC$ (Ax. 6). That is, $AB > AC$. Q.E.D.

124. Theorem. **The hypotenuse is the longest side of a right triangle.** (See 123.)

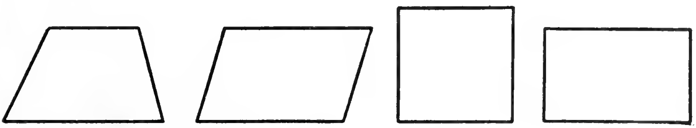
QUADRILATERALS

125. A **quadrilateral** is a portion of a plane bounded by four straight lines. These four lines are called the **sides.** The **vertices** of a quadrilateral are the four points at which the sides intersect. The **angles** of a quadrilateral are the four angles at the four vertices. The **diagonal** of a rectilinear figure is a line joining two vertices, not in the same side.

126. A **trapezium** is a quadrilateral having no two sides parallel.

A **trapezoid** is a quadrilateral having two and only two sides parallel.

A **parallelogram** is a quadrilateral having its opposite sides parallel (▱).



TRAPEZOID PARALLELOGRAM RHOMBOID SQUARE RECTANGLE

127. A **rectangle** is a parallelogram whose angles are right angles.

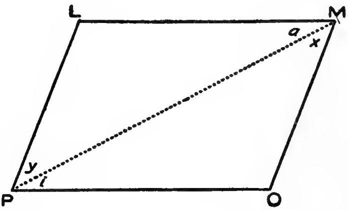
A **rhomboid** is a parallelogram whose angles are not right angles.

128. A **square** is an equilateral rectangle.

A **rhombus** is an equilateral rhomboid.

129. The side upon which a figure appears to stand is called its **base.** A trapezoid and all kinds of parallelograms are said to have **two bases,** — the actual base and the side parallel to it. The non-parallel sides of a trapezoid are sometimes called the **legs.** An **isosceles trapezoid** is a trapezoid whose legs are equal. The **median** of a trapezoid is the line connecting the midpoints of the legs. The **altitude** of a trapezoid and of all kinds of parallelograms is the perpendicular distance between the bases.

130. THEOREM. **The opposite sides of a parallelogram are equal.**



Given: ▱ $LMOP$.

To Prove: $LM = PO$ and $LP = MO$.

Proof: Draw diagonal PM.

In $\triangle$s LMP and OMP, $PM = PM$ (?); $\angle a = \angle i$ (?) (97); $\angle y = \angle x$ (?).

$\therefore \triangle LMP = \triangle OMP$ (?) (54).

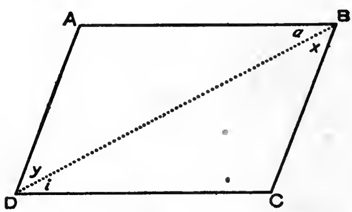
$\therefore LM = PO$ and $LP = MO$ (?) (27). Q.E.D.

131. COR. **Parallel lines included between parallel lines are equal.** (See 130.)

132. COR. **The diagonal of a parallelogram divides it into two equal triangles.**

133. COR. **The opposite angles of a parallelogram are equal.** (See 27.)

134. THEOREM. **If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.** [Converse of 130.]



Given: Quadrilateral $ABCD$; $AB = DC$; $AD = BC$.

To Prove: $ABCD$ is a ▱.

Proof: Draw diagonal BD.

In $\triangle$s ABD and CBD, $BD = BD$ (?); $AB = DC$ (?), and $AD = BC$ (?). $\therefore \triangle ABD = \triangle CBD$ (?) (58).

Hence $\angle a = \angle i$ (?). Therefore AB is $\parallel$ to DC (?) (101).

Also, $\angle y = \angle x$ (?). Therefore AD is $\parallel$ to BC (?).

Hence $ABCD$ is a parallelogram (Def. 126). Q.E.D.

135. THEOREM. **If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.**

Given: Quadrilateral $ABCD$; $AB = CD$ and $AB \parallel$ to CD.

To Prove: $ABCD$ is a $\square$.

Proof: Draw diagonal BD. In $\triangle ABD$ and CBD, $BD = BD$ (?); $AB = CD$ (?); and $\angle a = \angle i$ (?) (97).

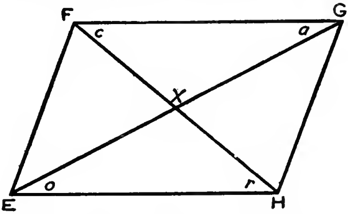
$\therefore \triangle ABD = \triangle CBD$ (?) (52).

Hence $\angle y = \angle x$ (?). $\therefore$ AD is $\parallel$ to BC (?) (101).

$\therefore ABCD$ is a parallelogram (?) (126). Q.E.D.

136. COR. **Any pair of adjoining angles of a parallelogram are supplementary.** (See 100.)

137. THEOREM. **The diagonals of a parallelogram bisect each other.**



Given: $\square$ $EFGH$; diagonals EG and FH intersecting at X.

To Prove: $FX = XH$ and $GX = XE$.

Proof: In $\triangle FXG$ and EXH, $FG = EH$ (?) (130); $\angle a = \angle o$ and $\angle c = \angle r$ (?) (97).

$\therefore \triangle FXG = \triangle EXH$ (?) (54).

$\therefore FX = XH$ and $GX = XE$ (?) (27). Q.E.D.

138. THEOREM. **If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.**

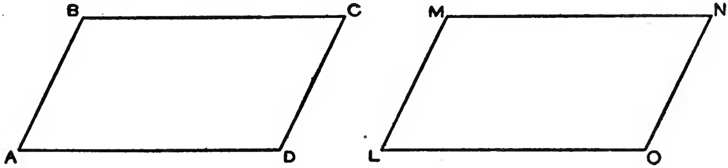
Given: (?). **To Prove:** (?). **Proof:** In $\triangle FXG$ and EXH show three parts of one = etc. Hence certain $\angle$s are = (?). Then two lines are $\parallel$ (?). Also = (?). Now use 135.

Ex. 1. In the figure of 137, if $\angle a = 20°$ and $\angle c = 30°$, find the four angles at X.

Ex. 2. If one angle of a parallelogram is 65°, find the other three. If one is 90°, find the others.

Ex. 3. State and prove the converse of 136.

139. Theorem. **Two parallelograms are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.**



Given: ▱s AC and LN; $AB = LM$; $AD = LO$; $\angle A = \angle L$.

To Prove: The ▱s are =.

Proof: Superpose ▱ $ABCD$ upon ▱ $LMNO$, so the equal angles A and L coincide, AD falling along LO and AB along LM.

Point D will coincide with point O [$AD = LO$ (Hyp.)].

Point B will coincide with point M [$AB = LM$ (Hyp.)].

BC and MN are both ‖ to LO (?) (126).

∴ BC falls along MN (?) (92).

CD and NO are both ‖ to LM (?).

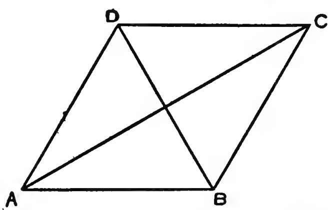
∴ CD falls along NO (?).

Hence C will fall exactly upon N (38).

∴ the figures coincide, and are equal (?) (28). Q.E.D.

140. Theorem. **Two rectangles are equal if the base and altitude of one are equal respectively to the base and altitude of the other.** (See 139.)

141. Theorem. **The diagonals of a rhombus (or of a square) are perpendicular to each other, bisect each other, and bisect the angles of the rhombus (or of the square).**



Given: Rhombus $ABCD$; diagonals AC and BD.

To Prove: $AC \perp$ to BD; AC and BD bisect each other; and they bisect ∠s DAB, ABC, etc.

Proof: Point A is equally distant from B and D (?) (128).

Point C is equally distant from B and D (?).

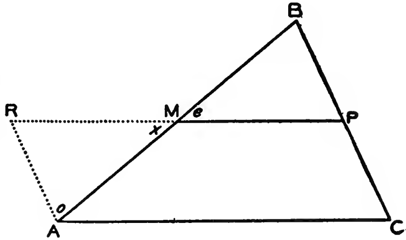
∴ AC is $\perp$ to BD (?) (70). Q.E.D.

Also AC and BD bisect each other (?) (137). Q.E.D.

The $\angle s$ at A are $=$ (?) (66, II). Etc. Q.E.D.

The proof if the figure is a square is exactly the same.

142. Theorem. The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.



Given: $\triangle ABC$; M, the midpoint of AB; P, the midpoint of BC; line MP.

To Prove: $MP \parallel$ to AC and $MP = \frac{1}{2} AC$.

Proof: Suppose AR is drawn through A, $\parallel$ to BC and meeting MP produced at R. In $\triangle s$ ARM and BPM, $AM = BM$ (Hyp.); $\angle x = \angle e$ (?) and $\angle o = \angle B$ (?) (97).

$\therefore \triangle ARM = \triangle BPM$ (?) (54).

Hence, $AR = BP$ (?). But $BP = PC$ (?). $\therefore AR = PC$ (?).

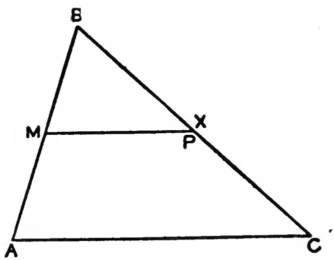
$\therefore ACPR$ is a $\square$ (?) (135).

Hence RP or MP is $\parallel$ to AC (?). Q.E.D.

Also, $RP = AC$ (?) (130). But $MP = RM$ (?) (27).

$\therefore MP = \frac{1}{2} RP = \frac{1}{2} AC$ (Ax. 6). Q.E.D.

143. Theorem. The line bisecting one side of a triangle and parallel to a second side, bisects also the third side.



Given: $\triangle ABC$; MP bisecting AB and $\parallel$ to AC.

To Prove: MP bisects BC also.

Proof: Suppose MX is drawn from M, the midpoint of AB to X, the midpoint of BC.

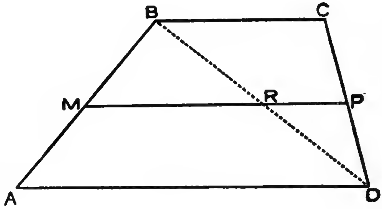
MX is $\parallel$ to AC (142); but MP is $\parallel$ to AC (Hyp.).

$\therefore MX$ and MP coincide (?).

That is, MP bisects BC. Q.E.D.

144. Theorem. The line bisecting one leg of a trapezoid and parallel to the base bisects the other leg, is the median, and is equal to half the sum of the bases.

Given: Trapezoid $ABCD$; M, the midpoint of AB; $MP \parallel$ to AD, meeting CD at P.



To Prove: I. P is the midpoint of CD.

II. MP is the median.

III. $MP = \frac{1}{2}(AD + BC)$.

Proof: I. Draw diagonal BD, meeting MP at R.

MP is $\parallel$ to BC (94). In $\triangle ABD$, MR bisects BD (143).

In $\triangle BDC$, RP bisects CD (?) (143).

That is, P is the midpoint of CD.

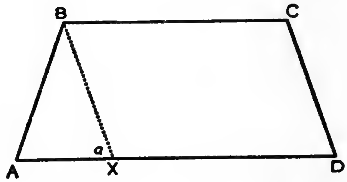
II. MP is the median (Def. 129).

III. $MR = \frac{1}{2} AD$ (142) and $RP = \frac{1}{2} BC$ (?).

$\therefore MP = \frac{1}{2}(AD + BC)$ (Ax. 2). Q.E.D.

145. Theorem. The angles adjoining each base of an isosceles trapezoid are equal.

Given: Trapezoid AC; $AB = CD$.



To Prove: $\angle A = \angle D$ and $\angle ABC = \angle C$.

Proof: Suppose BX is drawn through B and $\parallel$ to CD. $BX = CD$ (130); $AB = CD$ (Hyp.).

$\therefore AB = BX$ (?), and $\angle A = \angle a$ (?), and $\angle a = \angle D$ (?) (98).

$\therefore \angle A = \angle D$ (Ax. 1).

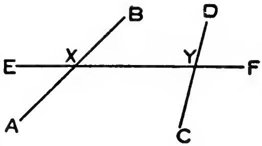
Again, $\angle C$ is supp. of $\angle D$ (?). Etc. Q.E.D.

146. Theorem. If the angles at the base of a trapezoid are equal, the trapezoid is isosceles.

Given: (?). **To Prove:** (?).

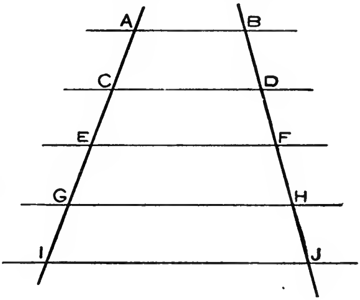
Proof: Suppose BX is drawn $\parallel$ to CD. $\angle a = \angle D$ (?); $\angle A = \angle D$ (?) $\therefore \angle A = \angle a$ (?). $\therefore AB = BX$ (?). Etc.

NOTE. The verb "*to intersect*" means merely "*to cut.*" In geometry, the verb "*to intercept*" means "*to include between.*" Thus the statement "AB and CD intercept XY on the line EF" really means, "AB and CD intersect EF and include XY, a part of EF, between them."



147. THEOREM. Parallels intercepting equal parts on one transversal intercept equal parts on any transversal.

Given: ‖s AB, CD, EF, GH, IJ intercepting equal parts AC, CE, EG, GI, on the transversal AI, and cutting transversal BJ.



To Prove: $BD = DF = FH = HJ$.

Proof: The figure $ABFE$ is a trapezoid (?). CD bisects AE and is ‖ to EF (Hyp.).

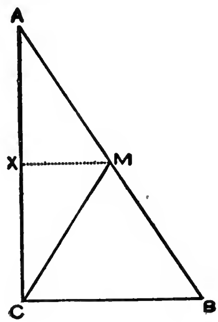
$\therefore D$ is midpoint of BF (?). That is, $BD = DF$.

Similarly, $CDHG$ is a trapezoid and EF bisects DH (?). That is, $DF = FH$.

Similarly, $FH = HJ$. $\therefore BD = DF = FH = HJ$ (Ax. 1). Q.E.D.

148. THEOREM. The midpoint of the hypotenuse of a right triangle is equally distant from the three vertices.

Given: Rt. $\triangle ABC$; M, the midpoint of hypotenuse AB.



To Prove: $AM = CM = BM$.

Proof: Suppose MX is drawn ‖ to BC, meeting AC at X. X is midpoint of AC (?) (143). MX is $\perp$ to AC (95).

That is, MX is $\perp$ to AC at its midpoint, and $AM = MC$ (?) (67).

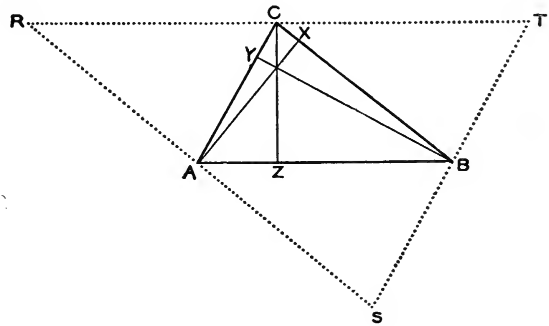
But $AM = BM$ (Hyp.).

$\therefore AM = CM = BM$ (Ax. 1). Q.E.D.

149. Theorem. The median of a trapezoid is parallel to the bases and equal to half their sum.

[This is another form of stating the theorem of 144.]

150. Theorem. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.



Given: $\triangle ABC$, $AX \perp$ to BC, $BY \perp$ to AC, and $CZ \perp$ to AB.

To Prove: These three $\perp$s meet in a point.

Proof: Through A suppose RS drawn $\parallel$ to BC; through B, $TS \parallel$ to AC; through C, $RT \parallel$ to AB, forming $\triangle RST$.

The figure $ABCR$ is a $\square$ (Const.) and $ABTC$ is a $\square$ (?).

$\therefore RC = AB$ and $CT = AB$ (?) (130). $\therefore RC = CT$ (Ax. 1).

Now CZ is $\perp$ to RT (?) (95).

That is, CZ is $\perp$ to RT at its midpoint.

Similarly AX is $\perp$ to RS at its midpoint.

And BY is $\perp$ to TS at its midpoint.

Therefore AX, BY, CZ meet at a point (?) (85). Q.E.D.

Ex. 1. Draw the three altitudes of an obtuse triangle and prolong them until they meet.

Ex. 2. Prove that each of the three outer triangles in the figure of 150 is equal to $\triangle ABC$.

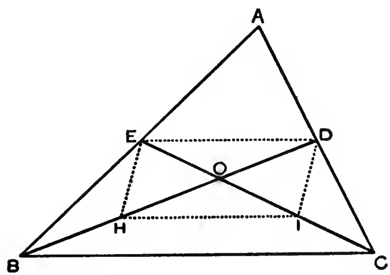
Ex. 3. Prove that any altitude of $\triangle RST$ is double the parallel altitude of $\triangle ABC$. [Use 143 and 130.]

151. THEOREM. **The point at which two medians of a triangle intersect is two thirds the distance from either vertex of the triangle to the midpoint of the opposite side.**

Given: $\triangle ABC$, BD and CE two medians intersecting at O.

To Prove: $BO = \frac{2}{3} BD$ and $CO = \frac{2}{3} CE$.

Proof: Suppose H is the midpoint of BO and I is the midpoint of CO. Draw ED, DI, IH, HE.



In $\triangle ABC$, DE is $\parallel$ to BC and $= \frac{1}{2} BC$ (?) (142).

In $\triangle OBC$, HI is $\parallel$ to BC and $= \frac{1}{2} BC$ (?).

$\therefore ED = HI$ (Ax. 1), and ED is $\parallel$ to HI (?) (94).

$\therefore EDIH$ is a $\square$ (?) (135).

$\therefore HO = OD$ and $IO = OE$ (?) (137).

$\therefore BH = HO = OD$ and $CI = IO = OE$ (Ax. 1).

That is, $BO = 2 \cdot OD = \frac{2}{3} BD$, and $CO = 2 \cdot EO = \frac{2}{3} CE$. Q.E.D.

152. THEOREM. **The three medians of a triangle meet in a point which is two thirds the distance from any vertex to the midpoint of the opposite side.**

Proof: Suppose AX is the third median of $\triangle ABC$ and meets BD at O'.

Then $BO' = \frac{2}{3} BD$ and $BO = \frac{2}{3} BD$ (?) (151).

$\therefore BO' = BO$ (?). That is O' coincides with O and the three medians meet at O which is $\frac{2}{3}$ the distance, etc. Q.E.D.

Ex. 1. In the figure of 151, prove $EH = \frac{1}{2} AO = DI$, by 142.

Ex. 2. In the figure of 151, if $BC > AC$, prove the angle BEC obtuse. [Use 87.]

Ex. 3. If one angle of a rhombus is 30°, find all the angles of the four triangles formed by drawing the diagonals.

Ex. 4. Show that any trapezoid can be divided into a parallelogram and a triangle by drawing one line.

Ex. 5. Prove that every right triangle can be divided into two isosceles triangles by drawing one line. [Use 148.]

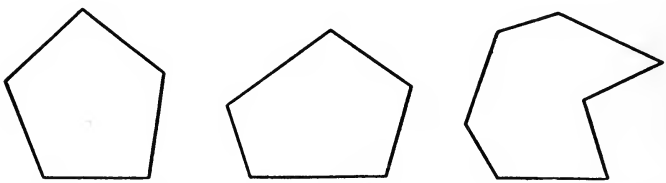
POLYGONS

153. A **polygon** is a portion of a plane bounded by straight lines. The lines are called the **sides.** The points of intersection of the sides are the **vertices.** The **angles** of a polygon are the angles at the vertices.

154. The **number** of sides of a polygon is the same as the number of its vertices or the number of its angles. An **exterior angle** of a polygon is an angle without the polygon, between one side of the polygon and another side prolonged.

155. An **equilateral polygon** has all of its sides equal to one another. An **equiangular polygon** has all of its angles equal to one another.

156. A **convex polygon** is a polygon no side of which if produced will enter the surface bounded by the sides of the polygon. A **concave polygon** is a polygon two sides of which if produced will enter the polygon.



EQUILATERAL EQUIANGULAR
CONVEX POLYGONS

CONCAVE, OR RE-ENTRANT

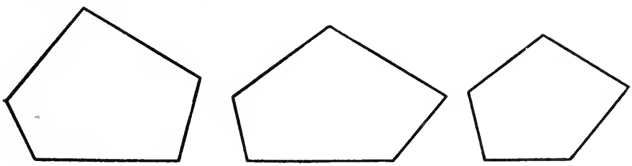
NOTE. A polygon may be equilateral and not be equiangular; or it may be equiangular and not be equilateral. The word "polygon" is usually employed to signify *convex* figures.

157. Two polygons are **mutually equiangular** if for every angle of the one there is an equal angle in the other and similarly placed. Two polygons are **mutually equilateral,** if for every side of the one there is an equal side in the other, and similarly placed.

158. **Homologous angles** in two mutually equiangular polygons are the pairs of equal angles. **Homologous sides** in two polygons are the sides between two pairs of homologous angles.

159. Two **polygons** are **equal** if they are mutually equiangular and their homologous sides are equal; or if they are composed of triangles, equal each to each and similarly placed. (Because in either case the polygons can be made to coincide.)

160. Two polygons may be mutually equiangular without being mutually equilateral; also, they may be mutually equilateral without being mutually equiangular — except in the case of triangles.

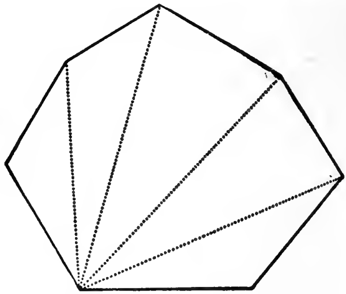


The first two figures are mutually equilateral but *not* mutually equiangular. The last two figures are mutually equiangular but *not* mutually equilateral.

161. A 3-sided polygon is a triangle.
A 4-sided polygon is a quadrilateral.
A 5-sided polygon is a pentagon.
A 6-sided polygon is a hexagon.
A 7-sided polygon is a heptagon.
An 8-sided polygon is an octagon.
A 10-sided polygon is a decagon.
A 12-sided polygon is a dodecagon.
A 15-sided polygon is a pentedecagon.
An n-sided polygon is called an n-gon.

Ex. Draw a pentagon and all the possible diagonals from one vertex. How many triangles are formed? Draw a decagon and the diagonals from one vertex. How many triangles are thus formed? Construct a 20-gon and the diagonals from one vertex. How many triangles are formed?

162. Theorem. The sum of the interior angles of an n-gon is equal to $(n-2)$ times $180°$.



Given: A polygon having n sides.

To Prove: The sum of its interior $\measuredangle s = (n-2)\cdot 180°$.

Proof: By drawing all possible diagonals from any vertex it is evident that there will be formed $(n-2)$ triangles.
The sum of the $\measuredangle s$ of one $\triangle = 180°$ (?) (110).

$\therefore$ the sum of the $\measuredangle s$ of $(n-2)$ $\triangle s = (n-2)\,180°$ (Ax. 3).

But the sum of the $\measuredangle s$ of the triangles = the sum of the $\measuredangle s$ of the polygon (Ax. 4).

$\therefore$ sum of $\measuredangle s$ of the polygon $= (n-2)\,180°$ (Ax. 1). Q.E.D.

163. Cor. The sum of the interior angles of an n-gon is equal to $180°n - 360°$.

164. Cor. Each angle of an equiangular n-gon $= \dfrac{(n-2)\,180°}{n}$.

165. Cor. The sum of the angles of any quadrilateral is equal to four right angles.

166. Cor. If three angles of a quadrilateral are right angles, the figure is a rectangle.

Ex. 1. How many degrees are there in each angle of an equiangular pentagon? of an equiangular pentedecagon? of a 30-gon?

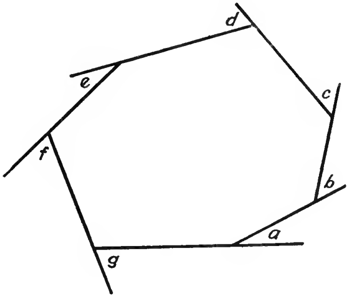
Ex. 2. If two angles of a quadrilateral are right angles, what is true of the other two?

Ex. 3. How many sides has that polygon the sum of whose interior angles is equal to 20 rt. $\measuredangle s$?

Ex. 4. How many sides has that equiangular polygon each of whose angles contains $160°$?

Ex. 5. If in the figure of 105, $\angle a = 65°$, how many degrees are there in each of the other angles of the figure?

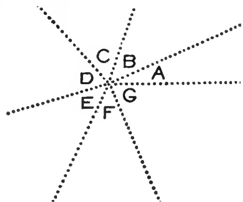
167. Theorem. If the sides of a polygon be produced, in order, one at each vertex, the sum of the exterior angles of the polygon will equal four right angles, that is, 360°.



Given: A polygon with sides prolonged in succession forming the several exterior angles a, b, c, d, etc.

To Prove: $\angle a + \angle b + \angle c + \angle d +$ etc. $= 4$ rt. $\angle s = 360°$.

Proof: Suppose at any point in the plane, lines are drawn parallel to the several sides of the given polygon, extending in the same direction, and forming angles A, B, C, D, etc.



Then $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G = 4$ rt. $\angle s$ (?) (47).

But $\angle a = \angle A$, $\angle b = \angle B$, $\angle c = \angle C$, $\angle d = \angle D$, etc. (?) (105).

$\therefore \angle a + \angle b + \angle c + \angle d + \angle e + \angle f + \angle g = 4$ rt. $\angle s = 360°$ (?) (Ax. 6).

Q.E.D.

168. Cor. Each exterior angle of an equiangular polygon is equal to $\frac{4 \text{ rt. } \angle s}{n}$; that is, $= \frac{360°}{n}$.

169. Cor. The sum of the exterior angles of a polygon is independent of the number of its sides.

Ex. 1. How many degrees are there in each exterior angle of an equiangular dodecagon? of an equiangular 40-gon?

Ex. 2. If any angle of an isosceles triangle is 60°, the triangle is equiangular.

Ex. 3. Prove the theorem of 162 by drawing lines from any point within the triangle to all the vertices.

Ex. 4. State the theorems which deal with concurrent lines.

Concurrent lines are lines that meet in a common point.

Ex. 5. How many sides has that polygon the sum of whose interior angles exceeds the sum of the exterior angles by 720°?

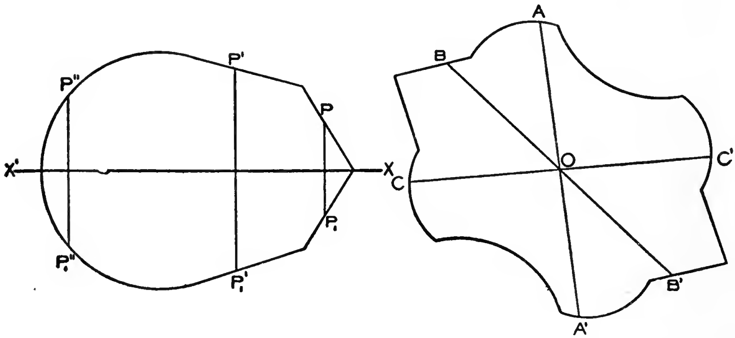
SYMMETRY

170. A figure is **symmetrical** with respect to a **line** if, by using that line as an axis, the part of the figure on one side of the line may be folded over, and will exactly coincide with the part on the other side. This line is an **axis of symmetry.**

171. A figure is **symmetrical** with respect to a **point** if this point bisects every line drawn through it and terminated (both ways) in the boundary of the figure.

This point is the **center of symmetry.**

172. It is evident that the axis of symmetry bisects at right angles every line joining two symmetrical points; and that the center of symmetry bisects every line joining any pair of points symmetrical with respect to it.



Examples of symmetry are given in these figures.

First figure is symmetrical with respect to XX' as an axis. (Why?)

Second figure is symmetrical with respect to O as a center. (Why?)

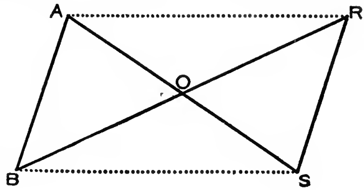
P' and P_1' are symmetrical with respect to XX' as an axis. (Why?)

A and A', B and B' etc. are symmetrical with respect to O as a center. (Why?) XX' is $\perp$ to $P'P_1'$ and bisects it. $AO = A'O$, $BO = B'O$, etc.

173. In order to prove that a line is an axis of symmetry it is necessary to show that it satisfies the condition of 170.

In order to prove that a point is a center of symmetry it is necessary to show that it satisfies the condition of 171.

174. THEOREM. If two lines are symmetrical with respect to a center, they are equal and parallel.

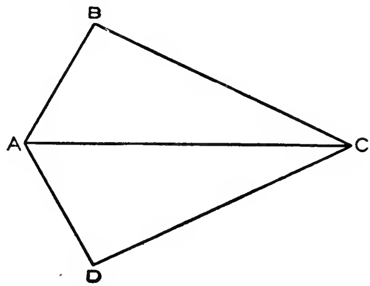


Given: AB and RS symmetrical with respect to O, that is, every line through O, terminated in AB and RS is bisected at O; AS and BR, two such lines.

To Prove: $AB = RS$ and $AB \parallel$ to RS.

Proof: Draw AR and BS. $AO = OS$ and $BO = OR$ (Hyp.).
$\therefore$ $ABSR$ is a ▱ (?) (138).
$\therefore AB = RS$ (?) and AB is $\parallel$ to RS (?). Q.E.D.

175. THEOREM. If a diagonal of a quadrilateral bisects two of its angles, this diagonal is an axis of symmetry.



Given: Quadrilateral $ABCD$; AC a diagonal bisecting $\angle BAD$ and $\angle BCD$.

To Prove: $ABCD$ symmetrical with respect to AC.

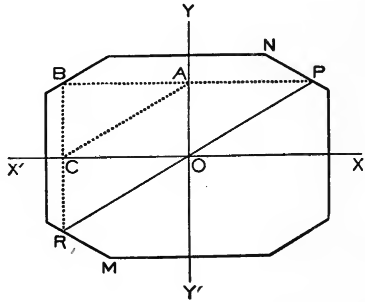
Proof: In $\triangle\!\triangle$ ABC and ADC, $AC = AC$ (?).
$\angle BAC = \angle DAC$ (?) and $\angle BCA = \angle DCA$ (?).
Hence $\triangle ABC = \triangle ADC$ (?).
$\therefore AC$ is an axis of symmetry (?). Q.E.D.

176. COR. The diagonal of a square or of a rhombus is an axis of symmetry. (Why?)

177. COR. The diagonal of a rectangle or of a rhomboid is not an axis of symmetry. (Why not?)

Ex. 1. Is the altitude of an equilateral triangle an axis of symmetry?
Ex. 2. Is the altitude of an isosceles triangle an axis of symmetry?
Ex. 3. Are all altitudes of all triangles axes of symmetry?
Ex. 4. Has an isosceles trapezoid an axis of symmetry?

178. THEOREM. **If a figure is symmetrical with respect to two perpendicular axes, it is symmetrical with respect to their intersection as a center.**



Given: Figure MN symmetrical with respect to the $\perp$ axes XX' and YY' which intersect at O.

To Prove: Figure MN is symmetrical with respect to O as a center.

Proof: Take any point P in the boundary. Draw PB $\perp$ to YY', intersecting YY' at A and meeting the boundary at B. Draw $BR \perp$ to XX', intersecting XX' at C and meeting the boundary at R. Draw AC, OP, OR.

[The demonstration is accomplished by proving ROP a straight line, bisected at O.]

PB is $\parallel$ to XX' and BR is $\parallel$ to YY' (?) (93).

Hence $ABCO$ is a $\square$ (?).

$\therefore BC = AO$ (?). But $BC = CR$ (?) (172).

$\therefore AO = CR$ (Ax. 1).

$\therefore ACRO$ is a $\square$ (?) (135). $\therefore RO$ is $=$ and $\parallel$ to AC (?).

Similarly CO is $=$ and $\parallel$ to AP; hence $ACOP$ is a $\square$ (?).

$\therefore PO$ is $=$ and $\parallel$ to AC (?).

$\therefore POR$ is a straight line (?) (92) and $PO = OR$ (?).

But P is *any* point, so POR is *any* line through O.

Hence O is a center of symmetry (171). Q.E.D.

LOCUS

179. The **locus** of a point is the series of positions the point must occupy in order that it may satisfy a given condition. It is the path of a point whose positions are limited or defined by a given condition, or given conditions.

180. Explanatory. I. If a point is moving so that it is always one inch from a given indefinite straight line, the point may occupy any position in either of two indefinite lines, one inch from the given line, parallel to it, and one on either side of it. And this point cannot occupy any position which is not in these lines. *Hence*, the locus of points at a given distance from a given line is a pair of parallels to the given line, one on either side of it, and at the given distance from it.

II. If a point is moving so that it is always equally distant from two parallels, it must move in a third parallel midway between them. *Hence*, the locus of points equally distant from two parallels is a third parallel midway between them.

III. The method of proving that a certain line or group of lines is the locus of points satisfying a given condition, consists in proving that every point in the line fulfills the given requirement, and that there is no other point that fulfills it. In the above illustrations it is evident that every point in the lines which were called the "locus," *did fulfill* the conditions of the case. It is also evident that there is no point outside these "loci" which does so fulfill the conditions. That is, these "loci" contain all the points described and no others.

IV. **Theorem. The locus of points equally distant from the extremities of a line is the perpendicular bisector of the line.**

Proof: Every point *in* the ⊥ bisector of a line is equally distant from its extremities (67). And *also*, there is no point outside the ⊥ bisector which is equally distant from the extremities of a line (69). Hence it is the **locus** of points equally distant from the extremities of the line.

V. **Theorem. The locus of points equally distant from the sides of an angle is the bisector of the angle.**

Proof: Like the preceding proof. [Use 79, 80, 81.]

VI. The locus of the vertices of all the isosceles triangles that can be constructed on a given base is the perpendicular bisector of the base. (Same as IV.)

CONCERNING ORIGINAL EXERCISES

181. In the original work which this text contains, the pupil is expected to state the hypothesis and conclusion of each theorem, and apply them to an appropriate figure. He is expected to state completely and logically the proof, giving a correct reason for every declarative statement.

In many of these exercises, suggestions are made and such assistance is given as experience has shown average pupils require. This is done in order that the learner may be encouraged toward definite accomplishment, which is one of the greatest incentives to further effort.

To apply the knowledge acquired from the preceding pages is now the student's task. His fascination for this science will depend largely upon the success of his efforts at proving originals. Therefore, many obstacles will be removed or modified, and no trouble will be spared in making the mastery of this department of geometry both agreeable and profitable.

The student should not draw a special figure for a general proposition. That is, if "triangle" is specified, he should draw a scalene and not an isosceles or a right triangle; and if "quadrilateral" is mentioned, he should draw a trapezium and not a parallelogram or a square.

SUMMARY. GENERAL DIRECTIONS FOR ATTACKING EXERCISES

182. A **triangle** is proved **isosceles** by showing that it contains two equal sides, or two equal angles.

183. A **triangle** is proved a **right** triangle by showing that one of its angles is a right angle, or two of its angles are complementary, or one of its angles is equal to the sum of the other two.

184. **Right triangles** are proved equal, by showing that they have:

(1) Hypotenuse and acute angle of one = etc.
(2) Hypotenuse and leg of one = etc.
(3) The legs of one = etc.
(4) Leg and adjoining angle of one = etc.
(5) Leg and opposite angle of one = etc.

185. Oblique triangles are proved equal, by showing that they have:
(1) Two sides and the included angle of one = etc.
(2) One side and the adjoining angles of one = etc.
(3) Three sides of one = etc.

186. Angles are proved equal, by showing that they are:
(1) Equal to the same or to equal angles.
(2) Halves or doubles of equals.
(3) Vertical angles.
(4) Complements or supplements of equals.
(5) Homologous parts of equal figures.
(6) Base angles of an isosceles triangle.
(7) Corresponding angles, alternate-interior angles, etc. of parallels.
(8) Angles whose sides are respectively parallel or perpendicular.
(9) Third angles of triangles which have two angles of one = etc.

187. Lines are proved equal, by showing that they are:
(1) Equal to the same or to equal lines.
(2) Halves or doubles of equals.
(3) Distances to the ends of a line from any point in its perpendicular bisector.
(4) Homologous parts of equal figures.
(5) Sides of an isosceles triangle.
(6) Distances to the sides of an angle from any point in its bisector.
(7) Opposite sides of a parallelogram.
(8) The parts of one diagonal of a parallelogram made by the other.

188. Two lines are proved **perpendicular**, by showing that they :
(1) Make equal adjacent angles with each other.
(2) Are legs of a right triangle.
(3) Have two points in one, each equally distant from the ends of the other.

189. Two lines are proved **parallel**, by :
(1) The customary angle-relations of parallel lines.
(2) Showing that they are opposite sides of a parallelogram.
(3) Showing that they are parallel or perpendicular to a third line.

190. Two lines, or two angles, are proved **unequal** by the usual axioms and theorems pertaining to inequalities.

[See especially, Ax. 5 *; Ax. 12; 68; 75; 76, III; 77; 86; 87 *; 90 109 *; 122 *; 124.]

* These refer to angles.

ORIGINAL EXERCISES

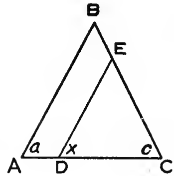
1. A line cutting the equal sides of an isosceles triangle and parallel to the base forms another isosceles triangle. [Use 186 (7), and 182.]

2. The bisectors of the equal angles of an isosceles triangle form, with the base, another isosceles triangle. [Use 186 (2).]

3. If the exterior angles at the base of a triangle are equal, the triangle is isosceles. [186 (4).]

4. If from any point in the base of an isosceles triangle a line be drawn parallel to one of the equal sides and meeting the other side, an isosceles triangle will be formed.

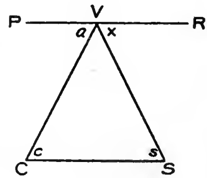
To Prove: $\triangle DEC$ isosceles.



5. If the median of a triangle is perpendicular to the base, the triangle is isosceles. [Use 187 (3).]

6. If a line through the vertex of a triangle and parallel to the base, makes equal angles with the sides, the triangle is isosceles.

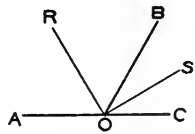
Given: $\angle a = \angle x$, etc.



7. The median of an isosceles triangle is perpendicular to the base. [Use 188 (3).]

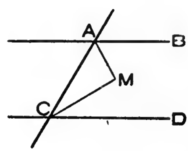
8. The bisectors of two supplementary-adjacent angles are perpendicular to each other.

Proof: $\angle AOB + \angle BOC = 180°$ (?); $\frac{1}{2}\angle AOB + \frac{1}{2}\angle BOC = 90°$ (Ax. 3). $\frac{1}{2}\angle AOB = \angle ROB$; etc.



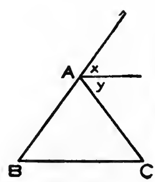
9. The bisectors of two adjoining-interior angles of two parallels meet at right angles.

Proof: $\angle MAC + \angle MCA = 90°$ as in No. 8. Then use 183.

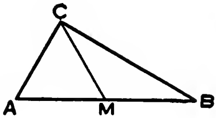


10. If the bisector of an exterior angle of a triangle is parallel to the base, the triangle is isosceles.

11. If the sum of two angles of a triangle is equal to the third, the triangle is a right triangle.
[Use 110.]



12. If the median of a triangle is equal to half the side to which it is drawn, it is a right triangle.



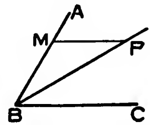
Given: $MA = MB = MC$.

To Prove: $\triangle ABC$ a rt. $\triangle$.

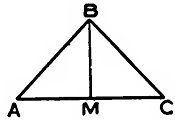
Proof: $\angle A = \angle ACM$ (?); $\angle B = \angle BCM$ (?).

$\therefore$ by adding, etc. (Use Ax. 2 and 183.)

13. If from any point in the bisector of an angle a line be drawn parallel to either side of the angle, an isosceles triangle will be formed. [182.]



14. If the bisector of the vertex-angle of a triangle is perpendicular to the base, the triangle is isosceles.



Proof: The rt. $\triangle$ are $=$ [184 (4)]. Then use 187 (4); etc.

15. Every isosceles right triangle can be divided by one line into two isosceles right triangles.

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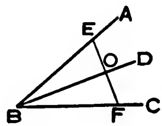
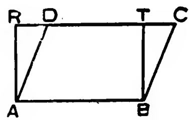
16. The diagonals of a rhombus divide the figure into four equal right triangles. [141.]

17. If a line be drawn perpendicular to the bisector of an angle terminating in the sides, the right triangles formed will be equal.



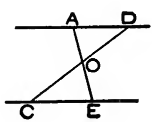
18. If from each point at which a transversal intersects two parallels a perpendicular to the other parallel be drawn, two equal right triangles will be formed.

19. If two perpendiculars be drawn to the upper base of a parallelogram from the extremities of the lower base, two equal right triangles will be formed.



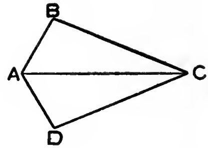
20. The perpendiculars to the equal sides of an isosceles triangle from the opposite vertices form two pairs of equal right triangles.

21. If two intersecting lines have their extremities in two parallels and their point of intersection bisects one of them, it bisects the other also.



Given: $AO = EO$; etc.

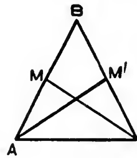
22. If two adjacent sides of a quadrilateral are equal and the diagonal bisects their included angle, the other two sides are equal.



23. If a diagonal of a quadrilateral bisects two of its angles, the quadrilateral has two pairs of equal sides.

24. The altitudes of an isosceles triangle upon the legs are equal.

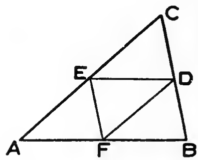
25. If a triangle has two equal altitudes, it is isosceles.



26. The diagonals of a rectangle are equal.

27. The medians drawn from the ends of the base of an isosceles triangle are equal.

28. The three lines joining the midpoints of the sides of a triangle divide the triangle into four equal triangles. [Use 142 and 132.]

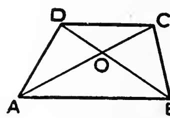


29. The bisector of the vertex-angle of an isosceles triangle bisects the base at right angles. [Use 185 (1); 27; 50.]

30. The bisectors of the equal angles of an isosceles triangle (terminating in the equal sides) are equal. [Use 185 (2).]

31. The median to the base of an isosceles triangle bisects the vertex-angle. [Use 185 (3).]

32. The diagonals of an isosceles trapezoid are equal. [Use 145.]



33. The diagonals of an isosceles trapezoid divide the figure into four triangles, of which one pair is isosceles and the other pair is equal.

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34. What is the complement of an angle containing 35°? 80°? 75° 25′? 8° 18′?

35. What is the supplement of 50°? 100°? 148°? 113° 48′?

36. In a right triangle ABC, if $\angle A$ is 47°, find $\angle B$.

37. In an isosceles triangle $\angle A = \angle B = 80°$; find $\angle C$.

38. In $\triangle ABC$, if $\angle A = 25°$, $\angle B = 88°$, find $\angle C$. Find the exterior angle at A.

39. In $\triangle ABC$, if $\angle A = 40°$, $\angle B = 70° 40'$; find $\angle C$ and the exterior angle at B.

40. The vertex-angle of an isosceles triangle is 44°. Find each base angle.

41. How many degrees are there in the sum of the angles of a pentagon? of a decagon? of a 9-gon?

42. How many degrees are there in each angle of an equiangular hexagon? of an equiangular octagon?

43. How many degrees are there in each exterior angle of an equiangular pentagon? hexagon? dodecagon? 16-gon?

44. If one acute angle of a right triangle is double the other, how many degrees are there in each? [Denote the less $\angle$ by x.]

45. If the acute angles of a right triangle are equal, how many degrees are there in each?

46. If one acute angle of a right triangle is five times the other, how many degrees are there in each?

47. If a base angle of an isosceles triangle is 60°, find the vertex-angle. What kind of triangle is this?

48. If the vertex-angle of an isosceles triangle is 60°, find each base angle. What kind of triangle is this?

49. If the vertex-angle of an isosceles triangle equals twice the sum of the two base angles, how many degrees are there in each angle?

50. If the vertex-angle of an isosceles triangle equals four times the sum of the base angles, find each angle.

51. If the vertex-angle of an isosceles triangle is half each of the base angles, find each angle.

52. If one angle of a parallelogram is 54°, how many degrees are there in each of the remaining angles?

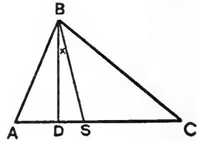
53. If a transversal cuts two parallels making one pair of alternate-interior angles each 40°, how many degrees are there in each of the other six angles formed?

54. Find the three angles formed by the bisectors of the angles of a triangle whose angles are 44°, 62°, and 74°.

55. If $\angle A$ of $\triangle ABC$ is 33° and the exterior angle at C is 110°, find $\angle B$.

56. If two angles of a triangle are 80° and 55°, how many degrees are there in the angle formed by their bisectors?

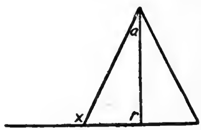
57. The vertex-angle of an isosceles triangle is one third of either exterior angle at the extremities of the base. Find each angle of the triangle.



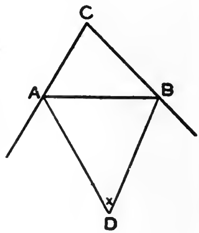
58. If two angles of a triangle are 30° and 40°, how many degrees are there in the angle formed by the bisector of the third angle and the altitude from the same vertex? **Solution**: $\angle x = \angle ABS - \angle ABD = \frac{1}{2} \angle ABC$ – comp. of $\angle A$.

59. Theorem. The angle between the altitude of a triangle and the bisector of the angle at the same vertex equals half the difference of the other angles of the triangle.

Proof: $\angle x = \angle ABS - \angle ABD = \frac{1}{2}\ (180° - \angle A - \angle C) - (90° - \angle A) =$ etc.



60. Theorem. The exterior angle at the base of an isosceles triangle equals half the vertex-angle plus 90°. **Proof**: $\angle x = \angle a + \angle r =$ etc.

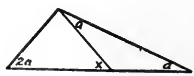


61. If in $\triangle\ ABC$, $\angle\ BAC = 80°$, $\angle\ ABC = 30°$, find the angle formed by the bisectors of the exterior angles at A and B. **Solution**: $\angle\ x = 180° - \angle BAD - \angle\ ABD$; $\angle\ BAD = \frac{1}{2}\ (180° - \angle\ A)$; etc.

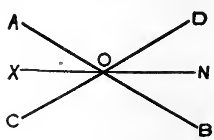
62. The angle formed by the bisectors of two exterior angles of a triangle equals half the sum of the interior angles at the same vertices.

63. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.

[Denote the less $\angle$ by x. Find the other. Draw the median from vertex of rt. $\angle$. Prove one $\triangle$ formed equilateral.]



64. If one angle of a triangle is double another, the line from the third vertex, making with the longer adjacent side an angle equal to the less given angle, divides the triangle into two isosceles triangles.

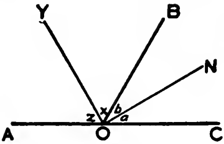


65. The bisector of an angle bisects, if produced, the vertical angle also.

Given: $\angle\ DON = \angle\ BON$.

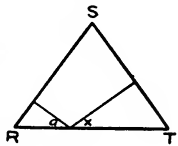
66. A line perpendicular to the bisector of an angle at the vertex bisects its supplementary-adjacent angle.

Given: $\angle a = \angle b$ and $OY \perp$ to ON.
(Use 48.)



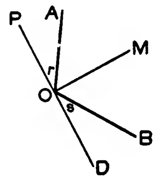
67. If the bisectors of two adjacent angles are perpendicular, the angles are supplementary.

Given: $\angle a = \angle b$; $\angle x = \angle z$; $\angle NOY =$ rt. $\angle$.



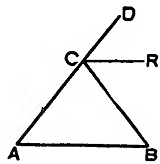
68. The bisectors of any two adjoining angles of a parallelogram meet at right angles. [Use 136; 183.]

69. If from any point in the base of an isosceles triangle perpendiculars to the equal sides be drawn, they will make equal angles with the base.
[Use 114; 48.]



70. If a line be drawn through the vertex of an angle and perpendicular to the bisector of the angle, it will make equal angles with the sides.

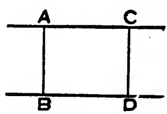
To Prove: $\angle r = \angle s$.



71. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

Proof: $\angle DCB = 2\ \angle A$ (?) and $= 2\ \angle DCR$ (?). Etc.

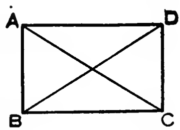
72. The line through the vertex of an isosceles triangle, parallel to the base, bisects the exterior angle.



73. Parallel lines are everywhere equally distant.

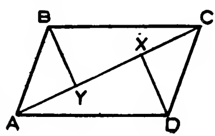
Given: $\|_s$ AC and BD; AB and CD $\perp_s$ to AC.

To Prove: $AB = CD$. (Use 93; 130.)



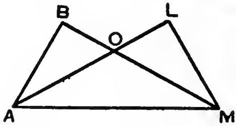
74. If two lines in a plane are everywhere equally distant, they are parallel. [Use 93; 135.]

75. If the diagonals of a parallelogram are equal, the figure is a rectangle.
[Use $\triangle$s ABC and DBC; 185 (3); 50.]



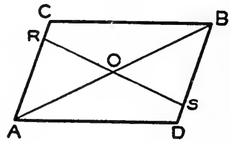
76. The perpendiculars upon a diagonal of a parallelogram from the opposite vertices are equal. [184 (1).]

77. The perpendiculars to the legs of an isosceles triangle from the midpoint of the base are equal.



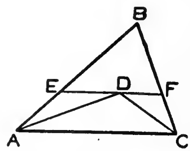
78. State and prove the converse of No. 77.

79. If $AB = LM$ and $AL = BM$, $\angle B = \angle L$ and $\angle BAO = \angle OML$ and $BO = OL$.



80. Any line terminated in a pair of opposite sides of a parallelogram and passing through the midpoint of a diagonal is bisected by this point. **To Prove:** $RO = OS$.

81. The midpoint of a diagonal of a parallelogram is a center of symmetry.



82. If the base angles of a triangle be bisected and through the intersection of the bisectors a line be drawn parallel to the base and terminating in the sides, this line will be equal to the sum of the parts of the sides it meets, between it and the base.

83. In two equal triangles, homologous medians are equal. Homologous altitudes are equal. Homologous bisectors are equal.

84. If two parallel lines are cut by a transversal, the two exterior angles on the same side of the transversal are supplementary.

85. If from a point a perpendicular be drawn to each of two parallels they will be in the same line. [Draw a third ∥ through the point.]

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86. One side of a triangle is less than the sum of the other two sides.

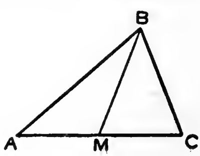
87. The sum of the sides of any polygon $ABCDE$ is greater than the sum of the sides of triangle ACE.

88. If X is a point in side AB of $\triangle ABC$, $AB + BC > AX + XC$.

89. In the figure of No. 88 $\angle AXC > \angle B$.

90. If lines be drawn from any point within a triangle to the ends of the base, they will include an angle which is greater than the vertex angle of the triangle. [Use 109 with figure of 75.]

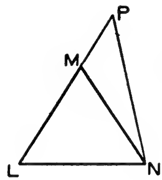
91. Any point (except the vertex) in either leg of an isosceles triangle is unequally distant from the ends of the base.



92. If two sides of a triangle are unequal and the median to the third side be drawn, the angles formed with the base will be unequal. [Use 87.]

93. State and prove the converse of No. 92.

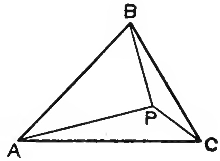
94. If the side LM, of equilateral triangle LMN, be produced to P, and PN be drawn, $\angle PNL > \angle L > \angle P$. Also $PL > PN > LN$.



95. If from any point within a triangle lines be drawn to the three vertices:

(1) Their sum will be less than the sum of the sides of the triangle. [Use 75 three times.]

(2) Their sum will be greater than half the sum of the sides of the triangle. [Use Ax. 12 three times.]



96. The sum of the diagonals of any quadrilateral is less than the sum of the four sides; but greater than half that sum.

97. The line drawn from any point in the base of an isosceles triangle to the opposite vertex is less than either leg.

---

98. The bisectors of a pair of corresponding angles are parallel.
[Use 98; 189, etc.]

99. If two lines are cut by a transversal and the exterior angles on the same side of the transversal are supplementary, the lines are parallel.

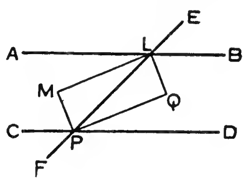
100. The bisectors of a pair of vertical angles are in the same straight line.

101. If one angle of a parallelogram is a right angle the figure is a rectangle.

102. The bisectors of the angles of a trapezoid form a quadrilateral two of whose angles are right angles.

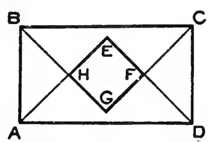
103. The bisectors of the four interior angles formed by a transversal cutting two parallels form a rectangle.

[Prove each $\angle$ of $LMPQ$ a rt. $\angle$.]

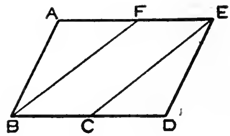


104. The bisectors of the angles of a parallelogram form a rectangle.

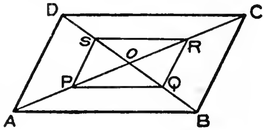
105. The bisectors of the angles of a rectangle form a square. [In order to prove $EFGH$ equilateral, the $\triangle$ AHB and CDF are proved equal and isosceles; similarly $\triangle$ BGC and AED.]



106. The lines joining a pair of opposite vertices of a parallelogram to the midpoints of the opposite sides are equal and parallel. [Prove $BCEF$ a ▱.]



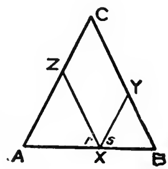
107. If the four midpoints of the four halves of the diagonals of a parallelogram be joined in order, another parallelogram will be formed.



108. If the points at which the bisectors of the equal angles of an isosceles triangle meet the opposite sides, be joined by a line, it will be parallel to the base.

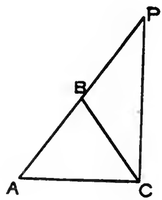
109. If two angles of a quadrilateral are supplementary, the other two are supplementary. [Use 165.]

110. If from any point in the base of an isosceles triangle parallels to the equal sides be drawn, the sum of the sides of the parallelogram formed will be equal to the sum of the legs of the triangle.



To Prove: $XY + YC + CZ + XZ = AC + BC.$

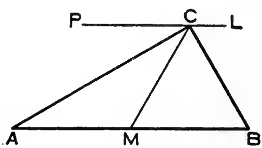
111. If one of the legs of an isosceles triangle be produced through the vertex its own length, and the extremity be joined to the nearer end of the base, this line will be perpendicular to the base. [Use 183.]



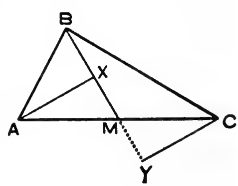
112. If the middle point of one side of a triangle is equally distant from the three vertices, the triangle is a right triangle.

[Proof and figure same as for No. 111.]

113. If through the vertex of the right angle of a right triangle a line be drawn parallel to the hypotenuse, the legs of the right triangle will bisect the angles formed by this parallel and the median drawn to the hypotenuse. [Use 148; 97; etc.]

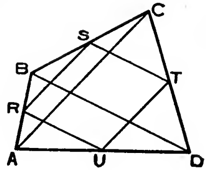


114. Any two vertices of a triangle are equally distant from the median from the third vertex.



115. If from any point within an angle perpendiculars to the sides be drawn, they will include an angle which is the supplement of the given angle.

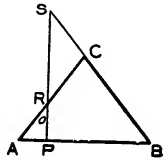
116. The lines joining (in order) the midpoints of the sides of a quadrilateral form a parallelogram the sum of whose sides is equal to the sum of the diagonals of the quadrilateral. [Use 142.]



117. The lines joining (in order) the midpoints of the sides of a rectangle form a rhombus. [Draw the diagonals.]

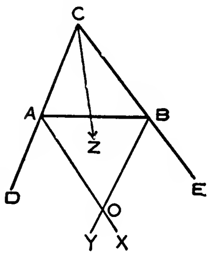
118. If a perpendicular be erected at any point in the base of an isosceles triangle, meeting one leg, and the other leg produced, another isosceles triangle will be formed.

[$\angle\ o$ and $\angle\ S$ are complements of $=$ $\angle$s A and B (?). Etc.]



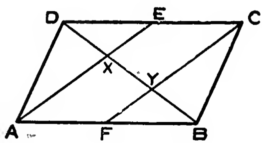
119. The difference between two sides of a triangle is less than the third side.

120. The bisectors of two exterior angles of a triangle and of the interior angle at the third vertex meet in a point.



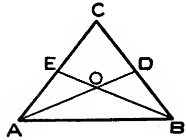
121. The bisectors of the exterior angles of a rectangle form a square.

122. If lines be drawn from a pair of opposite vertices of a parallelogram to the midpoints of a pair of opposite sides, they will trisect the diagonal joining the other two vertices. [Prove $AECF$ a $\square$ and use 143 in $\triangle$s DYC and ABX.]



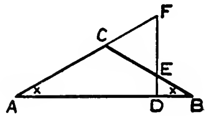
123. If two medians of a triangle are equal, the triangle is isosceles.

[Use 151. $AO = OB$ (Ax. 3). Hence prove $\triangle$s AEO and DBO equal.]

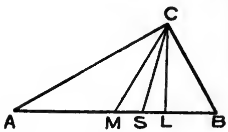


124. How many sides has the polygon the sum of whose interior angles exceeds the sum of its exterior angles by $900°$?

125. If the vertex-angle of an isosceles triangle is twice the sum of the base angles, any line perpendicular to the base forms with the sides of the given triangle (one side to be produced) an equilateral triangle. [Use 121.]



126. The lines bisecting two interior angles that a transversal makes with one of two parallels cut off equal segments on the other parallel from the point at which the transversal meets it. [The $\triangle$ formed are isosceles.]

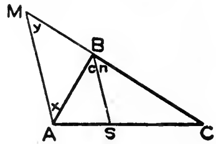


127. The bisector of the right angle of a right triangle is also the bisector of the angle formed by the median and the altitude drawn from the same vertex.

To Prove: $\angle MCS = LCS$.

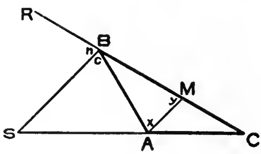
Proof: $\angle ACS = \angle BCS$ (?); $\angle ACM = \angle BCL$ (?). Now use Ax. 2.

128. If through the point of intersection of the diagonals of a parallelogram, two lines be drawn intersecting a pair of opposite sides (produced if necessary), the intercepts on these sides will be equal.

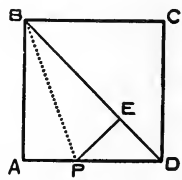


129. If ABC is a triangle, BS is the bisector of $\angle ABC$, and AM is parallel to BS meeting BC produced, at M, the triangle ABM is isosceles.

130. If ABC is a triangle and BS is the bisector of exterior $\angle ABR$ and AM is $\parallel$ to BS meeting BC at M, $\triangle ABM$ is isosceles.



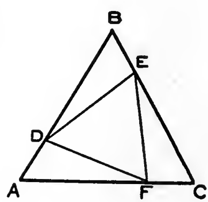
131. If A, B, C, and D are points on a straight line and $AB = BC$, the sum of the perpendiculars from A and C to any other line through D is double the perpendicular to that line from B. [Use 147; 144.]



132. If on diagonal BD, of square $ABCD$, BE be taken equal to a side of the square, and EP be drawn perpendicular to BD meeting AD at P, $AP = PE = ED$. [Draw BP.]

133. If in $\triangle ABC$, $\angle A$ is bisected by line meeting BC at M, $AB > BM$ and $AC > CM$. [Use 109; 123.]

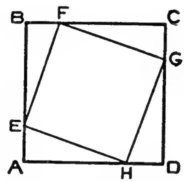
134. It is impossible to draw two straight lines from the ends of the base of a triangle terminating in the opposite side, so that they shall bisect each other. [Use 138.]



135. If ABC is an equilateral triangle and D, E, F are points on the sides, such that $AD = BE = CF$, triangle DEF is also equilateral.

[Prove the three small $\triangle$ $=$.]

136. If $ABCD$ is a square and E, F, G, H are points on the sides, such that $AE = BF = CG = DH$, $EFGH$ is a square.



[First, prove $EFGH$ equilateral; then one $\angle$ a rt. $\angle$.]

137. If ABC is an equilateral triangle and each side is produced (in order) the same distance, so that $AD = BE = CF$, the triangle DEF is equilateral.

138. If $ABCD$ is a square and the sides be produced (in order) the same distance, so that $AE = BF = CG = DH$, the figure $EFGH$ will be a square.

139. The two lines joining the midpoints of the opposite sides of a quadrilateral bisect each other. [Join the 4 midpoints (in order), etc.]

140. If two adjacent angles of a quadrilateral are right angles, the bisectors of the other angles are perpendicular to each other.

141. If two opposite angles of a quadrilateral are right angles, the bisectors of the other angles are parallel.

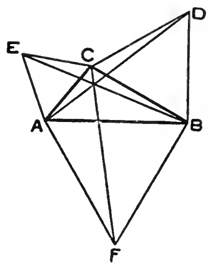
142. Two isosceles triangles are equal, if:

(1) The base and one of the adjoining angles in the one are equal respectively to the base and one of the adjoining angles in the other.

(2) A leg and one of the base angles in the one are equal respectively to a leg and one of the base angles in the other.

(3) The base and vertex-angle in one are equal to the same in the other.

(4) A leg and vertex-angle in one are equal to the same in the other.



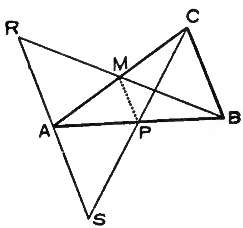
(5) A leg and the base in one are equal to the same in the other.

143. If upon the three sides of any triangle equilateral triangles be constructed (externally) and a line be drawn from each vertex of the given triangle to the farthest vertex of the opposite equilateral triangle, these three lines will be equal.

Proof: $\angle EAC = \angle BAF$ (?). Add to each of these, $\angle CAB$. $\therefore \angle EAB = \angle CAF$ (?).

Then prove $\triangle$ EAB and CAF equal.

Similarly, $\triangle CAD = \triangle CEB$. Etc.

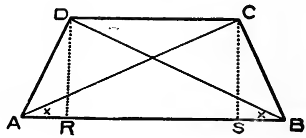


144. If two medians be drawn from two vertices of a triangle and produced their own length beyond the opposite sides and these extremities be joined to the third vertex, these two lines will be equal, and in the same straight line. [Draw MP and use 142.]

145. The median to one side of a triangle is less than half the sum of the other two sides.

Proof: (Fig. of No. 144.) Produce median BM its own length to R, draw RA. Prove $RA = CB$. Prove $BR < AB + BC$, etc.

146. The sum of the medians of a triangle is less than the sum of the sides of the triangle.

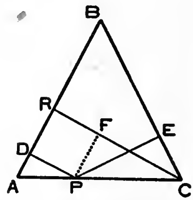


147. If the diagonals of a trapezoid are equal, it is isosceles.

[Draw DR and $CS \perp$ to AB; and prove rt. ⩟ ACS and BDR equal, to get $\angle x = \angle x$.]

148. If a perpendicular be drawn from each vertex of a parallelogram to any line outside the parallelogram, the sum of those from one pair of opposite vertices will equal the sum of those from the other pair.

[Draw the diagonals; use 144.]

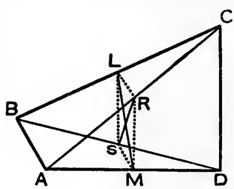


149. The sum of the perpendiculars to the legs of an isosceles triangle from any point in the base equals the altitude upon one of the legs. (That is, the sum of the perpendiculars from any point in the base of an isosceles triangle to the equal sides is constant for every point of the base.)

[Prove $PE = CF$ by 184 (1).]

150. The sum of the three perpendiculars drawn from any point within an equilateral triangle, to the three sides, is constant for all positions of the point.

[Draw a line through this point ∥ to one side; draw the altitude of the △ ⊥ to this line and side; prove the sum of the three ⊥s = this altitude and hence, = a constant.]



151. The line joining the midpoints of one pair of opposite sides of a quadrilateral and the line joining the midpoints of the diagonals bisect each other.

To Prove: LM and RS bisect each other.

152. If one leg of a trapezoid is perpendicular to the bases, the midpoint of the other leg is equally distant from the ends of the first leg. [Draw the median.]

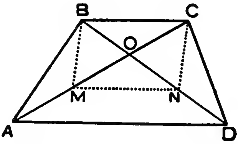
153. The median of a trapezoid bisects both the diagonals.

154. The line joining the midpoints of the diagonals of a trapezoid is a part of the median, is parallel to the bases, and is equal to half their difference.

155. If, in isosceles triangle XYZ, AD be drawn from A, the midpoint of YZ, perpendicular to the base XZ, $DZ = \frac{1}{4}XZ$. [Draw alt. from Y.]

156. If ABC is an equilateral triangle, the bisectors of angles B and C meet at D, DE be drawn parallel to AB meeting AC at E, and DF, parallel to BC meeting AC at F, then $AE = ED = EF = DF = CF$.

157. If A is any point in RS of triangle RST, and B is the midpoint of RA, C the midpoint of AS, D the midpoint of ST, and E the midpoint of TR, then $BCDE$ is a parallelogram.



158. In a trapezoid one of whose bases is double the other, the diagonals intersect at a point two thirds of the distance from each end of the longer base to the opposite vertex.

Proof: Take M, the midpoint of AO, etc.

159. If lines be drawn from any vertex of a parallelogram to the midpoints of the two opposite sides, they will divide the diagonal which they intersect, into three equal parts.

Proof: Draw the other diagonal and use 151.

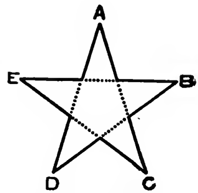
160. If the interior and exterior angles at two vertices of a triangle be bisected, a quadrilateral will be formed, two of whose angles are right angles and the other two are supplementary.

161. The angle between the bisectors of two angles of a triangle equals half the third angle plus a right angle.

162. If, in triangle ABC, the bisectors of the interior angle at B and of the exterior angle at C, meet at D, the angle BAC equals twice the angle BDC.

163. The four bisectors of the angles of a quadrilateral form a second quadrilateral whose opposite angles are supplementary.

Proof: Extend a pair of opposite sides of the given quadrilateral to meet at X. Bisect the base angles of the new $\triangle$ formed, meeting at O. Then show that $\angle O$ equals one of the $\measuredangle$s between the given bisectors, and $\angle O$ is supplementary to the angle opposite.



164. The sum of the angles at the vertices of a five-pointed star (pentagram) is equal to two right angles.

Proof: Draw interior pentagon. Find number of degrees in each of its angles. Hence find $\angle A$, etc.

165. The lines joining the midpoints of the opposite sides of an isosceles trapezoid are perpendicular to each other.

166. If the opposite sides of a hexagon are equal and parallel, the three diagonals drawn between opposite vertices meet in a point.

167. In triangle ABC, AD is perpendicular to BC, meeting it at D; E is the midpoint of AB, and F of AC; the angle EDF is equal to the angle EAF. [Use 148; 55.]

168. If the diagonals of a quadrilateral are equal, and also one pair of opposite sides, two of the four triangles into which the quadrilateral is divided by the diagonals are isosceles.

169. If angle A of triangle ABC equals three times angle B, there can be drawn a line AD meeting BC in D, such that the triangles ABD and ACD are isosceles.

170. If E is the midpoint of side BC of parallelogram $ABCD$, AE and BD meet at a point two thirds the distance from A to E and from D to B.

171. If in triangle ABC, in which AB is not equal to AC, AC' be taken on AB (produced if necessary) equal to AC, and AB' be taken on AC (produced if necessary) equal to AB, and $B'C'$ be drawn meeting BC at D, then AD will bisect angle BAC.

Proof: $\triangle ABC = \triangle AB'C'$ (?) (52). $\therefore$ their homologous parts are equal. Thus prove $\triangle BC'D = \triangle B'CD$ (54). Etc.

172. If a diagonal of a parallelogram bisects one angle, it also bisects the opposite angle.

173. If a diagonal of a parallelogram bisects one angle, the figure is equilateral.

174. Any line drawn through the point of intersection of the diagonals of a parallelogram divides the figure into two equal trapezoids. [See 159.]

175. If AR bisects angle A of triangle ABC and AT bisects the exterior angle at A, any line parallel to AB, having its extremities in AR and AT, is bisected by AC.

176. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram. [See 165.]

BOOK II

THE CIRCLE

191. A **curved line** is a line no part of which is straight.

192. A **circumference** is a curved line every point of which is equally distant from a point within, called the **center.**

193. A **circle** is a portion of a plane bounded by a circumference. [⊙.]

194. A **radius** is a straight line drawn from the center to the circumference.

A **diameter** is a straight line containing the center, and whose extremities are in the circumference.

CIRCUMFERENCE
CIRCLE
RADIUS
DIAMETER

SECANT
CHORD
TANGENT
POINT OF CONTACT

CENTRAL ANGLE
INSCRIBED ANGLE
ARC

SEMI-
CIRCUMFERENCES
SEMICIRCLES

A **secant** is a straight line cutting the circumference in two points.

A **chord** is a straight line whose extremities are in the circumference.

A **tangent** is a straight line which touches the circumference at only one point, and does not cut it, however far it may be extended. The point at which the line touches the circumference is called the **point of contact** or the **point of tangency**.

195. A **central angle** is an angle formed by two radii.

An **inscribed angle** is an angle whose vertex is on the circumference and whose sides are chords.

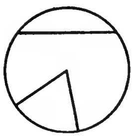
196. An **arc** is any part of a circumference.

A **semicircumference** is an arc equal to half a circumference.

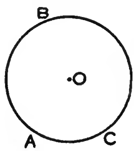
A **quadrant** is an arc equal to one fourth of a circumference.

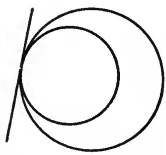
Equal circles are circles having equal radii.

Concentric circles are circles having the same center.

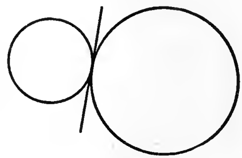


SECTOR
SEGMENT





CIRCLES INTERNALLY TANGENT



CIRCLES EXTERNALLY TANGENT

197. A **sector** is the part of a circle bounded by two radii and their included arc.

A **segment** of a circle is the part of a circle bounded by an arc and its chord.

A **semicircle** is a segment bounded by a semicircumference and its diameter.

198. Two **circles** are **tangent** to each other if they are tangent to the same line at the same point. Circles may be tangent to each other **internally**, if the one is within the other, or **externally**, if each is without the other.

199. POSTULATE. **A circumference can be described about any given point as center and with any given line as radius.**

Explanatory. A **circle** is **named** either by its center or by three points on its circumference, as "the ⊙ O," or "the ⊙ ABC."

The verb **to subtend** is used in the sense of "**to cut off.**"

A chord **subtends** an arc. Hence an arc is *subtended by* **a** chord.

An **angle** is said to **intercept** the arc between its sides. Hence an arc is *intercepted by* an angle.

The hypothesis is contained in what constitutes the subject of the principal verb of the theorem. (See 59.)

PRELIMINARY THEOREMS

200. Theorem. All radii of the same circle are equal. (See 192.)

201. Theorem. All radii of equal circles are equal. (See 196.)

202. Theorem. The diameter of a circle equals twice the radius.

203. Theorem. All diameters of the same or equal circles are equal. (Ax. 3.)

204. Theorem. The diameter of a circle bisects the circle and the circumference.

Given: Any ⊙ and a diameter.

To Prove: The segments formed are equal, that is, the diameter bisects the circle and the circumference.

Proof: Suppose one segment folded over upon the other segment, using the diameter as an axis. If the arcs do not coincide, there are points of the circumference unequally distant from the center. But this is impossible (?) (192).

∴ the segments coincide and are equal (?) (28). Q.E.D.

205. Theorem. With a given point as center and a given line as radius, it is possible to describe only one circumference. (See 192.)

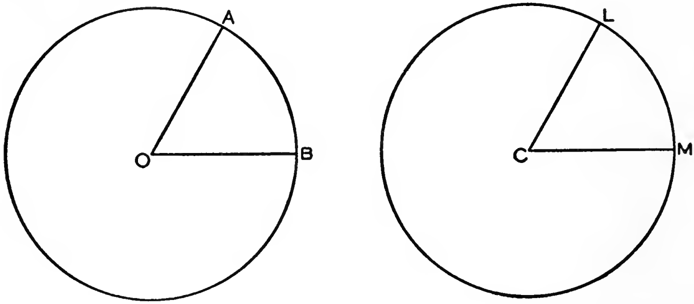
That is, a circumference is determined if its center and radius are fixed.

Note. The word "circle" is frequently used in the sense of "circumference." Thus one may properly speak of drawing a circle. The established definitions could not admit of such an interpretation save as custom makes it permissible.

Ex. Draw two intersecting circles and their common chord. Draw two circles which have no common chord. Draw figures to illustrate all the nouns defined on the two preceding pages.

THEOREMS AND DEMONSTRATIONS

206. Theorem. In the same circle (or in equal circles) **equal central angles intercept equal arcs.**



Given: $\odot O = \odot C$; $\angle O = \angle C$.

To Prove: Arc $AB =$ arc LM.

Proof: Superpose $\odot O$ upon the equal $\odot C$, making $\angle O$ coincide with its equal, $\angle C$. Point A will fall on L, and point B on M (?) (201).

Arc AB will coincide with arc LM (?) (192).

$\therefore$ AB arc $=$ arc LM (?) (28). Q.E.D.

207. Theorem. In the same circle (or in equal circles) **equal arcs are intercepted by equal central angles.** [Converse.]

Given: $\odot O = \odot C$; arc $AB =$ arc LM.

To Prove: $\angle O = \angle C$.

Proof: Superpose $\odot O$ upon the equal $\odot C$, making the centers coincide and point A fall on point L. Then arc AB will coincide with arc LM and point B will fall on point M. (Because the arcs are $=$.)

Hence OA will coincide with CL, and OB with CM (?) (39).

$\therefore \angle O = \angle C$ (?) (28). Q.E.D.

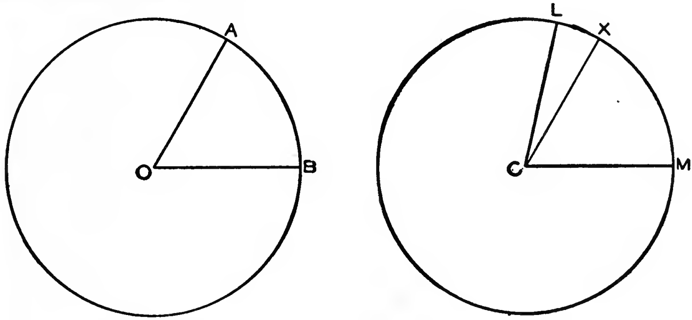
Ex. 1. Can arcs of unequal circles be made to coincide? Explain.

Ex. 2. If two sectors are equal, name the several parts that must be equal.

208. Theorem. In the same circle (or in equal circles):

I. If two central angles are unequal, the greater angle intercepts the greater arc.

II. If two arcs are unequal, the greater arc is intercepted by the greater central angle. [Converse.]



I. **Given:** $\odot O = \odot C$; $\angle LCM > \angle O$.

To Prove: Arc $LM >$ arc AB.

Proof: Superpose $\odot O$ upon $\odot C$, making sector AOB fall in position of sector XCM, OB coinciding with CM.

CX is within the angle LCM ($\angle LCM > \angle O$).

Arc AB will fall upon LM, in the position XM (192).

$\therefore$ arc $LM >$ arc XM (Ax. 5). That is, arc $LM >$ arc AB.

Q.E.D.

II. **Given:** (?). **To Prove:** $\angle LCM > \angle O$.

Proof: The pupil may employ either superposition, as in I, or the method of exclusion, as in 87.

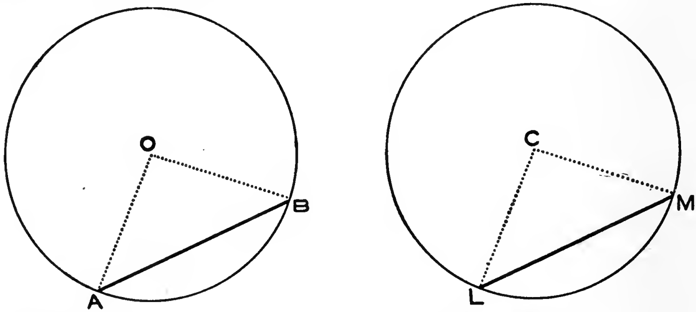
Note. Unless otherwise specified, the arc of a chord always refers to the *lesser* of the two arcs. If *two* arcs (in the same or equal circles) are concerned, it is understood either that each is less than a semicircumference, or each is greater.

Ex. 1. Two sectors are equal if the radii and central angle of one are equal respectively to the radii and central angle of the other.

Ex. 2. If in the figure of 206, arcs AB and LM were removed, how would the remaining arcs compare?

Ex. 3. If in the figure of 208, arcs AB and LM were removed, how would the remaining arcs compare?

209. Theorem. **In the same circle** (or in equal circles) **equal chords subtend equal arcs.**



Given: $\odot O = \odot C$; chord $AB =$ chord LM.

To Prove: Arc $AB =$ arc LM.

Proof: Draw the several radii to the ends of the chords. In $\triangle$ OAB and CLM, $OA = CL$, $OB = CM$ (?) (201).

Chord $AB =$ chord LM (Hyp.). $\therefore \triangle OAB = \triangle CLM$ (?).

Hence $\angle O = \angle C$ (?).

$\therefore$ arc $AB =$ arc LM (?) (206). Q.E.D.

210. Theorem. **In the same circle** (or in equal circles) **equal arcs are subtended by equal chords.**

Given: $\odot O = \odot C$; arc $AB =$ arc LM.

To Prove: Chord $AB =$ chord LM.

Proof: Draw the several radii to the ends of the chords. In $\triangle$ OAB and CLM, $OA = CL$, $OB = CM$ (?) (201).

$\angle O = \angle C$ (?) (207). $\therefore \triangle AOB = \triangle CLM$ (?).

$\therefore$ chord $AB =$ chord LM (?). Q.E.D.

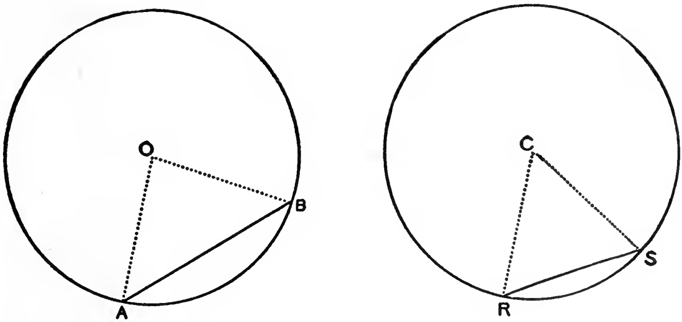
211. Theorem. **In the same circle** (or in equal circles):

I. If two chords are unequal, the greater chord subtends the greater arc.

II. If two arcs are unequal, the greater arc is subtended by the greater chord.

I. **Given:** $\odot O = \odot C$; chord $AB >$ chord RS.

To Prove: Arc $AB >$ arc RS.



Proof: Draw the several radii to the ends of the chords. In $\triangle\triangle$ AOB and RCS, $AO = RC$, $BO = SC$ (?) (201).

Chord $AB >$ chord RS (Hyp.). $\therefore \angle O > \angle C$ (?) (87).

$\therefore$ arc $AB >$ arc RS (?) (208, I). Q.E.D.

II. **Given:** $\odot O = \odot C$; arc $AB >$ arc RS.

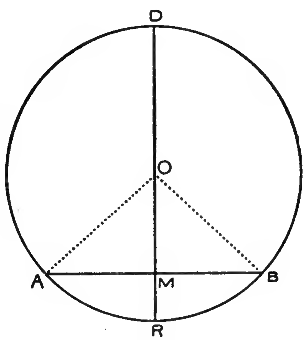
To Prove: Chord $AB >$ chord RS.

Proof: Draw the several radii. In $\triangle\triangle$ AOB and RCS, $AO = RC$, $BO = SC$ (?) (201).

But $\angle O > \angle C$ (?) (208, II).

$\therefore$ chord $AB >$ chord RS (?) (86). Q.E.D.

212. THEOREM. **The diameter perpendicular to a chord bisects the chord and both the subtended arcs.**



Given: Diameter $DR \perp$ to chord AB in $\odot O$.

To Prove: I. $AM = MB$; II. $AR = RB$ and $AD = DB$.

Proof: Draw radii to the ends of the chord.

I. In rt. $\triangle\triangle$ OAM and OBM, $OA = OB$ (?), $OM = OM$ (?).

$\therefore \triangle OAM = \triangle OBM$ (?).

Hence, $AM = MB$ (?). Q.E.D.

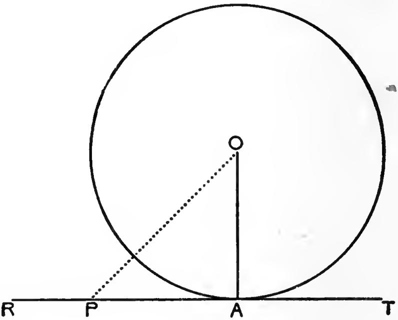
II. $\angle AOM = \angle BOM$ (27). $\therefore AR = RB$ (?) (206).

Also $\angle AOD = \angle BOD$ (?)(49). $\therefore AD = DB$ (?)(206). Q.E.D.

213. THEOREM. **The line from the center of a circle perpendicular to a chord bisects the chord and its arc.** **Proof:** The same as 212.

214. THEOREM. **The perpendicular bisector of a chord passes through the center of the circle.** [O is equidistant from A and B (?) (200). $\therefore$ it is in the $\perp$ bisector of AB (?) (69).]

215. THEOREM. **The line perpendicular to a radius at its extremity is tangent to the circle.**



Given: Radius OA of $\odot$ O, and $RT \perp$ to OA at A.

To Prove: RT tangent to the circle.

Proof: Take *any* point P in RT (except A) and draw OP.

$OP > OA$ (?) (77).

Hence P lies without the $\odot$. (Because $OP >$ radius.)

That is, every point (except A) of line RT is without the $\odot$.

Therefore, RT is a tangent (Def. 194). Q.E.D.

216. THEOREM. **If a line is tangent to a circle, the radius drawn to the point of contact is perpendicular to the tangent.**

Given: RT tangent to $\odot$ O at A; radius OA.

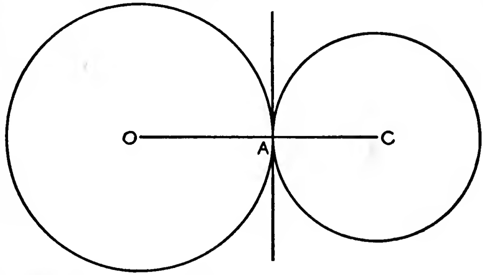
To Prove: $OA \perp$ to RT.

Proof: Every point (except A) in RT is without the $\odot$ (Def. 194).

Therefore a line from O to any point of RT (except A) is $> OA$. (Because it is $>$ a radius.) That is, OA is the shortest line from O to RT. $\therefore OA$ is $\perp$ to RT (?) (77). Q.E.D.

217. COR. **The perpendicular to a tangent at the point of contact passes through the center of the circle.** (See 43.)

218. THEOREM. **If two circles are tangent to each other, the line joining their centers passes through their point of contact.**



Given: ⓢ O and C tangent to a line at A, and line OC.

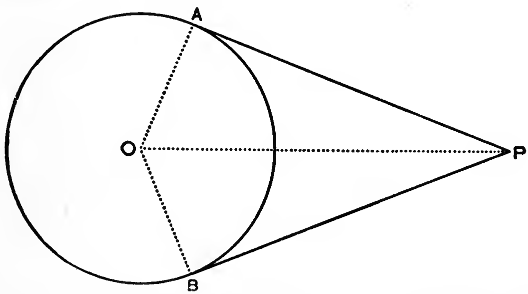
To Prove: OC passes through A.

Proof: Draw radii OA and CA. OA is $\perp$ to the tangent and CA is $\perp$ to the tangent (?) (216).

$\therefore$ OAC is a st. line (?) (43). $\therefore$ OAC and OC coincide and OC passes through A (39). Q.E.D.

Let the pupil apply this proof if the circles are tangent *internally.*

219. THEOREM. **Two tangents drawn to a circle from an external point are equal.**



NOTE. In this theorem the word "tangent" signifies the distance between the external point and the point of contact.

Given: ⊙ O; tangents PA, PB.

To Prove: Distance PA = distance PB.

Proof: Draw radii to the points of contact, and join OP.

$\angle$s OAP and OBP are rt. $\angle$s (?) (216).

In rt. $\triangle$s OAP and OBP, $OP = OP$ (?); $OA = OB$ (?).

$\therefore \triangle OAP = \triangle OBP$ (?). $\therefore PA = PB$ (?). Q.E.D.

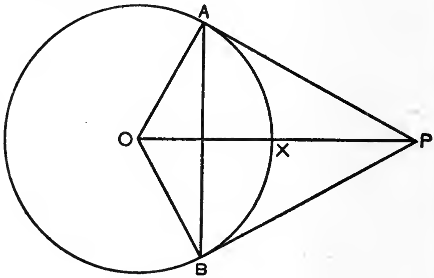
220. Theorem. **If from an external point tangents be drawn to a circle, and radii be drawn to the points of contact, the line joining the center and the external point will bisect:**

I. The angle formed by the tangents.

II. The angle formed by the radii.

III. The chord joining the points of contact.

IV. The arc intercepted by the tangents.



Proof: $\triangle OAP$ and OBP are rt. $\triangle$ (?).

They are $=$. (Explain.)

I. $\angle APO = \angle BPO$ (?).

II. $\angle AOP = \angle BOP$ (?).

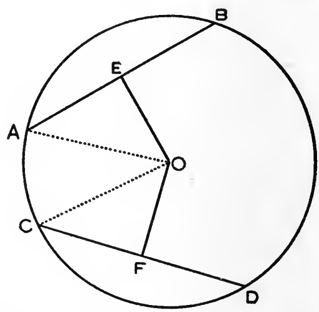
III. O is equidistant from A and B (?).

P is also (?) (219).

$\therefore OP$ is $\perp$ to AB at its midpoint (?) (70).

IV. Arc AX = arc BX (?) (206). Q.E.D.

221. Theorem. **In the same circle** (or in equal circles) **equal chords are equally distant from the center.**



Given: $\odot O$; chord AB = chord CD, and distances OE and OF.

To Prove: $OE = OF$.

Proof: Draw radii OA and OC.

In the rt. $\triangle AOE$ and COF, $AE = \frac{1}{2} AB$; $CF = \frac{1}{2} CD$ (213).

But $AB = CD$ (Hyp.).

Hence, $AE = CF$ (Ax. 3); and $AO = CO$ (?). $\therefore \triangle AOE = \triangle COF$ (?). $\therefore OE = OF$ (?). Q.E.D.

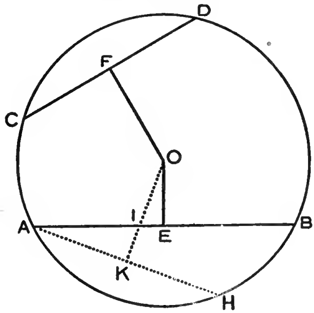
222. Theorem. **In the same circle** (or in equal circles) **chords which are equally distant from the center are equal.**

Given: $\odot O$; chords AB and CD; distance OE = distance OF.

To Prove: chord AB = chord CD.

Proof: Draw radii OA and OC. In rt. $\triangle$s AOE and COF, $AO = CO$ (?); $OE = OF$ (Hyp.). $\therefore \triangle AOE = \triangle COF$ (?). $\therefore AE = CF$ (?). AB is twice AE and CD is twice CF (?). $\therefore AB = CD$ (Ax. 3). Q.E.D.

223. Theorem. **In the same circle (or in equal circles) if two chords are unequal, the greater chord is at the less distance from the center.**



Given: $\odot O$; chord $AB >$ chord CD, and distances OE and OF.

To Prove: $OE < OF$.

Proof: Arc $AB >$ arc CD (?) (211, I). Suppose arc AH taken on arc AB = arc CD. Draw chord AH. Draw $OK \perp$ to AH cutting AB at I.

Chord AH = chord CD (?) (210).
Distance OK = distance OF (?) (221).
But $OE < OI$ (?) (77); and $OI < OK$ (?) (Ax. 5).
$\therefore OE < OK$ (Ax. 11). $\therefore OE < OF$ (Ax. 6). Q.E.D.

224. Theorem. **In the same circle (or in equal circles) if two chords are unequally distant from the center, the chord at the less distance is the greater.**

Given: $\odot O$; chords AB and CD; distance $OE <$ distance OF.

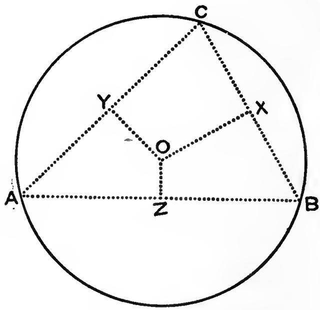
To Prove: Chord $AB >$ chord CD.

Proof: It is evident that chord $AB <$ chord CD, or = chord CD, or > chord CD. Proceed by the method of exclusion.

Another Proof: On OF take $OX = OE$. At X draw a chord $RS \perp$ to OX. Ch. RS is $\parallel$ to ch. CD (?). $\therefore$ arc $RS >$ arc CD (Ax. 5). $\therefore$ ch. $RS >$ ch. CD (?).
But ch. AB = ch. RS (?). $\therefore$ ch. $AB >$ ch. CD (Ax. 6).
Q.E.D.

225. Cor. **The diameter is longer than any other chord.**

226. Theorem. Through three points, not in the same straight line, one circumference can be drawn, and only one.



Given: Points A and B and C.

To Prove: I. (?). II. (?).

Proof: I. Draw lines AB, BC, AC. Suppose their ⊥ bisectors, OZ, OX, OY, be drawn. These ⊥s will meet at a point (?) (85). Using O as center and OA, OB, or OC as radius, a circumference can be described through A, B, C (85).

II. These ⊥s can meet at only one point (85); that is, there is only one center. The distances from O to A, O to B, O to C, are all equal (85); that is, there is only one radius. Therefore there is only one circumference (205). Q.E.D.

227. Cor. A circumference can be drawn through the vertices of a triangle, and only one.

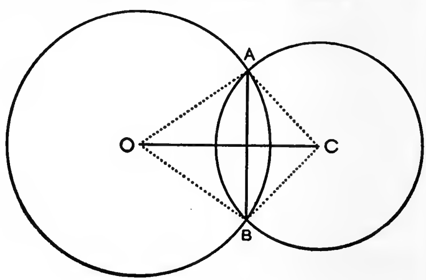
228. Cor. A circumference is determined by three points.

229. Cor. A circumference cannot be drawn through three points which are in the same straight line. [The ⊥s would be ∥.]

230. Cor. A straight line can intersect a circumference in only two points. (229.)

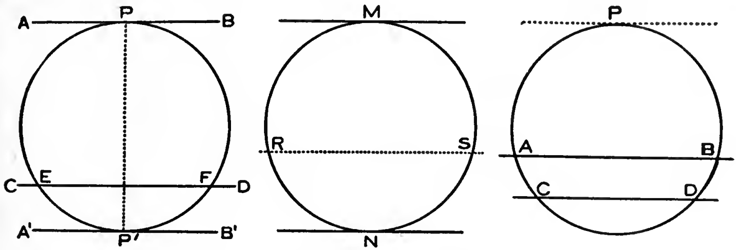
231. Cor. Two circumferences can intersect in only two points.

232. Theorem. If two circumferences intersect, the line joining their centers is the perpendicular bisector of their common chord.



Proof: Draw radii in each ⊙ to ends of AB. Point O is equally distant from A and B (?). Point C is equally distant from A and B (?). ∴ OC is the ⊥ bisector of AB (?) (70). Q.E.D.

233. Theorem. **Parallel lines intercept equal arcs on a circumference.**



Given: A circle and a pair of parallels intercepting two arcs.

To Prove: The intercepted arcs are equal.

There may be three cases:

I. If the $\parallel$s are a tangent (AB, tangent at P) and a secant (CD, cutting the circle at E and F).

Proof: Draw diameter to point of contact, P. This diameter is $\perp$ to AB (216). PP' is also $\perp$ to EF (?) (95). $\therefore$ arc $EP =$ arc FP (?) (212).

II. If the $\parallel$s are two tangents (points of contact being M and N).

Proof: Suppose a secant be drawn $\parallel$ to one of the tangents, cutting $\odot$ at R and S. RS will be $\parallel$ to other tangent (?) (94).

$\therefore$ arc $MR =$ arc MS; arc $RN =$ arc SN (proved in I). Adding, arc $MRN =$ arc MSN (Ax. 2).

III. If the $\parallel$s are two secants (one intersecting the $\odot$ at A and B; the other at C and D).

Proof: Suppose a tangent be drawn touching $\odot$ at P, $\parallel$ to AB. This tangent will be $\parallel$ to CD (?).

$\therefore$ arc $PC =$ arc PD; arc $PA =$ arc PB (by I).

Subtracting, arc $AC =$ arc BD (Ax. 2). Q.E.D.

234. A **polygon** is **inscribed** in a circle, or a **circle** is **circumscribed** about a polygon if the vertices of the polygon are in the circumference, and its sides are chords.

A **polygon** is **circumscribed** about a circle, or a **circle** is **inscribed** in a polygon	if the sides of the polygon are all tangent to the circle.

A **common tangent** to two circles is a line tangent to both of them.

The **perimeter** of a figure is the sum of all its bounding lines.

Exercises in Drawing Circles

1. Draw two unequal intersecting circles. Show that the line joining their centers is less than the sum of their radii.

2. Draw two circles externally (not tangent) and show that the line joining their centers is greater than the sum of their radii.

3. Draw two circles tangent externally. Discuss these lines similarly.

4. Draw two circles tangent internally. Discuss these lines similarly.

5. Draw two circles so that they can have only one common tangent.

6. Draw two circles so that they can have two common tangents.

7. Draw two circles so that they can have three common tangents.

8. Draw two circles so that they can have four common tangents.

9. Draw two circles so that they can have no common tangent.

Summary

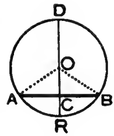
235. The following summary of the truths relating to magnitudes, which have been already established in Book II, may be helpful in attacking the original work following.

I. **Arcs are equal** if they are:
(1) Intercepted by equal central angles.
(2) Subtended by equal chords.
(3) Intercepted by parallel lines.
(4) Halves of the same arc, or of equal arcs.

II. **Lines are equal** if they are:
(1) Radii of the same or equal circles.
(2) Diameters of the same or equal circles.
(3) Chords which subtend equal arcs.
(4) Chords which are equally distant from the center.
(5) Tangents to one circle from the same point.

III. **Unequal arcs** and **unequal chords** have like relations. [See 208; 211; 223; 224.]

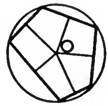
ORIGINAL EXERCISES



1. A diameter bisecting a chord is perpendicular to the chord and bisects the subtended arcs. [Use 70.]

2. A diameter bisecting an arc is the perpendicular bisector of the chord of the arc. [Draw AR and BR.]

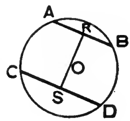
3. A line bisecting a chord and its arc is perpendicular to the chord.



4. The perpendicular bisectors of the sides of an inscribed polygon meet at a common point.

5. A line joining the midpoints of two parallel chords passes through the center of the circle.

[Suppose diam. drawn $\perp$ to AB; this will be $\perp$ to CD. Etc.]



6. The perpendiculars to the sides of a circumscribed polygon at the points of contact meet at a common point. [Use 217.]

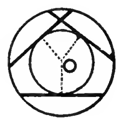
7. The bisector of the angle between two tangents to a circle passes through the center. [Use 80.]

8. The bisectors of the angles of a circumscribed polygon all meet at a common point.

9. Tangents drawn at the extremities of a diameter are parallel.

10. In the figure of 220, prove $\angle APO = \angle ABO$.

11. In the same figure, prove $\angle PAB = \angle POB$. [Use 48.]



12. If two circles are concentric, all chords of the greater, which are tangent to the less, are equal.

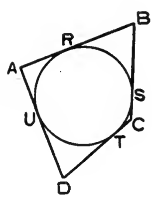
[Draw radii to points of contact. Use 216; 222.]

13. Prove 225 by drawing radii to the ends of the chord.

14. An inscribed trapezoid is isosceles. [Use 233.]

15. The line joining the points of contact of two parallel tangents passes through the center. [Draw radii to points of contact. Etc.]

16. A chord is parallel to the tangent at the midpoint of its subtended arc. [Draw radii to point of contact and to the ends of the chord. Also draw chords of the halves of the given arc.]



17. The sum of one pair of opposite sides of a circumscribed quadrilateral is equal to the sum of the other pair. [Use 219 four times, keeping R and T on the same side of the equations.]

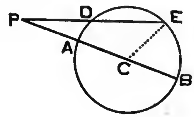
18. A circumscribed parallelogram is equilateral.

19. A circumscribed rectangle is a square.

20. If two circles are concentric and a secant cuts them both, the portions of the secant intercepted between the circumferences are equal. [Use 212.]

21. Of all secants that can be drawn to a circumference from a fixed external point, the longest passes through the center.

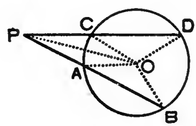
To Prove: $PB > PE$.



22. The shortest line from an external point to a circumference is that which, if produced, would pass through the center.

To Prove: $PA < PD$. Draw CD.

23. If two equal secants be drawn to a circle from an external point, their chord segments will be equal. [Draw OA, OP, OC, OB, OD. Prove ≜ POD and POB equal; then ≜ COD and AOB are equal.]



24. In No. 23 prove the external segments equal.

25. State and prove the converse of No. 23.

26. If two equal secants be drawn to a circle from an external point, they will be equally distant from the center.

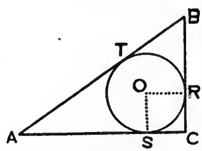
27. If two equal chords intersect on the circumference, the radius drawn to their point of intersection bisects their angle.

[Draw radii to the other extremities of the chords.]

28. Any two parallel chords drawn through the ends of a diameter are equal.

29. If a circle be inscribed in a right triangle, the sum of the diameter and hypotenuse will be equal to the sum of the legs.

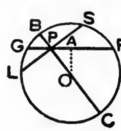
[Draw radii OR, OS; $ROSC$ is a square (?); then prove diameter $+ AB = AC + BC$.]



30. Of all chords that can be drawn through a given point within a circle, the chord perpendicular to the diameter through the given point is the shortest.

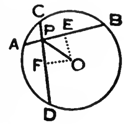
Given: P, the point; BOC the diam.; $LS \perp$ to BC at P; GR any other chord through P. **To Prove**: (?).

Proof: Draw $OA \perp$ to GR. Etc.



31. What is the longest chord that can be drawn through a given point within a circle?

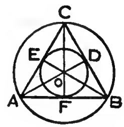
32. If the line joining the point of intersection of two chords and the center bisects the angle formed by the chords, they are equal. [Draw ⊥s OE and OF and prove them =. Etc.]



33. AB and AC are two tangents from A; in the less arc BC a point D is taken and a tangent drawn at D, meeting AB at E and AC at F; $AE + EF + AF$ equals a constant for all positions of D in arc BC.

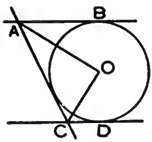
[Prove this sum $= AB + AC$.]

34. The radius of the circle inscribed in an equilateral triangle is half the radius of circle circumscribed about it. [Use 152.]



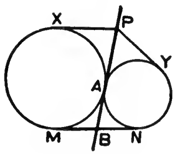
35. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

36. If two parallel tangents meet a third tangent and lines be drawn from the points of intersection to the center, they will be perpendicular.



37. Tangents drawn to two tangent circles from any point in their common interior tangent are equal.

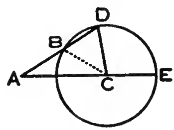
38. The common interior tangent of two tangent circles bisects their common exterior tangent.



39. Do the theorems of No. 37 and No. 38 apply if the circles are tangent internally? If so, prove.

40. In the adjoining figure if AE and AD are secants, AE passing through the center, and the external part of AD is equal to a radius, the angle $DCE = 3\angle A$.

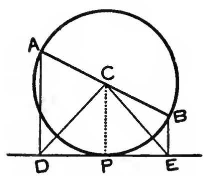
[Draw BC. $\angle DBC =$ ext. $\angle$ of $\triangle ABC = 2\angle A$ $= \angle D$ (explain). $\angle DCE =$ an ext. $\angle$, etc.]



41. If perpendiculars be drawn upon a tangent from the ends of any diameter:

(1) The point of tangency will bisect the line between the feet of the perpendiculars.

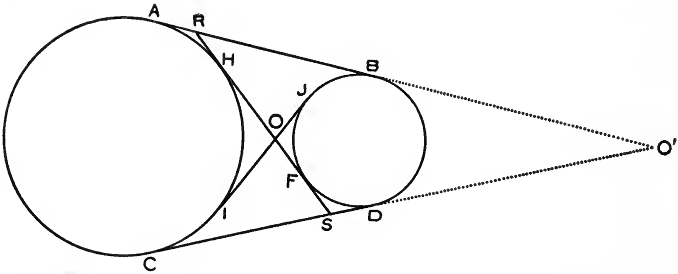
[Draw CP. Use 144.]



(2) The sum of the perpendiculars will equal the diameter.

(3) The center will be equally distant from the feet of the perpendiculars. [Use 67.]

42. The two common interior tangents of two circles are equal.



43. The common exterior tangents to two circles are equal. [Produce them to intersection.]

44. In the above figure, prove that $RH = SF$.

Proof: $AR + RB = CS + SD$; $\therefore$ $AR + (RH + HF) = (SF + HF) + SD$.

$\therefore$ $RH + RH + HF = SF + HF + SF$; $\therefore$ $2\ RH = 2\ SF$, etc. Give reasons and explain.

45. The common exterior tangents to two circles intercept on a common interior tangent (produced), a line equal to a common exterior tangent. **To Prove:** $RS = AB$.

46. Prove that in the figure of No. 42 the line joining the centers will contain O and O'.

47. Prove that in the figure of No. 42 if chords AC and BD are drawn, they are parallel.

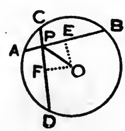
48. If a circle be described upon the hypotenuse of a right triangle as a diameter, it will contain the vertex of the right angle (148).

49. The median of a trapezoid circumscribed about a circle equals one fourth the perimeter of the trapezoid.

50. If the extremities of two perpendicular diameters be joined (in order), the quadrilateral thus formed will be a square.

51. If any number of parallel chords of a circle be drawn, their mid-points will be in the same straight line.

52. State and prove the converse of No. 35.



53. The line joining the center of a circle to the point of intersection of two equal chords bisects the angle formed by the chords.

KINDS OF QUANTITIES — MEASUREMENT

236. A **ratio** is the quotient of one quantity divided by another — both being of the same kind.

237. To **measure a quantity** is to find the number of times it contains another quantity of the same kind, called the **unit.** This *number* is the ratio of the quantity to the unit.

238. Two quantities are called **commensurable** if there exists a common unit of measure which is contained in each a whole (integral) number of times.

Two quantities are called **incommensurable** if there does not exist a common unit of measure which is contained in each a whole number of times.

Thus: \$17 and \$35 are commensurable, but \$17 and \$$\sqrt{35}$ are *not.*

Two lines $18\frac{1}{2}$ ft. and 13 yd. are commensurable, but $18\frac{1}{2}$ in. and $\sqrt[3]{13}$ mi. are *not.*

239. A **constant** quantity is a quantity whose value does not change (during a discussion). A constant may have only one value.

A **variable** is a quantity whose value is changing. A variable may have an unlimited number of values.

240. The **limit of a variable** is a constant, *to which* the variable cannot be equal, but *from which* the variable can be made to differ by less than any mentionable quantity.

241. Illustrative. The ratio of 15 yd. to 25 yd. is written either $\frac{15}{25}$ or $15 \div 25$ and is equal to three fifths. If we state that a son is two thirds as old as his father, we mean that the son's age divided by the father's equals two thirds. A ratio is a fraction.

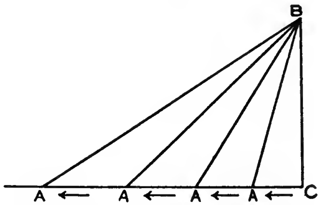
The statement that a certain distance is 400 yd. signifies that the unit (the yard), if applied to this distance, will be contained exactly 400 times.

Are \$7.50 and \$3.58 commensurable if the unit is \$1? 1 dime? 1 cent?

Are 10 ft. and $\sqrt{19}$ ft. commensurable?

The height of a steeple is a constant; the length of its shadow made by the sun is a variable. Our ages are variables. The length of a standard yard, mile, or meter, etc., is a constant. The height of a growing plant or child is a variable.

The **limit** of a variable may be illustrated by considering a right triangle ABC, and supposing the vertex A to move farther and farther from the vertex of the right angle. It is evident that the hypotenuse will become longer, that AC will increase, but BC will remain the same length. The angle A must decrease, the angle B must increase, but the angle C remains constantly a right angle. If we carry vertex A toward the left indefinitely, the $\angle A$ will become less and less but cannot become zero. [Because, then there could be no $\triangle$.]



Hence, the limit of the decreasing $\angle A$ is zero (240).

Likewise, the $\angle B$ will become larger and larger but cannot become equal to a right angle. [Because, then two sides of the triangle would be parallel, which is impossible.] But it may be made as nearly equal to a right angle as we choose.

Hence, the limit of $\angle B$ is a right angle (240).

To these limits we cannot make the variables equal, but **from** these limits we can make them differ by less than any mentionable angle, however small.

The following supplies another illustration of the limit of a variable. The sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$ etc. etc., will always be less than 2, no matter how many terms are collected. But by taking more and more terms we can make the actual difference between this sum and 2 less than any conceivable fraction, however small. Hence, 2 is the limit of the sum of the series. The limit is not 3 nor 4, because the difference between the sum and 3 cannot be made less than any assigned fraction. Neither is the limit $1\frac{1}{3}$. (Why not?) Similarly, the limit of the value of .333333 ····· *ad infinitum* is $\frac{1}{3}$.

Certain variables actually become equal to a fixed magnitude; but this fixed magnitude is not a limit (240). Thus the length of the shadow of a tower really becomes equal to a fixed distance (at noon). A man's age really attains to a definite number of years and then ceases to vary (at death).

Such variables have no limit in the mathematical sense of that word.

Hence: If a variable approaches a constant, and the difference between the two can be made indefinitely small but they cannot become equal, the constant is the **limit of the variable.** This is merely another definition of a limit.

242. THEOREM OF LIMITS. **If two variables are always equal and each approaches a limit, their limits are equal.**

Given: Two variables v and v'; v always $= v'$; also v approaching the limit l; v' approaching the limit l'.

To Prove: $l = l'$.

Proof: v is always $= v'$ (Hyp.). Hence they may be considered as a single variable. Now a single variable can approach only one limit (240). Hence, $l = l'$. Q.E.D.

NOTE. In order to make use of this theorem, one must have, *first*, two variables; *second*, these must be always equal; *third*, they must each approach a limit. *Then*, the limits are equal.

243. (1) **Algebraic principles concerning variables.**

If v is a variable and k is a constant:

I. $v + k$ is a variable.

II. $v - k$ is a variable.

III. $k \pm v$ is a variable.

IV. kv is a variable.

V. $\frac{v}{k}$ is a variable.

VI. $\frac{k}{v}$ is a variable.

These six statements are obvious.

(2) **Algebraic principles concerning limits.**

If v is a variable whose limit is l, and k is a constant:

I. $v \pm k$ will approach $l \pm k$ as a limit.

II. $k \pm v$ will approach $k \pm l$ as a limit.

III. kv will approach kl as a limit.

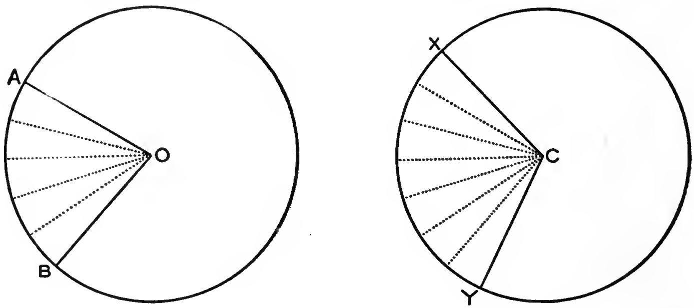
IV. $\frac{v}{k}$ will approach $\frac{l}{k}$ as a limit.

V. $\frac{k}{v}$ will approach $\frac{k}{l}$ as a limit.

NOTE. In these principles as applied to Plane Geometry, a variable is not added to, nor subtracted from, nor multiplied by, nor divided by another variable. These operations present little difficulty, however.

Proofs: I. v cannot $= l$ (240). $\therefore$ $v \pm k$ cannot $= l \pm k$.
Also, $v - l$ approaches zero (240).
$\therefore$ $(v \pm k) - (l \pm k)$ approaches zero. (Because it reduces to $v - l$.)
Hence, $v \pm k$ approaches $l \pm k$ (240).
II. Demonstrated similarly.
III. If $kv = kl$, then $v = l$ (Ax. 3). But this is impossible (240).
$\therefore$ kv cannot $= kl$.
Also $v - l$ approaches zero (240).
$\therefore$ $k(v - l)$ or $kv - kl$ approaches zero.
Therefore kv approaches kl (240).
IV and V. Demonstrated similarly.

244. THEOREM. **In the same circle** (or in equal circles) **the ratio of two central angles is equal to the ratio of their intercepted arcs.**



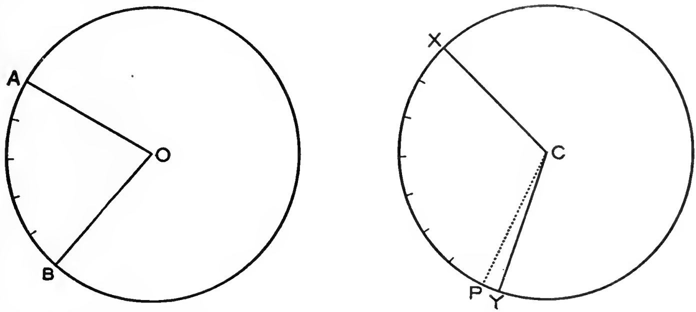
Given: $\odot O = \odot C$; central $\angle$s O and C; arcs AB and XY.

To Prove: $\dfrac{\angle O}{\angle C} = \dfrac{\text{arc } AB}{\text{arc } XY}$.

Proof: I. If the arcs are **commensurable.** There exists a common unit of measure of AB and XY (238). Suppose this unit, when applied to the arcs, is contained 5 times in AB and 7 times in XY. $\therefore \dfrac{\text{arc } AB}{\text{arc } XY} = \dfrac{5}{7}$ (Ax. 3). Draw radii to the several points of division of the arcs. $\angle O$ is divided into 5 parts, $\angle C$ into 7 parts; all of these twelve parts are equal (?) (207). $\therefore \dfrac{\angle O}{\angle C} = \dfrac{5}{7}$ (Ax. 3).

$\therefore \dfrac{\angle O}{\angle C} = \dfrac{\text{arc } AB}{\text{arc } XY}$ (Ax. 1). Q.E.D.

II. If the arcs are incommensurable. There does not exist a common unit (238). Suppose arc AB divided into equal parts (any number of them). Apply one of these as a unit of measure to arc XY. There will be a remainder PY left over. (Because AB and XY are incommensurable.)



Draw CP. Now $\dfrac{\angle O}{\angle XCP} = \dfrac{\text{arc } AB}{\text{arc } XP}$. (The case of commensurable arcs.)

Indefinitely increase the number of subdivisions of arc AB. Then each part, that is, our unit or divisor, will be indefinitely decreased. Hence PY, the remainder, will be indefinitely decreased. (Because the remainder < the divisor.)

That is, arc PY will approach zero as a limit
and $\angle PCY$ will approach zero as a limit.
$\therefore$ arc XP will approach arc XY as a limit (240)
and $\angle XCP$ will approach $\angle XCY$ as a limit (240).

$\therefore \dfrac{\angle O}{\angle XCP}$ will approach $\dfrac{\angle O}{\angle XCY}$ as a limit (243)

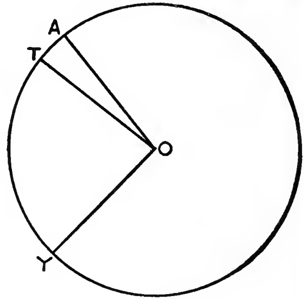
and $\dfrac{\text{arc } AB}{\text{arc } XP}$ will approach $\dfrac{\text{arc } AB}{\text{arc } XY}$ as a limit (243).

$$\therefore \frac{\angle O}{\angle XCY} = \frac{\text{arc } AB}{\text{arc } XY} \quad (?) \ (242).$$

Q.E.D.

Ex. How many degrees are there in a central angle which intercepts $\frac{1}{6}$ of the circumference? $\frac{1}{4}$ of the circumference? $\frac{1}{12}$ of the circumference? $\frac{5}{18}$ of the circumference?

245. Theorem. A central angle is measured by its intercepted arc.



Given: ⊙ O; $\angle AOY$; arc AY.

To Prove: $\angle AOY$ is measured by the arc AY, that is, they contain the same number of units.

Proof: The sum of all the $\angle s$ about $O = 4$ rt. $\angle s = 360°$ (?) (47).

If the circumference of this ⊙ be divided into 360 equal parts and radii be drawn to the several points of division, there will be 360 equal central $\angle s$ (207).

Each of these 360 central angles will be a *degree of angle* (21).

Suppose we call each of the 360 equal arcs, a *degree of arc*. Take $\angle AOT$, one of these degrees of angle, and arc AT, one of the degrees of arc. Then, $\dfrac{\angle AOY}{\angle AOT} = \dfrac{\text{arc } AY}{\text{arc } AT}$ (?) (244).

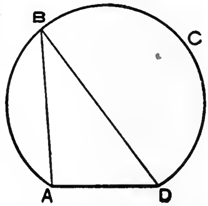
But $\dfrac{\angle AOY}{\angle AOT} = \angle AOY \div$ a unit of angle $=$ the number of units in $\angle AOY$ (237).

And $\dfrac{\text{arc } AY}{\text{arc } AT} =$ arc $AY \div$ a unit of arc $=$ the number of units in arc AY (237).

Hence, the number of units in $\angle AOY =$ the number of units in arc AY (Ax. 1).

That is, $\angle AOY$ is measured by arc AY. Q.E.D.

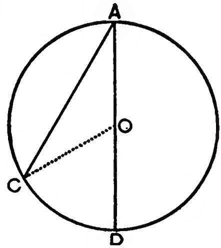
246. Cor. A central right angle intercepts a quadrant of arc. (Because each contains 90 units.)



247. Cor. A right angle is measured by half a semicircumference, that is, by a quadrant.

248. An **angle is inscribed** in a segment if its vertex is on the arc and its sides are drawn to the ends of the arc of the segment. Thus $ABCD$ is a segment and $\angle ABD$ is inscribed in it.

249. THEOREM. **An inscribed angle is measured by half its intercepted arc.**



Given : $\odot\, O$; inscribed $\angle A$; arc CD.

To Prove: $\angle A$ is measured by $\frac{1}{2}$ arc CD.

Proof: I. If one side of the $\angle$ is a diameter. Draw radius CO. $\triangle\ AOC$ is isosceles (?). $\therefore \angle A = \angle C$ (?).

$\angle COD = \angle A + \angle C$ (?) (108).

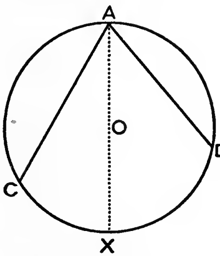
$\therefore \angle COD = \angle A + \angle A = 2 \angle A$ (Ax. 6).

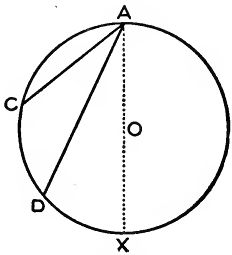
That is, $\angle A = \frac{1}{2} \angle COD$ (Ax. 3).

But $\angle COD$ is measured by arc CD (?) (245).

$\therefore \frac{1}{2} \angle COD$ is measured by $\frac{1}{2}$ arc CD (Ax. 3).

Therefore, $\angle A$ is measured by $\frac{1}{2}$ arc CD (Ax. 6).





II. If the center is *within* the angle. Draw diameter AX.

$\angle CAX$ is measured by $\frac{1}{2}$ arc CX (I).

$\angle DAX$ is measured by $\frac{1}{2}$ arc DX (I). Adding,

$\therefore \angle CAD$ is measured by $\frac{1}{2}$ arc CD (Ax. 2).

III. If the center is *without* the angle. Draw diameter AX.

$\angle CAX$ is measured by $\frac{1}{2}$ arc CX (I).

$\angle DAX$ is measured by $\frac{1}{2}$ arc DX (I). Subtracting,

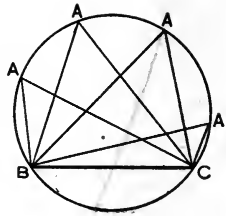
$\angle CAD$ is measured by $\frac{1}{2}$ arc CD (Ax. 2). Q.E.D.

NOTE. It is evident that angles measured by $\frac{1}{2}$ the same arc are equal.

Ex. 1. In the figure of 249, if arc CD is 56°, how many degrees are there in angle A? If arc CD is 108°, how many degrees are in angle A?

Ex. 2. If $\angle A$ contains 35°, how many degrees are there in arc CD?

250. Theorem. All angles inscribed in the same segment are equal.



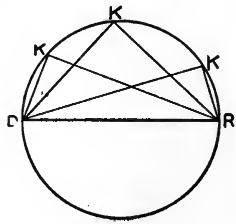
Given: The several $\angle s\ A$ inscribed in segment BAC.

To Prove: These angles all equal.

Proof: Each $\angle BAC$ is measured by $\frac{1}{2}$ arc BC (?) (249).

$\therefore$ they are equal. (Because they are measured by half the same arc.) Q.E.D.

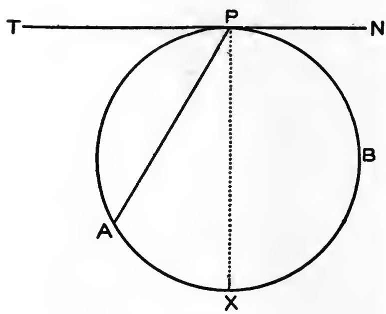
251. Cor. All angles inscribed in a semicircle are right angles.



Proof: Each is measured by half of a semicircumference (?) (249).

$\therefore$ each is a rt. $\angle$ (?) (247). Q.E.D.

252. Theorem. The angle formed by a tangent and a chord is measured by half the intercepted arc.



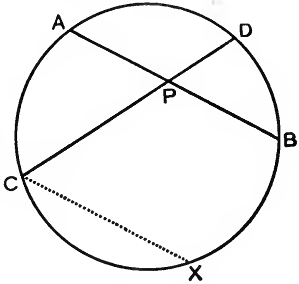
Given: Tangent TN; chord PA; $\angle TPA$; arc PA.

To Prove: $\angle TPA$ is measured by $\frac{1}{2}$ arc PA.

Proof: Draw diameter PX to point of contact.
$\angle TPX$ is a rt. $\angle$ (?) (216); arc PAX is a semicircumference (?) (204).

$\angle TPX$ is measured by $\frac{1}{2}$ arc PAX (?) (247).
$\angle APX$ is measured by $\frac{1}{2}$ arc AX (?) (249). Subtracting,
$\angle TPA$ is measured by $\frac{1}{2}$ arc PA (Ax. 2). Q.E.D
Similarly, $\angle NPA$ is measured by $\frac{1}{2}$ arc PBA.
(Use $\angle NPX$, and add.)

253. Theorem. The angle formed by two chords intersecting within the circumference is measured by half the *sum* of the intercepted arcs. (The *arcs* are those intercepted by the given angle and by its vertical angle.)



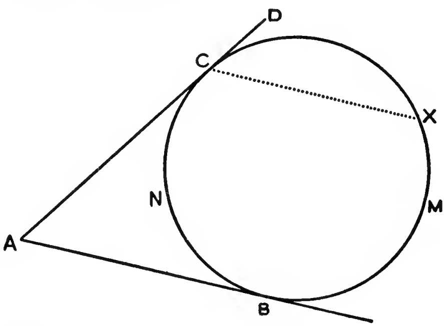
Given: Chords AB and CD intersecting at P; $\angle APC$; arcs AC and DB.

To Prove: $\angle APC$ is measured by $\frac{1}{2}$ (arc AC + arc DB).

Proof: Suppose CX drawn through $C \parallel$ to AB.
Now $\angle C$ is measured by $\frac{1}{2}$ arc DX (?) (249).
That is, $\angle C$ is measured by $\frac{1}{2}$ (arc BX + arc DB).
But $\angle C = \angle APC$ (?) (97) and arc BX = arc AC (?) (233).
$\therefore \angle APC$ is measured by $\frac{1}{2}$ (arc AC + arc DB) (?) (Ax. 6).
Q.E.D.

Note. This theorem may be proved by drawing chord AD. Then $\angle APC$ is an ext. $\angle$ of $\triangle ADP$ and $= \angle A + \angle D$ (?). [Use 249.]

254. Theorem. The angle formed by two tangents is measured by half the *difference* of the intercepted arcs.



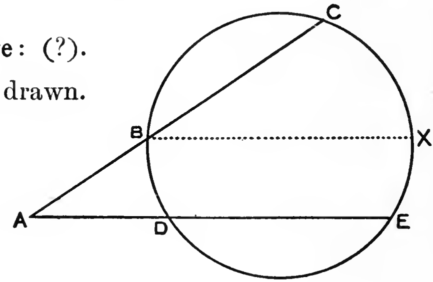
Given: The two tangents AC and AB; $\angle A$; arcs CMB and CNB.

To Prove: $\angle A$ is measured by $\frac{1}{2}$ (arc CMB − arc CNB).

Proof: Suppose CX drawn $\parallel$ to AB.

Now $\angle DCX$ is measured by $\frac{1}{2}$ arc CX (?) (252).
That is, $\angle DCX$ is measured by $\frac{1}{2}$ (arc CMB − arc BX).
But $\angle DCX = \angle A$ (?) (98); arc BX = arc CNB (?).
$\therefore \angle A$ is measured by $\frac{1}{2}$ (arc CMB − arc CNB) (Ax. 6). Q.E.D.

255. THEOREM. **The angle formed by two secants which intersect without the circumference is measured by half the *difference* of the intercepted arcs.**



Given: (?). **To Prove**: (?).

Proof: Suppose BX drawn. Where? How?

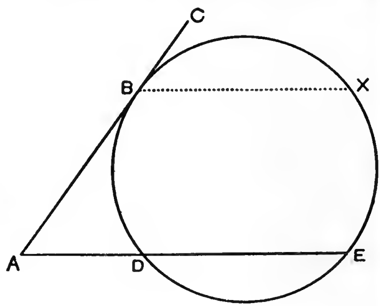
$\angle CBX$ is measured by $\frac{1}{2}$ arc CX (?).

That is, by $\frac{1}{2}$(arc CE − arc EX).

But $\angle CBX = \angle A$ (?); arc $EX =$ arc BD (?).

$\therefore \angle A$ is meas. by $\frac{1}{2}$ (arc CE − arc BD) (Ax. 6). Q.E.D.

256. THEOREM. **The angle formed by a tangent and a secant which intersect without the circumference, is measured by half the *difference* of the intercepted arcs.**



Given: (?).

To Prove: (?).

Proof: Suppose BX drawn etc.

$\angle CBX$ is measured by $\frac{1}{2}$ arc BX (?) (252).

That is, by $\frac{1}{2}$ (arc BXE − arc EX). Etc.

NOTE. The theorem of 254 may be proved by drawing chord BC. Then $\angle DCB = \angle A + \angle CBA$ (?); or, $\angle A = \angle DCB - \angle CBA$ (Ax. 2).

$\angle DCB$ is measured by $\frac{1}{2}$ arc CMB (?) and $\angle CBA$ is measured by $\frac{1}{2}$ arc CNB (?). Hence, $\angle A$ is measured by $\frac{1}{2}$ (arc CMB − arc CNB) (?).

Ex. 1. Prove the theorem of 253 for angle APD by drawing chord AC.

Ex. 2. Prove the theorem of 255 by drawing chord CD. Again, by drawing chord BE.

Ex. 3. Prove the theorem of 256 by drawing chord BD. Again, by chord BE.

ORIGINAL EXERCISES

1. If an inscribed angle contains 20°, how many degrees are there in its intercepted arc? How many degrees are there in the central angle which intercepts the same arc?

2. A chord subtends an arc of 74°. How many degrees are there in the angle between the chord and a tangent at one of its ends?

3. How many degrees are there in an angle inscribed in a segment whose arc contains 210°? in a segment whose arc contains 110°? 40°?

4. Two intersecting chords intercept opposite arcs of 28° and 80°. How many degrees are there in the angle formed by the chords?

5. The angle between a tangent and a chord contains 27°. How many degrees are there in the intercepted arc?

6. The angle between two chords is 30°; one of the arcs intercepted is 40°; find the other arc. [Denote the arc by x.]

7. If in figure of 252, arc AP contains 124°, how many degrees are there in $\angle APX$? in $\angle NPA$?

8. If in figure of 253, arc AC is 85°, $\angle APC$ is 47°, find arc DB.

9. If the arcs intercepted by two tangents contain 80° and 280°, find the angle formed by the tangents.

10. If the arcs intercepted by two secants contain 35° and 185°, find the angle formed by the secants.

11. If in figure of 254, arc CB is 135°, find the angle A.

12. If in figure of 255, angle $A = 42°$ and arc $BD = 70°$, find arc CE.

13. If in figure of 256, angle $A = 18°$, arc $BXE = 190°$, find arc BD.

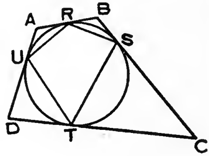
14. If the angle between two tangents is 80°, find the number of degrees in each intercepted arc. [Denote the arcs by x and $360° - x$.]

15. The circumference of a circle is divided into four arcs, 70°, 80°, 130°, and x. Find x and the angles of the quadrilateral formed by the chords of these arcs.

16. Find the angles formed by the diagonals in quadrilateral of No. 15.

17. Three of the intercepted arcs of a circumscribed quadrilateral are 68°, 98°, 114°. Find the angles of the quadrilateral. If the chords are drawn connecting (in order) the four points of contact, find the angles of this inscribed quadrilateral. Also find the angles between its diagonals.

18. If the angle between two tangents to a circle is 40°, find the other angles of the triangle formed by drawing the chord joining the points of contact.



19. The circumference of a circle is divided into four arcs, three of which are, $RS = 62°$, $ST = 142°$, $TU = 98°$. Find:

(1) Arc UR;

(2) The three angles at R; at S; at T; at U;

(3) The angles A, B, C, D of circumscribed quadrilateral;

(4) The angles between the diagonals RT and SU;

(5) The angle between RU and ST at their point of intersection (if produced);

(6) The angle between RS and TU at their intersection;

(7) The angle between AD and BC at their intersection;

(8) The angle between AB and DC at their intersection;

(9) The angle between RS and DC at their intersection;

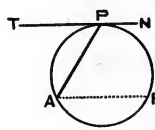
(10) The angle between AD and ST at their intersection.

20. If in the figure of No. 19, $\angle A = 96°$; $\angle B = 112°$; and $\angle C = 68°$, find the angles of the quadrilateral $RSTU$. [Denote arc RU by x. $\therefore$ in $\triangle ARU$, $96° + \frac{1}{2}x + \frac{1}{2}x = 180°$. $\therefore x =$, etc.]

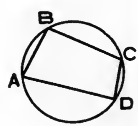
257. It is evident from the theorems relating to the measurement of angles, that:

1 Equal angles are measured by equal arcs (in the same circle).

2. Equal arcs measure equal angles.



21. Prove theorem of 252 by drawing chord parallel to the tangent.



22. The opposite angles of an inscribed quadrilateral are supplementary.

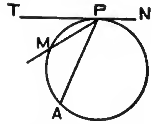
Proof: $\angle A + \angle C$ are meas. by $\frac{1}{2}$ circumference.

23. If two chords intersect within a circle, and at right angles, the sum of one pair of opposite arcs equals the sum of the other pair. [Use 253.]

24. If a tangent and a chord are parallel, and the chords of the two intercepted arcs be drawn they will make equal angles with the tangent. [Use 233; 252.]

25. The line bisecting an inscribed angle bisects the intercepted arc.

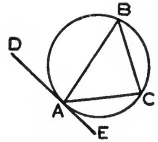
26. The line joining the vertex of an inscribed angle to the midpoint of its intercepted arc bisects the angle.



27. The line bisecting the angle between a tangent and a chord bisects the intercepted arc.

28. State and prove the converse of No. 27.

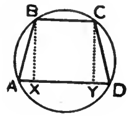
29. The angle between a tangent and a chord is half the angle between the radii drawn to the ends of the chord.



30. If a triangle be inscribed in a circle and a tangent be drawn at one of the vertices, the angles formed between the tangent and the sides will equal the other two angles of the triangle.

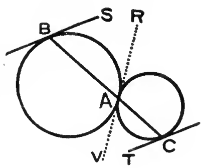
31. By the figure of No. 30 prove that the sum of the three angles of a triangle equals two right angles.

32. If one pair of opposite sides of an inscribed quadrilateral are equal, the other pair are parallel.



Proof: Draw $\perp$s BX, CY; arc $AB =$ arc CD (?).
$\therefore$ arc $ABC =$ arc BCD (Ax. 2).
Hence prove rt. $\triangle$s ABX and CDY equal.

33. If any pair of diameters be drawn, the lines joining their extremities (in order) will form a rectangle. [Use 251.]

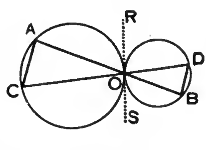


34. If two circles are tangent externally and any line through their point of contact intersects the circumferences at B and C, the tangents at B and C are parallel. [Draw common tangent at A. Prove: $\angle ACT = \angle ABS$.]

35. Prove the same theorem if the circles are tangent internally.

36. If two circles are tangent externally and any line be drawn through their point of contact terminating in the circumferences, the two diameters drawn to the extremities will be parallel.

37. Prove the same theorem if the circles are tangent internally.



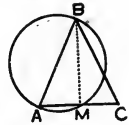
38. If two circles are tangent externally and any two lines be drawn through their point of contact intersecting their circumferences, the chords joining these points of intersection will be parallel. [Draw common tangent at O. Prove: $\angle C = \angle D$.]

39. Prove the same theorem if the circles are tangent internally.

40. The circle described on one of the equal sides of an isosceles triangle as a diameter bisects the base.

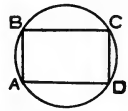
Proof: Draw line BM. The $\triangle$ are rt. $\triangle$ (?) and equal (?).

41. If the circle, described on a side of a triangle as diameter, bisects another side, the triangle is isosceles.



42. All angles that are inscribed in a segment greater than a semicircle are acute, and all angles inscribed in a segment less than a semicircle are obtuse.

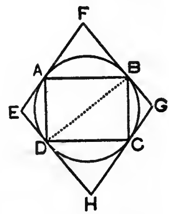
43. An inscribed parallelogram is a rectangle.



Proof: Arc $AB =$ arc CD (?) ; arc $BC =$ arc AD (?). $\therefore$ arc $ABC =$ arc ADC (?); that is, each = a semicircumference. Etc.

44. The diagonals of an inscribed rectangle pass through the center and are diameters.

45. The bisectors of all the angles inscribed in the same segment pass through a common point.

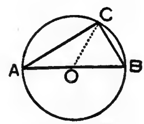


46. The tangents at the vertices of an inscribed rectangle form a rhombus.

[$\triangle$ ABF and HDC are isosceles (?) and equal? Etc.]

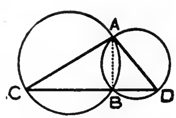
47. If a parallelogram be circumscribed about a circle, the chords joining (in order) the four points of contact will form a rectangle. [Prove BD a diameter.]

48. A circumference described on the hypotenuse of a right triangle as a diameter passes through the vertices of all the right triangles having the same hypotenuse.



49. If from one end of a diameter a chord be drawn, a perpendicular to it drawn from the other end of the diameter will intersect the first chord on the circumference. [Use 148.]

50. If two circles intersect and a diameter be drawn in each circle through one of the points of intersection, the line joining the ends of these diameters will pass through the other point of intersection. [Draw chord AB. Use 251; 43.]

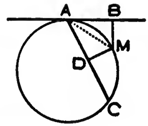


51. If $ABCD$ is an inscribed quadrilateral, AB and DC produced to meet at E, AD and BC produced to meet at F, the bisectors of angles E and F are perpendicular.

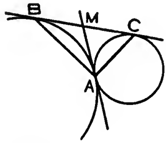
[The difference of one pair of arcs = difference of a second pair; the difference of a third pair = difference of a fourth pair. (Explain.) Transpose negative terms and add correctly, noting that the sum of 4 arcs = sum of 4 others, and hence = 180°. Half the sum of these 4 arcs measures the angle between the bisectors. (Explain.) Etc.]

52. If a tangent be drawn at one end of a chord, the midpoint of the intercepted arc will be equally distant from the chord and tangent.

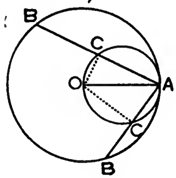
[Draw chord AM and prove the rt. $\triangle$s $=$.]



53. If two circles are tangent at A and a common tangent touches them at B and C, the angle BAC is a right angle. [Draw tangent at A. Use 219; 251.]



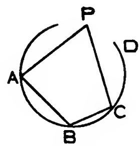
54. A circle described on the radius of another circle as diameter bisects all chords of the larger circle drawn from their point of contact. **To Prove:** AB is bisected at C. Draw chord OC. (Use 251; 213.)



55. If the opposite angles of a quadrilateral are supplementary, a circle can be drawn circumscribing it.

To Prove: A $\odot$ can be drawn through A, B, C, P.

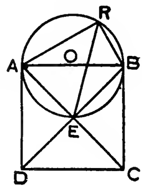
Proof: A $\odot$ can be drawn through A, B, C (?). It is required to prove that it will contain point P. $\angle P + \angle B$ are supp. (?). $\therefore$ must be meas. by half the entire circumf. $\angle B$ is meas. by $\frac{1}{2}$ arc ADC (?). Hence $\angle P$ is meas. by $\frac{1}{2}$ arc ABC. If $\angle P$ is *within* or *without* the circumf. it is not meas. by $\frac{1}{2}$ arc ABC. (Why not?)



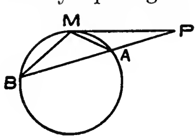
56. The circle described on the side of a square, or of a rhombus, as a diameter passes through the point of intersection of the diagonals. [Use 141; 148.]

57. The line joining the vertex of the right angle of a right triangle to the point of intersection of the diagonals of the square constructed upon the hypotenuse as a side, bisects the right angle of the triangle.

Proof: Describe a $\odot$ upon the hypotenuse as diameter and use 148; 209; 249.

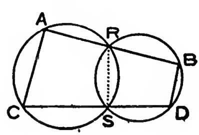


58. If two secants, PAB and PCD, meet a circle at A, B, and C, D respectively, the triangles PBC and PAD are mutually equiangular.

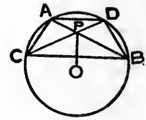


59. If PAB is a secant and PM is a tangent to a circle from P, the triangles PAM and PBM are mutually equiangular.

60. If two circles intersect and a line be drawn through each point of intersection terminating in the circumferences, the chords joining these extremities will be parallel. [Draw RS. $\angle A$ is supp. of $\angle RSC$ (?). Finally use 104.]



61. If two equal chords intersect within a circle,



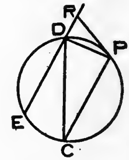
(1) One pair of intercepted arcs are equal.

(2) Corresponding parts of the chords are equal.

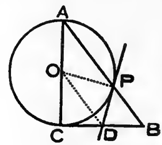
(3) The lines joining their extremities (in order) form an isosceles trapezoid.

(4) The radius drawn to their intersection bisects their angle.

62. If a secant intersects a circumference at D and E, PC is a parallel chord, and PR a tangent at P meeting secant at R, the triangles PCD and PRD are mutually equiangular. [$\angle R$ and $\angle CDP$ are measured by $\frac{1}{2}$ arc PC. (Explain.) Etc.]

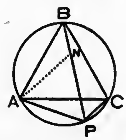


63. If a circle be described upon one leg of a right triangle as diameter and a tangent be drawn at the point of its intersection with the hypotenuse, this tangent will bisect the other leg.

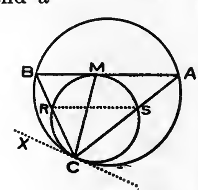


[Draw OP and OD. CD is tangent (?). OD bisects arc PC (?) (220). $\angle COD = \angle A$ (?) (257). $\therefore$ OD is $\parallel$ to AB (?). Etc.]

64. If an equilateral triangle ABC is inscribed in a circle and P is any point of arc AC, $AP + PC = BP$. [Take $PN = PA$; draw AN. $\triangle ANP$ is equilateral. (Explain.) $\triangle ANB = \triangle APC$ (?). Etc.]

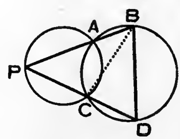


65. If two circles are tangent internally at C, and a chord AB of the larger circle is tangent to the less circle at M, the line CM bisects the angle ACB.



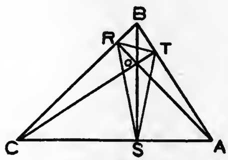
[Draw tangent CX and chord RS. $\angle RSC = \angle BCX = \angle A$. $\therefore$ AB is $\parallel$ to RS. (Explain.) Then use 233. Etc.]

66. If two circles intersect at A and C and lines be drawn from any point P, in one circumference, through A and C terminating in the other at points B and D, chord BD will be of constant length for all positions of point P.



[Draw BC. Prove $\angle BCD$, the ext. $\angle$ of $\triangle PBC$, = a constant. Etc.]

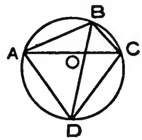
67. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of these perpendiculars.



To Prove: BS bisects $\angle RST$, etc.

Proof: If a circle be described on AO as diam., it will pass through T and S (?) (148). If a circle be described on OC as diam., it will pass through R and S (?). $\therefore \angle BAR = \angle BST$ (?); and $\angle BCT = \angle BSR$ (?). But $\angle BAR = \angle BCT$. (Each is the comp. of $\angle ABC$. $\therefore$ Etc.)

68. If ABC is a triangle inscribed in a circle, BD is the bisector of angle ABC, meeting AC at O and the circumference at D, the triangles AOB and COD are mutually equiangular. Also triangles BOC and AOD. Also triangles BOC and ABD. Also triangles AOD and ABD. Also triangles BCD and COD.



69. If two circles intersect at A and B, and from P, any point on one of them, lines AP and BP be drawn cutting the other circle again at C and D respectively, CD will be parallel to the tangent at P.

70. If two circles intersect at A and B, and through B a line be drawn meeting the circles at R and S respectively, the angle RAS will be constant for all positions of the line RS.

[Prove $\angle R + \angle S$ is constant. $\therefore \angle RAS$ is also constant.]

71. Two circles intersect at A and through A any secant is drawn meeting the circles again at M and N. Prove that the tangents at M and N meet at an angle which remains constant for all positions of the secant.

[Prove the angle between these tangents equal to the angle between the tangents to the circles, at A.]

72. Three unequal circles are each externally tangent to the other two. Prove that the three tangents drawn at the points of contact of these circles meet in a common point.

73. Two equal circles intersect at A and B, and through A any straight line MAN is drawn, meeting the circumferences at M and N respectively. Prove chord BM = chord BN.

74. If the midpoint of the arc subtended by any chord be joined to the extremities of any other chord,

(1) The triangles formed will be mutually equiangular.

(2) The opposite angles of the quadrilateral thus formed will be supplementary.

75. Two circles meet at A and B and a tangent to each circle is drawn at A, meeting the circumferences at R and S respectively; prove that the triangles ABR and ABS are mutually equiangular.

76. What is the locus of points at a given distance from a given point? Prove. (Review 179 and 180 now.)

77. What is the locus of the midpoints of all the radii of a given circle? Prove.

78. What is the locus of the midpoints of a series of parallel chords in a circle? Prove.

79. What is the locus of the midpoints of all chords of the same length in a given circle? Prove.

80. What is the locus of all points from which two equal tangents can be drawn to two circles which are tangent to each other?

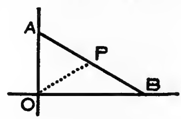
81. What is the locus of all the points at a given distance from a given circumference? Discuss if the distance is $>$ radius. If it is less.

82. What is the locus of the vertices of the right angles of all the right triangles that can be constructed on a given hypotenuse? Prove.

83. What is the locus of the vertices of all the triangles which have a given acute angle (at that vertex) and have a given base? Prove.

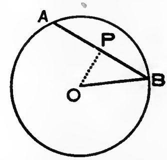
84. A line of given length moves so that its ends are in two perpendicular lines; what is the locus of its midpoint? Prove.

[Suppose AB represents one of the positions of the moving line. Draw OP to its midpoint. In all the positions of AB, $OP = \frac{1}{2} AB =$ a constant (148). $\therefore$ P is always a fixed distance from O. Etc.]



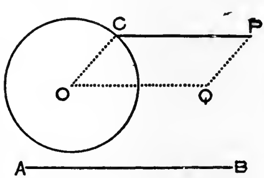
85. What is the locus of the midpoints of all the chords that can be drawn through a fixed point on a given circumference? Prove.

[Suppose AB represents one of the chords from B in circle O, with radius OB; and P the midpoint of AB. Draw OP. $\angle P$ is a rt. $\angle$ (?). That is, wherever the chord may be drawn, $\angle P$ is a rt. $\angle$.
$\therefore$ locus of P is, etc.]



86. A definite line which is always parallel to a given line moves so that one of its extremities is on a given circumference; find the locus of the other extremity.

[Suppose CP represents one position of the moving line CP. Draw $OQ =$ and $\parallel$ to CP from center O. Join OC and PQ. Wherever CP is, this figure is a ▱ (?). Its sides are of constant length (?). That is, P is always a fixed distance from Q, etc.]



CONSTRUCTIONS

258. Heretofore the figures we have used have been assumed. We have *supposed* such auxiliary lines to be drawn as conditions required. No methods have been given for drawing any lines, and only our three postulates have been assumed regarding such construction. But the lines that have been used were drawn as *aids* toward establishing truths, and precise drawings have not been essential. The following simple methods for constructing lines are given that mathematical precision may be employed if necessary in drawing diagrams of a more complex nature. The pupil should be very familiar with the use of the ruler and compasses.

259. A geometrical **construction** is a diagram made of points and lines.

260. A geometrical **problem** is the statement of a required construction. Thus: "it is required to bisect a line" is a problem. A problem is sometimes defined as "a question to be solved" and includes other varieties besides those involved in geometry.

261. The word **proposition** is used to include both theorem and problem.

262. The complete solution of a problem consists of *five* parts:

I. The **Given** data are to be described.

II. The **Required** construction is to be stated.

III. The **Construction** is to be outlined.

This part usually contains the verb only in the *imperative.* No reasons are necessary because no statements are made. The only limitation in this part of the process is, that every construction demanded shall have been shown possible by previous constructions or postulates. (See 32; 33; 199.)

IV. The **Statement** that the required construction has been completed.

V. The **Proof** of this declaration.

Sometimes a **discussion** of ambiguous or impossible instances will be necessary.

263. Notes. (1) A straight line is determined by two points.

(2) A circle is determined by three points.

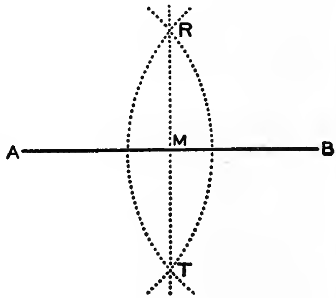
(3) A circle is determined by its center and its radius. Whenever a circumference, or even an arc, is to be drawn, it is *essential* that the **center** and the **radius** be mentioned.

(4) "Q. E. F." = *Quod erat faciendum* = "which was to be done." These letters are annexed to the statement that the construction which was required has been accomplished.

264. Problem. **It is required to bisect a given line.**

Given: The definite line AB.

Required: To bisect AB.



Construction: Using A and B as centers and *one* radius, sufficiently long to make the circumferences intersect, describe two arcs meeting at R and T. Draw RT meeting AB at M.

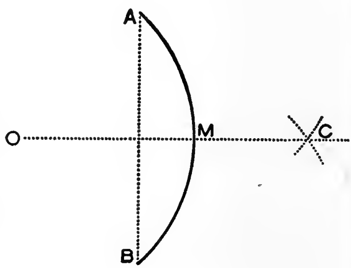
Statement: Point M bisects AB. Q.E.F.

Proof: R is equally distant from A and B (?) **(201)**.
T is equally distant from A and B (?).
Hence, RT is the $\perp$ bisector of AB (?) **(70)**.
That is, M bisects AB. Q.E.D.

265. Problem. **To bisect a given arc.**

Given: Arc AB whose center is O.

Required: To bisect arc AB.

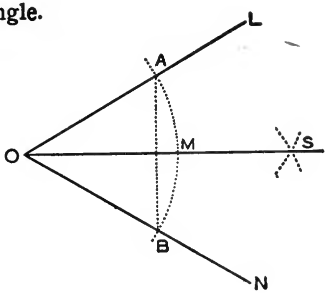


Construction: Draw chord AB; using A and B as centers and any sufficient radius, describe arcs meeting at C. Draw OC cutting arc AB at M.

Statement: The point M bisects arc AB. Q.E.F.

Proof: O and C are each equally distant from A and B (201).
$\therefore$ OC is the $\perp$ bisector of chord AB (?) (70).
$\therefore$ M bisects arc AB (?) (213). Q.E.D.

266. Problem. **To bisect a given angle.**



Given: $\angle LON$.

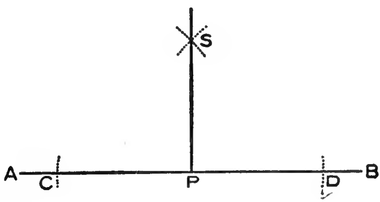
Required: To bisect $\angle LON$.

Construction: Using O as a center and any radius, draw arc AB, cutting LO at A and NO at B. Draw chord AB. Using A and B as centers and any sufficient radius, draw two arcs intersecting at S. Draw OS meeting arc AB at M.

Statement: OS bisects $\angle LON$. Q.E.F.

Proof: O and S are each equally distant from A and B (?).
$\therefore$ OS is the $\perp$ bisector of chord AB (?).
$\therefore$ M bisects arc AB (?) (213). $\therefore \angle AOM = \angle BOM$ (?) (207).
That is, OS bisects $\angle LON$. Q.E.D.

267. Problem. **At a fixed point in a straight line to erect a perpendicular to that line.**



Given: Line AB and point P within it.

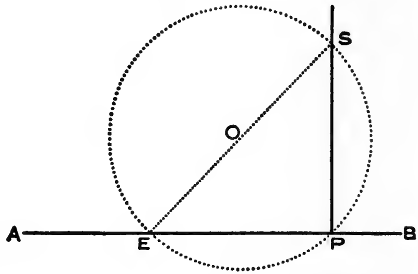
Required: To erect a line $\perp$ to AB at P.

Construction: Using P as a center and any radius, draw arcs meeting AB at C and D. Using C and D as centers and a radius longer than before, draw arcs meeting at S. Draw PS.

Statement: PS is $\perp$ to AB at P. Q.E.F.

Proof: PS is the $\perp$ bisector of CD (?) (70). Q.E.D.

Another Construction: Using any point O, without AB, as center, and OP as radius, describe a circumference, cutting AB at P and E. Draw diameter EOS. Join SP.

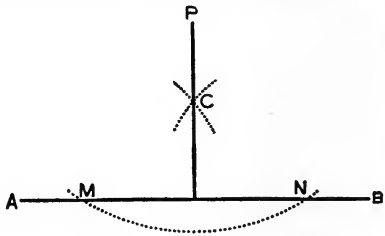


Statement: SP is $\perp$ to AB at P. Q.E.F.

Proof: Segment SPE is a semicircle (?) (204).

$\therefore \angle SPE$ is a rt. $\angle$ (?) (251). $\therefore SP$ is $\perp$ to AB (?) (17).

268. Problem. **Through a point without a line to draw a perpendicular to that line.**



Given: Line AB and point P without it.

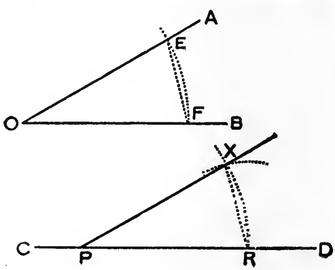
Required: (?).

Construction: Using P as a center and any sufficient radius, describe an arc intersecting AB at M and N. Using M and N as centers and any sufficient radius, describe arcs intersecting each other at C. Draw PC.

Statement: PC is $\perp$ to AB from P. Q.E.F.

Proof: PC is the $\perp$ bisector of MN (?) (70). Q.E.D.

269. Problem. **At a given point in a given line to construct an angle which shall be equal to a given angle.**



Given: $\angle AOB$; point P in line CD.

Required: To construct at P an $\angle = \angle AOB$.

Construction: Using O as a center with any radius, describe an arc cutting OA at E and OB

at F. Draw chord EF. Using P as a center and OE as a radius, describe an arc cutting CD at R. Using R as a center and chord EF as a radius, describe an arc cutting the former arc at X. Draw PX and chord RX.

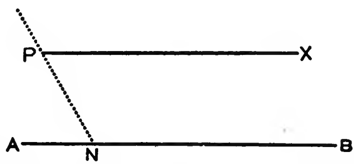
Statement: The $\angle XPD = \angle AOB$. Q.E.F.

Proof: Chord $EF =$ chord RX (?) (201).

$\therefore$ arc $EF =$ arc RX (?) (209). $\therefore \angle XPR = \angle O$ (?) (207).

That is, $\angle XPD = \angle AOB$. Q.E.D.

270. Problem. To draw a line through a given point parallel to a given line.



Given: Point P and line AB.

Required: To draw through P, a line $\parallel$ to AB.

Construction: Draw any line PN through P meeting AB at N.

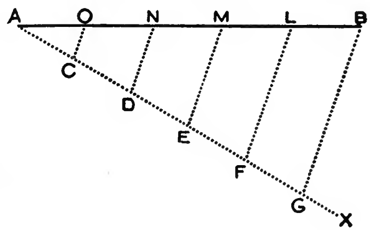
On this line, at P, construct $\angle NPX = \angle ANP$. (By 269.)

Statement: PX is $\parallel$ to AB. Q.E.F.

Proof: $\angle NPX = \angle ANP$ (?) (Const.).

$\therefore PX$ is $\parallel$ to AB (?) (101). Q.E.D.

271. Problem. To divide a line into any number of equal parts.



Given: Definite line AB.

Required: To divide it into five equal parts.

Construction: Draw through A any other line AX. On this take any length AC as a unit, and mark off on AX five of these units, AC, CD, DE, EF, FG. Draw GB.

Through F, E, D, C, draw $\parallel$ to GB, lines FL, EM, DN, CO.

Statement: Then, $AO = ON = NM = ML = LB$. Q.E.F.

Proof: $AC = CD = DE = EF = FG$ (Const.).

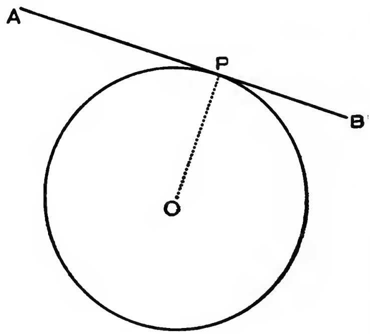
$\therefore AO = ON = NM = ML = LB$ (?) (147). Q.E.D.

272. Problem. To draw a tangent to a given circle through a given point:

I. If the point is on the circumference.

II. If the point is without the circle.

I. **Given:** ⊙ O; P, a point on the circumference.



Required: To draw a tangent through P.

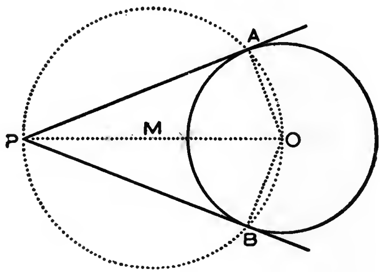
Construction: Draw the radius OP. Draw line $AB \perp$ to OP at P (by 267).

Statement: AB is tangent to ⊙ O at P. Q.E.F.

Proof: AB is $\perp$ to PO at P (Const.).

$\therefore AB$ is a tangent (?) (215). Q.E.D.

II. **Given:** ⊙ O; P, a point without it.



Required: To draw a tangent through P.

Construction: Draw PO; bisect it at M (by 264).

Using M as a center and PM as a radius, describe a circumference intersecting ⊙ O at A and B.

Draw PA, PB, OA, OB.

Statement: PA and PB are tangents through P. Q.E.F.

Proof: ⊙ M passes through O ($PM = MO$ by const.).

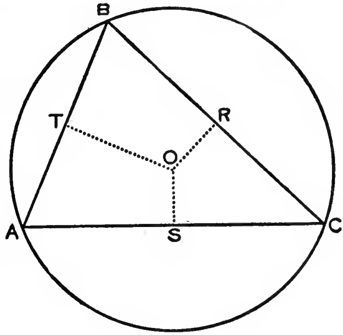
$\therefore \angle PAO$ is a rt. $\angle$ (?) (251). $\therefore PA$ is tangent (?) (215). Similarly PB is a tangent. Q.E.D.

Ex. 1. Show by two distinct methods how to bisect a line.

Ex. 2. Show how to construct the problem of 270 by use of 268.

273. PROBLEM. **To circumscribe a circle about a given triangle.**

Given: (?). **Required:** (?). (See 227.)



Construction: Bisect AB, BC, AC. Erect $\perp$s at these midpoints, meeting at O.

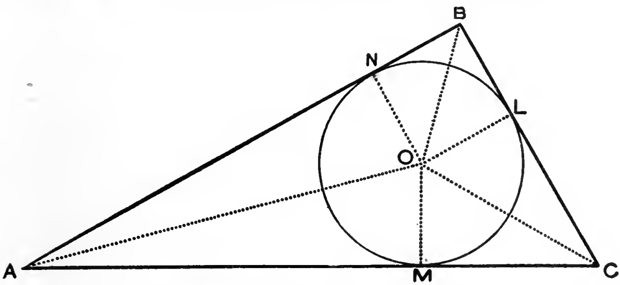
Using O as a center and OA as radius, draw a circle.

Statement: This $\odot$ will pass through vertices A, B, and C.

Q.E.F.

Proof: [Use 85.]

274. PROBLEM. **To inscribe a circle in a given triangle.**



Given: (?). **Required:** (?). **Construction:** Draw the three bisectors of the $\angle$s of $\triangle ABC$, meeting at O (by 266). Draw $\perp$s from O to the three sides.

Using O as a center and one of these $\perp$s as a radius, draw a $\odot$.

Statement: This $\odot$ will be tangent to the three sides of $\triangle ABC$.

Q.E.F.

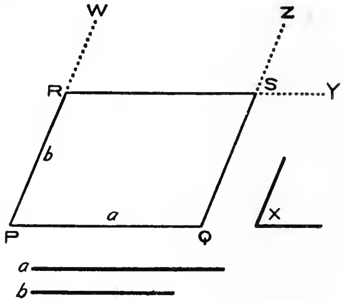
Proof: The bisectors of these angles meet in a point and the $\perp$s OL, OM, ON are equal (?) (84).

$\therefore$ the circumference passes through L, M, N (?) (192).

Therefore the three sides are tangent to the $\odot$ (?) (215).

That is, the $\odot$ O is inscribed in $\triangle ABC$ (?) (234). Q.E.D.

275. Problem. To construct a parallelogram if two sides and the included angle are given.



Given: The sides a and b and their included angle, x.

Required: To construct a ▱ containing these parts.

Construction: Take a straight line $PQ = a$.

At P construct $\angle P = \angle x$. On PW, the side of this $\angle$, take $PR = b$.

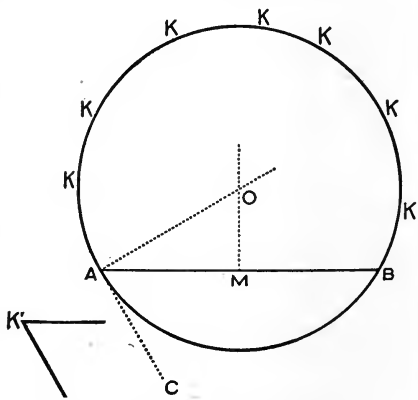
At R draw $RY \parallel$ to PQ; and at Q draw $QZ \parallel$ to PW. Denote the intersection of these lines by S.

Statement: $PQSR$ is the required parallelogram. Q.E.F.

Proof: First, it is *a* parallelogram (?).

Second, it is *the required* parallelogram. (Because it contains the given parts.) Q.E.D.

276. Problem. To construct a segment of a circle upon a given line, as chord, which shall contain angles equal to a given angle.



Given: Line AB and $\angle K'$.

Required: To construct a segment upon AB whose inscribed angles shall $= \angle K'$.

Construction: Construct at A, $\angle BAC = \angle K'$.

Bisect AB at M.

At M erect $OM \perp$ to AB.

At A erect $OA \perp$ to AC, meeting OM at O.

Using O as a center and OA as radius, describe ⊙ O.

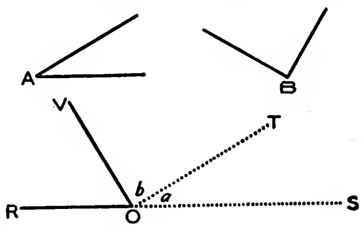
Statement: The $\angle$s inscribed in $AKB = \angle K'$. Q.E.F.

Proof: The circumference passes through B (?) (67).
$\therefore AB$ is a chord (?). AC is tangent to the $\odot$ (?) (215).
$\therefore \angle BAC$ is measured by half the arc AB (?) (252).
Any inscribed angle AKB is measured by half the arc AB (?).
Therefore, any angle $AKB = \angle BAC$ (?) (257, 2).
Consequently, any inscribed angle $AKB = \angle K'$ (Ax. 1).

Q.E.D.

[If the pupil will draw chords AK and BK, he will understand the proposition. These were purposely omitted.]

277. Problem. To construct the third angle of a triangle if two angles are known.



Given: $\angle$s A and B, two $\angle$s of a $\triangle$.

Required: To construct the third.

Construction: At point O in a line RS construct $\angle a = \angle A$.
At point O in OT construct $\angle b = \angle B$.

Statement: The $\angle VOR =$ the third $\angle$ of the $\triangle$. Q.E.F.

Proof: $\angle a + \angle b + \angle VOR = 2$ rt. $\angle$s (?) (46).
$\angle A + \angle B +$ the unknown $\angle = 2$ rt. $\angle$s (?) (110).
$\therefore \angle a + \angle b + \angle VOR = \angle A + \angle B +$ the unknown $\angle$ (?).
But $\angle a + \angle b \qquad = \angle A + \angle B$ (Const. and Ax. 2).
$\therefore \angle VOR \qquad =$ the unknown $\angle$ (Ax. 2).
That is, $\angle VOR =$ the third $\angle$ of the $\triangle$. Q.E.D.

Ex. 1. To circumscribe a triangle about a given circle.

Ex. 2. To construct the problem of 276 if the given angle is a right angle; if it is an obtuse angle.

Ex. 3. To construct the problem of 273 if the given triangle is obtuse.

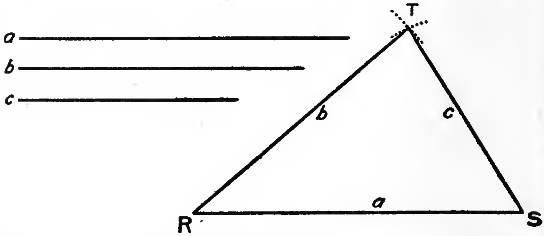
Ex. 4. Is the problem of 277 ever impossible? Explain.

Ex. 5. In the figure of 274, if $\angle A = 40°$ and $\angle B = 94°$, how many degrees are there in each of the six acute angles at O? If $\triangle LMN$ is constructed, how many degrees are there in each of its angles?

278. Problem. To construct a triangle if the three sides are known.

Given: Sides a, b, c of a $\triangle$.

Required: To construct the $\triangle$.



Construction: Draw $RS = a$.

Using R as a center and b as a radius, describe an arc; using S as a center and c as a radius, describe an arc intersecting the former arc at T. Draw RT and ST.

Statement: RST is the required $\triangle$. **Q.E.F.**

Proof: RST is *a* $\triangle$ (?) (23).

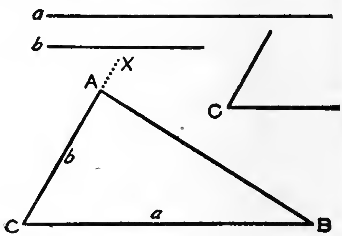
RST is *the required* $\triangle$. (It contains a, b, c.) **Q.E.D.**

Discussion: Is this problem ever impossible? When?

279. Problem. To construct a triangle if two sides and the included angle are known.

Given: The sides a and b, and their included $\angle C$ in a $\triangle$.

Required: To construct the $\triangle$.



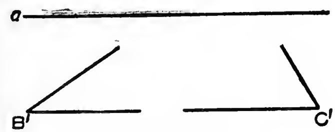
Construction: Draw $CB = a$. At C construct $\angle BCX =$ given $\angle C$. On CX take $CA = b$. Join AB.

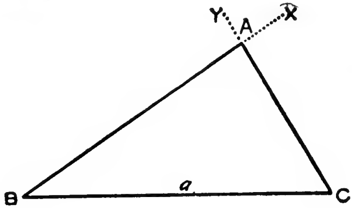
Statement: (?). **Proof**: (?).

Note. The student has probably observed that in constructions certain lines and angles *must precede* others. In such problems as 266, 267, 269, 272, and 276, the **order** of the successive steps is exceedingly important. Problems are not so numerous in geometry as theorems, but it must be apparent that problems are instructive, fascinating, and profitable.

Definition. If a circle is described, touching one side of a triangle and the prolongations of the other sides, it is called an **escribed circle**.

280. PROBLEM. To construct a triangle if a side and the two angles adjoining it are known.



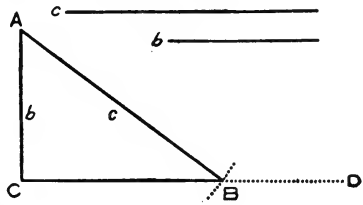


Given: (?). **Required**: (?).

Construction: Draw $BC = a$. At B construct $\angle CBX = \angle B'$; at C construct $\angle BCY = \angle C'$. Denote the point of intersection of BX and CY by A.

Statement: (?). **Proof**: (?). **Discussion**: (?).

281. PROBLEM. To construct a right triangle if the hypotenuse and a leg are known.



Given: Hypotenuse c; leg b.

Required: (?).

Construction: Draw an indefinite line CD and at C erect a $\perp = b$. Using A as a center and c as a radius, describe an arc cutting CD at B. Draw AB.

Statement: (?). **Proof**: (?). **Discussion**: (?).

NOTE. If it is required to construct a right triangle, having given the hypotenuse and another part, it is often advantageous to describe a semicircle upon the given hypotenuse as diameter. Every triangle whose base is this diameter and whose vertex is on this semicircumference is a right triangle (251). Hence if the triangle constructed contains the other given part, it is the required triangle.

Ex. 1. To construct the problem of 281 by use of the semicircle.

Ex. 2. Discuss the constructions of 279, 280, and 281 fully.

Ex. 3. To construct a triangle and its three escribed circles.

Ex. 4. To construct an isosceles right triangle having given the hypotenuse.

Ex. 5. To construct an isosceles right triangle having given the leg.

Ex. 6. To construct a quadrilateral having given the four sides and an angle.

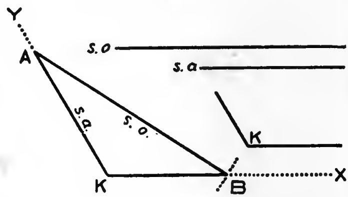
282. Problem. **To construct a triangle if an angle, a side adjoining it, and the side opposite it are known; that is, if two sides and an angle opposite one of them are known.**

The known angle may be obtuse, right, or acute. Consider:

First, If "side opposite" > "side adjoining."
Second, If "side opposite" = "side adjoining."
Third, If "side opposite" < "side adjoining."

Construction for all of these: Draw an indefinite line, KX, and at one extremity construct an $\angle = \angle K$; take on the side of this $\angle$ a distance from the vertex = "side adjoining." Using the end of this side as a center and the "side opposite" as a radius, describe an arc intersecting KX. Draw radius to the intersection just found.

If the known angle is obtuse or right.



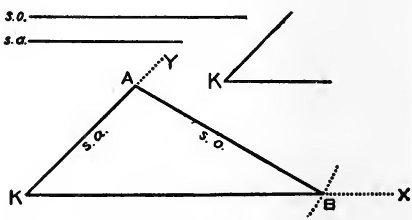
Given: $\angle K$, *s.a*, and *s.o.* of a $\triangle$.

Construction: As above.

Discussion: Case I. $s.o. > s.a.$ The $\triangle$ is always possible.

Case II. $s.o. = s.a.$ The $\triangle$ is never possible (55 and 112).

Case III. $s.o. < s.a.$ The $\triangle$ is never possible (?) (122).



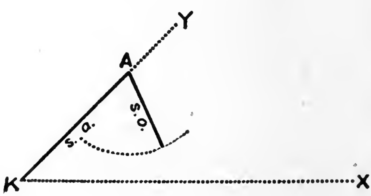
If the known angle is acute.

Case I. $s.o. > s.a.$ The $\triangle$ is always possible.

Case II. $s.o. = s.a.$ An isosceles $\triangle$.

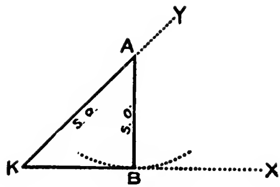
Case III. $s.o. < s.a.$

(1) If $s.o. <$ the $\perp$ from A to KX, the $\triangle$ is never possible.

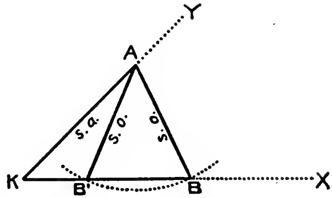


(2) If *s.o.* = the $\perp$ from A to KX, $\triangle$ is a rt.$\triangle$ (216).

(3) If *s.o.* > the $\perp$ from A to KX, there are *two* $\triangle$s.



In this instance the arc described, using A as center and "*s.o.*" as radius, intersects KX twice, at B and B'.



Hence, $\triangle AKB$ and $\triangle AKB'$ both contain the three given parts.

CONCERNING ORIGINAL CONSTRUCTIONS

Analysis

Many constructions are so simple that their correct solution will readily occur to the pupil. Sometimes, as in the case of complicated constructions, one requires the ability to put the given parts together, one by one.

The following outline may be found helpful if employed intelligently.

I. Suppose the construction made,—that is, suppose the figure drawn.

II. Study this figure in search of truths by which the **order** of the lines that have been drawn can be determined. This is essential.

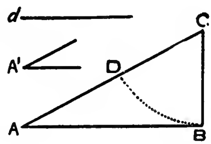
III. One or more auxiliary lines may be necessary.

IV. Finally, construct the figure and prove it correct.

Exercise. Given the base of a triangle, an adjacent acute angle, and the difference of the other sides, to construct the triangle.

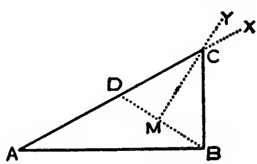
Given: Base AB; $\angle A'$; difference d.

Required: To construct the $\triangle$.



[**Analysis:** Suppose $\triangle ABC$ is the required $\triangle$. It is evident that if $CD = CB$, they may be sides of an isos. $\triangle$ and $AD = d$. This isosceles $\triangle$ may be constructed.]

Construction: At A on AB construct $\angle BAX = \angle A'$ and take on AX, $AD = d$. Join DB. At M, midpoint of DB, draw MY $\perp$ to DB meeting AX at C. Draw CB.



Statement: (?). **Proof:** (?). **Discussion:** (?).

ORIGINAL CONSTRUCTIONS

1. To construct a right angle.

2. To construct an angle containing 45°.

3. To construct the complement of a given angle; the supplement.

4. To construct an angle of 60°. [See 115.]

5. To construct an angle of 30°; of 15°; of 120°.

6. To construct an angle of 150°; of 135°; of 75°; of 165°.

7. To find the center of a given circle. [See 214.]

8. To construct a tangent to a given circle, parallel to a given line. [Draw a radius ⊥ to the given line.]

9. To construct a tangent to a given circle, perpendicular to a given line.

10. To construct the other acute angle of a right triangle if one is known.

11. To draw through a given point without a given line, another line which shall make a given angle with the line.

[Draw a ∥ to the given line through the given point.]

12. To trisect a right angle.

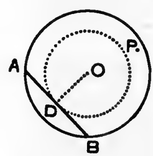
13. To find a point in one side of a triangle equally distant from the other sides. [Use 266.]

14. To construct a chord of a circle if its midpoint is known.

[Draw a radius through this point and use 267.]

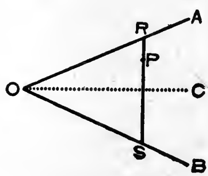
15. To construct the shortest chord that can be drawn through a given point within a circle.

Proof: Draw any other chord through the point, etc.



16. To construct through a given point within a circle two chords each equal to a given chord.

17. To construct in a given circle a chord equal to a second chord and parallel to a third.



18. To construct through a given point a line which shall make equal angles with the sides of a given angle. [Use 266; 268.]

19. To construct from a given point in a given circumference a chord which shall be at a given distance from the center.

[How many can be drawn from this point?]

I. To construct an **isosceles triangle**, having given:

20. The base and one of the equal sides.

21. The base and one of the equal angles.

22. One of the equal sides and the vertex-angle.

23. One of the equal sides and one of the equal angles.

24 The base and altitude upon it.

25. The base and the radius of the inscribed circle.

[Bisect the base; erect a $\perp =$ radius; describe $\odot$, etc.]

26. The base and the radius of the circumscribed circle.

[First, describe $\odot$ with given radius and any center.]

27. The altitude and the vertex-angle.

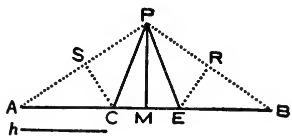
[Draw an indefinite line and erect a $\perp$ equal the given altitude. Bisect the given $\angle$; at the end of the altitude construct $\angle = \frac{1}{2}$ given $\angle$, etc.]

28. The base and the vertex-angle.

[Find the supplement of given $\angle$; bisect this; at each end of base construct an $\angle =$ this half; etc.]

29. The perimeter and the altitude.

Given: Perimeter $= AB$; alt. $= h$.
Required: (?). **Construction**: Bisect AB; erect at $M \perp = h$; draw AP and BP.
Bisect these; erect $\perp$s SC and RE; etc.



II. To construct a **right triangle**, having given:

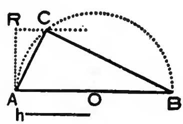
30. The two legs.

31. One leg and the adjoining acute angle.

32. One leg and the opposite acute angle.

33. The hypotenuse and an acute angle.

34. The hypotenuse and the altitude upon it.

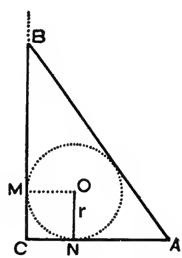


35. The median and the altitude upon the hypotenuse. [Same as No. 34.]

36. The radius of the circumscribed circle and a leg.

37. The radius of the inscribed circle and a leg.

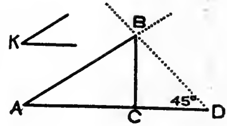
Given: Radius $= r$; leg $= CA$. **Required**: (?). **Analysis**: Consider that ABC is the completed figure; $CNOM$ is a square, whose vertex O is the center of the circle, and side ON is the given radius. AB is tangent from A. **Construction**: On CA take $CN = r$ and construct square, $CNOM$. Prolong CM indefinitely. Describe $\odot$, etc.



38. One leg and the altitude upon the hypotenuse.

39. An acute angle and the sum of the legs.

Given: $AD =$ sum; $\angle K$. **Required**: (?).



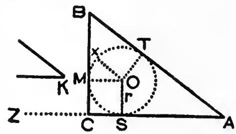
Construction: At A construct $\angle A = \angle K$; at D construct $\angle D = 45°$, the sides of these $\angle s$ intersecting at B. Draw $BC \perp$ to AD; etc.

40. The hypotenuse and the sum of the legs.

[Use A as center, hypotenuse as radius, etc.]

41. The radius of the circumscribed circle and an acute angle.

42. The radius of the inscribed circle and an acute angle.



Construction: Take CS on indefinite line $ZA = r$. On CS construct square $CSOM$. At O construct $\angle MOX = \angle K$. Draw radius OT $\perp$ to OX. Draw tangent at T. **Proof**: $\triangle ABC$ is *a* rt. $\triangle$ and it is *the* rt. $\triangle$. (Explain.)

III. To construct an **equilateral triangle**, having given:

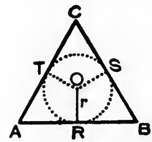
43. One side.

44. The altitude.

45. The perimeter.

46. A median.

47. The radius of the inscribed circle.



[Draw circle and radius; at center construct $\angle ROS = 120°$ and $\angle ROT = 120°$; etc.]

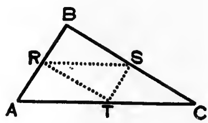
48. The radius of the circumscribed circle.

IV. To construct a **triangle**, having given:

49. The base, an angle adjoining it, the altitude upon it.

50. The midpoints of the three sides.

[Draw RS, RT, ST, etc.]



51. One side, altitude upon it, and the radius of the circumscribed circle.

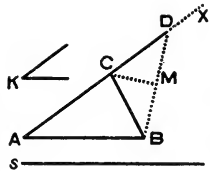
Construction: Draw $\odot$ with given radius and any center. Take chord $=$ given side; etc.

52. One side, an adjoining angle, and the radius of the circumscribed circle.

53. Two sides and the altitude from the same vertex.

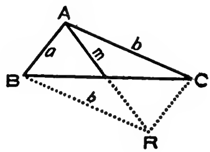
Construction: Erect $\perp =$ altitude, upon an indefinite line. Use the end of this altitude as center and the given sides as radii; etc.

54. One side, an angle adjoining it, and the sum of the other two sides.



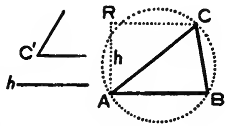
Construction: At A construct $\angle BAX =$ given $\angle K$. On AX take $AD = s$; draw DB; bisect DB at M, etc.

55. Two sides and the median to the third side.



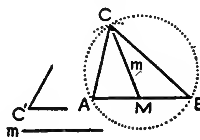
Given: a, b, m. **Construction**: Construct $\triangle ABR$ whose three sides, $AB = a$, $BR = b$, $AR = 2m$. Draw $AC \parallel$ to BR and $RC \parallel$ to AB meeting at C. Draw BC. **Statement**: (?). **Proof**: (?).

56. A side, the altitude upon it, and the angle opposite it.



Given: Side $= AB$, alt. $= h$; opposite $\angle = \angle C'$.
Construction: Upon AB construct segment ACB which will contain $\measuredangle = \angle C'$ (by 276). At A erect $AR \perp$ to AB and $= h$; etc.

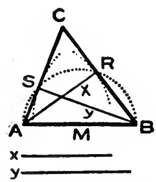
57. A side, the median to it, the angle opposite it.



[**Statement**: $\triangle ABC$ is the required $\triangle$.]

58. One side and the altitude from its extremities to the other sides.

Given: Side $= AB$, altitudes x and y.
Construction: Bisect AB; describe a semicircle. Using A as center and x as radius, describe arc cutting the semicircle at R; etc.



59. Two sides and the altitude upon one of them.
[**Given**: Sides $= AB$ and BC; alt. on $BC = x$.]

60. One side, an angle adjoining it, and the radius of the inscribed circle.

Construction: Describe $\odot$ with given radius, any center. Construct central $\angle =$ given $\angle$. Draw two tangents $\parallel$ to these radii.

V. To construct a **square**, having given:

61. One side.

62. The diagonal.

63. The perimeter.

64. The sum of a diagonal and a side.

VI. To construct a **rhombus**, having given:

65. One side and an angle adjoining it.

66. One side and the altitude.

67. The diagonals.

68. One side and one diagonal. [Use 278.]

69. An angle and the diagonal to the same vertex.

70. An angle and the diagonal between two other vertices.

71. One side and the radius of the inscribed circle.

VII. To construct a **rectangle**, having given:

72. Two adjoining sides.

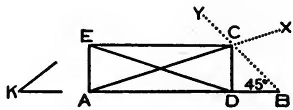
73. A diagonal and a side.

74. One side and the angle formed by the diagonals.

75. A diagonal and the sum of two adjoining sides. [See No. 40.]

76. A diagonal and the perimeter.

77. The perimeter and the angle formed by the diagonals.



Construction: Bisect the perimeter and take $AB =$ half it. Bisect $\angle K$. At A construct $\angle BAX =$ half $\angle K$. Etc.

VIII. To construct a **parallelogram**, having given:

78. One side and the diagonals. [Use 137 and 278.]

79. The diagonals and the angle between them.

80. One side, an angle, and the diagonal not to the same vertex.

81. One side, an angle, and the diagonal to the same vertex.

82. One side, an angle, and the altitude upon that side.

83. Two adjoining sides and the altitude.

IX. To construct an **isosceles trapezoid**, having given:

84. The bases and an angle adjoining the larger base.

85. The bases and an angle adjoining the less base.

86. The bases and the diagonal.

87. The bases and the altitude.

88. The bases and one of the equal sides.

89. One base, an angle adjoining it, and one of the equal sides.

90. One base, the altitude, and one of the equal sides.

91. One base, the radius of the circumscribed circle, and one of the equal sides. [First, describe a ⊙.]

92. One base, an angle adjoining it, and the radius of the circumscribed circle.

93. The bases and the radius of the circumscribed circle.

94. One base and the radius of the inscribed circle.
Construction: Bisect the base and erect a $\perp =$ radius; etc.

X. To construct a **trapezoid**,* having given:

95. The bases and the angles adjoining one of them.
Construction: Take $EC =$ longer base, and on it take $ED =$ less base. Construct $\triangle$ DBC (by 280).

96. The four sides.

97. A base, the altitude, and the non-parallel sides.
Construction: Construct a $\triangle$ two sides of which = the given non-$\parallel$ sides of the trapezoid, and the alt. from same vertex = given alt. (See No. 53.)

98. The bases, an angle, and the altitude.
Construction: Construct ▱ on ED, having given altitude and $\angle$.

99. A base, the angles adjoining it, and the altitude.

100. The longer base, an angle adjoining it, and the non-parallel sides.

101. The shorter base, an angle not adjoining it, and the non-parallel sides.

XI. To construct the **locus** of a point which will be:

102. At a given distance from a given point.

103. At a given distance from a given line.

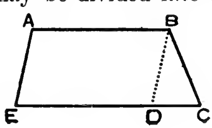
104. At a given distance from a given circumference:
(i) If the given radius is < the given distance;
(ii) If the given radius is > the given distance.

105. Equally distant from two given points.

106. Equally distant from two intersecting lines.

---

*NOTE. It is evident that every trapezoid may be divided into a parallelogram and a triangle by drawing one line (as BD) $\parallel$ to one of the non-$\parallel$ sides. Hence the construction of a trapezoid is often merely constructing a triangle and a parallelogram.



XII. To find (by intersecting loci) * the **point** P, which **will be:**

107. At two given distances from two given points.†

108. Equally distant from three given points.

109. In a given line and equally distant from two given points.

110. In a given line and equally distant from two given intersecting lines.

111. In a given circumference and equally distant from two given points.†

112. In a given circumference and equally distant from two intersecting lines.†

113. Equally distant from two given intersecting lines and equally distant from two given points.†

114. At a given distance from a given line and equally distant from two given points.†

115. At a given distance from a given line and equally distant from two other intersecting lines.†

116. Equally distant from two given points and at a given distance from one of them.†

117. Equally distant from two given intersecting lines and at a given distance from one of them.†

118. At a given distance from a point and equally distant from two other points.†

119. At given distances from two given intersecting lines.†

120. At given distances from a given line and from a given circumference.†

121. At given distances from a given line and from a given point.†

122. Equally distant from two parallels and equally distant from two intersecting lines.†

123. At a given distance from a given point and equally distant from two given parallels.†

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* It is well to draw the loci concerned as dotted lines. (See No. 124.)

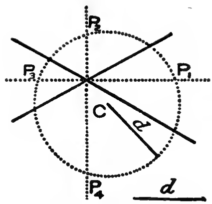
† In the **Discussion**, include the answers to questions like these:

(1) Is this ever impossible? (*i.e.* must there always be such a point?)

(2) Are there ever two such points? When?

(3) Are there ever more than two? When?

(4) Is there ever only one? When? Etc.



124. At a given distance from a given point and equally distant from two given intersecting lines.

Can C be so taken that there will be no point?

Can C be so taken that there will be only one point? Only two? Only three? More than four?

XIII. To find (by intersecting loci) the **center** of a circle which will:

125. Pass through three given points.*

126. Pass through a given point and touch a given line at a given point.*

127. Have a given radius and be tangent to a given line at a given point.*

128. Have a given radius, touch a given line, and pass through a given point.*

129. Pass through a given point and touch two given parallel lines.*

130. Touch two given parallels, one of them at a given point.*

131. Have a given radius and touch two given intersecting lines.*

132. Have a given radius and pass through two given points.*

133. Touch three given indefinite lines, no two of them being parallel.†

134. Touch three given lines, only two of them being parallel.

XIV. To construct a **circle** which will:

135. Pass through a given point and touch a given line at a given point.

136. Touch two given parallel lines, one of them at a given point.

137. Pass through a given point and touch two given parallels.

138. Have a given radius, touch a given line, and pass through a given point.

139. Have its center in one line, touch another line, and have a given radius.

---

* **Discussion:** Is this ever impossible? Are there ever two circles and hence two centers? Are there ever more than two? Etc.

† Four solutions. One is in 274.

140. Have a given radius and touch two given intersecting lines.

141. Have a given radius and pass through two given points.

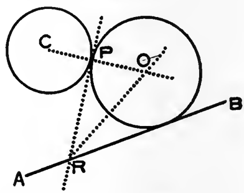
142. Have a given radius and touch a given circumference at a given point. [Draw tangent to the given ⊙ at the given point.]

143. Have a given radius and touch two given circumferences.

144. Touch three indefinite intersecting lines.*

145. Touch two given intersecting lines, one of them at a given point.

146. Touch a given line and a given circumference at a given point.



Given: Line AB; ⊙ C; point P.

Construction: Draw radius CP. Draw tangent at P meeting AB at R. Bisect $\angle PRB$, meeting CP produced at O; etc.

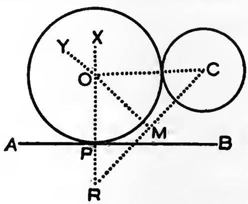
147. Be inscribed in a given sector.

Construction: Produce the radii to meet the tangent at the midpoint of the arc. In this △ inscribe a ⊙.

148. Have a given radius and touch two given intersecting circles.

149. Have a given radius, touch a given line, and a given circumference.

150. Touch a given line at a given point and touch a given circumference.



Given: Line AB; point P; ⊙ C.

Construction: At P erect $PX \perp$ to AB, and extend it below AB, so $PR =$ radius of ⊙ C.

Draw CR and bisect it at M.

Erect $MY \perp$ to CR at M, meeting PX at O; etc.

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151. What is the locus of the vertices of all right triangles having the same hypotenuse?

152. Through a given point on a given circumference to draw two equal chords perpendicular to each other.

153. To draw a line of given length through a given point and terminating in two given parallels.

Construction: Use any point of one of the ‖s as center and the given length as radius to describe an arc meeting the other ‖. Join these two points. Through the given point draw a line ‖, etc.

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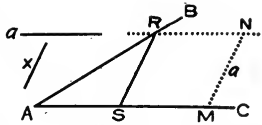
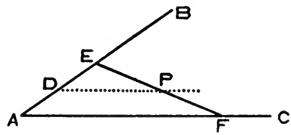
* Four solutions. One is in 274.

154. To draw a line, terminating in the sides of an angle, which shall be equal to one line and parallel to another.



Statement: RS is $= a$ and $\parallel$ to x.

155. To draw a line through a given point within an angle, which shall be terminated by the sides of the angle and bisected by the point.

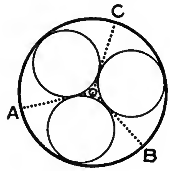


Construction: Through P draw $PD \parallel$ to AC. Take on AB, $DE = AD$. Draw EPF; etc.

156. To circumscribe a circle about a rectangle.

157. To construct three circles having the vertices of a given triangle as centers so that each touches the other two.

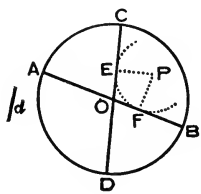
Construction: Inscribe a ⊙ in the △; etc.



158. To construct within a circle three equal circles each of which will touch the given circle and the other two.

Construction: Draw a radius, OA, and construct $\angle AOB = 120°$ and $\angle AOC = 120°$. In these sectors inscribe, etc.

159. Through a point without a circle to draw a secant having a given distance from the center.

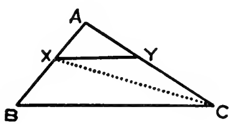


160. To draw a diameter to a circle at a given distance from a given point.

161. Through two given points within a circle to draw two equal and parallel chords.

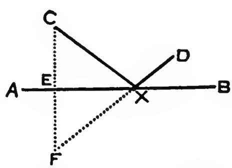
Construction: Bisect the line joining the given points and draw a diameter, etc.

162. To draw a parallel to side BC of triangle ABC, meeting AB in X and AC in Y, such that $XY = YC$.



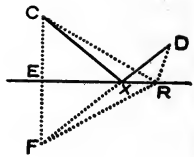
163. Find the locus of the points of contact of the tangents drawn to a series of concentric circles from an external point.

164. Given: Line AB and points C and D on the same side of it; find point X in AB such that $\angle AXC = \angle BXD$.



Construction: Draw $CE \perp$ to AB and produce to F so that $EF = CE$. Draw FD meeting AB in X. Draw CX.

165. To draw from one given point to another the shortest path which shall have one point in common with a given line.

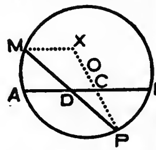


Statement: $CX + XD$ is $< CR + RD$.

166. To draw a line parallel to side BC of triangle ABC meeting AB at X and AC at Y, so that $XY = BX + YC$.

Construction: Draw bisectors of ∠s B and C, meeting at O, etc.

167. To draw in a circle, through a given point of an arc, a chord which will be bisected by the chord of the arc.



Construction: Draw radius OP meeting chord at C. Prolong PO to X so $CX = CP$. Draw $XM \parallel$ to AB meeting ⊙ at M. Draw PM cutting AB at D; etc. Is there any other chord from P bisected by AB?

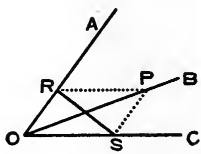
168. To inscribe in a given circle a triangle whose angles are given.

Construction: At the center construct three ∠s, doubles of the given ∠s.

169. To circumscribe about a given circle a triangle whose angles are given.

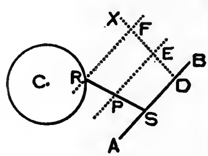
Construction: Inscribe △ (like No. 168) first, and draw tangents ∥ to the sides.

170. Three lines meet in a point; it is required to draw a line terminating in the outer two and bisected by the inner one.



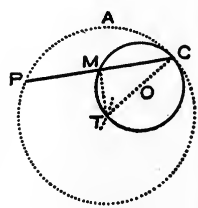
Construction: Through any point P, of OB, draw ∥s to the outer lines. Draw diagonal RS; etc.

171. To draw through a given point, P, a line which will be terminated by a given circumference and a given line and be bisected by P.



Construction: Draw any line DX meeting AB at D. Draw $PE \parallel$ to AB meeting DX at E. Take $EF = ED$; etc.

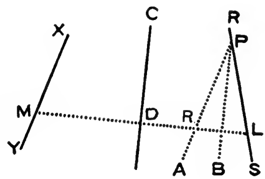
172. Through a given point without a circle to draw a secant to the circle which shall be bisected by the circumference.



Construction: Draw arc at T, using P as center and diam. of ⊙ O as radius. Using T as center and same radius as before, describe circumference touching ⊙ O at C and passing through P. Draw PC meeting ⊙ O at M.

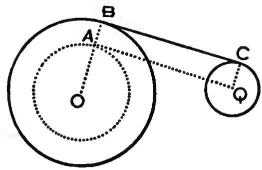
173. To inscribe a square in a given rhombus.
[Bisect the four $\triangle$s formed by the diagonals.]

174. To bisect the angle formed by two lines without producing them to their point of intersection.



Construction: At P, any point in RS, draw $PA \parallel$ to XY; bisect $\angle APS$ by PB. At any point in PB erect $ML \perp$ to PB, meeting the given lines in M and L. Bisect ML at D and erect $DC \perp$ to ML, etc.

175. To construct a common external tangent to two circles.

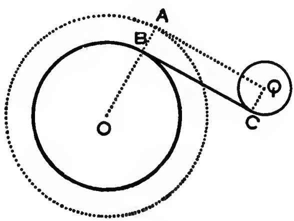


Construction: Using O as a center and a radius = difference of the given radii, construct (dotted) circle. Draw QA tangent to this ⊙ from Q; draw radius OA and produce it to meet given ⊙ at B. Draw radius $QC \parallel$ to OB. Join BC.

Statement: BC is tangent to both Ⓢ.

Proof: $AB = CQ$ (Const.). AB is $\parallel$ to CQ (?).
$\therefore ABCQ$ is a ▱ (?).
But $\angle OAQ$ is a rt. $\angle$ (?); etc.

176. To construct a common internal tangent to two circles.



Construction: Using O as a center and a radius = the sum of the given radii, construct (dotted) circle. Draw QA tangent to this ⊙ from Q; draw radius OA meeting given ⊙ at B, etc., as above.

BOOK III

PROPORTION. SIMILAR FIGURES

283. A **ratio** is the quotient of one quantity divided by another,—both being of the same kind.

284. A **proportion** is the statement that two ratios are equal.

285. The **extremes** of a proportion are the first and last terms.

The **means** of a proportion are the second and third terms.

286. The **antecedents** are the first and third terms.

The **consequents** are the second and fourth terms.

287. A **mean proportional** is the second or third term of a proportion in which the means are identical.

A **third proportional** is the last term of a proportion in which the means are identical.

A **fourth proportional** is the last term of a proportion in which the means are not identical.

288. A **series of equal ratios** is the equality of more than two ratios.

A **continued proportion** is a series of equal ratios in which the consequent of any ratio is the antecedent of the next following ratio.

289. Explanatory. A ratio is written as a fraction or as an indicated division; $\frac{a}{b}$, or $a \div b$, or $a:b$. A proportion is usually written $\frac{a}{b}=\frac{x}{y}$, or $a : b = x : y$, and is read: "a is to b as x is to y." In this proportion the extremes are a and y; the means are b and x; the antecedents are a and x; the consequents are b and y; and y is a fourth proportional to a, b, x. In the proportion $a : m = m : z$, the mean proportional is m, and the third proportional is z.

THEOREMS AND DEMONSTRATIONS

290. Theorem. In a proportion the product of the extremes is equal to the product of the means.

Given: $\frac{a}{b}=\frac{x}{y}$ or $a:b=x:y$. **To Prove:** $ay=bx$.

Proof: $\frac{a}{b}=\frac{x}{y}$ (Hyp.). Multiply by the common denominator, by and obtain, $ay=bx$ (Ax. 3). Q.E.D.

291. Theorem. If the product of two quantities is equal to the product of two others, one pair may be made the extremes of a proportion and the other pair the means.

Given: $ay=bx$.

To Prove: These eight proportions:

1. $a:b=x:y$,	5. $x:y=a:b$,
2. $a:x=b:y$,	6. $x:a=y:b$,
3. $b:a=y:x$,	7. $y:x=b:a$,
4. $b:y=a:x$,	8. $y:b=x:a$.

Proof: 1. $ay=bx$ (Hyp.). Divide each member by by, and obtain $\frac{ay}{by}=\frac{bx}{by}$ (Ax. 3). $\therefore \frac{a}{b}=\frac{x}{y}$, or $a:b=x:y$. Q.E.D.

2. Divide by xy; etc. 3. $bx=ay$ (Hyp.). Divide by ax; etc.

Numerical Illustration. Suppose in this paragraph $a=4, b=14$, $x=6$, $y=21$; the truth of the above proportions can be clearly seen by writing these equivalents. $4\times 21=14\times 6$ (True).
1. $4:14=6:21$ (True); 2. $4:6=14:21$ (True); etc.
They will all be recognized as true proportions.

292. Theorem. In any proportion the terms are also in proportion by alternation (that is, the first term is to the third as the second is to the fourth).

Given: $a:b=x:y$. **To Prove:** $a:x=b:y$.

Proof: $a:b=x:y$ (Hyp.). $\therefore$ $ay=bx$ (290).
Hence, $a:x=b:y$ (291). Q.E.D.

293. THEOREM. **In any proportion the terms are also in proportion by inversion** (that is, the second term is to the first as the fourth term is to the third).

[The proof is similar to the proof of 292.]

294. THEOREM. **In any proportion the terms are also in proportion by composition** (that is, the sum of the first two terms is to the first, or second, as the sum of the last two terms is to the third, or fourth).

Given: $a : b = x : y$. **To Prove**: $\begin{cases} a+b : a = x+y : x, \text{ or} \\ a+b : b = x+y : y. \end{cases}$

Proof: $a : b = x : y$ (Hyp.). $\therefore ay = bx$ (?) (290).

Add ax to each, and obtain, $ax + ay = ax + bx$ (Ax. 2).

That is, $a(x + y) = x(a + b)$.

Hence, $a + b : a = x + y : x$ (?) (291).

Similarly, by adding by, $a + b : b = x + y : y$. Q.E.D.

295. THEOREM. **In any proportion the terms are also in proportion by division** (that is, the difference between the first two terms is to the first, or second, as the difference between the last two terms is to the third, or fourth).

Given: $a : b = x : y$. **To Prove**: $\begin{cases} a - b : a = x - y : x, \text{ or} \\ a - b : b = x - y : y. \end{cases}$

Proof: $a : b = x : y$ (Hyp.). $\therefore ay = bx$ (?) (290).

Subtracting each side from ax, $ax - ay = ax - bx$ (Ax. 2).

That is, $a(x - y) = x(a - b)$.

Hence, $a - b : a = x - y : x$ (?) (291).

Likewise, $a - b : b = x - y : y$. Q.E.D.

NOTE I. The proportions of 294 and 295 may be written in many different forms (292, 293). Thus, (1) $a \pm b : a = x \pm y : x$; (2) $a \pm b : b = x \pm y : y$; (3) $a \pm b : x \pm y = a : x$, etc.

NOTE II. In any proportion the sum of the antecedents is to the sum of the consequents as either antecedent is to its consequent. (Explain.) Also, in any proportion the difference of the antecedents is to the difference of the consequents as either antecedent is to its consequent. (Explain.)

Thus: $a + x : b + y = a : b = x : y$.

Also, $a - x : b - y = a : b = x : y$.

296. **Theorem.** **In any proportion the terms are also in proportion by composition and division** (that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference).

Given: $a : b = x : y$. **To Prove:** $\frac{a+b}{a-b} = \frac{x+y}{x-y}$.

Proof: $\frac{a+b}{a} = \frac{x+y}{x}$ (?) (294).

$\frac{a-b}{a} = \frac{x-y}{x}$ (?) (295).

Divide the first by the second, $\frac{a+b}{a-b} = \frac{x+y}{x-y}$ (?). Q.E.D.

297. **Theorem.** **In any proportion, like powers of the terms are also in proportion, and like roots of the terms are in proportion.**

Given: $a : b = x : y$.

To Prove: $a^n : b^n = x^n : y^n$; and $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{x} : \sqrt[n]{y}$.

Proof: [Write the given proportion in fractional form, etc.]

298. **Theorem.** **In two or more proportions the products of the corresponding terms are also in proportion.**

Given: $a : b = x : y$, and $c : d = l : m$, and $e : f = r : s$.

To Prove: $ace : bdf = xlr : yms$.

Proof: [Write in fractional form and multiply.]

299. **Theorem.** **A mean proportional is equal to the square root of the product of the extremes.**

Given: $a : x = x : b$. **To Prove:** $x = \sqrt{ab}$.

Proof: [Use 290; etc.]

300. **Theorem.** **If three terms of one proportion are equal to the corresponding three terms of another proportion, each to each, the remaining terms are also equal.**

Given: $\left\{\begin{matrix} a : b = c : m, \text{ and} \\ a : b = c : r. \end{matrix}\right\}$. **To Prove:** $m = r$.

Proof: $am = bc$ and $ar = bc$ (?) (290).

$\therefore am = ar$ (Ax. 1). Hence, $m = r$ (Ax. 3). Q.E.D.

301. **Theorem.** **In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.**

Given: $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$

To Prove: $\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=$ etc.

Proof: Set each given ratio $= m$; thus,

$$\frac{a}{b}=m;\ \frac{c}{d}=m;\ \frac{e}{f}=m;\ \frac{g}{h}=m.$$

$$\therefore\ a=bm,\ c=dm,\ e=fm,\ g=hm \text{ (Ax. 3).}$$

Hence, $\frac{a+c+e+g}{b+d+f+h}=\frac{bm+dm+fm+hm}{b+d+f+h}$ (Substitution)

$$=\frac{m(b+d+f+h)}{b+d+f+h} \text{ (Factoring)}$$

$$=m \qquad \text{(Canceling).}$$

$$\therefore \frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h} \text{ (Ax. 1).}$$

Q.E.D.

EXERCISES

1. If $3:4=6:x$, find x.

2. If $8:12=12:x$, find x.

3. Find a fourth proportional to 6, 7, and 15.

4. Find a third proportional to 4 and 10.

5. If $11:15=x:25$, find x.

6. If $4:x=x:25$, find x.

7. Find a mean proportional between 8 and 18.

8. If $7:x=35:48$, find x.

9. Given, that $5:8=15:24$, write seven other true proportions containing these same four numbers.

10. If $5\times 6=2\times 15$, write eight proportions with these numbers.

11. If $7:12=21:36$, write the proportion resulting by alternation; inversion; composition; division; composition and division.

12. If $6:25=18:75$, write the proportions required in No. 11.

13. If $x+y:x-y=17:7$, write the proportions that result by virtue of composition; division; composition and division.

14. Apply 301 to the ratios, $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}$.

NOTE. We have seen that it is possible to add two lines and subtract one line from another. Now it is essential that we clearly understand the significance implied by indicating the multiplication or the division of one line by another.

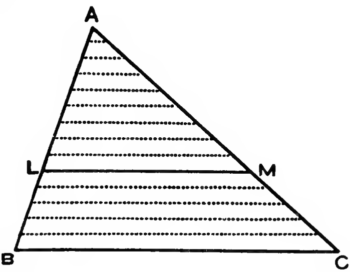
What is actually done is to multiply or divide the numerical measure of one line by the numerical measure of another. Thus if one line is 8 inches long and another is 18 inches long, we say that the ratio of the first line to the second is $\frac{8}{18}$ or $\frac{4}{9}$, meaning that the less line is four ninths of the larger.

Also, in referring to the product of two lines we merely understand that the product of their numerical measures is intended.

If a line is multiplied by itself, we obtain the square of the numerical measure of the line. The square of the line AB is written $\overline{AB}^2$ or $(AB)^2$ and the quantity that is squared is the numerical value of the length of AB.

In the preceding paragraphs of Book III, we have been considering numerical magnitudes. It should be distinctly understood that in the following geometrical propositions and demonstrations, the foregoing interpretation is implied in multiplication and division involving lines.

302. THEOREM. A line parallel to one side of a triangle divides the other sides into proportional segments.



Given: $\triangle ABC$ and line LM $\parallel$ to BC.

To Prove:
$AL : LB = AM : MC$.

Proof: I. If the parts AL and LB are **commensurable.**

There exists a common unit of measure of AL and LB (238).

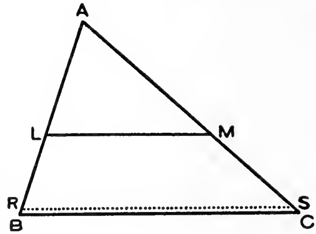
Suppose this is contained 9 times in AL and 5 times in LB.

Then, $\frac{AL}{LB} = \frac{9}{5}$ (Ax. 3).

Draw lines through the several points of division $\parallel$ to BC.

These will divide AM into 9 parts and MC into 5 parts. All of these 14 parts are equal (?) (147).

Hence, $\frac{AM}{MC} = \frac{9}{5}$ (Ax. 3). $\therefore \frac{AL}{LB} = \frac{AM}{MC}$ (Ax. 1). Q.E.D.



II. If the parts AL and LB are **incommensurable.**

There does not exist a common unit (238). Divide AL into several equal parts (by 271). Apply one of these as a unit of measure to LB. There will be a remainder, RB, left over (238).

Draw $RS \parallel$ to BC.

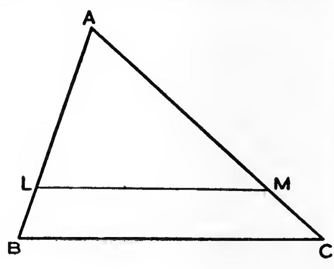
Now $\frac{AL}{LR} = \frac{AM}{MS}$ (The commensurable case).

Indefinitely increase the *number* of equal parts of AL. That is, indefinitely decrease each part, the unit or divisor. Hence, the remainder, RB, will be indefinitely decreased. (Because the remainder is < the divisor.)

That is, RB will approach zero as a limit,
and SC will approach zero as a limit.
$\therefore$ LR will approach LB as a limit (240),
and MS will approach MC as a limit (240).
$\therefore$ $\frac{AL}{LR}$ will approach $\frac{AL}{LB}$ as a limit (243),
and $\frac{AM}{MS}$ will approach $\frac{AM}{MC}$ as a limit (243).

Consequently, $\frac{AL}{LB} = \frac{AM}{MC}$ (?) (242). Q.E.D.

303. THEOREM. **If a line parallel to one side of a triangle intersects the other sides, these sides and their corresponding segments are proportional.**



Given: $\triangle ABC$; $LM \parallel$ to BC.

To Prove:

I. $AB : AC = AL : AM$.

II. $AB : AC = LB : MC$.

Proof:

$AL : LB = AM : MC$ (?) (302).

$\therefore$ I. $AL + LB : AL = AM + MC : AM$ (?) (294),

and II. $AL + LB : LB = AM + MC : MC$ (?).

But $AL + LB = AB$ and $AM + MC = AC$ (Ax. 4).

Therefore, I. $AB : AL = AC : AM$ (Ax. 6).

Hence, $AB : AC = AL : AM$ (?) (292).

II. $AB : LB = AC : MC$ (Ax. 6).

Hence, $AB : AC = LB : MC$ (?) (292). Q.E.D.

NOTE. Each of these proportions may be written eight ways (291). And they may be combined, thus, $\frac{AB}{AC} = \frac{AL}{AM} = \frac{LB}{MC}$ (Ax. 1).

304. Two lines are **divided proportionally**, if the ratio of the lines is equal to the ratios of corresponding segments.

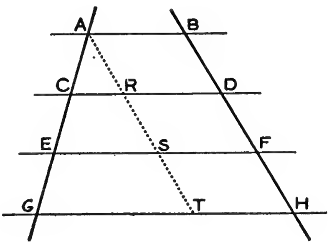
305. THEOREM. **If a line parallel to one side of a triangle intersects the other sides, it divides these sides proportionally.**

(Because the ratio of the sides = the ratio of corresponding segments (303). This theorem is the same as 303.)

306. THEOREM. **Three or more parallels intercept proportional segments on two transversals.**

Given: (?).

To Prove: $AC : BD = CE : DF = EG : FH$.



Proof: Draw from A, $AT \parallel$ to BH intersecting the $\parallel$s, etc.

In $\triangle AES$,

$$\frac{AE}{AS} = \frac{AC}{AR} = \frac{CE}{RS} \text{ (?) (305).}$$

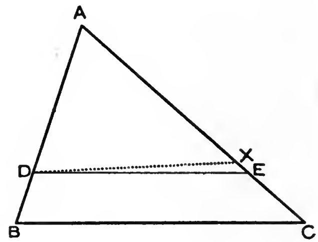
In $\triangle AGT$, $\frac{AE}{AS} = \frac{EG}{ST}$ (?) (305).

$$\therefore \frac{AC}{AR} = \frac{CE}{RS} = \frac{EG}{ST} \text{ (Ax. 1).}$$

But, $AR = BD$, $RS = DF$, $ST = FH$ (?) (130).

Hence, $AC : BD = CE : DF = EG : FH$ (Ax. 6). Q.E.D.

307. **THEOREM. If a line divides two sides of a triangle proportionally, it is parallel to the third side.**



Given: $\triangle ABC$; line DE; the proportion $AB : AC = AD : AE$.

To Prove: DE is $\parallel$ to BC.

Proof: Through D draw $DX \parallel$ to BC meeting AC at X.

$\therefore AB : AC = AD : AX$ (?) (305).

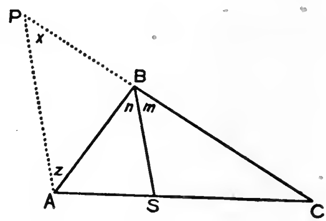
But $AB : AC = AD : AE$ (Hyp.).

$\therefore AX = AE$ (?) (300).

$\therefore DX$ and DE coincide (?) (39). That is, DE is $\parallel$ to BC.

Q.E.D.

308. **THEOREM. The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the other two sides.**



Given: $\triangle ABC$; BS the bisector of $\angle ABC$.

To Prove: $AS : SC = AB : BC$.

Proof: Through A draw $AP \parallel$ to BS, meeting BC, produced, at P.

Now, $\angle m = \angle x$ (?) (98) and $\angle n = \angle z$ (?).

But $\angle m = \angle n$ (Hyp.).

$\therefore \angle x = \angle z$ (Ax. 1). Hence $AB = BP$ (?) (120).

In $\triangle CAP$, BS is $\parallel$ to AP (Const.).

$\therefore AS : SC = BP : BC$ (?) (302).

$\therefore AS : SC = AB : BC$ (Ax. 6). Q.E.D.

Ex. If $AS = 3$, $AB = 4$, $BC = 9$, find SC.

Ex. If $AC = 20$, $AB = 9$, $BC = 21$, find AS and SC.

---

309. The **segments** of a line, made by one of its points, are the lines between this point and the extremities of the line.

Thus, in 308, the point S divides AC internally, into segments SA and SC.

But, if the point S' be in the prolongation of the given line, the segments are still $S'A$ and $S'C$, according to the definition, and the point S' divides AC, externally, into segments $S'A$ and $S'C$.

310. **Theorem. The bisector of an exterior angle of a triangle divides the opposite side** (externally) **into segments which are proportional to the other two sides.**

Given: $\triangle ABC$; BS', the bisector of exterior $\angle ABD$, meeting AC (externally) at S'. **To Prove:** $AS' : S'C = AB : BC$.

Proof: Through A draw $AP \parallel$ to BS' meeting BC at P.

Now, $\angle m = \angle x$ (?).

$\angle n = \angle z$ (?).

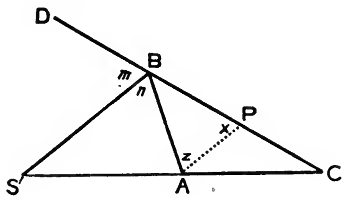
But $\angle m = \angle n$ (?).

$\therefore \angle x = \angle z$ (?).

Hence, $AB = BP$ (?). In $\triangle CBS'$, AP is $\parallel$ to BS' (?).

$\therefore AS' : S'C = BP : BC$ (?) (305).

$\therefore AS' : S'C = AB : BC$ (?). Q.E.D.



Ex. If $AS' = 10$, $AB = 7$, $BC = 16$, find $S'C$ and AC.

Ex. If $AC = 14$, $AB = 12$, $BC = 19$, find AS' and $S'C$.

311. A line is **divided harmonically** if it is divided internally and externally in the same ratio.

In 308, the line AC is divided internally by S, in the ratio $AB : BC$.

In 310, the line AC is divided externally by S' in the ratio $AB : BC$.

Theorem. The bisectors of the interior and exterior angles of a triangle (at a vertex) **divide the opposite side harmonically.**

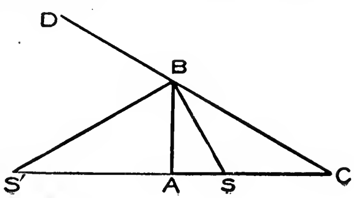
Given: $\triangle ABC$; BS bisecting $\angle ABC$; and BS' bisecting $\angle ABD$.

To Prove: $AS : SC = AS' : S'C$.

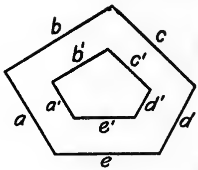
Proof: $\frac{AS}{SC} = \frac{AB}{BC}$ (?) (308).

$\frac{AS'}{S'C} = \frac{AB}{BC}$ (?) (310).

$\therefore \frac{AS}{SC} = \frac{AS'}{S'C}$ (?). Q.E.D.



312. **Similar polygons** are polygons that are **mutually** equiangular and whose homologous sides are proportional. That is, every pair of homologous angles are equal; and the ratio of one pair of homologous sides is equal to the ratio of every other pair of homologous sides, $a : a' = b : b' = c : c' = d : d' =$ etc.

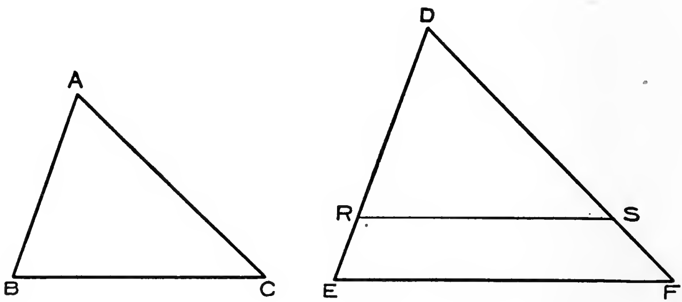


Triangles are **similar** if they are mutually equiangular and their homologous sides are proportional.

313. THEOREM. **Two triangles are similar if they are mutually equiangular.**

Given: $\triangle ABC, DEF$; $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

To Prove: The $\triangle$ are similar (that is, that their sides are proportional).



Proof: Place $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ coincides with its equal, $\angle D$, and $\triangle ABC$ takes the position of $\triangle DRS$.

Then $\angle DRS = \angle E$ (Hyp.). $\therefore RS$ is $\parallel$ to EF (?) (102).

Hence, $DE : DR = DF : DS$ (?) (305).

That is, $DE : AB = DF : AC$ (Ax. 6).

Likewise, by placing $\angle B$ upon $\angle E$, we may prove that, $DE : AB = EF : BC$.

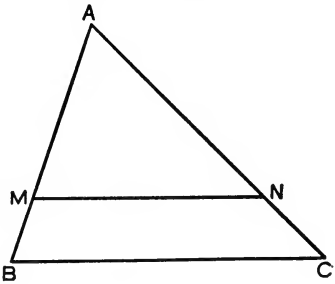
$\therefore DE : AB = DF : AC = EF : BC$ (Ax. 1).

Therefore, the $\triangle$ are similar (?) (312). Q.E.D.

314. Theorem. Two triangles are similar if two angles of one are equal to two angles of the other. [See 117 and 313.]

315. Theorem. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other. [See 314.]

316. Theorem. If a line parallel to one side of a triangle intersects the other sides, the triangle formed is similar to the original triangle.



Given: $MN \parallel$ to BC in $\triangle ABC$.

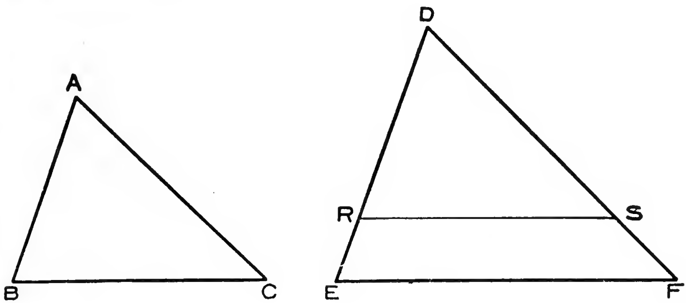
To Prove: $\triangle\ AMN$ similar to $\triangle\ ABC$.

Proof: $\angle A$ is common to both; $\angle AMN = \angle B$; $\angle ANM = \angle C$ (?) (98).

$\therefore$ ⟁ are similar (?) (313).

Q.E.D.

317. Theorem. If two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar.



Given: ⟁ ABC and DEF; $\angle A = \angle D$; $DE : AB = DF : AC$.

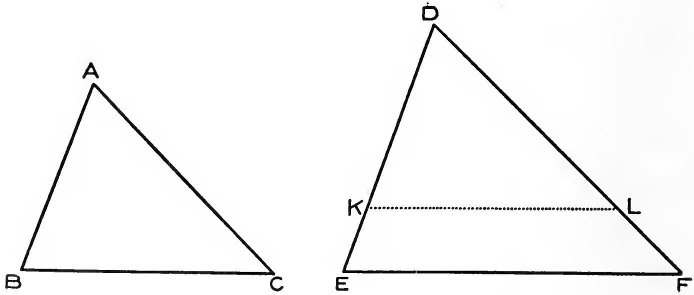
To Prove: The ⟁ similar.

Proof: Superpose $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ coincides with its equal, $\angle D$, and $\triangle ABC$ takes the position of $\triangle DRS$.

Then $DE : DR = DF : DS$ (Hyp.). $\therefore RS$ is $\parallel$ to EF (?) (307).

$\therefore \triangle\ DRS$ is similar to $\triangle\ DEF$ (?) (316). Q.E.D.

318. Theorem. If two triangles have their homologous sides proportional, they are similar.



Given: ⊿ ABC and DEF, and $DE : AB = DF : AC = EF : BC$.

To Prove: $\triangle ABC$ similar to $\triangle DEF$.

Proof: On DE take $DK = AB$; and on DF take $DL = AC$. Draw KL.

Now, $DE : AB = DF : AC$ (Hyp.).

$\therefore DE : DK = DF : DL$ (Ax. 6). $\therefore KL$ is $\parallel$ to EF (?) (307).

Therefore, $\triangle DKL$ is similar to $\triangle DEF$ (?) (316).

[Now $\triangle ABC$ is to be proven equal to $\triangle DKL$.]

$DE : DK = EF : KL$. (Definition of similar triangles, 312.)

That is, $DE : AB = EF : KL$ (Ax. 6).

But, $DE : AB = EF : BC$ (Hyp.). $\therefore BC = KL$ (300).

Hence, $\triangle ABC = \triangle DKL$ (?) (58).

But $\triangle DKL$ has been proved similar to $\triangle DEF$.

Therefore, $\triangle ABC$ is similar to $\triangle DEF$ (Ax. 6). Q.E.D.

Ex. 1. Are all equilateral triangles similar? Why?

Ex. 2. Are all squares similar? Why?

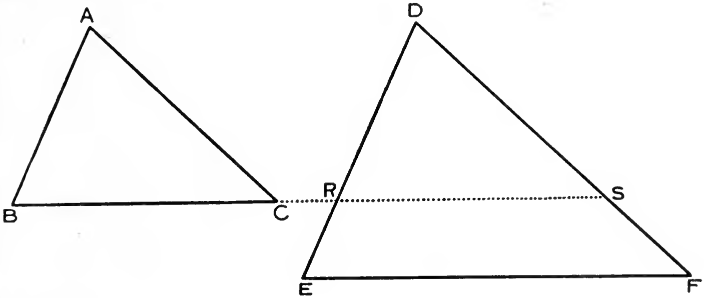
Ex. 3. Are all rectangles similar? Why?

Ex. 4. The sides of a triangle are 7, 8, and 12, and the longest side in a similar triangle is 30. Find the other sides.

Ex. 5. In the figure of 311, if $AB = 10$, $AC = 14$, $BC = 18$, find the four segments of AC made by S and S'.

Ex. 6. Prove the theorems of 142 and 143 by proportion.

319 THEOREM. If two triangles have their homologous sides parallel, they are similar.



Given: ⩓ ABC and DEF; $AB \parallel$ to DE; $AC \parallel$ to DF; and $BC \parallel$ to EF.

To Prove: $\triangle\ ABC$ similar to $\triangle\ DEF$.

Proof: Produce BC of $\triangle\ ABC$ until it intersects two sides of $\triangle\ DEF$ at R and S.

Now $\angle B = \angle DRS$, and $\angle DRS = \angle E$ (?) (98).

$\therefore \angle B = \angle E$ (?).

Likewise, $\angle ACB = \angle DSR$, and $\angle DSR = \angle F$ (?).

$\therefore \angle ACB = \angle F$ (?).

Therefore, $\triangle\ ABC$ is similar to $\triangle\ DEF$ (?) (314). Q.E.D.

320. THEOREM. If two triangles are similar to the same triangle, they are similar to each other.

Proof: The three angles of each of the first two triangles are respectively equal to the three angles of the third (312).

Hence, the first two ⩓ are mutually equiangular (Ax. 1).

Therefore they are similar (?) (313). Q.E.D.

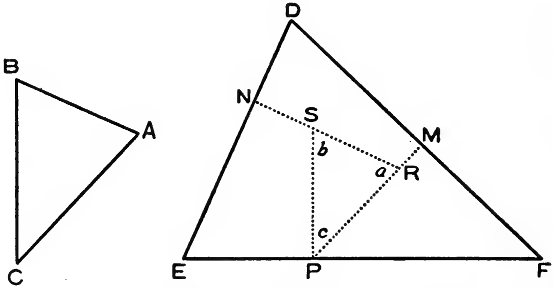
Ex. 1. Let the pupil prove the theorem of 319 if one triangle entirely surrounds the other.

Ex. 2. If one side of one triangle intersects two sides of the other.

Ex. 3. If they are so placed that no side of either, when prolonged, intersects any side of the other without being prolonged.

[Prolong any side of one and the sides not $\parallel$ to it in the other.]

321. Theorem. If two triangles have their homologous sides perpendicular, they are similar.



Given: $\triangle$ *ABC* and *DEF*; *AB* $\perp$ to *DE*; *AC* $\perp$ to *DF*; *BC* $\perp$ to *EF*.

To Prove: $\triangle$ *ABC* similar to $\triangle$ *DEF*.

Proof: Through *P*, any point in *EF*, construct *PR* $\parallel$ to *AC*, meeting *DF* at *M*. At *R*, any point in *PM*, draw *RS* $\parallel$ to *AB*, meeting *ED* at *N*. Draw *PS* $\parallel$ to *BC*, meeting *NR* at *S*, forming the $\triangle$ *PRS*. *PM* is $\perp$ to *DF* and *RN* is $\perp$ to *DE* (?). In quadrilateral *DMRN*, $\angle D + \angle M + \angle MRN + \angle N = 4$ rt. $\angle$s (?) (165).

But, $\angle M + \angle N = 2$ rt. $\angle$s (Const.).

$\therefore \angle D + \angle MRN = 2$ rt. $\angle$s (Ax. 2).

But, $\angle a + \angle MRN = 2$ rt. $\angle$s (?) (46).

$\therefore \angle D = \angle a$ (?) (49).

Similarly, by quadrilateral *EPSN*, it may be proved that $\angle E = \angle b$.

$\therefore \triangle$ *DEF* is similar to $\triangle$ *PRS* (?) (314).

But $\triangle$ *ABC* is similar to $\triangle$ *PRS* (?) (319).

$\therefore \triangle$ *ABC* is similar to $\triangle$ *DEF* (?) (320). Q.E.D.

Ex. 1. In the figure of 321, prove that $\angle F = \angle c$ by using 48.

Ex. 2. Draw the figure for the theorem of 321 if *P* is taken on *EF* prolonged. Prove the theorem with this figure.

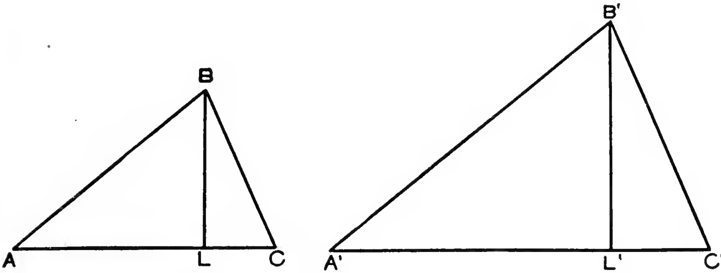
Ex. 3. Prove the same theorem if *P* is taken at a vertex.

Ex. 4. Prove, if *P* is taken within the triangle *DEF*.

322. Theorem. Two homologous altitudes of two similar triangles are proportional to any two homologous sides.

Given: (?).

To Prove: $\frac{BL}{B'L'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$.



Proof: $\triangle\ ABC$ is similar to $\triangle\ A'B'C'$ (?).

$\therefore \angle A = \angle A'$ (312). $\therefore \triangle\ ABL$ is similar to $\triangle\ A'B'L'$ (315).

$\therefore\ \frac{BL}{B'L'} = \frac{AB}{A'B'}$ (312). But, $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$ (312).

Hence, $\frac{BL}{B'L'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$ (Ax. 1). Q.E.D.

323. It is evident that: **In similar figures,**

1. **Homologous angles are equal.**
2. **Homologous sides are opposite equal angles** (in triangles).

Thus, shortest sides are homologous. [Opp. smallest $\measuredangle$s.]
Medium sides are homologous. [Opposite medium $\measuredangle$s.]
Longest sides are homologous. [Opposite largest $\measuredangle$s.]

3. **Homologous sides are proportional.**

The antecedents of this proportion belong to one of the similar figures and the consequents to the other.

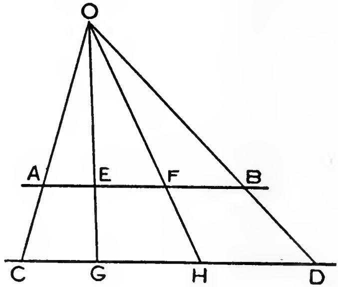
Ex. 1. Prove the theorem of 322 by use of triangles BLC and $B'L'C'$.

Ex. 2. State all the instances under which two triangles are similar.

Ex. 3. In the figure of 322, if $AB = 13$, $AC = 15$, $BL = 9$, $A'C' = 20$, find $A'B'$ and $B'L'$.

324. Theorem. If two parallel lines are cut by three or more transversals which meet at a point, the corresponding segments of the parallels are proportional.

Given: (?).



To Prove: $\frac{AE}{CG} = \frac{EF}{GH} = \frac{FB}{HD}$.

Proof: In $\triangle COG$, AE is $\parallel$ to CG (Hyp.). $\therefore \triangle OAE$ is similar to $\triangle OCG$ (?) (316).

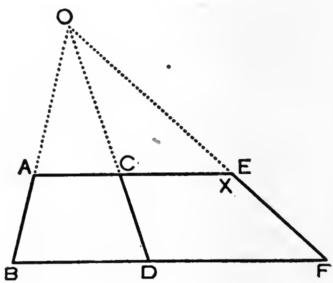
Likewise, $\triangle OEF$ is similar to $\triangle OGH$ and $\triangle OFB$ is similar to $\triangle OHD$ (316).

$\therefore \frac{AE}{CG} = \frac{OE}{OG}$ (?) (323, 3); also $\frac{EF}{GH} = \frac{OE}{OG}$ (?).

$\therefore \frac{AE}{CG} = \frac{EF}{GH}$ (?). Likewise, $\frac{EF}{GH} = \left(\frac{OF}{OH}\right) = \frac{FB}{HD}$ (?).

Therefore, $\frac{AE}{CG} = \frac{EF}{GH} = \frac{FB}{HD}$ (?). Q.E.D.

325. Theorem. If three or more non-parallel transversals intercept proportional segments on two parallels, they meet at a point.



Given: Transversals AB, CD, EF; parallels AE and BF; proportion, $AC : BD = CE : DF$.

To Prove: AB, CD, EF meet at a point.

Proof: Produce BA and CD until they meet, at O.

Draw OF cutting AE at X.

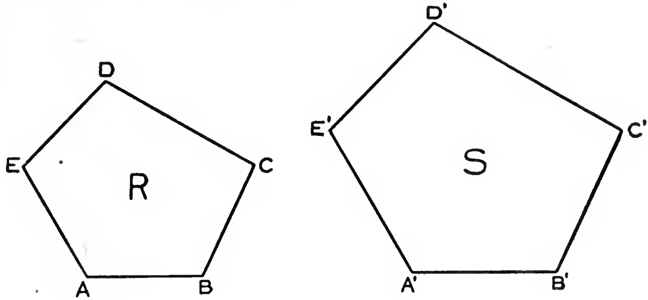
Now, $AC : BD = CX : DF$ (?) (324).

But $AC : BD = CE : DF$ (Hyp.). $\therefore CX = CE$ (?) (300).

Therefore, FE and FX coincide (?) (39).

That is, FE produced passes through O. Q.E.D.

326. Theorem. **The perimeters of two similar polygons are to each other as any two homologous sides.**



Given: Polygon R whose perimeter $= P$ and similar polygon S whose perimeter $= P'$.

To Prove: $P : P' = AB : A'B' = BC : B'C' =$ etc.

Proof: $AB : A'B' = BC : B'C' = CD : C'D' =$ etc. (323, 3).

$\therefore AB + BC + CD +$ etc. $: A'B' + B'C' + C'D' +$ etc. $= AB : A'B' = BC : B'C' =$ etc. (?) (301).

$\therefore P : P' = AB : A'B' = BC : B'C' =$ etc. (Ax. 6). Q.E.D.

Ex. 1. In the figure of 324, if $AE = EF = FB$, prove $CG = GH = HD$. State this truth in a theorem.

Ex. 2. The median of a triangle bisects every line that is parallel to the side to which the median is drawn and has its extremities in the other sides of the triangle.

Ex. 3. In the figure of 325, prove that the line bisecting AC and BD will pass through point O.

Ex. 4. Prove the theorem of 324 if the point O is between the parallels.

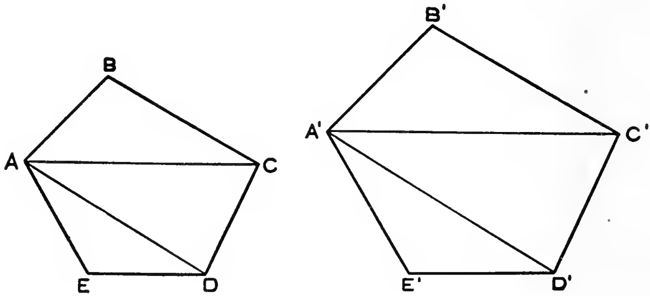
Ex. 5. If AB and CD are any two parallel lines whose midpoints are R and S respectively, prove that the lines AD, BC, RS meet in a point.

Ex. 6. Two homologous sides of two similar polygons are 8 and 15. The perimeter of the less polygon is 60. What is the perimeter of the larger?

Ex. 7. The perimeters of two similar polygons are 30 and 125 respectively. If the shortest side of the larger is $8\frac{1}{3}$, find the shortest side of the less.

Ex. 8. The sides of a polygon are 5, 6, 7, 8, 10 respectively. Find the perimeter of a similar polygon whose medium side is $17\frac{1}{2}$.

327. Theorem. If two polygons are similar, they may be decomposed into the same number of triangles similar each to each and similarly placed.



Given : Similar polygons BE and $B'E'$.

To Prove : $\triangle ABC$ similar to $\triangle A'B'C'$;
$\triangle ACD$ similar to $\triangle A'C'D'$;
$\triangle AED$ similar to $\triangle A'E'D'$.

Proof : First. $AB : A'B' = BC : B'C'$ (323, 3).
Also, $\angle B = \angle B'$ (323, 1).
Therefore, $\triangle ABC$ is similar to $\triangle A'B'C'$ (317).

Second. In $\triangle$s ABC and $A'B'C'$, $\dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$ (?) (323, 3).

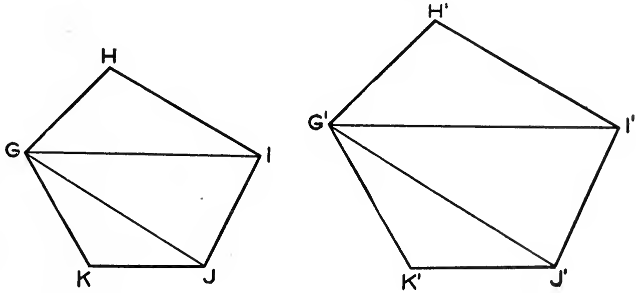
In the similar polygons, $\dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$ (?) (323, 3).

Consequently, $\dfrac{AC}{A'C'} = \dfrac{CD}{C'D'}$ (Ax. 1).

In the polygons, $\angle BCD = \angle B'C'D'$
In the $\triangle$s ABC and $A'B'C'$, $\angle BCA = \angle B'C'A'$ (323, 1).
Hence, by subtraction, $\angle ACD = \angle A'C'D'$ (Ax. 2).
Therefore, $\triangle ACD$ is similar to $\triangle A'C'D'$ (?) (317).

Third. $\triangle AED$ is proved similar to $\triangle A'E'D'$ in like manner. Q.E.D.

328. THEOREM. If two polygons are composed of triangles similar each to each and similarly placed, the polygons are similar.



Given: $\triangle\, GHI$ similar to $\triangle\, G'H'I'$;
$\triangle\, GIJ$ similar to $\triangle\, G'I'J'$;
$\triangle\, GJK$ similar to $\triangle\, G'J'K'$.

To Prove: The polygons HK and $H'K'$ similar.

Proof: First. In $\triangle\triangle\, HGI$ and $H'G'I'$, $\angle H = \angle H'$ (323, 1).
Also in these $\triangle\triangle$ $\angle HIG = \angle H'I'G'$ (?) (323, 1).
In $\triangle\triangle\, GIJ$ and $G'I'J'$, $\angle GIJ = \angle G'I'J'$ (?).
Adding, $\angle HIJ = \angle H'I'J'$ (Ax. 2).
Likewise, $\angle IJK = \angle I'J'K'$; etc.
That is, the polygons are mutually equiangular.

Second. In $\triangle\triangle\, GHI$ and $G'H'I'$, $\dfrac{GH}{G'H'} = \dfrac{HI}{H'I'} = \dfrac{GI}{G'I'}$ (323, 3).

In $\triangle\triangle\, GIJ$ and $G'I'J'$, $\dfrac{GI}{G'I'} = \dfrac{IJ}{I'J'}$ (?).

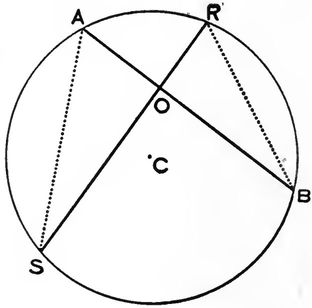
Hence, $\dfrac{GH}{G'H'} = \dfrac{HI}{H'I'} = \dfrac{IJ}{I'J'}$ (Ax. 1).

In the same way, we may prove $\dfrac{IJ}{I'J'} = \dfrac{JK}{J'K'} = \dfrac{KG}{K'G'}$.

$\therefore \dfrac{GH}{G'H'} = \dfrac{HI}{H'I'} = \dfrac{IJ}{I'J'} = \dfrac{JK}{J'K'} =$ etc. (Ax. 1).

That is, the homologous sides are proportional.
Therefore, the polygons are similar (?) (312). Q.E.D.

329. Theorem. If through a fixed point within a circle two chords be drawn, the product of the segments of one will equal the product of the segments of the other.



Given: Point O in circle C; chords AB and RS intersecting at O. (Review the note, p. 145.)

To Prove: $AO \cdot OB = RO \cdot OS$.

Proof: Draw AS and RB.

In $\triangle$ AOS and ROB, $\angle S = \angle B$ (?) (250).

And $\angle A = \angle R$ (?).

$\therefore$ these $\triangle$ are similar (?) (314).

Hence, $AO : RO = OS : OB$ (?) (323, 3).

$\therefore AO \cdot OB = RO \cdot OS$ (?) (290). Q.E.D.

330. Theorem. The product of the segments of any chord drawn through a fixed point within a circle is constant for all chords through this point. (See 329.)

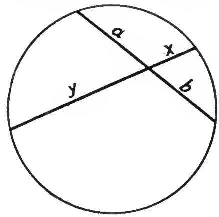
331. Direct proportion and reciprocal (or inverse) **proportion.**

Illustrations. I. If a man earns $\$4\frac{1}{2}$ each day, in 8 days he will earn \$36. In 12 days he will earn \$54. Hence, 8 da. : 12 da. = \$36 : \$54 is a proportion in which the antecedents belong to the same condition or circumstance, and the consequents belong to some other condition or circumstance. This is called a *direct proportion.*

II. If one man can build a certain wall in 120 days, 8 men can build it in 15 days; or 12 men in 10 days. Hence, 8 men : 12 men = 10 da. : 15 da. is a proportion in which the means belong to the same condition or circumstance, and the extremes belong to some other condition or circumstance. This is called a *reciprocal* (or inverse) *proportion.*

Definitions. A **direct proportion** is a proportion in which the antecedents belong to the same circumstance or figure, and the consequents belong to some other circumstance or figure. Thus the ordinary proportions derived from similar figures are direct proportions. (See 323, 3.)

A **reciprocal** (or **inverse**) **proportion** is a proportion in which the means belong to the same circumstance or figure, and the extremes belong to some other circumstance or figure.

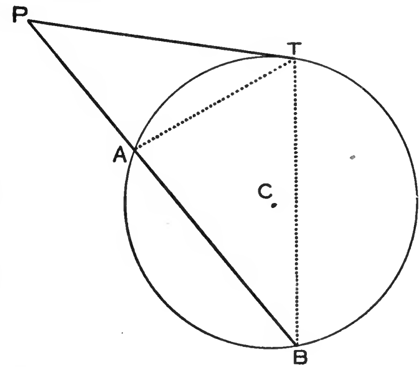


Thus, in the adjoining figure, $a \cdot b = x \cdot y$ (329). $\therefore\ a : x = y : b$ (291). This is a reciprocal proportion because the means are parts of *one* chord, and the extremes are parts of the *other* chord.

332. Theorem. If through a fixed point within a circle two chords be drawn, their four segments will be reciprocally (or inversely) **proportional.**

Proof: [Identical with proof of 329; omitting the last step.]

333. Theorem. If from a fixed point without a circle a secant and a tangent be drawn, the product of the whole secant and the external segment will equal the square of the tangent.



Given: $\odot\ C$; secant PAB; tangent PT.

To Prove:

$PB \cdot PA = \overline{PT}^2$.

Proof: Draw AT and BT.

In $\triangle$ PAT and PBT

$\angle P = \angle P$ (Iden.).

$\angle PTA$ is measured by $\frac{1}{2}$ arc AT (?). $\angle B$ is measured by $\frac{1}{2}$ arc AT (?).

$\therefore \angle PTA = \angle B$ (?).

Therefore, $\triangle PAT$ is similar to $\triangle PBT$ (?).

Hence, $PB : PT = PT : PA$ (?) (323, 3).

$\therefore\ PB \cdot PA = \overline{PT}^2$ (?) (290). Q.E.D.

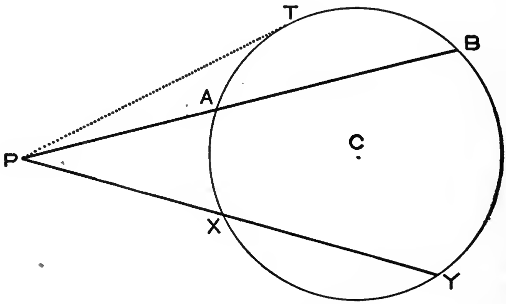
334. Theorem. If from a fixed point without a circle any secant be drawn, the product of the secant and its external segment will be constant for all secants.

Proof: Any secant $\times$ ext. seg. $=$ (tan.)$^2 =$ constant (333).

335. THEOREM. **If from a fixed point without a circle a secant and a tangent be drawn, the tangent will be a mean proportional between the secant and its external segment.**

Proof: [Identical with proof of 333; omitting the last step.]

336. THEOREM. **If from a fixed point without a circle two secants be drawn, these secants and their external segments will be reciprocally (or inversely) proportional.**

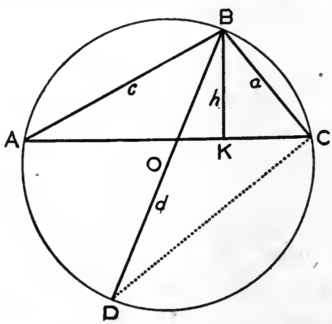


Proof: $PB \cdot PA = PY \cdot PX$ (?) (334).

$\therefore PB : PY = PX : PA$ (?) (291). Q.E.D.

Ex. If $PA = 3$ in., and $PB = 12$ in., find the length of PT.
Ex. If $PB = 21$ in., $PY = 15$ in., and $PA = 5$ in., find PX.

337. THEOREM. **In any triangle the product of two sides is equal to the diameter of the circumscribed circle multiplied by the altitude upon the third side.**



Given: $\triangle ABC$; circumscribed $\odot O$; altitude BK.

To Prove: $a \cdot c = d \cdot h$.

Proof: Draw chord CD.

$\angle BCD =$ rt. $\angle$ (?) (251).

In rt. $\triangle$ ABK and BCD,

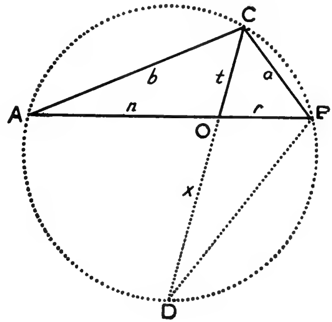
$\angle A = \angle D$ (?) (250).

$\therefore$ these $\triangle$ are similar (?).

$\therefore c : d = h : a$ (?) (323, 3).

Consequently, $a \cdot c = d \cdot h$ (?) (290). Q.E.D.

338. THEOREM. **In any triangle the product of two sides is equal to the square of the bisector of their included angle, plus the product of the segments of the third side formed by the bisector.**



Given: $\triangle$ *ABC*, *CO* the bisector of $\angle$ *ACB*.

To Prove: $a \cdot b = t^2 + n \cdot r$.

Proof: Circumscribe a $\odot$ about the $\triangle$ *ABC*.

Produce *CO* to meet $\odot$ at *D*; draw *BD*.

In $\triangle$ *ACO* and *BCD*, $\angle$ *ACO* = $\angle$ *BCD* (Hyp.).

And $\angle A = \angle D$ (?) (250).

$\therefore$ $\triangle$ *ACO* and *BCD* are similar (?) (314).

Hence, $b : (t + x) = t : a$ (?) (323, 3).

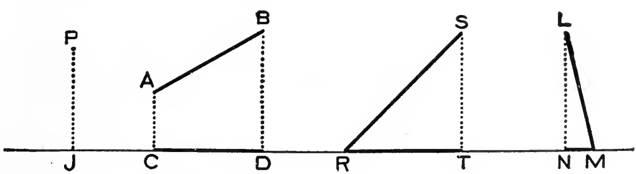
Therefore, $a \cdot b = t^2 + t \cdot x$ (?) (290).

CD and *AB* are chords (Const.). $\therefore t \cdot x = n \cdot r$ (?) (329).

Consequently, $a \cdot b = t^2 + n \cdot r$ (Ax. 6). Q.E.D.

339. The **projection** of a **point** upon a line is the foot of the perpendicular from the point to the line

Thus, the projection of *P* is *J*.



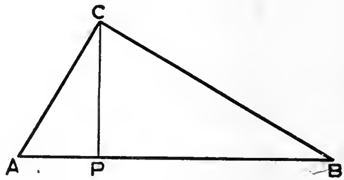
The **projection** of a definite **line** upon an indefinite line is the part of the indefinite line between the feet of the two perpendiculars to it, from the extremities of the definite line.

The projection of *AB* is *CD*; of *RS* is *RT*; of *LM* is *NM*.

340. THEOREM. **If in a right triangle a perpendicular be drawn from the vertex of the right angle upon the hypotenuse,**

I. The triangles formed will be similar to the given triangle and similar to each other.

II. The perpendicular will be a mean proportional between the segments of the hypotenuse.



Given: Rt. $\triangle$ ABC; $CP \perp$ to AB from C.

To Prove: I. $\triangle$s APC, ABC, and BPC similar.

II. $AP : CP = CP : PB$.

Proof: I. In rt. $\triangle$s APC and ABC, $\angle A = \angle A$ (Iden.).

$\therefore \triangle APC$ is similar to $\triangle ABC$ (?) (315).

In rt. $\triangle$s BPC and ABC, $\angle B = \angle B$ (?).

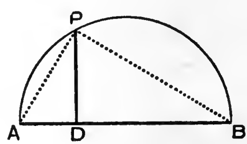
$\therefore \triangle BPC$ is similar to $\triangle ABC$ (?).

Therefore, $\triangle$s APC, ABC, and BPC are all similar (?) (320).

II. In the $\triangle$s APC and BPC, $AP : CP = CP : PB$ (?) (323, 3).

Q.E.D.

341. THEOREM. **If from any point in a circumference a perpendicular be drawn to a diameter, it will be a mean proportional between the segments of the diameter.**



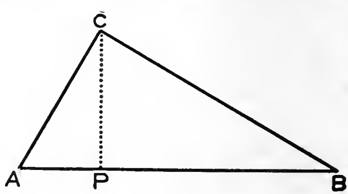
Given: (?). **To Prove:** (?).

Proof: Draw chords AP and BP.

$\triangle APB$ is a rt. $\triangle$ (?) (251).

$\therefore AD : PD = PD : DB$ (?). Q.E.D.

342. THEOREM. **The square of a leg of a right triangle is equal to the product of the hypotenuse and the projection of this leg upon the hypotenuse.**



Given: Rt. $\triangle ABC$; AC and BC the legs.

To Prove: I. $\overline{AC}^2 = AB \cdot AP$.

II. $\overline{BC}^2 = AB \cdot BP$.

Proof: I. The rt. $\triangle$ ABC and APC are similar (?) (340, I).

$\therefore AB : AC = AC : AP$ (323, 3). $\therefore \overline{AC}^2 = AB \cdot AP$ (?).

II. Rt. $\triangle$ ABC and BCP are similar (?).

$\therefore AB : BC = BC : BP$ (?). $\therefore \overline{BC}^2 = AB \cdot BP$ (?). Q.E.D.

Ex. 1. If, in 340, $AP = 3$, $PB' = 27$, find CP.

Ex. 2. If, in 342, $AP = 4$, $PB = 21$, find AC and BC.

Ex. 3. If, in 342, $AB = 20$, $AC = 6$, find AP, BP, CP, and BC.

---

343. Theorem. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.

Given: Rt. $\triangle$ ABC. **To Prove:** $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$.

Proof: Draw $CP \perp$ to the hypotenuse AB.

Then $\overline{AC}^2 = AB \cdot AP$ (?) (342).

And $\overline{BC}^2 = AB \cdot BP$ (?). Adding,

$\overline{AC}^2 + \overline{BC}^2 = AB \cdot AP + AB \cdot BP$ (Ax. 2).

$= AB\,(AP + BP) = AB \cdot AB = \overline{AB}^2$ (Ax. 4).

That is, $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$. Q.E.D.

344. Theorem. The square of either leg of a right triangle is equal to the square of the hypotenuse minus the square of the other leg.

That is, $\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2$; and $\overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2$ (?) (Ax. 2).

Ex. If $AC = 28$ and $BC = 45$, find AB.

Ex. If $AC = 21$ and $AB = 29$, find BC.

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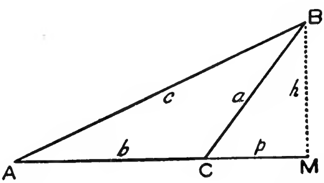
345. Theorem. In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these two sides and the projection of the other side upon that one.



Given: Obtuse $\triangle$ ABC; etc.

To Prove: $c^2 = a^2 + b^2 + 2\,bp$.

Proof: $c^2 = h^2 + (p + b)^2 = h^2 + p^2 + b^2 + 2\,bp$ (?) (343).

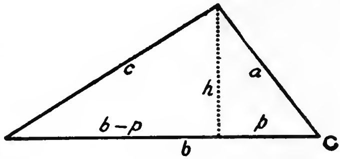
But $h^2 + p^2 = a^2$ (?) (343).

$\therefore c^2 = a^2 + b^2 + 2\,bp$ (Ax 6). Q.E.D.

346. THEOREM. **In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the product of one of these two sides and the projection of the other side upon that one.**

Given: (?).

To Prove: $c^2 = (?)$.



Proof: $c^2 = h^2 + (b - p)^2 = h^2 + p^2 + b^2 - 2\,bp$ (?) (343).

But, $h^2 + p^2 = a^2$ (?);

$\therefore c^2 = a^2 + b^2 - 2\,bp$ (Ax. 6).

Q.E.D.

NOTE. This theorem is equally true in case the triangle contains an obtuse angle. Thus, in the figure of 345, suppose the projection of AB is $AM = p$. Then, $a^2 = h^2 + (p - b)^2 = h^2 + p^2 + b^2 - 2\,bp =$ etc.

347. THEOREM. **I. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side increased by twice the square of the median upon that side.**

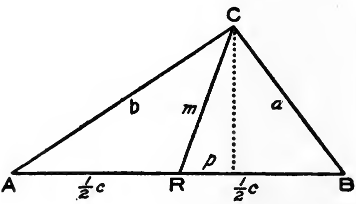
II. The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon that side.

Given: $\triangle ABC$; median $= m$; its projection $= p$; and side $b >$ side a.

To Prove:

I. $b^2 + a^2 = 2\left(\frac{1}{2}c\right)^2 + 2\,m^2$.

II. $b^2 - a^2 = 2\,cp$.



Proof: In $\triangle$ ARC and BRC, $AR = BR$ (Hyp.); CR is common; $AC > BC$ (Hyp.). $\therefore \angle ARC > \angle BRC$ (?) (87). That is, $\angle ARC$ is obtuse and $\angle BRC$ is acute.

$\therefore$ in $\triangle$ ARC, $b^2 = \left(\frac{1}{2}c\right)^2 + m^2 + cp$ (345),

and in $\triangle$ BRC, $a^2 = \left(\frac{1}{2}c\right)^2 + m^2 - cp$ (346).

I. Adding, $b^2 + a^2 = 2\left(\frac{1}{2}c\right)^2 + 2\,m^2$ (Ax. 2).

II. Subtracting, $b^2 - a^2 = 2\,cp$ (Ax. 2).

Q.E.D.

348. If the vertices of a triangle are denoted by A, B, C, the lengths of the sides opposite are denoted by a, b, c, respectively; the altitude upon these sides by h_a, h_b, h_c, respectively; the bisectors of the angles by t_a, t_b, t_c, respectively; the medians by m_a, m_b, m_c, respectively; the segments of the sides formed by the bisectors of the opposite angles by n_a and r_a, n_b and r_b, n_c and r_c; and the projections as follows: the projection of side a upon side b, by ${}_ap_b$; of side a upon side c, by ${}_ap_c$; of side b upon side c, by ${}_bp_c$; etc.

349. Formulas. It is assumed that a, b, c are known. The following values of the various lines in a triangle are obtained by solving the equations already derived.

I. **Projections.**

1. If $\angle C$ is obtuse, ${}_ap_b = \dfrac{c^2 - a^2 - b^2}{2b}$; ${}_bp_a = \dfrac{c^2 - a^2 - b^2}{2a}$; etc.

2. If $\angle C$ is acute, ${}_ap_b = \dfrac{a^2 + b^2 - c^2}{2b}$; ${}_bp_a = \dfrac{a^2 + b^2 - c^2}{2a}$; etc.

II. **Altitudes.** $h_b = \sqrt{a^2 - {}_ap_b{}^2}$; $h_a = \sqrt{b^2 - {}_bp_a{}^2}$; etc.

III. **Medians.** $m_c = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$; $m_a = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}$; etc.

IV. **Bisectors.** $t_c = \sqrt{ab - n_cr_c}$*; $t_a = \sqrt{bc - n_ar_a}$*; $t_b = \sqrt{ac - n_br_b}$.*

V. **Diameter** of circumscribed circle $= \dfrac{ac}{h_b}$; $= \dfrac{ab}{h_c}$; $= \dfrac{bc}{h_a}$.

VI. **Largest Angle.**

1. $\angle C$ is right if $c^2 = a^2 + b^2$ (343).
2. $\angle C$ is obtuse if $c^2 > a^2 + b^2$ (345).
3. $\angle C$ is acute if $c^2 < a^2 + b^2$ (346).

Ex. 1. If the sides of a triangle are $a = 7$, $b = 10$, $c = 12$, find the nature of $\angle C$.

Ex. 2. In the same triangle find m_a. Find m_b. Find m_c.

Ex. 3. In the same triangle find ${}_ap_b$. Find ${}_bp_a$. Find ${}_ap_c$. Find ${}_bp_c$. Find ${}_cp_a$. Find ${}_cp_b$.

Ex. 4. Find h_a. Find h_b. Find h_c.

Ex. 5. Find the diameter of the circumscribed circle.

Ex. 6. Find n_a and r_a. Find n_b and r_b. Find n_c and r_c.

Ex. 7. Find t_a. Find t_b. Find t_c.

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* The segments n and r can be found by 308; $n_b : r_b = c : a$, etc.

CONCERNING ORIGINALS

350. We should first determine from the nature of each numerical exercise upon which theorem it depends. By applying the truth of that theorem, the exercise is usually solved without difficulty.

ORIGINAL EXERCISES (NUMERICAL)

1. The legs of a right triangle are 12 and 16 inches; find the hypotenuse.

2. The side of a square is 6 feet; what is the diagonal?

3. The base of an isosceles triangle is 16 and the altitude is 15; find the equal sides.

4. The tangent to a circle from a point is 12 inches and the radius of the circle is 5 inches; find the length of the line joining the point to the center.

5. In a circle whose radius is 13 inches, what is the length of a chord 5 inches from the center?

[Draw chord, distance, and radius to its extremity.]

6. The length of a chord is 2 feet and its distance from the center is 35 inches; find the radius of the circle.

7. The hypotenuse of a right triangle is 2 feet 2 inches, and one leg is 10 inches; find the other.

8. The base of an isosceles triangle is 90 and the equal sides are each 53; find the altitude.

9. The radius of a circle is 4 feet 7 inches; find the length of the tangent drawn from a point 6 feet 1 inch from the center.

10. How long is a chord 21 yards from the center of a circle whose radius is 35 yards?

11. Each side of an equilateral triangle is 4 feet; find the altitude.

12. The altitude of an equilateral triangle is 8 feet; find the side. [Let x = each side; $\frac{1}{2}x$ = the base of each rt. $\triangle$.]

13. Each side of an isosceles right triangle is a; find the hypotenuse.

14. If the length of the common chord of two intersecting circles is 16, and their radii are 10 and 17, what is the distance between their centers?

15. The diagonal of a rectangle is 82 and one side is 80; find the other.

16. The length of a tangent to a circle whose diameter is 20, from an external point, is 24. What is the distance from this point to the center?

17. The diagonal of a square is 10; find each side.

18. Find the length of a chord 2 feet from the center of a circle whose diameter is 5 feet.

19. A flagpole was broken 16 feet from the ground, and the top struck the ground 63 feet from the foot of the pole. How long was the pole?

20. The top of a ladder 17 feet long reaches a point on a wall 15 feet from the ground. How far is the lower end of the ladder from the wall?

21. A chord 2 feet long is 5 inches from the center of a circle. How far from the center is a chord 10 inches long? [Find the radius.]

22. The diameters of two concentric circles are 1 foot 10 inches and 10 feet 2 inches. Find the length of a chord of the larger which is tangent to the less.

23. The lower ends of a post and a flagpole are 42 feet apart; the post is 8 feet high and the pole, 48 feet. How far is it from the top of one to the top of the other?

24. The radii of two circles are 8 inches and 17 inches, and their centers are 41 inches apart. Find the lengths of their common external tangents; of their common internal tangents.

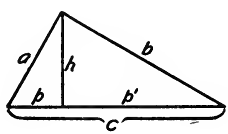
25. A ladder 65 feet long stands in a street; if it inclines toward one side, it will touch a house at a point 16 feet above the pavement; if to the other side, it will touch a house at a point 56 feet above the pavement. How wide is the street?

26. Two parallel chords of a circle are 4 feet, and 40 inches long, respectively, and the distance between them is 22 inches. Find the radius of the circle.

[Draw the radii to ends of chords; these = hypotenuses = R; the distances from the center = x and $22 - x$.]

27. The legs of an isosceles trapezoid are each 2 feet 1 inch long, and one of the bases is 3 feet 4 inches longer than the other. Find the altitude.

28. One of the non-parallel sides of a trapezoid is perpendicular to both bases, and is 63 feet long; the bases are 41 feet and 25 feet long. Find the length of the remaining side.



29. If $a = 10$, $h = 6$, find p, c, p', b.

30. If $h = 8$, $p' = 4$, find b, c, p, a.

31. If $a = 10$, $p' = 15$, find c, p, h, b.

32. If $a = 9$, $b = 12$, find c, p, p', h.

33. If $p = 3$, $p' = 12$, find a, h, b.

34. The line joining the midpoint of a chord to the midpoint of its arc is 5 inches. If the chord is 2 feet long, what is the diameter?

35. If the chord of an arc is 60 and the chord of its half is 34, what is the diameter?

36. The line joining the midpoint of a chord to the midpoint of its arc is 6 inches. The chord of half this arc is 18 inches. Find the diameter. Find the length of the original chord.

37. To a circle whose radius is 10 inches, two tangents are drawn from a point, each 2 feet long. Find the length of the chord joining their points of contact.

38. The sides of a triangle are 6, 9, 11. Find the segments of the shortest side made by the bisector of the opposite angle.

39. Find the segments of the longest side made by the bisector of the largest angle in No. 38.

40. The sides of a triangle are 5, 9, 12. Find the segments of the shortest side made by the bisector of the opposite exterior angle. Also of the medium side made by the bisector of its opposite exterior angle.

41. In the figure of 306, if $AC=3$, $CE=5$, $EG=8$, $BD=4$; find DF and FH.

42. If the sides of a triangle are 6, 8, 12 and the shortest side of a similar triangle is 15, find its other sides.

43. If the homologous altitudes of two similar triangles are 9 and 15 and the base of the former is 21, what is the base of the latter?

44. In the figure of 324, $AE=4$, $EF=6$, $FB=9$, $GH=15$. Find CG and CD.

45. The sides of a pentagon are 5, 6, 8, 9, 18, and the longest side of a similar pentagon is 78. Find the other sides.

46. A pair of homologous sides of two similar polygons are 9 and 16. If the perimeter of the first is 117, what is the perimeter of the second?

47. The perimeters of two similar polygons are 72 and 120. The shortest side of the former is 4, what is the shortest side of the latter?

48. Two similar triangles have homologous bases 20 and 48. If the altitude of the latter is 36, find the altitude of the former.

49. The segments of a chord, made by a second chord, are 4 and 27. One segment of the second chord is 6, find the other.

50. One of two intersecting chords is 19 in. long and the segments of the other are 5 in. and 12 in. Find the segments of the first chord.

51. Two secants are drawn to a circle from a point; their lengths are

15 inches and $10\frac{1}{2}$ inches. The external segment of the latter is 10; find the external segment of the former.

52. The tangent to a circle is 1 foot long and the secant from the same point is 1 foot 6 inches. Find the chord part of the secant.

53. The internal segment of a secant 25 inches long is 16 inches. Find the tangent from the same point to the same circle.

54. Two secants to a circle from a point are $1\frac{1}{2}$ feet and 2 feet long; the tangent from the same point is 12 inches. Find the external segments of the two secants.

55. The sides of a triangle are 5, 6, 8. Is the angle opposite 8 right, acute, or obtuse? Same for the triangle 8, 7, 4?

56. The sides of a triangle are 8, 9, 12. Is the largest angle right, acute, or obtuse? Same for the triangle 13, 7, 11?

57. The sides of a triangle are x, y, z. If z is the greatest side, when will the angle opposite be right? Obtuse? Acute?

58. The sides of a triangle are 6, 8, 9. Find the length of the projection of side 6 upon side 8; of side 8 upon side 9; of side 9 upon side 6.

59. The sides of a triangle are 5, 6, 9. Find the length of the projection of side 6 upon side 5; of side 9 upon side 6.

60. Find the three altitudes in triangle 9, 10, 17.

61. Find the three altitudes in triangle 11, 13, 20.

62. Find the diameter of circumscribed circle about triangle 17, 25, 26.

63. Find the length of the bisector of the least angle of triangle 7, 15, 20. Also of the largest angle.

64. Find the length of the bisector of the largest angle of triangle 12, 32, 33; also of the other angles.

65. Find the three medians in triangle 4, 7, 9.

66. Find the product of the segments of every chord drawn through a point 4 units from the center of a circle whose radius is 10 units.

67. The bases of a trapezoid are 12 and 20, the altitude is 8; the other sides are produced to meet. Find the altitude of the larger triangle formed.

68. The shadow of a yardstick perpendicular to the ground is $4\frac{1}{2}$ feet. Find the height of a tree whose shadow at the same time is 100 yards.

69. There are two belt-wheels 3 feet 8 inches and 1 foot 2 inches in diameter, respectively. Their centers are 9 feet 5 inches apart. Find the length of the belt suspended between the wheels if the belt does not cross itself. Also the length of the belt if it does cross.

Summary

351. Triangles are proved **similar** by showing that they **have:**

(1) Two angles of one equal to two angles of the other.

(2) An acute angle of one equal to an acute angle of the other. [In right triangles.]

(3) Homologous sides proportional.

(4) An angle of one equal to an angle of the other and the including sides proportional.

(5) Their sides respectively parallel or perpendicular.

352. Four lines are proved **proportional** by showing that they **are:**

(1) Homologous sides of similar triangles.

(2) Homologous sides of similar polygons.

(3) Homologous lines of similar figures.

353. The product of two lines is proved equal to the **product of two** other lines, by proving these four lines proportional and making the product of the extremes equal to the product of the means.

354. One line is proved a **mean proportional** between two others by proving that two triangles which contain this line in common are similar, and obtaining the required proportion from their sides.

355. In cases dealing with the **square of a line**, one **uses:**

(1) Similar triangles having this line in common, or,

(2) A right triangle containing this line as a part.

ORIGINAL EXERCISES (THEOREMS)

1. If two transversals intersect between two parallels, the triangles formed are similar. [Use 351 (1).]

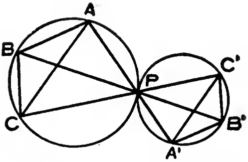
2. Two isosceles triangles are similar if a base angle of one is equal to a base angle of the other.

3. Two isosceles triangles are similar if the vertex-angle of one is equal to the vertex-angle of the other.

4. The line joining the midpoints of two sides of a triangle forms a triangle similar to the original triangle.

5. The diagonals of a trapezoid form, with the parallel sides, two similar triangles.

6. Two circles are tangent externally at P; through P three lines are drawn, meeting one circumference in A, B, C, and the other in A', B', C'. The triangles ABC and $A'B'C'$ are similar.



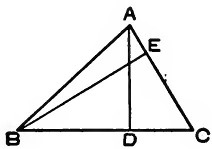
7. Prove the same theorem if the circles are tangent internally.

8. If two circles are tangent externally at P, and BB', CC' be drawn through P, terminating in the circumferences, the triangles PBC and $PB'C'$ will be similar.

[Draw the common tangent at P.]

9. Prove the same theorem if the circles are tangent internally.

10. If AD and BE are two altitudes of triangle ABC, the triangles ACD and BCE are similar.

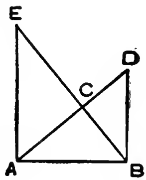


11. Two altitudes of a triangle are reciprocally proportional to the bases to which they are drawn.

To Prove: $AD : BE = AC : BC$.

12. The four segments of the diagonals of a trapezoid are proportional.

13. If at the extremities of the hypotenuse of a right triangle perpendiculars be erected meeting the legs produced, the new triangles formed will be similar.



14. In the figure of No. 13, prove:

(1) Triangle ABC similar to each of the triangles ACE and BCD.

(2) Triangle ABE similar to triangle ABD.

(3) Triangle ACE similar to triangle ABD.

(4) Triangle BCD similar to triangle ABE.

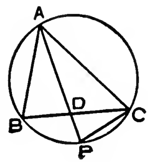
(5) Triangles ABC, ABD, ABE similar.

15. If AD and BE are two altitudes of triangle ABC (fig. of No. 11), meeting at O, the triangles BOD and AOE are similar.

16. Triangles CED and ABC (fig. of No. 11) are similar.

[First show $\triangle$ CAD and CEB similar.

$\therefore CA : CB = CD : CE$ (?). Then use 351 (4).]



17. Triangle ABC is inscribed in a circle and AP is drawn to P, the midpoint of arc BC, meeting chord CB at D. The triangles ABD and ACP are similar.

18. Two homologous medians in two similar triangles are in the same ratio as any two homologous sides.

[Prove a pair of the new triangles formed, similar, by 351 (4).]

19. Two homologous bisectors in two similar triangles are in the same ratio as any two homologous sides.

20. The radii of circles inscribed in two similar triangles are in the same ratio as any two homologous sides.

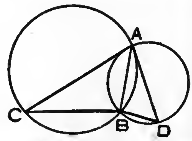
[Bisect two pairs of homol. ∠s; draw the altitudes of these new △; etc.]

21. The radii of circles circumscribed about two similar triangles are in the same ratio as any two homologous sides.

[Erect ⊥ bisectors; draw radius in each ⊙.]

22. In any right triangle the product of the hypotenuse and the altitude upon it is equal to the product of the legs.

23. If two circles intersect at A and B and AC and AD be drawn each a tangent to one circle and a chord of the other, the common chord AB will be a mean proportional between BC and BD.

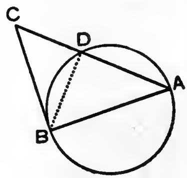


24. If two circles are tangent externally, the chords formed by a straight line drawn through their point of contact have the same ratio as the diameters of the circles.

[Draw com. tang. at point of contact; draw diameters from point of contact; prove △ sim.; etc.]

25. If AB is a diameter and BC a tangent, and AC meets the circumference at D, the diameter is a mean proportional between AC and AD.

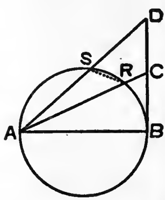
[Draw BD. Prove △ containing AB similar.]



26. If a tangent be drawn from one extremity of a diameter, meeting secants from the other extremity, these secants and their internal segments will be reciprocally proportional.

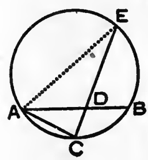
To Prove: $AC:AD = AS:AR$.

Proof: Draw RS. In △ ARS and ACD, $\angle A = \angle A$ and $\angle ARS = \angle D$. (Explain.) Etc.

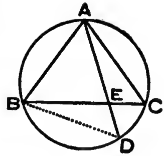


27. If AB is a chord and CE, another chord, drawn from C, the midpoint of arc AB, meeting chord AB at D, AC is a mean proportional between CD and CE.

Prove the above theorem and deduce that, $CE \cdot CD$ is constant for all positions of the point E on arc AEB.

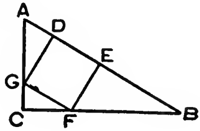


28. If chord AD be drawn from vertex A of inscribed isosceles triangle ABC, cutting BC at E, AB will be a mean proportional between AD and AE.



Prove the above theorem and deduce that, $AD \cdot AE$ is constant for all positions of the point D on arc BDC.

29. If a square be inscribed in a right triangle so that one vertex is on each leg of the triangle and the other two vertices on the hypotenuse, the side of the square will be a mean proportional between the other segments of the hypotenuse.



To Prove: $AD : DE = DE : EB$. First prove △ ADG and BEF similar.

30. If the sides of two triangles are respectively parallel, the lines joining homologous vertices meet in a point. (These lines to be produced if necessary.)

---

31. In each of the following triangles, is the greatest angle right, acute, or obtuse, 7, 24, 25? 13, 10, 8? 19, 13, 23?

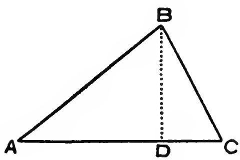
32. Prove theorem of 329 by drawing two other auxiliary chords.

33. Prove theorem of 325 if point O is between the parallels.

34. Prove theorem of 336 by drawing AY and BX.

35. In any triangle the difference of the squares of two sides is equal to the difference of the squares of their projections on the third side.

[$\overline{AB}^2 = (?)$; $\overline{BC}^2 = (?)$. Subtract, etc.]

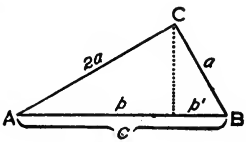


36. If the altitudes of triangle ABC meet at O, $\overline{AB}^2 - \overline{AC}^2 = \overline{BO}^2 - \overline{CO}^2$.

[Consult No. 35 and substitute.]

37. The square of the altitude of an equilateral triangle is three fourths the square of a side. [Let side $= a$, etc.]

38. If one leg of a right triangle is double the other, its projection upon the hypotenuse is four times the projection of the other.



Proof: $(2a)^2 = cp$; $a^2 = cp'$ (?).

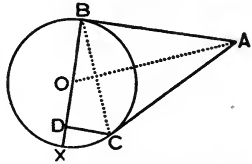
$\therefore p = \dfrac{4a^2}{c}$; $p' = \dfrac{a^2}{c}$ (Ax. 3).

$\therefore p = 4p'$ (?).

39. If the bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles.

40. The tangents to two intersecting circles from any point in their common chord produced are equal. [Use 333.]

41. If two circles intersect, their common chord, produced, bisects their common tangents. [Use 333.]



42. If AB and AC are tangents to a circle from A; CD is perpendicular to diameter BOX from C; then $AB \cdot CD = BD \cdot BO$.

[Use 351 (5).]

43. If the altitude of an equilateral triangle is h, find the side. [Denote the side by x and half the base by $\frac{1}{2}x$.]

44. If one side of a triangle be divided by a point into segments which are proportional to the other sides, a line from this point to the opposite angle will bisect that angle. [Converse of 308.]

To Prove: $\angle n = \angle m$ in fig. of 308.

Proof: Produce CB to P, making $BP = AB$; draw AP; etc.

45. State and prove the converse of 310.

46. Two rhombuses are similar if an angle of one is equal to an angle of the other.

47. If two circles are tangent internally and any two chords of the greater be drawn from their point of contact, they will be divided proportionally by the circumference of the less.

[Draw diameter to point of contact and prove the right $\triangle$ similar.]

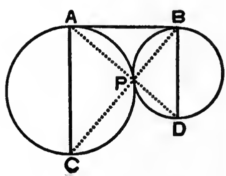
48. The non-parallel sides of a trapezoid and the line joining the midpoints of the bases, if produced, meet at a point. [Use Ax. 3 and 325.]

49. The diagonals of a trapezoid and the line joining the midpoint of the bases meet at a point.

50. If one chord bisects another, either segment of the latter is a mean proportional between the segments of the other.

51. Two parallelograms are similar if they have an angle of the one equal to an angle of the other and the including sides proportional.

52. Two rectangles are similar if two adjoining pairs of homologous sides are proportional.



53. If two circles are tangent externally, the common exterior tangent is a mean proportional between the diameters.

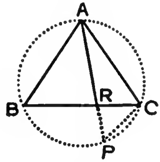
[Draw chords PA, PC, PB, PD. Prove, first, APD and BPC straight lines. Second, $\triangle ABC$ and ABD, similar.]

54. In any rhombus the sum of the squares of the diagonals is equal to the square of half the perimeter.

55. If in an angle a series of parallel lines be drawn having their ends in the sides of the angle, their midpoints will lie in one straight line.

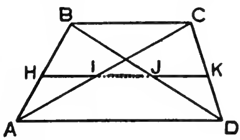
56. If ABC is an isosceles triangle and BX is the altitude upon AC (one of the legs), $\overline{BC}^2 = 2\,AC \cdot CX$. [Use 346.]

57. In an isosceles triangle the square of one leg is equal to the square of the line drawn from the vertex to any point of the base, plus the product of the segments of the base.



Proof : Circumscribe a ⊙; use method of 338.

58. If a line be drawn in a trapezoid parallel to the bases, the segments between the diagonals and the non-parallel sides will be equal.



Proof : $\triangle$s AHI and ABC are similar (?); $\triangle$s DJK and DCB also (?). $\therefore \frac{AH}{AB} = \frac{HI}{BC}$ (?).

$\frac{DK}{DC} = \frac{KJ}{BC}$ (?). But, $\frac{AH}{AB} = \frac{DK}{DC}$ (?). (Use **Ax. 1**); etc.

59. A line through the point of intersection of the diagonals of a trapezoid, and parallel to the bases, is bisected by that point.

60. If M is the midpoint of hypotenuse AB of right triangle ABC, $\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2 = 8\,\overline{CM}^2$.

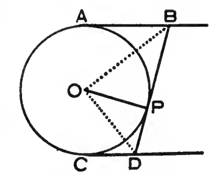
61. The squares of the legs of a right triangle have the same ratio as their projections upon the hypotenuse.

62. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares of one pair of opposite sides is equal to the sum of the squares of the other pair.

63. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals. [Use 347, I.]

64. If DE be drawn parallel to the hypotenuse AB of right triangle ABC, meeting AC at D and CB at E, $\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2$.

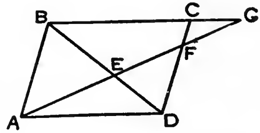
[Use 4 rt. $\triangle$s having vertex C.]



65. If between two parallel tangents a third tangent be drawn, the radius will be a mean proportional between the segments of the third tangent.

To Prove : $BP : OP = OP : PD$. **Proof :** $\triangle BOD$ is a rt. $\triangle$ (?). Etc.

66. If $ABCD$ is a parallelogram, BD a diagonal, AG any line from A meeting BD at E, CD at F, and BC (produced) at G, AE is a mean proportional between EF and EG.

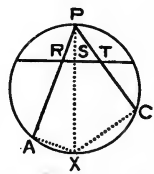


Proof: △ ABE and EDF are similar (?); also △ ADE and BEG (?). Obtain two ratios $= BE : ED$ and then apply Ax. 1.

67. An interior common tangent of two circles divides the line joining their centers into segments proportional to the radii.

68. An exterior common tangent of two circles divides the line joining their centers (externally) into segments proportional to the radii.

69. The common internal tangents of two circles and the common external tangents meet on the line determined by the centers of the circles.

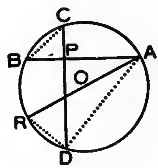


70. If from the midpoint P, of an arc subtended by a given chord, chords be drawn cutting the given chord, the product of each whole chord from P and its segment adjacent to P will be constant.

Proof: Take two such chords, PA and PC; draw diameter PX; etc. Rt. △ PST and PCX are similar. (Explain.)

71. If from any point within a triangle ABC, perpendiculars to the sides be drawn — OR to AB, OS to BC, OT to AC, $\overline{AR}^2 + \overline{BS}^2 + \overline{CT}^2 = \overline{BR}^2 + \overline{CS}^2 + \overline{AT}^2$. [Draw AO, BO, CO.]

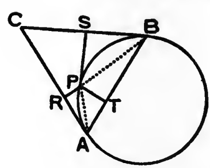
72. If two chords intersect within a circle and at right angles, the sum of the squares of their four segments equals the square of the diameter.



To Prove: $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 = \overline{AR}^2$.

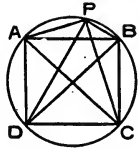
Proof: Draw BC, AD, RD. Ch. BR is $\perp$ to AB (?). $\therefore$ CD is $\parallel$ to BR (?). $\therefore$ arc BC = arc RD (?). Hence, ch. BC = ch. RD (?). Now, $\overline{RD}^2 = \overline{BC}^2 = \overline{BP}^2 + \overline{CP}^2$ (?). $\overline{AD}^2 =$ etc. (?). Finally, $\overline{AR}^2 = \overline{AD}^2 + \overline{RD}^2 =$ etc. (?).

73. The perpendicular from any point of an arc upon its chord is a mean proportional between the perpendiculars from the same point to the tangents at the ends of the chord.



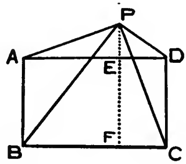
To Prove: $PR : PT = PT : PS$. **Proof:** Prove △ ARP and BTP are sim., also △ APT and PBS (?). Thus, get two ratios each $= PA : PB$.

74. If lines be drawn from any point in a circumference to the four vertices of an inscribed square, the sum of the squares of these four lines will be equal to twice the square of the diameter.



Proof: $\triangle$s APC, DPB, are rt. $\triangle$s; etc.

75. If lines be drawn from any external point to the vertices of a rectangle $ABCD$, the sum of the squares of two of them which are drawn to a pair of opposite vertices will be equal to the sum of the squares of the other two.

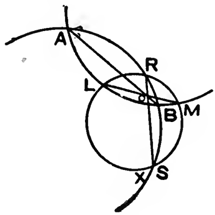


To Prove: $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.

Proof: Draw $PEF \perp$ to the base, etc.

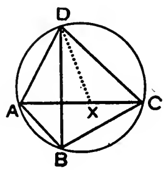
76. Is the theorem of No. 75 true if the point is taken within the rectangle?

77. If each of three circles intersects the other two, the three common chords meet in a point.



Given: (?). **To Prove:** AB, LM, RS meet at O.
Proof: Suppose AB and LM meet at O. Draw RO and produce it to meet the ⓢ at X and X'. Prove $OX = OX'$ (by 329). $\therefore$ X, X', S are coincident.

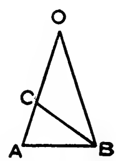
78. In an inscribed quadrilateral the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.



Proof: Draw DX making $\angle CDX = \angle ADB$; $\triangle$s ADB and CDX are sim. (?); also $\triangle$s BCD and ADX (?). Hence, $AB \cdot DC = DB \cdot XC$ (?), and $AD \cdot BC = DB \cdot AX$ (?). Adding; etc.

79. If AB is a diameter, BC and AD tangents, meeting chords AF and BF (produced) at C and D respectively, AB is a mean proportional between the tangents BC and AD.

80. If ABO is isosceles, $AB = CO$, and $AO : CO = CO : AC$, prove: (1) $AB = BC = CO$; (2) $\angle ABO = 2\angle O$; (3) $\angle O = 36°$.



Proofs: (1) $AO : AB = AB : AC$ (Ax. 6). $\therefore \triangle ABC$ is sim. to $\triangle ABO$ (?) (317). $\therefore \triangle ABC$ is isosceles (?). Etc.

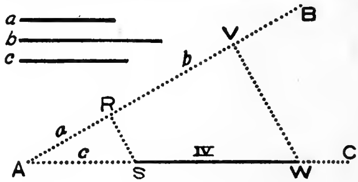
(2) $\angle ABC = \angle O$ (?), and $\angle CBO = \angle O$ (?). $\therefore \angle ABO = 2\angle O$ (Ax. 2).

(3) $\angle O = \frac{1}{5}$ the sum of $\angle$s of $\triangle AOB$ (?). Etc.

81. If from a point A on the circumference of a circle two chords be drawn and a line parallel to the tangent at A meet them, the chords and their segments nearer to A will be inversely proportional.

CONSTRUCTIONS

356. PROBLEM. To find a fourth proportional to three given lines.



Given: Three lines, a, b, c.

Required: To find a fourth proportional to a, b, c.

Construction: Take two indefinite lines, AB and AC, meeting at A. On AB take $AR = a$, $RV = b$. On AC take $AS = c$. Draw RS.

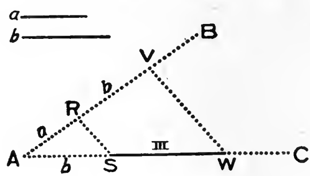
From V draw $VW \parallel$ to RS, meeting AC at W.

Statement: SW is the fourth proportional required. Q.E.F.

Proof: In $\triangle AVW$, RS is $\parallel$ to VW (Const.).

$\therefore a : b = c : SW$ (?) (302). Q.E.D.

357. PROBLEM. To find a third proportional to two given lines.



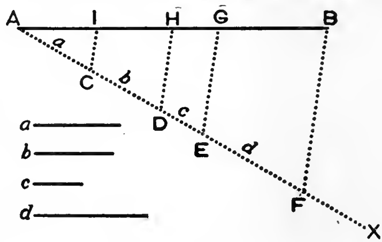
Given: (?).

Required: (?).

Construction:
Like that for 356.

Statement: (?). **Proof:** (?).

358. PROBLEM. To divide a given line into segments proportional to any number of given lines.



Given: AB; a, b, c, d.

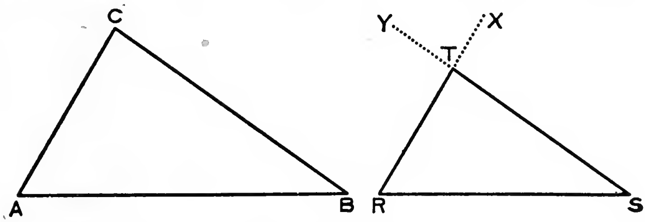
Required: To divide AB into parts which shall be proportional to a, b, c, d.

Construction: Draw AX, an indefinite line, oblique to AB from A. On AX take $AC = a$, $CD = b$, $DE = c$, $EF = d$. Draw FB. Then draw, through E, D, and C, lines $\parallel$ to BF, as EG, DH, and CI.

Statement: AI, IH, HG, GB are the required parts. Q.E.F.

Proof: $AI : a = IH : b = HG : c = GB : d$ (?) (306). Q.E.D.

359. PROBLEM. To construct a triangle similar to a given triangle and having a given side homologous to a side of the given triangle.



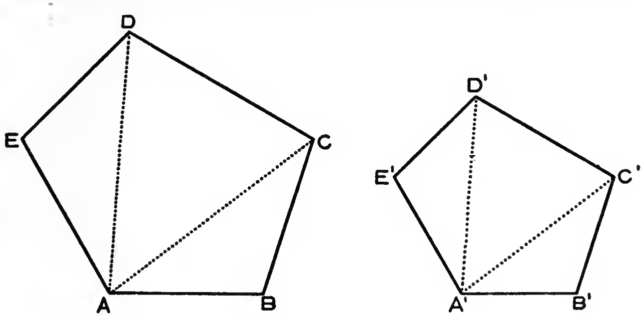
Given: $\triangle$ ABC and RS homologous to AB.

Required: To construct a $\triangle$ on RS similar to $\triangle$ ABC.

Construction: At R construct $\angle SRX = \angle A$; at S construct $\angle RSY = \angle B$, the sides of these angles meeting at T.

Statement: (?). **Proof**: (?) (314).

360. PROBLEM. To construct a polygon similar to a given polygon and having a given side homologous to a side of the given polygon.



Given: Polygon EB; line $A'B'$ homologous to AB.

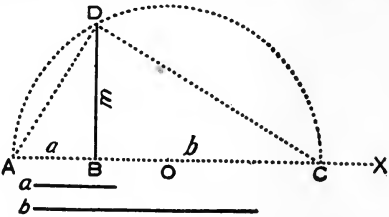
Required: To construct a polygon upon $A'B'$, similar to polygon EB.

Construction: From A draw diagonals AC and AD.
On $A'B'$ construct $\triangle$ $A'B'C'$ similar to $\triangle$ ABC (by 359).
On $A'C'$ construct $\triangle$ $A'C'D'$ similar to $\triangle$ ACD. Etc.

Statement: (?).

Proof: (?) (328).

361. Problem. To find the mean proportional between two given lines.



Given: Lines a and b.

Required: To find the mean proportional between them.

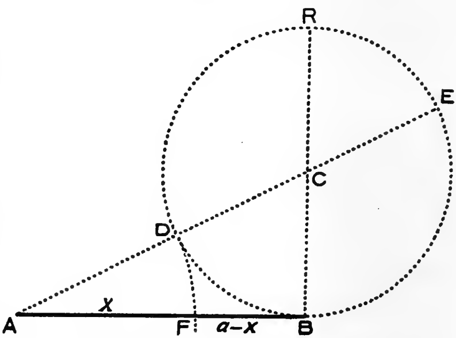
Construction: On an indefinite line, AX, take $AB = a$ and $BC = b$. Using O, the midpoint of AC, as center, and AO as radius, describe the semicircumference, ADC. At B erect $BD \perp$ to AC, meeting the arc at D. Draw AD and CD.

Statement: BD is the mean proportional required. Q.E.F.

Proof: $a : BD = BD : b$ (?) (341). Q.E.D.

362. A line is divided in **extreme and mean ratio** if one segment is a mean proportional between the whole line and the other segment. In other words, if a line is to one of its parts, as that part is to the other part, the line is divided in extreme and mean ratio.

363. Problem. To divide a line into extreme and mean ratio.



Given: Line $AB = a$.

Required: To divide AB into extreme and mean ratio, that is, so that $AB : AF = AF : FB$.

Construction: At B erect BR, $\perp$ to AB and $= AB$. Using C, the midpoint of BR, as center, and CB as radius, describe a $\odot$. Draw AC meeting $\odot$ at D and E. On AB take $AF = AD$; let $AF = x$.

Statement: F divides AB so that $AB : AF = AF : FB$. Q.E.F.

Proof: $DE = a$ (?) (203). $\therefore AE = a + x$ (Ax. 4).
AB is tangent to the $\odot$ (?) (215).

$$\therefore AE \cdot AD = \overline{AB}^2 \text{ (?) (333).}$$

$$\therefore (a+x)\,x = a^2 \text{(Ax. 6).}$$

That is, $x^2 = a^2 - ax$, or $x^2 = a(a-x)$.

$$\therefore a : x = x : a - x \text{ (?) (291).}$$

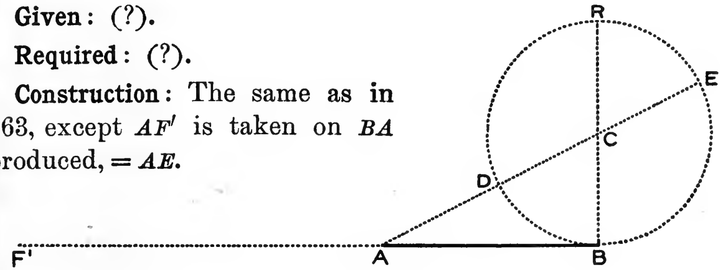
That is, $AB : AF = AF : FB$ (Ax. 6). Q.E.D.

364. Problem. To divide a line externally into extreme and mean ratio.

Given: (?).

Required: (?).

Construction: The same as in 363, except AF' is taken on BA produced, $= AE$.



Statement: $AB : AF' = AF' : BF'$. Q.E.F.

Proof: AB is tangent to the $\odot$ (?). $AB = BR = DE$ (?).
$\therefore AE : AB = AB : AD$ (?) (335).

Hence, $AE + AB : AE = AB + AD : AB$ (?) (294).

But $AE + AB = BF'$, and $AB + AD = AE = AF'$ (Const.).

$$\therefore BF' : AF' = AF' : AB \text{ (Ax. 6).}$$

That is, $AB : AF' = AF' : BF'$. Q.E.D.

365. The lengths of the several lines of 363 and 364 may be found by algebra, if the length of AB is known.

Thus, if $AB = a$, we know in 363, $a : x = x : a - x$.

Hence, $x^2 = a^2 - ax$. Solving this quadratic,

$x = AF = \frac{1}{2}a(\sqrt{5} - 1)$; also, $a - x = BF = \frac{1}{2}a(3 - \sqrt{5})$.

Likewise, in 364, if $AB = a$, $AF' = y$, $a : y = y : a + y$.

Solving for y, $y = AF' = \frac{1}{2}a(\sqrt{5} + 1)$.

Also $a + y = BF' = \frac{1}{2}a(3 + \sqrt{5})$.

ORIGINAL CONSTRUCTIONS

1. To construct a fourth proportional to lines that are exactly 3 in., 5 in., and 6 in. long. How long should this constructed line be?

2. To construct a mean proportional between lines that are exactly 4 in. and 9 in. How long should this constructed line be?

3. Will a fourth proportional to three lines 5 in., 8 in., and 10 in., be the same length as a fourth proportional to 5 in., 10 in., and 8 in.? to 8 in., 10 in., and 5 in.? to 10 in., 5 in., and 8 in.?

4. To construct a third proportional to lines that are exactly 3 in. and 6 in. long.

5. To produce a given line AB to point P, such that $AB : AP = 3 : 5$. [Divide AB into three equal parts, etc.]

6. To divide a line 8 in. long into two parts in the ratio of $5 : 7$. [Divide the given line into 12 equal parts.]

7. To solve No. 6 by constructing a triangle. [See 308.]

8. To divide one side of a triangle into segments proportional to the other two sides.

9. To divide one side of a triangle externally into segments proportional to the other sides.

10. To construct two straight lines having given their sum and ratio. [Consult No. 6.]

11. To construct two straight lines having given their difference and ratio. [Consult No. 5.]

12. To construct a triangle similar to a given triangle and having a given perimeter. [First, use 358.]

13. To construct a right triangle having given its perimeter and an acute angle. [Constr. a rt. $\triangle$ having the given acute $\angle$. Etc.]

14. To construct a triangle having given its perimeter and two angles. [Constr. a $\triangle$ having the two given $\angle$s. Etc.]

15. To construct a triangle similar to a given triangle and having a given altitude.

16. To construct a rectangle similar to a given rectangle and having a given base.

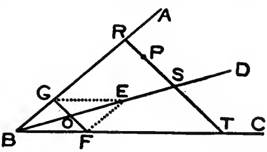
17. To construct a rectangle similar to a given rectangle and having a given perimeter.

18. To construct a parallelogram similar to a given parallelogram and having a given base.

19. To construct a parallelogram similar to a given parallelogram and having a given perimeter.

20. To construct a parallelogram similar to a given parallelogram and having a given altitude.

21. Three lines meet at a point; it is required to draw through a given point another line, which will be terminated by the outer two and be bisected by the inner one.



Construction: From E on BD draw ∥s. Etc. Through P draw RT ∥ to GF.

Statement: $RS = ST$.

22. To inscribe in a given circle a triangle similar to a given triangle.

Construction: Circumscribe a ⊙ about the given △; draw radii to the vertices; at center of given ⊙ construct 3 ∠s = to the other central angles.

23. To circumscribe about a given circle a triangle similar to a given triangle.

Construction: First, inscribe a △ similar to the given △.

24. To construct a right triangle, having given one leg and its projection upon the hypotenuse.

25. To inscribe a square in a given semicircle.

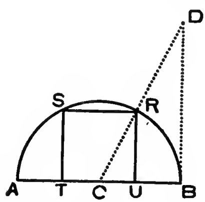
Construction: At B erect $BD \perp$ to AB and $= AB$; draw DC, meeting ⊙ at R; draw $RU \parallel$ to BD. Etc. **Statement**: (?).

Proof: $RSTU$ is a rectangle (?).

$\triangle CRU$ is similar to $\triangle CBD$ (?).

$\therefore CU : CB = RU : BD$ (?). But $CB = \frac{1}{2} BD$ (?).

$\therefore CU = \frac{1}{2} RU$ (?). Etc.

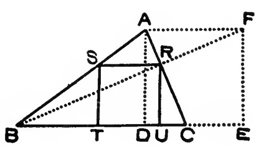


26. To inscribe in a given semicircle, a rectangle similar to a given rectangle.

Construction: From the midpoint of the base draw line to one of the opposite vertices. At given center construct an ∠ = the ∠ at the midpoint. Proceed as in No. 25.

Proof: First, prove a pair of △s similar. Etc.

27. To inscribe a square in a given triangle.



Construction: Draw altitude AD; construct the square $ADEF$ upon AD as a side; draw BF meeting AC at R.

Draw $RU \parallel$ to AD; $RS \parallel$ to BC. Etc.

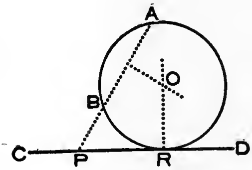
Statement: (?). **Proof**: △s BRU and BFE are similar (?). Also △s BRS and BAF (?).

Thence show that $SR = RU$. Etc.

28. To inscribe in a given triangle a rectangle similar to a given rectangle.

Construction: Draw the altitude. On this construct a rectangle similar to the given rectangle.

Proceed as in No. 27.



29. To construct a circle which shall pass through two given points and touch a given line.

Given: Points A and B; line CD.

Construction: Draw line AB meeting CD at P. Construct a mean proportional between PA and PB (by 361). On PD take PR = this mean. Erect $OR \perp$ to CD at R, meeting $\perp$ bisector of AB at O. Use O as center, etc.

30. To construct a line $= \sqrt{2}$ in. [Diag. of square whose side is 1 in.]

31. To construct a line $= \sqrt{5}$ in.

[Hyp. of a rt. $\triangle$, whose legs are 1 in. and 2 in. respectively.]

32. To divide a line into segments in the ratio of $1 : \sqrt{2}$.

33. To divide a line into segments in the ratio of $1 : \sqrt{5}$.

34. To construct a line x, if $x = \frac{ab}{c}$, and a, b, c are lengths of three given lines. [That is, to construct x, if $c : a = b : x$ (291).]

35. To construct a line x, if $x = \frac{ab}{3c}$. [$3c : a = b : x$.]

36. To construct a line x, if $x = \sqrt{ab}$. [$a : x = x : b$.]

37. To construct a line x, if $x = \frac{a^2}{c}$.

38. To construct a line x, if $x = \sqrt{a^2 - b^2}$. [$a + b : x = x : a - b$.]

39. To construct a line x, if $x = \frac{2a^2}{c}$.

40. To construct a line y, if $ay = \frac{2}{3} b^2$.

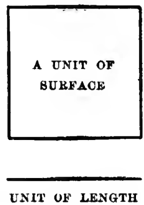
41. To construct a line $= \sqrt{10}$ in.

42. To construct a line $= 2\sqrt{6}$ in.

43. To construct a line $= \sqrt{a^2 + b^2}$, if a and b are given lines.

BOOK IV

AREAS



366. The **unit of surface** is a square whose sides are each a unit of length.

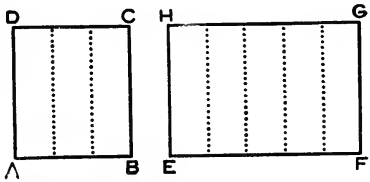
Familiar units of surface are the square inch, the square foot, the square meter, etc.

367. The **area of a surface** is the *number* of units of surface it contains. The area of a surface is the ratio of that surface to the unit of surface.

Equivalent (≎) figures are figures having equal areas.

NOTE. It is often convenient to speak of "triangle," "rectangle," etc., when one really means "the area of a triangle," or "the area of a rectangle," etc.

368. THEOREM. **If two rectangles have equal altitudes, they are to each other as their bases.**



Given: Rectangles AC and EG having = altitudes, and their bases being AB and EF.

To Prove:

$AC : EG = AB : EF.$

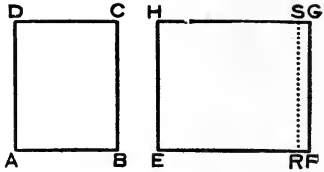
Proof: I. If AB and EF are **commensurable.**

There exists a common unit of measure of AB and EF (238). Suppose this unit is contained 3 times in AB and 5 times in EF. Hence, $AB : EF = 3 : 5$ (Ax. 3).

Draw lines through these points of division, ⊥ to the bases. These will divide rectangle AC into three parts and EG into 5 parts, and all of these eight parts are equal (?) (140).

Hence, $AC : EG = 3 : 5$ (?).

$\therefore AC : EG = AB : EF$ (Ax. 1). Q.E.D.



II. If AB and EF are **incommensurable.**

There does not exist a common unit (?) (238). Divide AB into several equal parts. Apply one of these as a unit of measure to EF. There will be a remainder, RF left over (Hyp.). Draw $RS \perp$ to EF. Now, $\frac{AC}{ES} = \frac{AB}{ER}$ (Case I).

Indefinitely increase the number of equal parts of AB; that is, indefinitely decrease each part, or the unit or divisor. Hence the remainder, RF, will be indefinitely decreased (?).

That is, RF will approach zero as a limit,
and, $RFGS$ will approach zero as a limit.
Hence, ER will approach EF as a limit (?),
and ES will approach EG as a limit (?).

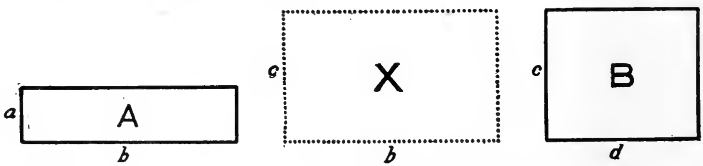
Therefore, $\frac{AC}{ES}$ will approach $\frac{AC}{EG}$ as a limit (?),

and $\frac{AB}{ER}$ will approach $\frac{AB}{EF}$ as a limit (?).

$$\therefore \frac{AC}{EG} = \frac{AB}{EF} \text{ (?) (242).}$$ Q.E.D.

369. Theorem. Two rectangles having equal bases are to each other as their altitudes. (Explain.)

370. Theorem. Any two rectangles are to each other as the products of their bases by their altitudes.



Given: Rectangles A and B whose altitudes are a and c and bases b and d respectively.

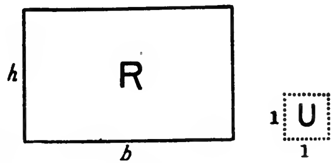
To Prove: $A : B = a \cdot b : c \cdot d$.

Proof: Construct a third rectangle X whose base is b and whose altitude is c.

Then $A : X = a : c$ (?) (369). Also, $X : B = b : d$ (?).

Multiplying, $A : B = a \cdot b : c \cdot d$ (?) (Ax. 3). Q.E.D.

371. Theorem. The area of a rectangle is equal to the product of its base by its altitude.



Given: Rectangle R, whose base is b and altitude, h.

To Prove: Area of $R = b \cdot h$.

Proof: Draw a square U, each of whose sides is a unit of length. This square is a unit of surface (366).

Now, $\frac{R}{U} = \frac{b \cdot h}{1 \cdot 1} = b \cdot h$ (370). But $\frac{R}{U} =$ area of R (367).

$\therefore$ area of $R = b \cdot h$ (Ax. 1). Q.E.D.

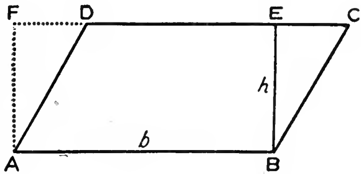
372. Theorem. The area of a square is equal to the square of its side. (See 371.)

373. Theorem. The area of a parallelogram is equal to the product of its base by its altitude.

Given: ▱ $ABCD$ whose base is b and altitude, h.

To Prove: Area of $ABCD = b \cdot h$.

Proof: From A and B, the extremities of the base, draw ⊥s to the upper base meeting it in F and E respectively.



In rt. △s ADF and BCE, $AF = BE$ (?), $AD = BC$ (?).

$\therefore \triangle ADF = \triangle BCE$ (?).

Now, from the whole figure take $\triangle ADF$ and parallelogram $ABCD$ remains; and from the whole figure take $\triangle BCE$ and rectangle $ABEF$ remains.

$\therefore$ ▱ $ABCD$ ≎ rect. $ABEF$ (Ax. 2).

Rect. $ABEF = b \cdot h$ (?). $\therefore$ ▱ $ABCD = b \cdot h$ (Ax. 1). Q.E.D.

374. Cor. All parallelograms having equal bases and equal altitudes are equivalent.

375. Theorem. Two parallelograms having equal altitudes are to each other as their bases.

Proof: $P = b \cdot h$; $P' = b' \cdot h$ (?).

$\therefore \frac{P}{P'} = \frac{b \cdot h}{b' \cdot h} = \frac{b}{b'}$ (Ax. 3).

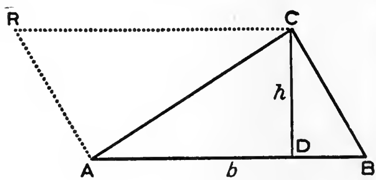
376. Theorem. Two parallelograms having equal bases are to each other as their altitudes.

Proof: (?).

377. Theorem. Any two parallelograms are to each other as the products of their bases by their altitudes.

Proof: (?).

378. Theorem. The area of a triangle is equal to half the product of its base by its altitude.



Given: $\triangle ABC$; base $= b$; altitude $= h$.

To Prove:
Area of $\triangle ABC = \frac{1}{2} b \cdot h$.

Proof: Through A draw $AR \parallel$ to BC and through C draw $CR \parallel$ to AB, meeting AR at R. Now $ABCR$ is a $\square$ (?).
Area $\square ABCR = b \cdot h$ (?). $\frac{1}{2} \square ABCR = \frac{1}{2} b \cdot h$ (Ax. 3).

Also $\triangle ABC = \frac{1}{2} \square ABCR$ (?) (132).

$\therefore \triangle ABC = \frac{1}{2} b \cdot h$ (Ax. 1). Q.E.D.

379. Cor. If a parallelogram and a triangle have the same base and altitude, the triangle is equivalent to half the parallelogram.

380. Cor. All triangles having equal bases and equal altitudes are equivalent.

381. Cor. All triangles having the same base and whose vertices are in a line parallel to the base are equivalent (?).

382. THEOREM. Two triangles having equal altitudes are to each other as their bases.

Proof: $\triangle T = \frac{1}{2} b \cdot h$; $\triangle T' = \frac{1}{2} b' h$ (?).

$$\therefore \frac{\triangle T}{\triangle T'} = \frac{\frac{1}{2} b h}{\frac{1}{2} b' h} = \frac{b}{b'} \text{ (?)}.$$

383. THEOREM. Two triangles having equal bases are to each other as their altitudes.

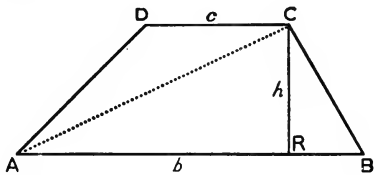
Proof: (?).

384. THEOREM. Any two triangles are to each other as the products of their bases by their altitudes.

385. THEOREM. The area of a right triangle is equal to half the product of the legs.

386. THEOREM. The area of a trapezoid is equal to half the product of the altitude by the sum of the bases.

Given: Trapezoid $ABCD$; altitude $= h$; bases $= b$ and c.



To Prove:
Area $ABCD = \frac{1}{2} h \cdot (b + c)$.

Proof: Draw diagonal AC. Now consider the $\triangle ABC$ and ADC as having the same altitude, h, and their bases b and c, respectively.

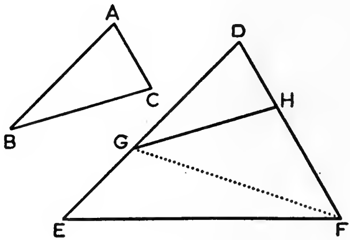
$$\begin{array}{lrl} \text{Now,} & \triangle ABC & = \frac{1}{2} b \cdot h \text{ (?)}, \\ \text{and} & \triangle ADC & = \frac{1}{2} c \cdot h \text{ (?)}. \\ \hline \text{Adding,} & \triangle ABC + \triangle ADC & = \frac{1}{2} b \cdot h + \frac{1}{2} c \cdot h \quad \text{(Ax. 2)}. \\ \text{That is,} & \text{trapezoid } ABCD & = \frac{1}{2} h \cdot (b + c) \quad \text{(Ax. 6)}. \end{array}$$

Q.E.D.

387. THEOREM. The area of a trapezoid is equal to the product of the altitude by the median.

Proof: Area $ABCD = \frac{1}{2} h \cdot (b + c) = h \cdot \frac{1}{2} (b + c)$.
But $\frac{1}{2} (b + c) =$ median (144).
Hence, area of trapezoid $ABCD = h \cdot m$ (Ax. 6). Q.E.D.

388. Theorem. If two triangles have an angle of one equal to an angle of the other, they are to each other as the products of the sides including the equal angles.



Given: $\triangle$s ABC and DEF, $\angle A = \angle D$.

To Prove:
$$\frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot AC}{DE \cdot DF}.$$

Proof: Superpose $\triangle$ ABC upon $\triangle$ DEF so that the equal $\angle$s coincide and BC takes the position denoted by GH. Draw GF.

Now $\triangle$s DGH and DGF have the same altitude (a $\perp$ from G to DF), and $\triangle$s DGF and DEF have the same altitude (a $\perp$ from F to DE).

$$\therefore \frac{\triangle DGH}{\triangle DGF} = \frac{DH}{DF} \text{ (?) (382). And } \frac{\triangle DGF}{\triangle DEF} = \frac{DG}{DE} \text{ (?).}$$

Multiplying, $$\frac{\triangle DGH}{\triangle DEF} = \frac{DH \cdot DG}{DE \cdot DF} \text{ (?).}$$

That is, $$\frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot AC}{DE \cdot DF} \text{ (Ax. 6).}$$ Q.E.D.

Ex. 1. Prove theorem of 386 by drawing through C a line parallel to AD, dividing the trapezoid into a parallelogram and a triangle.

Ex. 2. Which includes the other, the word "equal" or the word "equivalent"? Which of these words conveys no idea of shape?

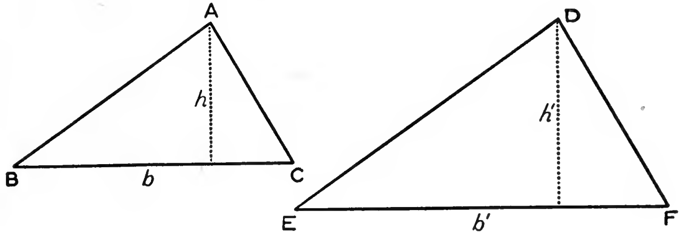
Ex. 3. What is the area of a parallelogram whose base is 8 inches and altitude is 5 inches? What is the area of a triangle having the same base and altitude?

Ex. 4. Is the area of a triangle equal to half the base multiplied by the whole altitude? Or half the altitude multiplied by the whole base? Or half the base multiplied by half the altitude? Or half the product of the base by the altitude?

Ex. 5. If, in the figure of 388, one triangle is four times as large as the other, $AB = 10$, $AC = 6$, $DE = 16$, find DF.

Ex. 6. The base of a triangle is 20 and its altitude is 15. The bases of an equivalent trapezoid are 13 and 11; find its altitude.

389. Theorem. **Two similar triangles are to each other as the squares of any two homologous sides.**



Given: Similar $\triangle$s ABC and DEF.

To Prove: $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\overline{AB}^2}{\overline{DE}^2} = \dfrac{\overline{AC}^2}{\overline{DF}^2} = \dfrac{\overline{BC}^2}{\overline{EF}^2}$.

Proof: One Method. $\angle BAC = \angle EDF$ (?) (323, 1).

$\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$ (?) (388).

That is, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB}{DE} \times \dfrac{AC}{DF}$. Now, $\dfrac{AC}{DF} = \dfrac{AB}{DE}$ (?).

$\therefore \dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB}{DE} \times \dfrac{AB}{DE} = \dfrac{\overline{AB}^2}{\overline{DE}^2}$ (Ax. 6).

But $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$ (?), and $\dfrac{\overline{AB}^2}{\overline{DE}^2} = \dfrac{\overline{AC}^2}{\overline{DF}^2} = \dfrac{\overline{BC}^2}{\overline{EF}^2}$ (297).

$\therefore \dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\overline{AB}^2}{\overline{DE}^2} = \dfrac{\overline{AC}^2}{\overline{DF}^2} = \dfrac{\overline{BC}^2}{\overline{EF}^2}$ (Ax. 1). Q.E.D.

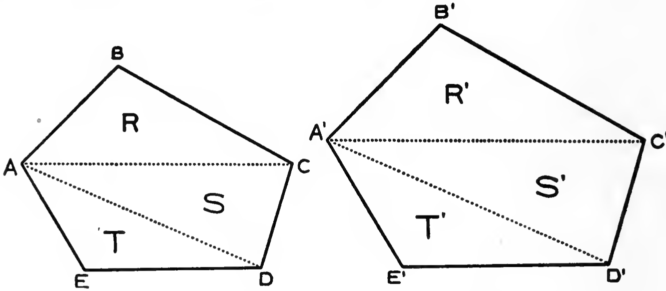
Another Method. Denote a pair of homologous altitudes by h and h', and the corresponding bases by b and b'.

$\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{b \cdot h}{b' \cdot h'} = \dfrac{b}{b'} \cdot \dfrac{h}{h'}$ (?) (384). But $\dfrac{h}{h'} = \dfrac{b}{b'}$ (?) (322).

Hence, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{b}{b'} \cdot \dfrac{b}{b'} = \dfrac{b^2}{b'^2}$ (Ax. 6).

That is, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\overline{BC}^2}{\overline{EF}^2} = \dfrac{\overline{AB}^2}{\overline{DE}^2} = \dfrac{\overline{AC}^2}{\overline{DF}^2}$. Q.E.D.

390. Theorem. Two similar polygons are to each other as the squares of any two homologous sides.



Given: Similar polygons $ABCDE$ and $A'B'C'D'E'$.

To Prove: $ABCDE : A'B'C'D'E' = \overline{AB}^2 : \overline{A'B'}^2 =$ etc.

Proof: Draw from homologous vertices, A and A', all the pairs of homologous diagonals, dividing the polygons into $\triangle$s. These $\triangle$s are similar, in pairs (?) (327).

$$\therefore \frac{\triangle R}{\triangle R'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \text{ (?) and } \frac{\triangle S}{\triangle S'} = \frac{\overline{CD}^2}{\overline{C'D'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \text{ (?) and}$$

$$\frac{\triangle T}{\triangle T'} = \frac{\overline{AE}^2}{\overline{A'E'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \text{ (?).}$$

Therefore, $$\frac{\triangle R}{\triangle R'} = \frac{\triangle S}{\triangle S'} = \frac{\triangle T}{\triangle T'} \text{ (Ax. 1).}$$

Hence, $$\frac{\triangle R + \triangle S + \triangle T}{\triangle R' + \triangle S' + \triangle T'} = \frac{\triangle R}{\triangle R'} \text{ (?) (301).}$$

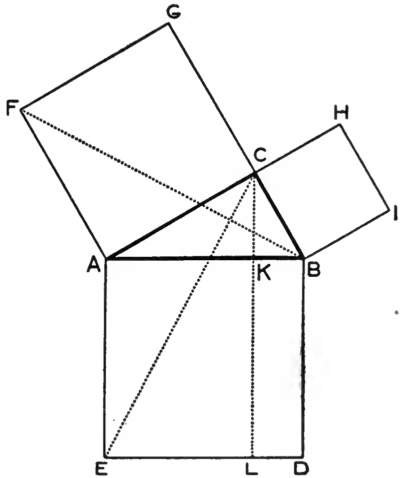
But $$\frac{\triangle R}{\triangle R'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \text{ (?).}$$

$$\therefore \frac{\text{polygon } ABCDE}{\text{polygon } A'B'C'D'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \text{etc. (?).}$$ Q.E.D.

Ex. 1. The base of a triangle is 6. Find the base of a similar triangle that is 9 times as large. Five times as large.

Ex. 2. The area of a polygon is 104 and its longest side is 12. What is the area of a similar polygon whose longest side is 15?

391. Theorem. **The square described upon the hypotenuse of a right triangle is equivalent to the sum of the squares described upon the legs.**



Given: (?).

To Prove: (?).

Proof: Draw *CL* ⊥ to *AB*, meeting *AB* at *K* and *ED* at *L*. Draw *BF* and *CE*.

Now, ∠s *ACB*, *ACG*, and *BCH* are all rt. ∠s (?).

Hence, *ACH* and *BCG* are straight lines (?) (45).

Also, *AELK* and *BDLK* are rectangles (?) (127).

In △s *ABF* and *ACE*, $AB = AE$, $AF = AC$ (128), and $\angle BAF = \angle CAE$. (Each = a rt. ∠ + ∠ *BAC*.)

∴ △ *ABF* = △ *ACE* (?) (52).

Also, △ *ABF* and square *AG* have the same base, *AF*, and the same altitude, *AC*.

Hence, square *AG* ⇔ 2 △ *ABF* (379).

Similarly, rectangle *AKLE* ⇔ 2 △ *ACE* (?).

Therefore, rectangle *AKLE* ⇔ square *ACGF* (Ax. 1).

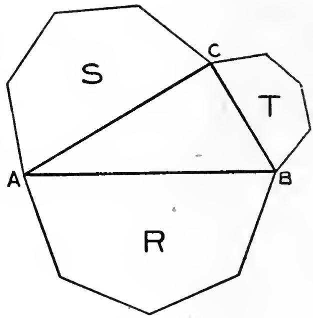
By drawing *AI* and *CD*, it is proved in like manner, that rectangle *BDLK* ⇔ square *BCHI*. Then, by adding, square *ABDE* ⇔ square *ACGF* + square *BCHI* (Ax. 2). Q.E.D.

392. Theorem. **The square described upon one of the legs of a right triangle is equivalent to the square described upon the hypotenuse minus the square described upon the other leg.** (Explain.)

Ex. 1. If the legs of a right triangle are 15 and 20, what is the hypotenuse? If the legs are $m^2 - 1$ and $2m$, what is the hypotenuse?

Ex. 2. What is the difference in the wording of the theorems of 343 and 391? Which proof is purely algebraic? Which is geometric?

393. THEOREM. *If the three sides of a right triangle are the homologous sides of three similar polygons, the polygon described upon the hypotenuse is equivalent to the sum of the two polygons described upon the legs.*



Proof: $\dfrac{S}{R} = \dfrac{\overline{AC}^2}{\overline{AB}^2}$ (?).

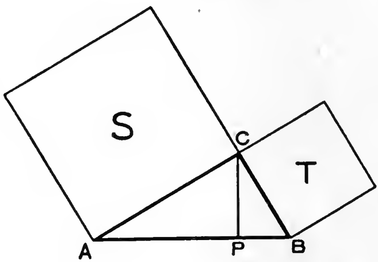
And $\dfrac{T}{R} = \dfrac{\overline{BC}^2}{\overline{AB}^2}$ (?). Adding,

$$\frac{S+T}{R} = \frac{\overline{AC}^2 + \overline{BC}^2}{\overline{AB}^2} = \frac{\overline{AB}^2}{\overline{AB}^2} = 1. \quad \text{(Explain.)}$$

$\therefore R \approx S + T$ (?). Q.E.D.

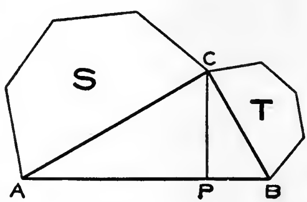
394. COR. *If the three sides of a right triangle are the homologous sides of three similar polygons, the polygon described upon one of the legs is equivalent to the polygon described upon the hypotenuse minus the polygon described upon the other leg.*

395. THEOREM. *The two squares described upon the legs of a right triangle are to each other as the projections of the legs upon the hypotenuse.*



Proof: $\dfrac{\text{Square } S}{\text{Square } T} = \dfrac{\overline{AC}^2}{\overline{BC}^2} = \dfrac{AB \cdot AP}{AB \cdot BP} = \dfrac{AP}{BP}$. (Explain.)

396. THEOREM. *If two similar polygons are described upon the legs of a right triangle as homologous sides, they are to each other as the projections of the legs upon the hypotenuse.*



Proof: $\dfrac{S}{T} = \dfrac{\overline{AC}^2}{\overline{BC}^2} = \dfrac{AP}{BP}$. (Explain.)

ORIGINAL EXERCISES (THEOREMS)

1. If one parallelogram has half the base and the same altitude as another, the area of the first is half the area of the second.

2 If one parallelogram has half the base and half the altitude of another, its area is one fourth the area of the second.

3. State and prove two analogous theorems about triangles.

4. If a triangle has half the base and half the altitude of a parallelogram, the triangle is one eighth of the parallelogram.

5. The area of a rhombus is equal to half the product of its diagonals.

6. The diagonals of a parallelogram divide it into four equivalent triangles.

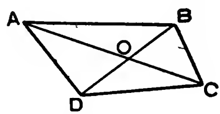
7. The diagonals of a trapezoid divide it into four triangles, two of which are similar and the other two are equivalent.

8. If a parallelogram has half the base and half the altitude of a triangle, its area is half the area of the triangle.

9. The line joining the midpoints of two sides of a triangle forms a triangle whose area is one fourth the area of the original triangle.

10. The line joining the midpoints of two adjacent sides of a parallelogram cuts off a triangle whose area is one eighth of the area of the parallelogram.

11. If one diagonal of a quadrilateral bisects the other, it also divides the quadrilateral into two equivalent triangles.



To Prove: $\triangle ABC \backsimeq \triangle ADC$.

12. Either diagonal of a trapezoid divides the figure into two triangles whose ratio is equal to the ratio of the bases of the trapezoid. Prove two ways. [By 382 and by 388.]

13. If, in triangle ABC, D and E are the midpoints of sides AB and AC respectively, $\triangle BCD \backsimeq \triangle BEC$.

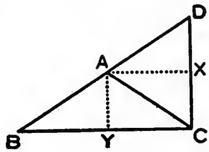
14. If the diagonals of quadrilateral $ABCD$ meet at E and $\triangle ABE \backsimeq \triangle CDE$, the sides AD and BC are parallel. [Prove $\triangle ABD \backsimeq \triangle ACD$.]

15. The square described upon the hypotenuse of an isosceles right triangle is equivalent to four times the triangle.

16. The square described upon the diagonal of a square is double the original square.

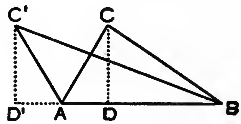
17. Any two sides of a triangle are reciprocally proportional to the altitudes upon them. [Use 378 and 291.]

18. In equivalent triangles the bases and the altitudes upon them are reciprocally proportional.



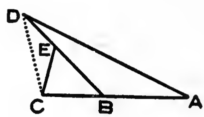
19. If two isosceles triangles have the legs of one equal to the legs of the other, and the vertex-angle of the one the supplement of the vertex-angle of the other, the triangles are equivalent.

Given: ▲ ABC and ACD, etc.



20. Two triangles are equivalent if they have two sides of one equal to two sides of the other and the included angles supplementary.

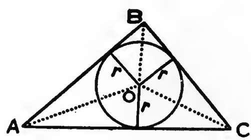
Proof: $\angle CAD = \angle C'AD'$ (?) and $CA = C'A$ (?). $\therefore$ the rt. ▲ are = (?). Etc.



21. If two triangles have an angle of one the supplement of an angle of the other, the triangles are to each other as the products of the sides including these angles.

Given: ▲ ABD and EBC, ∡ at B supplementary.

Proof: Draw DC, use $\triangle BCD$, and proceed as in 388.



22. The area of a triangle is equal to half the perimeter of the triangle multiplied by the radius of the inscribed circle.

Proof: Draw OA, etc. $\triangle AOC = \frac{1}{2}AC \cdot r$ (?); $\triangle AOB = \frac{1}{2}AB \cdot r$ (?), etc. Add.

23. The area of a polygon circumscribed about a circle is equal to half the product of the perimeter of the polygon by the radius of the circle.

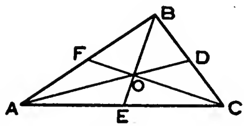
24. The line joining the midpoints of the bases of a trapezoid bisects the area of the trapezoid.

25. Any line drawn through the midpoint of a diagonal of a parallelogram, intersecting two sides, bisects the area of the parallelogram.

26. The lines joining (in order) the midpoints of the sides of any quadrilateral form a parallelogram whose area is half the area of the quadrilateral.

27. If any point within a parallelogram is joined to the four vertices, the sum of one pair of opposite triangles is equivalent to the sum of the other pair; that is, to half the parallelogram.

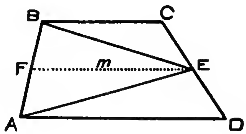
28. Is a triangle bisected by an altitude? By the bisector of an angle? By a median? By the perpendicular bisector of a side? Give reasons.



29. If the three medians of a triangle are drawn, there are six pairs of triangles formed, one of each pair being double the other.

To Prove: $\triangle AOB = 2 \triangle AOE$; etc.

30. If the midpoints of two sides of a triangle are joined to any point in the base, the quadrilateral formed is equivalent to half the original triangle.

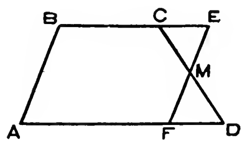


31. If lines are drawn from the midpoint of one leg of a trapezoid to the ends of the other leg, the middle triangle thus formed is equivalent to half the trapezoid.

Proof: Draw median $EF = m$. Then EF is ∥ to the bases (?). Denote the altitude of the trapezoid by h. Then EF bisects h (?). $\triangle BFE = \frac{1}{2} m \cdot \frac{1}{2} h$ (?).
$\triangle AEF = \frac{1}{2} m \cdot \frac{1}{2} h$ (?). $\therefore \triangle ABE = \frac{1}{2} mh$. Consult 387.

32. The area of a trapezoid is equal to the product of one of the non-parallel sides, by the perpendicular upon it from the midpoint of the other.

Proof: Prove that $\triangle ABE =$ half the trapezoid, by No. 31. But the $\triangle ABE = \frac{1}{2} AB \times$ the $\perp$ to AB from E (?).
$\therefore$ half the trapezoid $= \frac{1}{2} AB \times$ this $\perp$ (Ax. 1). Etc.



33. If through the midpoint of one of the non-parallel sides of a trapezoid a line is drawn parallel to the other side, the parallelogram formed is equivalent to the trapezoid.

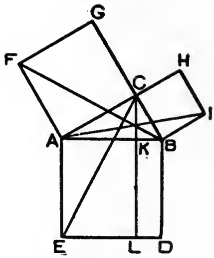
34. If two equivalent triangles have an angle of one equal to an angle of the other, the sides including these angles are reciprocally proportional.

35. The sum of the three perpendiculars drawn to the three sides of an equilateral triangle from any point within is constant (being equal to the altitude of the triangle).

Proof: Join the point to the vertices. Set the sum of the areas of the three inner $\triangle$s equal to the area of the whole $\triangle$. Etc.

36. In the figure of 391, prove:

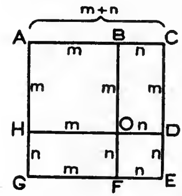
(*i*) Points I, C, and F are in a straight line.
(*ii*) CE and BF are perpendicular.
(*iii*) AG and BH are parallel.
(*iv*) $\triangle AEF \approx \triangle CGH \approx \triangle BDI \approx \triangle ABC$.
[See Ex. 20.]



37. The sum of the squares described upon the four segments of two perpendicular chords in a circle is equivalent to the square described upon the diameter. (Fig. is on page 178.)

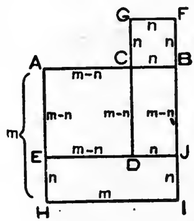
38. The square described upon the sum of two lines is equivalent to the sum of the squares described upon the two lines, plus twice the rectangle of these lines.

To Prove: Square $AE \approx m^2 + n^2 + 2\,mn$.



39. The square described upon the difference of two lines is equivalent to the sum of the squares described upon the two lines minus twice the rectangle of these lines.

To Prove: Square $AD = m^2 + n^2 - 2\,mn$.



40. A and B are the extremities of a diameter of a circle; C and D are the points of intersection of any third tangent to this circle, with the tangents at A and B respectively. Prove that the area of $ABDC$ is equal to $\frac{1}{2}\,AB \cdot CD$.

41. If the four points midway between the center and vertices of a parallelogram be joined in order, there will be a parallelogram formed; it will be similar to the original parallelogram; its perimeter is half of the perimeter of the original figure; and its area is one quarter of the area of the original figure.

42. If two equivalent triangles have the same base and lie on opposite sides of it, the line joining their vertices is bisected by the base.

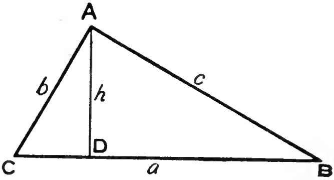
43. What part of a right triangle is the quadrilateral which is cut from the triangle by a line joining the midpoints of the legs?

44. Show by drawing a figure that the square on half a line is one fourth the square on the whole line.

45. From M, a vertex of parallelogram $LMNO$, a line MPX is drawn meeting NO at P and LO produced, at X. LP and NX are also drawn. Prove triangles LOP and XNP are equivalent.

FORMULAS

397. Problem. **To derive a formula for the area of a triangle in terms of its sides.**



Given: $\triangle ABC$, having sides $= a, b, c$.

Required: To derive a formula for its area, containing only a, b, and c.

Solution: Draw altitude AD.

Now $CD = {}_bp_a = \dfrac{a^2 + b^2 - c^2}{2a}$ (349),

and $\overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2$ (392).

$$\therefore h^2 = b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2 \text{ (Ax. 6).}$$

$$\text{Hence, } h^2 = \left\{b + \frac{a^2 + b^2 - c^2}{2a}\right\}\left\{b - \frac{a^2 + b^2 - c^2}{2a}\right\} \text{(by factoring),}$$

and
$$h^2 = \frac{2\,ab + a^2 + b^2 - c^2}{2a} \cdot \frac{2\,ab - a^2 - b^2 + c^2}{2a}$$

$$= \frac{(a + b + c)(a + b - c)(c + a - b)(c - a + b)}{4a^2}.$$

$$\therefore h = \sqrt{\frac{(a + b + c)(a + b - c)(a - b + c)(-a + b + c)}{4a^2}}.$$

Now, area of $\triangle = \dfrac{1}{2} a \cdot h =$

$$\frac{a}{2} \cdot \sqrt{\frac{(a + b + c)(a + b - c)(a - b + c)(-a + b + c)}{4a^2}}.$$

$$\therefore \text{area} = \tfrac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}. \quad \text{Q.E.F.}$$

To simplify this formula, let us call $a + b + c = 2\,s$.

Then, it is evident that $a + b - c = 2(s - c)$; $a - b + c = 2(s - b)$; $-a + b + c = 2(s - a)$.

Substituting, in the formula for area,

Area of $\triangle = \frac{1}{4}\sqrt{2s \cdot 2(s-c) \cdot 2(s-b) \cdot 2(s-a)}$.

That is,

$$\textbf{Area of } \triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

EXERCISE. Find the area of a triangle whose sides are **17, 25, 28.**

Here, $a = 17$, $b = 25$, $c = 28$, $s = 35$, $s - a = 18$, $s - b = 10$, $s - c = 7$.

Area $= \sqrt{35 \cdot 18 \cdot 10 \cdot 7} = \sqrt{7^2 \cdot 5^2 \cdot 2^2 \cdot 3^2} = 210$.

---

398. PROBLEM. **To derive formulas for the altitudes of a triangle in terms of the three sides.**

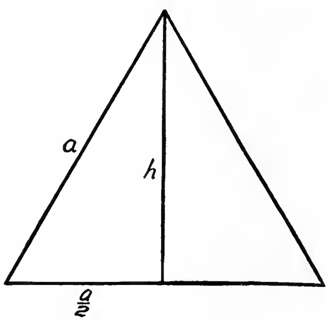
Solution: Area $= \frac{1}{2} a h_a = \sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore h_a = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}a}.$$

Similarly, $h_b = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}b}$; $h_c = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}c}$.

399. PROBLEM. **To derive the formulas for the altitude and the area of an equilateral triangle, in terms of its side.**

Solution: Let each side $= a$, and altitude $= h$.



Then, $h^2 = a^2 - \dfrac{a^2}{4} = \dfrac{3a^2}{4}$ (?).

$$\therefore h = \frac{a}{2}\sqrt{3}.$$

Also,

Area $= \dfrac{1}{2}$ base $\cdot h = \dfrac{1}{2} a \cdot \dfrac{a}{2}\sqrt{3}$.

$$\therefore \textbf{Area} = \frac{a^2\sqrt{3}}{4}.$$

Ex. 1. Find the area of the triangle whose sides are **7, 10, 11.**

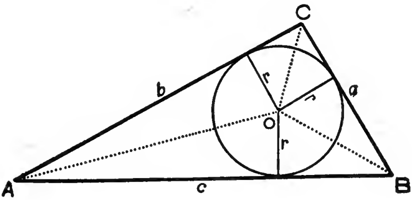
Ex. 2. Find the area of the triangle whose sides are 8, 15, 17.

Ex. 3. Find the area of the equilateral triangle whose side is 8.

Ex. 4. Find the side of the equilateral triangle whose area is $121\sqrt{3}$.

Ex. 5. Find the area of the equilateral triangle whose altitude is 10.

400. PROBLEM. To derive the formula for the radius of the circle inscribed in a triangle, in terms of the sides of the triangle.



Solution:

$$\left.\begin{array}{l}\text{Area of } \triangle AOB = \frac{1}{2} c \cdot r \\ \text{area of } \triangle AOC = \frac{1}{2} b \cdot r \\ \text{area of } \triangle BOC = \frac{1}{2} a \cdot r\end{array}\right\} (?).$$

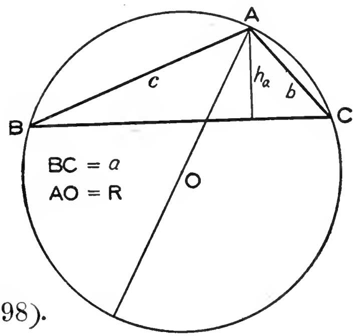
Adding, area of $\triangle ABC = \frac{1}{2}(a + b + c) r = sr$.

(Because, $\frac{1}{2}(a + b + c) = s$.)

Hence, $r = \dfrac{\text{area of } \triangle ABC}{s}$.

$$\therefore r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

401. PROBLEM. To derive the formula for the radius of the circle circumscribed about a triangle, in terms of the sides of the triangle.



Solution:

$2R \cdot h_a = b \cdot c$ (?) (337).

$$\therefore R = \frac{b \cdot c}{2 h_a}.$$

$$h_a = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2} a} \text{ (398)}.$$

$$\therefore R = \frac{a \cdot b \cdot c}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Ex. 1. Find the radius of the circle inscribed in, and the radius of the circle circumscribed about, the triangle whose sides are 17, 25, 28.

Ex. 2. Find for triangle whose sides are 11, 14, 17, the radii of the inscribed and circumscribed circles.

ORIGINAL EXERCISES (NUMERICAL)

1. The base of a parallelogram is 2 ft. 6 in. and its altitude is 1 ft. 4 in. Find the area. Find the side of an equivalent square.

2. The area of a rectangle is 540 sq. m. and its altitude is 15 m. Find its base and diagonal.

3. The base of a rectangle is 3 ft. 4 in. and its diagonal is 3 ft. 5 in. Find its area.

4. The bases of a trapezoid are 2 ft. 1 in., and 3 ft. 4 in., and the altitude is 1 ft. 2 in. Find the area.

5. The area of a trapezoid is 736 sq. in. and its bases are 3 ft. and 4 ft. 8 in. Find the altitude.

6. The area of a certain triangle whose base is 40 rd., is 3.2 A. Find the area of a similar triangle whose base is 10 rd. Find the altitudes of these triangles.

7. The base of a certain triangle is 20 cm. Find the base of a similar triangle four times as large; of one five times as large; twice as large; half as large; one ninth as large.

8. The altitude of a certain triangle is 12 and its area is 100. Find the altitude of a similar triangle three times as large. Find the base of a similar triangle seven times as large. Find the altitude and base of a similar triangle one third as large.

9. The area of a polygon is 216 sq. m. and its shortest side is 8 m. Find the area of a similar polygon whose shortest side is 10 m. Find the shortest side of a similar polygon four times as large; one tenth as large.

10. If the longest side of a polygon whose area is 567 is 14, what is the area of a similar polygon whose longest side is 12? of another whose longest side is 21?

11. Find the area of an equilateral triangle whose sides are each 6 in. Of another whose sides are each $10\sqrt{3}$ ft.

12. Find the area of an equilateral triangle whose altitude is 4 in.; of another whose altitude is 18 dm.

13. The area of an equilateral triangle is $64\sqrt{3}$. Find its side and its altitude.

14. The area of an equilateral triangle is 90 sq. m. Find its altitude.

15. Find the side of an equilateral triangle whose area is equal to a square whose side is 15 ft.

16. The equal sides of an isosceles triangle are each 17 in. and the base is 16 in. Find the area.

17. Find the area of an isosceles right triangle whose hypotenuse is 2 ft. 6 in.

18. Find the area of a square whose diagonal is 20 m.

19. There are two equilateral triangles whose sides are 33 and 56 respectively. Find the side of the third, equivalent to their sum. Find the side of the equilateral triangle equivalent to their difference.

20. There are two similar polygons two of whose homologous sides are 24 and 70. Find the side of a third similar polygon equivalent to their sum; the side of a similar polygon equivalent to their difference.

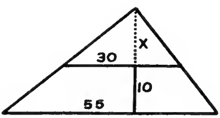
21. What is the area of the right triangle whose hypotenuse is 29 cm. and whose short leg is 20 cm.?

22. The base of a triangle is three times the base of an equivalent triangle. What is the ratio of their altitudes?

23. The bases of a trapezoid are 56 ft. and 44 ft. and the non-parallel sides are each 10 ft. Find its area. Also find the diagonal of an equivalent square.

24. The base of a triangle is 80 m., and its altitude is 8 m. Find the area of the triangle cut off by a line parallel to the base and at a distance of 3 m. from it. Another, cut off by a line parallel to the base and 6 m. from it.

25. The bases of a trapezoid are 30 and 55, and its altitude is 10. If the non-parallel sides are produced till they meet, find the area of the less triangle formed.



[The $\triangle$s are similar. $\therefore 30:55=x:x+10$. Etc.]

26. The diagonals of a rhombus are 2 ft. and 70 in. Find the area; the perimeter; the altitude.

27. The altitude (h) of a triangle is increased by n and the base (b) is diminished by x so the area remains unchanged. Find x.

28. The projections of the legs of a right triangle upon the hypotenuse are 8 and 18. Find the area of the triangle.

29. In triangle ABC, AB is 5, BC is 8, and AB is produced to P, making $BP=6$. BC is produced (through B) to Q and PQ drawn so the triangle BPQ is equivalent to triangle ABC. Find the length of BQ. [Use 388.]

30. The angle C of triangle ABC is right; $AC = 5$; $BC = 12$. BA is produced through A, to D making $AD = 4$; CA is produced through A, to E so triangle AED is equivalent to triangle ABC. Find AE.

31. Find the area of a square inscribed in a circle whose radius is 6.

32. Find the side of an equilateral triangle whose area is $25\sqrt{3}$.

33. Two sides of a triangle are 12 and 18. What is the ratio of the two triangles formed by the bisector of the angle between these sides?

34. The perimeter of a rectangle is 28 m. and one side is 5 m. Find the area.

35. The perimeter of a polygon is 5 ft. and the radius of the inscribed circle is 5 in. Find the area of the polygon.

In the following triangles, find the area, the three altitudes, radius of inscribed circle, radius of circumscribed circle:

36. $a = 13,\ b = 14,\ c = 15.$

37. $a = 15,\ b = 41,\ c = 52.$

38. 20, 37, 51. **39.** 25, 63, 74. **40.** 140, 143, 157.

41. The sides of a triangle are 15, 41, 52; find the areas of the two triangles into which this triangle is divided by the bisector of the largest angle.

42. Find the area of the quadrilateral $ABCD$ if $AB = 78$ m., $BC = 104$ m., $CD = 50$ m., $AD = 120$ m., and $AC = 130$ m.

43. One diagonal of a rhombus is $\frac{8}{15}$ of the other and the difference of the diagonals is 14. Find the area and perimeter of the rhombus.

44. A trapezoid is composed of a rhombus and an equilateral triangle; each side of each figure is 16 inches. Find the area of the trapezoid.

45. Find the side of an equilateral triangle equivalent to the square whose diagonal is $15\sqrt{2}$.

46. Which of the figures in No. 45 has the less perimeter?

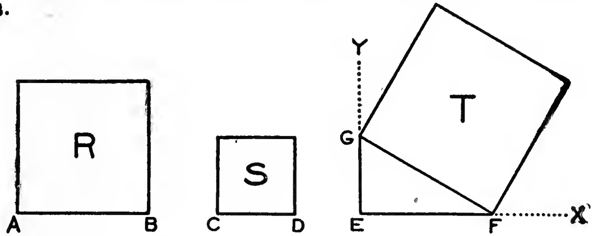
47. In a triangle whose base is 20 and whose altitude is 12, a line is drawn parallel to the base, bisecting the area of the triangle. Find the distance from the base to this parallel.

48. Parallel to the base of a triangle whose base is 30 and altitude is 18 are drawn two lines dividing the area of the triangle into three equal parts. Find their distances from the vertex.

49. Around a rectangular lawn 30 yards × 20 yards is a drive 16 feet wide. How many square yards are there in the drive?

CONSTRUCTIONS

402. PROBLEM. To construct a square equivalent to the sum of two squares.

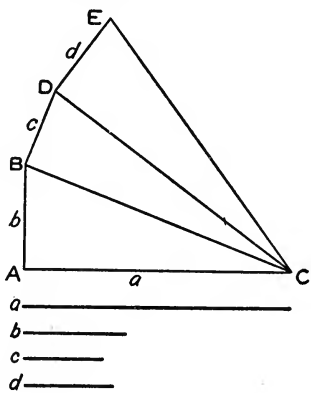


Given: (?). **Required:** (?). **Construction:** Construct a rt. $\angle E$, whose sides are EX and EY. On EX take $EF = AB$, and on EY take $EG = CD$. Draw FG. On FG construct square T.

Statement: $T \approx R + S$. Q.E.F.

Proof: $\overline{GF}^2 = \overline{EF}^2 + \overline{EG}^2$ (?). But $\overline{GF}^2 = T$; $\overline{EF}^2 = R$; $\overline{EG}^2 = S$ (?) (372). Hence, $T \approx R + S$ (Ax. 6). Q.E.D.

403. PROBLEM. To construct a square equivalent to the sum of several squares.



Given: Squares whose sides are a, b, c, d.

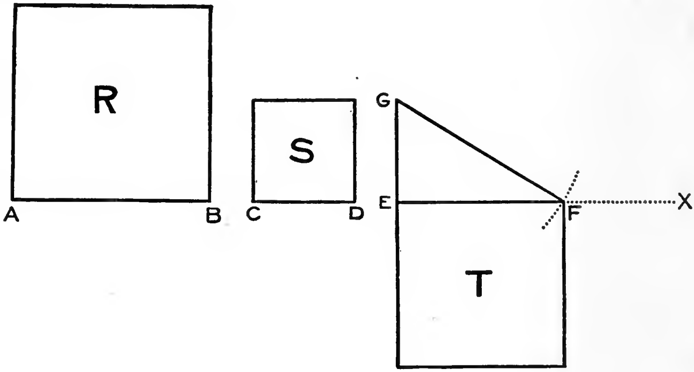
Required: To construct a square $\approx a^2 + b^2 + c^2 + d^2$.

Construction: Construct a rt. $\angle$ whose sides are equal to a and b. Draw hypotenuse BC. At B erect a $\perp = c$ and draw hypotenuse, DC. At D erect a $\perp = d$, etc.

Statement: The square constructed on EC is $\approx$ to the sum of the several given squares. Q.E.F.

Proof: $\overline{EC}^2 = \overline{DC}^2 + d^2 = (\overline{BC}^2 + c^2) + d^2$
$= a^2 + b^2 + c^2 + d^2$ (?) (391). Q.E.D.

404. Problem. **To construct a square equivalent to the difference of two given squares.**



Given: (?). **Required:** (?).

Construction: At one end of indefinite line, EX, erect EG $\perp$ to EX and $= CD$ (a side of the less square, S). Using G as center and AB as radius, describe arc intersecting EX at F. Draw GF. On EF construct square T.

Statement: $T \approx R - S$. Q.E.F. **Proof:** (?).

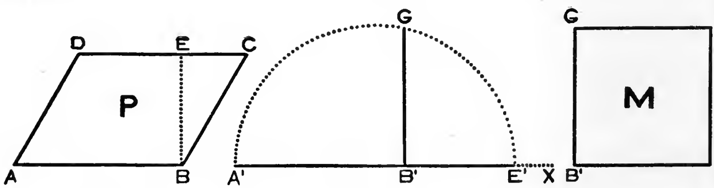
405. Problem. **To construct a polygon similar to two given similar polygons and equivalent to their sum.**

Construction: Like 402. **Proof:** (393).

406. Problem. **To construct a polygon similar to two given similar polygons and equivalent to their difference.**

Construction: Like 404. **Proof:** (394).

407. Problem. **To construct a square equivalent to a given parallelogram.**



Given: (?). **Required:** (?). **Construction:** Construct a mean proportional between the base, AB, and the altitude, BE; on this mean proportional, $B'G$, construct a square, M.

Statement: Square $M \Leftrightarrow$ parallelogram P. Q.E.F.

Proof: $AB : B'G = B'G : BE$ (Const.).

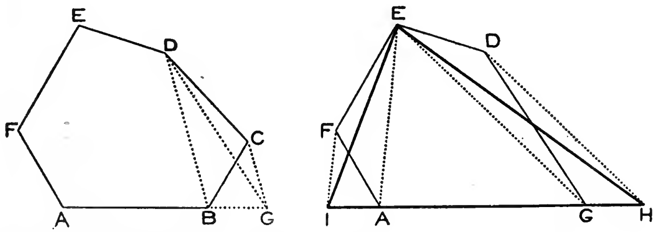
$\therefore \overline{B'G}^2 = AB \cdot BE$ (?). But $\overline{B'G}^2 = M$ (?).

And $AB \cdot BE = P$ (?) (371). Hence, $M \Leftrightarrow P$ (Ax. 6). Q.E.D.

408. Problem. **To construct a square equivalent to a given triangle.**

Construct a mean proportional between half the base and the altitude, and proceed as in 407.

409. Problem. **To construct a triangle equivalent to a given polygon.**



Given: Polygon AD. **Required:** To construct a $\triangle \Leftrightarrow AD$.

Construction: Draw a diagonal, BD, connecting any vertex (B) to the next but one (D). From the vertex between these (C), draw $CG \parallel$ to BD, meeting AB prolonged, at G. Draw DG. Repeat (2d figure) by drawing EG, then $DH \parallel$ to EG, and EH. Repeat again by drawing AE, $FI \parallel$ to AE, and EI.

Statement: $\triangle IEH \Leftrightarrow$ polygon $ABCDEF$. Q.E.F.

Proof: In 1st fig., $\triangle BGD \Leftrightarrow \triangle BCD$ (381; BD is the base).

Add $\underline{\text{polygon } ABDEF = \text{polygon } ABDEF.}$

$\therefore$ polygon $AGDEF \Leftrightarrow$ polygon $ABCDEF$ (Ax. 2).

Likewise, $AHEF \Leftrightarrow AGDEF$; and $\triangle IEH \Leftrightarrow AHEF$. (Explain.)

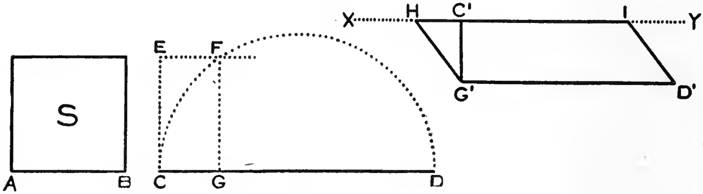
$\therefore$ $\triangle IEH \Leftrightarrow$ polygon $ABCDEF$ (Ax. 1). Q.E.D.

410. Problem. **To construct a square equivalent to a given polygon.** [Use 409 and 408.]

411. Problem. To construct a parallelogram (or a rectangle) equivalent to a given square, and having:

I. The sum of its base and altitude equal to a given line.

II. The difference of its base and altitude equal to a given line.



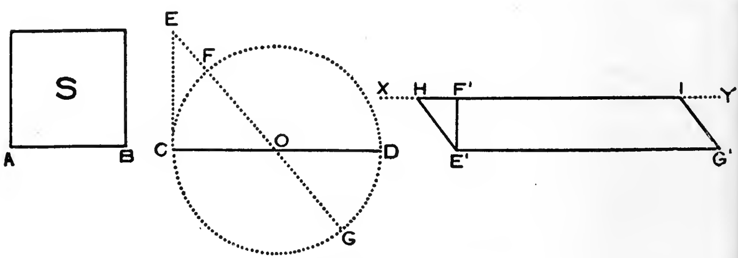
I. **Given:** Square S and line CD.

Required: To construct a $\square \Leftrightarrow S$, whose base + altitude shall $= CD$.

Construction: On CD as a diameter describe a semicircle. At C erect $CE \perp$ to CD and $= AB$. Through E draw $EF \parallel$ to CD, meeting the circumference at F. Draw $FG \perp$ to CD. Take $G'D' = GD$ and draw $XY \parallel$ to $G'D'$ at the distance from it $= CG$. On XY take $HI = GD$. Draw HG' and ID'.

Statement: $\square\ G'D'IH \Leftrightarrow S$ and base + alt. $= CD$. Q.E.F.

Proof: $G'D'IH$ is a $\square$ (?) (135). $GD \times CG = \overline{FG}^2$ (?) (341). But $GD \times CG =$ area $G'D'IH$ (?). $\overline{FG}^2 = \overline{EC}^2 =$ area S. (Explain.)
$\therefore \square\ G'D'IH \Leftrightarrow S$ (Ax. 6). Also $G'D' + G'C' = CD$ (?). Q.E.D.



II. **Given:** Square S and line CD.

Required: To construct a $\square \Leftrightarrow S$; base − altitude $= CD$.

Construction: On CD as diameter, describe a $\odot$, O. At C erect $CE \perp$ to CD and $= AB$. Draw $EFOG$ meeting $\odot$ at F and G. Take $E'G' = EG$ and draw $XY \parallel$ to $E'G'$ at a distance from it $= EF$. On XY take $HI = EG$. Draw HE' and IG'.

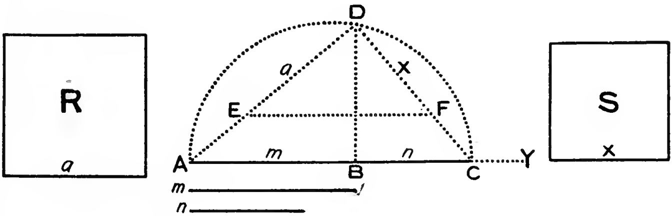
Statement: $\square\ E'G'IH \approx S$ and base $-$ alt. $= CD$. Q.E.F.

Proof: EC is tangent to $\odot\ O$ (?). $\therefore EG \cdot EF = \overline{EC}^2$ (?) (333). $EG \cdot EF =$ area $\square\ E'G'IH$ (?), and $\overline{EC}^2 = \overline{AB}^2 =$ area S (?). $\therefore \square\ E'G'IH \approx S$ (?). Also, $E'G' - E'F' = FG = CD$ (?). Q.E.D.

412. Problem. To find two lines whose product is given:

I. If their sum is also given.
II. If their difference is also given. } [The same as 411.]

413. Problem. To construct a square having a given ratio to a given square.



Given: Square R, and lines m and n.

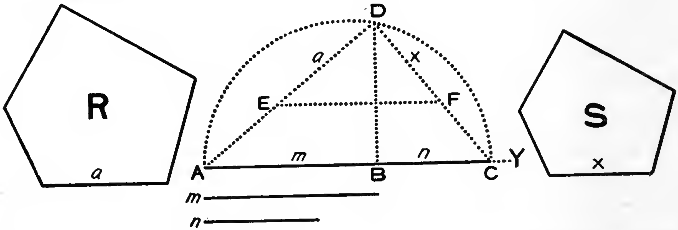
Required: To construct a square such that,
The square R : the unknown square $= m : n$.

Construction: On an indefinite line AY take $AB = m$, and $BC = n$. On AC as diameter describe a semicircle. At B erect $BD \perp$ to AC, meeting arc at D. Draw AD and DC. On AD take $DE = a$, and draw $EF \parallel$ to AC, meeting DC at F. Using $DF = x$, as a side, construct square S.

Statement: $R : S = m : n$. Q.E.F.

Proof: $\angle ADC$ is a rt.$\angle$ (?). $\therefore \overline{AD}^2 : \overline{DC}^2 = m : n$ (?) (395).
$\frac{a}{x} = \frac{AD}{DC}$ (?); $\therefore \frac{a^2}{x^2} = \frac{\overline{AD}^2}{\overline{DC}^2}$ (297). $\therefore \frac{a^2}{x^2} = \frac{m}{n}$ (Ax. 1).
$a^2 = R$; $x^2 = S$ (?). $\therefore R : S = m : n$ (Ax. 6). Q.E.D.

414. Problem. To construct a polygon similar to a given polygon and having a given ratio to it.



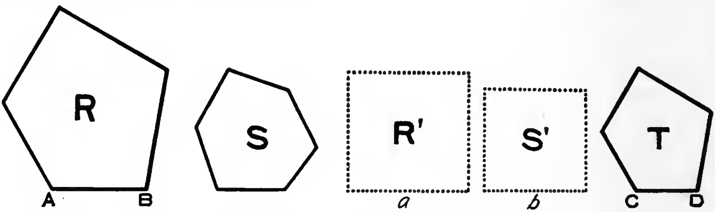
Given: (?). **Required:** (?). **Construction** and **Statement** are the same as in 413.

Proof: $\angle ADC$ is a rt. $\angle$ (?). $\therefore \overline{AD}^2 : \overline{DC}^2 = m : n$ (?)(395).

$\frac{a}{x} = \frac{AD}{DC}$(?); $\therefore \frac{a^2}{x^2} = \frac{\overline{AD}^2}{\overline{DC}^2} = \frac{m}{n}$. (Explain.) $\frac{R}{S} = \frac{a^2}{x^2}$ (?) (390).

$\therefore R : S = m : n$ (Ax. 1). Q.E.D.

415. Problem. To construct a polygon similar to one given polygon and equivalent to another.



Given: Polygons R and S. **Required:** (?).

Construction: Construct squares $R' \Leftrightarrow R$, and $S' \Leftrightarrow S$ (by 410). Find a fourth proportional to a, b, and AB. This is CD. Upon CD, homologous to AB, construct T similar to R.

Statement: $T \Leftrightarrow S$. Q.E.F.

Proof: $\frac{R}{T} = \frac{\overline{AB}^2}{\overline{CD}^2}$ (?) (390). $\frac{a}{b} = \frac{AB}{CD}$ (Const.). $\therefore \frac{a^2}{b^2} = \frac{\overline{AB}^2}{\overline{CD}^2}$ (?).

$\therefore \frac{R}{T} = \frac{a^2}{b^2}$ (Ax. 1). Now, $a^2 = R' \Leftrightarrow R$; $b^2 = S' \Leftrightarrow S$. (Explain.)

$\therefore \frac{R}{T} = \frac{R}{S}$ (Ax. 6). $\therefore T \Leftrightarrow S$ (Ax. 3). Q.E.D.

ORIGINAL CONSTRUCTIONS

1. To construct a square equivalent to a given right triangle.

2. To construct a right triangle equivalent to a given square.

3. To construct a right triangle equivalent to a given parallelogram.

4. To construct a square equivalent to the sum of two given right triangles.

5. To construct a square equivalent to the difference of two given right triangles.

6. To construct a square equivalent to the sum of two given parallelograms.

7. To construct a square equivalent to the difference of two given parallelograms.

8. To construct a square equivalent to the sum of several given right triangles.

9. To construct a square equivalent to the sum of several given parallelograms.

10. To construct a square equivalent to the sum of several given triangles.

11. To construct a square equivalent to the sum of several given polygons.

12. To construct a square equivalent to the difference of two given polygons.

13. To construct a square equivalent to three times a given square. To construct a square equivalent to seven times a given square.

14. To construct a right triangle equivalent to the sum of several given triangles.

15. To construct a right triangle equivalent to the difference of any two given triangles; of any two given parallelograms.

16. To construct a square equivalent to a given trapezoid; equivalent to a given trapezium.

17. To construct a square equivalent to a given hexagon.

18. To construct a rectangle equivalent to a given triangle, having given its perimeter.

19. To construct an isosceles right triangle equivalent to a given triangle.

20. To construct a square equivalent to a given rhombus.

21. To construct a rectangle equivalent to a given trapezium, and having its perimeter given.

22. To find a line whose length shall be $\sqrt{2}$ units. [See 402.]

23. To find a line whose length shall be $\sqrt{3}$ units.

24. To find a line whose length shall be $\sqrt{11}$ units.

25. To find a line whose length shall be $\sqrt{7}$ units.

26. To find a line whose length shall be $\sqrt{10}$ units.

27. To construct a square which shall be $\frac{3}{5}$ of a given square.

28. To construct a square which shall be $\frac{4}{9}$ of a given square.

29. To construct a polygon which shall be $\frac{2}{7}$ of a given polygon, and similar to it.

30. To construct a square which shall have to a given square the ratio $\sqrt{3} : 4$. If the given ratio is $4 : \sqrt{3}$.

31. To draw through a given point, within a parallelogram, a line which shall bisect the parallelogram.

32. To construct a rectangle equivalent to a given trapezoid, and having given the difference of its base and altitude.

33. To construct a triangle similar to two given similar triangles and equivalent to their sum.

34. To construct a triangle similar to a given triangle and equivalent to a given square. [See 415.]

35. To construct a triangle similar to a given triangle and equivalent to a given parallelogram.

36. To construct a square having twice the area of a given square. [Two methods.]

37. To construct a square having $3\frac{1}{2}$ times the area of a given square.

38. To construct an isosceles triangle equivalent to a given triangle and upon the same base.

39. To construct a triangle equivalent to a given triangle, having the same base, and also having a given angle adjoining this base.

40. To construct a parallelogram equivalent to a given parallelogram, having the same base and also having a given angle adjoining the base.

41. To draw a line that shall be perpendicular to the bases of a parallelogram and that shall bisect the parallelogram.

42. To construct an equilateral triangle equivalent to a given triangle. [See 415.]

43. To trisect (divide into three equivalent parts) the area of a triangle, by lines drawn from one vertex.

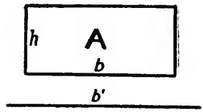
44. To construct a square equivalent to $\frac{2}{3}$ of a given pentagon.

45. To construct an isosceles trapezoid equivalent to a given trapezoid.

46. To construct an equilateral triangle equivalent to the sum of two given equilateral triangles.

47. To construct an equilateral triangle equivalent to the difference of two given equilateral triangles.

48. To construct upon a given base a rectangle that shall be equivalent to a given rectangle.

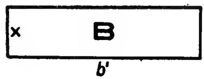


Analysis: Let us call the unknown altitude x. Then $b \cdot h = b' \cdot x$ (?). Hence, $b' : b = h : x$ (?).

That is, the *unknown altitude* is a fourth proportional to *the given base*, *the base of the given rectangle*, and *the altitude of the given rectangle.*

Construction: Find a fourth proportional, x, to b', b and h. Construct a rectangle having base $= b'$ and alt. $= x$.

Statement: This rectangle, $B \approx A$.



Proof: $b' : b = h : x$ (Const.). $\therefore b'x = bh$ (?). But $b'x =$ the area of B (?). Etc.

49. To construct a rectangle that shall have a given altitude and be equivalent to a given rectangle.

50. To construct a triangle upon a given base that shall be equivalent to a given triangle.

51. To construct a triangle that shall have a given altitude and be equivalent to a given triangle.

52. To construct a rectangle that shall have a given base, and shall be equivalent to a given triangle.

53. To construct a triangle that shall have a given base, and be equivalent to a given rectangle.

54. To construct a triangle that shall have a given base and be equivalent to a given polygon.

55. Construct the problems 49, 50, 51, 53, 54 if the first noun in each problem is the word "parallelogram."

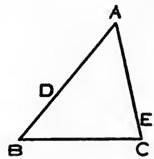
56. To construct upon a given hypotenuse, a right triangle equivalent to a given triangle.

57. To construct upon a given hypotenuse, a right triangle equivalent to a given square.

58. To construct a triangle which shall have a given base, a given adjoining angle, and be equivalent to a given triangle. Another, equivalent to a given square. Another, equivalent to a given polygon.

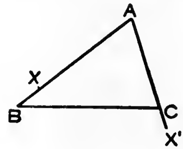
59. To construct a parallelogram which shall have a given base, a given adjoining angle, and be equivalent to a given parallelogram. Another, equivalent to a given triangle; to a given polygon.

60. To construct a line, DE, from D in AB of triangle ABC, so that DE bisects the triangle.



Analysis: After DE is drawn, $\triangle ABC = 2 \triangle ADE$ (Hyp.). But $\triangle ABC : \triangle ADE = AB \cdot AC : AD \cdot AE$ (?). Hence, $AB \cdot AC = 2\,(AD \cdot AE)$ (Ax. 6).
$\therefore 2\,AD : AB = AC : x$ (?). Thus x, (that is, AE) is a fourth proportional to three given lines.

61. To draw a line meeting two sides of a triangle and forming an isosceles triangle equivalent to the given triangle.



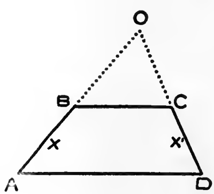
Analysis: Suppose AX a leg of isosceles $\triangle$.
$\therefore \triangle ABC : \triangle AXX' = AB \cdot AC : AX \cdot AX'$.
But the $\triangle$s are equivalent and $AX = AX'$ (Hyp.).
Hence, $AB \cdot AC = \overline{AX}^2$. $\therefore AX$ is a mean proportional between AB and AC.

62. To draw a line parallel to the base of a triangle which shall bisect the triangle. [See 389 and use 414.]

63. To draw a line meeting two sides of a triangle forming an isosceles triangle equivalent to half the given triangle.

64. To draw a line parallel to the base of a triangle forming a triangle equivalent to one third the original triangle.

65. To draw a line parallel to the base of a trapezoid so that the area is bisected.



Analysis: $\triangle OXX' \approx \frac{1}{2}\,(\triangle OAD + \triangle OBC)$ and is similar to $\triangle OBC$.

Construction: [Use 408, 402, 415.]

66. To construct two lines parallel to the base of a triangle, that shall trisect the area of the triangle.

67. To construct a triangle having given its angles and its area.

Analysis: The required $\triangle$ is similar to any $\triangle$ containing the given $\angle$s. The given area may be a square. This reduces the problem to 415.

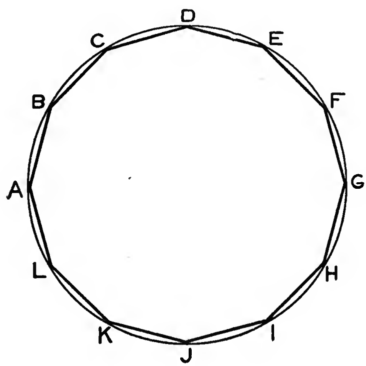
68. To find two straight lines in the ratio of two given polygons.

BOOK V

REGULAR POLYGONS. CIRCLES

416. A **regular polygon** is a polygon which is equilateral and equiangular.

417. THEOREM. **An equilateral polygon inscribed in a circle is regular.**



Given: *AG*, an equilateral inscribed polygon.

To Prove: *AG* is regular.

Proof: $\angle A$ is measured by $\frac{1}{2}$ arc *BGL* (?).

Also $\angle B$ is measured by $\frac{1}{2}$ arc *CHA* (?), etc.

Subtended arcs, *AB*, *BC*, *CD*, etc., are all = (?).

Hence, arcs *BGL*, *CHA*, *DIB*, etc., are all = (Ax. 2).

$\therefore \angle A = \angle B = \angle C =$ etc. (?).

That is, the polygon is equiangular.

Therefore, the polygon is regular (?) (416). Q.E.D.

418. THEOREM. **If the circumference of a circle be divided into any number of equal arcs, and the chords of these arcs be drawn, they will form an inscribed regular polygon.**

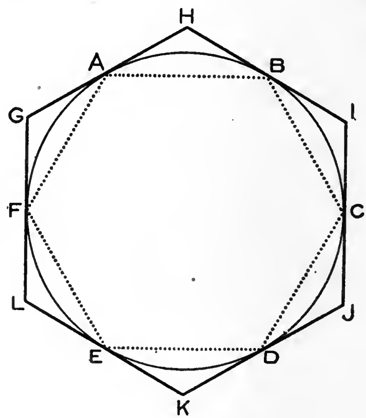
Proof: Chords *AB*, *BC*, *CD*, etc. are all = (?).

Therefore, the polygon is regular (?). Q.E.D.

Ex. 1. What is the usual name of a regular 3-gon? of a regular 4-gon? Is an equiangular inscribed polygon necessarily regular?

Ex. 2. In the figure of 417, how many degrees are there in each of the arcs, *AB*, *BC*, etc.? How many degrees are there in each angle?

419. THEOREM. **If the circumference of a circle be divided into any number of equal parts, and tangents be drawn, at the several points of division, they will form a circumscribed regular polygon.**



Given: (?). **To Prove**: (?).

Proof: Draw chords AB, BC, CD, etc.

In $\triangle ABH$, BCI, CDJ, etc., $AB = BC = CD =$ etc. (?).

$\angle HAB = \angle HBA = \angle IBC = \angle ICB = \angle JCD =$ etc. (?).

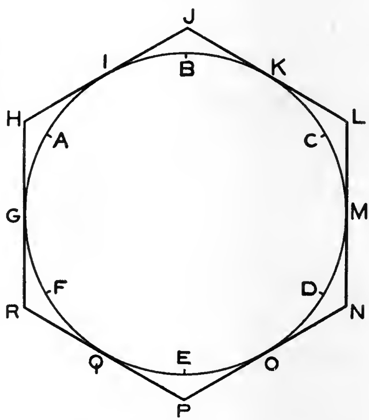
$\therefore$ these $\triangle$ are isosceles (?) (120), and $=$ (?).

$\therefore \angle H = \angle I = \angle J =$ etc. (?). That is, polygon GJ is equiangular.

Also, $AH = HB = BI = IC = CJ =$ etc. (?), and $HI = IJ = JK =$ etc. (?) (Ax. 3).

That is, polygon GJ is equilateral; $\therefore$ it is regular (?). Q.E.D.

420. THEOREM. **If the circumference of a circle be divided into any number of equal parts and tangents be drawn at their midpoints, they will form a circumscribed regular polygon.**



Given: (?). **To Prove**: (?).

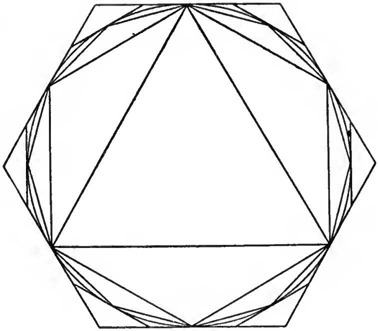
Proof: Arcs AB, BC, CD, etc. are all $=$ (?).

Also, arcs AI, IB, BK, KC, CM, etc. are all $=$ (?) (Ax. 3).

$\therefore$ arcs IK, KM, MO, etc. are all $=$ (?) (Ax. 3).

Therefore, the polygon is regular (?) (419). Q.E.D.

421. THEOREM. **If chords be drawn joining the alternate vertices of an inscribed regular polygon (having an even number of sides), another inscribed regular polygon will be formed.** (See 417.)

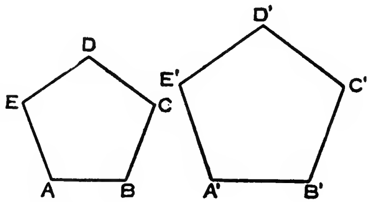


422. THEOREM. If the vertices of an inscribed regular polygon be joined to the midpoints of the arcs subtended by the sides, another inscribed regular polygon will be formed (having double the number of sides).

423. THEOREM. If tangents be drawn at the midpoints of the arcs between adjacent points of contact of the sides of a circumscribed regular polygon, another circumscribed regular polygon will be formed having double the number of sides (?).

424. THEOREM. The perimeter of an inscribed regular polygon is less than the perimeter of an inscribed regular polygon having twice as many sides, and the perimeter of a circumscribed regular polygon is greater than the perimeter of a circumscribed regular polygon having twice as many sides. (Ax. 12.)

425. THEOREM. Two regular polygons having the same number of sides are similar.



Given: Regular n-gons AD and $A'D'$.

To Prove: They are similar.

Proof: $\angle A = \dfrac{(n-2)\,180°}{n}$ (?) (164).

$\angle A' = \dfrac{(n-2)\,180°}{n}$ (?). $\therefore \angle A = \angle A'$ (?).

Similarly, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

That is, these polygons are mutually equiangular.

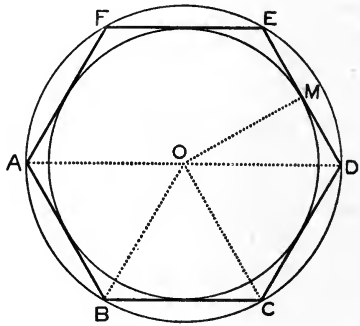
Also, $AB = BC = CD =$ etc.; $A'B' = B'C' = C'D' =$ etc. (?).

$\therefore AB : A'B' = BC : B'C' = CD : C'D' =$ etc. (Ax. 3).

That is, the homologous sides are proportional.

Therefore, the polygons are similar (?). Q.E.D.

426. Theorem. A circle can be circumscribed about, and a circle can be inscribed in, any regular polygon.



Given: Regular polygon *ABCDEF*.

To Prove: I. A circle can be circumscribed about the polygon.

II. A circle can be inscribed in the polygon.

Proof: I. Through three consecutive vertices, A, B, and C, describe a circumference, whose center is O. Draw radii OA, OB, OC, and draw line OD.

In $\triangle$ AOB and COD, $AB = CD$ (?) ; $BO = CO$ (?).

Now $\angle ABC = \angle BCD$ (?) (416). $\angle OBC = \angle OCB$ (?).
Subtracting, $\angle ABO = \angle OCD$ (Ax. 2).

$\therefore \triangle AOB = \triangle COD$ (?). $\therefore AO = OD$ (?).

Hence, the arc passes through D, and in like manner it may be proved that it passes through E and F.

That is, a circle can be circumscribed about the polygon.

II. AB, BC, CD, DE, etc. are = chords (?) (416).

Therefore they are equally distant from the center (?).

That is, a circle described, using O as a center and OM as a radius, will touch every side of the polygon.

Hence a circle can be inscribed. (Def. 234.) Q.E.D.

427. The **radius** of a regular polygon is the radius of the circumscribed circle. The radius of the inscribed circle is called the **apothem**. The **center** of a regular polygon is the common center of the circumscribed and inscribed circles.

428. The **central angle** of a regular polygon is the angle included between two radii drawn to the ends of a side.

429. Theorem. Each central angle of a regular n-gon $= \frac{360°}{n}$ (?).

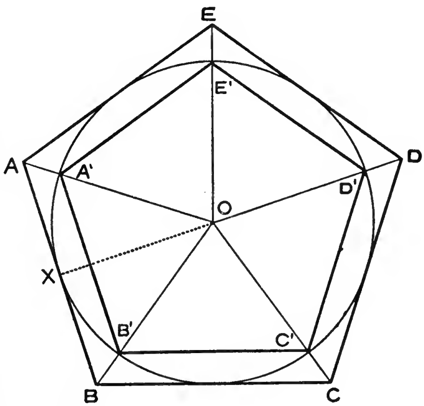
430. **Theorem.** Each exterior angle of a regular n-gon $= \frac{360°}{n}$ (?).

431. **Theorem.** The radius drawn to any vertex of a regular polygon bisects the angle at the vertex. (See 80.)

432. **Theorem.** The central angles of regular polygons having the same number of sides are equal. (See 429.)

433. **Theorem.** If radii be drawn to all the vertices of a circumscribed regular polygon, and chords be drawn connecting the points of intersection of these lines with the circumference, an inscribed regular polygon of the same number of sides will be formed and the sides of the two polygons will be respectively parallel.

Given: (?). **To Prove:** (?).



Proof: Central $\angle$s at O are all $=$ (?).

$\therefore$ the intercepted arcs are all $=$ (?).

$\therefore$ the chords $A'B'$, $B'C'$, etc. are all $=$ (?).

$\therefore$ the inscribed polygon is regular (?).

Also, $\triangle$s AOB, $A'OB'$, etc. are isosceles (?).

If a line OX be drawn from O, bisecting $\angle AOB$, it is $\perp$ to AB and to $A'B'$ (?) (57).

$\therefore AB$ is $\parallel$ to $A'B'$ (?). Q.E.D.

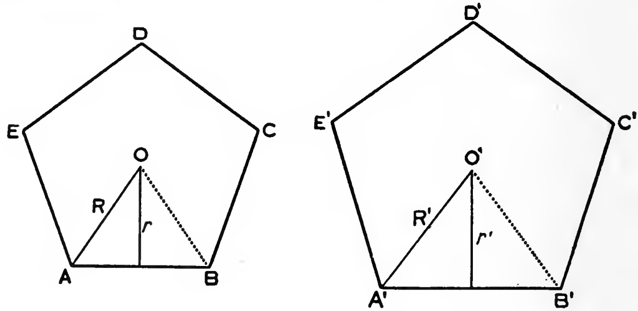
Ex. 1. How many degrees are there in the angle, in the central angle, and in the exterior angle of a regular hexagon? a regular decagon? a regular 15-gon?

Ex. 2. In the figure of 433, prove,

(*a*) Triangle AOB similar to triangle $A'OB'$.

(*b*) Triangle XOB similar to triangle $X'OB'$.

434. Theorem. The perimeters of two regular polygons having the same number of sides are to each other as their radii and also as their apothems.



Given: Regular n-gons, EC whose perimeter is P, radius R, apothem r; and $E'C'$ whose perimeter is P', radius R', apothem r'.

To Prove: $P : P' = R : R' = r : r'$.

Proof: Draw radii OB and $O'B'$. In $\triangle$s AOB and $A'O'B'$, $\angle AOB = \angle A'O'B'$ (?) (432); $AO = BO$ and $A'O' = B'O'$ (?) (200).

Hence, $\dfrac{AO}{A'O'} = \dfrac{BO}{B'O'}$ (Ax. 3).

$\therefore \triangle AOB$ is similar to $\triangle A'O'B'$ (?) (317).

$\therefore \dfrac{AB}{A'B'} = \dfrac{R}{R'} = \dfrac{r}{r'}$ (?). Also, the polygons are similar (?).

$\therefore \dfrac{P}{P'} = \dfrac{AB}{A'B'}$ (?) (326). Hence, $\dfrac{P}{P'} = \dfrac{R}{R'} = \dfrac{r}{r'}$ (Ax. 1). Q.E.D.

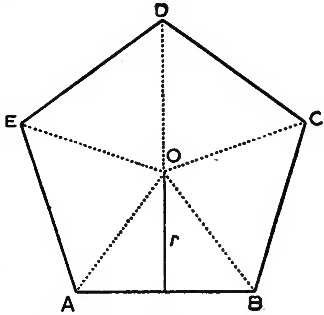
435. Theorem. The areas of two regular polygons having the same number of sides are to each other as the squares of their radii and also as the squares of their apothems.

Proof: If K and K' denote their areas, we have:

$\dfrac{K}{K'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$ (?) (390). But $\dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{R^2}{R'^2} = \dfrac{r^2}{r'^2}$. (Explain.)

Hence, $\dfrac{K}{K'} = \dfrac{R^2}{R'^2} = \dfrac{r^2}{r'^2}$ (Ax. 1). Q.E.D.

436. Theorem. The area of a regular polygon is equal to half the product of the perimeter by the apothem.



Given: (?).

To Prove: (?).

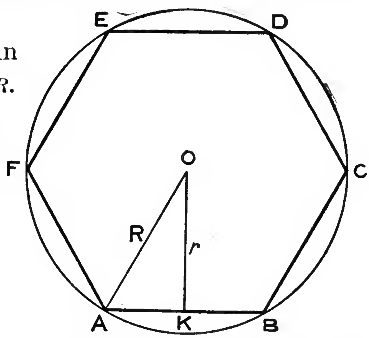
Proof: Draw radii to all the vertices, forming several isosceles triangles.

$$\left.\begin{array}{l}\text{Area of } \triangle AOB = \frac{1}{2} AB \cdot r \\ \text{Area of } \triangle BOC = \frac{1}{2} BC \cdot r \\ \text{Area of } \triangle COD = \frac{1}{2} CD \cdot r \\ \qquad \text{etc., etc.}\end{array}\right\} (?).$$

Area of polygon $= \frac{1}{2}(AB + BC + CD + \text{etc.}) \cdot r$ (?),

or, area $= \frac{1}{2}\ P \cdot r$ (Ax. 6). Q.E.D.

437. Theorem. If the number of sides of an inscribed regular polygon be increased indefinitely, the apothem will approach the radius as a limit.



Given: N-gon FC inscribed in ⊙ O; apothem $= r$; radius $= R$.

To Prove: That as the number of sides is indefinitely increased, r approaches R as a limit.

Proof: In the $\triangle AOK$,

$R < r + AK$ (?),

or $R - r < AK$ (?).

Now as the number of sides of the polygon is indefinitely increased, AB will be indefinitely decreased.

Hence, $\frac{1}{2}\ AB$, or AK, will approach zero as a limit.

$\therefore R - r$ will approach zero (because $R - r < AK$).

That is, r will approach R as a limit (240). Q.E.D.

Note. It is evident that if the difference between two variables approaches zero, either

(1) one is approaching the other as a limit; or

(2) both are approaching some third quantity as their limit.

Ex. 1. The apothem of a regular polygon perpendicular to a side bisects the side and the central angle of the polygon.

Ex. 2. Are all pairs of squares similar? Why? Are all pairs of rhombuses similar? Why? Is a rhombus a regular polygon? Is a rectangle a regular polygon?

Ex. 3. What is the area of a square whose perimeter is 40 inches?

Ex. 4. What is the area of a regular hexagon whose side is 8 and apothem is $4\sqrt{3}$?

Ex. 5. What is the area of a regular dodecagon whose side is $10\sqrt{2-\sqrt{3}}$ and whose apothem is $5\sqrt{2+\sqrt{3}}$?

Ex. 6. Prove that the lines joining the midpoints of the sides of a regular polygon, in order, form a regular polygon of the same number of sides.

Ex. 7. Prove that a polygon is regular if the inscribed and circumscribed circles are concentric.

Ex. 8. Draw a figure showing an inscribed equiangular polygon that is not regular.

Ex. 9. Draw a figure showing a circumscribed equilateral polygon that is not regular.

Ex. 10. Prove that the circumference of a circle is greater than the perimeter of any inscribed polygon.

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438. The theorems of 439 and 440 are considered so evident, and rigorous proofs are so difficult for young students to comprehend (like the demonstrations for many fundamental principles in mathematics) that it is advisable to omit the profound demonstrations and insert only simple explanations.

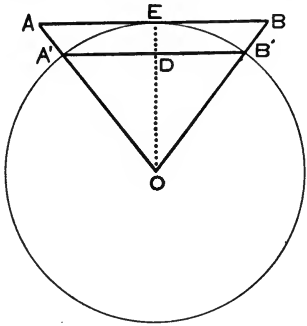
439. Theorem. **The circumference of a circle is less than the perimeter of any circumscribed polygon.**

By drawing tangents at the midpoints of the included arcs another circumscribed polygon is formed; the perimeter of this polygon is less than the perimeter of the given polygon (?). This can be continued indefinitely, decreasing the perimeter of the polygons. Hence, there can be no circumscribed polygon whose perimeter can be the least of all such polygons; because, by increasing the number of sides, the perimeter is lessened. Hence, the circumference must be less than the perimeter of any circumscribed polygon.

440. Theorem. If the number of sides of an inscribed regular polygon and of a circumscribed regular polygon be indefinitely increased,

I. The perimeter of each polygon will approach the circumference of the circle as a limit.

II. The area of each polygon will approach the area of the circle as a limit.



Given: A circle O, whose circumference is C and area is S; AB and $A'B'$, sides of regular circumscribed and inscribed polygons, having the same number of sides; P and P', their perimeters; K and K', their areas.

To Prove: That if the number of sides be indefinitely increased:

I. P will approach C and P' will approach C as limit.

II. K will approach S and K' will approach S as limit.

Proof: I. The polygons are similar (?) (425).

$\therefore \frac{P}{P'} = \frac{OE}{OD}$ (?) (434). Now, if the number of sides of these polygons be indefinitely increased, OD will approach OE (?) (437).

Hence, $\frac{OE}{OD}$ will approach 1. That is, $\frac{P}{P'}$ will approach 1, or P and P' will approach equality; that is, they will approach the same constant as a limit.

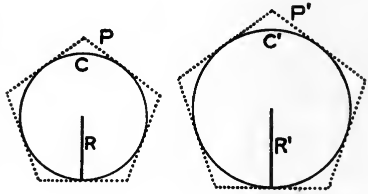
But $P > C$ and $C > P'$ and C is constant.

Hence, P will approach C and P' will approach C. Q.E.D.

II. $\frac{K}{K'} = \frac{\overline{OE}^2}{\overline{OD}^2}$ (?) (435). If the number of sides of these polygons be indefinitely increased, $\overline{OD}^2$ will approach $\overline{OE}^2$, and thus $\frac{\overline{OE}^2}{\overline{OD}^2}$ will approach unity.

(The argument continues the same as in I.)

441. Theorem. The circumferences of two circles are to each other as their radii.



Given: Two ⓢ whose radii are R and R' and circumferences, C and C' respectively.

To Prove: $C : C' = R : R'$.

Proof: Circumscribe regular polygons (having the same number of sides) about these ⓢ and let P and P' denote their perimeters. Then, $P : P' = R : R'$ (?) (434). Hence, $P \cdot R' = P' \cdot R$ (?). Now suppose the number of sides of these polygons to be indefinitely increased,

P will approach C (?) (440).

P' will approach C' (?).

$\therefore P \cdot R'$ will approach $C \cdot R'$,

and $P' \cdot R$ will approach $C' \cdot R$.

Hence, $C \cdot R' = C' \cdot R$ (?) (242).

Therefore $C : C' = R : R'$ (?) (291). Q.E.D.

442. Theorem. The ratio of any circumference to its diameter is constant for all circles. That is, any circumference divided by its diameter is the same as any other circumference divided by its diameter.

Proof: $\dfrac{C}{C'} = \dfrac{R}{R'}$ (?) (441). But $\dfrac{R}{R'} = \dfrac{2R}{2R'} = \dfrac{D}{D'}$ (?).

$$\therefore \frac{C}{C'} = \frac{D}{D'} \text{ (Ax. 1).}$$

Hence, $\dfrac{C}{D} = \dfrac{C'}{D'}$ (?) (292). That is, $\dfrac{C}{D} =$ constant. Q.E.D.

443. Definition of π (pi). The constant ratio of a circumference to its diameter is called π. That is, $\dfrac{C}{D} = \pi$.

The numerical value of $\pi = 3.141592 = 3\frac{1}{7}$, approximately. (This is determined in 470.)

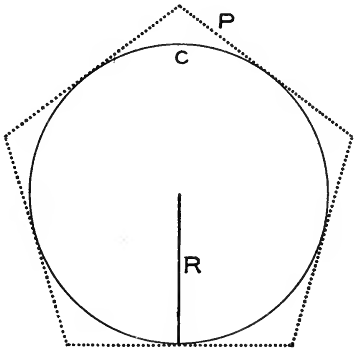
444. FORMULA. Let C = circumference and R = radius.

Then, $\frac{C}{2R} = \pi$ (443). $\therefore \boldsymbol{C = 2\pi R}$ (Ax. 3).

445. THEOREM. **The area of a circle is equal to half the product of its circumference by its radius.**

Given: ⊙ whose circumference $= C$, area $= S$, radius $= R$.

To Prove: $S = \frac{1}{2} C \cdot R$.



Proof: Circumscribe a regular polygon about the circle; denote its area by K and perimeter by P.

Now $K = \frac{1}{2} P \cdot R$ (?) (436).

Suppose the number of sides of the polygon be indefinitely increased.

K will approach S, and P will approach C (?).

$\frac{1}{2} P \cdot R$ will approach $\frac{1}{2} C \cdot R$ as a limit (?).

Hence, $S = \frac{1}{2} C \cdot R$ (?) (242). Q.E.D.

446. FORMULA. Let S = area of ⊙, C = its circumference, and R = its radius. Then, $S = \frac{1}{2} C \cdot R$ (445).

Now $C = 2\pi R$ (444). Substituting, $S = \frac{1}{2}(2\pi R) R$.

$$\therefore \boldsymbol{S = \pi R^2}.$$

Ex. 1. Could 445 be proved by inscribing a regular polygon? Why?

Ex. 2. The radius of a circle is 40. Find the circumference and area.

Ex. 3. The diameter of a circle is 25. Find the circumference and area.

Ex. 4. Prove that the area of a circle equals .7854 D^2.

447. THEOREM. **The areas of two circles are to each other as the squares of their radii, and as the squares of their diameters.**

To Prove: $S : S' = R^2 : R'^2 = D^2 : D'^2$.

Proof: $\left\{\begin{matrix} S = \pi R^2 \\ S' = \pi R'^2 \end{matrix}\right\}$ (?) (446). $\therefore \dfrac{S}{S'} = \dfrac{\pi R^2}{\pi R'^2} = \dfrac{R^2}{R'^2}$ (Ax. 3).

$$\text{And } \frac{S}{S'} = \frac{(\frac{1}{2} D)^2}{(\frac{1}{2} D')^2} = \frac{\frac{1}{4} D^2}{\frac{1}{4} D'^2} = \frac{D^2}{D'^2} \text{ (?).}$$

$$\therefore S : S' = R^2 : R'^2 = D^2 : D'^2 \text{ (Ax. 1).}$$ Q.E.D.

448. THEOREM. **The area of a sector is the same part of the circle as its central angle is of 360°.** (Ax. 1.)

449. FORMULA. An arc : circum. = central $\angle$: 360° (244).

$$\therefore \textbf{arc} : 2\pi \boldsymbol{R} = \angle : \textbf{360°}.$$

NOTE. If any two of the three quantities, arc, R, $\angle$, are known, the remaining one can be found by this proportion.

450. FORMULA. Sector : area of $\odot$ = central $\angle$: 360° (448).

$$\therefore \textbf{sector} : \pi \boldsymbol{R}^2 = \angle : \textbf{360°}.$$

NOTE. If any two of the three quantities, sector, R, $\angle$, are known, the remaining one can be found by this proportion.

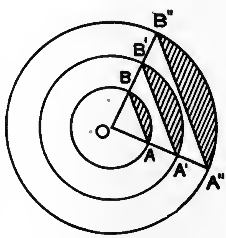
451. FORMULA. Sector : area of $\odot$ = arc : circum. (Ax. 1).

$\therefore$ sector : πR^2 = arc : $2\pi R$ (Ax. 6).

$$\therefore \textbf{sector} = \tfrac{1}{2} \boldsymbol{R} \cdot \textbf{arc (290).}$$

452. FORMULA. **Area of a segment of a circle = area of the sector minus* area of an isosceles triangle.**

453. **Similar arcs, similar sectors,** and **similar segments** are those which correspond to equal central angles, in unequal circles.



Thus, AB, $A'B'$, $A''B''$ are similar arcs; AOB, $A'OB'$, and $A''OB''$ are similar sectors; and the shaded segments are similar segments.

* If the segment is greater than a semicircle, the area of the triangle should be *added*.

454. THEOREM. Similar arcs are to each other as their radii.

Given: Arcs whose lengths are a and a', radii R and R'.

To Prove: $a : a' = R : R'$.

Proof: $\dfrac{a}{2\pi R} = \dfrac{\angle}{360°}$ and $\dfrac{a'}{2\pi R'} = \dfrac{\angle}{360°}$ (?) (449).

$\therefore \dfrac{a}{2\pi R} = \dfrac{a'}{2\pi R'}$ (Ax. 1).

$\therefore a : a' = 2\pi R : 2\pi R'$ (292); $\therefore a : a' = R : R'$. Q.E.D.

455. THEOREM. Similar sectors are to each other as the squares of their radii.

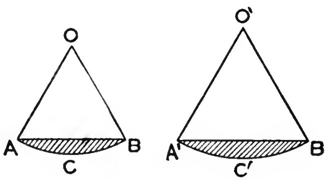
Given: Sectors whose areas are T and T', radii R and R'.

To Prove: $T : T' = R^2 : R'^2$.

Proof: $\dfrac{T}{\pi R^2} = \dfrac{\angle}{360}$ and $\dfrac{T'}{\pi R'^2} = \dfrac{\angle}{360}$ (?).

$\therefore \dfrac{T}{\pi R^2} = \dfrac{T'}{\pi R'^2}$ (?). $\therefore T : T' = R^2 : R'^2$. (Explain.) Q.E.D.

456. THEOREM. Similar segments are to each other as the squares of their radii.



Given: (?). **To Prove:** (?).

Proof: $\triangle$s AOB and $A'O'B'$ are similar (?) (317).

$\therefore \dfrac{\triangle AOB}{\triangle A'O'B'} = \dfrac{R^2}{R'^2}$ (?) (389),

and $\dfrac{\text{sector } OACB}{\text{sector } O'A'C'B'} = \dfrac{R^2}{R'^2}$ (?) (455).

$\therefore \dfrac{\text{sector } OACB}{\text{sector } O'A'C'B'} = \dfrac{\triangle AOB}{\triangle A'O'B'}$ (Ax. 1).

$\therefore \dfrac{\text{sector } OACB - \triangle AOB}{\text{sector } O'A'C'B' - \triangle A'O'B'} = \dfrac{\triangle AOB}{\triangle A'O'B'}$ (295, Note II).

Hence, $\dfrac{\text{segment } ABC}{\text{segment } A'B'C'} = \dfrac{\triangle AOB}{\triangle A'O'B'} = \dfrac{R^2}{R'^2}$ (Ax. 6). Q.E.D.

ORIGINAL EXERCISES (THEOREMS)

1. The central angle of a regular polygon is the supplement of the angle of the polygon.

2. An equiangular polygon inscribed in a circle is regular (if the number of its sides is odd).

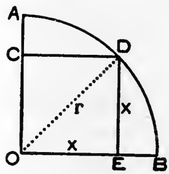
3. An equiangular polygon circumscribed about a circle is regular. [Draw radii and apothems.]

4. The sides of a circumscribed regular polygon are bisected at the points of contact.

5. The diagonals of a regular pentagon are equal.

6. The diagonals drawn from any vertex of a regular n-gon divide the angle at that vertex into $n-2$ equal parts.

7. If a regular polygon be inscribed in a circle and another regular polygon having the same number of sides be circumscribed about it, the radius of the circle will be a mean proportional between the apothem of the inner and the radius of the outer polygon.



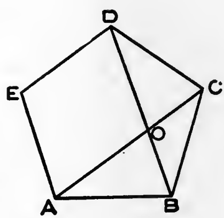
8. The area of the square inscribed in a sector whose central angle is a right angle is equal to half the square of the radius.

[Find x^2, the area of $OEDC$.]

9. The apothem of an equilateral triangle is one third the altitude of the triangle.

10. The chord which bisects a radius of a circle at right angles is the side of the inscribed equilateral triangle.

[Prove the central $\angle$ subtended is 120°.]



11. If $ABCDE$ is a regular pentagon, and diagonals AC and BD be drawn, meeting at O:

(*a*) AO will $= AB$.

(*b*) AO will be $\parallel$ to ED.

(*c*) $\triangle BOC$ will be similar to $\triangle BDC$.

(*d*) $\angle ACB$ will $= 36°$.

(*e*) AC will be divided into mean and extreme ratio at O.

12. The altitude of an equilateral triangle is three fourths the diameter of the circumscribed circle.

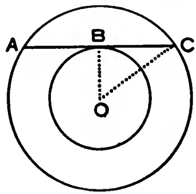
13. The apothem of an inscribed regular hexagon equals half the side of an inscribed equilateral triangle.

14. The area of a circle is four times the area of another circle described upon its radius as a diameter.

15. The area of an inscribed square is half the area of the circumscribed square.

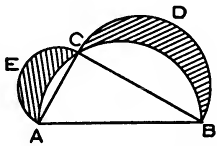
16. An equilateral polygon circumscribed about a circle is regular (if the number of its sides is odd).

17. The sum of the circles described upon the legs of a right triangle as diameters is equivalent to the circle described upon the hypotenuse as a diameter.



18. A circular ring (the area between two concentric circles) is equivalent to the circle described upon the chord of the larger circle, which is tangent to the less, as a diameter.

Proof: Draw radii OB, OC. $\triangle OBC$ is rt. $\triangle$ (?); and $\overline{OC}^2 - \overline{OB}^2 = \overline{BC}^2$ (?). Etc.



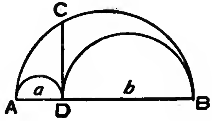
19. If semicircles be described upon the three sides of a right triangle (on the same side of the hypotenuse), the sum of the two crescents thus formed will be equivalent to the area of the triangle.

Proof: $\begin{cases} \text{Entire figure} \approx \frac{1}{8}\pi\overline{AB}^2 + \text{crescent } BDC + \text{crescent } AEC \text{ (?)}. \\ \text{Entire figure} \approx \frac{1}{8}\pi\overline{AC}^2 + \frac{1}{8}\pi\overline{BC}^2 + \triangle ABC \text{ (?)}. \end{cases}$

Now use Ax. 1; etc.

20. Show that the theorem of No. 19 is true in the case of a right triangle whose legs are 18 and 24.

21. If from any point in a semicircumference a line be drawn perpendicular to the diameter and semicircles be described on the two segments of the hypotenuse as diameters, the area of the surface bounded by these three semicircumferences will equal the area of a circle whose diameter is the perpendicular first drawn.



Proof: Area $= \frac{1}{2}\pi\left(\frac{a+b}{2}\right)^2 - \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 - \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 =$ etc.

22. Show that the theorem of No. 21 is true in the case of a circle whose diameter AB is 25 and AD is 5.

23. If the sides of a circumscribed regular polygon are tangent to the circle at the vertices of an inscribed regular polygon, each vertex of the outer lies on the prolongation of the apothems of the inner polygon, drawn perpendicular to the several sides.

24. The sum of the perpendiculars drawn from any point within a regular n-gon to the several sides is constant $[= n \cdot \text{apothem}]$.

Proof: Draw lines from the point to all vertices. Use 436 and 378.

25. The area of a circumscribed equilateral triangle is four times the area of the inscribed equilateral triangle.

26. If a point be taken dividing the diameter of a circle into two parts and circles be described upon these parts as diameters, the sum of the circumferences of these two circles equals the circumference of the original circle.

27. Show that the theorem of No. 26 is true in the case of a circle the segments of whose diameter are 7 and 12.

28. The area of an inscribed regular octagon is equal to the product of the diameter by the side of the inscribed square.

29. If squares be described on the six sides of a regular hexagon (externally), the twelve exterior vertices of these squares will be the vertices of a regular 12-gon.

30. If the alternate vertices of a regular hexagon be joined by drawing diagonals, another regular hexagon will be formed. Also its area will be one third the original hexagon.

31. Show that the theorem of No. 18 is true in the case of two concentric circles whose radii are 34 and 16.

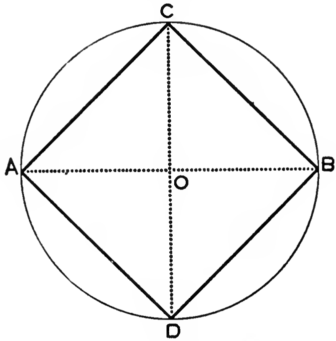
32. In the same or equal circles two sectors are to each other as their central angles.

33. If the diameter of a circle is 10 in. and a point be taken dividing the diameter into segments whose lengths are 4 in. and 6 in., and on these segments as diameters semicircumferences be described on opposite sides of the diameter, these arcs will form a curved line which will divide the original circle into two parts in the ratio of 2 : 3.

34. If the diameter of a circle is d and a point be taken dividing the diameter into segments whose lengths are a and $d - a$, and on these segments as diameters semicircumferences be described on opposite sides of the diameter, these arcs will form a curved line which will divide the original circle into two parts in the ratio of $a : d - a$.

CONSTRUCTIONS

457. PROBLEM. To inscribe a square in a given circle.



Given: The circle *O*. **Required:** To inscribe a square.

Construction: Draw any diameter, *AB*, and another diameter, *CD*, $\perp$ to *AB*. Draw *AC*, *BC*, *BD*, *AD*.

Statement: *ACBD* is an inscribed square. Q.E.F.

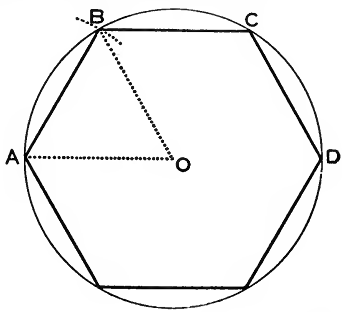
Proof: $\angle$s at *O* are = (?).

$\therefore$ arcs *AC*, *CB*, etc. are = (?).

$\therefore$ *ACBD* is an inscribed regular polygon (?) (418).

$\therefore$ *ABCD* is a square (?). Q.E.D.

458. PROBLEM. To inscribe a regular hexagon in a given circle.



Given: (?). **Required:** (?).

Construction: Draw any radius, *AO*. At *A*, with radius = *AO*, describe arc intersecting the given $\odot$ at *B*. Draw *AB*.

Statement: *AB* is the side of an inscribed regular hexagon.

Proof: Draw *BO*. $\triangle$ *ABO* is equilateral (Const.).

$\therefore \triangle$ *ABO* is equiangular (?) (56). $\therefore \angle$ *AOB* = 60° (?) (115).

$\therefore$ arc $AB = \frac{1}{6}$ of the circumference ($\frac{1}{6}$ of 360°).

$\therefore$ polygon *AD*, inscribed, having each side = *AB*, is an inscribed regular hexagon (?) (418). Q.E.D.

Ex. If the radius of a circle is 7 in., find:

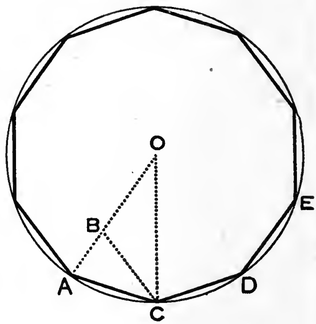
(*a*) The circumference and the area.

(*b*) The side and area of the inscribed square.

(*c*) The side and area of the inscribed regular hexagon.

459. Problem. To inscribe a regular decagon in a given circle.

Given: (?). **Required:** (?).



Construction: Draw any radius AO. Divide it into mean and extreme ratio (by 363), having the larger segment next the center. Take A as a center and OB as a radius, draw an arc cutting ⊙ at C. Draw AC, BC, OC.

Statement: AC is a side of the inscribed regular decagon. Q.E.F.

Proof: $AO : BO = BO : AB$ (Const.).

That is, $AO : AC = AC : AB$ (Ax. 6). Hence, $\triangle ABC$ and AOC have $\angle A$ common and are similar (?) (317).

$\therefore$ 1st, $\angle ACB = \angle O$ (?);

and 2d, $\triangle ABC$ is isosceles (being similar to $\triangle AOC$).

Hence, $AC = BC$ (?), but $AC = BO$ (?). $\therefore BC = BO$ (Ax. 1).

Therefore, $\angle BCO = \angle O$ (?) (55).

$\therefore \angle ACO = 2 \angle O$ (Ax. 2). Also, $\angle A = \angle ACO$ (?) (55).

$\therefore \angle A = 2 \angle O$ (Ax. 1).

$\underline{\angle O = 1 \angle O.}$ Adding,

$\angle$s of $\triangle ACO = 5 \angle O$ (Ax. 2).

Hence, $5 \angle O = 180°$ (?) (110).

$\therefore \angle O = 36°$ (Ax. 3); that is, arc $AC = \frac{1}{10}$ of the circumference ($\frac{1}{10}$ of 360°).

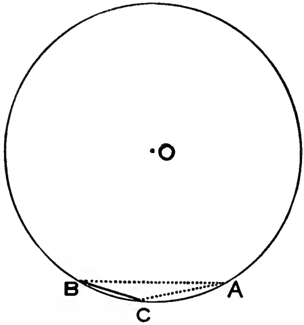
Hence, polygon AE, having each side $= AC$, is an inscribed regular decagon (?) (418). Q.E.D.

Ex. If the radius of a circle is 20 in., find:

(*a*) The circumference and area.
(*b*) The side and area of the inscribed square.
(*c*) The side and area of the inscribed regular hexagon.
(*d*) The side of the inscribed regular decagon. (See 365.)
(*e*) The area of sector AOC (fig. 459).
(*f*) The radius of a circle containing twice the area of this circle.

460. PROBLEM. To inscribe a regular 15-gon (pentedecagon) in a given circle.

Given: (?). **Required**: (?).



Construction: Draw AB, the side of an inscribed hexagon, and AC, the side of an inscribed decagon. Draw BC.

Statement: BC is the side of an inscribed regular 15-gon. Q.E.F.

Proof: Arc $BC =$ arc $AB -$ arc $AC = \frac{1}{6} - \frac{1}{10} = \frac{1}{15}$ of the circumference. (Const.)

Hence, the polygon having each side, a chord, $= BC$, is an inscribed regular 15-gon (?) (418). Q.E.D.

461. PROBLEM. To inscribe in a given circle:

I. A regular 8-gon, a regular 16-gon, a regular 32-gon, etc.

II. A regular 12-gon, 24-gon, etc.

III. A regular 30-gon, 60-gon, etc.

Construction: I. Inscribe a square; bisect the arcs; draw chords. **Statement**: (?). **Proof**: (?). (See 422.) Etc.

II. Inscribe a regular hexagon; bisect the arcs. Etc.

III. Inscribe a regular 15-gon, etc.

462. PROBLEM. To inscribe an equilateral triangle in a circle.

Construction: Join the alternate vertices of an inscribed regular hexagon. **Proof**: (?) (See 421.)

463. PROBLEM. To inscribe a regular pentagon in a given circle.

464. PROBLEM. To circumscribe a regular polygon about a circle.

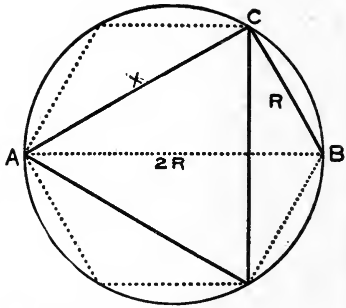
Construction: Inscribe a polygon having the same number of sides. At the several vertices draw tangents.

Statement: (?). **Proof**: (?). (See 419.)

FORMULAS

465. Sides of inscribed polygons.

1. **Side of inscribed equilateral triangle** $= R\sqrt{3}$.



Proof: $\angle ACB$ is a rt. $\angle$ (?).
$AB = 2R$ and $CB = R$ (?).
$\therefore x^2 = (2R)^2 - R^2$.
$\therefore x = R\sqrt{3}$.

2. **Side of inscribed square** $= R\sqrt{2}$.

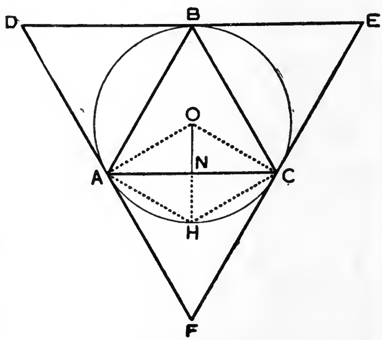
Proof: Use fig. of 457.

3. **Side of inscribed regular hexagon** $= R$ (?).

4. **Side of inscribed regular decagon** $= \frac{1}{2} R(\sqrt{5} - 1)$. (365.)

466. Sides of circumscribed polygons.

1. **Side of circumscribed equilateral** $\triangle = 2R\sqrt{3}$.



Proof: $\angle DAB = \angle DBA = \angle D = 60°$ (?).
$\therefore \triangle ABD$ is equilateral.
$AD = AB = R\sqrt{3}$ (?).
$\therefore DF = 2R\sqrt{3}$.

2. **Side of circumscribed square** $= 2R$ (?).

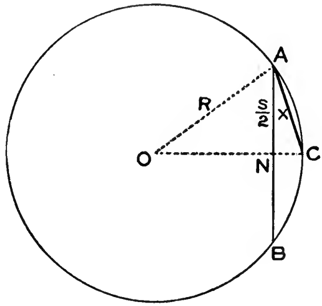
3. **Side of circumscribed regular hexagon** $= \frac{2}{3} R\sqrt{3}$. (Explain.)

467. In equilateral triangle, **apothem** $= \frac{1}{2} R$.

Proof: Bisect arc AC at H. Draw OA, OC, AH, CH. Figure $AOCH$ is a rhombus. (Explain.)
$ON = \frac{1}{2} OH = \frac{1}{2} R$ (?) (141).

468. Problem. **In a circle whose radius is R is inscribed a regular polygon whose side is s; to find the formula for the side of an inscribed regular polygon having double the number of sides.**

Given: $AB = s$, a side of an inscribed regular polygon in ⊙ whose radius is R; C, the mid-point of arc AB; chord AC.



Required: To find the value of AC, the side of a regular polygon having double the number of sides and inscribed in the same circle.

Construction: Draw radii OA and OC.

Computation: OC bisects AB at right ∠s (?) (70).

In rt. △ AON, O is an acute ∠. Hence in △ AOC,

$$\overline{AC}^2 = \overline{OA}^2 + \overline{OC}^2 - 2 \cdot OC \cdot ON \text{ (?) (346).}$$

But $AO = R$, $OC = R$, $ON = \sqrt{R^2 - (\frac{1}{2}s)^2} = \frac{1}{2}\sqrt{4R^2 - s^2}$ (?).

$$\therefore \overline{AC}^2 = 2R^2 - 2R \cdot \tfrac{1}{2}\sqrt{4R^2 - s^2} \text{ (Ax. 6);}$$

or
$$AC = \sqrt{2R^2 - R\sqrt{4R^2 - s^2}}. \qquad \text{Q.E.F.}$$

469. Formula. If $R = 1$, and given side $= s$, the side of a regular polygon having twice as many sides $= \sqrt{2 - \sqrt{4 - s^2}}$.

Ex. 1. If the radius of a circle is 4, find:

(*a*) The side of the inscribed equilateral triangle.
(*b*) The side of the circumscribed equilateral triangle.
(*c*) The side of the inscribed square.
(*d*) The side of the circumscribed square.
(*e*) The side of the inscribed regular hexagon.
(*f*) The side of the circumscribed regular hexagon.
(*g*) The apothem of the inscribed equilateral triangle.
(*h*) The apothem of the inscribed regular hexagon.
(*i*) The side of the inscribed regular dodecagon (468).
(*j*) The side of the inscribed regular octagon.

470. PROBLEM. **To find the approximate numerical value of π.**

Given: A circle whose diameter $= D$ and circumference $= C$.

Required: The value of π, that is, the value of $C \div D$.

Method: 1. We may select a $\odot$ of any diameter (442). Hence, for simplicity, we take the $\odot$ in which $D = 2$; $\therefore R = 1$.

2. We compute the perimeter of some inscribed regular polygon (by 465).

3. We compute the length of a side of the inscribed regular polygon having double the number of sides, by the formula $s = \sqrt{2 - \sqrt{4 - s^2}}$ (469). From this we can find the perimeter of this polygon.

4. Using the side of this polygon as known, we compute, by the same formula, a side of the inscribed regular polygon having still double the number of sides. Hence its perimeter can be found.

5. By continuing this process we may approximate the value of the circumference (440, I).

6. Thus we can find the value of $C \div D$, or π.

Computation: 1. Assume $R = 1$.

2. Consider the regular hexagon and let s_6 represent its side and P_6 its perimeter. Then $s_6 = 1$ and $P_6 = 6$ (?).

3. Then, a side of the inscribed regular 12-gon is $s_{12} = \sqrt{2 - \sqrt{4 - 1}} = 0.5176381$, and $P_{12} = 6.2116572$.

4. Thus we may find $s_{24} = 0.2610524$ and $P_{24} = 6.2652576$.

5. By continuing, $s_{3072} = 0.002045$, and $P_{3072} = 6.283184$.

6. $\therefore$ it is evident that $C = 6.283184$, approximately.

But, $\pi = \dfrac{C}{D}$ (?). $\therefore \pi = \dfrac{6.283184}{2} = 3.141592+$. Q.E.F.

This calculation is tabulated for reference.

$s_6 = 1,$	$\therefore P_6 = 6.$	$s_{192} = 0.032723,$	$\therefore P_{192} = 6.282904.$
$s_{12} = 0.517638,$	$\therefore P_{12} = 6.211657.$	$s_{384} = 0.016362,$	$\therefore P_{384} = 6.283115.$
$s_{24} = 0.261052,$	$\therefore P_{24} = 6.265257.$	$s_{768} = 0.008181,$	$\therefore P_{768} = 6.283169.$
$s_{48} = 0.130806,$	$\therefore P_{48} = 6.278700.$	$s_{1536} = 0.004091,$	$\therefore P_{1536} = 6.283180.$
$s_{96} = 0.065438,$	$\therefore P_{96} = 6.282063.$	$s_{3072} = 0.002045,$	$\therefore P_{3072} = 6.283184.$

ORIGINAL EXERCISES (NUMERICAL)

Mensuration of Regular Polygons and the Circle

1. Find the angle and the central angle of:
(*i*) a regular pentagon; (*ii*) a regular octagon; (*iii*) a regular dodecagon; (*iv*) a regular 20-gon.

2. Find the area of a regular hexagon whose side is 8.

3. Find the area of a regular hexagon whose apothem is 4.

4. In a circle whose radius is 10 are inscribed an equilateral triangle, a square, and a regular hexagon. Find the perimeter, apothem, and area of each.

5. About a circle whose radius is 10 are circumscribed an equilateral triangle, a square, and a regular hexagon. Find the perimeter and area of each.

6. Find the circumference and area of a circle whose radius is 5 inches. [Use $\pi = 3\frac{1}{7}$.]

7. Find the circumference and area of a circle whose diameter is 42 centimeters.

8. The radius of a certain circle is 9 meters. What is the radius of a second circle whose circumference is twice as long as the first? Of a third circle whose area is twice as great as the first?

9. If the circumference of a circle is 55 yards, what is its diameter?

10. If the area of a circle is $113\frac{1}{7}$ square meters, what is its radius?

11. In a circle whose radius is 35 there is a sector whose angle is 40°. Find the length of the arc and the area of the sector.

12. The area of a circle is $6\frac{1}{4}$ times the area of another. If the radius of the smaller circle is 12, what is the radius of the larger circle?

13. If the angle of a sector is 72° and its arc is 44 inches, what is the radius of the circle? What is the area of the sector?

14. In a circle whose radius is 7 find the area of the segment whose central angle is 120°. [See 451.]

15. If the radius of a circle is 4 feet, what is the area of a segment whose arc is 60°? of a segment whose arc is a quadrant?

16. Find the area of the circle inscribed in a square whose area is 75.

17. Find the area of an equilateral triangle inscribed in a circle whose area is 441π square meters.

18. If the length of a quadrant is 8 inches, what is the radius?

19. Find the length of an arc subtended by the side of an inscribed regular 15-gon if the radius is $4\frac{2}{3}$ inches.

20. The side of an equilateral triangle is 10. Find the areas of its inscribed and circumscribed circles.

21. Find the perimeter and area of a segment whose chord is the side of an inscribed regular hexagon, if the radius of a circle is $5\frac{1}{4}$.

22. A circular lake 9 rods in diameter is surrounded by a walk $\frac{1}{2}$ rod wide. What is the area of the walk?

23. A locomotive driving wheel is 7 feet in diameter. How many revolutions will it make in running a mile?

24. What is the number of degrees in the central angle whose arc is as long as the radius?

25. Find the side of the square equivalent to a circle whose diameter is 4.2 meters.

26. Find the radius of that circle equivalent to a square whose side is 5.5 inches.

27. Find the radius of the circumference which divides a given circle whose radius is $10\frac{1}{2}$ into two equal parts.

28. Three equal circles are each tangent to the other two and the diameter of each is 40 feet. Find the area between these circles.

[Required area = area of an eq. $\triangle$ minus area of three sectors.]

29. Find the area of the three segments of a circle whose radius is $5\sqrt{3}$, formed by the sides of the inscribed equilateral triangle.

30. If a cistern can be emptied in 5 hours by a 2-inch pipe, how long will be required to empty it by a 1-inch pipe?

31. Find the side, apothem, and area of a regular decagon inscribed in a circle whose radius is 6 feet.

32. What is the area of the circle circumscribed about an equilateral triangle whose area is $48\sqrt{3}$?

33. The circumferences of two concentric circles are 40 inches and 50 inches. Find the area of the circular ring between them.

34. A circle has an area of 80 square feet. Find the length of an arc of 80°.

35. Find the angle of a sector whose perimeter equals the circumference.

36. Find the angle of a sector whose area is equal to the square of the radius.

37. Find the area of a regular octagon inscribed in a circle whose radius is 20.

[Inscribe square, then octagon. Draw radii of octagon. Find area of one isosceles △ formed, whose altitude is half the side of the square.]

38. A rectangle whose length is double its width, a square, an equilateral triangle, and a circle all have the same perimeter, namely 132 meters. Which has the greatest area? the least?

39. Through a point without a circle whose radius is 35 inches two tangents are drawn, forming an angle of 60°. Find the perimeter and area of the figure bounded by the tangents and their smaller intercepted arc.

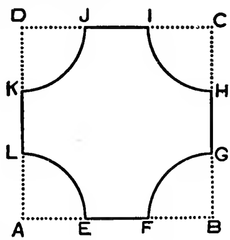
40. In a circle whose radius is 12 are two parallel chords which subtend arcs of 60° and 90° respectively. Find the perimeter and area of the figure bounded by these chords and their intercepted arcs.

41. A quarter mile race track is to be laid out, having parallel sides but semicircular ends whose radius is 105 feet. Find the length of the parallel sides.

42. If the diameter of the earth is 7920 miles, how far at sea can the light from a lighthouse 150 feet high be seen?

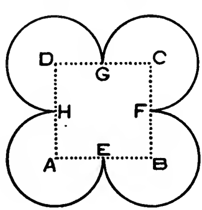
43. The diameter of a circle is 18 inches. Find the area of the figure between this circle and the circumscribed equilateral triangle.

44. How far does the end of the minute hand of a clock move in 20 minutes, if the hand is $3\frac{1}{2}$ inches long?



45. The diameter of a circle is 16 inches. What is the area of that portion of the circle outside the inscribed regular hexagon?

46. Using the vertices of a square whose side is 12, as centers, and radii equal to 4, four quadrants are described within the square. Find the perimeter and area of the figure thus formed.

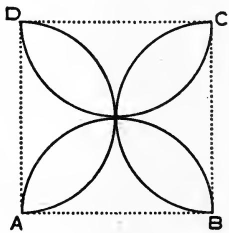


47. Using the four vertices of a square, whose side is 12, as centers and radii equal to 6, four arcs are described without the square (see figure). Find the perimeter and area of the figure bounded by these four arcs.

48. Using the vertices of an equilateral triangle, whose side is 16, as centers and radii equal to 8, three arcs are described within the triangle. Find the perimeter and area of the figure bounded by these arcs. Do the same if the three arcs are described *without* the triangle (terminating in the sides, in each case).

49. Using the vertices of a regular hexagon, whose side is 20, as centers and radii equal to 10, six arcs are described within the hexagon. Find the perimeter and area of the figure bounded by these arcs. Do the same if the six arcs are described without the hexagon (terminating in the sides, in each case).

50. If semicircumferences be described within a square, whose side is 8 inches, upon the four sides as diameters, find the areas of the four lobes bounded by the eight quadrants. Find the area of any one.



In the following exercises let n = the number of sides of the regular polygon; s = the length of its side; r = its apothem; R = its radius; K = its area.

51. If $n = 3$, show that $s = R\sqrt{3}$; $r = \frac{1}{2}R$; $K = \frac{3R^2\sqrt{3}}{4} = 3r^2\sqrt{3}$.

52. If $n = 4$, show that $s = R\sqrt{2} = 2r$; $K = 2R^2 = 4r^2$.

53. If $n = 6$, show that $s = R = \frac{2r\sqrt{3}}{3}$;

$$K = \frac{3R^2\sqrt{3}}{2} = \frac{3s^2\sqrt{3}}{2} = 2r^2\sqrt{3}.$$

54. If $n = 8$, show that $s = R\sqrt{2-\sqrt{2}} = 2r(\sqrt{2}-1)$; $r = \frac{R}{2}\sqrt{2+\sqrt{2}}$;

$$R = r\sqrt{4-2\sqrt{2}};\ K = 2R^2\sqrt{2} = 8r^2(\sqrt{2}-1).$$

55. If $n = 10$, show that $s = \frac{R}{2}(\sqrt{5}-1)$; $r = \frac{R}{4}\sqrt{10+2\sqrt{5}}$.

56. If $n = 5$, show that $s = \frac{R}{2}\sqrt{10-2\sqrt{5}}$; $r = \frac{R}{4}(\sqrt{5}+1)$.

57. If $n = 12$ show that $s = R\sqrt{2-\sqrt{3}} = 2r(2-\sqrt{3})$; $R = 2r\sqrt{2-\sqrt{3}}$;

$$r = \frac{R}{2}\sqrt{2+\sqrt{3}};\ K = 12r^2(2-\sqrt{3}) = 3R^2.$$

58. The apothem of a regular hexagon is $18\sqrt{3}$ inches. Find its side and area. Find the area of the circle circumscribed about it.

59. What is the radius of a circle whose area is doubled by increasing the radius 10 feet?

60. If an 8-inch pipe will fill a cistern in 3 hours 20 minutes, how long will it require a 2-inch pipe to fill it?

61. The radius of a circle is 12 meters. Find:

(*a*) The area of the inscribed square.
(*b*) The area of the inscribed equilateral triangle.
(*c*) The area of the inscribed regular hexagon.
(*d*) The area of the inscribed regular dodecagon.
(*e*) The area of the circumscribed square.
(*f*) The area of the circumscribed equilateral triangle.
(*g*) The area of the circumscribed regular hexagon.
(*h*) The area of the circumscribed regular dodecagon.

62. The radius of a circle is 18. Find:

(*a*) The side and apothem of the inscribed square.
(*b*) The side and apothem of the inscribed equilateral triangle.
(*c*) The side and apothem of the inscribed regular hexagon.
(*d*) The area of the inscribed square.
(*e*) The area of the inscribed equilateral triangle.
(*f*) The area of the inscribed regular hexagon.
(*g*) The area of the inscribed regular octagon.
(*h*) The area of the circumscribed regular hexagon.

63. Prove that the area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and the circumscribed equilateral triangles. [Find the three areas in terms of R.]

64. AB is one side of an inscribed equilateral triangle, and C is the midpoint of AB. If AB be prolonged to O making BO equal to BC, and OT be drawn tangent to the circle at T, OT will be $\frac{3}{2}$ the radius.

65. A square, an equilateral triangle, a regular hexagon, and a circle all have the same area, namely 5544 sq. ft. Which figure has the least perimeter? the greatest?

66. A square, an equilateral triangle, a regular hexagon, and a circle all have the same perimeter, namely 396 in. Find their areas and compare.

67. The circumferences of two concentric circles are 330 and 440 in. respectively. Find the radius of another circle equivalent to the ring between these two circumferences.

ORIGINAL CONSTRUCTIONS

1. To circumscribe a regular hexagon about a given circle.

2. To circumscribe an equilateral triangle about a given circle.

3. To circumscribe a regular decagon about a given circle; a regular 16-gon; a regular 24-gon; a square.

4. To construct an angle of 36°; of 18°; of 72°; of 24°; of 6°; of 48°; of 96°.

5. To construct a regular hexagon upon a given line as a side.

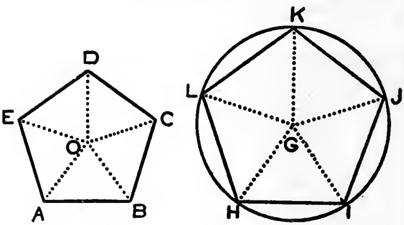
6. To construct a regular decagon upon a given line as a side.

7. To construct a regular octagon upon a given line as a side.

8. To construct a regular pentagon upon a given line as a side.

9. To construct a square which shall have double the area of a given square.

10. To inscribe in a given circle a regular polygon similar to a given regular polygon.



Construction: From the center of the polygon draw radii. At the center of the circle construct $\angle$s = these central $\angle$s of the polygon. Draw chords. Etc.

11. To construct a regular pentagon which shall have double the area of a given regular pentagon.

12. To construct a circumference equal to the sum of two given circumferences.

13. To construct a circumference which shall be three times a given circumference.

14. To construct a circumference equal to the difference of two given circumferences.

15. To construct a circle whose area shall be five times a given circle.

16. To construct a circle equivalent to the sum of two given circles; another, equivalent to their difference.

17. To construct a circle whose area shall be half a given circle.

18. To bisect the area of a given circle by a concentric circumference.

19. To divide a given circumference into two parts which shall be in the ratio of 3 : 7; into two other parts which shall be in the ratio of 5 : 7; into still two other parts, in the ratio of 8 : 7.

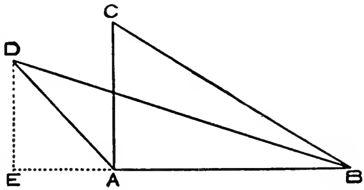
MAXIMA AND MINIMA

471. Of geometrical magnitudes which satisfy a given condition (or given conditions) the greatest is **maximum**, and the least is **minimum**.

Thus, of all chords that can be drawn through a given point within a circle, the diameter is the maximum, and the chord perpendicular to the diameter at the point is the minimum.

Isoperimetric figures are figures having equal perimeters.

472. Theorem. Of all triangles having two given sides, that in which these sides form a right angle is the maximum.



Given: $\triangle ABC$ and $\triangle ABD$ having AB common, and $AC = AD$, but $\angle CAB$ a rt. $\angle$ and $\angle DAB$ not a right $\angle$.

To Prove: $\triangle ABC > \triangle ABD$.

Proof: Draw altitude DE.

Now $AD > DE$ (?). $\therefore AC > DE$ (Ax. 6).

Multiply each member by $\frac{1}{2}$ AB.

Then $\frac{1}{2}$ $AB \cdot AC > \frac{1}{2}$ $AB \cdot DE$ (?).

Now $\frac{1}{2}$ $AB \cdot AC =$ area $\triangle ABC$ (?),

and $\frac{1}{2}$ $AB \cdot DE =$ area $\triangle ABD$ (?).

Therefore, $\triangle ABC > \triangle ABD$ (Ax. 6). Q.E.D.

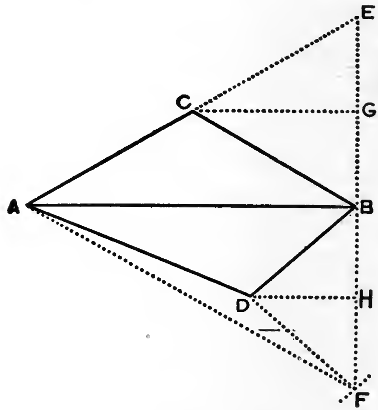
This theorem may be stated thus: *Of all triangles having two given sides, that triangle whose third side is the diameter of the circle which circumscribes it is the maximum.*

Therefore, *Of all n-gons having $n-1$ sides given, that polygon whose n^{th} side is the diameter of a circle which circumscribes the polygon is the maximum.*

Ex. 1. Of all parallelograms having two adjacent sides given, the rectangle is the maximum.

Ex. 2. Of all lines that can be drawn from an external point to a circumference, which is the maximum? the minimum?

473. THEOREM. **Of all isoperimetric triangles having the same base the isosceles triangle is the maximum.**



Given: $\triangle$s ABC and ABD isoperimetric, having the same base, AB, and $\triangle\ ABC$ isosceles.

To Prove:

$\triangle\ ABC > \triangle\ ABD$.

Proof: Prolong AC to E, making $CE = AC$, and draw BE. Using D as a center and BD as a radius, describe an arc cutting EB prolonged, at F. Draw CG and $DH \parallel$ to AB, meeting EF at G and H respectively. Draw AF.

Now, using C as a center and AC or BC or EC as a radius, the circle described will pass through A, B, and E (Hyp. and Const.). $\therefore \angle ABE =$ rt. $\angle$ (?).

That is, AB is $\perp$ to EF. Hence, CG and DH are $\perp$ to EF (?).

$AC + CE = AC + CB = AD + DB = AD + DF$ (Hyp. and Const.).

That is, $AE = AD + DF$ (Ax. 1).

But $AD + DF > AF$ (?). $\therefore AE > AF$ (Ax. 6).

$\therefore BE > BF$ (?) (90), and $\frac{1}{2}\ BE > \frac{1}{2}\ BF$ (?).

Now, $BG = \frac{1}{2}\ BE$ and $BH = \frac{1}{2}\ BF$ (?) (73, Cor.).

$\therefore BG > BH$ (Ax. 6).

Multiply each member by $\frac{1}{2}\ AB$.

Then, $\frac{1}{2}\ AB \cdot BG > \frac{1}{2}\ AB \cdot BH$ (?).

But, $\frac{1}{2}\ AB \cdot BG =$ area $\triangle\ ABC$ (?),

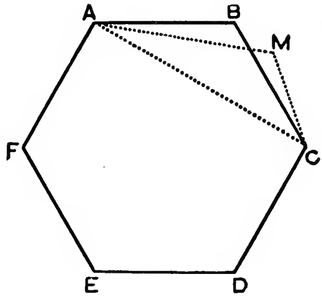
and $\frac{1}{2}\ AB \cdot BH =$ area $\triangle\ ABD$ (?).

$\therefore \triangle\ ABC > \triangle\ ABD$ (Ax. 6). Q.E.D.

474. THEOREM. **Of isoperimetric triangles the equilateral triangle is the maximum.**

[Any side may be considered the base.]

475. Theorem. Of isoperimetric polygons having the same number of sides the maximum is equilateral.



Given: Polygon *AD*, the maximum of all polygons having the same perimeter and the same number of sides.

To Prove: $AB = BC = CD = DE =$ etc.

Proof: Draw AC and suppose AB not $= BC$.

On AC as base, construct $\triangle ACM$ isoperimetric with $\triangle ABC$ *and isosceles;* that is, make $AM = CM$. Then $\triangle ACM > \triangle ABC$ (?) (473).

Add to each member, the polygon $ACDEF$.

$\therefore$ polygon $AMCDEF >$ polygon AD (?).

But the polygon AD is maximum (Hyp.).

$\therefore AB$ cannot be unequal to BC as we supposed (because that results in an impossible conclusion).

Hence, $AB = BC$. Likewise it is proved that $BC = CD =$ etc.

Q.E.D.

476. Theorem. Of isoperimetric polygons having the same number of sides the equilateral polygon is maximum.

Proof: Only one such polygon is maximum, and the maximum is equilateral (475).

Only one such polygon is equilateral, hence the equilateral polygon and the maximum polygon are the same. Q.E.D.

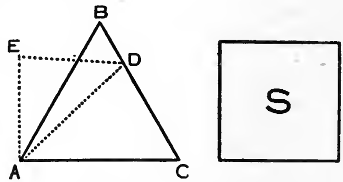
Ex. 1. Of isoperimetric triangles, the maximum is equilateral.

Ex. 2. Of all right triangles that can be constructed upon a given hypotenuse, which is maximum? Why?

Ex. 3. Of all triangles having a given base and a given vertex-angle, the isosceles is the maximum.

Ex. 4. Of all mutually equilateral polygons, that which can be inscribed in a circle is the maximum.

477. Theorem. Of isoperimetric regular polygons, the polygon having the greatest number of sides is maximum.



Given: Equilateral △ *ABC* and square *S*, having the same perimeter.

To Prove: Square *S* > △ *ABC*.

Proof: Take *D*, any point in *BC*, and draw *AD*. On *AD* as base, construct isosceles △ *ADE*, isoperimetric with △ *ABD*.

Now △ *AED* > △ *ABD* (?) (473).

Adding △ *ADC* to each member, *AEDC* > △ *ABC* (?).

AEDC is isoperimetric with △ *ABC* and *S* (Hyp. and Const.).

Hence, *S* > *AEDC* (?) (476).

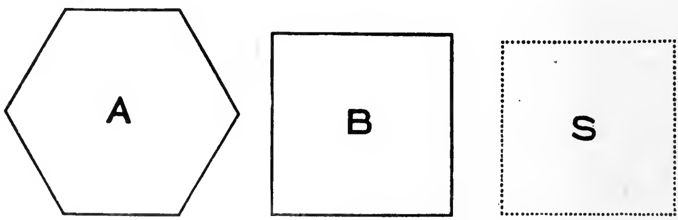
Therefore *S* > △ *ABC* (?) (Ax. 11).

Similarly we may prove that an isoperimetric regular pentagon is greater than *S*; and an isoperimetric regular hexagon is greater than this pentagon, etc.

Therefore, the regular polygon having the greatest number of sides is maximum. Q.E.D.

478. Theorem. Of all isoperimetric plane figures the circle is the maximum.

479. Theorem. Of equivalent regular polygons the perimeter of the polygon having the greatest number of sides is the minimum.



Given: Any two equivalent regular polygons, *A* and *B*, *A* having the greater number of sides.

To Prove: The perimeter of $A <$ the perimeter of B.

Proof: Construct regular polygon S, similar to B and isoperimetric with A.

Then $A > S$ (477), but $A \Leftrightarrow B$ (?). $\therefore B > S$ (?) (Ax. 6).

Hence, the perimeter of $B >$ perimeter of S (390).

But, the perimeter of $S =$ perimeter of A (?).

$\therefore$ perimeter of $B >$ perimeter of A (Ax. 6).

That is, the perimeter of $A <$ the perimeter of B. Q.E.D.

480. Theorem. Of all equivalent plane figures the circle has the minimum perimeter.

ORIGINAL EXERCISES

1. Of all equivalent parallelograms having equal bases the rectangle has the minimum perimeter.

2. Of all lines drawn between two given parallels (terminating both ways in the parallels), which is the minimum? Prove.

3. Of all straight lines that can be drawn on the ceiling of a room 12 feet long and 9 feet wide, what is the length of the maximum?

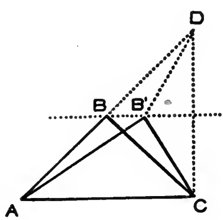
4. Find the areas of an equilateral triangle, a square, a regular hexagon, and a circle, the perimeter of each being 264 inches. Which is maximum? What theorem does this exercise illustrate?

5. Find the perimeters of an equilateral triangle, a square, a regular hexagon, and a circle, if the area of each is 1386 square feet. Which perimeter is the minimum? What theorem does this exercise illustrate?

6. Of isoperimetric rectangles which is maximum?

7. To divide a given line into two parts such that their product (rectangle) is maximum.

8. Of all equivalent triangles having the same base the isosceles triangle has the minimum perimeter.



To Prove: The perimeter of $\triangle ABC <$ the perimeter of $\triangle AB'C$.

Proof: $AD < AB' + B'C$; etc.

9. Of all rectangles inscribed in a circle which is maximum? Prove.

10. Of all rectangles inscribed in a semicircle which is maximum? Prove.

11. Of all equivalent rectangles, the square has the minimum perimeter.

12. Of all triangles having a given base and a given vertex-angle, the isosceles triangle has the maximum area.

13. Of all triangles having a given altitude and a given vertex-angle, the isosceles triangle is the minimum.

14. Of all triangles that can be inscribed in a given circle the equilateral triangle has the maximum area.

15. The cross section of a bee's cell is a regular hexagon. Would this be the most economical for the bee, if one cell in a hive were all he were to fill (that is, would he use the least wax)? Considering also the adjoining cells, does the form of the regular hexagon require the least wax? Explain. Does it also permit the storing of the most honey? Why?

16. Prove that a regular hexagon is greater than an isoperimetric square, by the method employed in 477.

17. Answer the questions of exercise 65 on page 243, without any computation. Give reasons.

18. Compare the areas of the figures mentioned in exercise 66, page 243, without performing any computation.

397
250
1475
30

SOLID GEOMETRY

BOOK VI

LINES, PLANES, AND ANGLES IN SPACE

481. A **solid** is any limited portion of space. The boundaries of a solid are **surfaces.**

A **plane** is a surface in which, if any two points are taken, the straight line connecting them lies wholly in that surface.

Solid Geometry is a science that treats of magnitudes, all of which are not in the same plane.

482. The **intersection** of two surfaces is the line, or the lines, all of whose points lie in both surfaces. The intersection of a line and a surface is the point, or points, common to both the line and the surface. The **foot of a line** intersecting a plane is their point of intersection.

483. A **straight line** is **perpendicular** to a plane if the line is perpendicular to every straight line in the plane drawn through its foot.

A **normal** is a straight line perpendicular to a plane.

484. A **straight line** is **parallel** to a plane if the line and the plane never meet, when indefinitely extended. A **straight line** is **oblique** to a plane if it is neither perpendicular nor parallel to the plane. Two **planes** are **parallel** if they never meet when indefinitely extended.

485. The **projection** of a **point** upon a plane is the foot of the perpendicular from the point to the plane.

The **projection** of a **line** upon a plane is the line formed by the projections of all the points of the given line.

486. A plane is **determined** if its position is fixed and if that position can be occupied by only one plane.

PRELIMINARY THEOREMS

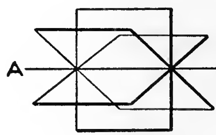
487. THEOREM. **If two points of a straight line are in a plane, the whole line is in the plane.** [Def. 481.]

488. THEOREM. **A straight line can intersect a plane in only one point.** [See 487.]

489. THEOREM. **If a line is perpendicular to a plane, it is perpendicular to every line in the plane drawn through its foot.** [See 483.]

490. THEOREM. **Through one straight line any number of planes may be passed.**

Because, if we consider a plane containing a line *AB* to revolve about *AB*, it may occupy an indefinitely great number of positions. Each of these will be a different plane containing *AB*.



491. THEOREM. **Through a fixed straight line and an external point a plane can be passed.**

Because, if we pass a plane containing this line *AB*, it may be revolved about *AB* until it contains the given point.

492. THEOREM. **A straight line and an external point determine a plane.** [See 491, 486.]

493. THEOREM. **Three points not in a straight line, determine a plane.**

Because two of the points may be joined by a line; then this line and the third point determine a plane. [See 492.]

494. THEOREM. **Two parallel lines determine a plane.** [See 91 and 492.]

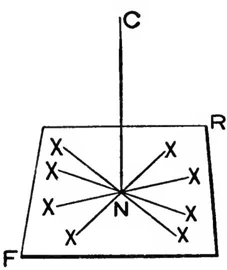
495. THEOREM. **Two intersecting straight lines determine a plane.**

Because one of these lines and a point in the second line determine a plane (492).

And this plane contains the second line (487).

496. THEOREM. **If two planes are parallel, no line in the one can meet any line in the other.** [Def. 484.]

NOTE. A plane is represented to the eye by a quadrilateral. In some positions it appears to be a parallelogram, and in others, a trapezoid. The eye, however, must be aided by the imagination in really understanding the diagrams of Solid Geometry. Thus, in the adjoining figure, the line *CN* is perpendicular to the plane *FR*, and it is perpendicular to every line in *FR* drawn through *N*. Consider several lines drawn through a point on the floor, and a cane, *CN*, occupying a vertical position, so that it is perpendicular to all of these lines. Then every angle *CNX* is a right angle, though to the unskilled eye they do not all appear to be right angles in the diagram. The object of all geometrical diagrams is that the eye may assist the mind in grasping truths or in developing logical demonstrations, and the student should thoroughly examine every figure until he completely understands the relative positions of its parts, and thus trains his eye to see three dimensions represented in a plane. Photography accomplishes this, and we should be as familiar with the significance of a geometrical diagram, as with a picture.

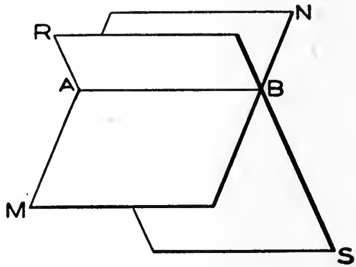


When, during the process of a demonstration or elsewhere, it becomes necessary to employ a plane not already indicated, it is customary to pass such a plane, or to conceive it constructed.

THEOREMS AND DEMONSTRATIONS

POINTS. LINES. PLANES

497. THEOREM. If two planes intersect, their intersection is a straight line.



Given: Intersecting planes *MN* and *RS*.

To Prove: Their intersection is a straight line.

Proof: Suppose *A* and *B* are two points common to both planes. Draw *AB*, a straight line. *AB* is in plane *RS* (?) (487). *AB* is in plane *MN* (?). That is, *AB* is common to both planes.

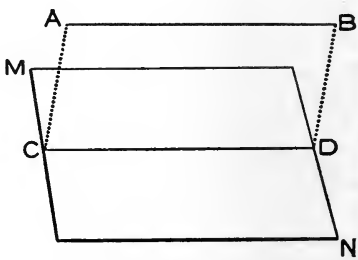
Now, if there were a point outside of *AB*, in both planes, these planes would coincide (?) (492).

That is, *AB* contains all points common to planes *MN* and *RS*.

Hence, *AB* is the intersection (482).

That is, the intersection is a straight line. Q.E.D.

498. THEOREM. If two straight lines are parallel, a plane containing one, and only one, is parallel to the other line.



Given: ∥ lines *AB* and *CD*; plane *MN* containing *CD*.

To Prove: plane *MN* ∥ to line *AB*.

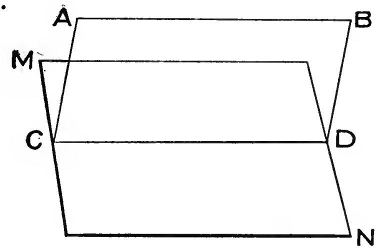
Proof: *AB* and *CD* are in the same plane *AD* (?) (91). Plane *AD* intersects plane *MN* in *CD* (?) (497).

If AB ever meets MN, it must meet MN in CD; but AB can never meet CD (Hyp.).

∴ AB can never meet MN, and AB is ∥ to MN (?) (484).

Q.E.D.

499. Theorem. If a straight line is parallel to a plane, and another plane containing this line intersects the given plane, the intersection is parallel to the given line.



Given: AB ∥ to MN; plane AD containing AB and intersecting plane MN in CD.

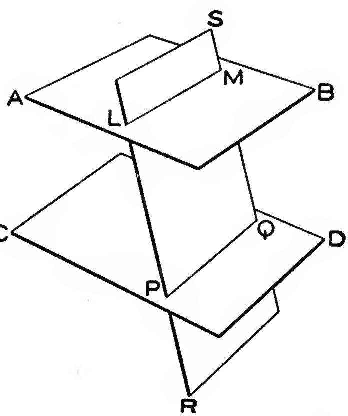
To Prove: AB ∥ to CD.

Proof: AB and CD are in the same plane AD (Hyp.). If AB ever meets CD, it must meet CD in plane MN; but AB can never meet MN (Hyp.).

∴ AB can never meet CD, and AB is ∥ to CD (?) (91).

Q.E.D.

500. Theorem. The intersections of two parallel planes by a third plane are parallel lines.



Given: ∥ planes AB and CD cut by plane RS in lines LM and PQ.

To Prove: LM ∥ to PQ.

Proof: LM and PQ are in the same plane RS (Hyp.). Also, LM and PQ can never meet (?) (496).

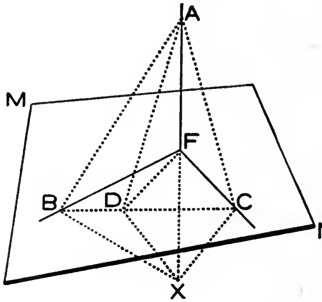
∴ LM is ∥ to PQ (?) (91).

Q.E.D.

Ex. 1. Why are the usual folds in a sheet of paper straight lines?

Ex. 2. If a rod is held parallel to the pavement, why is the shadow parallel to the rod?

501. Theorem. A straight line perpendicular to each of two straight lines at their intersection is perpendicular to the plane of the lines.



Given: $AF \perp$ to BF and CF at F; plane MN containing BF and CF.

To Prove: $AF \perp$ to plane MN.

Proof: In plane MN draw BC; draw also DF from F to any point, D, in BC.

Prolong AF to X, making $FX = AF$, and draw AB, AD, AC, BX, DX, CX.

Now, BF and CF are $\perp$ to AX at its midpoint (Hyp. and Const.).

In $\triangle$s ABC and BCX, $AB = BX$, $AC = CX$ (?) (67), and $BC = BC$ (?). $\therefore \triangle ABC = \triangle BCX$ (?) (58).

Also, in $\triangle$s ABD and BDX, $\angle ABC = \angle CBX$ (?) (27), $BD = BD$ (?), and $AB = BX$ (?).

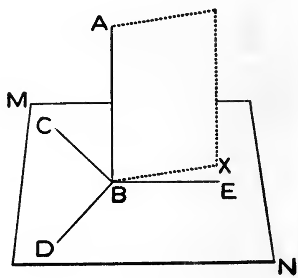
$\therefore \triangle ABD = \triangle BDX$ (?) (52) and $AD = DX$ (?).

$\therefore DF$ is $\perp$ to AX (?) (70).

That is, AF is $\perp$ to all lines in MN through F.

Consequently, AF is $\perp$ to plane MN (?) (483). Q.E.D.

502. Theorem. All straight lines perpendicular to a line at one point are in one plane, which is perpendicular to this line at this point.



Given: $AB \perp$ to BC, BD, BE, etc.; plane MN containing BC and BD.

To Prove: BE is in the plane MN and MN is $\perp$ to AB at B.

Proof: Pass plane AE containing AB and BE, and intersecting MN in line BX.

Now, AB is $\perp$ to MN (?) (501).

That is, plane MN is $\perp$ to AB.

$\therefore AB$ is $\perp$ to BX (?) (489).

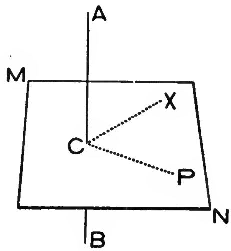
But AB is $\perp$ to BE (Hyp.). That is, BX and BE are both $\perp$ to AB, in the plane AE, at B.

Hence, BE and BX coincide (?) (43).

That is, BE is in the plane MN. Q.E.D.

503. Theorem. Through a point in a straight line one plane can be passed perpendicular to the line, and only one. (502.)

504. Theorem. Through an external point one plane can be passed perpendicular to a given straight line, and only one.



Given: The line AB and point P outside of AB.

To Prove: Through P, one plane can be passed $\perp$ to AB, and only one.

Proof: I. Draw from P, $PC \perp$ to AB, and at C draw CX, another line $\perp$ to AB.

PC and CX determine a plane MN (?).

Plane MN contains P and is $\perp$ to AB (?) (501).

II. Only one line $\perp$ to AB can be drawn from P (?) (71).

And only one plane $\perp$ to AB can be passed at C (?) (503).

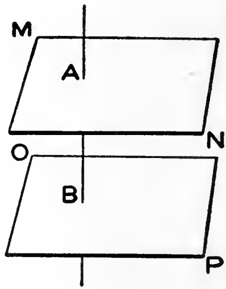
That is, MN is the only plane $\perp$ to AB, that can be passed through P. Q.E.D.

Ex. 1. If two lines are each parallel to a plane, are the lines necessarily parallel?

Ex. 2. If two planes intersect, how can a line be drawn through a given point that shall be parallel to both planes?

Ex. 3. How many planes are determined by two intersecting lines (a and b) and two points (R and S) not in the plane of the lines?

505. Theorem. Two planes perpendicular to the same straight line are parallel.



Given: Planes *MN* and *OP* ⊥ to *AB*.

To Prove: *MN* ∥ to *OP*.

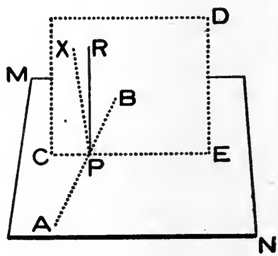
Proof: If the planes *MN* and *OP* are not ∥, they will meet when sufficiently extended (?) (Def. 484).

Then there would be two planes from the same point ⊥ to *AB* (⊥ by hyp.).

But this is impossible (?) (504).

∴ the planes never meet and therefore they are parallel (Def. 484). Q.E.D.

506. Theorem. At a given point in a plane one line can be drawn perpendicular to the given plane, and only one.



Given: Plane *MN* and point *P* within it.

To Prove: One line can be drawn ⊥ to *MN* at *P*, and only one.

Proof: I. In *MN* draw any line *AB*, through *P*.

Suppose plane *CD* be passed ⊥ to *AB* at *P*, meeting the plane *MN* in *CE*. In plane *CD* draw *PR* ⊥ to *CE*, from *P*. Now, *AB* is ⊥ to plane *CD* (Const.).

∴ *AB* is ⊥ to *PR* (?) (489).

PR is ⊥ to *CE* (Const.). ∴ *PR* is ⊥ to plane *MN* (?) (501).

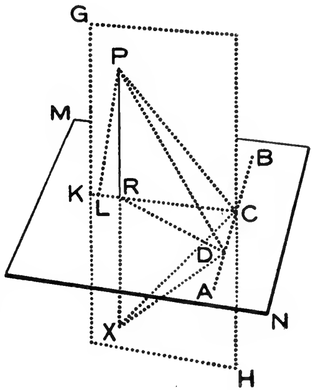
II. Suppose another line *PX* to be ⊥ to plane *MN* at *P*. Then *PX* and *PR* would determine a plane *CD*, intersecting plane *MN* in *CE* (?) (495 and 497).

PR and *PX* would then both be ⊥ to *CE* at *P* (?) (489).

But this is impossible (?) (43).

That is, *PR* and *PX* coincide, and *PR* is the only line ⊥ to plane *MN* at *P*. Q.E.D.

507. Theorem. **Through a given external point one line can be drawn perpendicular to a given plane, and only one.**



Given: Plane MN and point P outside of it.

To Prove: One line can be drawn through $P \perp$ to MN, and only one.

Proof: I. In plane MN draw any line AB. Suppose a plane GH be passed through $P \perp$ to AB, meeting plane MN in KC, and AB at C. In plane GH draw $PR \perp$ to KC and prolong PR to X, making $RX = PR$. Draw RD to any point in AB, except C. Draw PC, PD, CX, DX. Now RC is $\perp$ to PX at its midpoint (Const.).

Also AB is $\perp$ to plane GH (Const.).

Hence, $\angle$s DCP and DCX are rt. $\angle$s (?) (489).

In rt. $\triangle$s DCP and DCX, $DC = DC$ (?), $PC = CX$ (?) (67).

$\therefore \triangle DCP = \triangle DCX$ (?) (53). Hence, $PD = XD$ (?).

$\therefore$ RD is $\perp$ to PX (?) (70). That is, PR is $\perp$ to RC and RD, in plane MN.

Consequently, PR is $\perp$ to plane MN from P (?) (501).

II. Suppose there is another line $PL \perp$ to plane MN from P. Then PR and PL determine a plane GH (?).

This plane intersects plane MN in KC (?).

PR and PL would then both be $\perp$ to KC (?) (489).

But this is impossible (?) (71).

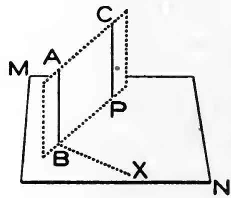
That is, PR and PL coincide and therefore PR is the only line $\perp$ to plane MN from P. Q.E.D.

Ex. In the figure of 507 name the six right angles at C.

508. **Theorem.** **If a plane is perpendicular to a line in another plane, any line in the first plane perpendicular to the intersection of the planes is perpendicular to the second plane.**

Proof: Identical with the proof of 507, I.

509. **Theorem.** **If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other also.**



Given: Plane $MN \perp$ to line AB, and $AB \parallel$ to CP.

To Prove: $CP \perp$ to plane MN.

Proof: Lines AB and CP determine a plane (?) (494).

Pass this plane BC, intersecting plane MN in line BP.

Draw $BX \perp$ to BP, in plane MN.

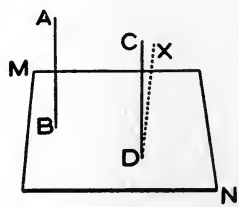
AB is $\perp$ to BX (?) (489). $\therefore BX$ is $\perp$ to plane BC (?) (501).

BP is $\perp$ to AB (?) (489). $\therefore BP$ is $\perp$ to CP (?) (95).

That is, plane BC is $\perp$ to BX and CP is $\perp$ to the intersection BP.

$\therefore CP$ is $\perp$ to plane MN (?) (508). Q.E.D.

510. **Theorem.** **Two lines perpendicular to the same plane are parallel.**



Given: Lines AB and $CD \perp$ to plane MN.

To Prove: $AB \parallel$ to CD.

Proof: Through D, the foot of CD, draw $DX \parallel$ to AB.

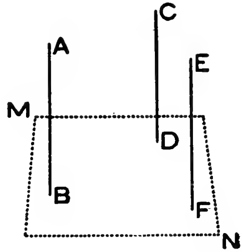
Then DX is $\perp$ to plane MN (?) (509).

But CD is $\perp$ to plane MN at D (Hyp.).

$\therefore DX$ and DC coincide (?) (506). That is, AB is $\parallel$ to CD.

Q.E.D.

511. Theorem. Two straight lines that are parallel to a third straight line are parallel to each other.



Given: Lines *CD* and *EF* each ∥ to *AB*.

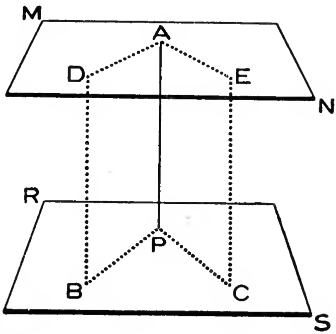
To Prove: *CD* ∥ to *EF*.

Proof: Suppose plane *MN* be passed ⊥ to *AB*.

∴ *MN* is ⊥ to *CD* and to *EF* (?) (509).

∴ *CD* is ∥ to *EF* (?) (510). Q.E.D.

512. Theorem. A line perpendicular to one of two parallel planes is perpendicular to the other also.



Given: Plane *MN* ∥ to plane *RS*; *AP* ⊥ to plane *RS*.

To Prove: *AP* ⊥ to *MN*.

Proof: Through *AP* pass any two planes, *AB* and *AC*, intersecting *MN* in *AD* and *AE*, and intersecting *RS* in *PB* and *PC*.

AD is ∥ to *PB*, and *AE* is ∥ to *PC* (?) (500).

AP is ⊥ to *PB* and *PC* (?) (489).

∴ *AP* is ⊥ to *AD* and *AE* (?) (95).

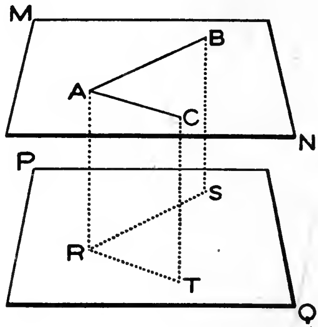
∴ *AP* is ⊥ to plane *MN* (?) (501). Q.E.D.

513. Theorem. If two planes are each parallel to a third plane, they are parallel to each other.

Proof: Draw a line ⊥ to the third plane. This line is ⊥ to each of the other planes (?). ∴ the planes are ∥ (?) (505).

Ex. Why are not *AB*, *CD*, *EF* in figure of 511 all parallel?

514. Theorem. If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to the given plane.



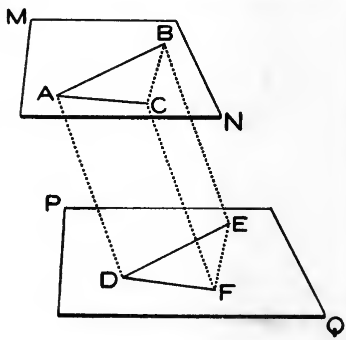
Given: Intersecting lines AB and AC in plane MN; each line ∥ to plane PQ.

To Prove: Plane MN ∥ to plane PQ.

Proof: Draw $AR \perp$ to MN at A, meeting PQ at R. Through AR and AB pass plane AS, and through AR and AC pass plane AT, intersecting plane PQ in RS and RT, respectively.

AB is ∥ to RS, and AC is ∥ to RT (?) (499).
But, AR is $\perp$ to AB and AC (?) (489).
∴ AR is $\perp$ to RS and RT (?) (95).
Hence, AR is $\perp$ to plane PQ (?) (501).
∴ plane MN is ∥ to plane PQ (?) (505). Q.E.D.

515. Theorem. If two angles, not in the same plane, have their sides parallel each to each, and extending in the same directions from their vertices, the angles are equal and the planes are parallel.



Given: $\angle BAC$ in plane MN and $\angle EDF$ in plane PQ; AB ∥ to DE; AC ∥ to DF, and extending in the same directions.

To Prove:
I. $\angle BAC = \angle EDF$.
II. Plane MN ∥ to plane PQ.

Proof: I. Take DE and AB equal, and DF and AC equal. Draw AD, BE, CF, BC, EF.

The figure $ABED$ is a ▱ (?) (135). ∴ $AD = BE$ (?).

Also $ACFD$ is a $\square$ (?). $\therefore AD = CF$ (?).

$\therefore BE = CF$ (?). Also AD is $\parallel$ to BE and AD is $\parallel$ to CF (?).

$\therefore BE$ is $\parallel$ to CF (?) (511). $\therefore BCFE$ is a $\square$ (?) (135).

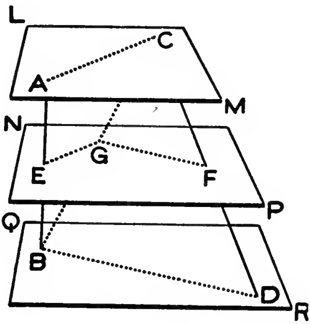
Now, in $\triangle$ ABC and DEF, $AB = DE$ (?); $AC = DF$ (?), and $BC = EF$ (?) (130). $\therefore \triangle ABC = \triangle DEF$ (?) (58).

$$\therefore \angle BAC = \angle EDF \text{ (?)}.$$

II. AB is $\parallel$ to plane PQ and AC is $\parallel$ to plane PQ (?) (498).

$\therefore$ plane MN is $\parallel$ to plane PQ (?) (514). Q.E.D.

516. Theorem. If three parallel planes intersect two straight lines, the corresponding intercepts are proportional.



Given: Parallel planes, LM, NP, QR, intersecting line AB at A, E, B, and CD at C, F, D, respectively.

To Prove: $AE : EB = CF : FD$.

Proof: Draw BC, meeting plane NP at G.

Through AB and BC pass a plane cutting LM in AC and NP in EG. Through BC and CD pass a plane cutting NP in GF and QR in BD. Now, EG is $\parallel$ to AC and GF is $\parallel$ to BD (?) (500).

$$\therefore \frac{AE}{EB} = \frac{CG}{BG} \text{ and } \frac{CG}{BG} = \frac{CF}{FD} \text{ (?) (302).}$$

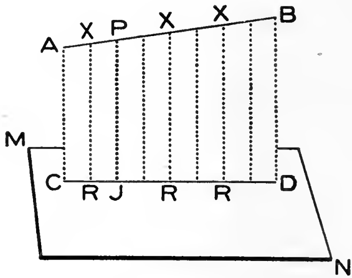
Consequently, $AE : EB = CF : FD$ (Ax. 1). Q.E.D

Also, AB and CD are intercepts between planes LM and QR

$$\text{And } \frac{AE}{AB} = \left(\frac{CG}{CB}\right) = \frac{CF}{CD} \text{ (?) (305).}$$

Ex. Prove, in figure of 516, $AB : CD = AE : CF = EB : FD$.

517. Theorem. The projection of a straight line upon a plane is a straight line.*



Given: Line AB and plane MN.

To Prove: The projection of AB on MN is a straight line.

Proof: Draw $PJ \perp$ to plane MN from any point P, in AB.

AB and PJ determine a plane (?) (495).

This plane AD intersects plane MN in a straight line CD (?).

Now in plane AD draw $XR \parallel$ to PJ from X, any other point in AB. XR is $\perp$ to MN (?) (509).

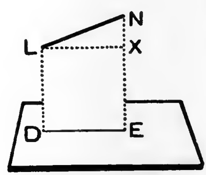
Now R is the projection of X (?) (485).

$\therefore$ CD is the projection of AB (?) (485).

That is, the projection of AB upon the plane MN is a straight line. Q.E.D.

518. Cor. A line and its projection upon a plane are in the same plane.

519. Theorem. A line not parallel to a plane is longer than its projection upon the plane.



Given: A plane and line LN not $\parallel$ to the plane, and DE the projection of LN upon the plane.

To Prove: $LN > DE$. **Proof:** Draw LD and NE. Draw $LX \perp$ to NE from L, in the plane LE. LD and NE are $\perp$ to the plane (Def. of projection, 485).

$LXED$ is a rectangle (?) (166).

Now $LN > LX$ (?) (77). But $LX = DE$ (?) (130).

Hence, $LN > DE$ (Ax. 6). Q.E.D.

* Except only if the given straight line is a normal to the given plane.

520. Theorem. Of all lines that can be drawn to a plane from a point:

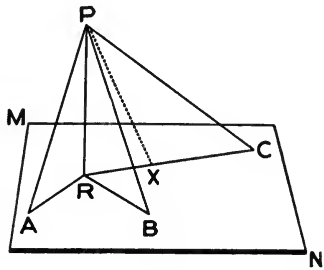
I. The perpendicular is the shortest.

II. Oblique lines having equal projections are equal.

III. Equal oblique lines have equal projections.

IV. Oblique lines having unequal projections are unequal, and the line having the greater projection is the longer.

V. Unequal oblique lines have unequal projections, and the longer line has the greater projection.



I. **Given:** Plane MN; point P; $PR \perp$ to MN; any other line from P to plane MN, as PA.

To Prove: $PR < PA$.

Proof: Draw AR. Now PR is $\perp$ to AR (?); PA is not $\perp$ to AR (?) (71). $\therefore PR < PA$ (?) (77).

II. **Given:** Oblique lines PA and PB, whose projections AR and BR are equal.

To Prove: $PA = PB$.

Proof: The rt. $\triangle$ PRA and PRB are $=$. (Explain.)

III. **Given:** Equal oblique lines PA and PB.

To Prove: Their projections, AR and BR, are equal.

Proof: The rt. $\triangle$ PRA and PRB are $=$. (Explain.)

IV. **Given:** Oblique lines PC and PA; proj. $RC >$ proj. RA

To Prove: $PC > PA$.

Proof: In $\triangle$ PCR, take on RC, $RX = RA$, and draw PX. Now $PC > PX$ (?) (76, III). But $PA = PX$ (?) (520, II)

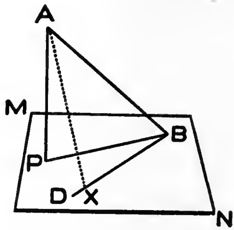
$\therefore PC > PA$ (Ax. 6).

V. **Given:** Unequal oblique lines, $PC > PA$.

To Prove: Projection $RC >$ projection RA.

Proof: By *reductio ad absurdum*. (See 88, 89).

521. Theorem. **The acute angle that a line makes with its own projection upon a plane is the least angle that the line makes with any line of the plane.**



Given: AB, any line meeting plane MN at B; BP, its projection upon MN; BD, any other line in MN, through B.

To Prove: $\angle ABP < \angle ABD$.

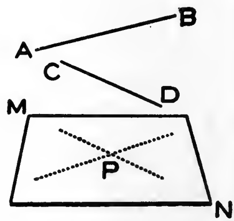
Proof: On BD take $BX = BP$ and draw AX. In $\triangle$ APB and ABX, $AB = AB$ (?); $BP = BX$ (?) (Const.).

But $AP < AX$ (?) (520, I).

$\therefore \angle ABP < \angle ABD$ (?) (87). Q.E.D.

522. Theorem. **Through a given point one plane can be passed parallel to any two given lines in space, and only one.**

Given: Point P; two lines, AB and CD.



To Prove: Through P one plane can be passed $\parallel$ to AB and CD, and only one.

Proof: I. Through P draw a line $\parallel$ to AB and another $\parallel$ to CD. Pass a plane MN, containing these lines.

MN is $\parallel$ to both AB and CD (?) (498).

II. Only one line can be drawn through P $\parallel$ to AB, and only one, $\parallel$ to CD (?) (92).

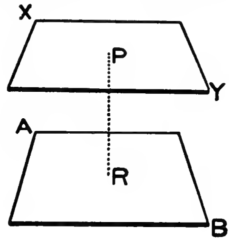
$\therefore$ there is only one plane (?) (495). Q.E.D.

Ex. 1. If a line meets a plane, with what line in the plane does this line make the greatest angle?

Ex. 2. In the figure of 515, prove AD parallel to the plane CE. Also BE parallel to the plane AF.

Ex. 3. In the figure of 516, if $AE = 6$, $EB = 5$, $CD = 16\frac{1}{2}$, find DF.

523. Theorem. Through a given point one plane can be passed parallel to a given plane, and only one.



Given: (?). **To Prove:** (?).

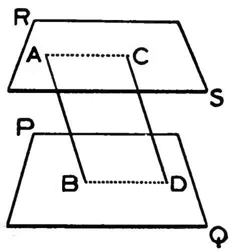
Proof: I. Suppose PR be drawn $\perp$ to AB; and XY be passed $\perp$ to PR at P.

Then XY is $\parallel$ to AB (?) (505).

II. Only one line $\perp$ to AB can be drawn from P (?) (507). Only one plane $\perp$ to PR can be passed at P (503).

$\therefore$ only one plane can contain P and be $\parallel$ to AB. Q.E.D.

524. Theorem. Parallel lines included between parallel planes are equal.



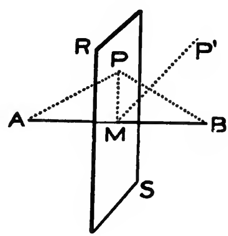
Given: (?). **To Prove:** (?).

Proof: AB and CD determine a plane (?). This plane intersects RS and PQ in lines AC and BD, which are $\parallel$ (?) (500).

$\therefore ABDC$ is a $\square$ (?).

Hence, $AB = CD$ (?). Q.E.D.

525. Theorem. The plane perpendicular to a line at its midpoint is the locus of points in space, equally distant from the extremities of the line.



Given: $RS \perp$ to AB at its midpoint, M.

To Prove: (?).

Proof: (1) Take P, any point in RS. Draw PM, PA, PB. PM is $\perp$ to AB (?) (489). $\therefore PA = PB$ (?) (67).

That is, any point in RS is equally distant from A and B.

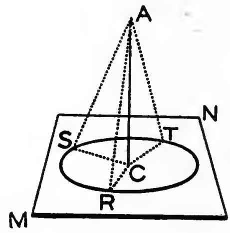
(2) Take P', any point outside of RS. Draw $P'M$.

$P'M$ is not $\perp$ to AB (?) (502). $\therefore P'$, any point outside of plane RS, is not equally distant from A and B (?) (68).

Hence, plane RS is the locus of points in space equally distant from A and B (?) (179). Q.E.D.

526. Theorem. The locus of points in space equally distant from all the points in the circumference of a circle is the line perpendicular to the plane of the circle at its center.

Given : (?). **To Prove** : (?).



Proof : I. Any point in AC is equally distant from all the points in the circumference of the circle (?) (520, II).

II. Any point equally distant from all points of the circumference of the circle is in AC (?) (520, III).

$\therefore$ AC is the required locus (179). Q.E.D.

Ex. 1. What is the locus of points equally distant from two given points?

Ex. 2. What is the locus of points equally distant from three given points?

527. The **distance** from a point to a plane is the length of the perpendicular from the point to the plane.

Thus, the word "distance," referring to the shortest line from a point to a plane, implies the perpendicular.

The **inclination** of a line to a plane is the angle between the line and its projection upon the plane.

ORIGINAL EXERCISES

1. Through one straight line a plane can be passed parallel to any other straight line in space, and only one.

Through a point of the first line draw a line $\parallel$ to the second.

2. Two parallel planes are everywhere equally distant.

3. If a line and a plane are parallel, another line parallel to the given line and through any point in the given plane lies wholly in the given plane.

Through the given line and the point P pass a plane cutting the given plane in PX. Use 499.

4. A straight line parallel to the intersection of two planes, but in neither, is parallel to both planes.

5. If two straight lines are parallel and two intersecting planes are passed, each containing one of the lines, the intersection of these planes is parallel to each of the given lines.

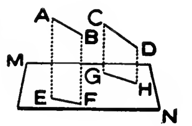
6. If three straight lines through a point meet the same straight line, these four lines all lie in the same plane.

7. If a straight line meets two parallel planes, its inclinations to the planes are equal.

8. Two parallel planes can be passed, each containing one of two given lines in space. Is this ever impossible? 522

9. If each of three straight lines intersects the other two, the three lines all lie in a plane.

10. The projections of two parallel lines on a plane are parallel.



Proof: AB is $\parallel$ to CD (?). AE is $\parallel$ to CG (?).

$\therefore$ planes AF and CH are $\parallel$ (?); etc.

11. If two lines in space are equal and parallel, their projections on a plane are equal and parallel.

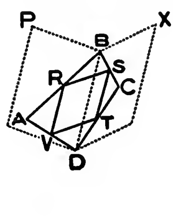
12. If a plane is parallel to one of two parallel lines, it is parallel to the other.

13. If a straight line and a plane are perpendicular to the same straight line, they are parallel.

14. Equal oblique lines drawn to a plane from one point have equal inclinations with the plane.

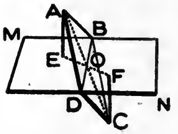
15. If a line and a plane are both parallel to the same line, they are parallel to each other.

16. Four points in space, A, B, C, D, are joined, and these four lines are bisected. Prove that the four lines joining (in order) the four midpoints of the first lines form a parallelogram.



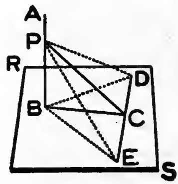
Proof: Pass plane DP through points A, D, B, and plane DX through points B, C, D,— these planes intersecting in BD. ST is $\parallel$ to BD and $= \frac{1}{2} BD$ (?); etc.

17. If a plane is passed containing a diagonal of a parallelogram and perpendiculars be drawn to the plane from the other vertices of the parallelogram, they are equal.



To Prove: $AE = CF$. **Proof:** Draw diagonal AC. Draw EO and OF in plane MN. EO, OF, and EOF are projections; etc.

18. If from the foot of a perpendicular to a plane, a line be drawn at right angles to any line in the plane, the line connecting this point of intersection with any point in the perpendicular is perpendicular to the line in the plane.



Given: $AB \perp$ to plane RS; $BC \perp$ to DE in the plane; PC drawn from C to P, in AB.

To Prove: PC is $\perp$ to DE.

Proof: Take $CD = CE$, draw PD, PE, BD, BE. BC is $\perp$ to DE at its midpoint (?). $\therefore$ $BD = BE$ (?). $PD = PE$ (?) (520, II).

$\therefore$ PC is $\perp$ to DE (?) (70).

19. A line PB is perpendicular to a plane at B, and a line is drawn from B meeting any line DE, of the plane, at C. If PC is perpendicular to DE, BC is perpendicular to DE.

20. Are two planes that are parallel to the same straight line necessarily parallel?

21. If each of two parallel lines is parallel to a plane, is the plane of these lines also parallel to the given plane?

22. Is a three-legged chair always stable on the floor? Why? Is a four-legged chair always stable? Why?

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23. What is the locus in space of points equally distant from two parallel planes? From two parallel lines?

24. What is the locus of points in space at a given distance from a given plane?

25. What is the locus of points in a plane at a given distance from an external point?

26. What is the locus of points in space equally distant from two points and equally distant from two parallel planes?

27. What is the locus of points in space, equally distant from the vertices of a given triangle?

28. What is the locus of all straight lines perpendicular to a given straight line at a given point?

29. What is the locus of all lines parallel to a given plane and drawn through a given point?

30. If the points in a line satisfy one condition and the points in a plane satisfy another condition, what will be true of their intersection? What will be true if they do not intersect?

31. If the points in one plane satisfy one condition and the points in another plane satisfy another condition, what is true of their intersection? What is true if the planes are parallel?

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32. To construct a plane perpendicular to a given line at a given point in the line.

33. To construct a plane perpendicular to a given line through a given external point.

34. To construct a line perpendicular to a given plane, through a given point in the plane.

35. To construct a line perpendicular to a given plane, through a given external point.

36. To construct a plane parallel to a given plane, through a given point.

37. To construct a number of equal oblique lines to a plane from a given external point.

38. To construct a line through a given point parallel to a given plane.

39. To construct a line through a given point and parallel to each of two given intersecting planes.

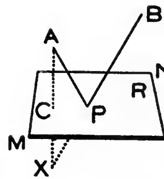
40. To construct a plane containing one given line and parallel to another.

41. To construct a plane through a given point parallel to any two given lines in space.

42. To construct a line through a given point in space which will intersect two given lines not in the same plane.

When is there no such line? Is there ever more than one?

43. To find a point in a plane such that the sum of the two lines joining it to two fixed points on one side of the plane shall be the least possible.



Construction: Draw $AC \perp$ to plane MN and prolong it to X, making $CX = AC$. Draw BX, meeting plane at P. Draw AP. Take any other point R in plane MN.

Statement: $AP + PB < AR + RB$. Etc.

44. To find a point in a given line equally distant from two given points. Is this ever impossible?

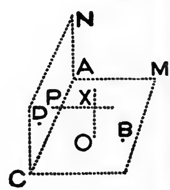
45. To find a point in a given plane equally distant from three given points. Is this ever impossible?

46. To find the line whose points are equally distant from two given points, and at a given distance from a given plane.

47. To find the point equally distant from two given points and equally distant from three other given points; or, to find the point equally distant from the ends of a given line, and also equally distant from the vertices of a given triangle.

When is there no such point?

48. To find the one point equally distant from four given points not in the same plane.



Given: The four points, A, B, C, D.

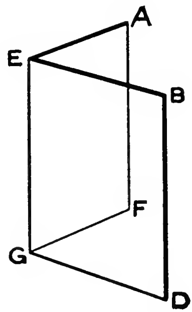
Required: To find a point equally distant from all of them.

Construction: Pass plane CM, containing A, B, C, and plane CN, containing A, D, C. Find O, the center of the ⊙ containing A, B, C. Find P, similarly. Draw the locus of points equally distant from A, B, C. (Consult 526.) Draw the locus of points equally distant from A, D, C. The plane $\perp$ to AC at its mid-point contains both these loci. (Explain.) Hence, OX and PX intersect (?) $\therefore X$ is the point.

DIHEDRAL ANGLES

528. A **dihedral angle** is the amount of divergence of two intersecting planes. The **edge** of the dihedral angle is the line of intersection of the planes. The **faces** of the dihedral angle are the planes.

The intersecting planes AG and ED form the dihedral angle whose edge is EG. This dihedral angle is named $A\text{-}GE\text{-}D$; or, when there is only one dihedral angle at the edge, it may be called "the angle EG."

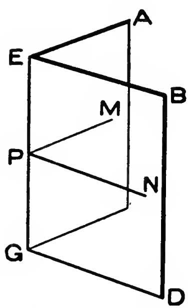


529. **Adjacent dihedral angles** are two dihedral angles that have the same edge and a common face between them.

Vertical dihedral angles are two dihedral angles that have the same edge, and the faces of one are the extensions of the faces of the other.

530. The **plane angle** of a dihedral angle is the angle formed by two straight lines, one in each face, and perpendicular to the edge at the same point.

If PM is in plane AG and perpendicular to EG, and PN is in plane ED and perpendicular to EG at P, the angle MPN is the plane angle of the dihedral angle EG.



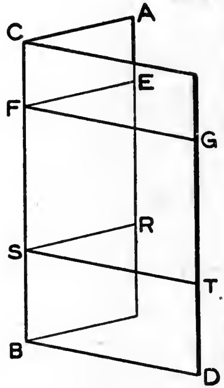
531. If one plane meets another, making the adjacent dihedral angles equal, these angles are **right dihedral angles.**

One plane is **perpendicular** to another plane if the dihedral angle formed by the two planes is a right dihedral angle.

532. Two dihedral angles are **equal** if they can be made to coincide.

A dihedral angle is **acute, right,** or **obtuse** according as its plane angle is acute, right, or obtuse.

Dihedral angles are **complementary** or **supplementary** according as their plane angles are complementary or supplementary.



533. Theorem. The plane angles of a dihedral angle are all equal.

Given: $\angle\ EFG$, the plane $\angle$ of dihedral $\angle\ BC$, at F, and $\angle\ RST$, the plane $\angle$ at S.

To Prove: $\angle\ EFG = \angle\ RST$.

Proof: EF is $\parallel$ to RS, and FG is $\parallel$ to ST (?) (93).

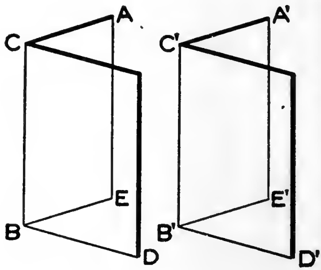
$\therefore \angle\ EFG = \angle\ RST$ (?) (515). Q.E.D.

534. Theorem. The plane of the plane angle of a dihedral angle is perpendicular to the edge. (See 501.)

535. Theorem. Two dihedral angles are equal if their plane angles are equal.

Given: Dihedral $\measuredangle s\ CB$ and $C'B'$ whose plane $\measuredangle s\ EBD$ and $E'B'D'$ are equal.

To Prove:
Dih. $\angle\ CB =$ dih. $\angle\ C'B'$.



Proof: CB is $\perp$ to plane EBD, and $C'B'$ is $\perp$ to plane $E'B'D'$ (?) (534).

Apply dih. $\angle\ C'B'$ to dih. $\angle\ CB$ so that the plane $\angle\ E'B'D'$ coincides with its equal $\angle\ EBD$.

$C'B'$ will coincide with CB (?) (506).

$\therefore$ plane $C'D'$ will coincide with plane CD and plane $C'E'$ will coincide with plane CE (?) (495).

Hence, dih. $\angle\ CB =$ dih. $\angle\ C'B'$ (?) (532). Q.E.D.

536. THEOREM. **If two dihedral angles are equal, their plane angles are equal.**

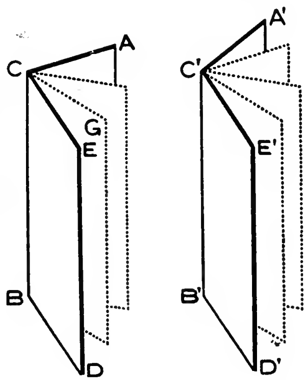
Proof: Superpose dih. $\angle C'B'$ upon its equal dih. $\angle CB$, making B' fall on B, and edge $B'C'$ on BC. Then face $C'D'$ will coincide with face CD, etc.

537. THEOREM. **Two vertical dihedral angles are equal.** (See 535.)

538. THEOREM. **The plane angle of a right dihedral angle is a right angle; and if the plane angle of a dihedral angle is a right angle, the dihedral angle is right.** (See 531.)

539. THEOREM. **Two dihedral angles have the same ratio as their plane angles.**

Given: Dihedral $\measuredangle s$ $A\text{–}BC\text{–}D$ and $A'\text{–}B'C'\text{–}D'$, having plane $\measuredangle s$ ACE and $A'C'E'$, respectively. **To Prove:** Dih. $\angle A\text{–}BC\text{–}D$: dih. $\angle A'\text{–}B'C'\text{–}D' = \angle ACE : \angle A'C'E'$.



Proof: I. If the plane angles are **commensurable,**

There exists a common unit of measure of the plane $\measuredangle s$ ACE and $A'C'E'$ (?) (238). Suppose this unit when applied to these angles is contained 3 times in $\angle ACE$ and 4 times in $\angle A'C'E'$.

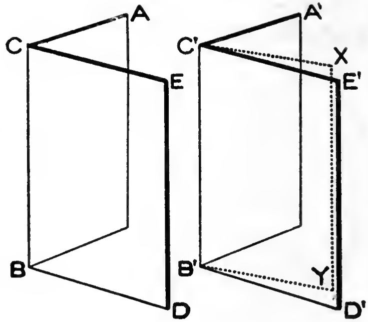
$$\therefore \frac{\angle ACE}{\angle A'C'E'} = \frac{3}{4} (?) \text{ (Ax. 3).}$$

Pass planes through the edges and the several lines of division of the angles. Dih. $\angle A\text{–}BC\text{–}D$ is divided into 3 parts; dih. $\angle A'\text{–}B'C'\text{–}D'$ is divided into 4 parts; all of these seven parts are equal (?) (535).

$$\therefore \frac{\text{dih. } \angle A\text{–}BC\text{–}D}{\text{dih. } \angle A'\text{–}B'C'\text{–}D'} = \frac{3}{4} (?) \text{ (Ax. 3).}$$

$\therefore$ dih. $\angle A\text{–}BC\text{–}D$: dih. $\angle A'\text{–}B'C'\text{–}D' = \angle ACE : \angle A'C'E'$ (?).

II. If the plane angles are **incommensurable.**



There does not exist a common unit (?). Suppose $\angle ACE$ to be divided into equal parts (any number of them).

Apply one of these as a unit of measure to $\angle A'C'E'$.

There will be a remainder, $XC'E'$, left over (because the $\angle$s are incommensurable).

Pass a plane $C'Y$, determined by $B'C'$ and $C'X$.

Now $\dfrac{\text{dih.} \angle A\text{–}BC\text{–}D}{\text{dih.} \angle A'\text{–}B'C'\text{–}Y} = \dfrac{\angle ACE}{\angle A'C'X}$ (Commensurable $\angle$s).

Indefinitely increase the number of subdivisions of $\angle ACE$.

Then each part, that is, our unit or divisor, will be indefinitely decreased. Hence $XC'E'$, the remainder, will be indefinitely decreased.

That is, $\angle XC'E'$ will approach zero as a limit; and dih. $\angle X\text{–}B'C'\text{–}D'$ will approach zero as a limit.

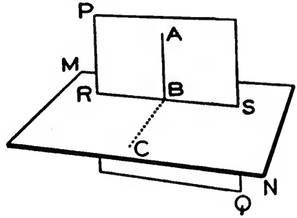
$\therefore \angle A'C'X$ will approach $A'C'E'$ as a limit (240); and dih. $\angle A'\text{–}B'C'\text{–}Y$ will approach dih. $\angle A'\text{–}B'C'\text{–}D'$ as a limit (240).

$\therefore \dfrac{\text{dih.} \angle A\text{–}BC\text{–}D}{\text{dih.} \angle A'\text{–}B'C'\text{–}Y}$ will approach $\dfrac{\text{dih.} \angle \text{–}ABC\text{–}D}{\text{dih.} \angle A'\text{–}B'C'\text{–}D'}$ as a limit and $\dfrac{\angle ACE}{\angle A'C'X}$ will approach $\dfrac{\angle ACE}{\angle A'C'E'}$ as a limit.

$$\therefore \frac{\text{dih.} \angle A\text{–}BC\text{–}D}{\text{dih.} \angle A'\text{–}B'C'\text{–}D'} = \frac{\angle ACE}{\angle A'C'E'} \quad (242).$$

Q.E.D.

540. Theorem. If a straight line is perpendicular to a plane, any plane containing this line is perpendicular to the given plane.



Given: Line $AB \perp$ to plane MN; plane PQ containing AB and intersecting plane MN in RS.

To Prove: PQ is $\perp$ to MN.

Proof: In MN draw $BC \perp$ to RS.

AB is $\perp$ to RS and to BC (?) (489).

$\therefore \angle ABC$ is the plane $\angle$ of dih. $\angle P\text{–}SR\text{–}N$ (?) (530).

But $\angle ABC$ is a rt. $\angle$ (?).

$\therefore PQ$ is $\perp$ to MN (?) (538). Q.E.D.

541. Theorem. If a plane is perpendicular to the edge of a dihedral angle, it is perpendicular to each face. (See 540.)

542. If one plane is perpendicular to another, any line in either plane, perpendicular to their intersection, is perpendicular to the other plane.

Given: Plane $PQ \perp$ to plane MN; AB in plane $PQ \perp$ to the intersection, RS.

To Prove: $AB \perp$ to plane MN.

Proof: In plane MN draw $BC \perp$ to RS.

Now, $\angle ABC$ is the plane angle of the dih. $\angle P\text{–}SR\text{–}N$ (?).

$\therefore \angle ABC$ is a rt. $\angle$ (?) (538). $\therefore AB$ is $\perp$ to BC (?) (17).

But AB is $\perp$ to RS (Hyp.).

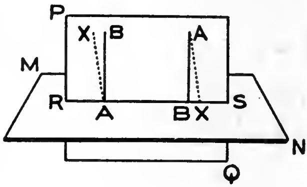
$\therefore AB$ is $\perp$ to plane MN (?) (501). Q.E.D.

Ex. 1. In the figure of 540 prove BC perpendicular to plane PQ. Also prove RS perpendicular to the plane ABC.

Ex. 2. In the figure of 540 prove the plane ABC perpendicular to planes MN and PQ.

Ex. 3. Under what condition will a line in one face of a dihedral angle meet a line in the other face?

543. Theorem. If one plane is perpendicular to another, a line drawn from any point in their intersection and perpendicular to one plane, lies in the other.



Given: Plane $PQ \perp$ to plane MN, intersecting in RS; $AB \perp$ to plane MN from A, in RS.

To Prove: AB is in plane PQ.

Proof: At A erect in plane PQ, $AX \perp$ to RS.

Then AX is $\perp$ to plane MN (?) (542).
But AB is $\perp$ to plane MN at A (Hyp.).
$\therefore$ AB and AX coincide (?) (506).
That is, AB is in plane PQ. Q.E.D.

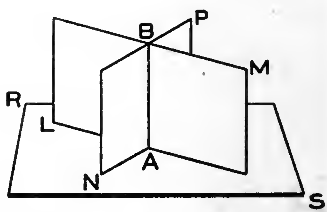
544. Theorem. If one plane is perpendicular to another, a line drawn from any point in one plane, and perpendicular to the other, lies in the first plane.

Given: Plane $PQ \perp$ to plane MN; $AB \perp$ to plane MN from A, any point in plane PQ.

To Prove: (?).

Proof: From A draw in plane PQ, $AX \perp$ to RS.
Then AX is $\perp$ to plane MN (?) (542).
$\therefore$ AB and AX coincide (?). Etc.

545. Theorem. If two planes are perpendicular to a third plane, their intersection also is perpendicular to that plane.



Given: Planes LM and NP, each $\perp$ to plane RS.

To Prove: The intersection AB is $\perp$ to plane RS.

Proof: If, at A, a line be erected $\perp$ to plane RS, it will lie in plane LM (?) (543).
This $\perp$ will lie also in plane NP (?).
$\therefore$ this $\perp$ is the intersection AB (?) (482).
That is, AB is $\perp$ to plane RS. Q.E.D.

546. Theorem. **If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection.** (The same truth as 545.)

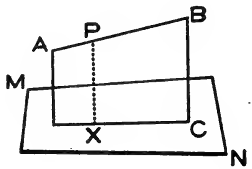
547. Theorem. **If each of three planes is perpendicular to the other two, each of the three intersections is perpendicular to the remaining plane, and perpendicular to the other two intersections.**

That is, if each of the three planes *RS*, *LM*, *NP* is ⊥ to the others, then,

$$\left.\begin{array}{l} BA \text{ is } \perp \text{ to plane } RS, \text{ to } LA, \text{ and to } NA; \\ LA \text{ is } \perp \text{ to plane } NP, \text{ to } AB, \text{ and to } NA; \\ NA \text{ is } \perp \text{ to plane } LM, \text{ to } AB, \text{ and to } LA. \end{array}\right\} \text{(545 and 489).}$$

This truth is illustrated at the corner of an ordinary box or room.

548. Theorem. **Through a given line not perpendicular to a plane, one plane can be passed perpendicular to that plane, and only one.**



Given: *AB* not ⊥ to plane *MN*.

To Prove: Through *AB* one plane can be passed ⊥ to *MN*, and only one.

Proof: I. From *P*, any point in *AB*, draw *PX* ⊥ to *MN*. Through *AB* and *PX* pass plane *AC*. Plane *AC* is ⊥ to plane *MN* (?) (540).

II. Suppose another plane containing *AB* is ⊥ to plane *MN*. Then the intersection *AB*, of these two planes, which are ⊥ to plane *MN*, will be ⊥ to plane *MN* (?) (545).

But *AB* is not ⊥ to plane *MN* (?) (Hyp.).

∴ there is only one plane containing *AB* and ⊥ to plane *MN*.

Q.E.D.

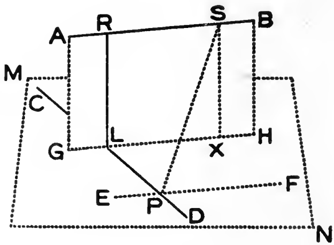
549. Cor. **The plane containing a straight line and its projection upon a plane is perpendicular to the given plane.**

550. Theorem. One line can be drawn perpendicular to both of two lines in space, not in the same plane, and only one.

Given: Lines *AB* and *CD* not in the same plane.

To Prove: One line can be drawn ⊥ to *AB* and *CD*, and only one.

Proof: I. At *P*, any point in *CD*, draw *EF* ∥ to *AB*. Pass plane *MN*, containing *CD* and *EF*. Pass plane *AH* through *AB* and ⊥ to *MN*, intersecting *MN* in *GH*, and *CD* at *L*. In plane *AH* draw *RL* ⊥ to *GH*.



Plane *MN* is ∥ to *AB* (498).

GH is ∥ to *AB* (?) (499).

RL is ⊥ to *GH* (Const.). ∴ *RL* is ⊥ to plane *MN* (?) (542).

∴ *RL* is ⊥ to *CD* (?). Also, *RL* is ⊥ to *AB* (?) (95).
That is, *RL* is ⊥ to both the given lines.

II. If another line can be drawn ⊥ to *AB* and *CD*, suppose *SP* is this ⊥. In plane *AH* draw *SX* ⊥ to *GH*. Then *SX* is ⊥ to plane *MN* (?) (542).

But if *SP* is ⊥ to *AB*, it is ⊥ to *EF* (?) (95).

∴ *SP* is ⊥ to plane *MN* (?) (501).

Thus there are two ⊥s from *S* to plane *MN* (*SX* and *SP*).

But this is impossible (?) (507).

∴ there can be no second ⊥ to these two given lines.

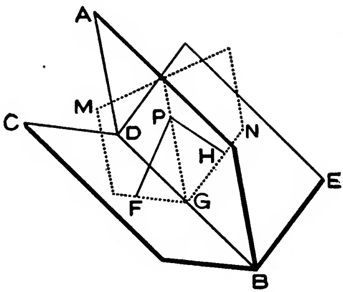
Q.E.D.

Ex. 1. In the figure of 550 prove that a plane perpendicular to *RL* at its midpoint will be parallel to *AB* and *CD*.

Ex. 2. Prove, also, that this plane will bisect *SP*.

Ex. 3. Prove that if *CD* is not perpendicular to *EF*, no plane can be passed through *AB*, perpendicular to *CD*.

551. THEOREM. Every point in a plane bisecting a dihedral angle is equally distant from the faces of the angle.



Given: Plane *AB*, bisecting the dih. $\angle$ *C–BD–E;* any point *P* in plane *AB*; *PF* $\perp$ to face *CB*; *PH* $\perp$ to face *DE*.

To Prove: $PF = PH$.

Proof: Pass plane *MN*, containing *PF* and *PH*, intersecting *CB* in *FG*, *AB* in *PG*, *DE* in *HG*, *BD* at *G*.

Now plane *MN* is $\perp$ to planes *CB* and *DE* (?) (540).

$\therefore$ plane *MN* is $\perp$ to *BD* (?) (546).

$\therefore$ *BG* is $\perp$ to *FG*, *PG*, and *HG* (?) (489).

Hence, $\angle PGF$ is the plane $\angle$ of dih. $\angle$ *A–BD–C* and $\angle PGH$ is the plane $\angle$ of dih. $\angle$ *A–BD–E* (?) (530).

These dih. $\angle$s are = (Hyp.). $\therefore \angle PGF = \angle PGH$ (?) (536).

$\angle$s *PFG* and *PHG* are rt. $\angle$s (?) (489).

In the right $\triangle$s *PFG* and *PHG*, $PG = PG$ (?),

and $\angle PGF = \angle PGH$ (?).

$\therefore$ these $\triangle$s are = (?). Consequently, $PF = PH$ (?). Q.E.D.

552. THEOREM. Any point in a dihedral angle and equally distant from the faces of the angle is in the plane bisecting the angle.

Given: $PF = PH$.

To Prove: Plane *AB*, determined by *P* and *DB*, is the bisector of dih. $\angle$ *C–BD–E*.

Proof: Similar to the proof of 551.

553. THEOREM. The plane bisecting a dihedral angle is the locus of points in space equally distant from the faces of the dihedral angle. (See 551, 552.)

ORIGINAL EXERCISES

1. Are two planes perpendicular to the same plane necessarily parallel?

2. A straight line and a plane perpendicular to the same plane are parallel.

3. A plane perpendicular to a line in another plane, is perpendicular to that plane.

4. If three planes, all perpendicular to a fourth, intersect in three lines, these lines are parallel, in pairs.

5. If the projection of any line (straight or curved) upon a plane is a straight line, the line is entirely in one plane.

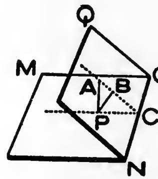
6. The angle between the normals drawn to the faces of a dihedral angle from a point within the angle is the supplement of the plane angle of the dihedral angle.

7. If a line is parallel to a plane, any plane perpendicular to the line is perpendicular also to the plane.

[Construct the projection of the given line upon the given plane.]

8. What is the locus of points in space equally distant from two intersecting planes?

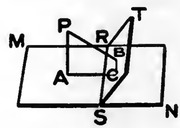
9. If from any point in a face of a dihedral angle, a normal is drawn to each face, the plane of these normals is perpendicular to the edge of the dihedral angle.



10. If from any point in a face of a dihedral angle, a normal is drawn to each face, the angle they form is equal to the plane angle of the dihedral angle.

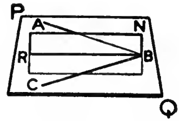
11. If a line is perpendicular to a plane, any plane parallel to the line is also perpendicular to the plane.

12. If PA is a normal to plane MN, PB a normal to plane ST, and BC a normal to plane MN, AC is perpendicular to RS, the intersection of planes MN and ST.



13. The plane perpendicular to the line that is perpendicular to two lines in space, at its middle point, bisects every straight line having its extremities in these lines.

14. The common perpendicular to two lines in space is the shortest line that can be drawn between them.

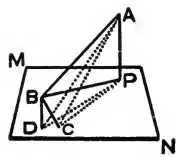


15. The plane perpendicular to the plane of an angle and containing the bisector of the angle is the locus of points equally distant from the sides of the angle.

Proof: The $\perp$s from any point in plane NR to AB and BC will have equal projections. (Explain by use of 79.)

$\therefore$ these perpendiculars are equal (?).

16. What is the locus of points in space equally distant from two intersecting lines?

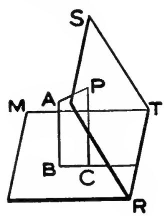


17. If AP is a normal to plane MN and if angle PBC, in plane MN, is a right angle, angle ABC also is a right angle.

[Prove BC is $\perp$ to plane APB].

18. If AP is a normal to plane MN and $\angle PBD$, in plane MN, is obtuse, $\angle ABD$ also is obtuse.

Proof: Take $BC = BD$; prove $PD > PC$. Then prove $AD > AC$, etc.



19. PA is perpendicular to plane RS; AB and PC are perpendicular to plane MR. Prove BC perpendicular to RT.

20. If two parallel planes are cut by a third plane, the alternate-interior dihedral angles are equal; the corresponding dihedral angles are equal; the alternate-exterior dihedral angles are equal; the adjoining interior dihedral angles are supplementary.

21. State and prove the converse theorems of those in No. 20.

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22. To construct a plane perpendicular to a given plane and containing a given line in that plane.

23. To construct a plane perpendicular to a given plane and containing a given line without that plane.

24. To construct through a given point a line which will intersect any two given lines in space.

Construction: Pass a plane through the point and one of the lines. This plane intersects the other line at a point. This point and the given point determine the required line. Explain. Discuss.

25. To bisect a given dihedral angle.

Construction: Pass a plane $\perp$ to the edge. Bisect the plane $\angle$ formed, pass the plane determined by this bisector and the given edge.

26. To construct a line whose points shall be equally distant from the ends of a given line and also equally distant from the faces of a dihedral angle.

27. To find the locus of points equally distant from two points and also equally distant from two intersecting planes. Discuss.

28. To find a point equally distant from three given points and equally distant from two intersecting planes.

Is this problem ever impossible? Will there ever be two points? When will there be only one?

29. To find a point equally distant from three given points and equally distant from two intersecting lines. Discuss fully.

POLYHEDRAL ANGLES

554. If three or more planes meet at a point, they form a **polyhedral angle.** The opening partially surrounded by the planes is the polyhedral angle.

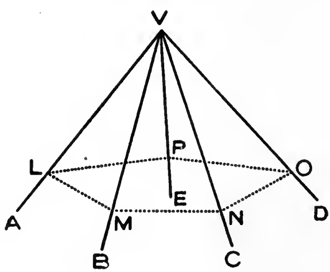
The point common to all the planes is the **vertex.**

The planes are the **faces.**

The intersections of adjacent faces are the **edges.**

The angles formed at the vertex, by adjacent edges, are the **face angles.**

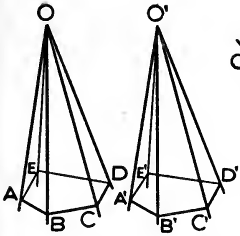
Thus, $V-ABCDE$ is a polyhedral angle; V is the vertex; AV, BV, etc., are edges; planes AVB, BVC, etc., are faces; $\angle s$ AVB, BVC, etc., are face angles.



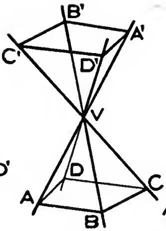
555. A **plane section** of a polyhedral angle is the plane figure bounded by the intersections of all the faces by a plane.

Polygon $LMNOP$ is a plane section of polyhedral angle $V-ABCDE$.

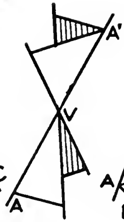
A **convex polyhedral angle** is one whose plane sections are all convex.



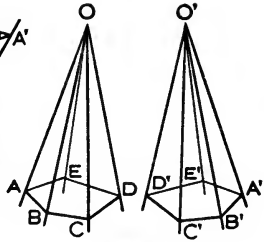
EQUAL POLYHEDRAL ANGLES



VERTICAL POLYHEDRAL ANGLES



VERTICAL DIHEDRAL ANGLES



SYMMETRICAL POLYHEDRAL ANGLES

556. Two polyhedral angles are **equal** if they can be made to coincide in all particulars. That is, if two polyhedral angles are equal, their homologous dihedral angles are equal; their homologous face angles are equal, and they are arranged in the same order. The length of the edges or the extent of the faces does not affect the size of the polyhedral angle.

Two polyhedral angles are **vertical** if the edges of one are the prolongations of the edges of the other.

Two polyhedral angles are **symmetrical** if all the parts of one are equal to the corresponding parts of the other, but arranged in opposite order.

NOTE. It is apparent from the definitions that equal polyhedral angles are mutually equiangular, as to the face angles and as to the dihedral angles.

Vertical polyhedral angles are mutually equiangular, as to their face angles and as to their dihedral angles, but the order is reversed.

Symmetrical polyhedral angles are also mutually equiangular as to their face angles and as to their dihedral angles, but the order is reversed.

Thus, if one follows around the polygon $A'D'$ in alphabetical order, he is moving as the hands of a clock — if the eye is at the vertex O'; but if he follows around AD alphabetically, he is moving in a direction opposite to the motion of the hands of a clock — if the eye is at the vertex O.

Hence, it is apparent that, in general, symmetrical polyhedral angles cannot be made to coincide.

557. A **trihedral angle** is a polyhedral angle having three and only three faces.

A trihedral angle is **rectangular** if it contains a right dihedral angle; **birectangular,** if it contains two right dihedral angles; **trirectangular,** if it contains three right dihedral angles.

A trihedral angle is **isosceles** if two of its face angles are equal.

558. THEOREM. **Two vertical polyhedral angles are symmetrical.**

Proof: Their homologous face angles are equal and arranged in reverse order, and their homologous dihedral angles are equal and arranged in reverse order.

$\therefore$ they are symmetrical (Def. 556).

559. THEOREM. **If two polyhedral angles are symmetrical, the vertical polyhedral angle of the one is equal to the other.**

Because the corresponding parts are equal and they are arranged in the same order.

560. THEOREM. **Provided two trihedral angles have their parts arranged in the same order, they are equal:**

I. If two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other.

II. If a face angle and the two dihedral angles adjoining it of the one are equal respectively to a face angle and the two dihedral angles adjoining it, of the other.

Proof: By method of superposition, as in plane $\triangle$s.

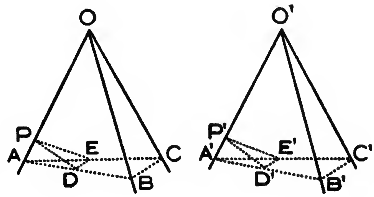
561. Theorem. **Provided two trihedral angles have their parts arranged in the same order, they are equal, if the three face angles of one are equal respectively to the three face angles of the other.**

Given: Trih. $\angle s$ O and O';

$\angle AOB = \angle A'O'B'$;

$\angle BOC = \angle B'O'C'$;

$\angle COA = \angle C'O'A'$.



To Prove: Trih. $\angle O =$ trih. $\angle O'$, that is, dih. $\angle OA$ = dih. $\angle O'A'$, etc.

Proof: Take $OA = OB = OC = O'A' = O'B' = O'C'$.

Draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

Take, on edges AO and $A'O'$, $AP = A'P'$ and in face AOB draw $PD \perp$ to AO.

$\angle OAB$ is acute ($\triangle AOB$ is isosceles). $\therefore PD$ will meet AB.

In face AOC draw $PE \perp$ to AO, meeting AC at E. Draw DE. Similarly draw $P'D'$, $P'E'$, $D'E'$.

Now $\angle s$ EPD and $E'P'D'$ are the plane $\angle s$ of the dihedral $\angle s$ AO and $A'O'$ (?) (530). To prove that these $\angle s$ are equal requires the proof that eight pairs of $\triangle s$ are equal.

(1) $\triangle OAB = \triangle O'A'B'$. (Explain.)
(2) $\triangle OBC = \triangle O'B'C'$. (Explain.)
(3) $\triangle OAC = \triangle O'A'C'$. (Explain.)
$\therefore AB = A'B'$, $BC = B'C'$, $AC = A'C'$, $\angle OAB = \angle O'A'B'$, $\angle OAC = \angle O'A'C'$ (?) (27).

(4) $\triangle APD = \triangle A'P'D'$. (Explain.)
(5) $\triangle APE = \triangle A'P'E'$. (Explain.)
$\therefore AD = A'D'$, $AE = A'E'$; $PD = P'D'$, etc. (?).

(6) $\triangle ABC = \triangle A'B'C'$. (Explain.) $\therefore \angle CAB = \angle C'A'B'$ (?).
(7) $\triangle AED = \triangle A'E'D'$. (Explain.) $\therefore ED = E'D'$ (?).
(8) $\triangle PED = \triangle P'E'D'$. (Explain.) $\therefore \angle EPD = \angle E'P'D'$ (?).

Hence, dih. $\angle AO =$ dih. $\angle A'O'$ (?) (535).

Similarly one may prove the other pairs of homologous dihedral angles equal.

$\therefore$ trihedral $\angle O =$ trihedral $\angle O'$ (?) (556). Q.E.D.

562. Theorem. **Provided two trihedral angles have their parts arranged in reverse order, they are symmetrical:**

I. If two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other.

II. If a face angle and the two dihedral angles adjoining it of the one are equal respectively to a face angle and the two dihedral angles adjoining it of the other.

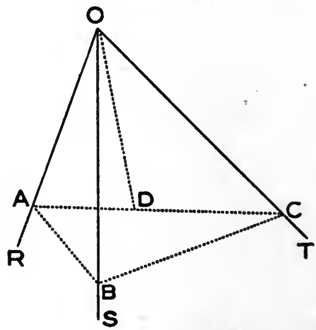
III. If the three face angles of one are equal respectively to the three face angles of the other.

Proof: In each case construct a third trihedral $\angle$ symmetrical to the first. This third figure will have its parts = to the parts of the second, and arranged in the same order (Def. 556).

$\therefore$ the third = the second (?) (560 and 561).

$\therefore$ the first is symmetrical to the second (Ax. 6). Q.E.D.

563. Theorem. **The sum of any two face angles of a trihedral angle is greater than the third face angle.**



Given: Trih. $\angle$ *O–RST* in which face angle *ROT* is the greatest.

To Prove:

$\angle$ *ROS* + $\angle$ *SOT* > $\angle$ *ROT*.

Proof: Construct, in face *ROT*, $\angle$ *ROD* = $\angle$ *ROS*.

Take *OD* = *OB*; draw *ADC*, meeting *OT* at *C*. Draw *AB* and *BC*.

$\triangle$ *AOD* = $\triangle$ *AOB*. (Explain.)

$\therefore$ *AB* = *AD* (?).

Now *AB* + *BC* > *AD* + *DC* (?) (Ax. 12).

But, *AB* = *AD* (?).

Subtracting, *BC* > *DC* (Ax. 7).

Now, *OB* = *OD* (?), *OC* = *OC* (?) and *BC* > *DC* (Just proved).

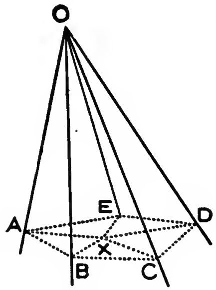
$\therefore$ $\angle$ *BOC* > $\angle$ *DOC* (?) (87).

But $\angle$ *AOB* = $\angle$ *AOD* (?).

Adding, $\angle$ *AOB* + *BOC* > $\angle$ *AOC* (Ax. 7).

That is, $\angle$ *ROS* + $\angle$ *SOT* > $\angle$ *ROT* (?) (Ax. 6). Q.E.D.

564. THEOREM. The sum of the face angles of any polyhedral angle is less than four right angles, or 360°.



Given: Polyhedral $\angle$ O, having n faces.

To Prove: The sum of the face $\angle$s at O $<$ 4 rt. $\angle$s, or 360°.

Proof: Pass a plane AD, intersecting all the faces, and the edges at A, B, C, etc.

In this section take any point X and join X to all the vertices of the polygon.

(1) There are n face $\triangle$s having their vertices at O (Hyp.).

(2) There are n base $\triangle$s having their vertices at X (Const.).

(3) The sum of the $\angle$s of the face $\triangle$s $= 2n$ rt. $\angle$s (110).

(4) The sum of the $\angle$s of the base $\triangle$s $= 2n$ rt. $\angle$s (110).

(5) $\therefore$ the sum of the $\angle$s of the face $\triangle$s $=$ the sum of the $\angle$s of the base $\triangle$s (Ax. 1.).

Now, $\angle OAE + \angle OAB > \angle EAB$ (?) (563).

And $\angle OBA + \angle OBC > \angle ABC$ (?), etc., etc.

Adding, the sum of the base $\angle$s of the face $\triangle$s $>$ the sum of the base $\angle$s of the base $\triangle$s (Ax. 8).

Subtracting this inequality from equation (5) above, the sum of the face $\angle$s at O $<$ the sum of the $\angle$s at X (Ax. 9).

But the sum of all the $\angle$s at $X = 4$ rt. $\angle$s (?) (47).

$\therefore$ the sum of the face $\angle$s at $O < 4$ rt. $\angle$s, or 360° (Ax. 6).

Q.E.D.

Ex. 1. Prove theorem of 563 for the case of an isosceles trihedral angle.

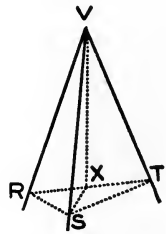
Ex. 2. In the figure of 564, as the vertex O approaches the base, does the sum of the face angles at O increase or decrease? What limit does this sum approach? Does the sum ever become equal to this limit?

ORIGINAL EXERCISES

1. The three planes bisecting the three dihedral angles of a trihedral angle intersect in a straight line.

2. The three planes containing the three bisectors of the three face angles of a trihedral angle and perpendicular to those faces intersect in a straight line.

3. If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.



Given: $\angle RVS = \angle SVT$.

To Prove: Dih. $\angle RV =$ dih. $\angle TV$.

Proof: Pass plane SVX bisecting dih. $\angle SV$. Prove trih. $\measuredangle$ V-RSX and V-TSX are sym. by 562, I.

4. An isosceles trihedral angle and its symmetrical trihedral angle are equal.

5. Find the locus of points equally distant from the three faces of a trihedral angle.

6. Find the locus of points equally distant from the three edges of a trihedral angle.

7. If the three face angles of a trihedral angle are equal, the three dihedral angles also are equal.

8. If the three face angles of a trihedral angle are right angles, the three dihedral angles also are right angles.

9. In any trihedral angle the greatest dihedral angle has the greatest face angle opposite it.

10. If the edges of one trihedral angle are perpendicular to the faces of a second trihedral angle, then the edges of the second are perpendicular to the faces of the first.

11. To construct the plane angles of the three dihedral angles of a trihedral angle, if the three face angles are known.

[To be accomplished by constructions in a plane.]

12. To construct, through a given point, a plane which shall make, with the faces of a polyhedral angle having four faces, a section that is a parallelogram.

Construction: Extend one pair of opp. faces to obtain their line of intersection. Similarly extend the other pair. Any plane section ∥ to these lines will be a ▱. (Explain.)

BOOK VII

POLYHEDRONS

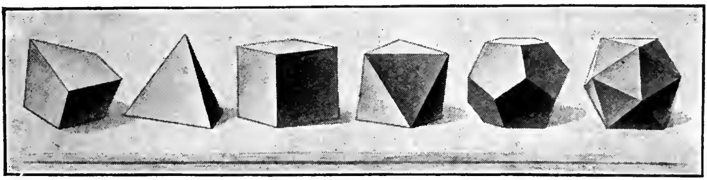
565. A **polyhedron** is a solid bounded by planes.

The **edges** of a polyhedron are the intersections of the bounding planes.

The **faces** are the portions of the bounding planes included by the edges.

The **vertices** are the intersections of the edges.

The **diagonal** of a polyhedron is a straight line joining two vertices not in the same face.



POLYHEDRON TETRAHEDRON HEXAHEDRON CUBE OCTAHEDRON DODECAHEDRON ICOSAHEDRON

566. A **tetrahedron** is a polyhedron having four faces.

A **hexahedron** is a polyhedron having six faces.

An **octahedron** is a polyhedron having eight faces.

A **dodecahedron** is a polyhedron having twelve faces.

An **icosahedron** is a polyhedron having twenty faces.

567. A polyhedron is **convex** if the section made by every plane is a convex polygon.

Only convex polyhedrons are considered in this book.

PRISMS

568. A **prism** is a polyhedron two of whose opposite faces are equal polygons in parallel planes, and whose other faces are all parallelograms.

The **bases** of a prism are the equal, parallel polygons.

The **lateral faces** of a prism are the parallelograms.

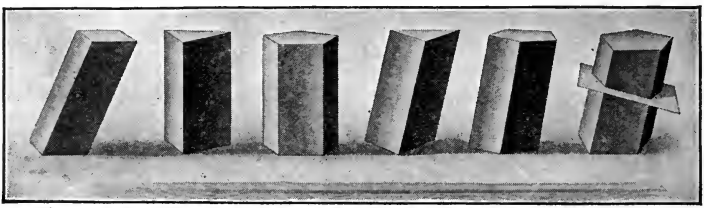
The **lateral edges** of a prism are the intersections of the lateral faces.

The **lateral area** of a prism is the sum of the areas of the lateral faces.

The **total area** of a prism is the sum of the lateral area and the areas of the bases.

The **altitude** of a prism is the perpendicular distance between the planes of the bases.

A **triangular prism** is a prism whose bases are triangles.



PRISM — TRIANGULAR PRISM — REGULAR PRISM — OBLIQUE PRISMS — RIGHT SECTION — TRUNCATED PRISMS

569. A **right prism** is a prism whose lateral edges are perpendicular to the planes of the bases.

A **regular prism** is a right prism whose bases are regular polygons.

An **oblique prism** is a prism whose lateral edges are not perpendicular to the planes of the bases.

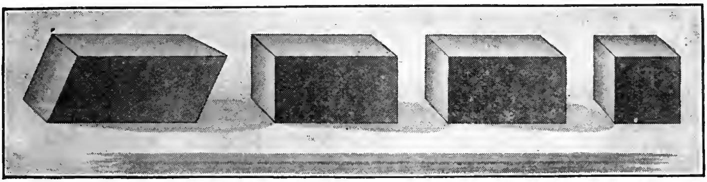
A **truncated prism** is the portion of a prism included between the base and a plane not parallel to the base.

A **right section** of a prism is the section made by a plane perpendicular to the lateral edges of the prism.

570. A **parallelepiped** is a prism whose bases are parallelograms.

A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to the planes of the bases.

A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.



PARALLELEPIPED — RIGHT PARALLELEPIPED — RECTANGULAR PARALLELEPIPED — CUBE

An **oblique parallelepiped** is a parallelepiped whose lateral edges are not perpendicular to the planes of the bases.

A **cube** is a rectangular parallelepiped whose six faces are squares.

571. The **unit of volume** is a cube whose edges are each a unit of length.

The **volume of a solid** is the number of units of volume it contains. The volume of a solid is the ratio of that solid to the unit of volume.

The three edges of a rectangular parallelepiped meeting at any vertex are the **dimensions** of the parallelepiped.

Equivalent solids are solids that have equal volumes.

Equal solids are solids that can be made to coincide.

Ex. What is the base of a rectangular parallelepiped? Of a right parallelepiped? Of an oblique parallelepiped?

NOTE. The space that is bounded by the surfaces of a solid, independent of the solid, is called a **geometrical solid.**

That is, if a material or physical body occupy a certain position and then be removed elsewhere, there is a definite portion of space that is the exact shape and size as the solid, and can be conceived as bounded by exactly the same surfaces as bounded the solid when in that original position. In order that we may pass planes and draw lines through solids, and superpose one solid upon another, it is convenient in studying the properties of solids to consider them usually as geometric solids, — the material body being removed for the time.

The three kinds of parallelepipeds can be illustrated by removing the cover and bottom of an ordinary cardboard box, and distorting the shape of the frame that remains.

PRELIMINARY THEOREMS

572. THEOREM. **The lateral edges of a prism are all equal (?).**

573. THEOREM. **Any two lateral edges of a prism are parallel.** (See 511.)

574. THEOREM. **Any lateral edge of a right prism equals the altitude.** (See 524.)

575. THEOREM. **The lateral faces of a right prism are perpendicular to the bases.** (See 540.)

576. THEOREM. **The lateral faces of a right prism are rectangles.** (Def. 569.)

577. THEOREM. **The faces and bases of a rectangular parallelepiped are rectangles.** (Def. 569.)

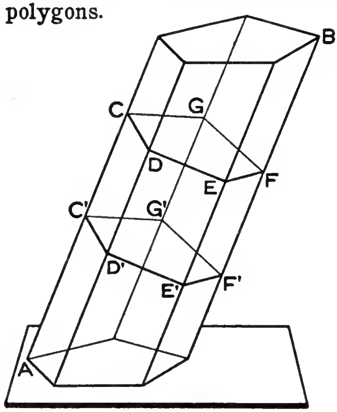
578. THEOREM. **All the faces of any parallelepiped are parallelograms (?).**

579. AXIOM. **A polyhedron cannot have fewer than four faces.**

580. AXIOM. **A polyhedron cannot have fewer than three faces at each vertex.**

THEOREMS AND DEMONSTRATIONS

581. Theorem. **The sections of a prism made by parallel planes cutting all the lateral edges are equal polygons.**



Given: Prism AB; $\parallel$ sections CF and $C'F'$.

To Prove: Polygon CF = polygon $C'F'$.

Proof: CD is $\parallel$ to $C'D'$, DE is $\parallel$ to $D'E'$, etc. (?) (500).

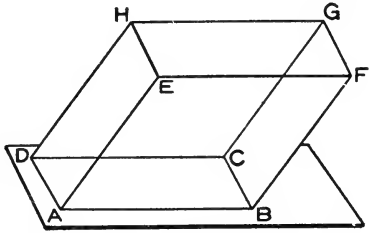
CD', DE', EF', etc., are ▱ (?).

$\therefore CD = C'D'$, $DE = D'E'$, $EF = E'F'$, etc. (?) (130).

$\angle GCD = \angle G'C'D'$, $\angle CDE = \angle C'D'E'$, etc. (?) (515).

$\therefore$ polygon CF = polygon $C'F'$ (?) (159). Q.E.D.

582. Theorem. **The opposite faces of a parallelepiped are equal and parallel.**



Given: (?). **To Prove:** Face AF = and $\parallel$ to face DG.

Proof: Faces AF and DG are ▱ (?).

$AB = DC$, $AE = DH$ (130).

AB is $\parallel$ to DC and AE is $\parallel$ to DH (?).

$\therefore \angle EAB = \angle HDC$ (?) (515).

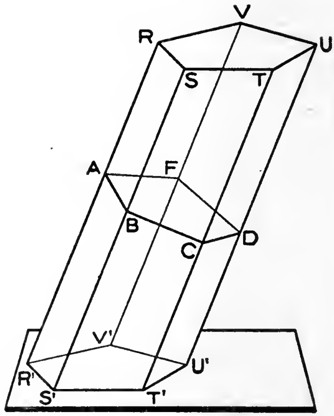
$\therefore$ face AF = face DG (?) (139).

Also face AF is $\parallel$ to face DG (?) (515). Q.E.D.

Ex. 1. All right sections of a prism are equal.

Ex. 2. Any section of a parallelepiped made by a plane cutting two pairs of opposite faces is a parallelogram.

583. THEOREM. The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.



Given: Prism RU'; edge $= E$; right section AD.

To Prove: Lateral area of $RU' = E \times$ perimeter of AD.

Proof: AB is $\perp$ to RR', BC is $\perp$ to SS', etc. (?) (489).

Area $\square$ $RS' = E \cdot AB$ (373).

Area $\square$ $ST' = E \cdot BC$ (?).

Area $\square$ $TU' = E \cdot CD$ (?).

etc. etc. Adding,

The lateral area $= E \cdot (AB + BC + CD +$ etc.) (Ax. 2).

$= E \cdot$ perimeter of rt. sect. (Ax. 6).

Q.E.D.

Ex. 1. Any section of a parallelepiped made by a plane parallel to any edge is a parallelogram.

Ex. 2. The sum of the face angles at all the vertices of any parallelepiped is equal to 24 right angles.

Ex. 3. The sum of the plane angles of all the dihedral angles of any parallelepiped is equal to 12 right angles.

Proof: Pass three planes $\perp$ to three intersecting edges. Prove these sections ▱ whose ∠s are the plane angles of the dihedral angles, etc.

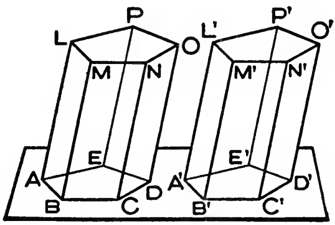
Ex. 4. Enunciate a theorem for the lateral area of a right prism.

Ex. 5. Find the lateral area of a right prism whose altitude is 8 feet and each side of whose triangular base is 5 feet. 120 sq. ft.

Ex. 6. Find the total area of a regular prism whose base is a regular hexagon, 10 inches on a side, if the altitude of the prism is 15 inches. 900+

Ex. 7. Find the lateral area of a prism whose edge is 12 inches and whose right section is a pentagon, the sides of which are 3, 5, 6, 9, and 11 inches. 4

584. Theorem. Two prisms are equal if three faces including a trihedral angle of one are equal, respectively, to three faces including a trihedral angle of the other, and similarly placed.



Given: Prisms AO and $A'O'$; face AM = face $A'M'$; face AP = face $A'P'$; face AD = face $A'D'$.

To Prove: Prism AO = prism $A'O'$.

Proof: The three face ∡ at A are respectively = to the three face ∡ at A' (?) (27).

$\therefore$ trih. $\angle A =$ trih. $\angle A'$ (?) (561).

Superpose prism AO upon prism $A'O'$, making the equal trihedral ∡ A and A' coincide.

Face AD will coincide with face $A'D'$, face AM with $A'M'$, face AP with $A'P'$. (They are = by hyp.)

That is, point L will fall on L'; M on M'; and P on P'.

$\therefore$ the plane LO will fall upon the plane $L'O'$ (?) (493).

Polygon $LO =$ polygon $L'O'$ (Ax. 1). $\therefore$ these bases coincide (?).

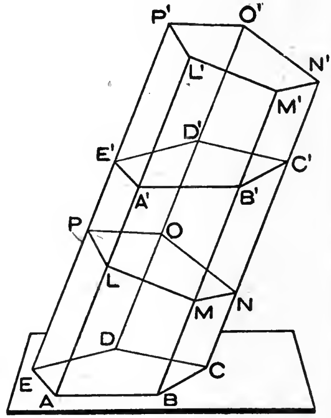
Similarly face BN coincides with $B'N'$, CO with $C'O'$. Etc.

Consequently, the prisms are equal (?) (571). Q.E.D.

585. Theorem. Two right prisms are equal if they have equal bases and equal altitudes. (Explain.)

586. Theorem. Two truncated prisms are equal if three faces including a trihedral angle of one are equal, respectively, to three faces including a trihedral angle of the other, and are similarly placed. (Explain.)

587. Theorem. An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to the lateral edge of the oblique prism.



Given: Oblique prism AC'; right prism PN' whose base is PN, a right section of AC', and whose altitude $PP' =$ edge EE'.

To Prove: Oblique prism AC' $\Leftrightarrow$ right prism PN'.

Proof: Edge $EE' = PP'$ (Hyp.). Subtract PE' from each, and $EP = E'P'$ (?) (Ax. 2).

Likewise $AL = A'L'$, $BM = B'M'$, $CN = C'N'$, etc.

(1) Face $AC =$ face $A'C'$ (?) (581).

(2) In faces AP and $A'P'$,
$EP = E'P'$, $AL = A'L'$ (Ax. 2), $AE = A'E'$, $PL = P'L'$ (?)(130).

That is, face AP and face $A'P'$ are mutually equilateral.

$$\left.\begin{array}{l}\text{Also } \angle EAL = \angle E'A'L',\ \angle PEA = \angle P'E'A', \\ \qquad \angle ALP = \angle A'L'P',\ \angle EPL = \angle E'P'L'\end{array}\right\} \text{(?) (98).}$$

That is, face AP and $A'P'$ are mutually equiangular.

$\therefore$ face $AP =$ face $A'P'$ (?) (159).

(3) Similarly, face $AM =$ face $A'M'$.

$\therefore$ truncated prism $AN =$ truncated prism $A'N'$ (?) (586).

Now, add, solid $PC' =$ solid PC' (Iden.).

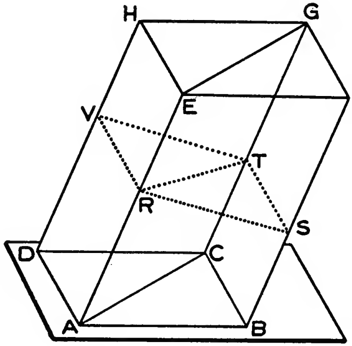
Oblique prism $AC' \Leftrightarrow$ right prism PN' (Ax. 2). Q.E.D.

Ex. Prove, in the figure of 587, that truncated prism AN is equal to truncated prism $A'N'$ by the method of superposition.

Proof: Polygon $PN =$ polygon $P'N'$ (?). Superpose solid AN upon solid $A'N'$ so that base PN will coincide with its equal $P'N'$. Etc.

588. Theorem. The plane containing two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equivalent triangular prisms.

Given: Parallelepiped BH and plane AG containing the opposite edges AE and CG.



To Prove: Prism $ABC-F$ $\Leftrightarrow$ prism $ADC-H$.

Proof: Pass a right section $RSTV$ intersecting the given plane in RT.

Face AF is $\parallel$ to DG (?).

$\therefore$ RS is $\parallel$ to VT (?) (500).

Likewise, RV is $\parallel$ to ST (?).

$\therefore$ $RSTV$ is a $\square$ (?).

$\therefore$ $\triangle RST = \triangle RVT$ (?) (132).

Prism $ABC-F$ $\Leftrightarrow$ a right prism whose base is RST and whose altitude $= EA$ (?) (587).

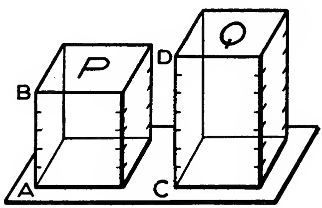
Prism $ADC-H$ $\Leftrightarrow$ a right prism whose base is RVT and whose altitude $= EA$ (?).

But these imaginary right prisms are equal (?) (585).

$\therefore$ prism $ABC-F$ $\Leftrightarrow$ prism $ADC-H$ (Ax. 1). Q.E.D.

589. Theorem. Two rectangular parallelepipeds having equal bases are to each other as their altitudes.

Given: Rectangular parallelepipeds P and Q, having $=$ bases, and their altitudes AB and CD, respectively.



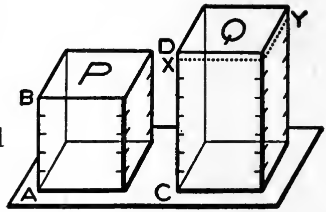
To Prove: $P : Q = AB : CD$.

Proof: I. If the altitudes are commensurable.

Consult 244, 302, 368, 539.

II. If the altitudes are **incommensurable.**

There does not exist a common unit (238).



Suppose AB divided into equal parts. Apply one of these as a unit of measure to CD. There will be a remainder, DX (?).

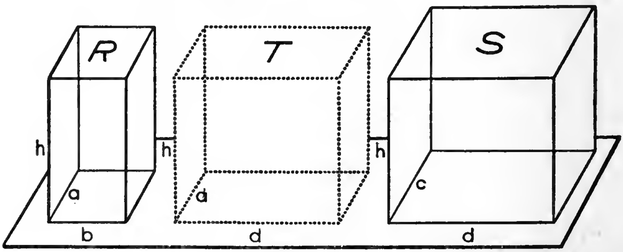
Pass plane XY, through X and ∥ to base. Now, $\frac{P}{CY} = \frac{AB}{CX}$ (?).

Indefinitely increase, etc., as in 244, 302, 368, 539. Mrs Alde

590. Theorem. Two rectangular parallelepipeds having two dimensions of the one equal respectively to two dimensions of the other, are to each other as their third dimension.

The faces having the sides of one equal to the sides of the other, respectively, may be considered the bases and the third dimensions the altitudes. Thus this statement is the same as 589.

591. Theorem Two rectangular parallelepipeds having equal altitudes are to each other as their bases.



Given: Rect. parallelepipeds R and S, having the same altitude h; and other dimensions a, b, and c, d, respectively.

To Prove: $$\frac{R}{S} = \frac{a \cdot b}{c \cdot d}.$$

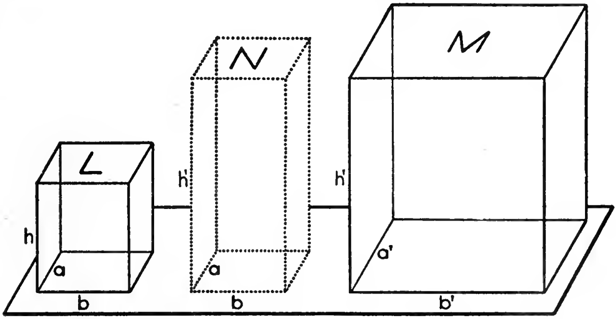
Proof: Construct a third rectangular parallelepiped, T, having altitude $= h$, another dimension $= a$, a third $= d$.

$\frac{R}{T} = \frac{b}{d}$ and $\frac{T}{S} = \frac{a}{c}$ (590). $\therefore \frac{R}{S} = \frac{a \cdot b}{c \cdot d}$ (Ax. 3). Q.E.D.

\\

592. Cor. Two rectangular parallelepipeds having one dimension in common, are to each other as the products of the other dimensions.

593. Theorem. Any two rectangular parallelepipeds are to each other as the products of their three dimensions.



Given: Rectangular parallelepipeds L and M, whose dimensions are a, b, h, and a', b', h', respectively.

To Prove: $\frac{L}{M}=\frac{a \cdot b \cdot h}{a' \cdot b' \cdot h'}$.

Proof: Construct N, whose dimensions are a, b, h'.

Then, $\frac{L}{N}=\frac{h}{h'}$ (?) (590), and $\frac{N}{M}=\frac{a \cdot b}{a' \cdot b'}$ (?) (592).

Multiplying, $\frac{L}{M}=\frac{a \cdot b \cdot h}{a' \cdot b' \cdot h'}$ (Ax. 3). Q.E.D.

594. Theorem. The volume of a rectangular parallelepiped is equal to the product of its three dimensions.

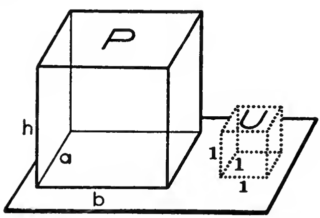
Given: (?) **To Prove:** (?).

Proof: Let U be a unit of vol.

$\frac{P}{U}=\frac{a \cdot b \cdot h}{1 \cdot 1 \cdot 1}=a \cdot b \cdot h$ (?)(593).

But $\frac{P}{U}=$ vol. of P (?) (571).

$\therefore$ vol. of $P = a \cdot b \cdot h$ (Ax. 1).

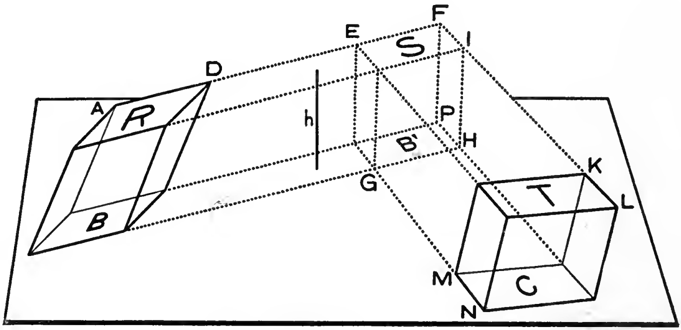


Q.E.D.

595. Theorem. **The volume of a rectangular parallelepiped is equal to the product of its base by its altitude.** (See 594.)

596. Cor. **The volume of a cube is equal to the cube of its edge.**

597. Theorem. **The volume of any parallelepiped is equal to the product of its base by its altitude.**



Given: Parallelepiped R, whose base $= B$ and alt. $= h$.

To Prove: Volume of $R = B \cdot h$.

Proof: Prolong the edge AD and all edges ∥ to AD. On the prolongation of AD, take $EF = AD$. Through E and F pass planes EG and FH, ⊥ to EF, forming the right parallelepiped S.

Again, prolong FI and all the edges ∥ to FI. On the prolongation of FI, take $KL = FI$. Through K and L pass planes KM and LN, ⊥ to KL, forming the rectangular parallelepiped T.

Consider EG the base of S, and EF its altitude, then $R \approx S$ (?) (587). Also $B \approx B'$ (?) (374).

Consider EP the base of S, and KM the base of T, and KL its altitude, then $S \approx T$ (?) (587). Also $B' = C$ (?)(140).

Hence, $R \approx T$ (Ax. 1); and $B \approx C$ (Ax. 1); and altitude of $T = h$ (?) (524).

But volume of $T = C \cdot h$ (?) (595).

∴ volume of $R = B \cdot h$ (Ax. 6). Q.E.D.

598. THEOREM. **Two parallelepipeds having equal altitudes and equivalent bases are equivalent (Ax. 1).**

599. THEOREM. **Two parallelepipeds having equal altitudes are to each other as their bases.**

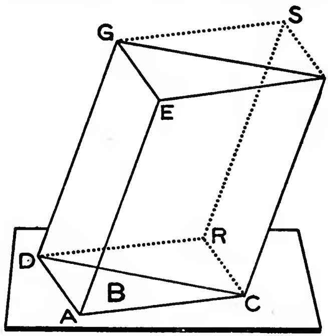
Proof: $Q = B \cdot h$ and $R = B' \cdot h$ (?) (597).

$\therefore$ by dividing, $\frac{Q}{R} = \frac{B}{B'}$ (Ax. 3). Q.E.D.

600. THEOREM. **Two parallelepipeds having equivalent bases are to each other as their altitudes (?).**

601. THEOREM. **Any two parallelepipeds are to each other as the products of their bases by their altitudes (?).**

602. THEOREM. **The volume of a triangular prism is equal to the product of its base by its altitude.**



Given: Triangular prism $ACD\text{–}F$; base $= B$; alt. $= h$.

To Prove: Volume of $ACD\text{–}F = B \cdot h$.

Proof: Construct parallelepiped AS having as three of its lateral edges AE, CF, DG.

Vol. $AS = ACRD \cdot h$ (?)(597).

Hence, $\frac{1}{2}$ volume of $AS = \frac{1}{2}$ $ACRD \cdot h$ (Ax. 3).

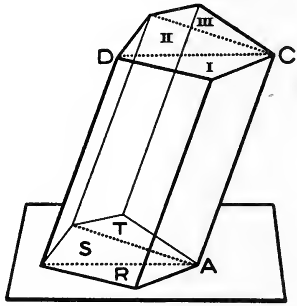
But $\frac{1}{2}$ volume of $AS =$ volume of prism $ACD\text{–}F$ (?) (588) and $\frac{1}{2}$ $ACRD = B$ (?) (132).

$\therefore$ volume of $ACD\text{–}F = B \cdot h$ (Ax. 6). Q.E.D.

Ex. 1. Which rectangular parallelepiped contains the greater volume, one whose edges are 5, 7, 9, or one whose edges are 4, 6, 13?

Ex. 2. The base of a prism is a right triangle whose legs are 8 and 12, and the altitude of the prism is 20. Find its volume.

603. THEOREM. **The volume of any prism is equal to the product of its base by its altitude.**



Given: Prism AD; base $= B$; altitude $= h$.

To Prove: Vol. of $AD = B \cdot h$.

Proof: Through any lateral edge, AC, and other lateral edges not adjoining AC, pass planes cutting the prism into triangular prisms I, II, III, having bases R, S, T, respectively.

$$\left.\begin{array}{l}\text{Vol. of prism I} = R \cdot h \\ \text{Vol. of prism II} = S \cdot h \\ \text{Vol. of prism III} = T \cdot h\end{array}\right\} (?)\ (602).$$

Adding,

$$\text{Vol. of prism } AD = (R + S + T)h = B \cdot h \text{ (Ax. 2).}$$

Q.E.D.

604. THEOREM. **Two prisms having equal altitudes and equivalent bases are equivalent.**

605. THEOREM. **Two prisms having equal altitudes are to each other as their bases.**

606. THEOREM. **Two prisms having equivalent bases are to each other as their altitudes.**

607. THEOREM. **Any two prisms are to each other as the products of their bases by their altitudes.**

ORIGINAL EXERCISES

1. How many faces has a parallelepiped? Edges? Vertices? How many faces has a hexagonal prism? Edges? Vertices?

2. Every lateral face of a prism is parallel to the lateral edges not in that face.

3. Every lateral edge of a prism is parallel to the faces that do not contain it.

4. Every plane containing one and only one lateral edge of a prism is parallel to all the other lateral edges.

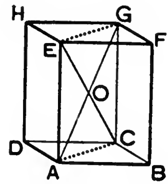
5. Any lateral face of a prism is less than the sum of the other lateral faces. [Use fig. of 583.]

6. The diagonals of a rectangular parallelepiped are equal.

Proof: Pass the plane $ACGE$. This is a rectangle (?), etc.

7. The four diagonals of a parallelepiped bisect each other.

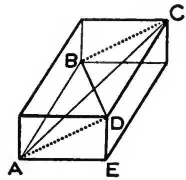
[First prove that one pair bisect each other, thus prove that any pair bisect each other, etc.]



8. Two triangular prisms are equal if their lateral faces are equal each to each.

9. Any prism is equivalent to the parallelepiped having the same altitude and an equivalent base.

10. The square of the diagonal of a rectangular parallelepiped is equal to the sum of the squares of its three dimensions.



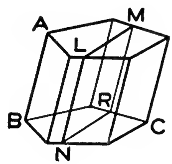
To Prove: $\overline{AC}^2 = \overline{AE}^2 + \overline{ED}^2 + \overline{DC}^2$.

Proof: AD is the hypotenuse of rt. $\triangle AED$, and AC, of rt. $\triangle ACD$.

11. The diagonal of a cube is equal to the edge multiplied by $\sqrt{3}$.

12. The volume of a triangular prism is equal to half the product of the area of any lateral face by the perpendicular drawn to that face from any point in the opposite edge. [Use the fig. of 602.]

13. Every section of a prism made by a plane parallel to a lateral edge is a parallelogram.



To Prove: $LMRN$ a $\square$. **Proof:** LM is $\parallel$ to NR (?). LN and MR are each $\parallel$ to any edge. (Explain.)

14. Every polyhedron has an even number of face angles.

Proof: Consider the faces as separate polygons. The number of sides of these polygons = double the number of edges of the polyhedron. (Explain.) But the number of sides of these polygons = the number of their angles, that is, the number of face angles. $\therefore$ the number of face angles = double the number of edges = an even number (?).

15. There is no polyhedron having fewer than 6 edges.

16. A room is 7 m. long, 5 m. wide, 3 m. high. Find its contents and its total area.

17. Find the volume, lateral area, and total area of an 8-in. cube.

18. A right prism whose height is 12 ft. has for its base a right triangle whose legs are 6 ft. and 8 ft. Find the volume, lateral area, and total area of the prism.

19. Find the altitude of a rectangular parallelepiped whose base is 21 in. × 30 in., equivalent to a rectangular parallelepiped whose dimensions are 27 in. × 28 in. × 35 in.

20. A cube and a rectangular parallelepiped whose edges are 6, 16, and 18, have the same volumes. Find the edge of the cube. 12 ✓

21. Find the volume of a rectangular parallelepiped whose total area is 620 and whose base is 14 × 9. $h = 8$, $V = 1008$ cu. units

22. How many bricks each $8 \times 2\frac{3}{4} \times 2$ in. will be required to build a wall 22 × 3 × 2 ft. (not allowing for mortar)? 5184

23. If a triangular prism is 20 in. high and each side of its base is 8 in., how many cubic inches does it contain? $320\sqrt{3}$

24. Find the lateral area, total area, and volume of a regular hexagonal prism each side of whose base is 10 and whose altitude is 15. LA = 900, TA =

25. A box is 12 × 9 × 8 in. What is the length of its diagonal? 17

26. Each edge of a cube is 8 in. Find its diagonal. $8\sqrt{3}$

27. The diagonal of a cube is $10\sqrt{3}$. Find its edge, volume, total area. 10 1000 600

28. A trench is 180 ft. long and 12 ft. deep, 7 ft. wide at the top and 4 ft. at the bottom. How many cubic yards of earth have been removed? 4

29. A metallic tank, open at the top, is made of iron 2 in. thick; the internal dimensions of the tank are, 4 ft. 8 in. long, 3 ft. 6 in. wide, 4 ft. 4 in. deep. Find the weight of the tank if empty; if full of water. [Water weighs $62\frac{1}{2}$ lb. to the cu. ft. and iron is 7.2 times as heavy as water.]

30. The base of a right parallelepiped is a rhombus whose sides are each 25, and the shorter diagonal is 14. The height of the parallelepiped is 40. Find its volume and total surface. $h = 24$ in. 

31. If the diagonal of a cube is 12 ft., find its surface. 48×6

32. If the total surface of a cube is 1 sq. yd., find its volume in cu. ft.

33. A right prism whose altitude is 25 has for its base a triangle whose sides are 11, 13, 20. Find its lateral area, total area, and volume.

$s = 22$, $A = \sqrt{22(11)(9)2} = 2 \cdot 11 \cdot 3 = 66$ area

PYRAMIDS

608. A **pyramid** is a polyhedron, one of whose faces is a polygon and whose other faces are all triangles having a common vertex.

The **lateral faces** of a pyramid are the triangles.

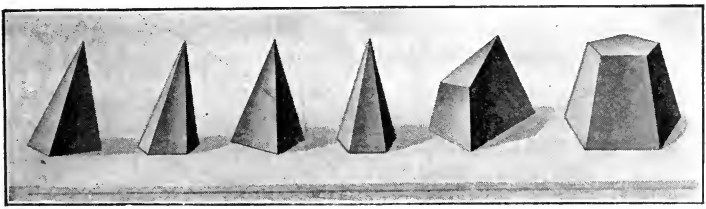
The **lateral edges** of a pyramid are the intersections of the lateral faces. The **vertex** of a pyramid is the common vertex of all the lateral faces.

The **base** of a pyramid is the face opposite the vertex.

The **lateral area** of a pyramid is the sum of the areas of the lateral faces. The **total area** of a pyramid is the sum of the lateral area and the area of the base.

The **altitude** of a pyramid is the perpendicular distance from the vertex to the plane of the base.

A **triangular pyramid** is a pyramid whose base is a triangle. It is called also a tetrahedron. (See 566.)



PYRAMIDS — REGULAR PYRAMIDS — TRUNCATED PYRAMID — FRUSTUM OF A PYRAMID

609. A **regular pyramid** is a pyramid whose base is a regular polygon and whose altitude, from the vertex, meets the base at its center.

The **slant height** of a regular pyramid is the line drawn in a lateral face, from the vertex perpendicular to the base of the triangular face. It is the altitude of any lateral face.

610. The **frustum** of a pyramid is the part of a pyramid included between the base and a plane parallel to the base.

The **altitude** of a frustum of a pyramid is the perpendicular distance between the planes of its bases.

The **slant height** of the frustum of a regular pyramid is the perpendicular distance, in a face, between the bases of that face.

A **truncated pyramid** is the part of a pyramid included between the base and a plane cutting all the lateral edges.

PRELIMINARY THEOREMS

611. THEOREM. **The lateral edges of a regular pyramid are all equal.** (See 520, II.)

612. COR. **The lateral faces of a regular pyramid are equal isosceles triangles.**

613. COR. **The lateral edges of the frustum of a regular pyramid are all equal.** (Ax. 2.)

614. THEOREM. **The lateral faces of the frustum of a regular pyramid are equal isosceles trapezoids.** (See 500.)

615. THEOREM. **The lateral faces of the frustum of any pyramid are trapezoids.** (?).

616. THEOREM. **The slant height of a regular pyramid is the same length in all the lateral faces.**

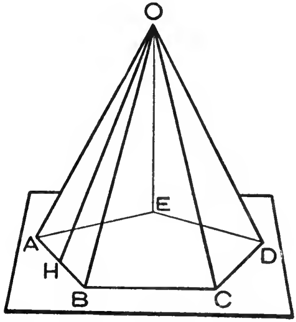
Ex. 1. Prove that the bases of any frustum of a pyramid are mutually equiangular.

Ex. 2. The foot of the altitude of a regular pyramid drawn from the vertex, coincides with the center of the circles inscribed in, and circumscribed about, the base.

Ex. 3. The sum of the medians of the lateral faces of the frustum of a trapezoid is equal to half the sum of the perimeters of the bases.

THEOREMS AND DEMONSTRATIONS

617. Theorem. The lateral area of a regular pyramid is equal to half the product of the perimeter of the base by the slant height.



Given: Regular pyramid O–$ABCDE$; lat. area $= L$; perimeter of base $= P$; slant height $OH = s$.

To Prove: $L = \frac{1}{2} P \cdot s$.

Proof:

$$\left.\begin{array}{l}\text{Area } \triangle AOB = \frac{1}{2} AB \cdot s \\ \text{Area } \triangle BOC = \frac{1}{2} BC \cdot s\end{array}\right\} \text{(?) (378).}$$

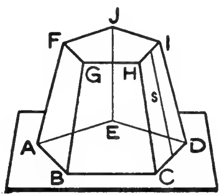
etc. etc. Adding,

Lateral area $= \frac{1}{2} AB \cdot s + \frac{1}{2} BC \cdot s +$ etc. (Ax. 2).

That is, $L = \frac{1}{2}(AB + BC + \text{etc.}) \cdot s$, or,

$L = \frac{1}{2} P \cdot s$ (Ax. 6). Q.E.D.

618. Theorem. The lateral area of the frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height.



Given: (?).

To Prove: $L = \frac{1}{2}(P + p) \cdot s$.

Proof: Area trapezoid $CI = \frac{1}{2}(CD + HI) \cdot s$ (?).

Area trapezoid $BH = \frac{1}{2}(BC + GH) \cdot s$ (?).

Area trapezoid $AG = \frac{1}{2}(AB + FG) \cdot s$ (?). Etc.

Adding, lateral area $AI = \frac{1}{2}(P + p) \cdot s$. (Explain.) Q.E.D.

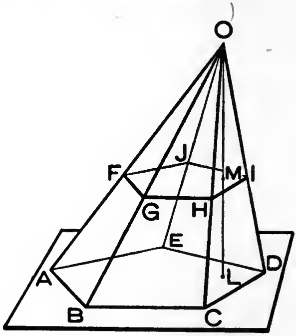
Ex. 1. The slant height of a regular pyramid whose base is a square, of which each side is 8 ft., is 15 ft. Find the lateral area; the total area.

Ex. 2. A regular pyramid stands on a hexagonal base 16 in. on a side, and the slant height is 2 ft. Find the lateral and total areas.

619. Theorem. If a pyramid is cut by a plane parallel to the base:

I. The lateral edges and altitude are divided proportionally.

II. The section is a polygon similar to the base.



Given: Pyr. $O\text{–}ABCDE$; plane $FI \parallel$ to the base; altitude $= OL$.

To Prove:

I. $\frac{OF}{OA} = \frac{OG}{OB} = \frac{OH}{OC} = \cdots = \frac{OM}{OL}.$

II. Section FI is similar to the base AD.

Proof: I. Imagine a plane through $O \parallel$ to plane AD.

This plane is $\perp$ to OL (512), and $\parallel$ to FI (?) (505).

$$\therefore \frac{OF}{OA} = \frac{OG}{OB} = \frac{OH}{OC} = \cdots = \frac{OM}{OL} \text{ (?) (516).}$$

II. FG is $\parallel$ to AB, GH is $\parallel$ to BC, etc. (?) (500).

$\therefore \angle FGH = \angle ABC$; $\angle GHI = \angle BCD$; etc. (?) (515).

That is, the polygons are mutually equiangular. Also, $\triangle OFG$ is similar to $\triangle OAB$; $\triangle OGH$ to $\triangle OBC$; etc. (?) (316).

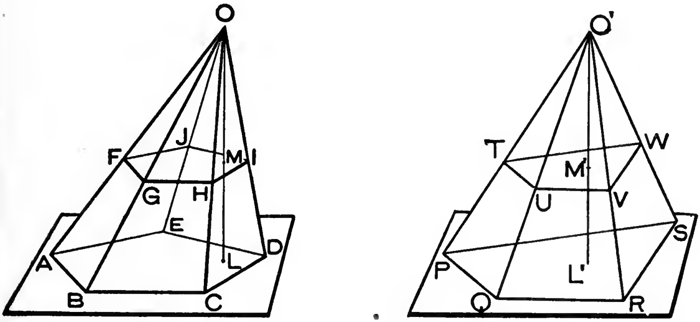
$$\therefore \frac{FG}{AB} = \left(\frac{OG}{OB}\right) = \frac{GH}{BC} = \left(\frac{OH}{OC}\right) = \frac{HI}{CD} = \text{etc. (?) (323, 3).}$$

$\therefore$ section FI is similar to base AD (?) (312). Q.E.D.

Ex. 1. The bases of the frustum of a regular pyramid are equilateral triangles whose sides are 12 in. and 20 in., respectively. The slant height is 40 in. Find the lateral area; the total area.

Ex. 2. The bases of frustum of a regular pyramid are regular hexagons whose sides are 8 and 18, respectively. The slant height is 25. Find the lateral area and total area.

620. Theorem. **If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases and at equal distances from the vertices are equivalent.**



Given: Pyramids $O\text{–}ABCDE$ and $O'\text{–}PQRS$; alt. $OL = O'L'$; base $AD \approx$ base PR; sections FI and $TV \parallel$ to bases; $OM = O'M'$.

To Prove: Section $FI \approx$ section TV.

Proof: FG is $\parallel$ to AB; TU is $\parallel$ to PQ, etc. (?) (500).

$\triangle$ OFG and OAB are similar, also $\triangle$ $O'TU$ and $O'PQ$ (?)(316).

$\therefore \dfrac{FG}{AB} = \dfrac{OF}{OA} = \dfrac{OM}{OL}$ (?) (323, 3 and 619, I).

And $\dfrac{TU}{PQ} = \dfrac{O'T}{O'P} = \dfrac{O'M'}{L'O'}$ (?). But $OM = O'M'$ and $OL = O'L'$ (Hyp.).

Hence, $\dfrac{FG}{AB} = \dfrac{TU}{PQ}$ (Ax. 1) and $\dfrac{\overline{FG}^2}{\overline{AB}^2} = \dfrac{\overline{TU}^2}{\overline{PQ}^2}$ (297).

Now section FI is similar to base AD, and section TV is similar to base PR (?) (619, II).

$\therefore \dfrac{\text{section } FI}{\text{base } AD} = \dfrac{\overline{FG}^2}{\overline{AB}^2}$ and $\dfrac{\text{section } TV}{\text{base } PR} = \dfrac{\overline{TU}^2}{\overline{PQ}^2}$ (?) (390).

Hence, $\dfrac{\text{section } FI}{\text{base } AD} = \dfrac{\text{section } TV}{\text{base } PR}$ (Ax. 1).

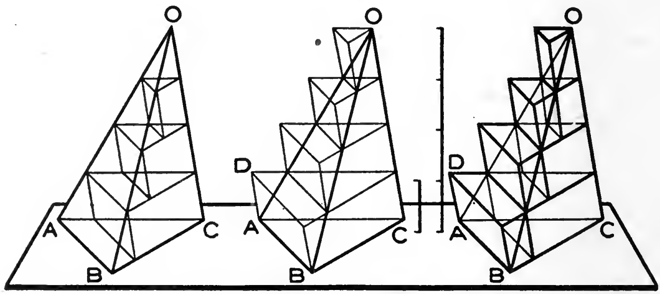
But base $AD \approx$ base PR (Hyp.).

$\therefore$ section $FI \approx$ section TV (?) (Ax. 3) Q.E.D.

621. If a plane be passed parallel to the base of a pyramid, intersecting all the lateral edges, and upon the section thus formed, as a base, a prism be constructed wholly inside the pyramid, but having one lateral edge in a lateral edge of the pyramid, this prism is called an **inscribed prism.** (See 622, fig. 1.)

If upon this section as a base a prism be constructed partly outside the pyramid, having one lateral edge in one of the lateral edges of the pyramid, this prism is called a **circumscribed prism.** (See 622, fig. 2.)

622. THEOREM. **The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms, having equal altitudes, if the number of prisms is indefinitely increased.**



Given: Triangular pyramid $O\text{–}ABC$, having a series of prisms inscribed in it, and another series circumscribed about it, all the prisms having equal altitudes.

To Prove: $O\text{–}ABC$ is the limit of the sum of each series of prisms as their number is indefinitely increased.

Proof: Denote the volume of the pyramid by V, the sum of the volumes of the series of inscribed prisms by S_i, and the sum of the volumes of the series of circumscribed prisms by S_c.

The uppermost circumscribed prism $\approx$ the uppermost inscribed prism (604).

The second pair of prisms also are equivalent (?).

And so on, until the last circumscribed prism, $D-ABC$, remains, for which there is no equivalent inscribed prism.

Hence, it is evident that $S_c - S_i = D-ABC$, the lowest prism.

Now by indefinitely increasing the number of the prisms, the altitude of $D-ABC$ will become indefinitely small, and hence the volume of $D-ABC$ will approach zero as a limit.

The altitude can never actually equal zero, nor can the volume equal zero. Hence, $S_c - S_i$ can be made less than any mentionable quantity, but cannot equal zero.

Now $S_c = S_c$

and $\underline{V > S_i}$ (Ax. 5).

$\therefore S_c - V < S_c - S_i$ (Ax. 9).

$V < S_c$ (Ax. 5)

and $\underline{S_i = S_i}$

$\therefore V - S_i < S_c - S_i$ (Ax. 7).

That is, $S_c - V$ and $V - S_i$ are each less than $S_c - S_i$, which itself approaches zero.

Hence, $S_c - V$ approaches zero and $V - S_i$ approaches zero.

$\therefore$ S_c approaches V as a limit, and S_i approaches V as a limit (?) (240). (See note on p. 223.) Q.E.D.

Ex. 1. If a plane is passed parallel to the base of a pyramid, cutting the lateral edges, the section is to the base as the square of the altitude of the pyramid cut away by this plane is to the square of the altitude of the original pyramid. (See proof of 620.)

Ex. 2. If two pyramids have equal altitudes and are cut by planes parallel to the bases and at equal distances from the vertices, the sections formed will be to each other as the bases of the pyramids.

Ex. 3. In the figure of 622 prove the planes of the faces of the prisms, that are opposite OC, are parallel to OC and to AB.

Ex. 4. State the theorems leading up to the theorem of 588.

Ex. 5. State the theorems leading up to the theorem of 597.

Ex. 6. State the theorems leading up to the theorem of 603.

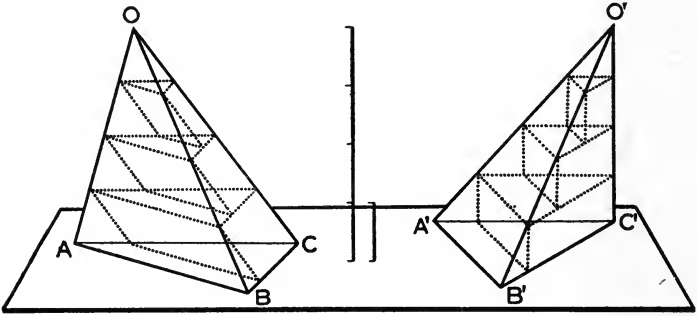
Ex. 7. The base of a pyramid is 180 sq. in. and its altitude is 15 in. What is the area of the section made by a plane parallel to the base, and 5 in. from the vertex? $15^2 : 5^2 = 180 : x$, $x = 20$

Ex. 8. The base of a pyramid is 200 sq. in., and its altitude is 12 in. At what distance from the vertex must a plane be passed so that the section shall contain half the area of the base?

$12^2 : x^2 = 200 : 100$

$x = 6\sqrt{2}$

623. Theorem. **Two triangular pyramids having equal altitudes and equivalent bases are equivalent.**



Given: Triangular pyramids $O\text{–}ABC$ and $O'\text{–}A'B'C'$ having equal altitudes and base $ABC \approx$ base $A'B'C'$.

To Prove: $O\text{–}ABC \approx O'\text{–}A'B'C'$.

Proof: Divide the altitude of each pyramid into any number of equal parts. Through these points of division pass planes ∥ to the bases, forming triangular sections.

Upon these sections as bases construct inscribed prisms.

Denote the volumes of the pyramids by V and V', and the sums of the volumes of these series of prisms by S and S'.

The corresponding sections are $\approx$ (?) (620).

∴ the corresponding prisms are $\approx$ (?) (604).

Hence, $S = S'$ (Ax. 2).

By indefinitely increasing the number of equal parts into which the altitudes are divided, the number of prisms becomes indefinitely great.

∴ S approaches V as a limit, (?) (622),

and S' approaches V' as a limit (?).

∴ $V \approx V'$ (?) (242). That is, $O\text{–}ABC \approx O'\text{–}A'B'C'$.

Q.E.D.

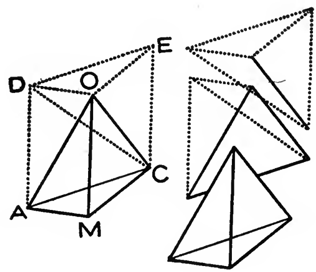
Note. As in plane geometry, $\triangle\, ABC$ is the same as $\triangle\, BAC$, so in solid geometry the pyramid $O\text{–}ABC$ is the same as the pyramid $A\text{–}BCO$ or $B\text{–}ACO$ or $C\text{–}ABO$.

624. **THEOREM. The volume of a triangular pyramid is equal to one third the product of its base by its altitude.**

Given: Triangular pyramid O–AMC, whose base $= B$ and altitude $= h$.

To Prove:

Volume $O\text{–}AMC = \frac{1}{3}\ B \cdot h$.



Proof: Construct a prism AMC–DOE, having AMC as its base, and OM as one of its lateral edges.

Pass a plane through DO and OC, cutting the face AE in line CD.

The prism is now divided into three triangular pyramids.

In pyramids O–AMC and C–ODE, the altitudes are $=$(?)(524).

The bases AMC and ODE are equal (?) (568).

$\therefore$ pyramid $O\text{–}AMC \Bumpeq$ pyramid C–ODE (?) (623).

In the pyramids C–AMO and C–AOD, the altitudes are the same line from $C \perp$ to plane DM (?) (507).

The bases AMO and AOD are $=$ (?) (132).

$\therefore$ pyramid $C\text{–}AMO \Bumpeq$ pyramid C–AOD (?) (623).

Hence, $O\text{–}AMC \Bumpeq C\text{–}ODE \Bumpeq C\text{–}AOD$ (Ax. 1).

That is, $O\text{–}AMC = \frac{1}{3}$ the prism.

But the volume of the prism $= B \cdot h$ (?).

$\therefore$ volume of pyramid $O\text{–}AMC = \frac{1}{3}\ B \cdot h$ (Ax. 6). Q.E.D.

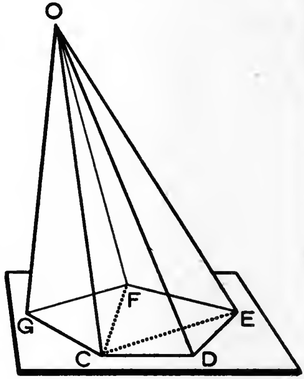
Ex. 1. In the figure of 624, prove pyramid $O\text{–}ACD \Bumpeq O\text{–}CDE$.

Ex. 2. The area of the base of a triangular pyramid is 30 sq. in., and its altitude is 20 in. Find the volume. Find the volume of the prism having the same base and altitude. 200 cu. in.; 600 cu. in.

Ex. 3. A pyramid whose base is b and altitude is h is equivalent to another pyramid whose base is d and altitude is x. Find x.

Ex. 4. If in the figure of 602, a plane is passed through E and CD, what part of the whole parallelepiped is the pyramid E–ACD?

625. THEOREM. The volume of any pyramid is equal to one third the product of its base by its altitude.



Given: Pyramid $O\text{–}CDEFG$, whose base $= B$ and altitude $= h$.

To Prove:
Volume of $O\text{–}CDEFG = \frac{1}{3} B \cdot h$.

Proof: Through any lateral edge, OC, and lateral edges not adjoining C, pass planes dividing the pyramid into triangular pyramids.

$$\left.\begin{aligned} \text{Vol. of } O\text{–}CDE &= \tfrac{1}{3} CDE \cdot h \\ \text{Vol. of } O\text{–}CEF &= \tfrac{1}{3} CEF \cdot h \\ \text{Vol. of } O\text{–}CFG &= \tfrac{1}{3} CFG \cdot h \end{aligned}\right\} \begin{array}{l} (?)(624). \\ \text{Adding,} \end{array}$$

Vol. of $O\text{–}CDEFG = \frac{1}{3} B \cdot h$ (Ax. 2 and Ax. 4). Q.E.D.

626. THEOREM. Any two pyramids having equal altitudes and equivalent bases are equivalent. (Ax. 1.)

627. THEOREM. Two pyramids having equal altitudes are to each other as their bases. (Prove.)

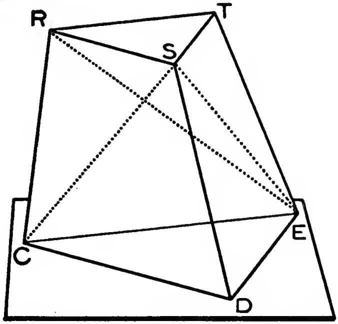
628. THEOREM. Two pyramids having equivalent bases are to each other as their altitudes. (Prove.)

629. THEOREM. Any two pyramids are to each other as the products of their bases by their altitudes. (Prove.)

630. THEOREM. The volume of the frustum of a triangular pyramid is equal to one third the altitude multiplied by the sum of the lower base, the upper base, and a mean proportional between the bases of the frustum.

Given: The frustum RD of a triangular pyramid whose lower base $= B$; upper base $= b$; altitude $= h$.

To Prove: Volume of $RD = \frac{1}{3} h \left[B + b + \sqrt{B \cdot b}\right]$.



Proof: Pass a plane through edge CE and vertex S, and another through edge RS and vertex E, dividing the frustum into three triangular pyramids, $S\text{–}CDE$, $E\text{–}RST$, $E\text{–}CRS$.

I. $S\text{–}CDE = \frac{1}{3}h \cdot B$ (?) (624).

II. $E\text{–}RST = \frac{1}{3}h \cdot b$ (?).

III. We shall now prove $E\text{–}CRS = \frac{1}{3}h \cdot \sqrt{B \cdot b}$.

$\frac{E\text{–}CSD}{E\text{–}CRS} = \frac{\triangle\ CSD}{\triangle\ CRS}$ (?) (627). $\frac{\triangle\ CSD}{\triangle\ CRS} = \frac{CD}{RS}$ (?) (382).

$$\therefore \frac{E\text{–}CSD}{E\text{–}CRS} = \frac{CD}{RS} \text{ (Ax. 1).}$$

Likewise, $\frac{S\text{–}CER}{S\text{–}ERT} = \frac{\triangle\ CER}{\triangle\ ERT}$ (?) and $\frac{\triangle\ CER}{\triangle\ ERT} = \frac{CE}{RT}$ (?).

$$\therefore \frac{S\text{–}CER}{S\text{–}ERT} = \frac{CE}{RT} \text{ (?).}$$

But $\triangle$ CDE and RST are similar (?) (619, II).

$$\therefore \frac{CD}{RS} = \frac{CE}{RT} \text{ (?).}$$

Hence, $\frac{E\text{–}CSD}{E\text{–}CRS} = \frac{S\text{–}CER \text{ or } E\text{–}CRS}{S\text{–}ERT}$ (Ax. 1).

That is, $\frac{\frac{1}{3}h \cdot B}{E\text{–}CRS} = \frac{E\text{–}CRS}{\frac{1}{3}h \cdot b}$ (Substituting from I and II).

$\therefore E\text{–}CRS = \sqrt{\frac{1}{3}h \cdot B \cdot \frac{1}{3}h \cdot b} = \frac{1}{3}h \cdot \sqrt{B \cdot b}$ (?) (299).

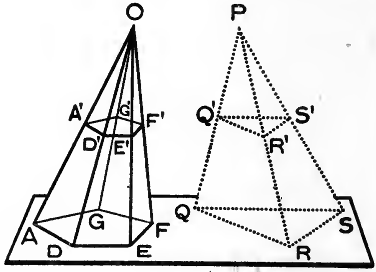
$\therefore$ volume of the frustum $= \frac{1}{3}h\,[B + b + \sqrt{B \cdot b}]$ (Ax. 2).

Q.E.D.

Note. Theorem 630 is sometimes stated thus:

The frustum of a triangular pyramid is equivalent to the sum of three pyramids whose altitudes are the same as the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.

631. Theorem. The volume of the frustum of any pyramid is equal to one third the altitude multiplied by the sum of the lower base, the upper base, and a mean proportional between the bases of the frustum.



Given: Pyr. $O\text{–}ADEFG$; frustum $A'F$, whose lower base $= B$, upper base $= b$, altitude $= h$.

To Prove: Vol. of frustum $= \frac{1}{3}h[B + b + \sqrt{B \cdot b}]$.

Proof: Construct a $\triangle\ QRS \approx$ polygon AF (by 409).

Upon $\triangle\ QRS$ as a base, construct a pyramid whose altitude = the altitude of $O\text{–}ADEFG$. Pass a plane $Q'R'S' \parallel$ to QRS and at a distance from $QRS = h$.

Vol. of $Q'R = \frac{1}{3}h[\triangle\ QRS + \triangle\ Q'R'S' + \sqrt{\triangle\ QRS \cdot \triangle\ Q'R'S'}]$ (630).

The alt. of $P\text{–}Q'R'S' =$ alt. of $O\text{–}A'D'E'F'G'$ (?) (Ax. 2).

Also, $QRS \approx B$ (Const.); and $Q'R'S' \approx b$ (?) (620).

$\therefore$ vol. of $O\text{–}ADEFG =$ vol. of $P\text{–}QRS$ (?) (626), and

vol. of $O\text{–}A'D'E'F'G' =$ vol. of $P\text{–}Q'R'S'$ (?). Subtracting,

vol. of frustum $A'F =$ vol. of frustum $Q'R$ (Ax. 2).

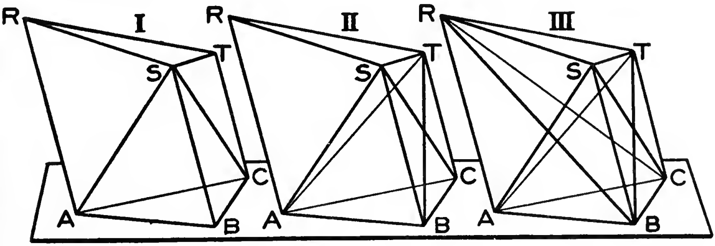
Vol. of frustum $A'F = \frac{1}{3}h[B + b + \sqrt{B \cdot b}]$ (Ax. 6). Q.E.D.

Ex. 1. Find the volume of a pyramid whose altitude is 18 in. and whose base is 10 in. square. 600 cu. in

Ex. 2. Find the volume of the frustum of a pyramid whose altitude is 20 and the areas of whose bases are 18 and 32. 24 2/3 cubic units
493 1/3 = 20 x

Ex. 3. The bases of the frustum of a pyramid are regular hexagons whose sides are 10 in. and 6 in., respectively. The altitude of the frustum is 2 ft. Find its volume. ~~146√3 = 333.+~~

632. Theorem. **A truncated triangular prism is equivalent to three triangular pyramids whose bases are the base of the prism and whose vertices are the three vertices of the face opposite the base (the inclined section).**



Given: The truncated triangular prism $ABC\text{–}RST$, whose base is ABC and whose opposite vertices are R, S, T. Let it be divided by the planes ACS, ABT, BCR.

To Prove: $ABC\text{–}RST \Leftrightarrow R\text{–}ABC + S\text{–}ABC + T\text{–}ABC$ (III).

Proof: In Fig. I, $S\text{–}ABC$ is obviously one of these pyramids.

In Fig. II, $A\text{–}CST \Leftrightarrow A\text{–}BCT$ (?) (626).

That is, $A\text{–}CST \Leftrightarrow T\text{–}ABC$.

In Fig. III, $T\text{–}ARS \Leftrightarrow T\text{–}ABR$ (?) (626).

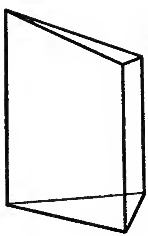
$T\text{–}ABR \Leftrightarrow C\text{–}ABR$ (?) (626).

$\therefore$ $T\text{–}ARS \Leftrightarrow R\text{–}ABC$ (Ax. 1).

Now, $ABC\text{–}RST \Leftrightarrow T\text{–}ARS + S\text{–}ABC + A\text{–}CST$ (Ax. 4).

Hence, $ABC\text{–}RST \Leftrightarrow R\text{–}ABC + S\text{–}ABC + T\text{–}ABC$ (Ax. 6).

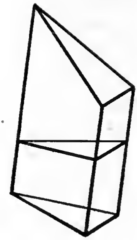
Q.E.D.



633. Cor. **The volume of a truncated triangular prism is equal to the product of the base by one third the sum of the three altitudes drawn to the base from the three vertices opposite the base.**

634. Cor. **The volume of a truncated right triangular prism is equal to the product of the base by one third the sum of its lateral edges.**

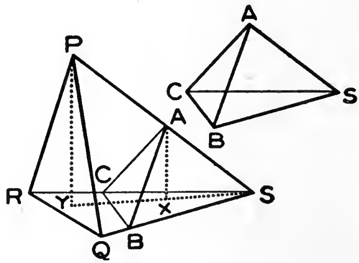
635. Theorem. The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.



Proof: The right section divides the solid into two truncated right prisms.

Hence, volume = right section × $\frac{1}{3}$ sum of lateral edges (634). Q.E.D.

636. Theorem. Two triangular pyramids (tetrahedrons) having a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges including the equal trihedral angles.



Given: Triangular pyramids $S\text{–}ABC$, $S\text{–}PQR$; having the trih. ∠s at S equal; and their volumes V and V'.

To Prove: $\dfrac{V}{V'} = \dfrac{SA \cdot SB \cdot SC}{SP \cdot SQ \cdot SR}$.

Proof: Place the pyramids so that the equal trihedral ∠s coincide. Draw the altitudes AX and PY and the projection SXY in plane SQR.

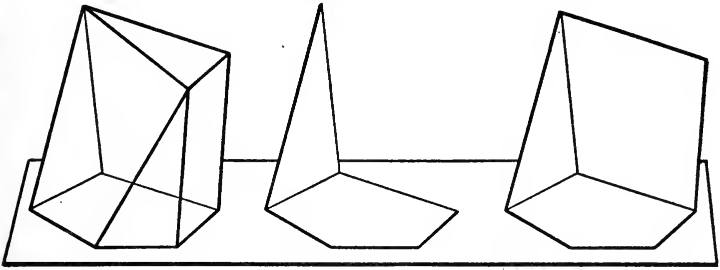
$\triangle SAX$ is similar to $\triangle SPY$ (?) (315).

Now, $\dfrac{V}{V'} = \dfrac{\triangle SBC \cdot AX}{\triangle SQR \cdot PY} = \dfrac{\triangle SBC}{\triangle SQR} \cdot \dfrac{AX}{PY}$ (?) (629).

But, $\dfrac{\triangle SBC}{\triangle SQR} = \dfrac{SB \cdot SC}{SQ \cdot SR}$ (?) (388); and $\dfrac{AX}{PY} = \dfrac{SA}{SP}$ (?) (323, 3).

Hence, $\dfrac{V}{V'} = \dfrac{SB \cdot SC}{SQ \cdot SR} \cdot \dfrac{SA}{SP} = \dfrac{SA \cdot SB \cdot SC}{SP \cdot SQ \cdot SR}$ (Ax. 6). Q.E.D.

637. Theorem. **In any convex polyhedion the number of edges increased by two is equal to the number of vertices increased by the number of faces.**



Given: A polyhedron; $E =$ number of edges; $F =$ number of faces; $V =$ number of vertices.

To Prove: $$E + 2 = V + F.$$

Proof: Suppose the surface of the polyhedron is put together, face by face.

For one face, $E = V$ (154). (Begin with the base.)

By attaching an adjoining face, the number of edges is one greater than the number of vertices.

That is, for two faces, $E = V + 1$.

Similarly, for three faces, $E = V + 2$,

and, for four faces, $E = V + 3$,

and, for five faces, $E = V + 4$,

. . . .

and, for n faces, $E = V + (n - 1)$,

and, for $F - 1$ faces, $E = V + (F - 2)$.

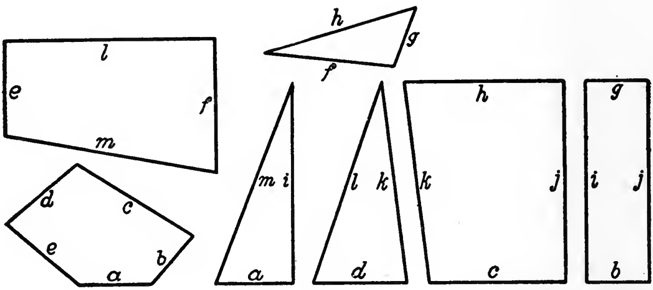
By attaching the last face, neither the number of edges nor the number of vertices is increased.

That is, for F faces, $E = V + F - 2$.

$\therefore$ for the complete solid, $E + 2 = V + F$ (Ax. 2). Q.E.D.

638. Theorem. **In any polyhedron the difference between the number of edges and the number of faces, is two less than the number of vertices, that is, $E - F = V - 2$.** (See 637.)

639. Theorem. **The sum of all the face angles of any polyhedron is equal to 4 right angles multiplied by two less than the number of vertices, that is, $S_{\measuredangle s} = (V - 2)\ 4$ rt. $\measuredangle s = (V - 2)\ 360°$.**



Given: A polyhedron; E = number of edges; F = number of faces; V = number of vertices.

To Prove: Sum of all the face $\measuredangle s = (V - 2)\ 4$ rt. $\measuredangle s$,
or $S_{\measuredangle s} = (V - 2)\ 360°$.

Proof: Considering the faces as separate polygons, it is obvious that each edge is a side of two polygons, that is, the number of sides of the several faces $= 2E$.

$\therefore$ the number of vertices of all the polygons $= 2E$ (154).
Suppose an exterior $\angle$ formed at each of these $2E$ vertices.
Then the sum of the exterior $\measuredangle s$ of each face $= 4$ rt. $\measuredangle s$ (?).
Sum of int. and ext. $\measuredangle s$ at each vertex $= 2$ rt. $\measuredangle s$ (?).
$\therefore$ the int. $\measuredangle s$ + ext. $\measuredangle s$ at all the $2E$ vertices $= 4E$ rt. $\measuredangle s$.
But, $\underline{\text{the ext. } \measuredangle s \text{ of all the } F \text{ faces} = 4F \text{ rt. } \measuredangle s}$ (?).
Hence, the int. $\measuredangle s$ of these polygons
$= 4E$ rt. $\measuredangle s - 4F$ rt. $\measuredangle s$ (?).
$= (E - F) \cdot 4$ rt. $\measuredangle s$.
But $E - F = V - 2$ (?) (638).
$\therefore S_{\measuredangle s} = (V - 2)\ 4$ rt. $\measuredangle s = (V - 2)\ 360°$ (Ax. 6). **Q.E.D.**

REGULAR AND SIMILAR POLYHEDRONS

640. A **regular polyhedron** is a polyhedron whose faces are equal regular polygons and whose polyhedral angles are all equal.

Similar polyhedrons are polyhedrons which have the same number of faces similar each to each and similarly placed, and which have their homologous polyhedral angles equal.

641. Theorem. There cannot exist more than five kinds of regular polyhedrons.

Proof: The faces must be equilateral $\triangle$, squares, regular pentagons, or some other regular polygons (?) (640).

There must be at least three faces at each vertex (?) (580).

Sum of the face $\angle$s at each vertex must be $< 360°$ (?) (564).

Let us now discuss the possible regular polyhedrons.

I. Each $\angle$ of an equilateral $\triangle = 60°$ (?). Hence, we may form a polyhedral $\angle$ by placing 3 equilateral $\triangle$ at a vertex, or 4 at a vertex, or 5 at a vertex. But not 6 at a vertex (?). That is, only three regular polyhedrons can be formed having equilateral triangles for faces.

II. Each $\angle$ of a square $= 90°$ (?). Hence, we may form a polyhedral $\angle$ by placing 3 squares at a vertex. But not 4 at a vertex (?). That is, only one regular polyhedron can be formed having squares for faces.

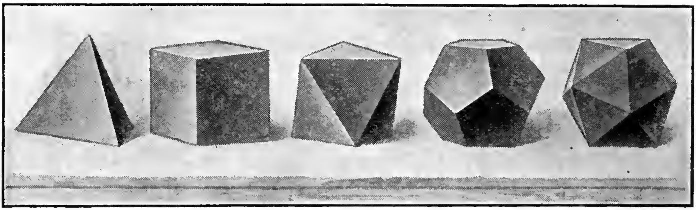
III. Each $\angle$ of a regular pentagon $= 108°$. (?) (164). Hence, we may form a polyhedral $\angle$ by placing 3 regular pentagons at a vertex. But not 4 at a vertex (?). That is, only one regular polyhedron can be formed having regular pentagons for faces.

IV. Each $\angle$ of a regular hexagon $= 120°$ (?). Hence, no polyhedral $\angle$ can be formed by hexagons (?).

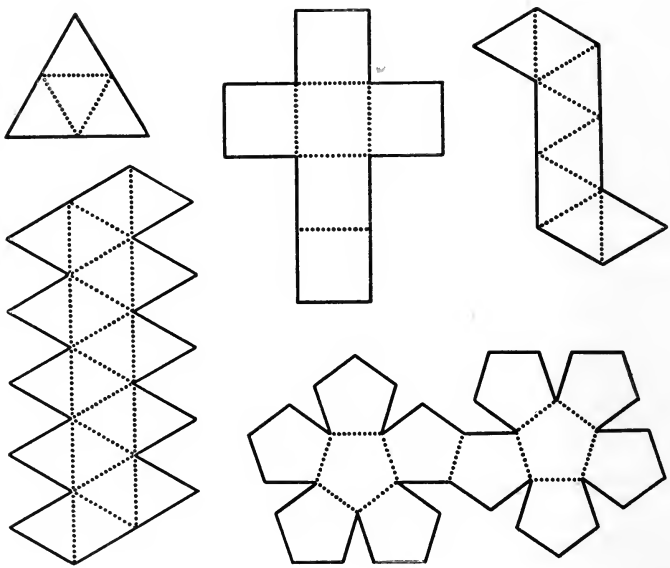
Consequently, there can be no more than five kinds of regular polyhedrons,—three kinds bounded by triangles, one kind by squares, and one by pentagons. Q.E.D.

642. The **names** of the regular polyhedrons.

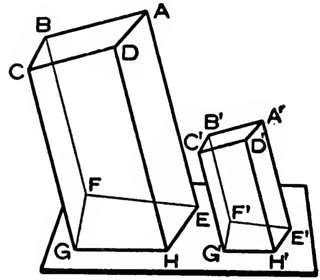
Names	Total Number of Faces	Number of Faces at Each Vertex	Kinds of Faces
Regular tetrahedron	4	3	Equilateral triangles
Regular hexahedron (cube)	6	3	Squares
Regular octahedron	8	4	Equilateral triangles
Regular dodecahedron	12	3	Regular pentagons
Regular icosahedron	20	5	Equilateral triangles



REGULAR TETRAHEDRON CUBE REGULAR OCTAHEDRON REGULAR DODECAHEDRON REGULAR ICOSAHEDRON



DIRECTIONS.—The regular polyhedrons may be constructed as follows: Mark on cardboard figures similar to the drawings on page 324, but much larger; with a knife cut the dotted lines half through and the solid lines entirely through; fold along the dotted lines, closing the solids up and forming the figures as illustrated on page 324; paste strips of paper along the edges to hold them in position.



643. THEOREM. In two similar polyhedrons:

I. Homologous edges are proportional.

II. Homologous faces are to each other as the squares of any two homologous edges.

III. Total surfaces are to each other as the squares of any two homologous edges.

Proof: I. Homologous faces are similar (?) (640).

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\frac{DH}{D'H'}=\frac{AE}{A'E'}=\frac{BF}{B'F'}=\text{etc. (?) (323, 3).}$$

$$\text{II.}\quad \frac{\text{Face } DG}{\text{Face } D'G'}=\frac{\overline{DH}^2}{\overline{D'H'}^2}=\frac{\text{face } AH}{\text{face } A'H'}=\frac{\overline{AE}^2}{\overline{A'E'}^2}=\text{etc. (?) (390).}$$

$$\text{III.}\quad \frac{\text{Face } DG}{\text{Face } D'G'}=\frac{\text{face } AH}{\text{face } A'H'}=\frac{\text{face } GE}{\text{face } G'E'}=\text{etc.}\left\{\begin{array}{l}\text{each being}\\ \text{equal to } \dfrac{\overline{DH}^2}{\overline{D'H'}^2}.\end{array}\right.$$

$$\frac{\text{Total surface of } AG}{\text{Total surface of } A'G'}=\frac{\text{face } DG}{\text{face } D'G'}=\frac{\overline{DH}^2}{\overline{D'H'}^2}=\frac{\overline{AE}^2}{\overline{A'E'}^2}=\text{etc. (301).}$$

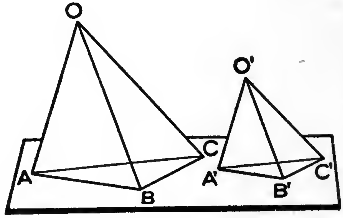
644. THEOREM. If a pyramid is cut by a plane parallel to the base, the pyramid cut away is similar to the original pyramid. (Def. of 640.)

Ex. 1. Show that the theorem of 637 holds true in the case of a cube.

Ex. 2. Show that the theorem of 639 holds true in the case of a cube.

Ex. 3. Show that the theorems of 637 and 639 are true in the case of a regular octahedron.

645. Theorem. Two similar tetrahedrons are to each other as the cubes of any two homologous edges.



Given: Similar tetrahedrons $O\text{–}ABC$ and $O'\text{–}A'B'C'$; whose volumes $= V$ and V'.

To Prove:

$V : V' = \overline{AB}^3 = \overline{A'B'}^3 = \text{etc.}$

Proof: Trihedral $\angle A =$ trihedral $\angle A'$ (?) (640).

$$\therefore \frac{V}{V'} = \frac{AB \cdot AC \cdot AO}{A'B' \cdot A'C' \cdot A'O'} \quad (?)\ (636).$$

$$= \frac{AB}{A'B'} \cdot \frac{AC}{A'C'} \cdot \frac{AO}{A'O'}$$

$$\text{But} \quad \frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{AO}{A'O'} \quad (?)\ (643, \text{I}).$$

$$\therefore \frac{V}{V'} = \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} \quad (\text{Ax. } 6).$$

That is, $V : V' = \overline{AB}^3 : \overline{A'B'}^3 = \overline{AC}^3 : \overline{A'C'}^3 = \text{etc.}$ Q.E.D.

Ex. 1. If, in the figure of 645, $AO = 4$ and $A'O' = 1$, what is the ratio of the total surfaces of the tetrahedrons? Of their volumes?

Ex. 2. Two homologous edges of two similar polyhedrons are 2 and 3. What is the ratio of their total surfaces?

Ex. 3. Two homologous edges of two similar tetrahedrons are 2 and 5. The area of the total surface of the less is 28, and its volume is 40. Find the area of the total surface and the volume of the other. a=175, V=625

Ex. 4. Show that the theorems of 637 and 639 are true in the cases of regular dodecahedrons and regular icosahedrons.

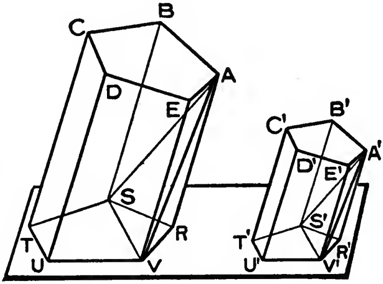
Ex. 5. A pyramid whose volume is V and altitude is h is bisected by a plane parallel to the base. Find the distance of this plane from the vertex.

Ex 6. Find the total surface of a regular tetrahedron whose edges are each equal to 4 in. $4 \times 4\sqrt{3} = 16\sqrt{3} = 27.2$

Ex. 7. Find the total surface of a regular octahedron whose edges are each equal to 18 in. $81\sqrt{3} \times 8 = 648\sqrt{3} = 1102+$

omit

646. THEOREM. Two similar polyhedrons can be decomposed into the same number of tetrahedrons similar each to each and similarly placed.



Given: (?).

To Prove: (?).

Proof: Suppose diagonals drawn in every face of AT, except the faces containing vertex A, dividing the faces into △. (The figure shows only SV.) Suppose lines drawn from A to the several vertices of these △. (The figure shows only AS, AV.)

Obviously, this process divides the solid (by planes) into tetrahedrons, each of which has a vertex at A.

Then construct homologous lines in solid $A'T'$. There will evidently be as many lines in $A'T'$ as in AT and as many tetrahedrons, and these will be similarly placed.

Now, in the tetrahedrons $A\text{–}SVR$ and $A'\text{–}S'V'R'$, $\triangle\ AVR$ is similar to $\triangle\ A'V'R'$; $\triangle\ ARS$ is similar to $\triangle\ A'R'S'$; $\triangle\ SVR$ is similar to $\triangle\ S'V'R'$ (?) (327).

$$\text{Also, } \frac{AV}{A'V'} = \frac{VR}{V'R'} = \frac{VS}{V'S'} = \frac{RS}{R'S'} = \frac{AS}{A'S'} \text{ (?) (323, 3).}$$

$$\therefore \frac{AV}{A'V'} = \frac{VS}{V'S'} = \frac{AS}{A'S'} \text{ (Ax. 1).}$$

Hence, $\triangle\ ASV$ is similar to $\triangle\ A'S'V'$ (?) (318).

Also, the trihedral ∠s R and R' are $=$; S and S' are $=$; V and V' are $=$, etc. (?) (561).

∴ the two tetrahedrons are similar.

Furthermore, after removing these tetrahedrons, the remaining polyhedrons are similar (Def. 640).

By the same process other tetrahedrons may be removed and proved similar, and the process continued until the polyhedrons are completely decomposed into tetrahedrons similar each to each and similarly placed. Q.E.D.

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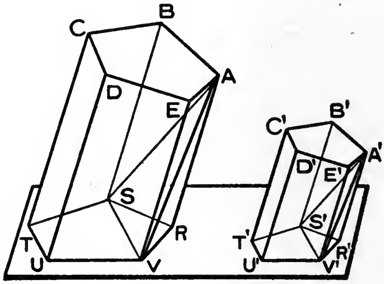
647. Theorem. The volumes of two similar polyhedrons are to each other as the cubes of any two homologous edges.



Given: Similar polyhedrons AT and $A'T'$; volumes V and V'; AR and $A'R'$, any two homologous edges.

To Prove: (?)

Proof: These solids may be decomposed, etc. (646). Denote vol. of tetrahedrons of AT by w, x, y, z, etc.; of $A'T'$ by w', x', y', z', etc.

Then, $\dfrac{w}{w'} = \dfrac{\overline{AR}^3}{\overline{A'R'}^3}$; $\dfrac{x}{x'} = \dfrac{\overline{AR}^3}{\overline{A'R'}^3}$; $\dfrac{y}{y'} = \dfrac{\overline{AR}^3}{\overline{A'R'}^3}$; etc. (?) (645).

Hence, $\dfrac{w}{w'} = \dfrac{x}{x'} = \dfrac{y}{y'} =$ etc. (Ax. 1). Therefore

$\dfrac{w + x + y + \text{etc.}}{w' + x' + y' + \text{etc.}} = \dfrac{w}{w'}$ (301). That is $\dfrac{V}{V'} = \dfrac{\overline{AR}^3}{\overline{A'R'}^3}$ (Ax. 6).

Q.E.D.

648. Theorem. The volumes of two similar pyramids are to each other as the cubes of their altitudes. (Explain.)

FORMULAS OF BOOK VII

Let B = area of base.	P = perimeter of base.
b = area of upper base.	P_r = perimeter of right section.
E = number of edges.	p = perimeter of upper base.
e, e' = homologous edges.	s = slant height.
F = number of faces.	T = total area.
h = altitude.	V = volume or number of vertices.
L = lateral area.	

Parallelepiped	$\boldsymbol{V = B \cdot h}$.
Prism	$\boldsymbol{L = P_r \cdot}$ **edge**; $\boldsymbol{T = L + 2B}$; $\boldsymbol{V = B \cdot h}$.
Regular pyramid . .	$\boldsymbol{L = \frac{1}{2} P \cdot s}$; $\boldsymbol{T = L + B}$.
Pyramid	$\boldsymbol{V = \frac{1}{3} B \cdot h}$.
Frustum of pyramid .	$\boldsymbol{L = \frac{1}{2}(P + p)s}$; $\boldsymbol{V = \frac{1}{3} h \cdot (B + b + \sqrt{B \cdot b})}$.
Polyhedron . . .	$\boldsymbol{E + 2 = V + F}$; **Sum of face** $\angle$s $\boldsymbol{= (V - 2)\,360°}$.
Similar polyhedrons	$\boldsymbol{T : T' = e^2 : e'^2}$; $\boldsymbol{V : V' = e^3 : e'^3}$.

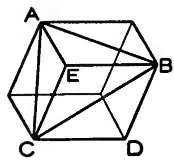
ORIGINAL EXERCISES

1. What plane through the vertex of a given tetrahedron will divide it into two equivalent parts? Prove.

2. The area of the base of any pyramid is less than the sum of the lateral faces.

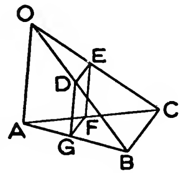
[Draw the altitudes of the lateral faces and the projections of the altitudes upon the base.]

3. Three of the edges of a parallelepiped that meet in a point are also the lateral edges of a pyramid. What part of the parallelepiped is this pyramid?



4. A plane is passed containing one vertex of a parallelepiped and a diagonal of a face not containing that vertex. What part of the volume of the parallelepiped is the pyramid thus cut off?

5. Any section of a tetrahedron made by a plane parallel to two opposite edges, is a parallelogram.

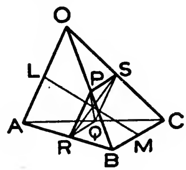


Given: Section $DEFG \parallel$ to OA and BC.

To Prove: $DEFG$ is a ▱.

Proof: EF is $\parallel$ to OA, DG is to OA (?). Also DE is $\parallel$ to BC and GF is $\parallel$ to BC (?) etc.

6. The three lines that join the midpoints of the opposite edges of a tetrahedron meet in a point and bisect one another.



Given: LM, PQ, RS, the three lines, etc.

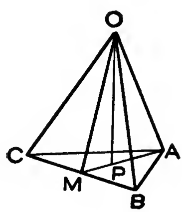
To Prove: (?).

Proof: Join PS, SQ, QR, PR. PS is $\parallel$ to and $= \frac{1}{2} BC$; RQ is $\parallel$ to and $= \frac{1}{2} BC$. (Explain.)

$\therefore$ fig. $PSQR$ is a ▱ (?).

Similarly, discuss LM and SR.

7. A pyramid having one of the faces of a cube for its base and the center of the cube for its vertex, contains one sixth of the volume of the cube.



8. A plane containing an edge of a regular tetrahedron and the midpoint of the opposite edge,

(*a*) contains the medians of two faces;

(*b*) is perpendicular to the opposite edge;

(*c*) is perpendicular to these two faces;

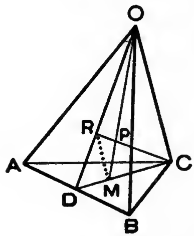
(*d*) contains two altitudes of the tetrahedron.

9. The altitude of a regular tetrahedron meets the base at the point of intersection of the medians of the base.

10. The altitude of a regular tetrahedron $= \frac{1}{3}\sqrt{6}$ times the edge.

11. The altitudes of a regular tetrahedron meet at a point.

12. The lines joining the vertices of *any* tetrahedron to the point of intersection of the medians of the opposite face meet in a point that divides each line into segments in the ratio 3 : 1.



Given: OM, CR, two such lines.

To Prove: The four such lines meet, etc.

Proof: OM and CR lie in the plane determined by OC and point D, the midpoint of AB.

$\therefore OM$ and CR intersect. Draw RM.

$\left.\begin{aligned} DR &= \tfrac{1}{2}\,RO \\ DM &= \tfrac{1}{2}\,MC \end{aligned}\right\}$ (?). $\therefore \dfrac{DR}{DM} = \dfrac{RO}{MC}$ (?).

$\therefore RM$ is $\parallel$ to OC (?).

$\therefore$ △ DMR and DCO are similar (?); and $DR : DO = RM : OC$ (?).

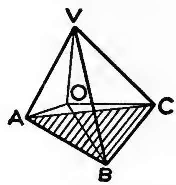
Thus $RM = \frac{1}{3}\,OC$. (Explain.)

Also △ PRM and OPC are similar (?).

$\left.\begin{aligned} &\text{Hence, } OP : PM = OC : RM = 3 : 1. \\ &\text{and } CP : PR = OC : RM = 3 : 1. \end{aligned}\right\}$ (Explain.) Q.E.D.

NOTE. This point P is called the **center of gravity** of the tetrahedron.

13. There can be no polyhedron having seven edges and only seven.



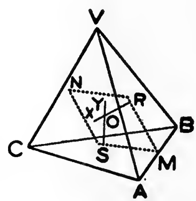
14. The planes bisecting the dihedral angles of any tetrahedron meet in a point that is equally distant from the faces.

15. The lines perpendicular to the faces of any tetrahedron, at the centers of the circles circumscribed about the faces, meet in a point that is equally distant from the vertices.

Proof: RX and SY are loci of points, etc. (526).

Plane MN, $\perp$ to AB at M, the midpoint of AB, is the locus of points, etc.

$\therefore RX$ and SY lie in MN and intersect at O, etc.



16. If a plane be passed through the midpoints of the three edges of a parallelepiped that meet at a vertex, what part of the whole solid is the pyramid thus cut off?

17. The plane bisecting a dihedral angle of a tetrahedron divides the opposite edge into two segments proportional to the areas of the faces that form the dihedral angle.

18. Two tetrahedrons are similar if a dihedral angle of one equals a dihedral angle of the other and the faces forming these dihedral angles are respectively similar.

19. If from any point within a regular tetrahedron perpendiculars to the four faces are drawn, their sum is constant and equal to the altitude of the tetrahedron.

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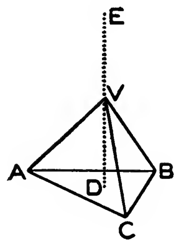
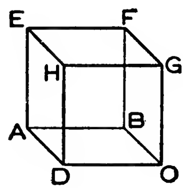
20. To construct a regular tetrahedron upon a given edge.



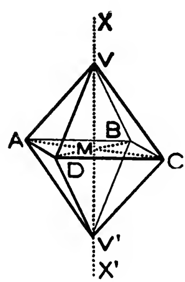
Construction: Upon AB, construct an equilateral $\triangle ABC$. Erect $ED \perp$ to plane of $\triangle ABC$, at D, the center of circumscribed $\odot$. Take V on ED such that $AV = BV = CV = AB$, etc.

21. To construct a regular hexahedron upon a given edge.



Construction: Upon AB, construct a square $ABCD$. At the vertices erect $\perp$s $= AB$ and join the extremities, etc.

22. To construct a regular octahedron upon a given edge.



Construction: Upon AB, construct a square $ABCD$. At M, the center of the square, erect XX' $\perp$ to plane of $ABCD$. On XX' take $MV = MV' = MD$. Draw the edges from V and V'.

Statement: VV' is a regular octahedron.

Proof: The right $\triangle$s DMV, DMC, DMV', are equal. (Explain.)

Thus the 12 edges are equal and the 8 faces are equal. (Explain.)

Figures $AVCV'$, $DVBV'$, $ABCD$ are equal squares. (Explain.)

Then, pyramids $V\text{–}ABCD$, $D\text{–}AVCV'$, etc., are equal and the 6 polyhedral angles are equal. (Explain.) $\therefore$ Etc.

23. To pass a plane through a cube so that the section will be a regular hexagon.

24. To pass planes through three given lines in space, no two of which are parallel, which shall inclose a parallelepiped.

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25. Find the lateral area and the total area of a regular pyramid whose slant height is 20 in. and whose base is a square, 1 ft. on a side.

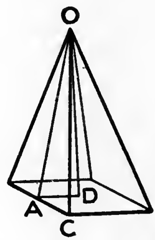
26. Find the volume of a pyramid whose altitude is 18 in. and whose base is an equilateral triangle each side of which is 8 in.

27. A regular hexagonal pyramid has an altitude of 9 ft. and each edge of the base is 6 ft. Find the volume.

28. The base of a pyramid is an isosceles triangle whose sides are 14, 25, 25, and the altitude of the pyramid is 12. Find its volume.

29. The altitude of the frustum of a pyramid is 25, and the bases are squares whose sides are 4 and 10, respectively. Find the volume of the frustum.

30. The frustum of a regular pyramid has hexagons for bases whose sides are 5 and 9, respectively. The slant height of the frustum is 14. Find its lateral area. Find its total area.



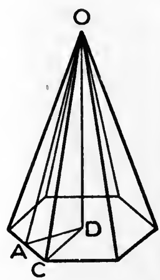
31. The altitude of a regular pyramid is 15, and each side of its square base is 16. Find the slant height, the lateral edge, the total area, and the volume.

$\overline{OA}^2 = \overline{OD}^2 + \overline{DA}^2 = (15)^2 + (8)^2 = 289.$

$\therefore AO = 17. \quad \overline{OC}^2 = \overline{OA}^2 + \overline{AC}^2 = 289 + 64 = 353.$

$\therefore OC = \sqrt{353} = 18.78 +.$

32. The slant height of a regular pyramid is 39, the altitude is 36, and the base is a square. Find the lateral area and volume.

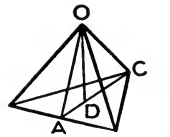


33. The lateral edge of a regular pyramid is 37 and each side of the hexagonal base is 12. Find the slant height, the lateral area, total area, and volume.

In rt. $\triangle ACD$, $CD = 12$, $AC = 6$, $\therefore AD = 6\sqrt{3}$.

In rt. $\triangle ACO$, $CO = 37$, $AC = 6$, $\therefore AO = \sqrt{1333}$.

In rt. $\triangle CDO$, $CO = 37$, $CD = 12$, $\therefore OD = 35$, etc.



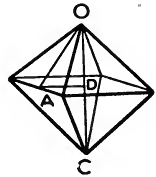
34. Find the total area and volume of a regular tetrahedron whose edge is six.

The four faces are equal equilateral △. ∴ $AO = AC = 3\sqrt{3}$; ∴ $AD = \sqrt{3}$ and $CD = 2\sqrt{3}$.

Hence, $OD = 2\sqrt{6}$. Area of any face $= 9\sqrt{3}$, etc.

35. Find the total area and volume of a regular tetrahedron whose edge is 10.

36. Find the total area and volume of a regular hexahedron whose edge is 8.



37. Find the total area and volume of a regular octahedron whose edge is 16.

The 8 faces are equal equilateral △. $AO = 8\sqrt{3}$. In $\triangle ADO$, one finds $OD = 8\sqrt{2}$. The volume of the octahedron = volumes of two pyramids, etc.

38. Find the total area and volume of a regular octahedron whose edge is 18.

39. The altitude of a regular pyramid is 16 and each side of the square base is 24. Find the lateral area and volume.

40. The slant height of a regular pyramid is 26 and its base is an equilateral triangle whose side is $20\sqrt{3}$. Find the total area and volume.

41. The altitude of a regular pyramid is 29 and its base is a regular hexagon whose side is 10. Find the total area and volume.

42. Find the total area and volume of a regular tetrahedron whose edge is 18.

43. Find the total area and volume of a regular octahedron whose edge is 20.

44. If the edge of a regular tetrahedron is e, show that the total area is $e^2\sqrt{3}$ and the volume is $\frac{1}{12}e^3\sqrt{2}$.

45. If the edge of a regular octahedron is e, show that the total area is $2e^2\sqrt{3}$ and the volume is $\frac{1}{3}e^3\sqrt{2}$.

46. A pyramid whose base is a square 9 in. on a side, contains 360 cu. in. Find its height.

47. A pyramid has for its base a hexagon whose side is $7\frac{1}{2}$ units and the pyramid contains 675 cu. units. Find the altitude.

48. The volume of a regular tetrahedron is $144\sqrt{2}$; find its edge.

49. The volume of a regular octahedron is $243\sqrt{2}$; find its edge.

50. The volume of a square pyramid is 676 cu. in. and the altitude is a foot. Find the side of the base. Find the lateral area.

51. The altitude of the Great Pyramid is 480 ft. and its base is 764 ft. square. It is said to have cost \$ 10 a cu. yd. and \$ 3 more for each sq. yd. of lateral surface (considered as planes). What was the cost?

52. The total surface of a regular tetrahedron is $324\sqrt{3}$ sq. in.; find its volume.

53. The base of a pyramid is a rhombus whose diagonals are 7 m. and 10 m. Find the volume if the altitude is 15 m.

54. The areas of the bases of the frustum of a pyramid are 3 sq. m. and 27 sq. m. The volume is 104 cu. m. Find the altitude.

55. The base of a pyramid is an isosceles right triangle whose hypotenuse is 8. The altitude of the pyramid is 15. Find the volume.

56. The altitude of a square pyramid, each side of whose base is 6 ft., is 10 ft. Parallel to the base and 2 ft. from the vertex a plane is passed. Find the area of the section. Find the volumes of the two pyramids concerned, and hence find the volume of the frustum.

57. Find the area of the section of a triangular pyramid each side of whose base is 8 in. and whose altitude is 18 in., made by a plane parallel to the base and a foot from the vertex.

58. The altitude of a frustum of a pyramid is 6, and the areas of the bases are 20 sq. in. and 45 sq. in. Find the altitude of the complete pyramid. Find the volume of this frustum by two distinct methods.

59. A granite monument in the form of a frustum of a pyramid, having rectangular bases one of which is 8 ft. wide and 12 ft long, and the other 6 ft. wide, is 30 ft. high. It is surmounted by a granite pyramid having the same base as the less base of the frustum, and 10 ft. in height. Find the entire volume. If one cu. ft. of water weighs $62\frac{1}{2}$ lb. and granite is three times as heavy as water, what is the weight of the entire monument?

60. If a square pyramid contains 40 cu. in. and its altitude is 15 in., find the side of its base.

61. A church spire in the form of a regular hexagonal pyramid whose base edge is 8 ft. and whose altitude is 75 ft. is to be painted at the rate of 18¢ per square yard. Find the cost.

62. Find the edge of a cube whose volume is equal to the volumes of two cubes whose edges are 4 and 6.

63. The base of a certain pyramid is an isosceles trapezoid whose parallel sides are 20 ft. and 30 ft. and the equal sides are each 13 ft. Find the volume of the pyramid if its altitude is 12 yards.

64. The lateral edge of the frustum of a regular square pyramid is 53 and the sides of the bases are 10 and 66. Find the altitude, the slant height, the lateral area, and the volume.

65. The sides of the base of a triangular pyramid are 33, 34, 65, and the altitude of the pyramid is 80. Find its volume.

66. The sides of the base of a tetrahedron are 17, 25, 26, and its altitude is 90. Find its volume.

67. If there are $1\frac{1}{4}$ cu. ft. in a bushel, what is the capacity (in bushels) of a hopper in the shape of an inverted pyramid, 12 ft. deep and 8 ft. square at the top?

68. In the corner of a cellar is a pyramidal heap of coal. The base of the heap is an isosceles right triangle whose hypotenuse is 20 ft. and the altitude of the heap is 7 ft. If there are 35 cu. ft. in a ton of coal, how many tons are there in this heap?

69. How many cubic yards of earth must be removed in digging an artificial lake 15 ft. deep, whose base is a rectangle 180×20 ft. and whose top is a rectangle 216×24 ft.? [The frustum of a pyramid.]

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70. One pair of homologous edges of two similar tetrahedrons are 3 ft. and 5 ft. Find the ratio of their surfaces. Of their volumes.

71. A pair of homologous edges of two similar polyhedrons are 5 in. and 7 in. Find the ratio of their surfaces. Of their volumes.

72. The edge of a cube is 3. What is the edge of a cube twice as large? Four times as large? Half as large?

73. An edge of a tetrahedron is 6. What is the edge of a similar tetrahedron three times as large? Eight times as large? Nine times as large? One third as large?

74. An edge of a regular icosahedron is 3 in. What is the edge of a similar solid five times as large? Ten times as large? Fifty times as large? A thousand times as large?

75. The edges of a trunk are 2 ft., 3 ft., 5 ft. Another trunk is twice as long (the other edges 2×3 ft.). How do their volumes compare? A third trunk has each dimension double those of the first. How does its volume compare with the first? How do their surfaces compare?

76. If the altitude of a certain regular pyramid is doubled, but the base remains unchanged, how is the volume affected? If each edge of the base is doubled, but the altitude unchanged, how is the volume affected? If the altitude and each edge of the base are all doubled, how is the volume affected?

77. If the slant height (only) of a regular pyramid is doubled, how is the lateral area affected? If each edge of the base is doubled, how is the lateral area affected? If both are doubled, what is the effect?

78. A pyramid is cut by a plane parallel to the base and bisecting the altitude. What part of the entire pyramid is the less pyramid cut away by this plane?

79. The volume of a certain pyramid, one of whose edges is 7, is 686. Find the volume of a similar pyramid whose homologous edge is 8.

80. A certain polyhedron whose shortest edge is 2 in. weighs 40 lb. What is the weight of a similar polyhedron whose shortest edge is 5 in.?

81. An edge of a polyhedron is 5 in. and the homologous edge of a similar polyhedron is 7 in. The entire surface of the first is 250 sq. in. and its volume is 375 cu. in. Find the entire surface and volume of the second.

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82. Find the edge of a cube whose volume equals that of a rectangular parallelepiped whose edges are $3 \times 4 \times 18$.

83. A pyramid and an equivalent prism have equivalent bases. How do their altitudes compare?

84. A pyramid and a prism have the same altitude and equivalent bases. Compare their volumes.

85. Solve: $V = \frac{1}{3} B \cdot h$ for B. For h.

86. Solve: $L = \frac{1}{2} P \cdot s$ for P. For s.

87. Solve: $L = \frac{1}{2} (P + p) \cdot s$ for s.

88. A prism whose altitude is 8 and whose base is an equilateral triangle whose side is 9 in. is transformed into a regular pyramid whose base is 10 in. square. Find its altitude.

89. An altitude of a pyramid is 10 m. How far from the vertex will a plane parallel to the base divide the pyramid into two equivalent parts?

90. The altitude of a pyramid is 12, and two planes are passed parallel to the base and dividing the pyramid into three equivalent parts. At what distances from the vertex are they?

BOOK VIII

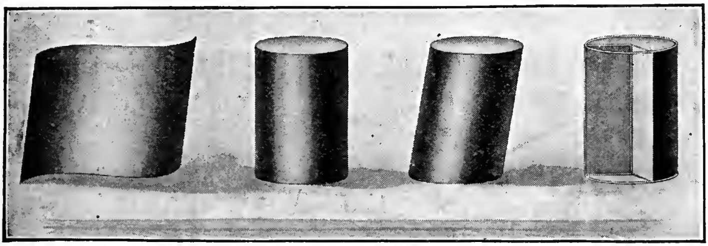
CYLINDERS, CONES

CYLINDERS

649. A **cylindrical surface** is a surface generated by a moving straight line which continually intersects a given curved line in a plane, and which is always parallel to a given straight line not in the plane of the curve.

The generating line is the **generatrix**. The directing curve is the **directrix**.

An **element** of a cylindrical surface is the generating line in any position.



CYLINDRICAL SURFACE — RIGHT CIRCULAR CYLINDER — OBLIQUE CYLINDER — CYLINDER OF REVOLUTION

650. A **cylinder** is a solid bounded by a cylindrical surface and two parallel planes.

The **bases** of a cylinder are the parallel plane sections.

The **lateral area** of a cylinder is the area of the cylindrical surface included between the planes of the bases.

The **total area** of a cylinder is the sum of the lateral area and the areas of the bases.

The **altitude** of a cylinder is the perpendicular distance between the planes of the bases.

651. A **right cylinder** is a cylinder whose elements are perpendicular to the planes of the bases.

A **circular cylinder** is a cylinder whose base is a circle.

An **oblique cylinder** is a cylinder whose elements are not perpendicular to the planes of the bases.

A **right circular cylinder** is a right cylinder whose base is a circle.

A **cylinder of revolution** is a cylinder generated by the revolution of a rectangle about one of its sides as an axis.

Similar cylinders of revolution are cylinders generated by similar rectangles revolving on homologous sides.

652. A **right section** of a cylinder is a section made by a plane perpendicular to all the elements.

A **plane** is **tangent** to a cylinder if it contains one element of the cylindrical surface and only one, however far it may be extended.

A **prism** is **inscribed** in a cylinder if its lateral edges are elements of the cylinder and the bases of the prism are inscribed in the bases of the cylinder.

A **prism** is **circumscribed** about a cylinder if its lateral faces are tangent to the cylinder and the bases of the prism are circumscribed about the bases of the cylinder.

PRELIMINARY THEOREMS

653. THEOREM. **Any two elements of a cylinder are equal and parallel.** (See 524 and 511.)

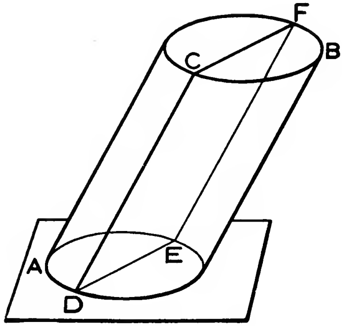
654. THEOREM. **A line drawn through any point in a cylindrical surface, parallel to an element, is itself an element.** (See 92.)

655. THEOREM. **A right circular cylinder is a cylinder of revolution.**

Ex. If a plane be defined as a surface generated by a moving straight line, what would the directrix be?

THEOREMS AND DEMONSTRATIONS

656. THEOREM. Every section of a cylinder made by a plane containing an element is a parallelogram.



Given: Cylinder AB; plane CE containing element CD.

To Prove: CE is a ▱.

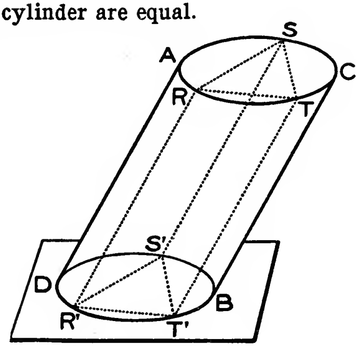
Proof: At E draw $EF \parallel$ to CD in plane CE. Also, EF is an element of the cylinder (654).

$\therefore$ EF is the intersection of the plane and the cylindrical surface (?) (482).

Also, CF is $\parallel$ to DE (?) (500).

$\therefore$ $CDEF$ is a ▱ (?) (126). Q.E.D.

657. THEOREM. The bases of a cylinder are equal.



Given: (?). **To Prove:** (?).

Proof: Suppose R, S, and T any three points in the perimeter of base AC. Draw elements RR', SS', TT'. Also, draw RS, ST, RT, $R'S'$, $S'T'$, $R'T'$. Now,

$$\left.\begin{array}{l} RR' \text{ is} = \text{and} \parallel \text{to } SS' \\ RR' \text{ is} = \text{and} \parallel \text{to } TT' \\ SS' \text{ is} = \text{and} \parallel \text{to } TT' \end{array}\right\} \text{(?) (653).}$$

$\therefore$ RS' is a ▱, RT' is a ▱, ST' is a ▱ (?) (135).

$\therefore$ $RS = R'S'$; $ST = S'T'$; $RT = R'T'$ (130); $\triangle RST = \triangle R'S'T'$ (?).

$\therefore$ base AC may be placed upon base BD so that R, S, and T coincide with R', S', and T', respectively. But S is *any point* on the perimeter; hence, *every point* on perimeter of AC will coincide with a corresponding point on perimeter of BD.

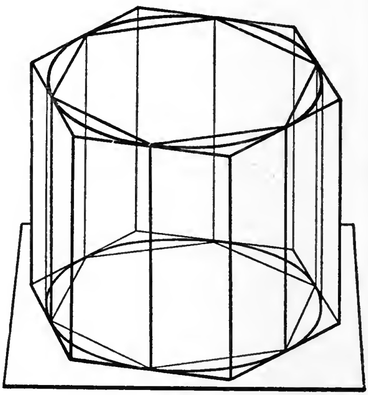
Therefore, base AC = base BD (?) (28). Q.E.D.

658. Cor. **Parallel plane sections of a cylinder (cutting all the elements) are equal.**

659. Cor. **Any section of a circular cylinder made by a plane parallel to the base is a circle.**

660. Theorem. **Every section of a right cylinder made by a plane containing an element is a rectangle (?).**

661. Theorem. **If a regular prism be inscribed in, or circumscribed about, a right circular cylinder and the number of sides of the base be indefinitely increased, the lateral area of the cylinder is the limit of the lateral area of the prism.**



Given: A regular prism inscribed in and a regular prism circumscribed about a right circular cylinder; the lateral area of the cylinder $= L$, and of the prisms, L_i and L_c, respectively.

To Prove: That as the number of sides of the bases of the prisms is indefinitely increased, L is the limit of both L_i and L_c.

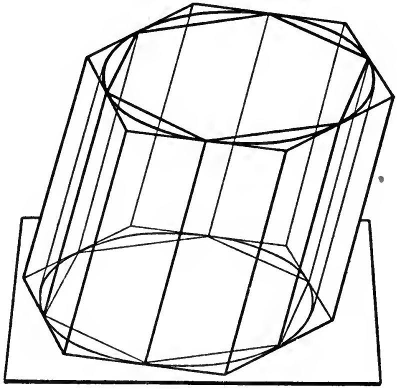
Proof: If the number of sides of the bases of the prisms is indefinitely increased, their perimeters will approach the circumference of the base of the cylinder as a limit (?).

Hence, it is obvious that the lateral area of the cylinder is the limit of the lateral area of either prism. Q.E.D.

Ex. 1. If the cylindrical surface of a cylinder be cut along an element, and this surface be placed in coincidence with a plane, what plane geometrical figure will it become?

Ex. 2. What two lines determine the size of a right circular cylinder?

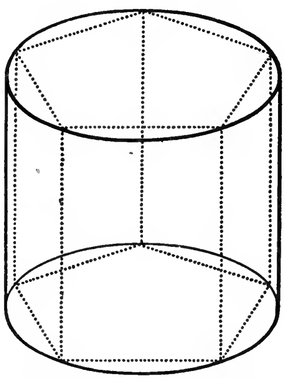
662. Theorem. If a prism having a regular polygon for a base be inscribed in, or circumscribed about, any circular cylinder and the number of the sides of the base of the prism be indefinitely increased, the volume of the cylinder is the limit of the volume of the prism.



Proof: If the number of sides of the base of either prism be indefinitely increased, the area of the base of the prism will approach the area of the base of the cylinder (440, II).

∴ it is obvious that the volume of the cylinder is the limit of the volume of either prism. Q.E.D.

663. Theorem. The lateral area of a right circular cylinder is equal to the product of the circumference of the base by an element.



Given: A right circular cylinder, the circumference of whose base $= C$, and whose element $= E$.

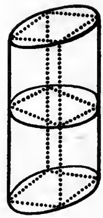
To Prove: Lateral area $L = C \cdot E$.

Proof: Inscribe in the cylinder a regular prism, the perimeter of whose base is P, whose lateral edge is E, and whose lateral area is L'.

Then $L' = P \cdot E$ (?). If the number of sides of the base of the prism is indefinitely increased, L' will approach L as a limit (?). P will approach C as a limit (?).

$$P \cdot E \text{ will approach } C \cdot E \text{ as a limit.}$$

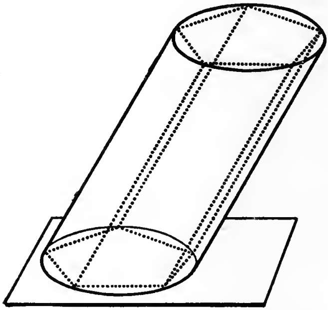
$$\therefore L = C \cdot E \ (?) \ (242).$$ Q.E.D.



NOTE. The lateral area of an oblique circular cylinder is equal to the product of the perimeter of a right section of the cylinder by an element.

The right section of an oblique circular cylinder is not a circle. The right section of an inscribed prism, having a regular polygon for a base, is not a regular polygon. Since elementary geometry does not deal with curves other than the circle, any proof or application of this theorem is omitted.

664. THEOREM. **The volume of a circular cylinder is equal to the product of its base by its altitude.**



Given: A circular cylinder whose base $= B$, altitude $= h$, and volume $= V$.

To Prove: $V = B \cdot h$.

Proof: Inscribe a prism having a regular polygon for its base, whose base $= B'$ and volume $= V'$.

Then, $V' = B' \cdot h$ (?) (603). If the number of sides of the base of the prism is indefinitely increased, V' approaches V as a limit (?), B' approaches B as a limit (?), and

$$B' \cdot h \text{ approaches } B \cdot h \text{ as a limit.}$$

$$\therefore V = B \cdot h \text{ (?) (242).}$$ Q.E.D.

FORMULAS

Let B = area of base. h = altitude. T = total area.
E = element. L = lateral area. V = volume.
C = circumference of base. R = radius of base.

665. **Lateral area** of right circular cylinder, $L = C \cdot E$ (?).

$$\therefore \boldsymbol{L = 2\pi Rh} \text{ (?).}$$

666. **Total area** of right circular cylinder, $T = L + 2B$ (?).

$$\therefore T = 2\pi Rh + 2\pi R^2 \text{ (?).} \quad \therefore \boldsymbol{T = 2\pi R(h + R)}.$$

667. **Volume** of circular cylinder, $V = B \cdot h$ (?).

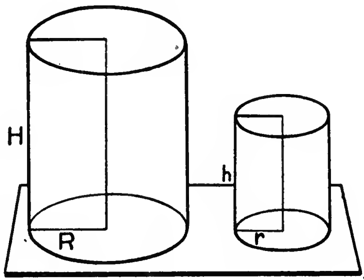
$$\therefore \boldsymbol{V = \pi R^2 h} \text{ (?).}$$

668. Theorem. **Of two similar cylinders of revolution:**

I. The lateral areas are to each other as the squares of their altitudes or as the squares of the radii of their bases.

II. The total areas are to each other as the squares of their altitudes or as the squares of the radii of their bases.

III. The volumes are to each other as the cubes of their altitudes or as the cubes of the radii of their bases.



Given: Two similar cylinders of revolution whose lateral areas $= L$ and l; whose total areas $= T$ and t; whose volumes $= V$ and v; whose altitudes $= H$ and h, and whose radii are R and r.

To Prove:

I. $L : l = H^2 : h^2 = R^2 : r^2$.

II. $T : t = H^2 : h^2 = R^2 : r^2$.

III. $V : v = H^3 : h^3 = R^3 : r^3$.

Proof: The generating rectangles of the cylinders are similar (?) (651).

$$\therefore H : h = R : r \text{ (?)}.$$

Hence, $H + R : h + r = H : h = R : r$ (?) (301).

I. $\frac{L}{l} = \frac{2\pi RH}{2\pi rh} = \frac{RH}{rh} = \frac{R}{r}\cdot\frac{H}{h} = \frac{H}{h}\cdot\frac{H}{h} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$. (Explain.)

II. $\frac{T}{t} = \frac{2\pi R(H+R)}{2\pi r(h+r)} = \frac{R(H+R)}{r(h+r)} = \frac{R}{r}\cdot\frac{H+R}{h+r} = \frac{H}{h}\cdot\frac{H}{h}$

$$= \frac{H^2}{h^2} = \frac{R^2}{r^2} \text{ (?)}.$$

III. $\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} = \frac{R^2 H}{r^2 h} = \frac{R^2}{r^2}\cdot\frac{H}{h} = \frac{H^2}{h^2}\cdot\frac{H}{h} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$ (?).

Q.E.D.

ORIGINAL EXERCISES (NUMERICAL)

$\pi = 3\frac{1}{7}$. 1 bu. $= 2150.42$ cu. in. 1 gal. $= 231$ cu. in.

In a cylinder of revolution,

1. If $R = 5$ in., $h = 14$ in., find L; T; V.
2. If $R = 7$ m., $h = 10$ m., find L; T; V.
3. If $R = 4\frac{2}{3}$ ft., $h = 18$ ft., find L; T; V.
4. If $R = 6$ in., $L = 792$ sq. in., find h; T; V.
5. If $R = 4$, $T = 352$, find h; L; V.
6. If $R = 2$, $V = 22$, find h; L; T.
7. If $h = 5.6$, $L = 352$, find R; T; V.
8. If $h = 9$, $T = 440$, find R; L; V.
9. If $h = 9\frac{1}{3}$, $V = 66$, find R; L; T.
10. If $L = 440$, $T = 1672$, find R; h; V.
11. If $L = 198$, $V = 594$, find R; h; T.

12. How many square inches of tin will be required to make a cylindrical pail 10 in. in diameter and a foot in height, without any lid? How many gallons will it contain?

13. The diameter of a well is $5\frac{1}{2}$ ft. and the water is 14 ft. deep. How many gallons of water in the well?

14. In a cylinder of revolution generated by a rectangle 30×14 in. revolving about its shorter side as an axis, find L; T; V.

15. In a cylinder of revolution generated by the rectangle of No. 14, revolving about its longer side as an axis, find L; T; V.

16. A cylindrical vessel 9 in. high, closed at one end, required $361\frac{3}{7}$ sq. in. of tin in its construction. Find its radius.

17. A cylindrical pail 12 in. high holds exactly two gallons. Find R.

18. How many cu. ft. of metal in a hollow cylindrical tube 42 ft. long, whose outer and inner diameters are 10 in. and 6 in., respectively?

19. A tunnel whose cross section is a semicircle 18 ft. high is one mi. long. How many cu. yd. of material were removed in the excavation?

20. An irregular stone is placed in a cylindrical vessel a in. in diameter and partly full of water. The water rises b in. Find volume of stone.

21. A rod of copper 18 ft. long and 2 in. square at the end is melted and formed into a wire $\frac{1}{8}$ in. in diameter. Find the length of the wire.

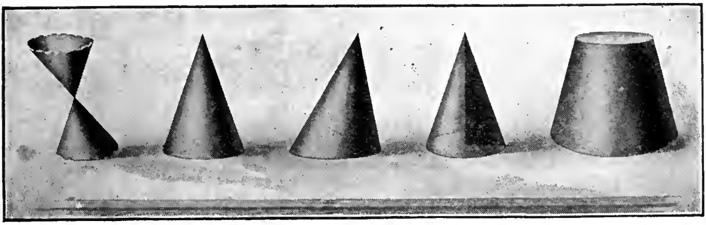
22. If a cylindrical bushel measure has for its altitude the diameter of the base, find the altitude.

CONES

669. A **conical surface** is a surface generated by a moving straight line that continually intersects a given curve in a plane, and passes through a fixed point not in this plane.

The generating line is the **generatrix**. The directing curve is the **directrix**. The fixed point is the **vertex** of the conical surface.

An **element** of a conical surface is the generating line in any position.



CONICAL SURFACE — RIGHT CIRCULAR CONE — OBLIQUE CONE — CONE OF REVOLUTION — FRUSTUM OF A CONE

670. A **cone** is a solid bounded by a conical surface and a plane cutting all the elements.

The **base** of a cone is its plane surface.

The **lateral area** of a cone is the area of the conical surface.

The **total area** of a cone is the sum of the lateral area and the area of the base.

The **altitude** of a cone is the perpendicular distance from the vertex to the plane of the base.

671. A **circular cone** is a cone whose base is a circle.

The **axis** of a circular cone is the line drawn from the vertex to the center of the base.

A **right circular cone** is a circular cone whose axis is perpendicular to the plane of the base.

An **oblique circular cone** is one whose axis is oblique to the plane of the base.

A **cone of revolution** is a cone generated by the revolution of a right triangle about one of the legs as an axis.

Similar cones of revolution are cones generated by the revolution of similar right triangles revolving about homologous sides.

The **slant height** of a cone of revolution is any one of its elements.

672. A **frustum** of a cone is the portion of a cone between the base and a plane parallel to the base.

The **altitude** of a frustum of a cone is the perpendicular distance between the planes of its bases. The **slant height** of a frustum of a cone is the portion of an element included between the bases. The **lateral area** of a frustum is the area of its curved surface. The **total area** of a frustum is the sum of the lateral area and the area of the bases.

673. A **plane** is **tangent** to a cone if it contains one element of the conical surface and only one, however far it may be extended.

A **pyramid** is **inscribed** in a cone if its base is inscribed in the base of the cone, and its vertex is the vertex of the cone.

A **pyramid** is **circumscribed** about a cone if its base is circumscribed about the base of the cone, and its vertex is the vertex of the cone.

The **frustum** of a pyramid is **inscribed** in, or **circumscribed** about, the frustum of a cone if the bases of the pyramid are inscribed in, or circumscribed about, the bases of the cone.

The **midsection** of a frustum of a cone is the section made by a plane parallel to the bases, and midway between them.

PRELIMINARY THEOREMS

674. THEOREM. The elements of a right circular cone are all equal. (See 520, II.)

675. THEOREM. A right circular cone is a cone of revolution.

The altitude, any element, and the radius of the base of a right circular cone form a right triangle, and in the same cone all such triangles are equal (53).

676. THEOREM. The altitude of a cone of revolution is the axis of the cone.

677. THEOREM. A straight line drawn from the vertex of a cone to any point in the perimeter of the base is an element. (See 39.)

678. THEOREM. The lateral edges of a pyramid inscribed in a cone are elements of the cone. (See 677.)

679. THEOREM. The lateral faces of a pyramid circumscribed about a cone are tangent to the conical surface. (Explain.)

680. THEOREM. The slant height of a regular pyramid circumscribed about a right circular cone is the same as the slant height of the cone.

681. THEOREM. The slant height of the frustum of a regular pyramid circumscribed about the frustum of a right circular cone is the same as the slant height of the frustum of the cone. (See 680.)

682. THEOREM. The radius of the mid-section of a frustum of a right circular cone is equal to half the sum of the radii of the bases.

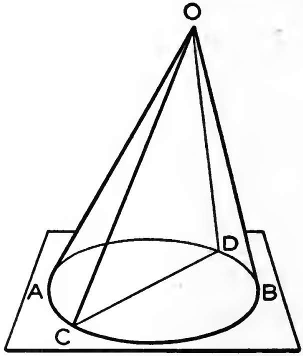
Proof: The radius of the mid-section is the median of a trapezoid whose bases are the radii of the bases of the frustum. That is, $m = \frac{1}{2}(R + r)$. (See 149.)

Ex. 1. What two lines determine the size of a right circular cone? What two lines determine the total area?

Ex. 2. Find the slant height of a right circular cone whose altitude is 8 and whose radius is 6.

THEOREMS AND DEMONSTRATIONS

683. THEOREM. Any section of a cone made by a plane passing through the vertex is a triangle.



Given: Cone O–AB; plane OCD.

To Prove: Section OCD is a $\triangle$.

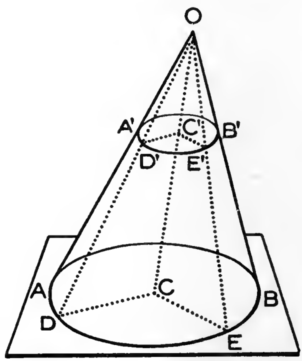
Proof: Draw straight lines OC, OD, in plane OCD. They are elements (?) (677).

$\therefore$ OC and OD compose the intersection of the plane and the conical surface (?) (482).

Also CD is a straight line (?).

$\therefore$ OCD is a $\triangle$ (?) (23). Q.E.D.

684. THEOREM. Any section of a circular cone made by a plane parallel to the base is a circle.



Given: Cone $O-AB$; circle C its base; section $A'B'$ ∥ to base.

To Prove: $A'B'$ also a $\odot$.

Proof: Draw the axis OC and pass planes OCD, OCE intersecting the base in CD, CE respectively, and the section in $C'D'$, $C'E'$.

In $\triangle$s OCD and OCE, $D'C'$ is ∥ to DC; $C'E'$ is ∥ to CE (?) (500).

$\therefore \triangle\ OC'D'$ is similar to $\triangle\ OCD$; $\triangle\ OC'E'$ is similar to $\triangle\ OCE$ (316).

$$\therefore \frac{OC'}{OC} = \frac{C'D'}{CD} \text{ and } \frac{OC'}{OC} = \frac{C'E'}{CE} \text{ (323, 3).} \quad \therefore \frac{C'D'}{CD} = \frac{C'E'}{CE} \text{ (?).}$$

But $CD = CE$ (?) (200). $\therefore C'D' = C'E'$ (Ax. 3). That is, all points on the perimeter of $A'B'$ are equally distant from C'.

$\therefore$ perimeter of section $A'B'$ is a circumference (?) (192).

Hence, $A'B'$ is a $\odot$ (?) (193). Q.E.D.

685. **Theorem.** **If a regular pyramid be inscribed in, or circumscribed about, a right circular cone and the number of sides of the base be indefinitely increased, the lateral area of the cone is the limit of the lateral area of the pyramid.** (See Fig. A.)

Demonstration is similar to that of 661.

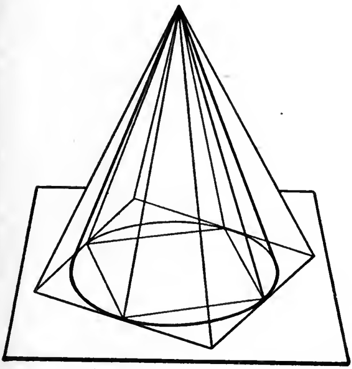


Fig. A

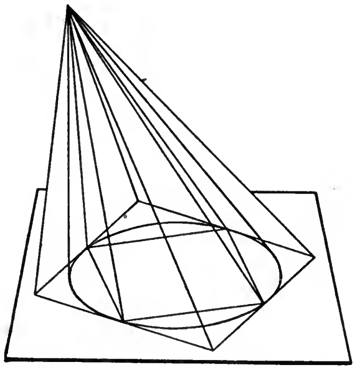


Fig. B

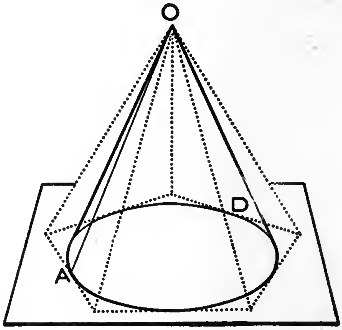
686. **Theorem.** **If a pyramid having a regular polygon for a base be inscribed in, or circumscribed about, any circular cone and the number of sides of its base be indefinitely increased, the volume of the cone is the limit of the volume of the pyramid.** (See Fig. B.)

Demonstration is similar to that of 662.

687. **Theorem.** **If a frustum of a regular pyramid be inscribed in, or circumscribed about, the frustum of a right circular cone and the number of sides of the bases be indefinitely increased, the lateral area of the frustum of the cone is the limit of the lateral area of the frustum of the pyramid.**

688. **Theorem.** **If the frustum of a pyramid having regular polygons for its bases be inscribed in, or circumscribed about, a frustum of any circular cone and the number of sides of the bases of the frustum be indefinitely increased, the volume of the frustum of the cone is the limit of the volume of the frustum of the pyramid.**

689. THEOREM. The lateral area of a right circular cone is equal to half the product of the circumference of the base by the slant height.



Given: Right circular cone O–AD, the circumference of whose base $= C$ and whose slant height $= s$.

To Prove: Lateral area $= \frac{1}{2} C \cdot s$.

Proof: Circumscribe a regular pyramid and denote the lateral area by L' and the perimeter of the base by P.

Slant height $OA = s$ (?) (680).

Then $L' = \frac{1}{2} P \cdot s$ (?) (617).

Now indefinitely increase the number of sides of the base of the pyramid and,

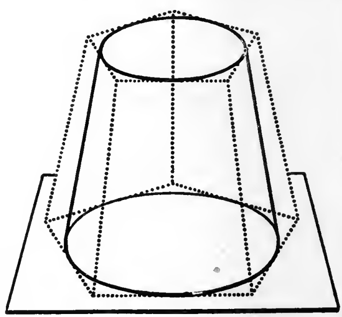
L' will approach L as a limit (?) (685).

P will approach C as a limit (?) (440, I).

$\frac{1}{2} P \cdot s$ will approach $\frac{1}{2} C \cdot s$ as a limit (?).

Hence, $L = \frac{1}{2} C \cdot s$ (?) (242). Q.E.D.

690. THEOREM. The lateral area of the frustum of a right circular cone is equal to half the sum of the circumferences of the bases multiplied by the slant height.



Given: Frustum of right circular cone, whose lateral area is L; whose slant height is s; and the circumferences of whose bases are C and c.

To Prove: $L = \frac{1}{2}(C + c) \cdot s$.

Proof: Circumscribe a frustum of a regular pyramid and denote its lateral area by L', the perimeters of its bases by P and p.

The slant height of frustum of pyramid $= s$ (?) (681).

Now, $L' = \frac{1}{2}(P + p) \cdot s$ (?) (618).

Indefinitely increase the number of the sides of the bases of the frustum of the pyramid, and

L' will approach L as a limit (?),

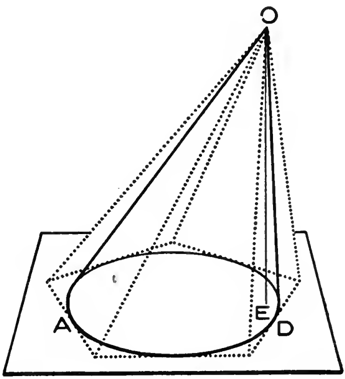
also P will approach C as a limit (?),

and p will approach c as a limit (?),

and $\frac{1}{2}(P + p) \cdot s$ will approach $\frac{1}{2}(C + c) \cdot s$ as a limit (?).

Hence, $L = \frac{1}{2}(C + c) \cdot s$ (?). Q.E.D.

691. Theorem. The volume of a circular cone is equal to one third the product of the area of the base by the altitude.



Given: Circular cone $O\text{–}AD$, whose volume $= V$; area of whose base $= B$; altitude $= OE = h$.

To Prove: $V = \frac{1}{3} B \cdot h$.

Proof: Circumscribe (or inscribe) a pyramid having a regular polygon for its base. Denote the volume of the pyramid by V', its base by B', its altitude $= OE = h$ (507).

$\therefore V' = \frac{1}{3} B' \cdot h$ (?).

Indefinitely increase the number of the sides, etc.

Then V' will approach V as a limit (?).

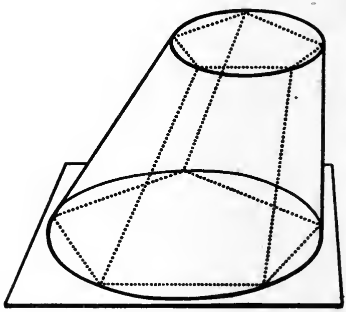
Also B' will approach B as a limit (?) (440, II).

$\frac{1}{3} B' \cdot h$ will approach $\frac{1}{3} B \cdot h$ as a limit (?).

Hence, $V = \frac{1}{3} B \cdot h$ (?). Q.E.D.

Ex. If the conical surface of a cone of revolution be cut along an element and the conical surface be placed in coincidence with a plane, what geometrical figure will the surface be?

692. Theorem. The volume of the frustum of a circular cone is equal to one third the product of the altitude by the sum of the lower base, the upper base, and a mean proportional between the bases of the frustum.



Given: A frustum of any circular cone, whose volume $= V$, whose bases are B and b, whose altitude is h.

To Prove:
$V = \frac{1}{3} h[B + b + \sqrt{B \cdot b}]$.

Proof: Inscribe (or circumscribe) a frustum of a pyramid having regular polygons for bases. Denote its volume by V', bases by B' and b', and altitude by h.

Then, $V' = \frac{1}{3} h[B' + b' + \sqrt{B' \cdot b'}]$ (?) (631).

Indefinitely, etc. V' will approach V as a limit (?).

B' and b' will approach B and b as limits (?).

$\sqrt{B' \cdot b'}$ will approach $\sqrt{B \cdot b}$ as a limit.

$\frac{1}{3} h[B' + b' + \sqrt{B' \cdot b'}]$ will approach $\frac{1}{3} h[B + b + \sqrt{B \cdot b}]$.

Hence, $V = \frac{1}{3} h[B + b + \sqrt{B \cdot b}]$ (?) (242). Q.E.D.

Ex. 1. Could you prove the theorem of 690 by inscribing a frustum of a pyramid? Could you prove the theorem of 692 by circumscribing a frustum of a pyramid? Give reasons for your answer.

Ex. 2. Could you prove the theorem of 689 by inscribing a pyramid? Could you prove the theorem of 691 by inscribing a pyramid? Give reasons.

Ex. 3. The frustum of a circular cone is 15 in. high and the bases are circles whose radii are 3 in. and 5 in. Find the volume.

Ex. 4. The frustum of a right circular cone has a slant height of 9 ft. and the radii are 5 ft. and 7 ft. Find the lateral area and the total area. What is the length of the altitude of this frustum?

FORMULAS

Let B = area of base.
b = area of less base.
C = circumference of base.
c = circumf. of less base.
h = altitude.
L = lateral area.
m = radius of mid-section of frustum.
R = radius of base.
r = radius of less base.
s = slant height.
T = total area.
V = volume.

693. Lateral area of right circular cone,

$$L = \frac{1}{2}C \cdot s = \frac{1}{2}(2\pi R)s \text{ (?)}.$$
$$\therefore \boldsymbol{L = \pi R s}.$$

694. Total area of right circular cone, $T = \pi Rs + \pi R^2$ (?).

$$\therefore \boldsymbol{T = \pi R(s + R)}.$$

695. Volume of circular cone, $V = \frac{1}{3}B \cdot h = \frac{1}{3}\pi R^2 \cdot h$ (?).

$$\therefore \boldsymbol{V = \tfrac{1}{3}\pi R^2 h}.$$

696. Lateral area of frustum of right circular cone,

$$L = \frac{1}{2}(C + c)s = \frac{1}{2}(2\pi R + 2\pi r)s \text{ (?)}.$$

$\therefore \boldsymbol{L = \pi(R + r)s}$. Also, $L = \pi(2m)s = \boldsymbol{2\pi m s}$ (682).

697. Total area of frustum of right circular cone,

$$T = \pi(R + r)s + \pi R^2 + \pi r^2 \text{ (?)}.$$
$$\therefore \boldsymbol{T = \pi[(R + r)s + R^2 + r^2]}.$$

698. Volume of frustum of circular cone,

$$V = \frac{1}{3}h[B + b + \sqrt{B \cdot b}] \text{ (?)}.$$
$$= \frac{1}{3}h[\pi R^2 + \pi r^2 + \sqrt{\pi R^2 \cdot \pi r^2}].$$
$$\therefore \boldsymbol{V = \tfrac{1}{3}\pi h[R^2 + r^2 + R \cdot r]}.$$

Ex. 1. State in words each of the final formulas of the paragraphs of this page.

Ex. 2. Find the radius of a circle having the same area as the lateral area of a cone of revolution whose radius is 4 and slant height, 9.

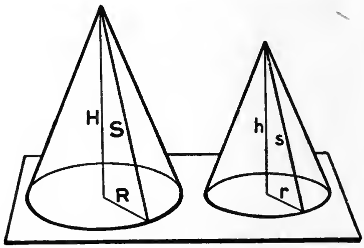
699. Theorem. Of two similar cones of revolution:

I. The lateral areas are to each other as the squares of their altitudes, or as the squares of their radii, or as the squares of their slant heights.

II. The total areas are to each other as the squares of their altitudes, or as the squares of their radii, or as the squares of their slant heights.

III. The volumes are to each other as the cubes of their altitudes, or as the cubes of their radii, or as the cubes of their slant heights.

Given: Two similar cones of revolution, whose respective lateral areas are L and l, total areas are T and t, volumes are V and v, altitudes are H and h, radii are R and r, slant heights are S and s.



To Prove:

I. $\frac{L}{l}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$.

II. $\frac{T}{t}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$.

III. $\frac{V}{v}=\frac{H^3}{h^3}=\frac{R^3}{r^3}=\frac{S^3}{s^3}$.

Proof: The generating $\triangle$ are similar (?) $\therefore \frac{H}{h}=\frac{R}{r}=\frac{S}{s}$ (?).

Hence, $\frac{R+S}{r+s}=\frac{R}{r}=\frac{S}{s}=\frac{H}{h}$ (?) (301 and Ax. 1).

I. $\frac{L}{l}=\frac{\pi RS}{\pi rs}=\frac{R}{r}\cdot\frac{S}{s}=\frac{H}{h}\cdot\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$. (Explain.)

II. $\frac{T}{t}=\frac{\pi R(R+S)}{\pi r(r+s)}=\frac{R}{r}\cdot\frac{R+S}{r+s}=\frac{H}{h}\cdot\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$. (Explain.)

III. $\frac{V}{v}=\frac{\frac{1}{3}\pi R^2H}{\frac{1}{3}\pi r^2h}=\frac{R^2}{r^2}\cdot\frac{H}{h}=\frac{H^2}{h^2}\cdot\frac{H}{h}=\frac{H^3}{h^3}=\frac{R^3}{r^3}=\frac{S^3}{s^3}$ (?).

Q.E.D.

ORIGINAL EXERCISES

In a cone of revolution,

1. If $h = 12$, $s = 13$, find R.

2. If $h = 15$, $R = 8$, find s.

3. If $R = 18$, $s = 30$, find h.

4. If $h = 6$, $s = 10$, find R; L; T; V.

5. If $h = 20$, $R = 21$, find s; L; T; V.

6. If $R = 7$, $s = 25$, find h; L; T; V.

7. If $L = 4070$, $s = 37$, find R; h; T; V.

8. If $L = 46.64$, $R = 2.8$, find s; h; T; V.

9. If $L = 400$, $T = 500$, find s; h; R; V.

10. If $T = 80\pi$, $R = 5$, find s; h; L; V.

11. If $T = 10\pi$, $s = 3$, find R; h; L; V.

12. If $V = 462$, $R = 21$, find h; s; L; T.

13. If $V = 8\frac{8}{175}$, $h = 3$, find R; s; L; T.

14. What would be the cost at 10¢ a square foot of painting a conical church steeple, 112 ft. high and 30 ft. in diameter at the base?

15. The sides of an equilateral triangle are each 12 in. Find the lateral surface, total surface, and volume of the solid generated by revolving this triangle about an altitude as an axis.

16. An isosceles right triangle whose legs are each 8 is revolved about the hypotenuse as an axis. Find the total surface and volume of the solid generated.

17. The sides of an equilateral triangle are each 10. Find the total surface and the volume of the solid generated by revolving this triangle about one of its sides as an axis.

18. Find the volume of a cone of revolution whose slant height is 16 and lateral area is 192π.

19. Find the lateral area of a cone of revolution whose altitude is 20 and volume is 240π.

20. How many bushels in a conical heap of grain whose base is a circle 35 ft. in diameter, and whose height is 25 ft.?

21. A regular hexagon whose side is 6 revolves about one of the longer diagonals. Find the surface and the volume of solid generated.

22. Find the volumes of the right circular cones inscribed in and circumscribed about a regular tetrahedron whose edge is a.

23. A right triangle whose legs are 15 and 20 is revolved about the hypotenuse as an axis. Find the surface and the volume of the solid generated.

In the frustum of a right circular cone,

24. If $h = 8$, $R = 10$, $r = 4$, find s; L; T; V.

25. If $h = 30$, $s = 34$, $r = 5$, find R; L; T; V.

26. If $s = 19\frac{1}{2}$, $R = 10\frac{1}{2}$, $r = 3$, find h; L; T; V.

27. How many square feet of tin are required to make a funnel 2 ft. long, if the diameters of the ends are 20 and 56 in., respectively?

28. A chimney 150 ft. high has a cylindrical flue 3 ft. in diameter. The bases of the chimney are circles whose diameters are 28 ft. and 7 ft. Find the number of cubic yards of masonry in the chimney.

29. A plane is passed parallel to the base of a right circular cone and $\frac{2}{5}$ the distance from the vertex to the base. Find the ratio of the smaller cone thus formed to the original cone. Compare the volume of the less cone with the frustum formed.

$$\frac{\text{Original cone}}{\text{Less cone}} = \frac{5^3}{2^3} \text{ (?)} = \text{etc.}$$

Hence, $$\frac{\text{Original cone} - \text{Less cone}}{\text{Less cone}} = \frac{125 - 8}{8} \text{ (?), etc.}$$

30. The altitude of a cone of revolution is 12 in. What is the altitude of the frustum of this cone that shall contain one fourth the volume of the whole cone?

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31. The altitudes of two similar cylinders of revolution are 3 and 5. What is the ratio of their lateral areas? Of their total areas? Of their volumes?

32. The altitudes of two similar cylinders of revolution are 5 and 6, and the lateral area of the first is 200. Find the lateral area of the second. If the volume of the first is 500, what is the volume of the second?

33. The total areas of two similar cones of revolution are 24π and 216π and the radius of the first is 3. Find the radius of the second. The slant height of the first is 5. Find the lateral area of the second. Find the altitude of the first and the volume of the second.

34. The volumes of two similar cones of revolution are 27π and 343π. The altitude of the first is 9. Find the altitude of the second. Find the radius of the base of each.

35. A cone of revolution whose radius is 10 and altitude 20, has the same volume as a cylinder of revolution whose radius is 15. Find the altitude of the cylinder.

36. A cylinder of revolution whose radius is 8 and altitude 30, is formed into a cone of revolution whose altitude is 40. Find the radius of its base.

37. The heights of two equivalent cylinders of revolution are in the ratio of $4:9$. If the diameter of the first is 12 ft., what is the diameter of the second?

38. A cylinder of revolution 8 ft. in diameter is equivalent to a cone of revolution 7 ft. in diameter. If the height of the cone is 16 ft., what is the height of the cylinder?

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39. Two circular cylinders having equal altitudes are to each other as their bases.

40. Two circular cylinders having equal bases are to each other as their altitudes.

41. Two circular cylinders having equal bases and equal altitudes are equivalent.

42. Two circular cones having equal altitudes are to each other as their bases.

43. Two circular cones having equal bases are to each other as their altitudes.

44. Two circular cones having equal bases and equal altitudes are equivalent.

45. If the altitude of a right circular cylinder is equal to the radius of the base, the lateral area is half the total area.

46. If the altitude of a right circular cylinder is half the radius of the base, the lateral area is equal to the area of the base.

47. If the slant height of a right circular cone is equal to the diameter of the base, the lateral area is double the area of the base.

48. The lateral area of a cone of revolution is equal to the area of a circle whose radius is a mean proportional between the slant height and the radius of the base.

49. The lateral area of a cylinder of revolution is equal to the area of a circle whose radius is a mean proportional between the altitude of the cylinder and the diameter of its base.

50. What relation does the section of a circular cone made by a plane parallel to the base have to the base? Prove.

51. At what distance from the vertex of a right circular cone whose altitude is h must a plane parallel to the base be passed, so as to bisect the lateral area? At what distance must it be passed so as to bisect the volume?

52. What does the volume V of a right circular cone become, if the altitude is doubled and the base undisturbed? Prove. What does the volume V become if the radius of the base is doubled but the altitude undisturbed? Prove. If both are doubled? Prove.

53. The intersection of two planes tangent to a cylinder is a line parallel to an element.

54. The intersection of two planes tangent to a cone is a line through the vertex.

55. One straight line can be drawn upon a cylindrical surface through a given point, and only one.

56. If two cylinders of revolution have equivalent lateral areas, their volumes are to each other as their radii.

57. If a rectangle be revolved about its unequal sides as axes, the volumes of the two solids generated are inversely proportional to the axes, and directly proportional to the radii of the bases.

58. Show that the formula for the volume of a circular cone can be derived from the formula for the volume of a frustum of a circular cone if one base of the frustum becomes a point.

59. Reduce the formula for the volume of a frustum of a circular cone if the radius of one base is double the radius of the other.

---

60. To pass a plane tangent to a circular cylinder and containing a given element.

Construction: Draw a line in plane of base tangent to the base at the end of the given element, etc.

61. To pass a plane tangent to a circular cone and containing a given element.

62. To pass a plane tangent to a circular cylinder and through a given point without it.

Construction: From the point draw a line $\parallel$ to an element, meeting the plane of the base. From this point of intersection draw a line tangent to the base of the cylinder. Through the point of contact draw an element, etc.

63. To pass a plane tangent to a circular cone and through a given point without it.

Construction: Connect this point with the vertex of the cone and prolong this line to meet the plane of the base, etc.

64. To divide the lateral surface of a cone of revolution into two equivalent parts by a plane parallel to the base.

65. Find the locus of points at a given distance from a given straight line.

66. Find the locus of points equally distant from two given points and at a given distance from a straight line. Discuss.

67. Find the locus of points at a given distance from a given plane and at a given distance from a given line. Discuss.

68. Find the locus of points at a given distance from a given cylindrical surface whose generatrix is a circle, and whose elements are perpendicular to the plane of the circle.

69. Find the locus of all lines making angles equal to a given angle, with a given line, at a given point.

70. Find the locus of all lines making angles equal to a given angle, with a given plane, at a given point.

71. Find the locus of a point at a given distance from a given line and equally distant from two given planes. Discuss.

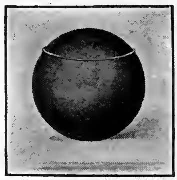
72. Find a point, X, at a given distance from a given line and equally distant from three given points. Discuss.

BOOK IX

THE SPHERE

700. A **sphere** is a solid bounded by a surface, all points of which are equally distant from a point within, called the **center.** The surface of a sphere is a **spherical surface.**

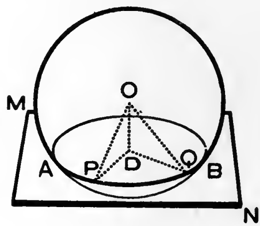
A **radius** of a sphere is a straight line drawn from the center to any point of the surface.



A **diameter** of a sphere is a straight line that contains the center and has its extremities in the surface.

701. THEOREM. **Every plane section of a sphere is a circle.**

Given: Sphere whose center is O; plane MN intersecting sphere in AB.



To Prove: The figure AB is a $\odot$.

Proof: Draw $OD \perp$ to plane MN, meeting the plane at D. Take P and Q, any two points on the perimeter of the section, and draw DP, DQ, OP, OQ.

$\triangle$ ODP and ODQ are rt. $\triangle$ (?).

In $\triangle$ ODP and ODQ, $OD = OD$ and $OP = OQ$ (?) (700).

$\therefore$ the $\triangle$ are equal (?).

Hence, $DP = DQ$ (?).

That is, all points of the perimeter of AB are equally distant from D.

$\therefore$ the perimeter of AB is a circumference (?) (192).

Hence, the section AB is a circle (?) (193). Q.E.D.

702. A **great circle** of a sphere is a section of the sphere made by a plane containing the center of the sphere.

A **small circle** of a sphere is a section of the sphere made by a plane that does not contain the center of the sphere.

The **axis of a circle** of a sphere is the diameter of the sphere perpendicular to the plane of the circle.

The **poles of a circle** of a sphere are the ends of its axis.

A **quadrant** (in Spherical Geometry) is one fourth of the circumference of a great circle.

Equal spheres are spheres having equal radii.

703 A **plane** is **tangent** to a sphere if it touches the sphere in only one point. Two **spheres** are **tangent** to each other if they are tangent to the same plane at the same point. They may be tangent internally or externally.

A **line** is tangent to a sphere if it touches the sphere in only one point and does not intersect it.

A **line** is tangent to the circle of a sphere if it lies in the plane of the circle and touches the circle at only one point. In all cases this common point is the **point of contact** or **point of tangency**.

704. A **sphere** is **inscribed** in a polyhedron if all the faces of the polyhedron are tangent to the sphere.

A **sphere** is **circumscribed** about a polyhedron if all the vertices of the polyhedron are in the spherical surface.

705. The **distance** between two points on the surface of a sphere is the less arc of a great circle passing through them.

The distance between a point on a circle of a sphere and the nearer pole of the circle is the **polar distance** of the point.

706. The **angle between two** intersecting **curves** is the angle formed by their tangents at the point of intersection.

A **spherical angle** is the angle between the circumferences of two great circles of a sphere.

PRELIMINARY THEOREMS

707. Theorem. All radii of a sphere are equal. (See 700.)

708. Theorem. All radii of equal spheres are equal. (See 702.)

709. Theorem. All diameters of the same sphere or of equal spheres are equal. (See Ax. 3.)

710. Theorem. All great circles of the same sphere or of equal spheres are equal. (See 196.)

711. Theorem. In the same sphere or in equal spheres:

I. Equal plane sections are equally distant from the center.

II. Plane sections equally distant from the center are equal.

III. Of two unequal plane sections, the greater is at the less distance from the center.

IV. Of two plane sections unequally distant from the center, the section at the greater distance is the less.

In each case the diameters of the sections are chords of great circles. These theorems follow from 221, 222, 223, 224.

712. Theorem. Two great circles of a sphere bisect each other. Because they have a common diameter. (See 204.)

713. Theorem. Every great circle of a sphere bisects the sphere and the spherical surface. (Proof is like the proof of 204.)

714. Theorem. A sphere may be generated by the revolution of a semicircle about the diameter as an axis.

715. Theorem. One and only one great circle can be drawn through two points, not the ends of a diameter, on the surface of a sphere. (See 493.)

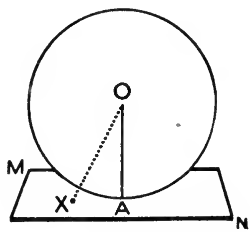
716. Theorem. One and only one circle can be drawn through three points on the surface of a sphere. (See 493.)

717. Theorem. A point is without a sphere if its distance from the center is greater than the radius, and if a point is without a sphere its distance from the center is greater than the radius. (See 700.)

718. Theorem. The axis of a circle of a sphere passes through its center. (This is proved in the proof of 701.)

THEOREMS AND DEMONSTRATIONS

719. Theorem. A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.



Given: Radius OA of sphere O, and plane $MN \perp$ to OA at A.

To Prove: MN tangent to the sphere.

Proof: Take *any* point X in MN (except A) and draw OX.

$OX > OA$ (?) (520, I).

$\therefore X$ lies without the sphere (717).

Hence, every point of plane MN, except A, is without the sphere; that is, plane MN is tangent to the sphere (?) (703).

Q.E.D.

720. Theorem. A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.

Given: Plane MN tangent to sphere O at A; radius OA.

To Prove: OA is $\perp$ to plane MN.

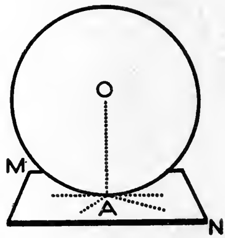
Proof: Every point in MN, except A, is without the sphere (?) (703).

Take any point X in MN and draw OX.

Now, OX is $> OA$ (?) (717). That is, OA is the shortest line from O to MN.

$\therefore OA$ is $\perp$ to MN (?) (520, I). Q.E.D.

721. THEOREM. A line tangent to a sphere lies in the plane tangent to the sphere at the same point.



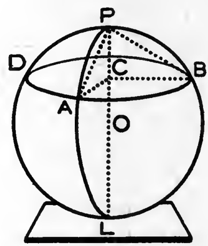
Proof: The line is $\perp$ to the radius drawn to the point of contact (?) (216).

The plane also is $\perp$ to the radius (?).

$\therefore$ the line is in the plane (502).

722. THEOREM. At a point on the surface of a sphere there can be only one tangent plane. (Explain.)

723. THEOREM. All points in the circumference of a circle of a sphere are equally distant from either pole; that is, the polar distances of all points in the circumference of a circle are equal.



Given: P and L, the poles of $\odot\ C$ on sphere O, and great Ⓢ PAL, PBL.

To Prove: Arc $PA =$ arc PB;
arc $AL =$ arc BL.

Proof: Draw the axis PL meeting plane of $\odot\ C$ at C.

Draw AC, AP, BC, BP.

PC is $\perp$ to plane DAB (?) (Def. of axis, 702).

$\therefore$ chord $PA =$ chord PB (?).

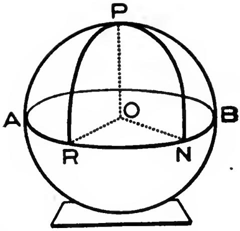
Hence, arc $PA =$ arc PB (?).

Likewise, arc $AL =$ arc BL. Q.E.D.

724. COR. The polar distance of a great circle is a quadrant.

725. THEOREM. If a point on the surface of a sphere is at the distance of a quadrant from two other points on the surface, not the ends of a diameter, it is the pole of the great circle containing these two points.

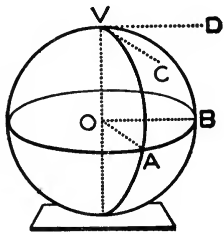
Given: P, a point, and R and N, two other points, — all on the surface of sphere O; arcs PR and PN, quadrants; great circle $ARNB$.



To Prove: P is the pole of ⊙ $ARNB$.

Proof: Draw the radii OP, OR, ON.
∠s PON and POR are rt. ∠s (245).
∴ PO is ⊥ to plane AB (?) (501).
∴ PO is the axis of ⊙ $ARNB$ (702).
∴ P is the pole of ⊙ $ARNB$ (702). Q.E.D.

726. Theorem. A spherical angle is measured by the arc of a great circle having the vertex of the angle as a pole and intercepted by the sides of the angle.



Given: Spherical ∠ AVB; arc AB of great ⊙ whose pole is V, on sphere O.

To Prove: ∠ AVB is measured by arc AB.

Proof: Draw radii OA, OB, OV, and at V draw VC tangent to ⊙ VA, and VD tangent to ⊙ VB. VB is a quadrant (724).
OV is ⊥ to VD (?), and
OV is ⊥ to OB (?) (245).
Likewise, OV is ⊥ to VC and to OA (?).
∴ VD is ∥ to OB, and VC is ∥ to OA (?) (93).
∴ ∠ CVD = ∠ AOB (?) (515).
And ∠ CVD is the spherical ∠ AVB (706).
But ∠ AOB is measured by arc AB (245).
∴ ∠ CVD is measured by arc AB (Ax. 6).
That is, ∠ AVB is measured by arc AB (Ax. 6). Q.E.D.

727. Cor. All arcs of great circles containing the pole of a great circle are perpendicular to the circumference of the great circle. (See 540.)

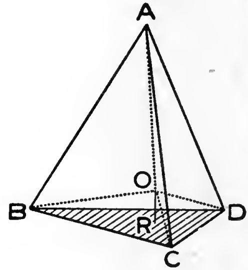
728. Cor. A spherical angle is equal to the plane angle of the dihedral angle formed by the planes of the sides of the angle. (See 530.)

729. Cor. If two great circles are perpendicular to each other, each of their circumferences contains the pole of the other. (See 543; 702.)

730. Theorem. A sphere may be inscribed in any tetrahedron.

Given: Tetrahedron *A–BCD*.

To Prove: (?).



Proof: Pass plane *OAB* bisecting dih. ∠ *AB*, and plane *OBC* bisecting dih. ∠ *BC*, and plane *OCD* bisecting dih. ∠ *CD*, the three planes meeting at point *O*.

Point *O*, in plane *OAB*, is equally distant from faces *ABC* and *ABD* (?) (551).

Point *O*, in plane *OBC*, is equally distant from faces *ABC* and *BCD* (?).

Point *O*, in plane *OCD*, is equally distant from faces *BCD* and *ACD* (?).

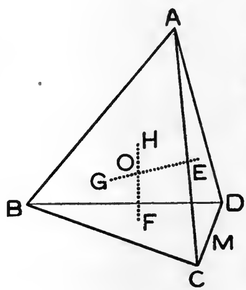
∴ *O* is equally distant from all four faces (Ax. 1).

Hence, using *O* as a center and the perpendicular distance *OR* as a radius, a sphere can be inscribed (?) (704). Q.E.D.

731. Cor. The six planes bisecting the six dihedral angles of any tetrahedron meet in a point.

732. Theorem. A sphere may be circumscribed about any tetrahedron.

Given: (?). **To Prove:** (?).



Proof: Take *E* and *F*, the centers of circles circumscribed about the faces *ACD* and *BCD*, respectively. Erect *EG*, and *FH*, ⊥ to these faces. Find *M*, the midpoint of edge *CD*.

EG is the locus of points equally distant from points *A*, *D*, *C* (?) (526).

FH is the locus of points equally distant from points *B*, *C*, *D* (?).

That is, all points in *EG* and *FH* are equally distant from *C* and *D* (Ax. 1).

But all points equally distant from C and D are in a plane $\perp$ to CD at M (?) (525).

$\therefore$ EG and FH are in one plane and are not parallel. (Not $\perp$ to the same plane).

That is, EG and FH must intersect at O.

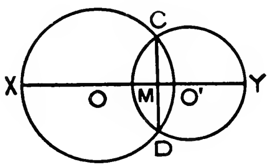
Hence, O is equally distant from A, B, C, and D (?) (Ax. 1).

That is, using O as a center and OA, or OB, or OC, or OD, as a radius, a sphere may be circumscribed about the tetrahedron A–BCD (?) (704). Q.E.D.

733. Cor. **Through any four points not in the same plane a sphere may be described.**

734. Cor. **The six planes perpendicular to the edges of any tetrahedron at their midpoints meet in a point.** (Explain.)

735. Theorem. **The intersection of two spherical surfaces is a circumference.**



Given: Two intersecting circumferences O and O'; common chord CD; line of centers XY, intersecting CD at M.

To Prove: The spherical surfaces generated by the revolution of these ⓢ, intersect in a circumference.

Proof: If these ⓢ be revolved upon XY as an axis, they will generate spheres (?) (714).

$$CM = MD \text{ (?) (232).}$$

Point C, common to both ⓢ, will generate the intersection of the spherical surfaces (?) (482).

$$CM \text{ is always } \perp \text{ to } XY \text{ (?) (232).}$$

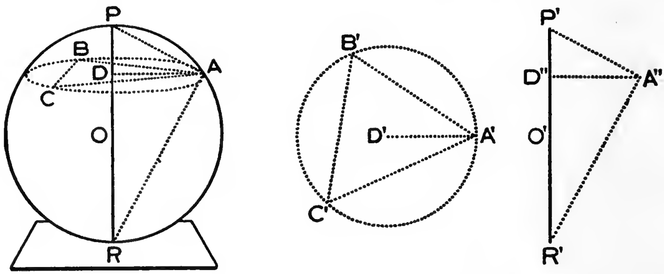
$\therefore$ the curved line generated by C is in one plane (?) (502).

$\therefore$ the intersection is a circumference (?) (192). Q.E.D.

Omit

CONSTRUCTIONS

736. Problem. To find the radius of a material sphere.



Given: A material sphere. **Required:** To find its radius.

Construction: *First*, place one point of the compasses at P, and using any opening of the compasses, as AP, with the other point draw a circumference on the surface of the sphere.

Upon this circumference take three points, A and B and C, and by means of the compasses measure the straight lines AB, AC, BC.

Second, construct a $\triangle\ A'B'C'$, whose sides are AB, AC, BC. Circumscribe a circle about this $\triangle$, and draw the radius $A'D'$.

Third, construct a right $\triangle\ P'A''D''$, whose hypotenuse is the known line PA and whose leg is the known radius $A'D'$. At A'' erect $A''R' \perp$ to $P'A''$ meeting $P'D''$, produced, at R'.

Bisect $P'R'$ at O'.

Statement: $O'P' =$ the required radius. Q.E.F.

Proof: Points P and O are equally distant from the points of the circumference ABC (Const. and 707).

$\therefore P$ is in the line $\perp$ to plane ABC at the center D (?) (526).

If this $\perp$ could be drawn within the solid sphere, it would be a diameter (?); and the $\angle\ PDA$ would be a rt. $\angle$ (?) (489).

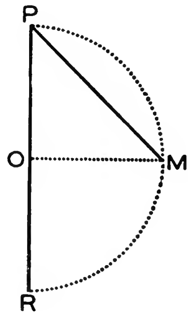
Now $\triangle\ ABC = \triangle\ A'B'C'$ (?) (58). $\therefore DA = D'A'$ (227, 201).

Also, $\triangle PAD = \triangle P'A''D''$ (?) (73). $\therefore \angle P = \angle P'$ (?)
$\angle PAR =$ a rt. $\angle$ (?). $\therefore \triangle PAR = \triangle P'A''R'$ (?) (74).
$\therefore PR = P'R'$ (?). Hence, $OP = O'P'$ (Ax. 3).

That is, $O'P' =$ the radius. Q.E.D.

737. PROBLEM. To find the chord of a quadrant of a material sphere.

Given: (?). **Required:** (?).



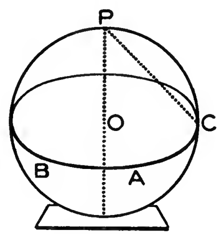
Construction: Find the radius of the sphere (by 736). Using this radius OP and any center O, describe a semicircumference PMR. Erect radius $OM \perp$ to diameter PR, and draw PM.

Statement: Arc PM is a quadrant of great circle of the given sphere and chord PM is the required chord. Q.E.F.

Proof: Arc PM is a quadrant (?).

738. PROBLEM. To describe the circumference of a great circle through two given points on the surface of a sphere.

Given: The points A and B on the surface of the sphere O.



Required: (?).

Construction: Find the chord of a quadrant of the given sphere (by 737).

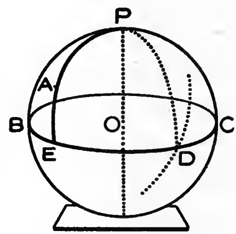
Place one point of the compasses at A, and using the chord just found as an opening, describe an arc on the surface of the sphere.

Similarly, place one point of the compasses at B, and using the same opening, describe an arc, meeting the former arc at P.

Now, place one point of the compasses at P and describe the circumference BAC, using the chord as before.

Statement: (?). **Proof:** (Use 725.)

739. PROBLEM. To draw an arc of a great circle through a given point on the surface of a sphere and perpendicular to a given great circle.



Given: Point A on sphere O, and great circle BC, whose pole is P.

Required: To draw through A an arc of a great circle $\perp$ to the great circle BC.

Construction: Place one point of the compasses at A, and with an opening equal to the chord of a quadrant of the given sphere describe an arc of a great circle intersecting the given great circle at D.

Now, place one point of the compasses at D and similarly draw arc of great circle PAE. Draw PD, the arc of great circle (by 738).

Statement: Arc PAE is $\perp$ to circumference $BEDC$. Q.E.F.

Proof: ED, EP, and PD are quadrants (?) (724).

$\therefore$ E is the pole of arc PD (?) (725).

Hence, $\angle PED$ is measured by quadrant PD (?) (726).

$\therefore$ $\angle PED$ is a right angle (247).

That is, arc PAE is $\perp$ to circumference $BEDC$. Q.E.D.

SPHERICAL TRIANGLES

740. A **spherical triangle** is a portion of the surface of a sphere bounded by three arcs of great circles.

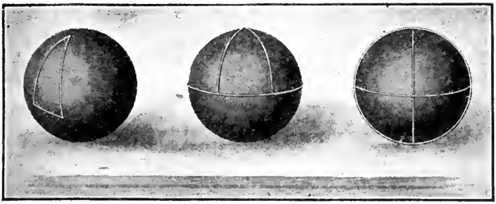
The bounding arcs are the **sides** of the triangle.

The intersections of the sides are the **vertices** of the triangle.

The angles formed by the sides are the **angles** of the triangle.

Spherical triangles are equilateral, equiangular, isosceles, scalene, acute, right, obtuse, under the same conditions as in plane triangles.

741. A **birectangular spherical triangle** is a spherical triangle, two of whose angles are right angles.

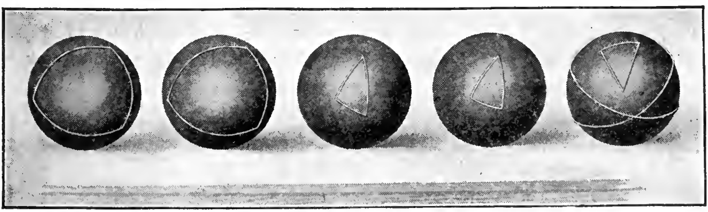


A **trirectangular spherical triangle** is a spherical triangle all of whose angles are right angles.

The **unit** usually employed in measuring the sides of a spherical triangle is the degree.

It is obvious that the circumferences of three great circles divide the surface of a sphere into eight spherical triangles.

742. Two spherical triangles are **mutually equilateral** if their sides are equal each to each; and they are **mutually equiangular** if their angles are equal each to each.

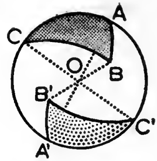


MUTUALLY EQUILATERAL SPHERICAL TRIANGLES — MUTUALLY EQUIANGULAR SPHERICAL TRIANGLES — POLAR TRIANGLES

743. If three great circles are described, having as their poles the vertices of a spherical triangle, one of the eight triangles thus formed is the **polar triangle** of the first.

The polar triangle is the one whose vertices are nearest the vertices of the original triangle.

744. If the diameters of a sphere are drawn to the vertices of a spherical triangle, the original triangle, and the triangle whose vertices are the opposite ends of these diameters, are **symmetrical spherical triangles.**



745. A **spherical polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles.

Two spherical polygons are **equal** if they can be made to coincide. The **diagonal** of a spherical polygon is the arc of a great circle connecting two vertices not in the same side.

Only *convex* spherical polygons are considered in this book.

PRELIMINARY THEOREMS

746. THEOREM. **The planes of the sides of a spherical triangle form a trihedral angle:**

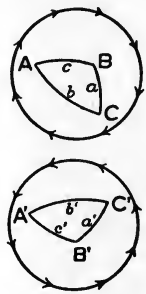
I. Whose vertex is the center of the sphere (?) (702).

II. Each of whose face angles is measured by the intercepted side of the triangle (?) (245).

III. Each of whose dihedral angles is equal to the corresponding angle of the triangle (?) (728).

747. THEOREM. **A pair of symmetrical spherical triangles are mutually equilateral and mutually equiangular.** (See 51, 206, 537, 728.)

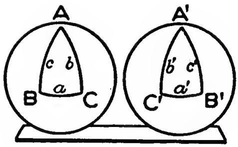
748. THEOREM. **The homologous parts of a pair of symmetrical spherical triangles are arranged in reverse order.**



Proof: If the eye is at the center of the sphere, the order of the vertices A, B, C is the same in direction as the motion of the hands of a clock. But the order of A', B', C' is in the opposite direction. (See 556, Note.) Hence, the parts are arranged in reverse order. Q.E.D.

749. THEOREM. **The homologous parts of two symmetrical spherical triangles are equal.** (747.)

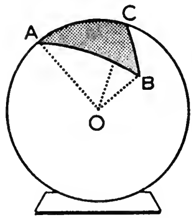
750. Theorem. Two symmetrical isosceles spherical triangles can be superposed and are equal.



Proof: The method of superposition, as in the case of plane triangles.

THEOREMS AND DEMONSTRATIONS

751. Theorem. One side of a spherical triangle is less than the sum of the other two.



Given: (?). **To Prove:** $AB < AC + BC$.

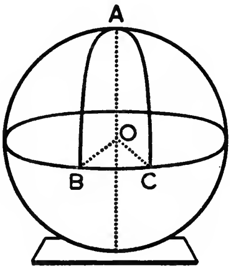
Proof: Draw radii OA, OB, OC.

In the trihedral $\angle\ O$, $\angle\ AOB < \angle\ AOC + \angle\ BOC$ (?) (563).

$\angle\ AOB$ is measured by arc AB, etc. (?).

$\therefore$ arc $AB <$ arc $AC +$ arc BC (Ax. 6). Q.E.D.

752. Theorem. In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the third angle is measured by the third side.



Given: Birectangular $\triangle\ ABC$; $\angle\ B$ and $\angle\ C$, right $\angle$s.

To Prove: I. AB and AC quadrants.

II. $\angle\ A$ is measured by arc BC.

Proof: I. Draw radii OA, OB, OC.

Arc AB is $\perp$ to arc BC and arc AC is $\perp$ to arc BC (Hyp.).

$\therefore A$ is the pole of arc BC (?) (729).

$\therefore AB$ and AC are quadrants (?) (724).

II. $\angle\ A$ is measured by arc BC (?) (726). Q.E.D.

753. Theorem. If two sides of a spherical triangle are quadrants, the triangle is birectangular.

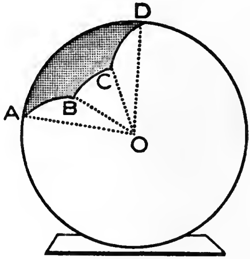
Proof: A is the pole of BC (?) (725).

$\therefore \angle$s B and C are rt. $\angle$s (?) (727). Q.E.D.

754. THEOREM. The three sides of a trirectangular spherical triangle are quadrants.

755. THEOREM. The sum of the sides of any spherical polygon is less than 360°.

Given: (?). **To Prove:** (?).



Proof: Draw radii to the several vertices of the polygon, forming the polyhedral $\angle O$.

Then, $\angle AOB + \angle BOC + \angle COD + \angle AOD < 360°$ (?) (564).

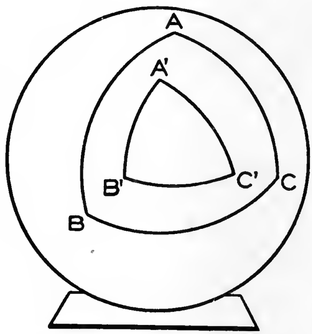
But $\angle AOB$ is measured by arc AB (?), etc.

$\therefore$ arc AB + arc BC + arc CD + arc $AD < 360°$ (Ax. 6). Q.E.D.

756. COR. The sum of the sides of any spherical polygon is less than the circumference of a great circle.

757. THEOREM. If one spherical triangle is the polar of a second triangle, then the second is the polar of the first.

Given: Spherical $\triangle ABC$ and its polar $\triangle A'B'C'$.



To Prove: $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

Proof: A is the pole of arc $B'C'$ (Hyp.).

$\therefore B'$ is the distance of a quadrant from A (?) (724).

C is the pole of arc $A'B'$ (?).

$\therefore B'$ is the distance of a quadrant from C (?).

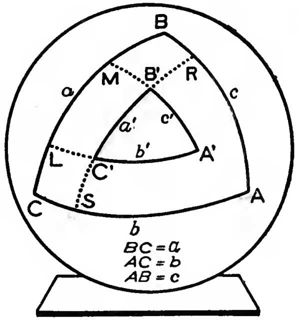
Hence, B' is the pole of arc AC (?) (725).

Likewise, A' is the pole of BC, and C' is the pole of AB.

$\therefore \triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$ (?) (743). Q.E.D

758. THEOREM. In two polar spherical triangles each angle of one and the opposite side of the other are supplementary.

Given: Polar $\triangle$ ABC and $A'B'C'$.



To Prove:

$\angle A + a' = 180°$; $\angle A' + a = 180°$;

$\angle B + b' = 180°$; $\angle B' + b = 180°$;

$\angle C + c' = 180°$; $\angle C' + c = 180°$.

Proof: Prolong arc $B'C'$ to meet arc AB at R and arc AC at S.

$B'S = 90°$ and $C'R = 90°$ (?) (724).

$\therefore B'S + C'R = 180°$ (Ax. 2).

That is, $C'S + B'C' + C'R$ or $RS + B'C' = 180°$ (Ax. 4).

Now, RS is the measure of $\angle A$ (?) (726), and $B'C' = a'$.

$$\therefore \angle A + a' = 180° \text{ (Ax. 6)}.$$

Similarly, $\angle B + b' = 180°$; $\angle C + c' = 180°$.

Again, prolong arcs $A'B'$ and $A'C'$ to meet arc BC at M and L. $BL = 90°$ and $CM = 90°$ (?) (724).

$$\therefore BL + CM = 180° \text{ (Ax. 2)}.$$

That is, $LM + MB + CM = 180°$, or $LM + BC = 180°$ (Ax. 4).

Now, LM is the measure of $\angle A'$ (?), and $BC = a$.

$$\therefore \angle A' + a = 180° \text{ (Ax. 6)}.$$

Similarly, $\angle B' + b = 180°$; $\angle C' + c = 180°$. Q.E.D.

759. THEOREM. In two polar spherical triangles each angle of one is measured by the supplement of the opposite side of the other.

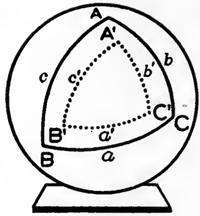
Proof: Identical with the proof of 758.

Ex. 1. What is the locus of the centers of those spherical surfaces that pass through two given points?

Ex. 2. What is the locus of the centers of the spherical surfaces of given radius, that contain two given points?

Ex. 3. What is the locus of the centers of the spherical surfaces that pass through three given points?

760. THEOREM. **The sum of the angles of a spherical triangle is greater than 180° and less than 540°.**



Given: A spherical $\triangle ABC$.

To Prove: I. $\angle A + \angle B + \angle C > 180°$;
II. $\angle A + \angle B + \angle C < 540°$.

Proof: I. Construct $\triangle A'B'C'$, the polar triangle of $\triangle ABC$.

$\angle A + a' = 180°, \angle B + b' = 180°, \angle C + c' = 180°$ (?) (758).
Adding, $\angle A + \angle B + \angle C + a' + b' + c' = 540°$ (Ax. 2).
But, $a' + b' + c' < 360°$ (?) (755).
Subtracting, $\angle A + \angle B + \angle C > 180°$ (Ax. 9).

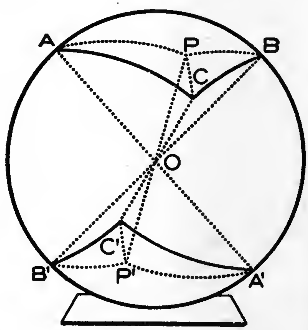
II. Again, $\angle A + \angle B + \angle C + a' + b' + c' = 540°$ (Ax. 2).
But, $a' + b' + c' > 0°$ (?) (740).
Subtracting, $\angle A + \angle B + \angle C < 540°$ (Ax. 9).

Q.E.D.

761. COR. **The sum of the angles of a spherical triangle is greater than two right angles and less than six right angles.**

762. COR. **A spherical triangle may have one, two, or three obtuse angles.**

763. THEOREM. **Two symmetrical spherical triangles are equivalent.**



Given: Two symmetrical spherical $\triangle$ ABC and $A'B'C'$.

To Prove: $\triangle ABC \approx \triangle A'B'C'$.

Proof: Suppose P is the pole of the circle containing A, B, C. Draw the diameters AA', BB', CC', PP', and the arcs of great circles, PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

$\angle POA = \angle P'OA'$ (?).
$\therefore$ arc $PA =$ arc $P'A'$ (?) (206).

Likewise, arc $PB =$ arc $P'B'$ and arc $PC =$ arc $P'C'$.

But $PA = PB = PC$ (?) (723). $\therefore P'A' = P'B' = P'C'$ (Ax. 1).

$$\left.\begin{aligned}\text{Hence, } \triangle APB &= \triangle A'P'B' \\ \triangle APC &= \triangle A'P'C' \\ \triangle BPC &= \triangle B'P'C'\end{aligned}\right\}\text{(?) (750).}$$

Adding, $\triangle ABC \approx \triangle A'B'C'$ (Ax. 2). Q.E.D.

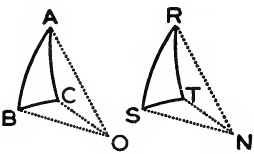
NOTE. If the pole P should be without the triangle ABC, one of the pairs of equal triangles would be without the original triangles and would be subtracted from the sum of the others to obtain triangles ABC and $A'B'C'$.

764. THEOREM. **Provided two spherical triangles on the same sphere** (or on equal spheres) **have their parts arranged in the same order, they are equal:**

I. If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

II. If a side and the two angles adjoining it of one are equal respectively to a side and the two angles adjoining it in the other.

III. If three sides of the one are equal respectively to three sides of the other.



Given: (?).

To Prove: (?).

Proof: I and II. Superposition as in plane triangles.

III. Draw radii of the sphere to all the vertices of the triangles.

The face $\measuredangle$s of the trih. $\angle O =$ the face $\measuredangle$s of the trih. $\angle N$, respectively. (Explain.)

Hence, trih. $\angle O =$ trih. $\angle N$ (?) (561).

$\therefore$ dih. $\angle OA =$ dih. $\angle NR$; dih. $\angle OB =$ dih. $\angle NS$; etc.

$\therefore$ the $\triangle$s are mutually equiangular (?) (746, III).

Hence, the $\triangle$s can be made to coincide.

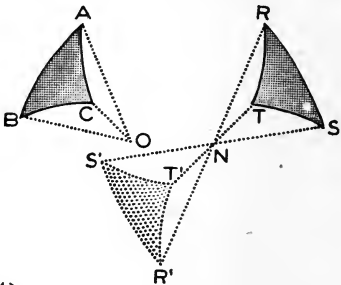
$\therefore \triangle ABC \doteq \triangle RST$ (?) (28). Q.E.D.

765. Theorem. Provided two spherical triangles on the same sphere (or on equal spheres) **have their parts arranged in reverse order, they are symmetrical:**

I. If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

II. If a side and the two angles adjoining it of one are equal respectively to a side and the two angles adjoining it of the other.

III. If three sides of one are equal respectively to three sides of the other.



Proof: In each of these cases construct a third spherical triangle, $R'S'T'$, symmetrical to the $\triangle\ RST$.

Then $\triangle\ R'S'T'$ will have its parts equal to the parts of $\triangle\ ABC$ and arranged in the same order. (Explain.)

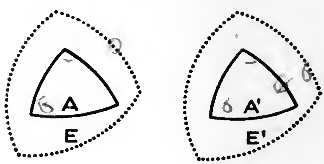
$\therefore\ \triangle\ R'S'T' = \triangle\ ABC$ (?) (764).

Hence, $\triangle\ RST$ is symmetrical to $\triangle\ ABC$ (Ax. 6). Q.E.D.

766. Cor. Two mutually equilateral spherical triangles are mutually equiangular and are equal or symmetrical.

When are they *equal?* When are they *symmetrical?*

767. Theorem. Two mutually equiangular spherical triangles on the same sphere (or on equal spheres) **are mutually equilateral, and are equal or symmetrical.**



Given: $\triangle$ A and A', mutually equiangular.

To Prove: $\triangle$ A and A' mutually equilateral, and equal or symmetrical.

Proof: Construct $\triangle$ E and E', the polar $\triangle$ of A and A'.

The sides of E are supplements of the $\angle$ of A and the sides of E' are supplements of the $\angle$ of A' $\Big\}$ (?) (758).

But the $\angle$s of A are $=$ respectively to the $\angle$s of A' (Hyp.).

$\therefore$ $\triangle$s E and E' are mutually equilateral (?) (49).

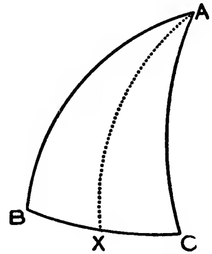
Hence, $\triangle$s E and E' are mutually equiangular (?) (766).

Again, $\triangle$s A and A' are the polar $\triangle$s of E and E' (?) (757).

$\left.\begin{array}{l}\therefore \text{ the sides of } A \text{ are supplements of the } \angle\text{s of } E \\ \text{and the sides of } A' \text{ are supplements of the } \angle\text{s of } E'\end{array}\right\}$ (?).

Hence, $\triangle$s A and A' are mutually equilateral (?). Also they are equal (when?); or symmetrical (when?). Q.E.D.

768. Theorem. The angles opposite the equal sides of an isosceles spherical triangle are equal.



Given: (?). **To Prove:** $\angle B = \angle C$.

Proof: Suppose X the midpoint of BC.

Draw AX, the arc of a great circle.

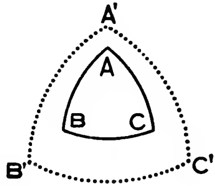
Now the two spherical $\triangle$s ABX and ACX are mutually equilateral. (Explain.)

$\therefore$ they are mutually equiangular and symmetrical (?) (766).

$\therefore \angle B = \angle C$ (?) (749). Q.E.D.

769. Cor. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the midpoint of the base bisects the vertex-angle and is perpendicular to the base. (See 749.)

770. Theorem. If two angles of a spherical triangle are equal, the sides opposite are equal.



Given: (?). **To Prove:** (?).

Proof: Construct $\triangle A'B'C'$, the polar triangle of $\triangle ABC$.

$\left.\begin{array}{l}\text{Then, } A'B' \text{ is the supplement of } \angle C \\ \text{and } A'C' \text{ is the supplement of } \angle B\end{array}\right\}$ (?).

$\therefore A'B' = A'C'$ (?) (49).

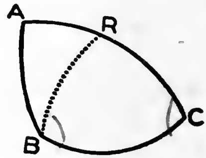
Therefore, $\angle B' = \angle C'$ (?) (768).

Again, $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$ (?) (757).

And AB is the supplement of $\angle C'$, and AC of $\angle B'$ (?).

$\therefore AB = AC$ (?). Q.E.D

771. THEOREM. **If two angles of a spherical triangle are unequal, the sides opposite are unequal and the greater side is opposite the greater angle.**



Given: $\triangle ABC$; $\angle ABC > \angle C$.

To Prove: $AC > AB$.

Proof: Suppose BR, the arc of a great circle, drawn, making $\angle CBR = \angle C$ and meeting AC at R.

Now $AR + BR > AB$ (?) (751). But $BR = CR$ (?) (770).

$\therefore AR + CR > AB$ (Ax. 6). That is, $AC > AB$. Q.E.D.

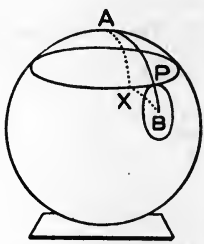
772. THEOREM. **If two sides of a spherical triangle are unequal, the angles opposite are unequal and the greater angle is opposite the greater side.**

To Prove: $\angle ABC > \angle C$.

Proof: $\angle ABC$ is either $< \angle C$ or $= \angle C$ or $> \angle C$.

Continue by method of exclusion (88).

773. THEOREM. **If the circumferences of two circles on a sphere contain a point on the arc of a great circle that joins their poles, they have no other point in common.**



Given: Point P on the arc AB of a great circle of a sphere, and P common to two circumferences whose poles are A and B.

To Prove: P is the only point common to these circumferences.

Proof: Suppose X is another common point.

Draw arcs of great ⓢ, AX and BX.

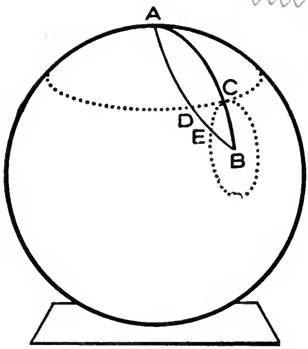
$$\begin{array}{lrl} & \text{Then } AX + BX > APB & \text{(?) (751).} \\ & \text{But } AX \qquad = AP & \text{(?) (723).} \\ \cline{2-2} \text{Subtracting,} & BX > BP & \text{(?) (Ax. 7).} \end{array}$$

That is, X is without the $\odot B$ and cannot be in both the circumferences. Q.E.D.

774. **Cor.** **If two circumferences on a sphere touch each other at one point, and only one, the arc of a great circle joining their poles contains their common point.**

775. **Theorem.** **The shortest line that can be drawn on the surface of a sphere, between two points on the surface, is the less arc of the great circle containing the two points.**

Omit



Given: Points *A* and *B*, and *AB* the arc of a great ⊙ joining them; line *ADEB*, any other line on the surface of the sphere, between *A* and *B*.

To Prove: Arc *AB* is the shortest line on the surface, that can be drawn connecting *A* and *B*.

Proof: Take on arc *AB* any point *C*, and describe two circumferences through *C*, having *A* and *B* as their poles, and intersecting *ADEB* at *D* and *E*. Point *C* is the only point common to these two ⓢ (?) (773).

No matter what kind of line *AD* is, a line of equal length can be drawn from *A* to *C*, on the surface; and a line can be drawn from *B* to *C* equal in length to *BE*.

[Imagine *AD* revolved on the surface of the sphere, using *A* as a pivot, and *D* will move along the circumference to coincide with point *C*. Similarly with *BE*.]

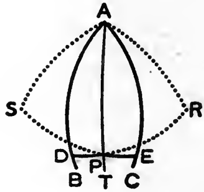
There is now a line from *A* to *B*, through *C*, < *ADEB*.

That is, whatever the nature of *ADEB*, there is a shorter line from *A* to *B*, which contains *C*, *any* point of arc *AB*.

Thus the shortest line contains all the points of *AB* and therefore *is the line AB*. Q.E.D.

Note. This theorem justifies the definition of the "distance" between two points, etc., in 705.

776. Theorem. Any point in the arc of a great circle that bisects a spherical angle is equally distant from the sides of the angle.



Given: Spherical $\angle BAC$; arc AT bisecting it; any point P, of arc AT; PD and PE, arcs of great Ⓢ ⊥ to AB and AC, respectively.

To Prove: Arc $PD =$ arc PE.

Proof: The arcs PD and PE, if prolonged, will pass through R and S, respectively, the poles of AB and AC (?) (729).

Draw arcs AR and AS.

Now, $\angle DAR = \angle EAS$. [Each is a right $\angle$; (727).]

$\angle DAP = \angle EAP$ (Hyp.). Subtracting,

$\angle RAP = \angle SAP$ (Ax. 2).

Also $RA = SA$ (?) (724), and $AP = AP$.

$\therefore \triangle RAP$ is symmetrical to $\triangle SAP$ (?) (765, I).

$\therefore RP = SP$ (?). But $RD = SE$. (Each is a quadrant.)

$\therefore PD = PE$ (Ax. 2). Q.E.D.

777. Theorem. Any point on the surface of a sphere and equally distant from the sides of a spherical angle is in the arc of a great circle that bisects the angle. (The proof is similar to the proof of 776.)

ORIGINAL EXERCISES

1. Vertical spherical angles are equal.

2. If two spherical triangles, on the same or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.

3. The polar triangle of an isosceles spherical triangle is isosceles.

4. The polar triangle of a birectangular spherical triangle is birectangular.

5. If two dihedral angles of a trihedral angle are equal, the opposite face angles also are equal.

Proof: Construct a sphere having the vertex as center, etc.

6. If two face angles of a trihedral angle are equal, the opposite dihedral angles also are equal.

7. A trirectangular spherical triangle is its own polar triangle.

8. Two symmetrical spherical polygons are equivalent.

9. Any side of a spherical polygon is less than the sum of the other sides. [Draw diagonals from a vertex.]

10. If the three face angles of a trihedral angle are equal, the three dihedral angles also are equal.

11. State and prove the converse of No. 10.

12. A straight line cannot meet a spherical surface in more than two points.

13. If two dihedral angles of a trihedral angle are unequal, the opposite face angles are unequal, and the greater face angle is opposite the greater dihedral angle.

14. State and prove the converse of No. 13.

15. Two lines tangent to a sphere at a point determine a plane tangent to a sphere at the same point.

16. All the tangent lines drawn to a sphere from an external point are equal.

17. The volume of any tetrahedron is equal to one third the product of its total surface by the radius of the inscribed sphere.

18. Every point in the circumference of a great circle that is perpendicular to an arc at its midpoint is equally distant from the ends of the arc.

19. The points of contact of all lines tangent to a sphere from an external point lie in the circumference of a circle.

20. The arcs of great circles perpendicular to the sides of a spherical triangle at their midpoints meet in a point equally distant from the vertices.

21. If the opposite sides of a spherical quadrilateral are equal, the opposite angles are equal.

22. If the opposite sides of a spherical quadrilateral are equal, the diagonals bisect each other.

23. If the diagonals of a spherical quadrilateral bisect each other, the opposite sides are equal.

24. The exterior angle of a spherical triangle is less than the sum of the opposite interior angles.

25. The sum of the angles of a spherical quadrilateral is more than four right angles and less than eight right angles.

26. If two spheres are tangent to each other, the straight line joining their centers passes through the point of contact.

27. The sum of the angles of a spherical polygon is more than $2n-4$ right angles and less than $2n$ right angles.

28. The arcs of great circles bisecting the angles of a spherical triangle meet in a point.

29. A circle may be inscribed in any spherical triangle.

30. If a tangent line and a secant be drawn to a sphere from an external point, the tangent is a mean proportional between the whole secant and the external segment.

31. The product of any secant that can be drawn to a sphere from an external point, by its external segment, is constant for all secants drawn through the same point.

32. If two spherical surfaces intersect and a plane be passed containing their intersection, tangents from any point in this plane to the two spherical surfaces are equal.

---

33. Find the distance from the center of a sphere whose radius is 15 to the plane of a small circle whose radius is 8.

34. The polar distance of a small circle is 60° and the radius of the sphere is 12 in. Find the radius of the circle.

35. The total surface of a tetrahedron is 90 sq. m., and the radius of the inscribed sphere is 4 m. Find the volume of the tetrahedron.

36. Find the radius of the sphere inscribed in a tetrahedron whose volume is 250 and total surface is 150.

37. Find the total surface of a tetrahedron whose volume is 320, if the radius of the inscribed sphere is 8.

38. Find the radius of the sphere inscribed in a regular tetrahedron whose edges are each 10 in.

39. Find the radius of the sphere circumscribed about a regular tetrahedron whose edges are each 18 in.

40. Find the radii of the spheres inscribed in and circumscribed about a cube whose edges are each 10 in.

41. The sides of a spherical triangle are 60°, 80°, 110°. Find the angles of its polar triangle.

42. The angles of a spherical triangle are 74°, 119°, 87°. Find the sides of its polar triangle.

43. The chord of the polar distance of the circle of a sphere is 12, and the radius of the sphere is 9. Find the radius of the circle.

44. The polar distance of a circle is 60° and the diameter of the circle is 8. Find the diameter of the sphere.

[Denote by R, each side of an equilateral triangle whose altitude is 4.]

45. The radii of two spherical surfaces are 11 in. and 13 in., and their centers are 20 in. apart. Find the radius of the circle of their intersection. Find also the distances from the centers of the spheres to the center of this circle.

46. The radii of two spherical surfaces are 20 m. and 37 m., and the distance between their centers is 19 m. What is the length of the diameter of their intersection?

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47. To bisect an arc of a great circle.

48. To draw an arc of a great circle perpendicular to a given arc of a great circle through a given point in the arc.

49. To bisect a spherical angle.

50. To bisect an arc of a small circle.

51. To circumscribe a circle about a given spherical triangle.

52. To construct a spherical angle equal to a given spherical angle at a given point on the same sphere.

53. To construct a spherical triangle having the three sides given.

54. To construct a spherical triangle having the three angles given.

55. To construct a plane tangent to a sphere at a given point on the surface.

56. To construct a spherical surface having the radius given and containing three given points.

57. To construct a spherical surface that shall have a given radius, touch a given plane, and contain two given points.

58. To construct a spherical surface that shall have a given radius, shall be tangent to a given sphere, and contain two given points.

59. To construct a spherical surface that shall contain four given points.

60. To construct a plane that shall contain a given line and be tangent to a given sphere.

61. To construct a plane tangent to a given sphere and parallel to a given plane.

62. What is the locus of points on the surface of a sphere:

(*a*) Equally distant from two given points on the surface?

(*b*) Equally distant from two given points not on the surface?

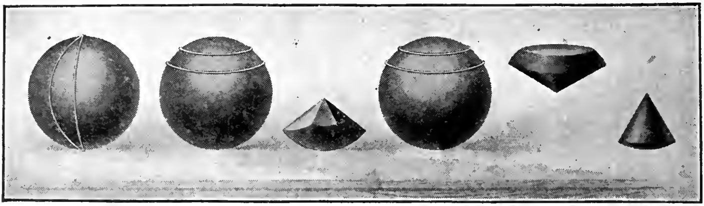
(*c*) Equally distant from two given intersecting arcs of great circles?

AREAS AND VOLUMES

778. A **lune** is a portion of the surface of a sphere bounded by two semicircumferences of great circles.

The points of intersection of the sides of a lune are the **vertices** of the lune.

The angles made at the vertices by the sides are the **angles** of the lune.



LUNE ZONE SPHERICAL PYRAMID SPHERICAL SEGMENT SPHERICAL SECTORS SPHERICAL CONE

779. A **zone** is a portion of the surface of a sphere bounded by the circumferences of two circles whose planes are parallel.

The **bases** of a zone are the circumferences bounding it.

The **altitude** of a zone is the perpendicular distance between the planes of its bases.

If one of the planes is tangent to the sphere the zone is a **zone of one base.**

780. A **spherical degree** is the one seven-hundred-and twentieth part of the surface of a sphere. If the surface of a sphere is divided into 720 equal parts, each part is a spherical degree.

The size of a spherical degree depends on the size of the sphere.

It may be easily conceived to be half a lune whose angle is a degree, that is, a birectangular spherical triangle whose third angle is 1°.

How many spherical degrees in a trirectangular spherical triangle?

781. The **spherical excess** of a spherical triangle is the sum of its angles less 180°. That is, $E = A + B + C - 180°$.

782. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of its sides.

The **vertex** of a spherical pyramid is the center of the sphere.

The **base** of a spherical pyramid is the spherical polygon.

783. A **spherical sector** is the solid generated by the revolution of the sector of a circle about any diameter of the circle as an axis.

The **base** of the spherical sector is the zone generated by the arc of the circular sector.

A **spherical cone** is a spherical sector whose base is a zone of one base.

784. A **spherical segment** is a portion of a sphere included between two parallel planes that intersect the sphere.

The **bases** of a spherical segment are the circles made by the parallel planes.

The **altitude** of a spherical segment is the perpendicular distance between the bases.

A **spherical segment of one base** is a segment, one of whose bounding planes is tangent to the sphere.

A **hemisphere** is a spherical segment of one base, and that base is a great circle.

A **spherical wedge** is a portion of a sphere bounded by a lune and the planes of its sides.

Ex. 1. What is the spherical excess of a spherical triangle whose angles are 60°, 70°, and 100°?

Ex. 2. Distinguish between a zone and a spherical segment.

Ex. 3. Find the area of a spherical degree on a sphere whose surface is 3600 sq. in.

Ex. 4. Find the area of a spherical triangle containing 80 spherical degrees, on a sphere whose surface is 450 sq. ft.

PRELIMINARY THEOREMS

785. Theorem. **Either angle of a lune is measured by the arc of a great circle described with the vertex of the lune as a pole, and included between the sides of the lune.** (See 726.)

786. Cor. **The angles of a lune are equal.**

787. Theorem. **Every great circle of a sphere divides the sphere into two equal hemispheres, and the surface into two equal zones.**

788. Theorem. **The spherical excess of a spherical n-gon is equal to the sum of its angles less $(n-2)$ 180°.**

Proof: By drawing diagonals from any vertex, the polygon is divided into $n-2$ ⟁.

For *one* △, $E = S - 180°$ (?) (781).

For *another* △, $E' = S' - 180°$ (?). Etc. for $(n-2)$ ⟁.

∴ by adding, the excess of all the ⟁ = the sum of all their ∡s $-(n-2)$ 180° (Ax. 2).

That is, the excess of a spherical polygon = the sum of its ∡s $-(n-2)$ 180°. Q.E.D.

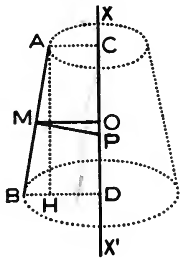
789. Theorem. **If a regular polygon having an even number of sides be inscribed in, or circumscribed about, a circle, and the figure be made to revolve about one of the longest diagonals of the polygon, the surface generated by the polygon will be composed of the surface of cones, cylinders, and frustums, and the surface generated by the circle will be a spherical surface.**

790. Theorem. **If a regular polygon having an even number of sides be inscribed in, or circumscribed about, a circle, and the figure be made to revolve about one of the longest diagonals of the polygon, the surface generated by the perimeter of the polygon will approach the surface of the sphere generated by the circle, as a limit, if the number of sides of the polygon is indefinitely increased.**

791. Theorem. **If a polyhedron be circumscribed about a sphere and the number of its faces be indefinitely increased, the surface of the polyhedron will approach the surface of the sphere as a limit, and the volume of the polyhedron will approach the volume of the sphere as a limit.**

THEOREMS AND DEMONSTRATIONS

792. Theorem. The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line upon the axis by the circumference of a circle whose radius is the line perpendicular to the revolving line at its midpoint, and terminating in the axis.



Given: Line AB revolving about axis XX'; $CD =$ projection of AB on XX'; $MP = a = \perp$ erected at midpoint of AB and terminating in XX'; $MO =$ radius of midsection.

To Prove:
Surface generated by $AB = CD \cdot 2\pi a$.

Proof: I. The surface generated by AB is the surface of the frustum of a right circular cone whose bases are generated by AC and BD, and the midsection, by MO.

Area of surface $= 2\pi MO \cdot AB$ (696).

Now, $\triangle ABH$ and MOP are similar (?) (321).

$\therefore MO : AH = MP : AB$ (?).

Hence, $MO \cdot AB = AH \cdot MP = CD \cdot a$ (?).

$\therefore$ area of surface $= 2\pi CD \cdot a = CD \cdot 2\pi a$ (Ax. 6).

II. If AB is $\parallel$ to XX', the surface is cylindrical and equals $CD \cdot 2\pi a$ (?) (665).

III. If AB meets XX' at C, the entire surface is conical and equals $\pi BD \cdot AB$ (?) (693).

Now $BD = 2MO$ (?) (142); and $MO \cdot AB = CD \cdot a$ (?).

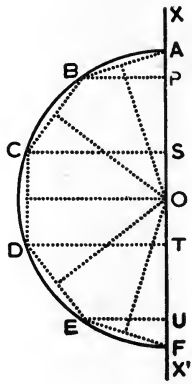
$\therefore \pi \cdot BD \cdot AB = \pi \cdot 2MO \cdot AB = \pi \cdot 2 \cdot CD \cdot a = CD \cdot 2\pi a$ (Ax. 6).

That is, the area of the surface $= CD \cdot 2\pi a$ (Ax. 6). Q.E.D.

Ex. 1. Find the spherical excess of a polygon whose angles are 80°, 110°, 140°, 130°, 160°.

Ex. 2. The spherical excess of a spherical polygon is the difference between the sum of its angles and the sum of the angles of a plane polygon having the same number of sides.

793. THEOREM. The surface of a sphere is equivalent to four great circles, that is, to $4\pi R^2$.



Given: Semicircle ACF; diameter AF; S = surface of sphere generated by revolving the semicircle about AF as an axis; R = radius of this sphere.

To Prove: $S = 4\pi R^2$.

Proof: Inscribe in this semicircle half of a regular polygon having an even number of sides. Draw the apothems and denote them by a. Draw the projections of the sides of the polygon on the diameter.

Now, if the figure revolve on AF as an axis,

$$\left.\begin{array}{l} \text{the surface } AB = AP \cdot 2\pi a \\ \text{the surface } BC = PS \cdot 2\pi a \\ \text{the surface } CD = ST \cdot 2\pi a \\ \quad \text{etc.} \qquad\qquad \text{etc.} \end{array}\right\} \text{(?) (792).}$$

Adding,

the entire surface $= (AP + PS + ST + \text{etc.}) \cdot 2\pi a$ (Ax. 2).

$= AF \cdot 2\pi a$ (Ax. 6).

Now if the number of sides of the polygon is indefinitely increased, the entire surface generated by the polygon will approach S (?) (790), and a will approach R as a limit (437).

Also $AF \cdot 2\pi a$ will approach $AF \cdot 2\pi R$.

$\therefore S = AF \cdot 2\pi R$ (?) (242). But $AF = 2R$ (?).

$\therefore \boldsymbol{S = 4\pi R^2}$ (Ax. 6). Q.E.D.

794. THEOREM. The area of a spherical degree equals $\frac{4\pi R^2}{720}$ (780).

795. THEOREM. The areas of the surfaces of two spheres are to each other as the squares of their radii and as the squares of their diameters. (See 793.)

Ex. 1. What is the area of the surface of a sphere whose radius is 10 in.? What is the area of a spherical degree on this sphere?

796. **THEOREM. The area of a zone is equal to the product of its altitude by the circumference of a great circle.**

Proof: The area generated by chord BC (Fig. of 793) $= PS \cdot 2\pi a$ (?) (792).

If the number of sides of the inscribed polygon is indefinitely increased, the length of chord BC will approach arc BC and the surface generated by chord BC will approach the area of a zone.

Also, $PS \cdot 2\pi a$ will approach $PS \cdot 2\pi R$ (?).

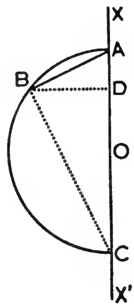
Hence, area of zone $BC = PS \cdot 2\pi R$ (?) (242).

If the altitude of the zone $= h$,

area of zone $= 2\pi R \cdot h$. Q.E.D.

797. **THEOREM. The area of a zone of one base is equal to the area of a circle whose radius is the chord of the generating arc.**

Given: Arc AB of semicircle ABC; diameter AC; chord AB.



To Prove:

Area of zone generated by arc $AB = \pi \overline{AB}^2$.

Proof: Area of zone $AB = AD \cdot 2\pi R$ (796).

That is, area of zone $AB = \pi \cdot AD \cdot 2R$.

Draw chord BC. $\triangle ABC$ is a rt. $\triangle$ (?).

$\therefore AD \cdot AC = \overline{AB}^2$ (?) (342).

That is, $AD \cdot 2R = \overline{AB}^2$ (Ax. 6).

Hence, area of zone $AB = \pi \overline{AB}^2$ (Ax. 6).

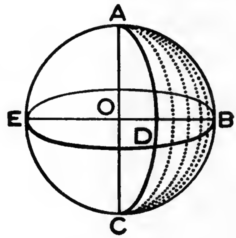
That is, **area of zone of one base $= \pi(\text{chord})^2$.** Q.E.D.

Ex. 1. On a sphere whose radius is 6 in., find the area of a zone $2\frac{1}{2}$ in. in height.

Ex. 2. What does the formula for the area of a zone become when the altitude becomes the diameter?

Ex. 3. What does the formula for the area of a zone of one base become when the generating arc becomes a semicircumference?

798. Theorem. **The area of a lune is to the area of the surface of its sphere as the angle of the lune is to 360°.**



Given: Lune $ABCDA$ on sphere O; L = area of lune; S = area of sphere; great ⊙ EB whose pole is A.

To Prove: $L : S = \angle A : 360$.

Proof: I. If arc BD and the circumference of ⊙ EB are **commensurable.**

There exists a common unit of measure. Suppose this unit contained 5 times in BD; 32 times in the circumference. $\therefore$ arc BD : circumference $= 5 : 32$ (?).

That is, arc $BD : 360 = 5 : 32$. Arc BD measures $\angle A$ (726).

$\therefore \angle A : 360 = 5 : 32$ (Ax. 6).

Pass great circles through the several points of division of circumference EB and vertex A, dividing the surface of the sphere into 32 equal lunes. Then, $L : S = 5 : 32$ (Ax. 3).

Hence, $L : S = \angle A : 360$ (Ax. 1). Q.E.D.

II. If the arc and circumference are **incommensurable.** The proof is similar to that found in 244, 302, 368, 539, etc.

799. Theorem. **The number of spherical degrees in the area of a lune is double the number of degrees in its angle.**

Proof: Let $L°$ denote the area of the lune, expressed in spherical degrees. Then, $L° : S = \angle A : 360$ (?) (798).

That is, $L° : 720 = \angle A : 360$ (Ax. 6).

Hence, $\boldsymbol{L° = 2\angle A}$. Q.E.D.

800. Formula. **The area of a lune** $= \dfrac{\pi R^2}{90} \times \angle A$.

Proof: $L : S = \angle A : 360$ (?) (798), and $S = 4\pi R^2$ (?).

$\therefore L : 4\pi R^2 = \angle A : 360$ (?). $\therefore \boldsymbol{L = \dfrac{\pi R^2}{90} \times \angle A}$. Q.E.D.

Note. The unit of measure in this formula is the square unit.

801. THEOREM. Two lunes on the same or equal spheres are to each other as their angles.

Proof: $L : S = \angle A : 360$, and $L' : S = \angle A' : 360$ (?) (798).

Dividing, $L : L' = \angle A : \angle A'$ (Ax. 3). Q.E.D.

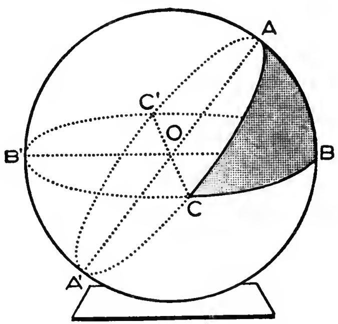
802. THEOREM. Two lunes on unequal spheres, having equal angles, are to each other as the squares of the radii of the spheres.

Proof: $L : S = \angle A : 360$, and $L' : S' = \angle A : 360$ (?).

Hence, $L : S = L' : S'$ (Ax. 1).

$\therefore L : L' = S : S' = R^2 : R'^2$ (292 and 795).

803. THEOREM. The number of spherical degrees in a spherical triangle is equal to the spherical excess of the triangle.



Given: Spherical $\triangle ABC$ on sphere O; spherical excess of the $\triangle = E$.

To Prove: Number of spherical degrees in $\triangle ABC = E$.

Proof: Continue the sides of the $\triangle ABC$ to form the lunes $ABA'CA$, $BAB'CB$, $CAC'BC$; draw diameters AA', BB', CC'.

$\triangle ABC' \approx \triangle A'B'C$ (?) (763).

Lune $CAC'BC = \triangle ABC + \triangle AC'B \approx \triangle ABC + \triangle A'B'C$ (Ax. 6).

Now, $\triangle ABC + \triangle A'B'C \approx$ lune $CAC'BC$,

and $\triangle ABC + \triangle A'BC =$ lune $ABA'CA$, $\Big\}$ (Ax. 4).

and $\triangle ABC + \triangle AB'C =$ lune $BAB'CB$ $\Big\}$ Adding,

$2\triangle ABC + \triangle ABC + \triangle A'B'C + \triangle A'BC + \triangle AB'C$

$\approx$ lune A + lune B + lune C (Ax. 2).

But, $\triangle ABC + \triangle A'B'C + \triangle A'BC + \triangle AB'C = 360$ spherical degrees (?) (713).

Lune A + lune B + lune $C = 2\angle A + 2\angle B + 2\angle C$ (799).

$\therefore 2\triangle ABC + 360 = 2\angle A + 2\angle B + 2\angle C$ (Ax. 6).

Hence, $\triangle ABC = \angle A + \angle B + \angle C - 180 = E$

(Ax. 2, Ax. 3, 781). Q.E.D.

804. FORMULA. **The area of a spherical triangle** $= \dfrac{4\pi R^2}{720} \times E$.

Proof: Area of one spherical degree $= \dfrac{4\pi R^2}{720}$ (794).

There are E spherical degrees in a spherical $\triangle$ (?) (803).

$\therefore$ **the area of a spherical triangle** $= \dfrac{4\pi R^2}{720} \times E$. Q.E.D.

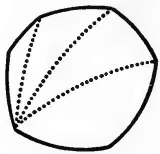
NOTES. The unit of measure in 803 is a spherical degree.

The unit of measure in 804 is a square unit (sq. in., sq. ft., etc.).

The formula of 804 reduces to $\dfrac{\pi R^2 E}{180}$.

The area of a spherical triangle is determined by its angles.

805. THEOREM. **The number of spherical degrees in a spherical polygon is equal to its spherical excess.**



Given: A spherical n-gon.

To Prove: The number of spherical degrees in this n-gon = the excess of the polygon.

Proof: From any vertex draw diagonals, dividing the polygon into $(n-2)$ △; the sums of the ∠s of these △ are denoted by s, s_1, s_2, . . . etc.

Now, the number of sph. degrees in one $\triangle = s - 180°$ (803);
the number of sph. degrees in another $\triangle = s_1 - 180°$ (?).
Etc., for $(n-2)$ △.

Adding, the number of sph. degrees in the n-gon
= the sum of its ∠s $- (n-2)\ 180°$ (Ax. 2).

Excess of the n-gon = sum of its ∠s $- (n-2)\ 180°$ (788).

$\therefore$ the number of spherical degrees in a spherical polygon = the excess of the polygon (Ax. 1). Q.E.D.

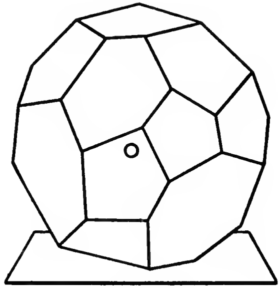
Ex. 1. Find the area of a spherical triangle whose angles are 80°, 125°, and 95°, on a sphere whose radius is 6.3 in.

Ex. 2. Find the area of a spherical polygon whose angles are 135°, 105°, 85°, 155°, 120°, on a sphere whose radius is 15 ft.

806. THEOREM. **The volume of a sphere** $=\frac{4\pi R^3}{3}$.

Given: Sphere O; radius $=R$; surface $=S$; volume $=V$.

To Prove: $V=\frac{4\pi R^3}{3}$.



Proof: Suppose a polyhedron circumscribed about the sphere, its surface denoted by S' and its volume by V'. Suppose planes be passed through the edges of the polyhedron and the center of the sphere, thus dividing the polyhedron into pyramids whose vertices are all at the center, and whose common altitude is R.

The volume of one such pyramid $=\frac{1}{3}R\cdot$ its base (?) (625).

$\therefore$ volume of *all* the pyramids $=\frac{1}{3}R\cdot$ sum of all their bases (Ax. 2); that is, $V'=\frac{1}{3}R\cdot S'$.

Indefinitely increase the number of faces of the polyhedron, thus indefinitely decreasing each face,

$$\left.\begin{array}{l}\text{and } V' \text{ will approach } V \text{ as a limit}\\ \text{and } S' \text{ will approach } S \text{ as a limit}\end{array}\right\}\text{(?) (791).}$$

Hence, $\frac{1}{3}R\cdot S'$ will approach $\frac{1}{3}R\cdot S$ as a limit (?).

Therefore, $V=\frac{1}{3}R\cdot S$ (?) (242). But $S=4\pi R^2$ (?).

$$\therefore V=\frac{4\pi R^3}{3} \text{ (Ax. 6).}$$ Q.E.D.

807. THEOREM. **The volumes of two spheres are to each other as the cubes of their radii or as the cubes of their diameters.**

Proof:

$$\frac{V}{V}=\frac{4\pi R^3}{3}\div\frac{4\pi R'^3}{3}=\frac{R^3}{R'^3}=\frac{(\frac{1}{2}D)^3}{(\frac{1}{2}D')^3}=\frac{D^3}{D'^3}.\quad\text{(Explain.)}$$

Q.E.D.

808. THEOREM. **The volume of a spherical pyramid is equal to one third the product of the polygon that is its base, by the radius of the sphere.**

Proof: Similar to the proof of 806.

809. **Theorem.** **The volume of a spherical wedge is to the volume of the sphere as the angle of its base is to 360.**

Proof: Similar to the proof of 798.

810. **Theorem.** **The volume of a spherical sector is equal to one third the product of the zone that is its base by the radius of the sphere.**

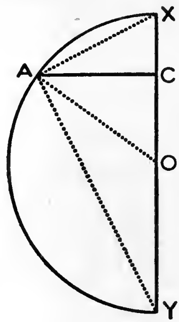
Proof: Similar to the proof of 806.

811. **Formulas.** Vol. of a spherical sector $= \frac{1}{3} R \cdot$ zone (810). But the zone $= 2\,\pi R \cdot h$ (?). Therefore,

1. **The volume of spherical sector** $= \frac{2}{3}\pi \boldsymbol{R^2 \cdot h}$ (Ax. 6).
2. **The volume of a spherical cone** $= \frac{2}{3}\pi \boldsymbol{R^2 \cdot h}$ (811, 1).
3. **The volume of a spherical wedge** $= \dfrac{\pi \boldsymbol{R^3} \times \angle A}{\mathbf{270}}$ (from 809).

812. **Problem.** **To find the volume of a spherical segment.**

1. Spherical segment of one base.



Given: Spherical segment generated by the figure ACX; semicircle XAY; $AC = r$; radius of sphere $= R$; altitude $= CX = h$.

Required: To find the volume of the spherical segment.

Computation: Draw chords AX, AY, and radius AO.

The right $\triangle\, ACO$ will generate a cone of revolution (?) (671).

The volume of spherical segment $ACX =$ volume of spherical cone OAX minus volume of cone ACO.

Volume of spherical cone $OAX = \frac{2}{3}\,\pi R^2 \cdot h$ (?) (811, 2);
volume of cone $ACO = \frac{1}{3}\,\pi r^2 \cdot CO$ (?) (695).

Now $r^2 = CX \cdot CY = h\,(2\,R - h)$ (?) (340, II); and $CO = R - h$. $\therefore$ vol. $ACO = \frac{1}{3}\,\pi h\,(2\,R - h)\,(R - h)$ (Ax. 6).

Hence, volume of spherical segment ACX

$= \frac{2}{3}\,\pi R^2 h - (\frac{2}{3}\,\pi R^2 h - \pi R h^2 + \frac{1}{3}\,\pi h^3)$ (Ax. 6). That is,

volume of spherical segment of one base $= \frac{1}{3}\,\pi \boldsymbol{h^2 (3\,R - h)}$.

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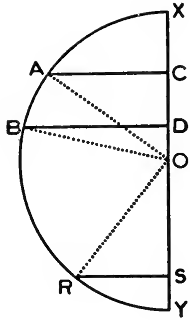


2. Spherical segment not including the center.

Given : Spherical segment generated by figure $ACDB$; semicircle $XABY$; $AC=r$; $BD=r'$; radius of sphere $=R$; altitude $=CD=h$.

Required: To find the volume of the spherical segment.

Computation : The △ ACO and BDO generate cones of revolution (?) (671).

The volume of spherical segment $ACDB$

= volume of spherical sector ABO
plus the volume of cone ACO
minus the volume of cone BDO.

Each of these volumes can be determined from formulas already established.

3. Spherical segment including the center.

Given : Spherical segment generated by figure $BDSR$; etc.

Required : (?). **Computation :** The same as that of case 2, except that both cones, BDO and RSO, are *added.*

ORIGINAL EXERCISES

1. Prove that the area of the surface of a sphere is equal to the square of the diameter multiplied by π, that is, $S=\pi D^2$.

2. Prove that the volume of a sphere is equal to one sixth the cube of the diameter multiplied by π, that is, $V=\frac{1}{6}\pi D^3$.

3. The surface of a sphere is equal to the cylindrical surface of the circumscribed cylinder.

4. The total surface of a hemisphere is three fourths the surface of the sphere.

5. The volume of a sphere is two thirds the volume of the circumscribed cylinder.

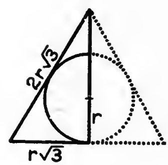
6. Upon the same circle as a base are constructed a hemisphere, a cylinder of revolution, and a cone of revolution, all having the same altitude. Prove that their total areas are $3\pi R^2$, $4\pi R^2$, $\pi R^2(1+\sqrt{2})$, respectively, and their volumes are $\frac{2}{3}\pi R^3$, πR^3, $\frac{1}{3}\pi R^3$, respectively.

7. Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.

8. The area of the surface of a sphere is equal to the area of the circle whose radius is the diameter of the sphere.

9. Show that the formula for the volume of a spherical segment of one base reduces to the correct formula for the volume of a hemisphere, when the base of the segment is a great circle; and to the correct formula for the volume of a sphere when the planes are both tangent.

10. In an equilateral triangle is inscribed a circle, and the figure is revolved about an altitude of the triangle as an axis. Prove,



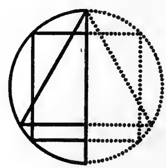
(*a*) That the surface generated by the circumference is two thirds the lateral surface generated by the triangle.

(*b*) That the volume generated by the circle is four ninths the volume generated by the triangle.

11. Derive a formula for the surface of a sphere, containing only V and π.

12. Derive a formula for the volume of a sphere, containing only S and π.

13. In a circle whose radius is R, there are inscribed a square and an equilateral triangle having their bases parallel; the whole figure is then revolved about the diameter perpendicular to the base of the triangle. Find, in terms of R,



(*a*) The total areas of the three surfaces generated;

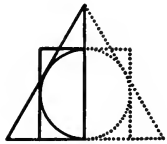
(*b*) The volumes of the three solids generated.

14. If a cylinder of revolution having its altitude equal to the diameter of its base, and a cone of revolution having its slant height equal to the diameter of its base are both inscribed in a sphere,

(*a*) The total area of the cylinder is a mean proportional between the area of the surface of the sphere and the total area of the cone;

(*b*) The volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

15. About a circle whose radius is R, there are circumscribed a square and an equilateral triangle having their bases in the same straight line. The whole figure is then revolved about an altitude of the triangle. Find, in terms of R,



(*a*) The total areas of the three surfaces generated.

(*b*) The volumes of the three surfaces generated.

16. If a cylinder of revolution having its altitude equal to the diameter of its base, and a cone of revolution having its slant height equal to the diameter of its base, be circumscribed about a sphere,

(*a*) The total area of the cylinder is a mean proportional between the area of the surface of the sphere and the total area of the cone;

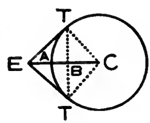
(*b*) The volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

17. The line joining the centers of two intersecting spherical surfaces is perpendicular to the plane of the intersection at the center of the intersection.

18. A cube and a sphere have equal surfaces; show that the sphere has the greater volume.

19. Prove that the parallel of latitude through a point having 30° north latitude bisects the surface of the northern hemisphere.

20. Prove that in order that the eye may observe one sixth of the surface of a sphere, it must be at a distance from the center of the sphere equal to $\frac{3}{2}$ of the radius.



Proof: Zone $TT = \frac{1}{6}$ surface of sphere (Hyp.).

$\therefore AB = \frac{1}{6}$ diam. $= \frac{1}{3}R$. Hence, $BC = \frac{2}{3}R$.

In rt. $\triangle ETC$, $\overline{TC}^2 = EC \cdot BC$ (?); $\therefore R^2 = EC \cdot \frac{2}{3}R$, or $EC = \frac{3}{2}R$ (Explain.) Q.E.D.

21. How many miles above the surface of the earth (diameter of earth $= 7960$ mi.) must a person be in order that he may see one sixth of the earth's surface?

22. If the area of a zone of one base is a mean proportional between the area of the remaining zone of the sphere and the area of the entire sphere, the altitude of the zone is $R(\sqrt{5} - 1)$.

23. The area of a lune is to the area of a trirectangular spherical triangle as the angle of the lune is to 45°.

24. A cone, a sphere, and a cylinder have the same diameters and altitudes. Prove that their volumes are in arithmetical progression.

25. The surface of a sphere bears the same ratio to the total surface of the circumscribed cylinder of revolution, as the volume of the sphere bears to the volume of the cylinder.

26. The smallest circle upon a sphere, whose plane passes through a given point within the sphere, is the circle whose plane is perpendicular to the diameter through the given point.

27. What part of the surface of the earth could one see if he were at the distance of a diameter above the surface?

28. Prove that if any number of lines in space be drawn through a point, and from any other point perpendiculars to these lines be drawn, the feet of all of these perpendiculars lie on the surface of a sphere.

29. The volume of a sphere is to the volume of the circumscribed cube as $\pi : 6$. The volume of a sphere is to the volume of the inscribed cube as $\pi : \frac{2}{3}\sqrt{3}$.

30. There are five spheres that touch the four planes of the faces of a tetrahedron.

31. If two angles of a spherical triangle are supplementary, the sides of the polar triangle, opposite these angles, are supplementary.

32. A square, whose side is a, is revolved about a diagonal, and also about an axis bisecting two opposite sides. Which of these figures contains the greater volume? Which has the greater surface?

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33. Find the area of the surface, and the volume of a sphere whose radius is 6.

34. Find the area of a zone whose altitude is 4 on a sphere whose radius is 14.

35. Find the area of a lune whose angle is 30° on a sphere whose radius is 8 in.

36. Find the area of a spherical triangle whose angles are 110°, 41°, 92°, on a sphere whose radius is 10.

37. Find the volume of a sphere whose radius is 5.

38. Find the volume of a spherical pyramid whose base is 35 sq. in., on a sphere whose radius is 12 in.

39. Find the area of a spherical polygon whose angles are 87°, 108°, 121°, 128°, on a sphere whose radius is 25.

40. What is the radius of a sphere whose surface is 1386 sq. yd.?

41. What is the radius of a sphere whose volume is $\frac{500\,\pi}{3}$ cu. in.?

42. What is the area of the surface of a sphere whose volume is $288\,\pi$ cu. ft.?

43. What is the volume of a sphere, the area of whose surface is 2464 sq. in.?

44. Find the area of a zone whose altitude is $3\frac{1}{2}$, if the radius of the sphere is $7\frac{1}{2}$.

45. Find the volume of a spherical sector the altitude of whose base is $5\frac{1}{4}$ in. if the radius of the sphere is 6 in.

46. Find the diameter, the circumference of a great circle, and the volume of a sphere the area of whose surface is $25\,\pi$ sq. ft.

47. By how many cubic inches is a 9-in. cube greater than a 9-in. sphere?

48. The radius of a sphere is 15, and the angles of the base of a spherical pyramid are 160°, 127°, 96°, 145°, and 117°. Find the volume of the pyramid.

49. A cylindrical vessel 10 in. in diameter contains a liquid. A metal ball is immersed in the liquid and the surface rises $\frac{5}{6}$ in. What is the diameter of the ball?

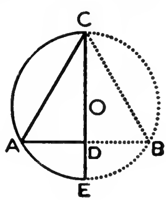
50. If a sphere 3 ft. in diameter weighs 99 lbs., what will a sphere of the same material 4 ft. in diameter weigh?

51. The radii of the bases of a frustum of a cone of revolution are 5 in. and 6 in., and the altitude of the frustum is $19\frac{1}{2}$ in. What is the diameter of an equivalent sphere?

52. What is the radius of a sphere whose surface is equivalent to the total surface of a right circular cylinder having an altitude equal to 21, and radius of the base equal to 6?

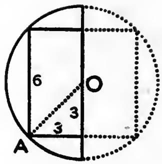
53. Find the volume generated by the revolution of an equilateral triangle inscribed in a circle whose radius is 8, about an altitude of the triangle as an axis. (See Fig. of Ex. 55.)

54. In the figure of No. 55, find the volume of the segment generated by the figure AED revolving about CD as an axis.



55. Find the area of the surface, and the volume of the sphere generated by a circle that is circumscribed about an equilateral triangle whose side is 10.

56. Circumscribing a sphere whose radius is 18, is a cylinder of revolution. Compare their total areas. Their volumes.



57. Circumscribing a cylinder of revolution whose altitude and diameter are each 6 in., is a sphere. Find the volume and area of the surface of the sphere.

58. Circumscribing a cylinder whose altitude is 4 and diameter is 3, is a sphere. Find the radius and volume of the sphere.

59. Each edge of a cube is 8 in. What is the area of the surface, and the volume of the circumscribed sphere?

60. Find the volume of one of the segments cut from a 10 in. sphere by the plane of one of the faces of the inscribed cube.

61. The volume of a certain sphere is $179\frac{2}{3}$ cu. ft. Find the radius of a sphere 8 times as large. Find the radius of a sphere 3 times as large.

62. The radius of a certain sphere is 5 in. What is the radius of a sphere twice as great? Half as great? Two thirds as great?

63. A hollow sphere has an outer diameter of 20 in., and an inner diameter of 16 in. Find the volume of the metal in the shell.

64. Find the diameter of that sphere whose volume is, numerically, equal to the area of its surface.

65. A projectile consists of a right circular cylinder having a hemisphere at each end. If the cylinder is 9 in. long and 7 in. in diameter, what is the volume of one projectile?

66. Inscribed in a regular tetrahedron whose edge is 4, and circumscribed about it are two spheres. Find their radii.

67. Find the radii of the spheres inscribed in and circumscribed about a regular hexahedron whose edge is 8 m.

68. Find the radii of the spheres inscribed in and circumscribed about a regular octahedron whose edge is 12 in.

69. How many spherical bullets $\frac{1}{2}$ in. in diameter can be made from a cube of lead 5 inches on each edge?

70. The area of a spherical triangle whose angles are 158°, 77°, 95°, is $288\frac{3}{4}$. Find the radius of the sphere.

71. The area of a spherical triangle whose excess is 75, is 135π. Find the radius of the sphere.

72. If the radius of a sphere is 2.5, and the sides of a triangle on it are 104°, 115°, 101°, find the area of the polar triangle.

73. In a trihedral angle the plane angles of the dihedral angles are 75°, 85°, 110°. Find the number of degrees of surface of a sphere, whose center is the vertex of the trihedral angle, inclosed by the faces of this trihedral angle.

74. What is the area of a spherical hexagon, each of whose angles is 145°, on a sphere whose radius is 15?

75. How many miles above the earth would a person have to be in order that he may see a third of its surface? One eighth of its surface?

76. Find the altitude of the zone whose area is equal to the area of a great circle of a sphere.

77. If the radius of a sphere is doubled, how is the amount of surface affected? The volume? The weight?

78. At a distance ($= d$) from the center of a sphere whose radius is r, is an illuminating point. What is the altitude of the zone illuminated?

79. On a sphere having a radius of 5 in. is an equiangular spherical triangle whose area is 5π sq. in. Find the angles of the triangle.

80. Find the area of the surface of a sphere whose volume is a cu. yd.

81. Find the volume of a sphere whose surface is a sq. yd.

82. If a circumference is described on the surface of a sphere, by a pair of compasses whose points are $2\frac{4}{5}$ in. apart, what is the area of the zone bounded by this circumference?

83. On a sphere the area of whose surface is 288 sq. ft. is a birectangular spherical triangle whose vertex angle is 100°. Find the area of this triangle.

84. Five inches from the center of a sphere whose diameter is two feet, a plane is passed. Find the areas of the two zones formed. Find the chords of their generating arcs.

85. The diameter of the moon is about 2000 mi.; that of the earth, about 8000 mi. How do their surfaces compare? Their volumes?

86. The radii of two concentric spheres are 12 and 13 in. A plane is tangent to the inner sphere. Find area of section of outer sphere.

87. If a solid sphere 4 ft. in diameter weighs 500 lbs., what is the weight of a spherical shell whose external diameter is 10 ft., made of the same material and a foot thick?

88. The sun's diameter is about 109 times the diameter of the earth. How do the areas of their surfaces compare? Their volumes?

89. How many quarter-inch spherical bullets can be made from a sphere of lead a foot in diameter?

90. The angles of a spherical triangle are 80°, 90°, 100°. Find the angle of an equivalent lune.

91. Find the angles of an equiangular spherical triangle equivalent to the sum of three equiangular spherical triangles (upon the same sphere) whose angles are each 75°.

92. What is the radius of a sphere equivalent to the sum of two spheres whose radii are 3 in. and 4 in., respectively?

93. What is the radius of a sphere equivalent to the difference of two spheres whose radii are 5 in. and 4 in., respectively?

94. The area of an equiangular spherical triangle is π, and the radius of the sphere is 4. Find the angles of the triangle.

95. The volumes of two spheres are to each other as 64 : 343. What is the ratio of their surfaces?

96. Find the volumes of the segments of a sphere whose radius is 12, formed by a plane whose distance from the center is 9.

97. If the radius of a sphere is 20, find:

(*a*) The area of its surface.

(*b*) The area of a zone whose altitude is 2.

(*c*) The edge of a cube inscribed in the sphere.

(*d*) The area of a lune whose angle is 80°.

(*e*) The area of a spherical triangle whose angles are 75°, 53°, 72°.

(*f*) The area of a spherical polygon whose angles are 68°, 119°, 128°, 147°, 150°.

(*g*) The area of a birectangular spherical triangle whose vertex-angle is 54°.

(*h*) The area of a zone of one base whose altitude is 5.

(*i*) The radius of a sphere whose surface is four times as large.

(*j*) The volume of the sphere.

(*k*) The volume of a wedge whose angle is 36°.

(*l*) The volume of a spherical pyramid whose base is the triangle of exercise (*e*).

(*m*) The volume of the spherical sector whose base is the zone of exercise (*b*).

(*n*) The volume of the spherical cone whose base is the zone of exercise (*h*).

(*o*) The volume of a spherical segment of one base, whose altitude is 6.

(*p*) The radius of a sphere whose volume is four times as large.

INDEX OF DEFINITIONS

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