Plane & Solid GEOMETRY

Plane and Solid
Geometry

Suggestive Method

REVISED EDITION

By

George C. Shutts

Instructor in Mathematics, State Normal School
Whitewater, Wisconsin

Atkinson, Mentzer & Company

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PREFACE.

In this revision the suggestive features of former editions have been retained, and it is hoped improved. The principal value in the study of geometry lies in the power developed by the individual student in working out as much as possible his own demonstrations.

Pupils in their early study of the subject of geometry sometimes fail to see the need of some of the steps in a rigorous deductive demonstration and, to satisfy the demands of the class room, resort to formal memory of text as a substitute for logical thinking. This text has been prepared, not so much to illustrate a logical development of mathematical science as to arrange and adapt the mathematical data herein contained to the comprehension and growth of the pupil. In the introduction of various subjects theorems are stated as postulates or preliminary propositions which in a strictly logical development would require proof. Many of these statements are of so fundamental a nature, have so long been accepted as obvious facts by the pupil, and yet are so difficult of demonstration, that the chief value in the use of them lies not so much in their demonstration as in the comprehension and use of them in the demonstration of truths based upon them. Theorems have been chosen for the earlier demonstrations such that the course of reasoning will appeal to the logical ability of the pupils at that stage of their reasoning power.

The protractor T square, parallel rulers, etc., have been brought into use before the principles upon which their construction is based have been developed, that through a mechanical construction of the various forms, the pupil may get a better understanding of their meaning than he usually gets from abstract definitions.

The rigor of the demonstrations has in no case been minimized and constructions, when the principles upon which they depend have been developed, as usual require the use of straight edge and dividers only.

The subject of measurement and proportion has been simplified, and the incommensurable cases placed in the latter part of the text in plane geometry, with the intent that in a short course in plane geometry this phase may be omitted if the teacher deems it wise to do so.

Many practical applications of the principles of geometry have been introduced in the exercises, but have been selected with reference to the interests of all the pupils, rather than the few who desire to enter technical occupations. Many of the old geometrical puzzles have been omitted and numerical problems of interest introduced. It is not expected that any given class or pupil will work all of the exercises, but the number is large enough and the variety sufficiently great to furnish interesting recreation and drill for the variety of tastes in any class of students. Many exercises follow the propositions to which they are related for application and drill, but many others are scattered throughout the text more or less unrelated to serve the purpose of review and test the pupil's power of independent thought.

The one suggestion the author would make to the teacher is that success depends upon letting the rate of progress be determined by the thorough comprehension by the pupils of the subject matter. The amount of text covered is not so important as the real development of the pupils.

The author wishes to acknowledge for their scholarly assistance, his obligation to H. D. Merrill of the chair of mathematics of the Evanston High School, who has made valuable suggestion throughout ; to A. W. Smith, Professor of Mathematics of Colgate University, Hamilton, N. Y., who assisted in the revision and read the proof, and to many students who through conscientious work in the class room have given valuable suggestions. To these old friends and to the prospective students, to whom it is hoped these pages will furnish inspiration, this work is dedicated.

GEORGE C. SHUTTS.

Whitewater, Wis., Aug. 1, 1912.

PLANE GEOMETRY

INTRODUCTION.

1. Physical Solid. Any material object is a *physical solid,* for example, a block of wood or a ball.

2. Geometric Solid. A limited portion of space is a *geometric solid.* A block of wood is a physical solid but the portion of space occupied by it is a geometric solid. Two reasons are suggested for this definition of a geometric solid: first, geometry is not concerned with the material of which an object is made but with its size and its shape; second, such solids can be made to coincide in whole or in part.

3. Surface. That which has length and breadth without thickness is a *surface.*

The boundaries of a solid are surfaces but these surfaces are not a part of the solid which they bound any more than one's shadow on a wall is a part of the wall. A surface may pass through a solid or another surface and may be unlimited in extent.

4. Line. That which has length without breadth or thickness is a *line.*

A line may pass through a solid, a surface, or another line and may be unlimited in extent.

5. Point. That which has position without dimensions is a *point.*

6. Magnitudes. Lines, surfaces, and solids are *geometric magnitudes.*

7. Magnitudes Generated. The path of a moving point is a line, or a point in motion *generates* a line. A line in motion (except upon itself) generates a surface. A surface in motion (except upon itself) generates a solid.

8. Magnitudes, Limited or Bounded. Solids are limited or bounded by surfaces, surfaces by lines, lines by points, unless in accordance with the definitions given above they are unlimited.

9. Geometric Figures. Any combination of points, lines, surfaces, and solids is a *geometric figure.* Some Geometric figures are mental images only. They cannot be constructed and the drawings used are but representations. For example, a point may be represented by a crayon or pencil dot, a line by a pencil mark or the edge of a ruler, a surface by a table top or the surface of a blackboard, etc.

10. Straight Line. A line is *straight* if any part of it will lie wholly in any other part when its extremities lie in that part, or if, when revolved with two points kept stationary, it occupies the same position as before. A straight line may be thought of as extending indefinitely in both directions. The first test may be applied to a pencil mark representing a straight line by using tracing paper to apply "any part" to "any other part" of it or by putting together the edges of two rulers as representing two parts of a straight line. The second test may well be illustrated with a piece of *straight* wire.

11. Line Segment. A limited portion of a line is a *line-segment.*

A line-segment is sometimes called a *sect.*

12. Ray. A portion of line limited at one end only is a *ray.* The limiting point of a ray is its origin. A segment differs from a ray in that it has two limiting points instead of one.

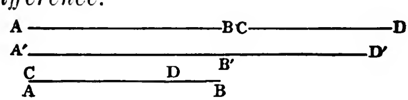
13. Notation. A point is read by naming a capital letter placed near it. A line is read by naming the letters which mark any two of its points. Lines, rays, and line-segments are at times indicated by single small letters instead of two capitals indicating points. For example in the following figures one reads: point A, line BC or l, segment DE or s, ray FG or r.

14. Equal Segments. If two line segments can be so placed that their extremities coincide the segments are said to be *equal.*

15. Addition and Subtraction of Segments.

(1) Two segments are added by placing them end to end. The resulting segment is called the sum.

(2) Two segments are subtracted by placing the shorter *upon* the longer so that one pair of end points coincide. That portion of the longer not covered by the shorter is called the *difference.*



For example, to find the sum of segment AB and segment CD, draw an indefinite line and on it lay off $A'B'$ equal to AB and $B'D'$ equal to CD. Then $A'D'$ is the sum of AB and CD or as it may be written in symbols,

$AB + CD = A'D'$. To find the difference between AB and CD lay CD upon AB with point C on A. The difference is the segment DB. The relation is here written as $AB - CD = DB$.

The measurements here necessary may be made with a ruler or scale or better yet with a pair of dividers.

16. Broken Line. A line made up of successive segments not forming a straight line is a *broken line.*

17. Curved Line. A line no part of which is straight is a *curved line* or simply a *curve.*

18. Plane. A surface such that a straight line through any two of its points lies wholly in the surface is a *plane.* A plane is sometimes spoken of as a flat surface and portions of a plane may be enclosed by broken lines or curves.

19. Plane Figures. A figure all parts of which lie in the same plane is a *plane figure.*

(1) A figure all lines of which are straight is a *rectilinear figure.*

(2) A portion of a plane entirely enclosed by a broken line is a plane polygon or simply a *polygon.*

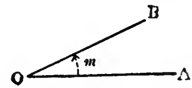
(3) A curve enclosing a portion of a plane in such a manner that every point of the curve is at the same distance from a point within is a *circle,* the point within is the *center,* the distance from the center to the circle is the *radius,* and any portion of the circle is an *arc.*

20. Plane Geometry. That part of geometry which treats of plane figures is called *plane geometry.*

21. Angle. Two rays proceeding from the same point contain or make an *angle.* The rays are the *sides*

of the angle and the point of meeting is the *vertex* of the angle.

An angle may be read by naming the letters which designate its sides, the vertex letter being read between the others, as the angle AOB. If no confusion results, an angle may be read by naming its vertex letter alone, as the angle O, or by naming a single small letter, as angle m.



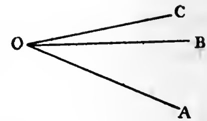
22. TURNING A LINE THROUGH AN ANGLE. If a line coincident with one side of an angle be revolved about the vertex as a pivot until it coincides with the other side of the angle, *the line turns through the angle.* This turning is usually conceived to be counter-clockwise.

23. SIZE OF ANGLE. The size of an angle is the amount of revolution necessary to turn a line through it. It bears no relation to the length of the sides.

24. EQUAL ANGLES. If two angles can be placed so that their vertices coincide and the sides of the one lie upon the sides of the other the angles coincide and the turning through both angles is effected at the same time. Such angles are said to be *equal.*

A pair of dividers may be used to represent an angle, the size of the angle depending upon the extent to which the dividers are opened. They may be used to compare two angles by placing them upon one of the angles with the legs of the dividers lying along the sides of the angle and then placing them upon the other angle, noticing whether or not it is necessary to open or close them in order that they fit the second angle. A similar comparison may be made by drawing one angle on a piece of tracing paper and placing it upon the other angle so that the vertex and one side of the trace lie upon the vertex and one side of the second angle. The relative position of the remaining sides of the two angles show which is the larger.

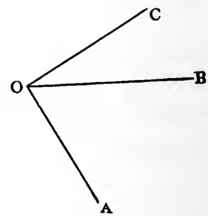
25. Adjacent Angles. Two angles that have the same vertex and one common side between them are *adjacent angles*.



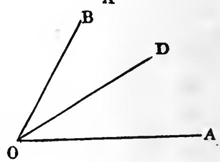
The angles AOB and BOC are adjacent.

26. Addition and Subtraction of Angles.

(1) Two angles are added by so placing them as to form adjacent angles. The sum is the large angle thus formed.

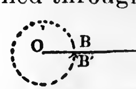


(2) Two angles are subtracted by placing the smaller within the larger so that they have one side and the vertex in common. The angle adjacent to the smaller is the difference. For example, the sum of angles AOB and BOC is AOC and the difference between AOB and DOB is AOD.



Are these results consistent with the statement in § 22? By means of tracing paper obtain the sum of the angles AOB of § 21 and BOC of § 25.

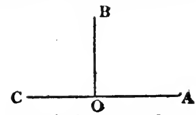
27. Perigon. The total angular magnitude about a point in a plane is a *perigon*. A line has turned through a perigon when it has made a complete revolution abouta point as BOB'.



28. Straight Angle. An angle the sides of which lie in a straight line and on opposite sides of the vertex is a *straight angle,* as angle AOB. A straight angle is one half a perigon. The student must distinguish between the straight line AOB and the straight angle AOB.

Let the student turn a ray through the angle AOB.

29. Right Angle. If a ray meets a line so that the two angles formed are equal, each angle is a *right angle*, as $\angle AOB$ and $\angle BOC$.



(1) A right angle is one half of a straight angle.

(2) A right angle is one fourth of a perigon.

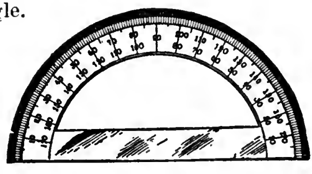
30. Degree. One ninetieth part of a right angle is a *degree*.

(1) One sixtieth part of a degree is a *minute* and one sixtieth part of a minute is a *second*.

(2) Degrees, minutes, and seconds are denoted by the symbols °, ′, ″ respectively, as 14° 27′ 30″.

(3) The degree is a unit of measure for angles. A protractor is sometimes used in constructing and measuring angles. It consists of a semicircular scale with degrees from 0° to 180° marked on it. To use the protractor in measuring an angle place the center O at the vertex of the angle and the zero of the protractor upon one side of the angle. The number of degrees and minutes in the angle is the number indicated upon the protractor at the point where the second side of the angle projects beyond the protractor. The protractor can also be used in constructing an angle of a given number of degrees or one that is equal to a given angle.

With the protractor construct a right angle, cut it out, and test the accuracy of the construction by superimposing it upon a known right angle.



31. Acute Angle. An angle less than a right angle is an *acute angle.*

With a protractor construct acute angles of 20°, 70°, 65°, 40°, 28°, 53′, etc.

32. Obtuse Angle. An angle greater than a right angle and less than a straight angle is an *obtuse angle.*

With a protractor construct obtuse angles of 95°, 115°, 170°. Add angles of 82° and 32°; of 47° and 53°.

33. Reflex Angle. An angle greater than a straight angle and less than a perigon is a *reflex angle.*

Reflex angles are seldom considered in elementary geometry.

With a protractor or tracing paper add angles of 98° and 71°; of 120° and 92°; of 72° and 65°. What kind of angles are the sums?

34. Oblique Angles. Acute and obtuse angles are *oblique angles.*

35. Complements. Two angles the sum of which is a right angle are *complements.*

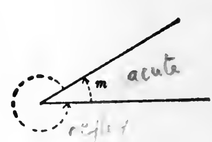
With a protractor construct the complements of 60°, 50°, 75°, 49°. Measure each and test the results.

36. Supplements. Two angles the sum of which is a straight angle are *supplements.* The complement or supplement of a given angle need not be adjacent to it.

Construct the supplement of 18°, 24°, 96°, 125°. Which are acute and which obtuse? Test the accuracy of the results with the protractor.

37. Conjugates. Two angles the sum of which is a perigon are *conjugates.*

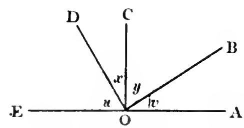
Two rays proceeding from a point really form two angles, as the angles m and n. These angles are conjugates.



Measure the degrees in m and n and test the result by this definition.

In referring to an angle the smaller of the two conjugates is meant unless otherwise stated.

The student should provide himself with a ruler, a protractor, a pair of dividers, parallel rulers, and if convenient a T square and drawing board.



1. In the adjacent figure AE is a straight line and CO is perpendicular to AE. Read a straight angle, two right angles, five acute angles, two obtuse angles.

2. Read ten pairs of adjacent angles in the above figure.

3. What is the sum of angles x and u? Of angles y and v? Of angles u and x? Of angles y and x?

4. Read two pairs of complementary angles in the above figure.

5. Read three pairs of supplementary angles in the above figure.

6. A perigon is equal to how many straight angles? A straight angle is equal to how many right angles? A perigon is equal to how many right angles? Compare a straight angle with a perigon; a right angle with a perigon; a right angle with a straight angle.

7. How many degrees are there in a straight angle? In a perigon?

8. How many degrees in the supplement of a right angle? In the supplement of two-thirds of a right angle? In the complement of three-fourths of a right angle?

9. Find the supplement and the complement of 54° 27′; of 57° 35′ 42″.

10. How many degrees in the smaller angle formed by the hands of a clock at 1 o'clock? At 5 o'clock? How many degrees are generated by the minute hand of a clock from two o'clock to twenty minutes past two?

11. A perigon is divided into six equal angles. How many degrees are there in each? Illustrate with a figure.

12. A perigon is divided into three angles, the second being twice the first and the third three times the first. Find the number of degrees in each angle. Illustrate with a figure.

Suggestion: Let x equal the number of degrees in the first.

13. Write an equality that will say that angles x and y are complements; that they are supplements.

14. Angles x and y are supplements and their difference is 50°. Find angle x and angle y. With a protractor draw x and y and check the results.

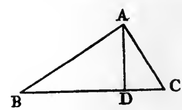
15. How can the supplement of an angle be found? Draw an angle, as x, with the protractor and find its supplement.

16. The supplement of an angle x is three times the complement of x. Find the angle x.

17. In the figure of example 1 measure with a protractor the angles x, y, and z; add them and test the result by measuring angle AOD.

18. Draw a straight angle. Divide it by a ray into two oblique angles. Measure each and add them. How many degrees should their sum be?

19. Divide a perigon into five angles and measure each. Test the accuracy of the work by adding the results.



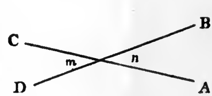
20. Measure angle A and angle CAD. Subtract the results and test the accuracy of the work by measuring angle DAB.

21. Draw a line-segment 1 inch long. Draw another $1\frac{1}{2}$ inches long so as to include between them an angle of 45°. Then measure the distance between the free ends of the segments. Make the construction a second time and with tracing paper test whether the two figures are congruent. § 64.

22. Draw an angle of 25°, 18°, 72°, 126°, 175°.

23. Draw a reflex angle of 197°, 240°, 315°, 345°.

38. Vertical Angles. When the sides of one angle are extensions of the sides of another angle, the two angles are *vertical angles*.



For example, m and n are vertical angles.

39. Perpendicular Lines. If two lines form a right angle, each line is *perpendicular* to the other. The word perpendicular is used also as a noun.

40. Oblique Lines. If two lines form an oblique angle, each line is oblique to the other.

41. Bisector. If a figure is divided into two equal parts it is *bisected.*

In plane geometry the bisector of a segment may be a point, a segment, a ray, or a line. The bisector of an angle may be a segment, a ray, or a line.

42. Proof. To prove a statement is to show by a logical course of reasoning that it must be true by means of other statements that have been accepted. In general the proof of a statement involves other statements which in turn require proof. Evidently as a foundation for reasonings some statements must be accepted without proof.

43. Axiom. A statement admitted without proof to be true and not limited to geometric figures is an *axiom.*

44. Postulate. A statement admitted without proof and limited to geometric figures is a *postulate.*

45. Theorem. A statement that is to be proved is a *theorem.*

46. Problem. The statement of a geometric construction to be made is a *problem.*

47. Proposition. Theorems and problems are *propositions.*

48. Corollary. A proposition easily deduced from another proposition is a *corollary.*

49. Axioms. In the following axioms the equalities and inequalities concerned are unconditional.

(1) Quantities that are equal to the same quantity, or to equal quantities, are equal to each other.

(2) If equals are added to or subtracted from equals the results are equal.

(3) If equals are multiplied or divided by equals (division by zero being excluded) the results are equal.

(4) If equals are added to or subtracted from unequals the results are unequal in the same order.

(5) If unequals are multiplied or divided by positive equals the results are unequal in the same order.

(6) If unequals are added to unequals in the same order the results are unequal in that order.

(7) If unequals are subtracted from equals the results are unequal in the reverse order.

(8) If the first of three quantities is greater than the second and the second is greater than the third, then the first is greater than the third.

(9) In considerations involving size only, the whole is greater than any of its parts and is always equal to the sum of its parts.

(10) Either of two equals may be substituted for the other in any process.

(11) Every magnitude, however small, can be divided into two or more parts.

50. POSTULATES.

(1) If two straight lines have two points in common they become one and the same straight line.

Between two points there is one and only one straight line.

(2) The shortest path between two points is the straight line segment joining them.

(3) A line-segment has one and but one mid-point. An angle has one and only one bisector.

(4) A figure can be moved without altering its size or its shape.

(5) A line in a plane divides the points of that plane

into two classes such that a line-segment connecting two points of the same class is not intersected by the line; and a line-segment connecting two points not of the same group is intersected by the line.

(6) Magnitudes which coincide are equal.

(7) All perigons are equal.

(8) A circle or an arc can be constructed with any radius and any given point as center.

(9) A straight line can be drawn between two points; a straight line can be terminated at any point; a sect can be extended any distance.

Symbols and Abbreviations.

The symbols +, −, ×, ÷ are used as in algebra.

> is greater than.
< is less than.
⊥ perpendicular.
‖ parallel.
= is equal to.
∼ is similar to.
≅ is congruent to.
⌶ is measured by.
≐ approaches as a limit.
∠ angle.
△ triangle.
▱ parallelogram.
Adj. adjacent.
Alt. alternate.
Auth. authority.
Ax. axiom.
Const. construction.
Cor. corollary.

Def. definition.
Ex. exercise.
Ext. exterior.
Fig. figure.
Hyp. hypothesis.
▭ rectangle.
⊙ circle.
§ article.
rt. right.
st. straight.
∴ therefore.
∵ hence.
Iden. identity.
Int. interior.
Post. postulate.
Prop. proposition.
Sug. suggestion.
Sup. supplementary.

The symbols ∠, △, ▱, ▭, ⊙ have the plural forms ∠s, △s, ▱s, ▭s, ⊙s.

Preliminary Theorems. The following are some important preliminary theorems:

51. Theorem. 1. *Straight angles are equal.*

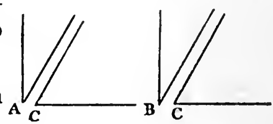
Sug. § 28, Post. 7, and Ax. 3.

52. Theorem 2. *Right angles are equal.*

Sug. § 29 and Ax. 3.

53. Theorem 3. (a) *Complements of equal angles are equal.* (b) *Supplements of equal angles are equal.*

a. **Given** angles A and B, each being a complement of $\angle C$. To prove $\angle A$ and $\angle B$ equal.



Sug. 1. How does the sum of ∠s A and C compare with the sum of ∠s B and C? Th. 2 and Ax. 1.

2. Compare $\angle A$ and $\angle B$. Ax. 2.

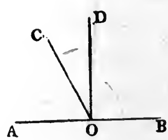
Therefore—

b. Follow the method of demonstration used for theorem 3 (a).

Note.—Before looking up references in the suggestions the pupil should endeavor to determine the authorities for himself.

54. Theorem 4. *Two right angles are supplements of each other.*

55. Theorem 5. *Only one line can be drawn perpendicular to a line at a given point on the line.*



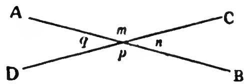
Sug. Can both OD and OC be ⊥ to AB at point O? §29, §39 and § 50 (3).

56. Theorem 6. *Two lines that intersect can have but one point in common.*

SUG. If they had two points in common, what would be true? Post. 1.

57. THEOREM 7. *If two straight lines intersect the vertical angles are equal.*

Given AB and CD intersecting at O with vertical angles m and p.



To Prove $\angle m = \angle p$.

SUG. 1. What relation does $\angle m$ bear to $\angle n$? § 36.

2. What relation does $\angle p$ bear to $\angle n$? § 36.

3. Compare $\angle m$ and $\angle p$. § 53 *(f)*.

Therefore—

Compare angles q and n.

1. Draw two supplementary adjacent angles and bisect each. How many degrees in the angle of the bisectors? Use the protractor and straight edge.

2. How many degrees in the angle formed by the bisectors of two complementary adjacent angles? By the bisectors of two adjacent angles of 63° and 70° respectively? Of 26° and 84° respectively?

3. If two lines intersect and one of the angles is a right angle, prove the others are right angles.

4. If one angle formed by two intersecting lines is 70° how many degrees are there in each of the others? If one angle is 50° how many in each of the others?

5. The difference between two complementary angles is 12°. What are the angles? If the angles are supplementary, what are they?

6. Solve problem 5 if the difference is 40°; 75°; 82° 16′.

7. The ratio of two complementary angles is 4:5. What are the angles? Solve for ratios of 3:6, $3\frac{1}{2}:5\frac{1}{2}$, 5:9.

8. Solve problem 7 with the supposition that the angles are supplementary.

9. What is the complement of 100°? The supplement of 212°? Of 198° 27′? If the student by use of algebra derives these results from the definitions of complements and supplements, he will have angles with the negative sign. The interpretation is this: Angles which are generated by counterclockwise notation are called positive and those generated by a clockwise notation are called negative. This plan is of great use whenever it is necessary to distinguish between these two ways of generating an angle.

58. SUMMARY.

1. Two angles are equal
   (1) if they can be made to coincide;
   (2) if they are equal to the same angle or to equal angles;
   (3) if they can be obtained by adding equal angles to equal angles or by subtracting equal angles from equal angles;
   (4) if they are doubles or halves of the same angle or of equal angles;
   (5) if they are straight angles, right angles, or vertical angles;
   (6) if they are complements or supplements of the same or of equal angles.
2. Two angles are complements if their sum is a right angle or 90°.
3. Two angles are supplements if their sum is a straight angle, two right angles, or 180°.
4. An angle is a right angle
   (1) if it is one of two equal adjacent angles formed by two straight lines;
   (2) if it is one half of a straight angle or contains 90°;
   (3) if its sides are perpendicular to each other;

(4) if it is one of two equal supplementary angles.

5. An angle is a straight angle
   (1) if its sides form a straight line;
   (2) if it is the sum of two right angles or 180°;
   (3) if it is one-half of a perigon.
6. Two lines are perpendicular to each other if they form a right angle.
7. Two lines form one straight line
   (1) if they have two points in common;
   (2) if they are the sides of a straight angle;
   (3) if they are perpendicular to the same line at the same point.
8. Line segments or sects can be proved equal by means of axioms 1, 2, 3 or by showing that they can be so placed that their end points coincide.
9. Angles or line segments can be proved unequal by means of axioms 4-9 inclusive or postulate 3.

Post. 3 — a line [illegible]
only one mid-point

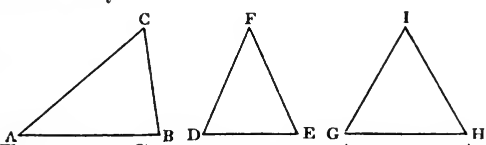
CHAPTER I.

RECTILINEAR FIGURES.

59. Triangle. A portion of a plane bounded by three sects is a *triangle.* The line-segments are the *sides* of the triangle, their intersections are the vertices, their sum is the *perimeter,* the angles formed by the sides are the *angles* of the triangle, and the three sides and three angles are the parts of the triangle.

60. Triangles Classified as to Sides. A triangle no two of the sides of which are equal is a *scalene triangle.* One with two equal sides is an *isosceles triangle.* One with three equal sides is an *equilateral triangle.* In the accompanying figures $\triangle ABC$ is scalene, $\triangle DEF$ is isosceles, and $\triangle GHI$ is equilateral.

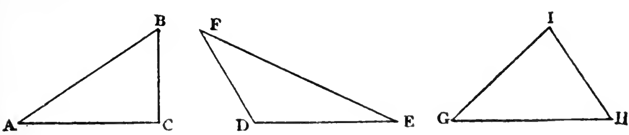
Measure to verify.



61. Triangles Classified as to Angles. A triangle one of the angles of which is a right angle is a *right triangle.* One with an obtuse angle is an *obtuse triangle.* If all the angles are acute the triangle is an *acute triangle.* If all the angles are equal the triangle is *equiangular.*

$\triangle ABC$ is right, $\triangle DEF$ is obtuse, $\triangle GHI$ is acute.

Verify with a protractor.



62. Base of a Triangle. Unless otherwise stated, the *base* of an isosceles triangle is the side which is not one of the two equal sides. In other cases the base of a triangle is that side upon which it is conceived to stand.

63. Vertex Angle of a Triangle. The angle opposite the base of a triangle is the *vertex angle* and the vertex of this angle is the *vertex of the triangle.*

64. Congruent Figures. Two figures that can be made to coincide throughout are *congruent.* Congruent means of the same shape and size, while *equal* means of the same size only.

1. Draw a triangle with two sides 2 and $2\frac{1}{2}$ inches respectively and included angle 75°. [30, (3)]. Draw another with the same conditions. Use tracing paper to compare them or cut one out and compare them by superposition. Are they congruent?

2. Draw a triangle at random. Measure two sides and the included angle. Construct another using the measurements obtained from the first and compare them by superposition.

3. Draw a triangle with two angles of 85° and 65° respectively with the included side 2 inches. Draw another with the same conditions. Compare them by superposition. What is the conclusion?

4. Draw a triangle at random. Measure two angles and the included side. Construct another triangle from these measurements and compare them by superposition.

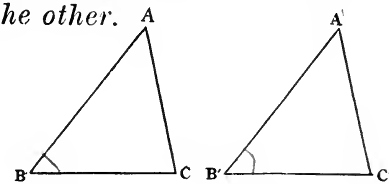
5. Draw a straight line $1\frac{1}{2}$ inches long. With the dividers take lines respectively $\frac{3}{4}$ and $1\frac{1}{4}$ inches and from the ends of the first line describe arcs that intersect. Connect the point of intersection with the ends of the first line. Take the same three lines in any order and construct a triangle. Compare these triangles as regards congruence.

6. Construct a triangle in which the sides are 2, 3, and 4 inches respectively. Construct a triangle with lines of $1\frac{1}{2}$, $1\frac{3}{4}$ and 4 inches respectively. Try again with lines of 2, 2, and 4 inches respectively; 2, 3, and 7; 4, 5, and 6.

7. State the conclusions of Ex. 6 as to the possibility of constructing a triangle of given sides.

PROPOSITION I.

65. Theorem. *Two triangles are congruent if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.*



Given $\triangle ABC$ and $\triangle A'B'C'$ with $BA = B'A'$, $BC = B'C'$, and $\angle B = \angle B'$.

To Prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that point B will fall on B' and BC will lie along $B'C'$.

Then C will fall on C' for $BC = B'C'$.

BA will fall along $B'A'$ for $\angle B = \angle B'$.

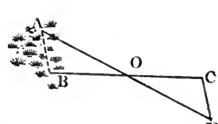
A will fall on A' for $BA = B'A'$,

Since C falls on C' and A falls on A', line AC will coincide with $A'C'$ for only one straight line can be drawn through two given points.

$\therefore \triangle ABC \cong \triangle A'B'C'$ for they can be made to coincide.

Therefore, Two triangles are congruent if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.

1. To find the distance across an impassable marsh: Stake off two lines AC and BD that cross at an accessible point O. Measure distances OD equal to OA and OB equal to OB. Then the required distance AB is equal to CD. Why?

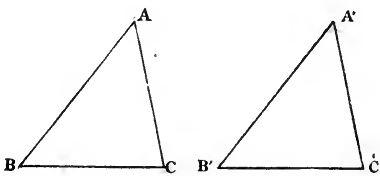


2. In the school grounds or elsewhere drive three stakes for A, B, and O of ex-

ample 1. Make the remaining measurements and test the accuracy of the work by comparing AB and CD.

PROPOSITION II.

66. **Theorem.** *Two triangles are congruent if two angles and the included side of the one are equal to two angles and the included side respectively of the other.*



Given $\triangle ABC$ and $\triangle A'B'C'$ with $\angle B = \angle B'$, $BC = B'C'$, $\angle C = \angle C'$.

To Prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that B shall fall on B' and BC shall fall along $B'C'$.

Then C will fall on C' for BC is given equal to $B'C'$.

BA will fall along $B'A'$ for $\angle B = \angle B'$.

$\therefore A$ will fall on $B'A'$ or $B'A'$ produced.

Also CA will fall along $C'A'$ for $\angle C = \angle C'$.

$\therefore A$ will fall on $C'A'$ or $C'A'$ produced.

$\therefore A$ must fall on A', the intersection of $B'A'$ and $C'A'$. Why?

$\therefore \triangle ABC \cong \triangle A'B'C'$ for they can be made to coincide.

Therefore, Two triangles are congruent if two angles and the included side of the one are equal to two angles and the included side respectively of the other.

67. Application of Propositions I and II. **If two triangles are congruent the angles are equal respectively**

and the sides are equal respectively. It is important to keep this in mind for usually the purpose in proving triangles congruent is to prove that a pair of angles or line-segments are equal. In congruent triangles pairs of equal sides lie opposite pairs of equal angles and pairs of equal angles lie opposite pairs of equal sides. The pairs of equal parts are called *corresponding* or *homologous parts.*

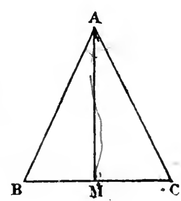
1. Draw a triangle at random and with a straight edge and protractor; construct a second triangle having two sides and their included angle of the same dimensions as the corresponding parts of the first. Why must the triangles be congruent? Compare the two, using tracing paper, as a test for accuracy of construction.

PROPOSITION III.

68. THEOREM. *If two sides of a triangle are equal, the angles opposite those sides are equal.*

Given $\triangle ABC$ with $AB = AC$.

To Prove $\angle B = \angle C$.



Proof. Draw AM to represent the bisector of $\angle A$ and extend it to meet BC as at M. $\angle s$ B and C will be proved equal if it can be shown that $\triangle$ AMB and AMC are congruent. A comparison of the two triangles shows

$AB = AC$ by hyp.

$\angle MAB = \angle MAC$ since AM bisects $\angle A$.

AM is common to the two triangles.

$\therefore \triangle AMB \cong \triangle AMC$, two sides and the included angle of the one being equal to two sides and the included angle of the other.

$\angle B$ is in $\triangle AMB$ and is opposite to side AM.

$\angle C$ is in $\triangle AMC$ and is opposite side AM.

$\therefore \angle B = \angle C$, being homologous angles in congruent triangles.

Therefore, If two sides of a triangle are equal, the angles opposite those sides are equal.

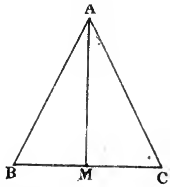
REMARK. Observe that $\angle B$ is an angle in each of two triangles. In $\triangle AMB$ it lies opposite the side AM, in $\triangle ABC$ it lies opposite side AC. Likewise $\angle C$ is an angle in each of two triangles.

69. COROLLARY. *An equilateral triangle is equiangular.*

Proof. Use proposition III.

PROPOSITION IV.

70. THEOREM. *The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base.*



Given $\triangle ABC$, with $AB = AC$ and $\angle MAB = \angle MAC$.

To Prove $BM = MC$ and $AM \perp BC$.

Proof. A comparison of the $\triangle AMB$ and $\triangle AMC$ shows

$AB = AC$ by hyp.

$\angle MAB = \angle MAC$ by hyp.

$\angle B = \angle C$, being opposite equal sides of the isosceles $\triangle ABC$.

$\therefore \triangle AMB \cong \triangle AMC$, having two angles and the included side of the one equal to the corresponding parts of the other.

In $\triangle AMB$ side BM is opposite $\angle MAB$ and in $\triangle MAC$ side MC is opposite $\angle MAC$.

$\therefore BM = MC$, being homologous sides in congruent triangles.

Also $\angle AMB = \angle AMC$, being homologous angles in congruent triangles.

$\therefore$ ∠s AMB and AMC are right angles. Why?

$\therefore AM \perp BC$. Why?

Therefore, The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base.

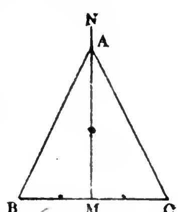
71. Distance. The length of the sect joining two points is the *distance* between these points.

It is to be noted that *distance* is always measured on the shortest line.

72. Perpendicular Bisector. A perpendicular erected at the mid-point of a line-segment is the *perpendicular bisector* of that segment,

PROPOSITION V.

73. Theorem. *Any point in the perpendicular bisector of a line-segment is equidistant from the extremities of the segment.*



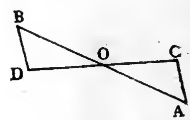
Given segment BC with $BM = MC$, $MN \perp BC$ and A any point in MN.

To Prove $AB = AC$.

Proof. SUG. If $\triangle BMA$ and $\triangle CMA$ can be proved congruent, segments AB and AC will be seen to be equal. Search the conditions given to see if there are enough elements equal to make the triangles congruent by Prop. I or II. See method of § 70. Compare AB and AC, giving authorities.

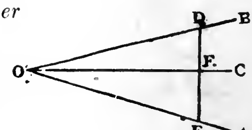
74. PARTS OF A THEOREM. From the propositions studied thus far it is seen that in a theorem certain conditions are given and a certain thing is to be proved. The conditions given constitute the *hypothesis* and the truth to be established is the *conclusion.* The course of reasoning by means of which the conclusion is established is the *proof* or *demonstration.* Each statement in a proof must be justified by reference to the hypothesis or to definition, axiom, postulate, or former proposition.

1. Two right triangles are congruent if the sides of the right angle of the one are equal respectively to the sides of the right angle of the other.



2. Lines AB and CD *bisect each other* at O.

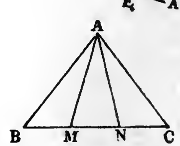
Prove $\triangle AOC \cong \triangle BOD$ and $AC = BD$.



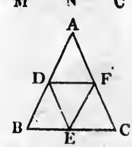
3. OC bisects $\angle AOB$ and $DE \perp OC$ at F.

Prove $\triangle OFD \cong \triangle OFE$, $OD = OE$, and $\angle ODF = \angle OEF$.

4. In $\triangle ABC$, $AB = AC$ and $\angle BAM = \angle CAN$. Prove $\triangle BAM = \triangle CAN$ and $\triangle AMN$ is isosceles.

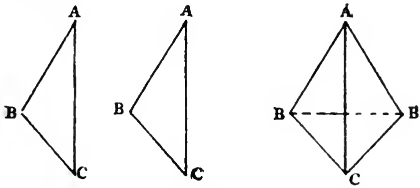


5. In $\triangle ABC$, $AB = AC$ and D, E, F are the mid-points of AB, BC, and CA respectively. Prove $\triangle DEF$ isosceles.



PROPOSITION VI.

75. THEOREM. *Two triangles are congruent if the sides of the one are equal respectively to the sides of the other.*



Given $\triangle ABC$ and $\triangle A'B'C'$, $AB = A'B'$, $BC = B'C$ and $CA = C'A'$.

To Prove $\triangle ABC \cong \triangle A'B'C'$.

SUG. 1. The conclusion follows at once if it can be shown that $\angle A = \angle A'$, $\angle B = \angle B'$, or $\angle C = \angle C'$.

2. Place $\triangle A'B'C'$ so that a side, as $A'C'$ coincides with its equal AC, the two points B and B' being on opposite sides of AC.

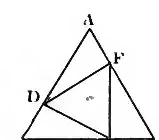
3. Draw BB'.
4. $\angle ABB' = \angle AB'B$. Why?
5. $\angle CBB' = \angle CB'B$. Why?
6. $\therefore \angle ABC = \angle AB'C$. Why?
7. $\therefore \triangle ABC \cong \triangle A'B'C'$. Why?

Demonstrate the theorem again, placing a second pair of equal sides in coincidence.

Therefore—

1. The straight lines which bisect the equal angles of an isosceles triangle and terminate in the opposite sides are equal.

2. $\triangle ABC$ is equilateral and $AD = BE = CF$. Prove $\triangle DEF$ equilateral.



PROPOSITION VII.

76. THEOREM. *Any point which is equidistant from the extremities of a line-segment is in the perpendicular bisector of the segment.*

The figure is to be constructed by the pupil.

Given the segment AB with mid-point C, and D any point such that $DA = DB$ and ME the line through C and D.

To Prove $ME \perp AB$ at C.

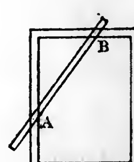
SUG. Compare $\triangle ACD$ and BCD and also the two adjacent angles at C.

77. COROLLARY. *Any two points each equidistant from the extremities of a line-segment determine the perpendicular bisector of the segment.*

SUG. How many points determine a line? § 50 (1).

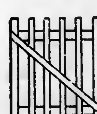
1. Nail together with one nail at each vertex three narrow strips of wood so as to form a triangle. Can it be racked out of shape without loosening the joints? Why?

2. Remove the bottom from a chalk box. Can the box now be racked out of shape? Nail a strip across two adjacent sides. Can it still be racked out of shape? Give authorities for your answers.



3. Why does the stay-lath AB which the carpenter nails across a window or door frame hold it in a fixed shape?

4. Observe at the next opportunity the board nailed obliquely across the studding as they are erected in putting up the frame of a building or the diagonal strip on a gate, as AB in the figure.

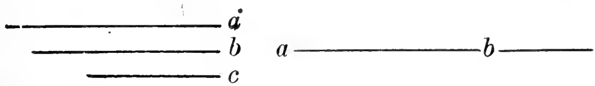


5. Name at least five other applications of prop. VI for insuring stability in mechanical constructions.

6. If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles. Use fig. of § 70.

PROPOSITION VIII.

78. PROBLEM. *Construct a triangle when the three sides are given.*



Given sides of a triangle as *a, b, c.*

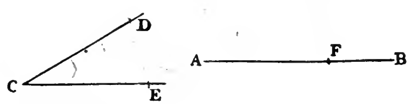
To Construct the triangle.

SUG. 1. Draw a line and with dividers lay off upon it one of the sides as $AB = c$.

2. With the dividers find a point C that is the distance a from B and the distance b from A. § 50 (8). $\triangle ABC$ is the required triangle.

PROPOSITION IX.

79. PROBLEM. *Construct upon a given ray an angle equal to a given angle.*



Given ray AB and $\angle C$.

To Construct an $\angle$ upon AB equal to $\angle C$ with its vertex at A.

SUG. 1. With dividers measure equal distances as CD and CE on the sides of $\angle C$.

2. Find F on AB so that $AF = CE$.

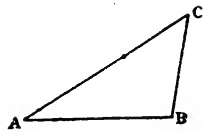
3. With the dividers find point X so that $AX = CD$ and $FX = ED$.

4. Then $\angle XAF = \angle DCE$ or $\angle C$.

For the $\triangle$s AXF and CDE have the three sides respectively equal.

PROPOSITION X.

80. **Problem.** *Construct a triangle congruent to a given triangle.*



Given △ *ABC.*

To Construct a △ congruent to △ *ABC.* Make three constructions.

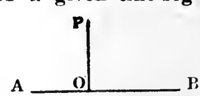
Sug. 1. Use the problem § 78.
2. Use § 65.
3. Use § 66.

In these problems use a straight edge and dividers only.

1. Prove the theorem of § 75, using two scalene triangles with a pair of the shorter sides coincident.

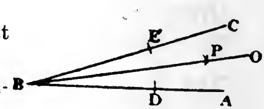
2. Upon a given segment as a base construct an equilateral triangle.

3. Construct the perpendicular bisector of a given line-segment. Sug. § 77.



4. Let *O* be a given point in *AB.* Construct the perpendicular to *AB* through *O.*

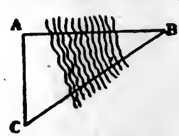
Sug. If *P* be a point in this perpendicular, how may *P* be obtained? Why must line *OP* when thus determined be the required perpendicular.



5. Solve example 4 assuming that *O* is outside *AB.*

6. Required to bisect a given angle, ∠ *ABC.*

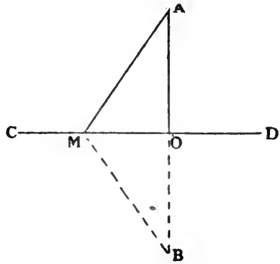
Sug. Obtain a point *P* such that when *BP* is drawn, ∠ *DBP* = ∠ *EBP.* § 75.



7. To measure a distance *AB* across a river. From *A* take any convenient line, *AC,* of known length, measure angles *A* and *C.* In some convenient place construct a △ congruent to △ *ABC* and measure the side corresponding to *AB.* State the authority for the conclusion. Sug. § 66.

PROPOSITION XI.

81. THEOREM. *Only one perpendicular can be drawn from a given point to a given line.*



Given line CD with A a point not in CD, $AO \perp CD$, and AM any other line from A to CD.

To Prove AM not perpendicular to CD.

SUG. 1. Produce AO to B, making $OB = OA$. Connect M and B.

2. Since AOB is a straight line by construction, AMB is not a straight line. Why?

3. $\therefore \angle AMB$ is not a straight angle.

4. Compare $\triangle AOM$ and $\triangle BOM$. Auth.

5. Compare $\angle AMO$ with $\angle BMO$.

6. $\therefore \angle AMO$ is one half of $\angle AMB$.

7. $\therefore \angle AMO$ cannot be a right angle.

8. $\therefore AM$ is not perpendicular to CD.

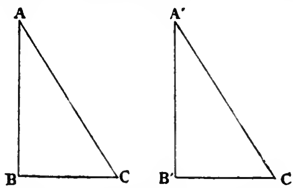
9. $\therefore AO$ is the only perpendicular from A to CD, since AM represented all other lines from A to CD.

82. HYPOTENUSE. The side of a right triangle opposite the right angle is the *hypotenuse*. The other sides are usually called the *sides* or *legs*.

1. Upon a given segment as a base construct an isosceles triangle.

PROPOSITION XII.

83. **Theorem.** *Two right triangles are congruent if the hypothenuse and an adjacent angle of the one are equal respectively to the hypothenuse and an adjacent angle of the other.*



Given $\triangle ABC$ and $\triangle A'B'C'$ with $\angle B$ and $\angle B'$ right angles, $AC = A'C'$, and $\angle C = \angle C'$.

To Prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that A falls on A and AC falls along $A'C'$. Where will C fall and why?

Also CB will fall along $C'B'$ for $\angle C = \angle C'$.

B will fall on $C'B'$ or $C'B'$ produced. Why?

Since A falls on A' and BC falls along $B'C'$, AB is the perpendicular from A' to $B'C'$.

But $A'B'$ is by hyp. perpendicular to $B'C'$. Why?

$\therefore AB$ must fall along $A'B'$. Why?

$\therefore B$ falls on B'. Why?

$\therefore \triangle ABC \cong \triangle A'B'C'$. Why?

Therefore—

1. If two isosceles triangles have their bases coincident and their vertices on opposite sides of the common base, prove:

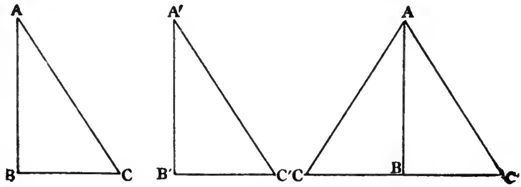
   (1) The entire figure is divided into two congruent triangles by a line connecting the opposite vertices.

   (2) This line is the perpendicular bisector of the common base.

2. If in example 1 the vertices are assumed to be on the same side of the common base, are the above conclusions true?

PROPOSITION XIII.

84. Theorem. *Two right triangles are congruent if the hypothenuse and a side of the one are equal respectively to the hypothenuse and a side of the other.*



Given $\triangle ABC$ and $\triangle A'B'C'$, with $\angle B$ and $\angle B'$ right angles, $AC = A'C'$, and $AB = A'B'$.

To Prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Sug. 1. The triangles are congruent provided $\angle C = \angle C'$ or $\angle A = \angle A'$. § 83.

2. Place the triangles so that A' falls on A, $A'B'$ along AB, and C and C' on opposite sides of AB.

3. Where will B' fall and why?

4. CBC' is a straight line. Why?

5. Figure ACC' is a triangle. Why?

6. What kind of a triangle is ACC'? Why?

7. Compare $\angle C$ and $\angle C'$. Auth.

8. Compare $\triangle ABC$ and $A'B'C'$. Auth.

Therefore—

Why is it necessary to prove CBC' a straight line?

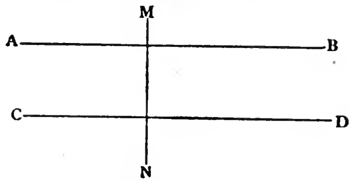
85. Cor. *The perpendicular from the vertex to the base of an isosceles triangle bisects the base and the vertex angle.*

86. Parallel Lines. Straight lines that lie in the same plane and cannot meet however far produced are *parallel lines.*

87. Postulate. Through a given point only one line can be drawn parallel to a given straight line.

PROPOSITION XIV.

88. Theorem. *Two lines in the same plane and perpendicular to the same line are parallel.*



Given AB and CD in the same plane and each perpendicular to MN.

To Prove $AB \parallel CD$.

Proof. Suppose AB and CD not parallel, i.e. suppose they meet if sufficiently produced. Then from this point of meeting there will be two perpendiculars to the line MN, viz. AB and CD. But this is impossible for only one perpendicular can be drawn from a given point to a given line. § 81.

Hence the supposition that AB and CD meet is false.

$\therefore AB \parallel CD$, for they are in the same plane and do not meet.

Therefore—

89. Indirect Proof. The proof in § 88 is an example of *indirect proof.* In this method the theorem is assumed to be *not true* and the assumption is then shown to be false in that it leads to the contradiction of known facts. § 90 is another example of indirect proof.

1. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides are equal.

2. Discuss exercise 1, when the vertex angle is obtuse; when it is right.

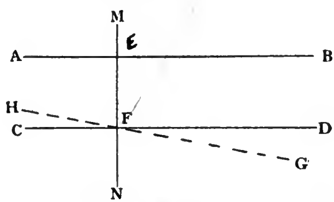
3. The perpendiculars drawn from the mid-point of the base of an isosceles triangle to the other sides are equal.

4. In $\triangle ABC$, D is the mid-point of BC. BE and CF are perpendiculars drawn from B and C respectively to AD (AD being produced through D). Prove $BE = CF$.

5. Discuss exercise 4, if $\triangle ABC$ is isosceles with $AB = AC$.

PROPOSITION XV.

90. THEOREM. *If one of two parallel lines is perpendicular to a given line, the other is perpendicular to the same line.*



Given $AB \parallel CD$, $AB \perp MN$ at E and CD intersecting MN at F.

To Prove $CD \perp MN$.

Proof. Suppose that CD is *not* perpendicular to MN. Let HG represent the line which is perpendicular to MN, at F.

Then $HG \parallel AB$. Why?

But $CD \parallel AB$. Hyp.

Then through F there are two lines parallel to AB.

This contradicts the postulate, § 87.

$\therefore$ the supposition that CD is not perpendicular to MN is false. $\therefore CD \perp MN$.

Therefore—

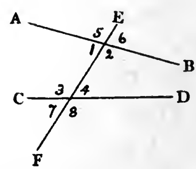
91. Discussion. Observe that HG is given the property which CD is *assumed* not to possess. This is legitimate since it is known that through point F there is a line perpendicular to MN and if CD is not this perpendicular we may represent it by HG.

Direct Proof. Without making any supposition as to CD, draw a perpendicular to MN through F. Then as before both HG and CD are parallel to AB. Whence HG, if correctly drawn, *must coincide with* CD for otherwise there would be two lines through a given point parallel to the same line.

Since $HG \perp MN$ by construction and CD coincides with HG, then $CD \perp MN$.

92. Transversal. A line that cuts two or more lines is a *transversal* or *secant* line.

93. Classification of Angles made by a Transversal. The angles made by two lines and a transversal are classified as follows:



1. The angles 1, 2, 3 and 4 are *interior* angles.

2. The angles 5, 6, 7 and 8 are *exterior* angles.

3. The pairs 1, 4 and 2, 3 are *alternate-interior* angles.

4. The pairs 5, 8 and 6, 7 are *alternate-exterior* angles.

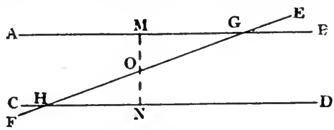
5. The pairs (1, 7), (2, 8), (6, 4), and (5, 3) are corresponding angles.

1. Any point in the bisector of an angle is equidistant from the sides of the angle.

Sug. Draw perpendiculars from a point in the bisector to the sides and prove them equal.

PROPOSITION XVI.

94. Theorem. *If two parallel lines are cut by a transversal the alternate-interior angles are equal.*



Given $AB \parallel CD$ and cut by the transversal EF at points G and H respectively.

To Prove $\angle AGF = \angle DHE$ and $\angle BGF = \angle CHE$.

Sug. 1. Through O, the mid-point of GH, draw $ON \perp CD$ and extend it to AB at M.

2. What relation does MN sustain to AB and why?

3. Compare $\triangle OMG$ and $\triangle ONH$. Auth.

4. Compare $\angle AGF$ and $\angle DHE$.

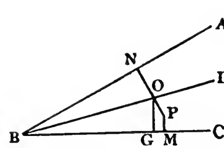
5. Compare $\angle BGF$ and $\angle CHE$.

Therefore—

Remark. In answering Sug. 4 observe that it is not known that OM and ON are equal. How then can it be shown that $\angle AGF$ and $\angle DHE$ are homologous angles in congruent triangles?

1. Any point equidistant from the sides of an angle is in the bisector of the angle.

2. Any point not in the bisector of an angle is not equidistant from the sides.



Sug. Draw $PM \perp BC$ and $PN \perp BA$. Let PN meet the bisector in O. Draw $OG \perp BC$. Then $OG = ON$. Why? $PG < PO + OG$. $\therefore PG < PN$. $PM < PG$. $\therefore PM < PN$.

Give the reason for each of these steps.

3. Demonstrate Ex. 2 by the indirect method.

The pupil is expected to make detailed proofs of the following three corollaries.

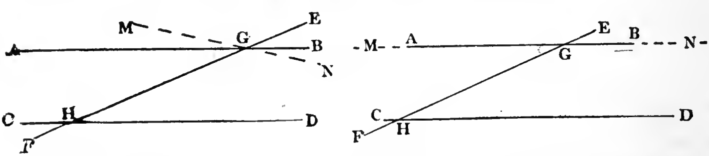
95. Cor. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

96. Cor. *If two parallel lines are cut by a transversal, the alternate-exterior angles are equal.*

97. Cor. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

PROPOSITION XVII.

98. Theorem. ***If two lines in the same plane are cut by a transversal and the alternate-interior angles are equal, the lines are parallel.***



Given AB and CD cut by the transversal EF at G and H respectively with $\angle AGF = \angle DHE$.

To Prove $AB \parallel CD$.

Sug. 1. Suppose AB not parallel to CD and let MN represent that line through G which is parallel to CD.

2. Compare $\angle MGF$ with $\angle DHE$. Auth.

3. Compare $\angle AGF$ with $\angle DHE$. Auth.

4. Compare $\angle MGF$ with $\angle AGF$. Auth.

5. Compare this conclusion with the facts.

6. What of the supposition that AB is not parallel to CD?

7. What is the true relation between AB and CD?

Therefore—

Prove this proposition by the direct method. See § 91.

Remark. The second of the above figures will be useful if chalks of different colors are used to represent the lines *AB* and *MN*.

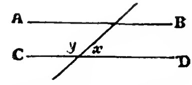
99. Cor. I. *If two lines in the same plane are cut by a transversal and the corresponding angles are equal, the lines are parallel.*

100. Cor. II. *If two lines in the same plane are cut by a transversal and the interior angles on the same side of the transversal are supplementary, the lines are parallel.*

101. Converse. If two propositions are so related that a condition of the first is the conclusion of the second and the conclusion of the first is a condition of the second, each proposition is called the *converse* of the other. For example, Prop. XVII is the converse of Prop. XVI for in the first of these the hypothesis is $AB \parallel CD$ and the conclusion is $\angle AGF = \angle DHE$; in the second the hypothesis is $\angle AGF = \angle DHE$ and the conclusion is $AB \parallel CD$. It is to be noted that there is one condition common to the two propositions, viz: two lines and a transversal are given:

The converse of a true statement is not always true. For example, all right angles are equal but not all equal angles are right angles.

1. If $AB \parallel CD$ and $x = 70°$, how many degrees in each one of the various angles in the figure?



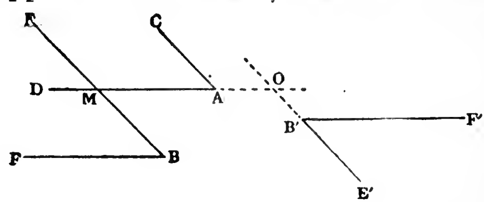
2. If $AB \parallel CD$ and $y - x = 80°$, how many degrees in each of the various angles? How many if $x = \frac{2y}{7}$? $\frac{y}{8}$? $\frac{2y}{3}$? $\frac{2y}{5}$?

3. Two angles are complementary and their difference is 38°. How many degrees in each?

4. Given a line *AB* and a point *C* not on *AB*. Construct a line through C and parallel to *AB*. Sug. Try to reproduce the conditions of Prop. XVII or of one of its corollaries.

PROPOSITION XVIII.

102. THEOREM. *Two angles the sides of which are respectively parallel and extend in the same or in opposite directions from the vertices are equal.*



Given ∠s *A*, *B*, *B′* with *AC* ∥ *BE*, *AD* ∥ *BF* and extending in the same directions from the vertices *A* and *B*; and with *AC* ∥ *B′E′*, *AD* ∥ *B′F′* and extending in opposite directions from the vertices *A* and *B′*.

To Prove ∠ *A* = ∠ *B* = ∠ *B′*.

SUG. 1. Extend *AD* and *B′E′* until they meet at *O*. (Why must they meet?)

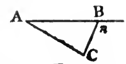
2. Compare ∠s *A* and *B′*.

3. Extend *AD* and *BE* until they meet at *M*.

4. Compare ∠s *A* and *B*.

Therefore—

In what kind of triangles does the bisector of the vertex angle coincide with the perpendicular from the vertex to the base and with the median to the base? The line from the vertex to the mid-point of the base is called the *median to the base.*

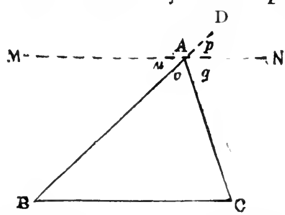


103. EXTERIOR ANGLE. An angle formed by one side of a rectilinear figure and an adjacent side extended is an *exterior angle*. In the figure n is an exterior angle of the triangle ABC.

104. OPPOSITE INTERIOR ANGLE. In the above figure with respect to the exterior angle n, A and C are the *opposite interior angles*.

PROPOSITION XIX.

105. THEOREM. *An exterior angle of a triangle is equal to the sum of the opposite interior angles.*



Given. $\triangle ABC$ with exterior angle DAC.

To Prove $\angle DAC = \angle B + \angle C$.

SUG. 1. Through the vertex A draw line $MN \parallel BC$.

2. Compare $\angle p$ with $\angle B$. Auth.

3. Compare $\angle q$ with $\angle C$. Auth.

4. Complete the proof.

Therefore—

1. Two lines in the same plane, each parallel to a third line in that plane, are parallel. Sug. Draw a transversal, a line perpendicular to the third line, or use the indirect method.

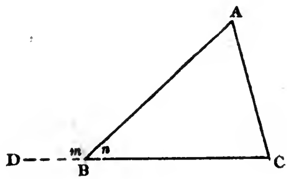
2. Two lines perpendicular to two intersecting lines respectively cannot be parallel. Sug. Use indirect proof.

3. Two angles are supplementary and their difference is 65°. Find the angles.

4. State all methods thus far given for proving two lines parallel.

PROPOSITION XX.

106. Theorem. *The sum of the interior angles of a triangle is equal to two right angles.*



Given △ ABC.

To Prove $\angle A + \angle B + \angle C = 2$ rt. ∠s.

Sug. 1. Extend one side as CB.

2. Compare $\angle m + \angle n$ with $\angle n + \angle A + \angle C$.

Therefore—

107. Cor. *A triangle can have at most one obtuse or one right angle.*

108. Cor. *The acute angles of a right triangle are complementary.*

1. Draw through B of the figure of § 105 a line parallel to AC and prove the theorem.

2. Extend side CA through A to a point R and prove the theorem using the angles p, RAD, RAM.

3. Make the constructions of exercises 1 and 2 for exterior angles at B and C and prove the theorem.

4. The bisector of one of two vertical angles is the bisector of the other.

5. The bisectors of two vertical angles are in the same straight line. Sug. Prove that the bisectors form a straight angle.

6. The bisectors of two supplementary adjacent angles are perpendicular to each other.

PROPOSITION XXI.

109. Theorem. *If two triangles have two angles of the one equal respectively to two angles of the other, the third angles are equal.*

Given $\triangle ABC$ and $\triangle A'B'C'$ with $\angle A = \angle A'$ and $\angle B = \angle B'$.

To Prove $\angle C = \angle C'$.

Sug. Use § 106.

110. Cor. *If two right triangles have an acute angle of the one equal to an acute angle of the other, the remaining angles are equal.*

1. How many degrees in each angle of an equilateral triangle?

2. Construct an angle of 60°; an angle of 30°; an angle of 45°. Use dividers and straight edge.

3. Construct a right triangle with acute angles of 60° and 30°.

4. Trisect a right angle.

As a *general* problem the trisection of an angle is impossible by the use of dividers and straight edge only. The pupil should note that this example is a *particular* case of the unsolved general problem.

5. How many degrees in each angle at the base of an isosceles right triangle?

6. The vertex angle of an isosceles triangle is 34° 40′. How many degrees in each angle at the base of the triangle?

7. An exterior angle at the base of an isosceles triangle is 100°. Find each angle of the triangle.

8. The angles of a triangle are $5x°$, $25x°$, and 30°. Find the unknown angles.

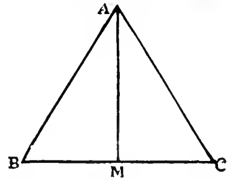
9. The acute angles of a right triangle are $x°$ and $8x°$. Find them.

10. The difference between the acute angles of a right triangle is 26°. Find them.

11. An exterior angle to one of the acute angles of a right triangle is 140°. Find the angles of the triangle.

PROPOSITION XXII.

111. THEOREM. *If two angles of a triangle are equal, the sides opposite are equal and the triangle is isosceles.*



Given $\triangle ABC$ with $\angle B = \angle C$.
To Prove $AB = AC$.

SUG. 1. Drop a perpendicular from A to BC, as AM.

2. Compare $\angle BAM$ and $\angle CAM$. Auth.

3. Compare $\triangle BAM$ and $\triangle CAM$. Auth.

4. Compare AB and AC.

Therefore—

1. The vertex angle of an isosceles triangle is ½ of an angle at the base. Find the angles of the triangle.

2. The angles of a triangle are denoted by $3x$, $7x$, and $5x$. Find each angle.

3. Two right triangles are congruent if a side and the opposite angle of the one are equal respectively to a side and the opposite angle of the other.

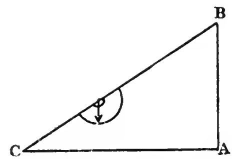
4. Perpendiculars to the sides drawn from any point in the base of an isosceles triangle make equal angles with the base.

5. Construct a protractor of wood or thick cardboard having a six to nine inch radius. To do this, place the small protractor upon the material to be used and extend the radii to the required length. Fasten a pendulum or marker at the center. This can be used to take vertical angles.

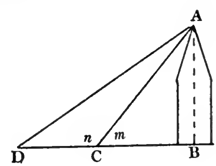
6. The acute angles of a right triangle have the ratio 2:5. Find them.

7. The angles of a triangle are x, y, z, with $x + y = 80°$ and $z - y = 50°$. Find x, y, z.

8. To find the height of a flag staff, measure back from the base of the staff on level ground a convenient distance as AC. From C sight along the straight edge of the protractor to B. The marker swinging freely will indicate upon the protractor the angle at B. (Why?) Then find $\angle C$ (How?) and on some convenient place construct a rt. triangle having the parts B, C, AC, and measure the side homologous to AB.

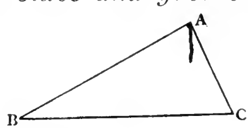


9. To find the height of a tower when the point directly under the top of the tower cannot be obtained. Measure $\angle m$ at some convenient point on level ground and then find $\angle n$. Measure a distance CD, the line BCD being straight, and at D determine the angle. On convenient ground construct a triangle with the given parts $\angle n$, $\angle D$, and side CD. Extend the side DC until it meets the perpendicular dropped to it from the point corresponding to A. The length of this perpendicular is the height of the tower.



PROPOSITION XXIII.

112. THEOREM. *Any side of a triangle is less than the sum of the two other sides and greater than their difference.*



Given $\triangle ABC$.

To Prove $AB < BC + CA$ and $AB > BC - CA$.

SUG. 1. $AB < BC + CA$. Why?

2. $AB + CA > BC$. Why?

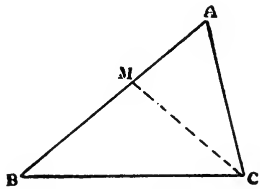
3. $\therefore AB > BC - CA$. § 49 (4).

Therefore—

1. Two sides of a triangle are 6 ft. and 8 ft. respectively. What are the limits to the length of the third side?

PROPOSITION XXIV.

113. THEOREM. *If two angles of a triangle are unequal, the sides opposite these angles are unequal and the side opposite the greater angle is the greater.*



Given $\triangle ABC$ with $\angle C > \angle B$.

To Prove $AB > AC$.

SUG. 1. Draw CM from C to AB so that $\angle BCM = \angle B$. Show that this is possible.

2. M must fall between A and B. Why?

3. Compare $CM + MA$ with AC. Auth.

4. Compare CM and BM. Auth.

5. Complete the proof.

Therefore—

114. COR. 1. *The hypotenuse is the longest side of a right triangle.*

115. COR. 2. *The perpendicular from a point to a line is shorter than any other line-segment drawn from that point to the same line.*

116. DISTANCE. The length of the perpendicular from a point to a line is the *distance from the point to the line.*

117. ALTITUDE OF A TRIANGLE. The distance from the vertex of a triangle to the base (produced if necessary) is the *altitude of the triangle.* The word "altitude" is often used to indicate the line itself as well as

its length. Since any one of the sides of a triangle may be considered as the base, every triangle has any one of three possible altitudes.

118. Median of a Triangle. The sect connecting the vertex of a triangle with the mid-point of the opposite side is a *median of the triangle.* There are in every triangle three medians.

1. The bisectors of the angles at the base of an isosceles triangle together with the base form another isosceles triangle.

2. If a line is drawn from any point in the bisector of an angle parallel to one side of the angle and is extended to meet the other side, an isosceles triangle is formed.

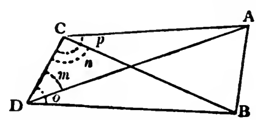
3. $\triangle ABC$ is a right triangle with the vertex of the right angle at C. Draw CD to AB so that $\angle ACD = \angle A$. Prove that D is the mid-point of AB.

Sug. Prove $AD = CD = BD$.

4. Prop. XXII is the converse of what proposition?

5. What relation does the mid-point of the hypothenuse of a right triangle sustain to the three vertices of the triangle? See Ex. 3.

6. To measure a distance between two points, A and B, both of which are inaccessible.



Sug. 1. Lay off a convenient line CD and measure CD, $\angle o$, $\angle p$, $\angle m$, and $\angle n$.

2. On CD as base construct $\triangle ACD$ and $\triangle BCD$.

3. This fixes two points corresponding to A and B, giving the required distance. It is to be noted that in these problems the method requires the construction of triangles which, if the distances are large, may be impracticable.

7. Prove that $AC > BC - AB$ in figure § 112. Why was the proof as given sufficient to cover this problem?

8. Using the dividers only show that $BC < AB + AC$.

9. Can a triangle be constructed with sides of 7 ft., 5 ft., and 13 ft.? Of 7 ft., 5 ft., and 12 ft.? Of 3 ft., 8 ft., and 10 ft.? Use the dividers in the construction.

PROPOSITION XXV.

119. THEOREM. *If two sides of a triangle are unequal, the angles opposite these sides are unequal and the angle opposite the greater side is the greater.*

Given $\triangle ABC$ with $AB > AC$.

To Prove $\angle C > \angle B$.

Proof. If $\angle C$ is not greater than $\angle B$ then $\angle C = \angle B$ or $\angle C < \angle B$.

Suppose $\angle C = \angle B$. What conclusion follows as to sides AB and AC? Auth.

What then of the supposition?

Suppose $\angle C < \angle B$. What conclusion as to the sides AB and AC? Auth.

What then of this supposition?

What possibility remains?

Therefore—

Discussion. Observe that one of three relations *must* exist between $\angle B$ and $\angle C$ and the true relation was determined by proving the impossibility of the two others. The argument does not differ from other indirect arguments that have been used except that there are three possible situations to be considered instead of two, as heretofore.

Prove Prop. XXV by a direct proof.

SUG. On AB lay off $AM = AC$ and draw CM.

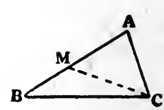
$\therefore \angle AMC = \angle ACM$. Why?

$\angle AMC > \angle B$. Why?

$\therefore \angle ACM > \angle B$.

$\angle ACB > \angle ACM$.

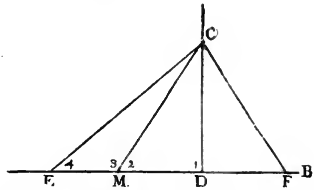
$\therefore \angle ACB > \angle B$. Why?



PROPOSITION XXVI.

120. THEOREM. *If sects are drawn from the same point in a perpendicular to a line and meet*

the line at unequal distances from the foot of the perpendicular, the sects are unequal and the more remote is the greater.



Given $OD \perp EB$ with O any point in OD and $ED > DF$.

To Prove $OE > OF$.

SUG. 1. Lay off $MD = DF$. M will lie between E and D. Why?

2. Draw OM. Then $OM = OF$. Why?

3. $\angle 1$ is a rt. angle. $\therefore \angle 2$ is acute. Why?

4. $\therefore \angle 3$ is obtuse. Why?

5. $\therefore \angle 4$ is acute. Why?

6. $\therefore \angle 3 > \angle 4$. Why?

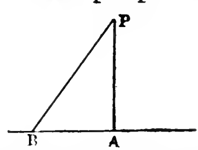
7. $\therefore OE > OM$. Why?

8. $\therefore OE > OF$. Why?

Therefore—

PROPOSITION XXVII.

121. THEOREM. *The shortest line from a point to a straight line is the perpendicular from the point to the line.*



Let PA represent the shortest line from P to line AB.

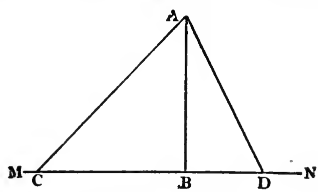
To Prove $PA \perp AB$.

Proof. If PA is not perpendicular to AB then some other line through P is perpendicular to AB. But by § 115 this new line would be shorter than PA, hence contrary to hypothesis.

Therefore—

PROPOSITION XXVIII.

122. THEOREM. *Two unequal line segments drawn from the same point in the perpendicular to a given line meet the line at unequal distances from the foot of the perpendicular, the longer segment meeting it at the greater distance.*



Given $AB \perp MN$ and $AC > AD$.

To Prove $BC > BD$.

SUG. 1. If CB is not greater than BD then $CB = BD$ or else $CB < BD$.

2. Suppose $CB = BD$, how does AC compare with AD? Why?

3. Suppose $CB < BD$, how does AC compare with AD? Why?

4. What then is the true relation of CB to BD?

Therefore—

Of what theorem is this the converse?

1. A line drawn from one end of the base of an isosceles triangle perpendicular to the opposite side makes with the base an angle equal to one-half of the vertex angle.

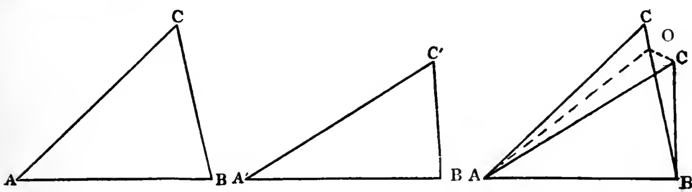
Suppose the vertex angle is obtuse.

PROPOSITION XXIX.

123. Theorem. *If two triangles have two sides of the one equal respectively to two sides of the other and the included angles unequal, the remaining sides are unequal and that side is the greater which is opposite the greater of the included angles.*

Given $\triangle ABC$ and $\triangle A'B'C'$ with $AB = A'B'$, $AC = A'C'$, and $\angle A > \angle A'$.

To Prove $BC > B'C'$.



Sug. 1. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $A'B'$ coincides with AB. Then since $\angle A > \angle A'$ side $A'C'$ will lie between AB and AC. The point C' may fall within $\triangle ABC$, on line BC, or beyond line BC.

The student should construct the figures for the first and second cases lettering them like the first figure.

2. Bisect $\angle CAC'$ and extend the bisector to meet BC at O.

3. Compare $\triangle AOC$ with $\triangle AOC'$ and OC with OC'. Auth's.

4. Compare BC' with $BO + OC'$.

5. Compare $BO + OC'$ with $BO + OC$ or its equal BC.

Therefore—

PROPOSITION XXX.

124. THEOREM. *If two triangles have two sides of the one equal respectively to two sides of the other and the third sides unequal, the included angles are unequal and that angle is the greater which is opposite the greater side.*

Given $\triangle ABC$ and $\triangle A'B'C'$, with $AB = A'B'$, $AC = A'C'$, and $BC > B'C'$.

To Prove $\angle A > \angle A'$.

SUG. 1. What three possible relations are there for $\angle$s A and A'?

2. Test each of these as was done in Prop. XXV.

3. Which is the only one not contradicted by the hypothesis?

Therefore—

125. QUADRILATERAL. A portion of a plane bounded by four sects is a *plane quadrilateral.*

126. QUADRILATERALS CLASSIFIED. A quadrilateral is a *parallelogram* if its opposite sides are parallel, a *trapezoid* if but one pair of opposite sides is parallel, a *trapezium* if no two sides are parallel. A trapezoid the non-parallel sides of which are equal is *isosceles.*

127. PARALLELOGRAMS CLASSIFIED. A parallelogram is a *rectangle* if all its angles are right angles, a *rhomboid* if all its angles are oblique, a *square* if it is an equilateral rectangle, a *rhombus* if it is an equilateral rhomboid.

With dividers (or scale) and protractor determine the character of each of the following figures.

128. Diagonal. A line-segment connecting any two non-adjacent vertices of a figure is a *diagonal.*

129. Base of a Quadrilateral. The side upon which a quadrilateral is assumed to stand is its *base.* In case of the trapezoid or parallelogram, one of the parallel sides is always considered as the base. The opposite side is the *upper base.*

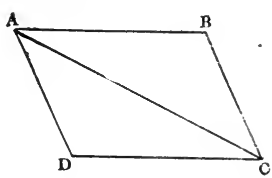
130. Altitude. The perpendicular distance between the bases of a parallelogram or a trapezoid is its *altitude.* The word "altitude" is often used to indicate the line itself as well as its length.

131. Diameter. The sect which connects the mid-points of two opposite sides of a parallelogram or the mid points of the non-parallel sides of a trapezoid is a diameter.

Construct each of the figures above defined and draw for each the altitude and indicate the bases and diameters.

PROPOSITION XXXI.

132. Theorem. *The opposite sides of a parallelogram are equal.*



Given ▱ $ABCD$.

To Prove $AB = DC$ and $AD = CB$.

Sug. 1. Draw a diagonal, as AB, and compare △ ABC with △ CDA. Auth.

2. Compare AD with CB AB with CD.

Therefore—

133. Cor. 1. *A diagonal divides a parallelogram into two congruent triangles.*

134. Cor. 2. *Segments of parallel lines intercepted between parallel lines are equal.*

135. Cor. 3. *Perpendicular segments between parallel lines are equal.*

136. Cor. 4. *The perpendicular segment intercepted between the base of a triangle and a line through the vertex parallel to the base is equal to the perpendicular from the vertex to the base.*

137. Distance. The length of the perpendicular segment intercepted by two parallel lines is the distance between them.

PROPOSITION XXXII.

138. Theorem. *The opposite angles of a parallelogram are equal and any two consecutive angles are supplementary.*

See § 97 and § 102.

PROPOSITION XXXIII.

139. Theorem. *The diagonals of a parallelogram bisect each other.*

The construction of the figure is left to the student.

Given ▱ $ABCD$ with diagonals AC and BD intersecting at O.

To Prove $AO = OC$ and $BO = OD$.

Sug. Select appropriate △ and compare them.

Therefore—

PROPOSITION XXXIV.

140. Theorem. *If a quadrilateral has two of its opposite sides equal and parallel, it is a parallelogram.*

The construction of the figure is left to the student.

Given a quadrilateral *ABCD,* with $AB = DC$ and $AB \parallel DC$.

To Prove *ABCD* a ▱.

SUG. 1. What is a parallelogram?

2. How much of the definition is given in the hypothesis and what remains to be proved?

3. Draw a diagonal as *AC,* compare $\angle BAC$ with $\angle ACD$ and then compare $\triangle ABC$ with $\triangle ACD$. Complete the demonstration.

Therefore—

141. A careful distinction must be drawn between statements which are definitions, i.e., which are necessary and at the same time sufficient to characterize the thing defined as distinct from all other things, and statements which point out certain properties of the thing in question without being in themselves complete enough to make this distinction. For example—since the hypothesis of Prop. XXXIV is sufficient to characterize the quadrilateral as a parallelogram according to the definition in § 126, it might be taken as the definition while Props. XXXI and XXXII could not be so used for the properties there indicated are possessed by other figures than parallelograms.

1. If one angle of a parallelogram is a right angle, the figure is a rectangle.

2. The sum of the angles of a quadrilateral equals four right angles.

3. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram. This bears what relation to § 139?

4. If the opposite angles of a quadrilateral are equal the figure is a parallelogram.

SUG. Compare the sum of two consecutive angles with four right angles.

PROPOSITION XXXV.

142. THEOREM. *A quadrilateral the opposite sides of which are equal is a parallelogram.*

The construction of the figure is left to the student.

Given ▱ $ABCD$ with $AB = DC$ and $AD = BC$.

To Prove $ABCD$ a ▱.

SUG. 1. Draw the diagonal AC.

2. The conditions of § 126 or § 140 may be used to identify the parallelogram.

3. What part of either is in the hypothesis of this theorem?

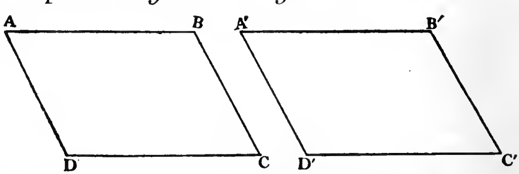
4. What remains to be proved? Complete the demonstration.

Therefore—

In the light of § 141 what relation does the statement of this theorem bear to the parallelogram?

PROPOSITION XXXVI.

143. THEOREM. *Two parallelograms which have two sides and the included angle of the one equal to two sides and the included angle of the other respectively are congruent.*



Given ▱s AC and $A'C'$, with $AB = A'B'$, $AD = A'D'$, and $\angle A = \angle A'$.

To Prove ▱ $AC \cong$ ▱ $A'C'$.

SUG. 1. Place ▱ AC upon ▱ $A'C'$ so that AB coincides with $A'B'$.

2. What direction does AD take? Why? Where does D fall? Why?

3. What direction does DC take and why?

4. Where does point C fall and why?

5. Complete the demonstration.

Therefore—

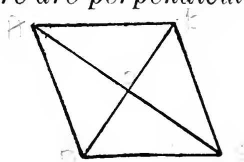
PROPOSITION XXXVII.

144. THEOREM. *The diagonals of the rhombus and of the square bisect the angles.*

The proof is left to the student.

PROPOSITION XXXVIII.

145. THEOREM. *The diagonals of the rhombus and of the square are perpendicular to each other.*



Given rhombus $ABCD$ (or a square) with **diagonals** AC and BD.

To Prove $AC \perp BD$.

SUG. 1. In $\triangle AOD$ and $\triangle AOB$ compare $\angle s$ AOD and AOB.

2. Complete the demonstration.

3. Note that no reference is made to the character of the angles A, B, C, D. What conclusion can be drawn as to the application of the demonstration to the square as well as the rhombus?

Therefore—

PROPOSITION XXXIX.

146. THEOREM. *A diameter of a parallelogram divides it into two congruent parallelograms.*

The proof is left to the student.

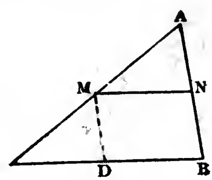
147. COR. 1. *A diameter of a parallelogram is parallel to the corresponding base.*

148. COR. 2. *The diameters of a parallelogram bisect each other.*

149. COR. 3. *A line which is parallel to the base of a parallelogram and which bisects one side bisects the other side also.*

PROPOSITION XL.

150. THEOREM. ***A line which bisects one side of a triangle and is parallel to the base bisects the other side and equals half the base.***



Given △ ABC with $MN \parallel CB$ and M mid-point of AC.

To Prove $AN = NB$ and $2\ MN = CB$.

SUG. 1. Draw $MD \parallel AB$.

2. Compare △ AMN and MCD; MD and AN; MD and NB; AN and NB. Auth.

3. Compare MN and CD; MN and DB; MN and CB. Auth.

Therefore—

1. The angle formed by the bisectors of the angles at the base of an isosceles triangle is equal to an exterior angle at the base of the triangle.

151. Cor. *The line connecting the mid-points of the sides of a triangle is parallel to the base and equal to one half the base.*

Sug. 1. Work out an indirect demonstration.

2. Work out a direct demonstration from the above figure, noting that $MN \parallel CB$ if $MNBD$ is a ▱ or if $\angle AMN = \angle MCD$. § 150.

PROPOSITION XLI.

152. Theorem. *If a line bisects one of the non-parallel sides of a trapezoid and is parallel to the base, it bisects the other side also, and equals half the sum of the bases.*

Sug. Draw a diagonal of the trapezoid.

The proof is left to the student.

153. Cor. *The diameter of a trapezoid is parallel to the bases and equal to half their sum.*

Sug. 1. Through one end of the diameter draw a line parallel to the bases and use the indirect method. Or

2. Through one end of the diameter draw a line parallel to the opposite side and use the direct method.

1. If the adjacent sides of a parallelogram are not equal the diagonals are not perpendicular to each other.

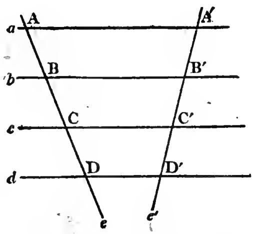
2. If the diagonals of a parallelogram intersect at right angles the figure is a rhombus or a square.

3. If the diagonals of a parallelogram are equal the figure is a rectangle.

4. The opposite sides of a window frame are equal. Before putting on stay laths the carpenter "squares it." At how many corners should he test it?

PROPOSITION XLII.

154. THEOREM. *If parallel lines intercept equal segments on one transversal, they intercept equal segments on all transversals.*



Given the parallel lines a, b, c, d, etc., with transversals e and e' such that $AB = BC = CD$.

To Prove $A'B' = B'C' = C'D'$.

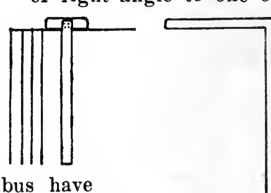
SUG. 1. e and e' may or may not be parallel. What figures are formed in each case?

2. In case e and e' are not parallel draw lines through A, B, C, etc., parallel to e' and terminating on the next following of the parallels.

3. Complete the proof.

Therefore—

1. Having laid out a plat or four sided frame so that the opposite sides are equal, how can one tell whether or not it is a rectangle without applying a "square" or right angle to one of the angles?



2. A draftsman wishing to draw parallel lines uses a T-square. Explain the principle of its use. How may parallel lines be drawn with a carpenter's square?

3. Can a rectangle and a rhombus have sides respectively equal? Construct a rectangle of wood, one nail at each vertex. Can it be distorted into a rhombus?

4. Explain the principle of the parallel rulers.

5. By experiment test which has the greater enclosed area, a rectangle or a rhomboid, the sides of the two being respectively equal. Squared paper can be used.

6. Construct a parallelogram having given two adjacent sides and the included angle, using dividers and straight edge. How could it be done with a protractor and straight edge?

7. Draw a trapezium connecting the mid-points of its adjacent sides. What kind of a quadrilateral is formed?

SUG. Draw a diagonal of the given figure.

8. Prove that the sects that connect the mid-points of opposite sides of a trapezium bisect each other.

The following problems are intended for use with pupils who have had or who are taking elementary Physics.

9. Two balls of the same size and rolling with the same speed at right angles to each other on a level surface strike another ball which is at rest. What direction will the latter ball take? If one ball has twice the force of the other, what direction will the third ball take?

NOTE. The force which would give to the third ball the same motion as is given to it by the two balls is called the resultant force or resultant of the forces due to the two balls independently.

10. Draw a parallelogram to represent the forces and directions of the three balls in the first part of Ex. 9. This parallelogram is *called the parallelogram* of forces.

11. Draw the parallelogram of forces for the second part of problem 9, representing a foot by ¼ inch.

12. Suppose the balls in Ex. 9 with equal force strike the third ball at an angle of 45°. Draw the parallelogram of forces and a line representing the resultant force.

13. Two forces, one of 5 lbs. and one of 8 lbs., strike a body at an angle of 60°. Draw the resultant. If they meet at an angle of 100°, draw the resultant. If one is a force of 7 lbs. and the other 21 lbs. and they meet at an angle of 100°, draw the resultant.

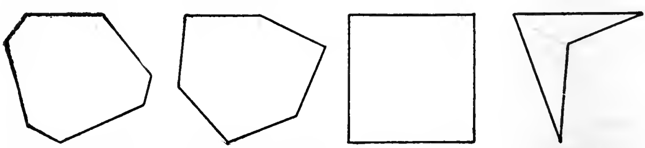
155. Polygons Classified. Beginning with the triangle and proceeding in order to polygons of four sides, five sides, etc., the plane polygons are the *triangle, quadrilateral, pentagon, hexagon, heptagon, octogon,* etc. A polygon of n sides is called an *n-gon.*

156. A polygon is *equilateral* if all its sides are equal, *equiangular* if all its angles are equal, and *regular* if it is both equilateral and equiangular.

Two polygons are *mutually equilateral* if the sides of the one are equal respectively to the sides of the other, *mutually equiangular* if the angles of one are equal respectively to the angles of the other.

The student should prove that two figures both mutually equilateral and mutually equiangular are congruent.

157. A polygon is *convex* if each of its angles is less than a straight angle and *concave* if one or more of its angles are reflex. In this book the term polygon will mean a convex polygon.



EQUIANGULAR EQUILATERAL REGULAR CONCAVE

158. Cor. *In any polygon the number of vertices is the same as the number of sides.*

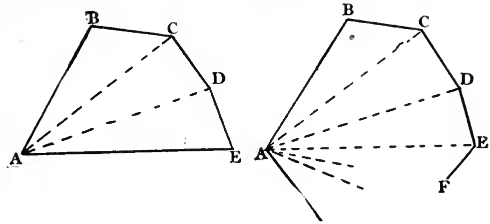
1. If ABC is a right triangle with the right angle at C and CD is drawn to the hypotenuse so that $\angle ACD = \angle B$, then $CD \perp AB$. Prove $\angle DCB = \angle A$.

2. If two parallel straight lines are cut by a transversal, the bisectors of the interior angles form a rectangle.

3. What relative positions of parallels and transversals will form a square?

PROPOSITION XLIII.

159. THEOREM. *The sum of the angles of a polygon is equal to $(n-2)$ straight angles, where n represents the number of sides.*



Given polygon $ABCD\ldots$, having n sides.

To Prove $\angle A + \angle B + \angle C +$ etc. $= (n-2)$ straight angles.

SUG. 1. Draw all possible diagonals from some one vertex as A.

2. Then $(n-2)$ triangles are formed. Explain why.

3. How many straight angles in the sum of all the interior angles in all these triangles?

4. Complete the proof.

Therefore—

160. COR. I. *The sum of the angles of a polygon is equal to $(2n-4)$ rt. angles, or $2n$ rt. $\measuredangle$ — 4 rt. $\measuredangle$.*

161. COR. II. *Each angle of an equiangular polygon is equal to* $\frac{2n-4}{n}$ *rt.* $\measuredangle$.

1. What is the sum of the angles of a quadrilateral? A pentagon? A heptagon? A decagon?

2. What is the size of each angle of a regular hexagon? A regular octagon? A regular decagon?

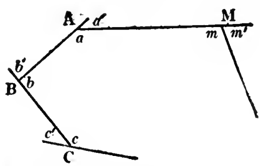
3. How many sides has a regular polygon if each angle is 156°?

4. Four of the angles of a pentagon are 100°, 170°, 95°, and 115° respectively. Find the fifth angle.

Sug. Form an equation from Prop. XLIII and solve algebraically.

PROPOSITION XLIV.

162. Theorem. *If one exterior angle be formed at each vertex of a polygon, the sum of these exterior angles is equal to four right angles.*



Given polygon MC, having n sides with one exterior angle formed at each vertex, as $\angle a'$ formed at vertex A with interior $\angle a$.

To Prove $\angle a' + \angle b' + \angle c' + \ldots = 4$ rt. $\angle s$.

Sug. 1. The polygon has n vertices. What is the sum of the exterior and the interior angle at each vertex?

2. What is the sum for all vertices?

3. What is the sum of the interior angles?

4. Complete the proof.

Therefore—

163. Cor. *The sum of all the exterior angles of any polygon is equal to eight straight angles.*

1. How many degrees in each exterior angle of a regular pentagon? Of a regular octogon?

2. How many sides has the polygon, the sum of the exterior angles of which is equal to the sum of the interior angles?

3. An exterior angle of a regular polygon is 60°. How many sides has the polygon? 45°?

4. An exterior angle of a regular polygon is $\frac{2}{3}$ of the adjacent angle. Find the number of sides of the polygon.

5. Each angle of a regular polygon is $\frac{5}{6}$ of a straight angle. Find the number of the sides of the polygon.

164. Locus. To locate a point definitely in a plane two conditions limiting its position must be given. If but one condition is given the point is limited to one or more lines. For example, if all that is known of a point is that it is four inches from a given straight line, the point is not definitely fixed in position but lies on either of two lines and may be any point on either. The lines in question are parallel to the given line, lying the one on one side and the other on the other side of the given line, at four inches distance, as will later be established.

The *locus* of a point is defined as the line or group of lines to which a point is limited, any point of which satisfies the given conditions. The locus of a point is both inclusive and exclusive. It includes all the points which satisfy the given condition and excludes all points which do not.

165. Locus Generated. The locus of a point may be regarded as the path of a point which moves according to a given law or condition. For example, the *law* in the illustration used in the preceding section is that the point must keep at the distance of four inches from the given line. The expressions, *locus of a point* and *locus of points* are both used.

Discuss without formal proof the following loci:

1. What is the locus of a ship at sea 20° N. Latitude?

2. A man lives one block west of the north and south street which passes the school house. What is the locus of his house?

3. A man is $\frac{1}{2}$ mile from the street or road in front of the school. What is his locus?

4. An article was lost ten feet west of the outer edge of a straight side walk. Where should one look for it, i. e. what is its locus?

5. In Ex. 4 change the words "west of" to "from." What is the locus?

6. What is the locus of the center of the headlight of a locomotive when in motion?

7. A farm is in range seven, what is its locus? In township fifteen, what is its locus?

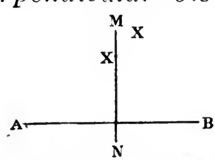
8. What is the locus of the tip of the pendulum of a clock?

9. What is the locus of the hub of a wheel on a moving auto?

166. To Verify Locus Conditions. To prove that a given line (or lines) is the *locus of points* satisfying a given condition it is necessary to prove two things; first —any point in the line satisfies the given condition and second—no point outside the line satisfies the given condition. For an example, see § 167.

PROPOSITION XLV.

167. Theorem. *The locus of points equally distant from the extremities of a given line segment is the perpendicular bisector of the line segment.*



Given line segment AB, with MN the perpendicular bisector of AB.

To Prove MN the locus of points equidistant from A and B.

Sug. 1. See Prop. V.

2. Let X' be a point not on MN. Compare $X'A$ and $X'B$. Auth.

3. Apply definition of locus.

Therefore—

168. Cor. *The locus of points equidistant from two given points is the perpendicular bisector of the line joining them.*

PROPOSITION XLVI.

169. Theorem. *The locus of points equidistant from the sides of an angle is the bisector of the angle.*

See examples 1 and 2 on p. 37.

170. Cor. *The locus of points equidistant from two intersecting lines is the pair of lines that bisect the angles formed by the given lines.*

What relation do these locus lines bear to each other?

PROPOSITION XLVII.

171. Theorem. *The locus of points at a given distance from a given line is the pair of lines parallel to the given line and at the given distance from it.*

Sug. 1. Through any point in the given line draw a perpendicular, on this line locate the two points which are at the required distance from the given line and through these points draw lines parallel to the given line.

2. What two facts must be established in order to prove these two constructed lines to be the desired locus?

Therefore—

1. If the opposite angles of a parallelogram are bisected by the diagonals, the figure is equilateral.

Sug. Draw one diagonal.

PROPOSITION XLVIII.

172. THEOREM. *The locus of points equally distant from two parallel lines is the line parallel to them and midway between them.*

SUG. 1. Construct a segment perpendicular to and intercepted by them.

2. Construct the perpendicular bisector of this segment.

3. Complete the proof.

Therefore—

173. USE OF LOCI. A common method of locating a point in a plane is to establish two loci of the point, the intersection of which is the required point. To use this method of attack successfully the student must bear in mind that two statements are made when a line, as *AB*, is said to be the locus of an unknown point *X*, viz: any point in *AB* can be *X* and no point outside of *AB* can be *X*, i.e. *X* must be in *AB*.

1. Find point *X* if it is to be in a given line and equally distant from two points.

GIVEN line *CD* and the points *A* and *B* (which may lie, one or both, on *CD*). *To find* point *X* in *CD* and equally distant from *A* and *B*.

SUG. 1. What is the locus of points equally distant from *A* and *B*?

2. Where then must *X* lie?

3. What position of the points *A* and *B* with respect to *CD* would make the problem impossible?

2. Find point *X* if it is in a given line and equally distant from two intersecting lines.

SUG. 1. What is the locus of points equally distant from two intersecting lines?

2. Complete the demonstration.

3. Is this problem ever impossible? What is the condition?

4. What relative position of the given lines would admit of an unlimited number of positions for point *X*?

3. Find point *X* if it is equally distant from two intersecting lines and equally distant from the extremities of a given line-segment. Is there any arrangement of the given lines which will make this problem impossible?

4. Find point *X* if equally distant from the sides of a given angle and equally distant from two parallel lines.

5. Find point *X* if equally distant from two points and equally distant from two parallel lines.

6. Find point *X* if it lies in one given line and is equally distant from another given line. Are there conditions which make the problem impossible?

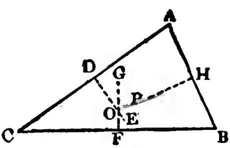
7. Find *X* if it is equally distant from two intersecting lines and also a given distance from a third line.

In general, how many such points are there? What is the least possible number? What is the greatest possible number?

174. CONCURRENT LINES. Two or more lines having one point in common are *concurrent lines.*

The construction of loci is a useful method of proving certain lines concurrent. Name the concurrent lines in the above exercises.

1. The perpendicular bisectors of the sides of a triangle are concurrent.



SUG. 1. Two of the bisectors as *DE* and *FG* must intersect as at *O*. Why? What loci are here involved?

2. $OA = OC$ and $OC = OB$. Why?

3. Therefore *O* is equidistant from *B* and *A*. Why?

4. In what third line must *O* then lie? Why?

2. The bisectors of the angles of a triangle are concurrent.

SUG. 1. Two of the bisectors must intersect. Why? What loci are involved?

2. Prove their intersection to be in the third bisector.

175. SUMMARY OF BOOK I.

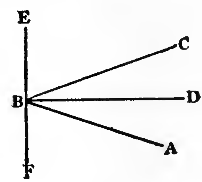
State all the authorities by which

(1) two triangles may be proved congruent.

(2) two line segments may be proved equal.
(3) two lines may be proved parallel.
(4) two lines may be proved perpendicular to each other.
(5) two angles may be proved unequal.
(6) two line segments may be proved unequal.
(7) a quadrilateral may be proved a parallelogram.
(8) a locus has been established.

These authorities should be carefully arranged, written out in a note book, and memorized. They are the "tools" to be used in future constructions and demonstrations. Geometry is not half learned if the student is unable to recall the various hypotheses which are available for deducing any desired conclusion.

1. BD bisects $\angle ABC$ and $EF \perp BD$ at B. Prove $\angle FBA = \angle CBE$. Prove the theorem when EF cuts BD at some point other than B.



2. If two straight line segments bisect each other at right angles any point in either is equidistant from the extremities of the other.

3. By how much does the supplement of an acute angle exceed the complement of the same angle?

4. If one of two supplementary adjacent angles is bisected, the perpendicular to this bisector at the vertex bisects the other angle.

5. If the bisectors of two adjacent angles are perpendicular to each other the angles are supplementary. Compare with Ex. 4.

6. The perimeter of a triangle is less than twice the sum of the medians.

7. The bisector of an angle of a triangle and the bisectors of the exterior angles of the two other vertices meet in a point which is equidistant from the sides of the triangle. Use loci.

8. If a line intersects the sides of an isosceles triangle at equal distances from the vertex, it is parallel to the base.

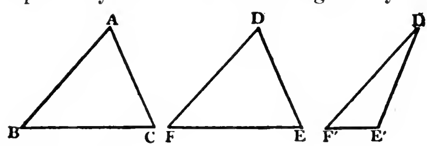
9. The line joining the feet of the perpendiculars from the extremities of the base of an isosceles triangle to the sides is parallel to the base.

10. Two angles having their sides respectively parallel, one pair of parallel sides having the same direction and the other pair having opposite directions, are supplementary.

11. Two triangles having two angles and a side opposite one of them in the first equal respectively to two angles and a corresponding side in the other are congruent.

12. Two isosceles triangles having equal bases and equal vertex angles are congruent.

13. Two triangles having two sides and an angle opposite one of them in the one triangle equal to two sides and an angle opposite one of them respectively in the other triangle may or may not be congruent.



GIVEN $\triangle ABC$ and $\triangle DEF$ or $\triangle D'E'F'$, with $AB = DF = D'F'$, $AC = DE = D'E'$, $\angle B = \angle F = \angle F'$.

TO PROVE $\triangle ABC \cong \triangle DEF$ but not congruent to $\triangle D'E'F'$.

It is to be noted that of the two triangles DEF and $D'E'F'$ one is acute and one is obtuse, and each have the required parts. Obviously $\triangle ABC$ is congruent to but one. The student should give detailed proof for the figures as drawn.

14. The sum of the exterior angles at the base of a triangle is equal to two right angles plus the vertex angle.

15. In $\triangle ABC$, $\angle C$ is twice the sum of $\angle A$ and $\angle B$ and $\angle B$ is twice $\angle A$. Find A, B, C.

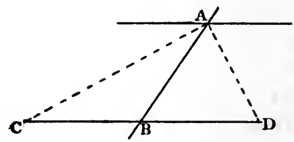
16. The exterior angles formed by producing the sides of an isosceles triangle beyond the base are equal.

17. Given a square $ABCD$. Draw the diagonal CD and on CD lay of $CE = CB$. Draw $EF \perp CD$ at E and extend to DB as at F. Prove $DE = EF = BF$.

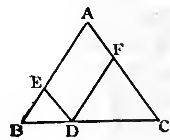
18. A straight line segment intercepted between parallels is bisected and another straight line is drawn through the point of bisection. Prove that the segment of this line intercepted between the parallels is bisected.

19. Find the sum of the interior angles of a concave polygon of 13 sides.

20. A segment is intercepted between two parallels and the adjacent angles formed by the segment and one of the parallels are bisected. Prove that the segments on the other parallel are equal.



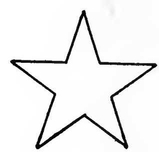
21. *ABC* is an isosceles triangle. *DE* and *DF* are parallel to *AC* and *AB* respectively. Prove the perimeter of *AFDE* equal to $AB + AC$.



22. If the legs of a trapezoid are equal, they make equal angles with the parallel sides.

23. Prove the converse of Ex. 22.

24. Prove that the sum of the angles at the vertices of the conventional five pointed star is equal to two right angles. Use the figure as drawn.



25. Connect the vertices of the star and give a different demonstration.

26. How many right angles in the sum of the vertex angles of a six pointed star? Of an eight pointed star?

27. If the vertex angle of an isosceles triangle is one-half as great as an angle at the base, the bisector of a base angle divides the given triangle into two isosceles triangles.

28. How many equiangular triangles can have a common vertex? How many rectangles? How many equiangular pentagons? Hexagons? Equiangular figures of a still greater number of sides?

29. Which of the above figures will exactly fill up the space about the common vertex?

30. Which of the above figures can be used for a patch work quilt or a mosaic design?

CHAPTER II.

CIRCLES.

176. CIRCLE. A closed curve in a plane all points of which are the same distance from a fixed point in the plane is a *circle*. The fixed point is the *center* of the circle. The circle completely encloses a portion of the plane. Thus a *circle* is the locus of points in a plane at a given distance from a fixed point in the plane.

It has been more or less customary for writers on geometry to define the circle as the portion of the plane enclosed by the curve and to call the curve the *circumference* of the circle. The above use of the term is more convenient and accords more perfectly with its general use.

177. LENGTH OF A CIRCLE. The length of the curve is called the *length of the circle*.

178. RADIUS. A straight sect joining the center and any point of the circle is a *radius*.

179. CHORD. A straight sect terminated at both ends by the circle is a *chord*.

180. DIAMETER. A chord which passes through the center is a *diameter*.

181. ARC. Any portion of a circle is an *arc*.

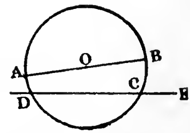
An arc which is half of a circle is a *semicircle*. An arc less than a semicircle is a *minor arc* and one which is greater than a semicircle is a *major arc*.

182. CENTRAL ANGLE. An angle the vertex of which

is the center of a circle and the sides of which are radii is a *central angle.*

183. Secant. A straight line that intersects a circle twice is a *secant.*

A secant is a chord produced.



OA and *OB* are radii, *AB* is a diameter, *CD* is a chord and *ED* is a secant.

184. Subtends. A chord *subtends* the two arcs which have the same extremities as itself. Unless otherwise stated the arc subtended by a chord is understood to be the minor arc.

185. Preliminary Theorems and Assumptions.

1. All radii of a circle are equal.
2. The diameter of a circle is twice the radius.
3. All diameters of a circle are equal.
4. The distance from any point in the plane to the center of a circle is greater than, less than, or equal to a radius according as the point is outside the circle, on the circle, or within the circle.
5. If the radii or diameters of two circles are equal the circles are equal. Since all circles are of the same shape this implies congruence as well.
6. If two circles are equal, the radii and the diameters of the two are equal.
7. An unlimited straight line that lies partly within a circle cuts the circle in two points.
8. If the end points of two minor arcs or two major arcs on the same or on equal circles can be made to coincide the arcs can be made to coincide.
9. If two arcs of two circles are equal the circles are equal.

186. Postulate. *A circle may be constructed about any given point in the plane as center with any given sect as radius.*

The actual construction of such a circle is done by aid of the dividers.

187. Cor. *A circle may be constructed upon any given sect as diameter.*

Sug. What must be done to the given diameter to find the center of the required circle?

188. Problem. *Given an arc of a circle, to find the center.*

Sug. If from the given arc MC two loci of the center can be determined, the center can be found.

2. Draw two chords as AB and AC. Where must the center lie with respect to A and B? With respect to A and C?

3. Construct the loci of points which satisfy these respective conditions. What can be said of the intersection of these loci?

4. A circle about this point O as a center with radius OA will pass through points A, B, C and therefore by § 185 (8) will embrace the given arc.

Therefore point O is the required center.

189. Problem. *To draw a circle through three points not in the same straight line.*

Sug. Make the construction from § 188, with A, B, C as given points in required ⊙.

190. Cor. I. *A circle cannot be drawn through three points which are in a straight line.*

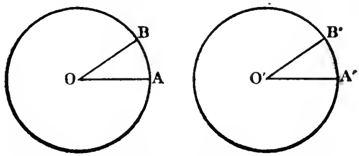
191. Cor. II. *Only one circle can be drawn through three points not in a straight line.*

Sug. It was seen in § 189 that at least one such circle can be drawn. From § 185 (8) it follows that there cannot be more than one. Why? Or, take a point X as any other center. Can X be equally distant from A and B? from A and C?

1. How many circles can be drawn through two points?
2. Two circles can intersect in two points at most.
3. Given arc M, find the center.

PROPOSITION I.

192. Theorem. *In the same circle, or in equal circles, equal central angles intercept equal arcs.*



Given circle O equal to circle O' and $\angle AOB = \angle A'O'B'$.

To Prove arc $AB =$ arc $A'B'$.

Proof. Sug. 1. Place $\odot O$ on $\odot O'$ with O on O'. Then the two circles will coincide, having a common center and equal radii.

2. Rotate $\odot O$ about O' until point A falls on A'.

3. Then since $\angle AOB = \angle A'O'B'$ the radii OB and $O'B'$ will coincide and B fall on B'.

4. Hence the arcs AB and $A'B'$ coincide. § 185 (8).

Therefore—

1. How can a carpenter test the accuracy of his "square" without using a previously made right angle?

PROPOSITION II.

193. THEOREM. *In the same circle, or in equal circles, equal arcs subtend equal central angles.*

Given ⊙ O and ⊙ O' with arc $AB =$ arc $A'B'$. (Use fig. Prop. I.)

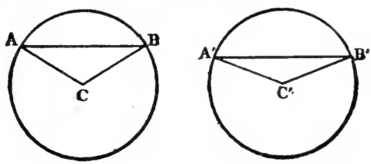
To Prove $\angle AOB = \angle A'O'B'$,

SUG. Place the circles together as in Prop. I and show that the two angles can be made to coincide.

Therefore—

PROPOSITION III.

194. THEOREM. *If, in the same or in equal circles, two central angles are unequal, the greater angle intercepts the greater arc.*



Given ⊙ C equal to ⊙ C' and $\angle ACB < \angle A'C'B'$.

To Prove arc $AB <$ arc $A'B'$.

Proof. SUG. 1. Place ⊙ C upon ⊙ C' as in Prop. I and rotate it until CA coincide with $C'A'$.

2. Where will CB fall with reference to $\angle A'C'B'$? Why?

3. Where will B fall with reference to the arc $A'B'$? Why?

4. Compare arcs $A'B$ and $A'B'$; arcs AB and $A'B'$.

Therefore—

1. A diameter is greater than any other chord.

SUG. Draw any chord not a diameter and draw radii to its extremities. Compare the chord with the sum of the radii.

PROPOSITION IV.

195. THEOREM. *If two arcs, in the same or in equal circles, are unequal, the greater arc subtends the greater central angle.*

Given $\odot C$ equal to $\odot C'$ and arc $AB <$ arc $A'B'$. (Use fig. of Prop. III.)

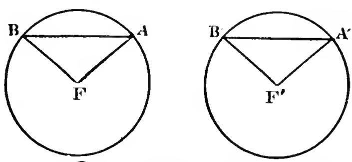
To Prove $\angle ACB < \angle A'C'B'$.

Proof. SUG. Place $\odot C$ upon $\odot C'$ as in Prop. III and show that $\angle ACB$ equals a part of $\angle A'C'B'$.

Therefore—

PROPOSITION V.

196. THEOREM. *In the same circle, or in equal circles, equal chords subtend equal arcs.*



Given $\odot F$ and $\odot F'$ equal with chord $AB =$ chord $A'B'$.

To Prove arc $AB =$ arc $A'B'$,

Proof. SUG. 1. Draw the radii $FA, FB, F'A'$ $F'B'$.

2. The arcs are equal provided $\angle F = \angle F'$. Why?

3. Prove these angles equal and complete the demonstration.

Therefore—

1. Napoleon and his engineer in exploring came to a river. Napoleon asked its width. The engineer sighted from the rim of his cap to the opposite side, swung upon his heel, and sighted to a point on the land, then paced to the point and said "Ten rods, Sire." Upon what proposition did his computation depend?

PROPOSITION VI.

197. THEOREM. *In the same circle, or in equal circles, chords which subtend equal arcs are equal.*

Given two equal ⓢ, F and F' with arc $AB =$ arc $A'B'$. (Use fig. of Prop. V.)

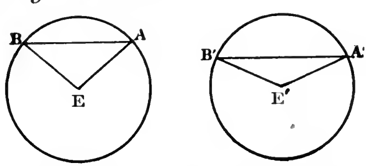
To Prove chord $AB =$ chord $A'B'$.

Proof. SUG. Compare the central angles F and F' and complete the demonstration.

Therefore—

PROPOSITION VII.

198. THEOREM. *In the same circle, or in equal circles, two chords which subtend unequal minor arcs are unequal and the greater chord subtends the greater arc.*



Given equal ⓢ E and E' with arc $AB <$ arc $A'B'$.

To Prove chord $AB <$ chord $A'B'$.

Proof. SUG. 1. Draw radii EA, EB, $E'A'$, $E'B'$.
2. Compare ∡ E and E'. Auth.?
3. Compare chords AB and $A'B'$.

Therefore—

1. If the mid-points of the three sides of a triangle be joined by straight lines, the triangle is divided into four congruent triangles.

2. If the mid-points of two opposite sides of a quadrilateral be joined to the mid-points of the diagonals, the joining lines form a parallelogram. As one particular case let the quadrilateral be a parallelogram. Is this case an exception?

PROPOSITION VIII.

199. THEOREM. *In the same circle, or in equal circles, two arcs which subtend unequal chords are unequal and the greater arc subtends the greater chord.*

Given equal ⓢ E and E' with chord $AB <$ chord $A'B'$. (Use fig. of Prop. VII.)

To Prove arc $AB <$ arc $A'B'$.

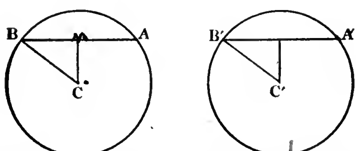
Proof. SUG. 1. Compare ∠s E and E'. By what theorem?

2. Complete the demonstration.

Therefore—

PROPOSITION IX.

200. THEOREM. *In the same circle, or in equal circles, equal chords are equally distant from the center.*



Given equal ⓢ C and C' with chord $AB =$ chord $A'B'$.

To Prove AB and $A'B'$ equally distant from C and C' respectively.

Proof. SUG. 1. Draw $CM \perp AB$ and $C'M' \perp A'B'$. Why? Also draw CB and $C'B'$. Why?

2. It is necessary to prove $CM = C'M'$. Why?

3. Complete the proof.

Therefore—

PROPOSITION X.

201. THEOREM. *In the same circle, or in equal circles, chords which are equally distant from the center are equal.*

Given ⓢ C and C' with chords AB and $A'B'$ such that their distances CM and $C'M'$ from the respective centers are equal. (Use fig. of Prop. IX.)

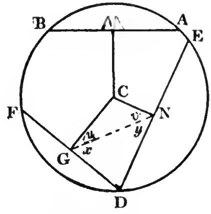
To Prove chord $AB =$ chord $A'B'$.

Proof. SUG. Compare MB and $M'B'$. Complete the demonstration.

Therefore—

PROPOSITION XI.

202. THEOREM. *In the same circle, or in equal circles, unequal chords are unequally distant from the center and the shorter chord is at the greater distance.*



Given ⊙ C with chord $AB <$ chord ED, CM and CN being the respective distances of the chords from the center.

To Prove $CM > CN$.

Proof. SUG. 1. Draw chord $FD = AB$ and $CG \perp FD$. Connect G and N.

2. Compare $\angle x$ with $\angle y$. Auth.
3. Hence $\angle u < \angle v$. Why?
4. $\therefore CG > CN$. Why?
5. $\therefore CM > CN$. Why?

Therefore—

PROPOSITION XII.

203. Theorem. *In the same circle, or in equal circles, chords which are unequally distant from the center are unequal and that chord which is at the greater distance is the shorter.*

Given ⊙ C, CM and CN being the respective distances of two chords AB and ED from the center. Also $CN < CM$. (Use fig. of Prop. XI.)

To Prove $AB < ED$.

Proof. Sug. 1. What three possibilities exist as to the relative sizes of AB and ED?

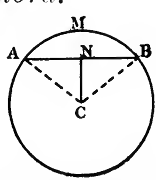
2. Assume each in turn to be true. Which is the only one which does not by a former theorem lead to a conclusion contradictory to the hypothesis?

Therefore—

Prove prop. § 203 by direct method (use fig. § 202).

PROPOSITION XIII.

204. Theorem. *A radius perpendicular to a chord bisects the chord.*



Given ⊙ C with chord AB and radius $CN \perp AB$ at N.

To Prove $AN = NB$.

Proof. Sug. 1. Draw radii CA and CB. Why?

2. Compare $\triangle CNA$ with $\triangle CNB$. Auth.

3. Compare AN and NB. Auth.

Therefore—

205. Cor. 1. *A radius perpendicular to a chord bisects the arc subtended by the chord.*

Sug. Compare the two arcs by means of the subtended central angles.

206. Cor. 2. *A radius which bisects a chord is perpendicular to the chord.*

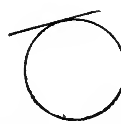
207. Cor. 3. *The perpendicular bisector of a chord of a circle bisects the arc subtended by the chord.*

208. Cor. 4. *The perpendicular bisector of a chord of a circle passes through the center of the circle.*

1. If the straight line connecting the mid-points of two chords of a circle passes through the center, the two chords are parallel.

2. The radius drawn to the mid-point of an arc is the perpendicular bisector of the subtended chord.

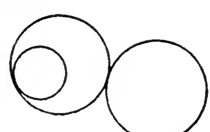
209. Tangent Line. A line that touches a circle at one point only and does not cut the circle is a *tangent line.*



210. Point of Tangency. The point which is common to the circle and a tangent line is the *point of tangency.*

211. Tangent Circle. If two circles have but one point in common they are *tangent circles.*

If one circle is within the other they are tangent *internally.* If they are without each other they are tangent *externally.*



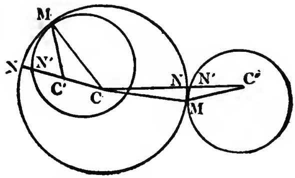
1. In a right triangle with acute angles of 30° and 60° respectively, one side is one-half the hypothenuse.

2. If the hypothenuse of a right triangle is equal to twice one of the sides the acute angles are 30° and 60° respectively.

3. The mid-point of the hypothenuse is equidistant from the three vertices.

PROPOSITION XIV.

212. Theorem. *The straight sect joining the centers of two tangent circles passes through the point of tangency.*



Given two circles C and C', tangent at M.

To Prove line CC' to pass through M.

Proof. Case I. *Internal tangency.*

Sug. 1. Assume that CC' does not pass through M.

Join C to M and C' to M. Extend CC' to meet $\odot$ C at N and $\odot$ C' at N'.

The pupil will note that the figure is distorted for the sake of the argument. The points C' C' are not the true centers.

2. Compare CM and CN. Auth?
3. $CN > CN'$. Why?
4. $CN' = CC' + C'N'$.
5. $C'N' = C'M$. Why?
6. $\therefore CM > CC' + C'M$. Is it possible?
7. What of the assumption in step 1?

Case II. *External tangency.*

1. Make the same assumption as in case I.
2. $CM = CN$ and $CN < CN'$. Why?
3. $CN' + C'N' = CC'$.
4. $CN + C'N' < CC'$. Why?

5. $\therefore CM + C'M < CC'$. Why? Is this possible?

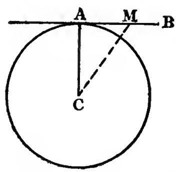
6. What of the assumption in step 1?

Therefore—

213. Cor. *If two circles are tangent a line tangent to one at their point of contact is tangent to the other.*

PROPOSITION XV.

214. Theorem. *A straight line perpendicular to a radius at its outer extremity is tangent to the circle.*



Given $\odot C$ with line $AB \perp CA$ at its outer extremity A.

To Prove AB tangent to $\odot C$.

Proof. Sug. 1. What must be known to prove AB tangent?

2. How much of this is included in the hypothesis?

3. What remains to be proved?

4. Let M represent any point on AB other than A and draw CM.

5. Compare CA and CM as to length.

6. Where then is point M with respect to $\odot C$?

Therefore—

215. Cor. I. *A line which is tangent to a circle is perpendicular to the radius at the point of contact.*

Sug. Prove that the radius is the shortest line from the center to the tangent.

216. Cor. II. *The perpendicular to a tangent at the point of contact passes through the center of the circle.*

Sug. Indirect proof.

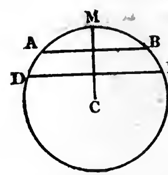
217. Cor. III. *The perpendicular dropped from the center of a circle to a tangent meets it at the point of tangency.*

218. Cor. IV. *At any point on a circle one and only one tangent can be drawn.*

PROPOSITION XVI.

219. Theorem. *Arcs of a circle intercepted by parallel lines are equal.*

Given ⊙ C, with two parallel chords AB and DE intercepting arcs EB and AD.



To Prove arc EB = arc AD.

Proof. Sug. 1. Drop a ⊥ from C to DE, extending it to meet the circle as at M. Why?

2. How does CM lie with reference to AB?

3. How does CM effect the arcs subtended by the two chords?

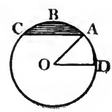
4. Complete the demonstration.

5. Assume one of the two parallel lines to be tangent to the circle and complete the demonstration.

6. Assume both lines to be tangents and complete the demonstration.

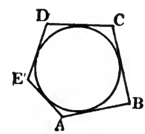
Therefore—

220. SEGMENT OF A CIRCLE. A portion of the plane enclosed by an arc and its subtended chord is a *segment of the circle.*



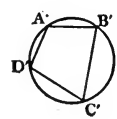
ABC is a minor segment and *CDA* is a major segment. By segment is usually meant the minor segment and it is indicated by its arc alone.

221. SECTOR OF A CIRCLE. A portion of the plane enclosed by two radii and the intercepted arc is a *sector of the circle.* *DOA* is a sector.



222. CIRCUMSCRIBED POLYGON. A polygon is *circumscribed* about a circle if all of its sides are tangent to the circle, as polygon *ABCDE*.

When a polygon is circumscribed about a circle, the circle is *inscribed in the polygon.*



223. INSCRIBED POLYGON. A polygon is *inscribed* in a *circle* if each of its sides is a chord of the circle, as *A'B'C'D'*.

When a polygon is inscribed in a circle, the circle is *circumscribed about the polygon.*

224. INSCRIBED ANGLE. An angle is *inscribed* in a circle when its vertex is on the circle and its sides are chords.

225. ANGLE INSCRIBED IN A SEGMENT. An angle is *inscribed in a segment* when it is subtended by the chord of the segment and its vertex is on the arc of the segment.

226. Whenever it is established that a certain relation of points and lines is possible, the rigor of a demonstration is not impaired by using a representation of that relation without performing the actual construc-

tion. This has been done directly in previous demonstrations and indirectly by the use of instruments, such as the protractor, the construction of which required the results of theorems at that time not proven. This has been done in order that the pupil may at as early a stage as possible learn to make careful constructions and more fully visualize the relations under consideration.

Plane geometry deals only with figures composed of straight lines and circles, the construction of which may usually be effected by means of the straight edge and the dividers. The postulates involving the simple use of these tools are restated below. Problems of construction are in no sense Pure Geometry but are applications of principles demonstrated in Pure Geometry.

Since constructions are based on previously demonstrated theorems some problems can be solved by several methods according as one or another theorem is used. For example, several methods for determining a perpendicular have already been indicated as well as for parallel lines. In the following problems it will be of added interest to make use of as many methods as possible in each.

227. POSTULATES.

(1) *A sect can be drawn between any two points and can be extended to any length through either extremity.*

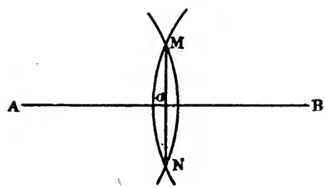
(2) *A circle can be drawn with any sect for a radius about any point as a center.*

The first postulate requires the straight edge and the second the dividers.

In solving problems of construction it is of advantage to represent the construction as completed in order that by a study of the lines involved and their relations to each other one may recall previous theorems and constructions from which as starting points the desired construction can be made.

PROPOSITION XVII.

228. Problem. *To bisect a given straight line segment.*



Construction. With A and B, the extremities, as centers and with equal radii greater than $\frac{1}{2}AB$ describe arcs intersecting as at M and N. Draw MN intersecting AB at O.

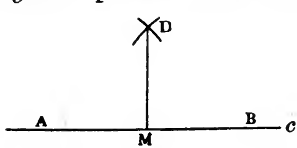
Then O is the mid-point of AB.

Proof. By the construction M and N are each equidistant from A and B and hence determine the perpendicular bisector of AB.

Note. A little experience will suggest what parts of lines may be omitted as unessential. For example, in the above construction all the circles may be omitted except such short arcs as are necessary to determine the intersection points M and N.

PROPOSITION XVIII.

229. Problem. *To erect a perpendicular to a given line at a given point on the line.*



Given point M in line c.

To Construct a perpendicular to c at M.

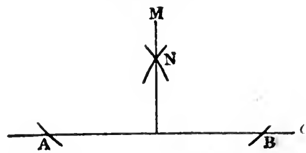
Construction. On c lay off $MA = MB$. With A and

B as centers and equal radii greater than *MA* describe arcs intersecting at *D*. Draw *DM* which is the required perpendicular.

Proof. *DM* is the perpendicular bisector of *AB* (Why?) and as *AB* is a part of line *c*, *DM* is perpendicular to *c*.

PROPOSITION XIX.

230. Problem. *From a given point to drop a perpendicular to a given line.*



Given the line *c* with a point *M* not on *c*.

To Construct a line through *M* perpendicular to *c*.

Construction. With *M* as a center and a radius of sufficient length describe an arc to cut line *c* in two points as *A* and *B*. Locate now a second point, *N*, equidistant from *A* and *B*. How can this be done? The line *MN* is the required line.

Proof. Left to the student.

PROPOSITION XX.

231. Problem. *To bisect a given arc.*

Given *AB*, the arc to be bisected.

Sug. 1. What propositions have been demonstrated involving the bisection of an arc?

2. If the center *O* is known, complete the demonstration.

3. If the center is not given, complete the demonstration.

Proof left to student.

1. Let a be a given straight sect with an unlimited straight line intersecting it. Construct a right triangle with hypothenuse a and the vertex of the right angle on the second line.

PROPOSITION XXI.

232. PROBLEM. *To bisect a given angle.*

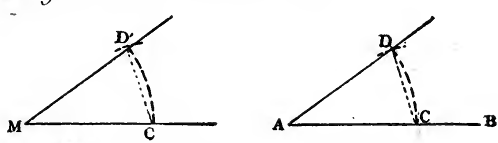
SUG. 1. Construct a circle with the given angle at the center and complete the demonstration.

Or 2. What propositions involve a bisected angle? Make a construction from one or more of these.

Proof

PROPOSITION XXII.

233. PROBLEM. *At a given point in a given line to construct an angle equal to a given angle with the given line as one side.*



Given $\angle M$ and the point A on line AB.

To Construct an angle at A equal to $\angle M$ and with AB as one side.

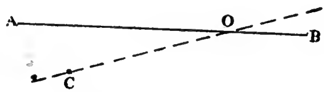
SUG. 1. $\angle M$ and the angle to be constructed will be equal if they are central angles in equal circles and are subtended by equal arcs. Complete the construction.

2. $\angle M$ and the angle to be constructed will be equal if they are opposite equal sides in congruent triangles. Complete the construction.

Proof

PROPOSITION XXIII.

234. PROBLEM. *Through a given point without a straight line to construct a line parallel tothe given line.*



Given any point C not on line AB.

To Construct a line through C parallel to AB.

SUG. 1. Draw through C any line cutting AB as a transversal.

2. With respect to AB and the transversal what conditions of the line to be constructed will make it parallel to AB?

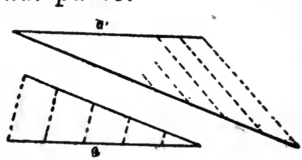
3. Complete a construction.

Other methods. Make constructions by other authorities.

Proof

PROPOSITION XXIV.

235. PROBLEM. *To divide a sect into any number of equal parts.*



Given sect a.

To divide a into 5, or more generally into n equal parts.

SUG. 1. From one extremity of a draw any line oblique to a.

2. Lay off on this line five (or n) equal sects of any convenient length.

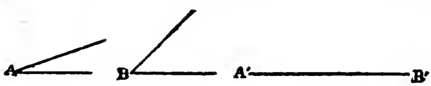
3. Join the extremity of the last of these to the free extremity of a and through each of the other points of division draw parallels to this join line extending them to meet a.

4. The five (or n) divisions thus made in a are equal. . Why?

Proof

PROPOSITION XXV.

236. PROBLEM. *Given two angles and the included side of a triangle, to construct the triangle.*



Given $\measuredangle$ A and B of a triangle with $A'B'$ as the included side.

To Construct the triangle.

SUG. Represent the triangle as already constructed and from the figure decide which of the preceding problems might be used to construct it.

Proof

QUERY. Can the line and angles be of such magnitude as to make the construction impossible?

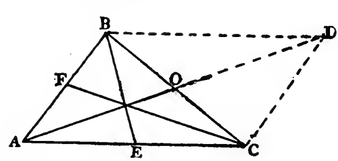
1. Upon a given base construct an isosceles triangle in which the sum of the two other sides equals a given line.

2. If from two opposite vertices of a parallelogram two lines be drawn to the middle points of two opposite sides, the lines will trisect the diagonal joining the other vertices.

The three medians of a triangle meet in a point.

SUG. CF cuts off one-third of diagonal AD. Why?

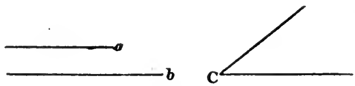
Find relation of BE to the diagonal.



Show that median AO lies in diagonal AD.

PROPOSITION XXVI.

237. PROBLEM. *Given two sides and the included angle of a triangle to construct the triangle.*



Given two sides, *a* and *b*, and the included angle *C* of a triangle.

To Construct the triangle.

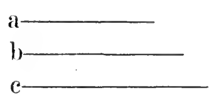
SUG. 1. At any point on an unlimited line construct an angle equal to *C*.

2. Complete the construction.

Proof

PROPOSITION XXVII.

238. PROBLEM. *To construct a triangle, given the three sides.*



Given *a, b, c* as the three sides of a triangle *ABC*.

To Construct $\triangle ABC$.

Construction. SUG. 1. Take a sect *AB* equal to *c*. *A* and *B* are then two vertices of the triangle.

2. If *b* is the side opposite vertex *B* what is the locus of the third vertex *C*?

3. With respect to vertex *B* where does *C* lie?

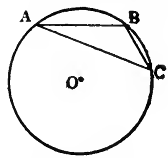
4. Complete the construction.

Proof

Discussion. What effect do the relative magnitudes of the given sides have upon the possibilities of construction?

PROPOSITION XXVIII.

239. PROBLEM. *To circumscribe a circle about a triangle.*



Given △ *ABC*.

To circumscribe a ⊙ about △ *ABC*.

SUG. 1. The problem is to find the center of a circle which passes through *A*, *B*, *C*.

2. Where must this center lie with respect to *A* and *B*? With respect to *B* and *C*?

3. Complete the construction and verify.

Proof

QUERY. How many circles can be circumscribed about a triangle? Why?

NOTE. Compare this problem with § 231.

PROPOSITION XXIX.

240. PROBLEM. *To inscribe a circle in a given triangle.*

SUG. 1. Study Ex. 1, P. 37, and write out the construction in full.

2. Prove the sides of the triangle to be tangents to the circle thus constructed.

241. REVIEW.

State all the theorems of Book II by which one can prove

(1) Two lines equal.
(2) Two angles equal.
(3) Two lines perpendicular.
(4) Two lines unequal.
(5) Two angles unequal.
(6) Two arcs equal.
(7) Two arcs unequal.

1. Construct an equilateral triangle. Will the problem as stated admit of more than one such triangle? What condition may be added to make the problem definite?

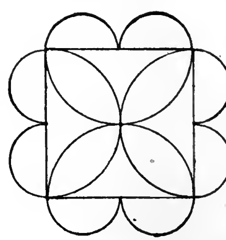
2. Construct an isosceles triangle with the base one third of a side.

3. Construct a triangle with a given base, given base angle, and a given altitude.

4. Construct an angle of 60° without a protractor.

5. Construct a triangle with an angle of 45°, an angle of 60°, and a given sect for the included side.

6. Construct an equilateral triangle on a two inch base. Circumscribe a circle about it and inscribe a circle within it. Show that the radii of the former is twice that of the latter. Show that this is true for any equilateral triangle.



7. Draw the pattern at the right the square having a two inch base.

8. Can a circle be circumscribed about a square? A rectangle? A rhombus? A rhomboid? Prove your conclusions.

9. Construct a checker-board design using a T-square. Construct another using a ruler and a right triangle. Give authorities.

10. Given the two diagonals of a rhombus, construct it. Do the same for a square.

11. Construct a rectangle having given one side and a diagonal.

12. Construct a rhomboid having given the two diagonals and their included angle.

13. Inscribe a square in a given circle; circumscribe one about the circle; and then circumscribe a second circle about the second square. Compare the length of the sides and diagonals of the two squares with the radii of the two circles.

CHAPTER III.

MEASUREMENT AND PROPORTION.

242. MAGNITUDE. *Magnitude* is the size, extent, or mass of anything and prompts the question, "How much?"

Magnitudes in geometry are lines, angles, areas, solids, etc.

243. MEASURE. To *measure* a magnitude is to find the number of times it contains a given *unit of measure.*

244. UNIT. A *unit* of measure is a selected magnitude of the same kind as the magnitude to be measured.

In everyday experience a unit of measure is a standard set by statute cr common consent, e.g. the yard, the gallon, the degree, the cubic foot, etc.

245. QUANTITY. *Quantity* is the result of measurement and *answers* the question "How much?"

The quantities of geometry are lengths, areas, contents, etc.

246. NUMERICAL MEASURE. The *numerical measure* of a magnitude is the *number* which expresses how many times the unit is contained in the magnitude measured.

If a line is measured and found to be 8 feet long, the *line* is the magnitude, the *foot* is the unit, *eight* is the numerical measure, and *8 feet* is the quantity.

Magnitude is indefinite, quantity is definite. To express quantity two elements, the numerical measure and the unit, are necessary. Careful distinction must be made between *quantity,* and *number* which is the measure of magnitude. These terms are often confused. If as units of measure, we consider the gallon, the degree, the square inch, the cubic foot, then the expressions 26 gallons, 5 degrees, 29 square inches, 18 cubic feet are *quantities* while 25, 5, 29, 18 are numbers.

247. RATIO. *The ratio* of one magnitude, quantity, or number to another of the same kind is the number which expresses how many times the first contains the second, or more generally the quotient of the first by the second. In other words, the *ratio* of one magnitude, quantity, or number to another is the numerical measure of the first, with respect to the second as the unit of measure, or, both being measured by the same unit, the ratio is the quotient of the numerical measure of the first by the numerical measure of the second.

(a) To illustrate—the ratio of sect a to sect b is the number of times a contains b as a unit of measure. The result may be obtained by laying off b upon a as many times as possible. If, however, a does not contain b an integral number of times they may each be measured by a common unit m, in which case the ratio is the quotient of the numerical measure of a by the numerical measure of b.

(b) From the definition of ratio, it follows that a ratio can exist only between magnitudes or quantities of the same kind and also that the ratio is always an arithmetical number. For example, if sect a contains unit m 8 times and sect b contains m 4 times, the ratio of a to b is $8 \div 4$ or 2. Hence if c be the number of times m is contained in sect a and d be the number of times m is contained in sect b, the ratio of the two sectsis that of c to d.

(c) Since *ratio* in the following discussions is based upon the algebraic conception of division and not upon the Euclidean definition (which is much too difficult for beginners), the division or fractional form of expression will be used. Hence the ratio of 6 ft. to 2 ft. will be written as $\frac{6 \text{ ft.}}{2 \text{ ft.}}$, $\frac{6}{2}$, or 3. The statement above as to the ratio of the sects a and b will be shortened to $\frac{a}{b} = \frac{c}{d}$.

248. Commensurable. Two magnitudes or quantities are *commensurable* if they each contain a common unit of measure a whole, or integral, number of times. The ratio of two commensurable quantities must then be an integral number or a quotient of two integral numbers.

In determining the common unit of measure of two commensurable magnitudes or quantities, the usual method for finding the greatest common divisor may be followed, which is to divide the greater of the two by the less, then the divisor by the remainder, this remainder by the second remainder and so on, until a remainder zero is obtained. This exact divisor is the desired common unit of measure.

249. Incommensurable Magnitudes. Two magnitudes or quantities which do not possess a common unit of measure are *incommensurable*. The ratio of two such magnitudes is neither an integral number nor a fraction.

The circle and its diameter are incommensurable, as are also the diagonal and side of a square. A proper discussion of this subject cannot be made at this point on account of certain difficulties necessarily involved.

250. Proportion. A *proportion* is an equality each member of which is a ratio. The four numbers or quantities a, b, c, d are in proportion if the ratio of a to b equals the ratio of c to d. The symbolic form of this statement is $\frac{a}{b}=\frac{c}{d}$. The four numbers or quantities are the *terms* of the proportion.

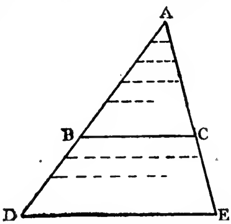
a. An equality of three or more equal ratios is a *continued proportion*.

1. Arrange the numbers 2, 5, 20, 8 in a proportion in as many ways as possible, verifying each by the definition of proportion.

2. Arrange the numbers $3\frac{1}{2}$, $5\frac{2}{3}$, $28\frac{1}{3}$, $17\frac{1}{2}$ in a proportion and verify it.

PROPOSITION I.

251. THEOREM. *If a line is parallel to the base of a triangle it divides the sides into proportional segments.*



Given $\triangle ADE$ with line $BC \parallel DE$.

To Prove $\frac{AB}{BD} = \frac{AC}{CE}$.

Proof. SUG. 1. To obtain the ratio of the sects AB and BD they must be measured. Let l be the unit sect and suppose it is contained m times in AB and n times in BD. What is the ratio $\frac{AB}{BD}$? Why?

2. Draw lines through the points of division in AB and BD $\parallel$ to DE and extend to AE. Compare the segments on AC and CE.

3. What is the ratio of AC to CE? Why?

4. Compare the ratio $\frac{AB}{BD}$ with $\frac{AC}{CE}$,

Therefore—

NOTE. In the above demonstration the case in which AB and BD have no common unit of measure is not considered. The conclusions in this case, however, are the same as above but the demonstration is omitted here as being too difficult for the beginning student.

252. An Important Consideration. Since all ratios of magnitudes or quantities, by the definition in use, can be considered only through the ratios of their numerical measures, all proportions herein will be treated as numerical proportions. And since the ratio of two magnitudes, as a and b, 247 (b) is a number and can be represented only by the ratio of their numerical measures, as $\frac{m}{n}$. the terms a and b will be considered, in the interests of brevity, to represent, as well, their numerical measures. In general the names and notations of magnitudes used in the operations of proportion will be synonymous with the notations for their numerical measures, and the treatment of the proportions used will be that of algebra.

For convenience of reference the propositions of proportion will be collected and briefly reviewed in the following sections.

1. What is the ratio of one side of an equilateral triangle to the perimeter? Of the perimeter to one side?

2. What is the ratio of a right angle to an angle of an equilateral triangle?

3. What is the ratio of a quadrant to a semi circle? To a circle?

253. The Terms of a Proportion. The first and third terms of a proportion are the *antecedents*.

The second and fourth terms are the *consequents*. The second and third terms are the *means*. The first and fourth terms are the *extremes*.

In the proportion, $\frac{a}{b} = \frac{c}{d}$, a and c are the antecedents; b and d are the consequents; b and c are the means; a and d are the extremes.

THEOREMS OF PROPORTION.

254. THEOREM I. *The product of the means equals the product of the extremes.*

SUG. Write the ratios as fractions and clear the equality of its fractional form.

255. THEOREM II. *If the product of two numbers equals the product of two others, the factors of one product may be made the means and the factors of the other product may be made the extremes of a proportion.*

Given $ab = cd$.

To Prove $\frac{a}{c} = \frac{d}{b}$.

SUG. By what must ab be divided to produce $\frac{a}{c}$? By what must cd be divided to produce $\frac{d}{b}$? Why are the resulting ratios equal?

1. From $ab = cd$ derive a proportion in which a and b are the means.

2. From $ab = cd$ make as many proportions as possible. Note in what respects they differ.

3. If the first three terms of a proportion are 5, 7, 15 what is the fourth term?

4. Given $\frac{7}{11} = \frac{8}{x}$. Find x.

5. Find x in the following proportions:

$\frac{7}{13} = \frac{x}{18}$; $\frac{x}{11} = \frac{5}{13}$; $\frac{11}{15} = \frac{x}{27}$; $\frac{15}{x} = \frac{11}{39}$; $\frac{3}{x} = \frac{5}{12}$; $\frac{6}{11} = \frac{18}{x}$.

6. Make four different proportions from the identity $8 \times 7 = 4 \times 14$.

7. Use theorem I to determine whether or not the ratios $\frac{7}{12}$ and $\frac{9}{15}$ will form a proportion.

256. THEOREM III. *If* $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{a}{c} = \frac{b}{d}$.

SUG. Use Theorems I and II.

257. ALTERNATION. The interchange of the two means of a proportion is *alternation.*

1. State theorem III in words without the use of symbols *a, b, c, d.*

2. In the fig. of § 251 take the proportion by alternation.

3. Construct a triangle with a line parallel to the base. Measure three of the four segments into which the two sides are cut and by § 251 determine the fourth. Check the calculation by measurement.

4. Show how the conditions of Ex. 3 may be used to measure an inaccessible distance, over a pond for example.

5. On the school grounds drive four stakes *A, B, C,* not in a straight line and *D* in the line *AC.* Find the point for a 5th stake so as to measure *AD* indirectly.

258. THEOREM IV. *If* $\frac{a}{b}=\frac{c}{d}$ *then* $\frac{b}{a}=\frac{d}{c}.$

SUG. $1 \div \frac{a}{b}=\frac{b}{a}.$

Complete the proof.

259. INVERSION. The interchange of first and second, third and fourth terms of a proportion is *inversion.*

1. State theorem IV in words without the use of the symbols *a, b, c, d.*

2. In Ex. 5, § 255, take the proportions by inversion. Which of those given may be obtained from the others by inversion?

260. THEOREM V. *If* $\frac{a}{b}=\frac{c}{d}$ *then* $\frac{a+b}{b}=\frac{c+d}{d}.$

SUG. Add 1 to each member of the given proportion. Complete the proof.

261. COMPOSITION. The second proportion in Theorem V is obtained from the first by *composition.* This process is sometimes termed *addition.*

1. State theorem V in words without the use of the symbols *a, b, c, d.*

2. By composition, alternation, etc., prove in the triangle of Prop. I the proportion $\frac{AD}{BD}=\frac{EA}{CE}$.

3. Prove $\frac{AD}{AB}=\frac{AE}{AC}$. Sug. § 256 and § 260.

4. Prove $\frac{AB}{AC}=\frac{AD}{AE}$.

5. Prove $\frac{BD}{CE}=\frac{AB}{AC}$.

6. Prove $\frac{AD}{AE}=\frac{BD}{CE}$.

262. THEOREM VI. *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a-b}{b}=\frac{c-d}{d}$.

SUG. Subtract 1 from each member of the given proportion. Complete the proof.

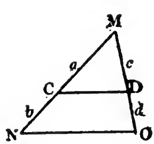
263. DIVISION. The second proportion in Theorem VI is obtained from the first by *division*. This is sometimes termed *subtraction*.

264. THEOREM VII. *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

SUG. The second proportion is said to be obtained from the first by *composition and division*. It may be obtained from the final proportions in Theorems V and VI. Complete the proof.

In the accompanying figure, *CD* being parallel to *NO*, show that $\frac{a}{c}=\frac{b}{d}$; $\frac{b}{a}=\frac{d}{c}$; $\frac{a+c}{a}=\frac{b+d}{b}$; $\frac{a+c}{c}=\frac{b+d}{d}$; $\frac{a}{a+b}=\frac{c}{c+d}$; $\frac{a-b}{a+b}=\frac{c-d}{c+d}$; $\frac{a-b}{c-d}=\frac{a+b}{c+d}$.

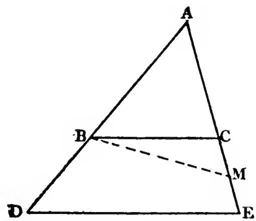
2. In the proportion $\frac{a}{b}=\frac{c}{d}$



$a=12$, $b=5$, $c=6$, find d.
$a=3$, $c=7$, $d=14$, find b.
$b=12$, $c=13$, $d=6$, find a.
$a=13$, $b=15$, $d=45$, find c.

PROPOSITION II.

265. THEOREM. *If a line divides two sides of a triangle proportionally, it is parallel to the base.*



Given $\frac{AB}{AD} = \frac{AC}{AE}$

To Prove $BC \parallel DE$.

Proof. SUG. 1. If BC is not parallel to DE, let BM represent the line through B which is.

2. Then $\frac{AB}{AD} = \frac{AM}{AE}$. Why?

3. From this proportion and the hypothesis compare AC and AM.

4. Where then must point M lie with respect to C?

Therefore—

1. Given Ex. 2, § 264. $MN = 17$, $MO = 24$, $DO = 8$; find a, b, c.
2. Given $c = 12$, $d = 6$, and $MN = 30$, find the other parts.

266. FOURTH PROPORTIONAL. The fourth term of a proportion of four different terms is a *fourth proportional* to the three others in order.

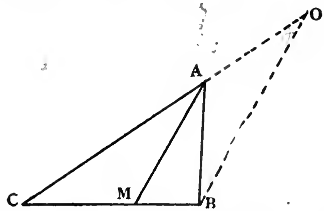
3. Given three sects a, b, c, find x a fourth proportional by construction. Sug. Construct an angle one side of which is the sum of the sects a and b. On the second side at the extremity of sect a lay off sect c. Complete the construction and verify it.

4. Divide a given sect in the ratio of three to four. Of two to three. Of five to three.

5. Divide a given sect a into two parts having the ratio of the given sects b and c.

PROPOSITION III.

267. **Theorem.** *The bisector of an angle of a triangle divides the side opposite into sects which are proportional to the adjacent sides.*



Given $\triangle ABC$, with AM bisecting $\angle A$, and M the point of division of CB.

To Prove $\frac{CM}{MB} = \frac{AC}{AB}$.

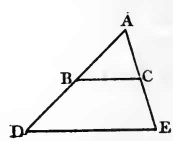
Proof. Sug. 1. Extend CA to O making $AO = AB$. Join O and B.

2. $OB \parallel AM$. Why?

3. Note that AM is parallel to BO and derive the required proportion.

Therefore—

1. If a line bisects one side of a triangle and is parallel to the base it bisects the other side and equals half the base.



Given $AB = BD$ and $BC \parallel DE$.

Prove $AC = CE$ and $BC = \frac{1}{2} DE$.

Sug. To prove $BC = \frac{1}{2} DE$, draw a line through B parallel to AE. Then use the first part of the exercise.

See § 150 for a different method.

2. If a line bisects two sides of a triangle, it is parallel to the base and equals one-half the base.

See § 151. Use another method here.

3. Divide sect *a* of Ex. 5 P. 105 into three equal parts. Into four equal parts.

268. Internal Division. A sect is divided into segments *internally* when the point of division lies in the segment.

AB is divided internally at *C* into the two segments *AC* and *CB*. Thus $AC + CB$ must equal *AB*. Ax. 9, § 49.

269. External Division. A sect is divided into segments *externally* when the point of division lies in the extension of the sect.

AB is divided externally at *C* into segments *AC* and *CB*.

In order that in this case also the sum $AC + CB$ may equal *AB*, the algebraic idea of positive and negative quantities may be introduced. In moving a point from *A* to *B* through the point of division *C*, the directions *AC* and *CB* are the same for internal division and opposite for external division. In the latter case, in moving from *C* to *B*, the point traverses a second time part of segment *AC* but with respect to the direction of *AC*, sect *CB* is then *negative*. Hence in this case also one may write $AC + CB = AB$. Unless expressly stated the notations used in this text will not involve this use of directed lines.

1. Draw a tangent to a circle at a given point on the circle.

2. Two tangents drawn to a circle from the same point are equal.

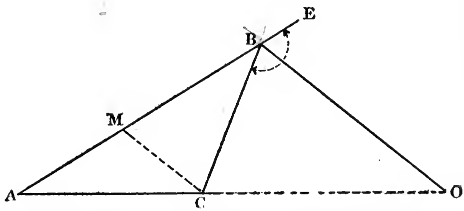
3. If two circles are concentric, all chords of the larger which are tangent to the smaller are equal.

4. The tangents to a circle at the extremities of a diameter are parallel.

5. Draw a triangle as large as may be on a given sheet of paper and draw a line parallel to the base and intersecting the sides. Measure three parts from which measurements the fourth part may be found. Check by measuring this fourth part.

PROPOSITION IV.

270. THEOREM. *The bisector of an external angle of a triangle divides the opposite side externally into sects proportional to the adjacent sides.*



Given $\triangle ABC$ with $\angle CBE$ bisected by a line dividing the opposite side AC externally in point O.

To Prove $\frac{AO}{OC}=\frac{AB}{BC}$.

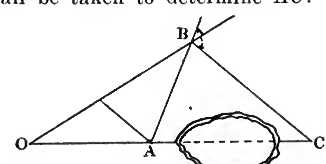
Proof. SUG. 1. Draw $CM \parallel BO$.

2. $\frac{AO}{CO}=\frac{AB}{MB}=\frac{AB}{BC}$. Why?

Therefore—

In the above figure and demonstration the inequality $AB > BC$ is assumed. Let the pupil prove the theorem for the case $AB \angle AC$ by lettering the figure the same and following the line of proof.

1. It is desired to measure an inaccessible distance on a plane. What use can be made of Proposition § 270? What measurements shall be taken to determine AC?



1. A line drawn through the vertex of a triangle dividing the opposite sides into segments proportional to the adjacent sides bisects the angle.

This is the converse of what proposition?

2. If a radius of one circle is a diameter of another, the circles are tangent and any line drawn from the point of contact to the outer circle is bisected by the inner one.

271. HARMONIC DIVISION. A line is divided *harmonically* when it is divided internally and externally in the same ratio.

1. AB is divided internally in the ratio of 6 to 4 at point M. $AB = 6 + 4 = 10$.

2. AB is divided externally at M' in the ratio of 6 to 4. $AB = 6 + (-4) = 6 - 4 = 2$.

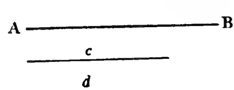
The unit used in the internal division is contained in AB ten times. In the case of external division the unit is contained in AB two times. In general, for internal division the number of divisions made in AB is the sum of the number made in AM and the number made in MB; for external division the number of divisions made in AB is the difference between the number made in AM' and the number made in $M'B$.

3. Divide sect CD harmonically in the ratio of 5 to 3.

SUG. To find M divide CD into 8 equal parts and to find M' divide CD into 2 equal parts.

4. Divide a given sect harmonically in the ratio of 7 to 5. into how many parts must it be divided?

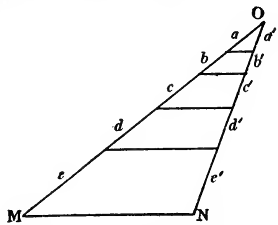
5. If both the interior and exterior angles at a vertex of a triangle are bisected, the opposite side is divided harmonically by the bisectors.



6. Divide harmonically sect AB in the ratio of c to d.

PROPOSITION V.

272. THEOREM. *If several lines are drawn parallel to the base of a triangle intersecting the sides, the corresponding segments of the sides form a continued proportion.*



Given △ *OMN* with lines parallel to the base *MN* cutting the sides into the sects *a, b, c, d, e* and *a′, b′, c′, d′, e′*, etc., respectively.

To Prove $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{d}{d'}=\frac{e}{e'}$, etc.

Proof. SUG. 1. $\frac{a}{a'}=\frac{b}{b'}$. Why?

2. $\frac{a}{a'}=\frac{a+b}{a'+b'}$. Why?

3. $\frac{a+b}{a'+b'}=\frac{c}{c'}$. Why?

4. Complete the demonstration.

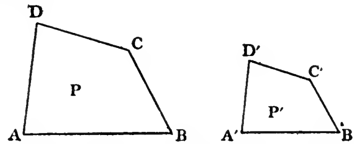
Therefore—

273. SIMILAR POLYGONS. Polygons which are mutually equiangular (§ 156) and which have their corresponding sides proportional are *similar polygons.*

274. HOMOLOGOUS. In similar polygons those points, lines, and angles which are similarly situated are *homologous.* In similar triangles the homologous sides are

those lying opposite equal angles and the equal angles are those lying opposite homologous sides.

275. Ratio of Similitude. In similar polygons the *ratio of similitude* is the ratio of any two homologous sides.



The polygons P and P' are similar, provided $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc., and $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc. Any one of the equal ratios $\frac{AB}{A'B'}$, etc., may be taken as the ratio of similitude.

276. From the definition of similar polygons, it follows that, if two polygons are known to be similar, the homologous angles are equal and the homologous sides are proportional.

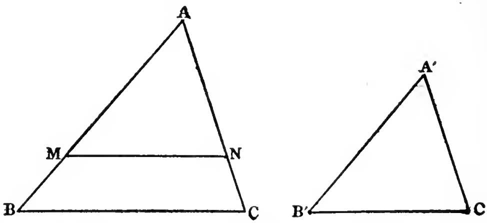
1. In the similar $\triangle$ ABC and $A'B'C'$, if $\angle A = \angle A'$, etc., which sides are homologous? If the sides AB and $A'B'$, etc. are homologous, which angles are equal?

277. Polygons may be mutually equiangular but may not have their sides proportional or they may have their sides proportional without being mutually equiangular. The first condition is illustrated by the rectangle and the square. The second condition is illustrated by a rhombus and a square or by a rectangle and a rhomboid if the sides are proportional.

It will be established *later* that triangles form an exception to the above statement, in that if either condition of the definition of similarity applies the other is a necessary consequence.

PROPOSITION VI.

278. THEOREM. *Two triangles which are mutually equiangular are similar.*



Given $\triangle$ ABC and $A'B'C'$ with $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

To Prove $\triangle$ ABC and $A'B'C'$ similar.

Proof. SUG. 1. What part of the definition of similar triangles remains to be proved?

2. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that A' falls on A, B' on AB at M and C' on AC at N. Can this be done? Why do it?

3. $MN \parallel BC$. Why?

4. $\frac{AB}{AM} = \frac{AC}{AN}$. Why? $\therefore \frac{AB}{A'B'} = \frac{AC}{A'C'}$.

5. What is yet to be proved?

6. Place $\triangle A'B'C'$ upon $\triangle ABC$ with B' on B, etc. Why?

7. What ratios can here be proved equal? Give all the steps.

8. Compare the three ratios $\frac{AB}{A'B'}$, $\frac{BC}{B'C'}$, $\frac{AC}{A'C'}$.

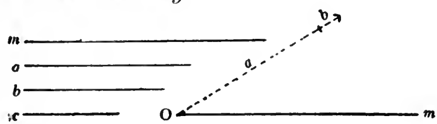
Therefore—

279. Cor. I. *Two triangles are similar if two angles of one are equal respectively to two angles of the other.*

280. Cor. II. *Two right triangles are similar if an acute angle of one equals an acute angle of the other.*

281. Cor. III. *Two triangles are similar if they are each similar to the same triangle.*

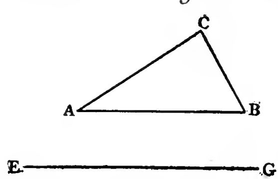
Given sect m. To divide m into segments proportional to a, b, c, etc..



Sug. At one extremity O of m draw any line oblique to m and on this line from O lay off in order the given sects a, b, c, etc. Why? Complete the construction by reproducing the conditions of proposition V Verify the results.

PROPOSITION VII.

282. Problem. *To construct upon a given line a triangle similar to a given triangle.*



Given $\triangle ABC$ and the line segment EG.

To Construct upon EG a triangle similar to $\triangle ABC$.

Sug. Use Prop. XXXV and make the required construction.

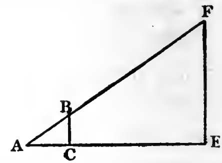
1. Draw a tangent to a given circle that shall make a given angle with a given line.

2. Two isosceles triangles are similar if the vertex angles of the two are equal.

3. Divide a sect into segments proportional to three or more given sects.

4. To measure the height of a nearby object, as EF, lie upon

the ground in such a position, *AC*, that the upper end, *B*, of a pole of known length placed vertically between the feet will appear in a line with the point *F*. The distances *AC*, *CB* and *AE* are known or easily measured. How may *EF* be determined?



As an illustration of the preceding problem, a woodsman in determining the height of trees uses a pole as long as his own height. What one distance will he need to measure? Why?

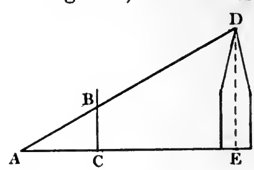
5. How may shadows be used to determine the height of objects?

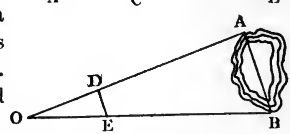
6. Determine as accurately as possible the height of some point on the school building.

NOTE. Not more than two should work together, the results being compared in class.

7. To measure a given height *ED*.

SUG. Set up a pole parallel to *ED* at some convenient point as *C* and while one person sights from a point *A* to *D* let a second person move a card upon the pole until a point *B* is found on the pole in the line *AD*. Make the required measurements and determine the height *ED*.





8. It is desired to find the distance *AB* indirectly. If *DE* is parallel to *AB* what measurements should be made?

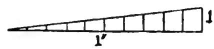
9. A triangle has two angles of 69° and 57° respectively, the included side being 26 rods in length. The length of the two other sides is required.

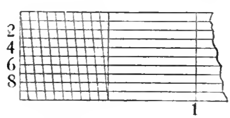
> SUG. Construct on paper a triangle similar to the given triangle, using the protractor for the construction of the angles. Measure the three sides. From this data determine the desired distances for the given triangle.
>
> State the various methods thus far used for the indirect measurement of distances. Which is the easiest?

10. To construct sects .01, .02, .03, etc., to .09 inches in length Divide a segment one inch long into 10 equal parts and at one

extremity erect a perpendicular .1″ in length. Connect the other extremities of these two sects forming a right triangle. The perpendiculars to the original sect at the points of division terminated by the hypotenuse are of the required lengths. Give the reasons for each step. Take upon the dividers .03, .05, .07 of an inch.

11. To construct a *diagonal scale* by which any segment may be measured in tenths and hundredths of an inch. The construction should be made on cardboard and preserved for future use, if the pupil has not already purchased a diagonal scale. Construct a square on a one inch segment, dividing each side into tenths. Connect one vertex of the square with the first point of division on the opposite side, and through the remaining division points of this side draw lines parallel to the first line. Through the division points of the second pair of opposite sides draw parallels to the first pair. Read from the scale .36, .42, .73, .85, .92. Give authority for all statements made.





12. With the dividers take .27 inches from the scale and on some given line lay off a sect .27 inches long:

13. Construct a sect 3.75 inches long; 3.56 inches; 2.05 inches.

14. With the dividers and diagonal scale measure the sects *a*, *b*, *c*, *d*.

15. Open a jointed two foot rule so that the ends are one foot apart. How long is the sect which connects points one inch from the joint? 2 inches? 5 inches? 9 inches? Use § 276.

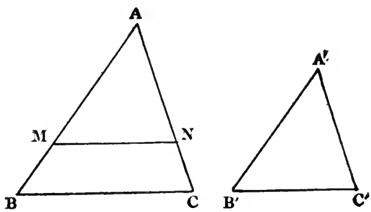
16. If the ends of the rule are six inches apart, how far apart are the pair of points marking divisions equally distant from the joint?

17. Open the rule various distances as above and determine the distances between corresponding divisions.

18. Work out the same exercises, using a one foot jointed rule. Also with a six inch jointed rule.

PROPOSITION VIII.

283. THEOREM. *If two triangles have an angle of one equal to an angle of the other and the sides including the equal angles proportional, the triangles are similar.*



Given $\triangle ABC$ and $\triangle A'B'C'$ with $\angle A = A'$ and $\frac{AB}{A'B'} = \frac{AC}{A'C'}$.

To Prove $\triangle ABC \sim \triangle A'B'C'$.

Proof. SUG. 1. What must be proved in addition to the hypothesis to make the triangles similar according to § 278?

2. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that A' falls on A, $A'B'$ on AB, and $A'C'$ upon AC. Is this possible? Why do so?

3. Where do B' and C' fall?

4. $B'C' \parallel BC$. Why?

5. Compare $\angle B'$ with $\angle B$. Complete the demonstration.

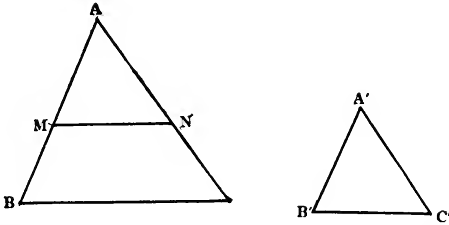
Therefore—

1. If triangles have their sides respectively parallel or perpendicular to each other they are similar.

2. Let ABC and $A'B'C'$ be two similar triangles. $AB = 7$ ft., $A'B' = 14$ ft., $AC = 5$. Find the length of $A'C'$. If AB is 13 ft., AC and $A'B'$ are 11 ft. and 10 ft. respectively; find the length of $A'C'$.

PROPOSITION IX.

284. THEOREM. *Two triangles are similar if the corresponding sides are proportional.*



Given $\triangle ABC$ and $A'B'C'$ with $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$.

To Prove $\triangle ABC \sim \triangle A'B'C'$.

Proof. SUG. 1. If any angle of $\triangle ABC$ equals the homologous angle of $\triangle A'B'C'$ the triangles are similar. Why?

2. Upon AB lay off AM equal to $A'B'$ and upon AC lay off AN equal to $A'C'$. Connect M and N. $\triangle AMN \sim \triangle ABC$. Why?

3. Compare the ratios $\frac{AM}{AB}$ and $\frac{MN}{BC}$; also $\frac{A'B'}{AB}$ and $\frac{B'C'}{BC}$.

4. Compare MN and $B'C'$. Auth.? $\triangle AMN \cong \triangle A'B'C'$. Auth.

5. $\triangle ABC \sim \triangle A'B'C'$. Why?

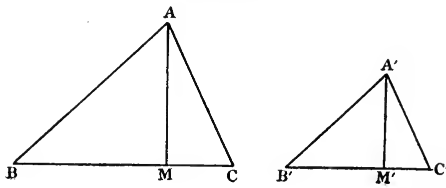
Therefore—

1. The sides of a triangle ABC are respectively 4, 8, and 11 feet. In a similar triangle, $A'B'C'$, the side homologous to the 4 foot side of ABC is 6 ft. Find the two other sides of $A'B'C'$.

2. In Prop. IX why not place $A'B'C'$ upon ABC as in Prop. VIII.

PROPOSITION X.

285. THEOREM. *The ratio of homologous altitudes of similar triangles is equal to the ratio of similitude of the triangles.*



Given $\triangle ABC \sim \triangle A'B'C'$ with $\frac{AB}{A'B'}$ as the ratio of similitude, and altitude MA homologous to $M'A'$.

To Prove $\frac{MA}{M'A'} = \frac{AB}{A'B'}$.

Proof. SUG. 1. Compare $\triangle AMB$ and $A'M'B'$. § 280.

2. Show that MA and $M'A'$ are homologous sides of $\triangle AMB$ and $A'M'B'$.

3. Complete the demonstration.

Therefore—

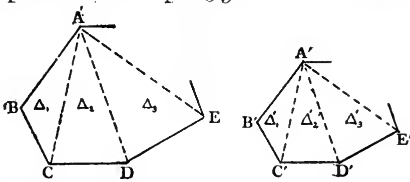
285. (1) COR. *In similar triangles homologous altitudes are proportional to the bases.*

SUG. Prove $\frac{MA}{M'A'} = \frac{BC}{B'C'}$.

NOTE. In deriving proportions from similar polygons in the early study of the subject it is usually best to select homologous sides for the terms of each ratio, taking the antecedent from one polygon and the consequent from the other. If any other form of proportion is desired it can be deduced by the theorems on proportion.

PROPOSITION XI.

THEOREM. *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.*



To Prove $P \sim$ to P'

Given polygons P and P'; P composed of $\triangle$ 1, 2, 3 . . . and P' of 1′ 2′ 3′ . . .; $\triangle 1 \sim \triangle 1'$, $\triangle 2 \sim \triangle 2'$. . . .

Proof. SUG. 1. What is the test of similarity of polygons?

2. Compare $<B$ and $<B'$; C and C', etc.

3. Compare $\frac{AB}{A'B'}$, $\frac{BC}{B'C'}$, $\frac{BC}{B'C'}$, $\frac{CD}{C''D'}$, etc. To compare $\frac{BC}{B'C'}$ and $\frac{CD}{C'D'}$ relate each to the ratio $\frac{AC}{A'C'}$.

4. Apply the difinition §273.

286 (1). COR. Two similar polygons can be divided into the same number of triangles similar each to each and similarly placed.

Use figure §286.

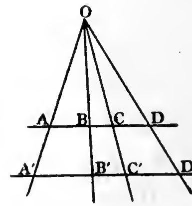
Prove that the polygons can be cut in the same number of $\triangle$; that $\triangle_1 \sim \triangle_1'$. $\triangle_2 \sim \triangle 2'$, etc.

Proof. SUG. 1. $\triangle_1 \sim \triangle_1'$ § 283.

2. To compare $\triangle_2$ and $\triangle_2'$. Compare ACD and $A'C'D'$. Compare ratios $\frac{AC}{A'C'}$ and $\frac{CD}{C'D'}$.

To compare these ratios note their common relation to the ratio $\frac{BC}{B'C'}$.

1. Straight lines drawn through any point intercept proportional segments upon two or more parallel lines.

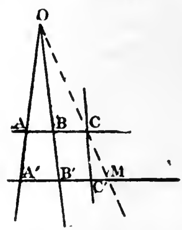


To Prove $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.

Sug. Compare the ratios $\frac{AB}{A'B'}, \frac{BC}{B'C'}$, with $\frac{OB}{O'B'}$, etc.

Complete the demonstration.

2. If two or more parallel lines are cut proportionally by a set of secant lines, prove that the secant lines pass through a common point. That is, given $\frac{AB}{A'B'} = \frac{BC}{B'C'}$, etc., prove that the lines $A'A$, $B'B$, $C'C$, etc., pass through a fixed point O.



Sug. Let O be the intersection point of two of the lines as $A'A$ and $B'B$. Connect O with C and extend OC to meet the second parallel at M. Compare the ratios $\frac{BC}{B'C'}$ and $\frac{BC}{B'M}$.

3. Given an isosceles triangle with vertex angle of 120°. Prove that the altitude from the vertex angle equals $\frac{1}{2}$ a leg of the triangle.

4. If tangents to a circle be drawn at the extremities of a chord, these tangents make with each other an angle which is twice the angle between the chord and that diameter of the circle drawn through an extremity of the chord.

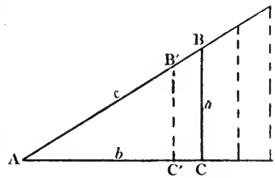
5. Given sect A the diagonal, and sect B the side of a square construct the square.

Work out through methods.

6. Points A and B are on the same side of line C. Find point X such that AX and BX make equal angles with C.

TRIGONOMETRIC RELATIONS.

287. It has been established that in a right triangle with an angle of 30° the side opposite this angle is one half the hypotenuse. This is true for every right triangle having an angle of 30°, for all such triangles are similar.



With a protractor carefully construct a right triangle with an angle of 40°, measure the hypotenuse and side opposite, and determine the ratio of the latter to the former. Repeat the construction several times and average the results. Do the same for a right triangle having an angle of 50°; of 60°; and tabulate the results.

288. Sine of an Acute Angle. The ratio of the side opposite an acute angle of a right triangle to the hypotenuse is the *sine of that angle*. Taking the notations from the fig. of § 287 this is written symbolically as $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$, or $c \sin A = a$ and $c \sin B = b$.

If $\sin A = .5$ then $a = .5\,c$. If $c = 40$ in., then $a = \frac{1}{2}$ of 40 in. $= 20$ in.

Note. The pupil should clearly understand that the trigonometric ideas here explained are introduced as an aid to the study of certain simple figures and that the complete study of trigonometry will require more general definitions than those here given.

1. In a right triangle the sine of an angle is .75 and the hypotenuse is 20 in. Find the length of the side opposite this angle.

2. If the hypotenuse is 20 in. and one side is 15 in. what is the sine of the angle opposite this side?

Note. Tables have been made in which the sines of angles for every degree and minute from 0° to 90° are recorded. A portion of such a table for intervals of one degree is to be found on p. 125

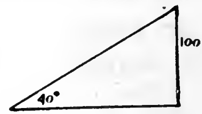
1. How many degrees in the angle found in Ex. 2 p, 121.

2. The hypotenuse of a triangle with an angle of 40° is 60 in. How long is the opposite side?

3. If the side opposite an angle of 35° is 34 in., how long is the hypotenuse?

4. The hypotenuse is 75 in., how many degrees in each angle if one side is 43 in?

7. A flag staff is 100 feet high. How long a rope is needed to reach from the top to a point on the ground at which the angle subtended by the pole is 40°?

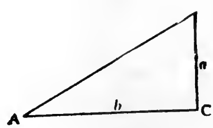


289. Cosine of an Acute Angle. The ratio of the side of a right triangle adjacent to an acute angle to the hypotenuse is the *cosine of that angle.*

6. If a right triangle has an angle of 60°, it has already been shown that the ratio of the adjacent side to the hypotenuse is ½. Draw such a triangle and verify the statement by measurement.

7. Draw a right triangle with an angle of 40°. Compute the cosine by measurement and check the result by the table. Do the same for 50° and 60°.

8. In the accompanying figure the relations involving cosines are written $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$, or $c \cdot \cos A = b$ and $c \cdot \cos B = a$. If $\cos A = .5$ then $b = .5\ c$. If $c = 40$ in. then $b = \frac{1}{2}$ of 40 in. = 20 in.



9. One angle of a right triangle is 45° and the hypotenuse is 60 ft. How long is the side adjacent to the given angle? How long is the side opposite?

10. If the adjacent side is 20 in. and the hypotenuse is 63 in. what is the cosine? How many degrees in the angle?

11. If the angle is 33° and the adjacent side is 37 ft. how long is the hypotenuse? How long is the side opposite?

290. The Tangent of an Acute Angle. In a right triangle the ratio of the side opposite an acute angle to the side adjacent is the *tangent of that angle.*

In the figure of § 289 the relations involving tangents are written $tan\ A = \frac{a}{b}$ and $tan\ B = \frac{b}{a}$ or $b \cdot tan\ A = a$ and $a \cdot tan\ B = b$.

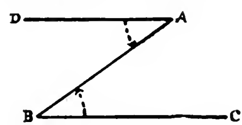
1. Construct a right triangle with an angle of 30°. Measure the two sides and compute the tangent of 30°.

2. What is the height of a flag staff if the angle subtended by the staff at a distance of 75 ft. is 35°?

3. In a right triangle the sides are 3 in. and 4 in. respectively. What are the tangents of the acute angles?

4. A tower 75 ft. high stands beside a river. The line from the top to a point on the opposite bank makes an angle of 52° with the tower. How wide is the river?

291. Angle of Elevation. If at any point in a horizontal or level line a line be drawn to a second point above the horizontal, the angle thus formed is the *angle of elevation* of the second point.



If BC is a horizontal or level line the angle B is the angle of elevation of point A from B.

292. Angle of Depression. If at any point in a horizontal line a line be drawn to a second point below the horizontal the angle thus formed is the *angle of depression* of the second point.

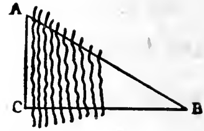
$\angle DAB$ is the angle of depression of B from A.

293. Horizontal Angles. Angles in a horizontal or level plane are *horizontal* angles.

294. Vertical Angles. Angles of elevation and depression are *vertical* angles as distinguished from horizontal angles.

5. From the top of a tower the angle of depression of a vessel at sea is 15°. If the tower is 87 ft. high how far away is the vessel?

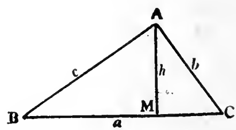
1. Points B and C are on opposite banks of a river. Line BC along the bank is 150 ft. long, $\angle A$ is 43°, $\angle C$ is 90°. How wide is the river?



295. In any triangle the sides of two acute angles have the same ratio as the sides opposite.

Given $\triangle ABC$ with sides a, b, c and acute angles B and C.

To Prove $\dfrac{\sin B}{\sin C} = \dfrac{b}{c}$.



SUG. Express $\sin B$ and $\sin C$. Complete the demonstration.

NOTE: This is a very important proposition as it gives certain relations connecting the sides and angles of triangles other than right triangles. It is proved here only for the case in which the two angles concerned are *acute*. It will be proved in trigonometry without exception and is known as the law of sines.

2. In $\triangle ABC$, $A = 45°$, $C = 76°$, and side $c = 60$ in. Find the other sides of the triangle and $\angle B$.

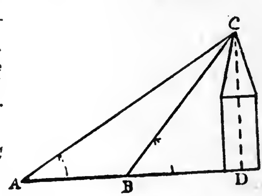
SUG. 1. $\dfrac{\sin A}{\sin C} = \dfrac{a}{c}$ and $\therefore a = c \cdot \dfrac{\sin A}{\sin C}$.

2. Since all angles of this triangle are acute this law may be likewise used for angles A and B.

3. Consult the table and complete the problem.

3. From a certain point the elevation of the top of a church tower is 43°. From a point 100 ft. nearer the base of the tower the angle of elevation is 55°. Find height of the tower.

SUG. Find sect BC from $\triangle ABC$ and then CD in $\triangle BCD$.



4. Knowing the distance AC to be 8 mi., the angle A to be 40°, and the angle C to be 82°; find the inaccessible distance AB across the lake.

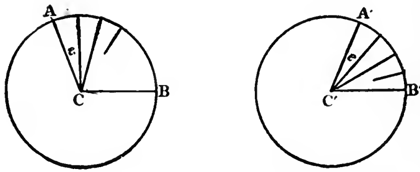
296. Trig. Table.

The Sines, Cosines and Tangents of Acute Angles.

Degrees	Sin.	Cos.	Tan.	Degrees	Sin.	Cos.	Tan.
1	.0175	.9998	.0175	46	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	49	.7547	.6561	1.1504
5	.0872	.9962	.0875	50	.7660	.6428	1.1918
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	.1736	.9848	.1763	55	.8192	.5736	1.4281
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	.2588	.9659	.2679	60	.8660	.5000	1.7321
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.8988	.4384	2.0503
20	.3420	.9397	.3640	65	.9063	.4226	2.1445
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	.4226	.9063	.4663	70	.9397	.3420	2.7475
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3.0777
28	.4695	.8829	.5317	73	.9563	.2924	3.2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	.5000	.8660	.5774	75	.9659	.2588	3.7321
31	.5150	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	.5736	.8192	.7002	80	.9848	.1736	5.6713
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	.6428	.7660	.8391	85	.9962	.0872	11.4301
41	.6561	.7547	.8693	86	.9976	.0698	14.3007
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0175	57.2900
45	.7071	.7071	1.0000				

PROPOSITION XII.

297. THEOREM. *In the same circle or in equal circles central angles are proportional to the arcs they intercept.*



Given $\odot C = \odot C'$ with central angles C and C' intercepting the arcs AB and $A'B'$ respectively.

To Prove $\dfrac{\angle C}{\angle C'} = \dfrac{\text{arc } AB}{\text{arc } A'B'}$.

Proof. SUG. 1. To obtain the ratio $\dfrac{\angle C}{\angle C'}$ the angles must be measured. Auth.? Suppose the unit angle e to be contained in $\angle C$ exactly m times and in $\angle C'$ exactly n times.

2. $\therefore \dfrac{\angle C}{\angle C'} = \dfrac{m}{n}$. Why?

3. To obtain the ratio $\dfrac{AB}{A'B'}$ the arcs must be measured. For a unit arc take the arcs intercepted by the unit angles. Auth?.

4. How many of these unit arcs in AB? Why? In $A'B'$? Why?

5. What then is the ratio of AB to $A'B'$?

6. Compare the ratios $\dfrac{\angle C}{\angle C'}$ and $\dfrac{AB}{A'B'}$.

Therefore—

NOTE. This demonstration does not cover the case in which the two angles are incommensurable. See note on § 251.

298. DEGREE OF ARC. The arc intercepted by a central angle of one degree is a *degree of arc.*

299. COR. *The number of degrees of angle in a central angle equals the number of degrees of arc in the intercepted arc.*

For: If central angle A intercepts arc a then §297 $\frac{\angle A}{1^\circ} = \frac{\text{arc } a}{1^\circ \text{ of arc}}$. But these two ratios are respectively the number of degrees in the angle and the arc.

This important theorem is usually stated thus: *A central angle is measured by its intercepted arc.* § 297.

If A and a are notations for the angle and arc the relation between them expressed in symbols is

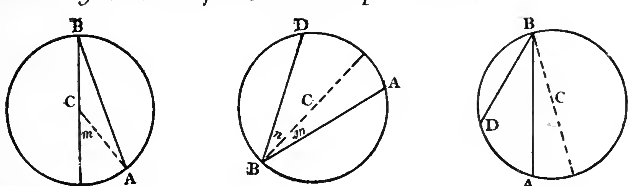
$\angle A \doteq$ arc a. (§ 50.)

1. Are all degrees of arc the same length?

2. How many degrees in a circle? In a semicircle? In a quadrant?

PROPOSITION XIII.

300. THEOREM. *An inscribed angle is measured by one half its intercepted arc.*



Given ⊙ C with the inscribed $\angle$ ABD intercepting the arc AD.

To Prove $\angle$ $ABD \doteq \frac{1}{2}$ arc AD.

Proof. There are three cases.

Case I. One side of the angle, BD, is a diameter.

Sug. 1. Connect A and C. $\angle m = 2 \angle B$. Why?

2. $\therefore \angle B = \frac{1}{2} \angle m$.

3. $\therefore \angle B \doteq \frac{1}{2}$ arc AD. Why?

Case II. The center of the circle lies between the sides of the angle.

Sug. 1. $\angle m$ is measured by what? Why?

2. $\angle n$ is measured by what? Why?

3. $\angle B$ is measured by what? Why?

Case III. The center of the circle is without the angle.

Proof is left to the pupil.

Therefore—

Note: This theorem may also be stated thus: The number of degrees in an inscribed angle is one-half the number in the intercepted arc and the same change in wording may be made throughout the proof.

301. Cor. I. *An angle inscribed in a semicircle is a right angle.*

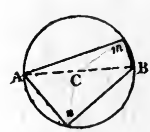
302. Cor. II. *A segment in which a right angle is inscribed is a semicircle.*

303. Cor. III. *All angles inscribed in the same or equal segments are equal.*

304. Cor. IV. *An angle inscribed in a segment greater than a semicircle is acute: in a segment less than a semicircle is obtuse.*

1. Construct a right triangle by means of Cor. I.

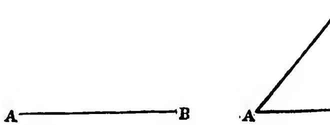
2. In $\odot C$ with diameter AB prove $\angle m = \angle n$.



3. If DE is a diameter of $\odot C$ which angle is the greater, m or n? n or o?

PROPOSITION XIV.

305. Problem. *To construct a right triangle when the hypotenuse and an acute angle are given.*

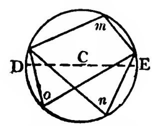


Given sect AB as the hypotenuse and $\angle A$ as an acute angle of a right triangle.

To Construct the triangle.

Sug. Construct the given angle at point A of AB. At B erect a $\perp$. Complete the problem.

1. To construct a perpendicular to a given line c from a point M on that line. See § 301.

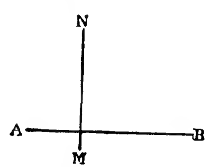


Sug. With a point C not on the line as center and with CM as radius draw a circle cutting line c in a second point A. Draw AC meeting the circle again in B. Then MB is the required line. Why?

2. From a given point B not on a line c draw a perpendicular to c. See § 301.

Sug. Draw any oblique line through B meeting line c in point A. On AB as diameter, construct a circle meeting c in point M. The required line is MB. Why?

3. Draw two lines from A and B meeting on the line MN so as to form a right angle. Is more than one construction possible?

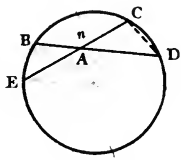


4. Measure the height of your school building using two angles of elevation and a distance. See Ex. 4, P. 124.

5. With the four vertices of a square as centers and with radii equal to one-half a side of the square draw four circles. Show that one circle can be drawn which is externally tangent to the four circles.

PROPOSITION XV.

306. THEOREM. *An angle formed by two intersecting chords is measured by one half the sum of the intercepted arcs.*



Given ⊙ O with chords CE and BD intersecting at A.

To Prove $\angle n = \frac{1}{2}$ (arc BC + arc DE).

Proof. SUG. 1. Draw CD. $\angle n = \angle C + \angle D$. Why?

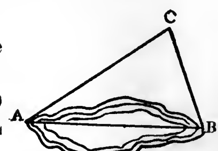
2. What is the measure of $\angle C$? of of $\angle D$?

3. Complete the proof.

Therefore—

1. Show two methods of finding the *center* of an equilateral triangle.

NOTE: The center of a polygon is that point which is equidistant from the vertices. Some polygons do not have centers.



2. On top of a hill 200 ft. high is a tower. From one point in the level plane at the foot of the hill the elevation of the top of the tower is 29°. At a point 200 ft. nearer the foot of the hill the elevation is 40°. How high is the tower?

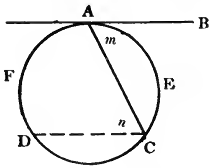
3. Inscribe a triangle in a given circle similar to a given triangle.

4. Circumscribe a triangle about a given circle similar to a given triangle.

5. If two circles are tangent internally and through the point of tangency a line is drawn, the chords intercepted by the circles are proportional to the radii of the circles.

PROPOSITION XVI.

307. Theorem. *An angle formed by a tangent and a chord drawn from the point of contact is measured by one half the intercepted arc.*



Given the ⊙ with $\angle m$ formed by the tangent AB and the chord AC.

To Prove $\angle m \doteq \frac{1}{2}$ arc AEC.

Proof. Sug. 1. Through C draw a line parallel to AB, as CD.

2. Compare $\measuredangle m$ and n; arcs AEC and AFD. Auth.

3. Complete the proof.

Therefore—

1. Two circles are tangent internally. Two lines are drawn from the point of tangency through the extremities of a diameter of one circle. Prove that they intersect the other circle in the extremities of a diameter.

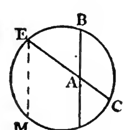
2. Prove Ex. 1 above if the circles are tangent externally.

3. AB and CD are two chords of a circle intersecting in the point O. Prove $\triangle AOD$ and $\triangle COB$ mutually equiangular. Prove the same for $\triangle AOC$ and $\triangle BOD$.

4. What is the locus of the vertex of the right angle of a right triangle with a given sect as hypotenuse? (§ 302.)

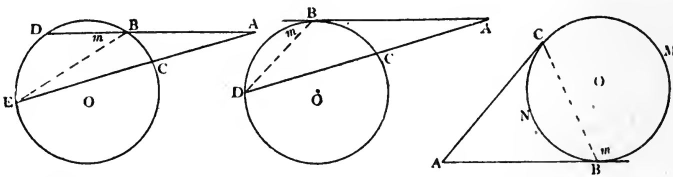
5. Construct a triangle having a given base, a given altitude, and a right vertex angle.

6. Prove Prop. XV from the accompanying figure in which $EM \parallel BD$.



PROPOSITION XVII.

308. Theorem. *An angle formed by two secants, a secant and a tangent, or by two tangents is measured by one half the difference of the intercepted arcs.*



Given $\odot$ O with I. two secants AD and AE; II. secant AD and tangent AB; III. two tangents AB and AC.

To Prove I. $\angle A = \frac{1}{2}$ (Arc DE — arc BC);

II. $\angle A = \frac{1}{2}$ (Arc BD — arc BC);

III. $\angle A = \frac{1}{2}$ (arc BMC — arc BNC).

Proof. Case I. Sug. 1. Compare $\angle A$ with $\angle m$ and $\angle E$.

2. What is the measure of $\angle m$, $\angle E$, $\angle A$?

Case II. Left to the pupil.

Case III. Left to the pupil.

Therefore—

1. All angles inscribed in the same segment are equal.

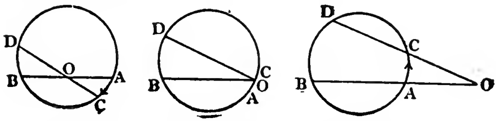
2. In the same or in equal circles an angle inscribed in the smaller of two segments is greater than an angle inscribed in the larger segment.

3. What is the locus of the vertex of a triangle having a given base and altitude?

4. A chord is met at one extremity by a tangent forming an angle of 75°. How many degrees in the arc that is subtended by the chord?

PROPOSITION XVIII.

309. THEOREM. *An angle formed by two lines either or both of which may be a secant or a tangent to a circle is measured by one half the sum of the intercepted arcs.*



Proof. SUG. 1. Is the theorem true if the lines intersect at the center? Why?

2. Is the theorem true if the intersection be anywhere within the circle? Why?

3. If the intersection points approach nearer and nearer to the circle what happens to one of the intercepted arcs? Suppose that when the point of intersection is on the circle this arc be considered as zero. Is the theorem true for this case?

4. A comparison of the figures shows that as O moves from a position within the ⊙ to a position on the circle the points O, A and C come together and as O passes without the circle they again separate but with this difference, that A and C, points in which the lines BA and DC meet the circle, in the third figure have their relative positions reversed, so that the arcs AC in the first and third figures are opposite in direction and if the idea of positive and negative lines be introduced the arc $-CA$ which enters the formula for the third case may be written as $+AC$. § 269.

5. For each case then the measure of the angle at O is $\frac{1}{2}$ (arc DB + arc AC).

The pupil should complete the demonstration for the case involving one or two tangent lines.

310. Continuity. A theorem can sometimes be stated in such a general way as to cover two or more particular theorems. Especially is this true in geometry when a distinction is made as to the direction of lines represented algebraically by the use of positive and negative quantities. Such a theorem is that of § 309 which is proved by the *principle of continuity*.

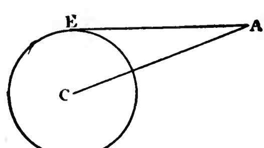
1. In making a pattern for a certain casting it is necessary to construct a true semicircle. How may this be done with a carpenter's square?

2. At a given point on a circle construct a tangent to the circle.

Sug. If a line were drawn from the center C to the point A what relation would it bear to the tangent through A? Complete the construction.

PROPOSITION XIX.

311. Problem. *From a given point without a circle to construct a tangent to the circle.*



Given $\odot$ C with point A without.

To Construct a tangent to $\odot$ C from A.

Sug. 1. Connect C and A. The problem is to determine the point of tangency.

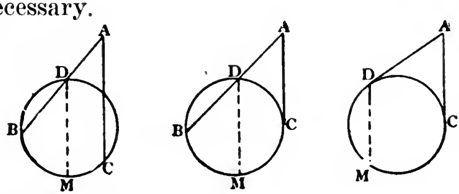
2. What relation exists between the tangent and the radius at the point of tangency?

3. What is the locus of a point with respect to AC satisfying this condition?

4. Complete the construction. § 302.

5. How many such tangents are there? Why?

6. Make a second construction drawing only such portions of the auxiliary lines as are necessary.



312. FIND THE LOCUS:

(1) Of centers of circles having a given radius and tangent to a given line.

(2) Of centers of circles tangent to a given line at a given point.

(3) Of centers of circles having a given radius and tangent to a given circle.

(4) Of centers of circles tangent to two intersecting lines.

(5) Of centers of circles having a given radius and passing through a given point.

(6) Of centers of circles passing through two fixed points.

1. Prove Prop. XVII by drawing from the point D in the accompanying figures a line $DM \parallel AC$.

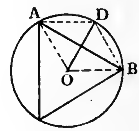
2. Two circles intersect in the points A and D. Lines AB and AC are diameters. Prove that B, D, and C lie in a straight line.

1. The sum of the distances from points in the base of an isosceles triangle to the legs is constant.

2. Given three non-parallel lines unlimited in length. Find points in one of them equally distant from the two others. How many such points are there?

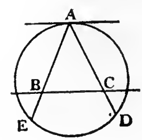
3. Two circles intersect in the points A and D. Sects AB and AC are chords of the two circles. Points B, D, C are in a straight line. Prove that the chords are diameters.

4. One side of an equilateral inscribed hexagon is equal to the radius of the circle.



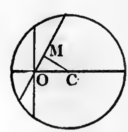
5. Inscribe an equilateral triangle in a circle and prove that the radius perpendicular to a side is bisected by the side.

SUG. $OABD$ is a ▱. Why?



6. Two chords drawn from a point on a circle are *inversely* proportional to the segments of the chords included between the tangent at their common point and a line parallel to this tangent.

TO PROVE $\frac{AD}{AE}=\frac{AB}{AC}$. See note, § 285 (1).



7. The locus of the middle points of all chords which pass through a given point is a circle the diameter of which is the line joining the given point and the center of the given circle.

SUG. Prove that the circle described on $\overline{OC}$ is the required locus.

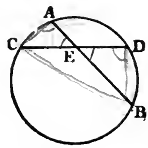
8. What has been done to the proportion $\frac{m}{n}=\frac{o}{s}$ to produce $\frac{m+o}{o}=\frac{n+s}{s}$? $\frac{m+n}{m}=\frac{o+s}{o}$? $\frac{m+n}{o+s}=\frac{n}{s}$?

9. Construct a circle having a given radius through a given point and tangent to a given line.

SUG. Find two loci of the centers.

PROPOSITION XX.

313. Theorem. *If two chords intersect the ratio of either segment of the one to either segment of the other is equal to the ratio of the remaining segment of the second to the remaining segment of the first.*



Given a circle with chords AB and CD intersecting at E.

To Prove $\frac{AE}{DE}=\frac{CE}{BE}$ or $\frac{AE}{CE}=\frac{DE}{BE}$.

Proof. Sug. 1. As no △ are given in the theorem, two △ must be constructed one of which contains the required antecedents and the other the required consequence. Note § 285 (1).

2. Prove the constructed triangles similar.

3. Establish the required proportions.

Therefore—

Make a second construction and proof for this proposition.

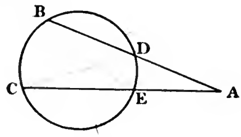
314. Cor. *The product of the segments of one of two intersecting chords of a circle equals the product of the segments of the other.*

To Prove $AE \times EB = CE \times ED$.

1. The line joining the center of a circle with an outside point bisects the angle made at that point by the two tangents to the circle.

PROPOSITION XXI.

315. Theorem. *If two secants intersect without a circle, the ratio of the first to the second is equal to the ratio of the external segment of the second to the external segment of the first.*



Given a circle with secants AB and AC intersecting the circle in the points D and E respectively.

To Prove $\frac{AB}{AC}=\frac{AE}{AD}$.

Proof. The desired conclusions will at once follow if two triangles can be constructed, one of them containing the required antecedents and the other the required consequents. Try such a construction and complete the demonstration. Note § 285 (1).

Therefore—

316. Cor. *If two secants meet without a circle the product of one secant and its external segment equals the product of the other and its external segment.*

1. In a given circle two radii are drawn to the center of two chords, and the angles made by them with the sect joining their feet upon the chords are equal; prove the chords are equal.

2. What line does the center of a ladder discribe as its foot is drawn directly away from the building against which it leans, the ground being level.

3. What is the locus of the center of a given sect whose end points continually touch the respective side of a right angle?

Sug. Ex. 2.

317. THEOREM. *In a series of equal ratios the ratio of the sum of the antecedents to the sum of the consequents equals any of the given ratios.*

Given $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{d}{d'}$, etc.

To Prove $\frac{a+b+c+d+\dots}{a'+b'+c'+d'+\dots}=\frac{a}{a'}$.

Proof. SUG. 1. Let n represent the common value of the given ratios. Then $a=na'$, $b=nb'$, etc.

2. $\therefore\ a+b+c+d+\dots=n(a'+b'+c'+d'+\dots)$

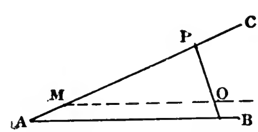
3. $\therefore\ \frac{a+b+c+d+\dots}{a'+b'+c'+d'+\dots}=n=\frac{a}{a'}$.

4. Give the authority for each step and verify by a particular set of numbers.

Therefore—

1. Give a summary of the tests for similarity of triangles.
2. Through a given point draw a line so that the sect intercepted between two given intersecting lines is bisected at the given point.

   SUG. Through the given point, o, draw a line parallel to one of the given lines.
3. Through a point included between the sides of an angle draw a line so that the sect intercepted between the sides shall be divided in the ratio of 1 to 4.



   SUG. Draw $MN \parallel AB$. Lay off on side AC a sect MP equal to $4AM$. Draw PO and extend to AB.
4 The sum of the three perpendiculars in the preceding example equals the altitude of the triangle.
5. The three altitudes upon the three sides of an equilateral triangle are equal.
6. The sum of the perpendiculars from any point in an equilateral triangle to the three sides is constant.

PROPOSITION XXII.

318. THEOREM. *The ratio of the perimeters of two similar polygons equals the ratio of similitude of the polygons.*

Given two similar polygons P and P' with sides a, b, c, ... and a', b', c', ... respectively, and perimeters p and p' respectively.

To Prove $\frac{p}{p'} = \frac{a}{a'}$.

Proof. SUG. 1. Write out in detail the relation involving the sides of P and P'.

2. Express the ratio $\frac{p}{p'}$ in terms of the sides.

3. Complete the proof.

Therefore—

319. MEAN PROPORTION. A proportion in which the means are the same is a *mean proportion.*

320. MEAN PROPORTIONAL. The second or third term in a mean proportion is a *mean proportional.*

321. THIRD PROPORTIONAL. The fourth term of a mean proportion is the *third proportional.*

$\frac{a}{b} = \frac{b}{c}$ is a mean proportion. The mean proportional is b and the third proportional is c.

1. Inscribe in a circle an isosceles triangle with a diameter of the circle as its base. How many degrees in the vertex angle?

2. If a right triangle is inscribed in a circle, the hypothenuse is always a diameter.

3. A line is drawn from the vertex of a triangle to the base. What is the locus of a point that divides it always in the same ratio?

PROPOSITION XXIII.

322. THEOREM. *A mean proportional of two numbers equals the square root of their product.*

Given $\frac{a}{b}=\frac{b}{c}$.

To Prove $b=\sqrt{ac}$.

Proof. $ac=b^2$. Why? Whence $b=$?

1. If $\frac{4}{x}=\frac{x}{9}$ find x.

2. If $\frac{3}{x}=\frac{x}{7}$ find x.

3. If $\frac{2}{5}=\frac{5}{x}$ find x.

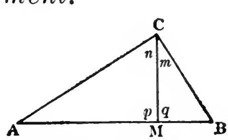
PROPOSITION XXIV.

323. THEOREM. *If the altitude on the hypotenuse of a right triangle be drawn:*

I. *the triangle is divided into two triangles, each similar to the given triangle and similar to each other;*

II. *the altitude is a mean proportional to the segments of the hypotenuse;*

III. *each side of the given triangle is a mean proportional between the whole hypotenuse and the adjacent segment.*



Given $\triangle ABC$ with altitude MC drawn to the hypotenuse AB.

To Prove I. $\triangle AMC \sim \triangle ACB$, $\triangle CMB \sim \triangle ACB$, $\triangle AMC \sim \triangle CMB$.

II. MC a mean proportional to AM and MB, i. e. $\frac{AM}{MC}=\frac{MC}{MB}$.

III. AC a mean proportional to AB and AM, i. e. $\frac{AB}{AC}=\frac{AC}{AM}$, or BC a mean proportional to AB and MB, i. e. $\frac{AB}{BC}=\frac{BC}{MB}$.

Proof. I. SUG. $\angle A$ is common to ⛛ AMC and ACB. $\therefore \triangle AMC \sim \triangle ACB$. Why? Complete the proof of I.

II. SUG. In ⛛ AMC and CMB sides AM and MC are homologous, also sides MC and MB. Why? Complete the proof.

Apply carefully § 285 (1) note. The difficulty in this case lies in the fact that MC is in two different triangles.

III. SUG. Use ⛛ AMC and ACB, noting the equal angles. Select the homologous sides for each ratio in accordance with the suggestion of § 285 (1).

Therefore—

324. COR. *In a semicircle a perpendicular from the arc upon the diameter is a mean proportional to the segments of the diameter.*

NOTE: In reading similar polygons care should be observed to follow a reading which gives the same order to homologous parts.

1. A perpendicular dropped from a circle upon a diameter is a mean proportional between the segments of the diameter.

2. Use the preceding exercise to construct a mean proportional to the sects a and b.

3. By means of this proposition, III, construct a mean proportional to two given sects. Also a third proportional.

4. If the altitude upon the hypotenuse divides the hypotenuse into two segments of 4 and 6 in. respectively, what is the length of the altitude? What is the length of each side? Express results accurate to nearest integer.

One segment is 8 in. and the adjacent side is 12 in. Find the hypotenuse and the two other sides.

5. Of all parallelograms having equal bases and altitudes the rectangle has the least perimeter.

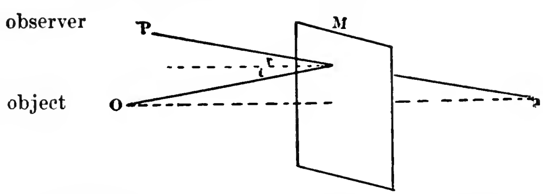
6. The diagonals of an isosceles trapezoid are equal.

7. If two parallel straight lines are cut by a transversal, the bisectors of the interior angles on the same side of the transversal are perpendicular to each other.

8. If the median drawn from the vertex angle of a triangle to the base equals half the base, the vertex angle is a right angle. Of which exercise is this the converse?

9. If a side of a triangle is parallel to the bisector of an exterior angle of the triangle, the triangle is isosceles.

10. In physics one learns that the angle of incidence equals the angle of reflection. In the figure these angles are *i* and *r* respectively. Prove that an object viewed in the mirror *M* seems as far behind the mirror as it is in front of it.



11. The bisectors of the angles of a parallelogram form a rectangle.

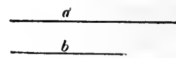
12. *a, b, c* are the angles of an inscribed triangle. $\angle a$ is four times *b* and *b* is one seventh of *c*. How many degrees in the arcs of the circle subtended by the respective sides of the triangle?

13. In the figure of Prop. XVI draw the diameter *AM* and prove the theorem.

PROPOSITION XXV.

325. PROBLEM. *Construct a mean proportional to two given sects.*

Given sects a and b.



To Construct sect x so that $\frac{a}{x} = \frac{x}{b}$.

Construction. SUG. 1. Make a drawing of the figure for Case II, Prop. XXIV to see which lines must be taken for a, b, x. Then construct the figure from the sects a and b.

2. Or in a similar manner make use of figure and conclusion of ex. 1 p. 142.

1. Construct a mean proportional to two given sects a and b by use of Case III of Prop. XXIV.

2. The bisectors of all angles inscribed in the same segment pass through a common point. Where is that point?

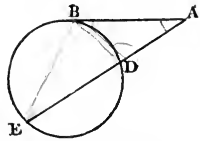
3. Draw any two equal non-intersecting chords of a circle and connect their adjacent extremities. Prove that the connecting chords are parallel.

4. Draw any two parallel chords and connect their extremities. Prove that the connecting chords are equal.

5. ABC is a triangle inscribed in a circle with center O. Line OD is perpendicular to BC. Prove $\angle DOC$, or its supplement, equal to $\angle A$.

PROPOSITION XXVI

326. THEOREM. *If from a common point a tangent and a secant be drawn to a circle, the tangent is a mean proportional between the secant and its external segment.*



Given a tangent AB and a secant AE of circle C.

To Prove $\frac{AD}{AB} = \frac{AB}{AE}$.

Proof. SUG. 1. Indicate a triangle having AE and AB as sides. Do the same for AB and AD. These △ are similar. Why?

2. Complete the demonstration.

Therefore—

327. EXTREME AND MEAN RATIO. A sect is divided into *extreme and mean ratio* when the greater segment is a mean proportional to the whole sect and the smaller segment.

AB is divided internally at M in extreme and mean ratio if $\frac{AB}{AM} = \frac{AM}{MB}$ and externally at M' in extreme and mean ratio if $\frac{AB}{AM'} = \frac{AM'}{M'B}$.

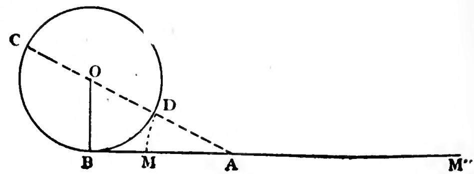
PROPOSITION XXVII.

328. PROBLEM. *To divide a sect internally and externally into extreme and mean ratio.*

Given sect AB.

To divide AB at M and at M' so that $\frac{AB}{AM} = \frac{AM}{BM}$ and

$\frac{AB}{AM'} = \frac{AM'}{M'B}$ respectively.



I. Sug. 1. Erect a perpendicular at B (or at A) equal to $\frac{1}{2}$ AB. With the free extremity O of this perpendicular as a center and OB as a radius describe a circle. Connect A and O and extend the join line to meet the circle again at C. Take $AM = AD$.

2. $\frac{AC}{AB} = \frac{AB}{AD} = \frac{AB}{AM}$. Why?

3. Form a new proportion from this by *division* and complete the proof.

4. Therefore M is the required point of *internal* division.

II. Sug. 1. Extend BA to M' so that $AM' = AC$.

2. $\frac{AB}{AC} = \frac{AD}{AB}$. Why?

3. Form a new proportion from this by *composition* and complete the proof.

4. Therefore M' is the required point of *external* division.

329. It must be remembered in the series of theorems following that in the operations of multiplication, division, squaring, etc., the symbols represent numbers, viz: The measures of the lines of the same name. *Sect* $a \times$ *sect* b means the product of the numerical measures of sect a and sect b. For brevity the operation of measuring will be assumed to have been done and the name or notation of sect will also represent the measure of it.

The operations performed in the demonstrations are then those of algebra. § 252.

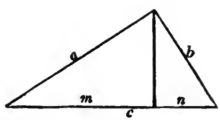
1. If the hypotenuse is divided by the altitude from the right angle, the ratio of the squares of the sides equals the ratio of the adjacent segments of the hypotenuse.

TO PROVE $\frac{a^2}{b^2} = \frac{m}{n}$. (Use fig. of Prop. XXIV.)

SUG. $a^2 = mc$. Why? $b^2 = ?$ Complete the proof.

PROPOSITION XXVIII.

330. THEOREM. *The square of the hypotenuse of a right triangle equals the sum of the squares of the two other sides.*



Given a right △ with sides a and b and hypotenuse c.

To Prove $a^2 + b^2 = c^2$.

Proof. SUG. 1. $a^2 = mc$. Why? $b^2 = ?$

2. $a^2 + b^2 = ?$

Notice that m and n are here coefficients of c.

Therefore—

Another proof is found in § 376.

331. COR. I. *The square of either side of a right triangle equals the difference of the squares of the hypotenuse and the other side.*

332. COR. II. *The ratio of a diagonal to the side of a square is* $\frac{\sqrt{2}}{1} = \sqrt{2}$

2. If the side of a square is 4 in., what is the diagonal? Find the result correct to two decimal places.

3. One side of a rectangle is 4 in. and a second side is 9 in. Find the diagonal correct to two decimal places.

4. A carpenter squares the frame of a building by measuring 6 ft. from the corner on one side, 8 ft. from the corner on the other, and then draws the frame until the distance across from the ends of the sects is 10 ft. Verify the correctness of his plan.

5. The distance from A to B is desired. A distance 70 rods is measured from B to C, so that the angle BAC is 90°. If AC is found to be 60 rods, how long is AB?

6. The diagonal of a rectangle is 12 in. Find the sum of the squares of the four sides.

7. Two sides of a rectangle are 6 and 8. Find the diagonal. Do the same for sides 5 and 12; 6 and 11.

8. Construct a right triangle and the altitude upon the hypotenuse. Measure the segments of the hypotenuse and compute the altitude. Check your work by measuring the altitude.

9. Find a mean proportional to 3 and 5; to 4 and 7; to 6 and 11. Use algebra and also geometric construction.

10. If $x = \sqrt{3 \times 4}$ find x. Sug. x is a mean proportional to 3 and 4. Why? Construct x geometrically and verify by measurement.

11. Find a line the length of which is $\sqrt{12}$.

Sug. $\sqrt{12} = \sqrt{3 \times 4}$. Make two other constructions. Use § 324.

12. Find $\sqrt{8}$ by the measurement of a line. Also $\sqrt{5}$, $\sqrt{9}$, $\sqrt{15}$.

Sug. Use § 324 and verify results by extracting the roots.

13. A boy in traveling from his home A to a town B on his wheel traveled 16 mi. in a straight line and then 24 mi. in a line at right angles to the first road. How many miles would he have saved by traveling along the railroad which ran in a straight line from A to B?

14. The sum of the squares of the two diagonals of a rectangle equals the sum of the squares on the four sides.

15. Find the length of the side of a square correct to two decimal places if the diagonal is 1. If the diagonal is 18; 2; 32; 26; 4; 12.

16. Find the diagonal of a square the side of which is 8. Do the same if the side is 25; 3.15; 4.18.

PROPOSITION XXIX.

333. THEOREM. *The square of the sum of two sects equals the sum of the squares of the sects plus twice the product of the sects.*

SUG. If a and b are the measures of the two given sects, § 329, it is necessary and sufficient to prove the formula $(a+b)^2 = a^2 + b^2 + 2ab$.

The pupil should do this by performing the indicated multiplication.

Another proof is found in P. 186.

PROPOSITION XXX.

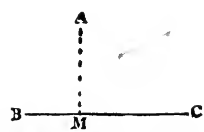
334. THEOREM. *The square of the difference of two sects equals the sum of the squares of the sects minus twice the product of the sects.*

Given a and b as the measures of the two sects.

To Prove $(a-b)^2 = a^2 + b^2 - 2\,ab$.

Proof left to the pupil.

335. PROJECTION OF A POINT. The *projection of a point* upon a line is the foot of the perpendicular drawn from the point to the line.



If AM is perpendicular to BC, then M is the projection of A upon BC.

336. PROJECTION OF A SECT. The *projection of a straight sect* upon a straight line is the sect between the projections of the extremities of the given sect upon the same line.

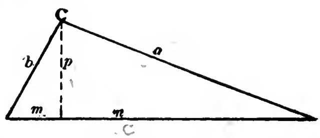
1. The product of the sum of two sects by their difference equals the difference of the squares of the sects.

TO PROVE $(a+b)\ (a-b) = a^2 - b^2$.

PROPOSITION XXXI.

337. Theorem. *The square of the side opposite an acute angle of a triangle equals the sum of the squares of the two other sides minus the product of one of those sides and the projection of the other upon it.*

twice



Given a △ with the sides a, b, c, side a being opposite an acute angle and m being the projection of b upon c.

To Prove $a^2 = b^2 + c^2 - 2mc$.

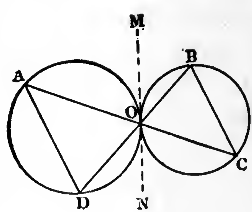
Proof. Sug. 1. Drop a perpendicular, p, from vertex C to side c.

2. $a^2 = p^2 + n^2 = p^2 + (c - m)^2 = b^2 - m^2 + (c - m)^2 = b^2 - m^2 + c^2 + m^2 - 2mc = b^2 + c^2 - 2mc$.

Therefore—

1. If a quadrilateral is circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.

2. If two circles are tangent and two secants are drawn through the point of contact the two chords joining the points in which the secants meet the respective circles are parallel.

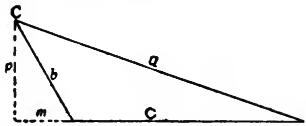


Prove $AD \parallel BC$.

Sug. Draw the common tangent MN. Compare ∠s AOM and CON; ∠s AOM and ADO; ∠s CON and OBC.

PROPOSITION XXXII.

338. THEOREM. *The square of the side opposite an obtuse angle of a triangle equals the sum of the squares of the two other sides plus twice the product of one of those sides and the projection of the other upon it.*



Given a △ with sides a, b, c, side a being opposite an obtuse angle, and m being the projection of b upon c.

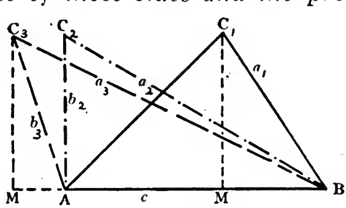
To Prove $a^2 = b^2 + c^2 + 2mc$.

Proof. SUG. 1. The projection m will in this case lie on the extension of c.

2. Complete the proof in a manner similar to the method of § 337.

339. The theorems of § 337 and § 338 may be included in one statement which may be proved by the *principle of continuity* (§ 310) as soon as a distinction is made as to the direction of the line representing the projected side. The theorem may be stated in general as follows:

The square of any side of a triangle equals the sum of the squares of the two other sides minus twice the product of one of those sides and the projection of the other upon it.



Given a △ with sides a, b, c, $\angle A$ being acute or obtuse, or right, and m, the projection of b upon c, being considered as positive when it extends in the direction AB and negative when it extends in the opposite direction.

To Prove $a^2 = b^2 + c^2 - 2mc$.

Discussion. As angle A increases from an acute angle to an obtuse angle, passing through the value 90°, point M, the projection of vertex C upon side c moves towards the vertex A, passes through this point when $\angle A = 90°$, and passes out on the extension of side c through A when $\angle A$ becomes obtuse. For $\angle A$ acute, m is positive; for $\angle A = 90°$, m is 0; and for $\angle A$ obtuse, m is negative.

The formula $a^2 = b^2 + c^2 - 2mc$ for these three values of m reduces to the formulas previously proved for these three cases in §§ 337, 330, and 338 respectively.

This theorem will be met again in trigonometry under the name of *The Law of Cosines.*

1. Fasten a rubber band to the two points of a pair of dividers. Let the rubber band represent side a in the triangles of § 339, and the arms of the dividers the sides b and c. The angle made by the arms then represents angle A. As the dividers are slowly opened, note the change of $\angle A$ from acute values to 90° and then to obtuse values and at the same time observe the shortening of the projection of b on c from a positive sect through zero on to increasingly large negative sects.

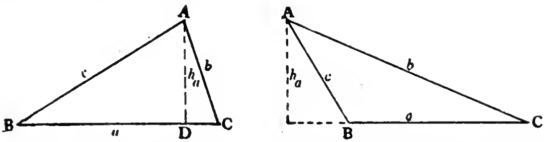
2. The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the four sides.

3. The sum of the squares of the diagonals of a quadrilateral equals the sum of the squares of the four sides plus four times the square of the sect joining the mid-points of the two diagonals.

4. A chord of a circle, AB, is 5 in. long. If it be produced to a point C so that the external segment BC is 10 in., how long is the tangent from C?

PROPOSITION XXXIII.

340. Problem. *To find any altitude of a triangle in terms of its sides.*



Given $\triangle$ with sides a, b, c, and h_a the altitude upon a.
To find h_a in terms of a, b, c.

Sug. 1. $h_a^2 = b^2 - \overline{DC}^2$ and $c^2 = a^2 + b^2 - 2a \times DC$.

2. $\therefore \quad DC = \dfrac{(a^2+b^2-c^2)}{2a}$.

3. $\therefore \quad h_a^2 = b^2 - \left[\dfrac{a^2+b^2-c^2}{2a}\right]^2$

$$= \left(b - \frac{a^2+b^2-c^2}{2a}\right)\left(b + \frac{a^2+b^2-c^2}{2a}\right)$$

$$= \frac{2ab-a^2-b^2+c^2}{2a} \quad \frac{2ab+a^2+b^2-c^2}{2a}$$

$$= \frac{(c-a+b)\ (c+a--b)\ (a+b-c)\ (a+b+c)}{4a^2}$$

4. Let $2s = a+b+c$;
$2(s-a) = c - a + b$, $2(s-b) = c+a-b$,
$2(s-c) = a + b - c$, whence,

$$h_a = \frac{2}{a}\sqrt{(s-a)\ (s-b)\ (s-c)s}$$

5. Similarly $h_b = \dfrac{2}{b}\sqrt{(s-a)(s-b)(s-c)s}$

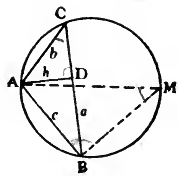
$$h_c = \frac{2}{c}\sqrt{(s-a)(s-b)(s-c)s}$$

1. By formula, find the three altitudes of triangle the sides of which are 12, 14, 18. Construct the triangle and measure the

altitudes as a check on the computations. Do the same for the triangle with sides 7, 9, and 12; also 13, 15, and 20.

PROPOSITION XXXIV.

341. Theorem. *In any triangle the product of any two sides equals the product of the altitude upon the third side and the diameter of the circumscribed circle.*



Given $\triangle ABC$ with sides a, b, c, altitude h_a upon side a and AM or d the diameter of the circumscribed circle.

To Prove $bc = dh_a$.

Proof. Sug. 1. Connect M and B. △ CAD and MAB are similar. Why?

2. Deduce a proportion from which the required equality may be obtained.

Therefore—

PROPOSITION XXXV.

342. Problem. *To find the length of the radius of the circumscribed circle in terms of the sides of the triangle.*

Given $\triangle ABC$ with sides a, b, c, altitude h_a upon side a, and AM or d the diameter of the circumscribed circle.

To find d in terms of a, b, c. (See fig. of Prop. XXXIV.)

Sug. 1. $h_a \times d = bc$. Why?

2. $\therefore d = \dfrac{bc}{h_a}$.

3. Substitute for h_a its value in terms of a, b, c.

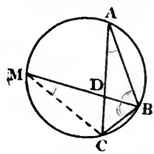
4. Simplify this result, obtaining

$$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

1. Find the radius of the circle circumscribed about a triangle with sides 8, 9, and 12. Construct the figure and check the computation by measurement.

PROPOSITION XXXVI.

343. Theorem. *In any triangle the square of the bisector of any angle equals the product of the sides including the angle minus the product of the segments of the third side made by the bisector.*



Given $\triangle ABC$ with BD bisecting $\angle B$, and meeting AC at D.

To Prove $\overline{BD}^2 = AB \times BC - AD \times DC$.

Proof. Sug. 1. Circumscribe a circle and extend BD to meet the circle at M. Draw MC (or MA).

2. $\triangle ABD \sim \triangle MBC$. Why?

3. $\frac{AB}{MB} = \frac{BD}{BC}$. Why?

4. $\therefore AB \times BC = MB \times BD$
$= (MD + DB)\ BD$
$= MD \times BD + \overline{BD}^2$
$= AD \times DC + \overline{BD}^2$.

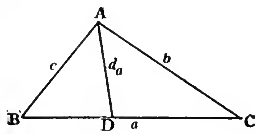
5. $\therefore \overline{BD}^2 = AB \times BC - AD \times DC$.

6. Verify each step.

Therefore—

PROPOSITION XXXVII

344. Problem. *To find the length of the bisector of an angle of a triangle in terms of the sides of the triangle.*



Given $\triangle ABC$ with sides a, b, c, AD or d_a being the bisector of $\angle A$, meeting BC at point D.

To find an expression for d_a in terms of a, b, c.

Sug. 1. By § 343 $d_a^2 = bc - BD \times DC$. Also $\frac{BD}{DC} = \frac{c}{b}$. Why?

2. $\therefore \frac{BD}{c} = \frac{DC}{b} = \frac{BD+DC}{c+b} = \frac{a}{c+b}$. Why?

3. $\therefore BD = \frac{ca}{b+c}$ and $DC = \frac{ab}{b+c}$.

4. $\therefore d_a^2 = bc - \frac{ac}{b+c} \times \frac{ab}{b+c} = bc - \frac{a^2bc}{(b+c)^2}$

$$= \frac{bc\,(b+c)^2 - a^2bc}{(b+c)^2} = \frac{bc\,[(b+c)^2 - a^2]}{(b+c)^2}$$

$$= \frac{bc(b+c-a)(a+b+c)}{(b+c)^2} = \frac{4bcs(s-a)}{(b+c)^2}.$$

5. $\therefore d_a = \frac{2}{b+c}\sqrt{bcs(s-a)}$, in which $2s = a+b+c$, as in § 340.

6. Verify each step and write corresponding expressions for d_b and d_c.

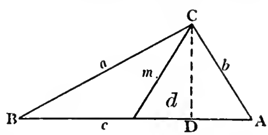
1. If the sides of a triangle are 7, 9, and 12; find the length of the bisector of each angle. Construct the triangle and check the computations by measurements.

2. In triangle ABC side $a = 24$, $b = 17$, and $c = 20$; find the bisector of $\angle C$.

3. Find the radius of the circle circumscribed about a triangle with sides of 12, 18, and 20. Check the results by construction.

PROPOSITION XXXVIII.

345. THEOREM. *In any triangle if a median be drawn to the base the sum of the squares of the two other sides is equal to twice the square of half the base plus twice the square of the median.*



Given a $\triangle$ with sides a, b, c, with D the projection of C and d the projection of m_c on c, and m_c the median on side c.

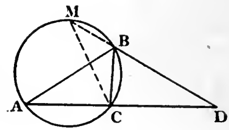
To Prove $a^2 + b^2 = 2m_c^2 + 2\left(\frac{c}{2}\right)^2$.

Proof. SUG. 1. If a and b are unequal, one angle between m_c and c is acute and the other is obtuse. Suppose then that a is opposite an obtuse angle.

2. Express a^2 and b^2 in terms of the remaining sides of the two triangles made by m_c. Add the resulting expressions.

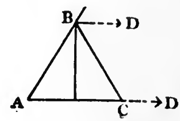
3. Prove the proposition when sides a and b are equal.

Therefore—



1. Prove § 343 for the bisector of an exterior angle.

Sug. Letter the figure as in the original proposition. Notice that AC is divided externally at D and $AC = AD - CD$.



2. Discuss the above problem when the angle bisected is exterior to the vertical angle of an isosceles triangle. In this case BD is parallel to AC. Why? No conclusions can be drawn as in the other cases. BD, AD, CD, all become infinitely large.

PROPOSITION XXXIX.

346. Problem. *To find the length of any median in terms of the sides.*

Given $\triangle ABC$ with sides a, b, c and m_a the median on side a.

To express m_a in terms of a, b, c.

From $2m_a^2 = b^2 + c^2 - 2\left(\frac{a}{2}\right)^2$ and

$m_a = (\frac{1}{2})\sqrt{2c^2 + 2b^2 - a^2)}$

Write the corresponding values of m_b and m_c.

*This proposition may be omitted.

PROPOSITION XL.

347. Theorem. *In any triangle, if the median be drawn to the base, the difference of the squares of the two sides equals twice the product of the base and the projection of the median on the base*

Given the conditions of Prop. XXXVIII.

To Prove $a^2 - b^2 = 2cd$.

Proof. Subtract b^2 from a^2.

Therefore—

1. In $\triangle ABC$ side $a = 8$, $b = 12$, and $c = 14$. Find the length of the median on c. Construct the triangle and median and check the computations by measurement.

2. In the above example find m_a and m_b and check computations.

Given the sides $a = 10$, $b = 6$, and $c = 8$. Find the median on a and check the computation.

3. Draw a triangle and a median with a straight edge. Measure the sides and the median. Check the results by computing the length of the median, and also by using other theorems involving the median and sides.

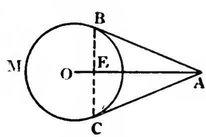
4. If two circles are tangent and a sect be drawn through the point of tangency terminated by the circles.

I. Tangents to the circles at the extremities of the sect are parallel.

SUG. Draw the common tangent at the point of tangency. Consider the two cases of internal and external tangency.

II. The diameters of the two circles through the extremities of the sect are parallel.

SUG. Use same construction as above, with two cases.



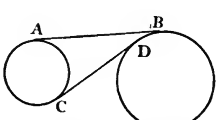
5. $\triangle ABC$ is equilateral and AB and AC are tangents to the circle. How many degrees in arc BC? In arc BMC?

6. If AB is 18 in., what is the diameter of the circle?

7. Show that OB is a mean proportional to OE and OA.

TO PROVE $\dfrac{OE}{OB} = \dfrac{OB}{OA}$.

318. EXTERNAL TANGENT. A line is an *external tangent* to two circles if it is tangent to each but does not cut the line of centers.

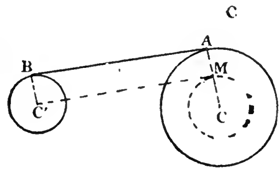


INTERNAL TANGENT. A line is an *internal tangent* to two circles if it is tangent to each and cuts the line of centers.

AB is an external tangent and CD an internal tangent.

PROPOSITION XLI.

349. PROBLEM. *To construct a common external tangent to two circles.*



Given two ⓢ C and C'.

To Construct AB a common external tangent to ⓢ C and C'.

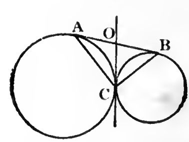
SUG. 1. Construct a circle concentric with the larger of the given circles with a radius equal to the difference between the given radii. Let this radius be CM. Why?

2. Draw a tangent to this auxiliary circle from C'. How may this be done?

3. Draw perpendiculars to this tangent line from C and C' meeting the circles in the points A and B respectively. Why?

4. AB is the required external tangent. Explain why.

5. Is AB the only external tangent? In what step of the construction might a second one be introduced?

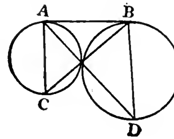


1. The common tangent to the two externally tangent circles at the common point meets an external tangent in a point equally distant from the three points of contact. To prove that $OA = OB = OC$.

2. If two circles are tangent externally the sects which join the point of contact to the two contact points of an external tangent form a right angle. To prove $\angle ACB = 90°$.

1. Two parallel tangents to a circle intercept a sect on a third tangent. Prove that this sect subtends an angle of 90° at the center.

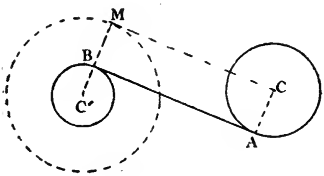
2. If two circles are externally tangent the sect which is the common external tangent is a mean proportional to the diameters.



3. Construct a mean proportional to sects *a* and *b* by means of Ex. 2.

PROPOSITION XLII.

350. Problem. *To construct an internal tangent to two circles.*



Given ⊙ C and C'.

To Construct AB, an internal tangent to ⊙ C and C'.

Sug. 1. Draw a circle concentric with the smaller of the given circles and with a radius equal to the sum of the radii of the given circles. Why?

2. The completion of the proof is left to the pupil.

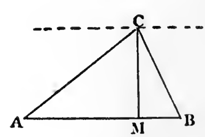
351. In the previous pages, certain principles of construction were stated and certain constructions made. In most of the problems thus far the pupil has been given more or less aid. In order that the pupil may become more independent in the solution of problems certain methods of analysis will now be studied in connection with the solution of a few problems.

Constructions are based upon previously demonstrated theorems, and the difficult thing in the solution of any problem is the recognition of the previous propositions which may be applied in effecting the desired construction. In the more difficult problems this can most easily be done by first making a sketch of the completed construction. This construction should next be analyzed in order to discover the theorems involved; then by reversing the order of the analysis, the construction can usually be made.

Each problem usually gives certain conditions for the determination of one or more points, the geometric representation of which conditions are loci, their intersections being the required points. The number of solutions, when a problem calls for the location of a point, is the same as the number of such intersections. In the case of no possible intersection, there is no solution.

352. Prob. I. To construct a triangle, given one side, an adjacent angle, and the altitude upon that side.

Given side AB, $\angle A$, and altitude MC.



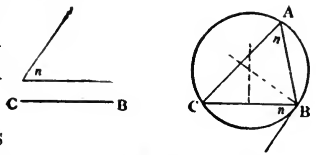
As AB is given it can be drawn and at one extremity the given angle A can be constructed.

The vertex C must now be determined. One locus of C is the unlimited side of $\angle A$, i.e. line AC. The other locus of C must be found from the other condition, viz: the given altitude MC. It is evident that the locus of the vertex of a triangle with a given altitude is a line parallel to the base and at a distance from the base equal to the given altitude. The construction can now be easily made.

Discussion: On a given side of line AB there is but one solution for two straight lines can have but one intersection. Also there is always a solution for if a line cuts one of two parallels it will cut the other.

PROB. II. Upon a given sect to construct a segment of a circle which shall contain a given angle.

Given $\angle\ n$ and sect BC. The problem is easily solved if the center of the required circle is found.

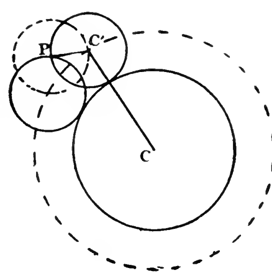


One locus of the center is the perpendicular bisector of sect BC. Why? It is observed from the figure that if the segment BAC contains $\angle A = \angle n$ that the arc BC is not only twice the measure of $\angle A$ but also of the angle at B between CB and a tangent to the circle. In other words if an angle equal to $\angle n$ be constructed at B, the second side will be tangent to the required circle. The perpendicular to this tangent at B is also a locus of the center. The center is thus determined. Does this problem have a solution for every sect BC and $\angle\ n$?

PROB. III. To construct a circle with a given radius that shall pass through a given point and be tangent to a given circle.

Given a circle C, a fixed point P, and a sect r.

To Construct a circle with radius r, passing through P, and tangent to C.

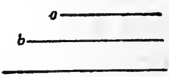


Draw a figure representing the completed construction. The center C' of the required circle is needed. What is the distance from C' *to* C? What then is a locus of C'? Since the distance PC' is r, what is a second locus of C'?

Discussion: In general how many intersections of these two loci and hence how many solutions? Is it possible for one solution only to exist? In this case how would the distance PC compare with the radii of the two circles? What relation of distances would make the intersection of the two loci an impossibility? Illustrate these special cases by figures.

PROB. IV. To construct a triangle having given the base, the altitude, and the radius of the circumscribed circle.

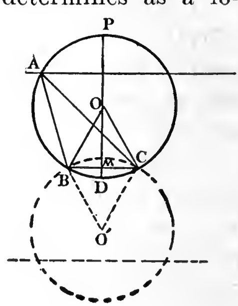
To Construct a triangle ABC with the sects a, b, and c as base, altitude, and radius of the circumscribed circle respectively.



The problem is solved as soon as one finds the third vertex, A, the extremities B and C of sect a being two of them.

What is the locus of vertex A as determined from the given altitude? The center of the circumscribed circle is at distance r from B and C. Locate its center O. The circumscribed circle is the second locus of vertex A.

Discussion. The first condition determines as a locus for A two straight lines. Why two? The Second condition determines two circles. Why? If the altitude is less than MD, each of the straight line loci meets each circle locus twice. How many solutions? If the altitude is greater than MD



but less than MP, each straight line locus meets one circle locus twice. How many solutions? If the altitude equals MD and is less than MP each straight line locus is tangent to one circle locus and meets the other in two points. How many solutions? If the altitude equals MP each straight line locus is tangent to one circle and does not meet the other. How many solutions? If the radius equals one half the base, then $MD = MP$, the problem is impossible or there would be two or four solutions, and any of the above cases in which MD and MP are assumed unequal are impossible. Discuss this in detail.

If the radius is less than half the base, there is no solution. Why?

1. The locus of the vertex of a given angle subtending a given sect is the arc of the segment which contains the angle.

SUG. 1. Construct the segment containing the given angle.

2. All angles inscribed in this segment equal the given angle. Why?

3. Any angle not inscribed in this segment and subtending the given sect is either greater or less than the given angle. Why?

2. Divide a sect 8 in. long internally into extreme and mean ratio. Measure the segments and arithmetically check the construction.

3. Divide 8 into two parts such that one part is a mean proportional between 8 and the other part.

4. Given a chord in a circle. From any point in the arc draw a second chord which shall be bisected by the first. Two solutions.

5. Construct a triangle, given the base, the altitude, and the vertex angle.

6. Construct a triangle given two angles and a side opposite one.

7. Construct a triangle, given two angles and the altitude upon the included side.

8. Construct a triangle, given the base, the median to the base, and an angle adjacent to the base.

9. Construct a triangle, given the base, the median to the base, and the vertex angle.

10. Construct a circle of given radius and tangent to two given circles. What conditions are necessary for a solution? How many solutions?

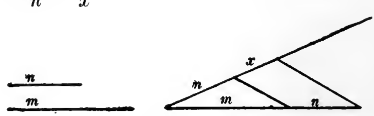
11. Construct a circle having a given radius, tangent to a given line, and passing through a given point. How many solutions and what conditions govern each case?

12. Construct a mean proportional to two sects, using Ex. 1 p. 142.

13. Construct a third proportional to two sects.

GIVEN sects m and n.

To construct sect x such that $\frac{m}{n} = \frac{n}{x}$.



SUG. 1. On an unlimited line lay off a sect $m + n$. Through one extremity of this sect draw an oblique line and from the intersection lay off on the line sect n. Join the extremities of the two sects of lengths m and n and through the free extremity of sect $m + n$ draw a line parallel to this join line. The sect intercepted on the oblique line by the parallels is x.

2. Verify the construction.

14. Construct a fourth proportional to sects a, b, c, using the method of Ex. 13.

15. Construct a fourth proportional to three sects using the problem for constructing a triangle similar to a given triangle.

16. Construct a fourth proportional to three sects using Ex. 1, p. 120.

NOTE: For such a problem any theorem in which a proportion of four different terms has been established may be used.

17. If a chord is 12 in. long and the perpendicular to that chord bisecting the subtended arc is 3 in., find the radius.

18. Draw accurately the figure of the preceding exercise, construct the circle and check the computations by measurements.

19. A bridge with a circular stone arch has a 39 ft. span and an 8 ft. rise. What is the diameter of the circle?

20. The altitude of an equilateral triangle is 12 in. Find the length of a side.

21. Prove that the perpendiculars from the center of a circle to the sides of an inscribed equilateral polygon are equal. This perpendicular is the *apothem* of the polygon.

22. Circumscribe a circle about an equilateral triangle and compare the apothem with the radius and the altitude.

23. Given an equilateral triangle of side 12 in. Find the apothem and altitude. Do the same for a triangle of side a.

24. Connect the vertices of an inscribed equilateral triangle with the mid-points of the arcs between them. Prove that the resulting hexagon is equilateral and equiangular.

25. Prove that the side of an inscribed equilateral hexagon equals the radius.

26. The sides of a triangle are 8, 11, 13. Find the altitudes, medians, bisectors of the angles, and the radius of the circumscribed circle.

27. In a circle of 10 in. radius there are two parallel chords, one being 6 in. from the center and the other 8 in. Find their lengths.

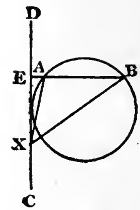
28. Find a fourth proportional to 2, 7, and 8 by arithmetic and geometric methods. Check the results by comparison.

29. Find a fourth proportional to 5, 7, and 9 by arithmetic method and by geometric construction. Check the results by comparison.

30. Find by arithmetic method and by geometric construction a sect equal to $\sqrt{9}$; $\sqrt{7}$; $\sqrt{10}$. Compare the results.

31. If two circles intersect, tangents drawn from any point in their common chord extended, are equal.

32. *CD* is tangent to circle *O*, *AB* is a secant cutting the circle at points *A* an *B* and perpendicular to *CD*. At what point *X* in *CD* is ∟ *AXB* the greatest?



33. If *A* and *B* are goal posts on a foot ball field and *E* the place of a touch down, how far back afield should the attempt be made to kick goal to insure the best opportunity?

34. The goal posts are 18½ ft. apart, and the touch down is made 49 feet from the nearest post. How many feet back is the best point from which to kick?

35. To find *AB*, the distance across the river. ∠ *A* is found to be 107°, ∠ *C* to be 53°, and line *AC* to be 24.7 rods.

CHAPTER IV.

AREA OF POLYGONS

353. **Area.** *Area* is a species of quantity (§ 245) and is obtained by measuring surface.

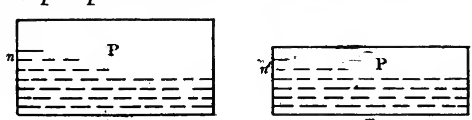
354. **Measurement of Surface.** *Measurement of surface* is the process of determining the number of times a unit of surface is contained in the given surface.

355. **Unit of Surface or Area.** A square having a given linear unit for one side is usually taken as the *unit of surface or area,* e. g. a square inch, a square rod, etc. There are exceptions to this, as the acre used in land measurement which consists of 160 square rods.

356. **Area.** The ratio of a given surface to the unit of surface is the *area* of the surface. It is expressed in terms of the unit used, e. g. 15 square inches, 24 square yards, etc. When thus expressed area tells two things; first, the number of times the unit is contained in the surface and second, the name of the unit used.

PROPOSITION I.

357. **Theorem.** *Two rectangles having equal bases are proportional to their altitudes.*



Given two rectangles P and P' with equal bases m and altitudes n and n' respectively.

To Prove $\frac{P}{P'} = \frac{n}{n'}$.

The altitudes n and n' are assumed to have a common unit of measure. The case in which they do not, i. e., in which they are incommensurable, will be considered later.

SUG. 1. Suppose n contains this common unit k times and that n' contains it k' times. What then is the ratio of n to n'? Auth.?

2. Through the points of division in n and n' draw lines parallel to the bases. The figures thus formed are rectangles. Why? They are congruent. Why? Therefore one of them can be used as a unit of measure.

3. How many times is this unit rectangle contained in P? In P'? What is the ratio of P to P'?

4. Compare the ratio of the altitudes with that of the rectangle. Auth.?

Therefore—

358. COR. *If two rectangles have equal altitudes they are proportional to their bases.*

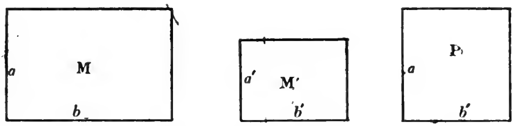
SUG. Either side of a rectangle may be regarded as the base.

359. COR. *If two rectangles have a common dimension, they are proportional to the other dimensions.*

The *dimensions* of a parallelogram or a triangle are its base and altitude.

PROPOSITION II.

360. THEOREM. *Two rectangles are proportional to the products of their dimensions.*



Given rectangles M and M' with dimensions a, b and a', b' respectively.

To Prove $\frac{M}{M'} = \frac{a \times b}{a' \times b'}$.

Proof. SUG. 1. Construct a rectangle P with one dimension of M, say a, and one dimension of M', say b'.

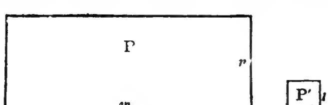
2. What is the ratio of M to P? Of P to M'? Why?

3. Find the product of these two ratios. What then is the ratio of M to M'?

Therefore—

PROPOSITION III.

361. THEOREM. *The area of a rectangle equals the product of its base and its altitude.*



Given rectangle P with dimensions m and n and unit P'.

To Prove area $P = m \times n$ times P'.

SUG. The ratio $\frac{P}{P'} = \frac{mn}{1 \times 1} = mn$. Why?

$\therefore P = mnP'$ or area $P = mn$ of the units of area.

The theorem is here stated in the usual abbreviated form. Literally interpreted it would imply that area is a number and that the "base and altitude" also are numbers. Stated in full and as it should be interpreted, it would read: *The area of a rectangle equals the product of the measures of the base and altitude times the unit of measure.* All the theorems of areas will be stated in the abbreviated form.

The name of a polygon is frequently used to designate its area.

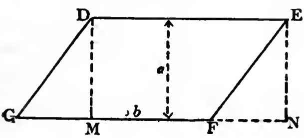
Polygons and circles are the surfaces to be measured in plane geometry.

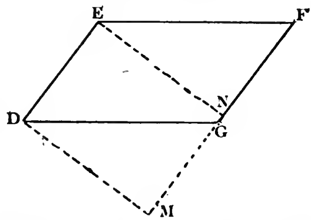
362. If the sides of the unit of measure are exact divisors of the corresponding sides of the rectangle a simple means of determining the area is to lay off the unit upon the rectangle, noting that the number of times the base of the unit is contained in the base of the rectangle is the number of area units in one row and that the number of times the altitude of the unit is contained in the altitude of the rectangle is the number of rows. Then by arithmetical analysis it is seen that the number of the rows times the number of the units in each row is the total number of units of area.

The demonstration of § 357 covers all cases of practical value for if the altitudes are incommensurable an approximate unit of altitude can be taken as small as the needs of the given case require. For instance, if the altitudes are 3 in. and $\sqrt{2}$ in. the ratio of the altitudes will be approximately $\frac{3}{1}$ if 1 in. be used as the unit; $\frac{30}{14}$ if the unit is .1 in.; $\frac{300}{141}$ if the unit is .01 in.; etc. The answer in this last case is correct to less than the area of a strip the length of the rectangle and .01 of the linear unit in width. Thus the accuracy of the area computed will depend upon the accuracy of the measurement of the dimensions, for the demonstration by § 357 can be carried rigorously to an approximation far beyond the skill of man to measure in any given case. All actual measurement gives but an approximation of the true value of the magnitude and the closeness of the approximation in measurement is usually determined by the degree of accuracy required.

PROPOSITION IV.

363. THEOREM. *The area of a parallelogram equals the product of its base and altitude.*





Given Parallelogram *DEFG* with base *b* and altitude *a*.

To Prove area of *DF* = *ab*.

Proof. SUG. 1. From two adjacent vertices as *D* and *E*, drop perpendiculars to the opposite side, as *DM* and *EN*.

2. Compare △ *DMG* with △ *ENF*.

3. Compare the areas of the parallelogram *DF* and the rectangle DN.

4. What is the area of *DF*? Of *DN*?

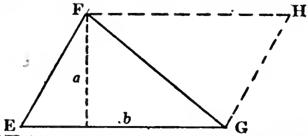
Therefore—

Does the above demonstration apply to both figures?

Construct carefully a parallelogram, *ABCD*, with sides from three to five inches in length. Construct the altitude upon *AB* as a base and compute the area. Construct the altitude upon *AD* as base and compute the area. Compare the results as a test for both accuracy of construction and accuracy of measurement.

PROPOSITION V.

364. Theorem. *The area of a triangle equals one-half the product of its base and altitude.*



Given a $\triangle EFG$, with base b and altitude a.

To Prove area of $EFG = \frac{ab}{2}$.

Proof. Sug. 1. Through vertex F draw a line parallel to base EG, and equal to EG. Connect its extremity H with G. What kind of a figure, EH, is thus formed? Why?

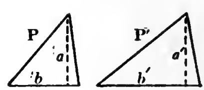
2. What is the area of EH? Compare the given triangle with EH. What is the area of the triangle?

3. Compare the bases and the altitudes of the triangle and EH.

Therefore—

365. Cor. I. *Two triangles having the same altitude are proportional to their bases.*

Sug. $\frac{\triangle P}{\triangle P'} = \frac{a \times b}{a \times b'} = \frac{b}{b'}$. Auth.



366. Cor. II. *Two triangles having equal bases are proportional to their altitudes.*

Proof left to the pupil.

Draw the three altitudes of a triangle and from each make the necessary construction for the proof of Prop. V.

Construct a triangle and its three altitudes. Make the measurements and compute the area from each altitude. Compare the results as a check.

PROPOSITION VI.

367. THEOREM. *Any two triangles are proportional to the products of their two dimensions.*

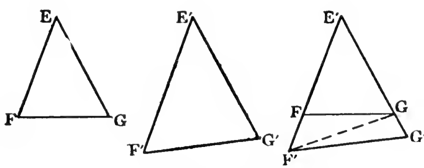
Given ⊿s P and P' with dimensions a, b and a', b' respectively.

To Prove $\frac{P}{P'}=\frac{ab}{a'b'}$.

Proof. SUG. Find the ratio of the areas and simplify the expression.

PROPOSITION VII.

368. THEOREM. *Two triangles having an angle in one equal to an angle in the other are proportional to the products of the sides including the equal angles.*



Given $\triangle EFG$ and $\triangle E'F'G'$ with $\angle E=\angle E'$.

To Prove $\frac{\triangle EFG}{\triangle E'F'G'}=\frac{EF\times EG}{E'F'\times E'G'}$.

Proof SUG. 1. Place $\triangle EFG$ upon $\triangle E'F'G'$ so that EF falls on $E'F'$, EG on $E'G'$, and E on E'.

2. Note that ⊿ $E'F'G'$ *and* $E'F'G$ have a common altitude, whence

$\frac{\triangle E'F'G}{\triangle E'F'G'}=$? Similarly $\frac{\triangle E'FG}{\triangle E'F'G}=$?

3. $\frac{\triangle E'FG}{\triangle E'F'G'}=$? $\frac{\triangle EFG}{\triangle E'F'G'}=$?

Therefore—

1. Compare the areas of three triangles formed by drawing lines through the vertex of a given triangle so as to trisect the base.

2. If the base of a triangle remain unchanged while the vertex is moved in a line parallel to the base, what is the effect upon the area of the triangle?

3. Divide a triangle into two parts by a line through the vertex so that one part is three fifths of the other.

4. Through a given point in one side of a triangle draw a line bisecting the triangle.

5. Through a given point in one side of a triangle draw a line which will cut off one third of the triangle.

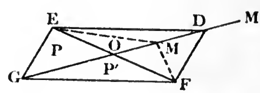
6. Divide a triangle into two equal parts by a line parallel to the base.

7. Divide a triangle into two parts by a line parallel to the base so that one part is one third the other.

8. If from any point in a parallelogram lines be drawn to the four vertices the sum of either pair of the opposite triangles thus formed equals one half the parallelogram.

9. The area of a square or a rhombus equals one half the product of the diagonals.

10. If EF and DG are diagonals of a parallelogram prove that any pair of triangles, as P and P', on the same side of a diagonal are equal.



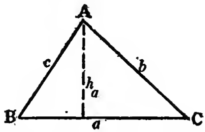
11. Let M (or M') be any point on diagonal EF of the preceding figure (or on EF produced). Connect M with E and F. Prove $\triangle MEO = \triangle MFO$ and $\triangle M'EO = \triangle M'FO$.

12. Drive three stakes several rods apart so as to form a triangle. Measure the three sides and compute the area. Measure an altitude and base and compute the area. Compare the results. If the several results differ to a considerable amount repeat the work to find the error.

13. Draw to some convenient scale the triangle of the preceding exercise. Compute the area by measurements according to one or other of the methods used before and compare the result with those derived from the direct measurements.

PROPOSITION VIII.

369. PROBLEM. *To determine the area of a triangle in terms of its sides.*



Given $\triangle ABC$ with sides a, b, c and altitude h_a on side a.

To find area of ABC in terms of a, b, c.

SUG. $\triangle ABC = \frac{ah_a}{2}$. In this substitute for h_a its value in terms of a, b, c, and simplify. §340

1. Write the formula for Prop. V using b and then c as base. Show that the simplified formulas are the same irrespective of the side used as base.

2. The sides of a triangular piece of ground are 7, 9, and 12 rods. Find the area.

3. Construct a triangle having sides of 8, 10, 12 inches. Find its area by § 369. Measure one altitude and compute the area. Check results by comparison.

4. The area of a triangle circumscribed about a circle equals one half the product of the perimeter and the radius of the circle. Prove the same theorem also for a circumscribed polygon.

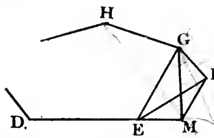
5. A number of triangles have as bases equal sects of the same straight line. If the opposite vertices are at the same point, which triangle is the larger?

6. Several triangles have as bases equal sects of a given straight line. If their areas are to be equal what is the locus of the opposite vertices?

7. Connect the mid-points of the adjacent sides of a parallelogram. Prove (1) that the included quadrilateral is a parallelogram, (2) that it equals one half the given parallelogram, (3) that the triangles at the four vertices are equal.

PROPOSITION IX.

370. PROBLEM. *To construct a triangle equal to a given polygon.*



Given any polygon as $DEFG \ldots$

To Construct a triangle equal to $DEFG \ldots$

Construction. SUG. 1. Draw a diagonal cutting off a triangle as EFG and through the vertex F draw a line parallel to the diagonal meeting DE produced at M. Join G and M.

2. Prove $\triangle EMG = \triangle EFG$.

3. Prove $DEFG \ldots = DEMG \ldots$

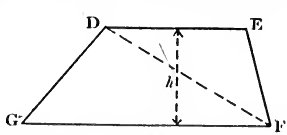
4. Compare the number of sides in $DEFG \ldots$ and $DEMG \ldots$

5. Proceed with the new polygon $DEMG \ldots$ as with $DEFG \ldots$

6. A triangle is finally obtained by this process. Why does it equal the given polygon?

PROPOSITION X.

371. THEOREM. *The area of a trapezoid equals one-half the product of the altitude and the sum of the bases.*



Given the trapezoid $DEFG$ with altitude h and bases DE and FG.

To Prove Area $DEFG = \frac{1}{2}(DE + FG)h$.

Proof. SUG. 1. Area of $\triangle DEF = \frac{h}{2} \cdot DE$. What is the area of $\triangle FGD$?

2. What is the area of $DEFG$?

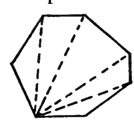
Therefore—

What is the unit of addition in step 2? What are the coefficients? Verify the addition algebraicly.

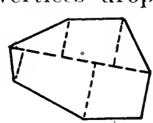
$$DE \times \frac{h}{2} + FG \times \frac{h}{2} = (DE + FG) \times \frac{h}{2}.$$

372. AREA OF A POLYGON. Various methods are used in finding the areas of irregular polygons, among which the following may well be noticed:

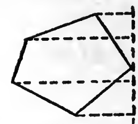
From any vertex of the polygon draw all possible diagonals. The polygon is by this means divided into triangles. The sides and altitudes of these triangles may be measured and their areas computed by any of the methods already shown. The area of the entire polygon is then found by addition.



Another method is to draw a diagonal, preferably the longest one and from the remaining vertices drop perpendiculars upon this diagonal. The polygon is thus divided into triangles and trapezoids. If the bases and altitudes of these triangles and trapezoids are measured their areas can be computed and the area of the polygon thus obtained.



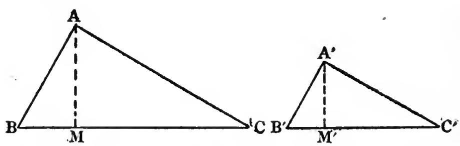
Another method is to draw through any vertex of the polygon a straight line exterior to the polygon upon which perpendiculars are dropped from the remaining vertices. In this way triangles and trapezoids are formed. The bases and altitudes of these are measured and their areas computed. The area of the given polygon is found by subtracting the areas of the parts exterior to the polygon from the sum of the areas of all the parts.



Solve Ex. 1 P. 184.

PROPOSITION XI.

373. THEOREM. *The ratio of the areas of two similar triangles is equal to the ratio of the squares of any two homologous sides or homologous altitudes.*



Given two similar △, ABC and $A'B'C'$ with homologous altitudes AM and $A'M'$.

To Prove $\frac{\triangle ABC}{\triangle A'B'C'}=\frac{\overline{BC}^2}{\overline{B'C'}^2}=\frac{\overline{CA}^2}{\overline{C'A'}^2}=\frac{\overline{AB}^2}{\overline{A'B'}^2}=\frac{\overline{AM}^2}{\overline{A'M'}^2}.$

Proof Sug. 1. $\frac{ABC}{A'B'C'}=\frac{BC\times AM}{B'C'\times A'M'}$ Why?

$=\frac{BC}{B'C'}\times\frac{AM}{A'M'}$ (factoring).

2. $\frac{BC}{B'C'}=\frac{CA}{C'A'}=\frac{AB}{A'B'}=\frac{AM}{A'M'}$ Why?

3. $\therefore \frac{ABC}{A'B'C'}=\frac{\overline{BC}^2}{\overline{B'C'}^2}=\frac{\overline{AM}^2}{\overline{A'M'}^2}$ etc. Why?

PROPOSITION XII.

374. THEOREM. *The ratio of the areas of two similar polygons is equal to the ratio of the squares of any two homologous sides.*

Given two similar polygons P and P' with AB and $A'B'$ any pair of homologous sides.

To Prove $\frac{P'}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$.

Proof. SUG. 1. From vertices A and A' draw all possible diagonals dividing P and P' each into the same number of triangles, similar each to each and similarly placed. § 286 (1).

2. The ratio of similitude of each pair of similar triangles is the same as the ratio of similitude of P and P'. For:

3. Let $\triangle_1, \triangle_1'$; $\triangle_2, \triangle_2'$; $\triangle_3, \triangle_3'$; etc., represent the pairs of similar triangles into which P and P′ are respectively divided. Then

$$\frac{\triangle_1}{\triangle_1'} = \frac{\triangle_2}{\triangle_2'} = \frac{\triangle_3}{\triangle_3'} = \ldots\ldots = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad \text{Why?}$$

4. Then

$$\frac{P}{P'} = \frac{\triangle_1 + \triangle_2 + \triangle_3 + \ldots\ldots}{\triangle_1' + \triangle_2' + \triangle_3' + \ldots\ldots} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad \text{See § 317.}$$

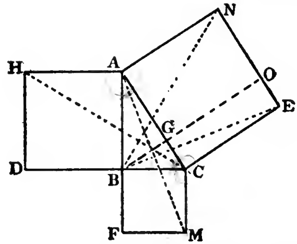
Therefore—

375. State each of the cases in which the ratio of areas has been found. Notice that the ratio of areas is always expressed by the product of two factors. If there is a common dimension, this factor can be cancelled out of both terms of the ratio. State all the cases thus far demonstrated in which this is true; also those

in which the ratio remains a product. Under what conditions may this product be written as a square?

PROPOSITION XIII.

376. THEOREM. *The square described upon the hypotenuse of a right triangle is equal to the sum of the squares described upon the two other sides.*



Given △ ABC with ∠ B a rt. angle and AE, CF, and BH squares upon the sides AC, CB, and BA respectively.

To Prove $AE = CF + BH$.

Proof. SUG. 1. Draw $BG \parallel$ to CE, meeting AC in G and NE in O. Draw BN, BE, CH and AM. What kind of polygons are $CEOG$ and $AGON$? Why?

2. Compare ⚠ BCE and ACM.

3. ABF is a straight sect. Why?

4. Compare the area of △ ACM with the area of the square BM; the area of △ BCE with the area of the rectangle CO.

5. Compare the area of the square BM with the area of the rectangle CO.

6. Compare ⚠ CAH and NAB.

7. CBD is a straight sect. Why?

8. Compare the square BH with the rectangle AO.

9. Compare the sum of the areas of the squares *BM* and *BH* with the sum of the areas of the rectangles *CO* and *AO*, i. e. with the area of the square *CN*.

Therefore—

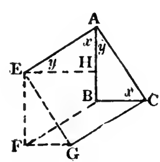
Compare this theorem with § 330 and show that the theorems are the same except that one is demonstrated by algebraic processes and the other by geometric processes. One deals purely with the number and the other with quantity.

377. Cor. *The square upon the leg of a right triangle equals the square upon the hypotenuse minus the square upon the other leg.*

$$\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2.$$

378. Scholium. This proposition is known as the Pythagorean proposition. It is named in honor of Pythagoras, 570 B. C., who is supposed to have given the first demonstration of it. The above demonstration is usually associated with his name.

The following is one among the many interesting proofs which have been devised for this proposition:



Let *ABC* be the right △, with *AG* the square upon the hypotenuse. Draw *EF* ∥ *AB*, *GF* ∥ *CB*, and *EH* ⊥ *AB*.

△ *EFG* = △ *ABC*. Why?

∴ ▱ *EB* + ▱ *BG* = sq. *EC*. Why?

Also △ *EAH* = △ *ABC*. Why?

∴ *EH* = *AB* and area of ▱ *EB* = area of sq. on *AB*. Why?

In the same way ▱ *BG* = sq. on *BC*.

Hence $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

1. Construct with instruments a pentagon. Compute the area by §§ 370 and 372. The comparison is a test of construction.

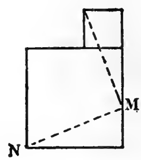
2. Find the area of a triangle with sides of 8, 12, 15.

3. If the diagonals of a quadrilateral intersect at right angles, show that the area of the quadrilateral is one half the area of the rectangle the sides of which are equal to the diagonals of the quadrilateral.

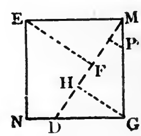
4. If three equal circles are tangent each to the two others, the lines joining their centers form an equilateral triangle.

5. Construct a triangle with an area 9 times that of a triangle the sides of which are 6, 7, 9.

6. A puzzle. Draw on a larger scale the accompanying figure. Cut it into 5 pieces along the dotted lines and rearrange the pieces forming a square on the line *MN*. This is an illustration of what theorem?



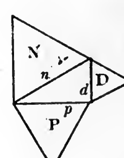
7. A puzzle. Construct a square *MN* of any size. Draw *MD* at random. Drop perpendiculars *EF* and *GH* to *MD*. Make *MP* equal to *ND* and drop a perpendicular from *P* to *MD*. The square *MN* can now be cut into 5 parts which can be rearranged into two squares. This illustrates § 376.



8. Verify the theorem of § 376 by making the drawing upon paper ruled in squares.

Sug. Construct the squares on the two legs and on the hypotenuse of the triangle and count the small squares included in each.

9. In the accompanying figure △ *M* is a rt. △, and △ *N*, *D*, and *P* are equilateral. Prove △ *N* = △ *D* + △ *P*.



Sug. $\frac{D}{P} = \frac{d^2}{p^2}$ Take proportions 2 and 3 by alternation and compare, hence $\frac{D+P}{P} = \frac{d^2+p^2}{p^2}$.

Also $\frac{N}{P}=\frac{n^2}{p^2}$ and hence by division $\frac{D+P}{N}=\frac{d^2+p^2}{n^2}=1.$

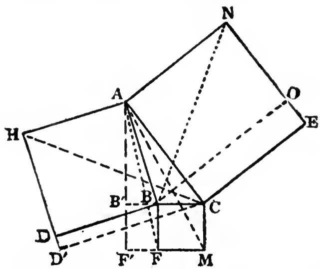
$\therefore \quad \triangle D+\triangle P=\triangle N.$

Is this relation still true if for N, D, P we substitute any similar polygons?

10. The square on the side of a triangle opposite an obtuse angle equals the sum of the squares upon the two other sides plus twice the rectangle formed by one side and the projection of the other side upon that side.

GIVEN $\triangle ABC$ with $\angle B$ obtuse, AE the sq. opposite $\angle B$, with BM and BH the squares on the two other sides, BB' the projection of AB upon BC, and $B'F$ the rectangle.

TO PROVE $AE=BM+BH+2\,B'F$.



SUG. 1. Assume the theorem in its algebraic form from § 338. Substitute for the algebraic products the areas which they represent in the above figure. (Draw BE, CD and extend AB to $D'C$.)

SUG. 2. From each vertex of the given triangle drop perpendiculars to the farther side of the opposite squares, or to these sides extended if need be. Connect the vertices of the triangle with the opposite vertices of their respective opposite squares. Work out a demonstration similar to § 376.

11. The square upon the side opposite an acute angle of a triangle equals the sum of the squares upon the other two sides minus twice the rectangle formed by one of those sides and the projection of the other upon it.

GIVEN $\triangle ABC$ with sides a, b, c.

TO PROVE the relation $a^2=b^2+c^2-2bm$, in which a^2, b^2, c^2

represent squares on the sides *a*, *b*, *c*, respectively and *bm* the rectangle formed by side *b* and the projection of *c* on *b*.

Sug. 1. Prove from § 376 as in preceding exercise.

Sug. 2. Draw the auxiliary lines and prove geometrically as in the preceding exercise.

12. What is the ratio of $\triangle MNO$ to $\triangle PMO$, $\angle M$ being 90°? Fig. 1.

13. Prove geometrically $(a+b)^2 = a^2 + 2ab + b^2$. Fig. 2.

14. Prove geometrically $(a-b)^2 = a^2 - 2ab + b^2$. Fig. 3.

15. Prove geometrically $(a+b)\ (a-b) = a^2 - b^2$. Fig. 4.

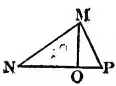


Fig. 1.

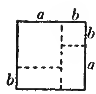


Fig. 2.

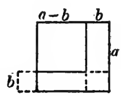


Fig. 3.

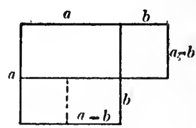


Fig. 4.

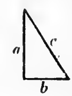


Fig. 5.

16. Prove by a figure the following formulas:

$a\ (a-b) = a^2 - ab$

$a\ (a+b) = a^2 + ab$

$a\ (b+c) = ab + ac$

$(a+b)\ (a+c) = a^2 + ab + ac + bc.$

$(a+b)\ (a-c) = a^2 + ab - ac - bc$

$(a-b)\ (a-c) = a^2 - ab - ac + bc.$

$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$

$(a+b)\ (c+d) = ac + ad + bc + bd.$

17. To construct a square equal to the sum of two given squares.

Given a square on *a* and a square on *b*. Fig. 5.

To construct a square equal to $a^2 + b^2$.

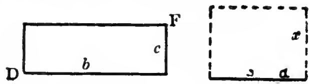
Sug. Construct a right triangle of which *a* and *b* are the legs. Complete construction. The proof is left to the pupil.

18. Construct a square equal to the sum of three given squares; of four given squares; of n given squares.

19. Construct a square equal to the difference between two given squares.

PROPOSITION XIV.

379. PROBLEM. *Upon a given base to construct a rectangle equal to a given rectangle.*



Given rectangle DF with adjacent sides b and c and sect a.

To Construct a rectangle on a equal to DF.

SUG. 1. Let x represent the altitude of the required rectangle. Then $b \times c = a \times x$.

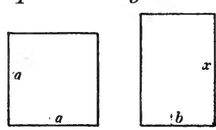
2. From this derive a proportion with x as the fourth and unknown term.

3. Find x and complete the construction.

4. Show that the constructed rectangle equals DF.

PROPOSITION XV.

380. PROBLEM. *Upon a given sect as base, to construct a rectangle equal to a given square.*



Given sect b and a square on sect a.

To Construct on b a rectangle equal to the given square.

SUG. 1. Let x be the altitude of the required rectangle. Then $a^2 = b \times x$.

2. From this derive a proportion with x as the fourth and unknown term. What problem is involved in the finding of x?

3. Complete the construction and prove the constructed rectangle equal to the given square.

1. Upon a given sect as base construct a rectangle the area of which shall equal the sum of the areas of a given square, a given trapezoid, and a given triangle.

2. The area of a square inscribed in a circle is one-half the area of any square circumscribed about the circle.

3. Construct a parallelogram with a given base and a given angle, the area of which shall equal that of a given rectangle.

4. Find the dimensions of a rectangle the perimeter of which is 26 in. with an area of 40 sq. in.

PROPOSITION XVI.

381. PROBLEM. *To construct a square equal to a given rectangle.*

Given a rectangle with adjacent sides a and b.

To Construct a square equal to rectangle ab.

SUG. 1. Let x represent the side of the required square. Then $x^2 = ab$.

2. Derive a proportion and find x.

3. Complete the construction.

5. Verify the constructions in §§ 379, 380, and 381 by measuring the dimensions and computing the areas.

6. Construct a square equal to a given trapezoid.

7. Construct a square equal to a given triangle.

8. Construct a square equal to a given parallelogram.

9. If the hypotenuse of a right triangle is 15 ft. and the ratio of its legs is $\frac{3}{4}$, what is its area?

10. Cut off one-third of a parallelogram by a line through a vertex. Cut off one-fourth in the same manner.

11. Construct an isosceles triangle having a given altitude and equal to a given triangle.

12. Bisect a parallelogram by a line parallel to a given line.

13. Construct a triangle equal to a given triangle, M, on base, a, and with its opposite vertex in a given line, x. Can this line be parallel to the base?

14. The diagonals of two squares are 5 ft. and 9 ft. respectively. What is the diagonal of a square equal to their sum?

15. Divide a triangle by a line through trisection points on one side into parts proportional to 1, 2, 3.

16. The side of an equilateral triangle is 10. What is the length of a side of a regular hexagon of the same area? Of a square of the same area?

17. The diagonal of a rectangle is 10 and the ratio of the sides is $\frac{3}{4}$. Find the area. If the diagonal is 40 and the ratio of the sides is $\frac{2}{5}$, find the area.

18. The difference between the squares of two sides of a triangle equals the difference between the squares of the projections of those sides upon the third side.

19. If two circles are tangent, internally or externally, chords of the one passing through the point of contact are divided proportionally by the other.

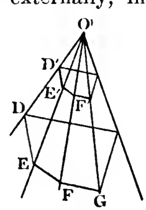
20. Draw a chord through a given point within a circle which shall be divided by this point in the ratio $\frac{1}{3}$. Also the ratio $\frac{m}{n}$.

SUG. Use preceding ex.

21. Draw a secant from a point without a circle terminated by the circle such that its ratio to its external segment shall be 3 to 1. Also $\frac{m}{n}$. Is this problem ever impossible?

22. Through a given point draw a line so that its distances from two given points shall be in a given ratio.

23. What is the locus of points which divide all chords of a circle passing through a given point, internally or externally, in a given ratio?



24. Let *DEFG*... be any polygon. Join the vertices to a point *O*. Through *D'*, any point on *OD*, draw a line parallel to *DE* terminating in *E'* on *DE*. Through *E'* draw a line parallel to *EF*, etc. Prove the polygon *D' E' F' G'* ...similar to *DEFG*....

Two similar polygons can always be placed in such a position with respect to some fixed point.

This property is characteristic and is sometimes taken as the definition of similarity. The point in which the lines joining homologous vertices meet is called the center of similitude.

24. *The area of a triangle equals one half the product of two sides and the sine of the included angle.*

This proof covers only the case in which the included angle is acute. The general theorem is proved in trigonometry.

GIVEN $\triangle ABC$ with sides a and c including the acute angle B.

TO PROVE area $ABC = \frac{1}{2} ac \sin B$.

PROOF. Area $ABC = \frac{1}{2} ah_a$ and $h_a = c \sin B$. Hence the theorem. Fig. § 369.

25. Drive three stakes in the School grounds so as to form an acute triangle. Measure two sides and the included angle and compute the area by ex. 24, also measure a base and altitude and compute and compare results. Review your work if not approximately the same.

CHAPTER V.

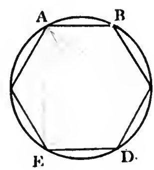
REGULAR POLYGONS.

382. Regular Polygon. A polygon which is both equilateral and equiangular is *regular*.

The equilateral triangle and the square are regular polygons.

PROPOSITION I.

383. Theorem. *An equilateral polygon inscribed in a circle is a regular polygon.*



Given *AD,* an equilateral polygon inscribed in a circle.

To Prove that *AD* is a regular polygon.

Proof. Sug. 1. According to the definition, § 382, what remains to be proved in order that *AD* be regular?

2. By what may the angles *A, B, C,* etc., be measured?

3. How do they compare with each other?

4. Test *AD* by the definition of a regular polygon.

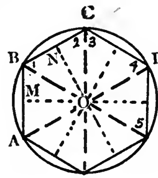
Therefore—

384. Cor. *If a circle be divided into any number of equal parts, the lines joining the points of division form a regular polygon.*

Proof left to the pupil.

PROPOSITION II.

385. THEOREM. *A circle can be circumscribed about a regular polygon. A circle can be inscribed in a regular polygon.*



Given a regular polygon, AD.

To Prove. I. A circle can be circumscribed about AD.

SUG. 1. At M and N the mid points of two adjacent sides erect perpendiculars extended until they meet as at O. Why do they meet?

2. Join O to the vertices A, B, C, etc. Compare $\triangle$ AOB and BOC; $\angle$ 1, 2 and 3.

3. $\triangle\ DOC \cong \triangle\ COB$. Why? Hence $OD = OB$. Why?

4. Similarly all the sects OA, OB, OC, OD, etc., are equal and hence a circle with center O can be circumscribed. Why?

To Prove. II. SUG. 1. What was proved concerning the successive $\triangle$ AOB, BOC, etc.?

2. Compare the sects OM, ON, etc. Auth.

3. Complete the demonstration.

Therefore—

Query. How many sides in the polygon AD? See the theorem.

386. Radius of a Regular Polygon. The radius of the circumscribed circle is the *radius of a regular polygon.*

387. Apothem of a Regular Polygon. The radius of the inscribed circle is the *apothem of a regular polygon.*

388. Center of a Regular Polygon. The center of the inscribed and circumscribed circles *is the center of a regular polygon.*

389. Angle at the Center of a Regular Polygon. The angle formed by two radii drawn to two adjacent vertices of the polygon *is the angle at the center of a regular polygon.*

From the definitions just given the following corollaries can be deduced. The pupil should prove them.

390. Cor. 1. *The angle at the center of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

391. Cor. II. *An interior angle of a regular polygon is equal to the sum of all the interior angles of the polygon divided by the number of sides of the polygon.*

392. Cor. III. *The angle at the center of a regular polygon equals an exterior angle and is the supplement of an interior angle of the polygon.*

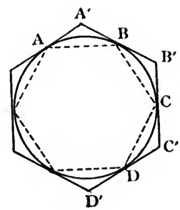
393. Cor. IV. *The radius of a regular polygon bisects the angle of the polygon to which it is drawn.*

1. A farm is 320 rods long, and rectangular in shape. From one end a square farm is cut off, leaving 120 acres. How wide is the farm and how many acres in the entire piece?

2. From any point M in side BC of $\triangle ABC$ draw a sect meeting AB produced in D, so that MD is bisected by AC. Also, so that MD is divided by AC in any given ratio.

PROPOSITION III.

394. THEOREM. *If a circle is divided into any number of equal parts, the tangents drawn through the points of division form a regular circumscribed polygon.*



Given the circle $ABC\ldots$, divided into equal parts $AB, CD, EF\ldots$, with tangents at the points $A, B, C,\ldots$ forming the polygon $A'B'C'\ldots$.

To Prove that $A'B'C'\ldots$ is regular.

SUG. 1. $\triangle A'AB, B'BC$, etc., are isosceles and congruent. Why?

2. Compare $AA', A'B, BB', B'C$, etc.

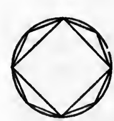
3. Prove $A'B' = B'C' = C'D'$, etc.

4. Prove $\angle A', B', C'$, etc., equal.

5. Apply the definition of a regular polygon.

Therefore—

395. COR. I. *If the vertices of a regular inscribed polygon are connected with the mid points of the arcs subtended by the sides, a regular inscribed polygon of double the number of sides is formed.* §394.



396. COR. II. *The perimeter and the area of a regu-*

lar inscribed polygon are less than the perimeter and area respectively of a regular inscribed polygon of double the number of sides.

397. COR. III. *If a regular polygon is circumscribed about a circle and tangents are drawn at the mid points of the intercepted arcs, a regular circumscribed polygon of double the number of sides is formed.* § 394.

398. COR. IV. *The perimeter and the area of a regular circumscribed polygon are greater than the perimeter and the area respectively of a regular circumscribed polygon of double the number of sides.*

1. The ratio of similitude of two similar polygons is $\frac{5}{7}$ and the sum of their areas is 518 sq. in. Find the area of each.

2. Find the base of a rectangle with an area of 108 sq. ft. and an altitude of 6 ft. Compare the method with that of 379.

3. Find the area of a right triangle with an hypotenuse of 1 ft. 8 in. and one leg of 1 ft. in length.

4. The area of a circumscribed polygon equals one-half the product of its perimeter and the radius of the circle.

5. If the mid points of two adjacent sides of a parallelogram be joined the area of the triangle thus formed is one-eighth that of the parallelogram.

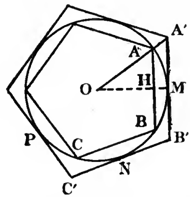
PROPOSITION IV.

399. THEOREM. *If a regular polygon is inscribed in a circle and tangents are drawn at the mid points of the arcs subtended by the sides:*

I. *A regular circumscribed polygon is formed.*

II. *The sides of the circumscribed polygon are parellel to the sides of the inscribed polygon, each to each.*

III. *The vertices of the circumscribed polygon lie in the extended radii of the inscribed polygon.*



Given a regular inscribed polygon $ABC\ldots.$, with $M, N, P,\ldots.$ the mid points of the arcs subtended by the sides AB, BC, CD, ... and tangents $A'B'$, $B'C'$, $C'D',\ldots.$ through the points $M, N, P,\ldots.$

To Prove. I. $A'B'C'\ldots$ is regular; **II.** $AB \parallel A'B'$, etc.; III. A' lies in OA produced, etc.

I. Sug. § 394.

II. Sug. Draw radius $OH \perp AB$ and extend it. Where does it meet arc AB? How do AB and $A'B'$ lie with reference to OH? Complete the argument.

III. Sug. 1. $\angle A = \angle A'$. Why?

2. OA bisects $\angle A$ and OA' bisects $\angle A'$. Why? § 393.

3. OHM is a straight line $\perp AB$ and $A'B'$. Hence $\angle MOA = \angle MOA'$. Why?

4. Hence A' lies on OA. Why?

Therefore

PROPOSITION V.

400. Problem. *To inscribe a regular hexagon in a circle.*

Given a circle with radius r.

To Inscribe a regular hexagon.

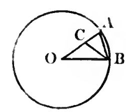
Solution. SUG. 1. The radius and side of the required hexagon are equal. Why?

2. With the dividers lay off six equal arcs. What is the length of the subtended chord?

3. Complete the constructior

PROPOSITION VI.

401. THEOREM. *If the radius of a circle is divided into extreme and mean ratio, the greater segment equals one side of a regular decagon inscribed in the circle.*



Given ⊙ O with radius OA divided at C into extreme and mean ratio, OC being the greater segment.

To Prove OC equal to one side of a regular decagon inscribed in the circle.

Proof.

1. Draw the chord AB equal to OC and join O and C to B.

2. It is to be shown that arc AB is one-tenth of the circle.

3. $\frac{OA}{OC}=\frac{OC}{AC}$. Why? § 327.

4. $\frac{OA}{AB}=\frac{AB}{AC}$ Why?

5. $\triangle OAB \sim \triangle BAC$. Why? § 283.

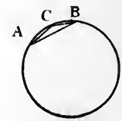
6. Compare $\angle ABC$ with $\angle O$; $\angle CBO$ with $\angle O$; $\angle ABO$ with $\angle O$.

7. Hence $\angle O$ is what part of 2 rt. ∠s? Of 4 rt. ∠s?

8. Hence AB subtends what fractional part of the circle?

Therefore—

402. Cor. *A regular decagon can be inscribed in a circle by dividing the radius in extreme and mean ratio and taking the greater segment as the side of the required polygon.*



1. Inscribe a regular polygon of 20 sides.
2. Inscribe a regular pentagon.
3. Inscribe a regular polygon of fifteen sides, called a pentadecagon.

Sug. 1. Draw a chord AB equal to the radius, and AC equal to the side of a regular inscribed decagon.

2. What part of the circle is subtended by AB? by AC?

3. Hence what part of the circle is subtended by CB?

4. Complete the construction.

4. What regular polygons can be inscribed the construction of which can be based upon that of the regular penta-decagon?
5. If the area of a regular inscribed triangle is 30, what is the area of the regular circumscribed triangle?
6. If the area of an inscribed square is 45, what is the area of the circumscribed square?
7. What is the numerical ratio of the apothem of a regular inscribed hexagon to that of a regular circumscribed hexagon?

PROPOSITION VII.

403. Problem. *To inscribe a square in a circle.*

Solution. Sug. 1. Two vertices must be the extremities of a diameter. Why?

2. How can the two other vertices be located?

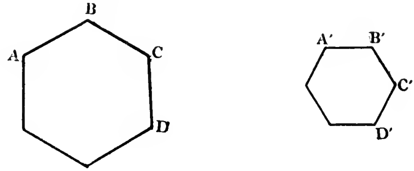
3. Draw a diameter and complete the construction.

Proof.

8. Inscribe a regular polygon of 8 sides; of 16 sides; of 32 sides.

PROPOSITION VIII.

404. THEOREM. *Regular polygons of the same number of sides are similar.*



Given AD and $A'D'$, two regular polygons of the same number of sides.

To Prove $AD \sim A'D'$.

Proof. SUG. 1. What must be established to prove the similarity?

2. Compare the corresponding angles A and A', B and B', etc.

3. $\frac{AB}{BC}=1$, $\frac{A'B'}{B'C'}=1$, etc. Why?

4. Compare the ratio $\frac{AB}{BC}$ with $\frac{A'B'}{B'C'}$; $\frac{AB}{A'B'}$ with $\frac{BC}{B'C'}$.

5. Compare $\frac{BC}{CD}$ with $\frac{B'C'}{C'D'}$; $\frac{BC}{B'C'}$ with $\frac{CD}{C'D'}$, etc.

6. $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}$, etc.

Therefore—

1. What is the locus of the centers of circles which are tangent to a given line at a given point?

2. Two parallel chords of a circle are respectively 36 and 48 inches long. The radius is 30 inches. Find the distance between the chords.

1. Find the dimensions of a rectangle which has a perimeter of 16 in. and an area of 15 sq. in.; the area of one with a perimeter of 28 ft. and an area of 10 ft.

2. If a parallelogram be inscribed in or circumscribed about a circle, the diagonals pass through the center.

PROPOSITION IX.

405. THEOREM. *The perimeters of two similar regular polygons have the same ratio as their radii and their apothems.*

Given AD and $A'D'$, two similar regular polygons with OA and $O'A'$ their respective radii, OM and $O'M'$ their respective apothems, AB and $A'B'$ two homologous sides, and p and p' their respective perimeters.

To Prove $\frac{p}{p'}=\frac{OA}{O'A'}=\frac{OM}{O'M'}$.

Proof. SUG. 1. Compare $\frac{p}{p'}$ with $\frac{AB}{A'B'}$. Auth.

2. Draw OB and $O'B'$ Then $\triangle AOB \cong \triangle A'O'B'$. Why?

3. Then $\frac{OA}{O'A'}=\frac{OM}{O'M'}=\frac{AB}{A'B'}$. Why?

4. Complete the proof.

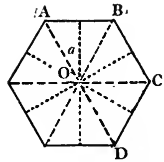
Therefore—

406. COR. *The areas of two similar regular polygons have the same ratio as the squares of their radii and the squares of their apothems.*

The proof similar to that of § 405 is left to the pupil.

PROPOSITION X.

407. THEOREM. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



Given a regular polygon AD, with area denoted by K, perimeter by p, and apothem by a.

To Prove $K = \frac{1}{2} p \times a$.

Proof. SUG. 1. What is the area of $\triangle$ AOB? Of BOC?

2. What is the area of each $\triangle$?

3. What is the sum of the areas of all the $\triangle$s?

Therefore—

In answering Sug. 3, let $\frac{1}{2}$ a be the unit of addition and AB, BC, etc., be the respective coefficients; or indicate the addition and divide by the common factor $\frac{1}{2}a$.

408. POSTULATE. *If the number of sides of a regular inscribed polygon be increased indefinitely, the apothem approaches indefinitely near the radius in length, likewise the perimeter approaches indefinitely near the circle and the area of the polygon approaches indefinitely near the area of the circle.*

A more exact statement of the postulate is as follows:

By sufficiently increasing the number of sides of a regular inscribed polygon the difference between the apothem and the radius can be made less than any given *number, however small. The same is true of the perimeter of the polygon and the circle and the areas of the two figures.*

1. What must be the nature of a parallelogram in order that a circle can be circumscribed about it?

2. If the radius of a circle is 12 inches, what is the length of a side of an inscribed square? What is the length of the apothem? What is the area of the square?

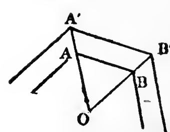
3. Find the area of the square in the preceding example by use of the apothem as a check upon the first computation.

4. If A and A' are two similar polygons, m and m' two homologous sides, and $A = 2A'$, then $m = m'\sqrt{2}$.

5. If A' is 16, find A. If A' is 12, find A. If A is 16, find A'. If A is 12, find A'. Find the coefficient of m' if a is $\frac{1}{3}a'$; if a is $\frac{1}{6}a'$; if a is $3a'$; if a is KA.

6. If one acute angle of a right triangle is 60°, prove that the area of the equilateral triangle constructed on the hypotenuse is equal to the area of a rectangle the adjacent sides of which are the two legs of the right triangle.

7. If two triangles have two sides of one equal to two sides of the other respectively and the included angles supplementary, they are equal.



8. The diagonals of a parallelogram divide it into four triangles equal in area.

9. Draw two concentric regular hexagons, one being twice the size of the other.

SUG. Having drawn one, with radius OA, let $x = OA'$ be the unknown radius of the other. Then x can be found from the proportion $\frac{\triangle AOB}{\triangle A'O'B'} = \frac{1}{2} = \frac{\overline{OA}^2}{x^2}$.

10. If 10 is the length of the radius of a regular hexagon, what is the length of the radius of a regular hexagon twice as large?

11. Divide a regular hexagon into four equal parts by concentric regular hexagons.

PROPOSITION XI.

409. THEOREM. *Two circles have the same ratio as their radii.*

Given two circles, c and c' with radii r and r', respectively, p and p' the respective perimeters of the similar inscribed polygons, p_2, p_2' being the respective perimeters of two similar inscribed polygons of double the number of sides, etc.

To Prove $\frac{c}{c'}=\frac{r}{r'}$.

Proof. 1. By § 405 $\frac{p_1}{p_1'}=\frac{p_2}{p_2'}=\frac{p_4}{p_4'}\dots\dots=\frac{r}{r'}$.

2. According to § 408 the difference between the circles and the perimeters of the inscribed polygons can be made as small as one chooses. Hence if x and x' be any small numbers there must be a point in the sequence of inscribed polygons where $c-p$ and $c'-p'$ are less than x and x' respectively. If these small differences be denoted by y and y', then $c-p=y$ and $c'-p'=y'$.

3. Hence $\frac{p}{r}=\frac{p'}{r'}$ may be written

$\frac{c-y}{r}=\frac{c'-y'}{r'}$.

4. From 3 follows $\frac{c}{r}=\frac{c'}{r'}-\left(\frac{y'}{r'}-\frac{y}{r}\right)$.

5. But since y and y' can be taken smaller than any given number the expression in parenthesis may for all practicable purposes be neglected.

6. Hence $\frac{c}{r}=\frac{c'}{r'}$.

Therefore—

410. Cor. I. *The ratio of a circle to its diameter is a fixed number whatever the radius.*

Proof. From the above proposition $\frac{c}{c'}=\frac{r}{r'}=\frac{2r}{2r'}=\frac{d}{d'}$, in which d and d' are the respective diameters. From this follows $\frac{c}{d}=\frac{c'}{d'}$. From this proportion follows the corollary. This ratio is represented by the Greek letter π.

411. Cor. II. $\frac{c}{d}=\pi$ or $c=\pi d=2\pi r$.

412. Cor. III. *The area of two circles have the same ratio as the squares of their radii.*

Let C and C' represent the two areas. Complete the proof in a manner similar to that of § 409.

In this proof the area of a circle can be approximated to any desired degree of accuracy, far beyond any demands of practical measurement.

413. Cor. IV. *The areas of two circles have the same ratio as the squares of their diameters.*

414. Cor. V. *Areas of similar sectors of two circles have the same ratio as the squares of their respective radii and diameters.*

1. Two tangents are drawn to a circle of 8 in. radius from a point 12 in. from the center. Find the length of the chord joining the points of tangency.

2. The legs of a trapezoid are each 15 inches and the bases are 12 and 30 inches respectively. Find the area.

3. Two registers similar in form are respectively 10 and 20 inches in width. What is the ratio of the amounts of air passing through them, the pressure being the same?

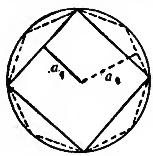
4. Two circular windows are respectively 24 and 30 inches in diameter. What is the ratio of their respective efficiencies for admitting light?

1. What is the locus of the vertices of all isosceles triangles on a given base?

2. Given △ ABC with D any point in BC extended. Find a point E in AB or in AB produced such that the area of △ EBD will equal that of △ ABC: will be one-half the area of △ ABC: will be double the area of △ ABC: will have any given ratio to the area of △ ABC.

PROPOSITION XII.

415. THEOREM. *The area of a circle is equal to one-half the product of its length and its radius.*



Given a circle o, with radius r, length c and area A.

To Prove $A = \frac{1}{2} cr$.

Proof. SUG. 1. If a sequence of regular polygons be inscribed, in each succeeding polygon of the sequence the number of sides being doubled, the apothem approaches the radius, the perimeter approaches the circle, and the area of the polygon approaches that of the circle. § 408.

2. The area of any of the polygons is $\frac{1}{2} ap$, a and p representing the apothem and perimeter respectively.

3. Substituting for these quantities those to which they, by this process, are made to approach, one obtains $A = \frac{1}{2} cr$.

Therefore—

NOTE. A more rigorous demonstration of the above theorem and of similar theorems which follow can be made by use of the theory of limits. In such a demonstration the possibility of making the substitution in step 3 of the above demonstration would be considered and proved in some detail. It is not thought wise to introduce such considerations at this point.

416. Cor. I. *The area of a circle equals* πr^2 or $\frac{\pi d^2}{4}$.

Proof. *Substitute for c, in* $\frac{1}{2}$ *cr, its value* $2\pi r$.

417. Cor. II. *The area of a sector equals one-half the product of its radius and its arc.*

Proof. Sug. 1. Prove by a method similar to that of § 297 that two sectors are to each other as their angles.

2. $$\frac{\text{Sector of arc } a}{\text{Sector of arc } 2\pi r} = \frac{a}{2\pi r}.$$

3. But a sector of arc $2\pi r$ has an area of πr^2. Substitute this and obtain a sector of arc $a = \frac{1}{2} a \times r$.

Note.—Since the formulas for area involve the number π, their accuracy, as well as that of the formula for length, depend upon the accuracy with which the number π is computed. The pupil should note that this computation has not yet been made. § 419.

1. The angle of a sector is 30° and the radius is 24 ft. What is the area of the sector? Express in terms of π.

2. The area of a sector is 88 sq. in. and its angle is 60°. Find the diameter of the circle.

3. The moon has approximately one-fourth the diameter of the earth. At a point equidistant from them what is the ratio of their respective "moonshine" powers?

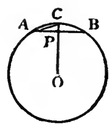
4. Two rectangular windows similar in form are respectively 2′ 6″ and 3′ in width. What is the ratio of the amounts of light they admit, other conditions being equal?

5. What is the locus of a point equidistant from two concentric circles?

6. Cut the sides of an equilateral triangle with three sects so as to form a regular hexagon.

PROPOSITION XIII.

418. Problem. *Given the radius of a circle and the side of a regular inscribed polygon, required to find the side of a regular inscribed polygon of double the number of sides, in terms of the given quantities.*



Given a circle O with radius r, AB a side of a regular inscribed polygon, and AC a side of a regular inscribed polygon of double the number of sides.

To Determine AC in terms of AB and r.

Sug. I. In $\triangle APC$ express AC in terms of AP and CP (§376) and then in terms of AB and CP.

2. Express CP in terms of r (i. e. OC) and OP. Express OP in terms of r (i. e. AO) and AP (§376) and then in terms of r and AB.

3. Express CP in terms of r and AB.

4. Express AC in terms of r and AB. This relation when simplified becomes

$$AC = \sqrt{2r^2 - r\sqrt{4r^2 - \overline{AB}^2}}$$

5. If r be taken equal to unity, this becomes $AC = \sqrt{2 - \sqrt{4 - \overline{AB}^2}}$

A partial statement of the details of the above steps is as follows:

$$AC = \sqrt{\overline{AP}^2 + \overline{CP}^2} = \sqrt{\frac{1}{4}\overline{AB}^2 + \overline{CP}^2} = \sqrt{\frac{1}{4}\overline{AB}^2 + (r - OP)^2}$$

$$=\sqrt{\frac{1}{4}\overline{AB}^2+\left(r-\sqrt{r^2-\frac{1}{4}\overline{AB}^2}\right)^2}$$

$$=\sqrt{\frac{1}{4}\overline{AB}^2+r^2-2r\sqrt{r^2-\frac{1}{4}\overline{AB}^2}+r^2-\frac{1}{4}\overline{AB}^2}$$

$$=\sqrt{2r^2-2r\sqrt{r^2-\frac{1}{4}\overline{AB}^2}}=\sqrt{2r^2-r\sqrt{4r^2-\overline{AB}^2}}$$

By means of this formula, if the length of the perimeter of any regular inscribed polygon is known, the length of the perimeter of a regular inscribed polygon of double the number of sides can be computed. From this result the perimeter can be computed in like manner for a polygon the number of sides of which is again doubled. This process can be continued indefinitely.

PROPOSITION XIV.

419. PROBLEM. *To compute approximately the ratio of a circle to its diameter.*

SUG. 1. For convenience the radius is taken as unity. Why may this be done?

2. The perimeter of a regular inscribed hexagon is 6 which may be regarded as a first approximate value of the circumference. The first approximation of π is then $\frac{p_6}{d}=\frac{6}{2}=3.$

3. From the formula of § 418, the value of s_{12}, which represents the length of a side of a regular inscribed polygon of 12 sides is

$$\sqrt{2-\sqrt{(4-1^2)}}=.51763809$$

and consequently p_{12}, or the perimeter, is $12s_{12}$, or 6.21165708. From this is obtained a second and closer approximation of

π, i. e $\frac{p_{12}}{d} = 6.21165708 = 3.10582854.$

4. Similarly

$$s_{24} = \sqrt{2 - \sqrt{4 - (5.51763809)^2}}$$

and $p_{24} = 24 \times s_{24}$.

5. Certain of the computed results are given in the following table:

No. Sid's	One Side	Perimeter	π
6	1	6.	3
12	$\sqrt{2-\sqrt{3}} = .51763809$	6.21165708	3.10582854
24	$\sqrt{2-\sqrt{4-(.51763809)^2}} = .26105238$	6.26525722	3.13262861
48	$\sqrt{1-\sqrt{4-(.26105238)^2}} = .13080626$	6.27870041	3.13935020
96	$\sqrt{2-\sqrt{4-(.13080626)^2}} = .06543817$	6.28206396	3.14103198
192	$\sqrt{2-\sqrt{4-(.06543817)^2}} = .03272346$	6.28290510	3.14145255

The approximation of π in common use is 3.1416. For ordinary work $3\frac{1}{7}$ is sufficiently accurate.

1. Inscribe a regular triangle. A regular polygon of twelve sides. Of twenty-four sides.

2. Measure the circumferences of several circular objects as a plate, the end of a pail, barrel, or stove pipe. Divide the circumference by the diameter. Average the several results to determine an approximate value of π.

3. Compute the length of the perimeter of a 24 sided polygon, P_{24}, with radius unity and compare the result with that given in the table of § 419.

4 If the radius is unity, what is the length of one side of a regular inscribed triangle?

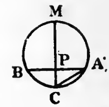
5. The radius is one and a side of a regular inscribed triangle is $\sqrt{3}$. Use the following formula to find the side of a

regular inscribed hexagon. Verify the result by § 418. Formula, $s_6 = \sqrt{2 - \sqrt{(4 - s_3{}^2)}}$

∴. A second method of solving the problem of § 418 is as follows:

To find AC in terms of r and AB.

SUG. $AC = \sqrt{MC \times PC}$.



420. SCHOLIUM. Archimedes (born 287 B. C.) found an approximate value for π. He proved that its value is between $3\frac{1}{7}$ and $3\frac{10}{71}$. In modern times the value of π has been computed to a large number of decimal places, two men, Clausen and Dase independently of each other having computed the value to the two-hundredth decimal place. Other computers have given the value to over five hundred decimal places but their results have not been verified. The number π is incommensurable with 1, i. e. it is neither an integer or a fraction, and hence cannot be expressed exactly by any number of decimal places.

1. The radius of a circle is 15 rods. Find its length, or circumference, and its area.

2. With a tape line measure the circumference of a tree and compute its diameter.

3. Draw a circle on paper or blackboard. Measure its diameter and compute its length and its area.

4. A circular silo is 30 ft. in diameter. How many square feet in its floor?

5. Given a circle with radius 5. What is its circumference or length and area?

6. The length of a circle is fourteen. What is its diameter and its area?

7. The area of a circle is 28. Find the length and the diameter.

8. Four 6 in. circles are tangent to each other. What is the area of the smallest square that can enclose them? Auth.

9. What is the area of the space enclosed by the circles of the preceding exercise?

10. What is the area of the circle externally tangent to the above four circles?

11. What is the area of the circle internally tangent to the above four circles.

12. Three 6 in. circles are tangent externally, what is the area of the triangle of their centers? What is the area of the space included between the circles?

13. What is the length of the apothem and what is the area of the triangle circumscribing them?

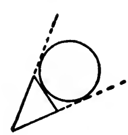
14. What is the area of the circle tangent to them internally?

15. If the area included by three equal circles tangent to each other is 5 acres, what is their radius?

16. Within a given circle construct 3 equal circles tangent to each other and to the given circle. Ex. 2, p. 37.

17. Construct three circles tangent to each other and externally tangent to a given circle. See ex. 2, p. 37.

18. Within a regular triangle construct three equal circles, each tangent to the two others and to two sides of the triangle.



19. A circle is escribed to a triangle when it is tangent to one side and to the two other sides produced. A triangle may have three escribed circles. Given a triangle, construct the three escribed circles.

Sug. Consider the principle of inscribing a circle in a triangle.

20. Find the area of the ring included between two concentric circles with radii of 8 and 10 inches respectively.

21. Find the length of a sect which is a chord of one of two concentric circles and a tangent to the other. Express the length in terms of the two radii.

22. Circles are described upon the three sides of a right triangle as diameters. Show that the one on the hypotenuse equals the sum of the others.

23. The diameter of a circle is one of the legs of an isosceles triangle. Show that the base is bisected by the circle.

24. A three inch and a five inch drain tile unite. What size of tile is necessary to continue the drain?

25. Three 5-in. tiles unite. What size of tile should be used at the union? Two methods.

26. If a straight line cuts two concentric circles, the segments included between the circles are equal.

27. If two diameters intercept at right angles, the sum of the squares upon the segments equals the square upon the diameter.

28. A saw log is 15 in. in diameter. What is the area of the cross section of the largest squared timber which can be sawn from this log? What proportion of the log goes into slabs?

29. If the above log were cut into a timber with rectangular cross section eight inches thick, what would be the cross section area? If the slabs are wasted, which of these two methods of sawing is more profitable?

30. The radius of a circle inscribed in a regular triangle is one-third its altitude.

31. Show that seven equal circles can be so drawn that one of them is "internally" tangent to the six others and that each of these six is tangent to two of the six.

32. Show that one circle can be so drawn as to include the circles of the preceding exercise and be tangent to six of them.

33. What part of the area of the large circle in above ex. is included in the area of the seven equal circles?

34. If the radius of the equal circles is 4, what is the area of that portion of the large circle which is external to the small circles?

35. Six dimes can be placed so that each is tangent to two others and also to a seventh dime. What is the ratio of the diameter of the dime to that of the circle which would circumscribe them all?

36. The sides of two regular hexagons are 3 and 5. If the area of the former is M, what is the area of the latter?

37. A regular hexagon of 63 square ft. is inscribed in a circle. Another is circumscribed about the circle. What is the area of the latter?

CHAPTER VI.

INCOMMENSURABLE MAGNITUDES.

421. In the discussion of quantities thus far only commensurable magnitudes have been considered, except in theorems involving π.

INCOMMENSURABLE MAGNITUDES. (§ 249.) Two magnitudes of the same kind that can not both be exactly measured by the same unit however small are incommensurable.

Thus $\sqrt{3}$ and 5 are incommensurable.

422. INCOMMENSURABLE RATIO. The ratio of two incommensurable quantities is an *incommensurable ratio* or *an incommensurable number.*

Thus the ratio $\sqrt{3}$ is an incommensurable number.

Approximations to the value of an incommensurable number may be obtained by neglecting fractional parts of the unit used. For example, the use of 1 as a unit gives as a first approximation of $\sqrt{3}$ the number 1. The use of .1 as a unit gives 17. The use of .01 as a unit gives 173. The use of .00001 as a unit gives 173205. This approximation may be carried to any desired degree of accuracy by the use of still smaller units.

An incommensurable number is neither a whole number nor a fraction, for if it were either the original quantities would not be incommensurable with respect to each other.

The pupil must not lose sight of the fact that an incommensurable number is as truly existent as any other. This fact is well illustrated by the consideration, for example, of the diagonal of a 1 in. square. That such a sect exists and that it has a length is not questioned. Geometry proves that its length expressed in inches is $\sqrt{2}$. This is an incommensurable number for if not it is an integer or a fraction. But it is neither of these because it is a number that when squared gives 2. It is quite evident that no integer when squared gives 2 and also that no fraction when squared will give an integer.

Thus if $\frac{a}{b} = \frac{2}{3}$, a and b cannot be incommensurable with respect to each other. This is seen as follows: By proportion $\frac{b-a}{a} = \frac{1}{2}$. Put $b - a = r$ and obtain $a = 2r$ and from this $b = 3r$.

The results in all *practical* measurements are but approximations of the magnitudes measured, depending for accuracy upon the skill of the operator and the precision of the instruments used. But when the results obtained are within the degree of accuracy required, they are used as if they were the actual magnitudes and not approximations.

Hence the previous demonstrations involving incommensurable magnitudes satisfy all demands of practical usage. For theoretical purposes it is interesting and important to know the absolute truths in the cases before considered and the discussion of such requires additional demonstrations of considerable rigor.

423. Constant. A quantity that is given the same value through a discussion is a *constant*.

424. Variable. A quantity which is given different values in a discussion is a variable. In general variables in changing values pass through a succession of values according to some law which serves to distinguish them from other variables.

425. Limit of a Variable. When a variable by its law of change differs from a constant by a variable quantity which may become and remain less than any assigned quantity however small, but not zero, this constant is the *limit of the variable.*

A variable may or may not attain its limit. This is a matter dependent upon the law under which the variable exists. The successive approximations of $\sqrt{2}$ may be considered as the successive values of a variable which has $\sqrt{2}$ as its limit. In this case the variable does not attain its limit. In the case of the sequence of regular inscribed polygons used in the theorems on the circle, the variable apothem, perimeter, and area do not attain their limits. On the other hand variable segments between the sides of a triangle may decrease in length till at the vertex they reach their limit zero; chords of a circle may increase in length until they reach their limit the diameter.

If a point move along a sect in such a manner as to traverse one half of it in a given period of time, one-half the remaining sect in a second period, and so on indefinitely, it will never traverse the entire sect. It is evident, however that by continuing the process for a sufficient number of periods of time the portion still untraversed by the point will become and remain less than any given sect, however small. The entire sect is then the limit of the variable distance traversed by the point.

By increasing the number of sides of an inscribed regular polygon (§ 408) indefinitely the perimeter of the polygon approaches the circle as its limit and the area of the polygon approaches the area of the circle as its limit, for it is evident that the difference between the perimeter and the circle and the difference between the two areas may, by this process, be made less than any assigned amount.

PROPOSITION I.

426. THEOREM OF LIMITS. *If two variables are always equal and each has a limit, their limits are equal.*

Proof. Let the two equal variables be x and y and let their respective limits be the constants a and b. Represent $a - x$ by v and $b - y$ by u. Then by subtraction follows $(a - x) - (b - y) = v - u$. Since $x = y$ this becomes $a - b = v - u$. By the definition of a limit (§ 405) a and b are constants and therefore $a - b$ is a constant. Hence $v - u$, which equals $a - b$ is constant. But since v and u may both be considered as small as one desires and are variables the expression $v - u$ cannot be constant unless it is zero in which case $a - b$ is zero and $a = b$.

427. The following statements are here assumed without proof, although demonstrations of them are easily made from the definitions on limits.

1. *The product of a variable by a constant is a variable and its limit is the product of the constant and the limit of the variable.*

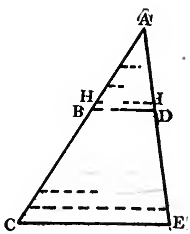
If the limit of x is a, then the limit of kx is ka, k being a finite constant.

2. *The quotient of a variable by a constant is a variable and its limit is the quotient of the limit of the variable by the constant.*

If the limit of x is a, then the limit of $\frac{x}{k}$ is $\frac{a}{k}$, k being a finite constant different from zero.

PROPOSITION II.

428. THEOREM. *If a line is parallel to the base of a triangle, the sides are divided into proportional segments.*



The following demonstration is for the case omitted in § 251.

Given $\triangle ACE$ with $BD \parallel CE$ and AB incommensurable with BC.

To Prove $\frac{AB}{BC} = \frac{AD}{DE}$.

Proof. SUG. 1. With some unit commensurable with BC divide AB and BC. A remainder, as HB, will occur in the division of AB. If parallel lines are drawn through the points of division DE will be divided into equal segments and the line through H must fall somewhere between the intersection of the preceding parallel line and D, as at I, otherwise HI would not be parallel to the other parallel lines.

2. Then by § 251, $\frac{AH}{BC} = \frac{AI}{DE}$.

3. A sequence of diminishing units makes AH a variable with AB as its limit. Hence

by § 427 $\frac{AH}{BC}$ is a variable with $\frac{AB}{BC}$ as its limit.

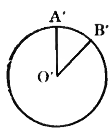
Similarly AI is a variable with AD as its limit and $\frac{AI}{DE}$ is a variable with $\frac{AD}{DE}$ as its limit.

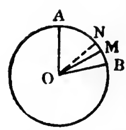
4. By § 426 the limits of the two variables $\frac{AH}{BC}$ and $\frac{AI}{DE}$ are equal and hence $\frac{AB}{BC}=\frac{AD}{DE}$.

Therefore—

PROPOSITION III.

429. THEOREM. *In the same or in equal circles, angles at the center have the same ratio as their intercepted arcs.*





The following demonstration is for the case omitted in § 297.

Given two circles O and O' with $\angle AOB$ incommensurable with respect to $\angle A'O'B'$, AB *and* $A'B'$ being the respective intercepted arcs.

To Prove $\frac{\angle O}{\angle O'}=\frac{\text{arc } AB}{\text{arc } A'B'}$.

Proof. SUG. 1. With an angle commensurable with $\angle O$ as a unit angle measure the two angles. In $\angle O$ there will be a remainder as $\angle MOB$ which is less than the unit angle, having the intercepted arc MB (the point M always falling between the end of the last unit arc and point B).

2. By § 297 $\frac{\angle AOM}{\angle A'O'B'}=\frac{AM}{A'B'}$.

3. A sequence of diminishing units makes $\angle AOM$ a variable with $\angle AOB$ as its limit. Hence by § 427 $\frac{\angle AOM}{\angle A'O'B'}$ is a variable with $\frac{\angle AOB}{\angle A'O'B'}$ as its limit. Similarly arc AM is a variable with arc AB as its limit and $\frac{AM}{A'B'}$ is a variable with $\frac{AB}{A'B'}$ as its limit.

4. By § 426 the limits of these variable quotients are equal and hence

$$\frac{\angle O}{\angle O'} = \frac{\angle AOB}{\angle A'O'B'} = \frac{\text{arc } AB}{\text{arc } A'B'}.$$

430. A second demonstration of § 429. Assume that $\frac{\angle AOB}{\angle A'O'B'} < \frac{AB}{A'B'}$ and take N so that $\frac{\angle AON}{\angle A'O'B'} = \frac{AB}{A'B'}$. (1)

Measure the given angles by a unit angle less than $\angle NOB$. One point of division will fall then between N and B, say at M and by 297 $\frac{\angle AOM}{\angle A'O'B'} = \frac{AM}{A'B'}$. (2)

From (1) and (2) follows $\frac{\angle AOM}{\angle AON} = \frac{AM}{AB}$ which is not true since $\angle AOM > \angle AON$ and arc $AM <$ arc AB. Hence the assumption is false. Similarly it can be shown that $\frac{\angle AQB}{\angle A'O'B}$, is not greater than $\frac{AB}{A'B'}$.

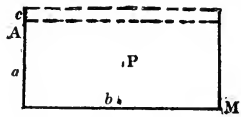
Therefore—

Apply this method of proof to Prop. II.

1. What part of the diameter of a circle is the apothem of an inscribed regular triangle? How can this conclusion be used as a basis for inscribing a regular triangle?

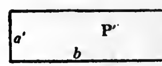
PROPOSITION IV.

431. THEOREM. *Two rectangles having equal bases are proportional to their altitudes.*



The following demonstration is for the case omitted in § 357.

Given two rectangles P and P' with equal bases and altitudes a and a' incommensurable with respect to each other.



To Prove $\frac{P}{P'}=\frac{a}{a'}$.

Proof. SUG. 1. Measure altitudes a and a' with a unit commensurable with a'. In the measurement of a there will then remain a segment c less than the unit. Why? Through the points of division pass lines parallel to the bases.

2. By § 357 Rect. $\frac{AM}{P'}=\frac{\text{Alt.}AM}{\text{Alt. }a'}$.

3. A sequence of diminishing units makes rectangle AM a variable with the limit P. Hence by § 427 the ratio of rectangle AM to P' is a variable with $\frac{P}{P'}$ as its limit. Similarly altitude AM is a variable with a as its limit and hence the ratio of altitude AM to a' is a variable with $\frac{a}{a'}$ as its limit.

4. By § 426 the limits of these variables are equal and hence $\frac{P}{P'}=\frac{a}{a'}$.

Therefore—

Adapt the demonstration of § 430 to this theorem.

PROPOSITION V.

432. THEOREM. *Two circles are to each other as their radii.*

This proof of § 409 is based on the theory of limits.

Given two circles c and c' with radii r and r' respectively.

To Prove $\frac{c}{c'}=\frac{r}{r'}$.

Proof. SUG. 1. Inscribe in the circles c and c' similar regular polygons with respective perimeters p and p'. Then $\frac{p}{p'}=\frac{r}{r'}$, and $p=p'\times\frac{r}{r'}$.

2. By increasing the number of sides p becomes a variable with c as its limit and p' becomes a variable with c' as its limit. Hence by § 407 the product $p'\times\frac{r}{r'}$ is a variable with $c\times\frac{r}{r'}$ as its limit.

3. By § 426 the limits of these variables are equal and $c=c'\times\frac{r}{r'}$ and hence $\frac{c}{c'}=\frac{r}{r'}$.

Therefore—

1. A regular hexagon with 6 in. side is inscribed in a circle. Find the area of the regular inscribed triangle.

2. If the angle of a parallelogram is bisected and the bisector extended to the opposite side an isosceles triangle is formed.

PROPOSITION VI.

433. THEOREM. *The areas of two circles are to each other as the squares of their radii.* (§ 412)

Prove this theorem in a manner like that of § 432.

434. A second proof of § 432. SUG. 1. Inscribe in c and c' similar regular polygons with perimeters p and p' respectively. Then $\frac{p}{p'} = \frac{r}{r'}$. Why?

2. Assume $\frac{c}{c'} > \frac{r}{r'}$ and hence $c > c' \times \frac{r}{r'}$.

Also $p < c$ and $p' < c'$. § 396.

3. Since in the sequence of inscribed polygons there is a point at which the inscribed polygon is nearer to c than any assigned number, consider a polygon q inscribed in c so that $q > c' \times \frac{r}{r'}$. Let q' be the similar polygon inscribed in c'. Then $\frac{q}{q'} = \frac{r}{r'}$, and $q = q' \times \frac{r}{r'}$.

4. Since $q' < c'$ it follows that

$q' \times \frac{r}{r'} < c' \times \frac{r}{r'}$ and hence $q < c' \times \frac{r}{r'}$.

But as this is contrary to fact the assumption in step 2 is false.

5. Similarly prove that $\frac{c}{c'}$ cannot be less than $\frac{r}{r'}$.

Therefore—

435. Adapt the demonstration § 434 to theorem § 433.

SOLID GEOMETRY.

CHAPTER VII.

Lines and Planes

436. Solid Geometry. That portion of Geometry which treats of figures the parts of which are not confined to a single plane (§ 19) is *Solid Geometry.*

The results of plane geometry furnish the basis for investigations in solid geometry, but it is to be remembered that the statements of plane geometry are made with respect to figures which are entirely in one plane and are not necessarily true in solid geometry. For instance, it has been proved in plane geometry that only one perpendicular can be erected to a line at a given point, but in solid geometry many perpendiculars can be so erected. This may be illustrated by the spokes of a wheel all of which are perpendicular to its axis. Therefore in solid geometry the theorems of plane geometry must not be applied unless the reference is to parts of a figure all of which are in one plane; but such theorems may be applied first to one plane, then to another, and so on. The pupil will be much helped in the study of solid geometry by noticing that most of the theorems are but extensions or generalizations of theorems previously studied in the plane geometry.

437. The relations of the parts of a figure or figures in a plane are not changed by moving the plane containing them from one position to another. § 50 (4).

438. A Plane. A surface such that the straight line joining any two of its points lies entirely in the surface is a *plane.* (§ 18.) A plane is unlimited in extent and from the definition it follows that if a straight line of a plane be indefinitely extended, it can never leave the

plane. A plane *embraces a line,* or is passed through a line, when the line lies wholly in the plane.

439. INTERSECTION OF PLANES. That portion of two planes which is common to both is their *intersection.*

440. PLANE DETERMINED. A plane is *determined* by certain lines or points when no other plane can embrace those lines or points without coinciding with the first plane.

441. POSTULATE. *A plane can be revolved about a line as an axis.*

Hence it may be inferred that a plane while embracing a line can take an infinite number of positions; and that, as a plane is unlimited in extent, all points in space can be embraced by the plane in the course of one complete revolution.

PROPOSITION I.

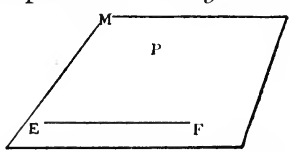
442. THEOREM. *A plane is determined—*

I. *By a straight line and a point without the line.*

II. *By three points not in a straight line.*

III. *By two intersecting straight lines.*

IV. *By two parallel straight lines.*



I. **Given** the straight line EF and the point P not in EF.

To Prove that EF and P determine a plane.

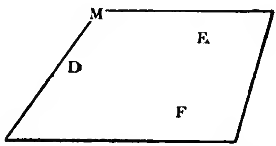
Proof. SUG. 1. Through line EF pass a plane and revolve it about EF as an axis until it contains the point P. § 441.

2. How much can the plane be revolved either way about EF and still contain point P? Why? §§ 4, 5.

3. How many planes can embrace the given point and given line?

4. $\therefore EF$ and P determine the plane MN. § 440.

Therefore—

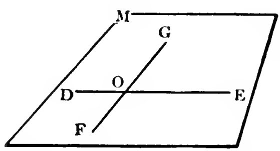


II. **Given** points D, E, F not in a straight line.

To Prove that D, E, F *determine a plane.*

Proof. Sug. Connect two of the points, and complete the demonstration.

Therefore—



III. **Given** DE and FG, two intersecting lines.

To Prove that DE and FG determine a plane.

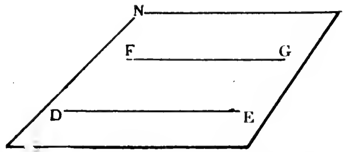
Proof. Sug. 1. Since the lines may be unlimited in extent, let E be any particular point in DE other than O. Then FG and E determine a plane. Why?

2. This plane contains the given lines. Why?

3. There is but one such, for if two planes contained the given lines they would each

contain FG and E. This is impossible. Case I.

Therefore—



IV. **Given** DE and FG, two parallel lines.

To Prove that DE and FG determine a plane.

Proof. SUG. 1. By definition of parallel lines there is at least one plane containing these lines. Represent it by N.

2. If there is more than one plane through these lines, there will be more than one plane containing DE and F, a point on FG. This is impossible by Case I.

Therefore—

1. A straight line can intersect a plane in but one point.

SUG. Suppose two points of the line to be on the plane.

2. Three straight lines each intersecting the two others, but not in a common point, lie in a plane.

3. What is the greatest number of planes that may be determined by two intersecting lines and a point? By three parallel lines?

4. A carpenter wishing to determine whether a board or other surface is a plane, places a straight edge (try-square) upon it in various positions. What is the test of the straight edge and why does it determine the question involved?

5. Can two lines be so placed as not to lie in one plane?

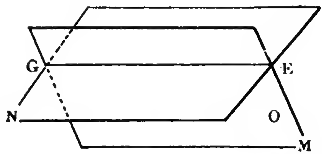
6. Can four points be so placed as not to lie in one plane? Give half a dozen illustrations of mechanical difficulties growing out of the answer.

7. Which is the more likely to stand firm, a three or a four legged stool? Why?

443. POSTULATE. *Two intersecting planes have at least two points in common.*

PROPOSITION II.

444. THEOREM. *The intersection of two planes is a straight line.*



Given two intersecting planes, *M* and *N* with two points *E* and *G* in common. § 443.

To Prove that the intersection of *M* and *N* is the straight line *EG*. § 439.

Proof. SUG. 1. Where does line *EG* lie with respect to each plane? Why?

2. Let *O* be any point in plane *M* outside of line *EG*. Can *O* lie in plane *N*? Why? § 442.

3. What are the only points common to planes *M* and *N*?

Therefore—

1. What is the locus of a point common to two planes?

445. THE FOOT OF A LINE. The point in which a line meets a plane is the *foot* of the line.

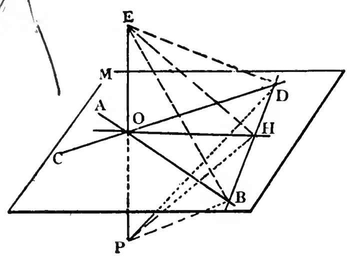
446. PERPENDICULAR TO A PLANE. A line is *perpendicular to a plane* when it is perpendicular to every line in the plane passing through its foot. The plane is then said to be perpendicular to the line.

447. OBLIQUE TO A PLANE. When a line is oblique to one or more lines of a plane, it is *oblique to the plane.*

2. Why is the crease formed in folding a piece of paper for an envelope a straight line?

PROPOSITION III.

448. THEOREM. *If a straight line is perpendicular to two lines of a plane at their point of intersection, it is perpendicular to the plane.*



Given $EO \perp AB$ and CD at O, and plane M determined by AB and CD.

To Prove $EO \perp$ plane M.

Proof. SUG. 1. Extend OE to P, making $OP = OE$, and let OH be any line of the plane through O. Draw BD and let H be the point in which it meets OH. Connect both E and P with the points B, H, and D.

2. In the $\triangle BEP$ compare BE and BP. In $\triangle DEP$ compare DE and DP. § 73.

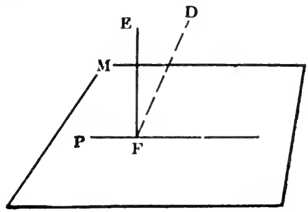
3. Compare $\triangle$s EBD and PBD; $\angle$s EBD and PBD. Auth.

4. Compare $\triangle$s EBH and PBH; lines EH and PH. Auth.

5. Compare $\triangle$s EOH and POH; $\angle$s EOH and POH. What relation does EO bear to OH? Or relate EO to OH by § 76.

Therefore—

449. Cor. *At a point in a plane only one perpendicular to the plane can be erected.*



Given $EF \perp$ plane M at point F.

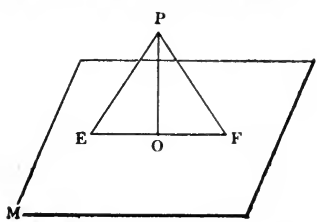
To Prove EF the only perpendicular to M at F.

Proof. If another perpendicular can be erected, represent it by DF. The lines EF and DF determine a plane (§ 442) which will intersect plane M in a straight line as PF. Then in this plane, both EF and DF are perpendicular to PF at point F. Is this possible? Why?

Therefore—

PROPOSITION IV.

450. Theorem. *From a point to a plane there is one line which is shorter than any other.*



Given a plane M and a point P without the plane.

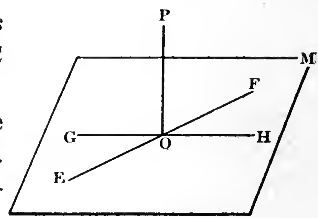
To Prove that from P to plane M there is one line shorter than any other.

Proof. Sug. 1. If there is not one shortest line, there must be a group of equal shortest lines. Let PE and PF represent two such lines. Connect E and F and let O be the mid-point of EF. Join P and O.

2. With respect to length, compare PO with PE or PF.

Therefore—

451. Cor. I. *The shortest line from a point to a plane is the perpendicular from that point to the plane.*



Given PO the shortest line from point P to the plane M.

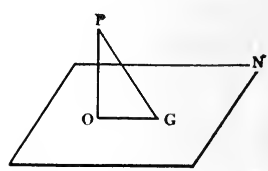
To Prove that PO is perpendicular to plane M.

Proof. Through the foot of PO draw in plane M any two lines as EF and GH. Then $PO \perp EF$ and $PO \perp GH$. Why?

$\therefore PO \perp$ plane M. Why?

Therefore—

452. Cor. II. *From a point without a plane only one perpendicular can be dropped to the plane.*



Given $PO \perp$ plane N from point P without the plane.

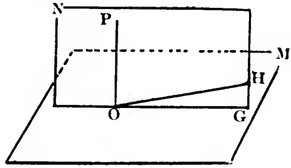
Proof. If there is a second perpendicular from P to N, represent it by PG. Then the plane PGO (§ 442) intersects plane N in line OG. Why?

What relation does PG bear to line OG? Why? What relation does PG bear to plane N? Why?

Therefore—

453. Distance from a Point to a Plane. The length of the perpendicular from a point to a plane is the *distance from the point to the plane.*

PROPOSITION V.



454. THEOREM. *All perpendiculars to a line at a given point lie in one plane perpendicular to the line at that point.*

Given plane $M \perp PO$ at O and line OH any line $\perp PO$ at O.

To Prove that OH lies in plane M.

Proof. SUG. 1. The plane N determined by PO and OH will intersect plane M in a straight line through O, as OG. Why?

2. $PO \perp OG$. Why?

3. OG and OH both lie in plane N and must therefore coincide. Why?

4. $\therefore$ OH must lie in plane M. Why?

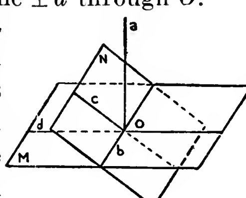
5. As OH was any line perpendicular to PO at O, the same is true of all such lines.

Therefore—

455. COR. I. *Through a point in a line only one plane can be erected perpendicular to the line.*

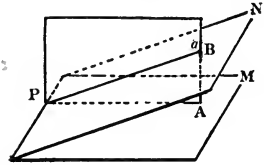
Given a line a and a plane $M \perp$ to a at point O.

To Prove that M is the only plane $\perp a$ through O.



Proof. Suppose that there is a second plane, as N, $\perp a$ at O, and let b be the intersection of planes M and N. Pass a plane P through line a which does not contain line b. This plane will cut M and N in two distinct lines, each perpendicular to a. But this, by § 454, is impossible. Why?

456. Cor. II. *From a point without a line only one plane can be drawn perpendicular to the line.*



Given point P not on line a and plane M through P and $\perp a$.

To Prove that M is the only plane through P and $\perp a$.

Proof. Let M meet line a in point A. If there is a second such plane, as N, let it meet a in point B. Pass a plane through P and line AB or a. This plane will cut M and N, each in a straight line. Complete the demonstration.

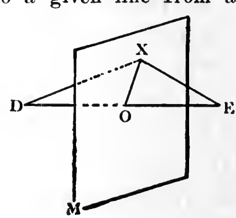
1. Erect a plane perpendicular to a line at a given point. § 454.

2. How can a carpenter erect a studding perpendicular to a floor with his square?

3. Set up in the school room a pole perpendicular to the floor, using a right angle (a carpenter's square or a book cover). § 448.

4. How many braces are required to hold the pole of the preceding exercise in position?

5. Construct a plane perpendicular to a given line from a point without the line. § 448.

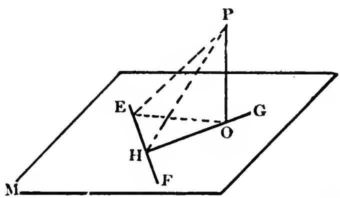


6. If a plane is perpendicular to a sect at its mid-point every point in the plane is equidistant from the extremities of the sect and every point outside the plane is not equidistant from the extremities.

SUG. Let X be any point in plane M, which is perpendicular to sect DE at its mid point O. Prove $DX = EX$.

PROPOSITION VI.

457. PROBLEM. *To drop a perpendicular from a point to a plane.*



Given a plane M and point P not on M.

To Construct a line PO from $P \perp M$.

Construction. SUG. 1. Draw any line EF in plane M and drop a perpendicular, PH, to EF in the plane of EF and P. Auth.

2. Through H in plane M draw $HG \perp EF$. Auth.

3. From P drop $PO \perp HG$. Auth.

4. To prove $PO \perp M$, draw a line from O to E, any point in EF except H, and join P and E.

5. $\overline{PH}^2 - \overline{OH}^2 = \overline{PO}^2$. Auth.

6. $\overline{EH}^2 + \overline{OH}^2 = \overline{OE}^2$. Auth.

7. Adding, $\overline{EP}^2 = \overline{PO}^2 + \overline{OE}^2$.

8. $\therefore PO \perp OE$.

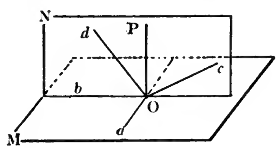
9. $\therefore PO \perp M$. Why?

1. What is the locus of points in space equidistant from two given points?

458. Locus in Space. The *locus of a point in space* satisfying certain given conditions consist of those geometric figures in space to which the point is limited and every point of which satisfies the conditions. §§ 164-165.

PROPOSITION VII.

459. Problem. *To erect a perpendicular to a plane at a given point in the plane.*



Given plane M and point O in M.

To Construct $OP \perp M$ at O.

Construction. Sug. 1. In plane M draw a line a and then two lines c and d each perpendicular to a at O.

2. What relation does the plane of c and d bear to a?

3. This plane cuts M in line b. Why?

4. In this new plane draw $OP \perp b$. Auth. How is OP related to a? Why?

5. How is OP related to plane M? Why?

Compare this construction with ex. 3, p. 232.

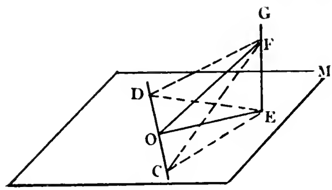
1. Another demonstration for § 448. Draw from D a line DB terminated in line AB and bisected by OH at H. Auth. (Use the fig. of § 448.)

$\overline{EB}^2+\overline{ED}^2 = 2\overline{EH}^2 + 2\,\overline{BH}^2$ $\overline{OB}^2+\overline{OD}^2 = 2\overline{OH}^2 + 2\,\overline{BH}^2$ § 345

Subtract and reduce. Apply § 311.

PROPOSITION VIII.

460. THEOREM. *If, from the foot of a perpendicular to a plane, a line is drawn perpendicular to any given line in the plane and, from this point of intersection, a line is drawn to any point of the perpendicular to the plane, the last line is perpendicular to the given line in the plane.*



Given line $EG \perp$ plane M and CD any line in M, with $EO \perp CD$ and F any point in EG.

To Prove $OF \perp CD$.

Proof. SUG. 1. On line CD take $OD = OC$. Join F with C, D and O, and E with C and D.

2. In $\triangle ECD$ compare EC and ED. Auth.

3. Compare FC and FD. § 65.

4. What relation does OF bear to CD? § 76.

Therefore—

1. In the fig. of § 460, given $EG \perp EO$, $EO \perp CD$, and $OF \perp CD$ at O, F being in EG. Prove $EG \perp$ plane M of EO and CD.

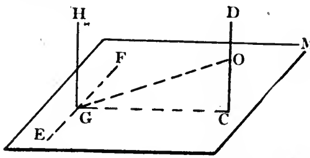
SUG. $EF^2 = OF^2 - EO^2$ and $EC^2 = OC^2 + OE^2$.

Add and complete the demonstration.

2. Show that by marking at right angles to the edges of a stick of timber with a carpenter's square continuously the line will end at its starting point.

PROPOSITION IX.

461. THEOREM. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



Given $GH \parallel CD$ and meeting plane M in G and C respectively, with $CD \perp M$.

To Prove $GH \perp M$.

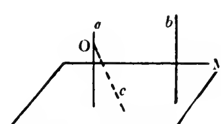
Proof. SUG. 1. GH and CD determine a plane which meets plane M in GC. Why? Join G with O, any point in CD. Draw EF in plane M and $\perp GC$.

2. What relation does EF sustain to GO? (§ 460.) To GC? To plane HC? § 448.

3. What relation does EF sustain to GH, or GH to EF? GH to GC? § 90. GH to plane M? Why?

PROPOSITION X.

462. THEOREM. *Two lines perpendicular to the same plane are parallel.*



Given two lines a and b, each perpendicular to plane M.

To Prove $a \parallel b$.

Proof. Sug. 1. Suppose a is not parallel to b and through a point O on a draw a line $c \parallel b$.

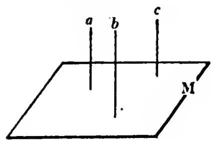
2. What relation does c bear to plane M? Why? § 461.

3. Complete the demonstration.

Therefore—

PROPOSITION XI.

463. Theorem. *Two straight lines each parallel to a third straight line are parallel to each other.*



Given $a \parallel c$ and $b \parallel c$.

To Prove $a \parallel b$.

Proof. Sug. 1. Construct plane $M \perp c$. Auth.

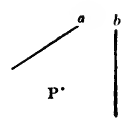
2. What relation do a and b bear to M? Auth.

3. What relation do a and b bear to each other? Auth.

Therefore—

1. To draw a straight line which shall intersect each of two lines not in the same plane and shall pass through a point not in either line.

Sug. Pass a plane through the given point P and one line a. Also pass a plane through P and the second line b. These planes intersect in a line x through P. Why? This line x also meets both a and b. Why?

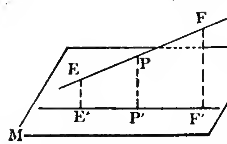


464. Projection of a Point on a Plane. The foot of the perpendicular from a point to a plane is the *projection of the point on the plane.*

465. PROJECTION OF A LINE UPON A PLANE. The locus of the projections of the points of a line upon a plane is the *projection of the line upon the plane.*

PROPOSITION XII.

466. THEOREM. *The projection of a straight line upon a plane is a straight line.*



Given a plane M and a straight line EF not in M, with E' and F' the projections of E and F respectively upon M.

To Prove the straight line $E'F'$ the projection of EF.

Proof. SUG. 1. It is necessary and sufficient to show that the projection P' of any third point in EF lies in $E'F'$.

2. Since PP', EE', FF' are each $\perp M$, they are parallel. Auth.

3. EE' and FF' determine a plane, N, cutting M in line $E'F'$. Why?

4. In plane N, it is possible to draw a line through P which is parallel to FF' and hence $\perp M$, but this line must coincide with PP'. Why?

5. Since PP' lies in N, its foot in M must lie in $E'F'$.

Therefore—

1. The area of a circle is 20 sq. in. What is the area of a circle of double the radius? Of $\frac{1}{2}$ the radius? Of m times the radius? Of $\frac{1}{m}$ times the radius?

2. If a is the radius of a circle with an area of 40 sq. in., what is the radius of a circle of 80 sq. in.? Of 10 sq. in.? Of 160 sq. in.?

PROPOSITION XIII.

467. THEOREM. *If from a point to a plane the perpendicular and oblique lines be drawn*

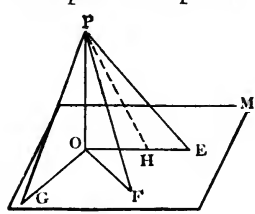
I. *The perpendicular is the shortest line from the point to the plane;*

II. *Oblique lines having equal projections upon the plane are equal;*

III. *If two oblique lines are equal, their projections upon the plane are equal;*

IV. *Of two unequal oblique lines, the line having the greater projection is the greater;*

V. *The greater of two oblique lines has the greater projection upon the plane.*



I. **Given** $PO \perp$ plane M.

To Prove PO shorter than any other line from P to M.

Proof. SUG. Compare $\angle$s POG and PGO.

II. **Given** PG and PE oblique to M with their respective projections OG and OE equal.

To Prove $PG = PE$.

Proof. The demonstration is left to the pupil.

III. **Given** $PG = PE$, with OG and OE their respective projections.

To Prove $OG = OE$.

Proof. The demonstration is left to the pupil.

IV. **Given** oblique lines PE and PF, with OE and OF their repective projections, and $OE > OF$.

To Prove $PE > PF$.

Proof. Sug. 1. On OE take $OH = OF$ and draw PH.

2. Compare PE with PH; PE with PF. Complete the demonstration.

V. **Given** $PE > PF$.

To Prove $OE > OF$.

Proof. Sug. Use the indirect method.

Therefore—

1. What is the locus of the foot of an oblique line in a plane when the oblique line is revolved with one end not in the plane stationary?

2. An oblique to a plane intersects its own projection upon the plane.

3. The locus of a point in space equidistant from all points in a circle is a straight line through the center of the circle and perpendicular to its plane.

4. A ceiling is 8 ft. high. A pole 10 ft. long is held at a given point on the ceiling and revolved with the lower end touching the floor. Find the diameter of the circle described.

5. How could the pole in the preceding exercise be used to find that point on the floor which is directly under the point on the ceiling?

6. Show that the angle that the pole makes with the perpendicular is constant as the pole revolves.

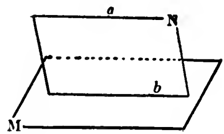
7. A horse is tied with a rope 100 ft. long to a point on a vertical pole 25 ft. from the ground. What is the shape of his pasture and how far can he get from the foot of the pole?

468. Line and Plane Parallel. *A straight line and a plane are parallel* if they can never meet, however far they may be extended.

469. Parallel Planes. Two planes which cannot meet, however far they may be extended, are *parallel planes*.

PROPOSITION XIV.

470. THEOREM. *A straight line is parallel to a plane if it is parallel to a line in the plane.*



Given line a outside plane M and parallel to line b in plane M.

To Prove $a \parallel M$.

Proof. SUG. 1. a and b determine a plane N. Why? § 442.

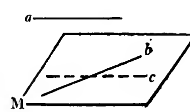
2. What is the intersection of M and N? Why? § 444.

3. If a and M have a point in common it must lie in b. Why? § 444.

4. This is impossible. Why? § 438.

Therefore—

471. COR. I. *A plane may be passed through one of two non-intersecting lines parallel to the other.*



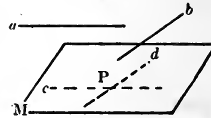
Given two non-intersecting lines a and b.

To Prove that a plane, M, can be passed through b parallel to a.

Proof. SUG. From any point in b draw the line c parallel to a. Then plane M determined by b and c is parallel to a. Why?

1. How many such planes (§ 471) are there?
2. If a line is parallel to a plane, it is parallel to its own projection upon the plane.

472. Cor. II. *Through a point not on either of two lines a plane can be passed parallel to both the lines.*



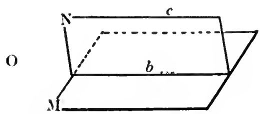
Given lines a and b and point P.

To Prove that a plane M parallel to a and to b can be passed through P.

Proof. Through P draw two lines, c and d, parallel to a and b respectively. Complete the demonstration.

PROPOSITION XV.

473. Theorem. *If a line is parallel to a plane, any plane that embraces the line and intersects the plane intersects it in a line parallel to the given line.*



Given line c parallel to plane M, and plane N through c and intersecting M in line b.

To Prove $b \parallel c$.

Proof. Sug. 1. If c and b are not parallel they will meet at some point as O. Why? This is impossible. Why?

Therefore—

1. Through a given point *in space* one and only one line can be drawn parallel to a given line. § 87.

2. Through a given point in space any number of lines can be drawn parallel to a given plane.

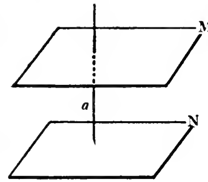
3. All lines through a given point and parallel to a given plane lie in one plane.

Sug. Drop a ⊥ from the given point to the given plane.

4. Through a given point only one plane can be passed parallel to a given plane.

PROPOSITION XVI.

474. THEOREM. *Planes perpendicular to the same straight line are parallel.*



Given two planes M and N each perpendicular to line a.

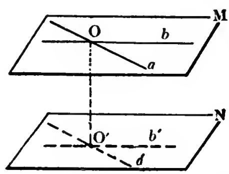
To Prove $M \parallel N$.

Proof. SUG. Use an indirect proof based on § 456.

Therefore—

PROPOSITION XVII.

475. THEOREM. *If two intersecting lines in one plane are each parallel to a second plane, the planes are parallel.*



Given two lines a and b in plane M, meeting in O, each line parallel to plane N.

To Prove $M \parallel N$.

Proof. SUG. 1. At O, the common point of a and b, erect a $\perp$ to plane M and extend it to meet plane N at O'. Line OO' is $\perp$ to a and to b. Why?

2. Lines a and OO' determine a plane which intersects N in a line as a' through O'. Likewise b and OO' determine a plane which intersects N in b'. Auth.

3. What relation does a' bear to a? b' to b? Auth.

4. What relation does OO' bear to a' and to b'? Auth.

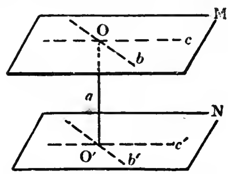
5. What relation does OO' bear to plane N? Auth.

6. What relation does N bear to M? Auth.

Therefore—

PROPOSITION XVIII.

476. *A line perpendicular to one of two parallel planes is perpendicular to the other.*



Given plane $M \parallel$ plane N and line $a \perp M$ at O.

To Prove $a \perp N$.

Proof. Sug. 1. Through any point in plane M draw two lines as b and c. Through a and b and through a and c pass planes and let these two planes intersect plane N in lines b' and c' respectively.

2. What relation does b' bear to b? c' to c? Auth.

3. What relation does a bear to b and to c? To b' and to c'? Auth.

4. What relation does a bear to N? Auth.

Therefore—

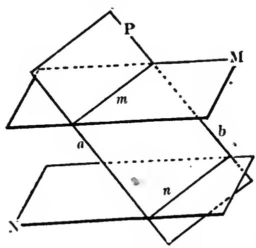
1. Two planes parallel to the same plane are parallel to each other. §§ 474, 476.

2. What is the locus of the foot of an oblique line 10 in. long drawn to a plane from a point eight in. from the plane? Compute the area of the figure bounded by the locus.

477. Distance Between Parallel Planes. The length of the sect intercepted between two parallel planes and perpendicular to both is *the distance between the parallel planes.*

PROPOSITION XIX.

478. Theorem. *If two parallel planes are cut by a third plane the intersections are parallel.*



Given two parallel planes M and N, cut by a third plane P in lines m and n respectively.

To Prove $m \parallel n$.

Proof. Sug. Show that m and n lie in the same plane and cannot meet.

Therefore—

479. Cor. I. *Parallel lines intercepted between parallel planes are equal.*

Sug. Assume lines a and b in § 478 to be parallel. Prove them equal.

480. Cor. II. *Parallel planes are everywhere equidistant.*

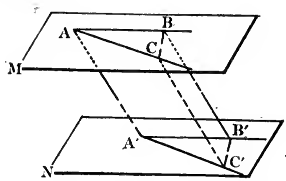
1. A straight line and a plane both perpendicular to the same straight line are parallel.

2. What is the locus of a line through a given point and parallel to a given plane?

3. If a straight line and a plane are parallel, any line parallel to the given line is parallel to the plane also.

PROPOSITION XX.

481. Theorem. *If two angles, not in the same plane, have their respective sides parallel and extending in the same directions from the vertices, the angles are equal.*



Given $\angle BAC$ in plane M and $\angle B'A'C'$ in plane N, with $AB \parallel A'B'$ and $AC \parallel A'C'$, the respective directions from A and A' being the same.

To Prove $\angle BAC = \angle B'A'C'$.

Proof. Sug. 1. Take points B, B', C, C' so that $AB = A'B'$ and $AC = A'C'$. Connect A and A', B and B', C and C', B and C, B' and C'.

2. Compare AA' and BB', AA' and CC', BB' and CC'.

3. Compare BC and $B'C'$; $\triangle BAC$ and $\triangle B'A'C'$; $\angle A$ and $\angle A'$. Auth.

Therefore—

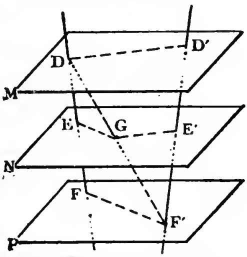
1. What is the locus of a point at a given distance from a given plane?

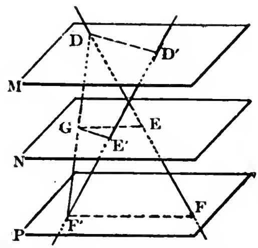
2. What is the locus of a point equidistant from two given points and also at a given distance from a given plane? When is there no solution?

3. From A, the mid-point of one side of a parallelogram draw lines dividing each of the adjacent sides into three equal parts and the opposite side into six equal parts. Prove that each of the twelve triangles thus formed equals one twelfth of the parallelogram.

PROPOSITION XXI.

482. THEOREM. *If three parallel planes intersect two straight lines, the corresponding segments are proportional.*





Given three parallel planes M, N, P intersecting the two lines a and a' in the points D, E, F and D', E', F' respectively.

To Prove $\frac{DE}{EF}=\frac{D'E'}{E'F'}$.

Proof. SUG. 1. Join D and F' and let G be the intersection of this line with plane N.

2. Plane $DD'F'$ intersects planes M and N in what lines? Plane DFF' intersects planes N and P in what lines?

3. What relation does EG bear to FF'? $E'G$ to DD'? Why?

4. Compare the ratios $\frac{DE}{EF}$, $\frac{DG}{GF'}$, $\frac{D'E'}{E'F'}$.

§ 251.

5. Complete the demonstration.

Therefore—

1. If two parallel planes intersect two parallel planes the four lines of intersection are parallel.

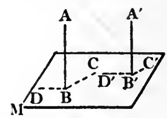
2. What is the locus of a point in space equidistant from two parallel planes?

1. Find a point X equidistant from two given points, equidistant from two other given points, and in a given plane. Is the problem ever impossible? Will any arrangement of the given parts permit an unlimited number of solutions?

2. Another demonstration for 461. Given $= AB \parallel A'B'$ and $A'B' \perp$ plane M.

To Prove $AB \perp$ plane M.

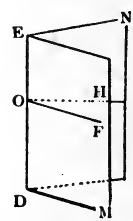
Sug. Draw in plane M any two lines through B' as $B'C'$ and $B'D'$. Through B draw BC and BD parallel to $B'C'$ and $B'D'$ *respectively.* $\angle s$ $A'B'C'$ and $A'B'D'$ are right angles. Why? They are equal to $\angle s$ ABC and ABD respectively. Why? Therefore $AB \perp M$. Why?



3. Find a point X equidistant from two given points, equidistant from two given planes, and at a given distance from a third plane. Discuss all possibilities.

4. What is the locus of a point equidistant from two given points A and B and also from two given points C and D? Discuss all possibilities due to the different positions of the pairs of points.

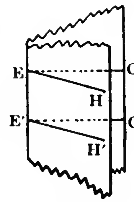
483. Dihedral Angle. Two planes which meet or intersect form a *dihedral angle.* The two planes M and N meeting in the line ED form a dihedral angle. The intersecting planes, M and N, are the *faces* of the angle and the line of intersection, ED, is the *edge.* A dihedral angle is read by reading in order one face, the edge, and the second face. When but one dihedral angle is formed at an edge, the angle may be read by naming the edge, as dihedral angle ED. A dihedral angle is often called a dihedral.



484. Plane Angle of a Dihedral. An angle formed by two lines, one in each face of a dihedral, perpendicular to the edge at the same point is the *plane angle of the dihedral.* Lines FO and HO lying in the faces M

and N respectively and each perpendicular to ED at O is a plane angle of the dihedral ED.

485. Cor. I. *The plane angle of a dihedral is the same size from whatever point of the edge it is drawn.*



Given dihedral EE' with plane angles GEH and $G'E'H'$.

To Prove $\angle GEH = \angle G'E'H'$.

Proof. Left to the pupil. § 481.

486. Cor. II. *The angle formed by the intersections of the faces of a dihedral with a plane perpendicular to the edge is a plane angle of the dihedral.*

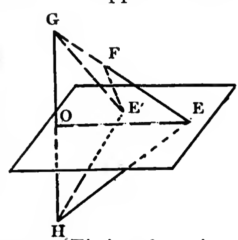
Proof left to the pupil. § 484.

487. Equal Dihedrals. Two dihedral angles are *equal* when they can be made to coincide.

The magnitude of a dihedral does not depend upon the extent of its faces. If a plane be made to revolve about the edge as an axis from the position of one face to the position of the other face, it revolves or turns through the dihedral angle and the greater the amount of the turning the greater the angle.

1. A and B are two points equally distant from a plane and upon the same side of it; prove that line AB is parallel to the plane.

2. Determine a point E in a plane such that the difference between its distances from two given points on opposite sides of the plane is a maximum.

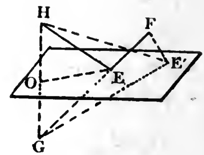


Sug. Drop a perpendicular to the plane from one of the points as H and extend it to G, an equal distance on the other side of the plane. Connect G with the second point F, this line meeting the plane in E. Prove E to be the required point, as follows:

$EH - EF = FG$. Take any other point as E' in the given plane, join it to F and H and prove $FG > E'H - E'F$.

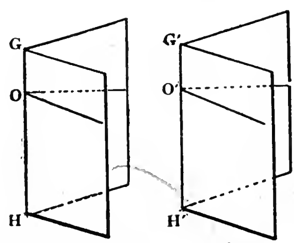
1. Determine a point E in a plane such that the sum of its distances from two points on the same side of the plane is the minimum.

SUG. Drop a perpendicular to the plane from one of the points as H and extend it to G, an equal distance beyond the plane. Join G to the second point F, this line meeting the plane in E. Prove E to be the required point. To do this, take any other point as E' in the given plane and prove $HE' + E'F > HE + EF$.



PROPOSITION XXII.

488. THEOREM. *Two dihedral angles are equal if their plane angles are equal.*



Given two dihedrals GH and $G'H'$, with equal plane angles O and O' respectively.

To Prove dihedrals GH and $G'H'$ equal.

Proof. SUG. 1. Place the dihedrals so that the plane angles coincide.

2. How will the edges lie with regard to each other? Auth.

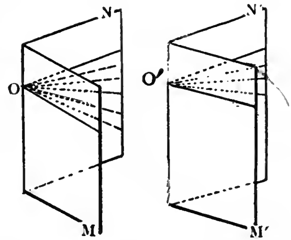
3. How will the respective faces lie with regard to each other? Auth.

Therefore—

489. COR. *If two dihedrals are equal their plane angles are equal.*

PROPOSITION XXIII.

490. THEOREM. *Two dihedral angles have the same ratio as their plane angles.*



Given two dihedral angles $M-O-N$ and $M'-O'-N'$, with plane angles O and O' respectively.

To Prove $\dfrac{\angle M-O-N}{\angle M'-O'-N'}=\dfrac{\angle O}{\angle O'}$.

Proof. CASE I. When the plane angles are commensurable.

SUG. 1. Divide the plane angles by a common unit of measure and suppose the unit to be contained in $\angle O$ and $\angle O'$ m and n times respectively. What then is the ratio of $\angle O$ to $\angle O'$?

2. Through the respective edges of the two dihedrals and the various division lines of the two plane angles pass planes. How do the dihedrals thus formed compare with one another? Auth. How many of them are there in $\angle M-O-N$? In $\angle M'-O'-N'$?

3. What is the ratio of the two dihedrals? Apply § 247.

4. Complete the demonstration.

CASE II. When the plane angles are incommensurable.

SUG. Use the method of § 429 in working out a demonstration.

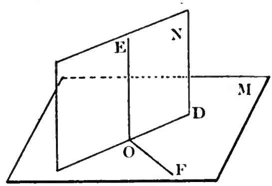
Therefore—

491. Measure of Dihedral Angles. Since dihedral angles are proportional to their plane angles, they are said to be *measured by their plane angles.* Thus a *right dihedral* angle is a dihedral the plane angle of which is a right angle. Similarly one of 27° is one the plane angle of which is 27°. Dihedral angles are *adjacent, vertical, acute, obtuse,* etc., according as the respective plane angles are adjacent, vertical, acute, obtuse, etc.

492. Perpendicular Planes. Planes which form a right dihedral angle are perpendicular planes.

PROPOSITION XXIV.

493. Theorem. *If a straight line is perpendicular to a plane, every plane containing that line is perpendicular to the plane.*



Given line $EO \perp$ plane M and a plane N containing EO and intersecting M in OD.

To Prove plane $N \perp$ plane M.

Proof. Sug. 1. What must be known to make $N \perp M$?

2. In plane M erect $OF \perp OD$.

3. How many degrees in $\angle EOF$? Why?

4. $\angle EOF$ is the plane angle of the dihedral. Why?

5. What relation does plane N bear to plane M? Why?

Therefore—

1. Vertical dihedral angles are equal.

SUG. Pass a plane through the angles perpendicular to the common edges and compare the resulting plane angles.

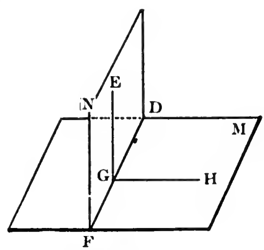
2. Adjacent dihedrals are supplementary if their exterior faces lie in the same plane.

3. Two planes which bisect respectively two vertical dihedrals form one and the same plane.

4. A plane which bisects one of two vertical dihedrals bisects the other.

PROPOSITION XXV.

494. *A line in one of two perpendicular planes perpendicular to their intersection is perpendicular to the other.*



Given plane $N \perp$ plane M with FD the line of intersection and in plane N a line $EG \perp FD$.

To Prove $EG \perp$ plane M.

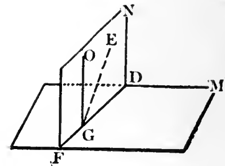
Proof. SUG. 1. What must be proved in addition to the hypothesis in order to show that EG is perpendicular to plane M?

2. In plane M draw $GH \perp FD$ at G. Angle EGH is the plane angle of the dihedral. Why? How many degrees in $\angle EGH$? Why?

3. Complete the demonstration by showing that EG must be perpendicular to M. § 448.

Therefore—

495. Cor. I. *If two planes are perpendicular to each other, a line perpendicular to one of them at any point of their intersection lies in the other.*



Given M and N, two perpendicular planes, FD their intersection, and $OG \perp M$ at G, a point in FD.

To Prove OG lies in plane N.

Proof. Sug. 1. Suppose OG does not lie in N and that EG is the line in N which is $\perp FD$ at G. What relation does EG bear to M by § 494?

2. How many perpendiculars can be erected to M at G?

3. What then of the supposition?

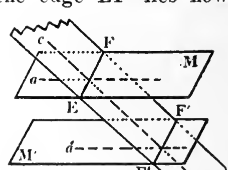
Therefore—

496. Cor. II. *If two planes are perpendicular, a perpendicular from any point in the first to the second lies in the first.*

Proof. Make a demonstration following the plan of § 495.

1. If a plane intersects two parallel planes the alternate interior dihedrals are equal.

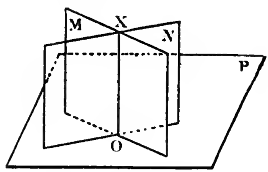
Sug. A plane perpendicular to the edge EF lies how with reference to edge $E'F'$? Why? It cuts the parallel planes M and M' in two lines as a and a'. How do a and a' lie with reference to each other? Why? Compare the plane angles of these dihedrals.



2. In the preceding problem prove the corresponding dihedrals equal and that the two dihedrals on the same side of the secant plane are supplementary.

PROPOSITION XXVI.

497. Theorem. *If two intersecting planes are each perpendicular to a third plane, their intersection is perpendicular to the third plane.*



Given two planes M and N, each $\perp$ plane P and intersecting in the line OX.

To Prove $OX \perp$ plane P.

Proof. Sug. 1. At point O which is common to all three planes erect the perpendicular to plane P.

2. Where does this perpendicular lie with reference to plane M? To plane N? § 495.

3. What relation does this perpendicular then bear to the intersection of M and N? Why?

Therefore—

1. A plane which is perpendicular to the intersection of two planes is perpendicular to the planes.

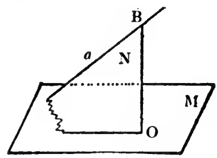
2. The plane formed by a line and its projection upon a given plane is perpendicular to the given plane.

3. The projections of a line upon two parallel planes are equal.

4. What condition must be added to the converse of the following theorem in order that it be true? If two parallel planes are cut by a third plane the interior angles on the same side of the secant plane are supplementary. See ex. 11, p. 268.

PROPOSITION XXVII.

498. THEOREM. *Through a given straight line, oblique or parallel to a given plane, one, and but one, plane can be passed perpendicular to the given plane.*



Given plane *M* with line *a* oblique or parallel to *M*.

I. **To Prove** that one plane can be passed through *a* and ⊥ *M*.

Proof. SUG. 1. From any point as *B* in line *a* drop a line ⊥ *M*, as *BO*.

2. *BO* and *a* determine a plane, *N*. Why?

3. What relation does plane *N* bear to plane *M?* § 493.

II. **To Prove** that *N* is the *only* plane through line *a* perpendicular to *M*.

SUG. 1. Suppose there is a second plane *P* embracing line *a* and ⊥ *M*. What relation does line *a* bear to planes *N* and *P?* § 439.

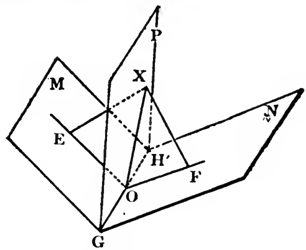
2. What relation must line *a* then bear to plane *M?* § 497.

3. Compare this conclusion with the hypothesis.

Therefore—

PROPOSITION XXVIII.

499. THEOREM. *Every point in a plane which bisects a dihedral angle is equidistant from the faces of the dihedral angle.*



Given plane P bisecting the dihedral formed by the planes M and N, X any point in plane P, and XE and XF perpendiculars from X to M and N respectively.

To Prove $XE = XF$.

Proof. SUG. 1. XE and XF determine a plane. Let O represent the intersection of this plane with GH, the edge of the dihedral. Then OE and OF will be the intersections of this plane with M and N respectively.

2. What relation does plane XEF sustain to plane M and to plane N? To their intersection, GH? § 497.

3. XOE and XOF are the plane angles of the dihedrals $M - O - P$ and $N - O - P$ respectively. Why?

4. Compare $\angle XOE$ and $\angle XOF$, § 489; $\triangle XOE$ and $\triangle XOF$; XE and XF.

Therefore—

Query. What statement in the theorem makes it necessary that XE and XF be perpendicular to M and N respectively?

500. Cor. I. *Every point equidistant from the faces of a dihedral is in the bisector of the angle.*

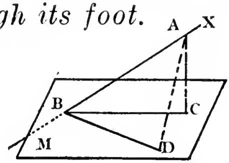
Sug. With the figure of § 499, assume $XE = XF$ and prove $\angle XOE = \angle XOF$. Make a full detailed demonstration.

501. Cor. II. *Every point not in the plane which bisects a dihedral is unequally distant from the faces of the dihedral.*

Sug. Use the indirect method or a direct method similar to that of ex. 3, p. 37.

PROPOSITION XXIX.

502. Theorem. *The acute angle which a straight line makes with its projection on a plane is the least angle the line makes with any line in the plane through its foot.*



Given line XB meeting plane M at B, BC its projection on M, and BD any line in M through B other than BC.

To Prove $\angle XBC < \angle XBD$.

Proof. Sug. 1. From any point in XB, as A, drop a perpendicular, AC, to M. Where will C lie? Why?

2. On the second line through B, take $BD = BC$ and join A and D.

3. Compare AC with AD. Auth.

4. Compare $\angle ABC$ with $\angle ABD$. § 124.

Therefore—

1. The locus of points in space equidistant from two intersecting planes is the pair of planes bisecting the dihedrals formed by the given planes.

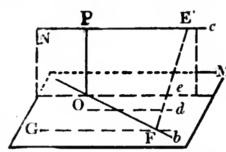
503. Angle of a Line to a Plane. The acute angle which a line makes with its projection on a plane is the *angle of the line to the plane.*

2. The supplement of the angle of a line to a plane is the greatest angle which the line makes with any line in the plane through its foot.

3. A line makes equal angles with parallel planes.

4. There is one and but one perpendicular which can be drawn to two non parallel lines which are not in the same plane.

Given two lines, b and c, not in the same plane and not parallel. Pass a line d through one of them, as b, parallel to the other, c. Pass through b and d the plane M. How does M lie with reference to line c? Through c pass a plane N perpendicular to M. Let line e be the intersection of M and N. Then $e \parallel c$. Why? Let the intersection of b and e be O and at O erect the line $OP \perp M$. This line lies in N. Why? Hence OP meets c and $OP \perp c$. Why?



Therefore there is at least one such common perpendicular to b and c.

Suppose there is another such perpendicular as EF. The plane of line c and EF will intersect M in a line GF and be $\perp$ to M. Why? Thus there will be two planes embracing c and each $\perp$ M, which is impossible? Why?

Therefore

5. In the preceding theorem the common perpendicular, OP, is the shortest line which can be drawn between the two given lines.

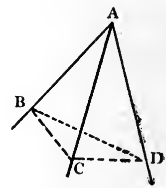
6. Find a point X equidistant from four given points not in the same plane.

Sug. Let the given points be A, B, C, D. What is the locus of points equidistant from A and B? From B and C?

From A, B, and C? From C and D? Complete the demonstration.

504. Polyhedral Angle. Three or more planes meeting at a common point form a *polyhedral angle;* or simply a *polyhedral.*

For example, A—BCD represents a polyhedral angle formed by three planes. The common point, A, is the vertex of the angle, the intersections of the planes, AB, AC, AD, are the edges of the angle, and the portions of the planes included between the successive edges are the faces of the angle. The plane angles formed by the successive edges are the face angles of the polyhedral. The face angles and the dihedrals of the successive faces are the parts of the polyhedral.



505. Convex Polyhedral Angle. If the intersections of a plane with all the faces of a polyhedral form a convex polygon the polyhedral is a *convex polyhedral angle.*

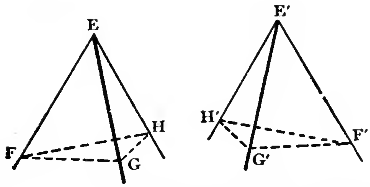
506. Classification of Polyhedral Angles. A polyhedral with three faces is a *trihedral angle,* or *a trihedral;* one having four faces is a *tetrahedral angle,* or *tetrahedral;* etc.

A trihedral angle with two equal face angles is *isosceles.*

507. Congruent Polyhedral Angles. Polyhedral angles which can be made to coincide are *congruent polyhedral angles.*

For two polyhedrals to be congruent, it is necessary that the respective parts of the two angles be equal and arranged in the same order.

508. Symmetrical Polyhedral Angles. Two polyhedral angles having the dihedrals and faces angles of the one equal respectively to the corresponding parts of the other but arranged in the reverse order are *symmetrical polyhedral angles.*

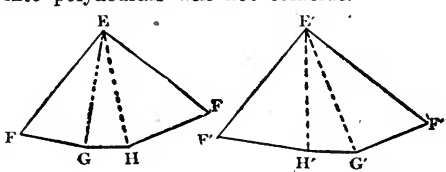


In general two symmetrical polyhedral angles cannot be made to coincide.

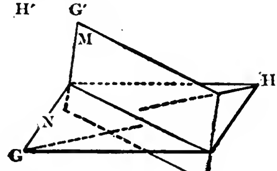
Polyhedrals $E—FGH$ and $E'—F'G'H'$ are symmetrical provided $\angle FEG = \angle F'E'G'$, $\angle GEH = \angle G'E'H'$, $\angle HEF = \angle H'E'F'$, dihedral EF = dihedral $E'F'$, etc., the arrangements of the respective parts in the two polyhedrals being opposite.

As an illustration of the symmetrical relation consider a pair of gloves. Two gloves for the same hand may be compared to equal polyhedrals and the pair to symmetrical polyhedrals.

1. Cut from pasteboard two figures like those in the accompanying figures, creasing them on the dotted lines. Note that all parts with corresponding notations are equal. The equality of the "parts" may be tested by superposition. The figures when bent into polyhedrals will not coincide.



2. If a plane be passed through either diagonal of a parallelogram, the perpendiculars to this plane from the extremities of the other diagonal are equal.

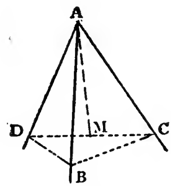


3. Having given a fixed straight line and two points not in the line, find a point in the fixed line equally distant from the given points.

4. What is the locus of a point equidistant from two given parallel planes and at the same time equidistant from two given points?

PROPOSITION XXX.

509. THEOREM. *The sum of any two face angles of a trihedral angle is greater than the third.*



Given the trihedral $A-BCD$, in which face angle DAC is the greatest.

To Prove $\angle CAB + \angle BAD > \angle DAC$.

Proof. SUG. 1. It is unnecessary to prove the theorem for the cases in which the greatest angle, $\angle DAC$, is one of the two angles added. Why?

2. In the face DAC draw a line AM equal to AB making $\angle MAD$ equal to $\angle DAB$, and through B and M pass a plane cutting the two other edges in points D and C.

3. Compare $\triangle BAD$ with $\triangle MAD$; DM with DB. Auth.

4. Compare $DB + BC$ and $DM + MC$, compare BC and MC. Auth.

5. Compare $\angle BAC$ with $\angle MAC$. Auth.

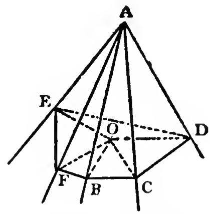
6. Compare $\angle BAC + \angle BAD$ with $\angle DAC$.

Therefore—

1. What is the locus of a point equidistant from two given parallel planes and at the same time equidistant from two other parallel planes?

PROPOSITION XXXI.

510. THEOREM. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given a polyhedral angle A with n face angles.

To Prove the sum of the face angles about vertex A to be less than 4 right angles.

Proof. SUG. 1. Pass a plane through the polyhedral cutting all the edges in the points B, C, D, F, etc. This cross section polygon will have n sides and the plane will form with the faces n "face" triangles with a common vertex A.

2. Let O be any point within this polygon and join O to the n vertices of the polygon, forming n "base" triangles with a common vertex O.

3. The sum of the angles of the face triangles equals the sum of the angles of the base triangles. Why?

4. $\angle ABC + \angle ABF > \angle CBF$, $\angle ACB + \angle ACD > \angle BCD$, etc.

5. Compare then the sum of all the base angles of the face $\triangle$ with the sum of all the base angles of the base $\triangle$.

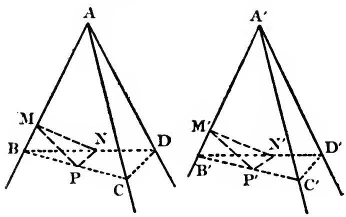
6. Compare the sum of the face angles at A with the sum of the angles about O.

7. Compare the sum of the angles at A with four rt. $\angle$s.

Therefore—

PROPOSITION XXXII.

511. THEOREM. *If two trihedral angles have the three face angles of the one equal respectively to the three face angles of the other, the corresponding dihedral angles are equal.*



Given two trihedrals, A and A', with $\angle BAC = \angle B'A'C'$, $\angle CAD = \angle C'A'D'$, $\angle DAB = \angle D'A'B'$.

To Prove dihedral $AB =$ dhl $A'B'$, dhl $AC =$ dhl $A'C'$, and dhl $AD =$ dhl $A'D'$.

Proof. SUG. 1. Take points on the six edges so that $AB = AC = AD = A'B' = A'C' = A'D'$ and $AM = A'M'$. Through B, C, D and B', C', D' pass planes. In the faces BAC and BAD respectively draw MN and MP *each* $\perp AB$. MP and MN can meet BC and BD in N and P respectively for $\angle$s ABC and ABD are acute. Why? Similarly draw lines $M'N'$ and $M'P'$ through point M'.

2. The dihedrals AB and $A'B'$ are equal if $\angle PMN = \angle P'M'N'$. Why?

3. Compare $\triangle ABC$ with $\triangle A'B'C'$; $\angle ABC$ with $\angle A'B'C'$, BC with $B'C'$. Auth.

4. Draw the similar conclusions from $\triangle ABD$ and $A'B'D'$. Also from $\triangle ACD$ and $A'C'D'$.

5. Compare $\triangle BMP$ with $\triangle B'M'P'$, BP with $B'P'$; MP with $M'P'$. Auth. Draw similar conclusions from $\triangle BMN$ and $\triangle B'M'N'$.

6. Compare $\triangle BCD$ with $\triangle B'C'D'$; $\angle CBD$ with $\angle C'B'D'$; $\triangle BPN$ with $\triangle B'P'N'$; NP with $N'P'$. Auth's.

7. Compare $\triangle NMP$ with $\triangle N'M'P'$, $\angle M$ with $\angle M'$. Auth.

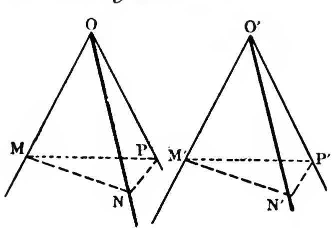
8. Compare dihedral AB with dihedral $A'B'$. Auth.

9. By similar arguments compare dihedrals AC and $A'C'$, AD and $A'D'$.

Therefore—

PROPOSITION XXXIII.

512. THEOREM. *If two trihedral angles have the three face angles of the one equal respectively to the three face angles of the other, they are either congruent or symmetrical.*



Given two trihedrals, O and O', with $\angle MOP = \angle M'O'P'$, $\angle PON = \angle P'O'N'$, and $\angle NOM = \angle N'O'M'$.

I. **To Prove** trihedrals O and O' congruent, the angles being arranged in the same order.

Proof. SUG. 1. Place O upon O' so that $\angle MOP$ coincides with $\angle M'O'P'$ and the two edges ON and $O'N'$ are on the same side of the plane MOP.

2. How does plane PON lie with reference to plane $P'O'N'$? Plane NOM with reference to plane $N'O'M'$? § 511.

3. Where does line ON lie with respect to $O'N'$? Why?

II. **To Prove** trihedrals O and O' symmetrical, the angles being arranged in reverse order.

Proof. By § 511 the corresponding dihedrals are equal. The parts are therefore respectively equal and by hypothesis the order of the angles is different in O and O'.

Therefore—§ 508.

If a line and a plane are parallel, a line drawn from any point in the plane parallel to the line will lie in the plane.

1. Parallel lines intersecting a plane make equal angles with the plane.

2. What is the locus in space of points equidistant from the sides of a plane angle?

3. Bisect a dihedral angle.

4. What is the locus of a point in a plane equidistant from two points without the plane?

5. Find a point X in a given line and equidistant from two points without the line.

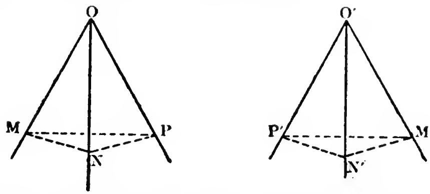
6. Find a point X in a plane and equidistant from three points without the plane.

7. What is the locus of a point X in a given plane, a given distance from a point not in that plane?

8. Find a point X in a plane and equidistant from the three edges of a trihedral.

PROPOSITION XIV.

513. Theorem. *Two symmetrical isosceles trihedral angles are congruent.*



Given two symmetrical isosceles trihedrals O and O' in which $\angle MOP = \angle M'O'P'$, $\angle PON = \angle P'O'N'$, $\angle NOM = \angle N'O'M'$, etc., and $\angle NOM = \angle PON$ and $N'O'M' = \angle P'O'N'$.

To Prove trihedrals O and O' congruent.

Proof. Sug. 1. On account of the symmetry, which face angles and which dihedrals are equal?

2. Because each is isosceles, which face angles are equal?

3. Compare $\angle NOM$ with $\angle P'O'N'$; $\angle PON$ with $\angle N'O'M'$.

4. From step 3, rearrange the equal parts so that the order is the same for O and O'. § 507.

Therefore—

514. Vertical Polyhedral Angles. When two polyhedrals are so placed that they have a common vertex and the sides of the one are extensions through the vertex of the sides of the other, they are vertical polyhedral angles.

1. Vertical trihedrals are symmetrical.
2. Vertical polyhedrals are symmetrical.
3. What is the locus of points equidistant from the faces of a trihedral? § 499.

4. What is the greatest number of equilateral triangles which can be put together to form a convex polyhedral angle?

5. How many different sized polyhedrals can be formed from equilateral triangles?

6. How many different sized polyhedrals can be formed from squares? From regular pentagons? From regular hexagons? § 509.

7. How many different polyhedrals can be constructed from regular polygons?

8. Can a mosaic or patch work pattern be constructed from regular triangles? From squares? From regular pentagons? From regular hexagons? From regular heptagons? From regular n-gons? Give reasons for each conclusion.

9. The foundation which bee keepers supply for the bees to build the honey comb upon is a mosaic constructed of regular polygons as near as possible to the shape of a circle. Of what polygons is it formed?

10. A circle would be a better shaped base for the body of the bee. Will the circle or polygon form of construction require the less material?

It is assumed in the preceding exercises that the amount of material used in making the cells is proportional to the amount used in making the base.

11. Are two planes perpendicular to the same plane necessarily parallel? Are two lines perpendicular to the same plane parallel? Are two planes perpendicular to the same line parallel?

12. Which is the longer, an oblique sect or its projection on a plane?

13. If three non parallel planes are each perpendicular to a fourth plane their three lines of intersection are parallel.

14. If two parallel planes intersect a dihedral, the respective lines of intersection form equal angles.

15. If the mid points of the adjacent sides of a skew quadrilateral (i. e. one the four vertices of which are not in the same plane) are joined by straight lines, the figure enclosed is a parallelogram.

515. Trirectangular Trihedral. A trihedral angle all of the face angles of which are right angles is a *trirectangular trihedral angle.*

16. In a trirectangular trihedral the dihedrals opposite the equal face angles are equal. Why? Is this true of any isosceles trihedral?

REVIEW

17. State in a theorem a possible condition by which one line can be proved parallel to another; a line can be proved parallel to a plane; a plane be proved parallel to a second plane.

18. State in a theorem a condition by which a line may be proved perpendicular to a second line; to a plane; a plane can be proved perpendicular to a second plane.

19. State in a theorem a condition by which two trihedrals can be proved congruent; can be proved symmetrical; two symmetrical trihedrals can be proved congruent.

20. What is the locus of points equidistant from the vertices of a triangle?

21. Locate a point in a given plane equidistant from all points of a circle which is in another plane.

22. Find a point X which is equidistant from two parallel planes, from two intersecting planes, and from two given points.

23. If two supplementary adjacent dihedrals are bisected by planes, the bisecting planes form a right dihedral.

24. If from any point within a dihedral angle perpendiculars are drawn to the faces of the dihedral, the angle formed is the supplement of the dihedral.

25. If from any point without a dihedral perpendiculars are drawn to the faces of the dihedral, the angle formed by the perpendiculars is equal to the dihedral.

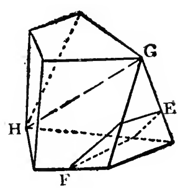
26. Two trihedrals having two face angles and the included dihedral of the one equal respectively to the corresponding parts of the other are either congruent or symmetrical.

27. Two trihedrals having two dihedrals and the included face angle of the one equal to the corresponding parts of the other are either congruent or symmetrical.

CHAPTER VIII.

POLYHEDRONS.

516. POLYHEDRON. A geometric solid bounded by planes is a *polyhedron.* The intersections of the planes are the *edges;* the intersections of the edges are the *vertices;* the portions of the planes bounded by the edges are the *faces;* the face upon which the polyhedron is supposed to rest is its *base.* Any face may be taken as the base. Any straight line connecting two vertices not in the same face is a *diagonal of the polyhedron.*



517. POLYHEDRONS CLASSIFIED. Polyhedrons are classified according to the number of faces. One of four faces is a *tetrahedron,* of six faces is a *hexahedron,* of eight faces is an *octohedron,* of twelve faces is a *dodecahedron,* of twenty faces is an *icosahedron,* etc.

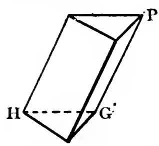
1. What is the least number of faces possible for a polyhedron?

518. PLANE SECTION OF A POLYHEDRON. The intersection of a plane and a polyhedron is a *plane section or a section* of the polyhedron. *FE* is a plane section of polyhedron.

519. Convex Polyhedron. A polyhedron such that every plane section is a convex polygon is a *convex polyhedron.*

Only convex polyhedrons will be considered in this book.

520. Prism. A polyhedron bounded by two parallel planes and a group of planes the intersections of which are parallel lines is *a prism.*



The faces formed by the parallel planes are the *bases,* usually designated as upper and lower base. The other faces are the *lateral faces.* The intersections of the lateral faces with each other are the *lateral edges* and their intersections with the bases are the *basal edges.*

521. Right Section. A section of a prism by a plane perpendicular to the lateral edges is a *right section.*

522. Oblique Section. A section of a prism by a plane which is oblique to the lateral edges is an *oblique section.*

523. Altitude of a Prism. The distance between the bases of a prism is its *altitude.* § 477.

524. Classification of Prisms. Prisms are classified as *triangular, quadrangular,* etc., according as their bases are triangles, quadrilaterals, etc.

Preliminary Theorems

525. Theorem I. *The lateral edges of a prism are equal.* § 479.

526. Theorem II. *The lateral faces of a prism are parallelograms.* § 520.

527. THEOREM III. *The acute angles made by the lateral edges with the planes of the bases are equal.* § 503.

528. RIGHT PRISM. A prism in which the lateral edges are perpendicular to the bases is a *right prism.*

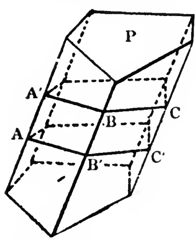
The edge of a right prism is also its *altitude.*

529. OBLIQUE PRISM. A prism in which the lateral edges are not perpendicular to the bases is an *oblique prism.*

530. REGULAR PRISM. A right prism the bases of which are regular polygons is a *regular prism.*

PROPOSITION I.

531. THEOREM. *Sections of a prism made by parallel planes are congruent polygons.*



Given ABC and $A'B'C'$, two parallel sections of a prism P.

To Prove $ABC \cong A'B'C'$.

Proof. SUG. 1. What two relations do the lines AB, BC, CD, ... bear to the lines $A'B'$, $B'C'$, $C'D'$, ... respectively? § 478.

2. Compare $\angle s\ ABC$, BCD, etc., with $\angle s\ A'B'C'$, $B'C'D'$, etc., respectively. Auth.

Therefore—

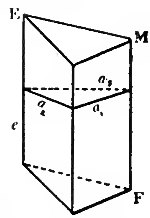
532. Cor. I. *The bases of a prism are congruent polygons.*

533. Cor. II. *A section of a prism made by a plane parallel to the base is congruent to the base.*

534. Cor. III. *All right sections of a prism are congruent polygons.*

PROPOSITION II.

535. Theorem. *The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.*



Given the prism M with p the perimeter and a_1, a_2, a_3, etc., the successive sides of a right section, e the length of the equal lateral edges, and S the lateral area.

To Prove $S = e \times p$.

Proof. Sug. 1. How do the successive sides of the right section lie with respect to the successive edges which they intersect?

2. What is the area of the face EF in terms of a_1 and e?

3. Express the area of each lateral face.

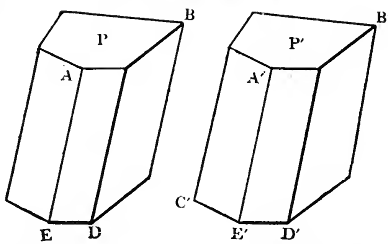
4. By adding these areas find the total lateral area, S, in terms of a_1, a_2, a_3, etc., and e and then in terms of e and p.

Therefore—

536. Cor. *The lateral area of a right prism equals the product of the perimeter of the base and a lateral edge.*

PROPOSITION III.

537. Theorem. *If two prisms have the three faces of a trihedral of one congruent to the three faces of a trihedral of the other and similarly placed, the prisms are congruent.*



Given two prisms P and P' in which the three faces AB, AC, AD forming the trihedral A are respectively congruent to the three faces $A'B'$, $A'C'$, $A'D'$ forming the trihedral A' and are similarly placed.

To Prove $P \cong P'$.

Proof. Sug. 1. Compare trihedrals A and A′, § 512.

2. Apply P' to P so that face $A'B'$ coincides with face AB. Why can this be done?

3. In what plane does face $A'C'$ fall? Face $A'D'$? Why? § 511.

4. Where does line $A'E'$ fall? Why? Where do points E', C', D' fall? Why? Where does plane $C'D'$ fall? Why?

5. Compare faces $C'D'$ and CD. § 532.

6. Complete the superposition. § 520.

Therefore—

538. Cor. I. *Two right prisms are congruent if their altitudes are equal and their bases congruent.*

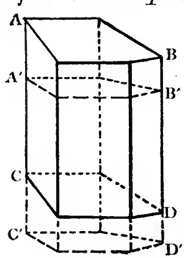
539. Truncated Prism. The portion of a prism included between the base and a section made by a plane not parallel to the base is a *truncated prism.*

540. Cor. II. *If two truncated prisms have the three faces about a trihedral of one congruent respectively to the three faces about a trihedral of the other and similarly placed, the truncated prisms are congruent.*

Proof. Draw figures and make a proof according to the method of § 537.

PROPOSITION IV.

541. Theorem. *An oblique prism is equal to a right prism the base of which is a right section of the oblique prism and the altitude of which is equal to the edge of the oblique prism.*



Given an oblique prism, BC, with a right section $A'B'$ and a lateral edge AC.

To Prove prism BC equal to a right prism with base $A'B'$ and an altitude equal to AC.

Proof. Sug. 1. Extend the lateral edges making $A'C' = AC$, and through C' pass a plane ∥ plane $A'B'$. Then $A'D'$ is a right prism.

2. Compare the faces about the trihedral A with the faces about the trihedral C.

3. Compare the truncated prism AB' with the truncated prism CD'.

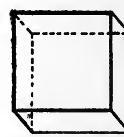
4. Compare prism BC with prism $B'C'$.

Therefore—

542. PARALLELOPIPEDS. A prism the bases of which are parallelograms is a *parallelopiped.*

543. RIGHT PARALLELOPIPED. A parallelopiped the edges of which are perpendicular to the bases is a *right parallelopiped.*

544. RECTANGULAR PARALLELOPIPED. A right parallelopiped the bases of which are rectangles is a *rectangular parallelopiped.*



545. CUBE. A rectangular parallelopiped all the faces of which are squares is a *cube.*

546. VOLUME OF A POLYHEDRON. The measure of a polyhedron in terms of some other polyhedron taken as the unit of measure is the *volume of the polyhedron.*

547. UNIT OF MEASURE FOR VOLUME. A cube with an edge equal to a given linear unit is the *unit of measure for volume.*

If a polyhedron contains a cubic inch twenty-five times, the volume of the polyhedron is twenty-five cubic inches.

548. PRELIMINARY THEOREMS.

THEOREM I. *All faces of a parallelopiped are parallelograms.*

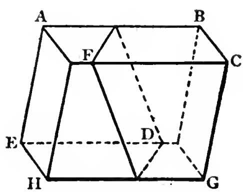
THEOREM II. *All faces of a rectangular parallelopiped are rectangles.*

THEOREM III. *All faces of a cube are congruent squares.*

THEOREM IV. *Any pair of opposite faces of a parallelopiped may be taken as bases.*

PROPOSITION V.

549. THEOREM. *Opposite faces of a parallelopiped are parallel and congruent parallelograms, and any section by a plane cutting four parallel edges is a parallelogram.*



Given a parallelopiped AG with bases AC and EG, AH and BG being one pair of opposite faces, and FHA a section made by a plane cutting four parallel edges.

To Prove faces AH and BG parallel and congruent parallelograms, and FD a ▱.

Proof. SUG. 1. AH and BG are parallelograms. Why?

2. Prove them congruent.
3. Prove them parallel. § 478.
4. Prove opposite sides of FD parallel.

Therefore—

1. Every section of a prism made by a plane parallel to a lateral edge is a parallelogram.

1. If from any point in space perpendiculars are drawn to the lateral faces of a prism, or to the lateral faces extended, these perpendiculars are all in the same plane.

2. Any straight line drawn through the middle point of any diagonal of a parallelopiped, terminating in two opposite faces is bisected at that point.

3. The four diagonals of a rectangular parallelopiped are equal to one another.

4. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares on three concurrent edges.

5. The sum of the squares upon the four diagonals of a rectangular parallelopiped is equal to the sum of the squares upon the twelve edges.

6. Prove the preceding exercise for any parallelopiped. §§ 337, 338.

7. Prove that the lateral area of a right prism is less than the lateral area of any oblique prism having the same base and an equal altitude.

> SUG. Draw an oblique and a right prism upon the same base with equal altitudes. No face of the oblique prism has a less altitude than the corresponding face of the right prism. Why? Some faces of the oblique prism must have altitudes greater than those of the corresponding faces of the right prism. Why? The altitudes of the faces are defined with respect to the common bases.

8. Make a list of theorems on polyhedrals and in connection with each write out a plane geometry theorem which closely corresponds to it.

9. From two points on the same side of a plane draw two lines to a point in the plane that shall make equal angles with the plane.

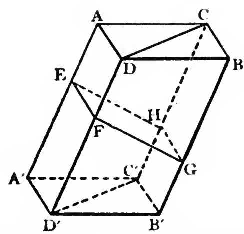
10. The lines joining the mid points of opposite sides of a skew quadrilateral bisect each other. Ex.

11. A plane perpendicular to a line in a plane is perpendicular to that plane.

12. If two or more planes intersect in one straight line, the perpendiculars drawn to them from any one point lie in the same plane.

PROPOSITION VI.

550. THEOREM. *A plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equal triangular prisms.*



Given parallelopipeds CD' divided by a plane through two opposite edges CC' and DD' into two triangular prisms $A'B'C'-A$ and $B'C'D'-B$.

To Prove $A'B'C'-A=B'C'D'-B$.

Proof. SUG. 1. A right section as HF intersects a ▭ on the parallelopiped and this is divided into two congruent $\triangle$s by the plane CD'. Why?

2. Compare $\triangle$-prism $A'C'D'-A$ with a right prism on EHF as base with an altitude equal to $D'D$. Compare $\triangle$-prism $B'C'D'-B$ with a right prism on GHF as base with an altitude equal to $C'C$. Auth.

3. Compare these two right prisms. § 538.

4. Compare $A'C'D'-A$ with $B'C'D'-B$.

Therefore—

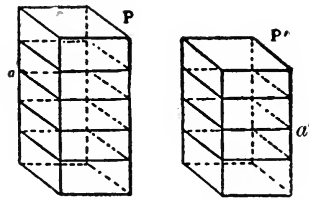
In the above figure which prisms are congruent and which are only equal?

1. In 550 prove $A'B'C'-A$ and $B'C'D'-B$ symmetrical.

2. The four diagonals of a parallelopiped bisect each other.

PROPOSITION VII.

551. THEOREM. *Two rectangular parallelopipeds having congruent bases are proportional to their altitudes.*



Given two rectangular parallelopipeds P and P' with congruent bases and altitudes a and a' respectively.

To Prove $\frac{P}{P'}=\frac{a}{a'}$.

Proof. *Case I. a and a′ commensurable.*

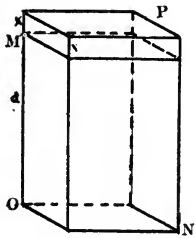
SUG. 1. Divide the altitudes by a common unit of measure, supposing it to be contained m and m' times in a and a' respectively. What is the ratio of the altitudes?

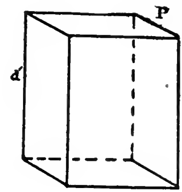
2. Through the points of division of the altitudes pass planes parallel to the bases. What kind of parallelopipeds are formed? Compare them in number and volume. Auth. What is the ratio of P and P'?

3. Compare the ratios of the altitudes and the parallelopipeds.

Case II. a and a′ incommensurable.

SUG. 1. Divide the altitudes a and a' by any unit commensurable with a'. In the division of a there will be a remainder x less than the unit. Why? By taking this unit smaller and smaller this remainder may be made to decrease indefinitely. Why?





2. Through this last point of division, M, pass a plane parallel to the base. As the unit in use is decreased what change takes place in the altitude OM? What change takes place in the parallelopiped MN? In the parallelopiped with altitude x? What is the limit of the variable altitude OM? Of the variable parallelopiped MN?

3. $\therefore \frac{OM}{a'} \doteq \frac{a}{a'}$ and $\frac{MN}{P'} \doteq \frac{P}{P'}$.

4. $\frac{MN}{P'} = \frac{OM}{a'}$. Why?

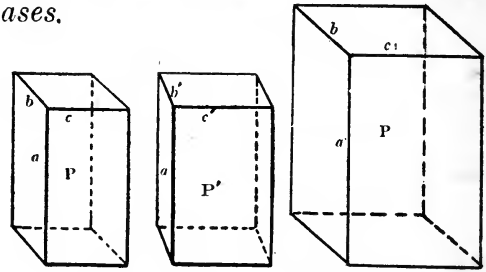
5. $\therefore \frac{P}{P'} = \frac{a}{a'}$. § 426.

Therefore—

552. Dimensions of a Parallelopiped. The *dimensions* of a parallelopiped are its altitude and the base and altitude of its base. Consequently the three dimensions of a rectangular parallelopiped are any three concurrent edges.

PROPOSITION VIII.

553. THEOREM. *Two rectangular parallelopipeds having equal altitudes are proportional to their bases.*



Given two rectangular parallelopipeds P and P' with equal altitudes a, their bases B and B' having the dimensions b, c and b', c' respectively.

To Prove $\dfrac{P}{P'} = \dfrac{B}{B'}$.

Proof. SUG. 1. Construct a third rectangular parallelopiped N with the altitude a and base with dimensions b', c.

2. $\dfrac{P}{N} = \dfrac{b}{b'}$ and $\dfrac{N}{P'} = \dfrac{c}{c'}$. Why?

3. Hence $\dfrac{P}{P'} = \dfrac{bc}{b'c'} = \dfrac{B}{B'}$. Why?

Therefore—

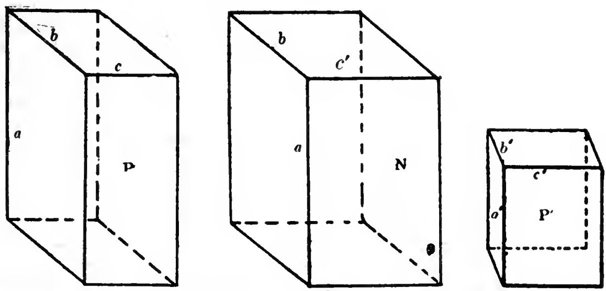
1. A gable for a dormer window is triangular in shape with an angle of 60° at the ridge. If the length of the rafters is 6 ft., what is the area of the largest circular window that can be inserted, making allowance for a four inch frame around the window?

2. The diagonals of a rectangular parallelopiped are equal.

3. Determine a point in one side of a triangle equally distant from the other two sides.

PROPOSITION IX.

554. THEOREM. *Two rectangular parallelopipeds are to each other as the product of their three dimensions.*



Given two rectangular parallelopipeds P and P', with dimensions a, b, c and a', b', c' respectively.

To Prove $\frac{P}{P'}=\frac{a\times b\times c}{a'\times b'\times c'}$.

Proof. SUG. 1. Let N be a rectangular parallelopiped with the dimensions a, b, c'.

2. Then $\frac{P}{N}=\frac{c}{c'}$ and $\frac{N}{P'}=\frac{a\times b}{a'\times b'}$. Why?

3. $\frac{P}{P'}=\frac{a\times b\times c}{a'\times b'\times c'}$. Why?

Therefore—

555. COR. I. *The volume of a rectangular parallelopiped is the product of its three dimensions.*

Given a rectangular parallelopiped P with dimensions a, b, c.

To Prove Vol. $P=abc$.

Proof. SUG. 1. Take as the unit of volume a cube U with an edge equal to the linear unit in which a, b, c are expressed.

2. $\frac{P}{U}=\frac{a\times b\times c}{1\times 1\times 1}=abc.$

3. $\therefore$ Vol. $P=abc.\ U.$

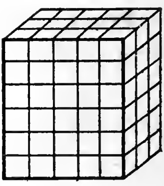
Therefore—

In the applications of this theorem the three dimensions must be expressed in terms of the same linear unit and the unit of volume must be a cube with an edge equal to this linear unit.

556. COR. *Two rectangular parallelopipeds having two dimensions respectively equal are to each other as their third dimensions.*

557. COR. II. *The volume of a cube equals the cube of its edge.*

558. By comparison of the theorem and the definition of volume (§546) it will be observed that the volume of a rectangular parallelopiped is equal to the product of the measures of the three edges meeting at any vertex times the unit of volume. The expression "product of the three dimensions" is understood to mean the product of the measures of these lines. § 361 note.



When each dimension of the rectangular parallelopiped is divisible by the linear unit which is the edge of the unit volume, the truth of the theorem on volume may be shown by dividing the parallelopiped into unit cubes.

1. What is the volume of a cube the edge of which is 7 in.?

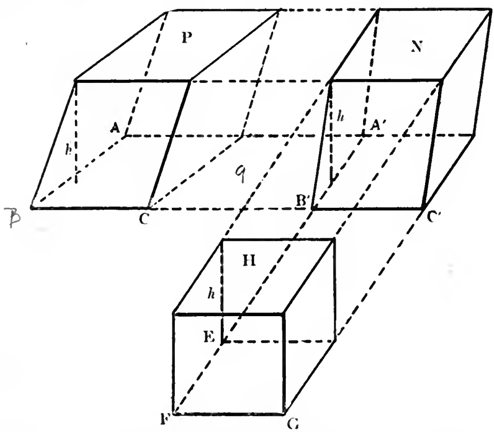
2. What is the volume of a rectangular parallelopiped the dimensions of which are 4 in., 7 in., 12 in.? 6 ft. $\sqrt{12}$ ft., $\sqrt{18}$ ft.? $\sqrt{12}$, $\sqrt{18}$, $\sqrt{24}$? $\sqrt{8}$, $\sqrt{24}$, $\sqrt{12}$?

3. The edges of a rectangular parallelopiped are 8, 12, 16. What is the length of a diagonal? What is its length if the dimensions are $\sqrt{8}$, $\sqrt{9}$, $\sqrt{18}$?

4. Find the total surface area of a regular triangular prism with basal edges of 4 ft. and lateral edges of 8 ft.

PROPOSITION X.

559. Theorem. *Any parallelopiped is equal to a rectangular parallelopiped having an equal base and altitude.*



Given a parallelopiped P with base ABC and altitude h.

To Prove $P =$ a rectangular parallelopiped H having a base $EFG = ABC$ and altitude h.

Proof. Sug. 1. Extend edge BC of P and the three edges parallel to BC and at some convenient point on line BC take $B'C' = BC$. Through B' and C' pass planes perpendicular to line $B'C'$, forming the parallelopiped N with base $A'B'C'$. Extend the edge $A'B'$ and the three edges parallel to $A'B'$ and at some convenient point take $EF = A'B'$. Through E and F pass planes perpendicular to EF forming the parallelopiped H.

2. Show that N is a right parallelo-

piped and H is a rectangular parallelopiped. Show that P, N, and H have the same altitude and equal bases.

3. Compare P and N. §§ 548 IV, 541.

Compare N and H.

4. Compare P and H.

Therefore—

560. Cor. I. *The volume of any parallelopiped is equal to the product of its three dimensions.*

Given a parallelopiped P with dimensions a, b, c.

To Prove Vol. $P = abc$.

Proof. P equals a rectangular parallelopiped with dimensions a, b, c. Why? The volume of this second parallelopiped is abc.

Therefore—

561. Cor. II. *Two parallelopipeds with equal bases are to each other as their altitudes.*

Proof. Let the dimensions be a, b, c and a, b, c', respectively. Use § 560 to find the ratio.

562. Cor. III. *Two parallelopipeds with equal altitudes are to each other as their bases.*

Proof. Let the dimensions be a, b, c, and a, b', c' respectively. Use § 560 to find the ratio.

563. Cor. IV. *Two parallelopipeds are to each other as the products of their three dimensions.*

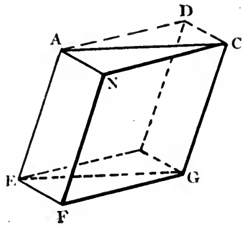
1. Find the lateral area of a regular pentagonal prism each edge of which is 3 in.

2. If a secant plane intersects two planes so that the lines of intersection are parallel and the corresponding dihedrals are equal, the two planes are parallel.

3. Prove the preceding example when the equal dihedrals are the alternate interior angles; prove it for equal alternate exterior angles.

PROPOSITION XI.

564. THEOREM. *The volume of a triangular prism is equal to the product of its base and its altitude.*



Given the triangular prism $EFG - N$, denoting its volume by V, its base by B, and its altitude by h.

To Prove $V = hB$.

Proof. SUG. 1. Extend the planes of the bases. Through the edges AE and CG pass planes parallel to the faces NG and NE respectively. The figure DF is a parallelopiped. Why?

2. What is the volume of DF in terms of B and h? Why?

3. What is the volume of the $\triangle$-prism in terms of DF? § 550.

4. What is the volume of the $\triangle$-prism in terms of B and h?

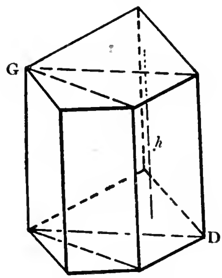
Therefore—

1. Compare the volumes of two rectangular parallelopipeds the respective edges of which are 2′, 3′, 7′ and 5′, 3′, 8′.

2. A parallelopiped has an altitude of 8 in. and its base is a rhombus with a 10 in. side, the shorter diagonal being 12 in. Find the volume.

PROPOSITION XII.

565. THEOREM. *The volume of any prism is equal to the product of its base and its altitude.*



Given the prism *GD* with *V, B,* and *h* denoting its volume, base, and altitude respectively.

To Prove $V = hB$.

Proof. SUG. 1. Through any one lateral edge pass diagonal planes. Into what kind of figures is the prism divided?

2. Denote the respective volumes and bases of these figures by $V_1, B_1; V_2, B_2; V_3, B_3$ etc. Then

$V_1 = h \times B_1$.
$V_2 = h \times B_2$.
$V_3 = h \times B_3$, etc. Why?

3. $V = V_1 + V_2 + V_3 \ldots$
$= hB_1 + hB_2 + hB_3 + \ldots$

4. Determine V in terms of h and B.

Therefore—

566. COR. I. *If two prisms have equal bases, their volumes are proportional to their altitudes.*

567. COR. II. *If two prisms have equal altitudes their volumes are proportional to their bases.*

1. Find the volume of a regular hexagonal prism, the lateral edge being 10 in. and the basal edge being 5 in.

2. Find the volume of a rectangular parallelopiped, the lateral edge being 20′ and the basal edges being 5′.

3. Find the volume of a parallelopiped, the base being 8″ square, the lateral edge 13″ and the perpendicular let fall from a vertex of one base striking the other base 5″ from the corresponding vertex.

4. Find the volume of a hexagonal prism, having for its base a regular polygon, the lateral edge being 25″, the basal edge 10″, and the inclination of the lateral edge to the base being 60°. Find the volume if the inclination is 45°.

5. The specific gravity of iron is 7.4. What is the weight of a rectangular tank including the cover $\frac{1}{2}$″ thick, made of iron the inside measurements being 20″ × 20″ × 4′? Make a diagram showing how the answer can be obtained with the least computation.

6. A right triangular tank has a lateral edge of 15′ and basal edges of 8′, 7′, and 5′, inside measurements. Find its total area and its capacity in gallons. $7\frac{1}{2}$ gal. = 1 cu. ft. approximately.

7. How can one obtain the volume of an irregular piece of rock by immersing it?

8. What is the ratio of two rectangular solids the dimensions of which are 3, 8, 12 and 4, 9, 20 respectively?

9. The apothem of a regular hexagonal prism is 10 and its lateral edge is 20. Find its volume and total area.

10. The apothem of a cube is 4. Find volume and total area.

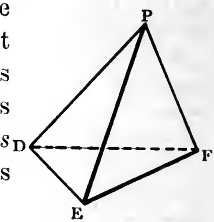
11. The edges of three cubes are 7″, 12″, and 13″ respectively. Find the volume of each and the total area of each.

12. A rectangular box 12″ × 18″ × 22″, outside measurement, is made of 1″ boards. What is its capacity in cubic inches? How many cubical boxes $2\frac{3}{4}$″ on an edge can be packed in it? What is its capacity in gallons?

13. A cistern is in the form of a regular hexagonal prism. The lateral edge is 7′ and the basal edge is 6′, inside measurements. What is its capacity in gallons?

14. The volume of a rectangular parallelopiped is 6,720 cubic inches and its edges are in the ratio of 3, 5, 7. Find the three edges.

568. Pyramid. A polyhedron all but one of the faces of which meet in the same point is a pyramid. The point in which these faces meet is the *vertex*. The face which does not meet the vertex is the *base* and the other faces are the *lateral faces*. The intersections of the lateral faces are the *lateral edges* and the intersections of the lateral faces with the base are the *basal edges*.



Point out the various parts of the pyramid above.

569. Altitude of a Pyramid. The length of the perpendicular from the vertex to the plane of the base is the *altitude* of the pyramid. It may also be taken as the perpendicular distance between the plane of the base and a plane through the vertex parallel to the base. Why?

570. Pyramids Classified. A pyramid is *triangular, quadrangular, pentagonal,* etc., according as its base is a triangle, quadrilateral, a pentagon, etc.

In a triangular pyramid any face may be taken for the base, the vertex of the opposite polyhedral angle then being the vertex of the pyramid.

571. Regular Pyramid. A pyramid with a regular base, such that the vertex lies in the perpendicular erected at the center of the base, is a *regular pyramid.*

572. Slant Height. The perpendicular from the vertex of a regular pyramid to any basal edge is the *slant height* of the regular pyramid.

573. Truncated Pyramid. That portion of a pyramid included between the base and a plane cutting all the lateral edges is a *truncated pyramid.*

574. Frustum of a Pyramid. A truncated pyramid

in which the cutting plane is parallel to the base is a *frustum of a pyramid.* The section made by the cutting plane is the *upper base* of the frustum.

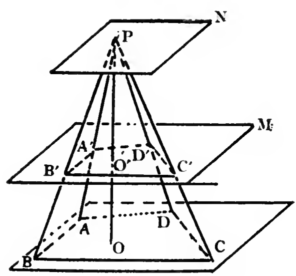
575. ALTITUDE OF A FRUSTUM. The perpendicular distance between the bases is the *altitude* of the frustum.

COROLLARIES TO THE DEFINITIONS.

576. COR. I. *The lateral faces of a pyramid are triangles.*

577. COR. II. *In a regular pyramid the lateral edges are equal, the lateral faces are congruent triangles, and the slant height is the same irrespective of the face in which it is drawn.*

578. COR. III. *In a frustum of a regular pyramid the lateral edges are equal, the lateral faces are congruent trapezoids, and the slant height is the same irrespective of the face in which it is drawn.*



PROPOSITION XIII.

579. THEOREM. *If a pyramid is cut by a plane parallel to the base,*

I. *The lateral edges and the altitude are cut proportionally;*

II. *The section is a polygon similar to the base.*

Given a pyramid $P-AC$ with base $ABC\ldots$ cut by a plane $M \parallel$ the base in the section $A'B'C'\ldots$ with altitude PO.

I. **To Prove** $\frac{PA}{PA'}=\frac{PB}{PB'}=\frac{PC}{PC'}$, etc., $=\frac{PO}{PO'}$.

Proof. Sug. 1. Through the vertex P pass plane $N \parallel$ plane M.

2. Compare the ratios

$\frac{PA}{PA'}, \frac{PB}{PB'}, \frac{PC}{PC'}, \frac{PO}{PO'}$. § 482.

II. **To Prove** $A'B'C' \sim ABC \ldots$

Proof. Sug. 1. What is the definition of similar polygons?

2. $\frac{AB}{A'B'}=\frac{PB}{PB'}=\frac{BC}{B'C'}$, etc. Why?

3. Complete the demonstration.

Therefore—

580. Cor. *The bases of a frustum of a pyramid are similar polygons.*

PROPOSITION XIV.

581. Theorem. *If a section of a pyramid is parallel to the base, the ratio of the section to the base equals the ratio of the squares of the distances from the vertex to the section and the base.*

Given a pyramid, as in § 579, with section $A'B'C'$ parallel to the base $ABC\ldots$ and with PO' and PO the respective distances of the section and the base from the vertex P.

To Prove $$\frac{A'B'C'\ldots}{ABC\ldots}=\frac{\overline{PO'}^2}{\overline{PO}^2}$$

Proof. SUG. $\frac{A'B'C' \ldots}{ABC \ldots} = \frac{\overline{A'B'}^2}{\overline{AB}^2} = \frac{\overline{PA'}^2}{\overline{PA}^2} = \frac{\overline{PO'}^2}{\overline{PO}^2}$.

Give the authority for each of these statements.

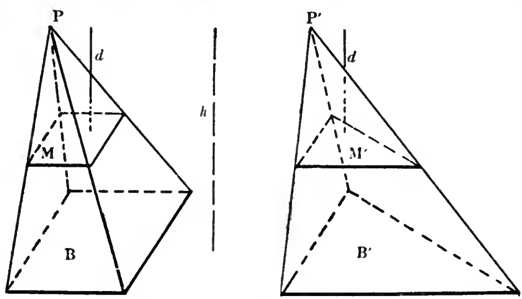
Therefore—

1. The three edges of a rectangular parallelopiped meeting in a point are 2′, 5′, and 10′. Find its lateral area and volume.

2. A right triangular prism has an altitude of 20″ and basal edges of 10′, 19′, and 12′. Find its lateral area and volume. Two methods.

PROPOSITION XV.

582. THEOREM. *In pyramids having equal bases and equal altitudes, sections made by planes parallel to the respective bases and at equal distances from the respective vertices are equal.*



Given two pyramids, P and P', with equal altitudes h, equal bases B and B' respectively; M and M' being sections of P and P' parallel to the respective bases and at the same distance d from the respective vertices.

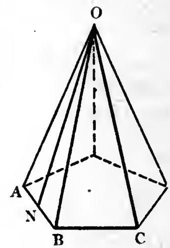
To Prove $M = M'$.

Proof. SUG. 1. $\frac{M}{B} = \frac{d^2}{h^2}$ and $\frac{M'}{B'} = \frac{d^2}{h^2}$. Why?

Therefore—

PROPOSITION XVI.

583. THEOREM. *The lateral area of a regular pyramid is equal to one-half the product of the perimeter of the base and the slant height.*



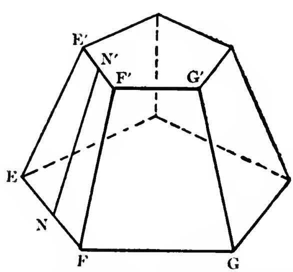
Given the regular pyramid O, its lateral area denoted by S, its slant height ON by l, and the perimeter of its base by p.

To Prove $S = \frac{1}{2} lp$.

Proof. Proof left to the student.

PROPOSITION XVII.

584. THEOREM. *The lateral area of the frustum of a regular pyramid is equal to the product of the slant height by one-half the sum of the perimeters of the bases.*



Given EG' the frustum of a regular pyramid, its lateral area denoted by S, its slant height, NN', by l, and the respective perimeters of the two bases by p and p'.

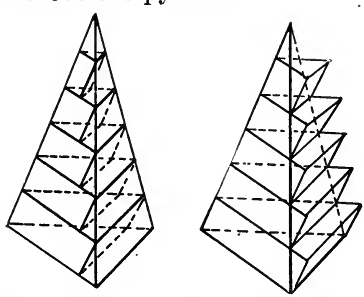
To Prove $S = \frac{1}{2} \times l (p + p')$.

Proof. SUG. 1. Find the area of each face and add these areas.

2. What is the coefficient in the addition? § 535.

Therefore—

585. INSCRIBED PRISMS. If a triangular pyramid is cut by any number of planes parallel to its base and through the lines in which these planes cut one of the lateral faces of the pyramid other planes are passed parallel to the opposite lateral edge of the pyramid, certain triangular prisms are formed. These are *inscribed prisms,* as *A, B, C,* etc., or *circumscribed prisms,* as *A′B′C′,* etc., according as this second set of planes meet the successive planes of the first set within the pyramid or without the pyramid.



586. POSTULATE. *A pyramid has a volume greater than the combined volumes of any set of inscribed prisms and less than any set of circumscribed prisms.*

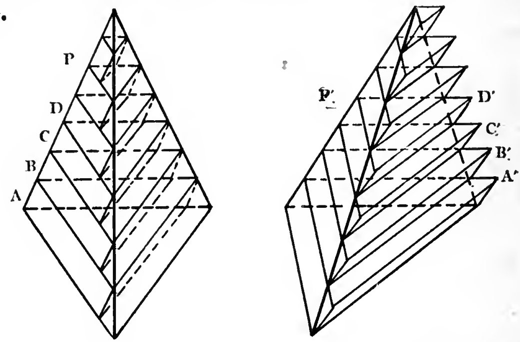
1. The diagonal of a rectangular parallelopiped is 24 and its edges are in the ratio of 2, 3, 4. Find the total area and the volume.

2. A cubical cistern holds 900 bbls. of 31½ gals. How many square feet of surface has it?

3. Find the interior surface of a rectangular cistern, the edges being in the ratio of 3, 4, 5 and its capacity 900 bbls.

PROPOSITION XVIII.

587. THEOREM. *The volumes of two triangular pyramids with equal altitudes and equal bases are equal.*



Given two triangular pyramids P and P', with equal altitudes h and equal bases.

To Prove $P = P'$.

Proof. SUG. 1. Divide the equal altitudes of P and P' into any number of equal parts x and through the points of division pass planes parallel to the bases and form in P a set of inscribed prisms, A, B, C . . . and in P' a set of circumscribed prisms A', B', C', . . .

2. Prisms $B' = A$, $C' = B$, $D' = C$, etc. Why? § 564.

3. Denote $A + B + C + \ldots$ by V and $A' + B' + C' + \ldots$ by V'. Then $V' - V = A'$. Why?

4. Suppose that $P' > P$, i. e. $P' - P$ equals some definite number K. By § 586 $P' < V'$ and $P > V$ so that $P' - P < V' - V$. § 49.

5. Hence $P' - P < A'$. Why? By

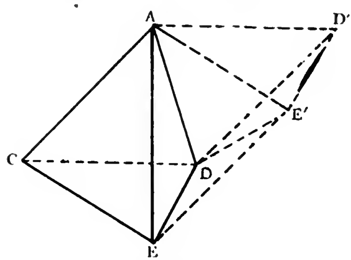
taking the length x small enough the prism A' can be made as small as is desired, even less than K, since its altitude x is decreased. § 564. Hence P' cannot be greater than P.

6. Form the inscribed prisms in P' and the circumscribed prisms in P. It can now be proved that P' cannot be less than P.

Therefore—

PROPOSITION XIX.

588. THEOREM. *The volume of a triangular pyramid is one-third the product of its base and its altitude.*



Given the triangular pyramid $A-ECD$, its volume denoted by V, its base by B, and its altitude by h.

To Prove $V = \frac{1}{3} h \times B$.

Proof. SUG. 1. Through A pass a plane parallel to the base, extend the planes of the faces ACE and ACD, and through ED pass a plane parallel to AC. The resulting figure is a prism. Why?

2. $A-EE'D'D$ is a quadrangular prism. The plane of $AE'D$ divides it into two equal pyramids, $A-EE'D$ and $A-E'D'D$.

3. $A-ECD = D-E'D'A = A-E'DE$

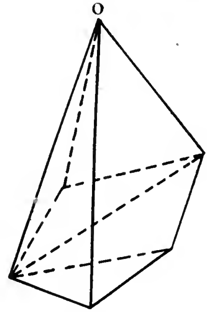
4. $A-ECD = \frac{1}{3}$ prism. What is its base and altitude? Authority for each statement.

6. Complete the demonstration.

Therefore—

PROPOSITION XX.

589. THEOREM. *The volume of any pyramid is equal to the product of its base and its altitude.*



Given pyramid O, denoting its volume by V, its base by B, and its altitude by h.

To Prove $V = \frac{1}{3} h \times B$.

Proof. The proof, similar to that of § 565, is left to the pupil.

590. COR. I. *If two pyramids have equal bases their volumes have the same ratio as their altitudes.*

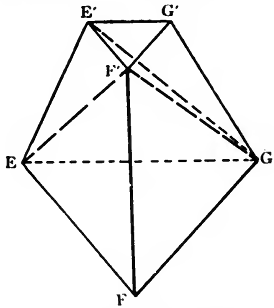
591. COR. II. *If two pyramids have the same altitude, their volumes have the same ratio as their bases.*

COR. III. *Any two pyramids are proportional to the products of their bases and altitudes.*

PROPOSITION XXI.

592. THEOREM. *The volume of a frustum of a triangular pyramid is equal to the sum of the volumes of three triangular pyramids each with the altitude of the frustum and with bases equal re-*

spectively to the upper base of the frustum, the lower base of the frustum, and a mean proportional to the bases of the frustum.



Given frustum of a triangular pyramid F, with upper base B_1 and lower base B_2.

To Prove $F = \frac{1}{3} h (B_1 + B_2 + \sqrt{B_1 B_2})$.

Proof. SUG. 1. By a plane through $GE'F'$ cut off a triangular pyramid P, with base B_1 and altitude h. What is its volume?

2. By a plane through GEF' cut off a triangular pyramid P_2 with base B_2 and altitude h. What is its volume?

3. The remaining portion is a triangular pyramid, P_3.

$$\frac{P_3}{P_1} = \frac{\triangle GEE'}{\triangle GE'G'} = \frac{GE}{G'E'} = \frac{\sqrt{B_2}}{\sqrt{B_1}}. \text{ Why?}$$

4. $P_3 = P_1 \frac{\sqrt{B_2}}{\sqrt{B_1}} = \frac{1}{3} h B_1 \frac{\sqrt{B_2}}{\sqrt{B_1}} = \frac{1}{3} h \sqrt{B_1 B_2}$

Why?

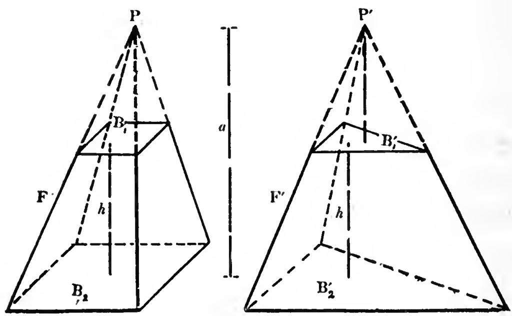
5. $F = P_1 + P_2 + P_3$.

$= \frac{1}{3} h (B_1 + B_2 + \sqrt{B_1 B_2})$.

Therefore—

PROPOSITION XXII.

593. THEOREM. *The volume of the frustum of any pyramid is equal to the sum of the volumes of three pyramids each with the altitude of the frustum and with bases equal respectively to the upper base of the frustum, the lower base of the frustum, and a mean proportional to the two bases of the frustum.*



Given the frustum F with upper base B_1, lower base B_2, and altitude h.

To Prove $F = \frac{1}{3} h (B_1 + B_2 + \sqrt{B_1 B_2})$.

Proof. SUG. 1. The lateral edges of F will if extended, meet in a point forming a pyramid P. Why? Construct a triangular pyramid P' with the same altitude a as P and a base B_2' equal to the base B_2 of P. Cut from P' a frustum F' with altitude h.

2. Compare the volumes of P and P'. Auth.

3. Compare B_1' and B_1. § 582.

4. The pyramids with bases B_1 and B_1' and altitudes $a - h$ are equal. Why?

5. Hence the frustum $F = F'$. Why?

6. $F' = \frac{1}{3} h (B_1' + B_2' + \sqrt{B_1'B_2'})$ Why?

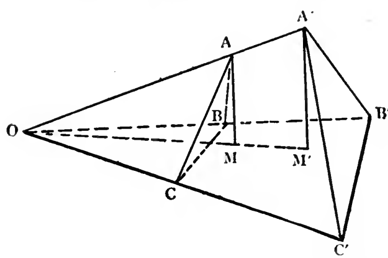
7. $\therefore F = \frac{1}{3} h (B_1 + B_2 + \sqrt{B_1 B_2})$.

Therefore—

1. A monument is 25′ high, 18″ square at one end, 30″ square at the other, and of uniform shape. What is its volume in cubic feet? If its specific gravity is $7\frac{1}{2}$ what does it weigh in tons?

PROPOSITION XXIII.

594. THEOREM. *Two tetrahedrons having a trihedral angle of one equal to a trihedral angle of the other have the same ratio as the products of the three edges including the equal trihedral angles.*



Given two tetrahedrons $O - ABC$ and $O' - A'B'C'$ with equal trihedrals O and O'.

To Prove $\dfrac{O-ABC}{O'-A'B'C'} = \dfrac{OA \times OB \times OC}{O'A' \times O'B' \times O'C'}$.

Proof. SUG. 1. Superimpose trihedral O' upon trihedral O and from points A and A' drop perpendiculars to the opposite face meeting it in points M and M' respectively.

2. Then $\dfrac{O-ABC}{O-A'B'C'} = \dfrac{AM \times \triangle OBC}{A'M' \times \triangle OB'C'}$

Why?

3. Also $\frac{\triangle OBC}{\triangle OB'C'} = \frac{OB \times OC}{OB' \times OC'}$. Why?

4. Points O, M, M' are in a straight line. Why?

5. $\frac{AM}{A'M'} = \frac{OA}{OA'}$. Why?

6. Complete the demonstration.

Therefore—

595. SIMILAR POLYHEDRONS. If two polyhedrons have the same number of faces similar each to each and similarly placed and have their corresponding polyhedral angles equal, the polyhedrons are *similar*.

The equal angles and the lines and faces in the two polyhedrons which are similarly situated are called *homologous* angles, lines, and faces, respectively.

PRELIMINARY THEOREMS.

596. THEOREM I. *Homologous lines in similar polyhedrons are proportional.*

597. THEOREM II. *Homologous faces in similar polyhedrons are proportional to the squares of homologous lines.*

598. THEOREM III. *The surfaces of similar polyhedrons are proportional to the squares of homologous lines.* § 374.

1. Find the volume of a regular triangular pyramid with basal edge of 4 ft. and altitude of $5\sqrt{3}$ ft.

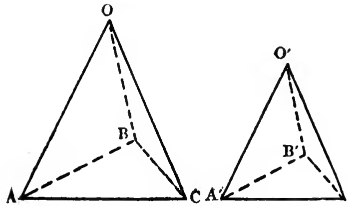
2. The point of meeting of the three medians of an equilateral triangle is $\frac{2}{3}$ of the length of the median from each vertex. Ex. 30, p. 212.

3. The edges of a regular tetrahedron are each a ft. Find its slant height, altitude, base area, lateral area, total area, and volume.

4. Find the volume of a regular hexagonal prism with a radius of 2 ft. and lateral edge of 2 yards.

PROPOSITION XXIV.

599. THEOREM. *Two similar tetrahedrons are proportional to the cubes of homologous edges.*



Given two similar tetrahedrons, O and O', edges OA, OB, OC . . . being homologous to $O'A'$, $O'B'$, $O'C'$ . . respectively.

To Prove $\frac{O}{O'}=\frac{\overline{OA}^3}{\overline{OA'}^3}$.

Proof. SUG. 1. Compare the trihedral angles O and O'. $\frac{O}{O'}=\frac{OA \times OB \times OC}{O'A' \times O'B' \times O'C'}=\frac{OA}{O'A'}\times\frac{OB}{O'B'}\times\frac{OC}{O'C'}$. Why?

2. Compare $\frac{OA}{O'A'}$, $\frac{OB}{O'B'}$, $\frac{OC}{O'C'}$. Auth.

3. Substitute for $\frac{OB}{O'B'}$ and $\frac{OC}{O'C'}$ in (1).

1. Volumes of similar solids are to each other as the square roots of the cubes of their surfaces.

2. If the amount of lumber used is to be the same in either case which provides the greater capacity, one barn or two, the barns being similar in shape?

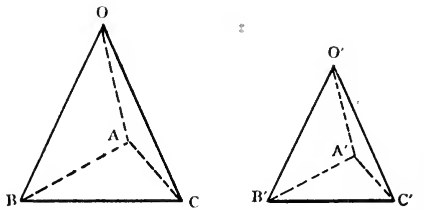
3. If the dimensions of a window be doubled, by what factor is its lighting capacity increased?

4. If the surface of a box be trebled in building a second box of the same shape, by what is the capacity increased?

5. What increase in material is required in building a new grain box having the same shape as the old one but of double the capacity?

PROPOSITION XXV.

600. THEOREM. *Two tetrahedrons are similar if three faces of one are respectively similar to three faces of the other.*



Given two tetrahedrons O and O' with the three faces about O similar to the three faces about O' respectively.

To Prove $O \sim O'$.

Proof. SUG. 1. Compare trihedrals O and O'. Auth.

2. Compare lines AB, BC, CA with $A'B'$, $B'C'$, $C'A'$ respectively. Auth.

3. Compare $\triangle$ ABC and $A'B'C'$. Auth.

4. The corresponding trihedrals and dihedrals are equal. Why?

5. The conditions for similarity are fulfilled. Why?

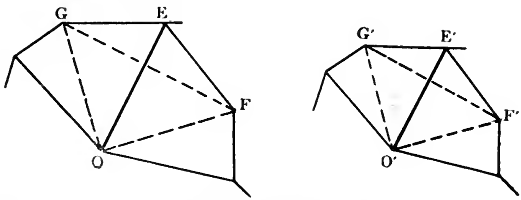
Therefore—

601. COR. *Two tetrahedrons having a dihedral of one equal to a dihedral of the other and the including faces of the first dihedral similar respectively to the including faces of the second dihedral and similarly placed are similar.*

Proof. SUG. Prove the faces opposite the equal dihedrals similar.

PROPOSITION XXVI.

602. Theorem. *Two similar polyhedrons can be divided into tetrahedrons similar each to each and similarly placed.*



Given P and P' similar polyhedrons.

To Prove that P and P' can be divided into tetrahedrons similar each to each and similarly placed.

Proof. Sug. 1. Through any two corresponding vertices as O and O' draw in each of the including faces all possible diagonals. The corresponding faces of P and P' are thus divided into the same number of triangles, similar each to each and similarly placed. Why?

2. Let OE and $O'E'$ be two corresponding edges and let OF and OG be the two diagonals including between them OE. Similarly let $O'F'$ and $O'G'$ be the corresponding lines of P'. Cut from P and P' the respective tetrahedrons $O-EFG$ and $O'-E'F'G'$. By Sug. (1) $\triangle OEF \sim \triangle O'E'F'$ and $\triangle OEG \sim \triangle O'E'G'$. Also dihedral $OE =$ dihedral $O'E'$. Why? Hence tetrahedron $O-EFG \sim$ tetrahedron $O'-E'F'G'$. § 601.

3. In the remaining portions of P and P' the new face $\triangle FOG$ and $\triangle EFG$ are similar respectively to $F'O'G'$ and $E'F'G'$. Why?

Also the ratio of the new edges *OF*, *OG*, *FG* to *O'F'*, *O'G'*, *F'G'* respectively, being equal to $\frac{OE}{O'E'}$ (Why?) the original ratio of similitude.

4. Since the trihedral angles *O* and *O'* of the removed tetrahedrons are equal, so are the remaining polyhedral angles *O* and *O'*.

5. Hence the remaining polyhedrals are similar. Why?

6. Repeat the process on the remaining portions of *P* and *P'*.

Therefore—

1. Similar polyhedrons have the same ratio as the cubes of homologous edges.

Sug. According to the demonstration of 602, the polyhedrons may be divided into tetrahedrons. Let them be respectively P_1 P_1', P_2 P_2' etc. Let a_1 and a_1' be homologous edges of P_1 and P_1', etc. Then by 599.

$\frac{P_1}{P_1'} = \frac{a_1^3}{a_1'^3}$, $\frac{P_2}{P_2'} = \frac{a_2^3}{a_2'^3}$, etc. But the ratios of the *a*'s are equal. Why? Hence the ratios of the *P*'s are equal and $\frac{P_1 + P_2 + P_3 + \dots}{P_1' + P_2' + P_3' + \dots} = \frac{P_1}{P_1'} = \frac{a_1^3}{a_1'^3}$.

603. Regular Polyhedron. A polyhedron in which all the faces are regular congruent polygons and in which the polyhedral angles are equal is a *regular polyhedron*. A regular polyhedron must be *convex*.

2. The perimeter of the mid section of a frustum of a pyramid equals one-half the sum of the perimeters of the bases.

3. A grain bin holds 100 bu. Another bin has each dimension equal to twice the corresponding dimension of the first. What is its capacity?

4. The surfaces of two similar solids are to each other as the cube roots of the squares of their volumes.

PROPOSITION XXVII.

604. THEOREM. *At most only five regular polyhedrons can be formed.*

Proof. SUG. 1. Show that three, four, or five regular triangles can be so used as to form a convex polyhedral angle.

2. Show that more than five regular triangles cannot form a convex polyhedral.

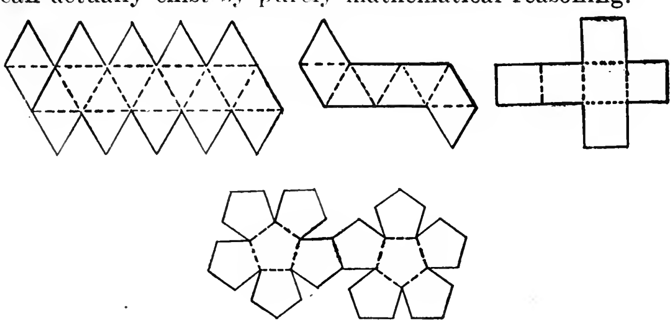
3. Make a similar test for the regular tetragon or square. § 510.

4. How many regular pentagons may be used? Auth.

5. Show that no regular polygon of more than five sides can be used to form a convex polyhedral.

Therefore—

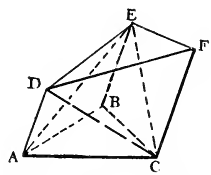
605. SCHOLIUM. That five regular polygons can all be constructed is illustrated as follows: Cut the patterns below from cardboard and cut the material half through along the dotted lines. Fold each so as to form a polyhedron, pasting strips of paper along the edges to keep them in shape. It is possible to prove that these figures can actually exist by purely mathematical reasoning.



1. The total areas of regular polyhedrons have the same ratios as the squares of their altitudes or as the squares of any two homologous edges.

2. Find the volume of a regular quadrangular pyramid with basal edge of 8′ and slant height of 5′.

3. The volume of a truncated triangular prism is equal to the sum of the volumes of three pyramids having the base of the prism as a common base and the vertices of the inclined section as respective vertices.



To Prove $EFD — ABC = E — ABC + D — ABC + F — ABC$.

Pass planes through each vertex of the truncating section, DEF, and the respective opposite edges of the base. Plane EAC cuts off the first pyramid. There is left $E — ADCF$. Plane EDC cuts off $E — ACD = B — ABC$ or $D — ABC$, § 570 the second pyramid. The remaining pyramid, $E — DCF$ or $D — FEC = D — FCB$ (equal bases). But $D — FCB = A — FCB$, the third pyramid

4. Find the volume of a truncated triangular right prism with basal edges of 5′, 5′, 8′ and lateral edges of 7′, 8′, 9′ respectively. Use two methods of finding the base.

5. The volume of any truncated triangular prism equals the product of a right section and one-third the sum of the lateral edges.

6. The sides of a right section of a truncated triangular pyramid are 8′, 9′, 15′ and the lateral edges are 5′, 9′, 12′ respectively. Find the lateral area and the volume.

7. The apothem of a cube is 7′. Find the volume by using the rule for pyramids.

8. Find the volume of a monolith 24′ high, 9′ square at one end and 4′ square at the other.

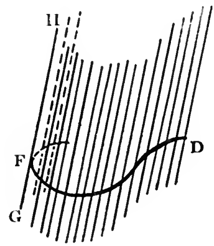
Solve without pencil.

9. Find the slant height, the apothem, the lateral surface, the total surface, and the volume of a regular tetrahedron with an edge of 5.

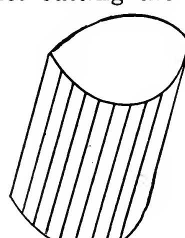
CHAPTER IX.

The Three Round Bodies.

606. Cylindrical Surface. A surface formed by a moving straight line which always remains parallel to its original position and continually touches a fixed curved line is a *cylindrical surface.* The moving straight line is the *generatrix* and the fixed curve is the *directrix.* The generatrix in any one of its positions is an *element* of the surface. If the directrix is a closed curve, the cylindrical surface is a *closed* cylindrical surface.



607. A Cylinder. A solid bounded by a closed cylindrical surface and two parallel planes cutting the elements is a *cylinder.* The plane surfaces are the *bases* of the cylinder and the cylindrical surface is the *lateral surface* of the cylinder. The distance between the two bases is the *altitude* of the cylinder.



608. Right Section. A section of a cylinder made by a plane perpendicular to an element is a *right section.*

609. Circular Cylinder. A cylinder the bases of which are circles is a *circular cylinder.* The line joining the centers of the bases is the *axis* of the cylinder.

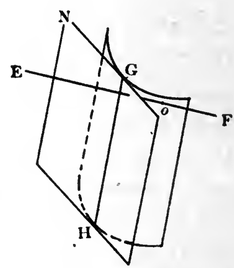
610. Right Cylinder. A cylinder the bases of which are perpendicular to the elements is a *right cylinder.*

611. Oblique Cylinder. A cylinder in which the elements are not perpendicular to the bases is an *oblique cylinder.*

612. Cylinder of Revolution. A cylinder generated by the revolution of a rectangle about one of its sides is a *cylinder of revolution.*

613. Similar Cylinders. Cylinders generated by the revolution of similar rectangles about homologous sides are *similar cylinders.*

614. Tangents to a Cylinder. If a *line* touches the lateral surface of a cylinder but does not intersect it, it is *tangent* to the cylinder. If a *plane* embraces an element of a cylinder but does not intersect the surface, it is *tangent* to the cylinder.



EF is tangent to the cylinder at O and plane N is tangent in the element GH.

Preliminary Theorems.

615. Theorem I. The elements of a cylinder are parallel and equal.

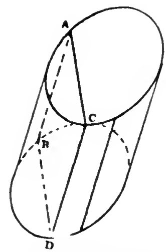
616. Theorem II. A right section of a cylinder is perpendicular to every element.

617. Theorem III. A cylinder of revolution is a right circular cylinder.

1. Sect a is perpendicular to sect b. If b is revolved about its free extremity in one plane what figure is formed by a?

PROPOSITION I.

618. THEOREM. *Every section of a cylinder made by a plane embracing an element is a parallelogram.*



Given the plane AD embracing the element AB.

To Prove that plane AD intersects the surface of the cylinder in ▱ $ABCD$.

Proof. SUG. 1. Let D be the point in which the plane cuts the perimeter of the base the second time and through D draw the element DC. How does DC lie with respect to BA? Why?

2. DC and BA determine a plane containing BA and D. Why?

3. How many planes can contain BA and D? Where then does DC lie with respect to the given plane ABD? With respect to the intersection of the plane and the cylinder?

4. Complete the proof by showing that $ABDC$ is a ▱.

Therefore—

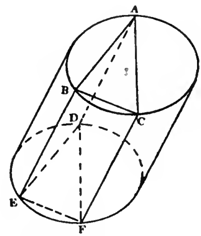
619. COR. *A plane containing an element of a cylindrical surface without being tangent intersects the surface in a second element also.*

620. COR. II. *Every section of a right cylinder by a plane which embraces an element is a rectangle.*

1. Cut a cylinder of revolution by a plane parallel to an element in such a manner that the section shall be a rectangle congruent to the rectangle which generates the cylinder.

PROPOSITION II.

621. THEOREM. *The bases of a cylinder are congruent.*



Given a cylinder *AD* with bases *ABC* and *DEF*.

To Prove bases *ABC* and *DEF* congruent.

Proof. SUG. 1. Let *A, B, C* be any three points in the perimeter of one base and draw the element *AD*. Pass planes through the element *AD* and the points *B* and *C*, intersecting the cylinder in the elements *BE* and *CF* respectively. *ADEB, ADFC, BCFE* are ▱s. Why?

2. Compare $\triangle$s *ABC* and *DEF*. Auth.

3. Superimpose $\triangle ABC$ upon $\triangle DEF$. As *A, B,* and *C* are any three points of the perimeter *ABC,* where does the perimeter *ABC* fall?

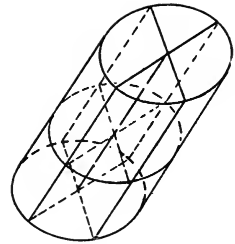
4. Compare the two bases.

Therefore—

622. COR. I. *Any two parallel sections cutting the elements of a cylinder are congruent.*

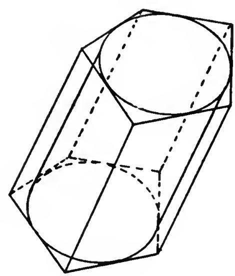
623. COR. II. *All sections of a cylinder parallel to the bases are congruent to the bases.*

624. COR. III. *The axis of a circular cylinder passes through the centers of all sections which are parallel to the bases.*

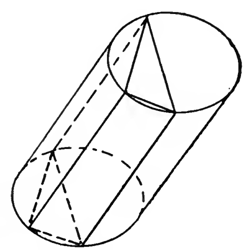


SUG. Draw any two diameters in one base. Pass planes through these two diameters and the elements at their extremities. Where will these two planes intersect the section? The other base? Where does the axis lie with respect to these planes?

1. A plane tangent to a cylinder of revolution is perpendicular to the plane of any right section.



625. CYLINDER INSCRIBED IN A PRISM. If each lateral face of a prism is tangent to a cylinder and the bases of the prism are circumscribed about the corresponding bases of the cylinder, the prism is *circumscribed about the cylinder* and the cylinder is *inscribed in the prism.*



626. CYLINDER CIRCUMSCRIBED ABOUT A PRISM. If each lateral edge of a prism is an element of a cylinder and the bases of the prism are inscribed in the corresponding bases of the cylinder, the prism is *inscribed in the cylinder* and the cylinder is *circumscribed about the prism.*

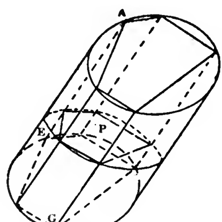
627. POSTULATE I. *The volume of a circumscribed prism, the areas and perimeters of its bases and sections, and its lateral area are greater than the corresponding parts of an inscribed cylinder.*

628. Postulate II. *The volume of an inscribed prism, the areas and perimeters of its bases and sections, and its lateral area are less than the corresponding parts of a circumscribed cylinder.*

629. Postulate III. *If the number of sides of a prism inscribed in a cylinder or circumscribed about a cylinder be indefinitely increased in such a manner that the sides of the prism be all indefinitely decreased, the volume, lateral surface, bases, sections, perimeters of bases and sections of the cylinder are the limits of the corresponding parts of the prism.*

PROPOSITION III.

630. Theorem. ***The area of the lateral surface of a cylinder is equal to the perimeter of a right section multiplied by an element of the surface.***



Given cylinder AG, its lateral area denoted by S, the perimeter of the right section by p, and the element AE by e.

To Prove $S = p \times e$.

Proof. Sug. 1. Inscribe in the cylinder a prism, denoting its lateral area by S', the perimeter of its right section p by p'. Its edge is e. Why?

2. Then $S' = p' \times e$. Why?

3. Indefinitely increase the number of

faces in the manner indicated in § 629. What are the limits of S', p', and $p' \times e$?

4. $\therefore S = p \times e$. Why?

Therefore—

631. Cor. I. *The lateral area of a cylinder of revolution is equal to the product of the circumference of the base and an element, or the altitude. i. e.* $S = 2\pi rh$.

632. Cor. II. *The lateral areas of similar cylinders are to each other as the squares of their altitudes and as the squares of the radii (or diameters) of their bases.*

Sug. Denote the lateral areas by S_1 and S_2, the altitudes by h_1 and h_2, the radii by r_1 and r_2, the diameters by d_1 and d_2 respectively.

Then $$\frac{S_1}{S_2} = \frac{2\pi r_1 h_1}{2\pi r_2 h_2} = \frac{r_1 h_1}{r_2 h_2} = \frac{r_1}{r_2} \times \frac{h_1}{h_2} = \frac{r_1{}^2}{r_2{}^2} = \frac{h_1{}^2}{h_2{}^2} = \frac{d_1{}^2}{d_2{}^2}.$$

Give the reasons for each statement.

633. Cor. III. *The total area, A, of a cylinder of revolution, which is the sum of the lateral area and the areas of the two bases, is given by the formula* $A = 2\pi rh + 2\pi r^2 = 2\pi r\,(r + h)$.

634. Cor. IV. *The total areas of two similar cylinders are to each other as the squares of their altitudes, and as the squares of their radii (or diameters).*

Proof left to the student.

1. How many feet of lumber will it take to build the walls of a circular silo 15′ in diameter and 25′ high, allowing $\frac{1}{5}$ for matching?

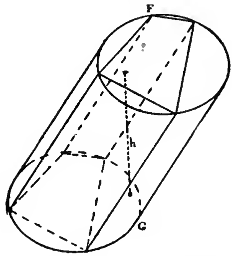
2. A cistern is 10′ deep and 8′ in diameter. How many square yards of cement is needed to line it?

3. The total area of a right circular cylinder is 80π sq. ft. and the radius of the base is 5 ft. Find the altitude.

4. Which generates the greater lateral area, the revolution of a rectangle about its longer side or about its shorter side?

PROPOSITION IV.

635. THEOREM. *The volume of a cylinder is equal to the product of its base and its altitude.*



Given a cylinder FG, its volume denoted by V, its base by B, and its altiude by h.

To Prove $V = B \times h$.

Proof. SUG. 1. Inscribe in the cylinder a prism, its volume denoted by V' and its base by B'. Its altitude will be h. Why?

2. Then $V' = B' \times h$. Why?

3. Indefinitely increase the number of sides of the prism in the manner indicated in § 629. What is the limit of V'? Of B'? Of the product $B' \times h$? Why?

4. Complete the demonstration.

Therefore—

636 COR. I. *The volume of a cylinder of revolution is expressed in terms of its altitude h and radius r by the formula $V = \pi r^2 h$.*

637. COR. II. *The volumes of two similar cylinders of revolution are to each other as the cubes of their altitudes and as the cubes of their radii (or their diameters).*

Proof left to the pupil. See § 632.

1. What must be the shape of a piece of paper which exactly covers the lateral surface of an oblique cylinder? Of a right cylinder?

2. A cistern holding 75 bbls. is 8 ft. deep. What is its diameter? How many square yards of cement are required to line it?

3. A cylindrical tank on a water works tower has a lateral area of 1,232 sq. ft. The radius of its base is $\frac{1}{4}$ the altitude. Find the altitude and the radius. Find the total area. What would be the lateral area if the altitude were doubled. Find its total area under the same condition.

4. What is the locus of a point in space at a given distance from an unlimited straight line?

5. A saw log is 16′ long and 18″ in diameter at the small end. How many board feet in the largest squared timber that can be sawed from it?

6. A cylindrical water reservoir is 110′ in diameter and 20′ deep. How many bbls. does it hold? How many square yards of cement are necessary to line it?

7. A rock submerged in a tank 29′ in diameter and 8′ deep raised the water 2′. What was the volume of the rock?

8. A farmer feeds his 60 cows 2 bu. of ensilage each a day. The silo is cylindrical, 30′ in diameter and 40′ deep. How long will the ensilage last?

9. There are wine tanks 15′ deep and 10′ in diameter. How many quart bottles can be filled from one of them?

10. What is the ratio of the lateral area of a right circular cylinder to the sum of the bases?

11. What is the locus of a sect at a given distance from a straight line to which it is parallel?

12. What is the locus of point X which is at a given distance from a straight line and equally distant from two fixed points?

13. What is the locus of a point at a given distance from the lateral surface of a cylinder? Is it possible for one element of the locus to lie within the cylinder? Under what condition?

14. From a given point without a cylinder draw a plane tangent to the cylinder.

Review.

638. State the formula for

1. The lateral area of a cylinder of revolution.
2. The total area of a cylinder of revolution.
3. The volume of a cylinder of revolution.
4. The ratio of the lateral area of two similar cylinders.
5. The ratios of the lateral areas of any two cylinders of revolution.
6. The ratios of the volumes of two similar cylinders.

1. A protecting wall for an embankment 500′ long is 12′ wide at the bottom, 3′ wide at the top, and 18′ high. How many cubic yards of stone in the wall?

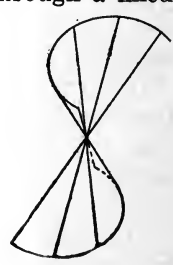
2. How many brick, 2″ × 4″ × 8″, are required to build a chimney two bricks thick with a flue 8″ × 12″ and 25′ high, if the mortar averages $\frac{1}{8}$″ thick? Make a drawing of two tiers of brick when properly laid.

3. If from any point in an equilateral triangle, perpendiculars be drawn to the sides, the sum of these three perpendiculars equals the altitude of the triangle.

4. If from any point within a regular tetrahedron perpendiculars be drawn to the four faces, the sum of these perpendiculars equals the altitude of the tetrahedron.

Cones.

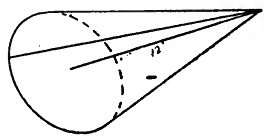
639. Conical Surface. A curved surface formed by a moving straight line which passes through a fixed point and continually touches a fixed curve is a *conical surface*. The moving straight line is the *generatrix* and the fixed curve is the *directrix*. The generatrix in any one of its positions is an *element* of the surface. The fixed point is the *vertex* of the conical surface. If the directrix is a closed



curve, the conical surface is a *closed* surface. The portions of the conical surface on the two sides of the vertex are the *nappes* of the conical surface and are designated as *upper* and *lower nappes.*

Usually only one nappe of a conical surface is considered.

640. A Cone. A solid bounded by one nappe of a closed conical surface and a plane cutting all the elements is a *cone.* The plane section is the *base* of the *cone.* The bounding portion of the conical surface is the *lateral surface* of the cone and the vertex of the conical surface is the vertex of the cone. The portions of the elements of the conical surface between the vertex and the base are the *elements* of the cone. The distance from the vertex to the plane of the base is the *altitude* of the cone.



641. Circular Cone. A cone with a circular base is a *circular cone,* and the straight line joining the vertex to the center of the base is the *axis of the circular cone.*

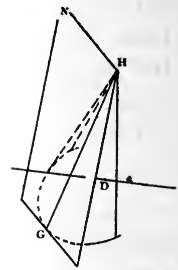
642. Right Cone. A cone such that the axis is perpendicular to the base is a *right cone.*

643. Oblique Cone. A cone such that the axis is not perpendicular to the base is an *oblique cone.*

644. Cone of Revolution. A cone generated by the revolution of a right triangle about one leg as an axis is a *cone of revolution.* Any element of a cone of revolution is its *slant height.*

645. Similar Cones. Cones of revolution that can be generated by the revolution of similar right triangles about homologous sides are *similar cones.*

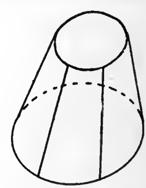
646. TANGENTS TO A CONE. A line which touches the conical surface in a point but does not intersect it is a *tangent line* to the cone. A plane which embraces an element but does not intersect the conical surface is a *tangent plane* to the cone.



Line *a* is tangent to the cone at the point of tangency *D* and plane *N* is tangent to the cone along the element *GH*.

647. A TRUNCATED CONE. That portion of a cone included between the base and a plane cutting all the elements is a *truncated cone.*

648. A FRUSTUM OF A CONE. A truncated cone such that the cutting plane is parallel to the base is a *frustum of a cone.* The base of the cone is the *lower base* of the frustum, the section is the *upper base,* the distance between the two parallel planes is the *altitude.* If the frustum be a portion of a cone of revolution, the portion of the slant height of the cone included between the bases of the frustum is the *slant height of the frustum*



PRELIMINARY THEOREMS.

649. THEOREM I. *Every cone of revolution is a right circular cone.*

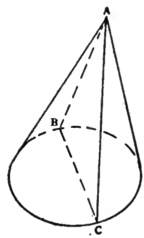
650. THEOREM II. *The axis of a cone of revolution is its altitude.*

651. THEOREM III. *All elements of a cone of revolution are equal (i. e. the slant height is constant to the hypotenuse of the generating triangle).*

1. Two circular cylinders have the same altitude, but the volume of one is four times that of the other. Find the ratio of their radii.

PROPOSITION V.

652. Theorem. *Every section of a cone embracing the vertex is a triangle.*



Given a plane embracing the vertex A and intersecting the perimeter of the base in the points B and C.

To Prove that the $\triangle ABC$ is the section made by the plane.

Proof. Sug. 1. The plane must cut the perimeter in two points as B and C.

2. AB and AC are each in both the plane and the conical surface.

3. AB and AC are intersections of the plane and the conical surface.

4. What kinds of lines are AB and AC?

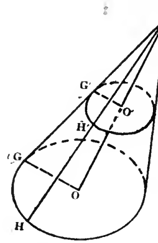
Therefore—

653. Cor. I. *A plane containing an element of a conical surface without being tangent intersects the surface in a second element also.*

2. In each of two right circular cylinders the altitude is equal to the diameter and the volume of one is $\frac{1}{27}$ that of the other. Find the ratio of the altitudes.

PROPOSITION VI.

654. THEOREM. *Every section of a circular cone made by a plane parallel to the base is a circle.*



Given $F - OGH$ a circular cone with base $O - GH$ and a section $O' - G'H'$ parallel to the base, O', G', H' being the points in which the plane intersects the elements FG, FH, and the axis FO respectively.

To Prove $O' - G'H'$ a ⊙.

Proof. SUG. 1. O is the center of the base and G and H are taken as any two points on the perimeter of the base. The figures $FG'GOO'$ and $FH'HOO'$ are plane figures. Why?

2. Compare the △ $FG'O'$ and FGO; △ $FH'O'$ and FHO; ratios $\frac{FO'}{FO}$, $\frac{O'G'}{OG}$ and $\frac{O'H'}{OH'}$. Auth.

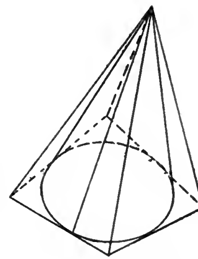
3. Compare $O'G'$ and $O'H'$.

4. ∴ $O' - GH$ is a circle. Why?

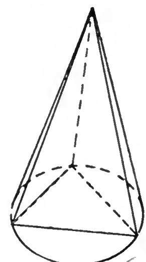
Therefore—

655. COR. *The axis of a circular cone passes through the centers of all sections parallel to the base.*

What is the character of the polygon which revolved about one of its sides generates a frustum of a cone of revolution?



656. Cone Inscribed in a Pyramid. If the vertex of a pyramid coincides with the vertex of a cone and the base of the pyramid is circumscribed about the base of the cone, the pyramid is *circumscribed about the cone* and the cone is *inscribed in the pyramid.*



657. Cone Circumscribed About a Pyramid. If the vertex of a pyramid coincides with the vertex of a cone and the base of the pyramid is inscribed in the base of the cone, the pyramid is *inscribed in the cone* and the cone is *circumscribed about the pyramid.*

Preliminary Theorems and Postulate.

658. Theorem I. *The faces of a pyramid circumscribed about a cone are tangent to the cone.*

659. Theorem II. *The edges of a pyramid inscribed in a cone are elements of the cone.*

660. Theorem III. *The slant height of a regular pyramid circumscribed about a right circular cone equals the slant height of the cone.*

661. Postulate I. *The volume of a circumscribed pyramid, the area and perimeter of its base, and its lateral area are greater than the corresponding parts of an inscribed cone.*

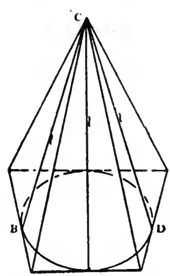
662. Postulate II. *The volume of an inscribed pyramid, the area and perimeter of its base, and its lateral area are less than the corresponding parts of a circumscribed cone.*

663. Postulate III. *If the number of sides of a pyramid inscribed in or circumscribed about a cone be indefinitely increased in such a manner that the faces of the pyramid be all indefinitely decreased, the volume, lateral surface, base, perimeters of the base and sections of the cone are the limits of the corresponding parts of the pyramid.*

State one kind of inscribed or circumscribed prisms by which the two conditions underlying the limiting process in Postulate III may be set up in one operation.

PROPOSITION VII.

664. Theorem. *The lateral area of a cone of revolution is equal to one-half the product of the perimeter of its base and its slant height.*



Given $C—BD$ a cone of revolution, its lateral area denoted by S, the perimeter of its base by p, and its slant height by l.

To Prove $S=\frac{1}{2}p\times l$.

Proof. Sug. 1. Circumscribe about the cone a regular pyramid, with lateral area denoted by S', perimeter of the base by p'. The slant height of the pyramid is l. Why?

2. Determine the lateral area S' in terms of p' and l.

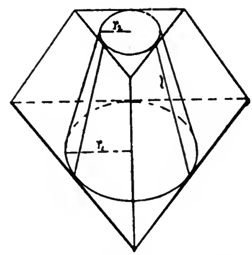
3. Complete the demonstration. See the method of § 630.

Therefore—

665. Cor. I. *If r be the radius of the base of a cone of revolution, then the lateral area is given by the formula* $S=\frac{1}{2}l\times 2\pi r=\pi rl$.

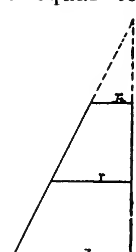
666. Cor. II. *The lateral areas of two similar cones are to each other as the squares of their altitudes and as the squares of the radii (or diameters) of their bases.*

Sug. See method of § 632.



667. Cor. III. *The lateral area of the frustum of a cone of revolution is equal to one-half the product of its slant height and the sum of the perimeters of its bases.*

Sug. If a regular pyramid be circumscribed about the cone from which the frustum is cut, the cutting plane will cut from the pyramid a frustum which will be circumscribed about the frustum of the cone. Denote the radii of the two bases by r_1 and r_2, the lateral area by S and the slant height by l. Prove that S is equal to $\pi l\,(r_1+r_2)$.

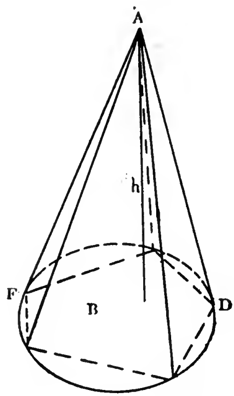


668. Cor. IV. *If r represent the radius of the section of a frustum of a cone of revolution midway between the bases, prove* $S=2\pi rl$.

Sug. Show $r=\frac{1}{2}(r_1+r_2)$.

PROPOSITION VIII.

669. Theorem. *The volume of a cone is equal to one-third the product of its base and its altitude.*



Given the cone $A-FD$ with altitude h, base B, and volume V.

To Prove $V=\frac{1}{3}B\times h$.

Sug. 1. Inscribe in the cone a pyramid, denoting its base by B', and its volume by V'. Its altitude will be h. Why?

2. Complete the proof by an adaptation of the method of Prop. VII.

Therefore—

670. Cor. I. *The volume of a circular cone is given by the formula* $V=\frac{1}{3}\pi r^2 h$.

671. Cor. II. *The volumes of similar cones are to each other as the cubes of the radii of their bases, as the cubes of their altitudes, and as the cubes of their slant heights.*

SUG. Adapt the method of § 666.

672. COR. III. *The volume of a frustum of a cone of revolution is equal to one-third the product of the altitude and the sum of the upper base, the lower base, and a mean proportional between the two bases, i. e.* $V=\frac{1}{3}h\,(B_1+B_2+\sqrt{B_1B_2})$ *in which* B_1 *and* B_2 *denote the two bases.*

SUG. Inscribe in the original cone a pyramid. The frustum of the pyramid made by the cutting plane of the cone will be inscribed in the frustum of the cone. Proceed as in Cor. III, Prop. VII.

673. COR. IV. *The volume of a frustum of a cone of revolution is given by the formula* $V=\frac{1}{3}\pi h\,(r_1{}^2+r_2{}^2+\sqrt{r_1\times r_2})$.

REVIEW.

674. State the formulas for

1. The area of a circle.
2. The lateral area of a cone of revolution.
3. The lateral area of the frustum of a cone of revolution.
4. The volume of a circular cone.
5. The ratio of the volumes of two similar cones.
6. The ratio of the lateral areas of two similar cones.

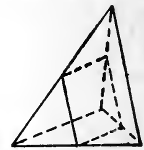
1. A granite monument is 30′ high. The base is 4′ in diameter. It diminishes gradually in size to a cross section circle 15″ in diameter, 6′ from the top. The remainder of the monument is conical. Find its volume in cubic yards and its weight in tons, the specific gravity being 2.65.

2. How many sq. yds. of canvas in a conical tent $12\frac{1}{2}'$ high with a base diameter of 12′?

3. Find the volume of the solid generated by the revolution upon one of its sides as an axis of an equilateral triangle with an edge of 6′.

4. Find the ratios of the volumes generated by the revolution of a right triangle about side a, about side b, and about hypotenuse c respectively.

5. The altitude of the frustum of a cone of revolution is $\frac{1}{3}$ the altitude of the cone. What is the ratio of their volumes?



6. A section of a tetrahedron cutting four edges at their mid points is a parallelogram.

7. A grain bin is $6' \times 12' \times 20'$. How many bushels does it hold?

8. The radii of the two bases of a frustum of a cone are 4′ and 9′ respectively. Find the volume.

Indicate and work mentally.

9. What is the locus of a rod 12′ long, one end being stationary on the ceiling of a room 8′ high and the other end resting on the floor?

10. Find the area of the plane figure enclosed on the floor in the preceding exercise and the volume enclosed by the floor and the locus.

11. A tent has a diameter of 16′ and a vertical side wall of 5′. The covering is conical in shape and the center pole is 12′ high. How many cubic feet of air does the tent contain?

12. Find the lateral area, total area, altitude, and volume of the cone that can be circumscribed about a regular tetrahedron with an 8″ edge.

13. A cylinder and a cone of revolution have the same altitude and concentric bases. The diameter of the base of the cone is three times that of the cylinder. How far from the vertex of the cone do the two lateral surfaces intersect? Suppose the diameter of the cylinder to be $\frac{1}{4}$ that of the cone, where do the surfaces intersect?

14. What is the locus of lines making a given angle with a given line at a given point in the line?

15. The altitude of a cone is trisected by planes parallel to the base. Compare the parts into which the cone is divided. Make a similar comparison when the altitude is divided into four equal parts.

16. What kind of a triangle is the section of a right cone through the vertex?

17. What is the volume of a piece of timber 15′ long, the bases being squares of 12″ and 14″ respectively?

18. If four similar cylinders have their altitudes proportional to 3, 4, 5, 6, prove that the volume of the largest one equals the sum of the volumes of the three others.

19. A log 20′ long has a diameter at the smaller end of 16″. What proportion of the log is cut into slabs if the largest possible squared stick of timber is sawed from it? What proportion will be slabs if the largest rectangular stick be sawed, the edges having the ratio of 3 to 4? Of 2 to 3?

20. A cylindrical vessel with a diameter equal to its altitude holds $1414\frac{2}{7}$ cu. ft. of water. What are its dimensions?

21. What are the dimensions of a cylindrical vessel holding $1414\frac{2}{7}$ cu. ft. if its altitude is twice its diameter? What are the dimensions if the ratio of the diameter and the altitude is $\frac{2}{5}$?

22. What are the dimensions of a quart cup if the ratio of its diameter and its altitude is $\frac{4}{5}$?

23. What is the diameter of a cylindrical peck measure 8″ deep?

24. Allowing two inches for the seam, how large a sheet of iron will be required for a joint of 8″ stove pipe? Of 6″ stove pipe?

25. Measure the diameter of a quart cup and compute its depth. Verify by measurement.

26. A rectangle is 5 x 8. Which side as an axis will produce the greater cylinder of revolution? What is the ratio of the two cylinders?

27. Find the lateral area of a regular pentagonal pyramid with basal edge of two feet and slant height of 1 yd.

28. Find the total area of a regular quadrangular pyramid with basal edge of 12′ and lateral edge of 10′. Find the volume.

29. The base of a regular pyramid is a square with a side of 6′ and the lateral area is $\frac{5}{8}$ of the total area. Find the altitude and the slant height.

30. What is the ratio of the four parts into which a pyramid is divided by parallel planes dividing the altitude into four equal parts?

31. How much of a cube is cut off by a plane passing through the mid-points of three concurrent edges? What part is removed when this is done for each set of three concurrent edges?

32. If a plane is passed through the extremities of three concurrent edges of a cube, prove (1) that the tetrahedron cut off is $\frac{1}{6}$ of the cube; (2) that four such tetrahedrons can be cut off of a cube; (3) that the remainder of the cube is a regular tetrahedron equal to $\frac{1}{3}$ of the cube.

Sug. Take a as the edge of the cube. Determine the bases and altitudes of the figures and compute the volumes.

33. Use the rule for finding the volume of a pyramid in order to find the volume of an 8″ cube.

34. How many cubic feet are removed in boring a 6″ well 180′ deep?

35. How large an object at a distance of 20′ will be obscured by an inch square placed $2\frac{1}{2}$′ from the eye?

36. How many inches from the vertex of a cone of revolution 12′ high must a plane be passed parallel to the base in order to bisect the cone? In order to trisect it, at what distances must the two planes be passed?

37. The interior of a rectangular bin with edges in the ratio of 2, 3, 4 has a surface of 672 sq. ft. How many bushels does it hold?

The Sphere.

675. A Sphere. A solid bounded by a surface all points of which are equally distant from a fixed point within is a *sphere*.

The fixed point is the *center* of the sphere and the bounding surface is the *surface* of the sphere.

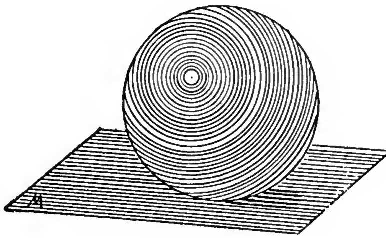
676. Radius of a Sphere. Any straight line drawn from the center of a sphere to its surface is a *radius* of the sphere.

677. Diameter of a Sphere. Any straight line drawn through the center of a sphere and terminated by the surface is a *diameter* of the sphere.

678. A Line Tangent to a Sphere. A line which

touches a sphere at one and only one point is a *tangent* to the sphere.

679. A Plane Tangent to a Sphere. A plane which touches a sphere at one and only one point is *tangent* to the sphere.



680. Point of Tangency. The point in which a tangent line or a tangent plane touches a sphere is the *point of tangency* or *point of contact*.

Two spheres are tangent when they have one and only one point in common.

Preliminary Theorems.

681. Theorem I. *All radii of a sphere are equal and all diameters of a sphere are equal.*

682. Theorem II. *Two spheres are equal if their radii are equal.*

In the case of spheres, equality implies congruence.

683. Theorem. III. *The radii of two equal spheres are equal.*

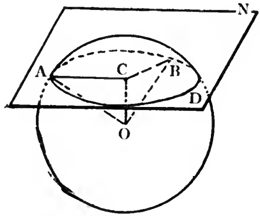
684. Theorem IV. *The revolution of a semicircle about its diameter generates a sphere.*

685. Theorem V. *A straight line which intersects a sphere intersects it in two points.*

686. Theorem VI. *A plane or a sphere which intersects a sphere intersects it in a closed line.*

PROPOSITION IX.

687. THEOREM. *Every section of a sphere by a plane is a circle.*



Given the plane N intersecting the sphere O in the closed line ABD.

To Prove ABD a circle.

Proof. SUG. 1. Drop a $\perp$ to plane N from the center O, meeting plane N in C. Let A and B be any two points in the perimeter of the section. Draw AC and BC.

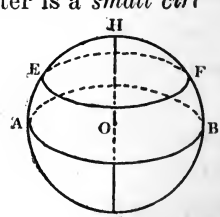
2. Compare AC and BC.

3. $\therefore ABD$ is a circle. Why?

Therefore—

688. A GREAT CIRCLE OF A SPHERE. A circle of a sphere which contains the center of the sphere is a *great circle of the sphere.*

689. A SMALL CIRCLE OF A SPHERE. A circle of a sphere which does not contain the center is a *small circ of the sphere.*



690. AXIS OF A CIRCLE OF A SPHERE. The diameter of a sphere perpendicular to a circle of a sphere is the *axis of the circle* of the sphere.

691. Poles of a Circle. The extremities of the axis of a circle are its *poles*.

AB is a great circle, *EF* is a small circle, *GH* is the axis of ⊙ *EF* and also of ⊙ *AB* if the two Ⓢ are parallel, *G* and *H* are the poles.

692. Cor. I. *The axis of a circle intersects it at its center.*

693. Cor. II. *Two circles of a sphere equally distant from the center are equal.*

694. Cor. III. *Of two circles of a sphere unequally distant from the center, that one which is nearer the center is the greater.*

695. Cor. IV. *The center of the sphere is the center of every great circle of the sphere.*

696. Cor. V. *All great circles of a sphere are equal.*

697. Cor. VI. *Two great circles of the same sphere intersect in a common diameter.*

698. Cor. VII. *Two great circles of the same sphere bisect each other, the sphere, and the surface of the sphere.*

699. Cor. VIII. *Any three points on the surface of a sphere determine a circle of the sphere.*

Sug. How is a plane determined?

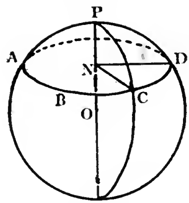
700. Cor. IX. *Through two points on the surface of a sphere, not the extremities of a diameter, one and only one great circle can be drawn.*

701. Distance on a Sphere. The *distance* between two points on the surface of a sphere is the shorter arc of the great circle joining them.

1. A plane embracing the axis of a circle is perpendicular to the circle. If the arc of a circle on a sphere is bisected by a plane embracing the axis, every point in the plane is equidistant from the extremities of the arc.

PROPOSITION X.

702. THEOREM. *All points on a circle of a sphere are equally distant from each of its poles.*



Given $ABCD$, a ⊙ of sphere O, P and P' being its poles, C and D any two points on the ⊙, PC and PD great circle arcs.

To Prove arc $PC =$ arc PD and arc $P'C =$ arc $P'D$.

Proof. SUG. 1. P and C determine a great circle which also passes through P'. The same is true of P and D. Why?

2. Draw CN and DN, radii of ⊙ ACD.

3. Compare the chords PC and PD.

4. Compare the arcs PC and PD.

5. Compare arcs PCP' and PDP'; arcs $P'C$ and $P'D$.

Therefore—

703. POLAR DISTANCE. The distance on the surface of a sphere from a circle to its nearer pole is the *polar distance of the circle.*

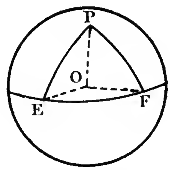
The polar distance of a circle is less than a great semicircle.

704. COR. I. *The polar distance of a great circle is a quadrant, i. e. an arc of ninety degrees.*

SUG. What angle at the center of the sphere subtends the polar distance of a great circle?

PROPOSITION XI.

705. THEOREM. *A point which is at the distance of a quadrant from each of two points on the surface of a sphere, not the extremities of a diameter, is a pole of the great circle embracing those points.*



Given two points E and F on sphere O, E and F not the extremities of a diameter, and a third point P such that arc $PE =$ arc $PF =$ a quadrant.

To Prove P is a pole of the great circle EF.

Proof. SUG. 1. E and F determine a great circle. Why?

2. Join E and F to O, the center of ⊙ EF, and P to O.

3. What are the angles POE and POF? Why?

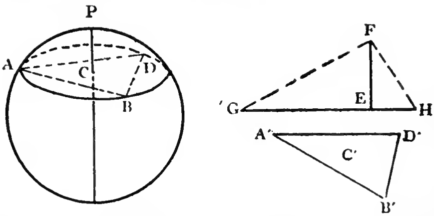
4. What is PO with regard to ⊙ EF? What is P? Why?

Therefore—

NOTE—By the truth of Prop. III arcs of great circles can be drawn on the surface of a sphere in a manner similar to that by which arcs can be drawn on a plane with given radii and centers. To do this a point is located at a quadrant's distance from the two points through which the great circle is to pass. If the dividers be now opened so as to span the chord subtending a great circle quadrant and one point placed on the fixed point as a pole, the free end can be made to describe the desired great circle. The means for determining the quadrant and its chord will be found in §706.

PROPOSITION XII.

706. Problem. *Given a material sphere, to find its radius or its diameter.*



Sug. 1. Take any point P on the surface of the sphere as a pole and with the dividers describe a circle C.

2. Take any three points A, B, D on this circle and by means of the dividers construct a $\triangle A'B'D'$ congruent to $\triangle ABD$.

3. Determine the center C' of the circle circumscribed about $\triangle A'B'D'$.

4. Draw EF equal to the radius $C'A'$ and through E draw an unlimited straight line $\perp EF$.

5. From F lay off $FH =$ chord PB and at F erect $FG \perp FH$, G being its intersection with EH.

6. Prove that GH is the diameter of the sphere and find the radius.

707. Cor. *Given a diameter of a sphere, a quadrant of the sphere can be determined.*

Proof left to the pupil.

1. Construct on a plane a circle equal to a great circle of a given sphere.

2. Determine the diameter of a base ball, a croquet ball, or some other spherical object.

3. Construct on a plane a circle equal to a given small circle on a sphere.

4. To draw a great circle that shall bisect an arc of a circle
Sug. Locate two points equidistant from the extremities.

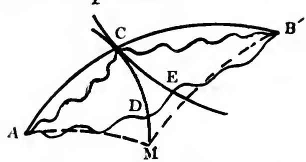
5. Describe a great circle through two points on the surface of a sphere.

6. Construct a small circle on the surface of a sphere through three given points.

7. Compare the polar distances of equal circles on the same sphere.

PROPOSITION XIII.

708. Theorem. *The shorter of the two great circle arcs joining two points on the surface of a sphere is less than any other line on the sphere which joins the two points.*



Given two points A and B on the surface of the sphere, AB the great circle arc joining them, and ADB any other line on the surface from A to B.

To Prove that arc AB is shorter than line ADB.

Proof. Sug. 1. Take C any point on the arc AB and with A and B as poles describe circles with AC and BC as their respective polar distances. Let M be any point on the first of these circles except point C. Join M to A and to B by great arcs. Then $AM + MB > AB$. Why? $AC = AM$ and hence $BM > BC$. Consequently M does not lie on the second circle and the two circles have but the one point C in common.

2. Let D and E be the points in which the two circles meet the line ADB. As A is the pole of the $\odot$ CD, a line may be drawn from A to C congruent to the line AD and one can be drawn for a like reason from B to C congruent to line BE. That is, a line can be constructed from A to B through C which is shorter than the given line ADB by the line DE. Thus, no matter what line be drawn from A to B other than arc AB, C, and hence every point of arc AB, lies in a line still shorter. Therefore every point of arc AB lies in the shortest line from A to B.

3. If now D be any point not on arc AB it cannot lie on the shortest line from A to B. This is seen if the circle about A as pole and with radius AD be drawn determining a point C by its intersection with arc AB. For then, as above, there can be drawn another line from A to B through C which is shorter than any line which can be drawn through D.

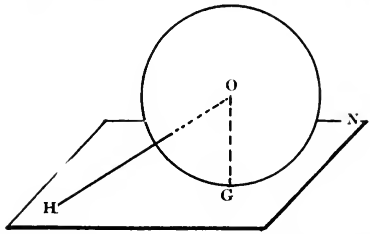
4. Hence AB must be the shortest line.

Therefore—

709. Inasmuch as the great circle arc is the shortest line between two points on the surface of a sphere, distances on a sphere are measured along great circle arcs. It will be natural then to expect on the surface of a sphere a surface or spherical geometry corresponding to plane geometry with figures formed from great circle arcs instead of straight lines. This will appear in the sequel.

PROPOSITION XIV.

710. Theorem. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*



Given sphere O, radius OG and plane $N \perp OG$ at G

To Prove plane N a tangent plane to the sphere.

Proof. Sug. 1. Let H be any point in plane N other than G and draw OH.

2. Where does H lie with respect to the sphere? Why?

3. Complete the demonstration.

Therefore—

711. Cor. I. *A plane tangent to a sphere is perpendicular to the radius at the point of contact.*

712. Cor. II. *Any straight line through the point of tangency and in the tangent plane is tangent to the sphere.*

713. Cor. III. *Any straight line perpendicular to a radius at its extremity is tangent to the sphere.*

Note. Of all figures in plane geometry the circle most closely resembles the sphere in its characteristics. It is interesting to note the resemblance of certain propositions and their proofs in plane and in solid geometry. The change of two words in Prop.XIV makes the theorem one of plane geometry. Compare the demonstrations of the two theorems.

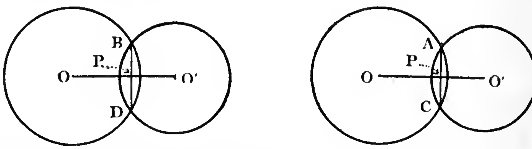
Such comparisons of plane and solid geometry theorems make an excellent review of much of the subject and will also in many instances serve to make more clear the meaning and the demonstrations of solid geometry theorems.

1. Construct a plane tangent to a sphere at a given point.

2. What is the locus of a point at a given distance from a given point and also equidistant from two other given points. Is the problem always possible? Is the locus ever a single point?

PROPOSITION XV.

714. THEOREM. *The intersection of the surfaces of two spheres is a circle.*



Given two spheres O and O' intersecting in a closed line, two points of which are A and B.

To Prove line AB is a circle.

Proof. SUG. 1. The plane AOO' intersects the two spheres in two great circles intersecting each other in the points A and C. Likewise the plane BOO' intersects the two spheres in two great circles which intersect each other in the points B and D. Why?

2. Chords AC and BD intersect OO' in the same point P. Why?

3. Compare $\triangle AOP$ with $\triangle BOP$. Auth. AP with BP.

4. AP and BP lie in the plane $\perp OO'$ at P. Why? Hence the closed line AB is a circle.

Therefore—

715. COR. I. *The line joining the centers of two intersecting spheres is perpendicular to the plane of their intersection and passes through the center of the circle of intersection.*

Cor. II. *The plane of the intersection of two equal spheres is the perpendicular bisector of their line of centers.*

1. What is the locus of points at given distances from two given points?

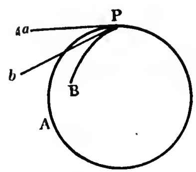
Discuss all possibilities.

2. Find a point X which is at given distances from two given points and equally distant from two other given points. Discuss all possibilities, showing that there are two, one, or no solutions.

3. Find a point X equidistant from two parallel lines, from two intersecting lines, and a given distance from a given point. What is the greatest number of solutions? What is the least number? Under what conditions is the problem impossible?

716. Angle of Two Arcs. The angle between the tangents of two arcs at their point of intersection is the *angle of the arcs.*

717. Spherical Angle. The angle formed by the arcs of two great circles is a *spherical angle.*



PA and *PB* are two great circles and *a* and *b* are tangent to the great circles respectively at *P*. The spherical angle *APB* equals the plane angle *aPb*.

718. Cor. *A spherical angle equals the dihedral angle formed by the planes of the great circles forming the spherical angle.*

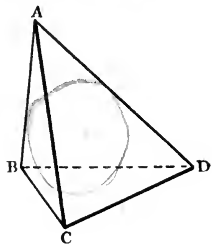
Sug. The tangents to the great circles are in the respective planes of the circles and perpendicular to the edge of the dihedral at the same point. Why?

719. A Circumscribed Sphere. When all the vertices of a polyhedron lie in the surface of a sphere, the sphere is *circumscribed* about the polyhedron and the polyhedron is *inscribed* in the sphere.

720. AN INSCRIBED SPHERE. When the faces of a polyhedron are all tangent to a sphere, the sphere is *inscribed* in the polyhedron and the polyhedron is *circumscribed* about the sphere.

PROPOSITION XVI.

721. THEOREM. *One and only one sphere can be circumscribed about any tetrahedron.*



Given a tetrahedron *ABCD.*

To Prove that one and only one sphere can be circumscribed about *ABCD,* i. e. to prove that there is one and only one point *X* equidistant from *A, B, C,* and *D.*

Proof. SUG. 1. What is the locus of points equidistant from *A* and *B*? From *B* and *C*? Auth.

2. Show that the two loci just found must intersect. What is then the locus of points equidistant from *A, B,* and *C*?

3. What is the locus of points equidistant from *C* and *D*?

4. Show that the loci in Sug. 2 and Sug. 3 intersect.

5. Show that this intersection is the required point *X*.

6. Show that only one such point exists.

Therefore—

PROPOSITION XVII.

722. Theorem. *One and only one sphere can be inscribed in a tetrahedron.*

Given a tetrahedron *ABCD*. § 721.

To Prove that one and only one sphere can be inscribed in *ABCD*.

Proof. Sug. 1. What is the locus of points equidistant from *ABC* and *ABD*? *ABC* and *ACD*? Auth.

2. Complete the demonstration on the outline of the demonstration § 721.

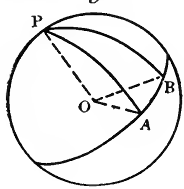
Therefore—

1. All lines tangent to a sphere from the same point are equal and touch the sphere in a circle of the sphere.

Sug. Connect the center of the sphere with the given point and with two or more points of contact.

PROPOSITION XVIII.

723. Theorem. *A spherical angle is measured by the arc of a great circle described from the vertex of the angle as a pole and intercepted between the sides of the angle.*



Given two great circle arcs *PA* and *PB* forming a spherical angle at *P*, *AB* being an arc of that great circle of which the pole is *P* intercepted by *AP* and *BP*.

To Prove that spherical angle *APB* is measured by arc *AB*.

Proof. Sug. 1. *OA* and *OB* are radii of the great circle *ABC* and lie respectively in the planes of the great circles *PA* and *PB*. What relation do they bear to *PO?* Auth.

2. What relation does $\angle AOB$ bear to the dihedral angle $A—PO—B$? Auth.

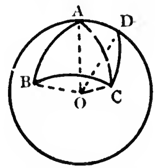
3. What relation does arc *AB* bear to $\angle AOB$? To $\angle APB$? Auth.

Therefore—

724. Spherical Polygon. A portion of the surface of a sphere bounded by arcs of great circles is a *spherical polygon.* The bounding arcs are the *sides* of the polygon, the intersections of the sides are the *vertices,* of the polygon, and the spherical angles are the *angles* of the polygon.

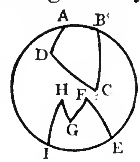
The planes of the sides of a spherical polygon form a polyhedral angle with vertex at the center of the sphere. The sides of a spherical polygon are great circle arcs and may be expressed in degrees of arc. They measure the corresponding face angles of the polyhedral. The spherical angles of a spherical polygon measure the dihedrals of the polyhedral. A *diagonal* of a spherical polygon is a great circle arc joining any two non-adjacent vertices.

A *diagonal* of a spherical polygon is a great circle arc joining any two non-adjacent vertices.



ABCD is a spherical polygon, *AB* is a side, *A* is a **vertex**, $O—ABCD$ is the subtended polyhedral angle, *AB* measures $\angle AOB$, $\angle A$ measures dihedral *OA* and *AC* is a diagonal.

725. SPHERICAL POLYGONS classified. Spherical polygons, like plane polygons, are classified as *triangular, quadrangular,* etc., according to the number of angles or sides. They may be *right-angled, isosceles, equilateral,* etc., as are plane triangles. They may be *congruent* in that they may be applied to each other as are congruent plane triangles for two great circle arcs coincide if two points are common to them, just as is true of straight lines. They may be *equal* in area without being congruent. In congruent spherical polygons the homologous parts are respectively equal. Equal spherical angles may be made to coincide.



726. CONVEX SPHERICAL POLYGON. A spherical polygon is *convex* when none of its sides if extended will cut the polygon.

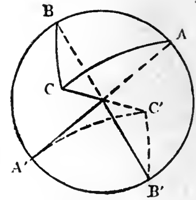
ABCD is convex.

727. CONCAVE SPHERICAL POLYGONS. A spherical polygon which is not convex is *concave* or *reentrant*. *EFGH* is concave.

728. SYMMETRICAL SPHERICAL TRIANGLES. Two spherical triangles such that the subtended trihedral angles are *symmetrical spherical trihedrals.* By 508 it is seen that the homologous parts of two symmetrical spherical trihedrals are equal but arranged in reverse order.

Symmetrical triangles exist in plane geometry but in that case either of them could be removed from the plane and turned over, thus reversing the order of its parts, and making the two triangles congruent. The distinction of congruent and symmetric was then unnecessary. On account of the curvature of the surface of a sphere, no portion of it can be turned over and applied to any other portion. A distinction between congruent and symmetrical is in this case necessary.

729. Opposite or Vertical Spherical Polygons. The polygons intercepted, or subtended, on the surface of a sphere by two opposite or vertical polyhedrals with their vertex at the center of the sphere are *opposite spherical polygons.*



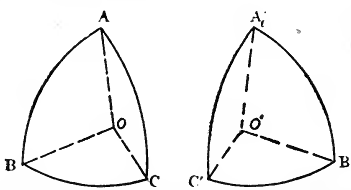
ABC and *A'B'C'* are opposite spherical polygons.

730. Cor. I. *Opposite spherical polygons are symmetrical.*

731. Cor. II. *Three planes not having a common line of intersection and all embracing the center of a sphere intersect the surface in two symmetrical spherical triangles.*

PROPOSITION XIX.

732. Theorem. *Two isosceles symmetrical triangles are congruent.*



Given two symmetrical spherical triangles *ABC* and *A'B'C'* isosceles with $AB = BC$ and $A'B' = B'C'$.

To Prove $ABC \cong A'B'C'$.

Proof. Sug. 1. Compare the face angles of the subtended trihedrals which are measured by *AB* and *BC*; by *A'B'* and *B'C'*. What kind of trihedrals are these?

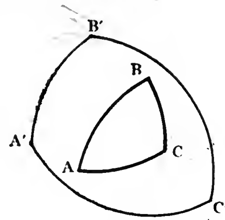
2. Compare the two trihedrals.

3. Put the two trihedrals in coincidence and compare the spherical triangles.

Therefore—

1. Demonstrate proposition § 732 by superposition.

733. Polar of a Triangle. In the spherical triangle ABC, let A' be that one of the two poles of arc BC which lies on the same side of BC as does the vertex A. In the same manner define B' and C' respective poles of arcs AB and CA. The triangle $A'B'C'$ is the *polar triangle* of ABC.

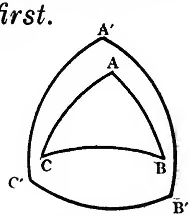


It is to be noted that the sides of ABC if extended will form eight spherical triangles including ABC. Of these but one answers the conditions of the definition of the polar.

2. Draw on a spherical blackboard or other sphere eight such spherical triangles and distinguish a triangle and its polar.

PROPOSITION XX.

734. Theorem. *If one spherical triangle is the polar of a second, then the second triangle is the polar of the first.*



Given △ $A'B'C'$ the polar of △ ABC.

To Prove △ ABC the polar of △ $A'B'C'$.

Proof. Sug. 1. What must be proved concerning A, B, C in order to show that △ ABC is the polar of △ $A'B'C'$?

2. By § 705 prove that A is a pole of $B'C'$, that B is a pole of $C'A'$, and that C is a pole of $A'B'$.

3. Suppose that A is not on the same side of $B'C'$ as is A' and draw the great circle AA'. Suppose it to meet BC in X and $B'C'$ in X'. Then both $A'AX$ and AX' are quadrants. Why? But as $AX' < A'AX$, the supposition is false and A and A' are on the same side of $B'C'$.

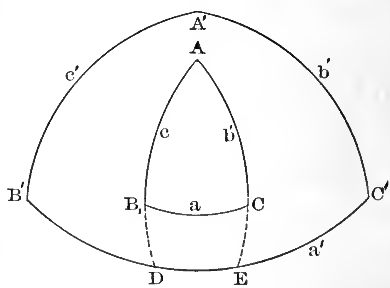
Therefore—

735. Polar Triangles. Two spherical triangles such that each is the polar of the other are *polar triangles.*

PROPOSITION XXI.

736. Theorem. *In two polar triangles each angle of one is measured by the supplement of the side opposite it in the other.*

Given two polar triangles ABC and $A'B'C'$, with corresponding sides a, b, c and a', b', c' opposite the respective angles A, B, C, A', B', C'.



To Prove $\angle A = 180° - a'$, etc.

Proof. Sug. 1. By what is a spherical angle measured?

2. Extend the sides of $\angle A$ to meet a' in D and E. Which arc measures $\angle A$?

3. $a' = B'E + EC' = B'E + C'D - DE$.

4. How many degrees in arc $B'E$? In arc $C'D$? Why?

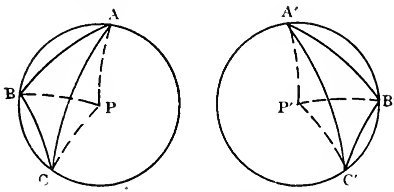
5. Express arc DE in terms of a'.

Therefore—

1. What is the locus of a point in space such that the sum of the squares of its distances from two fixed points equals the square of the distance between the two fixed points?

PROPOSITION XXII.

737. THEOREM. *Two symmetrical spherical triangles are equal.*



Given two symmetrical spherical triangles ABC and $A'B'C'$ on the same, or on equal spheres.

To Prove $\triangle ABC = \triangle A'B'C'$.

Proof. SUG. 1. Let P and P' be the respective poles of the small circles ABC and $A'B'C'$, lying on the same hemisphere as the respective $\triangle$s. Draw the great circle arcs $PA, PB, PC, P'A', P'B', P'C'$.

2. Compare the chords AB, $A'B'$, etc. Compare the circles ABC and $A'B'C'$. Auth.

3. Compare the arcs PA, PB, PC, $P'A'$, $P'B'$, $P'C'$. Auth.

4. Compare $\triangle PAB$ with $\triangle P'A'B'$; $\triangle PBC$ with $\triangle P'B'C'$; $\triangle PCA$ with $\triangle P'C'A'$. Auth.

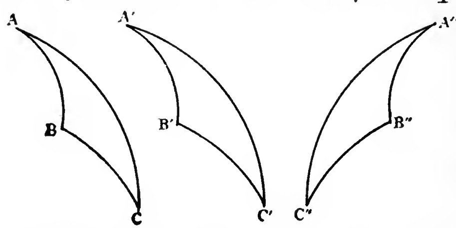
5. If P lies within $\triangle ABC$ then $\triangle ABC = \triangle PAB + \triangle PBC + \triangle PCA$. If P lies without $\triangle ABC$ then $\triangle ABC = PAB + \triangle PBC - \triangle PCA$. Similarly for $\triangle A'B'C'$.

6. Compare $\triangle ABC$ with $\triangle A'B'C'$.

Therefore—

PROPOSITION XXIII.

738. THEOREM. *Two triangles on the same sphere, or on equal spheres, having two sides and the included angle of one equal to two sides and the included angle of the other are either congruent or else symmetrical and therefore equal.*



Given spherical triangles ABC and $A'B'C'$ with $AB = A'B'$, $BC = B'C'$, and $\angle B = \angle B'$.

To Prove $\triangle ABC \cong \triangle A'B'C'$ when the order of arrangement of the respective parts is the same and $\triangle ABC$ symmetrical to $\triangle A'B'C'$ when the order of arrangement is different.

Proof. CASE I. SUG. Superimpose $\triangle ABC$ on $\triangle A'B'C'$. See § 725.

CASE II. SUG. 1. Construct $\triangle A''B''C''$ symmetrical to $\triangle ABC$.

2. Compare $\triangle A''B''C''$ with $\triangle A'B'C'$. Case I.

3. Compare $\triangle ABC$ with $\triangle A'B'C'$. § 728.

Therefore—

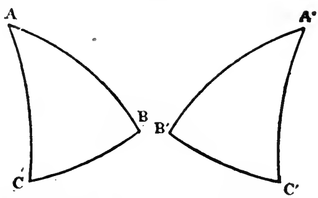
739. EQUAL POLYHEDRALS. Two polyhedrals which when placed with their vertices at the center of the same sphere intersect on the surface equal spherical polygons are *equal polyhedrals*, or *equal solid angles*.

740. Cor. *Two trihedrals having a dihedral and the including face angles of one equal respectively to a dihedral and the included face angles of the other are either congruent or else symmetrical and equal.*

Sug. Use Prop. § 738, and § 718.

PROPOSITION XXIV.

741. Theorem. ***Two triangles on the same, or on equal, spheres having two angles and the included side of one equal to two angles and the included side respectively of the other are either congruent or else symmetrical and therefore equal.***



Given △ ABC and $A'B'C'$ on the same or on equal spheres, with $\angle A = \angle A'$, $\angle B = \angle B'$, and arc $AB =$ arc $A'B'$.

To Prove △ ABC and $A'B'C'$ either congruent or else symmetrical and equal.

Proof. Sug. 1. Let △ EFG and $E'F'G'$ be the respective polars of ABC and $A'B'C'$ with sides e, f, g, e', f', g', opposite ∠s E, F, G, E', F', G' respectively. The pupil may construct the polar triangles.

2. Then $e = e'$, $f = f'$, $\angle G = \angle G'$.

3. Then by § 738 △ EFG and $E'F'G'$ are either congruent or symmetrical. Homologous parts of △ EFG and $E'F'G'$ are then equal.

4. Since $\angle E = \angle E'$, $\angle F = \angle F'$ and $g = g'$, it follows that $a = a'$, $b = b'$, and $\angle C = \angle C'$.

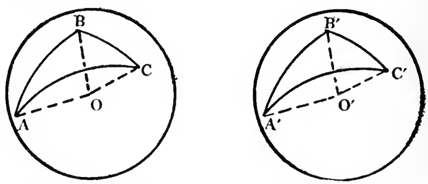
5. Compare △ ABC and $A'B'C'$. Auth.

Therefore—

742. Cor. *Two trihedrals having two dihedrals and included face angle of one equal to two dihedrals and the included face angle of the other respectively are either congruent or symmetrical.*

PROPOSITION XXV.

743. Theorem. *Two triangles on the same sphere, or on equal spheres, having the three sides of one equal respectively to the three sides of the other are either congruent or else symmetrical and therefore equal.*



Given two spherical triangles ABC and $A'B'C'$ on the same or equal spheres with $AB = A'B'$, $BC = B'C'$, $CA = C'A'$.

To Prove △ ABC and $A'B'C'$ either congruent or else symmetrical and equal.

Proof. Sug. 1. Connect the vertices with the respective centers O and O'.

2. Compare the face angles of the respective trihedrals O and O'.

3. Compare the trihedrals O and O'.

4. Compare the triangles ABC and $A'B'C'$.

Therefore—

PROPOSITION XXVI

744. THEOREM. *Two triangles on the same sphere, or on equal spheres, having the three angles of one equal respectively to the three angles of the other are either congruent or else symmetrical and therefore equal.*

Given two triangles ABC and EFG on the same or equal spheres, with $\angle A = \angle E$, $\angle B = \angle F$, $\angle C = \angle G$ with respective sides a, b, c, e, f, g.

The pupil may construct the figure from description in the text.

To Prove △ ABC and EFG congruent or else symmetrical.

Proof. SUG. 1. Let △ $A'B'C'$ and $E'F'G'$ be the respective polars of △ ABC and EFG, with respective sides a', b', c', e', f', g'. Compare a' and e', b' and f', c' and g'. Auth.

2. Compare △ $A'B'C'$ and $E'F'G'$. Auth.

3. Compare ∠s A', B', C' with ∠s E', F', G' respectively. Auth.

4. Compare sides a, b, c with e, f, g respectively. Auth.

5. Compare △ ABC and EFG. Auth.

Therefore—

745. COR. *Two trihedrals having the three dihedrals of one equal to the three dihedrals of the other respectively are either congruent or symmetrical and therefore equal.*

1. A straight line tangent to a circle of a sphere lies in the plane which is tangent to the sphere at the point of contact.

2. What is the locus of a line tangent to a sphere at a given point?

3. By planes parallel to the base divide a pyramid into two equal parts.

4. If two angles of a spherical triangle are equal, the triangle is isosceles.

Sug. Construct the polar of the given triangle.

5. The arc of a great circle drawn from the vertex of an isosceles triangle to the middle of the base is perpendicular to the base and bisects the vertex angle.

6. Determine a point X at a given distance from a fixed point, equidistant from two parallel planes, and equidistant from two given points.

7. Prove Prop. § 741 by superposition.

8. If one circle of a sphere passes through the poles of a second circle of a sphere, the planes of the two circles are perpendicular to each other.

9. Find a point X at a distance m from one point, a distance n from a second point, and a distance p from a third point. When is there no solution? When is there one? When two? Is there any other possibility?

10. The angles opposite the equal sides of an isosceles spherical triangle are equal.

11. In a given plane, find a point which is equidistant from the vertices of a triangle which is in another plane. Use locus.

PROPOSITION XXVII.

746. Theorem. I. *Each side of a spherical triangle is less than the sum of the other two.*

Sug. Construct the subtended trihedral angle at the center of the sphere and use § 509.

PROPOSITION XXVIII.

747. Theorem. *The sum of the sides of a convex spherical polygon is less than* 360° *of arc.*

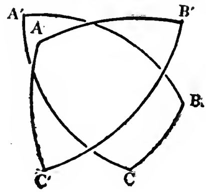
Sug. Construct the subtended polyhedral and use § 510.

748. Cor. *The sum of the sides of a convex spherical polygon is less than a great circle.*

749. Cor. *No side or diagonal of a convex spherical polygon is as great as* 180° *of arc.*

PROPOSITION XXIX.

750. Theorem. *The sum of the angles of a spherical triangle is greater than* 180° *and less than* 540°.



Given $\triangle ABC$.

To Prove $\angle A + \angle B + \angle C > 180°$ and $\angle A + \angle B + \angle C < 540°$.

Proof. Sug. 1. Let $\triangle A'B'C'$ be the polar of $\triangle ABC$. Then $A = 180° - a'$, $B = 180° - b'$, $C = 180° - c'$. Why?

2. Then $A + B + C = 540° - (a' + b' + c')$ Why?

3. $a' + b' + c' < 360°$. Why?

4. Hence $\angle A + \angle B + \angle C$ greater than 180° and $\angle A + \angle B + \angle C < 540°$. Why?

Therefore—

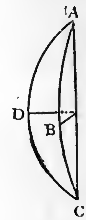
1. The angles of a spherical triangle are 70°, 96°, and 84°. How many degrees in the respective sides of the polar triangle?

2. The sides of a triangle are 90°, 65°, and 125°. How many degrees in the respective angles of the polar triangle?

751. Spherical Excess. A spherical triangle, unlike a plane triangle, may have two or three right angles, or two or three obtuse angles. The number of degrees in the angles of a spherical triangle in excess of two right angles is the *spherical excess* of the triangle.

752. Cor. *The spherical excess of a spherical triangle is less than four right angles.*

753. Lune. A portion of the surface of a sphere included between two semicircles is a lune. The angle between the great circles is the angle of the lune.



ABCDA is a lune, *A* and C are its angles and *ABC, ADC* are its edges.

Preliminary Theorems.

754. Theorem I. *The angle of a lune is equal to the dihedral angle of the planes of the great circles forming the lune.*

755. Theorem II. *The angle of a lune is measured by the arc of a great circle described from either vertex as a pole and intercepted between the sides of the lune.*

756. Theorem III. *Two lunes on the same sphere, or on equal spheres, are equal if their angles are equal.*

1. What is the locus of a point which is at a given distance, *a*, from a given plane, and at a given distance greater than *a* from a given point in the given plane?

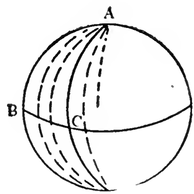
757. Tri-rectangular Triangles. The eight spherical triangles into which a spherical surface is divided by three great circles the planes of which are perpendicular to one another are *tri-rectangular* triangles, i. e., each contains three right angles.

758. Spherical Degree. If a tri-rectangular triangle be divided into ninety equal parts, one of these parts is a *degree of surface* or *spherical degree.*

2. One three hundred sixtieth part of the surface of a hemisphere, or one seven hundred twentieth part of the surface of a sphere, is a spherical degree.

PROPOSITION XXX.

759. THEOREM. *The surface of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*



Given a sphere O with surface S and a lune with angle BAC and surface S'.

To Prove $\frac{S'}{S} = \frac{\angle BAC}{4 \text{ rt. } \angle s}$.

Proof. CASE I. $\angle BAC$ commensurable with 4 rt. $\angle s$ or 360°.

SUG. 1. From A as a pole draw the great circle BC. Since the angles at A are measured by the arcs which they intercept on this great circle, arc BC and the great circle BC are commensurable. Divide the arc BC and the circle BC by a common unit of measure and through each point of division and A pass a great circle. Compare the small lunes into which lune BAC and the spherical surface are divided.

2. Using one of these lunes as a unit compare the ratio $\frac{S'}{S}$ with the ratio $\frac{\text{arc } BC}{\odot BC}$.

3. Using the angle of the unit lune as a unit, compare the ratio $\frac{\angle BAC}{360°}$ with the ratio

$\dfrac{\text{arc } BC}{\odot BC}$.

4. Compare the ratio $\dfrac{S'}{S}$ with the ratio $\dfrac{\angle BAC}{360°}$.

CASE II. $\angle BAC$ not commensurable with 4 rt. ∠s or 360°.

SUG. Proceed as in 428.

760. COR. I. *The number of spherical degrees in a lune is double the number of angular degrees in its angle.*

SUG. Let S denote the number of spherical degrees in the lune and A the number of angular degrees in the angle. Then $\dfrac{S}{720}=\dfrac{A}{360}$, whence $S=2A$.

761. COR. II. *The surface of a lune equals the product of a tri-rectangular triangle by twice the number of right angles in the angle of the lune.*

SUG. Denote the tri-rectangular triangle by T and the number of right angles in the angle of the lune by A. Then $\dfrac{A \text{ rt. ∠s}}{4 \text{ rt. ∠s}}=\dfrac{S}{8T}$.

$\therefore S=2A \times T$.

1. The angle of a lune is 72° and the area of the sphere is 95 sq. in. Find the area of the lune.

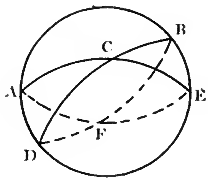
SUG. By Cor. I, $S = 2a$ sph. deg. $= 2 \times 72$ sph. degree.

Cor. II, $S = 2A \times T = \dfrac{2 \times 72}{90}$ T.

2. The angle of a lune is 130° and the area of the sphere is 45 sq. yds. Find the number of square yards in the lune.

PROPOSITION XXXI.

762. THEOREM. *The sum of the areas of the two vertical spherical triangles formed on a hemisphere by two intersecting great circles equals a lune with an angle equal to the angle of the intersecting circles.*



Given a hemisphere $O — ABCD$ with two great circles ACE and BCD intersecting at C and forming two vertical spherical ⨺ ACD and BCE.

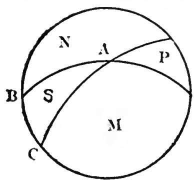
To Prove $\triangle ACD + \triangle ECB =$ lune $CAFD$.

Proof. SUG. Compare $\triangle ECB$ with its opposite, $\triangle AFD$.

Therefore—

PROPOSITION XXXII.

763. THEOREM. *The number of spherical degrees in a spherical triangle equals the number of angular degrees in its spherical excess.*



Given $\triangle ABC$, S denoting its area in spherical degrees, A, B, and C denoting the number of angular degrees in the angles of $\triangle ABC$, and E denoting its spherical excess.

To Prove $S = E$.

Proof. SUG. 1. The surface of the hemisphere $A - BCC'B'$, or 360 sph. deg., equals $\triangle S + \triangle M + \triangle N + \triangle P, M, N, P$ and S, denoting the number of spherical degrees in the respective triangles.

2. $\triangle S + \triangle M =$ lune B, $\triangle S + \triangle N =$ lune C, $\triangle S + \triangle P =$ lune A. Why?

3. Hence $S + M = 2B$, $S + N = 2C$, $S + P = 2A$, $\therefore 2S + 360 = 2(A+B+C)$. Why?

4. Whence $S = A + B + C - 180$, i. e. the number of spherical degrees in $\triangle ABC$ equals the number of angular degrees in the triangle less 180°, i. e. $S = E$.

Therefore—

764. COR. I. *The ratio of a spherical triangle to the surface of its sphere is* $\frac{E}{720}$ *and its ratio to a tri-rectangular triangle is* $\frac{E}{8}$.

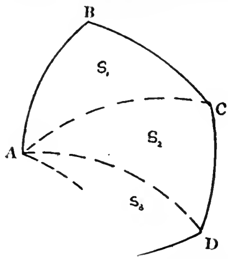
1. The area of a spherical triangle can be obtained when the area of the sphere and the angles of the triangle are known.

2. The angles of a triangle are 96°, 72° and 120°, and the surface of the sphere is 360 sq. in. How many sq. in. in the surface of the triangle?

765. SPHERICAL EXCESS OF A SPHERICAL POLYGON. The number of degrees by which the sum of the angles of a spherical polygon of n sides exceeds $(n - 2)\ 180°$ is the *spherical excess* of the polygon.

PROPOSITION XXXIII.

766. THEOREM. *The number of spherical degrees in a spherical polygon equals the number of angular degrees in its spherical excess.*



Given a spherical polygon $ABCD\ldots$, S denoting its area in spherical degrees and E its spherical excess.

To Prove $S = E$.

Proof. SUG. 1. Draw all possible diagonals from some one vertex, thus dividing the polygon into $n - 2$ △. Denote their areas in sph. deg. by S_1, S_2, S_3, etc., and their respective spherical excesses by E_1, E_2, E_3, etc.

2. $S_1 = E_1, S_2 = E_2, S_3 = E_3$, etc. Why?

3. $E_1 + E_2 + E_3 + \ldots$
$= (\angle\text{s of } \triangle S_1 - 180°) + (\angle\text{s of } \triangle S_2 - 180°) + \ldots.$
$= \angle\text{s of } ABCD - (n-2)\,180° = E.$

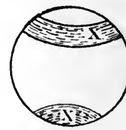
4. $S = S_1 + S_2 + S_3 + \ldots$
$= E_1 + E_2 + E_3 + \ldots = E.$

Therefore—

767. A ZONE. A portion of the surface of a sphere included between two parallel planes is a *zone.* The circles of the sphere formed by the bounding planes are the *bases* of the zone and the distance between them is the *altitude* of the zone.

768. Spherical Segment. A portion of a sphere included between two parallel planes is a *spherical segment.* The sections of the sphere formed by the two planes are the *bases* of the segment and the distance between the planes is the *altitude* of the segment.

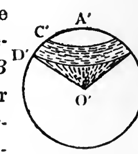
In the figure X is a spherical segment. The spherical surface of a segment is a zone. If a portion of a sphere be cut off by a plane, this portion is a segment and its spherical surface is a zone, for they are included between the cutting plane and a tangent plane parallel to the cutting plane. In this case the segment and the zone have but one base each. Find illustrations of zones of both kinds from geography.

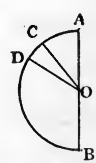


769. If a semicircle be revolved about its diameter as an axis a spherical surface is generated. Any arc of the semicircle generates a zone.

770. Spherical Sector. A portion of a sphere generated by the revolution of a circular sector about a diameter is a *spherical sector.*

It is important to form mental pictures of the different varieties of spherical sectors and describe them. For example, if the semi-circle ADB is revolved about the diameter AB, the circular sector AOC generates a spherical sector the surface of which consists of a zone of one base generated by the arc AC and a convex conical surface generated by the radius OC. The sector COD generates a spherical sector b bounded by a zone of two bases, a convex conical surface generated by OD, and a concave conical surface generated by OC.





Generate many spherical sectors and describe them. Make many drawings to illustrate spherical sectors.

1. Construct a semi-circle and in it a circular sector which, if revolved, will generate a spherical sector having two concave conical surfaces. Describe the zone.

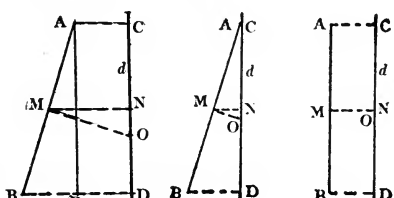
1. Construct a circular sector which generates a spherical sector the surface of which consists of a concave conical surface, a plane surface, and a zone.

2. Is a hemisphere a spherical sector? Why? A spherical segment? Why?

3. Describe the sectors that have for their spherical surface the torrid zone, the N. frigid zone, and the S. temperate zone.

PROPOSITION XXXIV.

771. THEOREM. *The area of the surface generated by the revolution of a straight line segment about an axis in its plane but not crossing the axis, is equal to the product of the projection of the segment upon the axis and the circumference of a circle the radius of which is a perpendicular erected at the mid point of the segment and terminated by the axis.*



Given axis d with a line segment AB in the same plane as d, but not crossing it, CD the projection of AB on the axis, M the mid point of AB, MO a perpendicular to AB at M intersecting the axis at O and S the area of the surface generated by the revolution of AB about the axis.

To Prove $S = 2\pi \times MO \times CD$.

Proof. CASE I. AB oblique to CD not intersecting it.

SUG. 1. Draw $MN \perp CD$, join A and C, B and D, draw $AP \parallel CD$. AC and BD are $\perp CD$. Why?

2. $S = 2\pi \times MN \times AB$. Why?

3. $\triangle ABP \sim \triangle OMN$. Why?

$\therefore AB \times MN = CD \times MO$. Why?

4. $\therefore S = 2\pi \times MO \times CD$.

CASE II. When AB meets CD.

SUG. Follow the plan of Case I.

CASE III. When $AB \parallel CD$.

Proof left to the pupil.

Therefore—

1. How many spherical degrees are there in a spherical triangle with angles of 200°, 140°, and 100°?

2. What part of the surface of a sphere is a spherical triangle with angles of 120°, 140°, and 160°?

772. POSTULATE. *If half of a regular polygon be inscribed in a semicircle and the number of sides be indefinitely increased, and if the semicircle be revolved about its diameter the surface generated by the semipolygon is a variable which approaches the sphere generated by the semicircle as its limit and the apothem of the polygon approaches the radius of the sphere as its limit.*

3. A boiler is $4\frac{1}{2}$ ft. in diameter and 18 ft. long. It is penetrated by 48 3-inch cylindrical tubes. How many gallons does it hold?

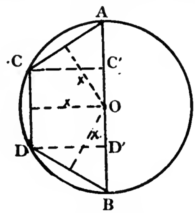
4. A pyramid with an altitude of $\frac{3}{16}$ ft. is cut into two parts of equal volume by a plane parallel to the base. Find the distance of the cutting plane from the vertex.

5. A plane parallel to the base of a cone bisects the altitude. Compare the volumes of the two parts.

6. A circular water tank with a diameter of $10\frac{1}{2}'$ is $2\frac{1}{2}'$ deep. How many barrels will it hold? (Indicate operation and abbreviate by cancellation.)

PROPOSITION XXXV.

773. THEOREM. *The area of a sphere is equal to the product of its diameter and a great circle.*



Given a sphere O generated by the revolution of the semicircle $ACDB$, its surface denoted by S, its radius by r.

To Prove $S = 2\pi r \times 2r$.

Proof. SUG. 1. Divide arc $ACDB$ into equal parts and draw the chords forming the regular semi-polygon $ACDB$. Draw the projections of the chords upon the diameter AB represented AC', $C'D'$.... Draw the apothem x of the polygon to each chord.

2. The surface generated by AC equals $2\pi \times AC' \times x$. Why? What is the surface generated by CD? By DB?

3. If S' be the surface generated by the semi-polygon, then $S' = 2\pi \times AB \times x$. Why?

4. If the number of sides be indefinitely increased then $S' \doteq S$, $x \doteq r$ $2\pi\, AB \times \doteq 2\pi\, AB \times r$ $\therefore S = 2\pi r \times 2r$. Why?

Therefore—

774. COR. I. *The surface of a sphere equals* $4\pi r^2$ *or* πd^2.

775. COR. II. *The area of the surface of a sphere equals that of four great circles.*

776. Cor. III. *The area of the surface of a sphere equals that of a circle with a radius equal to the diameter of the sphere.*

777. Cor. IV. *The surfaces of two spheres have the same ratio as the squares of their radii and the squares of their diameters.*

778. Cor. V. *The area of a spherical degree is* $\frac{4\pi r^2}{720}$.

779. Cor. VI. *The area of a spherical triangle is* $\frac{4\pi r^2 \times E}{720}$, *in which E is the spherical excess of the triangles.*

780. Cor. VII. *The area of a spherical polygon is* $\frac{4\pi r^2 \times E}{720}$, *in which E is the spherical excess of the polygon.*

781. Cor. VIII. *The area of a zone is* $2\pi r \times h$, *in which h is the altitude of the zone.*

Sug. Demonstrate according to the method of 773, using an arc less than a semicircle.

782. Cor. IX. *Zones on the same sphere or on equal spheres are proportional to their altitudes.*

783. Postulate. *If a polyhedron be circumscribed about a sphere and the number of its sides be indefinitely increased in such a manner that the areas of each of the faces of the polyhedron are at the same time indefinitely decreased, the surface of the polyhedron is a variable which approaches the surface of the sphere as its limit and the volume of the polyhedron is a variable which approaches the volume of the sphere as its limit.*

784. Spherical Pyramid. A solid bounded by a spherical polygon and the polyhedral angle which it subtends at the center of the sphere is a *spherical pyramid.* Its *vertex* is the center of the sphere and its *base* is the spherical polygon.

PROPOSITION XXXVI.

785. THEOREM. *The volume of a sphere is equal to the area of its surface multiplied by one-third its radius.*

Given a sphere, denoting its volume by V, its surface by S, and its radius by r.

To Prove $V = \frac{1}{3} r \times S$.

Proof. SUG. 1. Circumscribe a polyhedron about the sphere and join each vertex to the center. Each face forms the base of a pyramid, the edges of which are the lines from its vertices to the center. What is the altitude of each pyramid?

2. What is the volume of each pyramid? What is the volume of the polyhedron?

3. Apply the limiting process of §783.

Therefore—

786. COR. I. *The volume of a sphere is* $\frac{4}{3}\pi r^3$ (774), *r being the radius, or* $\frac{1}{6}\pi D^3$.

787. COR. II. *The volumes of two spheres have the same ratio as the cubes of their radii and the cubes of their diameters.*

788. COR. III. *The volume of a spherical sector is equal to the product of the area of its zone and one-third the radius of the sphere, i. e.* $\frac{2}{3}\pi R^2 h$.

SUG. Base a demonstration upon the method of 785.

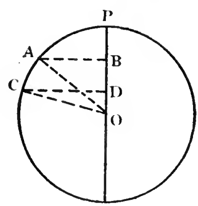
789. COR. IV. *The volume of a spherical pyramid equals the product of the area of its base and one third the radius of the sphere, i. e.* $\frac{\frac{4}{3}\pi r^3 E}{720}$.

SUG. 1. Make a demonstration based on the method of 785.

$$2. \quad V = \tfrac{1}{3} r \times B = \frac{\tfrac{4}{3}\pi r^3 \times E}{720}.$$

PROPOSITION XXXVII

790. PROBLEM. *To find the volume of a spherical segment.*



Given a spherical segment generated by the revolution of arc AC about the diameter OP, r_1 and r_2 denoting the radii of its respective bases, h its altitude, r the radius of the sphere, and V the volume of the segment.

CASE I. **To find** V in terms of r_1, r_2, and h_1, when both bases are on the same side of the center.

SUG. 1. Let m represent OD, the distance from the center to the nearest base, and let V_1, V_2, V_3 denote the volumes generated by the triangles OAB, OCD, and the sector COA respectively.

2. $V = V_3 + V_1 - V_2$. Why?

$= \tfrac{2}{3}\pi r^2 h + \tfrac{1}{3}\pi r_1{}^2 (m+h) - \tfrac{1}{3}\pi r_2{}^2 m.$

$= \tfrac{1}{3}[2\pi r^2 h + (r^2 - \overline{m+h}^2)(m+h) -$
$(r^2 - m^2)m]$, which by expanding and removing parentheses becomes

$= \tfrac{1}{3}\pi[3r^2 - 3m^2 - 3hm - h^2]h$

$= \pi h\,[r^2 - m^2 - hm - \tfrac{1}{3}h^2]$

$= \tfrac{1}{2}\pi h[r^2 - m^2 - 2hm - \tfrac{2}{3}h^2 + r^2 - m^2]$

3. In this last expression put the first $r^2 - m^2$ equal to r_2^2 (Auth.) and the second one equal to $r_1^2 + 2hm + h^2$, for $r^2 - m^2 = r_1^2 + (h+m)^2 - m^2$.

4. $\therefore \quad V = \frac{1}{2}h[\pi r_2^2 + \pi r_1^2] + \frac{1}{6}\pi h^3$.

Therefore—

The volume of a spherical segment equals one-half the product of its altitude and the sum of its bases increased by one-sixth the volume of a sphere with a diameter equal to the altitude of the segment.

Case II. When the center is between the bases.

Sug. Adapt the work of case I to this case.

Case III. When the segment has but one base.

Sug. Adapt the work of case I to this case by letting r_1 be zero. What term will vanish from the final formula? Translate the formula into words.

Use $3\frac{1}{7}$ for π in the following exercise:

1. Given a sphere 20′ in diameter. Find the volume of a segment, the radii of the bases being 6 and 8 ft. respectively.

2. Given a sphere 20′ in diameter. Find the volume of a segment with one base having a radius of 8′.

3. A 4-in. hole is bored through a sphere 10″ in diameter. How much of the volume of the sphere is cut away?

4. Find the volume of a spherical sector having a zone with an altitude of 10″ on a sphere with a radius of 20″.

5. Find the volume and area of a sphere 40″ in diameter.

6. A sphere is cut by parallel planes so that the diameter is divided into ten equal parts. Compare the areas of the zones and also the volumes of the spherical sectors the spherical surfaces of which are the respective zones.

7. If the average specific gravity of the earth is 5.6, what is its weight in tons?

8. Find the angles of an equiangular spherical triangle, the surface begin $\frac{1}{12}$ that of the sphere.

9. The radius of a sphere is 3 in. and the area of a spherical triangle *ABC* on the sphere is 12 sq. in. The angles *A* and *B* are 140° and 115° respectively. Find ∠ *C*.

10. The dimensions of a rectangular parallelopiped are 3′, 4′, and 12′. Find the length of a great circle of the circumscribing sphere.

11. Assume the diameters of the earth and moon to be 8,000 and 2,000 miles respectively. What is the ratio of their volumes?

12. A triangle on a 12″ globe has angles of 140°, 119°, and 196°. Compute the area.

13. The angles of a pentagonal spherical pyramid are 47°, 96°, 120°, 142°, and 87°, and the diameter of the sphere is 20″. Find the area of the base and the volume of the pyramid.

14. Prove that the volume of a portion of a sphere included by the planes of two great circles and the intercepted lune is to the volume of the sphere as the angle of the lune is to 4 rt. angles. Such a solid is a *wedge* or *ungula*. The lune is its *base*.

15. A cone of revolution and a sphere are inscribed in a cylinder of revolution. Compare the volumes of the three solids.

16. If the area of the convex surface of a right circular cone is twice the area of its base, prove that the slant height of the cone is equal to the diameter of the base.

17. If the specific gravity of an iron ball 10″ in diameter is 8.1, what is its weight?

18. An iron kettle in the shape of a hemisphere has a diameter of $3\frac{1}{2}$ ft. How many gallons does it hold when full? When filled to a depth of 10 in.?

19. A cube and a sphere have equal areas. Which has the greater volume?

20. Two balls are of the same material. One, weighing 12 lbs., has a diameter equal to $\frac{1}{2}$ that of the other. What is the weight of the second ball?

21. The radius of the base of a right circular cone is 5″ its volume is 31 cu. in. and the number of square inches in the area of its convex surface is equal to the number of cubic inches in the volume of the cone. What is its altitude and its slant height?

22. Determine the locus of the center of a sphere with given radius, such that

1. its surface passes through a given point.

2. it is tangent to a given plane.

3. it is tangent to a given sect.

4. it is tangent to a given sphere.

23. Determine the locus of a point which is

1. at a given distance from a fixed point.

2. at given distances from two fixed points.

3. equally distant from two parallel planes and at a fixed distance from a given point.

4. equally distant from two points and at a given distance from a third point.

24. Make up exercises involving loci which are the intersections of other loci.

25. Find the center of a sphere with a given radius, such that

1. its surface passes through three given points.

2. it is tangent to a given plane and its surface passes through two given points.

3. it is tangent to two given planes and its surface passes through a given point.

4. it is tangent to three given planes.

5. it is tangent to a given sphere, to a given plane, and its surface passes through a given point.

6. it is tangent to a given line, a given plane, and a given sphere.

7. Make other exercises in this group.

26. Find the pole of a circle determined by three points on the surface of a sphere.

Sug. It can be done in a manner similar to that by which the center of a circle is determined by three points in a plane.

27. The diameter of a sphere is equal to the altitude of a cone of revolution and of a cylinder of revolution, the radii of the three solids being the same. Prove that the volumes of the cone, sphere, and cylinder are proportional to 1, 2, 3 respectively.

28. An orange, $3\frac{1}{2}$ in. in diameter, pulp measure, sells at 50 cents a dozen. One 3 in. in diameter sells at 40 cents. Which is the better one to buy if the quality is the same? If the larger orange is worth 50 cents, what is the smaller one worth?

29. If a 3 in. orange sells for 36 cents, what is the real value of a 4 in. orange of same quality?

30. What part of a spherical surface is a spherical triangle each angle of which is 90°? How many spherical degrees in such a triangle?

31. The radius of a sphere is 10′ and the angles of a spherical triangle are 95°, 117°, and 92°. What is the area of the triangle?

32. What is the area of the earth's surface, assuming the earth to be a sphere with a radius of 7,912 miles?

33. A sphere can be inscribed in a cube.

34. By planes parallel to the base divide a pyramid into four equal parts.

35. What is the locus of the vertex of a pyramid having a fixed volume and a fixed base?

Review.

791. State the formula for

1. The volume of a cone of revolution.
2. The volume of a cylinder of revolution.
3. The volume of a frustum of a cone of revolution.
4. The lateral area of the above three solids.
5. The area of a sphere.
6. The area of a spherical triangle.
7. The area of a spherical polygon of n sides.
8. The volume of a sphere.
9. The area of a zone.
10. The volume of a spherical sector.
11. The volume of a spherical segment.
12. The volume of a spherical pyramid.
13. The volume of an ungula.