PLANE AND SPHERICAL TRIGONOMETRY

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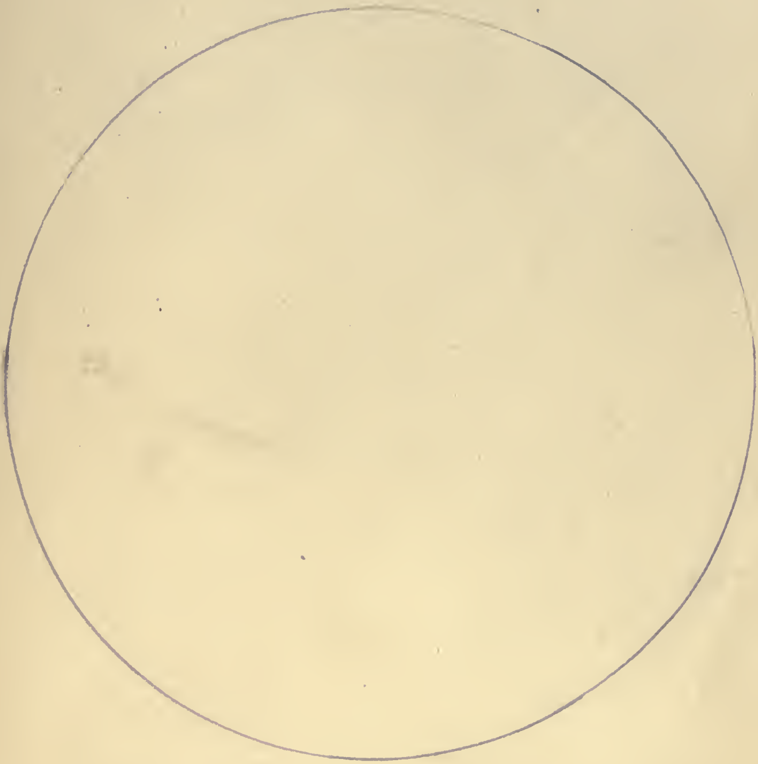

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PLANE AND SPHERICAL TRIGONOMETRY

THE MARSH AND ASHTON MATHEMATICAL SERIES.

BY

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PLANE AND SPHERICAL TRIGONOMETRY

AN ELEMENTARY TEXT-BOOK

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PREFACE

There have been two distinct methods of presenting the subject of Trigonometry. In one the treatment is purely geometrical, no attention being paid to the direction of lines. Here the algebraic signs of the functions are assigned in a purely arbitrary manner, and none of the proofs hold except for the particular figure. When this method is employed, the result must be shown to be general by some algebraic process.

In the second method all lines have direction as well as magnitude, and the proofs are given in such a form that they are general and hold for every possible figure. This would seem to be the logical method of developing the subject; for Trigonometry is the connecting link between elementary Geometry and those subjects in which Algebra and Geometry are combined in such a way that the directed line must be used constantly. This second method has been employed in this work, which is intended as a text-book for a fifty hour course in high schools and the ordinary first year classes in college, and is, therefore, made as elementary as possible. All matter not required for such a course has been excluded.

We have attempted to avoid the usual mistake of mixing the two methods mentioned above. Two distinct proofs of the Addition Theorem are given. The first employs the method of projection, the formulas for which have been simplified by employing the French symbol for

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an angle between two directed lines; the second is a geometric proof, similar to the one usually given, which has also been made perfectly general. The second demonstration may be preferred in a shorter course.

Carefully selected exercises are given at the end of almost every article. In solving triangles, the natural functions are used where the solution may be obtained easily without the aid of logarithms. This method has been followed because, where logarithms are used exclusively, the student often does not know the meaning of the operation he is performing. The subject of trigonometric equations, which is usually accorded little attention, is here given in a separate chapter, for it is thought that in no other way can the student acquire so good a knowledge of general principles or so great skill in applying the formulas.

August, 1902.

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PLANE AND SPHERICAL TRIGONOMETRY

PART I

PLANE TRIGONOMETRY

CHAPTER I

FUNCTIONS OF ACUTE ANGLES

1. Definitions. — Let ABC (Fig. 1) be a right triangle, right-angled at C; and let $A'B'C'$ be any second right triangle which is similar to the first.

From the definition of similar triangles it follows that

$$\angle A = \angle A', \text{ and } \angle B = \angle B',$$

$$\frac{CB}{AB} = \frac{C'B'}{A'B'}, \quad \frac{AC}{AB} = \frac{A'C'}{A'B'}, \text{ and } \frac{CB}{AC} = \frac{C'B'}{A'C'}.$$

These five equations between the angles and the ratios of the sides are true for any pair of similar right triangles. But any two right triangles are similar, if an acute angle of the one is equal to an acute angle of the other, or if any pair of homologous sides are proportional. Hence if any one of these five equations hold, the remaining four also hold and the triangles are similar.

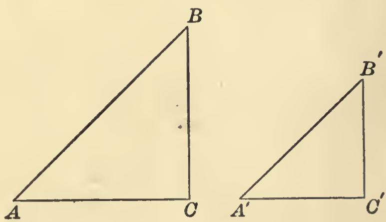


Fig. 1.

It follows then that in every triangle equiangular with ABC the ratios of the sides are the same as in ABC, and that in every right triangle not equiangular with ABC these ratios are not the same as in ABC; and, conversely, if any right triangle has any one of these ratios equal to the corresponding ratio of ABC, the triangle must be equiangular with ABC.

The ratio, then, of any two sides of a right triangle is a number which is entirely independent of the lengths of those sides, and, if the angle A is fixed, each of these ratios has a determinate value which is different from the value of the corresponding ratio for any other angle. These three ratios of the sides of the triangle may then be spoken of as *functions* of either of the acute angles of the triangle.

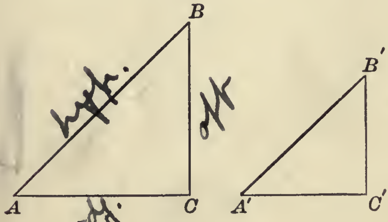


FIG. 1.

(One quantity, y, is said to be a function of another, x, when y has a determinate value, or values, for every value given to x.)

It has been found convenient to give names to these ratios as follows: If A is either of the acute angles of a right triangle (see Fig. 1),

$$\textbf{sine} \quad \text{of } A = \sin A = \frac{CB}{AB} = \frac{\text{opposite leg}}{\text{hypotenuse}},$$

$$\textbf{cosine} \quad \text{of } A = \cos A = \frac{AC}{AB} = \frac{\text{adjacent leg}}{\text{hypotenuse}},$$

$$\textbf{tangent} \quad \text{of } A = \tan A = \frac{CB}{AC} = \frac{\text{opposite leg}}{\text{adjacent leg}}.$$

The abbreviations in the second column should always be read the same as the first column.

Names are also given to the reciprocals of these ratios as follows:

$$\textbf{cosecant} \text{ of } A = \csc A = \frac{AB}{CB} = \frac{\text{hypotenuse}}{\text{opposite leg}},$$

$$\textbf{secant} \text{ of } A = \sec A = \frac{AB}{AC} = \frac{\text{hypotenuse}}{\text{adjacent leg}},$$

$$\textbf{cotangent} \text{ of } A = \cot A = \frac{AC}{CB} = \frac{\text{adjacent leg}}{\text{opposite leg}}.$$

From these definitions it will be seen at once that

$$\csc A = \frac{1}{\sin A}, \quad \sec A = \frac{1}{\cos A}, \quad \text{and} \quad \cot A = \frac{1}{\tan A}.$$

The last three ratios are used much less frequently than the first three, and will usually be studied as the reciprocals of the first set. It is in this way that the student is advised to memorize their definitions.

There are also two other functions which are occasionally used. They are defined by the equations,

$$\textbf{versed sine} \text{ of } A = \text{vers } A = 1 - \cos A.$$

$$\textbf{coversed sine} \text{ of } A = \text{covers } A = 1 - \sin A.$$

The student should note that these eight quantities are all *abstract numbers*, since they are the ratios of two lines. They are called the **trigonometric functions** of the angle A.

EXERCISE I

In the following problems, C is the right angle of the right triangle ABC, and the small letters a, b, c are used to represent the lengths of the sides opposite the corresponding angles; that is, c represents the hypotenuse, a and b the legs.

1. Find all the functions of the angle A, if

(a) $a = 5$, $b = 12$. (c) $a = 8$, $c = 17$.
(b) $b = 4$, $c = 5$. (d) $a = 10$, $c = 15$.

2. What are the functions of B in the same triangles?

3. What relations do you notice between the functions of A and B?

4. Construct the angle whose (a) sine is $\frac{3}{4}$,

(b) tangent is 5, (d) cotangent is 2,
(c) cosine is $\frac{1}{3}$, (e) secant is 3.

5. Draw, on a large sheet of paper, a line AC 10 in. long. At C erect a perpendicular, and at A lay off (with a protractor) angles of 10°, 20°, etc. Measure carefully the sides of the triangles thus formed, and from these measurements determine (to two decimal places) the functions of the angles given in the following table:

A	$\sin A$	$\cos A$	$\tan A$	$\cot A$
10°	.1737	.9848	.1763	5.6713
20°	.3420	.9397	.3640	2.7475
30°	.5000	.8660	.5774	1.7321
40°	.6428	.7660	.8391	1.1918
50°	.7660	.6428	1.1918	.8391
60°	.8660	.5000	1.7321	.5774
70°	.9397	.3420	2.7475	.3640
80°	.9848	.1737	5.6713	.1763

6. By the aid of the table given above find the legs of a right triangle, if $A = 40°$ and $c = 12$.

SOLUTION. — By definition,

$$\sin A = \frac{a}{c}, \quad \text{and} \quad \cos A = \frac{b}{c}.$$

Hence $a = c \sin A = 12 \sin 40° = 12 \times .6428 = 7.7136,$

and $b = c \cos A = 12 \cos 40° = 12 \times .7660 = 9.192.$

7. Find c, if $A = 30°$ and $a = 8$.

8. Find A, if $a = 171$ and $c = 500$.

9. Find A, a, and c, if $B = 50°$ and $b = 6$.

10. Find A, B, and c, if $a = 91$ and $b = 250$.

11. Find B, c, and a, if $A = 70°$ and $b = 10$.

12. Find A, a, and b, if $B = 20°$ and $c = 20$.

2. Functions of complementary angles. — Using the definitions of Art. 1 for the functions of angle B of the triangle ABC, we have (see Fig. 2)

$$\sin B = \frac{AC}{AB} = \cos A = \cos (90° - B),$$

$$\cos B = \frac{CB}{AB} = \sin A = \sin (90° - B),$$

$$\tan B = \frac{AC}{CB} = \cot A = \cot (90° - B),$$

$$\cot B = \frac{CB}{AC} = \tan A = \tan (90° - B),$$

$$\sec B = \frac{AB}{CB} = \csc A = \csc (90° - B),$$

$$\csc B = \frac{AB}{AC} = \sec A = \sec (90° - B).$$

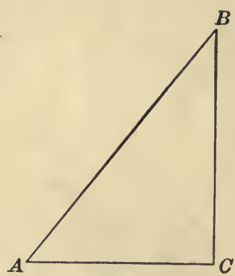


FIG. 2.

From which it appears that any function of an acute angle is equal to the co-named function of its complementary angle.

EXERCISE II

1. Express as functions of angles less than 45°:

   sin 70°, cos 85°, sec 63°, tan 56° 48′, cot 89° 56

2. If $\cos x = \sin x$, find a value of x.

3. If $\tan x = \cot 2x$, find a value of x.

4. If $\cos x = \sin (45° + 2x)$, find a value of x.

5. If A, B, and C are the angles of a triangle, show that $\cos \frac{1}{2} A = \sin \frac{1}{2} (B + C)$.

3. Variation of the functions as the angle varies. — Since the trigonometric functions have been defined by the aid of a right triangle, the angle A must be between 0° and 90°, not including either of these. Later we shall give definitions of the functions which will include all angles, but for the present we shall confine our discussion to angles between 0° and 90°.

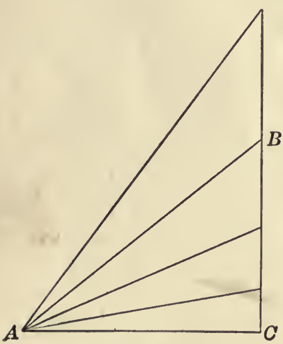


FIG. 3.

Let the angle A begin with small values and increase toward 90°, and consider the changes which will occur in each function. To fix our ideas, let AC remain the same, while AB and CB vary. For small angles CB is small, and AB nearly equal to AC. Hence the sine

$\left(\frac{CB}{AB}\right)$ and the tangent $\left(\frac{CB}{AC}\right)$ are small, the cosine $\left(\frac{AC}{AB}\right)$ nearly unity, the secant (reciprocal of the cosine) a little more than unity, while the cotangent and cosecant (reciprocals of the tangent and sine) are very large.

As the angle increases, both BC and AB increase and approach equality. Hence the sine, tangent, and secant increase, while the cosine, cotangent, and cosecant decrease. As the angle A approaches 90°, AB and CB are very large and nearly equal. Hence the sine approaches unity, and the tangent and secant increase indefinitely. The cosecant decreases toward unity, while the cosine and cotangent decrease indefinitely.

It appears, then, that for angles less than 90° the

sine and cosine are less than unity,
secant and cosecant are greater than unity,
tangent and cotangent may have any positive value.

4. Functions of 45°, 30°, and 60°.

(a) Functions of 45°.

In the right triangle ABC (Fig. 4) let $AC = CB = a$. Then angle A = angle $B = 45°$, and $AB = \sqrt{a^2 + a^2} = a\sqrt{2}$.

It follows at once from the definitions of the functions that

$$\sin 45° = \cos 45° = \frac{a}{a\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\tan 45° = \cot 45° = \frac{a}{a} = 1,$$

$$\sec 45° = \csc 45° = \frac{a\sqrt{2}}{a} = \sqrt{2}.$$

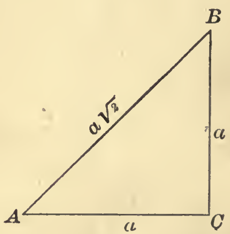


FIG. 4

(*b*) Functions of 30° and 60°.

Let each side of the equilateral triangle ABD (Fig. 5) be represented by $2a$. Bisect the angle B by the line BC. Then this line bisects the base AD and is perpendicular to it. A right triangle ABC is thus formed, in which angle $A = 60°$ and angle $B = 30°$. Also $AB = 2a$, $AC = a$, and $BC = \sqrt{4a^2 - a^2} = a\sqrt{3}$. It follows at once from the definitions of the functions that

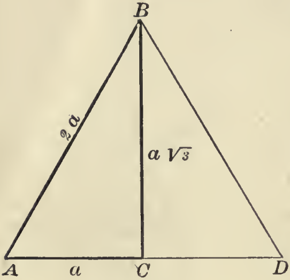


FIG. 5.

$$\sin 60° = \cos 30° = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},$$

$$\cos 60° = \sin 30° = \frac{a}{2a} = \frac{1}{2},$$

$$\tan 60° = \cot 30° = \frac{a\sqrt{3}}{a} = \sqrt{3},$$

$$\cot 60° = \tan 30° = \frac{a}{a\sqrt{3}} = \frac{1}{3}\sqrt{3},$$

$$\sec 60° = \csc 30° = \frac{2a}{a} = 2,$$

$$\csc 60° = \sec 30° = \frac{2a}{a\sqrt{3}} = \frac{2}{3}\sqrt{3}.$$

EXERCISE III

Find the numerical value of

1. $2 \sin 30° \cos 30° \cot 60°$.
2. $\tan^2 60° + 2 \tan^2 45°$.

NOTE. — $\tan^2 60°$ is equivalent to $(\tan 60°)^2$, or the square of the tangent of 60°.

3. $\tan^3 45° + 4 \cos^3 60°$.

4. $4 \cos^2 45° + \tan^2 60° + 3 \sec^2 30°$.

5. $\sin 60° \cos 30° + \cos 60° \sin 30°$.

6. $\sin^2 45° + \cos^2 60° - \sin^2 30°$.

Prove that

7. $\tan 60° = \dfrac{2 \tan 30°}{1 - \tan^2 30°}$.

8. $\sin 60° = 2 \sin 30° \cos 30°$.

9. $\cos 60° = 1 - 2 \sin^2 30°$.

10. $\cos^2 60° : \cos^2 45° : \cos^2 30° = 1 : 2 : 3$.

5. Relations between the functions. — It appeared at once from the definitions of the six trigonometric functions that they were not all independent; for three of them were defined as the reciprocals of the other three. From these definitions we have the following relations:

$$\sin A \csc A = 1, \qquad [1]$$

$$\cos A \sec A = 1, \qquad [2]$$

$$\tan A \cot A = 1. \qquad [3]$$

We shall now show that there are other relations between these functions; that, in fact, each depends on the others, so that, if one is given, all the others may be found.

In the right triangle ABC,

$$\overline{CB}^2 + \overline{AC}^2 = \overline{AB}^2.$$

Dividing by $\overline{AB}^2$,

we have $$\frac{\overline{CB}^2}{\overline{AB}^2} + \frac{\overline{AC}^2}{\overline{AB}^2} = 1,$$

or $$\sin^2 A + \cos^2 A = 1. \qquad [4]$$

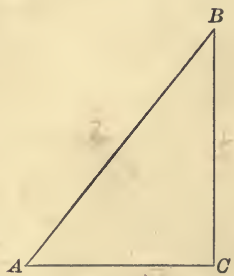


FIG. 6.

Again dividing the same equation by $\overline{AC}^2$,

we have $$\frac{\overline{CB}^2}{\overline{AC}^2} + 1 = \frac{\overline{AB}^2}{\overline{AC}^2},$$

or $$\mathbf{1 + \tan^2 A = \sec^2 A.} \qquad [5]$$

Dividing the same equation by $\overline{CB}^2$,

we have $$1 + \frac{\overline{AC}^2}{\overline{CB}^2} = \frac{\overline{AB}^2}{\overline{CB}^2},$$

or $$\mathbf{1 + \cot^2 A = \csc^2 A.} \qquad [6]$$

Dividing $\sin A$ by $\cos A$, we have

$$\frac{\sin A}{\cos A} = \frac{\dfrac{CB}{AB}}{\dfrac{AC}{AB}} = \frac{CB}{AC}.$$

But $$\tan A = \frac{CB}{AC}.$$

Hence $$\mathbf{\tan A = \frac{\sin A}{\cos A}.} \qquad [7]$$

Since $$\cot A = \frac{1}{\tan A},$$

$$\mathbf{\cot A = \frac{\cos A}{\sin A}.} \qquad [8]$$

These formulas should be carefully memorized by the student, as they are in constant use throughout the study of the subject. The student should also be perfectly familiar with the following forms, which may be obtained easily from the formulas given above:

$$\sin A = \sqrt{1 - \cos^2 A}, \qquad \cos A = \sqrt{1 - \sin^2 A},$$

$$\sec A = \sqrt{1 + \tan^2 A}, \qquad \csc A = \sqrt{1 + \cot^2 A},$$

$$\tan A = \sqrt{\sec^2 A - 1}, \qquad \cot A = \sqrt{\csc^2 A - 1}.$$

One of the simplest applications of these formulas is in finding the remaining functions, when any one function of an angle is given.

For example, let $\sin A = \frac{1}{2}$.

Then $$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \tfrac{1}{4}} = \tfrac{1}{2}\sqrt{3},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \tfrac{1}{3}\sqrt{3},$$

$$\cot A = \frac{1}{\tan A} = \frac{1}{\frac{1}{3}\sqrt{3}} = \sqrt{3},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\frac{1}{2}\sqrt{3}} = \tfrac{2}{3}\sqrt{3},$$

$$\csc A = \frac{1}{\sin A} = \frac{1}{\frac{1}{2}} = 2.$$

Again, let $\tan A = 5$.

Then $$\cot A = \frac{1}{\tan A} = \tfrac{1}{5},$$

$$\sec A = \sqrt{1 + \tan^2 A} = \sqrt{26},$$

$$\csc A = \sqrt{1 + \cot^2 A} = \tfrac{1}{5}\sqrt{26},$$

$$\sin A = \frac{1}{\csc A} = \tfrac{5}{26}\sqrt{26},$$

$$\cos A = \frac{1}{\sec A} = \tfrac{1}{26}\sqrt{26}.$$

EXERCISE IV

Find all the other functions of A, if

1. $\sin A = \frac{3}{4}$.
2. $\cos A = \frac{1}{5}$.
3. $\tan A = 3$.
4. $\cot A = 6$.
5. $\sec A = 2$.
6. $\csc A = 10$.
7. $\sin A = \frac{12}{13}$.
8. $\cos A = \frac{3}{5}$.
9. $\tan A = \frac{3}{8}$.
10. $\sec A = a$.

11. Find all the other functions of 45° from the fact that $\tan 45° = 1$.

12. Find all the other functions of 60° from the fact that $\sec 60° = 2$.

13. Find all the other functions of 30° from the fact that $\cot 30° = \sqrt{3}$.

14. Find all the other functions of A in terms of

(a) $\sin A$, (b) $\cos A$, (c) $\tan A$, (d) $\sec A$.

15. Transform each of the following expressions into another form which shall contain $\sin A$ only:

(a) $\sin A \cos^2 A + \dfrac{\tan A}{\cos A} - \dfrac{\cot A}{\cos A \sin A}$.

(b) $\sin A \csc A - \dfrac{\sin A}{\cos^2 A} + \dfrac{\cos A}{\cot A}$.

(c) $\sec^2 A + \cos^2 A - \tan^2 A \cot^2 A$.

16. Transform each of the following expressions into another form which shall contain $\cos A$ only:

(a) $\sin A \cos A \tan A \cot A$.

(b) $\sec A - \dfrac{1}{\cos A} + \dfrac{\tan A}{\sin A}$.

(c) $\sin^2 A + \cos^2 A - \tan^2 A - \cot^2 A$.

17. Transform each of the following expressions into another form which shall contain tan A only:

(*a*) $\sin A \cos A + \cos A \tan A + \sin A \cot A$.

(*b*) $\sin A \cot A - \dfrac{1 - \sin A \cos A}{1 + \sin A \cos A}$.

(*c*) $\sec A - \csc A + \cot A$.

6. Trigonometric identities. — Another important use of the formulas obtained in Art. 5 is in proving the identity of certain trigonometric expressions.

The student should here make himself familiar with the distinction between identities and conditional equations, such as usually occur in algebra. An identity is an equation in which the two members are equal for every possible value of the variables in it; while a conditional equation holds only for certain values of these variables. For example, $2x + 3x = 5x$ and $\sin^2 x + \cos^2 x = 1$ are identities; while $x^2 - 2x = 5$ and $\sin x = \cos x$ are conditional equations.

We are at present concerned only with the proof of identities. The method to be employed is to change the form of one of the members of the equation, by the application of the formulas of Art. 5, until it has been made to assume the form of the other member. Skill in choosing the proper formulas to accomplish this result with the least labor is acquired only after considerable practice; but the form which we wish to obtain will soon suggest to the student which formulas he should use.

The following examples illustrate the mode of procedure:

Example 1. Prove that $(1-\cos^2 A)\sec^2 A = \tan^2 A$.

If we replace $1-\cos^2 A$ by its equal $\sin^2 A$, and $\sec^2 A$ by $\dfrac{1}{\cos^2 A}$, the first member becomes $\dfrac{\sin^2 A}{\cos^2 A}$, which is evidently equal to $\tan^2 A$.

Example 2. Prove that $\csc^2 A \tan^2 A - 1 = \tan^2 A$.

Here it is probably simplest to express the first member in terms of the sine and cosine. This is often advisable. It becomes

$$\frac{1}{\sin^2 A}\cdot\frac{\sin^2 A}{\cos^2 A}-1=\frac{1}{\cos^2 A}-1=\sec^2 A-1=\tan^2 A.$$

Example 3. Prove that $\sec A - \tan A \sin A = \cos A$.

As in the previous example, express the first member in terms of the sine and cosine. It becomes

$$\frac{1}{\cos A}-\frac{\sin A}{\cos A}\sin A=\frac{1-\sin^2 A}{\cos A}=\frac{\cos^2 A}{\cos A}=\cos A.$$

EXERCISE V

Prove the following identities:

1. $\cos A \tan A = \sin A$.
2. $\sin A \sec A = \tan A$.
3. $\cos A \csc A = \cot A$.
4. $\sin A \sec A \cot A = 1$.
5. $\sin^2 A \sec^2 A = \sec^2 A - 1$.
6. $\dfrac{\sin A}{\csc A}+\dfrac{\cos A}{\sec A}=1$.
7. $\dfrac{\tan A}{\cot A}-\dfrac{\sec A}{\cos A}=-1$.

8. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A.$

9. $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A.$

10. $\tan A + \cot A = \sec A \csc A.$

11. $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A.$

12. $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = 2 \csc A.$

13. $\dfrac{1}{\tan A + \cot A} = \sin A \cos A.$

14. $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}.$

15. $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \sin A + \cos A.$

16. $\sec^2 A \csc^2 A = \tan^2 A + \cot^2 A + 2.$

17. $\sin A (\tan A - 1) - \cos A (\cot A - 1) = \sec A - \csc A.$

18. $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A).$

19. $\sec A - \tan A = \dfrac{\cos A}{1 + \sin A}.$

20. $(\sin A + \cos A)(\tan A + \cot A) = \sec A + \csc A.$

21. $\tan^2 A - \cot^2 A = \sec^2 A \csc^2 A (\sin^2 A - \cos^2 A).$

22. $(1 + \cot A - \csc A)(1 + \tan A + \sec A) = 2.$

23. $2 \operatorname{vers} A + \cos^2 A = 1 + \operatorname{vers}^2 A.$

24. $\cot^4 A + \cot^2 A = \csc^4 A - \csc^2 A.$

25. $(\sin A + \csc A)^2 + (\cos A + \sec A)^2 = \tan^2 A + \cot^2 A + 7$

CHAPTER II

RIGHT TRIANGLES

7. Solution of right triangles. — A right triangle may be solved (that is, the unknown parts may be found) when any two parts, one of which is a side, are given.

It is convenient to letter the triangle as in Fig. 7, the small letters a, b, c representing the lengths of the sides opposite the corresponding angles.

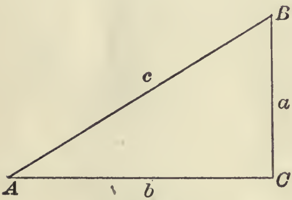


FIG. 7.

When either A or B is known, the other angle may be determined by subtracting the given angle from 90°. When any two of the sides are known, the third side may be found by the aid of the equation $a^2 + b^2 = c^2$. But when it is necessary to find the angles, having given two sides, or to find the other sides, having given one side and an angle, the trigonometric functions must be introduced. The unknown parts may then be found by the aid of the definitions of the functions,

$$\sin A = \cos B = \frac{a}{c},$$

$$\cos A = \sin B = \frac{b}{c},$$

$$\tan A = \cot B = \frac{a}{b}.$$

Choose that one of these formulas which contains the two given parts and one unknown part, and solve for the unknown part; its numerical value may then be found by the aid of a table of trigonometric functions.

If necessary, repeat the operation with one of the other formulas to find the remaining parts.

Each of the unknown parts should be determined directly from the given parts, without using the results of a previous operation. The accuracy of the work may then be tested by determining one of the given parts from the parts just found.

The different cases which may arise might be separated, and the particular formulas to be used in each indicated; but it is thought best to let the student determine the best method of solving each problem, only giving a few typical examples.

EXERCISE VI

1. Given $c = 16$ and $A = 26° 15'$; find the remaining parts.

To obtain a, use

$$\sin A = \frac{a}{c}, \quad \text{or} \quad a = c \sin A = 16 \times .44229 = 7.0766.$$

To obtain b, use

$$\cos A = \frac{b}{c}, \quad \text{or} \quad b = c \cos A = 16 \times .89687 = 14.35.$$

The accuracy of these results may be tested by determining whether they satisfy the equation $a^2 + b^2 = c^2$.

The angle B is the complement of A, or $63° 45'$.

2. Given $a = 8$ and $b = 12$; find the remaining parts.

Here c may be found at once from the equation $c^2 = a^2 + b^2$.

From this equation, $c = \sqrt{64 + 144} = 14.44$.

To obtain A, use $\tan A = \frac{a}{b} = \frac{8}{12} = .66666$.

From this, by the aid of the tables,

$$A = 33° 41'. \quad B = 90° - A = 56° 19'.$$

These results may be tested by determining whether they satisfy either the equation for $\sin A$ or $\cos A$.

Find the remaining parts in each of the following problems:

3. $a = 3, \quad b = 4.$
4. $a = 9, \quad b = 40.$
5. $a = 12, \; c = 13.$
6. $a = 60, \; c = 61.$
7. $b = 13, \; c = 85.$
8. $a = 12, \; A = 34° 42'.$
9. $c = 15, \; B = 23° 34'.$
10. $b = 17, \; A = 13° 52'.$

8. Solution by the aid of logarithms. — The student should now make himself familiar with the use of logarithms by reading the explanation of the tables, and by doing some of the problems given there. Logarithms are not of great use in solving right triangles, since they do not greatly decrease the labor involved; but it is best to become familiar with their use in solving simple problems.

EXERCISE VII

1. Given $a = 12.73$ and $c = 43.18$; find the remaining parts.

$$\sin A = \frac{a}{c}. \qquad \sin B = \frac{b}{c}, \text{ or } b = c \sin B.$$

Hence $\log \sin A = \log a - \log c$. Hence $\log b = \log c + \log \sin B$.

$$\begin{aligned} \log a &= 1.10483 \\ \log c &= \underline{1.63528} \\ \log \sin A &= 9.46955 \\ A &= 17° \; 8' 47''. \\ B = 90° - A &= 72° 51' 13''. \end{aligned} \qquad \begin{aligned} \log c &= 1.63528 \\ \log \sin B &= \underline{9.98026} \\ \log b &= 1.61554 \\ b &= 41.26. \end{aligned}$$

2. Given $A=43°48'$ and $b=67.92$; find the remaining parts.

$$\cos A = \frac{b}{c}, \text{ or } c = \frac{b}{\cos A}. \qquad \tan A = \frac{a}{b}, \text{ or } a = b \tan A.$$

Hence $\log c = \log b - \log \cos A$. Hence $\log a = \log b + \log \tan A$.

$$\begin{aligned} \log b &= 1.83200 \\ \log \cos A &= \underline{9.85839} \\ \log c &= 1.97361 \\ c &= 94.10. \end{aligned} \qquad \begin{aligned} \log b &= 1.83200 \\ \log \tan A &= \underline{9.98180} \\ \log a &= 1.81380 \\ a &= 65.13. \end{aligned}$$

Find the remaining parts in each of the following problems:

3. $a=5678$, $b=6789$.
4. $a=2222$, $c=3333$.
5. $a=.4545$, $c=.5454$.
6. $a=4567$, $A=23°52'$.
7. $c=.8765$, $B=27°25'$.
8. $a=206.14$, $A=24°24'$.
9. $c=2.383$, $B=32°42'$.
10. $a=1758$, $b=1312.7$.
11. $a=.581$, $b=13.5$.
12. $b=75.84$, $A=87°32'$.
13. $b=8.4$, $c=14$.
14. $a=795$, $b=164$.
15. $c=543.3$, $B=17°20'35''$.
16. $a=1.456$, $A=3°26'42''$.
17. $a=.0065$, $c=.0094$.
18. $c=765$, $A=84°16'$.
19. $c=1000$, $A=75°$.
20. $a=1.006$, $c=1.06$.

9. Heights and distances. — One of the applications of Trigonometry is in finding the height of objects or the distance between points, when that height or distance cannot be easily measured. This is accomplished by measuring other easily accessible lines and certain angles, and computing the desired lines from these data.

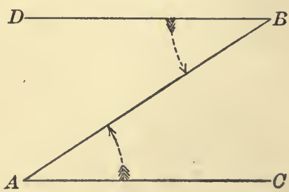


FIG. 8.

It is necessary to explain a few terms used in such problems. Let A and B be two points not in the same horizontal plane, and let AC and BD

be horizontal lines through these points. If B is observed from A, the angle CAB is called the **angle of elevation** of B from A. If A is observed from B, the angle DBA is called the **angle of depression** of A from B.

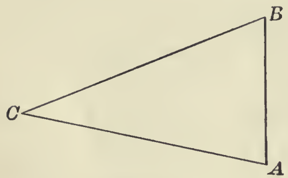


FIG. 9.

Again, let AB be any object, and let C be the point of observation. The angle ACB is spoken of as the **angle which AB subtends at C.**

The earlier problems of the following exercise give simple right triangles, and can be solved by the student without further assistance.

EXERCISE VIII

1. A vertical pole 13 ft. high casts a shadow 21 ft. long. What is the angle of elevation of the sun at this moment?

2. How high is the tower which casts a shadow 125 ft. long, when the angle of elevation of the sun is $28° 20'$?

3. What is the angle of elevation of the top of a tower 150 ft. high at a point in the same horizontal plane as its foot and 200 ft. distant?

4. From the top of a tower 100 ft. high, the angle of depression of an object in the same horizontal plane as its foot is found to be $31°$. How far is the object from the tower?

5. The angle of elevation of the top of a tower from a point in the same horizontal plane and 57 ft. from its foot is found to be $22° 14'$. How high is the tower?

6. A building 95 ft. high stands on the bank of a river. The angle of elevation of the top of the building from the opposite bank of the river is found to be $25° 10'$. Find the breadth of the river.

7. The two equal legs of an isosceles triangle are each 10 ft. and the opposite side is 6 ft. Find the three angles.

Note. — Draw the altitude of the triangle. Then there will be two equal right triangles formed, in which the base is 3 ft. and the hypotenuse is 10 ft.

8. From a point directly in front of the middle of a building and 100 ft. distant, the length of the building is found to subtend an angle of 34° 15′. How long is the building?

9. Two trees stand directly opposite each other on a straight road 80 ft. wide. From a point in the centre of the road the line joining their trunks subtends an angle of 5° 28′. How far is the point from the trees?

10. A circular balloon 10 yd. in diameter is noted by an observer to subtend an angle of 40′. At the same time the angle of elevation of its apparent lowest point is 50° 10′. Find, approximately, the height of the balloon.

11. A flagstaff 20 ft. long stands on the corner of a building 150 ft. high. Find the angle subtended by the flagstaff at a point 100 ft. from the foundation of the corner.

12. A strip of river bank is straight. It is 300 ft. long and it subtends a right angle at a point on the opposite shore. The angle between a line drawn from the point to one end of the strip and the perpendicular from the point to the strip is 15°. Find the width of the river.

13. A ladder 30 ft. long leans against a house on one side of a street making an angle of 60° with the street. On turning the ladder about its foot till the top touches the house on the opposite side, the angle is found to be 30°. Find the width of the street.

14. To find the height of a chimney a distance of 125 ft. is measured from its base. From the point thus reached the angle of elevation of the top of the chimney is found to be 48° 25′. What is the height of the chimney?

15. From the top of a telegraph pole 35 ft. tall a wire 50 ft. long is stretched to the ground. Find the angle which the wire makes with the ground.

16. A man lies on the ground with his eye to the edge of a well and, looking into the well, he sees the reflection of the opposite edge in the water. The direction in which he looks makes with the vertical an angle of 17°. The well is 6 ft. broad. Find the distance from the edge of the well to the surface of the water.

Note. — The angle of reflection equals the angle of incidence.

17. From one point of observation the angle of elevation of the top of a building is found to be 35°. The observer walks 100 ft. directly away from the building in the same horizontal plane and then finds the angle of elevation of the top of the building to be 25°. Find the height of the building.

Solution. — Let AB (Fig. 10) represent the face of the building, and let C be the first point of observation and D the second point. Then $ACB = 35°$, $CDB = 25°$, and $CD = 100$ ft. Draw CE perpendicular to BD. Then in the right triangle CDE,

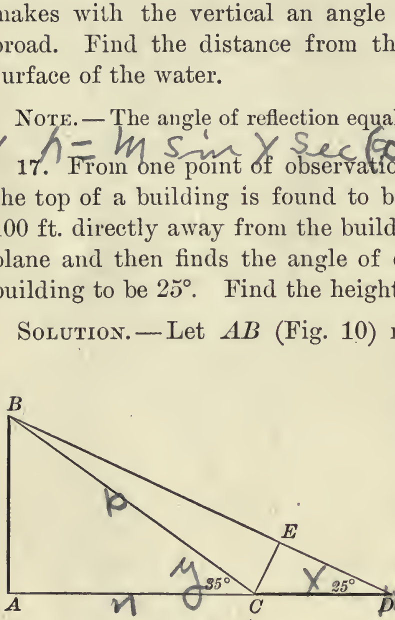


Fig. 10.

$$CE = CD \sin 25°,$$
$$= 100 \sin 25°.$$

In the right triangle BCE, the angle $CBE = 10°$.

Hence $$CB = \frac{CE}{\sin 10°} = \frac{100 \sin 25°}{\sin 10°}.$$

In the right triangle BAC,

$$AB = CB \sin 35° = \frac{100 \sin 25° \sin 35°}{\sin 10°}.$$

Applying logarithms,

$$\begin{aligned} \log 100 &= 2.00000 \\ \log \sin 25° &= 9.62595 \\ \log \sin 35° &= \underline{9.75859} \\ & 11.38454 \\ \log \sin 10° &= \underline{9.23967} \\ \log AB &= 2.14487 \\ AB &= 139.6. \end{aligned}$$

Second Solution. — The following is another method of solution in which natural functions are used.

In the right triangle ABC, $\frac{x}{y} = \tan 35°.$

In the right triangle ABD, $\frac{x}{y+100} = \tan 25°.$

Solving this pair of equations for x, we have

$$x = \frac{100 \tan 35° \tan 25°}{\tan 35° - \tan 25°} = \frac{100 \times .70021 \times .46631}{.2339}.$$

Applying logarithms,

$$\begin{aligned} \log 100 &= 2.00000 \\ \log .70021 &= 9.84522 \\ \log .46631 &= \underline{9.66868} \\ & 11.51390 \\ \log .2339 &= \underline{9.36903} \\ \log AB &= 2.14487 \\ AB &= 139.6. \end{aligned}$$

18. The shadow of a tower standing on a level plane is found to be 60 ft. longer when the sun's altitude is 30° than when it is 45°. Prove that the height of the tower is $30(1 + \sqrt{3})$.

Note. — Use second method.

19. Find the height of a chimney if the angle of elevation of its top changes from 31° to 40° on walking toward it 80 ft. in a horizontal line through its base.

20. From the top of a cliff 100 ft. high the angles of depression of two buoys, which are in the same vertical plane as the observer, are found to be 5° and 15°. Find the distance between the buoys.

CHAPTER III

FUNCTIONS OF ANY ANGLE

10. Directed lines. — If a point moves from A to B in a straight line, we shall say that it generates the line AB; if it moves from B to A, it generates the line BA. The position from which the generating point starts is called the **initial** point of the line; the point where it stops, the **terminal** point. In our study of Geometry, AB and BA meant the same thing,—the line joining A and B without regard to its direction. *But we shall now find it convenient to distinguish between AB and BA by calling one of them positive and the other negative.*

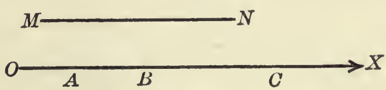


Fig. 11.

When either direction along a line has been chosen as the positive direction (as OX in Fig. 11), then all distances measured along this line, or any line parallel to it, in this direction, shall be represented by positive numbers, and those in the opposite direction by negative numbers.

In Fig. 11, if OX is chosen as the positive direction, AB and MN are positive lines, while CB is a negative line. The measures of AB and MN must, therefore, be positive numbers; but CB must be represented by a negative number.

The lines which we shall use in our further study will all be directed lines, unless the opposite is expressly stated, and we shall be concerned not so much with the lines themselves as with the measure of those lines. We shall therefore find it convenient to use the symbol AB to represent "the measure of the line AB" (its absolute magnitude with its proper sign attached); while if we wish to speak of the line itself, we shall write "the line AB."

Since the lines AB and BA are equal in magnitude but opposite in direction,

$$AB = -BA.$$

11. THEOREM. *If A, B, and C are any three points on a straight line,*

$$AB + BC = AC.$$

When the three points are situated as in Fig. 11, the theorem is evident, since all the numbers are positive, and the measure of AC equals the sum of the measures of AB and BC.

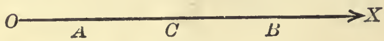


FIG. 12.

In Fig. 12, for the same reason as above

$$AB = AC + CB.$$

But, since we are dealing with numbers, we may treat this as an ordinary equation. Hence

$$AC = AB - CB$$
$$= AB + BC.$$

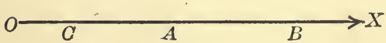


FIG. 13.

In Fig. 13, $CA + AB = CB.$

Hence $-AC + AB = -BC$, or $AC = AB + BC.$

Let the student place the points in other positions, and show that the theorem holds in all cases.

This theorem may be readily extended into the following:

If A, B, C ⋯ J, K are any number of points on a line, $AB + BC + \cdots IJ + JK = AK.$

12. Angles. — In the previous chapters only angles less than 90° have been considered; but the student is already familiar, in his study of plane geometry, with angles between 90° and 180°. In the further study of mathematics it is convenient *to regard an angle as formed by a straight line revolving in the plane about some point in the line.* If a line starts from the position OA and revolves in a fixed plane about the point O into the position OB, it is said to generate the angle AOB. The position from which the moving line starts is called the **initial** side of the angle; the position where it stops, the **terminal** side. We shall regard the *amount of such rotation* as the *measure* of the angle. In this sense there is no restriction on the size of an angle, since there is no limit to the possible amount of rotation of the moving line; after performing a complete revolution in either direction, it may continue to rotate as many times as we please, generating angles of any magnitude in either direction.

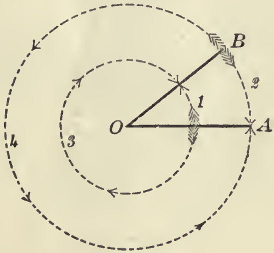


Fig. 14.

Since there are two directions in which the moving line may be made to rotate, it is found convenient to distinguish between them by using positive and negative signs, just as in algebra it is found convenient to attach signs to the ordinary arithmetic number. There is no special reason for choosing either direction rather than the other as positive; but the usual convention is *to regard as positive an angle formed by a line revolving in the direction opposite to the direction of rotation of the hands of a clock; the clockwise direction of rotation is then negative.* In reading an angle in the ordinary way a letter on the initial line is read first; for example, AOB means the angle formed by a line in rotating from OA to OB, while BOA means the angle formed by a line in rotating from OB to OA. This method of reading an angle is evidently ambiguous, since there is an indefinite number of positive and negative angles, all of which must be read AOB. For the rotating line, starting from OA, may make any number of complete revolutions in either the positive or negative direction and then continue to the position OB, and any one of these angles must still be read AOB. Such angles which have the same initial and terminal sides are called **congruent** angles. It will be seen later that it is usually unnecessary to distinguish between congruent angles, since their trigonometric functions will be found to be the same; but we shall understand that the smallest of the congruent angles is meant unless another angle is indicated by an arrow in the figure.

13. The measure of angles. — The angular unit of measure with which the student is familiar is the degree, or $\frac{1}{90}$th part of one right angle. When we wish to measure an angle, we say that it contains a certain number of these units. This system is convenient for numerical problems in the solution of triangles, etc. But there is another unit which is almost universally used in higher mathematics. On the circumference of any circle lay off an arc AB equal in length to the radius, and join its extremities to the centre of the circle. Since the ratio of a circumference to its radius is constant (and equal to 2π), the arc AB is always the same fractional part of a complete circumference, namely $\frac{1}{2\pi}$. Then, since angles at the centre of a circle are proportional to the arcs which they subtend, the angle AOB is the same fractional part of four right angles, and is, therefore, constant.

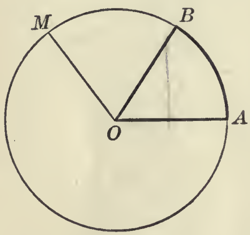


FIG. 15.

This angle AOB is the unit angle of circular measure, and is sometimes called a **radian**. *The* **circular measure** *of any angle is the ratio of that angle to this unit angle, or the number of times the given angle contains the unit angle.*

It is usually written without the name of the unit.

Since a complete revolution, or four right angles, has been shown to contain 2π radians, the circular measure of a right angle is $\frac{\pi}{2}$; of an angle of 60°, $\frac{\pi}{3}$; of 45°, $\frac{\pi}{4}$; etc.

If α is the circular measure of any angle AOM and r is the radius of the circle,

$$\alpha = \frac{AOM}{AOB} = \frac{\text{arc } AM}{\text{arc } AB} = \frac{\text{arc } AM}{r}.$$

Hence *the circular measure of any angle is equal to the arc it subtends divided by the radius, and, conversely, the* *length of any arc equals the radius multiplied by the number expressing the circular measure of the angle.* Or arc $AM = \alpha r$. Since there are 2π radians in a complete revolution, or 360°,

$$\text{one radian} = \frac{360°}{2\pi} = 57° \, 17' \, 45'', \text{ approximately.}$$

We have seen that it is convenient to call angles formed in the counterclockwise direction of rotation positive; the circular measures of such angles are, therefore, expressed by positive numbers, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc.; while the circular measures of angles formed in the clockwise direction of rotation are negative numbers, $-\frac{\pi}{3}$, $-\frac{3\pi}{4}$, etc.

EXERCISE IX

1. Express in circular measure the angles 15°, 30°, 40°, 120°, 250°, 300°.

2. Express in degrees, minutes, and seconds the angles $\frac{\pi}{5}$, $\frac{\pi}{9}$, $\frac{2\pi}{3}$, $\frac{5\pi}{6}$, 5π.

3. In a circle whose radius is 10 inches, what is the length of an arc which subtends at the centre of the circle an angle of $\frac{3\pi}{4}$, $\frac{\pi}{6}$, $\frac{3\pi}{2}$?

4. In a circle whose radius is 5 inches, what is the circular measure of an angle at the centre which subtends an arc of 10 inches?

14. Addition of directed angles. — If the moving line starts from OA (in any one of these figures) and rotates first through the angle AOB, and then through the angle BOC, it is evident that the position OC which the line finally reaches is the same as if, starting from

OA, it had rotated through the single angle AOC. The angle AOC is called the sum of the angles AOB and BOC. That is $\angle AOB + \angle BOC = \angle AOC$.

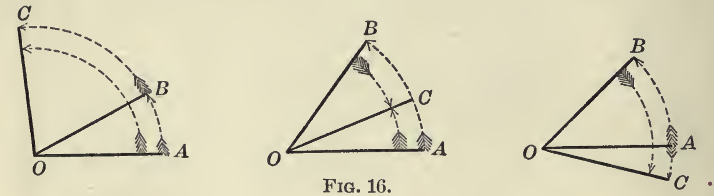
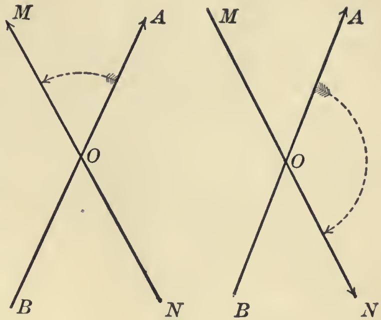


Fig. 16.

By a process similar to that used in Art. 11, it may be shown that the measure of the sum of any number of angles is equal to the sum of their measures, or that the above equation holds when, instead of the angle, we mean, in each case, the measure of the angle. It will not be necessary to distinguish further between an angle and its measure.

15. Angles between directed lines. — We shall have occasion to speak of the angle from the positive direction of one line to the positive direction of another line, and it is convenient to have a symbol to represent it, since the ordinary method of reading an angle is not sufficient. We shall adopt the following notation: (AB, MN) *shall indicate the angle from the positive direction of AB to the positive direction of MN.* (Sometimes it is more convenient to use single letters to represent the lines, as a and b, when the symbol (a, b) will be used for the same purpose.) In this symbol it is entirely immaterial whether we write AB or BA, MN or NM; since it is always to indicate *the angle between their positive directions without reference to the way they are read.*

Here again, as in the ordinary method of reading an angle, the symbol is ambiguous, since it may represent any one of the congruent angles; but the smallest of these will be understood unless another is indicated in the figure. If the arrows indicate the positive directions of the lines, in Fig. 17 (a) $(AB, MN) = AOM$, a positive acute angle; while in Fig. 17 (b) $(AB, MN) = AON$, a negative obtuse angle.



(a) FIG. 17. (b)

16. Functions of angles of any magnitude. — We must now define the trigonometric functions of an angle of any magnitude. For this purpose, let the plane be divided into four **quadrants** by a pair of indefinite lines perpendicular to each other, — one horizontal, $X'X$, called the **X-axis**, and the other vertical, $Y'Y$, called the **Y-axis.** Let the positive direction of the X-axis be from left to right, and let the positive direction of the Y-axis be upward. Then all lines drawn parallel to these axes must have the same positive directions; that is, *a line drawn to the right or upward is positive, while lines drawn to the left or downward are negative.* The positive direction of any line not parallel to one of the axes will be determined by other conventions.

The point where the axes cross is called the **origin.** The quadrants are numbered as in the figure.

For the purpose of defining the trigonometric functions of any angle, let the angle be placed with its vertex at the origin and with its initial line coincident with OX, the positive segment of the X-axis. It is called an angle in the first, second, third, or fourth quadrants, according to the position of its terminal line; that is, angles from O to $\frac{\pi}{2}$ are angles in the first quadrant, $\frac{\pi}{2}$ to π in the second quadrant, etc.

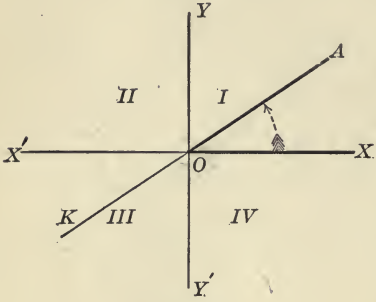


FIG. 18.

The positive direction of segments measured along the terminal line of an angle will always be from the origin along that terminal line.

For example, if we are considering the angle XOA, OA is positive and OK is negative; while if we are considering the angle XOK, OK is positive and OA is negative.

From any point P on the terminal line of the angle XOA drop the perpendicular MP to the X-axis. Then the trigonometric functions of the angle XOA are defined as follows:

$$\sin XOA = \frac{MP}{OP};$$

$$\cos XOA = \frac{OM}{OP};$$

$$\tan XOA = \frac{MP}{OM}.$$

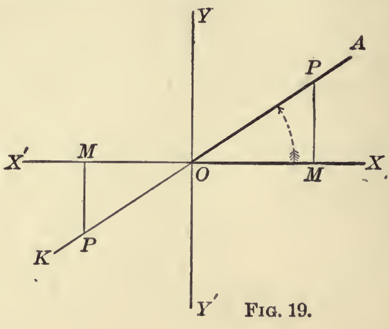


FIG. 19.

The secant, cosecant, and the cotangent are defined as the reciprocals of the cosine, sine, and tangent.

In each of these definitions the direction as well as the magnitude of the lines is to be considered.

The point P may be taken as any point either on the positive segment, OA, of the terminal line, or on OK, the negative segment of that line. For the similarity of the triangles shows that the ratios are the same numerically for all positions of P; and the signs of the ratios are also unchanged by a change of P from the positive segment OA to the negative segment OK, since this change simply reverses the signs of MP, OP, and OM.

From these definitions it appears that the value of any function is the same for all congruent angles, since we are concerned only with the positions of the initial and terminal lines. The functions of $\angle XOA$ and $\angle XOK$, which differ by π, are numerically equal, but may differ in sign.

17. Algebraic signs of the functions. — These definitions, when applied to angles less than $\frac{\pi}{2}$, will be seen to agree with the definitions in Chap. I. In the first quadrant all the functions are positive, since OM, MP, and OP are all positive. But in the other quadrants the signs of some of the functions will be seen to be negative. It will be simplest always to place P on the positive segment

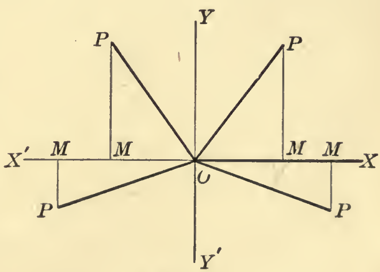


Fig. 20.

of the terminal line of the angle, so that OP shall always be positive, and it will only be necessary to consider the signs of OM and MP.

In the second quadrant MP is positive and OM is negative. *The sine and cosecant of angles in the second quadrant are, therefore, positive and all the other functions negative.*

In the third quadrant both MP and OM are negative, so that *all the functions of angles in the third quadrant are negative except the tangent and the cotangent.*

In the fourth quadrant OM is positive and MP negative. *The cosine and secant of angles in the fourth quadrant are, therefore, positive and all the other functions negative.*

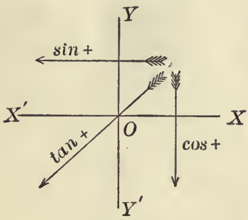


FIG. 21.

The student may find Fig. 21 useful in remembering the signs of the functions in the different quadrants. The function written at the head of each arrow is positive in the quadrants through which the arrow passes.

EXERCISE X

1. Determine the signs of the functions of the following angles: 100°, 200°, 300°, 400°, 500°, 600°, 700°, −50°, −150°, −350°

2. In which quadrant must an angle lie, if
   (*a*) its sine and cosine are negative,
   (*b*) its cosine and tangent are negative,
   (*c*) its sine is positive and its tangent is negative,
   (*d*) its cosine is negative and its tangent is positive?

3. In which quadrants may an angle lie, if its cosine and secant are negative?

4. For what angles in each quadrant are the absolute values of the sine and cosine the same? For which of these angles are they also alike in sign?

5. Determine the limits for x between which $\sin x + \cos x$ is positive.

18. Functions of the quadrantal angles. — The angles 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π, which have their terminal lines coincident with one of the axes, are called quadrantal angles. It is necessary to consider the definitions of the functions of these angles separately; for in some cases these definitions will be found to have no meaning.

Let XOP be a small angle and let it decrease. If the length of OP remains fixed, MP will decrease indefinitely, and the length of OM will approach that of OP. The sine $\left(\frac{MP}{OP}\right)$ and the tangent $\left(\frac{MP}{OM}\right)$ evidently decrease indefinitely and approach zero as a limit; while the cosine $\left(\frac{OM}{OP}\right)$ approaches unity as a limit. Then $\sin 0 = 0$, $\cos 0 = 1$, and $\tan 0 = 0$. Also, since $\sec A = \frac{1}{\cos A}$, $\sec 0 = 1$. But when we consider the other reciprocal functions, the cosecant and cotangent, we meet with a difficulty, since the reciprocal of zero does not exist. *There is, then, no cosecant or cotangent of a zero angle.* For small values of

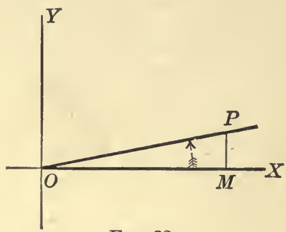


FIG. 22.

the angle XOP, the cosecant $\left(\frac{OP}{MP}\right)$ and the cotangent $\left(\frac{OM}{MP}\right)$ are large, and they continue to increase without limit as XOP approaches zero. This fact is usually represented by writing $\csc 0 = \infty$, and $\cot 0 = \infty$. But these equations must be distinctly understood to be abbreviations of the statement that, as an angle approaches zero, its cotangent and cosecant increase indefinitely; they must never be understood to mean that the cotangent and cosecant of a zero angle exists.

It is in this sense only that the symbol ∞ will be used throughout this work. If, in the expression $\frac{1}{x}$, x is made to decrease indefinitely, the value of $\frac{1}{x}$ will increase indefinitely. This may be abbreviated into $\left.\frac{1}{x}\right|_{x \doteq 0} = \infty$, which should be read "$\frac{1}{x}$ increases indefinitely, as x approaches zero." This is sometimes abbreviated still further into $\frac{1}{0} = \infty$. But this must never be interpreted as an ordinary equation, in which one member is equal to the other. It has no meaning whatever except as an abbreviation of the sentence above.

When it is necessary to express the fact that the negative values of a variable increase numerically without limit, the symbol $-\infty$ will be used.

When the angle XOP increases toward $\frac{\pi}{2}$, OM approaches zero and MP approaches equality with OP. Then the sine $\left(\frac{MP}{OP}\right)$ and the cosecant $\left(\frac{OP}{MP}\right)$ approach unity; the cosine $\left(\frac{OM}{OP}\right)$ and the cotangent $\left(\frac{OM}{MP}\right)$ approach zero; while the tangent $\left(\frac{MP}{OM}\right)$ and the secant $\left(\frac{OP}{OM}\right)$ increase indefinitely. Then

$$\sin\frac{\pi}{2}=1,\quad \cos\frac{\pi}{2}=0,\quad \cot\frac{\pi}{2}=0,\quad \csc\frac{\pi}{2}=1,\ \tan x\Big|_{x \doteq \frac{\pi}{2}}=\infty,$$

and $\sec x\Big|_{x \doteq \frac{\pi}{2}}=\infty.$

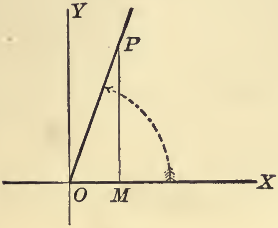


FIG. 23.

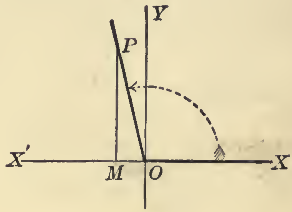


FIG 24.

It must be here noted that, if x is an angle in the second quadrant (as XOP in Fig. 24), OM is negative, and hence the tangent and secant are negative. If x is now made to decrease toward $\frac{\pi}{2}$, the negative values of these functions increase numerically without limit. This will be found to be true in every case where any trigonometric function becomes infinite. When the angle is made to approach this value from one side, the function has positive values which increase indefinitely, while if the angle is made to approach this value from the opposite side, the function has negative values which increase numerically without limit. This fact is usually expressed by writing $\tan\frac{\pi}{2}=\pm\infty.$

But the student must keep constantly in mind the fact that this is only an abbreviation for the statement made above.

Let the student show in the same manner that the functions of the other quadrantal angles are as follows:

$$\sin\pi = 0,\ \cos\pi = -1,\ \tan\pi = 0,\ \sec\pi = -1,\ \csc\pi = \pm\infty,$$
$$\cot\pi = \pm\infty.$$

$$\sin\tfrac{3}{2}\pi = -1,\quad \cos\tfrac{3}{2}\pi = 0,\quad \tan\tfrac{3}{2}\pi = \pm\infty,\quad \sec\tfrac{3}{2}\pi = \pm\infty,$$
$$\csc\tfrac{3}{2}\pi = -1,\ \cot\tfrac{3}{2}\pi = 0.$$

19. Line values of the functions. — The trigonometric functions have been defined as the ratios of lines to each other, and are, therefore, abstract numbers. But, by the aid of a circle drawn about the origin with unit radius, lines may be found which will represent, in magnitude and direction, the values and signs of the various functions. This method of representing the functions will be found to be an aid in remembering the changes in sign and value of the functions.

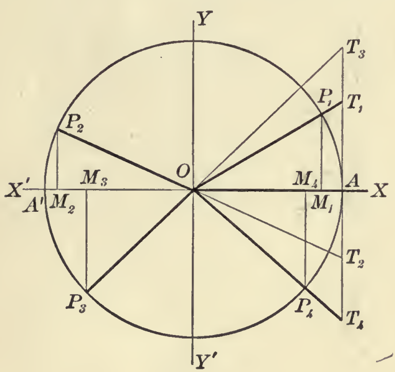


Fig. 25.

Construct a circle with its centre at the origin, having as radius a line which we shall use as the unit of length. Drop a perpendicular from the point P where the terminal line of the angle meets the circle. Then

$\sin x$, or $\frac{MP}{OP}$, is represented by MP both in magnitude and sign, for the denominator OP is always positive unity.

In like manner, OM will always represent $\cos x$.

To represent $\tan x$, it is necessary to erect a perpendicular at A, in order that the denominator of the ratio shall be the unit of measure. Then $\tan x = \frac{AT}{OA}$, and AT will represent $\tan x$. If the angle is in the second or third quadrant, the line which is to represent the tangent must still be drawn at A to meet the terminal line of the angle, produced in the negative direction; for, if it is drawn at A', the denominator of the fraction would be OA', which is equal to -1, and cannot, therefore, be used as the unit of length.

The other functions may also be represented by lines, but it seems best to think of them as the reciprocals of the three given above. Thus, the secant of any angle has the same sign as the cosine, and in magnitude is the reciprocal of the cosine.

20. Variations of the functions. — The changes in value of the functions were partially considered in Art. 18. But the student will find it much easier to remember the changes in both sign and value by making use of the line values discussed in the previous section.

The sine. — The line MP (Fig. 26) which represents the sine of XOP is seen, as the angle increases, to increase from 0 to 1 in the first quadrant; to decrease from 1 to 0 in the second quadrant; to continue to decrease from 0 to -1 in the third quadrant; and to increase from -1 to 0 in the fourth quadrant. This variation in value and sign may be represented by the aid of a curve as follows:

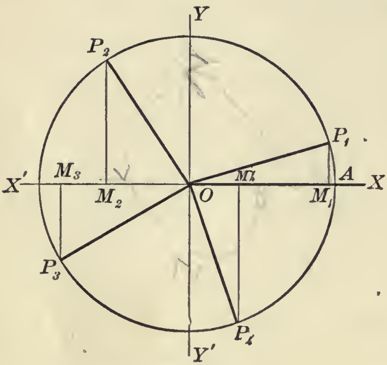


Fig. 26.

Let $y = \sin x$, where the angle x is expressed in circular measure. Then a graph of this equation may be formed just as is done in the case of an ordinary algebraic equation. Using a pair of perpendicular axes, lay off from O along OX various values of x, and at the points thus determined erect perpendiculars, whose lengths are the corresponding values of y. For example, take any convenient distance, as OA, to represent an angle of $\frac{\pi}{2}$, and at A erect a perpendicular of unit length to represent the fact that $\sin\frac{\pi}{2} = 1$. At B, midway between O and A, erect a perpendicular equal to $\frac{1}{2}\sqrt{2}$, or $\sin\frac{\pi}{4}$, etc. Do this for various values of x

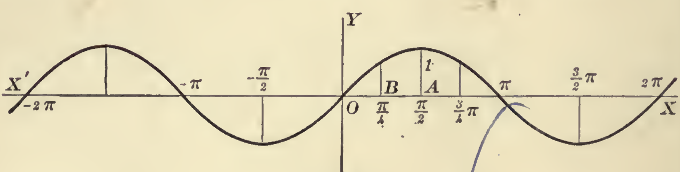


Fig. 27.

between 0 and 2π, and then pass a smooth curve through their extremities. This curve will form a picture of all the changes in value and sign as the angle changes from 0 to 2π. It will be seen that the curve crosses the axis at $x = \pi$, since $\sin \pi = 0$; that it remains below the axis from $x = \pi$ to $x = 2\pi$, forming a curve like that formed above the axis from $x = 0$ to $x = \pi$; and that this curve will be repeated indefinitely both to the right and left, if x is allowed to take on all possible values.

The horizontal distance which represents a radian, and the vertical distance which represents unity are usually chosen equal.

The cosine. — The line OM (Fig. 26), which represents the cosine of XOP, is seen, as the angle increases, to decrease from 1 to 0 in the first quadrant; to decrease

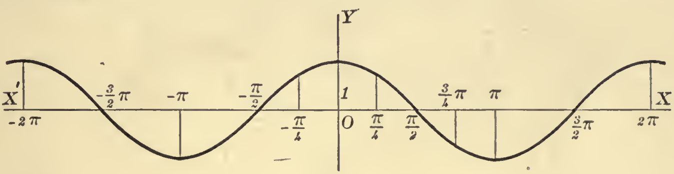


FIG. 28.

from 0 to -1 in the second quadrant; to increase from -1 to 0 in the third quadrant; and to continue to increase from 0 to 1 in the fourth quadrant. Let the student show that this variation is represented by the figure given above. The form of this curve is seen to be the same as that of the sine curve, but it is moved along the axis a distance $\frac{\pi}{2}$ to the left.

The tangent. — It has been shown that the tangent is always represented by a tangent to the unit circle drawn from A to meet the moving radius produced. Using the symbol $\pm\infty$ in the sense explained in Art. 18, it is easily seen that, in the first quadrant, AT_1 increases from 0 to $+\infty$; in the second quadrant, AT_2 increases from $-\infty$ to 0; in the third quadrant, AT_3 increases from 0 to $+\infty$; and in the fourth quadrant, AT_4 increases from $-\infty$ to 0. If we let $y = \tan x$, the variation of y will be represented by a curve starting at O (Fig. 30) and going upward indefinitely, as x approaches $\frac{\pi}{2}$. Erect at the point $x = \frac{\pi}{2}$ a line MN

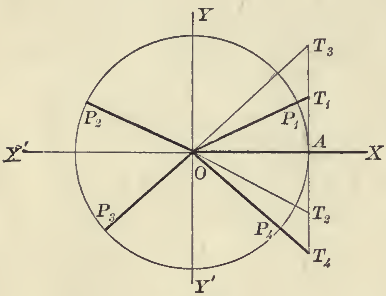


Fig. 29.

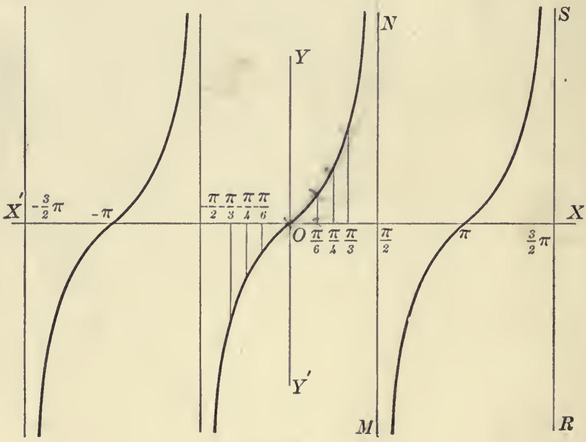


Fig. 30.

perpendicular to the axis. For every value of x less than $\frac{\pi}{2}$ there are finite values of the tangent, growing indefinitely large as x approaches $\frac{\pi}{2}$. But when $x=\frac{\pi}{2}$, there is no tangent. The curve, therefore, goes upward indefinitely at the left of MN, never touching or crossing this line, but constantly approaching it. Such a line is called an **asymptote** of the curve. When x is slightly greater than $\frac{\pi}{2}$, the tangent has a large negative value; it increases toward O as x approaches π; and increases indefinitely as x approaches $\frac{3}{2}\pi$. This branch of the curve has, then, the two lines MN and RS as asymptotes, and crosses the axis at the point $x=\pi$. When x increases from $\frac{3}{2}\pi$, the tangent again starts with a large negative value and passes through the same changes as before. If x is allowed to take all positive and negative values, there will be a series of such branches, just alike, at intervals of π from each other.

The cotangent. — Since the cotangent is the reciprocal of the tangent, it decreases from ∞ to 0 in the first quadrant; decreases from 0 to $-\infty$ in the second quadrant; decreases from $+\infty$ to 0 in the third quadrant; and decreases from 0 to $-\infty$ in the fourth quadrant.

The secant. — Since the secant is the reciprocal of the cosine, it increases from 1 to $+\infty$ in the first quadrant; increases from $-\infty$ to -1 in the second quadrant; decreases from -1 to $-\infty$ in the third quadrant; and decreases from $+\infty$ to 1 in the fourth quadrant.

The cosecant. — Since the cosecant is the reciprocal of the sine, it decreases from $+\infty$ to 1 in the first quadrant; increases from 1 to $+\infty$ in the second quadrant; increases from $-\infty$ to -1 in the third quadrant; and decreases from -1 to $-\infty$ in the fourth quadrant.

EXERCISE XI

1. How many angles less than 360° have their cosine equal to $-\frac{2}{3}$? In which quadrants do they lie?

2. How many angles less than 720° have their tangents equal to 5? In which quadrants do they lie?

3. Are there two angles less than 180° which have the same sine? the same cosine? the same tangent?

4. Construct the curve which represents the change in value and sign of the cotangent.

5. Construct the curve which represents the change in value and sign of the secant; the cosecant.

CHAPTER IV

RELATIONS BETWEEN THE FUNCTIONS

21. Relations between the functions of any angle. — The relations between the functions of an acute angle which were proved in Art. 5 may easily be shown to hold for all angles. In the triangle MOP, it is always true that

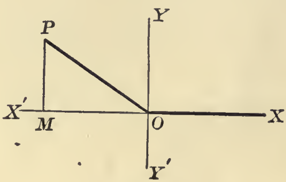


FIG. 31.

$$\overline{OM}^2 + \overline{MP}^2 = \overline{OP}^2.$$

By dividing successively by $\overline{OP}^2$, $\overline{OM}^2$, and $\overline{MP}^2$, we obtain, as in Art. 5,

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

$$1 + \tan^2 \alpha = \sec^2 \alpha,$$

$$1 + \cot^2 \alpha = \csc^2 \alpha.$$

It is also apparent from their definitions that

$$\tan \alpha = \frac{MP}{OM} = \frac{\frac{MP}{OP}}{\frac{OM}{OP}} = \frac{\sin \alpha}{\cos \alpha}, \text{ and that } \cot \alpha = \frac{\cos \alpha}{\sin \alpha}.$$

EXERCISE XII

1. If $\sin x = -\frac{1}{5}$, find the possible values of the other functions of x.

2. If $\cos x = \frac{1}{3}$, and $\sin x$ is negative, find the other functions of x.

3. If $\tan x = -3$, and x is an angle in the fourth quadrant, find the other functions of x.

4. If $\sec x = 4$, find the possible values of the other functions of x.

5. If $\sec A = \dfrac{x}{\sqrt{x^2 - y^2}}$, find the possible values of the other functions of A.

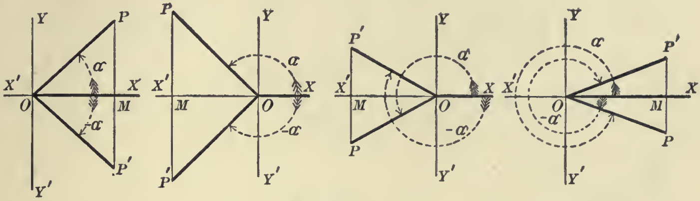
6. If $\sin A$ equal $\dfrac{x^2 - y^2}{x^2 + y^2}$, find the values of $\cos A$ and $\cot A$.

7. If
$$\sin A = \frac{m^2 + 2\,mn}{m^2 + 2\,mn + 2\,n^2},$$
prove that
$$\tan A = \pm \frac{m^2 + 2\,mn}{2\,mn + 2\,n^2}.$$

22. Functions of $-\alpha$, $\frac{\pi}{2} \pm \alpha$, $\pi \pm \alpha$, $\frac{3}{2}\pi \pm \alpha$. — In this article we shall determine the values of the functions of $-\alpha$, $\frac{\pi}{2} \pm \alpha$, etc., in terms of functions of α. In proving these relations, it is necessary to consider four cases according to the quadrant in which α lies; but the student should first go over the demonstration in the simplest case where α is an angle in the first quadrant, and then assure himself that the demonstration applies to all values of α.

The formulas will be obtained for the sine, cosine, and tangent only; the three reciprocal functions are seldom used and, if needed, formulas for them may be obtained easily from those given.

(a) Functions of $-\alpha$.



(a) (b) FIG. 32. (c) (d)

In any of these figures, let $XOP = \alpha$, and $XOP' = -\alpha$. Drop a perpendicular from P on the X-axis and continue it to meet OP'. In either figure, the right triangles MOP and MOP' are equal, since OM is common and the geometrical angles MOP and MOP' are equal. In every case OM is identical for the two angles, $OP' = OP$, and $MP' = -MP$.

$$\text{Hence}\quad \sin(-\alpha) = \frac{MP'}{OP'} = \frac{-MP}{OP} = -\sin\alpha,$$

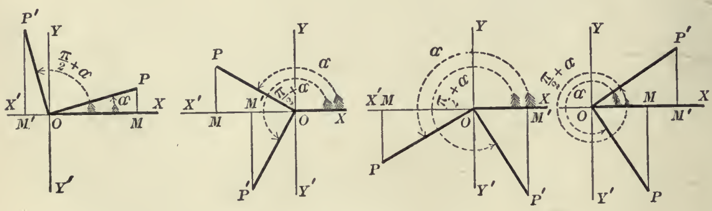
$$\cos(-\alpha) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\alpha,$$

$$\tan(-\alpha) = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\alpha.$$

Here XOP may be the angle indicated by the arrow, or it may be any one of the positive or negative congruent angles which are read in the same way. We must then understand by XOP' that one of the congruent angles which is equal to $-XOP$. For example, XOP may be the negative angle formed by revolving from OX to OP in the negative direction. Then XOP' is the positive angle formed by revolving from OX to OP'.

The above demonstration is therefore general, and α may be any angle, positive or negative. These statements apply also to the demonstrations which follow.

(*b*) Functions of $\frac{\pi}{2} \pm \alpha$.



(*a*) (*b*) Fig. 33. (*c*) (*d*)

In any of these figures, let $XOP = \alpha$ and $XOP' = \frac{\pi}{2} + \alpha$. If OP' is taken equal to OP, the triangles MOP and $M'OP'$ are equal; for the geometrical angles MOP and $M'P'O$ are equal, having their sides perpendicular. Then the sides of these triangles are equal in magnitude, but, when their signs are considered, $OM' = -MP$, and $M'P' = OM$.

Hence $\sin\left(\frac{\pi}{2} + \alpha\right) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\alpha,$

$$\cos\left(\frac{\pi}{2} + \alpha\right) = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\sin\alpha,$$

$$\tan\left(\frac{\pi}{2} + \alpha\right) = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot\alpha.$$

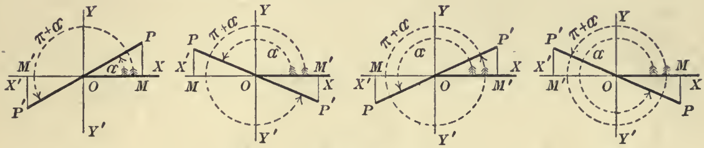
Since the demonstration just given holds for all values of α, we may replace α by $-\alpha$.

Hence $\sin\left(\frac{\pi}{2}-\alpha\right)= \cos(-\alpha)=\cos\alpha,$

$$\cos\left(\frac{\pi}{2}-\alpha\right)=-\sin(-\alpha)=\sin\alpha,$$

$$\tan\left(\frac{\pi}{2}-\alpha\right)=-\cot(-\alpha)=\cot\alpha.$$

(c) Functions of $\pi\pm\alpha$.



(a) (b) FIG. 34. (c) (d)

In either of the given figures let $XOP=\alpha$, and $XOP'=\pi+\alpha$. If OP' is taken equal to OP, the triangles MOP and $M'OP'$ are equal, and $M'P'=-MP$, and $OM'=-OM$.

Hence $\sin(\pi+\alpha)=\frac{M'P'}{OP'}=\frac{-MP}{OP}=-\sin\alpha,$

$$\cos(\pi+\alpha)=\frac{OM'}{OP'}=\frac{-OM}{OP}=-\cos\alpha,$$

$$\tan(\pi+\alpha)=\frac{M'P'}{OM'}=\frac{-MP}{-OM}=\tan\alpha.$$

Replacing α by $-\alpha$, we have

$$\sin(\pi-\alpha)=-\sin(-\alpha)=\sin\alpha,$$

$$\cos(\pi-\alpha)=-\cos(-\alpha)=-\cos\alpha,$$

$$\tan(\pi-\alpha)=\tan(-\alpha)=-\tan\alpha.$$

(*d*) Functions of $\frac{3}{2}\pi \pm \alpha$.

Let the student construct figures for the other values of α and show that

$$\sin\left(\tfrac{3}{2}\pi + \alpha\right) = -\cos\alpha,$$

$$\cos\left(\tfrac{3}{2}\pi + \alpha\right) = \sin\alpha,$$

$$\tan\left(\tfrac{3}{2}\pi + \alpha\right) = -\cot\alpha$$

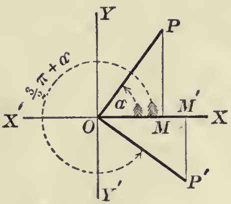


Fig. 35.

Also that

$$\sin\left(\tfrac{3}{2}\pi - \alpha\right) = -\cos\alpha,$$

$$\cos\left(\tfrac{3}{2}\pi - \alpha\right) = -\sin\alpha,$$

$$\tan\left(\tfrac{3}{2}\pi - \alpha\right) = \cot\alpha.$$

(*e*) Functions of $2\pi \pm \alpha$.

The functions of $2\pi - \alpha$ are equal to the same functions of $-\alpha$, since these angles are congruent. For the same reason the functions of $2n\pi + \alpha$ (where n is any integer and α is any angle, positive or negative) are equal to the same functions of α.

23. Reduction of the functions of any angle to functions of an angle less than $\frac{\pi}{4}$. — It appears from the results of the previous article that the functions of any angle may be obtained in terms of the functions of an angle less than, or equal to, 45°. This may be done by the aid of the formulas there derived; but since the functions of angles in the first quadrant are all positive, it is best to use the following simple rule, which is easily derived from the formulas of the previous article:

If 180° *or* 360° *is subtracted from a given angle, or if the given angle is subtracted from* 180° *or* 360° (so as to obtain in either case an acute angle), *the functions of the resulting angle will be numerically equal to the same named functions of the given angle; while if the given angle is combined in the same way with* 90° *or* 270°, *the functions of the resulting angle will be numerically equal to the co-named functions of the given angle.*

In any case, *attach to the result the proper sign of the function of the given angle, according to the quadrant in which it lies.*

For example, to obtain sin 290°. This is equal numerically to cos 20°; but since 290° is an angle in the fourth quadrant and the sine is negative in that quadrant, sin 290° = − cos 20°.

Any multiple of 360° may be added to, or subtracted from, an angle without changing the value or sign of any of its functions.

EXERCISE XIII

1. Express each of the following functions in terms of functions of angles not greater than 45°.

(*a*) sin 100°,	(*e*) sin −110°,	(*i*) cos 395°,
(*b*) cos 245°,	(*f*) cos −125°,	(*j*) tan 560°,
(*c*) tan 310°,	(*g*) tan −335°,	(*k*) sin $\frac{3}{4}\pi$,
(*d*) sec 190°,	(*h*) csc −25°,	(*l*) cos $\frac{4}{3}\pi$.

2. Find all the functions of each of the following angles. (See Art. 4.)

(*a*) 120°,	(*e*) 225°,	(*i*) 330°,
(*b*) 135°,	(*f*) 240°,	(*j*) −30°,
(*c*) 150°,	(*g*) 300°,	(*k*) −45°,
(*d*) 210°,	(*h*) 315°,	(*l*) −60°.

3. Prove geometrically the formulas for the functions of $\left(\frac{\pi}{2}-\alpha\right)$, $(\pi-\alpha)$, and $\left(\frac{3\pi}{2}-\alpha\right)$ in terms of functions of α.

4. From the formulas for the functions of $-\alpha$ and $\left(\frac{\pi}{2}+\alpha\right)$, derive algebraically the formulas for the functions of $(\pi \pm \alpha)$ and $\left(\frac{3\pi}{2}\pm\alpha\right)$.

Suggestion.—$\sin(\pi-\alpha)=\sin\left[\frac{\pi}{2}+\left(\frac{\pi}{2}-\alpha\right)\right]=\cos\left(\frac{\pi}{2}-\alpha\right)$ $=\sin\alpha$.

5. Obtain the functions of $\left(\alpha-\frac{\pi}{2}\right)$ in terms of the functions of α.

6. Obtain the functions of $(\alpha-\pi)$ in terms of the functions of α.

24*. Projection.—The projection of a point on a line is the foot of the perpendicular dropped from the point to the line.

The projection of one line on another is the locus of the projections of its points, or the distance measured along the second line from the projection of the initial point of the first line to the projection of its terminal point.

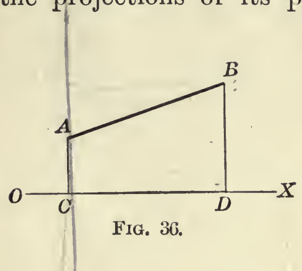


Fig. 36.

In Fig. 36, CD is the projection of AB on OX. The direction as well as the magnitude of CD must be considered, and CD will be positive or negative according as it is drawn in the positive or negative direction of OX.

Throughout the present work the projection will always be upon two perpendicular axes, $X'X$ and $Y'Y$.

When we wish to speak of the projection of a line, as AB, on the X-axis, it will be written $\textbf{proj}_x\, AB$; on the Y-axis, $\textbf{proj}_y\, AB$. These symbols will always mean *the distance measured along the axes from the projection of A to the projection of B*, and will be positive or negative according as they are drawn to the right or upward, or to the left or downward.

25*. Projection of a broken line. — *The sum of the projections on any axis of any series of lines, AB, BC, CD, etc., in which the initial point of each line is joined to the terminal point of the preceding line, is equal to the projection on the same axis of AD, the line which joins the initial point of the first line with the terminal point of the last line.*

In Fig. 37, $ab = \text{proj}_x AB$, $bc = \text{proj}_x BC$, etc. But by Art. 11,

$$ab + bc + cd = ad.$$

Hence $$\text{proj}_x AB + \text{proj}_x BC + \text{proj}_x CD = \text{proj}_x AD.$$

If the terminal D of any such broken line coincides with the initial point A, $ad = 0$, and we see that *the projection of any closed contour is zero.*

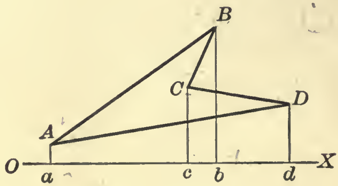


FIG. 37.

The lines AB, BC, etc., are directed lines, but it is not necessary that their positive direction should be from A to B, etc. The measure of some of the lines as read may be positive and of others negative without affecting the truth of the theorem.

26*. Projections on the axes of any line through the origin. — The definitions of the functions may be given very concisely by the aid of projection. Let P be any point on either the positive or negative segment of the terminal side of the angle XOA. The angle (OX, OP) is then the same as the angle XOA. Also $OM = \text{proj}_x OP$, and $MP = \text{proj}_y OP$. Substituting these expressions in the definitions of the sine and cosine given in Art. 16, they become

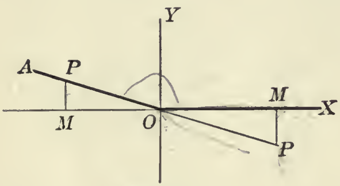


Fig. 38.

$$\sin(OX, OP) = \frac{\text{proj}_y OP}{OP},$$

and

$$\cos(OX, OP) = \frac{\text{proj}_x OP}{OP}.$$

Clearing of fractions, we have

$$\text{proj}_x OP = OP \cos(OX, OP),$$

and

$$\text{proj}_y OP = OP \sin(OX, OP).$$

The student must remember that in these formulas $\text{proj}_x OP$ and $\text{proj}_y OP$ are the measures of the distances from the projection of O to the projection of P, the signs being determined by the directions of the axes; that OP is the measure of the line OP; and that (OX, OP) is the angle from the positive direction of OX to the positive direction of OP.

27*. Projections of any line on the axes. — We shall now proceed to obtain the more general formulas for the projection of any line, AB, on the axes.

Through A, which may be either the positive or negative extremity of the line AB, draw a pair of axes, AX'

and AY', parallel to the given axes. The projections of AB on these new axes are evidently equal in magnitude and direction to its projections on the original axes, OX and OY; and the angles made by AB with these axes are the same as those made by it with the original axes. Then, from the formulas obtained in the previous article,

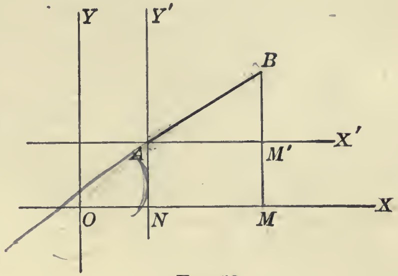


FIG. 39.

$$\textbf{proj}_x AB = AB \cos (OX, AB), \qquad [9]$$

$$\textbf{proj}_y AB = AB \sin (OX, AB). \qquad [10]$$

EXERCISE XIV

1. What are the projections on the axes of a line 5 inches long which makes an angle with the X-axis of 30°? of 100°? of 200°?

2. If $\text{proj}_x AB = 3$, and $\text{proj}_y AB = -4$, find the length of AB and the angle (OX, AB).

3. If $AB = 10$, and $\text{proj}_y AB = -3$, find the angle (OX, AB).

4. Describe an equilateral triangle (each side being 10 inches in length) by going from A to B, then to C, and back to A. What is the angle (AB, BC)? (AB, AC)? (BC, AB)?
What is the projection of AC on AB? of BC on AB? of CA on BC?

5. In the triangle of problem 4 drop a perpendicular CD from C on AB. Let its positive direction be from D to C. What is the projection of BC on this perpendicular? of CA? of AB? What is the projection of DC on BC? on AC?

28*. Functions of the sum and difference of two angles. — We shall now proceed to find the functions of the sum and difference of two angles in terms of the functions of those angles. Let the first angle α be placed in the position XOA, and let the second angle β ($\equiv AOB$) be added to α, making $\alpha + \beta$ ($\equiv XOB$). From any point P of the

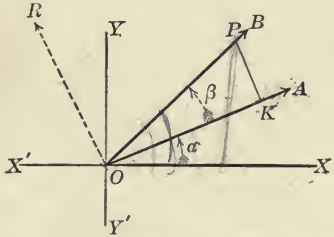


FIG. 40.

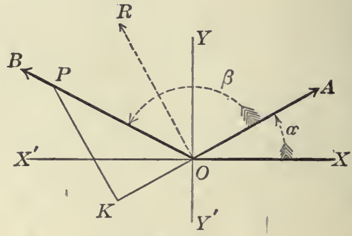


FIG. 41.

terminal line OB of β, drop a perpendicular KP on OA. The positive direction of all lines in the figure except KP is fixed by our former conventions; for we have agreed that the positive direction of the initial and terminal lines of an angle shall be from the vertex along those lines. To determine the positive direction of KP, draw from O a line OR, making with OA a positive angle $\frac{\pi}{2}$. Then OR bears the same relation to OA that OY bears to OX; and for any angle having OA as its initial line, the positive direction of KP, a perpendicular to OA, must be taken parallel to OR.

From Art. 26 we have

$$\sin(\alpha + \beta) = \frac{\text{proj}_y OP}{OP}.$$

But by Art. 25 the projection of OP on the Y-axis is equal to the sum of the projections of OK and KP on the same axis. Hence

$$\sin(\alpha+\beta)=\frac{1}{OP}[\text{proj}_y OK+\text{proj}_y KP], \qquad \text{or by [10],}$$

$$=\frac{1}{OP}[OK\sin(OX, OK)+KP\sin(OX, KP)].$$

The angle (OX, OK) is seen at once to be α, and since the positive direction of KP is the same as that of OR,

$$(OX, KP)=XOR=\alpha+\frac{\pi}{2}.$$

Hence $$\sin(OX, KP)=\cos\alpha.$$

Noting that $\dfrac{OK}{OP}=\cos\beta$ and $\dfrac{KP}{OP}=\sin\beta$, we have

$$\mathbf{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.} \qquad [11]$$

In like manner,

$$\cos(\alpha+\beta)=\frac{\text{proj}_x OP}{OP},$$

$$=\frac{1}{OP}[\text{proj}_x OK+\text{proj}_x KP],$$

$$=\frac{1}{OP}[OK\cos(OX, OK)+KP\cos(OX, KP)].$$

Here $$\cos(OX, OK)=\cos\alpha,$$

and $$\cos(OX, KP)=\cos\left(\alpha+\frac{\pi}{2}\right)=-\sin\alpha.$$

Hence $$\mathbf{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.} \qquad [12]$$

Since the magnitudes and directions of all the lines and angles in the figure have been carefully considered in this demonstration, it holds for all cases. But in order that the student may assure himself of this fact, let him draw various figures according to the same directions, using the angles of different magnitudes (for example, see Fig. 42), and go through this demonstration carefully, making sure that every statement made above applies to every figure.

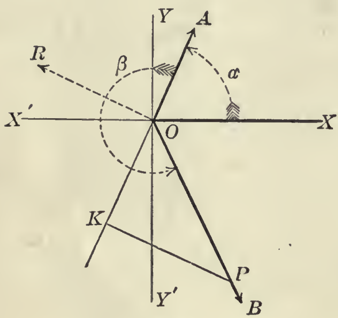


Fig. 42.

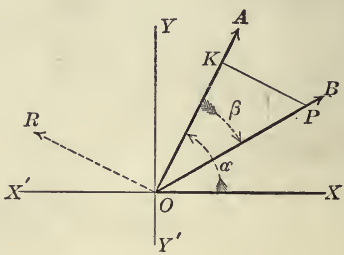


Fig. 43.

Let him also take note that the demonstration applies letter for letter when β is a negative angle (see Fig. 43). We may, therefore, replace β by $-\beta$ in formulas [11] and [12] and obtain the following formulas for the sine and cosine of the difference of two angles:

$$\sin [\alpha + (-\beta)] = \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta),$$

$$\mathbf{\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta,} \qquad [13]$$

$$\cos [\alpha + (-\beta)] = \cos \alpha \cos (-\beta) - \sin \alpha \sin (-\beta),$$

$$\mathbf{\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.} \qquad [14]$$

29. Second method of finding sin (α + β) and cos (α + β). — The following method of finding the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ avoids the use of projection and may be preferred by some teachers.

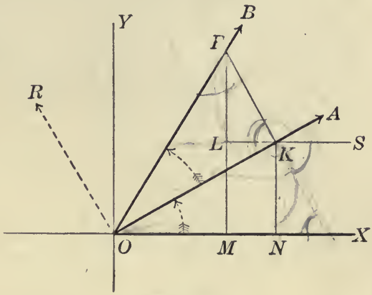


FIG. 44.

Let $\angle XOA = \alpha$, and $\angle AOB = \beta$. Then $\angle XOB = \alpha + \beta$. Through any point P of the line OB draw PM perpendicular to the X-axis and PK perpendicular to the line OA. Through K draw KN perpendicular to the X-axis and LK parallel to the same axis. Prolong LK toward the right to S. Then $\angle SKA = \alpha$ and $\angle SKP = \alpha + 90°$.

Then $$\sin(\alpha+\beta) = \frac{MP}{OP} = \frac{NK}{OP} + \frac{LP}{OP},$$

$$= \frac{NK}{OK}\cdot\frac{OK}{OP} + \frac{LP}{KP}\cdot\frac{KP}{OP}.$$

But $$\frac{NK}{OK} = \sin XOA = \sin\alpha,$$

$$\frac{OK}{OP} = \cos AOB = \cos\beta,$$

$$\frac{LP}{KP} = \sin SKP = \sin(\alpha + 90°) = \cos\alpha,$$

$$\frac{KP}{OP} = \sin AOB = \sin\beta.$$

Hence $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \qquad [11]$$

Also $$\cos(\alpha+\beta)=\frac{OM}{OP}=\frac{ON}{OP}+\frac{NM}{OP},$$
$$=\frac{ON}{OK}\cdot\frac{OK}{OP}+\frac{KL}{KP}\cdot\frac{KP}{OP}.$$

But $$\frac{ON}{OK}=\cos XOA=\cos\alpha,$$
$$\frac{OK}{OP}=\cos ACB=\cos\beta,$$
$$\frac{KL}{KP}=\cos SKP=\cos(\alpha+90^\circ)=-\sin\alpha,$$
$$\frac{KP}{OP}=\sin AOB=\sin\beta.$$

Hence $$\boldsymbol{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.} \quad [12]$$

In Fig. 44 not only are α and β acute, but their sum $\alpha+\beta$ is also acute. But the proof applies without change to Fig. 45, in which $\alpha+\beta$ is obtuse. The only difficulty which the student is likely to meet is the equation

$$OM=ON+NM,$$

which is used in finding $\cos(\alpha+\beta)$; but this is seen to hold in Fig. 45, when the direction as well as the magnitudes of the lines is considered.

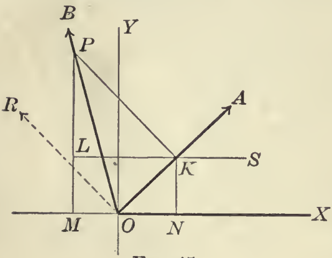


Fig. 45.

By a very slight change, this proof may be made perfectly general, so that it will apply to all values of α and β. Through O draw OR, making a positive right angle with OA, and having its positive direction from O toward R. Let KP, which is parallel to OR, have the same positive direction as OR.

If, in the above demonstration, we replace $\angle SKP$ by

(SK, KP), we shall have taken account of the directions of all lines and angles, and the demonstration will, therefore, be perfectly general, holding for all values of α and β, both positive and negative.

For example, consider Fig. 46, in which α and β are both obtuse. The construction is the same as above. From P, any point of the terminal line of β, draw PM perpendicular to the X-axis and PK perpendicular to the line OA, produced through the origin. Through K draw KN perpendicular to the X-axis and LK parallel to the same axis, prolonging LK to the right to S.

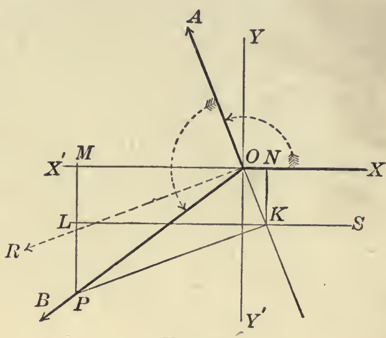


Fig. 46.

Draw OR, making a positive right angle with OA. Let the student now follow the demonstration on page 59. The first statements are evident. In considering the functions of α, since K is on the terminal line extended through the origin, OK is negative, but by definition $\sin\alpha = \frac{NK}{OK}$, and $\cos\alpha = \frac{ON}{OK}$. In determining the functions of β, turn the book so that OR points upward. Then it is seen that, by definition, $\sin\beta = \frac{KP}{OP}$, and $\cos\beta = \frac{OK}{OP}$.

In determining the functions of (SK, KP), think of K as a new origin and LS as a new X-axis. Then

$$\sin(SK, KP) = \frac{LP}{KP}, \text{ and } \cos(SK, KP) = \frac{KL}{KP}.$$

The student should have no difficulty in seeing that all the other equations hold for this figure. Let him also construct other figures according to the same directions and go through the demonstration carefully, making sure that every statement made above applies to every figure.

Since this demonstration holds for all values of α and β, we may replace β by $-\beta$ and obtain the following formulas for the sine and cosine of the difference of two angles:

$$\sin[\alpha+(-\beta)] = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta),$$

$$\mathbf{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,} \qquad [13]$$

$$\cos[\alpha+(-\beta)] = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta),$$

$$\mathbf{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.} \qquad [14]$$

30. $\tan(\alpha \pm \beta)$.

Since $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$,

we have $\tan(\alpha+\beta) = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$.

But this may be expressed entirely in terms of the tangent by dividing both numerator and denominator by $\cos\alpha\cos\beta$. This gives

$$\mathbf{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.} \qquad [15]$$

In like manner, let the student show that

$$\mathbf{\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.} \qquad [16]$$

EXERCISE XV

1. Find sin 75° from the functions of 30° and 45°.

Solution. — $\sin 75° = \sin(30° + 45°)$,

$$= \sin 30° \cos 45° + \cos 30° \sin 45°,$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{3}\cdot\tfrac{1}{2}\sqrt{2}$$

$$= \tfrac{1}{4}(\sqrt{2}+\sqrt{6}).$$

2. Find cos 75°.

3. Find tan 75°.

4. Find sin 15°, cos 15°, tan 15°.

5. If $\sin\alpha = \frac{1}{3}$ and $\sin\beta = \frac{1}{2}$, find $\sin(\alpha+\beta)$, when α and β are both acute; when they are both obtuse.

6. By the aid of formulas [11] to [14], prove the various formulas for the functions of $\left(\frac{\pi}{2}+\alpha\right)$, $(\pi \pm \alpha)$, etc. (See Art. 22.)

7. Find $\sin(\alpha+\beta+\gamma)$.

Solution. — Replacing β in formula [11] by $\beta+\gamma$, we have

$$\sin[\alpha+(\beta+\gamma)] = \sin\alpha\cos(\beta+\gamma) + \cos\alpha\sin(\beta+\gamma)$$

$$= \sin\alpha(\cos\beta\cos\gamma - \sin\beta\sin\gamma) + \cos\alpha(\sin\beta\cos\gamma + \cos\beta\sin\gamma)$$

$$= \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma.$$

8. Find $\cos(\alpha+\beta-\gamma)$.

9. Find $\tan(\alpha-\beta+\gamma)$.

Transform the first member into the second.

10. $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$.

11. $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$.

12. $\cos\left(\frac{\pi}{4}-\alpha\right)\cos\left(\frac{\pi}{4}-\beta\right)-\sin\left(\frac{\pi}{4}-\alpha\right)\sin\left(\frac{\pi}{4}-\beta\right)$
$= \sin(\alpha+\beta).$

13. $\cos(\alpha+\beta)\cos\alpha+\sin\alpha\sin(\alpha+\beta)=\cos\beta.$

14. $\dfrac{\tan\alpha-\tan\beta}{\cot\alpha+\tan\beta}=\tan(\alpha-\beta)\tan\alpha.$

31. Functions of twice an angle. — If in formulas [11] and [12] we place $\beta=\alpha$, they become

$$\sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha,$$

or

$$\mathbf{\sin 2\alpha = 2\sin\alpha\cos\alpha,} \qquad [17]$$

and

$$\cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha,$$

or

$$\mathbf{\cos 2\alpha=\cos^2\alpha-\sin^2\alpha.} \qquad [18, a]$$

By replacing $\cos^2\alpha$ by $1-\sin^2\alpha$, [18, *a*] becomes

$$\mathbf{\cos 2\alpha = 1-2\sin^2\alpha.} \qquad [18, b]$$

Also, by replacing $\sin^2\alpha$ by $1-\cos^2\alpha$, it becomes

$$\mathbf{\cos 2\alpha = 2\cos^2\alpha - 1.} \qquad [18, c]$$

These three forms for $\cos 2\alpha$ should be remembered, and the one most convenient for the purpose in hand should be used.

If we place $\beta=\alpha$ in formula [15], it becomes

$$\tan(\alpha+\alpha)=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha},$$

or

$$\mathbf{\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}.} \qquad [19]$$

The functions of 3α, 4α, etc., may be found in a similar manner by letting $\beta = 2\alpha$, etc.

These equations express the relations between the functions of any angle and the functions of twice that angle. It is not necessary that the angle in the first member should be expressed as 2α and that in the second as α. It is only necessary that the angle in the first member should be twice that in the second. We may then replace 2α by α, if at the same time we replace α by $\frac{1}{2}\alpha$. These formulas may then be written

$$\sin\alpha = 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha, \qquad [20]$$

$$\cos\alpha = \cos^2\tfrac{1}{2}\alpha - \sin^2\tfrac{1}{2}\alpha, \qquad [21, a]$$

$$= 1 - 2\sin^2\tfrac{1}{2}\alpha, \qquad [21, b]$$

$$= 2\cos^2\tfrac{1}{2}\alpha - 1, \qquad [21, c]$$

$$\tan\alpha = \frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha}. \qquad [22]$$

EXERCISE XVI

1. Find $\sin 2\alpha$, if $\cos\alpha = \frac{1}{2}$.
2. Find $\cos 2\alpha$, if $\tan\alpha = 3$.
3. Find $\tan 2\alpha$, if $\sin\alpha = .6$.
4. Find $\sin 3\alpha$.

Solution.

$$\sin 3\alpha = \sin(\alpha + 2\alpha) = \sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha,$$
$$= \sin\alpha(1 - 2\sin^2\alpha) + \cos\alpha\cdot 2\sin\alpha\cos\alpha,$$
$$= \sin\alpha(1 - 2\sin^2\alpha) + 2\sin\alpha(1 - \sin^2\alpha),$$
$$= 3\sin\alpha - 4\sin^3\alpha.$$

5. Find $\cos 3\alpha$ and $\tan 3\alpha$.

6. Find $\sin 4\alpha$, $\cos 4\alpha$, and $\tan 4\alpha$.

7. Find $\sin 5\alpha$, and $\cos 5\alpha$.

Transform the first member into the second.

8. $2\cot 2\alpha = \cot\alpha - \tan\alpha$.

9. $2\csc 2\alpha = \tan\alpha + \cot\alpha$.

10. $\csc 2\alpha + \cot 2\alpha = \cot\alpha$.

11. $\dfrac{1+\tan^2\left(\frac{\pi}{4}-\alpha\right)}{1-\tan^2\left(\frac{\pi}{4}-\alpha\right)} = \csc 2\alpha$.

12. $\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha} - \dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} = 2\tan 2\alpha$.

13. $\tan 2\alpha = \dfrac{2\cot\alpha}{\cot^2\alpha - 1}$.

14. $\dfrac{2\cot 2\alpha}{1-\tan\alpha} = (1+\tan\alpha)\cot\alpha$.

15. $\tan\left(\dfrac{\pi}{4}\pm\alpha\right) = \dfrac{1\pm\sin 2\alpha}{\cos 2\alpha}$.

16. $\tan\left(\dfrac{\pi}{4}+\alpha\right) + \tan\left(\dfrac{\pi}{4}-\alpha\right) = 2\sec 2\alpha$.

32. Functions of half an angle. — It has been shown in the previous article that

$$\cos\alpha = 1 - 2\sin^2\tfrac{1}{2}\alpha.$$

Solving this for $\sin\frac{1}{2}\alpha$, we have

$$\sin\tfrac{1}{2}\alpha = \pm\sqrt{\frac{1-\cos\alpha}{2}}. \qquad [23]$$

Also from formula [21, c],

$$\cos\alpha = 2\cos^2 \tfrac{1}{2}\alpha - 1.$$

Solving for $\cos\frac{1}{2}\alpha$, we have

$$\cos\tfrac{1}{2}\alpha = \pm\sqrt{\frac{1+\cos\alpha}{2}}. \qquad [24]$$

Dividing [23] by [24], we have

$$\tan\tfrac{1}{2}\alpha = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}, \qquad [25,\ a]$$

$$= \pm\sqrt{\frac{(1-\cos\alpha)^2}{1-\cos^2\alpha}} = \frac{1-\cos\alpha}{\sin\alpha}, \qquad [25,\ b]$$

$$= \pm\sqrt{\frac{1-\cos^2\alpha}{(1+\cos\alpha)^2}} = \frac{\sin\alpha}{1+\cos\alpha}. \qquad [25,\ c]$$

These three forms for $\tan\frac{1}{2}\alpha$ should all be remembered, and the one most convenient for the purpose in hand should be used.

It must be noted that the sign $\pm$ before the radicals does not mean that it has two values for any given angle α, but that it is impossible to determine in general what sign should be used. In any particular case, determine the quadrant in which $\frac{1}{2}\alpha$ lies, and affix the proper sign for each function. The ambiguous sign is omitted before the last two forms, since $1-\cos\alpha$ and $1+\cos\alpha$ are always positive, and $\tan\frac{1}{2}\alpha$ may easily be shown always to have the same sign as $\sin\alpha$.

It is sometimes convenient to express the functions of α in terms of the functions of 2α. This may be done by replacing $\frac{1}{2}\alpha$ by α, and α by 2α in the above formulas. They then become

$$\sin\alpha = \sqrt{\frac{1-\cos 2\alpha}{2}}, \qquad [26]$$

$$\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}}, \qquad [27]$$

$$\tan\alpha = \sqrt{\frac{1-\cos 2\alpha}{1+\cos 2\alpha}}, \qquad [28, a]$$

$$= \frac{1-\cos 2\alpha}{\sin 2\alpha}, \qquad [28, b]$$

$$= \frac{\sin 2\alpha}{1+\cos 2\alpha}. \qquad [28, c]$$

EXERCISE XVII

1. Find $\sin\frac{1}{2}\alpha$ and $\cos\frac{1}{2}\alpha$, if $\sin\alpha = .6$; find $\tan\frac{1}{2}\alpha$, if $\tan\alpha = 4$.

2. Determine the functions of 22° 30′ from the functions of 45°.

3. Determine the functions of 15° from the functions of 30°; compare the results with those obtained in Exercise XV, problem 4.

4. Determine the functions of 165° from the functions of 330°.

5. Obtain the formula for $\tan\frac{1}{2}\alpha$ by solving formula [22].

Transform the first member into the second.

6. $\sin\frac{1}{2}\alpha + \cos\frac{1}{2}\alpha = \pm\sqrt{1+\sin\alpha}$.

7. $\sin\frac{1}{2}\alpha - \cos\frac{1}{2}\alpha = \pm\sqrt{1-\sin\alpha}$.

8. $\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) = \sec\alpha + \tan\alpha$.

9. $\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=1.$

10. $\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\sin\alpha.$

11. $\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\cos\alpha.$

12. $\tan\frac{\alpha}{2}+\cot\frac{\alpha}{2}=2\csc\alpha.$

13. $1+\cot\alpha\cot\frac{\alpha}{2}=\csc\alpha\cot\frac{\alpha}{2}.$

14. $\tan\frac{1}{2}x+2\sin^2\frac{1}{2}x\cot x=\sin x.$

15. $\tan^3\frac{1}{2}x(1+\cot^2\frac{1}{2}x)^3=8\csc^3 x.$

33. Sum and difference of the sines and of the cosines of two angles. — In Art. 28, it has been shown that

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta,$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

From these, by addition and subtraction, we have

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta,$$
$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta,$$
$$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta.$$

If we let $\alpha+\beta=A$, and $\alpha-\beta=B$,

then $\alpha=\frac{1}{2}(A+B)$, and $\beta=\frac{1}{2}(A-B)$.

Substituting these values in the equations above, they become

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B), \qquad [29, a]$$

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B), \qquad [29, b]$$

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B), \qquad [29, c]$$

$$\cos A - \cos B = - 2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B). \qquad [29, d]$$

EXERCISE XVIII

Prove the following identities:

1. $\dfrac{\sin 3\alpha + \sin 5\alpha}{\cos 3\alpha - \cos 5\alpha} = \cot \alpha.$

2. $\dfrac{\sin 7\alpha - \sin \alpha}{\sin 8\alpha - \sin 2\alpha} = \dfrac{\cos 4\alpha}{\cos 5\alpha}.$

3. $\dfrac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \dfrac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}.$

4. $\dfrac{\sin 5^\circ + \sin 47^\circ}{\cos 5^\circ - \cos 47^\circ} = \tan 69^\circ.$

5. $\sin 2\alpha + \sin 4\alpha + \sin 6\alpha = 4 \cos \alpha \cos 2\alpha \sin 3\alpha$

6. $\dfrac{\sin \alpha + \sin 2\alpha + \sin 3\alpha}{\cos \alpha + \cos 2\alpha + \cos 3\alpha} = \tan 2\alpha.$

7. $\dfrac{\sin 2\alpha + \sin 2\beta}{\sin 2\alpha - \sin 2\beta} = \dfrac{\tan(\alpha + \beta)}{\tan(\alpha - \beta)}.$

8. $\dfrac{\sin \alpha + \sin \beta}{\cos \alpha + \cos \beta} = \tan \tfrac{1}{2}(\alpha + \beta).$

9. $\dfrac{\sin 5\alpha + \sin \alpha}{\sin 4\alpha + \sin 2\alpha} = 2 \cos \alpha - \sec \alpha.$

10. $\dfrac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \dfrac{\sqrt{3}}{3}.$

EXERCISE XIX

Prove the following identities:

1. $\sec(\alpha+\beta)=\dfrac{\sec\alpha\sec\beta}{1-\tan\alpha\tan\beta}$.

2. $(\sin\alpha+\cos\alpha)^2=1+\sin 2\alpha$.

3. $\cos^4\alpha-\sin^4\alpha=\cos 2\alpha$.

4. $\tan 3\alpha-\tan\alpha=2\sin\alpha\sec 3\alpha$.

5. $\cot\alpha-2\cot 2\alpha=\tan\alpha$.

6. $\tan^2\left(\dfrac{\pi}{4}+\dfrac{\beta}{2}\right)=\dfrac{2\csc 2\beta+\sec\beta}{2\csc 2\beta-\sec\beta}$.

7. $\left(\sin\dfrac{\alpha}{2}-\cos\dfrac{\alpha}{2}\right)^2=1-\sin\alpha$.

8. $\sin\left(\dfrac{\pi}{4}+\alpha\right)\sin\left(\dfrac{\pi}{4}-\alpha\right)=\frac{1}{2}\cos 2\alpha$.

9. $1+\tan\alpha\tan\dfrac{\alpha}{2}=\sec\alpha$.

10. $\tan\left(\dfrac{\pi}{4}+\alpha\right)-\tan\left(\dfrac{\pi}{4}-\alpha\right)=2\tan 2\alpha$.

11. $\sin 3\alpha+\sin 2\alpha-\sin\alpha=4\sin\alpha\cos\dfrac{\alpha}{2}\cos\dfrac{3\alpha}{2}$.

12. $\sec^2\alpha(1+\sec 2\alpha)=2\sec 2\alpha$.

13. $\csc\alpha-2\cot 2\alpha\cos\alpha=2\sin\alpha$.

14. $\cos 6\alpha=32\cos^6\alpha-48\cos^4\alpha+18\cos^2\alpha-1$.

15. $\dfrac{\sin 3\alpha}{\sin\alpha}=1+2\cos 2\alpha$.

16. $\dfrac{\csc 2\alpha}{1+\csc 2\alpha}=\dfrac{1+\tan^2\alpha}{(1+\tan\alpha)^2}$.

17. $(\cos 2\alpha+\cos 2\beta)^2+(\sin 2\alpha+\sin 2\beta)^2=4\cos^2(\alpha-\beta)$.

18. $\cot\frac{1}{2}(\alpha-\beta)=-\dfrac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta}$.

19. $\sin\frac{3}{2}\alpha=\pm(1+2\cos\alpha)\sqrt{\dfrac{1-\cos\alpha}{2}}$.

20. $3\sin 2\alpha-\sin 6\alpha=32\sin^3\alpha\cos^3\alpha$.

CHAPTER V

INVERSE FUNCTIONS AND TRIGONOMETRIC EQUATIONS

34. General values. — If $\sin x = \frac{1}{2}$, $x = \frac{\pi}{6}$, or $\frac{5\pi}{6}$, or any of the other angles which have the same terminal lines. There are, then, an indefinite number of angles which satisfy this equation, or any similar one. It is convenient to have general formulas to express all angles which have the same sine, cosine, or tangent, and we shall now proceed to find such formulas. We shall first obtain general expressions for all angles which have their terminal lines along one of the axes. Let n be any integer, positive, negative, or zero. Then all angles which have their terminal lines in coincidence with the initial line OX are represented by $2n\pi$; for this expression represents the series of angles 0, 2π, 4π, etc., and -2π, -4π, etc., which evidently contains all the angles which have their terminal lines in coincidence with OX.

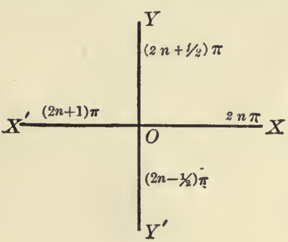


FIG. 47.

In like manner $(2n+1)\pi$ represents all the angles which have their terminal lines in coincidence with OX'; for π is the smallest of these angles and the addition or subtraction of any multiple of 2π will evidently give the same terminal line.

Let the student show that the angles which have their terminal lines along OY are represented by $(2n+\frac{1}{2})\pi$; and along OY' by $(2n-\frac{1}{2})\pi$.

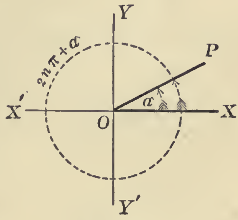


FIG. 48.

If OP is the terminal line of any angle α, all the angles (positive and negative) which have OP as their terminal line are represented by $2n\pi + \alpha$. All angles which differ from each other by any multiple of 2π have the same terminal line, and hence the same functions. This fact is expressed by saying that they are **periodic** functions, having a **period** of 2π.

Let the student show that the period of the tangent is π.

All angles which have the same sine or cosecant as α *may be expressed by* $2n\pi + \alpha$ *and* $(2n+1)\pi - \alpha$.

For it was shown in Art. 22 that $\sin\alpha = \sin(\pi - \alpha)$. Then all angles which have the same terminal lines as α and $\pi - \alpha$ are expressed by $2n\pi + \alpha$, and $2n\pi + (\pi - \alpha)$ or $(2n+1)\pi - \alpha$. Hence the theorem.

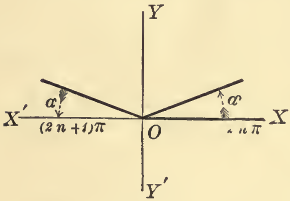


FIG. 49.

This result may also be shown easily from Fig. 49, where α is taken as an angle in the first quadrant. Similar figures may be drawn for any value of α.

Since $\csc\alpha = \dfrac{1}{\sin\alpha}$, these are also the formulas for all angles which have the same cosecant.

Example 1. Give general expressions for α, if $\sin\alpha = \frac{1}{2}\sqrt{2}$.

The smallest positive value of α is evidently $\frac{\pi}{4}$. Then the general expressions for α are

$$2n\pi + \tfrac{1}{4}\pi \text{ and } (2n+1)\pi - \tfrac{1}{4}\pi,$$

or

$$(2n + \tfrac{1}{4})\pi \text{ and } (2n + \tfrac{3}{4})\pi.$$

These are seen to represent the series of angles

$$\tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi,\ \tfrac{9}{4}\pi,\ \tfrac{11}{4}\pi, \text{ etc.},$$

and

$$-\tfrac{5}{4}\pi,\ -\tfrac{7}{4}\pi,\ -\tfrac{13}{4}\pi,\ -\tfrac{15}{4}\pi, \text{ etc.}$$

Example 2. Give general expressions for α, if $\sin\alpha = -\frac{1}{2}$.

The smallest positive value of α is $\frac{7}{6}\pi$. Then the general expressions for α are

$$2n\pi + \tfrac{7}{6}\pi \text{ and } (2n+1)\pi - \tfrac{7}{6}\pi,$$

or

$$(2n + \tfrac{7}{6})\pi \text{ and } (2n - \tfrac{1}{6})\pi.$$

All angles which have the same cosine or secant as α may be expressed by $2n\pi \pm \alpha$.

For it was shown in Art. 22 that $\cos\alpha = \cos(-\alpha)$. Then all angles which have the same terminal lines as α and $-\alpha$ are expressed by $2n\pi + \alpha$ and $2n\pi - \alpha$; or in one formula, $2n\pi \pm \alpha$. Hence the theorem.

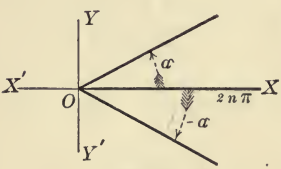


Fig. 50.

This result may also be shown easily by the aid of Fig. 50.

EXAMPLE. Give the general expression for α, if $\cos\alpha = -\frac{1}{2}$. The smallest value of α is evidently $\frac{2}{3}\pi$. Then the general expression for α is $2n\pi \pm \frac{2}{3}\pi$, or $(2n \pm \frac{2}{3})\pi$.

All angles which have the same tangent or cotangent as α may be expressed by $n\pi + \alpha$.

For it was shown in Art. 22 that $\tan\alpha = \tan(\pi + \alpha)$. All angles which have the same terminal lines as α and $\pi + \alpha$ are expressed by $2n\pi + \alpha$ and $2n\pi + (\pi + \alpha)$, or $(2n+1)\pi + \alpha$. But since $2n$ and $2n+1$ include all integers, these two formulas may be written as the one, $n\pi + \alpha$. Hence the theorem.

This result may also be shown by the aid of Fig. 51, since it is evident that all angles which have the same tangent have their terminal lines in the same line through the origin. We shall evidently reach one or the other of these terminal lines when we add any multiple of π to α.

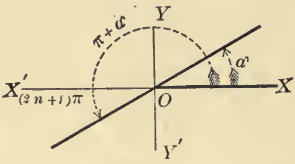


FIG. 51.

EXERCISE XX

Give general expressions for x, if

1. $\sin x = 0$.
2. $\cos x = 0$.
3. $\tan x = 0$.
4. $\sin x = 1$.
5. $\sin x = -1$.
6. $\cos x = 1$.
7. $\cos x = -1$.
8. $\sec x = 1$.
9. $\sec x = -1$.
10. $\tan x = 1$.
11. $\tan x = -1$.
12. $\sin x = -\frac{1}{2}\sqrt{2}$.
13. $\sin x = \frac{1}{2}$.
14. $\cos x = \frac{1}{2}$.

15. $\cos x = \frac{1}{2}\sqrt{2}$.

16. $\cos x = -\frac{1}{2}\sqrt{2}$.

17. $\sin x = \frac{1}{2}\sqrt{3}$.

18. $\cos x = -\frac{1}{2}\sqrt{3}$.

19. $\tan^2 x = \frac{1}{3}$.

20. $\cos^2 x = \frac{1}{2}$.

21. $\tan x = 2 - \sqrt{3}$.

22. $\sec x = -2$.

23. $\cos^2 x = 1$.

24. $\sin x = \frac{1}{3}$.

25. $\tan x = 2$.

26. $\cos x = -\frac{1}{5}$.

27. $\sec x = 10$.

28. $\csc^2 x = 2$.

29. $\tan^2 x = 3$.

30. $\tan^2 x = 7 - 4\sqrt{3}$.

35. Inverse trigonometric functions. — Throughout all the previous work the trigonometric ratios have been considered as functions of the angle; but it is also possible to think of the angle as a function of the trigonometric ratios. For this purpose, if $y = \sin x$, it is convenient to have a short method of writing the fact that "x is an angle whose sine is y." The usual method employed in this country is $x = \sin^{-1} y$, which may be read "x equals anti-sine y," or "inverse-sine y"; but the student must remember that this is only an abbreviation for the longer statement "x is an angle whose sine is y." With the same meaning, we use $x = \cos^{-1} y$, $x = \tan^{-1} y$, $x = \sec^{-1} y$, etc. These are called **inverse trigonometric** or **inverse circular** functions.

We have seen that when an angle is given, its trigonometric functions are determined. For example, if $y = \cos x$, y has a single determinate value for every given value of x. But if a value is given to y, it has been shown in the previous article that x has an indefinite number of values. Thus, if $y = \frac{1}{2}$, $x = \cos^{-1}\frac{1}{2} = 2n\pi \pm \frac{\pi}{3}$. We then may use $x = \cos^{-1}\frac{1}{2}$ and $x = 2n\pi \pm \frac{\pi}{3}$ as different methods of expressing the same idea. But in working with inverse trigonometric functions we shall understand that the *least positive angle* of the set is meant, unless the contrary is stated. This least positive value is called the **principal value** of the function.

With this meaning we may prove the equality of such expressions as $\sin^{-1} x$ and $\cos^{-1}\sqrt{1-x^2}$. But these two expressions do not represent the same series of angles, as may readily be seen by giving x a numerical value. If $x=\frac{1}{2}$, $\sqrt{1-x^2}=\frac{1}{2}\sqrt{3}$, and

$$\sin^{-1} x = 30°,\ 150°,\ 390°,\ 510°, \text{ etc.},$$

while $$\cos^{-1}\sqrt{1-x^2} = 30°,\ 330°,\ 390°,\ 690°, \text{ etc.}$$

Example 1. Prove that $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$.

Let $y=\sin^{-1} x$. Then $x=\sin y$. From this we see that $\cos y = \sqrt{1-x^2}$. Hence $y=\cos^{-1}\sqrt{1-x^2}$, and $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$.

Example 2. Prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$.

Let $\tan^{-1} x = z$ and $\tan^{-1} y = w$.

Then $$x = \tan z \text{ and } y = \tan w.$$

But $$\tan(z+w) = \frac{\tan z + \tan w}{1-\tan z \tan w} = \frac{x+y}{1-xy}.$$

Hence $$z + w = \tan^{-1}\frac{x+y}{1-xy},$$

and $$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}.$$

EXAMPLE 3. Prove that $2\tan^{-1}x = \cos^{-1}\dfrac{1-x^2}{1+x^2}$.

Let $\tan^{-1}x = y$. Then $x = \tan y$.

We wish to show that $2y = \cos^{-1}\dfrac{1-x^2}{1+x^2}$.

But
$$\cos 2y = \cos^2 y - \sin^2 y,$$
$$= \frac{1}{1+\tan^2 y} - \frac{\tan^2 y}{1+\tan^2 y},$$
$$= \frac{1-x^2}{1+x^2}.$$

Hence
$$2y = \cos^{-1}\frac{1-x^2}{1+x^2},$$

and
$$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}.$$

EXERCISE XXI

Prove the following identities for the principal values of the inverse functions.

1. $\sin(\cos^{-1}\frac{1}{2}) = \frac{1}{2}\sqrt{3}$.
2. $\tan(\sin^{-1}\frac{12}{13}) = \frac{12}{5}$.
3. $\tan(2\tan^{-1}\frac{1}{3}) = \frac{3}{4}$.
4. $\tan(\frac{1}{2}\tan^{-1}\frac{12}{5}) = \frac{2}{3}$.
5. $\tan(\tan^{-1}\frac{4}{3} - \tan^{-1}1) = \frac{1}{7}$.
6. $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$.
7. $2\cos^{-1}x = \cos^{-1}(2x^2-1)$.
8. $2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$.
9. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \dfrac{\pi}{4}$.
10. $3\tan^{-1}x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$.

11. $\tan(2\tan^{-1}a) = \dfrac{2a}{1-a^2}$.

12. $\cos(2\tan^{-1}a) = \dfrac{1-a^2}{1+a^2}$.

13. $\tan(2\sec^{-1}a) = \dfrac{2\sqrt{a^2-1}}{2-a^2}$.

14. $\sin(\sin^{-1}a + \sin^{-1}b) = a\sqrt{1-b^2} + b\sqrt{1-a^2}$.

15. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{2}{9} = \dfrac{\pi}{4}$.

36. Solution of trigonometric equations. — In all the previous work, except Art. 34, the equations with which we have dealt have been true for all values of the angles, and the student has been asked to prove this fact. There is another important class of problems in which the student is given an equation which is true only for certain values of the angle, and he is asked to determine the values for which it is true. In other words, he is asked to "solve the equation." The first step toward this end is usually *to transform the given equation into one which contains a single trigonometric function of a single angle.* This function may then be looked upon as the unknown quantity and its value may be obtained by the algebraic solution of the equation for this unknown. If the equation can be reduced to either a simple or a quadratic equation, the solution may be obtained by elementary methods, and only such equations will be considered here. The following problems illustrate the method of procedure in the simpler cases, where the equation contains functions of a single angle only.

EXAMPLE 1. Solve the equation $\sin x + \csc x = 2$.

Since $\csc x = \dfrac{1}{\sin x}$, the equation becomes

$$\sin x + \frac{1}{\sin x} = 2.$$

Clearing, $\sin^2 x - 2\sin x + 1 = 0$.

Solving, $\sin x = 1$,

$$x = \sin^{-1} 1,$$

or $x = \dfrac{\pi}{2}$.

All values of x are then represented by

$$x = (2n + \tfrac{1}{2})\pi.$$

EXAMPLE 2. Solve the equation $\sin x = \tan^2 x$.

Since $\tan x = \dfrac{\sin x}{\cos x}$, the equation becomes

$$\sin x = \frac{\sin^2 x}{\cos^2 x}$$

Clearing, $\sin x \cos^2 x = \sin^2 x$,

or $\sin x(\cos^2 x - \sin x) = 0$.

Hence $\sin x = 0$,

and $\cos^2 x - \sin x = 0$.

The first of these equations gives at once

$$x = 0, \text{ or } \pi,$$

$$= n\pi.$$

The second equation, $\cos^2 x - \sin x = 0$, may be written

$$1 - \sin^2 x - \sin x = 0.$$

Solving as an affected quadratic in $\sin x$, we have

$$\sin x = \frac{-1 \pm \sqrt{5}}{2} = 0.61803, \text{ or } -1.61803.$$

The second of these values is evidently impossible, since $\sin x$ cannot be numerically greater than 1. Impossible solutions of this nature are of frequent occurrence in solving trigonometric equations. This value of $\sin x$ satisfies the original equation, but since there is no angle whose sine is -1.61805, it does not give a possible solution of that equation.

From the other value of $\sin x$, we have $x = \sin^{-1} .61803 = 38° 10' 21''$. There will also be an angle in the second quadrant, $180° - 38° 10' 21''$. The general answers are then $n\pi$, $2n\pi + 38° 10' 21''$, and $(2n+1)\pi - 38° 10' 21''$.

EXERCISE XXII

1. $\cos^2 x = \sin^2 x$.
2. $2 \cos x = \sqrt{3} \cot x$.
3. $\tan x + \cot x = 2$.
4. $\sec x + 2 \cos x = 3$.
5. $\sin x = \tan x$.
6. $\sec^2 x = 4 \tan x$.
7. $\tan^2 x - \sec x = 1$.
8. $\tan^2 x + \csc^2 x = 3$.
9. $4 \tan x - \cot x = 3$.
10. $\tan^2 x + \cot^2 x = \frac{10}{3}$.

37. There are many equations in which it is convenient to introduce expressions containing radicals when we attempt to write the equation in terms of a single function. It will then be necessary to square the equation in the solution; and in doing this an ambiguity will be introduced, and the solutions of the resulting equation will contain not only the solutions of the original equation, but also the solutions of the equation obtained by changing the sign before the radical. It then becomes necessary, as in any radical equation, to determine, by actual substitution, which of the results satisfy the given equation. This difficulty may often be avoided by using formulas which do not contain radicals, but it is not always convenient to do this.

Example. Solve the equation $\sqrt{3}\sin x - \cos x = 1$.

Replacing $\sin x$ by $\sqrt{1-\cos^2 x}$,

$$\sqrt{3}\sqrt{1-\cos^2 x} = 1 + \cos x.$$

Squaring, $3(1-\cos^2 x) = 1 + 2\cos x + \cos^2 x$.

Uniting, $4\cos^2 x + 2\cos x - 2 = 0$.

Solving, $\cos x = \frac{1}{2}$, or -1.

$$x = \pm\frac{\pi}{3}, \text{ or } \pi.$$

If $\cos x = \frac{1}{2}$, $\sin x = \pm\frac{1}{2}\sqrt{3}$, and the equation is evidently satisfied if $\sin x = +\frac{1}{2}\sqrt{3}$, but not if $\sin x = -\frac{1}{2}\sqrt{3}$. The solution $-\frac{\pi}{3}$ must, therefore, be discarded. It is easily seen that the equation is satisfied when $x = \pi$. The general solutions are, then, $(2n+\frac{1}{3})\pi$, and $(2n+1)\pi$.

Second Solution. — This difficulty may be avoided in any problem which is in the form $a\sin x + b\cos x = c$ by the following device: Divide the equation through by $\sqrt{a^2+b^2}$. Then $\frac{a}{\sqrt{a^2+b^2}}$ may always be expressed as the sine of some angle, and $\frac{b}{\sqrt{a^2+b^2}}$ as the cosine of the same angle. In this problem $\sqrt{a^2+b^2}=2$. Dividing by 2,

$$\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x=\frac{1}{2}.$$

Substituting $\frac{\sqrt{3}}{2}=\sin\frac{\pi}{3}$, and $\frac{1}{2}=\cos\frac{\pi}{3}$, we have

$$\sin\frac{\pi}{3}\sin x-\cos\frac{\pi}{3}\cos x=\frac{1}{2}, \text{ or } \cos\left(\frac{\pi}{3}+x\right)=-\frac{1}{2}.$$

Hence

$$\frac{\pi}{3}+x=\left(2n\pm\frac{2}{3}\right)\pi, \text{ and } x=(2n-1)\pi, \text{ or } \left(2n+\frac{1}{3}\right)\pi.$$

EXERCISE XXIII

1. $\sin x-\cos x=0.$
2. $\sin x+\cos x=1.$
3. $\sin x-\cos x=-\sqrt{\frac{3}{2}}.$
4. $\csc x-\cot x=\sqrt{3}.$
5. $\frac{1}{3}\cos x-\frac{1}{4}\sin x=\frac{1}{5}.$
6. $\cot x-\tan x=\sin x+\cos x.$
7. $\cos x+\tan x=\sec x.$
8. $\csc x=1+\cot x.$
9. $\tan x+\sec x=\sqrt{3}.$
10. $5\sin x+2\cos x=5.$

38. In the previous problems only functions of x have occurred. If the equation contains functions of multiples of x, as $2x$, $3x$, $\frac{1}{2}x$, etc., these may all be replaced by their values in terms of functions of x, and the equation solved as in the previous problems. But many equations may be solved more readily by various devices, and some may be solved by these devices which, if expressed in functions of x, would give equations of the third or higher degree, which the student could not solve. There is an excellent chance for the display of the ingenuity of the student in discovering methods for shortening the work in many of these problems. A few of these are illustrated in the following examples.

EXAMPLE 1. Solve the equation

$$\sin 3x + \sin 2x + \sin x = 0.$$

By formula [29, a],

$$\sin 3x + \sin x = 2 \sin 2x \cos x.$$

The given equation may then be written

$$\sin 2x = -2 \sin 2x \cos x.$$

From which $\sin 2x = 0$, and $\cos x = -\frac{1}{2}$.

Hence $2x = n\pi$, $x = (2n \pm \frac{2}{3})\pi.$

or $x = \dfrac{n\pi}{2}.$

The values of x less than 360° are seen to be 90°, 120°, 180°, 240°, 270°.

EXAMPLE 2. Solve the equation

$$\csc x - \cot x = \sqrt{3}.$$

By substitution this becomes

$$\frac{1}{\sin x} - \frac{\cos x}{\sin x} = \sqrt{3},$$

or

$$\frac{1 - \cos x}{\sin x} = \sqrt{3}.$$

This reduces, by formula [25, b], to

$$\tan \tfrac{1}{2} x = \sqrt{3}.$$

Hence $\frac{1}{2} x = n\pi + \dfrac{\pi}{3},$

and $x = (2n + \frac{2}{3})\pi.$

The only value of x less than 360° is seen to be 120°.

EXAMPLE 3. Solve the equation $\cos 3x = \cos 2x$.

From Art. 34, $\cos 2x = \cos(2n\pi \pm 2x)$.

Hence $$3x = 2n\pi \pm 2x,$$

and $x = 2n\pi$, and $5x = 2n\pi$, or $x = \frac{2}{5}n\pi$.

The values of x less than 360° are seen to be 0, 72°, 144°, 216°, 288°.

EXAMPLE 4. Solve the equation

$$2 \sin x \sin 3x = 1.$$

By formula [29, d] this becomes

$$\cos 2x - \cos 4x = 1.$$

By formula [18, c],

$$\cos 2x - (2\cos^2 2x - 1) = 1,$$

or $$2\cos^2 2x = \cos 2x.$$

Solving,

$\cos 2x = 0,$ and $\cos 2x = \frac{1}{2}.$

Hence $2x = 2n\pi \pm \frac{\pi}{2},$ $2x = 2n\pi \pm \frac{\pi}{3},$

and $x = n\pi \pm \frac{\pi}{4},$ or $x = n\pi \pm \frac{\pi}{6}.$

The values of x less than 360° are seen to be 30°, 45°, 135°, 150°, 210°, 225°, 315°, 330°.

Example 5. Solve the equation

$$\sec x = 2(\sin x + \cos x).$$

Replacing $\sec x$ by $\frac{1}{\cos x}$, and reducing, we have

$$1 = 2 \sin x \cos x + 2 \cos^2 x.$$

This might be solved by replacing $\sin x$ by $\sqrt{1 - \cos^2 x}$, but we have seen that it is best to avoid the introduction of radicals. But if we notice that $2 \sin x \cos x = \sin 2x$, and $2 \cos^2 x - 1 = \cos 2x$, the equation becomes

$$\sin 2x + \cos 2x = 0.$$

Dividing by $\cos 2x$, $\quad \tan 2x = -1.$

Hence $\quad 2x = n\pi + \frac{3}{4}\pi,$

and $\quad x = \frac{n\pi}{2} + \frac{3}{8}\pi.$

EXERCISE XXIV

1. $\sin 2x = 2 \cos x.$
2. $4 \cos 2x + 3 \cos x = 1.$
3. $\tan 2x = \tan x.$
4. $2 \sin^2 2x = 1 - \cos 2x.$
5. $\cos 3x - \cos 5x = \sin x.$
6. $\cos 5x + \cos 3x + \cos x = 0.$
7. $2 \cos x \cos 3x + 1 = 0.$
8. $\tan 3x = 3 \tan x.$
9. $\cos 2x - \sin x = \frac{1}{2}.$
10. $\tan 2x \tan 3x = 1.$
11. $2 \sin^2 x + \sin^2 2x = 2.$
12. $\sin x + \cos x + \sin 2x = 2.$
13. $\tan x = 4 \sin \frac{1}{2} x.$
14. $\tan x - \tan 3x = 2.$
15. $\cot x - \csc 2x = 1.$
16. $\cos 3x + 2 \cos x = 0.$

EXERCISE XXV. REVIEW

1. If $\sec\theta = 3\frac{4}{7}$, find the value of $\tan\frac{\theta}{2}$.

2. If $\cot\alpha = 2 - \sqrt{3}$, prove that $\sec\alpha = \sqrt{6} + \sqrt{2}$, and that $\csc\alpha = \sqrt{6} - \sqrt{2}$, numerically.

3. Prove that $\cos(135° + A) + \cos(135° - A) = -\sqrt{2}\cos A$.

4. Prove that $(\sec 15° + \csc 15°)^2 = 24$.

5. Find the value of $\sin 15° + \cos 15°$.

6. Prove that $\tan 75° + \cot 75° = 4$.

7. Find the value of $\tan 105°$.

8. Prove that $\tan 60° - \tan 165° = 2$.

9. If $\text{vers}\, A = \frac{1}{5}$, find $\tan\frac{A}{2}$.

10. Prove that $\sin\left(\frac{\pi}{4} + \theta\right) = \cos\left(\frac{\pi}{4} - \theta\right)$.

11. If $\tan A = \frac{\sqrt{3}}{4 - \sqrt{3}}$, and $\tan B = \frac{\sqrt{3}}{4 + \sqrt{3}}$, find $\tan(A - B)$.

12. Prove that $\cos 20° + \cos 100° + \cos 140° = 0$.

13. The cosines of two of the angles of a triangle are $\frac{3}{5}$ and $\frac{5}{13}$ respectively; find the tangent of the third angle.

14. Solve the equation $\sin x + \cos x = \sec x$.

15. Construct geometrically an angle whose secant is -3.

16. Prove that $\cos^2 A + \cos^2(A + 120°) + \cos^2(A - 120°) = \frac{3}{2}$.

17. Prove that $\text{vers}(270° + A)\,\text{vers}(270° - A) = \cos^2 A$.

18. Solve the equation $\cot^2\theta - \tan^2\theta = 2\csc\theta\sec\theta$.

19. Prove that $2\,\text{vers}\left(\frac{\pi}{2} + \frac{\theta}{2}\right)\text{vers}\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \text{vers}(\pi - \theta)$.

20. Prove that $4\sin 20°\sin 40°\sin 80° = \sin 60°$.

21. Prove that $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$.

22. If $\sin A = \frac{2n}{n^2+1}$, and $\sin B = \frac{2p}{p^2+1}$, find $\tan(A+B)$.

23. Express in terms respectively of the secant and cosecant of A and B: (1) $\sec(A+B)$, (2) $\csc(A-B)$.

24. Prove that $\sec 105° = -\sqrt{2}(\sqrt{3}+1)$.

25. Prove that $\tan\frac{\pi - 2A}{4} + \tan\frac{\pi + 2A}{4} = 2\sec A$.

26. Prove that $\csc 2\alpha + \cot 2\alpha = \cot\alpha$.

27. Solve the equation $\sin^2 2x - \sin^2 x = \frac{1}{4}$.

28. Prove that $\sec^{-1} 3 = 2\cot^{-1}\sqrt{2}$.

29. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

30. If $\tan A + \tan 2A = \tan 3A$, prove that A must be a multiple of 60° or 90°.

CHAPTER VI

OBLIQUE TRIANGLES

39. We shall now proceed to prove three theorems, connecting the sides and angles of a triangle, which will enable us to solve any triangle. In every case let the triangle be lettered ABC, and let a, b, c represent the lengths of the sides opposite the corresponding angles. In these proofs no account is taken of the positive or negative direction of the sides or angles. The letters a, b, and c simply represent the positive magnitudes of the sides, and A, B, and C the interior angles of the triangle. These forms of the theorems are, therefore, only suited to the solution of triangles; they cannot be used when the directions as well as the magnitudes of the sides and angles need to be considered. For the general form of these theorems and their proof, see Art. 43.

40. Law of the sines. — *In any triangle, the sides are proportional to the sines of the opposite angles.*

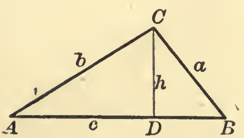


Fig. 52 (*a*).

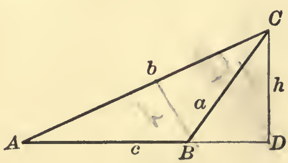


Fig. 52 (*b*).

In either Fig. 52 (*a*) or 52 (*b*), let the length of the perpendicular DC be represented by h. Then in triangle ACD,

$$\sin A = \frac{h}{b}.$$

In triangle DCB, $\sin B = \frac{h}{a}.$

(In Fig. 52 (*b*), $\frac{h}{a} = \sin(\pi - B) = \sin B.$)

Hence by division,

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \qquad [30]$$

By drawing perpendiculars from the other vertices, the same relations may be shown to hold between any pair of angles and sides.

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \quad \frac{a}{c} = \frac{\sin A}{\sin C}.$$

These three formulas may be written in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r,$$

where r may be shown to be the radius of the circumscribed circle.

41. Law of the cosines. — *The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of these sides and the cosine of the included angle.*

Since we are making no use of the directions of the sides, the proof, when the included angle is obtuse, will differ slightly from the proof when the angle is acute. It seems best, therefore, to give separate proofs of the two cases.

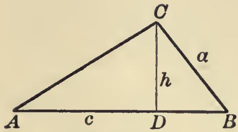


Fig. 53 (a).

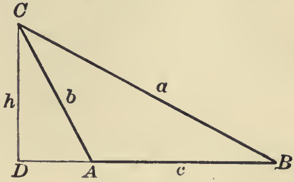


Fig. 53 (b).

In Fig. 53 (a), $\overline{BC}^2 = \overline{CD}^2 + (AB - AD)^2,$

$$= \overline{CD}^2 + \overline{AB}^2 + \overline{AD}^2 - 2\,AB \cdot AD.$$

But $\overline{CD}^2 + \overline{AD}^2 = \overline{AC}^2$, and $AD = AC \cos A$.

Hence $$a^2 = b^2 + c^2 - 2\,bc \cos A.$$

In Fig. 53 (b), $\overline{BC}^2 = \overline{CD}^2 + (DA + AB)^2,$

$$= \overline{CD}^2 + \overline{DA}^2 + \overline{AB}^2 + 2\,DA \cdot AB.$$

But $\overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2,$

and $$DA = AC \cos CAD = -\,AC \cos A.$$

Hence $$\boldsymbol{a^2 = b^2 + c^2 - 2\,bc \cos A.} \qquad [31]$$

We see then that the formula holds for sides opposite either acute or obtuse angles. In fact, the first proof is sufficient, if account is taken of the directions of the lines AB and AD.

By dropping perpendiculars from the other vertices, similar expressions may be found for the other sides.

$$b^2 = c^2 + a^2 - 2\,ca \cos B,$$

$$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

42. Law of the tangents. — *The sum of any two sides of a triangle is to their difference as the tangent of one-half of the sum of the opposite angles is to the tangent of one-half their difference.*

From formula [30],

$$\frac{a}{b}=\frac{\sin A}{\sin B}.$$

By composition and division, this proportion becomes

$$\frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}.$$

But, by formula [29, a] and [29, b],

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)},$$

$$=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

Hence
$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}. \qquad [32]$$

In like manner, it may be shown that

$$\frac{b+c}{b-c}=\frac{\tan\frac{1}{2}(B+C)}{\tan\frac{1}{2}(B-C)},$$

$$\frac{c+a}{c-a}=\frac{\tan\frac{1}{2}(C+A)}{\tan\frac{1}{2}(C-A)}.$$

43.* General form of the law of the sines and the law of the cosines. — In the three laws just obtained only the magnitude of the sides and of the interior angles have been considered. No attention has been paid to the signs of either the sides or angles. But in using either the law of the sines or the law of the cosines in other branches of mathematics, the sides and angles of the triangles are often directed lines and angles, and it is convenient to have a form of these laws which will apply to such cases.

Law of the sines.

Let the sides of the triangle ABC be directed lines. If the positive direction of AC is from A to C, place the

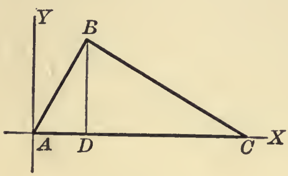


FIG. 54 (*a*).

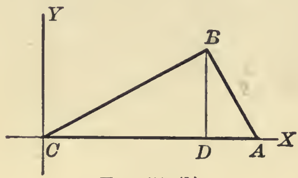


FIG. 54 (*b*).

triangle with A at the origin, as in Fig. 54 (*a*); if its positive direction is from C to A, place it as in Fig. 54 (*b*).

In either figure,

$$DB = \text{proj}_y AB = \text{proj}_y CB.$$

Hence, by [10],

$$AB \sin(AC, AB) = CB \sin(AC, CB),$$

or
$$\frac{AB}{CB} = \frac{\sin(AC, CB)}{\sin(AC, AB)},$$

or, changing the sign of the denominator of the first member and of the numerator of the second,

$$\frac{\mathbf{AB}}{\mathbf{BC}} = \frac{\sin(\boldsymbol{a}, \boldsymbol{b})}{\sin(\boldsymbol{b}, \boldsymbol{c})}. \qquad [33]$$

In like manner, $\dfrac{BC}{CA} = \dfrac{\sin(b, c)}{\sin(c, a)}$, etc.

Law of the cosines. — Placing the triangle as in the previous demonstration,

in either figure, $DA = DC + CA.$

Squaring, $\overline{DA}^2 = \overline{DC}^2 + \overline{CA}^2 + 2\,DC \cdot CA.$

Adding $\overline{DB}^2$, and noting that $\overline{DA}^2 + \overline{BD}^2 = \overline{AB}^2$, $\overline{DC}^2 + \overline{BD}^2 = \overline{BC}^2$, and that

$DC = \text{proj}_x\, BC = BC \cos(CA, BC) = BC \cos(BC, CA),$

we have $\mathbf{\overline{AB}^2 = \overline{BC}^2 + \overline{CA}^2 + 2\,BC \cdot CA \cos(BC, CA).}$ [34]

In like manner

$$\overline{BC}^2 = \overline{CA}^2 + \overline{AB}^2 + 2\,CA \cdot AB \cos(CA, AB), \text{ etc.}$$

In any numerical case these laws will be found to give the same results as the special forms given in the previous articles. To illustrate, consider the triangle in Fig. 55, in which the directions of the sides are shown by arrows, and the lengths of the sides are 2, 3, and 4. Let the letters A, B, and C represent, as usual, the magnitudes of the interior angles of the triangle.

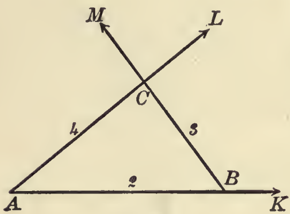


FIG. 55.

Then $AB = 2$, $BC = 3$, $CA = -4$, $(BC, CA) = \angle MCL = -C$, $(CA, AB) = \angle LAK = -A$, $(AB, BC) = \angle KBM = \pi - B$.

$\sin \tfrac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

$\cos \tfrac{1}{2}A = \sqrt{s(s-a)}$

The law of the sines gives $\frac{2}{3} = \frac{\sin(-C)}{\sin(-A)} = \frac{\sin C}{\sin A}$.

The law of the cosines gives

$$2^2 = 3^2 + (-4)^2 + 2(3)(-4)\cos(-C),$$
$$= 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos C.$$

44. Solution of oblique triangles. — The three sides and the three angles of a triangle are spoken of as the six parts of the triangle. It is, in general, possible to find the remaining parts, when any three of these are known, if one of the known parts is a side. The process of finding the three unknown parts is called solving the triangle.

Four cases are distinguished. There may be given

(1) two angles and one side,

(2) two sides and the included angle,

(3) two sides and an angle opposite one of them,

(4) three sides.

45. CASE I. **Given two angles and one side.** — Let A, B, and a be the known parts; it is required to find C, b, and c. The third angle C is determined at once from the equation

$$C = 180° - (A + B).$$

In the law of the sines,

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{or} \quad \frac{c}{a} = \frac{\sin C}{\sin A},$$

three of the four parts are known; the fourth part (b or c) may be found at once by the aid of the table of natural sines.

The accuracy of the result may be tested by obtaining

$\text{Tan } \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

one of the given parts from the three which have just been found, by the aid of a formula which has not been previously used.

EXERCISE XXVI

Find the remaining parts of the triangle, when

1. $A = 7° 21', B = 50° 30', a = 9.$

Solution.

$$C = 180° - (7° 21' + 50° 30') = 122° 9'.$$

$$b = \frac{a \sin B}{\sin A} = \frac{9 \times .7716}{.1279} = 54.29.$$

$$c = \frac{a \sin C}{\sin A} = \frac{9 \times .8467}{.1279} = 59.57.$$

2. $A = 82° 22', B = 43° 20', a = 4.79.$

3. $B = 10° 12', C = 46° 36', b = 5.$

4. $A = 12° 49', C = 141° 59', a = 82.$

5. $A = 99° 55', B = 45° 1', a = 804.$

6. $B = 4° 20', C = 136° 14', b = 51.$

Solution. — Logarithms may be used to advantage in applying the law of the sines.

$$A = 180° - (4° 20' + 136° 14') = 39° 26'.$$

$$a = \frac{b \sin A}{\sin B}, \text{ or } \log a = \log b + \log \sin A - \log \sin B.$$

$$c = \frac{b \sin C}{\sin B}, \text{ or } \log c = \log b + \log \sin C - \log \sin B.$$

$$\begin{array}{rr} \log b = & 1.70757 \\ \log \sin A = & \underline{9.80290} \\ & 11.51047 \\ \log \sin B = & \underline{8.87829} \\ \log a = & 2.63218 \end{array} \qquad \begin{array}{rr} \log b = & 1.70757 \\ \log \sin C = & \underline{9.83993} \\ & 11.54750 \\ \log \sin B = & \underline{8.87829} \\ \log c = & 2.66921 \end{array}$$

$$a = 428.73. \qquad c = 466.88.$$

Tan ½A (r / s − a)

7. $A = 38° 21' 47''$, $B = 54° 6' 8''$, $a = 13.509$.

8. $B = 125° 53' 52''$, $C = 15° 44' 21''$, $c = 5.904$.

9. $B = 44° 22' 49''$, $C = 106° 11' 53''$, $a = 1879.4$.

10. $A = 41° 13' 22''$, $C = 71° 19' 5''$, $a = 55$.

11. $A = 48° 24' 15''$, $C = 31° 13'$, $c = 926.74$.

12. $B = 16° 21' 18''$, $C = 24° 17'$, $b = 43.24$.

46. CASE II. **Given two sides and the included angle.** — Let a, b, and C be the known parts; it is required to find A, B, and c.

The law of the cosines may be used to find the third side, if the given sides are expressed in numbers which may be easily squared. The angles may then be found by the law of the sines, as in the previous case.

EXERCISE XXVII

Find the third side of the triangle, when

1. $A = 31°$, $b = 6$, $c = 10$.

SOLUTION.
$$a = \sqrt{b^2 + c^2 - 2\,bc \cos A},$$
$$= \sqrt{36 + 100 - 120 \times .8572},$$
$$= 5.756.$$

2. $C = 50°$, $a = 10$, $b = 11$.

3. $A = 60°$, $b = 8$, $c = 15$.

4. $C = 135°$, $a = \sqrt{3} - 1$, $b = \sqrt{2}$.

5. $B = 30°$, $a = 3$, $c = 2\sqrt{3}$.

47. Solution by logarithms. — When the numbers which express the length of the sides of the triangle contain a number of figures, and it is necessary to find the angles, the method given above is long, since the formula is not adapted to the use of logarithms.

For the general treatment of this case by the aid of logarithms, the law of the tangents is best suited. It may be written

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A + B).$$

The second member of this equation is completely known, since a and b are given, and $A + B = 180° - C$. We may, therefore, determine $\frac{1}{2}(A - B)$. From this, and the value of $\frac{1}{2}(A + B)$, the values of A and B can be obtained by addition and subtraction. The remaining sides may then be found by the aid of the law of the sines.

EXERCISE XXVIII

Find the remaining parts of the triangle, if

1. $a = 601$, $b = 289$, and $C = 100° 19' 6''$.

SOLUTION. — Apply the law of the tangents,

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A + B),$$

in which $a + b = 890, \; a - b = 312,$

and $\frac{1}{2}(A + B) = \frac{1}{2}(180° - 100° 19' 6'') = 39° 50' 27''.$

$$\begin{aligned} \log(a-b) &= 2.49415 \\ \log \tan \tfrac{1}{2}(A+B) &= \underline{9.92136} \\ & 12.41551 \\ \log(a+b) &= \underline{2.94939} \\ \log \tan \tfrac{1}{2}(A-B) &= 9.46612 \end{aligned}$$

$$\tfrac{1}{2}(A - B) = 16^\circ\, 18'\, 14''$$

But $\quad \tfrac{1}{2}(A + B) = 39^\circ\, 50'\, 27''$

Hence $\quad A = 56^\circ\ \ 8'\, 41''$,

and $\quad B = 23^\circ\, 32'\, 13''$.

The remaining side may now be found by the law of the sines, as in Case I.

2. $a = 23.34, \ b = 55.72, \ C = 18^\circ\, 23'$.
3. $a = 576, \ b = 431, \ C = 73^\circ\, 16'\, 10''$.
4. $c = .523, \ a = .726, \ B = 50^\circ\, 28''$.
5. $b = .0073, \ c = .008, \ A = 100^\circ$.
6. $a = 54.734, \ c = 65.791, \ B = 105^\circ\, 54'$.
7. $a = 1673, \ b = 2432, \ C = 98^\circ\, 5'\, 15''$.
8. $a = 3184, \ b = 917, \ C = 34^\circ\, 9'\, 16''$.
9. $a = 31.84, \ b = 12.34, \ C = 88^\circ\, 12'\, 40''$.
10. $a = 14.59, \ b = 3.99, \ C = 92^\circ\, 11'\, 18''$.

48. CASE III. **Given two sides and the angle opposite one of them.** — Let the given parts be A, a, and b.

It is shown in plane geometry that, when two sides of a triangle and the angle opposite one of them are given, there may be constructed, sometimes two triangles, sometimes one, and sometimes no triangle, according to the relation existing between the given sides and angles.

If the given angle A is acute, and the opposite side a is less than the adjacent side b but greater than the perpendicular CD $(= b \sin A)$, there are two possible constructions. Fig. 56 (a).

area = ½ ab sin C h = a sin C

If A is acute, and $a = b \sin A$ or $a > b$, there is one construction. Fig. 56 (b).

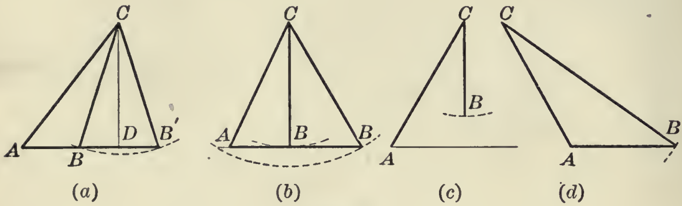


Fig. 56.

If A is acute, and $a < b \sin A$, no construction is possible. Fig. 56 (c).

If A is obtuse, there will be one construction if $a > b$; otherwise there will be no construction. Fig. 56 (d).

Many of these results appear also in the trigonometric solution of the triangle. From the law of the sines,

$$\sin B = \frac{b \sin A}{a}.$$

Since $\sin B$ cannot be greater than 1, we see at once that there will be no solution if $a < b \sin A$. Also that there will be only one solution ($B = 90°$) when $a = b \sin A$.

When $a > b \sin A$, there will be apparently two values of B, one acute and the other obtuse, which are supplementary. But both of these apparent values are not always possible; for, if $a > b$, plane geometry tells us that $A > B$, and hence B cannot be obtuse.

Again, if A is obtuse, B must be acute, and there can be only one solution. But here there will be no solution when $a < b$; for, if $a < b$, $A < B$, and this is impossible when A is obtuse.

This discussion may be condensed into the following table:

When $A < 90°$, and	$b \sin A < a < b$,	there are two solutions.
When $A < 90°$, and	$a \geqq b$,	there is one solution.
When $A < 90°$, and	$a = b \sin A$,	there is one solution.
When $A < 90°$, and	$a < b \sin A$,	there is no solution.
When $A > 90°$, and	$a > b$,	there is one solution.
When $A > 90°$, and	$a < b$,	there is no solution.

Before beginning the solution of any problem, the student should determine the number of possible solutions. If there is one solution, the law of the sines $\left(\sin B = \frac{b \sin A}{a}\right)$ gives the value of B. Then $C = 180° - (A + B)$. The side c may be found from the formula

$$c = \frac{a \sin C}{\sin A}.$$

When there are two solutions, the method of procedure is the same, except that there will be a second value of B, which we shall call B' $(= 180 - B)$. Then

$$C' = 180° - (A + B'), \text{ and } c' = \frac{a \sin C'}{\sin A}.$$

It is always possible to determine the number of solutions by attempting to solve the triangle. If there is no solution, we shall obtain a positive value of $\log \sin B$, which is impossible, since $\sin B$ cannot be greater than unity. Again, if we attempt to obtain two solutions where is only one, we shall find that the sum of two of the angles of the triangle is greater than 180°.

EXERCISE XXIX

Find the remaining parts of the triangle, if

1. $a=6$, $b=8$, and $A=40°$.

SOLUTION. — Since A is acute and $b>a>b \sin A$, there two solutions.

$$\sin B=\frac{b \sin A}{a}, \text{ or } \log \sin B=\log b+\log \sin A-\log a.$$

$$\begin{aligned} \log b &= 0.90309 \\ \log \sin A &= 9.80807 \\ & \overline{10.71116} \\ \log a &= 0.77815 \\ \log \sin B &= 9.93301 \end{aligned}$$

Hence $B= 58° 59' 15''$,

and $B'=121° \ 0' 45''$.

$$C=180°-(A+B)=81° \ 0' 45'',$$

$$C'=180°-(A+B')=18° 59' 15''.$$

$$c=\frac{a \sin C}{\sin A}. \qquad\qquad c'=\frac{a \sin C'}{\sin A}.$$

$$\begin{aligned} \log a &= 0.77815 \\ \log \sin C &= 9.99464 \\ & \overline{10.77279} \\ \log \sin A &= 9.80807 \\ \log c &= 0.96472 \\ c &= 9.2198. \end{aligned} \qquad\qquad \begin{aligned} \log a &= 0.77815 \\ \log \sin C' &= 9.51236 \\ & \overline{10.29051} \\ \log \sin A &= 9.80807 \\ \log c' &= 0.48244 \\ c' &= 3.037. \end{aligned}$$

2. $a=77.04$, $b=91.06$, $B=51° \ 9' \ 6''$.

3. $a=80$, $b=401$, $B=84° \ 16' \ 31''$.

4. $a=319$, $c=481$, $A=41° \ 32' \ 40''$.

5. $a=695$, $b=345$, $B=21° \ 14' \ 25''$.

6. $a = 4.32,\ b = 7.61,\ B = 59°\ 14'$.

7. $a = 704,\ b = 302,\ B = 25°\ 14'\ 13''$.

8. $a = 49,\ b = 45,\ B = 17°\ 41'\ 9''$.

9. $a = 242,\ b = 767,\ B = 36°\ 53'\ 2''$.

10. $c = 104.2,\ b = 55.8,\ B = 32°\ 22'\ 42''$.

49. CASE IV. **Given the three sides.** — If we solve the equation $a^2 = b^2 + c^2 - 2\,bc \cos A$ for $\cos A$,

we have
$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

The corresponding formulas for the other angles are

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca}, \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

When the sides of a triangle are expressed by small numbers, the angles may be found easily from these formulas, with the aid of a table of cosines. Each angle should be found in this way, and the accuracy of the result tested by adding the three angles.

EXERCISE XXX

1. Find the three angles, if $a = 8,\ b = 5,\ c = 7$.

SOLUTION.
$$\cos A = \frac{25 + 49 - 64}{70} = .1429. \qquad A = 81°\ 47'.$$
$$\cos B = \frac{64 + 49 - 25}{112} = .7857. \qquad B = 38°\ 13'.$$
$$\cos C = \frac{25 + 64 - 49}{80} = .5. \qquad C = 60°.$$

Adding, $A + B + C = 180°$, and the solution is accurate as far as minutes; but since we are using a four-place table, we cannot determine the seconds. With a five-place table, the seconds may be determined with fair accuracy; but even with a five-place table the results may often differ a number of seconds from the true value.

Find the three angles, if

2. $a = 13,\ b = 8,\ c = 15.$

3. $a = 26,\ b = 31,\ c = 21.$

4. $a = 25,\ b = 26,\ c = 27.$

5. $a = 17,\ b = 20,\ c = 27.$

50. Solution by logarithms. — When the sides of the triangle are expressed by large numbers, this method is long, since the formulas are not adapted to the use of logarithms. They may, however, be changed into a different form, to which logarithms may be applied.

From formula [23],

$$\sin \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{2}}.$$

Substituting in this the value of $\cos A$ obtained in Art. 49,

$$\sin \tfrac{1}{2} A = \sqrt{\frac{1 - \dfrac{b^2 + c^2 - a^2}{2\,bc}}{2}},$$

$$= \sqrt{\frac{a^2 - (b^2 - 2\,bc + c^2)}{4\,bc}},$$

$$= \sqrt{\frac{(a - b + c)(a + b - c)}{4\,bc}}.$$

For convenience, let $a + b + c = 2\,s$. Then $a - b + c = 2(s - b)$, and $a + b - c = 2(s - c)$.

Substituting these values,

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

In like manner,

$$\cos \tfrac{1}{2} A = \sqrt{\frac{1 + \frac{b^2 + c^2 - a^2}{2\,bc}}{2}},$$

$$= \sqrt{\frac{(b + c + a)(b + c - a)}{4\,bc}},$$

$$= \sqrt{\frac{s(s - a)}{bc}}.$$

Either of these formulas might be used for solving triangles, when the three sides are given, and are well adapted to the use of logarithms. But for values of $\frac{1}{2} A$ near $\frac{\pi}{2}$, the first is inaccurate (since the sine of such angles changes very slowly) and the same is true of the second for very small values of $\frac{1}{2} A$. It is, therefore, best to obtain the formula for $\tan \frac{1}{2} A$, which may be used for all angles.

$$\tan \tfrac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}},$$

$$= \frac{1}{s - a}\sqrt{\frac{(s - a)(s - b)(s - c)}{s}}.$$

If we let $\sqrt{\frac{(s - a)(s - b)(s - c)}{s}} = r,$

this formula becomes

$$\tan \tfrac{1}{2} A = \frac{r}{s - a}. \qquad [35, a]$$

Since r is not changed by interchanging the letters, the corresponding formulas for the other angles are

$$\tan\tfrac{1}{2}B = \frac{r}{s-b}, \qquad [35, b]$$

$$\tan\tfrac{1}{2}C = \frac{r}{s-c}. \qquad [35, c]$$

These formulas are evidently well adapted to the use of logarithms, and, since the tangent varies rapidly for angles of any magnitude, they may be used in all cases. Each of the angles should be found by these formulas, and the accuracy of the work tested by adding the three values. With a five-place table the sum should not differ by more than a few seconds from 180°.

EXERCISE XXXI

1. Find the angles of the triangle, if $a = 15.47$, $b = 17.39$, $c = 22.88$.

SOLUTION.	$a = 15.47$	$\log(s-a) = 1.09342$
	$b = 17.39$	$\log(s-b) = 1.02036$
	$c = 22.88$	$\log(s-c) = 0.69810$
	$2s = 55.74$	2.81188
	$s = 27.87$	$\log s = 1.44514$
	$s-a = 12.40$	$\log r^2 = 1.36674$
	$s-b = 10.48$	$\log r = 0.68337$
	$s-c = 4.99$	

Subtracting $\log(s-a)$, $\log(s-b)$, and $\log(s-c)$ successively from $\log r$, we have $\log\tan\frac{1}{2}A = 9.58995$, or $A = 42° 30' 44''$,

$\log\tan\frac{1}{2}B = 9.66301$, or $B = 49° 25' 49''$,

$\log\tan\frac{1}{2}C = 9.98527$, or $C = 88° 3' 27''$.

180° 00 00.

Find the three angles, if

2. $a = 17$, $b = 113$, $c = 120$.

3. $a = 3359.4$, $b = 4216.3$, $c = 4098.7$.

4. $a = 3.9009$, $b = 2.7147$, $c = 3.0012$.

5. $a = 289$, $b = 601$, $c = 712$.

6. $a = 354.4$, $b = 277.9$, $c = 401.3$.

7. $a = 5.134$, $b = 7.268$, $c = 9.313$.

8. $a = 0.099$, $b = 0.101$, $c = 0.158$.

9. $a = 33.112$, $b = 44.224$, $c = 55.336$.

51. Area of an oblique triangle. — Let K denote the area of the triangle ABC. Draw CD perpendicular to AB.

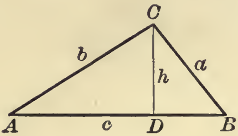


FIG. 57 (*a*).

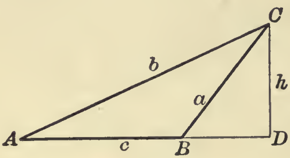


FIG. 57 (*b*).

Then $$K = \frac{1}{2} AB \times CD.$$

But in either figure $$CD = a \sin B.$$

Hence $$\mathbf{K = \frac{1}{2} ac \sin B.} \quad [36]$$

In like manner it may be shown that

$$K = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A.$$

Or, the area of any triangle is equal to one-half of the product of any two sides and the sine of the included angle.

When the three sides of a triangle are given, the area may be obtained as follows:

By formula [20] $\sin B = 2 \sin \frac{1}{2} B \cos \frac{1}{2} B.$

Hence $K = ac \sin \frac{1}{2} B \cos \frac{1}{2} B.$

Substituting in this expression the values of $\sin \frac{1}{2} B$ and $\cos \frac{1}{2} B$ obtained in Art. 50, we have

$$K = ac\sqrt{\frac{(s-a)(s-c)}{ac}} \cdot \sqrt{\frac{s(s-b)}{ac}}.$$

$$\boldsymbol{K = \sqrt{s(s-a)(s-b)(s-c)}}. \qquad [37]$$

When any other three parts are given, find either two sides and the included angle or the three sides, and apply one of the above formulas.

EXERCISE XXXII

Find the area of each of the following triangles:

1. $a = 5,\ b = 6,\ C = 78° 9'$.
2. $a = 45.34,\ c = 56.45,\ B = 100° 10'$.
3. $b = .1001,\ c = .3204,\ A = 30° 33' 25''$.
4. $a = 14,\ b = 14,\ c = 15$.
5. $a = .39,\ b = .8,\ c = .89$.
6. $a = 56,\ b = 90,\ c = 106$.
7. $a = 318,\ b = 181,\ A = 64° 58'$.
8. $b = 34.51,\ c = 183.94,\ A = 23° 53' 17''$.
9. $a = .7845,\ b = .07859,\ C = 120° 43' 50''$.
10. $a = 23,\ A = 76° 53' 25'',\ B = 13° 29' 15''$.

EXERCISE XXXIII.

1. The diagonals of a parallelogram are 73 and 95, and they cross each other at an angle of 35° 28′. Find the sides and angles of the parallelogram.

2. At one point of observation the horizontal angle subtended by a round fort is 4° 35′. On going 500 ft. directly toward the fort, it is found to subtend an angle of 6°. Find the diameter of the fort.

3. The parallel sides of a trapezoid are 16 and 23 ft. The angles at the extremities of the longer side are 35° 54′ and 76° 20′. Find the non-parallel sides.

4. A tower stands at the foot of an inclined plane whose inclination to the horizon is 9°; a line 100 ft. in length is measured straight up the inclined plane from the foot of the tower, and at the upper extremity of this line the tower subtends an angle of 54°. Find the height of the tower.

5. Looking out of a window, with his eye at the height of 15 ft. above the roadway, an observer finds that the angle of elevation of the top of a telegraph pole is 17° 18′ 35″, and the angle of depression of its foot is 8° 32′ 15″. Find the height of the pole and its distance from the observer.

6. A and B are two points, 200 yards apart, on the bank of a river, and C is a point on the opposite bank. The angles ABC and BAC are respectively 54° 30′ and 65° 30′. Find the breadth of the river.

7. A ship sails due east past two headlands which are two miles apart and bear in a line due south. Half an hour later one of the headlands bears 15° south of west and the other 30°. What is the rate of the ship?

8. What is the approximate distance at which a boy must hold a coin one inch in diameter from his eye to conceal the moon, if its apparent angular diameter is half a degree?

9. A balloon rises vertically at a horizontal distance of 3000 yards from an observer, who finds the angle of elevation to be 15° when he first sights the balloon. When he again measures the angle he finds it to be 30°. Through what distance has the balloon risen between the two observations?

10. A pole 10 ft. high stands upright in the ground. The angle of elevation of the top of a tree from the foot of the pole is 32° 27′, while the angle of elevation of the top of the pole from the foot of the tree is 22° 44′. Find the distance between the pole and the tree, also the height of the tree.

11. A telegraph pole stands on the bank of a stream. Its angle of elevation from a point directly opposite on the other bank is 36° 53′, and from a point 60 ft. from the bank in a straight line with the first point and the pole, the angle is 16° 42′. Find the width of the stream and the height of the telegraph pole.

12. A church stands on the bank of a river. From the opposite side of the river the angle of elevation of the top of the spire is found to be 57° 25′. The observer moves back 200 ft. in a direct line with the foot of the spire and there finds the angle of elevation to be 48° 30′. Find the width of the river.

13. A man wishing to determine roughly the length of a pond finds that a line joining two stakes, driven one at each end of the pond, runs N.W. He then takes 150 paces from one of the stakes toward the N.E., turns and takes 200 paces to the other stake. What is the length of the pond, if one of the man's paces is $2\frac{1}{2}$ ft.? What is the angle through which the man turns?

14. The angle of elevation of a steeple is 71° 34′, when the observer's eye is on a level with the bottom. From a window 25 ft. above the place where the observer stands the angle of elevation is 69° 27′. Find the observer's distance from the steeple and the height of the steeple.

15. A man at a station B at the foot of a mountain observes the elevation of the summit A to be 50°. He then walks one mile toward the summit up an incline, making an angle of 30° with the horizon to a point C, and observes the angle ACB to be 150°. Find the height of the mountain.

16. A tower l ft. high stands in a plane. The angles of depression from the top of the tower of two objects lying in the plane in a direct line from the foot of the tower are for the nearer α and for the more remote β. Find the distance between the objects.

17. Two inaccessible objects P and Q lie in a horizontal plane. To find the distance PQ a base line AB of 500 yards is measured in the plane. At its extremities A and B, the following angles are measured: $\angle BAQ = 36° 12'$, $\angle QAP = 50° 46'$, $\angle ABP = 43° 22'$, and $PBQ = 72° 9'$. What is the distance PQ?

18. There is a tower on the top of a hill. From a point in the plane on which the hill stands the angle of elevation of the base of the tower is 37°, and of the top of the tower 50°. From another point straight away from the hill in a line through the first point and 200 ft. from that point the angle of elevation of the top of the tower is 31° 22′. Find the height of the tower.

19. An obelisk stands on a hill whose slope is uniform. A man measured from the foot of the obelisk a distance of 32 ft. directly down the hill and found the angle between the incline and the top of the obelisk to be 45°; after measuring downward an additional distance of 68 ft., the angle found in the same manner was 21° 47′. What is the height of the obelisk and the inclination of the hillside?

20. An observer in a ship sees two rocks, A and B, in the same straight line N. 25° E. He then sails northwest for 4 miles, and observes A to bear due east and B northeast of his new position. Find the distance from A to B.

21. The circular basin of a fountain subtends an angle of 25° at a distance of 44 ft. from the edge of the basin, measured on a diameter produced. Find the radius of the basin. 12.15

22. From the top of a vertical tower whose height is 100 ft., the angle of depression of an object is observed to be 60°, and from the base to be 30°; show that the vertical height of the base of the tower above the object is 50 ft.

23. A flagstaff 40 ft. tall stands on a castle wall. At a horizontal distance of 20 ft. from the foot of the wall the flagstaff subtends an angle of 15°. Find the height of the wall.

24. The angle of elevation of a tower at a distance of 20 yards from its foot is three times as great as the angle of elevation 100 yards from the same point. Show that the height of the tower is $\frac{300}{\sqrt{7}}$ ft.

25. Two parallel chords of a circle, lying on the same side of the centre, subtend respectively 72° and 144° at the centre. Show that the distance between the chords is half the radius of the circle.

26. A person standing due south of a lighthouse observes that his shadow, cast by the light at the top, is 24 ft. long; on walking 100 yards due east, he finds his shadow to be 30 ft. Supposing him to be 6 ft. high, find the height of the light from the ground.

27. A man 5 ft. tall stands on the edge of a pond. The angle of elevation of a tree on the opposite bank is 45° and the angle of depression of its reflection is 60°. Find the height of the tree.

Note. — The reflection of the top of the tree appears as far below the surface of the water as the top of the tree is above the water.

28. One end of a pole rests on the ground and the other end touches the top of a window. When the lower end of the pole is moved away 16 ft. farther from the wall, the top rests on the sill of the window. If the first angle the pole makes with the ground is 71° 25′ and the second 48° 35′, find the length of the window.

29. Two captive balloons are floating at equal heights in calm air. A man standing in the straight line between their points of attachment finds the angle of elevation of the nearer balloon to be $\tan^{-1} 1$. He then walks a distance of 240 ft. at right angles to the straight line joining the points of attachment and finds the angle of elevation of the same balloon to be $\tan^{-1} \frac{3}{5}$, and that of the other $\tan^{-1} \frac{9}{20}$. Find the height of the balloons and the distance between their centres.

30. Wishing to find the inclination of a roadway rising from a level park, a man walked 100 ft. up the incline and observed the angle of depression of an object in the park to be 30°. After walking up the plane 100 ft. farther, the angle of depression of the same object was 45°. Show that the angle of inclination is $\cot^{-1}(2-\sqrt{3})$.

31. Two persons stand facing each other on opposite sides of a pool. They are of such heights that the eye of one is 5 ft. above the ground and that of the other 6 ft. When the line of vision of each makes an angle of 54° with vertical, the reflection of the eye of either is visible to the eye of the other. What is the width of the pool?

32. Viewed from the S.E. corner of a room the N.E. corner has an elevation of θ and the S.W. corner of ϕ. Find the elevation of the N.W. corner, also the angles the diagonals make with the edges of the room.

33. The angle of elevation of a tower from a point A due south is α, and from a point B due west of the first station it is β. If the distance, AB, between the two stations is b, show that the height of the tower is

$$\frac{b \sin \alpha \sin \beta}{\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}}.$$

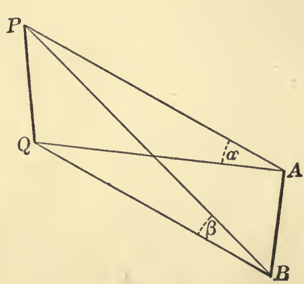


FIG. 58.

34. A man walked up an inclined plane a feet and observed the angle of depression of an object in a horizontal plane to be α. When he had walked a distance of $2a$ ft. further up the incline the angle of depression was 2α. Show that the angle the incline makes with the horizontal plane is

$$\tan^{-1}\frac{2\sin 2\alpha}{1-4\sin^2\alpha}.$$

35. A man walks along the bank, AB, of a straight stream and at A observes the greatest angle subtended by two objects, P and Q, on the opposite side to be 75°; he then walks a distance of 300 ft. to B and finds that the objects are in a straight line, which makes an angle of 15° with the bank. Find the distance between the objects.

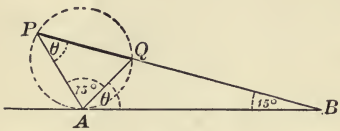


Fig. 59.

Note. — The point on the bank where the greatest angle, subtended by the objects, is made, will be the point of tangency of a circle passing through them. $\angle BPA = \angle QAB = \theta$ can be expressed in terms of 15°, 75°, and 90°.

36. A person on the top of a tower observes the angles of depression of two objects in the plane on which the tower stands to be 60° and 30°. He knows the distance between the two objects to be 500 ft. The angle subtended at his eye by the line joining the two objects is 30°. Find the height of the tower.

37. From two stations a ft. apart a balloon is observed. At one of the stations the horizontal angle between the balloon and the other station is γ, and the angle of elevation of the balloon is α; at the other station the corresponding angles are δ and β. Show that the height of the balloon is

$$\frac{a\sin\gamma\tan\beta}{\sin(\delta+\gamma)}.$$

Also show that $\tan\alpha\sin\delta = \tan\beta\sin\gamma.$

38. An inclined plane AB, of length l, has a vertical rod, PQ, fastened to its surface at such a distance from its foot that the upper end of the rod and the top of the plane are in the same straight line with a point D at a distance l from the foot of the plane. The angles subtended by the rod and the part of the plane below are each equal to α. Find the distance at the foot of the rod from the lower end of the plane, also the length of the rod.

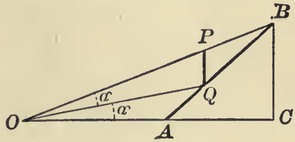


Fig. 60.

PART II

SPHERICAL TRIGONOMETRY

CHAPTER VII

RIGHT AND QUADRANTAL TRIANGLES

52. A spherical triangle is a portion of the surface of a sphere bounded by the arcs of three great circles which intersect. Spherical Trigonometry is concerned with the relations of the sides and angles of a spherical triangle, and the computation of the unknown parts when any three parts are given. In the following treatment a knowledge of Solid Geometry is presupposed, but it is thought best to begin with a statement of the definitions and theorems which are most important for our subject.

Let ABC be a spherical triangle on a sphere whose centre is at O. Join O to the vertices A, B, and C, and pass planes through O and the sides of the triangle. These sides are measured in degrees, and their measures are, therefore, equal to the measures of the corresponding plane angles AOB, BOC, COA.

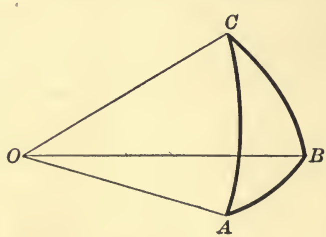


Fig. 1.

A spherical angle is equal to the angle between tangents to the sides of the angle, drawn at the vertex. Hence the angle A is equal to the angle formed by drawing, in the faces AOB and AOC, any two lines perpendicular to OA at the same point.

A spherical angle may also be measured by the arc of a great circle having the vertex of the angle as a pole and intercepted between its sides.

We shall restrict our study of triangles to those in which each of the angles and sides is less than 180°. The sum of the three sides may have any value less than 360°, while the sum of the angles must lie between 180° and 540°.

A triangle may contain one, two, or three right angles, or each of the angles may be greater than 90°. If each of the angles is a right angle, each of the sides is a quadrant, and it is called a tri-rectangular triangle.

Polar triangles are so related that the vertices of each are the poles of the corresponding sides of the other. Each angle of either triangle is the supplement of the side lying opposite it in its polar triangle.

When one or more of the sides of a spherical triangle are quadrants, it is called a *quadrantal triangle*.

EXERCISE XXXIV

1. Prove that, if a triangle has three right angles, it is its own polar.

2. Prove that the polar of a right triangle is a quadrantal triangle.

3. Prove that, if a triangle has two right angles, the sides of the polar triangle opposite these are quadrants, and that the third side measures the third angle of the given triangle.

4. Prove that, if one side of a triangle is a quadrant, either of the other sides and the angle opposite it are either both less or both greater than 90°.

5. Prove that, if a triangle has one right angle, each of its remaining angles is in the same quadrant as the side opposite it.

6. Prove that, if the sides about the right angle of a right spherical triangle are in the same quadrant, the hypotenuse is less than 90°; while if they are in different quadrants the hypotenuse is greater than 90°.

7. Prove that, if one of the sides of a right triangle is equal to the opposite angle, the remaining parts are each equal to 90°.

8. The angles of a triangle are 80°, 75°, and 105°. Find the number of degrees in each side of the polar triangle, and, if the radius of the sphere be 90 ft., compute their lengths in feet.

9. The sides of a spherical triangle are 70°, 80°, and 110°. Find the angles of its polar triangle.

10. In an equiangular spherical triangle each of the angles is 120°. Find the value of each side of the polar triangle. If the angles increase to 180° each, state the limits of the triangle and its polar.

53. Right triangles. — Let the spherical triangle ABC (Fig. 2) be right-angled at C, and let each of the other parts be less than 90°. Pass planes through the sides of the triangle and O, the centre of the sphere. Represent the measure of the sides opposite A, B, and C by the corresponding small letters a, b, c. Then these are also the measures of the corresponding angles at O. That is,

$$\angle BOC = a, \ \angle COA = b, \ \angle AOB = c.$$

Through B pass a plane BED perpendicular to OA. Then EB and ED are perpendicular to OA, and the angle BED is the measure of the spherical angle A.

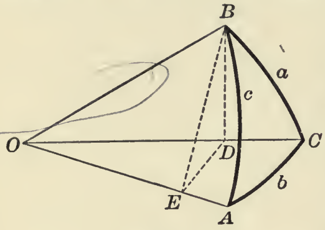


FIG. 2.

Also BD is perpendicular to the plane AOC, since it is the line of intersection of two planes which are perpendicular to that plane. Then the triangles BOE, BOD, DOE, and BDE are all right triangles.

Then $$\cos c = \frac{OE}{OB}.$$

Also $$OE = OD \cos b = OB \cos a \cos b.$$

Hence $$\mathbf{\cos c = \cos a \cos b.} \qquad [1]$$

Again $$\sin A = \frac{DB}{EB} = \frac{OB \sin a}{OB \sin c} = \frac{\sin a}{\sin c}.$$

Hence $$\mathbf{\sin a = \sin c \sin A.} \qquad [2]$$

Interchanging a and b, A and B,

$$\mathbf{\sin b = \sin c \sin B.} \qquad [3]$$

Again $$\cos A = \frac{ED}{EB} = \frac{OE \tan b}{OE \tan c} = \frac{\tan b}{\tan c},$$

or $$\mathbf{\cos A = \tan b \cot c.} \qquad [4]$$

Also $$\mathbf{\cos B = \tan a \cot c.} \qquad [5]$$

Formula [4] may be written

$$\cos A = \frac{\sin b \cos c}{\cos b \sin c}.$$

But $$\frac{\sin b}{\sin c} = \sin B, \text{ and } \frac{\cos c}{\cos b} = \cos a.$$

Hence $$\cos A = \cos a \sin B. \qquad [6]$$

Also $$\cos B = \cos b \sin A. \qquad [7]$$

Substituting the values of cos a and cos b obtained from these last equations in formula [1], we have

$$\cos c = \cot A \cot B. \qquad [8]$$

Again $$\sin b = \frac{ED}{OD} = \frac{DB \cot A}{DB \cot a} = \frac{\cot A}{\cot a}.$$

Hence $$\sin b = \tan a \cot A. \qquad [9]$$

Also $$\sin a = \tan b \cot B. \qquad [10]$$

In deriving these formulas we have used a triangle in which none of the parts is greater than 90°; but they may be easily shown to hold for any right triangle. Let ABC (Fig. 3) be a right triangle in which $a < 90°$ and $b > 90°$. Then by Problem 6, Exercise xxxiv, $c > 90°$. Continue the sides AB and AC until they meet at A'. $A'BC$ is a right triangle in which each of the five parts is less than 90°.

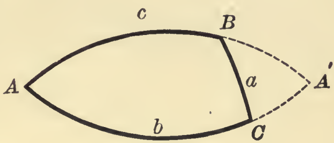


FIG. 3.

Since each of the arcs ABA' and ACA' is a semicircumference,

$$A'B = 180° - c, \text{ and } CA' = 180° - b.$$

Then by formula [1],

$$\cos(180° - c) = \cos a \cos(180° - b),$$

or $$\cos c = \cos a \cos b.$$

Let the student show that all the other formulas hold for this case. Let him also construct a figure, and prove that these formulas hold when both a and b are greater than 90°.

54. Napier's rules of circular parts. — As it is difficult to remember these ten formulas, the following device may be used as an aid to the memory. The two legs of the right triangle, the complements of the two angles, and the complement of the hypotenuse are called the *circular parts.* Place these on the triangle as shown in Fig. 4, omitting the right angle. Any one of these five parts may be called the middle part; then the two parts on each side of it are called the adjacent parts and the remaining two, the opposite parts. Then the ten formulas obtained above may be condensed into the two following rules:

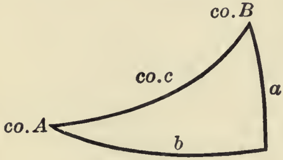


Fig. 4.

The sine of the middle part is equal to the product of the tangents of the adjacent parts.

The sine of the middle part is equal to the product of the cosines of the opposite parts.

If we apply these rules, using each part successively as the middle part, they will be found to give the ten formulas of the previous article.

For example, if a is taken as the middle part, b and co. B are the adjacent parts, and co. A and co. c are the opposite parts. Then the first rule gives $\sin a = \tan b \tan \text{co.} B = \tan b \cot B$, which is formula [10]. The second rule gives $\sin a = \cos \text{co.} A \cos \text{co.} c = \sin A \sin c$, which is formula [2].

55. Solution of right triangles. — When any two of the parts of a right spherical triangle are given, the remaining parts may be determined by the aid of the ten formulas of Art. 53. Six cases may occur. There may be given

1. *the two legs,*
2. *one leg and the hypotenuse,*
3. *the two angles,*
4. *one angle and the adjacent side,*
5. *one angle and the opposite side,*
6. *one angle and the hypotenuse.*

There will be one determinate solution in all of these cases except case 5, where there may be two triangles which satisfy the given conditions. This appears geometrically if the sides AC and AB of the triangle ABC are extended to form a lune. The angle A' equals angle A, and the triangles ABC and $A'BC$ are right triangles which contain a given angle A and the opposite side a.

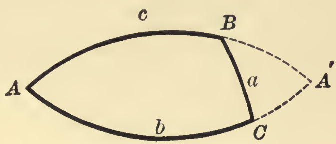


FIG. 5.

The formula required for the solution of any problem is best obtained by marking the two given parts and the one required on Fig. 4. Then choose for the middle part that one of the three which has the other two either as adjacent or opposite parts. For example, if a and B are given and c is wanted, co. B should be chosen as the middle part. Then sin co. $B =$ tan co. c tan a, or cos B $=$ tan a cot c, from which c may be obtained.

If a and c are given and A is wanted, a must be chosen as the middle part; then co. A and co. c are the opposite parts.

When each of the parts of the spherical triangle which are given is less than 90°, the solution by the aid of logarithms presents no new difficulty. But if either of the parts is greater than 90°, careful attention must be paid to the signs of the functions. The following example illustrates the method of procedure.

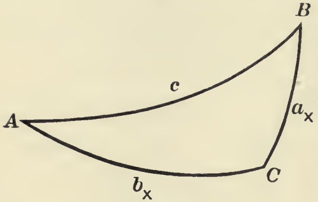


Fig. 6.

Example 1. Find the remaining parts of the right spherical triangle in which $a = 125°$ and $b = 60°$.

By Napier's rules we find that

$$\cos c = \cos a \cos b,$$

$$\sin b = \tan a \cot A, \text{ or } \tan A = \frac{\tan a}{\sin b},$$

and $$\sin a = \tan b \cot B, \text{ or } \tan B = \frac{\tan b}{\sin a}.$$

Since a is between 90° and 180°, $\cos a$ and $\tan a$ are negative, while $\sin a$ is positive. Then c and A are obtuse, and B is acute.

$$\begin{aligned} \log\cos a &= 9.75859 \\ \log\cos b &= 9.69897 \\ \log\cos c &= 9.45756 \\ 180° - c &= 73°\ 20'\ 3'', \\ c &= 106°\ 39'\ 57''. \end{aligned} \qquad \begin{aligned} \log\tan a &= 10.15477 \\ \log\sin b &= 9.93753 \\ \log\tan A &= 10.21724 \\ 180° - A &= 58°\ 46'\ 2'', \\ A &= 121°\ 13'\ 58''. \end{aligned}$$

$$\begin{aligned} \log\tan b &= 10.23856 \\ \log\sin a &= 9.91336 \\ \log\tan B &= 10.32520 \\ B &= 64°\ 41'\ 20''. \end{aligned}$$

EXAMPLE 2. Find the remaining parts of the right spherical triangle in which $a = 21° 39'$ and $A = 42° 10' 10''$.

The formulas needed for the solution in this case are found to be $\sin c = \frac{\sin a}{\sin A}$, $\sin b = \tan a \cot A$, $\sin B = \frac{\cos A}{\cos a}$.

Since each of the unknown parts is to be determined by its sine, there will be two values of each less than 180°. From Example 5, Exercise xxxiv, it appears that a and A must be in the same quadrant; it is also evident from the above formulas that there will be no solution unless $\sin A > \sin a$.

$$\begin{aligned} \log \sin a &= 9.56695 \\ \log \sin A &= 9.82693 \\ \log \sin c &= 9.74002 \\ c &= 33° 20' 15'', \\ \text{or} \quad &= 146° 39' 45''. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 9.59872 \\ \log \cot A &= 10.04298 \\ \log \sin b &= 9.64170 \\ b &= 25° 59' 28'', \\ \text{or} \quad &= 154° \ 0' 32''. \end{aligned}$$

$$\begin{aligned} \log \cos A &= 9.86991 \\ \log \cos a &= 9.96823 \\ \log \sin B &= 9.90168 \\ B &= 52° 53', \\ \text{or} \quad &= 127° \ 7'. \end{aligned}$$

These values must be combined according to the laws stated in Examples 5 and 6 in Exercise xxxiv. In this problem all the values less than 90° form one solution and those greater than 90°, the other. This will be true only in case the given parts are less than 90°.

EXERCISE XXXV

Find the remaining parts of the right spherical triangle in which

1. $a = 9° 45' 19'', b = 12° 16' 42''$.
2. $a = 28° 26' 56'', b = 29° 37' 36''$.
3. $c = 41° 5' 6'', A = 41° 32' 38''$.
4. $a = 12° 16' 42'', B = 79° 29' 45''$.
5. $A = 15° 38' 6'', B = 80° 14' 41''$.
6. $b = 48° 27' 22'', c = 56° 15' 43''$.
7. $B = 56° 15' 43'', c = 58° 40' 13''$.
8. $A = 47° 37' 21'', B = 61° 33' 4''$.
9. $a = 48° 27' 22'', c = 64° 9' 43''$.
10. $b = 74° 21' 54'', A = 38° 57' 12''$.
11. $a = 35°, A = 61°$.
12. $b = 75° 45', B = 65° 38'$.
13. $b = 105° 30', c = 80° 25'$.
14. $b = 98° 35', A = 47° 38'$.
15. $b = 79° 35', B = 80° 25' 20''$.

56. Quadrantal triangles. If one side of a spherical triangle is a quadrant, the angle opposite that side in the polar triangle is a right angle.

From the two given parts of a quadrantal triangle two parts of its polar triangle may be obtained. This right triangle may then be solved and, from these solutions, the unknown parts of the quadrantal triangle may be obtained.

EXERCISE XXXVI

Find the remaining parts of the spherical triangle in which $c = 90°$ and

1. $C = 163° \ 53' \ 38''$, $A = 169° \ 29' \ 45''$.
2. $C = 141° \ 2' \ 48''$, $B = 142° \ 5' \ 54''$.
3. $B = 140° \ 2' \ 56''$, $a = 163° \ 53' \ 38''$.
4. $A = 148° \ 40' \ 13''$, $b = 127° \ 54' \ 6''$.
5. $a = 138° \ 54' \ 54''$, $b = 100° \ 30' \ 15''$.
6. $a = 65°$, $A = 48° \ 35'$.
7. $B = 50° \ 38' \ 20''$, $b = 75° \ 37' \ 30''$.
8. $C = 70° \ 30' \ 28''$, $b = 128° \ 35' \ 12''$.

CHAPTER VIII

OBLIQUE TRIANGLES

57. Law of the sines. — In the oblique spherical triangle ABC draw the arc CD perpendicular to AB, forming the two right spherical triangles ADC and CDB.

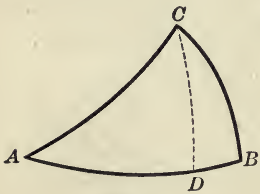


FIG. 7.

Then by formula [2],

$$\sin DC = \sin a \sin B,$$

and $$\sin DC = \sin b \sin A.$$

Hence

$$\sin a \sin B = \sin b \sin A,$$

or $$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}. \qquad [11, a]$$

In like manner $$\frac{\sin b}{\sin c} = \frac{\sin B}{\sin C}, \qquad [11, b]$$

and $$\frac{\sin c}{\sin a} = \frac{\sin C}{\sin A}. \qquad [11, c]$$

In this proof we have used a figure in which D falls between A and B; but all of the statements will be seen to be true for Fig. 8, in which D falls without AB, if we note that, in this case, $\sin DBC = \sin (\pi - B) = \sin B$.

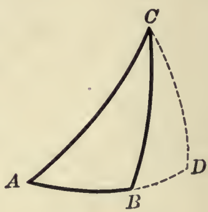


FIG. 8.

58. Law of the cosines. — Let ABC be a spherical triangle in which two of the sides, b and c, are each less than a quadrant. Join its vertices to O, the centre of the sphere. Through D, any point on OA, pass a plane DEF perpendicular to OA, cutting the planes OAC and OAB in the lines DE and DF. Then DE and DF are perpendicular to OA, and $\angle EDF$ is the measure of the spherical angle A.

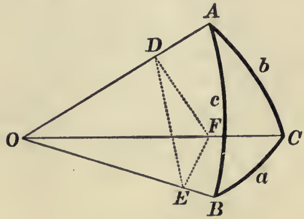


FIG. 9.

In the triangle OEF, by [31], Pt. I,

$$\overline{EF}^2 = \overline{OE}^2 + \overline{OF}^2 - 2\,OE \cdot OF \cos EOF.$$

In the triangle DEF, by [31], Pt. I,

$$\overline{EF}^2 = \overline{DE}^2 + \overline{DF}^2 - 2\,DE \cdot DF \cos EDF.$$

Equating these two values of EF and reducing, we have

$$OE \cdot OF \cos EOF = \overline{OD}^2 + DE \cdot DF \cos EDF.$$

Hence $$\cos EOF = \frac{OD}{OE} \cdot \frac{OD}{OF} + \frac{DE}{OE} \cdot \frac{DF}{OF} \cos A,$$

or $$\mathbf{\cos a = \cos b \cos c + \sin b \sin c \cos A.} \qquad [12, a]$$

In the above proof b and c were each taken less than a quadrant. Let the student show by a method similar to that used in Art. 53 that the formula holds for all values of b and c.

By interchanging letters in the cyclic order we have

$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \qquad [12,\ b]$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \qquad [12,\ c]$$

Let $A'B'C'$ be the polar triangle of ABC. Since the formulas just found hold for all triangles,

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A',$$

$$\cos b' = \cos c' \cos a' + \sin c' \sin a' \cos B',$$

$$\cos c' = \cos a' \cos b' + \sin a' \sin b' \cos C'.$$

But $a' = \pi - A$, $A' = \pi - a$, etc. Substituting these values and noting that $\sin(\pi - x) = \sin x$, and $\cos(\pi - x) = -\cos x$, we have

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a, \qquad [13,\ a]$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \qquad [13,\ b]$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \qquad [13,\ c]$$

59. Formulas [12] and [13] are unsuited to logarithmic computation; but by a transformation very similar to that used on page 104 of the Plane Trigonometry, we may obtain from them formulas which are well adapted to the use of logarithms.

From [12, a],

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

But $\sin \frac{1}{2} A = \sqrt{\frac{1 - \cos A}{2}}$, [23], Pt. I,

$$= \sqrt{\frac{\sin b \sin c - \cos a + \cos b \cos c}{2 \sin b \sin c}},$$

$$= \sqrt{\frac{\cos (b - c) - \cos a}{2 \sin b \sin c}}, \text{ by [14], Pt. I,}$$

$$= \sqrt{\frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c}},$$

by [29, d], Pt. I.

Let $a + b + c = 2s$. Then $\frac{1}{2}(a + b - c) = s - c$, etc.

Hence $\sin \frac{1}{2} A = \sqrt{\frac{\sin (s - c) \sin (s - b)}{\sin b \sin c}}$.

Also $\cos \frac{1}{2} A = \sqrt{\frac{1 + \cos A}{2}}$, [24], Pt. I,

$$= \sqrt{\frac{\cos a - \cos (b + c)}{2 \sin b \sin c}},$$

by [12] and [29, d], Pt. I,

$$= \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.$$

By division,

$$\tan \frac{1}{2} A = \sqrt{\frac{\sin (s - c) \sin (s - b)}{\sin s \sin (s - a)}},$$

$$= \frac{1}{\sin (s - a)} \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}$$

If we let $$k=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan\tfrac{1}{2}A=\frac{k}{\sin(s-a)}, \qquad [14, a]$$

$$\tan\tfrac{1}{2}B=\frac{k}{\sin(s-b)}, \qquad [14, b]$$

$$\tan\tfrac{1}{2}C=\frac{k}{\sin(s-c)}. \qquad [14, c]$$

Let the student obtain in a similar manner the following formulas from formula [13]:

$$\tan\tfrac{1}{2}a=K\cos(S-A), \qquad [15, a]$$

$$\tan\tfrac{1}{2}b=K\cos(S-B), \qquad [15, b]$$

$$\tan\tfrac{1}{2}c=K\cos(S-C), \qquad [15, c]$$

in which $2S=A+B+C$, and

$$K=\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}.$$

60. Napier's analogies. — Dividing [14, a] by [14, b], we have

$$\frac{\sin\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}A\sin\frac{1}{2}B}=\frac{\sin(s-b)}{\sin(s-a)}.$$

By composition and division,

$$\frac{\sin\frac{1}{2}A\cos\frac{1}{2}B+\cos\frac{1}{2}A\sin\frac{1}{2}B}{\sin\frac{1}{2}A\cos\frac{1}{2}B-\cos\frac{1}{2}A\sin\frac{1}{2}B}=\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)}.$$

Applying [11], [13], [29, a], [29, b], Pt. I, we have

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}. \qquad [16]$$

Multiplying [14, a] by [14, b], we have

$$\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}B}=\frac{k^2}{\sin(s-a)\sin(s-b)}=\frac{\sin(s-c)}{\sin s}.$$

By inversion; then by division and composition,

$$\frac{\cos\frac{1}{2}A\cos\frac{1}{2}B-\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}B+\sin\frac{1}{2}A\sin\frac{1}{2}B}=\frac{\sin s-\sin(s-c)}{\sin s+\sin(s-c)}.$$

Applying [12], [14], [29, b], [29, a], Pt. I, we have

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}. \qquad [17]$$

Let the student obtain in a similar manner the following formulas from [15, a] and [15, b]:

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}, \qquad [18]$$

$$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)}. \qquad [19]$$

Let the student also obtain from each of the above four formulas two others by interchanging the letters.

EXERCISE XXXVII

1. Show that $\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ may have any value less than 180°; but that each of the other angles used in Napier's analogies must be less than 90°.

2. Show by the aid of formulas [17] and [19] that $A+B$ is less than, equal to, or greater than 180°, according as $a+b$ is less than, equal to, or greater than 180°; and the converse.

3. Prove that in any spherical triangle each angle is greater than the difference between 180° and the sum of the other two angles.

4. Deduce from Napier's Rules for the right-angled spherical triangle the relation

$$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}.$$

5. Show what changes occur in formulas [11, a, b, c], (1) when $C = 90°$, (2) when $c = 90°$, (3) when $B = C = 90°$.

6. If a perpendicular be dropped from vertex C of an oblique spherical triangle upon the opposite side AB, to meet it at X, prove by Napier's Rules that, disregarding signs,

$$\frac{\sin AX}{\sin BX} = \frac{\cot A}{\cot B}.$$

7. By means of Napier's Rules, derive formulas for finding in an oblique spherical triangle the parts required in the following cases:

(*a*) Given A, C, a, required b.
(*b*) Given B, C, c, required b.

8. A certain point X on a sphere is joined to three points P, Q, R on an arc of a great circle. By application of the law of cosines to the triangles formed and by reduction, deduce the formula, $\sin PQ \cos RX + \sin QR \cos PX - \sin PR \cos QX = 0$.

61. Solution of oblique spherical triangles. — When any three parts of a spherical triangle are given, the remaining parts may be determined by the aid of the seven formulas, [11] and [14] to [19].

Six cases may occur. There may be given

1. *the three sides,*
2. *the three angles,*
3. *two sides and the included angle,*
4. *two angles and the included side,*
5. *two sides and the angle opposite one of them,*
6. *two angles and the side opposite one of them.*

There will be one determinate solution in each of the first four cases, but in 5 and 6 there may be two solutions.

62. Case I. **Given the three sides *a*, *b*, and *c*.**

The angles may be determined by formula [14].

$$\tan \tfrac{1}{2} A = \frac{k}{\sin (s-a)}, \quad \tan \tfrac{1}{2} B = \frac{k}{\sin (s-b)},$$

$$\tan \tfrac{1}{2} C = \frac{k}{\sin (s-c)}.$$

Example 1. Find the three angles of the spherical triangle in which

$$a = 16° 6' 22'', \ b = 52° 5' 54'', \text{ and } c = 61° 33' 4''.$$

$s = 64° 52' 40''$,

$s - a = 48° 46' 18''$,	$\log \sin (s-a) = 9.87627$
$s - b = 12° 46' 46''$,	$\log \sin (s-b) = 9.34478$
$s - c = 3° 19' 36''$.	$\log \sin (s-c) = 8.76364$
	27.98469
	$\log \sin s = 9.95684$
	$\log k^2 = 18.02785$
	$\log k = 9.01392$

$\log k = 9.01392$	$\log k = 9.01392$
$\log \sin (s-a) = 9.87627$	$\log \sin (s-b) = 9.34478$
$\log \tan \tfrac{1}{2} A = 9.13765$	$\log \tan \tfrac{1}{2} B = 9.66914$
$A = 15° 38' 6''$.	$B = 50° 2' 52''$.

$\log k = 9.01392$
$\log \sin (s-c) = 8.76364$
$\log \tan \tfrac{1}{2} C = 10.25028$

$C = 121° 19' 46''$.

EXERCISE XXXVIII

Find the three angles of the spherical triangle in which

1. $a = 22° 35' 52''$, $b = 40° 9' 21''$, $c = 56° 52' 23''$.
2. $a = 35° 30' 24''$, $b = 38° 57' 12''$, $c = 56° 15' 43''$.
3. $a = 39° 20' 24''$, $b = 41° 5' 6''$, $c = 60° 22' 24''$.
4. $a = 41° 32' 38''$, $b = 44° 44' 17''$, $c = 57° 10' 4''$.
5. $a = 52° 5' 54''$, $b = 61° 33' 4''$, $c = 83° 34' 56''$.
6. $a = 56° 52' 23''$, $b = 80° 14' 41''$, $c = 103° 59' 30''$.
7. $a = 68° 12' 58''$, $b = 80° 14' 41''$, $c = 128° 11' 15''$.
8. $a = 95° 38' 20''$, $b = 108° 26' 30''$, $c = 56° 27' 48''$.
9. $a = 120° 22' 40''$, $b = 111° 34' 27''$, $c = 96° 28' 35''$.
10. $a = 56° 20' 20''$, $b = 56° 20' 20''$, $c = 60° 28' 38''$.

63. CASE II. **Given the three angles *A*, *B*, and *C*.** The sides may be determined by formula [15].

$$\tan \tfrac{1}{2} a = K \cos (S - A), \quad \tan \tfrac{1}{2} b = K \cos (S - B),$$

$$\tan \tfrac{1}{2} c = K \cos (S - C).$$

The logarithmic work is very similar to that of Case I.

EXERCISE XXXIX

Find the three sides of the spherical triangle in which

1. $A = 50° 2' 56''$, $B = 56° 52' 23''$, $C = 86° 34' 33''$.
2. $A = 47° 37' 21''$, $B = 74° 18' 19''$, $C = 77° 43' 18''$.
3. $A = 45° 26' 42''$, $B = 47° 37' 21''$, $C = 102° 16' 42''$.
4. $A = 15° 38' 6''$, $B = 16° 6' 22''$, $C = 159° 44' 26''$.

64. CASE III. **Given two sides and the included angle.** The two remaining angles may be determined by formulas [18] and [19].

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C,$$

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C.$$

The third side may then be found by either [11], [16], or [17].

EXAMPLE 1. Find the remaining parts of the spherical triangle in which $a = 40° \, 9' \, 21''$, $c = 79° \, 29' \, 45''$, $B = 50° \, 2' \, 56''$.

Here we must use

$$\tan \tfrac{1}{2}(C - A) = \frac{\sin \frac{1}{2}(c - a)}{\sin \frac{1}{2}(c + a)} \cot \tfrac{1}{2} B,$$

and
$$\tan \tfrac{1}{2}(C + A) = \frac{\cos \frac{1}{2}(c - a)}{\cos \frac{1}{2}(c + a)} \cot \tfrac{1}{2} B.$$

$\tfrac{1}{2}(c - a) = 19° \, 40' \, 12''$, and $\tfrac{1}{2}(c + a) = 59° \, 49' \, 33''$.

$\log \sin \frac{1}{2}(c-a) =$	9.52712	$\log \cos \frac{1}{2}(c-a) =$	9.97389
$\log \cot \frac{1}{2} B =$	10.33084	$\log \cot \frac{1}{2} B =$	10.33084
	19.85796		20.30473
$\log \sin \frac{1}{2}(c+a) =$	9.93677	$\log \cos \frac{1}{2}(c+a) =$	9.70125
$\log \tan \frac{1}{2}(C-A) =$	9.92119	$\log \tan \frac{1}{2}(C+A) =$	10.60348
$\frac{1}{2}(C-A) =$	$39° \, 49' \, 46''$	$\frac{1}{2}(C+A) =$	$76° \, 0' \, 28''$.

Hence $A = 36° \, 10' \, 42''$,

and $C = 115° \, 50' \, 14''$.

To determine b, use $\sin b = \dfrac{\sin B \sin a}{\sin A}$.

$$\begin{aligned}
\log \sin B &= 9.88456 \\
\log \sin a &= \underline{9.80947} \\
&\ 19.69403 \\
\log \sin A &= \underline{9.77106} \\
\log \sin b &= 9.92297
\end{aligned}$$

$$b = 56° \ 52' \ 27''.$$

EXERCISE XL

Find the remaining parts of the spherical triangle in which

1. $b = 68° \ 12' \ 58''$, $c = 80° \ 14' \ 41''$, $A = 17° \ 20' \ 54''$.
2. $a = 27° \ 59' \ 4''$, $b = 41° \ 5' \ 6''$, $C = 123° \ 44' \ 17''$.
3. $a = 29° \ 6' \ 11''$, $c = 77° \ 43' \ 18''$, $B = 38° \ 57' \ 12''$.
4. $a = 41° \ 5' \ 6''$, $b = 60° \ 20' \ 54''$, $C = 77° \ 43' \ 18''$.
5. $c = 125° \ 20'$, $b = 175° \ 36'$, $A = 20° \ 28' \ 46''$.
6. $c = 98° \ 35' \ 26''$, $a = 39° \ 48' \ 30''$, $B = 47° \ 28' \ 42''$.
7. $b = 85° \ 35' \ 20''$, $c = 73° \ 24' \ 26''$, $A = 95° \ 28' \ 40''$.
8. $b = 140° \ 38'$, $a = 130° \ 28'$, $C = 150° \ 34'$.

65. Case IV. **Given two angles and the included side.** The two remaining sides may be determined by formulas [16] and [17].

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c,$$

The third angle may then be found by either [10], [18], or [19].

The logarithmic work is very similar to that of Case III.

EXERCISE XLI

Find the remaining parts of the spherical triangle in which

1. $c = 107° 37' 55''$, $A = 50° 2' 56''$, $B = 64° 9' 43''$.
2. $a = 50° 2' 56''$, $B = 61° 33' 4''$, $C = 84° 53' 48''$.
3. $b = 56° 52' 23''$, $A = 41° 32' 38''$, $C = 111° 47' 4''$.
4. $a = 41° 5' 6''$, $B = 56° 15' 43''$, $C = 109° 45' 36''$.
5. $A = 48° 39' 20''$, $B = 69° 28' 30''$, $c \doteq 58° 24' 36''$.
6. $A = 110° 48' 24''$, $C = 60° 25' 48''$, $b = 98° 59' 30''$.
7. $B = 98° 35' 28''$, $C = 99° 52' 48''$, $a = 50° 50' 50''$.
8. $A = 110° 45' 38''$, $B = 99° 37' 18''$, $c = 120° 28' 20''$.

66. CASE V. **Given two sides and the angle opposite one of them.** — If a, b, and A are given, B may be found from formula [11].

$$\sin B = \frac{\sin b \sin A}{\sin a}.$$

If $\sin b \sin A > \sin a$, there is no solution. But if $\sin b \sin A < \sin a$, there are sometimes two solutions. After the two values of B have been obtained, the number of solutions may be determined from the fact that the greater side is opposite the greater angle. It will also be necessary to see that the theorem of Problem 2, Exercise xxxvii, is satisfied.

The remaining parts, c and C, may now be found from formulas [16] and [18], or from [17] and [19].

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b),$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B).$$

EXAMPLE 1. Find the remaining parts of the spherical triangle in which $a = 103° 10'$, $b = 120° 12'$, $B = 131° 40'$.

$$\sin A = \frac{\sin a \sin B}{\sin b}.$$

$$\begin{aligned} \log \sin a &= 9.98843 \\ \log \sin B &= \underline{9.87334} \\ & 19.86177 \\ \log \sin b &= \underline{9.93665} \\ \log \sin A &= 9.92512 \end{aligned}$$

$$A = 57° 18' 45'', \text{ or } 122° 41' 15''.$$

Both of these values of A will be seen to satisfy the conditions stated above. There are, therefore, two solutions. Using the first value of A, we shall proceed to find the corresponding values of c and C.

$$\tfrac{1}{2}(a+b) = 111° 41', \qquad \tfrac{1}{2}(b-a) = 8° 31',$$
$$\tfrac{1}{2}(A+B) = 94° 29' 22'', \quad \tfrac{1}{2}(B-A) = 37° 10' 37''.$$

Since $\frac{1}{2}(A+B)$ is near 90°, more accurate results will be obtained by using formulas [17] and [19], which contain the cosine.

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(B+A)}{\cos \frac{1}{2}(B-A)} \tan \tfrac{1}{2}(b+a),$$

$$\cot \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(b+a)}{\cos \frac{1}{2}(b-a)} \tan \tfrac{1}{2}(B+A).$$

$$\begin{aligned} \log \cos \tfrac{1}{2}(B+A) &= 8.89363 \\ \log \tan \tfrac{1}{2}(b+a) &= \underline{10.40054} \\ & 19.29417 \\ \log \cos \tfrac{1}{2}(B-A) &= \underline{9.90133} \\ \log \tan \tfrac{1}{2} c &= 9.39284 \\ c &= 27° 45' 26''. \end{aligned}
\qquad
\begin{aligned} \log \cos \tfrac{1}{2}(b+a) &= 9.56759 \\ \log \tan \tfrac{1}{2}(B+A) &= \underline{11.10504} \\ & 20.67263 \\ \log \cos \tfrac{1}{2}(b-a) &= \underline{9.99518} \\ \log \cot \tfrac{1}{2} C &= 10.67745 \\ C &= 23° 44' 14''. \end{aligned}$$

In each of the above formulas two factors in the second member are negative. The first members are, therefore, positive, and the acute values of $\frac{1}{2}c$ and $\frac{1}{2}C$ must be chosen.

Using the second value of A we find, by the aid of the same formulas, $c = 113° 28' 14''$, $C = 127° 32' 56''$.

EXERCISE XLII

Find the remaining parts of the spherical triangle in which

1. $a = 16° 6' 22''$, $c = 52° 5' 54''$, $A = 15° 38' 7''$.
2. $b = 38° 57' 12''$, $c = 56° 15' 43''$, $B = 47° 37' 21''$.
3. $b = 50° 2' 56''$, $c = 56° 52' 23''$, $B = 64° 9' 43''$.
4. $b = 28° 35' 30''$, $c = 30° 28' 15''$, $B = 85° 38' 40''$.
5. $B = 86° 9'$, $a = 72° 18' 15''$, $b = 71° 54' 15''$.
6. $A = 120° 35' 28''$, $b = 98° 48' 24''$, $a = 105° 30' 30''$.
7. $A = 103° 28' 12''$, $b = 20° 25' 35''$, $a = 28° 58' 25''$.

Show that the following triangle is impossible; also find a value for c such that B shall be equal to 90°.

8. $C = 98° 35' 28''$, $b = 70° 35' 24''$, $c = 50° 28' 22''$.

67. CASE VI. **Given two angles and the side opposite one of them.**

If A, B, and a are given, b may be found from formula [11].

$$\sin b = \frac{\sin B \sin a}{\sin A}.$$

The number of solutions may be determined as in Case V.

The remaining parts c and C may now be found as in Case 5, by formulas [16] and [18], or by [17] and [19].

The logarithmic work is very similar to that of Case V.

EXERCISE XLIII

Find the remaining parts of the spherical triangle in which

1. $b = 56° 15' 43''$, $B = 38° 57' 12''$, $C = 138° 54' 54''$.

2. $b = 48° 20'$, $A = 76° 50'$, $B = 59° 48'$.

3. $A = 70° 30' 28''$, $a = 45° 28' 32''$, $B = 60° 20' 32''$.

4. $A = 78° 47' 20''$, $a = 63° 49' 10''$, $C = 80° 25' 30''$.

5. $c = 112° 49' 24''$, $C = 152° 49' 27.5''$, $A = 29° 42' 13.7''$.

6. $C = 8° 48' 48''$, $c = 85° 26' 45''$, $B = 23° 49' 15''$.

7. $A = 57° 48' 23''$, $B = 120° 38' 27''$, $a = 48° 25' 20''$.

8. $A = 70° 28'$, $a = 80° 25' 40''$, $C = 125° 28'$.

REVIEW EXERCISE

1. If a median be drawn from the vertex C of a spherical triangle to the opposite side c, and the parts of the angle adjacent to sides a and b of the triangle be named α and β respectively, show that $\frac{\sin \alpha}{\sin \beta} = \frac{\sin b}{\sin a}$.

2. The city of Quito is situated nearly on the equator and its longitude is 78° 50′ west of Greenwich. The latitude of Greenwich is 51° 28′. Find the distance from Greenwich to Quito (on the arc of a great circle). Assume the radius of the earth to be 4000 miles.

3. In a spherical triangle whose sides are 48°, 57°, and 65°, respectively, a median is drawn to the side whose length is 48° from the opposite vertex. Find the length of the median and also the parts into which it divides the angle.

4. If the values of two sides of a spherical triangle are α and β and the angle between them n, find the length of the perpendicular upon the third side from the vertex of the angle n.

5. If a lune whose angle is α be drawn upon a sphere whose radius is c feet, and an arc of a great circle be drawn to intersect at equal angles the sides of the lune, making the part of the arc so included β, find the distance from each vertex of the lune to the transverse arc in feet.

6. A ship starting from a point on the equator in longitude 130° West sailed for 3 days, arriving at a point whose latitude is 20° North and longitude 150° West. What was its rate per hour, allowing the radius of the earth to be 4000 miles?

7. The sides of a spherical triangle enclosing an angle of 75° are respectively 60° and 54°. Find the length of the bisector of the angle and the angles it makes with the base.

8. There is a regular tetrahedron each of whose face angles is 60°. Find the angle between any two faces.

NOTE. — Suppose one vertex of the tetrahedron to be at the centre of a sphere whose radius is an edge of the tetrahedron. The other three vertices of the solid will determine upon the surface of the sphere the vertices of a spherical triangle whose sides are measured by the face angles of the tetrahedron.

9. If the face angle at the vertex of a regular four-sided pyramid is 50°, find the angle between any two lateral faces.

10. Find the area of a spherical triangle whose sides are 45° 26′, 53° 44′, 68° 46′, respectively, on a sphere whose radius is 10 ft.

NOTE. — The formula for the area of a spherical triangle is: area $= \frac{\pi R^2 E}{180°}$, where E denotes the excess in degrees of the sum of the angles of the triangle over 180°. This excess may be found when the three sides of the triangle are given by l'Huilier's Formula,

$$\tan \tfrac{1}{4} E = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2} (s-a) \tan \tfrac{1}{2} (s-b) \tan \tfrac{1}{2} (s-c)},$$

in which a, b, and c denote the sides of the triangle, and s, as usual, the half sum of the sides.

11. In a sphere of radius 12 is a spherical pyramid whose base is a spherical triangle of which the sides are 85°, 65°, and 120°. The vertex of the pyramid being at the centre of the sphere, find its volume.

12. If a line makes an angle θ with its projection on a plane passing through one end of the line and if the projection makes an angle ϕ with a second line drawn in a plane which intersects the first plane in the line of the projection at an angle of 30°, show that the angle between the first line and the second is $\cos^{-1}\frac{1}{2}(2\cos\theta\cos\phi \ \sin\theta\sin\phi)$.

13. A flight of stone steps faces due south. A rod rests with one end on a step and leans against the edge of the step above, in a plane perpendicular to the steps. At noon the horizontal part of the shadow is marked on the step and also the vertical part. If the rod makes an angle θ with the step upon which its foot rests, show that t hours after noon the angle the horizontal part of the shadow makes with its position at noon may be determined by the equation $\tan x = \sin\theta\tan t$, and the angle the vertical part of the shadow makes with its position at noon, by the equation $\tan y = \cos\theta\tan t$.

NOTE. — This example illustrates the principle of both the horizontal and the vertical sun dial. θ represents the latitude of the place. Let the lower end of the rod be the centre of a sphere whose surface is pierced by the rod and its two horizontal shadows in three points which are the vertices of a spherical right triangle. By means of this triangle the first relation may be proved.

It should be remembered that each hour of time corresponds to 15°.

14. If the longitude of New York is 40° 43′, find the angles which the shadows the sun would cast at three o'clock P.M. upon a dial constructed according to the principle of Ex. 13, would make with the shadows cast at noon.

ANSWERS

ANSWERS

Exercise I, page 4

1. (*a*) $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$, $\sec A = \frac{13}{12}$, $\csc A = \frac{13}{5}$, $\cot A = \frac{12}{5}$, $\text{vers } A = \frac{1}{13}$, $\text{covers } A = \frac{8}{13}$.

(*b*) $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$, $\sec A = \frac{5}{4}$, $\csc A = \frac{5}{3}$, $\cot A = \frac{4}{3}$.

(*c*) $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$, $\sec A = \frac{17}{15}$, $\csc A = \frac{17}{8}$, $\cot A = \frac{15}{8}$.

(*d*) $\sin A = \frac{2}{3}$, $\cos A = \frac{1}{3}\sqrt{5}$, $\tan A = \frac{2}{5}\sqrt{5}$, $\sec A = \frac{3}{5}\sqrt{5}$, $\csc A = \frac{3}{2}$, $\cot A = \frac{1}{2}\sqrt{5}$.

7. $c = 16$. 8. $A = 20°$. 9. $A = 40°$, $a = 5.0346$, $c = 7.83$.

10. $A = 20°$, $B = 70°$, $c = 266$. 11. $B = 20°$, $c = 29.24$, $a = 27.475$.

12. $A = 70°$, $a = 18.794$, $b = 6.84$.

Exercise II, page 6

1. $\cos 20°$, $\sin 5°$, $\csc 27°$, $\cot 33°\ 12'$, $\tan 4'$.

2. $x = 45°$. 3. $x = 30°$. 4. $x = 15°$.

Exercise III, page 8

1. $\frac{1}{2}$. 2. 5. 3. $\frac{3}{2}$. 4. 9. 5. 1. 6. $\frac{1}{2}$.

Exercise IV, page 12

1. $\cos A = \frac{1}{4}\sqrt{7}$, $\tan A = \frac{3}{7}\sqrt{7}$, $\sec A = \frac{4}{7}\sqrt{7}$, $\csc A = \frac{4}{3}$, $\cot A = \frac{1}{3}\sqrt{7}$.

2. $\sin A = \frac{2}{5}\sqrt{6}$, $\tan A = 2\sqrt{6}$, $\sec A = 5$, $\csc A = \frac{5}{12}\sqrt{6}$, $\cot A = \frac{1}{12}\sqrt{6}$.

3. $\sin A = \frac{3}{10}\sqrt{10}$, $\cos A = \frac{1}{10}\sqrt{10}$, $\sec A = \sqrt{10}$, $\csc A = \frac{1}{3}\sqrt{10}$, $\cot A = \frac{1}{3}$.

4. $\sin A = \frac{1}{37}\sqrt{37}$, $\cos A = \frac{6}{37}\sqrt{37}$, $\tan A = \frac{1}{6}$, $\sec A = \frac{1}{6}\sqrt{37}$, $\csc A = \sqrt{37}$.

5. See Art. 4.

6. $\sin A = \frac{1}{10}$, $\cos A = \frac{3}{10}\sqrt{11}$, $\tan A = \frac{1}{33}\sqrt{11}$, $\sec A = \frac{10}{33}\sqrt{11}$, $\cot A = 3\sqrt{11}$.

7. $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$, $\sec A = \frac{13}{5}$, $\csc A = \frac{13}{12}$, $\cot A = \frac{5}{12}$.

8. $\sin A = \frac{4}{5}$, $\tan A = \frac{4}{3}$, $\sec A = \frac{5}{3}$, $\csc A = \frac{5}{4}$, $\cot A = \frac{3}{4}$.

9. $\sin A = \frac{3}{73}\sqrt{73}$, $\cos A = \frac{8}{73}\sqrt{73}$, $\sec A = \frac{1}{8}\sqrt{73}$, $\csc A = \frac{1}{3}\sqrt{73}$, $\cot A = \frac{8}{3}$.

10. $\sin A = \dfrac{\sqrt{a^2-1}}{a}$, $\cos A = \dfrac{1}{a}$, $\tan A = \sqrt{a^2-1}$, $\csc A = \dfrac{a}{\sqrt{a^2-1}}$, $\cot A = \dfrac{1}{\sqrt{a^2-1}}$.

14. (*a*) $\cos A = \sqrt{1-\sin^2 A}$, $\tan A = \dfrac{\sin A}{\sqrt{1-\sin^2 A}}$, $\cot A = \dfrac{\sqrt{1-\sin^2 A}}{\sin A}$, $\sec A = \dfrac{1}{\sqrt{1-\sin^2 A}}$, $\csc A = \dfrac{1}{\sin A}$.

(*b*) $\sin A = \sqrt{1-\cos^2 A}$, $\tan A = \dfrac{\sqrt{1-\cos^2 A}}{\cos A}$, $\cot A = \dfrac{\cos A}{\sqrt{1-\cos^2 A}}$, $\sec A = \dfrac{1}{\cos A}$, $\csc A = \dfrac{1}{\sqrt{1-\cos^2 A}}$.

(*c*) $\sin A = \dfrac{\tan A}{\sqrt{1+\tan^2 A}}$, $\cos A = \dfrac{1}{\sqrt{1+\tan^2 A}}$, $\cot A = \dfrac{1}{\tan A}$, $\sec A = \sqrt{1+\tan^2 A}$, $\csc A = \dfrac{\sqrt{1+\tan^2 A}}{\tan A}$.

(*d*) $\sin A = \dfrac{\sqrt{\sec^2 A-1}}{\sec A}$, $\cos A = \dfrac{1}{\sec A}$, $\tan A = \sqrt{\sec^2 A-1}$, $\cot A = \dfrac{1}{\sqrt{\sec^2 A-1}}$, $\csc A = \dfrac{\sec A}{\sqrt{\sec^2 A-1}}$.

15. (*a*) $\dfrac{\sin^7 A - 2\sin^5 A + 2\sin^3 A + \sin^2 A - 1}{\sin^2 A - \sin^4 A}$.

(*b*) $\dfrac{1-\sin^2 A - \sin^3 A}{1-\sin^2 A}$. (*c*) $\dfrac{1-\sin^2 A+\sin^4 A}{1-\sin^2 A}$.

16. (*a*) $\cos A\sqrt{1-\cos^2 A}$. (*b*) $\dfrac{1}{\cos A}$. (*c*) $\dfrac{1-3\cos^2 A+3\cos^4 A}{\cos^4 A-\cos^2 A}$.

17. (*a*) $\dfrac{\tan A}{1+\tan^2 A}+\dfrac{1+\tan A}{\sqrt{1+\tan^2 A}}$. (*b*) $\dfrac{1}{\sqrt{1+\tan^2 A}}-\dfrac{1-\tan A+\tan^2 A}{1+\tan A+\tan^2 A}$.

(*c*) $\dfrac{(\tan A-1)\sqrt{1+\tan^2 A}+1}{\tan A}$.

Exercise VI, page 18

3. $A=36^\circ 52'$, $B=53^\circ\ 8'$, $c=5$.
4. $A=12^\circ 41'$, $B=77^\circ 19'$, $c=41$.
5. $A=67^\circ 23'$, $B=22^\circ 37'$, $b=5$.
6. $A=79^\circ 37'$, $B=10^\circ 23'$, $b=11$.
7. $A=81^\circ 12'$, $B=8^\circ 48'$, $a=84$.
8. $B=55^\circ 18'$, $b=17.33$, $c=21.08$.
9. $A=66^\circ 26'$, $a=13.75$, $b=5.997$.
10. $B=76^\circ\ 8'$, $a=4.197$, $c=17.51$.

Exercise VII, page 19

3. $A = 39^\circ 54' 28''$, $c = 8850.6$.
4. $A = 41^\circ 48' 35''$, $b = 2484.3$.
5. $A = 56^\circ 26' 27''$, $b = 0.3015$.
6. $b = 10322$, $c = 11287$.
7. $a = 0.778$, $b = 0.4036$.
8. $b = 454.43$, $c = 499$.
9. $a = 2.005$, $b = 1.287$.
10. $A = 53^\circ 15' 6''$, $c = 2194$.
11. $A = 2^\circ 27' 52''$, $c = 13.48$.
12. $a = 1760.5$, $c = 1762.2$.
13. $A = 53^\circ 7' 48''$, $a = 11.2$.
14. $A = 78^\circ 20' 39''$, $c = 811.74$.
15. $a = 518.61$, $b = 161.95$.
16. $b = 24.187$, $c = 24.23$.
17. $A = 43^\circ 44' 51''$, $b = 0.00679$.
18. $a = 761.17$, $b = 76.42$.
19. $a = 965.93$, $b = 258.82$.
20. $A = 71^\circ 38'$, $b = 0.334$.

Exercise VIII, page 20

1. $31^\circ 45' 33''$.
2. 67.4 ft.
3. $36^\circ 52' 12''$.
4. 166.43 ft.
5. 23.3 ft.
6. 202.2 ft.
7. $34^\circ 54' 54''$, $72^\circ 32' 33''$, $72^\circ 32' 33''$.
8. 61.6 ft.
9. 838.8 ft.
10. 660 yd.
11. $3^\circ 13' 29''$.
12. 75 ft.
13. 40.98 ft.
14. 140.88 ft.
15. $44^\circ 25' 37''$.
16. 9.81 ft.
19. 169.3 ft.
20. 769.8 ft.

Exercise IX, page 29

1. $\frac{1}{12}\pi$, $\frac{1}{6}\pi$, $\frac{2}{9}\pi$, $\frac{2}{3}\pi$, $\frac{25}{18}\pi$, $\frac{5}{3}\pi$.
2. 36°, 20°, 120°, 150°, 900°.
3. $\frac{15}{2}\pi$, $-\frac{5}{3}\pi$, 15π.
4. 2.

Exercise XII, page 46

1. $\cos x = \pm \frac{2}{5}\sqrt{6}$, $\tan x = \pm \frac{1}{12}\sqrt{6}$, $\sec x = \pm \frac{5}{12}\sqrt{6}$, $\csc x = -5$, $\cot x = \pm 2\sqrt{6}$.

2. $\sin x = -\frac{2}{3}\sqrt{2}$, $\tan x = -2\sqrt{2}$, $\sec x = 3$, $\csc x = -\frac{3}{4}\sqrt{2}$, $\cot x = -\frac{1}{4}\sqrt{2}$.

3. $\sin x = -\frac{3}{10}\sqrt{10}$, $\cos x = \frac{1}{10}\sqrt{10}$, $\sec x = \sqrt{10}$, $\csc x = -\frac{1}{3}\sqrt{10}$, $\cot x = -\frac{1}{3}$.

4. $\sin x = \pm \frac{1}{4}\sqrt{15}$, $\cos x = \frac{1}{4}$, $\tan x = \pm \sqrt{15}$, $\csc x = \pm \frac{4}{15}\sqrt{15}$, $\cot x = \pm \frac{1}{15}\sqrt{15}$.

5. $\sin A = \pm \frac{y}{x}$, $\cos A = \frac{\sqrt{x^2 - y^2}}{x}$, $\tan A = \pm \frac{y}{\sqrt{x^2 - y^2}}$, $\csc A = \pm \frac{x}{y}$, $\cot A = \pm \frac{\sqrt{x^2 - y^2}}{y}$.

6. $\cos A = \pm \frac{2xy}{x^2 + y^2}$, $\cot A = \pm \frac{2xy}{x^2 - y^2}$.

Exercise XIV, page 55

1. $\frac{5}{2}\sqrt{3}$, $\frac{5}{2}$; -0.868, 4.924; -4.698, -1.71.

2. 5, $-53° 8'$; or -5, $126° 52'$. 3. $197° 27' 28''$, or $-17° 27' 28''$.

4. If the triangle is described in the positive direction of rotation, the angles are $120°$, $-120°$, $-120°$, 5; -5; -5.

5. $5\sqrt{3}$; $-5\sqrt{3}$; 0; $7\frac{1}{2}$; $-7\frac{1}{2}$.

Exercise XV, page 63

2. $\frac{1}{4}(\sqrt{6} - \sqrt{2})$. 3. $2 + \sqrt{3}$. 4. $\frac{1}{4}(\sqrt{6} - \sqrt{2})$, $\frac{1}{4}(\sqrt{6} + \sqrt{2})$, $2 - \sqrt{3}$.

5. $\frac{1}{6}\sqrt{3} + \frac{1}{3}\sqrt{2}$; $-\frac{1}{6}\sqrt{3} - \frac{1}{3}\sqrt{2}$.

8. $\cos\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma + \sin\alpha\cos\beta\sin\gamma$.

9. $\frac{\tan\alpha - \tan\beta + \tan\gamma + \tan\alpha\tan\beta\tan\gamma}{1 + \tan\alpha\tan\beta - \tan\alpha\tan\gamma + \tan\beta\tan\gamma}$.

Exercise XVI, page 65

1. $\pm \frac{1}{2}\sqrt{3}$. 2. $-\frac{4}{5}$. 3. 3.43.

5. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$. $\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$.

6. $\sin 4\alpha = 8\sin\alpha\cos^3\alpha - 4\sin\alpha\cos\alpha$.
$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$.
$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$.

7. $\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha$,
$\cos 5\alpha = 5\cos\alpha - 20\cos^3\alpha + 16\cos^5\alpha$.

Exercise XVII, page 68

1. 0.316, 0.9487; 0.78.

2. $\sin 22° 30' = \frac{1}{2}\sqrt{2 - \sqrt{2}}$, $\cos 22° 30' = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $\tan 22° 30' = \sqrt{2} - 1$.

4. $\sin 165° = \frac{1}{4}(\sqrt{6} - \sqrt{2})$, $\cos 165° = -\frac{1}{4}(\sqrt{2} + \sqrt{6})$, $\tan 165° = \sqrt{3} - 2$.

Exercise XX, page 75

1. $n\pi$. 2. $(n + \frac{1}{2})\pi$. 3. $n\pi$. 4. $(2n + \frac{1}{2})\pi$.
5. $(2n - \frac{1}{2})\pi$. 6. $2n\pi$. 7. $(2n + 1)\pi$. 8. $2n\pi$.
9. $(2n + 1)\pi$. 10. $(n + \frac{1}{4})\pi$. 11. $(n + \frac{3}{4})\pi$.
12. $(2n - \frac{1}{4})\pi$, $(2n - \frac{3}{4})\pi$. 13. $(2n + \frac{1}{6})\pi$, $(2n + \frac{5}{6})\pi$.
14. $(2n \pm \frac{1}{3})\pi$. 15. $(2n \pm \frac{1}{4})\pi$. 16. $(2n \pm \frac{3}{4})\pi$.

Exercise XXII, page 81

1. $45°, 135°, 225°, 315°$; $n\pi \pm \frac{\pi}{4}$.
2. $60°, 90°, 120°, 270°$; $2n\pi \pm \frac{\pi}{2}$, $2n\pi + \frac{\pi}{3}$, $(2n + 1)\pi - \frac{\pi}{3}$.
3. $45°, 225°$; $n\pi + \frac{\pi}{4}$.
4. $0°, 60°, 300°$; $2n\pi$, $2n\pi \pm \frac{\pi}{3}$. 5. $0°, 180°$; $n\pi$.
6. $15°, 75°, 195°, 255°$; $n\pi + \frac{\pi}{12}$, $n\pi + \frac{5\pi}{12}$.
7. $60°, 180°, 300°$; $2n\pi \pm \frac{\pi}{3}$, $(2n + 1)\pi$.
8. $45°, 135°, 225°, 315°$; $n\pi \pm \frac{\pi}{4}$.
9. $45°, 165°\,58', 225°, 345°\,58'$; $n\pi + \frac{\pi}{4}$, $n\pi + \tan^{-1}(-\frac{1}{4})$.
10. $30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°$; $n\pi \pm \frac{\pi}{6}$, $n\pi \pm \frac{\pi}{3}$.

Exercise XXIII, page 83

1. $45°, 225°$; $n\pi + \frac{\pi}{4}$. 3. $285°, 345°$; $2n\pi - \frac{5\pi}{12}$, $2n\pi - \frac{\pi}{12}$.
2. $0°, 90°$; $2n\pi$, $2n\pi + \frac{\pi}{2}$. 4. $120°$; $2n\pi + \frac{2\pi}{3}$.
5. $24°\,27', 261°\,49'$; $n \cdot 360° - 36°\,52' \pm 61°\,19'$.
6. $27°\,58', 135°, 242°\,2', 315°$; $n\pi + \frac{3\pi}{2}$, $\frac{1}{2}\sin^{-1}(2\sqrt{2} - 2)$.
7. $0°, 90°, 180°$; $n\pi$, $2n\pi + \frac{\pi}{2}$.
8. $0°, 90°$; $2n\pi$, $2n\pi + \frac{\pi}{2}$.
9. $30°, 270°$; $2n\pi + \frac{\pi}{6}$, $2n\pi - \frac{\pi}{2}$.
10. $46°\,24', 90°$; $2n\pi + \frac{\pi}{2}$, $n \cdot 360° + 46°\,24'$.

Exercise XXIV, page 86

1. 90°, 270°; $n\pi + \frac{\pi}{2}$.

2. 51° 19′, 180°, 308° 41′; $(2n+1)\pi$, $\cos^{-1}\frac{5}{8}$. **3.** 0°, 180°; $n\pi$.

4. 0°, 60°, 120°, 180°, 240°, 300°; $n\pi$, $(n+\frac{1}{2})\pi \pm \frac{\pi}{6}$.

5. 0°, $7\frac{1}{2}$°, $37\frac{1}{2}$°, $97\frac{1}{2}$°, $127\frac{1}{2}$°, 180°, etc.; $n\pi$, $\frac{1}{4}\left(2n\pi + \frac{\pi}{6}\right)$, $\frac{1}{4}\left(2n\pi + \frac{5\pi}{6}\right)$.

6. 30°, 60°, 90°, 120°, 150°, 210°, 240°, 270°, 300°, 330°; $(n+\frac{1}{2})\pi$, $\frac{1}{4}\left(2n\pi \pm \frac{2\pi}{3}\right)$.

7. 45°, 60°, 120°, 135°, 225°, 240°, 300°, 315°; $n\pi \pm \frac{\pi}{4}$, $n\pi \pm \frac{\pi}{3}$.

8. 0°, 180°; $n\pi$.

9. 18°, 162°, 234°, 306°; $\sin^{-1}\frac{-1 \pm \sqrt{5}}{4}$.

10. 18°, 54°, 90°, 126°, 162°, etc.; $\frac{1}{5}\left(n\pi + \frac{\pi}{2}\right)$.

11. 45°, 90°, 135°, 225°, 270°, 315°; $n\pi \pm \frac{\pi}{4}$, $n\pi + \frac{\pi}{2}$.

12. 22° 6′, 67° 54′; $\frac{1}{2}\sin^{-1}\frac{1}{2}(5 - \sqrt{13})$.

13. 0°, 65° 4′, 252° 45′; $2n\pi$, $2\cos^{-1}\frac{1 \pm \sqrt{33}}{8}$.

14. 45°, $67\frac{1}{2}$°, 90°, $157\frac{1}{2}$°, 225°, $247\frac{1}{2}$°, 270°, $337\frac{1}{2}$°; $n\pi + \frac{\pi}{4}$, $n\pi + \frac{\pi}{2}$, $\frac{n\pi}{2} + \frac{3\pi}{8}$.

15. 22° 30′, 112° 30′; $\frac{1}{2}\left(n\pi + \frac{\pi}{4}\right)$.

16. 60°, 90°, 120°, 240°, 270°, 300°; $n\pi \pm \frac{\pi}{3}$, $n\pi + \frac{\pi}{2}$.

Exercise XXV, page 87

1. $\frac{3}{4}$. **5.** $\frac{1}{2}\sqrt{6}$. **7.** $-2-\sqrt{3}$. **9.** $\frac{1}{3}$. **11.** $\frac{3}{8}$. **13.** $\frac{56}{33}$.

14. 0°, 45°, 180°, 225°; $n\pi$, $n\pi + \frac{\pi}{4}$.

18. $22\frac{1}{2}$°, $112\frac{1}{2}$°, $202\frac{1}{2}$°, $292\frac{1}{2}$°; $\frac{1}{2}\left(n\pi + \frac{\pi}{4}\right)$.

22. $\frac{2(np-1)(n+p)}{(np+n+p-1)(np-n-p-1)}$.

23. (1) $\frac{\sec A \sec B}{1-\sqrt{(\sec^2 A-1)(\sec^2 B-1)}}$; (2) $\frac{\csc A \csc B}{\sqrt{\csc^2 B-1}-\sqrt{\csc^2 A-1}}$.

27. 18°, 54°, 126°, 162°, 198°, 234°, 306°, 342°; $\sin^{-1} \pm \frac{1}{4}(\sqrt{5} \pm 1)$.

Exercise XXVI, page 96

2. $C = 54° 18'$, $b = 3.317$, $c = 3.925$.
3. $A = 123° 12'$, $a = 23.63$, $c = 20.51$.
4. $B = 25° 12'$, $c = 227.7$, $b = 157.4$.
5. $C = 35° 4'$, $b = 577.3$, $c = 468.9$.
7. $C = 87° 32' 5''$, $b = 17.632$, $c = 21.746$.
8. $A = 38° 21' 47''$, $a = 13.509$, $b = 17.632$.
9. $A = 29° 25' 18''$, $b = 2675.9$, $c = 3674$.
10. $B = 67° 27' 33''$, $b = 77.08$, $c = 79.06$.
11. $B = 100° 22' 45''$, $a = 1337.2$, $b = 1758.9$.
12. $A = 139° 21' 42''$, $a = 100$, $c = 63.15$.

Exercise XXVII, page 97

2. $c = 8.9$. 3. $a = 13$. 4. $c = 2$. 5. $b = \sqrt{3}$.

Exercise XXVIII, page 99

2. $A = 12° 22'$, $B = 149° 15'$, $c = 34.37$.
3. $A = 64° 19' 28''$, $B = 42° 24' 22''$, $c = 612.06$.
4. $A = 84° 12' 33''$, $C = 45° 46' 59''$, $b = 0.5591$.
5. $B = 37° 48' 5''$, $C = 42° 11' 55''$, $a = 0.0117$.
6. $A = 33° 5' 18''$, $C = 41° 0' 42''$, $b = 96.42$.
7. $A = 31° 50' 20''$, $B = 50° 4' 25''$, $c = 3139.9$.
8. $A = 133° 51' 34''$, $B = 11° 59' 10''$, $c = 2479.2$.
9. $A = 70° 22' 38''$, $B = 21° 24' 42''$, $c = 33.787$.
10. $A = 72° 40' 41''$, $B = 15° 8' 1''$, $c = 15.272$.

Exercise XXIX, page 102

2. $A = 41° 13' 0''$, $C = 87° 37' 54''$, $c = 116.82$.
3. $A = 11° 26' 58''$, $C = 84° 16' 31''$, $c = 401$.
4. $B = 48° 27' 20''$, $C = 90°$, $b = 360$.
5. $A = 46° 52' 10''$, $C = 111° 53' 25''$, $c = 883.65$.
   $A' = 133° 7' 50''$, $C' = 25° 37' 45''$, $c = 411.92$.
6. $A = 29° 11' 39''$, $C = 91° 34' 21''$, $c = 8.853$.
7. $A = 83° 40'$, $C = 71° 5' 47''$, $c = 670.1$.
   $A' = 96° 20'$, $C' = 58° 25' 47''$, $c' = 603.5$.
8. $A = 19° 19' 3''$, $C = 142° 59' 48''$, $c = 89.15$.
   $A' = 160° 40' 57''$, $C' = 1° 37' 54''$, $c' = 4.218$.
9. $A = 10° 54' 58''$, $C = 132° 12'$, $c = 946.68$.
10. $A = 57° 37' 18''$, $C = 90°$, $a = 88$.

Exercise XXX, page 103

2. $A = 60°$, $B = 32° 12'$, $C = 87° 48'$.
3. $A = 56° 7'$, $B = 81° 47'$, $C = 42° 6'$.
4. $A = 56° 15'$, $B = 59° 51'$, $C = 63° 54'$.
5. $A = 38° 57'$, $B = 47° 41'$, $C = 93° 22'$.

Exercise XXXI, page 106

2. $A = 7° 37' 42''$, $B = 61° 55' 38''$, $C = 110° 26' 40''$.
3. $A = 47° 38'$, $B = 68° 1' 6''$, $C = 64° 20' 54''$.
4. $A = 85° 55' 7''$, $B = 43° 57' 33''$, $C = 50° 7' 20''$.
5. $A = 23° 32' 12''$, $B = 56° 8' 42''$, $C = 100° 19' 6''$.
6. $A = 59° 39' 30''$, $B = 42° 35' 20''$, $C = 77° 45' 10''$.
7. $A = 33° 15' 39''$, $B = 50° 56'$, $C = 95° 48' 21''$.
8. $A = 37° 22' 19''$, $B = 38° 15' 41''$, $C = 104° 22'$.
9. $A = 36° 45' 14''$, $B = 53° 3' 8''$, $C = 90° 11' 38''$.

Exercise XXXII, page 108

1. 14.68.
2. 1259.6.
3. 0.00815.
4. 88.66.
5. 0.156.
6. 2520.
7. 28621.
8. 1285.3.
9. 0.0265.
10. 63.34.

Exercise XXXIII, page 109

1. 27.65, 80.08; 65° 19′ 58″, 114° 40′ 2″.
2. 169.45 ft.
3. 4.43 ft.; 7.35 ft.
4. 114.41 ft.
5. 46.14 ft.; 99.92 ft.
6. 171.08 yd.
7. 13 miles per hour, nearly.
8. 114.6 in.
9. $2000(2\sqrt{3} - 3)$ yd.
10. 23.87 ft.; 15.18 ft.
11. 39.97 ft.; 29.99 ft.
12. 520.44 ft.
13. 330.72 ft.; 138° 35′ 30″.
14. 75.13 ft.; 225.4 ft.
15. 11646 ft.
16. $l(\cot\beta - \cot\alpha)$.
17. 753.1 yd.
18. 91.772 ft.
19. 47.168 ft.; 16° 20′.
20. 8.574 miles.
21. 12.15 ft.
23. $20\sqrt{3}$ ft.
26. 106 ft.
27. $5(2 + \sqrt{3})$ ft.
28. 9.24 ft.
29. 180 ft.; 500 ft.
31. 15.14 ft.
32. $\tan^{-1}\dfrac{\tan\theta\tan\phi}{\sqrt{\tan^2\theta + \tan^2\phi}}$; $\tan^{-1}(\tan\theta\cot\phi\sec\phi)$; $\tan^{-1}(\tan\phi\cot\theta\sec\theta)$; $\tan^{-1}(\cot\theta\cot\phi\sqrt{\tan^2\theta + \tan^2\phi})$.
35. $50\sqrt{6}$ ft.
36. $250\sqrt{3}$ ft.
38. $\dfrac{l}{3 - 4\sin^2\alpha}$; $\dfrac{2l\sin 2\alpha}{3 - 4\sin^2\alpha}$

Exercise XXXIV, page 118

8. 100°, 105°, 75°; 157.08, 164.934, 117.81. **9.** 110°, 100°, 70°.
10. 60°; a great circle and its pole.

Exercise XXXV, page 126

1. $c = 15° 38' 6''$, $A = 38° 57' 12''$, $B = 52° 5' 54''$.
2. $c = 40° 9' 21''$, $A = 47° 37' 21''$, $B = 50° 2' 56''$.
3. $a = 25° 50' 17''$, $b = 33° 7' 37''$, $B = 56° 15' 43''$.
4. $b = 48° 54' 54''$, $c = 50° 2' 56''$, $A = 16° 6' 22''$.
5. $a = 12° 16' 42''$, $b = 51° 2' 48''$, $c = 52° 5' 54''$.
6. $a = 33° 7' 37''$, $A = 41° 5' 6''$, $B = 64° 9' 43''$.
7. $a = 42° 22' 39''$, $b = 45° 15' 43''$, $A = 52° 5' 54''$.
8. $a = 39° 57' 4''$, $b = 49° 50' 39''$, $c = 60° 22' 24''$.
9. $b = 48° 54' 54''$, $A = 56° 15' 43''$, $B = 56° 52' 23''$.
10. $a = 37° 54' 6''$, $c = 77° 43' 18''$, $B = 80° 14' 41''$.
11. $c = 40° 58' 50''$, $b = 22° 50' 19''$, $B = 36° 17' 17''$,
$c = 139° 1' 10'$, $b = 157° 9' 41''$, $B = 143° 42' 43''$.
12. Impossible.
13. $A = 127° 30' 11''$, $a = 128° 32'$, $B = 102° 14' 30''$.
14. $B = 96° 19' 51.5''$, $c = 95° 48' 28''$, $a = 47° 18' 44''$.
15. $a = 66° 37'$, $c = 85° 52'$, $A = 66° 57' 48''$.
$a = 113° 23'$, $c = 94° 8'$, $A = 113° 2' 12''$.

Exercise XXXVI, page 127

1. $B = 167° 43' 18''$, $a = 138° 54' 54''$, $b = 129° 57' 4''$.
2. $A = 170° 14' 41''$, $a = 164° 21' 54''$, $b = 102° 16' 42''$.
3. $C = 138° 54' 54''$, $A = 169° 29' 45''$, $b = 102° 16' 42''$.
4. $C = 138° 15' 43''$, $B = 146° 15' 43''$, $a = 132° 22' 39''$.
5. $C = 102° 16' 42''$, $A = 140° 2' 56''$, $B = 106° 6' 22''$.
6. $B = 31° 54' 40''$, $C = 55° 50' 7''$, $b = 39° 42' 23''$.
$B = 148° 5' 20''$, $C = 124° 9' 53''$, $b = 140° 17' 37''$.
7. $A = 18° 12' 24''$, $C = 52° 57' 12''$, $a = 23° 2' 44''$.
$A = 161° 47' 36''$, $C = 127° 2' 48''$, $a = 156° 57' 16''$.
8. $B = 132° 31' 45''$, $A = 60° 25' 39''$, $a = 67° 18' 38''$.

Exercise XXXVIII, page 136

1. $A = 20° 35' 37''$, $B = 36° 10' 39''$, $C = 129° 57' 4''$.
2. $A = 43° 2' 7''$, $B = 47° 37' 21''$, $C = 102° 16' 42''$.
3. $A = 45° 26' 42''$, $B = 47° 37' 21''$, $C = 102° 16' 42''$.
4. $A = 52° 5' 54''$, $B = 56° 52' 23''$, $C = 88° 42' 27''$.

5. $A = 50° 2' 56''$, $B = 58° 40' 13''$, $C = 105° 6' 41''$.
6. $A = 52° 5' 54''$, $B = 68° 12' 58''$, $C = 113° 53' 57''$.
7. $A = 52° 5' 54''$, $B = 56° 52' 23''$, $C = 138° 5' 42''$.
8. $A = 84° 26' 48''$, $B = 108° 24' 54''$, $C = 56° 28' 32''$.
9. $A = 126° 18' 42''$, $B = 119° 42' 8''$, $C = 111° 51' 42''$.
10. $A = 67° 9' 28''$, $B = 67° 9' 28''$, $C = 74° 27' 56''$.

Exercise XXXIX, page 136

1. $a = 36° 10' 39''$, $b = 40° 9' 21''$, $c = 50° 13' 58''$.
2. $a = 38° 57' 12''$, $b = 55° 1' 2''$, $c = 56° 15' 43''$.
3. $a = 39° 20' 24''$, $b = 41° 5' 6''$, $c = 60° 22' 24''$.
4. $a = 50° 2' 56''$, $b = 52° 5' 54''$, $c = 99° 57' 42''$.

Exercise XL, page 138

1. $a = 20° 32' 33''$, $B = 52° 5' 54''$, $C = 123° 7' 37''$.
2. $c = 60° 22' 24''$, $A = 26° 40' 20''$, $B = 38° 57' 12''$.
3. $b = 56° 15' 43''$, $A = 21° 34' 28''$, $C = 132° 22' 39''$.
4. $c = 60° 22' 24''$, $A = 47° 37' 21''$, $B = 77° 39' 26''$.
5. $C = 21° 41' 25''$, $B = 178° 0' 29''$, $a = 50° 33' 38''$.
6. $C = 129° 53' 4''$, $A = 29° 47' 28''$, $b = 71° 45' 15''$.
7. $B = 84° 11' 45''$, $C = 72° 59' 41''$, $a = 93° 58' 18''$.
8. $B = 161° 46' 32''$, $A = 157° 58' 8''$, $c = 85° 19' 46''$.

Exercise XLI, page 139

1. $a = 56° 52' 23''$, $b = 79° 29' 45''$, $C = 119° 15' 56''$.
2. $b = 52° 5' 54''$, $c = 63° 21' 53''$, $A = 58° 40' 13''$.
3. $a = 44° 44' 17''$, $c = 80° 14' 41''$, $B = 52° 5' 54''$.
4. $b = 60° 22' 24''$, $c = 79° 39' 38''$, $A = 38° 57' 12''$.
5. $a = 40° 12' 34''$, $b = 53° 38' 28''$, $C = 82° 9'$.
6. $a = 112° 25' 37''$, $c = 59° 19' 25''$, $B = 87° 14'$.
7. $b = 108° 20' 51''$, $c = 108° 58' 5''$, $A = 53° 53' 6''$.
8. $a = 108° 32' 10''$, $b = 88° 35' 18''$, $C = 121° 47' 14''$.

Exercise XLII, page 141

1. $b = 40° 32' 33''$, $B = 39° 9' 35''$, $C = 129° 57' 4''$.
   $b = 61° 33' 4''$, $B = 121° 19' 47''$, $C = 50° 2' 56''$.
2. $a = 35° 30' 24''$, $A = 43° 2' 7''$, $C = 102° 16' 42''$.
   $a = 55° 1' 2''$, $A = 74° 18' 19''$, $C = 77° 43' 18''$.
3. $a = 21° 7' 35''$, $A = 25° 26' 16''$, $C = 100° 30' 15'$.
   $a = 46° 0' 59''$, $A = 47° 39' 0''$, $C = 79° 29' 25''$.

4. Impossible.
5. $A = 90°$, $C = 12° 29' 4''$, $c = 11° 53' 42''$.
6. $B = 61° 59'$, $c = 13° 38' 16''$, $C = 12° 9' 24''$.
   $B = 118° 1'$, $c = 132° 29' 46''$, $C = 138° 48' 6''$.
7. $B = 44° 28' 46''$, $c = 16° 35' 58''$, $C = 34° 59' 24''$.
8. $c = 68° 50' 36''$.

Exercise XLIII, page 142

1. $a = 5° 21' 59''$, $c = 60° 22' 24''$, $A = 4° 3' 15''$.
   $a = 77° 43' 18''$, $c = 119° 37' 37''$, $A = 47° 37' 21''$.
2. $A = 57° 18' 43''$, $C = 66° 31' 42''$, $c = 52° 27' 4''$.
   $A = 122° 41' 17''$, $C = 152° 14' 42''$, $c = 156° 15' 54''$.
3. $b = 41° 5' 17''$, $c = 42° 55' 48''$, $C = 64° 14'$.
4. $c = 64° 26' 20''$, $b = 40° 48' 50''$, $B = 45° 35' 50''$.
   $c = 115° 33' 40''$, $b = 176° 34' 16''$, $B = 176° 15' 4''$.
5. $a = 90°$, $b = 25° 57' 12''$, $B = 12° 28' 38''$.
6. Impossible.
7. $b = 49° 30' 48''$, $c = 2° 5' 26''$, $C = 2° 21' 54''$.
   $b = 130° 29' 12''$, $c = 118° 5' 56''$, $C = 94° 4' 4''$.
8. $c = 121° 33' 9''$, $b = 77° 39' 20''$, $B = 69° 0' 18''$.

Review Exercise

2. 5799.3 miles.
3. median $= 58° 3' 15''$.
   angles $= 26° 12' 29''$ and $28° 31' 6''$.
4. $\tan^{-1}\left(\dfrac{\sin n \tan \alpha}{\sqrt{\sin^2 n + (\tan \alpha - \tan \beta \cos n)^2}}\right)$.
5. $\dfrac{\pi c \sin^{-1}\left(\tan \frac{\beta}{2} \cot \frac{\alpha}{2}\right)}{180}$.
6. 27 miles (nearly).
7. bisector $= 50° 35' 21''$.
   angles $= 84° 34'$ and $95° 26'$.
8. $70° 31' 43''$.
9. $87° 15' 2''$.
10. 42.43 ft.
11. 904.7808.
14. $33° 7' 2''$; $37° 9' 37''$.

$0 < \sin A < 1$ VQ 17136 $\sin A = \frac{opp}{hyp}$

$1 > \cos A > 0$

$0 < \tan A$ $\cos A = \frac{adj.}{hyp}$

$\tan A = \frac{opp}{adj.}$

$\csc A = \frac{1}{\sin A}$

$\sec A = \frac{1}{\cos A}$

$\cot A = \frac{1}{\tan A}$

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