PLANE AND SPHERICAL TRIGONOMETRY

BY

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PREFACE

The plan and scope of this work may be indicated briefly by the following characteristic features:

Directed lines and Cartesian coordinates are introduced as a working basis.

Each subject, when first introduced, is treated in a general manner and is presented as fully as the character of the work demands. This avoids the necessity of treating the same subjects several times for the purpose of modifying and extending certain conceptions.

Statements in the form of theorems and problems are used freely to indicate the aim of various articles and to define the data clearly.

Inverse functions are treated more fully than is customary.

The general principles governing the solution of triangles, the solution of trigonometric equations, and the proof of identities are carefully presented.

Special attention is given to the arrangement of computations.

The sine and cosine series are obtained from De Moivre's Theorem, thus completing the line of development which leads to the calculation of the trigonometric functions.

The work on spherical trigonometry contains the development of all the formulas that are generally used in practical astronomy.

The right spherical triangle is treated from two points of view: as a special case of the oblique triangle, and directly from geometric figures. The work is so arranged that either view may be presented independently.

The solutions of the oblique spherical triangle by means of auxiliary quantities, characteristic of astronomy, are included as interesting mathematical problems and as preparation for astronomical work.

PREFACE TO THE SECOND REVISED EDITION

The second edition embodies such modifications, rearrangements, and additions as have been suggested by experience in the classroom.

Due to numerous requests tables have been added. Simple three-place tables have been included. By their use the numerical work is reduced to a minimum, thus leaving the student free to give more attention to the principles involved in the solution of a triangle. Accordingly a few triangles, suitable for three-place table or slide rule computation, have been introduced at the beginning of the list of problems under the right triangle and also under the oblique triangle.

In order to give a wider range to computational work four-place tables as well as five-place tables have been included.

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PLANE TRIGONOMETRY

CHAPTER I

RECTANGULAR COORDINATES AND ANGLES

1. Introduction. The word *trigonometry* is derived from the two Greek words for triangle (τρίγωνον) and measurement (μετρία). Originally trigonometry was concerned chiefly with the solution of triangles. At present this is but one part of the subject.

Certain preliminary considerations, concerning directed lines and angles, are necessary before introducing the fundamental definitions of trigonometry.

RECTANGULAR COORDINATES

2. Directed lines. A positive and a negative direction may be assigned arbitrarily to every line.

If the direction from A to C is *positive*, the opposite direction from C to A is *negative*.

If we let the order of the letters indicate the direction in which a segment is measured, it is evident that AC and CA represent the same segment measured in opposite directions; hence

$$AC = -CA \text{ or } -AC = CA.$$

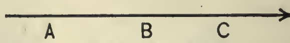
Also, if B be a third point on the line, the segments AB and BA are opposite in sign; likewise the segments BC and CB. Hence

$$AB = -BA \text{ or } -AB = BA,$$

and $$BC = -CB \text{ or } -BC = CB.$$

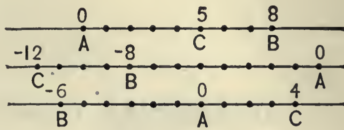
Then for all positions of A, B, and C on a line it follows that

$$AC = AB + BC.$$

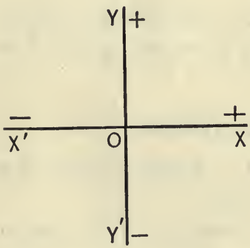


Thus for the following figures:

$$5 = 8 + (-3) = 5$$
$$-12 = -8 + (-4) = -12$$
$$4 = -6 + (+10) = 4$$



3. Lines of reference. Two directed lines, perpendicular to each other, may be taken as **lines of reference** or **axes.** They are usually designated by $X'X$ and $Y'Y$ and are called the X-axis and the Y-axis respectively.



The X-axis is positive from left to right and negative from right to left.

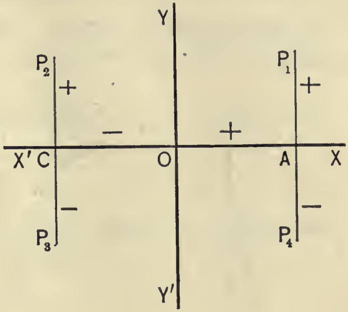
The Y-axis is positive upward and negative downward.

The point of intersection of the axes is the **origin.** The origin serves as a convenient starting point from which to measure distances.

4. Quadrants. The axes produced divide the plane into four parts called **quadrants.** The quadrants are designated by number. The first quadrant is indicated by XOY, the second by YOX', the third by $X'OY'$, and the fourth by $Y'OX$.

5. Coordinates of a point. From any given point P_1 in the plane, draw a line parallel to the Y-axis intersecting the X-axis in some point A.

Then the distance from the origin to the point of intersection, or OA, is the **abscissa** of the given point P_1.



The distance from the point of intersection to the given point, or AP_1, is the **ordinate** of the given point P_1.

The abscissa and the ordinate of the point P_2 are OC and CP_2 respectively.

The abscissa and the ordinate of the point P_3 are OC and CP_3 respectively.

The abscissa and the ordinate of the point P_4 are OA and AP_4 respectively.

6. Signs of coordinates. Abscissas are positive when measured from the origin to the right, and negative when measured from the origin to the left.

Ordinates are positive when measured upward from the X-axis and negative when measured downward.

Thus OA, AP_1, and CP_2 are positive; OC, CP_3, and AP_4 are negative.

7. EXERCISES

1. Locate the point whose abscissa is 4 and whose ordinate is 7. This point is designated (4, 7).

2. Locate the point whose abscissa is 2 and whose ordinate is 5, *i.e.* the point (2, 5).

3. Locate the points $(-3, 4)$, $(-6, -3)$, $(5, -1)$, $(0, 4)$, $(-7, 0)$, and $(-8, 10)$.

4. Locate the points $(1\frac{1}{2}, 3)$, $(-\frac{1}{2}, 0)$, (m, n), (x, y), $(x, 0)$.

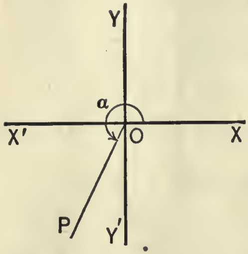
5. What is the locus of the points whose abscissas are 6?

6. What is the locus of the points whose ordinates are -3?

7. What is the locus of the points whose abscissas are twice their ordinates?

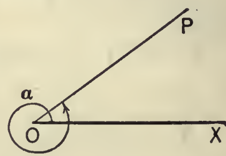
ANGLES

8. Magnitude of angles. In elementary geometry the angles considered are usually less than two right angles; but in trigonometry it is necessary to introduce angles of *any* magnitude, positive or negative.

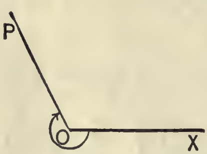


To extend the conception of an angle, suppose a line to revolve in a fixed plane about a fixed point O, from the initial position OX to the successive positions OY, OX', and OP, generating α*, an angle greater than two right angles.

If the line continues to revolve, making more than one complete revolution, it generates an angle α which is greater than four right angles. Evidently by continuing the rotation an angle of any magnitude may be generated. Thus the size of the angle α depends upon the amount of rotation of OP and is designated by an arc.



9. Direction of rotation. Positive and negative angles. As a positive and negative direction may be assigned arbitrarily to a line, so a positive and negative sense of generation may be assigned arbitrarily to an angle.



The direction of rotation indicated by the arrows in the figures of Art. 8 is the positive direction; the opposite direction, indicated by the arrow in the adjoining figure, is the negative direction.

* Angles will usually be designated by Greek letters:

α	. . . Alpha	δ	. . . Delta
β	. . . Beta	θ	. . . Theta
γ	. . . Gamma	ϕ	. . . Phi

A **positive angle** is an angle generated by a line rotating in the positive direction.

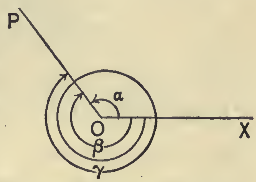
A **negative angle** is an angle generated by a line rotating in the negative direction.

Negative angles, like positive angles, may have unlimited magnitude.

10. Initial and terminal lines. The fixed line from which both positive and negative angles are measured is the **initial line.** It usually coincides with that part of the X-axis lying to the right of the origin. Thus, in the last three figures, OX is the initial line.

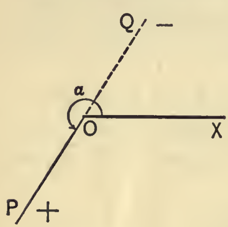
The final position of the revolving line, marking the termination of the angle, is the **terminal line.** Thus, in the last three figures, OP is the terminal line.

Two or more unequal angles may have the same terminal line. Thus the positive angle α and the negative angles β and γ have the same terminal line.



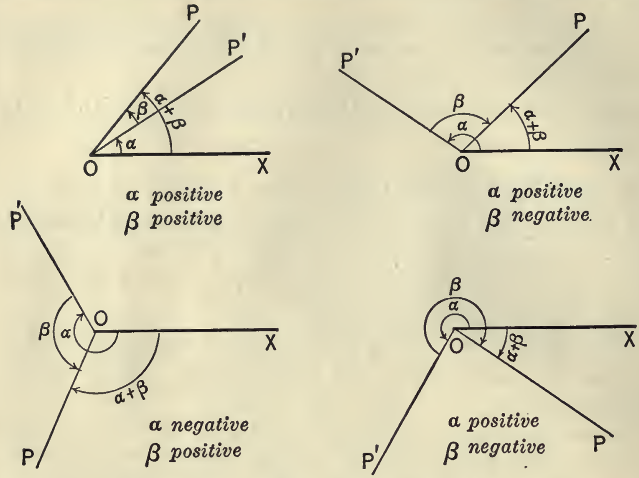
Angles having the same terminal line, and the same initial line, are called **coterminal angles.**

11. Sign of terminal line. The terminal line, drawn from the origin in the direction of the extremity of the measuring arc, is positive; the terminal line produced, drawn from the origin in the opposite direction, is negative.



Thus the terminal line OP, of the angle α, is positive, and the terminal line produced, OQ, is negative. As the terminal line OP revolves it retains its positive sign.

12. The algebraic sum of two angles. To construct the algebraic sum of two angles, conceive OP to rotate from the position OX, through the angle α, to OP'; then from this



position let it rotate through the angle β, whether positive or negative, to OP. Then XOP is the desired angle $\alpha + \beta$.

13. Measurement of angles. Various units may be employed in the measurement of angles. In elementary geometry the right angle is frequently used. Two other units in common use are the degree and the radian. The degree is generally used in practical problems involving numerical computations, while the radian is essential in many theoretical considerations.

The degree is defined as one ninetieth of a right angle. The degree is divided into sixty equal parts called minutes. The minute is divided into sixty equal parts called seconds.

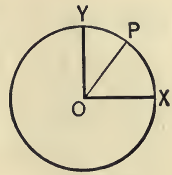
Then $\quad 60 \text{ seconds } (60'') = 1 \text{ minute.}$

$\quad\quad 60 \text{ minutes } (60') = 1 \text{ degree.}$

$\quad\quad 90 \text{ degrees } (90°) = 1 \text{ right angle.}$

The angle 26 degrees, 39 minutes, and 57 seconds is written $26° \, 39' \, 57''$.

14. Circular or radian measure. A **radian** is an angle of such magnitude that, if placed with its vertex at the center of any circle, it will intercept an arc equal in length to the radius of the circle.



Thus, if the arc XP is equal to the radius OX, the angle XOP is a radian, or $\angle XOP = 1^r$, ^{r} being used to designate radian.

15. Value of the radian. Since, in the same circle or in equal circles, angles at the center are proportional to their intercepted arcs, it follows that

$$\frac{\angle XOP}{\angle XOY} = \frac{\text{arc } XP}{\text{arc } XY},$$

or
$$\frac{\text{one radian}}{\text{one right angle}} = \frac{r}{\frac{1}{4}(2\pi r)} = \frac{2}{\pi}.$$

Therefore,
$$\textbf{one radian} = \frac{2}{\pi} \cdot \textbf{one right angle.}$$

or
$$\pi \textbf{ radians} = 180°.$$

It is clear that the value of the radian is independent of the radius, depending only upon the constant π and the right angle, and hence is an invariable unit.

16. Relation between degree and radian. From the preceding article it follows that

$$1^r = \frac{180°}{\pi}, \tag{1}$$

or
$$1^r = \frac{180°}{3.1416} = 57° \, 17' \, 44''. \tag{2}$$

Also, from equation (1),

$$1° \doteq \frac{\pi^r}{180}, \tag{3}$$

or
$$1° = 0.01745^r. \tag{4}$$

Equations (1) and (2) are used to convert radians into degrees, and equations (3) and (4) are used to convert degrees into radians. Thus,

from (1), $\frac{2}{3}\pi^r = \frac{2\pi}{3}\left(\frac{180°}{\pi}\right) = 120°,$

from (2), $4^r = 4\,(57°\ 17'\ 44'') = 229°\ 10'\ 56'',$

from (3), $20° = 20\left(\frac{\pi^r}{180}\right) = \frac{\pi^r}{9},$

from (4), $3° = 3\,(0.01745^r) = 0.05235^r.$

Using equation (3) to convert degrees into radians introduces π into the numerical value of the angle. Thus π becomes associated with radian measure. Since the radian is the angular unit with which π is commonly associated, no ambiguity arises by omitting to state with each angle, expressed in terms of π, that the radian is the unit.

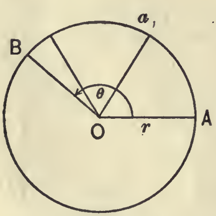
Thus, $90° = \frac{\pi}{2}$ radians, $180° = \pi$ radians, $29° = \frac{29\pi}{180}$ radians

are usually written

$$90° = \frac{\pi}{2}, \qquad 180° = \pi, \qquad 29° = \frac{29\pi}{180}.$$

It must be especially noted that when no unit is specified the radian is always understood. The constant π is always equal to 3.1416, and can never equal 180°, but π radians are equal to 180°.

17. Relation between angle, radius, and arc. It is evident from the definition of a radian that if any arc of a circle AB be divided by the radius, the quotient indicates the number of radians contained in the central angle subtended by the given arc, hence



$$\frac{\text{arc } AB}{\text{radius}} = \text{angle } AOB$$

where angle AOB is expressed in radians. Representing the length of the arc AB by a, the radius by r, and the angle AOB by θ, the relation above may be written

$$\frac{a}{r} = \theta$$

or

$$a = r\theta \tag{1}$$

It should be especially noted that the angle θ is expressed in terms of radians.

In the above figure θ is approximately equal to $2\frac{1}{3}$ radians.

Equation (1) expresses the relation between a, r, and θ, and determines any one of these elements when the other two are known.

PROBLEM. What is the length of the arc subtended by a central angle of 112° in a circle whose radius is 15 feet?

Solution. Reducing the angle to radians it is seen that

$$112° = \frac{28\,\pi^r}{45} = 1.95^r.$$

Hence $$\theta = 1.95.$$

Substituting in Equation (1) we have

$$a = 15 \times 1.95 = 29.3$$

Hence the length of the arc is 29.3 ft.

18. Linear and angular velocity. Equation (1) of Art. 17 leads directly to the relation between *linear velocity* and *angular velocity* in uniform motion in a circle. Suppose a point P moves along the circumference of a circle at a constant velocity v, describing an arc a in time t; then a/t is called the *linear velocity* and is represented by v. During the same time t, the angle θ is generated; and θ/t is called the *angular velocity*, and is represented by the Greek letter ω (omega).

Dividing Equation (1), Art. 17, by t gives

$$\frac{a}{t}=\frac{r\theta}{t} \text{ or } \boldsymbol{v = r\omega}$$

i.e. the *linear velocity* is equal to r *times the angular velocity.*

While in general angular velocity may be expressed in any units whatsoever, in the equation $v = r\omega$ the angular velocity must be expressed in radians per unit of time.

PROBLEM 1. If a point moves 26 feet in the arc of a circle of radius 7 feet in 3 seconds, what is its angular velocity?

Solution. The linear velocity of the point is $\frac{26}{3}$ feet per second.

Substituting in equation (1) we have

$$\tfrac{26}{3} = 7\,\omega$$

$$\therefore \omega = \tfrac{26}{21} = 1.238 \text{ radians per second.}$$

PROBLEM 2. A flywheel makes 200 revolutions per minute. Show that its angular velocity is 72,000° per minute or $\frac{20\,\pi}{3}$ radians per second.

If the radius of the flywheel is 3 feet, show that the velocity of a point on the rim is 42.84 miles per hour.

19. EXAMPLES

1. Construct the following angles: 30°, 45°, 60°, 135°, 300°, – 60°, – 90°, – 390°, – 420°.

2. Construct approximately the following angles: 2 radians, $3\frac{1}{2}$ radians, $-\frac{1}{2}$ radian, – 4 radians, 9 radians.

3. Construct the following angles:

$$\frac{\pi}{2},\ -\frac{\pi}{3},\ \frac{\pi}{4},\ \pi,\ -\frac{5\,\pi}{4},\ \frac{5\,\pi}{2}.$$

4. Reduce the following angles to radian or circular measure: 10°, 30°, 45°, 60°, 135°, 225°, – 270°, – 12°, – 18°, 24° 15′, – 612° 19′ 25″.

5. Reduce to degree measure: 2 radians, 5 radians, -3 radians, $\frac{1}{2}$ radian, $-\frac{1}{4}$ radian, b radians.

6. Reduce to degrees:

$$\frac{\pi}{3}, \frac{3\pi}{4}, -\frac{5\pi}{3}, \frac{3.1416}{\pi}, \frac{2}{\pi}, \frac{\pi+1}{2}, \frac{2}{\pi+3}.$$

7. If an arc of 30 ft. subtends an angle of 4 radians, find the radius of the circle.

8. In a circle whose radius is 5, the length of an intercepted arc is 12. Find the angle (a) in radians, (b) in degrees.

9. If two angles of a plane triangle are respectively equal to 1 radian and $\frac{1}{2}$ radian, express the third angle in degrees.

10. In a circle whose radius is 12 ft., find the length of the arc intercepted by a central angle of 16°.

11. Find the angle between the tangents to a circle at two points whose distance apart measured on the arc of the circle is 378 ft., the radius of the circle being 900 ft.

12. An automobile whose wheels are 34 inches in diameter travels at the rate of 25 miles per hour. How many revolutions per minute does a wheel make? What is its angular velocity in radians per second?

13. Assuming the earth's orbit to be a circle of radius 92,000,000 miles, what is the velocity of the earth in its path in miles per second?

14. The rotor of a steam turbine is two feet in diameter and makes 25,000 revolutions per minute. The blades of the turbine, situated on the circumference of the rotor, have one-half the velocity of the steam which drives them. What is the velocity of the steam in feet per second?

15. A belt travels around two pulleys whose diameters are 3 feet and 10 inches respectively. The larger pulley makes 80 revolutions per minute. Find the angular velocity of the smaller pulley in radians per second, also the speed of the belt in feet per minute.

CHAPTER II

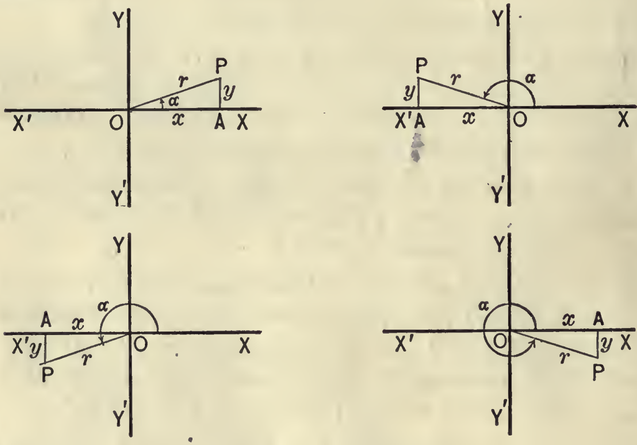
TRIGONOMETRIC FUNCTIONS

20. The trigonometric functions, upon which trigonometry is based, are functions of an angle.

These functions are the sine, cosine, tangent, cotangent, secant, cosecant, versed sine, and coversed sine of an angle.

For any angle α they are written $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\cot\alpha$, $\sec\alpha$, $\csc\alpha$, $\text{vers}\,\alpha$, and $\text{covers}\,\alpha$.

21. Definitions of the trigonometric functions. Let OX and OP be the initial and terminal lines respectively of any angle α.



Let P be any point on the terminal line,

OP or r the **distance** from the origin to the point P,
OA or x the **abscissa** of the point P, and
AP or y the **ordinate** of the point P.

It should be noted that OP, OA, and AP are directed lines and hence

$$r = OP \text{ and not } PO$$
$$x = OA \text{ and not } AO$$
$$y = AP \text{ and not } PA.$$

Then the trigonometric functions are defined as follows:

$$\sin \alpha = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}$$

$$\cos \alpha = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}$$

$$\tan \alpha = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}$$

$$\cot \alpha = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}$$

$$\sec \alpha = \frac{r}{x} = \frac{\text{distance}}{\text{abscissa}}$$

$$\csc \alpha = \frac{r}{y} = \frac{\text{distance}}{\text{ordinate}}$$

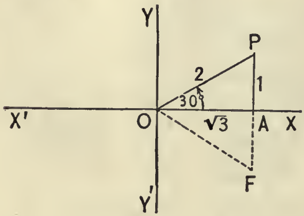
$$\text{vers}\, \alpha = 1 - \cos \alpha$$
$$\text{covers}\, \alpha = 1 - \sin \alpha$$

It will be observed that each of the first six functions is defined as a **ratio** between two line segments. These are the fundamental **trigonometric functions**, and the ratios defining them are called the **trigonometric ratios.**

The first six functions are called **trigonometric ratios.** The expression **trigonometric functions** is more general and embraces the versed sine, coversed sine, and the trigonometric ratios. It is evident, from the definitions, that the values of the trigonometric functions are *abstract numbers.*

22. Values of the trigonometric functions of 30°, 45°, and 60°. A few concrete illustrations serve to show the nature of the trigonometric functions, to fix ideas, and to prepare the way for more general considerations.

Functions of 30°. Let OPF be an equilateral triangle having its sides equal to 2 units. Place the triangle with a vertex at the origin so that OX bisects the angle POF. Then, by geometry, the angle $AOP = 30°$, the ordinate $AP = 1$, and the abscissa $OA = \sqrt{3}$.



Hence, applying definitions,

$$\sin 30° = \frac{1}{2} = .500$$

$$\cos 30° = \frac{\sqrt{3}}{2} = .866$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = .577$$

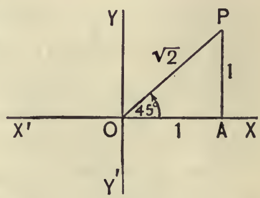
$$\cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3} = 1.732$$

$$\text{vers } 30° = 1 - \frac{1}{2}\sqrt{3} = .134 \qquad \sec 30° = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3} = 1.155$$

$$\text{covers } 30° = 1 - \frac{1}{2} = \frac{1}{2} = .500 \qquad \csc 30° = \frac{2}{1} = 2.000$$

Problem. Find the values of the trigonometric functions of 30°, as above, taking $OP = 1$.

Functions of 45°. Let OAP be a right-angled isosceles triangle having its sides OA and AP each equal to 1. Then $\angle AOP = 45°$ and the distance $OP = \sqrt{2}$.



Hence, applying definitions,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} \qquad \sec 45° = \frac{\sqrt{2}}{1} = \sqrt{2}$$

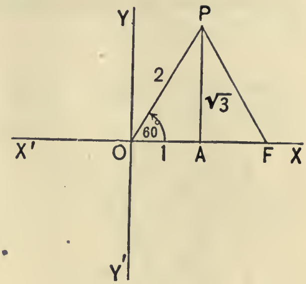
$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} \qquad \csc 45° = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\tan 45° = \frac{1}{1} = 1 \qquad \text{vers } 45° = 1 - \frac{1}{2}\sqrt{2}$$

$$\cot 45° = \frac{1}{1} = 1 \qquad \text{covers } 45° = 1 - \frac{1}{2}\sqrt{2}$$

PROBLEM. Find the values of the trigonometric functions of 45°, taking $OA = 3$.

Functions of 60°. Let OPF, an equilateral triangle having each side equal to 2, be placed as in the figure. Then the abscissa and ordinate of P are 1 and $\sqrt{3}$, respectively.



Hence, applying definitions,

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \sec 60° = \frac{2}{1} = 2$$

$$\cos 60° = \frac{1}{2} \qquad \csc 60° = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$$

$$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \text{vers } 60° = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\cot 60° = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} \qquad \text{covers } 60° = 1 - \frac{1}{2}\sqrt{3}$$

PROBLEM. Find the values of the functions of 60°, as above, taking $OP = 4a$.

The values of the sines and cosines of 30°, 45°, and 60° are used frequently and should be memorized. The following table may be found helpful:

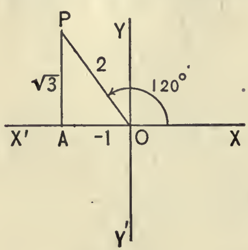
$$\sin 30° = \frac{1}{2}\sqrt{1} = \cos 60°$$

$$\sin 45° = \frac{1}{2}\sqrt{2} = \cos 45°$$

$$\sin 60° = \frac{1}{2}\sqrt{3} = \cos 30°$$

23. Values of the trigonometric functions of 120°, 135°, and 150°. By using the magnitudes of the figures of Art. 22 and properly placing them with respect to the axes, the values of the trigonometric functions of various angles may be obtained.

Functions of 120°. From the figure and definitions it is evident that



$$\sin 120^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 120^\circ = \frac{-1}{2} = -\frac{1}{2}$$

$$\tan 120^\circ = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\cot 120^\circ = \frac{-1}{\sqrt{3}} = -\frac{1}{3}\sqrt{3}$$

$$\text{vers } 120^\circ = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2} \qquad \sec 120^\circ = \frac{2}{-1} = -2$$

$$\text{covers } 120^\circ = 1 - \frac{\sqrt{3}}{2} \qquad \csc 120^\circ = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$$

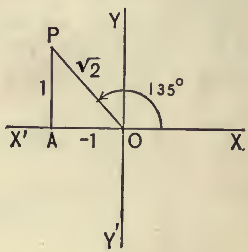
Functions of 135°. From the figure and definitions it is evident that

$$\sin 135^\circ = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

$$\cos 135^\circ = \frac{-1}{\sqrt{2}} = -\frac{1}{2}\sqrt{2}$$

$$\tan 135^\circ = \frac{1}{-1} = -1$$

$$\cot 135^\circ = \frac{-1}{1} = -1$$



$$\sec 135^\circ = \frac{\sqrt{2}}{-1} = -\sqrt{2} \qquad \text{vers } 135^\circ = 1 + \frac{1}{2}\sqrt{2}$$

$$\csc 135^\circ = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \text{covers } 135^\circ = 1 - \frac{1}{2}\sqrt{2}$$

Functions of 150°. From the figure and definitions it is evident that

$$\sin 150° = \frac{1}{2}$$

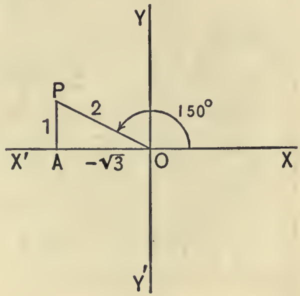
$$\cos 150° = \frac{-\sqrt{3}}{2} = -\frac{1}{2}\sqrt{3}$$

$$\tan 150° = \frac{1}{-\sqrt{3}} = -\frac{1}{3}\sqrt{3}$$

$$\cot 150° = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

$$\sec 150° = \frac{2}{-\sqrt{3}} = -\frac{2}{3}\sqrt{3}$$

$$\csc 150° = \frac{2}{1} = 2$$



$$\text{vers } 150° = 1 + \frac{1}{2}\sqrt{3}$$

$$\text{covers } 150° = 1 - \frac{1}{2} = \frac{1}{2}$$

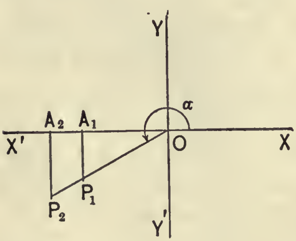
PROBLEM. Find the values of the trigonometric functions of 210°, 225°, 240°, 300°, 315°, and 330°.

24. Signs of the trigonometric ratios. The signs of the trigonometric ratios of any angle depend upon the signs of the ordinate, the abscissa, and the distance of any point on its terminal line. As the terminal line passes from one quadrant to another, there is always a change of sign in either the abscissa or the ordinate of any point on that line, but the distance remains positive. When a coördinate changes its sign, every trigonometric ratio dependent upon it must also change its sign.

The following table is constructed by taking account of the signs of the abscissa x and of the ordinate y, and remembering that the distance r is always positive. It gives the sign of each trigonometric ratio of an angle terminating in any quadrant.

	1st Quadrant	2d Quadrant	3d Quadrant	4th Quadrant
sin α	$\frac{+}{+}=+$	$\frac{+}{+}=+$	$\frac{-}{+}=-$	$\frac{-}{+}=-$
cos α	$\frac{+}{+}=+$	$\frac{-}{+}=-$	$\frac{-}{+}=-$	$\frac{+}{+}=+$
tan α	$\frac{+}{+}=+$	$\frac{+}{-}=-$	$\frac{-}{-}=+$	$\frac{-}{+}=-$
cot α	$\frac{+}{+}=+$	$\frac{-}{+}=-$	$\frac{-}{-}=+$	$\frac{+}{-}=-$
sec α	$\frac{+}{+}=+$	$\frac{+}{-}=-$	$\frac{+}{-}=-$	$\frac{+}{+}=+$
csc α	$\frac{+}{+}=+$	$\frac{+}{+}=+$	$\frac{+}{-}=-$	$\frac{+}{-}=-$

25. Theorem. *For every given angle there is one and only one value of each trigonometric function.*



The theorem is demonstrated for the sine of an angle. The same method is applicable to each of the remaining functions.

Let α be any angle. Referring to Art. 21, it is clear that if it be possible to obtain two or more values for sin α they must be obtained by taking different points on the terminal line.

Let P_1 and P_2 be any two points on the terminal line. Then by definition

$$\sin\alpha = \frac{A_1P_1}{OP_1}, \text{ or } \sin\alpha = \frac{A_2P_2}{OP_2}.$$

But since the right triangles OA_1P_1 and OA_2P_2 are similar, and have their corresponding sides in the same direction, it follows that

$$\frac{A_1P_1}{OP_1}=\frac{A_2P_2}{OP_2}.$$

Hence the value of sin α is independent of the position of the point chosen on the terminal line, but depends solely upon the position of the terminal line, *i.e.* upon the angle.

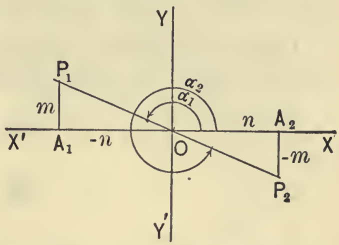
The above theorem may be stated as follows: *The trigonometric functions are single-valued functions of the angle.*

26. Theorem. *Every given value of a trigonometric function determines an unlimited or infinite number of positive and negative angles, among which there are always two positive angles less than* 360°.

The theorem is demonstrated for a given tangent. A similar method is applicable to the remaining functions.

Let $\tan \alpha = -\frac{m}{n}$ where m and n are positive numbers. Then

$$\tan \alpha = \frac{+m}{-n} = \frac{-m}{+n}.$$



Locate the point P_1, whose abscissa is $-n$, and whose ordinate is m. This point and the origin determine the terminal line of the angle α_1, whose tangent is $-\frac{m}{n}$.

Likewise the point P_2 is located by using $-m$ and n as ordinate and abscissa respectively. Drawing the terminal line OP_2, a second angle α_2 is found, which also has the given tangent.

The angles α_1 and α_2, are evidently less than 360°, and have the given tangent $-\frac{m}{n}$. There is an unlimited number of positive and negative angles coterminal with α_1 and α_2, all of which have the given tangent. Hence the theorem.

27. EXAMPLES

In what quadrants does α terminate when

1. $\sin\alpha = -\frac{1}{2}$. **3.** $\tan\alpha = 5$. **5.** $\sin\alpha = \frac{2}{3}$.

2. $\cos\alpha = \frac{1}{3}$. **4.** $\cot\alpha = -8$. **6.** $\cos\alpha = -\frac{3}{5}$.

7. $\sin\alpha$ is positive and $\cos\alpha$ is negative.

8. $\tan\alpha$ is positive and $\cos\alpha$ is negative.

9. $\operatorname{cosec}\alpha$ is negative and $\cos\alpha$ is negative.

10. $\tan\alpha$ is negative and $\sin\alpha$ is positive.

11. $\cos\alpha$ is negative and $\sin\alpha$ is negative.

Give the signs of the trigonometric functions of the following angles:

12. $750°$. **14.** $560°$. **16.** $-15°$. **18.** $-470°$.

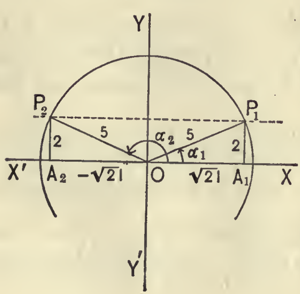
13. $\frac{8\pi}{3}$. **15.** $5\frac{2}{3}\pi$. **17.** $-\frac{5\pi}{4}$. **19.** $-\frac{7\pi}{6}$.

Find the negative angles, numerically less than 360°, that are coterminal with the following angles:

20. $\frac{10\pi}{3}$. **22.** $300°$. **24.** $\frac{3\pi}{2}$.

21. $\frac{2\pi}{3}$. **23.** π. **25.** $-495°$

26. Construct the positive angles, less than 360°, for which the sine is equal to $\frac{2}{5}$, and find the values of the other functions of both angles.



SOLUTION. Determine the points whose ordinates are 2 and whose distances are 5, as follows:

With O as center and a radius 5, describe a circle. Through a point on the Y-axis, 2 units above the origin, draw a line parallel to the X-axis, intersecting the circle in the two required points P_1 and P_2. Draw the terminal lines OP_1 and OP_2, giving the angles α_1 and α_2, for which we have

$$\sin\alpha_1 = \frac{2}{5} \qquad \text{and} \qquad \sin\alpha_2 = \frac{2}{5}$$

$$\cos\alpha_1 = \frac{\sqrt{21}}{5} \qquad\qquad \cos\alpha_2 = \frac{-\sqrt{21}}{5}$$

$$\tan\alpha_1 = \frac{2}{\sqrt{21}} = \frac{2}{21}\sqrt{21} \qquad\qquad \tan\alpha_2 = \frac{2}{-\sqrt{21}} = -\frac{2}{21}\sqrt{21}$$

$$\cot\alpha_1 = \frac{\sqrt{21}}{2} \qquad\qquad \cot\alpha_2 = \frac{-\sqrt{21}}{2}$$

$$\sec\alpha_1 = \frac{5}{\sqrt{21}} = \frac{5}{21}\sqrt{21} \qquad\qquad \sec\alpha_2 = \frac{5}{-\sqrt{21}} = -\frac{5}{21}\sqrt{21}$$

$$\csc\alpha_1 = \frac{5}{2} \qquad\qquad \csc\alpha_2 = \frac{5}{2}$$

27. Construct the positive angles, less than 360°, for which the cosine is $\frac{3}{5}$, and find the values of the other functions of both angles.

Find the values of the functions of all angles less than 360° determined by

28. $\tan\alpha = -\frac{6}{7}$.

29. $\cot\alpha = 1\frac{1}{2}$.

30. $\csc\alpha = 3\frac{1}{2}$.

31. $\cot\alpha = -5$.

32. $\sin\alpha = -\frac{2}{3}$.

33. $\sin\alpha = \frac{a}{b}$.

34. $\cos\alpha = .4$.

35. Given $\tan\alpha = -4$, find the value of $\frac{\sin\alpha\cos\alpha}{\cot\alpha}$.

36. Given $\sec\alpha = 6$; find the value of $\frac{\sin^2\alpha + \cos^2\alpha}{\cos^2\alpha}$.

37. Given $\sin\alpha = .3$, find the value of $\tan\alpha\sec\alpha\cos\alpha$.

38. Given $\csc\alpha = 8$ and $\tan\beta = 3$, find the value of $\sin\alpha\cos\beta + \cos\alpha\sin\beta$.

39. Given $\tan\alpha = \frac{1}{2}$ and $\cos\beta = -\frac{1}{5}\sqrt{5}$, α terminating in the first quadrant and β in the second, show that the angle between the terminal lines of α and β is a right angle.

CHAPTER III

RIGHT TRIANGLES

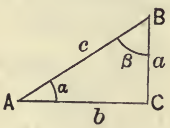
28. In every triangle there are six **elements** or **parts.** These are the three sides and the three angles.

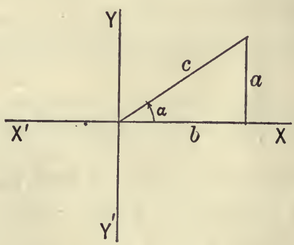
When three elements are given, one of which is a side, the other three elements can be determined.

The solution of a triangle is the process of determining the unknown parts from the given parts.

In the present chapter it will be shown how the trigonometric functions can be employed to solve the right triangle.

29. Applications of definitions of the trigonometric functions to the right triangle. Let ABC be any right triangle. Place the triangle in the first quadrant with the vertex of one acute angle coinciding with the origin, and one side, not the hypotenuse, coinciding with the X-axis, as in the figure.





Then, by definitions,

$$\sin \alpha = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}$$

$$\cos \alpha = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\tan\alpha = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}$$

$$\cot\alpha = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}$$

$$\sec\alpha = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\csc\alpha = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{side opposite}}$$

It is seen that the functions of any acute angle of a right triangle are expressed in terms of the *side adjacent*, the *side opposite*, and the *hypotenuse*, *without reference to the coordinate axes.*

It follows that

$$\sin\beta = \frac{b}{c} \qquad \cot\beta = \frac{a}{b}$$

$$\cos\beta = \frac{a}{c} \qquad \sec\beta = \frac{c}{a}$$

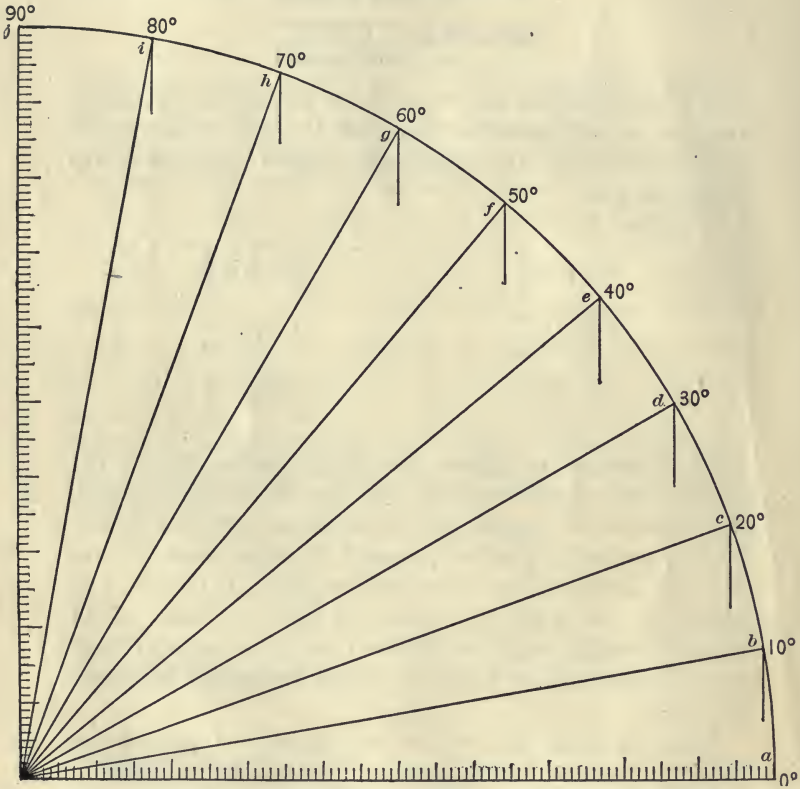
$$\tan\beta = \frac{b}{a} \qquad \csc\beta = \frac{c}{b}$$

30. Trigonometric tables. In the preceding chapter the trigonometric functions of 30°, 45°, and 60° were calculated. By processes too complicated to introduce here, tables have been computed, giving the values of the trigonometric functions of acute angles. These tables generally contain two parts. In one part the values of the functions, called **natural functions**, are given; in the other part the logarithms of the trigonometric functions, called **logarithmic functions**, are given.

When an angle is given its trigonometric functions can be taken from the tables, and *vice versa.* The functions of known angles thus become known numbers and can be used in problems of computation.

Approximate values of the trigonometric functions can be obtained by graphical methods.

Problem. Measure the distance, abscissas and ordinates of the points $a, b, c, \cdots, j$. From these measurements compute to two figures the sine, cosine, and tangent of 0°, 10°, 20°, $\cdots$, 90°. By arranging the results in tabular form, a two place table is constructed.



This table may be used to solve Examples 1 to 6, Art. 36. On account of its extreme simplicity the use of this table allows the attention to be focused upon the fundamental processes involved in the solution of triangles.

31. Formulas used in the solution of right triangles. A right triangle can always be solved when, in addition to the right angle γ, two independent parts are given. The formulas usually employed are:

$$\sin\alpha = \frac{a}{c} \qquad \sin\beta = \frac{b}{c} \qquad \alpha + \beta = 90^\circ$$
$$a^2 + b^2 = c^2$$
$$\cos\alpha = \frac{b}{c} \qquad \cos\beta = \frac{a}{c}$$
$$\tan\alpha = \frac{a}{b} \qquad \tan\beta = \frac{b}{a}$$

When the computations are made without logarithms, the formulas involving the cotangent, secant, and cosecant may sometimes be used advantageously.

32. Selection of formulas.

(a) If an angle and a side are given, it is always possible to find the unknown parts directly from the given parts without the use of the formula $a^2 + b^2 = c^2$. To find the unknown side, select that formula which contains the given parts and the desired unknown side. The unknown angle can always be found from $\alpha + \beta = 90^\circ$.

(b) If two sides are given, the third side would naturally be found by the use of $a^2 + b^2 = c^2$, but in practice it is generally preferable first to compute an angle by the use of a formula involving the given sides and an angle. To find the third side, select a formula containing one of the given sides, the angle already computed, and the required side.

33. Check formulas. In all computations it is necessary constantly to guard against numerical errors. However carefully the computations are made, errors may still occur, and therefore computed parts should be checked by means of check formulas. Any formula which was not used in the solution of the triangle may be used for this purpose.

For the right triangle the formula

$$a^2 = c^2 - b^2 = (c - b)(c + b)$$

may conveniently be used, and in general it is sufficient.

To Here.

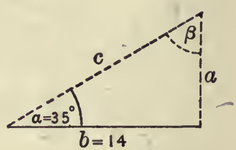
34. Suggestions on solving a triangle. Make a careful free-hand construction of the required triangle, and write down an estimate of the values of the unknown parts. Large errors will be detected readily, without the use of check formulas, when the computed parts are compared with the estimates.

Before entering the tables, and before making any computations, select all the formulas to be used, solve the formulas for the required parts, and make an outline in which a place is provided for every number to be used in the computation. This will often lessen the actual work, for frequently several required numbers are found on the same page of the table.

The arrangement of the work is of considerable importance in every extended computation.

35. Illustrative examples.

1. In a right triangle, given $b = 14$, $\alpha = 35°$, to find a, c, and β.



SOLUTION. Approximate construction.
Estimate $a = 9$, $c = 17$.

By natural functions

$\tan\alpha = \dfrac{a}{b}$ $\qquad \cos\alpha = \dfrac{b}{c}$

$\therefore a = b\tan\alpha \qquad \therefore c = \dfrac{b}{\cos\alpha} \qquad \beta = 90° - \alpha$

$a = 14 \times .7002 \qquad c = 14 \div .8192 \qquad \beta = 55°$

$a = 9.803 \qquad c = 17.09$

Check

$$c^2 = a^2 + b^2$$
$$a^2 = 96.10$$
$$b^2 = 196.0$$
$$292.1$$
$$c^2 = (17.09)^2 = 292.1$$

By logarithms

$a = b\tan\alpha$

b	14
α	35°
$\log b$	
$\log\tan\alpha$	
$\log a$	
a	

$c = \dfrac{b}{\cos\alpha}$

$\log b$	
$\log\cos\alpha$	
$\log c$	
c	

$\beta = 90° - \alpha$

$\beta =$

Check

$$a^2 = c^2 - b^2 = (c - b)(c + b)$$

c		$\log(c-b)$	
b		$\log(c+b)$	
$c-b$		$2\log a$	
$c+b$		$\log a$	

Filling in the above outline, the completed work appears as follows:

$$a = b \tan\alpha \qquad c = \frac{b}{\cos\alpha} \qquad \beta = 90° - \alpha$$

b	14	$\log b$	1.14613	$\beta = 55°$
α	35°	$\log\cos\alpha$	9.91336 − 10	
$\log b$	1.14613	$\log c$	1.23277	
$\log\tan\alpha$	9.84523 − 10	c	17.091	
$\log a$	0.99136			
a	9.8030			

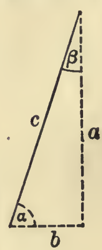
Check

$$a^2 = c^2 - b^2 = (c - b)(c + b)$$

c	17.091	$\log(c-b)$	0.49010
b	14.	$\log(c+b)$	1.49263
$c-b$	3.091	$2\log a$	1.98273
$c+b$	31.091	$\log a$	0.99136

Since the check formula gives the same value for log a as that found in the solution, the computation is in all probability correct.

2. In a right triangle, given $c = 6.275$, $\beta = 18° 47'$, to find a, b, and α.



SOLUTION. Approximate construction.
Estimated $a = 5$, $b = 2$.

By natural functions

$a = c\cos\beta$ $\qquad$ $b = c\sin\beta$
$a = 6.275 \times .9468$ $\qquad$ $b = 6.275 \times .3220$
$a = 5.941$ $\qquad$ $b = 2.021$
$\alpha = 90° - \beta$
$\alpha = 71° 13'$

Check

$c^2 = a^2 + b^2$
$a^2 = 35.30$
$b^2 = 4.084$
39.384
$c^2 = (6.275)^2 = 39.38$

By logarithms

$a = c \cos \beta$

c	6.275
β	18° 47′
$\log c$	0.79761
$\log \cos \beta$	9.97623 − 10
$\log a$	0.77384
a	5.9407

$b = c \sin \beta$

$\log c$	0.79761
$\log \sin \beta$	9.50784 − 10
$\log b$	0.30545
b	2.0205

$\alpha = 90° - \beta$

$\alpha = 71° 13'$

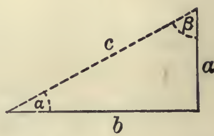
Check

$$a^2 = c^2 - b^2 = (c - b)(c + b)$$

c	6.275
b	2.0205
$c - b$	4.2545
$c + b$	8.2955

$\log (c - b)$	0.62885
$\log (c + b)$	0.91884
$2 \log a$	1.54769
$\log a$	0.77384

3. Given $a = .064873$, $b = .12574$, to find α, β, and c.



Solution. Approximate construction. Estimate $\alpha = 30°$, $\beta = 60°$, $c = .15$.

$\tan \alpha = \dfrac{a}{b}$

a	.064873
b	.12574
$\log a$	8.81206 − 10
$\log b$	9.09948 − 10
$\log \tan \alpha$	9.71258 − 10
α	27° 17′ 23″
β	62° 42′ 37″

$c = \dfrac{a}{\sin \alpha}$

$\log a$	8.81206 − 10
$\log \sin \alpha$	9.66133 − 10
$\log c$	9.15073 − 10
c	.14149

Check

$$a^2 = (c - b)(c + b)$$

c	.14149
b	.12574
$c - b$	.01575
$c + b$	.26723

$\log (c - b)$	8.19728 − 10
$\log (c + b)$	9.42689 − 10
$2 \log a$	17.62417 − 20
$\log a$	8.81208 − 10

36. EXAMPLES

Solve the following right triangles, γ being the right angle. It is recommended that the first ten problems be solved by the use of a three-place table of natural functions, or by means of a slide rule.

1. $a = 6$, $\alpha = 20°$
2. $c = 2.5$, $\alpha = 35°$
3. $b = .84$, $\beta = 75°$
4. $a = 25$, $b = 60$
5. $c = 82$, $a = 37$
6. $c = .0091$, $a = .0029$
7. $b = 371$, $\alpha = 43°$
8. $c = 7.72$, $b = 6.87$
9. $\alpha = 18°$, $c = .0938$
10. $\beta = 49° 30'$, $c = 12.47$

The following problems should be solved by the use of a four-place * or a five-place table.

11. $a = 1870$, $\alpha = 19° 55'$
12. $c = .3194$, $\alpha = 25° 41'$
13. $b = .9292$, $\beta = 32° 43'$
14. $a = .00006$, $b = .000019$
15. $c = 1200.7$, $a = 885.6$
16. $c = 12.145$, $a = 9.321$
17. $b = 78.545$, $\alpha = 31° 41' 6''$
18. $b = 3.4572$, $\beta = 57° 57' 57''$
19. $c = 20.082$, $b = 16.174$
20. $\alpha = 78° 0' 3''$, $a = 271.82$

* Whenever sides are given to four significant figures and angles to minutes.

21. $a = 5987.2$
$\beta = 88° \, 53' \, 2''$

22. $c = .09008$
$a = .07654$

23. $\alpha = 46° \, 39' \, 50''$
$a = 26.434$

24. $a = 30.008$
$b = 29.924$

25. $a = 111.45$
$b = 121.69$

26. $\beta = 83° \, 15' \, 6''$
$c = 7000$

27. $\alpha = 66° \, 6' \, 18''$
$c = 8070.6$

28. $a = 978.45$
$b = 1067.2$

29. $a = 5280$
$b = 5608$

30. $\alpha = 17° \, 26' \, 34''$
$c = 46.474$

31. Solve the isosceles triangle, one of the equal sides being 690.13, and one of the base angles being 15° 20′ 25″.

32. Solve the isosceles triangle whose altitude is 606.6, one of the equal sides being 955.7.

33. Solve the isosceles triangle whose base is 2558, and whose vertical angle is 104° 0′ 46″.

34. Solve the isosceles triangle whose base is 161.4, and whose altitude is 204.4.

35. Find the length of a side of a regular octagon inscribed in a circle whose radius is 49.

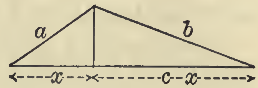
37. Oblique triangles. When any three independent parts of an oblique triangle are given it can be solved by means of right triangles. The oblique triangle may be divided into two right triangles by drawing a perpendicular from one vertex to the opposite side, or the opposite side produced.

When one of the given parts is an angle, the perpendicular must be selected so that one of the resulting right triangles will contain two of the given parts.

In case the three sides are given, a second part of one of the right triangles can be obtained by equating the expressions for the length of the perpendicular obtained from each right triangle.

Thus $\quad a^2 - x^2 = b^2 - (c - x)^2,$

from which $\quad x = \dfrac{a^2 - b^2 + c^2}{2c}.$



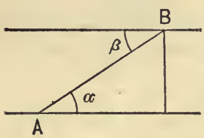
38. APPLICATIONS

1. To find the distance from B to C, two points on opposite sides of a river, a line BA, 200 feet long, was laid off at right angles to the line BC, and the angle BAC was measured and found to equal $55° \, 29'$. What was the required distance?

2. A railway is inclined $4° \, 23' \, 20''$ to the horizontal. How many feet does it rise per mile, measured along the horizontal?

3. From the top of a tower 120 feet high the angle of depression of an object in the horizontal plane of the base of the tower is $24° \, 27'$. How far is the object from the foot of the tower? How far from the observer?

Given any point A and a second point B at a greater elevation than A.



The angle of *elevation* of B from A is the angle that the line AB makes with its orthogonal projection upon the horizontal plane through A. In the figure it is the angle α.

The angle of *depression* of A from B is the angle that the line BA makes with its orthogonal projection upon the horizontal plane through B. In the figure it is the angle β. It is clear that $\alpha = \beta$.

4. Find the length of one side of a regular pentagon inscribed in a circle whose radius is 18.24 feet.

5. Find the perimeter of a regular polygon of n sides inscribed in a circle whose radius is r.

6. Find the perimeter of a regular pentagon circumscribed about a circle whose radius is 18.24 feet.

7. Find the perimeter of a regular polygon of n sides circumscribed about a circle whose radius is r.

8. Find the length of a chord subtending a central angle of $63° 14' 20''$ in a circle whose radius is 124.93 feet.

9. A straight road, PR, makes an angle of $19° 27' 30''$ with another straight road, PS. Having given $PR = 640$ feet, find RS, the perpendicular distance from R to PS.

10. At a certain point the angle of elevation of a mountain is $34° 28'$. At a second point 500 feet farther away, the angle of elevation is $31° 12'$. Find the height of the mountain above the table-land.

11. One side of the square base of a right pyramid is 15 inches, and the altitude is 20 inches. Find the slant height and the edge. Find the inclination of a face to the base of the pyramid.

12. How far can you see from an elevation of 2000 feet, assuming the earth to be a sphere with a radius of 3960 miles?

13. Two forces of 95.75 pounds and 120.25 pounds, at right angles to each other, act at a point; find the magnitude of their resultant and the angle it makes with the greater force.

14. Find the velocity of a point, whose latitude is $44° 30' 20''$, due to the rotation of the earth. Assume the radius of the earth to be 3960 miles.

15. Find the length of a belt running around two pulleys whose radii are 12 inches and 4 inches, respectively, the distance between the centers of the pulleys being 6 feet.

16. A diagonal of a cube and a diagonal of a face of a cube intersect at a vertex. Find the angle between them.

17. A force of 2000 pounds applied at the origin makes an angle of 33° 25′ with the positive X-axis. Find its components along the X and Y axes respectively.

18. A force of 185 pounds applied at the origin makes an angle of 82° 12′ with the positive X-axis. Another force of 327 pounds applied at the origin makes an angle of 11° 32′ with the positive X-axis. Find the X and Y components of each of these forces. Add the X-components and also the Y-components and then find the resultant in magnitude and direction.

19. A cylindrical tank, whose axis is horizontal, is 8 feet in diameter and 12 feet long. The tank is partly filled with water, so that the depth of the water at the deepest point is 3 feet. How many gallons of water are in the tank, there being $7\frac{1}{2}$ gallons in a cubic foot?

20. A man who can paddle his canoe 5 miles per hour in still water paddles at his usual rate directly across a river one-half mile wide. If the river flows 4 miles per hour, where will the canoe land and what is its speed in the water?

$x \qquad x = \dfrac{500}{\cot B - \cot \phi}$

B 500 ϕ y

Per. regular polygon = $2nr \tan \dfrac{180°}{n}$

when r = radius n = no. sides

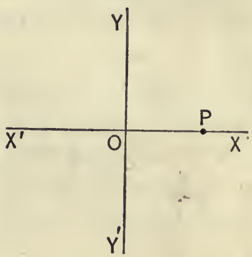
Quiz on 1st 4. chaps.
Wednesday

CHAPTER IV

VARIATIONS OF THE TRIGONOMETRIC FUNCTIONS REDUCTION OF FUNCTIONS OF $n\,90° \pm a$

39. In the study of the variations of the trigonometric functions, their values at 0°, 90°, 180°, 270°, and 360° are of special importance.

Values of functions of 0°. The terminal line of 0° coincides with the positive X-axis. For a point on this line, at a distance r from the origin, we have



$$x = r, \qquad y = 0$$

Hence

$$\sin 0° = \frac{0}{r} = 0 \qquad \cot 0° = \frac{r}{0}{}^{*} = \infty$$

$$\cos 0° = \frac{r}{r} = 1 \qquad \sec 0° = \frac{r}{r} = 1$$

$$\tan 0° = \frac{0}{r} = 0 \qquad \csc 0° = \frac{r}{0} = \infty$$

Values of functions of 90°. The terminal line of 90° coincides with the positive Y-axis, therefore

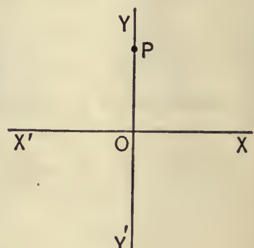
$$x = 0, \qquad y = r.$$

Hence

$$\sin 90° = \frac{r}{r} = 1 \qquad \cot 90° = \frac{0}{r} = 0$$

$$\cos 90° = \frac{0}{r} = 0 \qquad \sec 90° = \frac{r}{0} = \infty$$

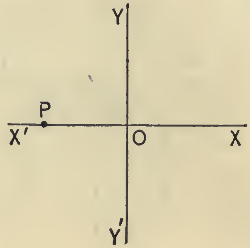
$$\tan 90° = \frac{r}{0} = \infty \qquad \csc 90° = \frac{r}{r} = 1$$



* For the interpretation of $\frac{r}{0}$ see any College Algebra.

Values of functions of 180°. The terminal line of 180° coincides with the negative X-axis, therefore

$$x = -r, \qquad y = 0.$$



Hence

$$\sin 180° = \frac{0}{r} = 0$$

$$\cos 180° = \frac{-r}{r} = -1$$

$$\tan 180° = \frac{0}{-r} = 0 \qquad \sec 180° = \frac{r}{-r} = -1$$

$$\cot 180° = \frac{-r}{0} = \infty \qquad \csc 180° = \frac{r}{0} = \infty$$

Values of functions of 270°. The terminal line of 270° coincides with the negative Y-axis, therefore

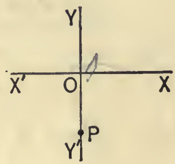
$$x = 0, \qquad y = -r$$

Hence

$$\sin 270° = \frac{-r}{r} = -1 \qquad \cot 270° = \frac{0}{-r} = 0$$

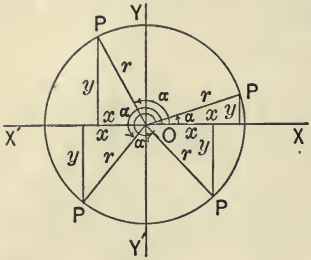
$$\cos 270° = \frac{0}{r} = 0 \qquad \sec 270° = \frac{r}{0} = \infty$$

$$\tan 270° = \frac{-r}{0} = \infty \qquad \csc 270° = \frac{r}{-r} = -1$$



The values of the functions of 360° are identical with the values of the functions of 0°, since these angles are coterminal.

40. Variation of the functions. It has been shown that the trigonometric functions are functions of an angle. As the angle varies, the trigonometric functions depending upon the angle also vary.



Let the line OP, of fixed length r, revolve about O from the initial position OX. Then the angle XOP or α increases from 0° to 360°. The variations of the trigonometric functions can be traced by observing the changes in the abscissa and ordinate of the point P.

a. As the angle α increases from 0° to 90°,
y increases from 0 to r,
and x decreases from r to 0.

Hence

$\sin \alpha$ or $\frac{y}{r}$ is positive and increases from 0 to 1

$\cos \alpha$ or $\frac{x}{r}$ is positive and decreases from 1 to 0

$\tan \alpha$ or $\frac{y}{x}$ is positive and increases from 0 to ∞

$\cot \alpha$ or $\frac{x}{y}$ is positive and decreases from ∞ to 0

$\sec \alpha$ or $\frac{r}{x}$ is positive and increases from 1 to ∞

$\csc \alpha$ or $\frac{r}{y}$ is positive and decreases from ∞ to 1

As the angle α increases through 90°, x passes through zero, changing from a positive number to a negative number. Then, immediately before α becomes 90°, $\cos \alpha$ or $\frac{x}{r}$ is a very small positive number; while immediately after α has passed 90°, $\cos \alpha$ is a very small negative number. This may be expressed by saying that $\cos \alpha$ passes through zero and changes sign as α passes through 90°.

Likewise, immediately before α becomes $90°$, $\tan\alpha$ or $\frac{y}{x}$ is a very large positive number; while immediately after α has passed $90°$, $\tan\alpha$ is a very large negative number. It has been seen that $\tan 90° = \infty$. Hence $\tan\alpha$ passes through ∞ and changes sign as α passes through $90°$. Hence we may say that $\tan 90° = \pm\infty$, choosing the positive sign when associating $90°$ with the first quadrant and the negative sign when associating $90°$ with the second quadrant.

Similarly, whenever a trigonometric function passes through ∞ *it changes sign.*

b. As the angle α increases from $90°$ to $180°$,

y decreases from r to 0,

and x decreases from 0 to $-r$.

Hence

$\sin\alpha$ or $\frac{y}{r}$ is positive and decreases from 1 to 0

$\cos\alpha$ or $\frac{x}{r}$ is negative and decreases from 0 to -1

$\tan\alpha$ or $\frac{y}{x}$ is negative and increases from $-\infty$ to 0

$\cot\alpha$ or $\frac{x}{y}$ is negative and decreases from 0 to $-\infty$

$\sec\alpha$ or $\frac{r}{x}$ is negative and increases from $-\infty$ to -1

$\csc\alpha$ or $\frac{r}{y}$ is positive and increases from 1 to $+\infty$

c. As the angle α increases through $180°$, y passes through zero, changing from a positive number to a negative number.

As the angle α increases from $180°$ to $270°$,

y decreases from 0 to $-r$,

and x increases from $-r$ to 0.

Hence

$\sin\alpha$ or $\frac{y}{r}$ is negative and decreases from 0 to -1

$\cos \alpha$ or $\frac{x}{r}$ is negative and increases from -1 to 0

$\tan \alpha$ or $\frac{y}{x}$ is positive and increases from 0 to $+\infty$

$\cot \alpha$ or $\frac{x}{y}$ is positive and decreases from $+\infty$ to 0

$\sec \alpha$ or $\frac{r}{x}$ is negative and decreases from -1 to $-\infty$

$\csc \alpha$ or $\frac{r}{y}$ is negative and increases from $-\infty$ to -1

d. As the angle α increases through $270°$, x passes through zero, changing from a negative number to a positive number.

As the angle α increases from $270°$ to $360°$,
y increases from $-r$ to 0,
and x increases from 0 to r.

Hence

$\sin \alpha$ or $\frac{y}{r}$ is negative and increases from -1 to 0

$\cos \alpha$ or $\frac{x}{r}$ is positive and increases from 0 to 1

$\tan \alpha$ or $\frac{y}{x}$ is negative and increases from $-\infty$ to 0

$\cot \alpha$ or $\frac{x}{y}$ is negative and decreases from 0 to $-\infty$

$\sec \alpha$ or $\frac{r}{x}$ is positive and decreases from $+\infty$ to 1

$\csc \alpha$ or $\frac{r}{y}$ is negative and decreases from -1 to $-\infty$

The above results are presented in tabular form.

	1st Quadrant (0° to 90°)	2d Quadrant (90° to 180°)	3d Quadrant (180° to 270°)	4th Quadrant (270° to 360°)
$\sin \alpha = \frac{y}{r}$	$+0$ inc. $+1$	$+1$ dec. $+0$	-0 dec. -1	-1 inc. -0
$\cos \alpha = \frac{x}{r}$	$+1$ dec. $+0$	-0 dec. -1	-1 inc. -0	$+0$ inc. $+1$
$\tan \alpha = \frac{y}{x}$	$+0$ inc. $+\infty$	$-\infty$ inc. -0	$+0$ inc. $+\infty$	$-\infty$ inc. -0
$\cot \alpha = \frac{x}{y}$	$+\infty$ dec. $+0$	-0 dec. $-\infty$	$+\infty$ dec. $+0$	-0 dec. $-\infty$
$\sec \alpha = \frac{r}{x}$	$+1$ inc. $+\infty$	$-\infty$ inc. -1	-1 dec. $-\infty$	$+\infty$ dec. $+1$
$\csc \alpha = \frac{r}{y}$	$+\infty$ dec. $+1$	$+1$ inc. $+\infty$	$-\infty$ inc. -1	-1 dec. $-\infty$

It is thus seen that the sine and cosine can never be greater than $+1$ nor less than -1, while the secant and cosecant have no values between $+1$ and -1, but have values ranging from $+1$ to $+\infty$ and from -1 to $-\infty$.

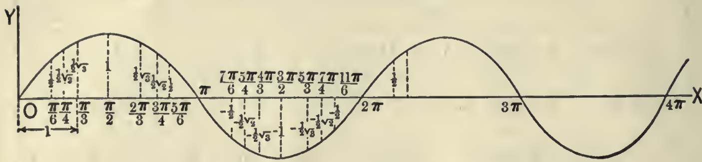
The tangent and cotangent may have any value from $+\infty$ to $-\infty$.

41. Graphical representation. A graphical representation of the trigonometric functions is effected by first locating points using the different values of the angle as abscissas and the corresponding function-values as ordinates, and then drawing a smooth curve through these points taken in the order of increasing angles.

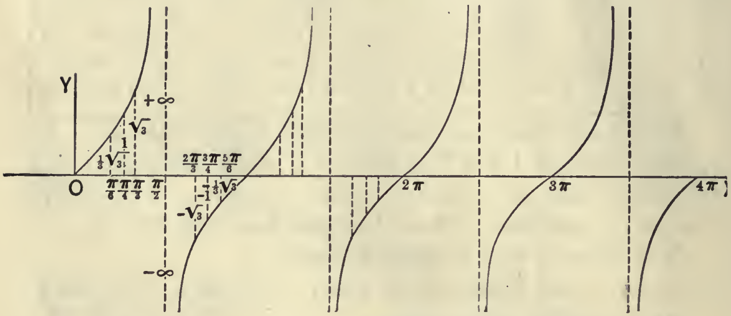
The values of the functions of the angles previously calculated are sufficient to determine an approximate graph. For greater accuracy the values of the functions may be taken from the table of natural functions.

α	$\sin \alpha$	α	$\tan \alpha$
0	0	0	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{1}{3}\sqrt{3}$
$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	∞
$\frac{2\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{2\pi}{3}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{3}$
π	0	π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\frac{7\pi}{6}$	$\frac{1}{3}\sqrt{3}$
etc.		etc.	

Sine Curve $y = \sin x$



Tangent Curve $y = \tan x$



The sine and tangent curves illustrate the truth of the theorems of Arts. 25 and 26.

Thus for any given value of the angle, as $\frac{3\pi}{4}$, there is but one value of each function, the curves showing the sine and tangent to be $\frac{1}{2}\sqrt{2}$ and -1 respectively.

But for any given value of a trigonometric function, as $\sin\alpha = \frac{1}{2}$, there are an unlimited number of angles, as

$$\frac{\pi}{6},\ \frac{5\pi}{6},\ 2\tfrac{1}{6}\pi,\ 2\tfrac{5}{6}\pi, \text{ etc.}$$

Also for $\tan\alpha = \sqrt{3}$ we see that α may have the values

$$\frac{\pi}{3},\ 1\tfrac{1}{3}\pi,\ 2\tfrac{1}{3}\pi,\ 3\tfrac{1}{3}\pi, \text{ etc.}$$

42. Periodicity of the trigonometric functions. From the sine curve it is readily seen that the sine is 0 when the angle is 0, and that it increases to 1 as the angle increases to $\frac{\pi}{2}$. At this point the sine begins to decrease and has the value 0 when the angle is π, and finally reaches the value -1 when the angle is $\frac{3\pi}{2}$; then the sine again begins to increase and has the value 0 when the angle has the value 2π.

When the angle increases from 2π to 4π, the sine repeats its values of the interval from 0 to 2π. If the angle were increased indefinitely, the sine would repeat its values for each interval of 2π. For this reason the sine is called a **periodic function**, and 2π is its period.

A study of the tangent curve shows that the tangent has the same values between 0 and π that it has between π and 2π or between 2π and 3π. Hence the tangent is also a periodic function having the period π.

43. EXAMPLES

1. Plot $y = \cos x$ and give the period of the cosine.

2. Plot $y = \cot x$ and give the period of the cotangent.

3. Plot $y = \sec x$ and give the period of the secant.

4. Plot $y = \csc x$ and give the period of the cosecant.

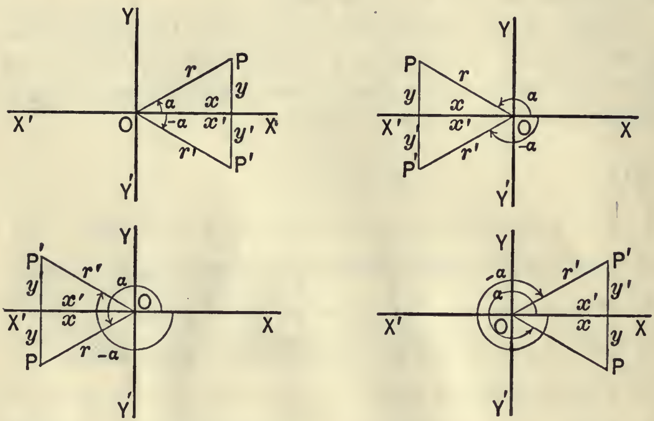
5. Plot $y = \sin x + \cos x$.

6. Plot $y = x + \sin x$.

REDUCTION OF FUNCTIONS OF n 90° ± α

44. The formulas to be developed in this section enable us to express any function of any angle in terms of a function of an angle differing from the given angle by any multiple of 90°. They may be used to express any function of any angle in terms of a function of an angle less than 90° or less than 45°.

45. Functions of $-\alpha$ in terms of functions of $\alpha (n = 0)$. Let XOP be any positive angle and XOP' a numerically equal negative angle.



Then taking $OP' = OP$ we have, for each figure,

$$x' = x, \quad y' = -y, \quad r' = r,$$

where x, y, r and x', y', r' are associated with P and P' respectively.

Then in each quadrant

$$\sin(-\alpha) = \frac{y'}{r'} = \frac{-y}{r} = -\sin\alpha$$

$$\cos(-\alpha) = \frac{x'}{r'} = \frac{x}{r} = \cos\alpha$$

$$\tan(-\alpha)=\frac{y'}{x'}=\frac{-y}{x}=-\tan\alpha$$

$$\cot(-\alpha)=\frac{x'}{y'}=\frac{x}{-y}=-\cot\alpha$$

$$\sec(-\alpha)=\frac{r'}{x'}=\frac{r}{x}=\sec\alpha$$

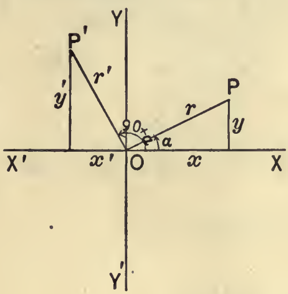
$$\csc(-\alpha)=\frac{r'}{y'}=\frac{r}{-y}=-\csc\alpha$$

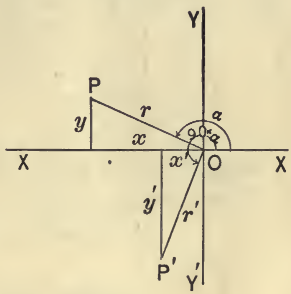
46. Functions of $90° + \alpha$ in terms of functions of $\alpha (n = 1)$. Let XOP be any positive angle α and XOP' the angle $90° + \alpha$.

Then taking $OP' = OP$ we have, for each figure,

$$x' = -y,\ y' = x,\ r' = r,$$

where x, y, r and x', y', r' are associated with P and P' respectively.





Similar figures can be constructed for the other quadrants. Then in each quadrant

$$\sin(90° + \alpha)=\frac{y'}{r'}=\frac{x}{r}=\cos\alpha$$

$$\cos(90° + \alpha)=\frac{x'}{r'}=\frac{-y}{r}=-\sin\alpha$$

$$\tan(90° + \alpha)=\frac{y'}{x'}=\frac{x}{-y}=-\cot\alpha$$

$$\cot(90° + \alpha) = \frac{x'}{y'} = \frac{-y}{x} = -\tan\alpha$$

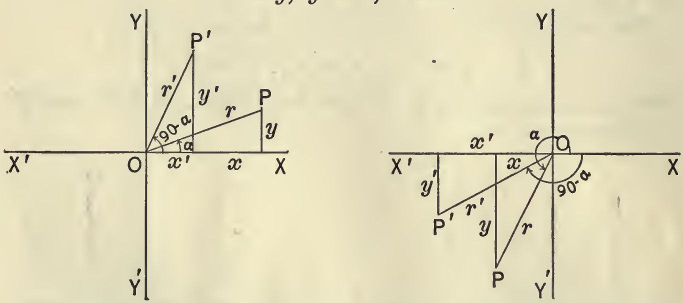
$$\sec(90° + \alpha) = \frac{r'}{x'} = \frac{r}{-y} = -\csc\alpha$$

$$\csc(90° + \alpha) = \frac{r'}{y'} = \frac{r}{x} = \sec\alpha$$

47. Functions of $90° - \alpha$ in terms of functions of α ($n = 1$). Let XOP be any positive angle and XOP' the angle $90° - \alpha$.

Then taking $OP' = OP$ we have, for each figure,

$$x' = y,\ y' = x,\ r' = r.$$



Similar figures can be constructed for the other quadrants. Then in each quadrant

$$\sin(90° - \alpha) = \frac{y'}{r'} = \frac{x}{r} = \cos\alpha$$

$$\cos(90° - \alpha) = \frac{x'}{r'} = \frac{y}{r} = \sin\alpha$$

$$\tan(90° - \alpha) = \frac{y'}{x'} = \frac{x}{y} = \cot\alpha$$

$$\cot(90° - \alpha) = \frac{x'}{y'} = \frac{y}{x} = \tan\alpha$$

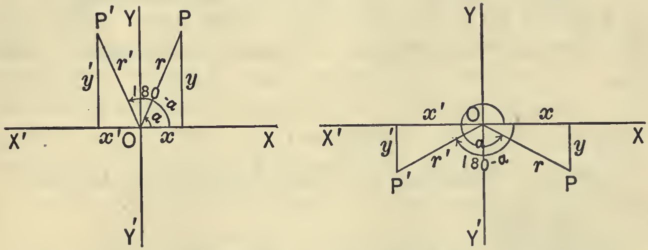
$$\sec(90° - \alpha) = \frac{r'}{x'} = \frac{r}{y} = \csc\alpha$$

$$\csc(90° - \alpha) = \frac{r'}{y'} = \frac{r}{x} = \sec\alpha$$

48. Functions of 180° − α in terms of functions of α (n=2). Let XOP be any positive angle and XOP' the angle $180° - \alpha$.

Then taking $OP' = OP$ we have, for each figure,

$$x' = -x,\ y' = y,\ r' = r.$$



Similar figures can be constructed for the other quadrants. Then in each quadrant

$$\sin(180° - \alpha) = \frac{y'}{r'} = \frac{y}{r} = \sin\alpha$$

$$\cos(180° - \alpha) = \frac{x'}{r'} = \frac{-x}{r} = -\cos\alpha$$

$$\tan(180° - \alpha) = \frac{y'}{x'} = \frac{y}{-x} = -\tan\alpha$$

$$\cot(180° - \alpha) = \frac{x'}{y'} = \frac{-x}{y} = -\cot\alpha$$

$$\sec(180° - \alpha) = \frac{r'}{x'} = \frac{r}{-x} = -\sec\alpha$$

$$\csc(180° - \alpha) = \frac{r'}{y'} = \frac{r}{y} = \csc\alpha$$

49. Laws of reduction. The method of the last four articles may be applied to any angle of the form $n\,90° \pm \alpha$ to obtain formulas of reduction for all positive and negative integral values of n, α being any angle. A complete investigation would show the following laws to be true:

When n is even, any function of $n\ 90° \pm \alpha$ is numerically equal to the same function of α; when n is odd, any function of $n\ 90° \pm \alpha$ is numerically equal to the cofunction of α.

If the function of $n\ 90° \pm \alpha$ is positive, α being considered acute, the members of the equation have like signs; if the function of $n\ 90° \pm \alpha$ is negative, α being considered acute, the members of the equation have unlike signs. The results thus obtained are valid for all positive and negative values of α.

50. EXAMPLES

Express as functions of a positive angle less than $90°$:

1. $\sin 130°$.

SUGGESTION. $130° = 90° + 40°$, or $130° = 180° - 50°$.

2. $\cos 170°$.

3. $\tan 110°$.

4. $\cot 160°$.

5. $\cos(-20°)$.

6. $\tan(-80°)$.

7. $\sin(-120°)$.

Express as functions of θ:

8. $\sin(810° - \theta)$.

9. $\tan(360° - \theta)$.

10. $\cot(270° + \theta)$.

11. $\sin(\theta - 90°)$.

12. $\tan(\theta - 180°)$.

13. $\sec(-180° - \theta)$.

14. $\csc(-630° + \theta)$.

15. $\cos(990° - \theta)$.

Express each of the trigonometric functions of the following angles as functions of α without using the laws of reduction.

16. $180° + \alpha$.

17. $270° - \alpha$.

18. $270° + \alpha$.

19. $360° - \alpha$.

Test on Chapt V
Mon.
learn it

CHAPTER V

FUNDAMENTAL RELATIONS. LINE VALUES

51. In the present chapter eight fundamental relations between the trigonometric functions are developed. It is then shown that each trigonometric function may be represented geometrically by a single line, giving the so-called line values. This gives a second view of the trigonometric functions. The line values offer a simple method of demonstrating the fundamental relations, and of developing the properties already derived by means of the trigonometric ratios. In fact the line values might serve as the fundamental definitions of the trigonometric functions and thus trigonometry could be based upon these values. Incidentally the line values suggest the origin of some of the terms used to designate the several trigonometric functions.

FUNDAMENTAL RELATIONS

52. Certain relations of fundamental importance exist between the trigonometric functions of an angle. These will now be developed.

From definitions,

$$\sin\alpha = \frac{y}{r},$$

$$\csc\alpha = \frac{r}{y}. \qquad \therefore \sin\alpha = \frac{1}{\csc\alpha}. \tag{1}$$

$$\cos\alpha = \frac{x}{r},$$

$$\sec\alpha = \frac{r}{x}. \qquad \therefore \cos\alpha = \frac{1}{\sec\alpha}. \tag{2}$$

$$\tan \alpha = \frac{y}{x},$$

$$\cot \alpha = \frac{x}{y}. \qquad \therefore \tan \alpha = \frac{1}{\cot \alpha}. \tag{3}$$

$$\sin \alpha = \frac{y}{r},$$

$$\cos \alpha = \frac{x}{r},$$

$$\tan \alpha = \frac{y}{x}. \qquad \therefore \frac{\sin \alpha}{\cos \alpha} = \tan \alpha. \tag{4}$$

$$\cot \alpha = \frac{x}{y}, \quad \text{and} \frac{\cos \alpha}{\sin \alpha} = \cot \alpha. \tag{5}$$

From geometry, $y^2 + x^2 = r^2.$

Dividing successively by r^2, x^2, and y^2,

$$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1, \quad \therefore \sin^2 \alpha + \cos^2 \alpha = 1. \tag{6}$$

$$\frac{y^2}{x^2} + 1 = \frac{r^2}{x^2}, \quad \therefore \tan^2 \alpha + 1 = \sec^2 \alpha. \tag{7}$$

$$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2}, \quad \therefore \cot^2 \alpha + 1 = \csc^2 \alpha. \tag{8}$$

These eight formulas are called the **fundamental relations.** They are used frequently, and should be memorized. In doing this, it is well to have the method of derivation clearly in mind.

53. The use of exponents. In affecting trigonometric functions with exponents, the exponent is usually placed immediately after the function. Thus, in the formulas of the preceding article, $\sin^2 \alpha$ is identical in meaning with $(\sin \alpha)^2$, and $\tan^2 \alpha$ is identical with $(\tan \alpha)^2$.

An exception to the above is made for the exponent -1, in which case it is always necessary to make use of parentheses. Thus, $(\cos\alpha)^{-1}$ is used to express $\frac{1}{\cos\alpha}$, while $\cos^{-1}\alpha$ has an entirely different meaning, as will be explained later.

54. Trigonometric identities. An identical equation, or an **identity,** is an equation which is satisfied for *all* values of the unknown quantities. The eight fundamental relations are trigonometric identities. There are many other identities depending upon these.

The truth of an identity can be established in two ways:

First. Begin with one of the fundamental relations, and produce the given identity by means of the fundamental relations and algebraic principles.

Second. Begin with the given identity and transform it to one of the fundamental relations, or reduce one member of the equation to the other by means of the fundamental relations and algebraic principles.

The choice of the trigonometric transformations necessary to effect the reductions is often suggested by the functions involved in the given problem. When no transformation is thus suggested, the problem may generally be simplified by expressing each of the functions in terms of sines and cosines, and making use of the relation $\sin^2\alpha + \cos^2\alpha = 1$.

Avoid the use of radicals whenever possible.

55. Trigonometric equations. A conditional equation is an equation which is not satisfied for all values of the unknown quantity, but is satisfied only for particular values.

(a) Trigonometric equations involving different trigonometric functions of the same angle may often be solved by simplifying the equation by the use of the fundamental relations. Thus

$$\sin\alpha = \cos\alpha$$

may be written

$$\frac{\sin\alpha}{\cos\alpha} = 1 \text{ or } \tan\alpha = 1$$

$$\therefore \alpha = 45^\circ \text{ or } 225^\circ.$$

(*b*) Some equations may conveniently be solved by transposing all of the terms to the first member of the equation, factoring, and then placing each factor equal to zero. Thus

$$2\sin^2\alpha+\sqrt{3}\cos\alpha=2\sqrt{3}\sin\alpha\cos\alpha+\sin\alpha$$

may be written

$$\sin\alpha(2\sin\alpha-1)+\sqrt{3}\cos\alpha(1-2\sin\alpha)=0$$

or

$$(\sin\alpha-\sqrt{3}\cos\alpha)(2\sin\alpha-1)=0.$$

This equation is satisfied if either

$$\sin\alpha-\sqrt{3}\cos\alpha=0 \text{ or } 2\sin\alpha-1=0$$

whence $\tan\alpha=\sqrt{3}$ or $\sin\alpha=\frac{1}{2}$.

$$\therefore \alpha=60°,\ 240° \text{ or } \alpha=30°,\ 150°.$$

(*c*) When no other method suggests itself, the equation may be transformed, by the use of the fundamental relations, into an equation containing only one function. The equation thus obtained may be solved algebraically for the function involved, from which the values of the angle may be obtained. Thus

$$10\cos^3\alpha\tan\alpha-9\cos^2\alpha-\frac{2}{\csc\alpha}-10\sin\alpha+9=0$$

$$10\sin\alpha(1-\sin^2\alpha)-9(1-\sin^2\alpha)-12\sin\alpha+9=0$$

$$10\sin\alpha-10\sin^3\alpha+9\sin^2\alpha-12\sin\alpha=0$$

$$\therefore \sin\alpha=0 \text{ or } 10\sin^2\alpha-9\sin\alpha+2=0$$

from which $\sin\alpha=0,\ \frac{2}{5}, \text{ or } \frac{1}{2}$.

$$\therefore \alpha=0°,\ 23°\ 35',\ 30°,\ 150°,\ 156°\ 25', \text{ or } 180°.$$

A trigonometric equation is considered completely solved when every positive angle less than 360° which satisfies it has been determined. All other angles coterminal with these angles also satisfy the equation.

In the solution of trigonometric equations extraneous roots may occur. These may be detected by substitution in the original equation.

56. EXAMPLES

Prove the following trigonometric identities:

1. $\sin\theta = \cos\theta \tan\theta.$

2. $\tan\theta = \sin\theta \sec\theta.$

3. $\cos\theta = \dfrac{\cot\theta}{\csc\theta}.$

4. $(1-\sin x)(1+\sin x) = \cos^2 x.$

5. $(\sec x - \tan x)(\sec x + \tan x) = 1.$

6. $(\sin x + \cos x)^2 = 2\sin x \cos x + 1.$

7. $\dfrac{\sqrt{1+\cot^2 x}}{\cot x} = \sec x.$

8. $\cos^2\alpha \tan^2\alpha + \sin^2\alpha \cot^2\alpha = 1.$

9. Given $\cos\alpha = \frac{5}{9}$, α being in the fourth quadrant, find the values of the remaining trigonometric functions.

SOLUTION. Since $\sin^2\alpha + \cos^2\alpha = 1$,

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \tfrac{25}{81} = \tfrac{56}{81}.$$

$$\therefore \sin\alpha = -\sqrt{\tfrac{56}{81}} = -\tfrac{2}{9}\sqrt{14}.$$

$$\text{Also } \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-\frac{2}{9}\sqrt{14}}{\frac{5}{9}} = -\tfrac{2}{5}\sqrt{14};$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{-\frac{2}{5}\sqrt{14}} = -\frac{5}{2\sqrt{14}} = -\tfrac{5}{28}\sqrt{14};$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\frac{5}{9}} = \frac{9}{5};$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{1}{-\frac{2}{9}\sqrt{14}} = -\frac{9}{2\sqrt{14}} = -\tfrac{9}{28}\sqrt{14}.$$

Compare with the method of examples 26 and 27, Art. 27.

10. Given $\tan y = 4$, y being in the third quadrant, find the values of the remaining trigonometric functions.

SOLUTION. Since $\sec^2 y = 1 + \tan^2 y$,

$$\sec y = -\sqrt{1+16} = -\sqrt{17}.$$

$$\text{Also } \cot y = \frac{1}{\tan y} = \frac{1}{4};$$

$$\csc^2 y = 1 + \cot^2 y = 1 + \tfrac{1}{16} = \tfrac{17}{16};$$

$$\csc y = -\tfrac{1}{4}\sqrt{17};$$

$$\sin y = \frac{1}{\csc y} = \frac{1}{-\frac{1}{4}\sqrt{17}} = -\tfrac{4}{17}\sqrt{17};$$

$$\cos y = \frac{1}{\sec y} = \frac{1}{-\sqrt{17}} = -\tfrac{1}{17}\sqrt{17}.$$

11. Given $\sin\theta = \frac{2}{3}$, θ terminating in the second quadrant, find the values of the remaining trigonometric functions.

12. Given $\cot\theta = \frac{7}{10}$, θ terminating in the first quadrant, find the values of the remaining trigonometric functions.

13. Given $\sec\theta = -2$, θ being in the third quadrant, find the values of the remaining trigonometric functions.

14. Given $\tan\alpha = -\frac{4}{7}$, find the remaining functions of the angles less than 360° which satisfy the equation.

Solve the following trigonometric equations for angles less than 90°:

15. $5\sin x + 8 = 3(4 - \sin x)$.

16. $\sin u - \cos u = 0$.

17. $2\cos^2 x - 3\cos x + 1 = 0$.

18. $(\tan\theta - 1)(\tan\theta - \sqrt{3}) = 0$.

19. Solve $\sin x = \cot x$ for $\sin x$.

Some of the following examples are identities and others are equations. Establish the identities and solve the equations for angles less than 180°.

20. $\sin^4 x = 1 - 2\cos^2 x + \cos^4 x$.

21. $\sin^4 x = 2 - 6\cos^2 x + \cos^4 x$.

22. $(\sqrt{3}+1)\dfrac{\sin\theta}{\tan\theta} + \dfrac{2\cos\theta}{\cot\theta} = \sin\theta + \cos\theta$.

23. $2\dfrac{\sin\theta}{\tan\theta} + \dfrac{\cos\theta}{\cot\theta} = \sin\theta + \cos\theta + \dfrac{\sqrt{3}}{2}$.

24. $\tan u + \cot u = \sec u \csc u$.

25. $2\sin^2 y \csc y + 3\csc y = 7$.

Prove the following trigonometric identities:

26. $\tan^2\theta + \cot^2\theta = \sec^2\theta \csc^2\theta - 2.$

27. $\sec^2\beta + \cos^2\beta = \tan^2\beta \sin^2\beta + 2.$

28. $\csc^2\gamma - \sec^2\gamma = \cos^2\gamma \csc^2\gamma - \sin^2\gamma \sec^2\gamma.$

29. $\sin\alpha \tan\alpha = \dfrac{\sin\alpha + \tan\alpha}{\cot\alpha + \csc\alpha}.$

30. $\cot^2\alpha - \cos^2\alpha = \cot^2\alpha \cos^2\alpha.$

31. $\dfrac{1 - 2\cos^2\theta}{\sin\theta \cos\theta} = \tan\theta - \cot\theta.$

32. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2.$

33. $\sin^2 x + \operatorname{vers}^2 x = 2(1 - \cos x).$

34. $\sec^2 a \csc^2 a - \dfrac{(1 - \tan^2 a)^2}{\tan^2 a} = 4.$

Solve the following trigonometric equations for angles less than 180°:

35. $2\cos^2\alpha - 3\sin\alpha = 0.$

36. $\sec^2 a + \cot^2 a = 4\frac{1}{3}.$

37. $1 + \tan^2 x - 4\cos^2 x = 0.$

38. $\tan x + \cot x = 2.$

39. $2\sin^2 x + 3\cos x = 0.$

40. $\sqrt{3}\cos x + \sin x = \sqrt{2}.$

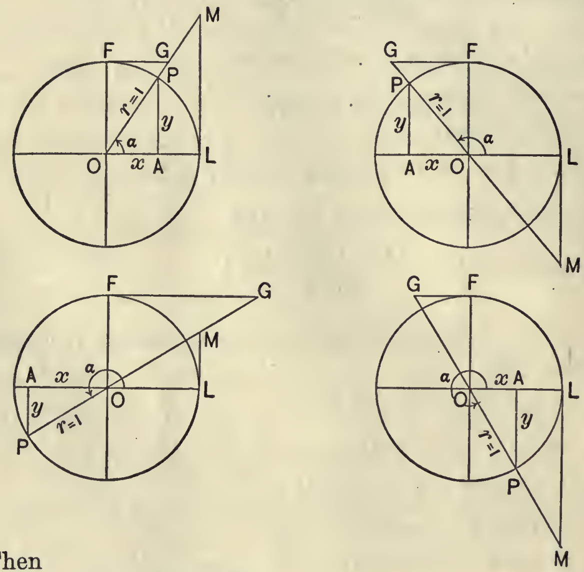
41. $\csc^2 x - 4\sin^2 x = 0.$

42. $\cot x + \csc^2 x = 3.$

LINE VALUES

57. Representation of the trigonometric functions by lines. The trigonometric functions of an angle have been studied solely from the standpoint of ratios; but each function can also be represented in magnitude and sign by a single line.

Let LOP be any angle. Take $OP = 1$. Draw the ordinate and abscissa of the point P. About O as the center construct a circle with OP as a radius. Draw the tangents to this circle at L and F, the *beginning of the first and second quadrants* respectively, and produce them to intersect the terminal line, or the terminal line produced, in M and G.



Then

$$\sin\alpha = \frac{y}{r} = \frac{AP}{OP} = \frac{AP}{1} = AP \qquad \text{or} \qquad \sin\alpha = AP$$

$$\cos\alpha = \frac{x}{r} = \frac{OA}{OP} = \frac{OA}{1} = OA \qquad \text{or} \qquad \cos\alpha = OA$$

$$\tan\alpha = \frac{y}{x} = \frac{AP}{OA} = \frac{LM}{OL} = LM \qquad \text{or} \qquad \tan\alpha = LM$$

$$\cot\alpha = \frac{x}{y} = \frac{OA}{AP} = \frac{FG}{OF} = FG \qquad \text{or} \qquad \cot\alpha = FG$$

$$\sec\alpha = \frac{r}{x} = \frac{OP}{OA} = \frac{OM}{OL} = OM \qquad \text{or} \qquad \sec\alpha = OM$$

$$\csc\alpha = \frac{r}{y} = \frac{OP}{AP} = \frac{OG}{OF} = OG \qquad \text{or} \qquad \csc\alpha = OG$$

Thus the trigonometric functions are represented by the segments AP, OA, LM, FG, OM and OG. It is evident that these segments represent the trigonometric functions *in magnitude*. They also represent the trigonometric functions *in sign*. In accordance with the conventions of Arts. 6 and 11, the segments AP and LM are positive when drawn upward from the X-axis and negative when drawn downward; the segments OA and FG are positive when drawn from the Y-axis to the right and negative when drawn to the left; the segments OM and OG are positive when they coincide with the terminal line of the angle and negative when they coincide with the terminal line produced.

It is evident that the sign of each trigonometric function is determined by the segment representing the function, since in each quadrant the segment is positive whenever the ratio defining the function is positive, and negative whenever the ratio is negative. Thus in each quadrant LM has the same sign as the ratio $\frac{y}{x}$, and similarly for the other functions. Therefore the segments represent the trigonometric functions in sign as well as in magnitude.

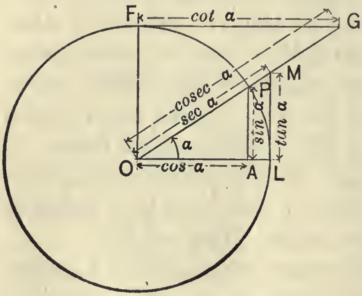
The segments which represent the trigonometric functions are called the **line values** of the functions.

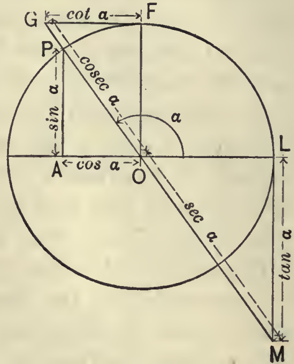
PROBLEM. Represent vers α and covers α by line values.

58. Variations of the trigonometric functions as shown by line values. The line values of the trigonometric functions give a simple method of tracing the variations in the functions as the angle varies from 0° to 360°. Thus as the point P, in the figures of Art. 57, describes the circle of radius unity, the changes in the segments AP, OA, LM, etc., represent the variations in the sine, cosine, tangent, etc., respectively.

Numerous figures, with three or four values of the angle in each quadrant, serve to suggest these variations.

59. Fundamental relations by line values. By the use of line values the fundamental relations of Art. 52 may be simply obtained and easily memorized.





Thus

From the triangle OAP,

$$\overline{AP}^2 + \overline{OA}^2 = \overline{OP}^2 \quad \text{or} \quad \sin^2 a + \cos^2 a = 1.$$

From the triangle OLM,

$$\overline{LM}^2 + \overline{OL}^2 = \overline{OM}^2 \quad \text{or} \quad \tan^2 a + 1 = \sec^2 a.$$

From the triangle OGF,

$$\overline{FG}^2 + \overline{OF}^2 = \overline{OG}^2 \quad \text{or} \quad \cot^2 a + 1 = \csc^2 a.$$

From definitions,

$$\tan \alpha = \frac{AP}{OA} = \frac{\sin \alpha}{\cos \alpha} \quad \text{or} \quad \tan a = \frac{\sin a}{\cos a}$$

$$\cot \alpha = \frac{OA}{AP} = \frac{\cos \alpha}{\sin \alpha} \quad \text{or} \quad \cot a = \frac{\cos a}{\sin a}$$

$$\sec \alpha = \frac{OP}{OA} = \frac{1}{\cos \alpha} \quad \text{or} \quad \sec a = \frac{1}{\cos a}$$

$$\csc \alpha = \frac{OP}{AP} = \frac{1}{\sin \alpha} \quad \text{or} \quad \csc a = \frac{1}{\sin a}$$

$$\cot \alpha = \frac{OA}{AP} = \frac{OL}{LM} = \frac{1}{\tan \alpha} \quad \text{or} \quad \boldsymbol{\cot \alpha = \frac{1}{\tan \alpha}}$$

The process of derivation of these formulas applies equally well to an angle of the third or fourth quadrant; hence the formulas are true for all values of α.

To memorize these formulas most easily, form a clear mental picture of the figure for the first quadrant and read each formula directly from the figure, applying the ratio definitions and the Pythagorean theorem.

60. EXAMPLES

1. Construct the line values of an angle terminating in the third quadrant.

2. Construct the line values of an angle terminating in the fourth quadrant.

3. Obtain the fundamental relations for an angle in the third quadrant, by the use of line values.

4. Obtain the fundamental relations for an angle terminating in the fourth quadrant, by the use of line values.

5. Deduce the relations between the functions of $90° + x$ and the functions of x by means of line values.

Suggestion. Construct two figures, one for the angle $90° + x$ and the other for the angle x.

6. Deduce the relations between the functions of $180° - x$ and the functions of x by means of line values.

7. Deduce the relations between the functions of $90° - x$ and the functions of $270° + x$ by means of line values.

By means of line values, trace the variations in the following functions as α varies from 0° to 360°:

8. $\sin \alpha$.	**11.** $\cot \alpha$.
9. $\cos \alpha$.	**12.** $\sec \alpha$.
10. $\tan \alpha$.	**13.** $\csc \alpha$.

Prove the following trigonometric identities:

14. $\dfrac{1}{\cot x + \tan x} = \sin x \cos x.$

15. $\cos^2 \alpha (1 + \cot^2 \alpha) = \cot^2 \alpha.$

16. $\sin \theta \cot^2 \theta = \csc \theta - \sin \theta.$

17. $\tan \alpha - \tan \alpha \sin^2 \alpha = \sin \alpha \cos \alpha.$

18. $\cos \alpha \tan^2 \alpha = \sec \alpha - \cos \alpha.$

19. $\dfrac{\cos \alpha}{\tan \alpha} + \dfrac{\sin \alpha}{\cot \alpha} = \dfrac{\sin^3 \alpha + \cos^3 \alpha}{\sin \alpha \cos \alpha}.$

20. $\dfrac{\sin^3 \alpha + \cos^3 \alpha}{\sin \alpha + \cos \alpha} = 1 - \sin \alpha \cos \alpha.$

21. $\dfrac{\sin x}{\cos x} + \dfrac{\tan x}{\cot x} + \dfrac{\sec x}{\csc x} = \dfrac{2 \cot x + 1}{\cot^2 x}.$

22. $\dfrac{1 - \cos x}{1 + \cos x} = \dfrac{\sec x - 1}{\sec x + 1}.$

23. $\dfrac{1 - \cos x}{1 + \cos x} = \dfrac{\sin^2 x}{(1 + \cos x)^2}.$

24. $\dfrac{1 - \cos x}{1 + \cos x} = \dfrac{(1 - \cos x)^2}{\sin^2 x}.$

25. $2 \sin y \cos^2 y + (2 \cos^2 y - 1) \sin y = 3 \sin y - 4 \sin^3 y.$

26. $\cos^4 \alpha - \sin^4 \alpha = \cos^2 \alpha - \sin^2 \alpha.$

27. $\sec^2 \alpha + \operatorname{cosec}^2 \alpha = \sec^2 \alpha \csc^2 \alpha.$

28. $\dfrac{\tan x + \tan y}{\sec x - \sec y} = \dfrac{\sec x + \sec y}{\tan x - \tan y}.$

29. $\tan^4 u - \cot^4 u = (\tan^2 u + \cot^2 u)(\sec^2 u - \csc^2 u).$

30. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y.$

31. $\dfrac{\tan x - \tan y}{\cot x - \cot y} = -\tan x \tan y.$

32. $\dfrac{\tan x + \cot y}{\cot x + \tan y} = \dfrac{\tan x}{\tan y}.$

33. Express $\sin^2 x - \cos^2 x$ in terms of $\tan x$.

34. Express $\text{vers}\, x - \text{covers}\, x$ in terms of $\sin x$.

35. Express $\cot^2 x + \csc^2 x$ in terms of $\cos x$.

36. Express $\dfrac{\text{vers}\, x}{\text{covers}\, x}$ in terms of $\tan x$.

37. Express $\tan^2 x + \sec^2 x$ in terms of $\cot x$.

38. Express $(4 \sin^3 \theta - 3 \sin \theta)(2 \cos^2 \theta - 1)$
$+ (3 \cos \theta - 4 \cos^3 \theta)(2 \sin \theta \cos \theta)$ in terms of $\sin \theta$.

39. Given $\sin \theta = a$, find the remaining functions of θ.

40. Given $\cot \theta = b$, find the remaining functions of θ.

41. Express each trigonometric function of θ in terms of $\cos \theta$.

42. Express each trigonometric function of θ in terms of $\tan \theta$.

43. Simplify $(a + b) \sin 30° - b \cos 60° + a \tan 180°$.

44. Simplify $l \sin (270° - x) + m \cos (180° + x)$
$+ n \sin (90° - x)$.

45. Simplify $a \sin 135° + (a - b) \cos 225° + b \cos 315°$.

46. Simplify $\tan (-120°) + \cot 150° - \tan 210° + \cot 240°$.

Find the positive values of θ, less than 180°, that satisfy the following equations:

47. $\sin \theta \cos \theta = 0$.

48. $\sin \theta + \cos \theta = 0$.

49. $\sin \theta (2 \sin \theta - 1)(2 \cos \theta - 1) = 0$.

50. $\sin \theta + \cos \theta \cot \theta = 2$.

51. $\sin\theta\cos\theta = \frac{1}{2}$.

52. $2\sin\theta\cos\theta + \sin\theta - 2\cos\theta = 1$.

53. $2\cos\theta + \sec\theta = 3$.

54. $\sec\theta + \tan\theta = 2$.

55. $2\sin\theta + 5\cos\theta = 2$.

56. $1 + \sin^2\theta = 3\sin\theta\cos\theta$.

57. $\tan^4\theta - 4\tan^2\theta - 5 = 0$.

58. What negative angles, numerically less than 180°, satisfy the equation $\sin^2 x + \sin x = \cos^2 x - 1$?

59. Given $9\cos^2 u + 9\sin u = 11$, find $\tan u$.

60. Given $\tan^2 x - 5\sec x + 7 = 0$, find $\sin x$.

Find the positive values of θ, less than 360°, that satisfy the following equations:

61. $\sin\theta = -\cos 285°$.

62. $\tan\theta = \cot(-144°)$.

63. $\cos(-\theta) = \sin 190°$.

64. $\sin\theta = -\sin 50°$.

65. If $\sin 122° = k$, prove that $\tan 32° = \dfrac{\sqrt{1-k^2}}{k}$.

66. If $\cot 255° = a$, prove that $\cos 345° = \dfrac{1}{\sqrt{1+a^2}}$.

67. If $\cos(-100°) = k$, prove that $\tan 80° = -\dfrac{\sqrt{1-k^2}}{k}$.

Solve the following equations for *any trigonometric function* of α.

68. $2\sin\alpha + \csc\alpha = 1$.

69. $\csc\alpha\cot\alpha = \frac{5}{3}$.

70. $2\sin\alpha + \cos^2\alpha = 1$.

71. $\tan^2\alpha + 4\sec\alpha = 5$.

72. $\tan\alpha + \cot\alpha = 2$.

73. Given $\sin u = k\sin v$ and $\tan u = l\tan v$, find $\sin u$ and $\sin v$.

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CHAPTER VI

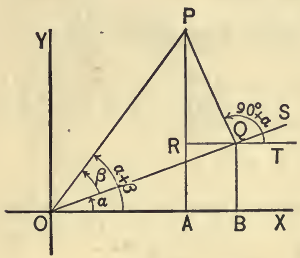
FUNCTIONS OF THE SUM OF TWO ANGLES

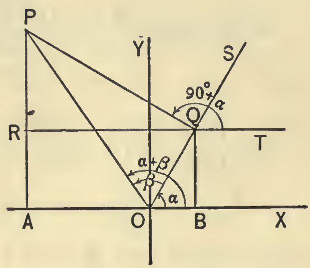
DOUBLE ANGLES. HALF ANGLES

61. We have thus far considered the properties and relations of the functions of a single angle. We have also shown that the functions of the angles $n\,90 \pm \alpha$ depend upon the functions of the angle α.

In the present chapter we consider the relations between the functions of the sum of two independent angles and the functions of the separate angles, and develops some related formulas.

62. The sine of the sum of two acute angles expressed in terms of the sines and cosines of the angles.





Let α and β be any given acute angles.

Construct the angle $(\alpha+\beta)$, Art. 12.

Then $(\alpha+\beta)$ may be acute as in Fig. 1, or obtuse as in Fig. 2.

From any point P, in the terminal line of the angle $(\alpha+\beta)$, draw PA perpendicular to OX, and PQ perpendicular to OS. Through Q draw RQT parallel to OX, and QB perpendicular to OX.

Then the angle TQP is equal to $90° + \alpha$.

By definition

$$\sin(\alpha+\beta) = \frac{AP}{OP} = \frac{AR+RP}{OP} = \frac{BQ}{OP} + \frac{RP}{OP} = \frac{BQ}{OQ}\cdot\frac{OQ}{OP} + \frac{RP}{QP}\cdot\frac{QP}{OP}.$$

But $\frac{BQ}{OQ} = \sin\alpha$, $\frac{OQ}{OP} = \cos\beta$, $\frac{RP}{QP} = \sin(90° + \alpha) = \cos\alpha$,

and $\frac{QP}{OP} = \sin\beta$,

therefore $\boldsymbol{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$. (1)

Problem 1. Show that formula (1) is true when either angle is 90°.

Problem 2. Show that formula (1) is true when each angle is 90°.

63. The cosine of the sum of two acute angles expressed in terms of the sines and cosines of the angles. Referring to the figures of the previous article, we have by definition

$$\cos(\alpha+\beta) = \frac{OA}{OP} = \frac{OB+BA}{OP} \text{ (Art. 2)} = \frac{OB}{OP} + \frac{QR}{OP} = \frac{OB}{OQ}\cdot\frac{OQ}{OP} + \frac{QR}{QP}\cdot\frac{QP}{OP}.$$

But $\frac{OB}{OQ} = \cos\alpha$, $\frac{OQ}{OP} = \cos\beta$,

$\frac{QR}{QP} = \cos(90° + \alpha) = -\sin\alpha$, and $\frac{QP}{OP} = \sin\beta$,

therefore $\boldsymbol{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$. (1)

Problem 1. Show that the formula (1) is true when either angle is 90°.

Problem 2. Show that formula (1) is true when each angle is 90°.

64. The formulas developed in the last two articles are of the utmost importance, since many other formulas are derived from them. They form the basis of the present chapter. In developing these formulas α and β were considered positive acute angles, but the formulas are true for all values of α and β, as may be shown either geometrically or analytically.

It is shown geometrically by constructing figures in which α and β are of any magnitude, and following the method of proof given above for acute angles.

We present the analytic demonstration as the more satisfactory.

65. *To prove that*

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad (1)$$

$$\textit{and}\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad (2)$$

are true for all values of a and β.

First. *To show that α can be replaced by α', where $\alpha' = \alpha + 90°$.*

Let $\alpha' = \alpha + 90°$ where α is acute.

As α varies from 0° to 90°, α' varies from 90° to 180°.

$$\text{Also } \sin\alpha' = \sin(\alpha+90°) = \cos\alpha, \qquad (3)$$

$$\cos\alpha' = \cos(\alpha+90°) = -\sin\alpha. \qquad (4)$$

$$\text{Now } \sin(\alpha'+\beta) = \sin(90°+\alpha+\beta) = \cos(\alpha+\beta).$$

Since α and β are both acute, $\cos(\alpha+\beta)$ may be expanded by (2), hence

$$\sin(\alpha'+\beta) = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Therefore, by (3) and (4),

$$\sin(\alpha'+\beta) = \sin\alpha'\cos\beta + \cos\alpha'\sin\beta. \qquad (5)$$

Similarly,

$$\cos(\alpha'+\beta) = \cos(90°+\alpha+\beta) = -\sin(\alpha+\beta)$$
$$= -\sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Therefore, by (3) and (4),

$$\cos(\alpha' + \beta) = \cos\alpha' \cos\beta - \sin\alpha' \sin\beta. \tag{6}$$

Formulas (5) and (6) show that one angle, α, in (1) and (2) may be extended to 180°.

Second. *To show that α can be replaced by α'' where $\alpha'' = \alpha + 180°$.*

Let $\alpha'' = \alpha + 180°$.

As α varies from 0° to 180°, α'' varies from 180° to 360°.

Also $\sin\alpha'' = \sin(\alpha + 180°) = -\sin\alpha,$ (7)

$\cos\alpha'' = \cos(\alpha + 180°) = -\cos\alpha.$ (8)

Now $\sin(\alpha'' + \beta) = \sin(180° + \alpha + \beta) = -\sin(\alpha + \beta)$.

Since α is less than 180° and β acute, $\sin(\alpha + \beta)$ may be expanded by (1), as extended in the first part of the proof, hence

$$\sin(\alpha'' + \beta) = -\sin(\alpha + \beta) = -\sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Therefore, by (7) and (8),

$$\sin(\alpha'' + \beta) = \sin\alpha''\cos\beta + \cos\alpha''\sin\beta. \tag{9}$$

Similarly,

$$\begin{aligned}\cos(\alpha'' + \beta) &= \cos(180° + \alpha + \beta) = -\cos(\alpha + \beta)\\ &= -\cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

Therefore, by (7) and (8),

$$\cos(\alpha'' + \beta) = \cos\alpha''\cos\beta - \sin\alpha''\sin\beta. \tag{10}$$

Formulas (9) and (10) show that one angle, α, of (1) and (2) may be extended to 360°.

Third. In a similar manner it may be shown that (1) and (2) are true when β varies from 0° to 360°, α having any value from 0° to 360°, from which it easily follows that (1) and (2) are true for *all positive* values of α and β.

Fourth. Formulas (1) and (2) may be shown to be true when either α or β or both are negative, by the use of the substitutions

$$\alpha' = \alpha - n\,360^\circ$$
$$\beta' = \beta - n\,360^\circ;$$

hence (1) and (2) are true for all positive and negative values of α and β.

66. *To find the tangent of the sum of any two given angles in terms of the tangents of the given angles.*

Let α and β be the given angles.

Then $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$

Dividing numerator and denominator of the last fraction by $\cos\alpha\cos\beta$ and simplifying, we have

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

67. *To find the cotangent of the sum of any two given angles in terms of the cotangents of the given angles.*

Let α and β be the given angles.

Then $\cot(\alpha+\beta) = \dfrac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}.$

Dividing numerator and denominator of the last fraction by $\sin\alpha\sin\beta$ and simplifying we have

$$\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}.$$

68. Addition formulas. Collecting the formulas for the sum of two angles for convenience of reference, we have

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad (1)$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (2)$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{3}$$

$$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}. \tag{4}$$

69. *To find the sine, cosine, tangent, and cotangent of the difference of two given angles.*

Since the formulas of Art. 68 are true for all values of α and β, they are true when $-\beta$ is substituted for β.

Hence

$$\sin(\alpha-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$

$$\cos(\alpha-\beta)=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$$

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot(-\beta)-1}{\cot(-\beta)+\cot\alpha}$$

But remembering that

$$\sin(-\beta)=-\sin\beta,\ \cos(-\beta)=\cos\beta,$$

$$\tan(-\beta)=-\tan\beta,\text{ and }\cot(-\beta)=-\cot\beta,$$

these formulas become

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \tag{1}$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \tag{2}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{3}$$

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}. \tag{4}$$

70. EXERCISES

1. Find $\sin 75°$.

SOLUTION. $75° = \sin(45°+30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$$= \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\sqrt{2}.$$

$$= \tfrac{1}{4}\sqrt{2}(\sqrt{3}+1).$$

2. Find cos 75°.

3. Find sin 15°.

4. Find cos 15°.

5. Find tan 75°.

6. Find cot 15°.

7. Find sin $(90° - \beta)$.

8. Find tan $(90° + \alpha)$.

9. Find cos $(180° - \alpha)$.

10. Find sec 15°.

11. Find csc $(90° + \alpha)$.

71. Double angles. *To find the sine, cosine, tangent, and cotangent of twice a given angle in terms of the functions of the given angle.*

From the formulas of Art. 68, letting $\beta = \alpha$, we have, after reduction,

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha \tag{1}$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \tag{2}$$

$$= 1 - 2 \sin^2 \alpha \tag{3}$$

$$= 2 \cos^2 \alpha - 1 \tag{4}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \tag{5}$$

$$\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}. \tag{6}$$

To express clearly the real significance of these formulas, we may state them from two points of view.

If α be the angle under consideration, the formula

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

may be stated: *The sine of twice an angle is equal to twice the sine of the angle times the cosine of the angle.*

If 2α be the angle under consideration, the same formula may be stated: *The sine of an angle is equal to twice the sine of half the angle times the cosine of half the angle.*

It then follows that

$$\sin \alpha = 2 \sin \tfrac{1}{2} \alpha \cos \tfrac{1}{2} \alpha. \tag{7}$$

Similarly, if α be the angle under consideration, the formula $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

may be stated: *The cosine of twice an angle is equal to the square of the cosine of the angle minus the square of the sine of the angle.*

If 2α be the angle under consideration, the same formula may be stated: *The cosine of an angle is equal to the square of the cosine of half the angle minus the square of the sine of half the angle.*

It then follows that

$$\cos\alpha = \cos^2 \tfrac{1}{2}\alpha - \sin^2 \tfrac{1}{2}\alpha. \tag{8}$$

Similarly,

from $\cos 2\alpha = 1 - 2\sin^2\alpha$ we have

$$\cos\alpha = 1 - 2\sin^2 \tfrac{1}{2}\alpha, \tag{9}$$

from $\cos 2\alpha = 2\cos^2\alpha - 1$ we have

$$\cos\alpha = 2\cos^2 \tfrac{1}{2}\alpha - 1, \tag{10}$$

from $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$ we have

$$\tan\alpha = \frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha}, \tag{11}$$

from $\cot 2\alpha = \dfrac{\cot^2\alpha - 1}{2\cot\alpha}$ we have

$$\cot\alpha = \frac{\cot^2\frac{1}{2}\alpha - 1}{2\cot\frac{1}{2}\alpha}. \tag{12}$$

72. Half angles. *To find the sine, cosine, tangent, and cotangent of one half a given angle in terms of functions of the given angle.*

From formula (9), Art. 71, we have

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2},$$

or
$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}. \tag{1}$$

From formula (10), Art. 71, we have

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1,$$

or
$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}. \tag{2}$$

From (1) and (2),

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}, \tag{3}$$

and
$$\cot\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}. \tag{4}$$

In each of these formulas the positive or negative sign is chosen to agree with the sign of the function of $\frac{\alpha}{2}$, depending on the quadrant in which $\frac{\alpha}{2}$ lies.

These formulas may be considered from two view points. For example, formula (1) may be stated: *The sine of half an angle is equal to the square root of the fraction whose numerator is one minus the cosine of the angle, and whose denominator is two;* or, *The sine of an angle is equal to the square root of the fraction whose numerator is one minus the cosine of twice the angle, and whose denominator is two.*

73. *To find the sum and difference of the sines of any two angles, also the sum and difference of the cosines of any two angles.*

From the formulas of Arts. 68 and 69 we have, by addition and subtraction,

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$
$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$
$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta.$$

Let $\alpha+\beta=\gamma$ and $\alpha-\beta=\partial$. Solving for α and β, we have $\alpha=\frac{1}{2}(\gamma+\partial)$ and $\beta=\frac{1}{2}(\gamma-\partial)$. Substituting these values in the above equation, it follows that

$$\sin\gamma+\sin\partial=2\sin\tfrac{1}{2}(\gamma+\partial)\cos\tfrac{1}{2}(\gamma-\partial), \tag{1}$$

$$\sin\gamma-\sin\partial=2\cos\tfrac{1}{2}(\gamma+\partial)\sin\tfrac{1}{2}(\gamma-\partial), \tag{2}$$

$$\cos\gamma+\cos\partial=2\cos\tfrac{1}{2}(\gamma+\partial)\cos\tfrac{1}{2}(\gamma-\partial), \tag{3}$$

$$\cos\gamma-\cos\partial=-2\sin\tfrac{1}{2}(\gamma+\partial)\sin\tfrac{1}{2}(\gamma-\partial). \tag{4}$$

Equations (1), (2), (3), and (4) may be read from two view points.

Regarding γ and ∂ as the given angles, (1) may be stated: *The sum of the sines of two angles is equal to twice the product of the sine of half the sum of the given angles into the cosine of half the difference of the given angles.*

Thus, $$\sin 6x+\sin 4x=2\sin 5x\cos x.$$

Regarding $\frac{1}{2}(\gamma+\partial)$ and $\frac{1}{2}(\gamma-\partial)$ as the given angles, it is clear that their sum is γ and their difference ∂. Then, by reading the second member first, equation (1) may be stated: *Twice the sine of any angle times the cosine of any other angle is equal to the sine of the sum of the angles plus the sine of the difference of the angles.*

Thus, $$2\sin 20°\cos 5°=\sin 25°+\sin 15°.$$

74. Equations and identities. The formulas developed in the present chapter are true for all values of the angles involved; hence they are trigonometric identities. By the use of these identities many others may be established. The remarks of Art. 54, concerning the use of the fundamental relations in establishing identities, apply here.

The identities of the present chapter are also useful in solving trigonometric equations. By their aid an equation involving functions of multiple angles may be transformed into an equation containing functions of a single angle (see Ex. 33, Art. 75). This transformed equation can then be solved as indicated in Art. 55. Frequently equations may be much simplified by reducing sums or differences of sines and cosines to products by the relations of Art. 73 (see Ex. 41, Art. 75).

75. EXAMPLES

1. Find $\sin 22\frac{1}{2}°$, $\cos 22\frac{1}{2}°$, $\tan 22\frac{1}{2}°$, and $\cot 22\frac{1}{2}°$.

2. Given $\cos\alpha = \frac{1}{4}$; find $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$, and $\cot 2\alpha$.

3. Given $\tan\alpha = 3$; find $\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$, $\tan\frac{1}{2}\alpha$, and $\cot\frac{1}{2}\alpha$.

Prove the following identities:

4. $(\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha) = \cos 2\alpha$.

5. $(\sin\alpha + \cos\alpha)^2 = 1 + \sin 2\alpha$.

6. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$.

Suggestion: $\sin 3\alpha = \sin(2\alpha + \alpha)$.

7. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$.

8. $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$.

9. $\tan\frac{1}{2}x = \dfrac{\sin x}{1 + \cos x}$.

10. $\cot\frac{1}{2}x = \dfrac{\sin x}{1 - \cos x}$.

11. $\dfrac{1 + \tan\frac{1}{2}x}{1 - \tan\frac{1}{2}x} = \dfrac{1 + \sin x}{\cos x}$.

12. $\tan^2\alpha = \dfrac{1 - \cos 2\alpha}{1 + \cos 2\alpha}$.

13. $\sec 2\alpha = \dfrac{\sec^2\alpha}{2 - \sec^2\alpha}$.

14. $4\sin^2\frac{1}{2}\alpha\cos^2\frac{1}{2}\alpha = 1 - \cos^2\alpha$.

15. $\sin 4\alpha + \sin 2\alpha = 2\sin 3\alpha\cos\alpha$. (See Art. 73.)

16. $\cos 6\alpha - \cos 2\alpha = -2\sin 4\alpha\sin 2\alpha$.

17. $\sin(45^\circ + \alpha) - \sin(45^\circ - \alpha) = \sqrt{2} \sin \alpha$.

18. $\sin(30^\circ + \alpha) + \sin(30^\circ - \alpha) = \cos \alpha$.

19. Express $\cos 4\alpha \sin 3\alpha$ as the difference of two sines.

20. Express $\cos 5\alpha \cos \alpha$ as the sum of two cosines.

Prove the following identities:

21. $\cos^2 \alpha - \cos^2 \beta = -\sin(\alpha + \beta) \sin(\alpha - \beta)$.

22. $\dfrac{\sin 2\alpha + \sin \alpha}{\cos 2\alpha + \cos \alpha} = \tan \dfrac{3\alpha}{2}$.

23. $\dfrac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \dfrac{\alpha + \beta}{2}$.

24. $\sin \theta = \dfrac{2 \tan \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}$.

25. $\sin \frac{1}{2}x + 2 \sin^2 \frac{1}{2}x \cos \frac{1}{2}x \cot x = \sin x \cos \frac{1}{2}x$.

26. $2 \tan \frac{1}{2}x \csc x = 1 + \tan^2 \frac{1}{2}x$.

27. $\cos(\alpha + \beta) \cos \beta - \cos(\alpha + \gamma) \cos \gamma =$
$\sin(\alpha + \gamma) \sin \gamma - \sin(\alpha + \beta) \sin \beta$.

28. $\cos 2\alpha = 8 \sin^4 \frac{1}{2}\alpha - 8 \sin^2 \frac{1}{2}\alpha + 1$.

29. $\cos 2\alpha - \cos \alpha = 2(4 \sin^4 \frac{1}{2}\alpha - 3 \sin^2 \frac{1}{2}\alpha)$.

30. $(1 - \cos 2\alpha) \cot^2 \frac{1}{2}\alpha = \cos 2\alpha + 4 \cos \alpha + 3$.

31. $2 \sin^2 \frac{1}{2}\theta \cot \theta - \csc 2\theta + \csc \theta = \cot 2\theta + \sin \theta$.

32. Given $\tan x = a \tan \frac{1}{3}x$; find $\tan \frac{1}{3}x$ in terms of a.

Solve the following equations for all positive values of α less than 180°:

33. $\sin 2\alpha - 4 \cos^2 \frac{1}{2}\alpha - \sin \alpha + 3 = 0$.

Solution. $2 \sin \alpha \cos \alpha - 4\left(\dfrac{1 + \cos \alpha}{2}\right) - \sin \alpha + 3 = 0,$

or $2 \sin \alpha \cos \alpha - 2 \cos \alpha - \sin \alpha + 1 = 0,$

or $(2 \cos \alpha - 1)(\sin \alpha - 1) = 0,$

$\therefore$ $\cos \alpha = \frac{1}{2}$ or $\sin \alpha = 1.$

and $\alpha = 60^\circ$ or 90°.

34. $\sin 2\alpha - \cos \alpha = 0.$

35. $\sin 2\alpha + \sin \alpha = 0.$

36. $\sin 2\alpha + \cos 2\alpha = 0.$

37. $\sin \alpha + \cos 2\alpha = 1.$

38. $\cos 3\alpha + 2\cos 2\alpha - 8\cos^2\alpha + 2\sin^2\alpha + 5\cos\alpha = 0.$

39. $\cos 2\alpha + 2\sin^2 \frac{1}{2}\alpha - 1 = 0.$

40 $\sin 2\alpha + \cos 2\alpha + \sin \alpha = 1.$

41. $\sin 5\alpha - \sin 3\alpha + \sin \alpha = 0.$

SOLUTION. $2\sin 3\alpha \cos 2\alpha - \sin 3\alpha = 0.$ Art. 73.

or $\sin 3\alpha(2\cos 2\alpha - 1) = 0.$

$\therefore$ $\sin 3\alpha = 0$, or $\cos 2\alpha = \frac{1}{2}.$

$\therefore$ $3\alpha = 0°, 180°, 360°$, and $2\alpha = 60°, 300°.$

$\therefore$ $\alpha = 0°, 60°, 120°, 30, 150°.$

42. $\cos 3\alpha + \sin 2\alpha - \cos \alpha = 0.$

43. $\sin 3\alpha + \sin 2\alpha + \sin \alpha = 0.$

44. $\cos 3\alpha - \sin 2\alpha + \cos \alpha = 0.$

45. $\sin 3\alpha + \cos 3\alpha - \sin \alpha - \cos \alpha = 0.$

46. $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x.$

47. $\dfrac{1 - \tan x}{1 + \tan x} = 2\cos 2x.$

Prove the following identities:

48. $\tan(30° + \alpha)\tan(30° - \alpha) = \dfrac{2\cos 2\alpha - 1}{2\cos 2\alpha + 1}.$

49. $\tan \alpha + \cot \alpha = 2\csc 2\alpha.$

50. $\sin 80° = \sin 40° + \sin 20°.$

51. $\dfrac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha} = \tan\dfrac{\alpha}{2}.$

52. $\cos^2\alpha - \sin(30° + \alpha)\sin(30° - \alpha) = \frac{3}{4}.$

53. $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \tan\dfrac{\alpha + \beta}{2}.$

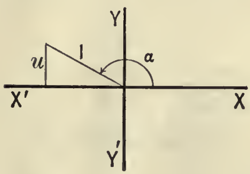
CHAPTER VII

INVERSE FUNCTIONS

76. Inverse trigonometric functions are closely related to the trigonometric functions previously studied.

After introducing the fundamental idea of an inverse function, it will be shown that they lead to new relations, closely allied to the relations already developed.

77. Fundamental idea of an inverse function. From the equation $\sin\alpha = u$, it is evident that α is an angle whose sine is u. The statement *α is an angle whose sine is u* is abbreviated into



$$\alpha = \sin^{-1} u.$$

The equation $u = \sin\alpha$ expresses u in terms of α.

The equation $\alpha = \sin^{-1} u$ expresses α in terms of u.

We thus have two methods of expressing the relation between an angle and its sine.

The symbol $\sin^{-1} u$ is the **inverse sine of u.** It may be read the *inverse sine of u, the anti sine of u, arc sine u,* or **an angle whose sine is u.** The last form should be used until the conception of an inverse function is perfectly clear.

Corresponding to each direct function there is an inverse function. Thus,

$$\alpha = \sin^{-1} u \text{ corresponds to } \sin\alpha = u,$$
$$\alpha = \cos^{-1} u \text{ corresponds to } \cos\alpha = u,$$
$$\alpha = \tan^{-1} u \text{ corresponds to } \tan\alpha = u, \text{ etc.}$$

CAUTION. Since $\sin^{-1} u$ is an inverse function, the -1

cannot be an exponent. It is simply part of a symbol for an inverse function. When a function is affected by -1 as an exponent, it *must* be written with parentheses, thus $(\sin x)^{-1}$.

78. Multiple values of an inverse function. It has been shown that the trigonometric functions are *single-valued* functions. Thus, if α is given, there is only one value for $\sin \alpha$. See Arts. 25 and 41.

On the contrary, the inverse functions are *multiple-valued* functions. Thus, if u is given there are an infinite number of values for $\sin^{-1} u$. See Arts. 26 and 41. A few examples will illustrate this property of inverse functions, and show that all the values of any given inverse function may be combined in a single expression.

To find all the values of $\tan^{-1}\frac{1}{\sqrt{3}}$.

$$\begin{aligned}
\tan^{-1}\frac{1}{\sqrt{3}} &= 30^\circ \\
&= 180^\circ + 30^\circ \\
&= 360^\circ + 30^\circ \\
&\cdots\cdots\cdots\cdots \\
&= -180^\circ + 30^\circ, \text{ or } -150^\circ \\
&= -360^\circ + 30^\circ, \text{ or } -330^\circ \\
&= -540^\circ + 30^\circ, \text{ or } -510^\circ \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

All these angles may be expressed by $n\,180^\circ + 30^\circ$, n being any positive or negative integer. Hence

$$\tan^{-1}\frac{1}{\sqrt{3}} = n\pi + \frac{\pi}{6}.$$

In general, if α is an angle whose tangent is u, it may be shown that

$$\tan^{-1} u = n\pi + \alpha.$$

To find all the values of $\cos^{-1}\frac{1}{2}$.

$$
\begin{aligned}
\cos^{-1}\tfrac{1}{2} &= 60^\circ \\
&= -60^\circ \\
&= 360^\circ + 60^\circ \\
&= 360^\circ - 60^\circ \\
&= 720^\circ + 60^\circ \\
&= 720^\circ - 60^\circ \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&= -360^\circ + 60^\circ \\
&= -360^\circ - 60^\circ \\
&= -720^\circ + 60^\circ \\
&= -720^\circ - 60^\circ \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{aligned}
$$

All these angles may be expressed by $n\,360^\circ \pm 60^\circ$, n being any positive or negative integer. Hence

$$\cos^{-1}\tfrac{1}{2} = 2\,n\pi \pm \frac{\pi}{3}.$$

In general, if α is an angle whose cosine is u, it may be shown that

$$\boldsymbol{\cos^{-1} u = 2\,n\pi \pm \alpha.}$$

To find all the values of $\sin^{-1}\frac{1}{2}$.

$$
\begin{aligned}
\sin^{-1}\tfrac{1}{2} &= 30^\circ \\
&= 180^\circ - 30^\circ \\
&= 360^\circ + 30^\circ \\
&= 540^\circ - 30^\circ \\
&= 720^\circ + 30^\circ \\
&= 900^\circ - 30^\circ \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&= -180^\circ - 30^\circ \\
&= -360^\circ + 30^\circ \\
&= -540^\circ - 30^\circ \\
&= -720^\circ + 30^\circ \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{aligned}
$$

All these angles may be expressed by $n\,180° + (-1)^n 30°$, n being any positive or negative integer. Hence

$$\sin^{-1}\tfrac{1}{2} = n\pi + (-)1^n\frac{\pi}{6}.$$

In general, if α is an angle whose sine is u, it may be shown that

$$\sin^{-1} u = n\pi + (-1)^n \alpha.$$

In a similar manner it may be shown that

$$\cot^{-1} u = n\pi + \alpha,$$
$$\sec^{-1} u = 2\,n\pi \pm \alpha,$$
$$\csc^{-1} u = n\pi + (-1)^n \alpha.$$

79. Principal values. The smallest numerical value of an inverse function is called its **principal value**, preference being given to positive angles in case of ambiguity.

The principal values of the inverse sine and the inverse cosecant lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$; of the inverse cosine and the inverse secant, between 0 and π; of the inverse tangent and the inverse cotangent, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The principal values of an inverse function are sometimes distinguished from the general values by the use of a capital letter.

Thus $\qquad \mathrm{Sin}^{-1}\tfrac{1}{2} = \frac{\pi}{6},$

while $\qquad \sin^{-1}\tfrac{1}{2} = n\pi + (-1)^n\frac{\pi}{6}.$

80. *To interpret* $\sin \sin^{-1} u$ *and* $\sin^{-1} \sin \alpha$.

The expression $\sin \sin^{-1} u$ is read: the sine of the angle whose sine is u. This sine is evidently u, hence

$$\sin \sin^{-1} u = u.$$

The expression $\sin^{-1}\sin\alpha$ is read: the angle whose sine is the sine of α. This angle is evidently α, hence

$$\sin^{-1}\sin\alpha = \alpha,$$

or more generally,

$$\sin^{-1}\sin\alpha = n\pi + (-1)^n\alpha.$$

Similar relations exist between any direct function and the corresponding inverse function.

Thus $\cos\cos^{-1}u = u;$

$$\cos^{-1}\cos\alpha = \alpha, \text{ or } \cos^{-1}\cos\alpha = 2n\pi \pm \alpha;$$

$$\tan\tan^{-1}u = u;$$

$$\tan^{-1}\tan\alpha = \alpha, \text{ or } \tan^{-1}\tan\alpha = n\pi + \alpha, \text{ etc.}$$

81. Application of the fundamental relations to angles expressed as inverse functions. The fundamental relations, being true for all angles, must necessarily be true when the angles are expressed as inverse functions.

Thus, letting $\alpha = \tan^{-1}u$ in the identity

$$\sin^2\alpha + \cos^2\alpha = 1,$$

we have $(\sin\tan^{-1}u)^2 + (\cos\tan^{-1}u)^2 = 1.$

Similarly $(\sin\csc^{-1}u)^2 + (\cos\csc^{-1}u)^2 = 1;$

$$(\tan\sec^{-1}u)^2 + 1 = (\sec\sec^{-1}u)^2;$$

$$\sin\cos^{-1}u = \frac{1}{\csc\cos^{-1}u}.$$

By expressing the angle of the fundamental relations as an inverse function, we may develop relations between the inverse functions.

82. Given an angle, expressed as an inverse function of *u*, to find the value of any function of the angle in terms of *u*.

By the application of one or more of the fundamental relations, it is always possible to solve the stated problem.

Several illustrations are given below. The method employed can be readily applied to the other functions.

1. *To find the value of* $\tan \cos^{-1} u$ *in terms of* u.

If $\tan \cos^{-1} u$ is expressed in terms of the cosine of $\cos^{-1} u$, the problem is solved, since $\cos \cos^{-1} u = u$.

This may be done as follows:

$$\tan \cos^{-1} u = \frac{\sin \cos^{-1} u}{\cos \cos^{-1} u} = \frac{\pm\sqrt{1-(\cos \cos^{-1} u)^2}}{u} = \frac{\pm\sqrt{1-u^2}}{u}.$$

This result may be obtained geometrically. Since u is given, it is evident that $\cos^{-1} u$ represents, among others, two positive angles, α_1 and α_2, each less than 360°.

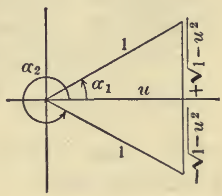
Let us assume u positive and let us construct these angles defined by $\cos^{-1} u$.

Then from the figure and the definition of the tangent,

$$\tan \alpha_1 = \frac{+\sqrt{1-u^2}}{u}$$

and

$$\tan \alpha_2 = \frac{-\sqrt{1-u_2}}{u}.$$



Since the tangent of any angle coterminal with α_1 is $\dfrac{+\sqrt{1-u^2}}{u}$, and the tangent of any angle coterminal with α_2 is $\dfrac{-\sqrt{1-u^2}}{u}$, and since $\cos^{-1} u$ represents all angles coterminal with either α_1 or α_2, we have

$$\tan \cos^{-1} u = \frac{\pm\sqrt{1-u^2}}{u}.$$

2. *To find the value of* $\sec \cot^{-1} u$ *in terms of* u.

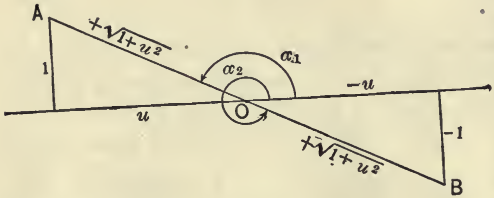
To solve this problem it is only necessary to express $\sec \cot^{-1} u$ in terms of $\cot \cot^{-1} u$.

Thus

$$\sec \cot^{-1} u = \pm\sqrt{1+(\tan \cot^{-1} u)^2} = \pm\sqrt{1+\frac{1}{(\cot \cot^{-1} u)^2}}$$

$$= \pm\sqrt{1+\frac{1}{u^2}} = \frac{\sqrt{u^2+1}}{u}.$$

This result may be obtained geometrically. Construct the angles given by $\cot^{-1} u$. Let us assume in this problem that u is negative and hence that $-u$ is positive. If the cotangent of an angle is negative the angle must terminate in either the second or fourth quadrant. Since OA and OB are the terminal lines of α_1 and α_2 respectively, and since the terminal line is always positive, we have



$$OA = OB = +\sqrt{1+u^2}.$$

Then $\quad \sec \alpha_1 = \dfrac{+\sqrt{1+u^2}}{u}$ and $\sec \alpha_2 = \dfrac{+\sqrt{1+u^2}}{-u}$,

or by considerations similar to those in the previous example, we have

$$\sec \cot^{-1} u = \frac{+\sqrt{1+u^2}}{\pm u} = \frac{\pm\sqrt{1+u^2}}{u}.$$

3. *To find the value of* $\sin \cos^{-1} u$ *in terms of* u.

We have $\sin \cos^{-1} u = \pm\sqrt{1-(\cos \cos^{-1} u)^2} = \pm\sqrt{1-u^2}$.

4. *To find the value of* $\cot \operatorname{vers}^{-1} u$ *in terms of* u.

We have

$$\cot \operatorname{vers}^{-1} u = \frac{\cos \operatorname{vers}^{-1} u}{\pm\sqrt{1-(\cos \operatorname{vers}^{-1} u)^2}}$$

$$= \frac{1-\operatorname{vers} \operatorname{vers}^{-1} u}{\pm\sqrt{1-(1-\operatorname{vers} \operatorname{vers}^{-1} u)^2}} = \frac{1-u}{\pm\sqrt{1-(1-u)^2}}$$

$$= \frac{1-u}{\pm\sqrt{2u-u^2}}.$$

83. Some inverse functions expressed in terms of other inverse functions.

1. *To express* $\cos^{-1} u$ *in terms of an inverse tangent.*

From $\tan \cos^{-1} u = \frac{\pm\sqrt{1-u^2}}{u}$, (Art. 82, prob. 1) by taking the inverse tangent of each member (Art. 80), there results

$$\cos^{-1} u = \tan^{-1}\frac{\pm\sqrt{1-u^2}}{u}.$$

2. *To express the* $\cot^{-1} u$ *in terms of an inverse secant.*

From $\sec \cot^{-1} u = \frac{\pm\sqrt{1+u^2}}{u}$ (Art. 82, prob. 2) there results

$$\cot^{-1} u = \sec^{-1}\frac{\pm\sqrt{1+u^2}}{u}.$$

3. Similarly from $\sin \cos^{-1} u = \pm\sqrt{1-u^2}$ there results

$$\cos^{-1} u = \sin^{-1}(\pm\sqrt{1-u^2}).$$

4. Also from $\cot \operatorname{vers}^{-1} u = \frac{1-u}{\pm\sqrt{2u-u^2}}$ there results

$$\operatorname{vers}^{-1} u = \cot^{-1}\frac{1-u}{\pm\sqrt{2u-u^2}}.$$

By the method exemplified in Arts. 82 and 83 it is possible to express any inverse function in terms of any other inverse function.

In applying the above formulas care must be exercised in selecting the angles, since each inverse function represents an infinite number of angles and one member of the equation may represent angles not represented by the other. For example, in problem 1, if u be positive the $\cos^{-1} u$ represents angles terminating in the first and fourth quadrants; but $\tan^{-1}\frac{\pm\sqrt{1-u^2}}{u}$ represents angles terminating in the second and third quadrants as well as angles terminating in the first and fourth quadrants.

84. Some relations between inverse functions derived from the formulas for double angles, half angles, and the addition formulas.

The general method applicable to this class of problems will be illustrated by a few examples.

1. *To express* $\cos(2\sec^{-1}u)$ *in terms of* u.

$$\cos(2\sec^{-1}u) = 2(\cos\sec^{-1}u)^2 - 1 \quad \text{Art. 71, Eq. 4.}$$
$$= \frac{2}{(\sec\sec^{-1}u)^2} - 1 = \frac{2}{u^2} - 1.$$

From this relation it follows that,

$$2\sec^{-1}u = \cos^{-1}\left(\frac{2}{u^2} - 1\right).$$

2. *To express* $\tan(\tfrac{1}{2}\cos^{-1}u)$ *in terms of* u.

$$\tan(\tfrac{1}{2}\cos^{-1}u) = \pm\sqrt{\frac{1-\cos\cos^{-1}u}{1+\cos\cos^{-1}u}} = \pm\sqrt{\frac{1-u}{1+u}}.$$

Art. 72, Eq. 3.

From this relation it follows that,

$$\tfrac{1}{2}\cos^{-1}u = \tan^{-1}\left(\pm\sqrt{\frac{1-u}{1+u}}\right).$$

3. *To express* $\sin(\sin^{-1}u + \cos^{-1}v)$ *in terms of* u *and* v.

$$\sin(\sin^{-1}u + \cos^{-1}v) = \sin\sin^{-1}u \cdot \cos\cos^{-1}v + \cos\sin^{-1}u \cdot \sin\cos^{-1}v$$

Art. 62, Eq. 1.

$$= uv \pm \sqrt{1-u^2}\sqrt{1-v^2}.$$

From this relation it follows that,

$$\sin^{-1}u + \cos^{-1}v = \sin^{-1}(uv \pm \sqrt{1-u^2}\sqrt{1-v^2}).$$

85. EXAMPLES

Find the value of each of the following:

1. $\sin^{-1}\tfrac{1}{2}\sqrt{3}$.

2. $\cos^{-1}(-\tfrac{1}{2}\sqrt{3})$.

3. $\tan^{-1}1$.

4. $\tan\cot^{-1}4$.

5. $\sin\cot^{-1}4$.

Express the following in terms of u and v:

6. $\cos\cot^{-1}u$.

7. $\sec\cot^{-1}u$.

8. $\csc\cot^{-1}u$.

9. $\cos\sin^{-1}u$.

10. $\cos\tan^{-1}u$.

11. $\cos\sec^{-1}u$.

12. $\cos\csc^{-1}u$.

13. $\sin(2\cos^{-1}u)$.

14. $\tan(2\tan^{-1}u)$.

15. $\tan(2\sec^{-1}u)$.

16. $\tan(2\cos^{-1}u)$.

17. $\cos(2\cos^{-1}u)$.

18. $\cos(2\sin^{-1}u)$.

19. $\cos(2\tan^{-1}u)$.

20. $\sin(\sin^{-1}u+\sin^{-1}v)$.

21. $\cos(\sin^{-1}u+\sin^{-1}v)$.

22. $\tan(\tan^{-1}u+\cot^{-1}v)$.

23. $\tan(\sec^{-1}u+\sec^{-1}v)$.

24. $\cos(\sec^{-1}u+\csc^{-1}v)$.

25. $\sin(\frac{1}{2}\cos^{-1}u)$.

26. $\cos(\frac{1}{2}\cos^{-1}u)$.

27. $\sin(\frac{1}{2}\sec^{-1}u)$.

28. $\sin(\frac{1}{2}\tan^{-1}u)$.

Find x in terms of a.

29. $\tan^{-1}x=\cot^{-1}a$.

30. $\sin^{-1}x=\tan^{-1}a$.

31. $\cos^{-1}x=2\sin^{-1}a$.

32. $\cos^{-1}x=\sin^{-1}a+\tan^{-1}a$.

33. $\sin^{-1}x=\frac{1}{2}\sec^{-1}a$.

Find x in terms of a and b.

34. $\tan^{-1}x=\sin^{-1}a+\sin^{-1}b$.

35. $\cos^{-1}x=\sec^{-1}a-\sec^{-1}b$.

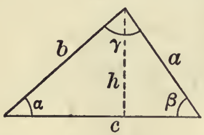
36. $\sin^{-1}x=2\cos^{-1}a+\frac{1}{2}\cos^{-1}b$.

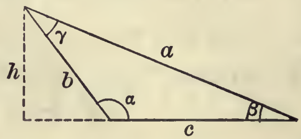
CHAPTER VIII

OBLIQUE TRIANGLE

86. In the present chapter we develop the formulas by means of which a triangle may be completely solved when any three independent parts are given.

87. Law of sines. *In a plane triangle any two sides are to each other as the sines of the opposite angles.*





Let a, b, c be the sides of a triangle and α, β, γ the angles opposite these sides, respectively.

From the vertex of γ draw h perpendicular to the side c, or c produced.

Then, for each figure, $\sin\alpha = \dfrac{h}{b}$, Art. 21

and $\sin\beta = \dfrac{h}{a}$. Art. 29

Dividing the first equation by the second, we have

$$\frac{a}{b} = \frac{\sin\alpha}{\sin\beta}.$$

In a similar manner, dropping a perpendicular from the vertex of α, it is seen that

$$\frac{b}{c} = \frac{\sin\beta}{\sin\gamma}.$$

The last two equations may be written

$$\frac{\boldsymbol{a}}{\sin\boldsymbol{\alpha}} = \frac{\boldsymbol{b}}{\sin\boldsymbol{\beta}} = \frac{\boldsymbol{c}}{\sin\boldsymbol{\gamma}}.$$

88. Law of tangents. *The tangent of half the difference of two angles is to the tangent of half their sum, as the difference of the corresponding opposite sides is to their sum.*

From the law of sines we have

$$\frac{a}{b}=\frac{\sin\alpha}{\sin\beta}.$$

By division and composition, this becomes

$$\frac{a-b}{a+b}=\frac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta},$$

which reduces to

$$\frac{a-b}{a+b}=\frac{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)} \quad \text{Art. 73}$$

$$=\cot\tfrac{1}{2}(\alpha+\beta)\tan\tfrac{1}{2}(\alpha-\beta)$$

or

$$\boldsymbol{\frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}=\frac{a-b}{a+b}}. \tag{1}$$

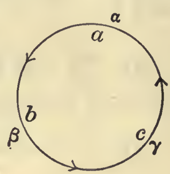
Similarly

$$\boldsymbol{\frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}=\frac{b-c}{b+c}}, \tag{2}$$

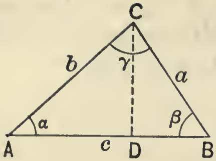
and

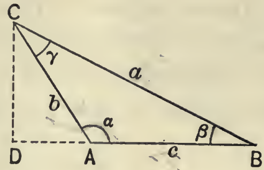
$$\boldsymbol{\frac{\tan\frac{1}{2}(\gamma-\alpha)}{\tan\frac{1}{2}(\gamma+\alpha)}=\frac{c-a}{c+a}}. \tag{3}$$

89. Cyclic interchange of letters. Each formula pertaining to the oblique triangle gives rise to two other formulas of the same type by a cyclic interchange of letters. A cyclic interchange of letters may be accomplished by arranging the letters around the circumference of a circle as in the figure, and then replacing each letter by the next in order as indicated by the arrows. Thus by this cyclic interchange of letters formula (1) of Art. 88 gives rise to formula (2); likewise formula (2) gives rise to formula (3).



90. Law of cosines. *The square of any side of a plane triangle is equal to the sum of the squares of the other sides minus twice the product of those sides into the cosine of the included angle.*





For each figure

$$AB = AD + DB, \quad \text{or} \quad DB = AB - AD. \qquad \text{Art. 2}$$

But $\quad AB = c, \ AD = b\cos\alpha, \quad$ hence $DB = c - b\cos\alpha.$

Also $\quad DC = b\sin\alpha.$

From the right triangle CDB we have

$$\overline{BC}^2 = \overline{DC}^2 + \overline{DB}^2.$$

Substituting values, this equation becomes

$$\begin{aligned} a^2 &= (b\sin\alpha)^2 + (c - b\cos\alpha)^2 \\ &= b^2\sin^2\alpha + c^2 - 2\,bc\cos\alpha + b^2\cos^2\alpha \\ &= b^2(\sin^2\alpha + \cos^2\alpha) + c^2 - 2\,bc\cos\alpha. \end{aligned}$$

Hence $\quad \boldsymbol{a^2 = b^2 + c^2 - 2\,bc\cos\alpha.}$

Similarly $\quad \boldsymbol{b^2 = c^2 + a^2 - 2\,ca\cos\beta,}$

and $\quad \boldsymbol{c^2 = a^2 + b^2 - 2\,ab\cos\gamma.}$

91. *To find the sine of half an angle of a plane triangle in terms of the sides of the triangle.*

From equation (1), Art. 72,

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}},$$

whence
$$2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha. \tag{1}$$

From the cosine law
$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2\,bc}. \tag{2}$$

From equations (1) and (2)

$$2\sin^2\frac{\alpha}{2} = 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc}. \tag{3}$$

Let $$a+b+c=2s. \tag{4}$$

Subtracting $2a$, $2b$, and $2c$ from each member of (4) we have respectively

$$-a+b+c=2(s-a)$$

$$a-b+c=2(s-b)$$

$$a+b-c=2(s-c).$$

Then equation (3) becomes

$$\sin\frac{\alpha}{2} = +\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Similarly $$\sin\frac{\beta}{2} = +\sqrt{\frac{(s-c)(s-a)}{ca}},$$

and $$\sin\frac{\gamma}{2} = +\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

In these formulas the positive sign is given to the radical, since it is known that half an angle of any plane triangle is less than 90°. The same applies to the corresponding formulas for the cosine and the tangent of half an angle.

92. *To find the cosine of half an angle of a plane triangle in terms of the sides of the triangle.*

From equation (2), Art. 72, we have

$$2\cos^2\frac{\alpha}{2}=1+\cos\alpha.$$

Then $$2\cos^2\frac{\alpha}{2}=1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(a+b+c)(-a+b+c)}{2bc}$$

and $$\cos^2\frac{\alpha}{2}=\frac{s(s-a)}{bc}.$$

Hence $$\cos\frac{\alpha}{2}=+\sqrt{\frac{s(s-a)}{bc}}.$$

Similarly $$\cos\frac{\beta}{2}=+\sqrt{\frac{s(s-b)}{ca}},$$

and $$\cos\frac{\gamma}{2}=+\sqrt{\frac{s(s-c)}{ab}}.$$

93. *To find the tangent of half an angle of a plane triangle in terms of the sides of the triangle.*

Since $$\tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}},$$

$$\tan\frac{\alpha}{2}=+\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \tag{1}$$

Similarly $$\tan\frac{\beta}{2}=+\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \tag{2}$$

and $$\tan\frac{\gamma}{2}=+\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \tag{3}$$

Formula (1) may be written

$$\tan\frac{\alpha}{2}=\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

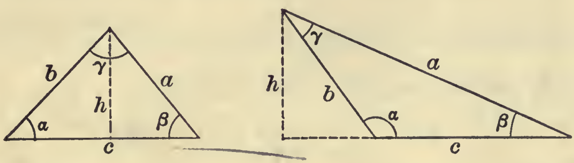
Letting
$$r=+\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},\tag{4}$$

$$\tan\frac{\alpha}{2}=\frac{r}{s-a}.\tag{5}$$

Similarly
$$\tan\frac{\beta}{2}=\frac{r}{s-b},\tag{6}$$

and
$$\tan\frac{\gamma}{2}=\frac{r}{s-c}.\tag{7}$$

94. *To find the area of a plane triangle in terms of two sides and the included angle.*



Let A be the area and h the altitude. Then in each figure

$$A=\tfrac{1}{2}ch,$$

and
$$h=b\sin\alpha.$$

Therefore
$$\boldsymbol{A=\tfrac{1}{2}bc\sin\alpha.}$$

Similarly
$$\boldsymbol{A=\tfrac{1}{2}ca\sin\beta,}$$

and
$$\boldsymbol{A=\tfrac{1}{2}ab\sin\gamma.}$$

95. *To find the area of a plane triangle in terms of a side and two adjacent angles.*

From Art. 94, $A=\frac{1}{2}bc\sin\alpha.$

From the sine law $b = \dfrac{c \sin \beta}{\sin \gamma}$,

$$A = \frac{c^2 \sin \alpha \sin \beta}{2 \sin \gamma}.$$

Then since $\alpha + \beta + \gamma = 180°$,

$$\boldsymbol{A = \frac{c^2 \sin \alpha \sin \beta}{2 \sin (\alpha + \beta)}}.$$

96. *To find the area of a plane triangle in terms of the three sides.*

From Art. 94.

$$A = \tfrac{1}{2}\, bc \sin \alpha.$$

Since $\sin \alpha = 2 \sin \dfrac{\alpha}{2} \cos \dfrac{\alpha}{2}$,

$$A = bc \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}.$$

Substituting the values of $\sin \dfrac{\alpha}{2}$ and $\cos \dfrac{\alpha}{2}$ as found in Arts. 91 and 92, we have, after reduction,

$$\boldsymbol{A = \sqrt{s(s-a)(s-b)(s-c)}}.$$

97. Formulas for solving an oblique triangle. The formulas developed in the present chapter are sufficient to solve a plane triangle when three independent parts are given.

The law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

is used when two of the given parts are an angle and the opposite side.

The law of tangents

$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(\alpha + \beta)$$

is used when two sides and the included angle are given.

If a, b, and γ are the given parts, $\frac{1}{2}(\alpha+\beta)$ is obtained from the relation $\alpha+\beta+\gamma=180°$. The formula then gives the value of $\frac{1}{2}(\alpha-\beta)$, which, united with the value of $\frac{1}{2}(\alpha+\beta)$, gives α and β.

The laws of half angles

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

$$\tan\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{r}{s-a},$$

are used when the three sides are given. The last formula is the most accurate, since the tangent varies more rapidly than either the sine or the cosine. The formula involving r is advantageous when *all* the angles are to be computed.

The law of cosines

$$a^2=b^2+c^2-2\,bc\cos\alpha$$

may be used to determine the third side when two sides and an angle are given. It may also be used to determine an angle when the three sides are given.

This formula is used with *natural functions,* not being adapted to logarithmic computation.

98. Check formulas. Any formula which was not used in the solution of the triangle may be used as a check formula.

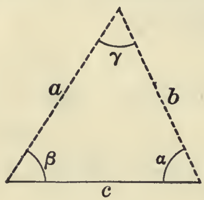
The relation $\alpha+\beta+\gamma=180°$ cannot be used as a check when a problem has been solved by the law of tangents, since the law of tangents involves this relation.

When two equations from the sine law have been used to find two elements of a triangle, the third equation from the sine law cannot be used as a check, since the first two equations involve the third.

99. Illustrative problems.

1. Given $c = 127.32$ to find a
$\alpha = 71^\circ\, 58'\, 22''$ b
$\beta = 52^\circ\, 19'\, 40''$ γ.

SOLUTION. *Construction and estimates.*



$a = 140$
$b = 120$
$\gamma = 60^\circ$

Outline

$$a = \frac{c \sin \alpha}{\sin \gamma}$$

c	
α	
β	
$\alpha + \beta$	
γ	
$\log \sin \alpha$	
$\log c$	
$\text{colog} \sin \gamma$	
$\log a$	
a	

Check

$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(\alpha + \beta).$$

$a - b$	
$a + b$	
$\frac{1}{2}(\alpha + \beta)$	
$\log a - b)$	
$\text{colog}\ (a + b)$	
$\log \tan \frac{1}{2}(\alpha + \beta)$	
$\log \tan \frac{1}{2}(\alpha - \beta)$	
$\frac{1}{2}(\alpha - \beta)$	
$\frac{1}{2}(\alpha + \beta)$	
α	
β	

$$b = \frac{c \sin \beta}{\sin \gamma}$$

$\log \sin \beta$	
$\log c$	
$\text{colog} \sin \gamma$	
$\log b$	
b	

Or, more compactly,

$\log a$		$\log \sin \alpha$	
$\log a$	$\log b$	$\log c$	
$\log a$	$\log b$	$\text{colog} \sin \gamma$	
	$\log b$	$\log \sin \beta$	
		$\log a$	
		$\log b$	
		a	
		b	

Filling in the above outline, the completed work appears as follows:

$$a = \frac{c \sin \alpha}{\sin \gamma}$$

c	127.32
α	71° 58′ 22″
β	52° 19′ 40″
$\alpha + \beta$	124° 18′ 2″
γ	55° 41′ 58″
$\log \sin \alpha$	9.97814 − 10
$\log c$	2.10490
$\operatorname{colog} \sin \gamma$	0.08297
$\log a$	2.16601
a	146.56

$$b = \frac{c \sin \beta}{\sin \gamma}$$

$\log \sin \beta$	9.89846 − 10
$\log c$	2.10490
$\operatorname{colog} \sin \gamma$	0.08297
$\log b$	2.08633
b	121.99

Check

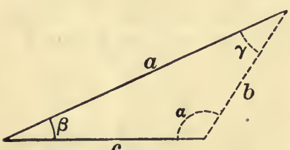
$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(\alpha + \beta)$$

$a - b$	24.57
$a + b$	268.55
$\frac{1}{2}(\alpha + \beta)$	62° 9′ 1″
$\log (a - b)$	1.39041
$\operatorname{colog} (a + b)$	7.57098
$\log \tan \frac{1}{2}(\alpha + \beta)$	0.27708
$\log \tan \frac{1}{2}(\alpha - \beta)$	9.23847 − 10
$\frac{1}{2}(\alpha - \beta)$	9° 49′ 28″
$\frac{1}{2}(\alpha + \beta)$	62° 9′ 1″
α	71° 58′ 29″
β	52° 19′ 33″

Or, in the more compact form,

$\log a$	$\log \sin \alpha$	9.97814 − 10
$\log a$, $\log b$	$\log c$	2.10490
$\log a$, $\log b$	$\operatorname{colog} \sin \gamma$	0.08297
$\log b$	$\log \sin \beta$	9.89846
	$\log a$	2.16601
	$\log b$	2.08633
	a	146.56
	b	121.99

2. Given $a = 1674.3$, $c = 1021.7$, $\beta = 28° 44' 39''$ to find α, γ, b.



Estimates

$\alpha = 120°$
$\gamma = 30°$
$b = 900.$

To Here

$$\tan \tfrac{1}{2}(\alpha - \gamma) = \frac{a - c}{a + c} \tan \tfrac{1}{2}(\alpha + \gamma)$$

a	1674.3
c	1021.7
β	28° 44′ 39″
$a - c$	652.6
$a + c$	2696.0
$\alpha + \gamma$	151° 15′ 21″
$\frac{1}{2}(\alpha + \gamma)$	75° 37′ 40″
$\log (a - c)$	2.81465
$\operatorname{colog} (a + c)$	6.56928 − 10
$\log \tan \frac{1}{2}(\alpha + \gamma)$	0.59135
$\log \tan \frac{1}{2}(\alpha - \gamma)$	9.97528 − 10
$\frac{1}{2}(\alpha - \gamma)$	43° 22′ 12″
$\frac{1}{2}(\alpha + \gamma)$	75° 37′ 40″
α	118° 59′ 52″
γ	32° 15′ 28″

$$b = \frac{c \sin \beta}{\sin \gamma}$$

$\log c$	3.00932
$\log \sin \beta$	9.68205 − 10
$\operatorname{colog} \sin \gamma$	0.27268
$\log b$	2.96405
b	920.56

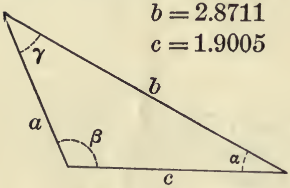
Check

$$b = \frac{a \sin \beta}{\sin \alpha}$$

$\log a$	3.22384
$\log \sin \beta$	9.68205 − 10
$\operatorname{colog} \sin \alpha$	0.05817
$\log b$	2.96406

3. Given $a = 1.4932$, $b = 2.8711$, $c = 1.9005$ to find α, β, γ.

Estimates
$\alpha = 25°$
$\beta = 120°$
$\gamma = 35°$



$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\tan \frac{\alpha}{2} = \frac{r}{s - a}$$

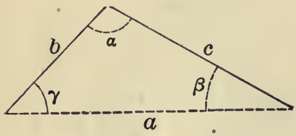
$$\tan \frac{\beta}{2} = \frac{r}{s - b}$$

$$\tan \frac{\gamma}{2} = \frac{r}{s - c}$$

a	1.4932
b	2.8711
c	1.9005
$2s$	6.2648
s	3.1324
$s - a$	1.6392
$s - b$	0.2613
$s - c$	1.2319
s	3.1324

$\log (s - a)$	0.21463
$\log (s - b)$	9.41714 − 10
$\log (s - c)$	0.09058
$\operatorname{colog} s$	9.50412 − 10
$\log r^2$	19.22647 − 20
$\log r$	9.61324 − 10
$\log \tan \frac{\alpha}{2}$	9.39861 − 10
$\log \tan \frac{\beta}{2}$	0.19610
$\log \tan \frac{\gamma}{2}$	9.52266 − 10
$\frac{\alpha}{2}$	14° 3′ 26″
$\frac{\beta}{2}$	57° 31′ 2″
$\frac{\gamma}{2}$	18° 25′ 34″
α	28° 6′ 52″
β	115° 2′ 4″
γ	36° 51′ 8″
Check: $\alpha + \beta + \gamma$	180° 00′ 4″

4. Given $b = .0060041$ to find β
$c = .0093284$ α
$\gamma = 44°\,47'\,58''$ a.



Estimates

$\beta = 30°$
$\alpha = 105°$
$a = .012.$

Check

$$\sin\beta = \frac{b\sin\gamma}{c}$$

b	.0060041
c	.0093284
γ	$44°\,47'\,58''$
$\log b$	$7.77845 - 10$
$\operatorname{colog} c$	2.03019
$\log\sin\gamma$	$9.84796 - 10$
$\log\sin\beta$	$9.65660 - 10$
β	$26°\,58'\,12''$
$\beta + \gamma$	$71°\,46'\,10''$
α	$108°\,13'\,50''$

$$a = \frac{c\sin\alpha}{\sin\gamma}$$

$\log c$	$7.96981 - 10$
$\operatorname{colog}\sin\gamma$	0.15204
$\log\sin\alpha$	$9.97764 - 10$
$\log a$	$8.09949 - 10$
a	0.012574

$$\tan\tfrac{1}{2}(\alpha - \gamma) = \frac{a - c}{a + c}\tan\tfrac{1}{2}(\alpha + \gamma)$$

a	.012574
c	.0093284
$a - c$	.0032456
$a + c$	.021902
$\alpha + \gamma$	$153°\;\;1'\,48''$
$\frac{1}{2}(\alpha + \gamma)$	$76°\,30'\,54''$
$\log(a - c)$	$7.51129 - 10$
$\operatorname{colog}(a + c$	1.65952
$\log\tan\frac{1}{2}(\alpha + \gamma)$	0.62015
$\log\tan\frac{1}{2}(\alpha - \gamma)$	$9.79096 - 10$
$\frac{1}{2}(\alpha - \gamma)$	$31°\,42'\,51$
$\frac{1}{2}(\alpha + \gamma)$	$76°\,30'\,54''$
α	$108°\,13'\,45''$
γ	$44°\,48'\;\;3''$

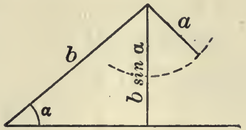
100. The ambiguous case. When two sides and an angle opposite one of them are given, the triangle may admit of no solution, of one solution, or of two solutions.

Let a, b, α be given. The formula

$$\sin\beta = \frac{b\sin\alpha}{a}$$

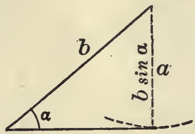
determines β.

If the calculated $\sin\beta$ is greater than 1, there can be no solution.



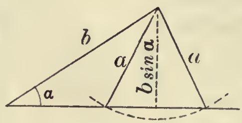
$a < b \sin\alpha$

If the calculated $\sin\beta$ equals 1, $\beta = 90°$ and there is one solution.



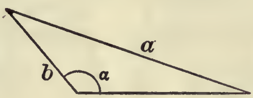
$a = b \sin\alpha$

If the calculated $\sin\beta$ is less than 1, two supplementary values of β are determined, giving two solutions unless the larger value of β plus α is equal to or greater than 180°.



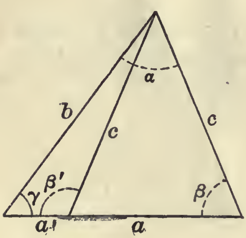
$a > b \sin\alpha$
$a < b$

$a \overset{=}{>} b$



$a > b$

Given $\quad b = 420 \quad$ to find β
$c = 389.73 \quad \alpha$
$\gamma = 53° 47' 20'' \quad a.$



Estimates	*Two solutions*
$\beta = 65°$	$\beta' = 115°$
$\alpha = 60°$	$\alpha' = 10°$
$a = 390.$	$a' = 90.$

$$\sin \beta = \frac{b \sin \gamma}{c}$$

b	420
c	389.73
γ	53° 47′ 20″
$\log b$	2.62325
$\operatorname{colog} c$	7.40924 − 10
$\log \sin \gamma$	9.90679 − 10
$\log \sin \beta$	9.93928 − 10
β	60° 24′ 9″
$180° - \beta = \beta'$	119° 35′ 51″
$\beta + \gamma$	114° 11′ 29″
$\beta' + \gamma$	173° 22′ 11″
α	65° 48′ 31″
α'	6° 37′ 49″

Check of 1st solution

$$\tan \tfrac{1}{2}(\alpha - \gamma) = \frac{a - c}{a + c} \tan \tfrac{1}{2}(\alpha + \gamma)$$

a	440.61
c	389.73
$a - c$	50.88
$a + c$	830.34
$\alpha + \gamma$	119° 35′ 51″
$\frac{1}{2}(\alpha + \gamma)$	59° 47′ 50″
$\log (a - c)$	1.70655
$\operatorname{colog} (a + c)$	7.08074 − 10
$\log \tan \frac{1}{2}(\alpha + \gamma)$	0.23502
$\log \tan \frac{1}{2}(\alpha - \gamma)$	9.02231 − 10
$\frac{1}{2}(\alpha - \gamma)$	6° 0′ 34″
$\frac{1}{2}(\alpha + \gamma)$	59° 47′ 50″
α	65° 48′ 24″
γ	53° 47′ 16″

$$a = \frac{c \sin \alpha}{\sin \gamma}$$

$$a' = \frac{c \sin \alpha'}{\sin \gamma}$$

$\log \sin \alpha$	9.96008 − 10
$\log c$	2.59076
$\operatorname{colog} \sin \gamma$	0.09321
$\log \sin \alpha'$	9.06244 − 10
$\log a$	2.64405
$\log a'$	1.74641
a	440.61
a'	55.771

Check of 2d solution

$$\tan \tfrac{1}{2}(\gamma - \alpha') = \frac{c - a'}{c + a'} \tan \tfrac{1}{2}(\gamma + \alpha')$$

c	389.73
a'	55.771
$c - a'$	333.96
$c + a'$	445.50
$\gamma + \alpha'$	60° 25′ 9″
$\frac{1}{2}(\gamma + \alpha')$	30° 12′ 34″
$\log (c - a')$	2.52370
$\operatorname{colog} (c + a')$	7.35115 − 10
$\log \tan \frac{1}{2}(\gamma + \alpha')$	9.76509 − 10
$\log \tan \frac{1}{2}(\gamma - \alpha')$	9.63994 − 10
$\frac{1}{2}(\gamma - \alpha')$	23° 34′ 45″
$\frac{1}{2}(\gamma + \alpha')$	30° 12′ 34″
γ	53° 47′ 19″
α'	6° 37′ 49″

FIRST SOLUTION

$\beta = 60° 24' 9''$
$\alpha = 65° 48' 31''$
$a = 440.61$

SECOND SOLUTION

$\beta' = 119° 35' 51''$
$\alpha' = 6° 37' 49''$
$a' = 55.771$

101. EXAMPLES

Solve the following triangles, using a three-place table or a slide-rule.

1. $a = 26$, $\alpha = 53°$, $\beta = 49°$

2. $a = 48$, $\beta = 61°$, $\gamma = 69°$

3. $\alpha = 73°$, $\beta = 59°$, $b = 55$

4. $\alpha = 69°$, $a = 80$, $b = 64$

5. $\alpha = 54° 40'$, $b = 122$, $c = 110$

6. $a = 163$, $b = 241$, $\gamma = 34° 20'$

7. $a = 42$, $b = 28$, $\gamma = 72°$

8. $b = 115$, $c = 96$, $\alpha = 110°$

9. $a = 51$, $b = 63$, $c = 70$

10. $a = 8.03$, $b = 6.42$, $c = 7.15$

11. $\alpha = 20°$, $a = 15$, $b = 25$

12. $\gamma = 43° 50'$, $a = .34$, $c = .30$

Solve the following triangles:

13. $b = 63.67$, $\beta = 100° 10'$, $\alpha = 40° 0' 10''$

14. $b = 20.007$, $\alpha = 40° 27' 30''$, $\gamma = 42° 30' 15''$

15. $a = 238.61$, $b = 216.77$, $c = 98.435$

16. $a = 8.0038$, $b = 4.6259$, $c = 4.3167$

17. $b = .76328$, $c = 2.4359$, $\gamma = 120° 46' 18''$

18. $b = 85.249$, $c = 105.63$, $\alpha = 50° 40' 24''$

19. $a = 1.4562$
$c = .45296$
$\beta = 74° 19' 38''$

20. $a = 83.831$
$b = 56.479$
$c = 74.025$

21. $a = 2.1469$
$b = 3.2824$
$c = 4.0026$

22. $c = 7.2693$
$a = .54871$
$\alpha = 5° 41' 30''$

23. $b = .06532$
$c = .01846$
$\gamma = 8° 0' 20''$

24. $b = 10.246$
$c = 18.075$
$\beta = 33° 30' 5''$

Hand in

25. $a = 764.38$
$\alpha = 143° 18' 31''$
$\beta = 13° 34' 26''$

26. $a = 962.27$
$b = 637.34$
$c = 655.80$

Find the areas of the following triangles:

27. $a = 15$
$b = 20$
$c = 25$

28. $a = 172$
$b = 103$
$c = 141$

29. $a = 20.46$
$b = 19.72$
$c = 15.04$

30. $a = 18.3$
$b = 22.4$
$\gamma = 32°$

31. $b = 3.46$
$c = 4.09$
$\alpha = 56° 10'$

32. $c = 435.3$
$a = 289.6$
$\beta = 31° 7'$

33. $a = 48$
$\beta = 26°$
$\gamma = 43°$

34. $b = 10.34$
$\alpha = 83° 22'$
$\gamma = 60° 40'$

35. The horizontal distance from a point on top of a tower to a distant flagpole is 468 ft. The angle of elevation of the top of the flagpole is 5° 8′ 30″. The angle of depression of the foot of the pole is 15° 36′. Find the height of the flagpole.

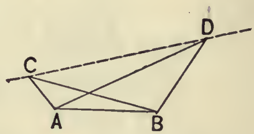
36. A tower is situated on a hill which inclines at an angle of 23° 19′ 10″ to the horizontal. The angle of elevation of the top of the tower, from a point on the hillside, was measured and found to be 43° 39′ 50″. At a point 75.5 ft. farther down, the angle of elevation was found to be 39° 23′ 20″. How high is the tower?

37. From a point 5 miles from one end of a lake and 3 miles from the other end, the lake subtends an angle of 47° 34′ 30″. Find the length of the lake.

38. Find the altitudes of a triangle whose sides are 125.4, 230.6, and 179.8.

39. A flagpole stands on the summit of a hill. The hill inclines 32° 18′ 20″ to the horizontal. At a point 25.5 ft. from the base of the flagpole, measured along the incline, the angle subtended by the flagpole was found to be 41° 24′. Find the height of the flagpole.

40. Two observers, A and B, stationed 4000 ft. apart, at the same instant observe the angles BAC and CBA to an automobile traveling on a straight road. Three minutes later they measure the angles DAB and ABD to the second position of the automobile.

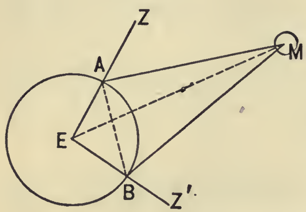


$$\text{If } \angle BAC = 136°\ 28',$$
$$\angle CBA = 32°\ 8',$$
$$\angle DAB = 40°\ 12',$$
$$\angle ABD = 118°\ 44',$$

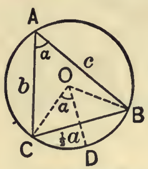
what is the rate of the automobile?

41. A man wishes to measure the length of a lake from his position on a hill top 185 ft. above the level of the lake. He finds the angles of depression of the ends of the lake to be 6° 18′ and 2° 30′. The angle subtended by the lake, formed by the two lines of sight, is 66° 27′. Find the length of the lake.

42. The angles of elevation of a cloud, directly above a straight road, from two points of the road on opposite sides of the cloud, are 78° 15′ 20″ and 59° 47′ 40″. Find the height of the cloud, the distance between the two points of observation being 5000 ft.



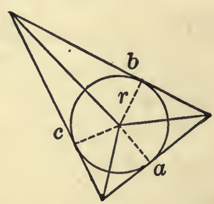
43. Two observers at A and B, whose longitudes are the same, simultaneously observe the moon and find the angles ZAM and $Z'BM$ to be 35° 2′ 20″ and 51° 17′ 10″ respectively. The latitude of A (Greenwich) is N. 51° 17′ 15″ and the latitude of B (Cape of Good Hope) is S. 33° 45′ 16″. The moon is in the plane determined by A, E, and B. Find EM, the distance from the center of the earth to the moon, the radius of the earth being 3960 miles.



44. Show that twice the radius, $2R$, of a circle circumscribing a triangle is given by the equations

$$2R = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Suggestion. $\angle BAC = \angle DOC$.



45. Find the radius of a circle inscribed in a triangle whose sides are given.

Solution. Representing the area of the triangle by A, we have

$$A = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = rs.$$

But $\quad A = \sqrt{s(s-a)(s-b)(s-c)}$. Art. 96.

Therefore $\quad r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$. See Art. 93.

MISCELLANEOUS EXERCISES

102. **1.** An angle of 4 radians, having its vertex at the center of a circle, intercepts an arc of 7 inches; find the radius of the circle.

2. Express the following in degrees: $\frac{\pi}{3}$ radians, $\frac{2\pi}{5}$ radians, $\frac{3\pi}{4}$ radians.

3. Express the following in radian measure: 40°, 55°, 38°, 52° 16′.

4. Reduce $\frac{3}{4}$ radians to degrees.

5. Find the number of degrees in a central angle which intercepts an arc of 5 feet in a circle whose radius is 8 feet.

6. An arc of 12 inches subtends a central angle of 50°; find the radius of the circle.

7. The number of minutes in an angle is $7\frac{1}{2}$ times the number of degrees in its supplement; find the number of radians in the angle.

8. An angle exceeds another by $\frac{\pi}{8}$ radians, and their sum is 160°; express each angle in radian measure.

9. The angles of a triangle are to each other as 2 : 3 : 4; express each angle in radian measure.

10. The angles of a triangle are in arithmetical progression, and the mean angle is twice the smallest; express each angle in radian measure.

11. The circumference of a circle is divided into 7 parts in arithmetical progression, the greatest part being 10 times the least; express in radians the angle which each arc subtends at the center.

12. An arc of 40° on a circle whose radius is 6 inches is equal in length to an arc of 25° on another circle; what is the radius of the latter circle?

Prove each of the following:

13. $\sin\alpha\cos\alpha = \sin^3\alpha\cos\alpha + \cos^3\alpha\sin\alpha.$

14. $\cot\alpha\csc\alpha = \dfrac{1}{\sec\alpha - \cos\alpha}.$

15. $\dfrac{1+\tan^2\alpha}{1+\cot^2\alpha} = \dfrac{\sin^2\alpha}{\cos^2\alpha}.$

16. $\sin\beta + \dfrac{\sin\beta}{\cot\beta - 1} = \dfrac{\cos\beta}{1-\tan\beta} - \cos\beta.$

17. $\sin^6\alpha + \cos^6\alpha = 1 - 3\sin^2\alpha\cos^2\alpha.$

18. $\cot\alpha - \tan\alpha = \csc\alpha\sec\alpha\,(1-2\sin^2\alpha).$

19. $\dfrac{1-\tan\beta}{1+\tan\beta} = \dfrac{\cot\beta - 1}{\cot\beta + 1}.$

20. $\sec^4\alpha - \tan^4\alpha = 2\sec^2\alpha - 1.$

21. $\tan 15° = 2 - \sqrt{3}.$

22. $\sin 4x = 4(\cos^3 x\sin x - \sin^3 x\cos x).$

23. $\cos 4x = 4\cos^4 x + 4\sin^4 x - 3.$

24. $\sin 5x = 16\sin^5 x - 20\sin^3 x + 5\sin x.$

25. $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x.$

26. $$\begin{aligned}\sin(\alpha+\beta+\gamma) = {} & \sin\alpha\cos\beta\cos\gamma \\ & + \cos\alpha\sin\beta\cos\gamma \\ & + \cos\alpha\cos\beta\sin\gamma \\ & - \sin\alpha\sin\beta\sin\gamma.\end{aligned}$$

27. $\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\beta\tan\gamma - \tan\gamma\tan\alpha - \tan\alpha\tan\beta}.$

28. $2\sin(\alpha-\beta)\cos\alpha=\sin(2\alpha-\beta)-\sin\beta.$

29. $\cos 2\alpha=\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}.$

30. $\sin\left(\dfrac{\pi}{4}+\theta\right)=\dfrac{\sin\theta+\cos\theta}{\sqrt{2}}.$

31. $\cos\left(\dfrac{\pi}{4}-\theta\right)=\dfrac{\sin\theta+\cos\theta}{\sqrt{2}}.$

32. $\tan\left(\dfrac{\pi}{4}+\theta\right)=\dfrac{1+\tan\theta}{1-\tan\theta}.$

33. $\tan\alpha+\tan\beta=\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}.$

34. $\cot\alpha+\cot\beta=\dfrac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}.$

35. $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta.$

36. $\cos\theta=\dfrac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}.$

37. $\sec 2\theta=\dfrac{\cot\theta+\tan\theta}{\cot\theta-\tan\theta}.$

38. $\tan\theta+\cot\theta=2\operatorname{cosec}2\theta.$

39. $\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=\tan 2\alpha+\sec 2\alpha.$

40. $\dfrac{\cos(45^\circ+\theta)}{\cos(45^\circ-\theta)}=\sec 2\theta-\tan 2\theta.$

41. $\sin(\alpha+\beta)\cos(\alpha-\beta)+\cos(\alpha+\beta)\sin(\alpha-\beta)=\sin 2\alpha.$

42. $\dfrac{\sin\alpha-\sin\beta}{\cos\alpha+\cos\beta}=\tan\left(\dfrac{\alpha-\beta}{2}\right).$

43. $\tan 2\alpha - \tan \alpha = \tan \alpha \sec 2\alpha.$

44. $\sin \alpha = \dfrac{2\cos^2 \frac{1}{2}\alpha}{\cot \frac{1}{2}\alpha}.$

45. $\cos 3\alpha \cos \alpha + \sin 3\alpha \sin \alpha = \cos 2\alpha.$

46. $$\left[\sin\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\right]\left[\sin\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)\right] = \sin\alpha + \cos\beta.$$

47. $4\sin\theta \sin(60° - \theta)\sin(60° + \theta) = \sin 3\theta.$

48. $\tan^2 \frac{1}{2}\theta = \dfrac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta}.$

Solve each of the following equations for all values of the unknown quantity less than 360°.

49. $\tan\theta = \sin\theta.$

50. $(3 - 4\cos^2\alpha)\cos 2\alpha = 0.$

51. $\sin\alpha + \cos\alpha \cot\alpha = 2.$

52. $\tan(45° + \theta) = 3\tan(45° - \theta).$

53. $5\sin\theta = \tan\theta.$

54. $1 + \sin^2\alpha = 3\sin\alpha\cos\alpha.$

55. $2\cos x + \sec x = 3.$

56. $\tan^4 y - 4\tan^2 y + 3 = 0.$

57. $\sec\beta + \tan\beta = 2.$

58. $2\sin\alpha + 5\cos\alpha = 2.$

59. $\sin 5x + \sin 3x = 0.$

60. $\cos 7x - \cos x = 0.$

61. $\tan 6\theta = 1.$

62. $\tan 2\theta \tan\theta = 1.$

63. $\sin 4\theta + \sin 2\theta + \cos\theta = 0.$

64. $\sin 2\theta - 2\cos\theta + 2\sin\theta - 2 = 0.$

65. $\sin^2\theta + \sin 3\theta = 2\sin\theta\cos 2\theta - \frac{1}{4}.$

66. $\tan 2\theta\cot\theta - \tan 2\theta + \cot\theta - 1 = 0.$

67. $\cos 7\theta + \cos 5\theta + \cos 3\theta = 0.$

68. $\sec^2\theta\csc^2\theta + 4 = 4\csc^2\theta + \sec^2\theta.$

69. Given $\tan\beta = u$, find $\sin 2\beta$.

70. Given $\tan\theta = \csc 2\theta$, find $\cos\theta$.

71. Given $\cos x = \frac{1}{2}$, find $\cos\frac{1}{2}x$.

72. Given $\cos x = \frac{3}{7}$, find $\tan\frac{1}{2}x$ and $\tan 2x$.

73. Given $\tan 2x = m$, find $\tan x$.

74. If $\alpha + \beta + \gamma = 180°$, show that

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma.$$

75. If $\alpha + \beta + \gamma = 180°$, show that

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}.$$

76. If $\alpha + \beta + \gamma = 180°$, show that

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

Prove the following:

77. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} = \frac{\pi}{4}.$

78. $\tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{14}{27} = \frac{\pi}{4}.$

79. $\tan^{-1}k + \tan^{-1}l = \tan^{-1}\frac{k+l}{1-kl}.$

80. $3\tan^{-1}u = \tan^{-1}\frac{3u-u^3}{1-3u^2}.$

81. $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}.$

82. $\sin^{-1}\sqrt{\dfrac{m}{m+n}}=\tan^{-1}\sqrt{\dfrac{m}{n}}.$

83. $\cos^{-1}\dfrac{m}{n}=\operatorname{cosec}^{-1}\sqrt{\dfrac{n}{n-m}}.$

84. $\cot^{-1}\dfrac{k^2-2}{2\sqrt{k^2-1}}=2\csc^{-1}k.$

Solve the following for y:

85. $2\sin^{-1}\dfrac{1}{2}+\cot^{-1}y=\dfrac{\pi}{2}.$

86. $\sin^{-1}y+\sin^{-1}2y=\dfrac{\pi}{2}.$

87. $\dfrac{\pi}{2}-\cot^{-1}3y=2\tan^{-1}y.$

88. $\tan^{-1}y=\sin^{-1}a+\cos^{-1}b.$

89. Prove that in a plane triangle, right-angled at γ,

$$\sin 2\alpha=\frac{2ab}{b^2+a^2};\quad \cos 2\alpha=\frac{b^2-a^2}{b^2+a^2}.$$

90. In a right triangle, c being the hypothenuse, prove that

$$\sin^2\frac{1}{2}\alpha=\frac{c-b}{2c};\quad \cos^2\frac{1}{2}\alpha=\frac{c+b}{2c}.$$

91. In an isosceles triangle in which $a=b$, prove that

$$\cos\alpha=\frac{c}{2a};\quad \text{vers}\ \gamma=\frac{c^2}{2a^2}.$$

92. In a triangle in which $\gamma=60°$, prove that

$$\cos(60°-\beta)=\frac{a+b}{2c}.$$

93. Show that the area of a regular polygon, inscribed in a circle whose radius is r, is $\dfrac{nr^2}{2}\sin\dfrac{2\pi}{n}.$

94. Show that the area of a regular polygon, circumscribed about a circle whose radius is r, is $nr^2 \tan \frac{\pi}{n}$.

95. Show that the area of a regular polygon of n sides is $\frac{na^2}{4} \cot \frac{\pi}{n}$, a being the length of a side.

96. Prove that the areas of an equilateral triangle and of a regular hexagon, of equal perimeters, are to each other as $2:3$.

97. A flagpole 125 ft. high, standing on a horizontal plane, casts a shadow 250 ft. long; find the altitude of the sun.

98. From the top of a rock that rises vertically 325.6 ft. out of the water, the angle of depression of a boat was found to be $24° \, 35'$; find the distance of the boat from the rock.

99. A balloon is directly above a station A. From a second station B, in the same horizontal plane with A, the angle of elevation of the balloon is $60° \, 30'$. If $AB = 300$ ft., find the height of the balloon.

100. From a tower 64 ft. high the angles of depression of two objects in the same horizontal line with the base of the tower and on the same side of the tower, are measured and found to be $28° \, 14'$ and $42° \, 47'$ respectively; find the distance between the objects.

101. Find the area of a triangular field whose sides are 48 rods, 62 rods, and 74 rods.

102. The earth subtends an angle of $17\frac{1}{2}''$ at the sun; find the distance of the sun from the earth, the radius of the earth being 3960 mi.

103. A vertical tower makes an angle of $113° \, 12'$ with the inclined plane on which it stands; and at a distance of 89 ft. 8 in. from its base, measured down the plane, the angle subtended by the tower is $23° \, 26'$. Find the height of the tower.

104. In a circle of radius 8, find the area of a sector with an arc of 50°.

105. In a circle of radius 12, find the area of a segment with an arc of 132°.

106. In a circle of radius r, find the area of a sector with an arc of 1 radian.

107. The area of a square inscribed in a circle is 200 sq. ft.; find the area of an equilateral triangle inscribed in the same circle.

108. Find the area of a triangle of which two sides are $6+\sqrt{5}$ and $6-\sqrt{5}$, the included angle being 32° 12′.

109. At what latitude is the radius of the circle of latitude equal to $\frac{1}{3}$ the radius of the earth?

110. From the top of a lighthouse 85 feet high, standing on a rock, the angle of depression of a ship was 3° 38′, and at the bottom of the lighthouse the angle of depression was 2° 43′; find the horizontal distance of the vessel and the height of the rock.

111. At a point directly south of a flagpole, in the horizontal plane of its base, I observed its elevation, 45°; then going east 200 ft. its elevation was 35°. Find the height of the flagpole.

112. A castle and a monument stand on the same horizontal plane. The angles of depression of the top and the bottom of the monument viewed from the top of the castle are 40° 32′ 18″ and 80° 17′ 46″, and the height of the castle is $104\frac{3}{4}$ feet. Find the height of the monument.

113. At the distance a from the foot of a tower the angle of elevation α of the top of the tower is the complement of the angle of elevation of a flagstaff on top of the tower; show that the length of the staff is $2\,a\cot 2\,\alpha$.

114. A flagstaff a feet high is on a tower $3a$ feet high; the observer's eye is on a level with the top of the staff, and the staff and tower subtend equal angles. How far is the observer from the top of the staff?

115. Two towers on a horizontal plane are 120 feet apart. A person standing successively at their bases observes that the angular elevation of one is double that of the other; but, when he is halfway between them, the elevations are complementary. Find the heights of the towers.

116. An observer sailing north sees two lighthouses 8 miles apart, in a line due west; after an hour's sailing one lighthouse bears S. W., and the other S. S. W. Find the ship's rate.

117. From the top of a house 42 ft. high, the angle of elevation of the top of a pole is $14° 26' 9''$; at the bottom of the house it is $23° 21' 33''$; find the height of the pole.

118. From the top of a hill I observe that the angles of depression of two successive milestones in the horizontal plain below, in a straight line before me, are $14° 20' 24''$ and $5° 31' 14''$. Find the height of the hill.

119. Along the bank of a river is measured a line 500 ft. in length; the angles between this line and the lines of sight from its extremities to an object on the opposite bank are $53° 1' 7''$ and $79° 44' 55''$. Find the breadth of the river.

120. A cape bears N. by E. ($\frac{1}{8}$ of 90° E. of N.), as seen from a ship. The ship sails N. W. 30 miles and then the cape bears E. How far is it from the second point of observation?

121. Find the height of a precipice, its angles of elevation at two stations in a horizontal line with its base being $39° 30'$ and $34° 15'$ and the distance between the stations being 145 ft.

122. From the summit of a hill, 360 ft. above a plain, the angles of depression of the top and bottom of a tower standing on the plain were 41° and 54°; required the height of the tower.

123. At one side of a canal is a flagstaff 21 feet high fixed on the top of a wall 15 feet high; on the other side of the canal, at a point on the ground directly opposite, the flagstaff and the wall subtend equal angles. Find the width of the canal.

124. From the top C of a cliff 600 feet high, the angle of elevation of a balloon B was observed to be 47° 22′, and the angle of depression of its shadow S upon the sea was 61° 10′; find the height of the balloon, the altitude of the sun being 65° 31′ and B, S, C being in the same vertical plane and the sun being behind the observer.

125. At each extremity of a base $AB = 758$ yards, the angles between the other extremity and two objects C and D were observed, viz. $CAB = 103° \ 50' \ 41''$, $DAB = 53° \ 17' \ 24''$, $DBA = 85° \ 47' \ 30''$, and $CBA = 46° \ 13' \ 27''$; find CD.

126. Show that if r be the radius of the earth, h the height of the observer above the sea, and d the angle of depression of the horizon; then $\tan d = \frac{\sqrt{(2r+h)h}}{r}$.

127. A privateer lies 12.75 miles S. W. of a harbor, and a merchantman leaves the harbor in a direction E. by S., at the rate of 10 miles an hour; on what course and at what rate must the privateer sail in order to overtake the merchantman in $1\frac{1}{2}$ hours?

128. From the top of a hill the angles of depression of two objects in the plain at its base were observed to be 45° and 30°, and the horizontal angle between them is also 30°; find the height of the hill in terms of the distance a between the objects.

129. The topmast, 120 feet above the water line of a man-of-war coming into port at the rate of 10 miles an hour, was first seen on the horizon at 8.45 A.M. by a person swimming near the water's edge; and at 10.06 A.M. she cast anchor. Find an approximate value for the radius of the earth.

130. A railway curve which is a circular quadrant has telegraph poles at its extremities and at equal distance along the arc, the whole number of poles being 10. A person in one of the extreme radii produced and at a distance of 300 feet from its extremity, sees the third and sixth poles in line. Find the radius of the curve.

CHAPTER IX

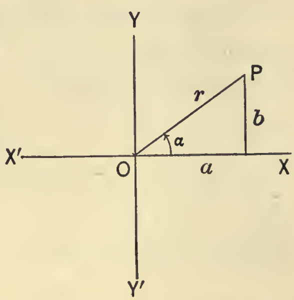
DE MOIVRE'S THEOREM WITH APPLICATIONS

103. In the present chapter De Moivre's theorem is introduced with some of its applications, including the demonstration of the fundamental series by means of which we may calculate the trigonometric tables.

A few preliminary considerations pertaining to the complex number furnish the necessary basis.

104. Geometric representation of a complex number. In algebra it is shown that every complex number can be reduced to the form $a+b\sqrt{-1}$ or $a+ib$ where a and b are any real numbers, and $i=\sqrt{-1}$.

Having given a complex number $a+ib$, we can construct a point P whose coordinates are a and b. This point may be considered the geometric representation of the complex number. It is thus seen that to every complex number there corresponds a point in the plane.



Conversely, to every point in the plane there corresponds a complex number, since the coordinates of the point represent the two elements, a and b, of the complex number.

The line OP, instead of the point P, may be considered the geometric representation of the complex number.

From the figure it is evident that

$$a = r \cos \alpha,$$

and $$b = r \sin \alpha.$$

Therefore $$a + ib = r \cos \alpha + ir \sin \alpha.$$

Hence any complex number, $a + ib$, can be reduced to the form

$$r(\cos \alpha + i \sin \alpha). \qquad (1)$$

The form (1) includes all real and all pure imaginary numbers as special cases. Letting $\alpha = 0°$ or $180°$, (1) reduces to r or $-r$, which represents any real number since r represents any real positive number. Letting $\alpha = 90°$ or $270°$, (1) reduces to $\pm ri$, any pure imaginary number.

The angle XOP or α is the **argument** of the complex number $a + ib$. It is determined by the equation

$$\tan \alpha = \frac{b}{a}.$$

The distance OP or r is the **modulus** of the complex number $a + ib$. It is determined by the equation

$$r = \sqrt{a^2 + b^2}.$$

105. *To show that*

$$[r(\cos \alpha + i \sin \alpha)]^n = r^n(\cos n\alpha + i \sin n\alpha),$$

n being a positive integer.

Squaring the quantity $r(\cos \alpha + i \sin \alpha)$, we have

$$\begin{aligned}[r(\cos \alpha + i \sin \alpha)]^2 &= r^2(\cos^2 \alpha - \sin^2 \alpha + 2i \sin \alpha \cos \alpha) \\ &= r^2(\cos 2\alpha + i \sin 2\alpha) \text{ by Art. 71.} \qquad (1)\end{aligned}$$

Multiplying each member of (1) by $r(\cos \alpha + i \sin \alpha)$ we have

$$\begin{aligned}[r(\cos \alpha + i \sin \alpha)]^3 &= r^3[(\cos 2\alpha \cos \alpha - \sin 2\alpha \sin \alpha) \\ &\quad + i(\sin 2\alpha \cos \alpha + \sin \alpha \cos 2\alpha)] \\ &= r^3(\cos 3\alpha + i \sin 3\alpha) \text{ by Art. 68.} \qquad (2)\end{aligned}$$

From (2) we have, similarly,

$$[r(\cos \alpha + i \sin \alpha)]^4 = r^4(\cos 4\alpha + i \sin 4\alpha). \qquad (3)$$

Equations (1), (2), and (3) have the form

$$[r(\cos \alpha + i \sin \alpha)]^n = r^n(\cos n\alpha + i \sin n\alpha). \qquad (4)$$

Multiplying both members of (4) by $r(\cos\alpha + i\sin\alpha)$, we have

$$[r(\cos\alpha+i\sin\alpha)]^{n+1} = r^{n+1}[\cos(n+1)\alpha + i\sin(n+1)\alpha]. \quad (5)$$

Hence, assuming the law expressed in (4) to be true, equation (5) shows that it is true when n is increased by unity. But the law is true for $n=4$, by equation (3); hence it is true for $n=5$. Being true for $n=5$, it must also be true for $n=6$, etc. Hence equation (4) is true for all positive integral values of n.

It can be shown that equation (4) is still true when n is a negative integer, or a fraction.

From equation (4) it is seen that the nth power of a complex number is a complex number having an argument n times the argument of the given number and a modulus equal to the nth power of the given modulus.

Letting $r=1$, equations (1), (2), (3), and (4) become

$$(\cos\alpha + i\sin\alpha)^2 = \cos 2\alpha + i\sin 2\alpha \quad (5)$$

$$(\cos\alpha + i\sin\alpha)^3 = \cos 3\alpha + i\sin 3\alpha \quad (6)$$

$$(\cos\alpha + i\sin\alpha)^4 = \cos 4\alpha + i\sin 4\alpha \quad (7)$$

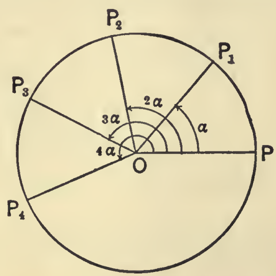
$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha. \quad (8)$$

The last equation is known as De Moivre's theorem.

PROB. Show De Moivre's theorem is true when $n = -3$.

106. Geometric Interpretation. Since each of the complex numbers

$$\cos\alpha + i\sin\alpha$$
$$\cos 2\alpha + i\sin 2\alpha$$
$$\cos 3\alpha + i\sin 3\alpha$$
$$\cos 4\alpha + i\sin 4\alpha$$



has a modulus equal to unity, the lines representing these numbers terminate in points lying on the circumference of a circle whose radius is unity. The arguments of any two consecutive integral powers of $\cos\alpha + i\sin\alpha$ differ by α, hence the lines representing any two consecutive powers differ in direction by α.

107. Applications of De Moivre's theorem. De Moivre's theorem may be used to find the various roots of unity, to extract any root of a complex number, to obtain the sine and cosine of any multiple of an angle, and to expand the sine and cosine of an angle in a series of powers of the angle.

108. *To find the cube roots of unity.*

If the cube roots of unity are real numbers, or complex numbers, we may assume, by Art. 104, that

$$\sqrt[3]{1} = r(\cos\alpha + i\sin\alpha). \tag{1}$$

Then, cubing, $1 = r^3(\cos 3\alpha + i\sin 3\alpha)$. (2)

Also $$1 = r^3\cos(3\alpha - 2n\pi) + ir^3\sin(3\alpha - 2n\pi), \tag{3}$$

n being an integer.

Equating the real and imaginary parts of equation (3), we have

$$r^3\cos(3\alpha - 2n\pi) = 1$$

and $$r^3\sin(3\alpha - 2n\pi) = 0.$$

These equations of condition are satisfied when

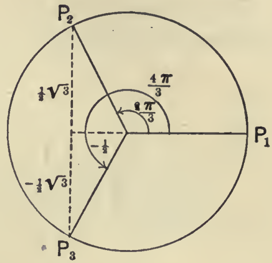
$$3\alpha - 2n\pi = 0 \text{ and } r = 1,$$

whence $$\alpha = \frac{2n\pi}{3}.$$

When $n = 0,\ 1,\ 2,\ 3,\ 4$, etc.

$$\alpha = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \frac{8\pi}{3}, \text{ etc., respectively.} \tag{4}$$

Every angle in this series is coterminal with either 0, $\frac{2\pi}{3}$, or $\frac{4\pi}{3}$; hence all the values of $\sin\alpha$ and $\cos\alpha$ are obtained by using only the first three values of α. Substituting these values of α in equation (1), and remembering that $r=1$, we have



$\sqrt[3]{1}=1$ for $n=0$,

$\sqrt[3]{1}=-\frac{1}{2}+i\frac{1}{2}\sqrt{3}$ for $n=1$,

$\sqrt[3]{1}=-\frac{1}{2}-i\frac{1}{2}\sqrt{3}$ for $n=2$.

The three cube roots of unity are represented geometrically by P_1, P_2, and P_3.

We can now write the three cube roots of any real number a, for letting a_1, a_2, and a_3 be the cube roots of a, we have

$$a_1=\sqrt[3]{a}\cdot 1,\ a_2=\sqrt[3]{a}\left(-\tfrac{1}{2}+i\tfrac{1}{2}\sqrt{3}\right),\ a_3=\sqrt[3]{a}\left(-\tfrac{1}{2}-i\tfrac{1}{2}\sqrt{3}\right),$$

where $\sqrt[3]{a}$ is the arithmetical cube root.

Problem. Show that $(-\frac{1}{2}+i\frac{1}{2}\sqrt{3})^3=1$, thus justifying the assumption made in Eq. 1.

109. *To find the fifth roots of unity.*

Let $\sqrt[5]{1}=r(\cos\alpha+i\sin\alpha)$. (1)

Then, raising each member to the fifth power,

$$1=r^5(\cos 5\alpha+i\sin 5\alpha). \tag{2}$$

Also $1=r^5\cos(5\alpha-2n\pi)+ir^5\sin(5\alpha-2n\pi)$, (3)

n being an integer.

Equating the real and imaginary parts of equation (3), we have

$$r^5\cos(5\alpha-2n\pi)=1,$$

and

$$r^5\sin(5\alpha-2n\pi)=0.$$

These equations of condition are satisfied when

$$5\alpha-2n\pi=0, \text{ and } r=1.$$

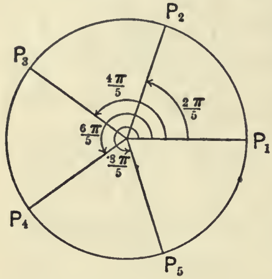
Therefore

$$\alpha=\frac{2n\pi}{5}.$$

When $n=0,\ 1,\ 2,\ 3,\ 4,$ etc.

$$\alpha=0,\ \frac{2\pi}{5},\ \frac{4\pi}{5},\ \frac{6\pi}{5},\ \frac{8\pi}{5}, \text{ etc., respectively.} \tag{4}$$

Substituting the values from (4) in equation (1), and remembering that $r=1$, we have



$$\sqrt[5]{1}=1 \qquad \text{for } n=0,$$
$$\sqrt[5]{1}=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5} \text{ for } n=1,$$
$$\sqrt[5]{1}=\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5} \text{ for } n=2,$$
$$\sqrt[5]{1}=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5} \text{ for } n=3,$$
$$\sqrt[5]{1}=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5} \text{ for } n=4.$$

110. *To extract the square root of $a+ib$.*

Let $\sqrt{a+ib}=r(\cos\alpha+i\sin\alpha)$. (1)

Squaring, $a+ib=r^2(\cos 2\alpha+i\sin 2\alpha)$. (2)

Also $a+ib=r^2[\cos(2\alpha-2n\pi)+i\sin(2\alpha-2n\pi)]$, (3)

where n is any integer.

Equating the real and imaginary parts,

$$r^2\cos(2\alpha-2n\pi)=a, \qquad (4)$$
$$r^2\sin(2\alpha-2n\pi)=b. \qquad (5)$$

Squaring and adding, we have

$$r^4=a^2+b^2, \qquad (6)$$

or
$$r^2=\sqrt{a^2+b^2}, \qquad (7)$$

and
$$r=\sqrt[4]{a^2+b^2}. \qquad (8)$$

Equation (8) gives the value of r in terms of the known numbers a and b.

From equations (4) and (7)

$$\cos(2\alpha-2n\pi)=\frac{a}{r^2}=\frac{a}{\sqrt{a^2+b^2}},$$

whence
$$2\alpha-2n\pi=\cos^{-1}\frac{a}{\sqrt{a^2+b^2}},$$

the quadrant of $2\alpha-2n\pi$ being determined by (4) and (5).

Then $$\alpha = \tfrac{1}{2}\cos^{-1}\frac{a}{\sqrt{a^2+b^2}} \qquad \text{for } n=0, \tag{9}$$

and $$\alpha = \pi + \tfrac{1}{2}\cos^{-1}\frac{a}{\sqrt{a^2+b^2}} \qquad \text{for } n=1. \tag{10}$$

The values of α for $n = 2, 3, 4$, etc., are coterminal with the values of α in equations (9) or (10). Hence there are only two values for $\sin\alpha$ and $\cos\alpha$; namely those for which $n=0$ and $n=1$.

Substituting these values of α in equation (1), we have, finally,

$$\sqrt{a+ib} = \sqrt[4]{a^2+b^2}\left[\cos\left(\tfrac{1}{2}\cos^{-1}\frac{a}{\sqrt{a^2+b^2}}\right) + i\sin\left(\tfrac{1}{2}\cos^{-1}\frac{a}{\sqrt{a^2+b^2}}\right)\right], \tag{11}$$

and $$\sqrt{a+ib} = \sqrt[4]{a^2+b^2}\left[\cos\left(\pi+\tfrac{1}{2}\cos^{-1}\frac{a}{\sqrt{a^2+b^2}}\right) + i\sin\left(\pi+\tfrac{1}{2}\cos^{-1}\frac{a}{\sqrt{a^2+b^2}}\right)\right]. \tag{12}$$

These values of $\sqrt{a+ib}$, while more complicated than $\sqrt{a+ib}$ itself, are nevertheless in the standard form of complex numbers,

$$r(\cos\alpha + i\sin\alpha).$$

111. *To extract the kth root of $a+ib$.*

Let $$\sqrt[k]{a+ib} = r(\cos\alpha + i\sin\alpha). \tag{1}$$

Then $$a+ib = r^k(\cos k\alpha + i\sin k\alpha) \tag{2}$$

$$= r^k[\cos(k\alpha - 2n\pi) + i\sin(k\alpha - 2n\pi)], \tag{3}$$

where n is an integer.

Equating the real and imaginary parts of equation (3), we have $$r^k\cos(k\alpha - 2n\pi) = a, \tag{4}$$

and $$r^k\sin(k\alpha - 2n\pi) = b. \tag{5}$$

Squaring and adding equations (4) and (5), we have

$$r^{2k} = a^2 + b^2. \tag{6}$$

Then
$$r^k = \sqrt{a^2 + b^2}, \tag{7}$$

and
$$r = \sqrt[2k]{a^2 + b^2}. \tag{8}$$

From equations (4) and (7), we have

$$\cos(k\alpha - 2n\pi) = \frac{a}{r^k} = \frac{a}{\sqrt{a^2 + b^2}},$$

or
$$k\alpha - 2n\pi = \cos^{-1}\frac{a}{\sqrt{a^2 + b^2}},$$

the quadrant of $k\alpha - 2n\pi$ being determined by the signs of (4) and (5).

Therefore
$$\alpha = \frac{2n\pi}{k} + \frac{1}{k}\cos^{-1}\frac{a}{\sqrt{a^2 + b^2}}, \tag{9}$$

whence
$$\alpha = \frac{1}{k}\cos^{-1}\frac{a}{\sqrt{a^2 + b^2}} \text{ for } n = 0, \tag{10}$$

$$\alpha = \frac{2\pi}{k} + \frac{1}{k}\cos^{-1}\frac{a}{\sqrt{a^2 + b^2}} \text{ for } n = 1, \tag{11}$$

$$\alpha = \frac{4\pi}{k} + \frac{1}{k}\cos^{-1}\frac{a}{\sqrt{a^2 + b^2}} \text{ for } n = 2. \tag{12}$$

. . . . .

Substituting the values for r and α as found in equations (8), (10), (11), (12), etc., in equation (1), we have the several kth roots of $a + ib$.

When k is an integer there are k roots. Consecutive values of α, as given by (9), differ by $\frac{2\pi}{k}$, hence all the values of α after the kth are coterminal with one of the first k values.

112. *To express* $\sin n\alpha$ *and* $\cos n\alpha$ *in terms of* $\sin\alpha$ *and* $\cos\alpha$.

We have

$$\cos n\alpha + i\sin n\alpha = (\cos\alpha + i\sin\alpha)^n.$$

Expanding the second member by the binomial theorem and equating the real and imaginary parts, the problem is solved.

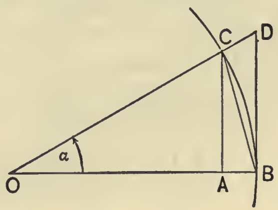
Thus, for $\sin 4\alpha$ and $\cos 4\alpha$ we have

$$\cos 4\alpha + i \sin 4\alpha = (\cos\alpha + i\sin\alpha)^4 = \cos^4\alpha,$$
$$+ 4i\cos^3\alpha\sin\alpha - 6\cos^2\alpha\sin^2\alpha - 4i\cos\alpha\sin^3\alpha + \sin^4\alpha.$$

Therefore $\quad \sin 4\alpha = 4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha.$

and $\quad \cos 4\alpha = \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha.$

113. Comparison of the values of sin α, α, and tan α, α being an acute angle. Let α be any acute angle expressed in radians. With the vertex O as a center and any radius OB, describe the arc BC. Draw AC and BD perpendicular to OB, and join B with C.



The area of the triangle OBC is less than the area of the sector OBC, and the sector OBC is less than the triangle OBD.

But since

$AC = OC\sin\alpha$, $BD = OB\tan\alpha$, and arc $BC = OB\cdot\alpha$, Art. 17,

the area of the triangle OBC is equal to $\frac{1}{2}\cdot OB\cdot OC\sin\alpha$,
the area of the sector OBC is equal to $\frac{1}{2}\cdot OB\cdot OB\cdot\alpha$,
and the area of the triangle OBD is equal to $\frac{1}{2}\cdot OB\cdot OB\tan\alpha$
Hence

$$\tfrac{1}{2}\cdot OB\cdot OC\sin\alpha < \tfrac{1}{2}\cdot OB\cdot OB\cdot\alpha < \tfrac{1}{2}\cdot OB\cdot OB\tan\alpha,$$

or

$$\sin\alpha < \alpha < \tan\alpha.$$

114. Value of $\frac{\sin\alpha}{\alpha}$ for small values of α. In Art. 40 we saw that $\sin\alpha$ approaches 0 as α approaches 0. The value of $\frac{\sin\alpha}{\alpha}$ therefore approaches $\frac{0}{0}$ as α approaches 0.

But from the previous article

$$\sin \alpha < \alpha < \tan \alpha.$$

Dividing by $\sin \alpha$, we have

$$1 < \frac{\alpha}{\sin \alpha} < \frac{1}{\cos \alpha}$$

or

$$1 > \frac{\sin \alpha}{\alpha} > \cos \alpha.$$

Since $\frac{\sin \alpha}{\alpha}$ lies between 1 and $\cos \alpha$, $\frac{\sin \alpha}{\alpha}$ must approach 1 as α approaches 0, since $\cos \alpha$ approaches 1.

Then for very small angles $\sin \alpha$ may be replaced by α, expressed in radians. The error thus introduced is so small that it may be neglected in many problems. Thus, to five decimal places,

$\sin 1° = 0.01745$	$1° = 0.01745$ radians
$\sin 2° = 0.03490$	$2° = 0.03491$ radians
$\sin 3° = 0.05234$	$3° = 0.05236$ radians
$\sin 4° = 0.06976$	$4° = 0.06981$ radians

115. *To develop* $\sin \alpha$ *and* $\cos \alpha$ *in terms of* α.

By De Moivre's theorem,

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.$$

On expanding the second member by the binomial theorem, we have

$$\cos n\theta + i \sin n\theta = \cos^n \theta + i\, n \cos^{n-1} \theta \sin \theta.$$

$$- \frac{n(n-1)}{\lfloor 2} \cos^{n-2} \theta \sin^2 \theta - i \frac{n(n-1)(n-2)}{\lfloor 3} \cos^{n-3} \theta \sin^3 \theta$$

$$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} \cos^{n-4} \theta \sin^4 \theta + \cdots. \qquad (1)$$

Equating the imaginary parts, we have

$$\sin n\theta = n \cos^{n-1} \theta \sin \theta - \frac{n(n-1)(n-2)}{\lfloor 3} \cos^{n-3} \theta \sin^3 \theta$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5} \cos^{n-5} \theta \sin^5 \theta - \cdots. \qquad (2)$$

Let $n\theta = \alpha$. Then equation (2) may be written

$$\sin\alpha = \frac{\alpha}{\theta}\cos^{n-1}\theta\sin\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta$$
$$+\frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)\left(\frac{\alpha}{\theta}-4\right)}{\lfloor 5}\cos^{n-5}\theta\sin^5\theta - \cdots, \quad (3)$$

$$\text{or } \sin\alpha = \alpha\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{\lfloor 3}\cos^{n-3}\theta\left(\frac{\sin^3\theta}{\theta^3}\right)$$
$$+\frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)(\alpha-4\theta)}{\lfloor 5}\cos^{n-5}\theta\left(\frac{\sin^5\theta}{\theta^5}\right) - \cdots. \quad (4)$$

Let α remain constant while n increases indefinitely. Then θ necessarily decreases indefinitely, since $n\theta = \alpha$, a constant. By Art. 114, when θ approaches 0, $\frac{\sin\theta}{\theta}$ approaches 1, and $\cos\theta$ approaches 1. Making these substitutions in equation (4), we have

$$\sin\alpha = \alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \frac{\alpha^7}{\lfloor 7} + \cdots.$$

Equating the real parts of equation (1), we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\lfloor 2}\cos^{n-2}\theta\sin^2\theta$$
$$+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cos^{n-4}\theta\sin^4\theta - \cdots. \quad (5)$$

By the same process as above, equation (5) becomes

$$\cos\alpha = 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \frac{\alpha^6}{\lfloor 6} + \cdots.$$

The series for $\sin\alpha$ and $\cos\alpha$ are convergent for all finite values of α.* They enable us to compute the sine and cosine of any angle. It is then possible to construct a table of natural functions, from which the logarithmic functions may be obtained. In using these series α must, of course, be expressed in radian measure.

* See any College Algebra on the convergency of series.

116. EXAMPLES

1. Find the four fourth roots of unity by De Moivre's theorem.

2. Find the six sixth roots of unity by De Moivre's theorem.

3. Find the square root of $5-3i$.

SOLUTION. — Let $\sqrt{5-3i}=r(\cos\alpha+i\sin\alpha)$. (1)

Then $$5-3i=r^2(\cos 2\alpha+i\sin 2\alpha)$$
$$=r^2[\cos(2\alpha-2n\pi)+i\sin(2\alpha-2n\pi)].$$

Equating the real and the imaginary parts,

$$r^2\cos(2\alpha-2n\pi)=5, \quad (2)$$

and $$r^2\sin(2\alpha-2n\pi)=-3. \quad (3)$$

Squaring and adding (2) and (3), we have

$$r^4=34,\ \therefore\ r^2=\sqrt{34},\ \text{and } r=\sqrt[4]{34}. \quad (4)$$

Then $$\cos(2\alpha-2n\pi)=\frac{5}{\sqrt{34}},$$

and $$2\alpha-2n\pi=329^\circ\ 2', \quad (5)$$

the quadrant being determined by (2) and (3).

When $$n=0,\ \alpha=164^\circ\ 31'; \quad (6)$$
$$n=1,\ \alpha=344^\circ\ 31'. \quad (7)$$

Substituting from (4) and (6) in (1), we have

$$\sqrt{5-3i}=-2.3271+.6446\,i.$$

Substituting from (4) and (7) in (1), we have

$$\sqrt{5-3i}=2.3271-.6446\,i.$$

4. Find the square root of $3+4i$.

5. Find the square root of $-3-4i$.

6. Find the square root of $1+2i$.

7. Find the square root of i.

8. Find the square root of $-i$.

9. Find the cube root of $2-3i$.

SOLUTION. — Let $\sqrt[3]{2-3i}=r(\cos\alpha+i\sin\alpha)$. (1)

Then $$2-3i=r^3(\cos 3\alpha+i\sin 3\alpha)$$
$$=r^3[\cos(3\alpha-2n\pi)+i\sin(3\alpha-2n\pi)].$$

Equating the real and imaginary parts,

$$r^3 \cos(3\alpha - 2n\pi) = 2; \tag{2}$$

and

$$r^3 \sin(3\alpha - 2n\pi) = -3. \tag{3}$$

Squaring and adding (2) and (3), we have

$$r^6 = 13, \therefore r^3 = \sqrt{13}, \text{ and } r = \sqrt[6]{13}. \tag{4}$$

Then

$$\cos(3\alpha - 2n\pi) = \frac{2}{\sqrt{13}},$$

and

$$3\alpha - 2n\pi = 303° 41'; \tag{5}$$

the quadrant being determined by (2) and (3).

When

$$n = 0, \alpha = 101° 14'; \tag{6}$$

$$n = 1, \alpha = 221° 14'; \tag{7}$$

$$n = 2, \alpha = 341° 14'. \tag{8}$$

Substituting from (4) and (6) in (1), we have

$$\sqrt[3]{2 - 3i} = -.2987 + 1.5041 i.$$

Substituting from (4) and (7) in (1), we have

$$\sqrt[3]{2 - 3i} = -1.1530 - 1.0107 i.$$

Substituting from (4) and (8) in (1), we have

$$\sqrt[3]{2 - 3i} = 1.4518 - .4933 i.$$

We have thus found the three cube roots of $2 - 3i$.

10. Find the cube root of $1 + i$.

11. Find the cube root of $-1 + i$.

12. Find the cube root of $2 + 3i$.

13. Find the values of $\sin 3x$ and $\cos 3x$ in terms of $\sin x$ and $\cos x$.

14. Find the values of $\sin 5x$ and $\cos 5x$ in terms of $\sin x$ and $\cos x$.

15. Prove by De Moivre's theorem that

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}, \text{ also } \cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}.$$

Suggestion. $\cos\alpha + i\sin\alpha = \left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)^2.$

16. Show that $\cos\alpha = \cos^3\frac{\alpha}{3} - 3\cos\frac{\alpha}{3}\sin^2\frac{\alpha}{3},$

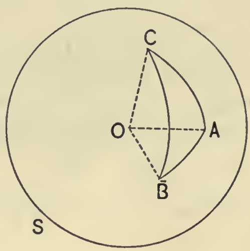
$$\sin\alpha = 3\cos^2\frac{\alpha}{3}\sin\frac{\alpha}{3} - \sin^3\frac{\alpha}{3}.$$

SPHERICAL TRIGONOMETRY

CHAPTER X

FUNDAMENTAL FORMULAS

117. The spherical triangle.* Spherical trigonometry treats of the relations between the various parts of a spherical triangle and of the methods of solving the spherical triangle.



The sides of a spherical triangle are always arcs of great circles.

Having given a spherical triangle ABC, situated upon a sphere S, a triedral angle O–ABC may be formed by passing planes through O, the center of the sphere, and through the sides of the triangle.

It is known from geometry that the arc AB and the angle AOB contain the same number of degrees, and that the angle CAB and the diedral angle C–AO–B contain the same number of degrees.

The sides and angles of a spherical triangle may have any values between 0° and 360°.

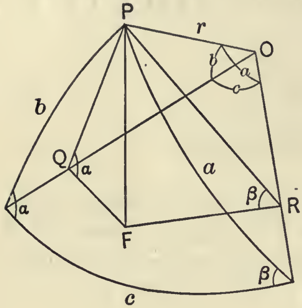
A triangle having one or more of its parts greater than 180° is called a **general spherical triangle**.

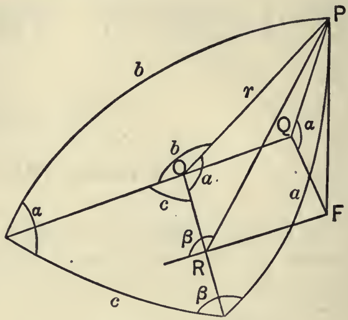
A triangle having each of its parts less than 180° is called a **spherical triangle**.

We shall consider only those triangles whose parts are each less than 180°.

* For a course on the right spherical triangle read Arts. 117 and 126 and from Art. 128 to end of Chapter XI.

118. Law of sines. *To find the relation between two sides of a spherical triangle and the angles opposite.*





$\beta > 90°,\ b > 90°.$

Given a spherical triangle and its accompanying triedral angle; through the vertex P pass planes perpendicular to OR and OQ, intersecting in PF a line perpendicular to the plane ORQ.

Then $\qquad \sin\alpha = \dfrac{FP}{QP}$ and $\sin\beta = \dfrac{FP}{RP}$,

therefore $\qquad \dfrac{\sin\alpha}{\sin\beta} = \dfrac{RP}{QP}.$

Also $\qquad \sin a = \dfrac{RP}{OP}$ and $\sin b = \dfrac{QP}{OP}$,

therefore $\qquad \dfrac{\sin a}{\sin b} = \dfrac{RP}{QP}.$

Hence $\qquad \dfrac{\sin\alpha}{\sin\beta} = \dfrac{\sin a}{\sin b}.$

Likewise $\qquad \dfrac{\sin\beta}{\sin\gamma} = \dfrac{\sin b}{\sin c}.$

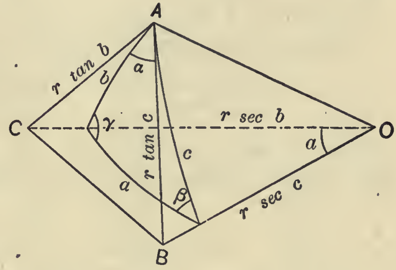
Uniting these equations, we have

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma}. \tag{1}$$

This demonstration applies to similar figures drawn for all possible cases,* hence the theorem is always true.

119. Law of cosines. *To find the relation between the three sides and an angle.*

Given a spherical triangle and its accompanying triedral angle; pass a plane through the vertex A perpendicular to OA, intersecting the planes of the triedral angle in the lines AB, AC, and BC.



$b < 90°$, $c < 90°$; $a < 180°$, $\alpha < 180°$.

Then $AB = r \tan c$, $OB = r \sec c$, $AC = r \tan b$, $OC = r \sec b$. From the triangle OBC, by Art. 90,

$$\overline{BC}^2 = (r \sec b)^2 + (r \sec c)^2 - 2(r \sec b)(r \sec c) \cos a. \quad (1)$$

Likewise from the triangle ABC,

$$\overline{BC}^2 = (r \tan b)^2 + (r \tan c)^2 - 2(r \tan b)(r \tan c) \cos \alpha. \quad (2)$$

Subtracting (2) from (1), we have

$$0 = r^2(\sec^2 b - \tan^2 b) + r^2(\sec^2 c - \tan^2 c)$$
$$- 2r^2 \sec b \sec c \cos a + 2r^2 \tan b \tan c \cos \alpha,$$

which reduces to

$$0 = 1 - \sec b \sec c \cos a + \tan b \tan c \cos \alpha,$$

or $$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha. \quad (3)$$

Also $$\cos b = \cos c \cos a + \sin c \sin a \cos \beta, \quad (4)$$

and $$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma. \quad (5)$$

* The following seven cases can arise:

(1) 3 sides $< 90°$, 3 angles $< 90°$
(2) 3 sides $< 90°$, 2 angles $< 90°$
(3) 2 sides $< 90°$, 2 angles $< 90°$
(4) 1 side $< 90°$, 2 angles $< 90°$
(5) 1 side $< 90°$, 1 angle $< 90°$
(6) 1 side $< 90°$, 0 angle $< 90°$
(7) 0 side $< 90°$, 0 angle $< 90°$.

It is to be understood that all parts not mentioned are greater than 90°.

120. *To extend the law of cosines.*

In the derivation of the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha,$$

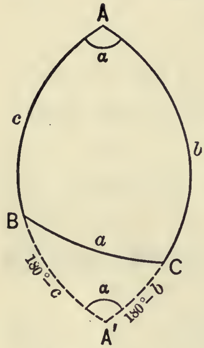
b and c were less than 90°, while a and α were less than 180°.

To show that the formula is true in general, it is necessary to consider two additional cases:

1st. Both b and c greater than 90°.

2d. Either b or c greater than 90°.

Since β and γ do not enter the formula, they may have any value consistent with the above conditions.



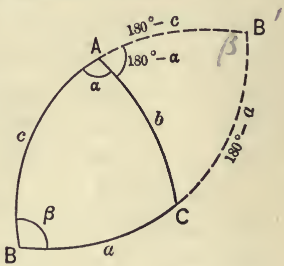
First. Given the triangle ABC in which $b > 90°$ and $c > 90°$.

Extend the sides b and c of the triangle ABC, forming the lune whose angle is α. Then in the triangle $A'BC$, the sides $A'B$ and $A'C$ are each less than 90°, hence by Art. 119

$$\cos a = \cos (180° - b) \cos (180° - c) + \sin (180° - b) \sin (180° - c) \cos \alpha,$$

or $\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$

Hence the law of cosines holds when both b and c are greater than 90°.



Second. Given the triangle ABC, in which $b < 90°$ and $c > 90°$.

Extend the sides a and c of the triangle ABC, forming the lune whose angle is β. Then in the triangle $AB'C$, the sides AB' and AC are each less than 90°, and the angle $B'AC$ is equal to $180° - \alpha$.

Then, by Art. 119,

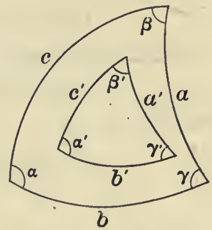
$$\cos (180° - a) = \cos b \cos (180° - c) + \sin b \sin (180° - c) \cos (180° - \alpha),$$

or $$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

Hence the law of cosines holds when either b or c is greater than 90°. The law of cosines is therefore true in general.

121. *To find the relation between one side and the three angles.*

Let a, b, and c be the sides of any spherical triangle, and a', b', c' the sides of its polar triangle.



Applying the law of cosines to the polar triangle, we have

$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos \alpha'.$

But

$$a' = 180° - \alpha, \quad b' = 180° - \beta, \text{ etc.}$$

Therefore

$$\cos(180° - \alpha) = \cos(180° - \beta)\cos(180° - \gamma) + \sin(180° - \beta)\sin(180° - \gamma)\cos(180° - a),$$

or $$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a. \qquad (1)$$

Also $$\cos\beta = -\cos\gamma\cos\alpha + \sin\gamma\sin\alpha\cos b, \qquad (2)$$

and $$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c. \qquad (3)$$

122. The sine-cosine law. *To find the relation between three sides and two angles.*

We have

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha, \qquad (1)$$

and $$\cos b = \cos c\cos a + \sin c\sin a\cos\beta. \qquad (2)$$

Eliminating $\cos a$ by substitution,

$$\cos b = \cos b\cos^2 c + \sin b\sin c\cos c\cos\alpha + \sin c\sin a\cos\beta.$$

Transposing and factoring,

$$\cos b(1 - \cos^2 c) = \sin b\sin c\cos c\cos\alpha + \sin a\sin c\cos\beta.$$

Replacing $(1 - \cos^2 c)$ by $\sin^2 c$, and dividing by $\sin c$, we have

$$\cos b\sin c = \sin b\cos c\cos\alpha + \sin a\cos\beta.$$

Rearranging terms,

$$\sin a\cos\beta = \cos b\sin c - \sin b\cos c\cos\alpha. \qquad (3)$$

Also $\quad \sin b \cos \gamma = \cos c \sin a - \sin c \cos a \cos \beta,$ (4)

and $\quad \sin c \cos \alpha = \cos a \sin b - \sin a \cos b \cos \gamma.$ (5)

Interchanging β and γ and consequently b and c, we have from (3)

$$\sin a \cos \gamma = \cos c \sin b - \sin c \cos b \cos \alpha. \qquad (6)$$

Similarly from (4)

$$\sin b \cos \alpha = \cos a \sin c - \sin a \cos c \cos \beta, \qquad (7)$$

and from (5)

$$\sin c \cos \beta = \cos b \sin a - \sin b \cos a \cos \gamma. \qquad (8)$$

123. *To find the relation between two sides and the three angles.*

Applying the sine-cosine law to the polar of the given triangle, we have

$$\sin a' \cos \beta' = \cos b' \sin c' - \sin b' \cos c' \cos \alpha'.$$

But $a' = 180° - \alpha$, $\beta' = 180° - b$, etc.

Then

$$\sin (180° - \alpha) \cos (180° - b) = \cos (180° - \beta) \sin (180° - \gamma) - \sin (180° - \beta) \cos (180° - \gamma) \cos (180° - a).$$

Therefore $\sin \alpha \cos b = \cos \beta \sin \gamma + \sin \beta \cos \gamma \cos a.$ (1)

Also $\quad \sin \beta \cos c = \cos \gamma \sin \alpha + \sin \gamma \cos \alpha \cos b,$ (2)

$$\sin \gamma \cos a = \cos \alpha \sin \beta + \sin \alpha \cos \beta \cos c, \qquad (3)$$

$$\sin \alpha \cos c = \cos \gamma \sin \beta + \sin \gamma \cos \beta \cos a, \qquad (4)$$

$$\sin \beta \cos a = \cos \alpha \sin \gamma + \sin \alpha \cos \gamma \cos b, \qquad (5)$$

$$\sin \gamma \cos b = \cos \beta \sin \alpha + \sin \beta \cos \alpha \cos c. \qquad (6)$$

124. *To find the relation between two sides and two angles, one of the angles being included between the given sides.*

From Art. 122

$$\sin a \cos \beta = \cos b \sin c - \sin b \cos c \cos \alpha.$$

Dividing this equation by

$$\sin a \sin \beta = \sin b \sin \alpha, \qquad \text{Art. 118}$$

member by member, we have

$$\cot \beta = \frac{\cot b \sin c - \cos c \cos \alpha}{\sin \alpha}.$$

Therefore $\sin \alpha \cot \beta = \cot b \sin c - \cos c \cos \alpha.$ (1)

Similarly $\sin \beta \cot \gamma = \cot c \sin a - \cos a \cos \beta,$ (2)

and $\sin \gamma \cot \alpha = \cot a \sin b - \cos b \cos \gamma.$ (3)

Interchanging α and β and consequently a and b, we have from (1)

$$\sin \beta \cot \alpha = \cot a \sin c - \cos c \cos \beta. \quad (4)$$

Similarly from (2)

$$\sin \gamma \cot \beta = \cot b \sin a - \cos a \cos \gamma, \quad (5)$$

and from (3)

$$\sin \alpha \cot \gamma = \cot c \sin b - \cos b \cos \alpha. \quad (6)$$

125. Formulas independent of the radius of the sphere. It will be noticed that r, the radius of the sphere, does not enter any of the formulas thus far developed; hence they are independent of the radius of the sphere, and may be applied, without modification, to triangles on any sphere. The fact is serviceable in problems of Astronomy and Geodesy where the formulas are applied to triangles situated upon the celestial and terrestrial spheres.

CHAPTER XI

SPHERICAL RIGHT TRIANGLE

126. The spherical right triangle is a spherical triangle one of whose angles is a right angle. The other parts may have any values between 0° and 180°.

In the work that follows the angle γ will be taken as the right angle.

127. Formulas for the solution of right triangles. The formulas for the solution of any spherical right triangle are obtained from the general formulas of Chapter X by letting $\gamma = 90°$. We thus have

from eq. (1) Art. 118	$\sin a = \sin c \sin \alpha$
from eq. (1) Art. 118	$\sin b = \sin c \sin \beta$
from eq. (5) Art. 119	$\cos c = \cos a \cos b$
from eq. (1) Art. 121	$\cos \alpha = \sin \beta \cos a$
from eq. (2) Art. 121	$\cos \beta = \sin \alpha \cos b$
from eq. (3) Art. 121	$\cos c = \cot \alpha \cot \beta$
from eq. (2) Art. 124	$\cos \beta = \tan a \cot c$
from eq. (3) Art. 124	$\sin b = \cot \alpha \tan a$
from eq. (5) Art. 124	$\sin a = \cot \beta \tan b$
from eq. (6) Art. 124	$\cos \alpha = \tan b \cot c.$

128. Direct geometric derivation of formulas. Let a and b be the sides of a given spherical right triangle, α and β the angles opposite, and c its hypotenuse.

Let $O\text{–}ABC$ be its accompanying triedral angle.

Through the vertex B pass a plane perpendicular to OA, intersecting the planes of the triedral angle in AB, BC, and CA.

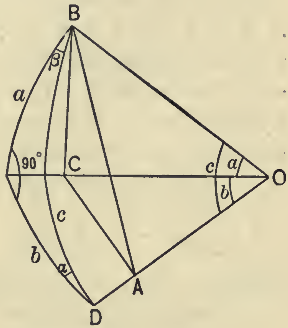
Then $\angle BAC=\alpha$, $\angle BOC=a$, $\angle AOC=b$, and $\angle AOB=c$. Also $\angle BCA$, $\angle BCO$, $\angle CAO$, $\angle BAO$ are each a right angle.

From the triangle ABC we have

$$\sin\alpha=\frac{CB}{AB}=\frac{\frac{CB}{OB}}{\frac{AB}{OB}}=\frac{\sin a}{\sin c},\qquad (1)$$

$$\cos\alpha=\frac{AC}{AB}=\frac{\frac{AC}{OA}}{\frac{AB}{OA}}=\frac{\tan b}{\tan c},\qquad (2)$$

$$\tan\alpha=\frac{CB}{AC}=\frac{\frac{CB}{OC}}{\frac{AC}{OC}}=\frac{\tan a}{\sin b}.\qquad (3)$$



By interchanging α and β and consequently a and b, or by passing a plane through $D \perp$ to OB and proceeding as above, we have

$$\sin\beta=\frac{\sin b}{\sin c},\qquad (4)$$

$$\cos\beta=\frac{\tan a}{\tan c},\qquad (5)$$

$$\tan\beta=\frac{\tan b}{\sin a}.\qquad (6)$$

Also from the figure,

$$\cos c=\frac{OA}{OB}=\frac{\frac{OA}{OC}}{\frac{OB}{OC}}=\frac{\cos b}{\sec a}=\cos a\cos b.\qquad (7)$$

Dividing (1) by (5) and reducing by (7) we have

$$\sin\alpha=\frac{\cos\beta}{\cos b},\qquad (8)$$

and by interchange of letters as before

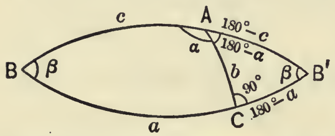
$$\sin\beta=\frac{\cos\alpha}{\cos a}.\qquad (9)$$

Substituting the values of $\cos a$ and $\cos b$ from (8) and (9) in (7), we have, after reduction,

$$\cos c = \cot \alpha \cot \beta. \tag{10}$$

In the demonstration of formulas (1) to (10) the parts α, β, a, b, and c were assumed less than 90°. To show that these formulas are true in general, it is necessary to consider two additional cases: viz. (1) when one side and the hypotenuse are each greater than 90°, and (2) when the two sides are each greater than 90°.

1. *When $a > 90°$ and $c > 90°$.*



Let ABC be the given spherical right triangle. Draw the lune BB'. Then in the right triangle $AB'C$ each part is less than 90°, and formulas (1) to (10) are applicable.

From (1) $\quad \sin(180° - \alpha) = \dfrac{\sin(180° - a)}{\sin(180° - c)}$

or $$\sin \alpha = \frac{\sin a}{\sin c},$$

which shows that (1) holds when a and c are each greater than 90°.

From (2) $\quad \cos(180° - \alpha) = \dfrac{\tan b}{\tan(180° - c)}$

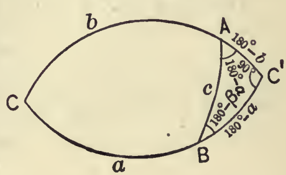
or $$\cos \alpha = \frac{\tan b}{\tan c},$$

which shows that (2) also holds in this case.

Similarly it may be shown that formulas (3) to (10) hold when a and c are each greater than 90°.

2. *When $a > 90°$ and $b > 90°$.*

Let ABC be the given spherical right triangle. Draw the lune CC'. Then in the right triangle ABC' each part is less than 90° and formulas (1) to (10) are applicable.



$$\text{From (1)}\quad \sin(180° - \alpha) = \frac{\sin(180° - a)}{\sin c}$$

or

$$\sin \alpha = \frac{\sin a}{\sin c},$$

which shows that (1) holds when a and b are each greater than 90°.

$$\text{From (2)}\quad \cos(180° - \alpha) = \frac{\tan(180° - b)}{\tan c}$$

or

$$\cos \alpha = \frac{\tan b}{\tan c},$$

which shows that (2) also holds in this case.

Similarly it may be shown that formulas (3) to (10) hold when a and b are each greater than 90°.

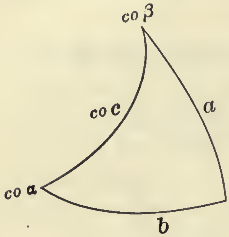
129. Sufficiency of formulas. It will be noticed that the ten formulas of Arts. 127 and 128 contain all possible combinations of the five parts of a spherical right triangle, taken three at a time; hence they are sufficient to solve any spherical right triangle directly from two given parts.

130. Comparison of formulas of plane and spherical right triangles. By rearranging the formulas of the previous article, the analogy between the formulas of the plane and the spherical right triangles is made apparent.

In Plane Right Triangles*		In Spherical Right Triangles	
$\sin \alpha = \frac{a}{c}$	$\sin \beta = \frac{b}{c}$	$\sin \alpha = \frac{\sin a}{\sin c}$	$\sin \beta = \frac{\sin b}{\sin c}$
$\cos \alpha = \frac{b}{c}$	$\cos \beta = \frac{a}{c}$	$\cos \alpha = \frac{\tan b}{\tan c}$	$\cos \beta = \frac{\tan a}{\tan c}$
$\tan \alpha = \frac{a}{b}$	$\tan \beta = \frac{b}{a}$	$\tan \alpha = \frac{\tan a}{\sin b}$	$\tan \beta = \frac{\tan b}{\sin a}$
$\sin \alpha = \cos \beta$	$\sin \beta = \cos \alpha$	$\sin \alpha = \frac{\cos \beta}{\cos b}$	$\sin \beta = \frac{\cos \alpha}{\cos a}$
$c^2 = a^2 + b^2$		$\cos c = \cos a \cos b$	
$1 = \cot \alpha \cot \beta.$		$\cos c = \cot \alpha \cot \beta.$	

*The above comparison is taken from Chauvenet's "Plane and Spherical Trigonometry."

131. Napier's Rules. The ten formulas used in the solution of spherical right triangles can all be expressed by means of two rules, known as Napier's rules of circular parts.

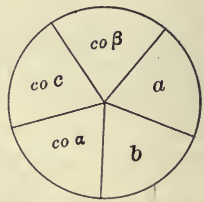


Napier's circular parts are the sides a and b, the complements of the angles opposite or $90° - \alpha$, $90° - \beta$, and the complement of the hypotenuse or $90° - c$.

They are usually written

$$a, \quad b, \quad \text{co}\,\alpha, \quad \text{co}\,\beta, \quad \text{co}\,c.$$

It will be noticed that the right angle is not one of the circular parts.



Let the five circular parts be placed in the sectors of a circle in the order in which they occur in the triangle. Whenever any three parts are considered, it is always possible to select one of them in such a manner that the other two parts will either be *adjacent* to this part, or *opposite* this part. The part having the other two parts adjacent to it or opposite it is called the **middle part**.

Thus let co α, b, and a be the parts under consideration. Then b is the middle part and co α and a are adjacent parts.

If co c, co β, and b are the parts under consideration, b is the middle part and co c and co β are opposite parts.

Napier's rules may now be stated as follows:

The sine of the middle part is equal to the product of the cosines of the opposite parts.

*The sine of the middle part is equal to the product of the tangents of the adjacent parts.**

* To associate *cosine* with *opposite* and *tangent* with *adjacent*, it may be noticed that the words *cosine* and *opposite* have the same vowels; likewise the words *tangent* and *adjacent*.

132. Theorem. *In a spherical right triangle, a side and the angle opposite terminate in the same quadrant.*

From the equation

$$\cos \alpha = \cos a \sin \beta$$

it is seen that $\cos \alpha$ and $\cos a$ must always have the same sign, since $\sin \beta$ is always positive. Hence α and a terminate in the same quadrant.

133. Theorem. *Of the three parts a, b, c, if any two terminate in the same quadrant, the third terminates in the first quadrant; if any two terminate in different quadrants, the third terminates in the second quadrant.*

This follows directly from the equation

$$\cos c = \cos a \cos b$$

by noticing that if any two of the quantities $\cos a$, $\cos b$, and $\cos c$ have like signs, the third is positive; if any two have unlike signs, the third is negative.

134. Two parts determine a triangle. In order to solve a spherical right triangle two parts, in addition to the right angle, must be given. *Each of the required parts should be obtained directly from the given parts.*

Thus, given

$$b = 50°, \ c = 110°$$

we have

$$\cos a = \frac{\cos 110°}{\cos 50°},$$

$$\cos \alpha = \tan 50° \cot 110°,$$

$$\sin \beta = \frac{\sin 50°}{\sin 110°},$$

using the formulas for spherical right triangles, or Napier's Rules. If, in the solution of a problem, the sine of any required part is found to be negative, no triangle is possible, since no part of a spherical triangle can be greater than 180°.

Likewise if the logarithmic sine or cosine of any required part is found to be greater than zero, the triangle is impossible, since no sine or cosine is numerically greater than unity.

135. The quadrant of any required part. Since the parts of a spherical triangle may have any value between 0° and 180°, it is always necessary to determine whether the required parts are greater or less than 90°. This can be done by the theorems of Arts. 132 and 133.

Thus, given

$$b = 50°, \quad c = 110°,$$

we have $a > 90°$, $\alpha > 90°$, and $\beta < 90°$.

The quadrant in which any required part terminates may also be determined from the formula used in calculating that part, by observing the signs of the functions involved. But when the unknown part is determined from its sine, the part terminates in both quadrants, giving two solutions, unless limited by the theorems of Arts. 132 and 133.

Thus, given

$$b = 50°, \quad c = 110°,$$

by writing the signs of each function above the function, we have

$$\overset{-}{\cos}\, a = \frac{\overset{-}{\cos}\, 110°}{\overset{+}{\cos}\, 50°},$$

$$\overset{-}{\cos}\, \alpha = \overset{+}{\tan}\, 50° \; \overset{-}{\cot}\, 110°,$$

$$\overset{+}{\sin}\, \beta = \frac{\overset{+}{\sin}\, 50°}{\overset{+}{\sin}\, 110°}.$$

Then $a > 90°$ and $\alpha > 90°$, since their cosines are negative, and $\beta < 90°$ by Art. 132.

136. Check formula. The formula containing the three computed parts may always be used as a check formula.

Thus, having given

$$b = 50°, \quad c = 110°,$$

the check formula is

$$\cos \alpha = \cos a \sin \beta.$$

137. Solution of a right triangle.

Given $b = 77° \; 35' \; 16''$ and $\alpha = 112° \; 19' \; 42''$

$$\overset{-}{\tan} a = \frac{\overset{+}{\sin} b}{\overset{-}{\cot} \alpha}$$

b	77° 35′ 16″
α	112° 19′ 42″
log sin b	9.98973 – 10
log cot α	9.61353 – 10
log tan a	0.37620
	67° 11′ 30″
a	112° 48′ 30″

$$\overset{+}{\cos} \beta = \overset{+}{\cos} b \; \overset{+}{\sin} \alpha$$

log cos b	9.33233 – 10
log sin α	9.96615 – 10
log cos β	9.29848 – 10
β	78° 31′ 53″

$$\overset{-}{\cot} c = \frac{\overset{-}{\cos} \alpha}{\overset{+}{\tan} b}$$

log cos α	9.57969 – 10
log tan b	0.65740
log cot c	8.92229 – 10
	85° 13′ 13″
c	94° 46′ 47″

Check

$$\overset{+}{\cos} \beta = \overset{-}{\tan} a \; \overset{-}{\cot} c$$

log tan a	0.37620
log cot c	8.92229 – 10
log cos β	9.29849 – 10

138. Two solutions or the ambiguous case. Whenever a side and the angle opposite are given, there are two solutions.

Thus if a and α are given, the only formulas by which b, β, and c can be determined are

$$\sin b = \tan a \cot \alpha,$$

$$\sin \beta = \frac{\cos \alpha}{\cos a},$$

$$\sin c = \frac{\sin a}{\sin \alpha}.$$

Since the unknown parts are obtained from their sines, each may have two values, giving two solutions, the theorems of Arts. 132 and 133 not restricting the values of the parts to one solution.

Having found the two values for each part, the theorems of Arts. 132 and 133 determine the values that belong to each solution.

Thus, given $a = 155° \, 27' \, 45''$ to find b
$\alpha = 100° \, 21' \, 50''$ c
β

SOLUTION

$$\overset{+}{\sin} b = \overset{-}{\tan} a \overset{-}{\cot} \alpha$$

a	$155° \, 27' \, 45''$
α	$100° \, 21' \, 50''$
$\log \tan a$	$9.65945 - 10$
$\log \cot \alpha$	$9.26217 - 10$
$\log \sin b$	$8.92162 - 10$
b	$4° \, 47' \, 20''$ and
	$175° \, 12' \, 40''$

$$\overset{+}{\sin} \beta = \frac{\overset{-}{\cos} \alpha}{\overset{-}{\cos} a}$$

$\log \cos \alpha$	$9.25503 - 10$
$\log \cos a$	$9.95889 - 10$
$\log \sin \beta$	$9.29614 - 10$
β	$11° \, 24' \, 22''$ and
	$168° \, 35' \, 38''$

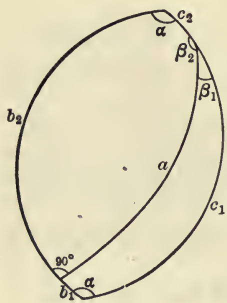
$$\overset{+}{\sin} c = \frac{\overset{+}{\sin} a}{\overset{+}{\sin} \alpha}$$

$\log \sin a$	$9.61835 - 10$
$\log \sin \alpha$	$9.99286 - 10$
$\log \sin c$	$9.62549 - 10$
c	$24° \, 58' \, 18''$ and
	$155° \; 1' \, 42''$

Check

$$\overset{+}{\sin} b = \overset{+}{\sin} c \overset{+}{\sin} \beta$$

$\log \sin c$	$9.62549 - 10$
$\log \sin \beta$	$9.29614 - 10$
$\log \sin b$	$8.92163 - 10$



FIRST SOLUTION	SECOND SOLUTION
$b_1 = 4° \, 47' \, 20''$	$b_2 = 175° \, 12' \, 40''$
$\beta_1 = 11° \, 24' \, 22''$	$\beta_2 = 168° \, 35' \, 38''$
$c_1 = 155° \; 1' \, 42''$	$c_2 = 24° \, 58' \, 18''$

139. EXAMPLES

Solve the following spherical right triangles, right-angled at γ.

1. $b = 10° 32'$
   $\alpha = 12° 3'$

2. $a = 25° 18'$
   $b = 32° 41'$

3. $c = 120° 37'$
   $\beta = 9° 49'$

4. $c = 46° 40'$
   $\alpha = 20° 50'$

5. $a = 115° 6'$
   $b = 123° 14'$

6. $a = 112° 43' 30''$
   $c = 85° 10' 10''$

7. $a = 15° 18' 20''$
   $c = 21° 30' 40''$

8. $b = 168° 13' 45''$
   $c = 150° 9' 20''$

9. $a = 132° 25'$
   $\alpha = 107° 30'$

10. $c = 80° 3' 20''$
    $\beta = 135° 16' 30''$

11. $b = 171° 3' 15''$
    $c = 12° 20' 30''$

12. $a = 35° 54' 20''$
    $\alpha = 47° 6' 10''$

13. $b = 15° 2' 30''$
    $\beta = 20° 11' 40''$

14. $\alpha = 20° 26' 20''$
    $\beta = 84° 41' 40''$

15. $a = 25° 41' 30''$
    $\alpha = 34° 25' 40''$

140. Quadrantal triangles. A quadrantal triangle is a triangle one of whose sides is 90°. Its polar triangle is then a right triangle. The solution of a quadrantal triangle is effected through the solution of its polar triangle.

141. Isosceles triangles. The solution of an isosceles triangle is effected by solving the two equal right triangles formed by dropping a perpendicular from the vertex to the base.

142. EXAMPLES

Solve the following triangles.

1. $\alpha = 117° 54' 30''$
   $b = 95° 42' 20''$
   $c = 90°$

2. $\alpha = 69° 45'$
   $\beta = 94° 40'$
   $c = 90°$

3. $\beta = 153° 16'$
   $a = 19° 3'$
   $c = 90°$

4. $a = 159° 33' 40''$
   $b = 95° 18' 20''$
   $c = 90°$

5. The base of an isosceles triangle is 51° 8′. The equal angles are each 41° 57′. Find the equal sides and the angle at the vertex.

6. The base angles of an isosceles triangle are each 100° 12′ 30″, the vertical angle is 50° 19′ 40″. Find the equal sides and the base.

CHAPTER XII

OBLIQUE SPHERICAL TRIANGLE

143. In the present chapter the general formulas of spherical trigonometry, already developed (Chap. X) are transformed into standard formulas adapted to logarithmic computation; and the problem of the solution of the spherical triangle is discussed.

GENERAL SOLUTION

144. *To find the angles when the three sides are given.*

From Art. 119, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

Therefore $$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{1}$$

But $$2 \sin^2 \tfrac{1}{2} \alpha = 1 - \cos \alpha$$

$$= \frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c}$$

$$= -\frac{\cos a - \cos (b - c)}{\sin b \sin c}.$$

Applying formula (4) of Art. 73, we have

$$\sin^2 \tfrac{1}{2} \alpha = \frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c}.$$

Letting $2s = a + b + c$, we have

$$\sin \tfrac{1}{2} \alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}. \tag{2}$$

Similarly uniting (1) with

$$2\cos^2\tfrac{1}{2}\alpha = 1 + \cos\alpha,$$

we have
$$2\cos^2\tfrac{1}{2}\alpha = \frac{\sin b\sin c + \cos a - \cos b\cos c}{\sin b\sin c}$$

$$= -\frac{\cos(b+c) - \cos a}{\sin b\sin c}$$

$$= \frac{2\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(-a+b+c)}{\sin b\sin c}.$$

Therefore
$$\cos\tfrac{1}{2}\alpha = \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}. \qquad (3)$$

Uniting (2) and (3),

$$\tan\tfrac{1}{2}\alpha = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} = \frac{\tan r}{\sin(s-a)}. \qquad (4)$$

Similarly
$$\tan\tfrac{1}{2}\beta = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}} = \frac{\tan r}{\sin(s-b)}, \qquad (5)$$

and
$$\tan\tfrac{1}{2}\gamma = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}} = \frac{\tan r}{\sin(s-c)}, \qquad (6)$$

where
$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \qquad (7)$$

145. *To find the sides when the three angles are given.*

Following the method of the last article, the equation

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$

gives
$$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-\alpha)}{\sin\beta\sin\gamma}},$$

and
$$\cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S-\beta)\cos(S-\gamma)}{\sin\beta\sin\gamma}},$$

and
$$\tan\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-\alpha)}{\cos(S-\beta)\cos(S-\gamma)}} = \tan R\cos(S-\alpha)$$

where $2S = \alpha + \beta + \gamma,$

and $\tan R = \sqrt{\dfrac{-\cos S}{\cos (S-\alpha)\cos (S-\beta)\cos (S-\gamma)}}.$

146. Delambre's or Gauss's formulas express relations between the six parts.

By Art. 68

$$\sin \tfrac{1}{2}(\alpha+\beta) = \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta + \cos \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta.$$

Substituting for $\sin \frac{1}{2}\alpha$, $\cos \frac{1}{2}\alpha$, $\sin \frac{1}{2}\beta$, and $\cos \frac{1}{2}\beta$ their values in terms of the sides of the triangle, and simplifying, we have

$$\sin \tfrac{1}{2}(\alpha+\beta) = \frac{\sin (s-b) + \sin (s-a)}{\sin c}\sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}$$

$$= \frac{2 \sin \frac{1}{2}(2s-a-b) \cos \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} \cdot \cos \tfrac{1}{2}\gamma.$$

Then $$\sin \tfrac{1}{2}(\alpha+\beta) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c} \cdot \cos \tfrac{1}{2}\gamma. \tag{1}$$

Similarly $$\sin \tfrac{1}{2}(\alpha-\beta) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c} \cdot \cos \tfrac{1}{2}\gamma, \tag{2}$$

and $$\cos \tfrac{1}{2}(\alpha+\beta) = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c} \cdot \sin \tfrac{1}{2}\gamma, \tag{3}$$

and $$\cos \tfrac{1}{2}(\alpha-\beta) = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c} \cdot \sin \tfrac{1}{2}\gamma. \tag{4}$$

147. Napier's analogies express relations between five parts of a triangle. They are easily obtained from Gauss's formulas.

Dividing (1) by (2), Art. 146,

$$\frac{\sin \frac{1}{2}(\alpha+\beta)}{\sin \frac{1}{2}(\alpha-\beta)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a-b)}. \tag{1}$$

Dividing (3) by (4),

$$\frac{\cos \frac{1}{2}(\alpha+\beta)}{\cos \frac{1}{2}(\alpha-\beta)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a+b)}. \tag{2}$$

Dividing (4) by (2),

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}\gamma}{\tan\frac{1}{2}(\alpha-\beta)}. \tag{3}$$

Dividing (3) by (1),

$$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}\gamma}{\tan\frac{1}{2}(\alpha+\beta)}. \tag{4}$$

148. Formulas collected. The following formulas are sufficient to solve a spherical triangle when any three parts are given:

$$*\tan\tfrac{1}{2}\alpha=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}=\frac{\tan r}{\sin(s-a)} \qquad \mathrm{I}$$

$$\tan\tfrac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-\alpha)}{\cos(S-\beta)\cos(S-\gamma)}}=\tan R\cos(S-\alpha) \qquad \mathrm{II}$$

$$\frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)} \qquad \mathrm{III}$$

$$\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)} \qquad \mathrm{IV}$$

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}\gamma}{\tan\frac{1}{2}(\alpha-\beta)} \qquad \mathrm{V}$$

$$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}\gamma}{\tan\frac{1}{2}(\alpha+\beta)} \qquad \mathrm{VI}$$

$$\frac{\sin a}{\sin\alpha}=\frac{\sin b}{\sin\beta}. \qquad \mathrm{VII}$$

Formula I is used to determine an angle when the three sides are given.

Formula II is used to determine a side when the three angles are given.

Formulas III and IV are used when two angles and the

* These formulas are typical. Other formulas of the same type are obtained by a cyclic change of letters.

included side are given. Formula III determines $\frac{1}{2}(a-b)$ and formula IV determines $\frac{1}{2}(a+b)$, from which a and b are obtained. Either formula may also be used to determine the side c when the other two sides and their opposite angles are given.

Formulas V and VI are used when two sides and the included angle are given. Formula V determines $\frac{1}{2}(\alpha-\beta)$ and formula VI determines $\frac{1}{2}(\alpha+\beta)$, from which α and β are obtained. Either formula may also be used to determine the angle γ when the other two angles and their opposite sides are given.

Formula VII is used when an angle and the side opposite are among the given parts.

149. Whenever the formulas I to VI are employed in the solution of a spherical triangle as indicated above, the quadrant in which any part terminates may always be determined by noticing the signs of the functions involved.

But when the law of sines is employed, two values are found for the required part. This leads to two solutions unless limited to one solution by the following principles.

150. **Theorem.** *Half the sum of any two angles is in the same quadrant as half the sum of the sides opposite.*

This follows from a consideration of the signs of the functions involved in

$$\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}.$$

Since each part is less than 180°, $\tan\frac{1}{2}c$ and $\cos\frac{1}{2}(\alpha-\beta)$ are always positive. Hence $\cos\frac{1}{2}(\alpha+\beta)$ and $\tan\frac{1}{2}(a+b)$ must always have the same sign. Hence $\frac{1}{2}(\alpha+\beta)$ and $\frac{1}{2}(a+b)$ terminate in the same quadrant.

151. **Theorem.** *A side which differs more from 90° than another side, terminates in the same quadrant as its opposite angle.*

We have from Art. 119

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

If a differs more from 90° than b, $\cos a$ is numerically greater than $\cos b$. $\operatorname{Cos} a$ is also greater than $\cos b \cos c$, since $\cos c$ is not greater than unity. Hence the numerator of this fraction has the same sign as $\cos a$. The denominator being always positive, $\cos a$ and $\cos \alpha$ have the same sign. Hence a and α terminate in the same quadrant.

The negative of this theorem is not true.

Thus given, $a = 165°$, $b = 120°$, $\beta = 135°$, α terminates in the second quadrant, since a differs more from 90° than b.

152. Theorem. *An angle which differs more from 90° than another angle, terminates in the same quadrant as its opposite side.*

This follows from

$$\cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}$$

by considerations similar to those of the previous article.

Thus given, $\alpha = 80°$, $\gamma = 140°$, and $a = 120°$, c terminates in the second quadrant, since γ differs more from 90° than α.

153. Illustrative examples.

1. Given the three sides, $a = 105° 27' 20''$, $b = 83° 14' 40''$, $c = 96° 53' 10''$, to find α, β, and γ.

a	105° 27′ 20″
b	83° 14′ 40″
c	96° 53′ 10″
$2s$	284° 94′ 70″
s	142° 47′ 35″
$s - a$	37° 20′ 15″
$s - b$	59° 32′ 55″
$s - c$	45° 54′ 25″
Check: s	142° 47′ 35″

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$\tan \tfrac{1}{2}\alpha = \frac{\tan r}{\sin(s-a)}$$

$$\tan \tfrac{1}{2}\beta = \frac{\tan r}{\sin(s-b)}$$

$$\tan \tfrac{1}{2}\gamma = \frac{\tan r}{\sin(s-c)}$$

$\log \sin (s-a)$	$9.78284-10$
$\log \sin (s-b)$	$9.93553-10$
$\log \sin (s-c)$	$9.85623-10$
$\operatorname{colog} \sin s$	0.21846
$2 \log \tan r$	$9.79306-10$
$\log \tan r$	$9.89653-10$
$\log \tan \frac{1}{2}\alpha$	0.11369
$\log \tan \frac{1}{2}\beta$	$9.96100-10$
$\log \tan \frac{1}{2}\gamma$	0.04030
$\frac{1}{2}\alpha$	$52° 24' 55''$
$\frac{1}{2}\beta$	$42° 25' 51''$
$\frac{1}{2}\gamma$	$47° 39' 17''$
α	$104° 49' 50''$
β	$84° 51' 42''$
γ	$95° 18' 34''$

Check

$$\cot \tfrac{1}{2}\gamma = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(\alpha - \beta)$$

$a+b$	$188° 42'$ 0
$a-b$	$22° 12' 40'$
$\frac{1}{2}(a+b)$	$94° 21' 0'$
$\frac{1}{2}(a-b)$	$11° 6' 20'$
$\frac{1}{2}(\alpha-\beta)$	$9° 59' 4'$
$\log \sin \frac{1}{2}(a+b)$	$9.99875-10$
$\log \tan \frac{1}{2}(\alpha-\beta)$	$9.24563-10$
$\operatorname{colog} \sin \frac{1}{2}(a-b)$	0.71531
$\log \cot \frac{1}{2}\gamma$	$9.95969-10$
$\log \tan \frac{1}{2}\gamma$	0.04031

2. Given two sides and the included angle, $a = 29° 18'$, $b = 37° 30'$, $\gamma = 51° 52'$, to find α, β, and c.

$$\tan \tfrac{1}{2}(\beta - \alpha) = \frac{\sin \frac{1}{2}(b-a)}{\sin \frac{1}{2}(b+a)} \cot \tfrac{1}{2}\gamma$$

$$\tan \tfrac{1}{2}(\beta + \alpha) = \frac{\cos \frac{1}{2}(b-a)}{\cos \frac{1}{2}(b+a)} \cot \tfrac{1}{2}\gamma$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(\beta+\alpha)}{\sin \frac{1}{2}(\beta-\alpha)} \tan \tfrac{1}{2}(b-a)$$

b	$37° 30'$
a	$29° 18'$
γ	$51° 52'$
$b-a$	$8° 12'$
$b+a$	$66° 48'$
$\frac{1}{2}(b-a)$	$4° 6'$
$\frac{1}{2}(b+a)$	$33° 24'$
$\frac{1}{2}\gamma$	$25° 56'$
$\log \sin \frac{1}{2}(b-a)$	$8.85429-10$
$\log \cot \frac{1}{2}\gamma$	0.31310
$\operatorname{colog} \sin \frac{1}{2}(b+a)$	0.25926
$\log \tan \frac{1}{2}(\beta-\alpha)$	$9.42665-10$
$\frac{1}{2}(\beta-\alpha)$	$14° 57' 14''$

$\log \cos \frac{1}{2}(b-a)$	$9.99889-10$
$\log \cot \frac{1}{2}\gamma$	0.31310
$\operatorname{colog} \cos \frac{1}{2}(b+a)$	0.07839
$\log \tan \frac{1}{2}(\beta+\alpha)$	0.39038
$\frac{1}{2}(\beta+\alpha)$	$67° 51' 8''$
$\frac{1}{2}(\beta-\alpha)$	$14° 57' 14''$
β	$82° 48' 22''$
α	$52° 53' 54''$

$\log \tan \frac{1}{2}(b-a)$	$8.85540-10$
$\log \sin \frac{1}{2}(\beta+\alpha)$	$9.96671-10$
$\operatorname{colog} \sin \frac{1}{2}(\beta-\alpha)$	0.58831
$\log \tan \frac{1}{2}c$	$9.41042-10$
$\frac{1}{2}c$	$14° 25' 43''$
c	$28° 51' 26''$

Check

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

$\log \sin a$	$9.68965 - 10$
$\log \sin \alpha$	$9.90177 - 10$
	$9.78788 - 10$

$\log \sin b$	$9.78445 - 10$
$\log \sin \beta$	$9.99656 - 10$
	$9.78789 - 10$

$\log \sin c$	$9.68361 - 10$
$\log \sin \gamma$	$9.89574 - 10$
	$9.78787 - 10$

3. Given two sides, and an angle opposite one of them, $\alpha = 63° 24' 50''$, $b = 17° 36' 40''$, $a = 44° 48' 20''$, to find β, γ, and c.

$$\sin \beta = \frac{\sin \alpha}{\sin a} \sin b$$

$$\cot \tfrac{1}{2}\gamma = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(\alpha - \beta)$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(\alpha+\beta)}{\sin \frac{1}{2}(\alpha-\beta)} \tan \tfrac{1}{2}(a - b)$$

α	$63° 24' 50''$
a	$44° 48' 20''$
b	$17° 36' 40''$
$a + b$	$62° 25' \ 0''$
$a - b$	$27° 11' 40''$
$\frac{1}{2}(a+b)$	$31° 12' 30''$
$\frac{1}{2}(a-b)$	$13° 35' 50''$
$\log \sin \alpha$	$9.95147 - 10$
$\log \sin b$	$9.48081 - 10$
$\operatorname{colog} \sin a$	0.15200
$\log \sin \beta$	$9.58428 - 10$
β	$22° 34' 44''$
$\alpha - \beta$	$40° 50' \ 6''$
$\alpha + \beta$	$85° 59' 34''$
$\frac{1}{2}(\alpha-\beta)$	$20° 25' \ 3''$
$\frac{1}{2}(\alpha+\beta)$	$42° 59' 47''$

$\log \sin \frac{1}{2}(a+b)$	$9.71445 - 10$
$\log \tan \frac{1}{2}(\alpha-\beta)$	$9.57083 - 10$
$\operatorname{colog} \sin \frac{1}{2}(a-b)$	0.62876
$\log \cot \frac{1}{2}\gamma$	$9.91404 - 10$
$\frac{1}{2}\gamma$	$50° 38' 0''$
γ	$101° 16' 0'$

$\log \sin \frac{1}{2}(\alpha+\beta)$	$9.83375 - 10$
$\log \tan \frac{1}{2}(a-b)$	$9.38359 - 10$
$\operatorname{colog} \sin \frac{1}{2}(\alpha-\beta)$	0.45735
$\log \tan \frac{1}{2} c$	$9.67469 - 10$
$\frac{1}{2} c$	$25° 18' 20''$
c	$50° 36' 40'$

Check

$$\frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

$\log \sin b$	$9.48081 - 10$
$\log \sin \beta$	$9.58428 - 10$
	$9.89653 - 10$

$\log \sin c$	$9.88810 - 10$
$\log \sin \gamma$	$9.99155 - 10$
	$9.89655 - 10$

154. Two solutions. There are two solutions, if any, whenever two sides and an angle opposite one of them, or two angles and a side opposite one of them, are given, unless limited to one solution by the principles of Arts. 150, 151, and 152.

Thus, having given $\beta = 45° 15' 12''$, $b = 56° 49' 46''$, $a = 68° 52' 48''$, to find α, γ, and c.

$$\sin \alpha = \frac{\sin \beta}{\sin b} \sin a$$

a	68° 52′ 48″	α	52° 19′ 33″	127° 40′ 27″
b	56° 49′ 46″	$\alpha + \beta$	97° 34′ 45″	172° 55′ 39″
β	45° 15′ 12″	$\alpha - \beta$	7° 4′ 21″	82° 25′ 15″
log sin β	9.85140	$\frac{1}{2}(\alpha + \beta)$	48° 47′ 22″	86° 27′ 50″
log sin a	9.96980	$\frac{1}{2}(\alpha - \beta)$	3° 32′ 10″	41° 12′ 38″
colog sin b	0.07725	$a + b$	125° 42′ 34″	
log sin α	9.89845	$a - b$	12° 3′ 2″	
α	52° 19′ 33″, or	$\frac{1}{2}(a + b)$	62° 51′ 17″	
	127° 40′ 27″	$\frac{1}{2}(a - b)$	6° 1′ 31″	

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}(\alpha - \beta)} \tan \tfrac{1}{2}(a - b)$$

log sin $\frac{1}{2}(\alpha + \beta)$	9.87639	9.99917
log tan $\frac{1}{2}(a - b)$	9.02346	9.02346
colog sin $\frac{1}{2}(\alpha - \beta)$	1.20984	0.18124
log tan $\frac{1}{2} c$	0.10969	9.20387
$\frac{1}{2} c$	52° 9′ 35″	9° 5′ 7″
c	104° 19′ 10″	18° 10′ 14″

$$\cot \tfrac{1}{2} \gamma = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(\alpha - \beta)$$

log sin $\frac{1}{2}(a + b)$	9.94932	9.94932
log tan $\frac{1}{2}(\alpha - \beta)$	8.79099	9.94238
colog sin $\frac{1}{2}(a - b)$	0.97895	0.97895
log cot $\frac{1}{2} \gamma$	9.71926	0.87065
$\frac{1}{2} \gamma$	62° 21′ 1″	7° 40′ 16″
γ	124° 42′ 2″	15° 20′ 32″

First Solution

$\alpha = 52° 19' 33''$
$c = 104° 19' 10''$
$\gamma = 124° 42' 2''$

Check

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)} \tan \tfrac{1}{2}(a + b)$$

$\log \cos \frac{1}{2}(\alpha + \beta)$	9.81877	8.79013
$\log \tan \frac{1}{2}(a + b)$	0.29012	0.29012
$\text{colog} \cos \frac{1}{2}(\alpha - \beta)$	0.00083	0.12361
$\log \tan \frac{1}{2} c$	0.10972	9.20386

Second Solution

$\alpha = 127° 40' 27''$
$c = 18° 10' 14''$
$\gamma = 15° 20' 32''$

$$\cot \tfrac{1}{2} \gamma = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(\alpha + \beta)$$

$\log \cos \frac{1}{2}(a + b)$	9.65920	9.65920
$\log \tan \frac{1}{2}(\alpha + \beta)$	0.05762	0.20904
$\text{colog} \cos \frac{1}{2}(a - b)$	0.00241	0.00241
$\log \cot \frac{1}{2} \gamma$	9.71923	0.87065

155. Area of spherical triangle. Representing the area of a sphere, S, by 720 spherical degrees, it is demonstrated in geometry that the area of a spherical triangle, A, *in terms of spherical degrees*, is equal to its spherical excess; or

$$\boldsymbol{A = (\alpha + \beta + \gamma - 180)} \textbf{ spherical degrees}$$

Hence
$$\frac{A}{S} = \frac{\alpha + \beta + \gamma - 180}{720}.$$

Let A' and S' represent the area of the triangle and the sphere respectively, in terms of the unit in which r is expressed. Since the ratio of the area of the spherical triangle to the area of the sphere is independent of the units used, we have
$$\frac{A'}{S'} = \frac{A}{S}.$$

Therefore
$$\frac{A'}{S'} = \frac{\alpha + \beta + \gamma - 180}{720}.$$

But the area of the sphere expressed in terms of r is $4\pi r^2$, therefore the area of the triangle is given by
$$A' = \pi r^2 \frac{\alpha + \beta + \gamma - 180}{180}$$

156. EXAMPLES

Solve the following spherical triangles and check the results:

1. $b = 10° \ 0' \ 10''$
 $c = 114° \ 40' \ 40''$
 $\alpha = 92° \ 28' \ 20''$

2. $b = 85° \ 4' \ 19''$
 $c = 139° \ 58' \ 25''$
 $\alpha = 12° \ 20' \ 31''$

3. $c = 82° \ 3' \ 4''$
 $a = 70° \ 14' \ 12''$
 $\beta = 84° \ 20' \ 9''$

4. $a = 95° \ 3' \ 30''$
 $b = 128° \ 38' \ 50''$
 $\gamma = 170° \ 52' \ 20''$

5. $a = 29° \ 18'$
 $b = 37° \ 30'$
 $\gamma = 51° \ 52'$

6. $\alpha = 11° \ 21' \ 10''$
 $\gamma = 19° \ 0' \ 20''$
 $b = 66° \ 19' \ 30''$

7. $\gamma = 179° \ 22' \ 11''$
 $\alpha = 148° \ 17' \ 17''$
 $b = 25° \ 39' \ 34''$

8. $a = 108° \ 5' \ 18''$
 $b = 170° \ 30' \ 46''$
 $c = 85° \ 50' \ 22''$

9. $a = 105° \ 27' \ 20''$
 $b = 83° \ 14' \ 40''$
 $c = 96° \ 53' \ 10''$

10. $a = 34° \ 19' \ 30''$
 $b = 28° \ 37' \ 10''$
 $c = 22° \ 44' \ 40''$

11. $\alpha = 61° \ 8'$
 $\beta = 59° \ 12'$
 $\gamma = 78° \ 25'$

12. $a = 76° \ 43' \ 15''$
 $b = 83° \ 35' \ 27''$
 $c = 98° \ 26' \ 38''$

13. $\alpha = 110° \ 35'$
 $\beta = 135° \ 42'$
 $\gamma = 146° \ 8'$

14. $\gamma = 11° \ 34' \ 10''$
 $b = 82° \ 56' \ 30''$
 $c = 27° \ 9' \ 40''$

15. $\alpha = 32° \ 4' \ 10''$
 $\beta = 128° \ 56' \ 20''$
 $a = 39° \ 50' \ 30''$

16. $\beta = 80° \ 40' \ 2''$
 $c = 75° \ 54' \ 0''$
 $b = 100° \ 21' \ 28''$

17. $\alpha = 21° \ 1' \ 10''$
 $\gamma = 17° \ 22' \ 50''$
 $a = 14° \ 13' \ 30''$

18. $\beta = 77° \ 44' \ 55''$
 $\gamma = 92° \ 17' \ 24''$
 $a = 26° \ 29' \ 39''$

19. $\alpha = 114° \ 23' \ 9''$
 $\beta = 88° \ 41' \ 11''$
 $\gamma = 79° \ 0' \ 4''$

20. $\beta = 99° \ 4' \ 12''$
 $\gamma = 106° \ 0' \ 9''$
 $a = 161° \ 2' \ 10''$

21. $c = 100° \, 10' \, 40''$
$a = 65° \, 20' \, 30''$
$\gamma = 94° \, 30' \, 10''$

22. $a = 103° \, 19' \, 50''$
$\beta = 92° \, 37' \, 30''$
$\gamma = 128° \, 54' \, 20''$

23. $b = 42° \, 15' \, 20''$
$c = 127° \; 3' \, 30''$
$\beta = 31° \, 44' \, 20''$

24. $\gamma = 127° \; 4' \, 10''$
$a = 88° \, 12' \; 0''$
$c = 141° \, 20' \, 30''$

SOLUTION WHEN ONLY ONE PART IS REQUIRED

157. In many problems of astronomy and geodesy, it is required to find only one or two of the unknown parts of a spherical triangle, the other unknown parts being of no importance in the problem. It then becomes desirable to have a method whereby the required parts can be computed without being under the necessity of first computing any part not desired.

It has already been shown that any angle can be found directly from three given sides (Art. 144), and that any side can be found directly from three given angles (Art. 145).

It is evident that any part can be found from any three given parts by the use of the general formulas containing the required part and the three given parts. By the introduction of auxiliary quantities, these formulas will now be adapted to logarithmic computation.

158. *Given two sides and the included angle, to find any one of the remaining parts.*

Let a, b, γ be the given parts.

First. To find c.

The relation between a, b, γ, and c is (Art. 119),

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma. \qquad (1)$$

To adapt this formula to logarithmic computation, let

$$m \sin M = \sin b \cos \gamma, \qquad (2)$$

and

$$m \cos M = \cos b. \qquad (3)$$

Then eliminating b by uniting (1), (2), and (3), we have

$$\cos c = m(\cos a \cos M + \sin a \sin M),$$

or

$$\cos c = m \cos(a - M). \tag{4}$$

From (2) and (3)

$$\tan M = \tan b \cos \gamma, \tag{5}$$

and from (3) and (4)

$$\cos c = \frac{\cos b \cos(a - M)}{\cos M}. \tag{6}$$

Equations (5) and (6) enable us to find c.

ILLUSTRATION. Given $a = 75° 38' 20''$, $b = 54° 54' 38''$, and $\gamma = 30° 17' 43''$.

$\tan M = \tan b \cos \gamma$

a	75° 38′ 20″
b	54° 54′ 38″
γ	30° 17′ 43″
log tan b	0.15333
log cos γ	9.93623
log tan M	0.08956
M	50° 51′ 58″
$a - M$	24° 46′ 22″

$\cos c = \dfrac{\cos b \cos(a - M)}{\cos M}$

log cos b	9.75956
log cos $(a - M)$	9.95808
colog cos M	0.19987
log cos c	9.91751
c	34° 12′ 27″

Second. To find β.

The relation between a, b, γ, and β is given by Art. 124, (equation 5), from which we have

$$\cot \beta = \frac{\cot b \sin a - \cos a \cos \gamma}{\sin \gamma}. \tag{7}$$

Multiplying numerator and denominator by $\sin b$,

$$\cot \beta = \frac{\cos b \sin a - \sin b \cos a \cos \gamma}{\sin b \sin \gamma}. \tag{8}$$

Again using equations (2) and (3) we have

$$\cot \beta = \frac{m(\cos M \sin a - \sin M \cos a)}{\sin b \sin \gamma}, \tag{9}$$

or

$$\cot \beta = \frac{m \sin(a - M)}{\sin b \sin \gamma}. \tag{10}$$

As before $\tan M = \tan b \cos \gamma$. (11)

From (2) and (10)

$$\cot \beta = \frac{\cot \gamma \sin (a - M)}{\sin M}. \tag{12}$$

Equations (11) and (12) enable us to find β.

Third. To find γ.

By interchanging a and b and consequently α and β in (11) and (12) we have, calling the auxiliary angle N,

$$\tan N = \tan a \cos \gamma \tag{13}$$

and

$$\cot \alpha = \frac{\cot \gamma \sin (b - N)}{\sin N} \tag{14}$$

to determine α.

159. Two parts required. It will be noticed that the same auxiliary quantity M is used to find both c and β. We thus have a convenient method, much used in astronomy, for finding a side and an angle when two sides and the included angle are given.

For finding c and β we have, collecting our formulas,

$$\tan M = \tan b \cos \gamma$$

$$\cos c = \frac{\cos b \cos (a - M)}{\cos M}$$

$$\cot \beta = \frac{\cot \gamma \sin (a - M)}{\sin M}.$$

Dividing equation (10) of Art. 158 by equation (4), we have

$$\frac{\cot \beta}{\cos c} = \frac{\tan (a - M)}{\sin b \sin \gamma},$$

which serves as a check upon c and β.

Similarly, for finding c and α, we have

$$\tan N = \tan a \cos \gamma$$

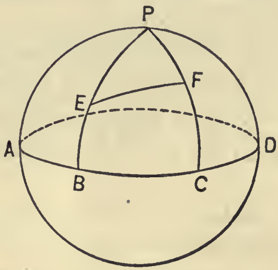
$$\cos c = \frac{\cos a \cos (b - N)}{\cos N}$$

$$\cot \alpha = \frac{\cot \gamma \sin (b - N)}{\sin N}.$$

160. PROBLEMS

An arc of 1′ on the earth's surface is equal to one English geographical mile.

1. Find the distance between Boston, latitude 42° 21′ N., longitude 71° 41′ W., and San Francisco, latitude 37° 48′ N. and longitude 122° 28′ W.



SOLUTION. — Let APD be the meridian of Greenwich from which longitude is measured, $ABCD$ the equator, and P the north pole.

Let the positions of San Francisco and Boston be represented by E and F respectively. The desired distance is EF.

Then
$$\text{angle } DPE = 122° \, 28',$$
$$\text{angle } DPF = 71° \, 41',$$
$$\text{arc } EB = 37° \, 48',$$
$$\text{arc } CF = 42° \, 21',$$
$$\text{arc } PB = \text{arc } PC = 90°.$$

But
$$PE = PB - EB = 52° \, 12',$$
$$PF = PC - FC = 47° \, 39',$$
$$\angle FPE = \angle DPE - \angle DPF = 50° \, 47'.$$

Letting
$$\angle FPE = \gamma, \; PE = a, \; PF = b,$$

we may find EF or c by the formulas

$$\tan M = \tan b \cos \gamma, \; \cos c = \frac{\cos b \cos (a - M)}{\cos M}.$$

$\tan M = \tan b \cos \gamma$

a	52° 12′
b	47° 39′
γ	50° 47′
$\log \tan b$	0.04023
$\log \cos \gamma$	9.80089 − 10
$\log \tan M$	9.84112 − 10
M	34° 44′ 23″
$a - M$	17° 27′ 37″

$$\cos c = \frac{\cos b \cos (a - M)}{\cos M}$$

$\log \cos b$	9.82844 − 10
$\log \cos (a - M)$	9.97951 − 10
$\operatorname{colog} \cos M$	0.08526
$\log \cos c$	9.89321 − 10
c	38° 33′ 20″, or
	2313⅓ miles

2. Find the distance between New York (lat. 40° 43′ N., long. 74° 0′ W.) and San Francisco (lat. 37° 48′ N., long. 122° 28′ W.).

3. Find the distance between Calcutta (lat. 22° 33′ N., long. 88° 19′ E.) and Greenwich (lat. 51° 29′ N.).

4. Find the distance between Baltimore (lat. 39° 17′ N., long. 76° 37′ W.) and Calcutta.

161. *Given two angles and the included side, to find any one of the remaining parts.*

Let α, β, c be the given parts.

First. To find γ.

The relation between α, β, c, and γ is, Art. 121, eq. (3),

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c. \qquad (1)$$

Let $m \sin M = \cos \alpha$, (2)

and $m \cos M = \sin \alpha \cos c$. (3)

Uniting (1), (2), and (3)

$$\cos \gamma = m\,(-\sin M \cos \beta + \cos M \sin \beta),$$

or

$$\cos \gamma = m \sin (\beta - M). \qquad (4)$$

From (2) and (3)

$$\cot M = \tan \alpha \cos c, \qquad (5)$$

and from (2) and (4)

$$\cos \gamma = \frac{\cos \alpha \sin (\beta - M)}{\sin M}. \tag{6}$$

Equations (5) and (6) enable us to find γ.

Second. To find a.

The relation between α, β, c, and a is given by Art. 124, equation (4), from which

$$\cot a = \frac{\sin \beta \cot \alpha + \cos c \cos \beta}{\sin c}, \tag{7}$$

or
$$\cot a = \frac{\sin \beta \cos \alpha + \sin \alpha \cos c \cos \beta}{\sin \alpha \sin c}. \tag{8}$$

Uniting equations (2), (3), and (8),

$$\cot a = \frac{m(\sin \beta \sin M + \cos \beta \cos M)}{\sin \alpha \sin c}, \tag{9}$$

or
$$\cot a = \frac{m \cos (\beta - M)}{\sin \alpha \sin c}. \tag{10}$$

From (2), (3), and (10)

$$\cot M = \tan \alpha \cos c \tag{11}$$

and
$$\cot a = \frac{\cot c \cos (\beta - M)}{\cos M}, \tag{12}$$

from which a is found.

Third. To find b.

From (11) and (12) by interchanging α and β, and consequently a and b, we have, calling the auxiliary quantity N,

$$\cot N = \tan \beta \cos c \tag{13}$$

and
$$\cot b = \frac{\cot c \cos (\alpha - N)}{\cos N} \tag{14}$$

to determine b.

162. *Given two sides and an angle opposite one of them, to find any one of the remaining parts.*

Let a, b, α be the given parts.

First. To find c.

The relation between a, b, α and c is

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha. \tag{1}$$

Let $$m \sin M = \sin b \cos \alpha, \tag{2}$$

and $$m \cos M = \cos b. \tag{3}$$

Then $$\cos a = m \cos (c - M). \tag{4}$$

From equations (2), (3), and (4),

$$\tan M = \tan b \cos \alpha \tag{5}$$

and $$\cos (c - M) = \frac{\cos a \cos M}{\cos b}. \tag{6}$$

Equation (5) determines M and equation (6) determines $c - M$. Adding these values, we have c.

In general there are 2 solutions for c. We may limit M to positive values less than 180°. By equation (6) $c - M$ may have two values, numerically equal but opposite in sign, giving two values for c unless the sum $M + (c - M)$ is greater than 180° or negative, in which case there is but one solution.

Second. To find γ.

The relation between a, b, α, and γ is

$$\sin \gamma \cot \alpha = \cot a \sin b - \cos b \cos \gamma.$$

Multiplying by $\sin \alpha$ and rearranging, we have

$$\sin \gamma \cos \alpha + \cos \gamma \sin \alpha \cos b = \sin \alpha \cot a \sin b. \tag{7}$$

Let $$n \cos N = \cos \alpha, \tag{8}$$

and $$n \sin N = \sin \alpha \cos b. \tag{9}$$

From (7), (8), and (9) we have

$$n \sin (\gamma + N) = \sin \alpha \cot a \sin b. \tag{10}$$

Then from (8), (9), and (10),

$$\tan N = \tan \alpha \cos b, \tag{11}$$

and $$\sin (\gamma + N) = \sin N \cot a \tan b. \tag{12}$$

Equation (12) determines $\gamma + N$ and equation (11) determines N. Subtracting the second value from the first gives γ.

In general there are two solutions, since $\gamma + N$ may have two values.

Third. To find β.

The angle β is found from

$$\sin\beta = \frac{\sin b\ \sin\alpha}{\sin a}, \tag{13}$$

which in general gives two values.

163. *Given two angles and a side opposite one of them, to find any one of the remaining parts.*

Let α, β, a be the given parts.

First. To find γ.

The relation between α, β, a, and γ is

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a. \tag{1}$$

Let $$m\sin M = \cos\beta, \tag{2}$$

and $$m\cos M = \sin\beta\cos a. \tag{3}$$

Then $$\cos\alpha = m\sin(\gamma - M). \tag{4}$$

From (2), (3), and (4)

$$\cot M = \tan\beta\cos a, \tag{5}$$

and $$\sin(\gamma - M) = \frac{\cos\alpha\ \sin M}{\cos\beta}. \tag{6}$$

Equations (5) and (6) determine M and $\gamma - M$, from which γ is found.

Two solutions may be possible, as in Art. 162.

Second. To find c.

The relation between α, β, a, and c is

$$\sin\beta\cot\alpha = \cot a\ \sin c - \cos c\ \cos\beta. \tag{7}$$

Multiplying by $\sin a$ and transposing,

$$\cos a\ \sin c - \sin a\ \cos c\ \cos\beta = \sin\beta\ \sin a\ \cot\alpha. \tag{8}$$

Let $$n\cos N = \cos a, \tag{9}$$

and $$n\sin N = \sin a\ \cos\beta. \tag{10}$$

Then uniting (8), (9), and (10)

$$n \sin(c - N) = \sin\beta \sin a \cot\alpha. \tag{11}$$

From (9), (10), and (11)

$$\tan N = \tan a \cos\beta, \tag{12}$$

and
$$\sin(c - N) = \tan\beta \cot\alpha \sin N. \tag{13}$$

Equations (12) and (13) determine N and $c - N$, from which c is found.

There may be two solutions.

Third. To find b.

We have
$$\sin b = \frac{\sin\beta \sin a}{\sin\alpha}. \tag{14}$$

There are two values of b unless restricted to one solution by the principles of Arts. 150, 151, and 152.

164. The general triangle. The parts of the *general spherical* triangle are not restricted to values less than 180°. It can be shown that all the formulas developed for the oblique spherical triangle are true for the general spherical triangle if the double sign is introduced in the formulas of Arts. 144, 145, and 146.

ANSWERS

Art. 19; Page 10

4. $\frac{\pi}{18}, \frac{\pi}{6}, \frac{\pi}{4}$, etc.

5. 114° 35′ 28″, 286° 28′ 40″, etc.

6. 60°, 135°, − 300°, 57° 17′ 44″, 36° 28′ 31″, etc.

7. $7\frac{1}{2}$ ft.

8. $2\frac{2}{5}$ radians, 137°.30′ 34″.

9. 94° 3′ 24″.

10. $\frac{16}{15}\pi$ ft.

11. 2.7216 radians.

12. 247.16 R. P. M.
25.882.

13. 18.33 mi. per sec.

14. 5236.

15. 9.6π
240π ft. per min.

Art. 27; Page 20

1. 3d and 4th. 2. 1st and 4th. 3. 1st and 3d.

7. 2d. 8. 3d. 20. $-\frac{2\pi}{3}$. 21. $-\frac{4\pi}{3}$.

28. $\sin\alpha_1 = \frac{6}{85}\sqrt{85}$, $\cos\alpha_1 = -\frac{7}{85}\sqrt{85}$, $\cot\alpha_1 = -\frac{7}{6}$,
$\sec\alpha_1 = -\frac{1}{7}\sqrt{85}$, $\csc\alpha_1 = \frac{1}{6}\sqrt{85}$,
$\sin\alpha_2 = -\frac{6}{85}\sqrt{85}$, $\cos\alpha_2 = \frac{7}{85}\sqrt{85}$, $\cot\alpha_2 = -\frac{7}{6}$,
$\sec\alpha_2 = \frac{1}{7}\sqrt{85}$, $\csc\alpha_2 = -\frac{1}{6}\sqrt{85}$.

35. $\frac{16}{17}$.

Art. 36; Page 29

1. $b = 16.5$
$c = 17.5$
$\beta = 70°$.

2. $a = 1.44$
$b = 2.05$
$\beta = 55°$.

3. $c = .869$
$a = .225$
$\alpha = 15°$.

4. $c = 65$
$\alpha = 23°$
$\beta = 67°$

5. $\alpha = 27°$
$\beta = 63°$
$b = 72.6$

6. $\alpha = 18\frac{1}{2}°$
$\beta \doteq 71\frac{1}{2}°$
$b = .00867$

7. $a = 346$ $c = 507$ $\beta = 47°$	**8.** $\alpha = 27°$ $\beta = 63°$ $a = 3.50$	**9.** $a = .029$ $b = .089$ $\beta = 72°$	**10.** $\alpha = 40\frac{1}{2}°$ $a = 8.11$ $b = 9.48$
11. $b = 5161$ $c = 5489$ $\beta = 70° 5'$	**12.** $a = .1384$ $b = .2878$ $\beta = 64° 19'$	**13.** $a = 1.446$ $c = 1.719$ $\alpha = 57° 17'$	
14. $c = .00006294$ $\alpha = 72° 26'$ $\beta = 17° 34'$	**15.** $b = 810.80$ $\alpha = 47° 31' 32''$ $\beta = 42° 28' 28''$	**16.** 17. etc. check your results.	

31. Base 1331.1, vertical angle 149° 19′ 10″.

32. Base angles 39° 23′ 56″, base 1477.0.

33. Equal sides 1622.9, base angles 37° 59′ 37″.

34. Equal sides 219.75, base angles 68° 27′ 19″.

35. 37.504.

Art. 38; Page 31

1. 290.83 ft.

2. 405.24 ft.

3. 263.92 ft.
289.93 ft.

4. 21.442 ft.

5. $2\,nr \sin \frac{180°}{n}$.

6. 132.52 ft.

7. $2\,nr \tan \frac{180°}{n}$.

8. 130.99 ft.

9. 226.11 ft.

10. 2572.5 ft.

11. 21.360 in.
22.638 in.
69° 26′ 36″.

12. 54.775 mi.

13. 153.72 lbs.
38° 31′ 46″.

14. 739.38 mi. per hr.

15. 16 ft. $3\frac{5}{32}$ in.

16. 35° 16′.

Art. 50; Page 46

2. $-\cos 10°$, $-\sin 80°$.

3. $-\cot 20°$, $-\tan 70°$.

4. $-\cot 20°$, $-\tan 70°$.

5. $\cos 20°$, $\sin 70°$.

6. $-\tan 80°$, $-\cot 10°$.

7. $-\sin 60°$, $-\cos 30°$.

8. $\cos \theta$. **9.** $-\tan \theta$. **10.** $-\tan \theta$. **11.** $-\cos \theta$.

Art. 56; Page 51

11. $\cos\theta = -\frac{1}{3}\sqrt{5}$, $\tan\theta = -\frac{2}{5}\sqrt{5}$, $\cot\theta = -\frac{1}{2}\sqrt{5}$, $\sec\theta = -\frac{3}{5}\sqrt{5}$, $\csc\theta = \frac{3}{2}$.

12. $\sin\theta = \frac{10}{149}\sqrt{149}$, $\cos\theta = \frac{7}{149}\sqrt{149}$, $\tan\theta = \frac{10}{7}$, $\sec\theta = \frac{1}{7}\sqrt{149}$, $\csc\theta = \frac{1}{10}\sqrt{149}$.

15. $x = 30°$. **16.** $u = 45°$. **17.** $x = 0°, 60°$.

18. $\theta = 45°, 60°$. **19.** $\sin x = \pm\sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{5}}$.

20. Identity. **22.** $\theta = 120°$. **24.** Identity.

21. $x = 60°, 120°$. **23.** $\theta = 30°$. **25.** $y = 30°, 150$.

35. $\alpha = 30°, 150°$. **36.** $\alpha = 30°, 60°, 120°, 150°$.

37. $x = 45°, 135°$. **38.** $x = 45°$.

Art. 60; Page 57

33. $\dfrac{\tan^2 x - 1}{\tan^2 x + 1}$. **34.** $\sin x \mp \sqrt{1 - \sin^2 x}$. **35.** $\dfrac{1 + \cos^2 x}{1 - \cos^2 x}$.

39. $\cos\theta = \pm\sqrt{1 - a^2}$, $\tan\theta = \dfrac{a}{\pm\sqrt{1 - a^2}}$,

$\cot\theta = \dfrac{\pm\sqrt{1 - a^2}}{a}$, $\sec\theta = \dfrac{1}{\pm\sqrt{1 - a^2}}$, $\csc\theta = \dfrac{1}{a}$.

41. $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$, $\tan\theta = \dfrac{\pm\sqrt{1 - \cos^2\theta}}{\cos\theta}$,

$\cot\theta = \dfrac{\cos\theta}{\pm\sqrt{1 - \cos^2\theta}}$, $\sec\theta = \dfrac{1}{\cos\theta}$,

$\csc\theta = \dfrac{1}{\pm\sqrt{1 - \cos^2\theta}}$.

43. $\dfrac{a}{2}$. **47.** 0°, 90°. **48.** 135°. **49.** 0°, 30°, 60°, 150°.

50. 30°, 150°. **51.** 45°. **54.** 36° 52′. (See tables.)

55. 46° 24′, 90°. **57.** 65° 54′, 114° 6′.

58. 0°, − 30°, − 150°.

59. $\tan u = \pm\frac{2}{5}\sqrt{5}$, $\pm\frac{1}{4}\sqrt{2}$. **61.** 195°, 345°.

60. $\sin x = \pm\frac{1}{2}\sqrt{3}$, $\pm\frac{2}{3}\sqrt{2}$. **62.** 54°, 234°.

Art. 70; Page 66

2. $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$. **3.** $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$. **4.** $\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$.

5. $\dfrac{3+\sqrt{3}}{3-\sqrt{3}}$. **7.** $\cos\beta$. **8.** $-\cot\alpha$. **9.** $-\cos\alpha$.

Art. 75; Page 71

1. $\frac{1}{2}\sqrt{2-\sqrt{2}}$, $\frac{1}{2}\sqrt{2+\sqrt{2}}$, $\sqrt{3-2\sqrt{2}}$, $\sqrt{3+2\sqrt{2}}$.

2. $\pm\frac{1}{8}\sqrt{15}$, $-\frac{7}{8}$, $\pm\frac{1}{7}\sqrt{15}$, $\pm\frac{7}{15}\sqrt{15}$.

3. $\pm\sqrt{\dfrac{10\pm\sqrt{10}}{20}}$, etc.

32. $0, \pm\sqrt{\dfrac{3-a}{1-3a}}$. **36.** $67° 30'$, $157° 30'$.

40. 0°, 65° 42′, 114° 18′. (See tables.)

34. 30°, 90°, 150°. **42.** 0°, 30°, 90°, 150°.

35. 0°, 120°. **43.** 0°, 90°, 120°.

Art. 85; Page 82

1. 60°, 120°, 420°, etc. **2.** 150°, 210°, −150°, etc.

3. 45°, 225°, −135°, etc. **4.** $\frac{1}{4}$.

5. $\pm\frac{1}{17}\sqrt{17}$. **6.** $\dfrac{u}{\pm\sqrt{u^2+1}}$.

7. $\dfrac{\pm\sqrt{u^2+1}}{u}$. **9.** $\pm\sqrt{1-u^2}$.

10. $\dfrac{1}{\pm\sqrt{1+u^2}}$. **11.** $\dfrac{1}{u}$.

13. $\pm 2u\sqrt{1-u^2}$.

14. $\dfrac{2u}{1-u^2}$. **15.** $\dfrac{\pm 2\sqrt{u^2-1}}{2-u^2}$.

20. $\pm u\sqrt{1-v^2}\pm v\sqrt{1-u^2}$. **21.** $\pm\sqrt{1-u^2}\sqrt{1-v^2}-uv$.

25. $\pm\sqrt{\dfrac{1-u}{2}}$.

27. $\pm\sqrt{\dfrac{u-1}{2u}}$.

29. SOLUTION. $x = \tan\cot^{-1} a = \dfrac{1}{\cot\cot^{-1} a} = \dfrac{1}{a}$.

30. $x = \dfrac{\pm a}{\sqrt{1+a^2}}$.

31. $1-2a^2$.

32. $\dfrac{\sqrt{1-a^2}-a^2}{\sqrt{1+a^2}}$.

34. $\dfrac{a\sqrt{1-b^2}+b\sqrt{1-a^2}}{\sqrt{1-a^2}\sqrt{1-b^2}-ab}$.

35. $\dfrac{1}{ab}(1+\sqrt{a^2-1}\sqrt{b^2-1})$.

Art. 101; Page 98

1. $b = 24.5$
$c = 31.8$
$\gamma = 78°$.

2. $\alpha = 50°$
$b = 54.9$
$c = 58.6$

3. $a = 61.4$
$c = 47.7$
$\gamma = 48°$.

4. $\beta = 48° \, 20'$
$\gamma = 62° \, 40'$
$c = 76.1$

5. $\beta = 68° \, 22'$
$\gamma = 56° \, 58'$
$a = 107$

6. $\alpha = 40° \, 48'$
$\beta = 104° \, 52'$
$c = 141$.

7. $\alpha = 69° \, 22'$
$\beta = 38° \, 38'$
$c = 42.7$

8. $\beta = 38° \, 37'$
$\gamma = 31° \, 23'$
$c = 173$.

9. $\alpha = 44° \, 42'$
$\beta = 60° \, 20'$
$\gamma = 74° \, 54'$

10. $\alpha = 72° \, 21'$
$\beta = 49° \, 38'$
$\gamma = 58° \, 1'$

11. $\beta = 34° \, 44'$ or $146° \, 16'$
$\gamma = 125° \, 16'$ or $13° \, 44'$
$c = 35.8$ or 10.4

12. $\alpha = 51° \, 44'$ or $128° \, 16'$
$\beta = 84° \, 26'$ or $7° \, 54'$
$b = .431$ or $.059$

Art. 101; Page 98

13. $\gamma = 39° \, 49' \, 50''$
$a = 41.581$
$c = 41.432$

14. $a = 13.081$
$c = 13.620$
$\beta = 97° \, 2' \, 15''$

15. $\alpha = 90° \, 20' \, 34''$
$\beta = 65° \, 17' \, 34''$
$\gamma = 24° \, 21' \, 50''$

16. $\alpha = 126° \, 59' \, 18''$
$\beta = 27° \, 29' \, 38''$
$\gamma = 25° \, 31' \, 4''$

17. $a = 1.9555$
$\alpha = 43° \, 36' \, 35''$
$\beta = 15° \, 37' \, 7''$

18. $\gamma = 77° \, 22' \, 16''$
$\beta = 51° \, 57' \, 20''$
$a = 83.732$

19. $\alpha = 87° \, 33' \, 58''$
$\gamma = 18° \, 6' \, 24''$
$b = 1.4033$

20. $\alpha = 78° \, 40' \, 32''$
$\beta = 41° \, 20' \, 47''$
$\gamma = 59° \, 58' \, 41''$

21. $\alpha = 32° \, 24' \, 0''$
$\beta = 55° \, 0' \, 28''$
$\gamma = 92° \, 35' \, 32''$

22. Impossible.

23. $\alpha = 142° \, 28' \, 9''$ or $21° \, 31' \, 11''$
$\beta = 29° \, 31' \, 31''$ or $150° \, 28' \, 29''$
$a = .080746$ or $.04862$

24. $\gamma = 76° \, 50' \, 20''$ or $103° \, 9' \, 40''$
$\alpha = 69° \, 39' \, 35''$ or $43° \, 20' \, 15''$
$a = 17.405$ or 12.739

25. $c = 502.28$
$b = 300.25$
$\gamma = 23° \, 7' \, 3''$

26. $\alpha = 96° \, 9' \, 32''$
$\beta = 41° \, 11' \, 10''$
$\gamma = 42° \, 39' \, 18''$

27. $A = 150.$

30. $A = 108.61$

33. $A = 368.91.$

35. 172.8 ft.

36. 106.1 ft.

37. 3.710 mi.

38. 97.14
124.59
178.64

39. 60.1 ft.

40. 57.93 mi. per hr.

41. 3888.0 ft.

42. 6328.7 ft.

43. 239600 mi.

Art. 102; Page 102

1. $1\frac{3}{4}$ in. **2.** $60°, 72°, 135°$. **3.** $\frac{2\pi}{9}, \frac{11\pi}{36}, \frac{19\pi}{90}, \frac{196\pi}{675}$.

4. $42° 58' 18''$. **5.** $35° 48' 35''$. **6.** 13.754 in.

7. $\frac{\pi}{9}$. **8.** $\frac{55\pi}{144}, \frac{73\pi}{144}$. **9.** $\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}$.

10. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$. **11.** $\frac{4\pi}{77}, \frac{10\pi}{77}, \frac{16\pi}{77}, \frac{22\pi}{77}, \frac{28\pi}{77}, \frac{34\pi}{77}, \frac{40\pi}{77}$.

12. $9\frac{3}{5}$ in. **49.** $0°, 180°$.

50. $30°, 150°, 210°, 330°, 45°, 135°, 225°, 315°$.

51. $30°, 150°$. **52.** $\tan^{-1}(2 \pm \sqrt{3})$. **53.** $0°, 180°, \cos^{-1}\frac{1}{5}$.

54. $45°, 135°, 225°, 215°, \sin^{-1}\sqrt{\frac{1}{5}}$. **55.** $60°, 300°, 90°, 270°$.

56. $60°, 120°, 240°, 300°, 45°, 135°, 225°, 315°$.

57. $\tan^{-1}\frac{3}{4}$. **58.** $90°, 270°, \sin^{-1}(-\frac{21}{29})$.

59. $0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°$.

60. $0°, 45°, 60°, 90°, 120°, 180°, 225°, 240°, 270°, 300°, 315°$.

61. $7\frac{1}{2}°, 37\frac{1}{2}°, 67\frac{1}{2}°, 97\frac{1}{2}°, 127\frac{1}{2}°, 157\frac{1}{2}°$, etc.

62. $30°, 150°, 210°, 330°$.

63. $90°, 270°, 70°, 110°, 190°, 230°, 310°, 350°$.

64. $90°, 180°$. **65.** $210°, 330°$.

66. $45°, 215°, 67\frac{1}{2}°, 157\frac{1}{2}°, 247\frac{1}{2}°, 337\frac{1}{2}°$.

67. $60°, 120°, 240°, 300°, 18°, 54°, 90°, 126°, 162°$, etc.

68. $60°, 90°, 120°, 240°, 270°, 300°$.

69. $\frac{2u}{1+u^2}$. **70.** $0, \pm\sqrt{\frac{1}{2}}$. **71.** $\pm\frac{\sqrt{3}}{2}$. **72.** $\frac{1}{5}\sqrt{10}, -\frac{12}{31}\sqrt{10}$.

73. $\frac{-1 \pm \sqrt{1+m^2}}{m}$. **85.** $\sqrt{3}$. **86.** $\frac{1}{5}\sqrt{5}$. **87.** $0, \frac{1}{3}\sqrt{3}$.

88. $\frac{ab + \sqrt{1-a^2}\cdot\sqrt{1-b^2}}{b\sqrt{1-a^2} + a\sqrt{1-b^2}}$. **97.** $26° 34'$. **98.** 711.7.

99. 532. **100.** 50.0. **101.** 1478.5.
102. 93,470,000 mi. **103.** 51.9 ft. **104.** 27.925.
105. 112.36. **106.** $\frac{r^2}{2}$. **107.** 129.90 ft.
108. 8.2596. **109.** 70° 31′ 43″. **110.** 5296 ft., 251 ft.
111. 196. **112.** 89.431. **114.** $a\sqrt{2}$.
115. 90 ft., 40 ft. **116.** 13.66. **117.** 103.97.
118. 820.54. **119.** 535.4. **120.** 25.43.
121. 567.3. **122.** 132.6. **123.** 36.7 ft.
124. 641. **125.** 962.605. **127.** 16.33 N. 75° 36′ E.
128. a. **129.** 4009. **130.** 1674.3.

Art. 116; Page 124

1. $1, i, -1, -i$.

2. $1,\quad \frac{1}{2}+\frac{1}{2}\sqrt{3}\,i,\quad -\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i,\quad -1,\quad -\frac{1}{2}-\frac{1}{2}\sqrt{3}\,i,$ $\frac{1}{2}-\frac{1}{2}\sqrt{3}\,i$.

4. $\pm(2+i)$. **5.** $\pm(1-2\,i)$.

6. $\pm(1.272+.786\,i)$.

7. $\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}\,i,\ -\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\,i$.

8. $-\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}\,i,\ \frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\,i$.

10. $1.0842+.29051\,i,$
$-.79370+.79370\,i,$
$-.29051-1.0842\,i.$

Art. 139; Page 143

1. $\beta=78°\,9'\,22''$
$c=10°\,45'\,55''$
$a=2°\,14'\,5''$

2. $\alpha=41°\,11'\,53''$
$\beta=56°\,19'\,56''$
$c=40°\,27'\,11''$

3. $b=8°\,26'\,14''$
$a=120°\,59'\,19''$
$\alpha=95°\,2'\,10''$

4. $\beta=75°\,21'\,53''$
$b=44°\,43'\,49''$
$a=14°\,59'\,33''$

5. $\alpha = 111° 23' 47''$
$\beta = 120° 40' 56''$
$c = 76° 33' 24''$

6. $\beta = 101° 38' 28''$
$\alpha = 112° 13' 48''$
$b = 102° 35' 26''$

7. $\beta = 46° 1' 28''$
$b = 15° 18' 0''$
$\alpha = 46° 2' 40''$

8. $\alpha = 68° 42' 11''$
$\beta = 155° 48' 0''$
$a = 27° 37' 26''$

9. $\beta = 153° 31' 29''$ or $26° 28' 31''$
$c = 50° 43' 22''$ or $129° 16' 38''$
$b = 159° 48' 44''$ or $20° 11' 16''$

Art. 142 ; Page 144

1. $a = 117° 45' 28''$
$\beta = 96° 27' 1''$
$\gamma = 93° 0' 51''$

2. $\gamma = 88° 23' 11''$
$a = 69° 48' 42''$
$b = 94° 22' 46''$

3. $\alpha = 8° 49' 46''$
$\gamma = 28° 3' 4''$
$b = 106° 56' 53''$

4. $\alpha = 160° 13' 48''$
$\beta = 105° 21' 16''$
$\gamma = 104° 25' 45''$

5. Sides, $32° 45' 6''$
angles, $105° 49' 32''$

6. Sides, $112° 32' 20''$
base, $46° 15' 12''$

Art. 156 ; Page 154

1. $\beta = 11° 0' 47''$
$\gamma = 92° 8' 27''$
$a = 114° 42' 50''$

2. $\beta = 14° 53' 47''$
$\gamma = 170° 26' 51''$
$a = 55° 56' 0''$

3. $\alpha = 71° 1' 23''$
$\gamma = 84° 22' 25''$
$b = 82° 1' 30''$

6. $a = 24° 35' 10''$
$c = 43° 29' 48''$
$\beta = 154° 19' 20''$

8. $\alpha = 133° 28' 34''$
$\beta = 169° 38' 12''$
$\gamma = 132° 6' 14''$

9. $\alpha = 104° 49' 50''$
$\beta = 84° 51' 42''$
$\gamma = 95° 18' 24''$

10. $\alpha = 84° 57' 8''$
$\beta = 57° 47' 44''$
$\gamma = 43° 4' 36''$

14. $\alpha = 155° \, 14' \, 24''$ or $21° \, 52' \, 40''$
$\beta = 25° \, 50' \, 58''$ or $154° \, 9' \, 2''$
$a = 107° \, 34' \, 50''$ or $58° \, 0' \, 44''$

15. Two solutions.

24. No solution.

Art. 160; Page 159

2. $2229\frac{5}{6}$ miles.

3. $4291\frac{1}{3}$ miles.

Tawny [illegible]

$$X = \frac{500}{\cot\angle B - \cot\angle A}$$

Characteristic log of nos less than 1 — subtract the no of zeros from 9, writing −10 after the mantissa.

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