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PLANE AND SPHERICAL

TRIGONOMETRY

BY

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W. P. I

PREFACE

In this work the author has attempted to produce a text-book which should present in a concise and yet thorough manner an adequate treatment of both the theoretical and the practical sides of elementary trigonometry. The material here presented has been gathered and tested during the course of many years of experience in the class room, and the arrangement and method of presentation are the result of numerous experiments made for the purpose of ascertaining what could be done most effectively in the limited time usually devoted to this subject.

The problems given in connection with the different cases under the solution of triangles are nearly all new, and are well graded and sufficiently numerous to give the student ample preparation for the various problems that arise in plane surveying and in elementary astronomical and geodetic work. That portion of the book which treats of theoretical trigonometry has been written in the attempt to present this aspect of the subject in the simplest and clearest manner, and at the same time with the design of equipping the student for the more advanced work in pure and applied mathematics which is pursued in the later years of his college course.

The best English, French, and Italian text-books have been consulted, as well as those published in this country. For assistance in the preparation of the work thanks are due to my colleague, Professor Arthur D. Butterfield, to Professor W. B. Fite of Cornell University, to Professor O. S. Stetson of Syracuse University, to Mr. C. G. Brown, head of the department of mathematics in the Englewood, New Jersey, High School, and to Mr. J. A. Bullard, instructor in mathematics in the Worcester Polytechnic Institute.

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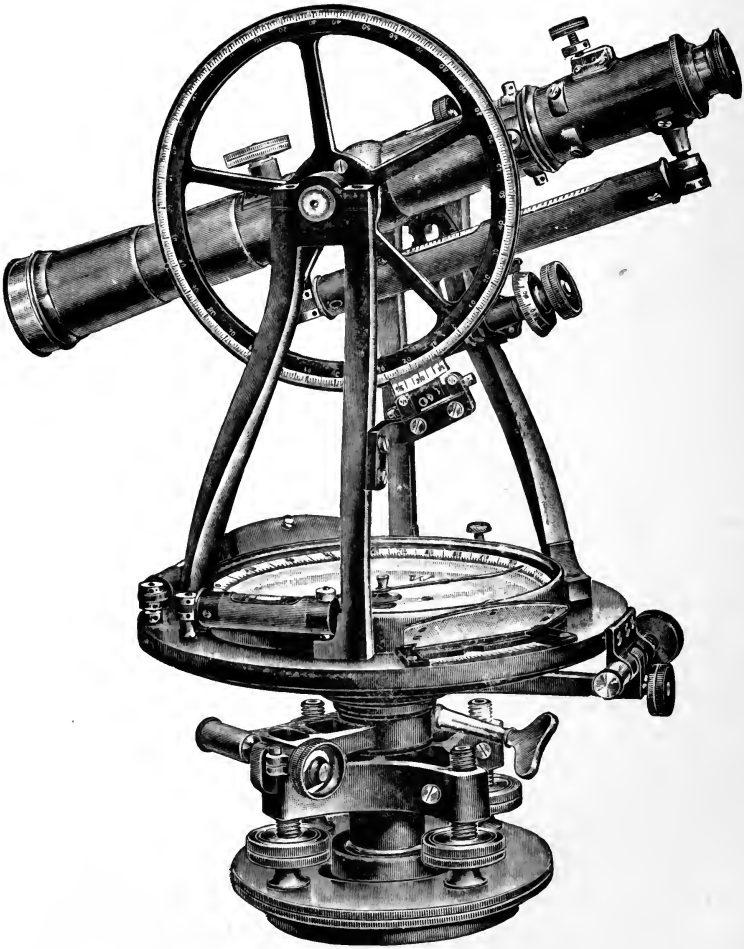


Engineer's Transit, with Gradienter

CONTENTS

PLANE TRIGONOMETRY

SPHERICAL TRIGONOMETRY

GREEK ALPHABET

Greek is written with the following twenty-four letters:

Form		Name	Latin Equivalent
Α	α	*alpha*	a
Β	β	*beta*	b
Γ	γ	*gamma*	g
Δ	δ	*delta*	d
Ε	ε	*epsilon*	ĕ
Ζ	ζ	*zeta*	z
Η	η	*eta*	ē
Θ	θ ϑ	*theta*	th
Ι	ι	*iota*	i
Κ	κ	*kappa*	c, k
Λ	λ	*lambda*	l
Μ	μ	*mu*	m
Ν	ν	*nu*	n
Ξ	ξ	*xi*	x
Ο	ο	*omicron*	ŏ
Π	π	*pi*	p
Ρ	ρ	*rho*	r
Σ	σ ς	*sigma*	s
Τ	τ	*tau*	t
Υ	υ	*upsilon*	y
Φ	φ	*phi*	ph
Χ	χ	*chi*	ch
Ψ	ψ	*psi*	ps
Ω	ω	*omega*	ō

PLANE TRIGONOMETRY

CHAPTER I

THE MEASUREMENT OF ANGULAR MAGNITUDE

1. The size and shape of a plane triangle can be completely determined when any three of its six parts are known, provided at least one of the known parts is a side.

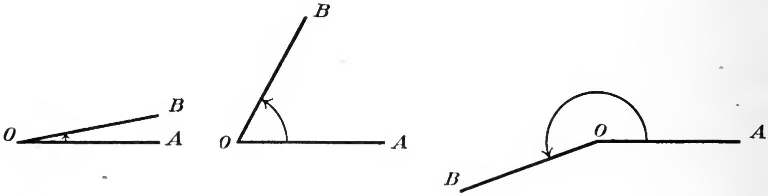
By means of certain ratios called **trigonometric functions,** which will be defined later, trigonometry enables us to investigate and to determine the unknown parts and the area of a triangle when any three of the parts are known, provided at least one of the known parts is a side. Hence, in its most elementary sense,

Trigonometry is that branch of mathematics which treats of the solution of triangles. During the past two centuries the sense in which the word "trigonometry" is used has been greatly extended, and it is now understood to include the general subject of mathematical investigation by means of trigonometric functions.

Plane trigonometry treats of plane triangles, and of plane angles and their functions.

2. Angles. In its geometric sense the word "angle" is defined as the difference in direction of two intersecting lines. In trigonometry, however, this word receives an extension of meaning, which must be fully understood at the outset.

Suppose two straight lines, OA and OB, are drawn from the point O in such a manner that they very nearly coincide. Let one of the lines, OA, remain fixed in position, while the other, OB, revolves on the point O as a pivot. We are now free to revolve OB, either back into actual coincidence with OA, or forward, so as to enlarge the opening between the lines. At any point of the revolution the angle AOB may be said to have been formed, or generated, by the revolution of the line OB.



In plane geometry angles greater than 180° are seldom employed, but in trigonometry the freest possible use is made of such angles. Trigonometry even considers angles greater than 360°, meaning by an angle of that magnitude merely the amount of revolution that has been performed by the moving or generating line.

As an illustration of the meaning of the word "angle" used in this sense, consider the movement of one of the hands of the clock. Let the minute hand start from the position it occupies at noon. In fifteen minutes it will move over or generate an angle of 90°; in thirty minutes an angle of 180°; in forty-five minutes an angle of 270°; and in one hour an angle of 360°. Continuing, we may say that in two hours the minute hand will move over an angle of 720°, in three hours an angle of 1080°, in four hours an angle of 1440°, in n hours an angle of $n \times 360°$, etc.

Again, suppose a runner to be competing in a two-mile race on a circular track a quarter of a mile in length. If we suppose a line to be drawn connecting the position of the runner with the center of the circle formed by the track, the position of the runner both on the track and in the race can be described at any instant with perfect accuracy by giving the magnitude of the angle through which this line has revolved since the beginning of the race.

Thus, when the line has revolved through an angle, and hence the runner has traversed an arc, of 180°, he has completed one eighth of a mile; when he has traversed an arc of 360°, he has completed one fourth of a mile; and when he has finished the race, he has run around the track eight times. In other words, when he has finished the race the line that connects him with the center of the track has revolved through an angle of $8 \times 360°$, or $2880°$. During this time the runner has traversed an arc of the same magnitude, *i.e.* of $2880°$.

It is at once seen that an idea is here introduced which is an extension of the idea of the angle as it is ordinarily used in geometry. This idea, which is fundamental in all work in trigonometry involving angles, is the idea of formation or generation in connection with the angle. Evidently a definition of this word is required which differs from that to which the student has become accustomed in geometry; and in the extended sense here used, the term "angle" may be defined as follows:

An **angle** is that relation of two lines which is measured by the amount of revolution necessary to make one coincide with the other.

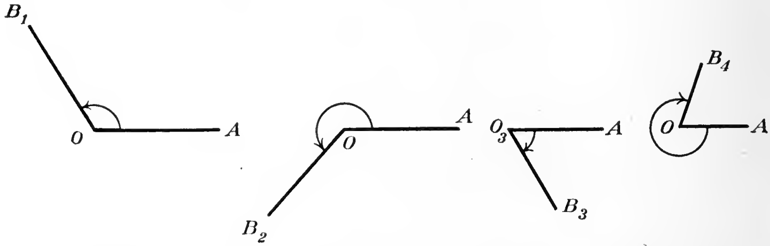
3. The point about which the generating line revolves is called the **origin.** The generating line is called the **radius vector.** The line with which the radius vector coincides when in its original position is called the **initial line**; and the line with which it coincides when in its final position is called the **terminal line.**

4. Positive and negative angles. It is convenient, and often necessary, to know not only the size of an angle, but also the direction in which the radius vector has moved while generating the angle. For this reason it is customary to speak of angles as being either positive or negative.

If the radius vector moves in a direction opposite to that of the hands of a watch when the face of the watch is toward the observer, the angle it generates is said to be **positive.** The motion of the radius vector as it generates the angle is then said to be *counter-clockwise.*

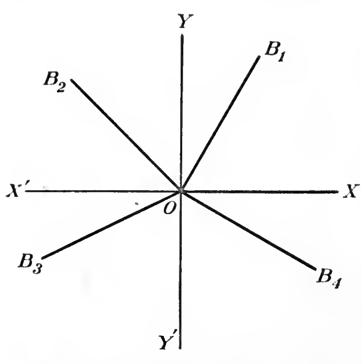
If the radius vector moves in the same direction as the hands of a watch when the face of the watch is toward the observer, the angle it generates is said to be **negative.** The motion of the radius vector is then called *clockwise.*

The angles AOB_1 and AOB_2 are positive angles, and the angles AOB_3 and AOB_4 are negative angles. The initial line in each case is OA, and the terminal lines are OB_1, OB_2, OB_3, OB_4, respectively. The direction of rotation for each angle is indicated by the arrowhead.



5. Angles are often described by referring them to some position with reference to two intersecting lines, at right angles to each other, of which one is horizontal and the other vertical. It is customary to regard the horizontal line extending toward the right as the initial line for all angles, when nothing is said to the contrary.

If the radius vector, as shown in the figure, occupies any position between OX and OY, then the angle XOB_1 is said to be in the first quadrant. If the radius vector is between OY and OX', the angle XOB_2 is said to be in the second quadrant. Similarly, XOB_3 is said to be in the third quadrant, and XOB_4 in the fourth quadrant. These expressions only mean, of course, that the terminal lines lie in the first, second, third, and fourth quadrants respectively.



6. In practical work the unit of measure that is always employed in dealing with angular magnitudes is the **right angle** or some fraction of the right angle. This unit is chosen because:

(i) The right angle is a constant angle.

(ii) It is easy to draw or to construct in a practical manner.

(iii) It is the most familiar of all angles, entering as it does most frequently into the practical uses of life.

In geometry the right angle is the unit universally used.

In trigonometry two systems of measurement, involving the use of two different units, are in common use.

7. The sexagesimal system. In this system the unit of measure is the right angle. The right angle is divided into 90 equal parts, called *degrees;* each degree is divided into 60 equal parts, called *minutes;* and each minute is divided into 60 equal parts, called *seconds.* The symbols $1°$, $1'$, $1''$, are employed to denote one degree, one minute, and one second respectively.

60 seconds ($60''$) = one minute.
60 minutes ($60'$) = one degree.
90 degrees ($90°$) = one right angle.

This system is almost universally employed where numerical measurements are to be made. It is, however, inconvenient because of the multipliers, 60 and 90, which it introduces into computations.

Another system, called the **centesimal system**, was proposed in France a little over a century ago. In this system the right angle is divided into 100 equal parts called grades, the grade is divided into 100 equal parts called minutes, and the minute is divided into 100 equal parts called seconds. The centesimal system has been used to some extent in France, but its use has never been looked upon with favor in other countries. If its use were to become general, an enormous amount of labor would have to be expended in the re-computation of existing tables. For this reason the centesimal system, in spite of its intrinsic advantage over the sexagesimal system, will probably never come into general use.

EXERCISE I

Express the following angles in terms of a right angle:

1. $30°$. **3.** $68° 14'$. **5.** $228° 46'$.

2. $120°$. **4.** $114° 38' 12''$. **6.** $321° 14' 22''$.

7. The angles of a right triangle are in arithmetical progression, and the greatest angle is three times the least; what is the number of degrees in each angle?

Show by a figure the position of the revolving line when it has generated each of the following angles:

8. $\frac{3}{5}$ rt. angle.	**11.** $2\frac{1}{3}$ rt. angles.	**14.** $-150°$.
9. $-\frac{2}{3}$ rt. angle.	**12.** $4\frac{2}{3}$ rt. angles.	**15.** $275°$.
10. $-1\frac{1}{2}$ rt. angles.	**13.** $17\frac{1}{2}$ rt. angles.	**16.** $1225°$.

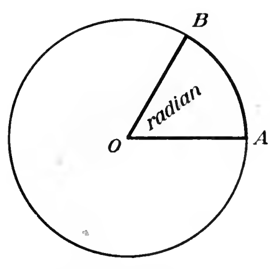
17. The angles of a triangle are such that the first contains a certain number of degrees, the second 10 times as many minutes, and the third 120 times as many seconds; find each angle.

18. How many degrees are passed over by each of the hands of a watch in one hour?

Represent by a figure each of the following angular magnitudes:

19. $1\frac{1}{2}+2\frac{1}{3}$ rt. angles.	**23.** 4 rt. angles.
20. $2\frac{1}{2}-1\frac{1}{3}$ rt. angles.	**24.** $4n$ rt. angles (n integral).
21. -4 rt. angles.	**25.** $(4n+1)$ rt. angles.
22. $-6\frac{1}{2}$ rt. angles.	**26.** $(4n-2)$ rt. angles.

8. Circular measure. Another system for the measurement of angles has, in modern times, come into vogue. It is extensively used in work connected with higher branches of mathematics, and is the almost universal unit employed in theoretical investigations.



The unit of circular measure is the **radian**, which is obtained as follows:

On the circumference of a circle lay off an arc, AB, equal in length to the radius of the circle, OA. The angle AOB is called a radian. Accordingly:

A radian is an angle at the center of a circle, subtended by an arc equal in length to the radius of the circle.

In order to use the radian as a unit of measure, it is necessary to prove that it is a constant angle; or, in other words, it is necessary to prove that the magnitude of the radian is the same for all circles.

9. THEOREM. *The radian is a constant angle.*

By definition the radian is measured by an arc equal in length to the radius. Also,

An angle of two right angles is measured by an arc equal to one half the circumference.

Therefore, since angles at the center of a circle are to each other as the arcs by which they are subtended (Geom.),

$$\frac{\text{a radian}}{\text{2 rt. angles}} = \frac{\text{radius}}{\text{semi-circumference}} = \frac{R}{\pi R} = \frac{1}{\pi}.$$

$$\therefore \text{ a radian} = \frac{1}{\pi} \text{ of 2 right angles} = \frac{1}{\pi} \times 180° = 57.2958°$$

$$= 57°\ 17'\ 44.8'' \text{ nearly.}$$

Therefore the radian is a constant angle.

10. The reason for the use of this unit may now be readily understood.

Since $$1 \text{ radian} = \frac{2 \text{ rt. } \measuredangle s}{\pi},$$

$$\therefore\ \pi \text{ radians} = 2 \text{ rt. } \measuredangle s = 180°.$$

Similarly, $$\frac{\pi}{2} \text{ radians} = 1 \text{ rt. } \angle = 90°.$$

$$\frac{\pi}{6} \text{ radians} = \tfrac{1}{3} \text{ rt. } \angle = 30°.$$

$$\frac{\pi}{3} \text{ radians} = 60°.$$

$$\tfrac{2}{3}\pi \text{ radians} = 120°.$$

$$\tfrac{3}{2}\pi \text{ radians} = 270°.$$

$$2\pi \text{ radians} = 4 \text{ rt. } \measuredangle s = 360°.$$

$$5\pi \text{ radians} = 10 \text{ rt. } \measuredangle s = 900°.$$

$$18\pi \text{ radians} = 36 \text{ rt. } \measuredangle s = 3240°.$$

This gives a method for the expression of the value of an angle that is often far more convenient than that furnished by the sexagesimal system. It is especially useful in dealing with angles of great magnitude, and it greatly simplifies many of the investigations and formulas of trigonometry.

11. The symbol r is often used as the symbol to denote radians. Thus, 6^r would stand for 6 radians, θ^r for θ radians, π^r for π radians, etc.

When the value of the angle is expressed in terms of π, and when the unit is the radian, it is customary to omit the r and to give the value of the angle in terms of π alone, the r being understood. Thus, when referring to angular magnitude, π means π radians, $\frac{\pi}{2}$ means $\frac{\pi}{2}$ radians, 6π means 6π radians, etc. When the word "radians" is omitted, the student should mentally supply it, or he may readily fall into the error of supposing that π alone means 180°. The value of π is the same here as in geometry, *i.e.* 3.14159. Neither π nor any multiple of π can by itself ever denote an angle. It simply tells how many radians the angle contains. Too great care cannot be exercised in keeping this distinction clear.

12. *To find the number of degrees in an angle containing a given number of radians, and vice versa.*

Since $$180^\circ = \pi \text{ radians},$$

$$1^\circ = \frac{\pi}{180} \text{ of a radian},$$

and $$1^r = \frac{180}{\pi} \text{ of a degree}.$$

Hence,

To convert radians into degrees, *multiply the number of radians by* $\frac{180}{\pi}$.

To convert degrees into radians, *multiply the number of degrees by* $\frac{\pi}{180}$.

EXERCISE II

1. How many degrees are there in 3 radians?

$$1^r = \frac{180^\circ}{\pi},$$

$$3^r = 3 \times \frac{180}{\pi} = \frac{540}{\pi} = 171.89 \text{ nearly}$$

$$= 171^\circ\, 53'\, 24'' \text{ nearly}.$$

2. How many radians are there in 113° 15′?

$$113° \, 15' = 113.25°.$$

Since
$$1° = \frac{\pi^r}{180},$$

$$113.25° = 113.25 \times \frac{\pi^r}{180}$$
$$= \frac{113.25 \times 3.14159}{180}$$
$$= 1.976 + \text{radians.}$$

Express in degrees, minutes, and seconds the following angles:

3. $\frac{\pi^r}{6}$. **5.** $\frac{\pi^r}{5}$. **7.** $\frac{3\,\pi^r}{2}$. **9.** $3\,\pi^r$.

4. $\frac{2\,\pi^r}{3}$. **6.** $\frac{3\,\pi^r}{10}$. **8.** $\frac{5\,\pi^r}{6}$. **10.** $15\,\pi^r$.

Express in radians the following angles:

11. 45°. **14.** 225°. **17.** 286° 38′. **20.** $A°$.

12. 120°. **15.** 60° 30′. **18.** 684° 26′. **21.** $\frac{90°}{\pi}$.

13. 135°. **16.** 115° 45′. **19.** $n°$. **22.** 78.126°.

23. The difference between two acute angles of a right triangle is $\frac{\pi}{5}$ radians; find the value of each of the angles in degrees.

24. If one of the angles of a triangle is 56° and a second angle is $\frac{2\,\pi}{5}$ radians, find the value of the third angle.

25. The angles of a triangle are in A. P., and the smallest is an angle of 36°; find the value of each in radians.

26. The value of the angles of a triangle are in A. P., and the number of degrees in the least is to the number of radians in the greatest as $60 : \pi$; find each angle in degrees.

27. The value of one of the interior angles of one regular polygon is to the value of one of the interior angles of another regular polygon as $3 : 4$, and the number of sides in the first is to the number of sides in the second as $2 : 3$; find the number of sides in each.

28. Find the number of radians in one of the interior angles of a regular pentagon; a regular heptagon; a regular nonagon.

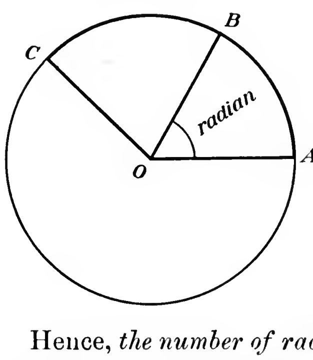
29. The angles of a triangle are in A. P., and the number of radians in the least angle is to the number of degrees in the mean angle as 1 : 120; find the value of each angle in radians.

30. The angles of a quadrilateral are in A. P., and the greatest is double the least; find the value of each angle in radians.

31. Express in degrees and in radians the angle between the hour hand and the minute hand of a clock at (1) five o'clock; (2) quarter-past nine; (3) half-past ten.

32. At what time between four and five o'clock are the hour and the minute hands of a clock 90° apart? At what time are they 180° apart?

13. THEOREM. *The circular measure of an angle whose vertex is at the center of a circle is the ratio of its intercepted arc to the radius of the circle.*



By geometry,

$$\frac{\angle AOC}{\angle AOB} = \frac{\text{arc } AC}{\text{arc } AB} = \frac{\text{arc } AC}{\text{a radius}},$$

$$\therefore \angle AOC = \frac{\text{arc } AC}{\text{radius}} \times \angle AOB$$

$$= \frac{\text{arc } AC}{\text{radius}} \times \text{a radian.}$$

Hence, *the number of radians in any angle is found by dividing the arc which subtends that angle by the radius of the circle.*

The formula just obtained is often expressed in the following convenient, though somewhat incorrect, form:

$$\textbf{arc} = \textbf{angle} \times \textbf{radius}. \quad (1)$$

The meaning of this formula is, that the length of any arc of a circle is equal to the length of the radius of the circle multiplied by the number of radians in the angle subtended by the arc.

EXERCISE III

1. Find in degrees the angle subtended at the center of a circle whose radius is 30 ft. by an arc whose length is 46 ft. 6 in.

In this circle the arc which subtends an angle equal to a radian is 30 ft. in length, and the required angle is subtended by an arc whose length is 46.5 ft.

$$\therefore \frac{46.5}{30} \text{ radians} = \frac{465}{300} \times \frac{180}{\pi} = 88.8^\circ. \quad \textit{Ans.}$$

2. In a circle whose radius is 8 ft., what is the length of the arc subtended by an angle at the center, of $26^\circ\ 38'$?

Let $x =$ the length of the required arc.

Then, $\frac{x}{8} =$ the number of radians in $26^\circ\ 38'$

$$= 26\tfrac{38}{60} \times \frac{\pi}{180}. \quad \text{(See Art. 13.)}$$

$$= \tfrac{1598}{10800} \times \tfrac{22}{7} \text{ ft.}$$

$$x = 3.72 \text{ ft. nearly.}$$

3. In running at a uniform speed on a circular track, a man traverses in one minute an arc which subtends at the center of the track an angle of $3\frac{1}{7}$ radians. If each lap is 880 yd., how long does it take him to run a mile?

Let $x =$ the number of yards traversed during each minute.

Then, $x = 3\frac{1}{7} \times R.$ (See Art. 13.)

$$= \frac{22\,R}{7} = \frac{22}{7} \times 140 = 440 \text{ yards.}$$

Since $\frac{1760}{440} = 4,$

therefore he can run a mile in 4 min.

4. The radius of a circle is 15 ft.; find the number of radians in an angle at the center subtended by an arc of $26\frac{1}{2}$ ft.

5. The radius of a circle is 32 ft.; find the number of degrees in a central angle subtended by an arc of 5π ft.

6. The fly wheel of an engine makes 3 revolutions per second; how long will it take it to turn through 5 radians?

7. The minute hand of a tower clock is 2 ft. 4 in. long; through how many inches does its extremity move in half an hour?

8. A horse is picketed to a stake; how long must the rope be to enable the horse to graze over an arc of 104.72 yd., the angular measurement of this distance being 150°?

9. What is the difference between the latitude of two places, one of which is 150 mi. north of the other, the radius of the earth being reckoned as 4000 mi.?

10. The angle subtended by the sun's diameter at the eye of an observer is 32′; find approximately the diameter of the sun, if its distance from the observer is 92,500,000 mi.

NOTE. In this example the diameter of the sun, which is really the chord of an arc of which the observer's eye is the center, may be regarded as coinciding with the arc which it subtends.

11. Calling the earth a sphere, and the arc of a great circle on its surface subtended by an angle of 1° at the center $69\frac{1}{6}$ mi., what is the radius of the earth?

12. A railway train is traveling at the rate of 60 mi. an hour on a circular arc of two thirds of a mile radius; through what angle does it turn in 10 sec.?

13. The radius of a circle is 3 m.; find approximately, in radians, the arc subtended by a chord whose length is also 3 m.

14. How many seconds are there in an angle at the center of a circle subtended by an arc one mile in length, the radius of the circle being 4000 mi.?

15. In the circle of Ex. 14, what is the length of an arc that subtends an angle of 3′ at the center?

16. What is the ratio of the radii of two circles, if the semi-circumference of the greater is equal in length to an arc of the smaller which subtends an angle of 225° at the center?

17. If an arc 1.309 m. long subtends at the center of a circle whose radius is 10 m. an angle of 7° 30′, what is the ratio of the circumference of a circle to its diameter?

18. The circumference of a circle is divided into four parts which are in A. P., and the greatest part is twice the least; find the number of radians in the central angle subtended by each of the respective arcs into which the circumference is divided.

19. The diameter of a circle is 80 m., and an arc whose length is 15.75 m. subtends a central angle of 22° 30′; find the value of π to four decimal places.

20. How many radians are there in a central angle subtended by an arc of 20″?

21. The semicircumference of a certain circle is equal to its diameter plus a given arc; find the central angle subtended by that arc.

22. Find the radius of a globe such that the distance of 3 in. on its surface, measured on an arc of a great circle, may subtend at the center an angle of 1° 45′.

23. At what distance does a telegraph pole, 24 ft. high, subtend an angle of 10′, the eye of the observer being on the same level as the foot of the pole?

NOTE. The suggestion made in connection with Ex. 10 applies to this problem also. When a chord and its arc differ but little from each other it is often convenient to use the arc in place of the chord.

24. At what distance will a church spire 100 ft. high subtend an angle of 9′, the angle being measured from the level on which the church stands?

25. The difference between two angles is $\frac{2\pi}{9}$ radians, and their sum is 76°; what is the value of each of the angles?

26. If an incline rises 5 ft. in 300 ft., find the angle it makes with its projection on the horizontal plane.

27. How many radians are there in an angle of $a°$?

28. How many radians are there in an angle of 10″?

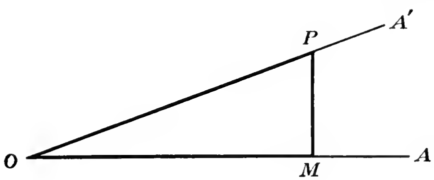
$(\pi = \frac{355}{113})$.

CHAPTER II

TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

14. In the present chapter only acute angles will be considered. In Chapter V the definitions here given will be extended to angles of any magnitude.

Let any line having a given initial position OA begin to revolve on O as a pivot, in a direction opposite to the direction in which the hands of a clock move. Let the angle which it generates be the acute angle AOA'.



From any point in either side of the angle, as P in the side OA', let fall a perpendicular PM to the other side of the angle.

The trigonometric functions, or ratios, of the angle AOA' are then defined as follows:

The sine of the angle AOA' is the ratio $\dfrac{MP}{OP} = \dfrac{\text{side opposite}}{\text{hypotenuse}}$.

The cosine of the angle AOA' is the ratio $\dfrac{OM}{OP} = \dfrac{\text{side adjacent}}{\text{hypotenuse}}$.

The tangent of the angle AOA' is the ratio $\dfrac{MP}{OM} = \dfrac{\text{side opposite}}{\text{side adjacent}}$.

The cotangent of the angle AOA' is the ratio $\dfrac{OM}{MP} = \dfrac{\text{side adjacent}}{\text{side opposite}}$.

The secant of the angle AOA' is the ratio $\dfrac{OP}{OM} = \dfrac{\text{hypotenuse}}{\text{side adjacent}}$.

The cosecant of the angle AOA' is the ratio $\dfrac{OP}{MP} = \dfrac{\text{hypotenuse}}{\text{side opposite}}$.

In addition to these there are two other functions, less frequently used,

$$\text{versed sine of } AOA' = 1 - \text{cosine of } AOA',$$
$$\text{coversed sine of } AOA' = 1 - \text{sine of } AOA'.$$

In writing, it is customary to abbreviate the words "sine," "cosine," "tangent," etc., and to express the functions of any given angle, x, as follows:

$$\sin x,\quad \cos x,\quad \tan x,\quad \cot x,\quad \sec x,\quad \csc x,\quad \text{vers}\, x,\quad \text{covers}\, x.$$

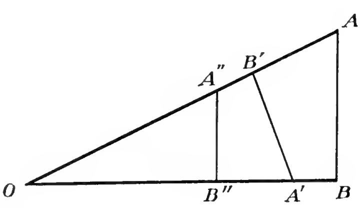
It should be noted at the very beginning that these functions are mere numbers, and their values can be expressed numerically whenever the angle to which they belong is known. Thus, $\sin x$ may equal $\frac{1}{2}$, $\frac{1}{3}$, or any other proper fraction; $\tan x$ may equal 2, 5, 18, or any other real number whatever. The expression $\sin x$, for example, is a single symbol, and is to be regarded as the name of the number which expresses the value of the particular ratio in question. The expressions sin, cos, etc., have no meaning unless some angle is associated with them.

15. *The trigonometric functions are always constant for the same angle.*

From any points in either side of the angle x, as A, A', A'', drop perpendiculars AB, $A'B'$, $A''B''$ to the other side. Then, by geometry, the triangles AOB, $A'OB'$, $A''OB''$ are similar, and their homologous sides are proportional. Therefore,

$$\frac{BA}{OA} = \frac{B'A'}{OA'} = \frac{B''A''}{OA''} = \sin x,$$

$$\frac{OB}{OA} = \frac{OB'}{OA'} = \frac{OB''}{OA''} = \cos x,$$



and similarly for the other functions.

Hence, the value of any function of x remains unchanged as long as the value of the angle itself remains unchanged.

Any increase or decrease in the size of the angle produces a change in the value of the function, or ratio. From this it is readily seen why these ratios are called *functions of the angle.*

From the last paragraph the following important results may now be stated:

(1) *To every acute angle there corresponds one and only one value of each trigonometric function.*

(2) *Two unequal acute angles have different trigonometric functions.*

(3) *To each value of any trigonometric function there is but one corresponding acute angle.*

16. Fundamental relations between the trigonometric functions of an acute angle. From the definitions given in Art. 14 it follows immediately that the sine of the angle x is the reciprocal of the cosecant x; also that cosine x is the reciprocal of secant x, and that tangent x is the reciprocal of cotangent x. That is,

$$\left.\begin{aligned} \sin x &= \frac{1}{\csc x}, \quad \text{or} \quad \sin x \csc x = 1, \\ \cos x &= \frac{1}{\sec x}, \quad \text{or} \quad \cos x \sec x = 1, \\ \tan x &= \frac{1}{\cot x}, \quad \text{or} \quad \tan x \cot x = 1. \end{aligned}\right\} \qquad (1)$$

Also, it follows from the definitions that

$$\tan x = \frac{\sin x}{\cos x}, \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x}. \qquad (2)$$

In the right triangle ABC, $a^2 + b^2 = c^2$. Therefore,

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1,$$

$$\frac{c^2}{b^2} = 1 + \frac{a^2}{b^2},$$

and

$$\frac{c^2}{a^2} = 1 + \frac{b^2}{a^2}.$$

From these equations it follows that

$$\sin^2 x + \cos^2 x = 1, \qquad (3)$$

$$\sec^2 x = 1 + \tan^2 x, \qquad (4)$$

$$\csc^2 x = 1 + \cot^2 x. \qquad (5)$$

17. From the definitions of the trigonometric functions, p. 20, it follows that in any right triangle any function of either of the acute angles is equal to the corresponding co-function of the other acute angle. For example,

$$\sin A = \frac{a}{c}, \text{ and } \cos B = \frac{a}{c}. \quad \therefore \sin A = \cos B = \cos(90° - A).$$

Similarly,

$$\cos A = \sin B = \sin(90° - A),$$
$$\tan A = \cot B = \cot(90° - A),$$
$$\cot A = \tan B = \tan(90° - A),$$
$$\sec A = \csc B = \csc(90° - A),$$
$$\csc A = \sec B = \sec(90° - A),$$
$$\text{vers } A = \text{covers } B = \text{covers }(90° - A),$$
$$\text{covers } A = \text{vers } B = \text{vers }(90° - A).$$

Hence,

Any function of an acute angle is equal to the corresponding co-function of its complement.

The meaning of the prefix *co*, in cosine, cotangent, cosecant, and coversed sine appears from the above. The cosine of an angle is the *complement-sine*, i.e. *the sine of the complement of that angle:* the tangent of an angle is the *cotangent of its complementary angle;* and a similar statement may be made for the secant and for the versed sine of an angle.

ORAL EXERCISES

Prove the following relations:

1. $\sin A \cot A = \cos A$.

Solution. Using only the left number of the equation, we proceed as follows:

$$\sin A \cot A = \sin A \frac{\cos A}{\sin A} \quad \text{(Art. 16, (2).)}$$
$$= \cos A.$$
$$\therefore \sin A \cot A = \cos A.$$

2. $\cos A \tan A = \sin A$.

3. $(\sec A - \tan A)(\sec A + \tan A) = 1$.

4. $(\csc A - \cot A)(\csc A + \cot A) = 1$.

5. $(\tan A + \cot A) \sin A \cos A = 1$.

6. $(\tan A - \cot A) \sin A \cos A = \sin^2 A - \cos^2 A.$

7. $\sin^2 \theta \div \csc^2 \theta = \sin^4 \theta.$

8. $\sin^4 \theta - \cos^4 \theta = \sin^2 \theta - \cos^2 \theta.$

9. $(\sin \theta - \cos \theta)^2 = 1 - 2 \sin \theta \cos \theta.$

10. $(\sin \theta - \cos \theta)^2 + (\sin \theta + \cos \theta)^2 = 2.$

11. $\sec \theta \cot \theta = \csc \theta.$

12. $(\tan \theta + \cot \theta)^2 = \sec^2 \theta + \csc^2 \theta.$

13. $\cot^2 \theta \cos^2 \theta = \cot^2 \theta - \cos^2 \theta.$

14. $\sin^2 \theta + \csc^2 \theta + 2 = (\sin \theta + \csc \theta)^2.$

15. $\text{vers}\, \theta (1 + \cos \theta) = \sin^2 \theta.$

16. $\sin^2 \theta + \text{vers}^2 \theta = 2(1 - \cos \theta).$

17. $\sec \theta - \sin \theta \tan \theta = \cos \theta.$

18. $\csc \theta - \cos \theta \cot \theta = \sin \theta.$

19. $\sec^2 \theta - \tan^2 \theta = \sin^2 \theta + \cos^2 \theta.$

20. $\csc^2 \theta - \cot^2 \theta = \sin^2 \theta + \cos^2 \theta.$

EXERCISE IV

Prove the following identities:

1. $\cos^4 \theta - \sin^4 \theta = 2 \cos^2 \theta - 1.$

SOLUTION. Using only the left member of the equation, we proceed as follows:

$$\begin{aligned} \cos^4 \theta - \sin^4 \theta &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) \\ &= (1)(\cos^2 \theta - \sin^2 \theta) \\ &= \cos^2 \theta - (1 - \cos^2 \theta) \quad \text{(Art. 16, (3).)} \\ &= 2 \cos^2 \theta - 1. \end{aligned}$$

$$\therefore \cos^4 \theta - \sin^4 \theta = 2 \cos^2 \theta - 1.$$

2. $\sin^3 \theta + \cos^3 \theta = (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta).$

3. $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = 2 \csc A.$

4. $(1 + \sin a + \cos a)^2 = 2(1 + \sin a)(1 + \cos a).$

5. $(\cos^3 \theta - \sin^3 \theta) = (\cos \theta - \sin \theta)(1 + \sin \theta \cos \theta).$

6. $\cos^2 \beta (\sec^2 \beta - 2 \sin^2 \beta) = \cos^4 \beta + \sin^4 \beta.$

7. $\dfrac{\sin\beta}{1-\sin\beta}+\dfrac{1+\sin\beta}{\sin\beta}=\sec^2\beta\,(\csc\beta+1).$

8. $\tan\alpha+\tan\beta=\tan\alpha\tan\beta\,(\cot\alpha+\cot\beta).$

9. $\cot\alpha+\tan\beta=\cot\alpha\tan\beta\,(\tan\alpha+\cot\beta).$

10. $\cos^6\alpha+\sin^6\alpha=1-3\sin^2\alpha\cos^2\alpha.$

11. $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A.$

12. $\sin^2\theta\tan^2\theta+\cos^2\theta\cot^2\theta=\tan^2\theta+\cot^2\theta-1.$

13. $\dfrac{\csc A}{\csc A-1}+\dfrac{\csc A}{\csc A+1}=2\sec^2 A.$

14. $\dfrac{\csc A}{\cot A+\tan A}=\cos A.$

15. $(1-\sin\alpha-\cos\alpha)^2(1+\sin\alpha+\cos\alpha)^2=4\sin^2\alpha\cos^2\alpha.$

16. $(\sec A+\cos A)(\sec A-\cos A)=\tan^2 A+\sin^2 A.$

17. $\dfrac{1}{\cot A+\tan A}=\sin A\cos A.$

18. $\dfrac{1-\tan A}{1+\tan A}=\dfrac{\cot A-1}{\cot A+1}.$

19. $\sin^3 A\cos A+\cos^3 A\sin A=\sin A\cos A.$

20. $\sin^2 A\cos^2 A+\cos^4 A=1-\sin^2 A.$

21. $\dfrac{\csc\alpha-\sec\alpha}{\cot\alpha+\tan\alpha}=\dfrac{\cot\alpha-\tan\alpha}{\csc\alpha+\sec\alpha}.$

22. $\dfrac{1+\tan^2 A}{1+\cot^2 A}=\dfrac{\sin^2 A}{\cos^2 A}.$

23. $\dfrac{\sec A-\tan A}{\sec A+\tan A}=1-2\sec A\tan A+2\tan^2 A.$

24. $\tan^2\alpha\sec^2\alpha+\cot^2\alpha\csc^2\alpha$
$=\sec^4\alpha\csc^4\alpha-3\sec^2\alpha\csc^2\alpha.$

25. $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}=\sec A\csc A+1.$

26. $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\sin A+\cos A.$

27. $\cot^4 A + \cot^2 A = \csc^4 A - \csc^2 A.$

28. $\sqrt{\csc^2 A - 1} = \cos A \csc A.$

29. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A.$

30. $(1 + \cot A - \csc A)(1 + \tan A + \sec A) = 2.$

31. $\dfrac{1}{\csc A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} - \dfrac{1}{\csc A + \cot A}.$

32. $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}.$

33. $2 - \text{vers}^2 \theta = \sin^2 \theta + 2 \cos \theta.$

34. $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2 \sin^2 A \cos^2 A).$

35. $\dfrac{\cos A \csc A - \sin A \sec A}{\cos A + \sin A} = \csc A - \sec A.$

36. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}.$

37. $(\tan \alpha + \csc \beta)^2 - (\cot \beta - \sec \alpha)^2$
$= 2 \tan \alpha \cot \beta (\csc \alpha + \sec \beta).$

38. $2 \sec^2 \alpha - \sec^4 \alpha - 2 \csc^2 \alpha + \csc^4 \alpha = \cot^4 \alpha - \tan^4 \alpha.$

39. $(\sin \alpha + \csc \alpha)^2 + (\cos \alpha + \sec \alpha)^2 = \tan^2 \alpha + \cot^2 \alpha + 7.$

40. $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\csc^2 A} - \dfrac{\csc A}{\sec^2 A}.$

41. $2 \text{ vers } A + \cos^2 A = 1 + \text{vers}^2 A.$

42. $\dfrac{\sec x - \tan x}{\sec x + \tan x} = 1 + 2 \tan x (\tan x - \sec x).$

43. $\dfrac{2 \sin \theta \cos \theta - \cos \theta}{1 - \sin \theta + \sin^2 \theta - \cos^2 \theta} = \cot \theta.$

44. $(\sin \alpha \cos \beta + \cos \alpha \sin \beta)^2 + (\cos \alpha \cos \beta - \sin \alpha \sin \beta)^2 = 1.$

45. $(\tan \theta + \sec \theta)^2 = \dfrac{1 + \sin \theta}{1 - \sin \theta}.$

18. Limits of the values of the trigonometric functions of an acute angle.

Since $$\sin^2 A + \cos^2 A = 1,$$

and since each term, being a square, is positive, neither $\sin^2 A$ nor $\cos^2 A$ can be greater than unity. Hence, neither $\sin A$ nor $\cos A$ can be numerically greater than unity.

Since $\csc A$ is the reciprocal of $\sin A$, and $\sec A$ is the reciprocal of $\cos A$, both $\sec A$ and $\csc A$ can have any values numerically greater than unity, but neither can ever be numerically less than unity.

Since $$\sec^2 A = 1 + \tan^2 A,$$

$$\tan A = \sqrt{\sec^2 A - 1}.$$

Hence, $\tan A$ can have any value between 0 and ∞. And since $\cot A$ is the reciprocal of $\tan A$, therefore $\cot A$ can have any value between ∞ and 0.

These results are summarized as follows:

When A is an acute angle,

$\sin A$ can take any value between 0 and $+1$,

$\cos A$ can take any value between $+1$ and 0,

$\tan A$ can take any value between 0 and $+\infty$,

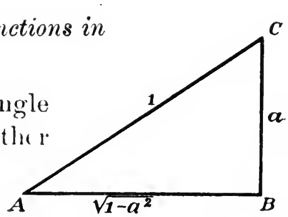
$\cot A$ can take any value between $+\infty$ and 0,

$\sec A$ can take any value between $+1$ and $+\infty$,

$\csc A$ can take any value between $+\infty$ and $+1$.

These results also follow directly from the definitions of the functions of an acute angle, p. 20.

19. *To express all the trigonometric functions in terms of any one of them.*



From any point in either side of the angle A let fall a perpendicular upon the other side. Let the hypotenuse of the right triangle thus formed be taken as unity, and call the perpendicular a. Then the remaining side of the right triangle is $\sqrt{1-a^2}$. Then,

$$\sin A = \frac{a}{1} = a = \sin A,$$

$$\cos A = \sqrt{1-a^2} = \sqrt{1-\sin^2 A},$$

$$\tan A = \frac{a}{\sqrt{1-a^2}} = \frac{\sin A}{\sqrt{1-\sin^2 A}},$$

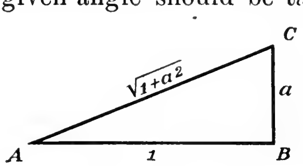
$$\cot A = \frac{\sqrt{1-a^2}}{a} = \frac{\sqrt{1-\sin^2 A}}{\sin A},$$

$$\sec A = \frac{1}{\sqrt{1-a^2}} = \frac{1}{\sqrt{1-\sin^2 A}},$$

$$\csc A = \frac{1}{a} = \frac{1}{\sin A}.$$

This gives the value of each of the functions, except the vers A and the covers A, in terms of $\sin A$.

To express the values of the functions in terms of $\cos A$, $\tan A$, or of any other given function of A, proceed in a similar manner. It is not best, however, to assume the hypotenuse equal to unity for all cases. Sometimes the side opposite the given angle should be taken as unity, and sometimes the side adjacent. For example, to find the other functions of A in terms of $\tan A$, assume the side adjacent A equal to unity, and let the side opposite the same angle equal a; then the hypotenuse of the right triangle equals $\sqrt{1+a^2}$, and the required values are found as follows :



$$\tan A = \frac{a}{1} = a = \tan A,$$

$$\sin A = \frac{a}{\sqrt{1+a^2}} = \frac{\tan A}{\sqrt{1+\tan^2 A}},$$

$$\cos A = \frac{1}{\sqrt{1+a^2}} = \frac{1}{\sqrt{1+\tan^2 A}},$$

$$\cot A = \frac{1}{a} = \frac{1}{\tan A},$$

$$\sec A = \frac{\sqrt{1+a^2}}{1} = \sqrt{1+\tan^2 A},$$

$$\csc A = \frac{\sqrt{1+a^2}}{a} = \frac{\sqrt{1+\tan^2 A}}{\tan A}.$$

In this work it will be noticed that the side adjacent to A is assumed equal to unity, while in the preceding the hypotenuse was assumed to be unity. Any other supposition might be made with equal correctness, but no other would be equally convenient.

EXERCISE V

1. Express all the other functions of θ in terms of $\cos\theta$.

This problem can be solved, and the required values found, in a manner similar to that employed in finding the values of each of the other functions in terms of $\sin\theta$, or $\tan\theta$, which has just been illustrated. Or the values can be found by means of the relations deduced in Art. 16. Thus:

$$\sin\theta = \sqrt{1-\cos^2\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta},$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\sqrt{1-\cos^2\theta}}, \text{ etc.}$$

2. Express all the other functions of θ in terms of $\cot\theta$.
3. Express all the other functions of θ in terms of $\sec\theta$.
4. Express all the other functions of θ in terms of $\csc\theta$.
5. Given $\sin\theta = \frac{2}{5}$, find $\cos\theta$ and $\tan\theta$.

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\tfrac{4}{25}} = \tfrac{1}{5}\sqrt{21}.$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2}{5} \div \frac{1}{5}\sqrt{21} = \frac{2}{5} \times \frac{5}{\sqrt{21}} = \frac{2}{\sqrt{21}} = \frac{2}{21}\sqrt{21}.$$

4 13

6. Construct the angle θ if $\tan\theta = \frac{2}{7}$.

The angle θ may be considered one of the acute angles of a right triangle. Hence, to construct θ we have only to construct a right triangle whose legs are respectively 2 and 7. Since $\tan\theta = \frac{2}{7}$, θ is the acute angle opposite the side 2.

7. If $\sin\theta = \frac{3}{11}$, find $\sec\theta$.

8. If $\sin A = \frac{60}{61}$, find vers A.

9. If $\cos\theta = \frac{3}{5}$, find $\csc\theta$ and $\tan\theta$.

10. If $\cos\theta = \frac{3}{7}$, find $\cot\theta$ and $\sec\theta$.

11. If $\tan A = \frac{11}{60}$, find $\sec A$ and $\cos A$.

12. If $\tan A = \frac{3}{4}$, find $\csc A$ and $\cos A$.

13. If $\cot A = \frac{5}{6}$, find $\sin A$ and $\cos A$.

14. If $\sec B = 5$, find $\sin B$ and $\tan B$.

15. If $\sec B = \frac{61}{60}$, find $\tan B$ and vers B.

16. If $\csc A = 8$, find $\cos A$ and $\tan A$.

17. If $\csc A = \frac{9}{5}$, find $\sin A$ and $\sec A$.

18. Find all the functions of each of the acute angles, A, B, of the right triangle whose sides are 8, 15, 17.

19. Find all the functions of each of the acute angles, A, B, of the right triangle whose sides are $x + y$, $\frac{2xy}{x-y}$, $\frac{x^2+y^2}{x-y}$.

20. If $\sin^2\theta + \cos\theta = \frac{29}{25}$, find $\tan\theta$.

21. If $\tan^2\theta - \sec\theta = 5$, find $\cos\theta$.

22. If $10\sin^2\theta - 5\cos\theta = -\frac{2}{5}$, find $\csc\theta$.

23. If $\sin\theta + \cot\theta = \frac{319}{136}$, find $\cos\theta$.

24. If $\sin\theta = a$ and $\tan\theta = b$, prove $(1 - a^2)(1 + b^2) = 1$.

CHAPTER III

VALUES OF THE FUNCTIONS OF CERTAIN USEFUL ANGLES

20. Functions of an angle of 0°. If the angle A is very small, then in considering the value of $\sin A$, that is, the ratio $\frac{CB}{AB}$, it is at once seen that the numerator, CB, is very small in comparison with the denominator, AB. Hence, the numerical value of $\sin A$ is very small when the angle A is very small. Also, if A decreases, the numerator of the fraction will also decrease, while the denominator will remain constant; and as the angle approaches 0° as a limit, the sine of the angle will also approach 0 as a limit. When the angle becomes 0°, that is, when AB coincides with AC, we shall have $CB = 0$, and $AB = AC$. Hence,

$$\sin 0° = \frac{CB}{AB} = 0, \qquad \tan 0° = \frac{CB}{AC} = 0, \qquad \sec 0° = \frac{AB}{AC} = 1,$$

$$\cos 0° = \frac{AC}{AB} = 1, \qquad \cot 0° = \frac{AC}{CB} = \infty, \qquad \csc 0° = \frac{AB}{CB} = \infty.$$

When we say that $\sin A = 0$ when $A = 0°$, we simply mean that, if A is made small enough, we can make the value of CB, and hence the value of $\sin A$ as small as we please; or, to express the same statement in different words, we can make $\sin A$ smaller than any assignable quantity.

Hence, as stated above, $\sin A$ approaches 0 as a limit when A approaches 0° as a limit.

In a similar manner, we interpret the statements $\cos 0° = 1$, $\tan 90° = \infty$, etc., as meaning that $\cos A$ approaches 1 as a limit, $\tan A$ approaches ∞ as a limit, etc., when A approaches 0° as a limit, when A approaches 90° as a limit, etc.

21. Functions of an angle of 30°. Let OAC be an equilateral triangle; then is it also equiangular. From the vertex O draw OB perpendicular to AC. Then in the right triangle OAB the angle $A = 60°$, and the angle $AOB = 30°$. Also,

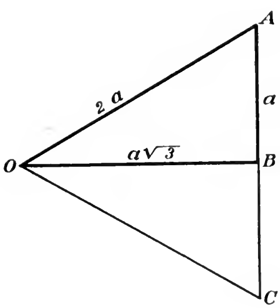
$$\text{the leg } AB = \tfrac{1}{2}AC = \tfrac{1}{2}OA.$$

Let $AB = a$. Then $OA = 2a$, and

$$OB = \sqrt{\overline{OA}^2 - \overline{AB}^2} = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = a\sqrt{3}.$$

The trigonometric functions of 30° can now be found as follows:

$$\sin 30° = \frac{BA}{OA} = \frac{a}{2a} = \frac{1}{2},$$



$$\cos 30° = \frac{OB}{OA} = \frac{a\sqrt{3}}{2a} = \frac{1}{2}\sqrt{3},$$

$$\tan 30° = \frac{BA}{OB} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3},$$

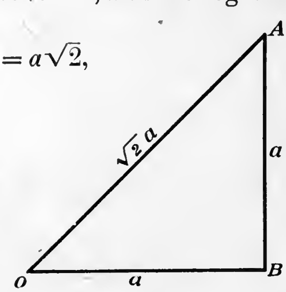
$$\cot 30° = \frac{OB}{BA} = \frac{a\sqrt{3}}{a} = \sqrt{3},$$

$$\sec 30° = \frac{OA}{OB} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3},$$

$$\csc 30° = \frac{OA}{BA} = \frac{2a}{a} = 2.$$

22. Functions of an angle of 45°. Let OAB be an isosceles right triangle. Each of the acute angles is 45°, and the leg OB equals the leg AB.

Let $AB = a$. Then $OB = a$ and $OA = a\sqrt{2}$, and we have:



$$\sin 45° = \frac{BA}{OA} = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\cos 45° = \frac{OB}{OA} = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\tan 45° = \frac{BA}{OB} = \frac{a}{a} = 1,$$

$$\cot 45° = \frac{OB}{BA} = \frac{a}{a} = 1,$$

$$\sec 45° = \frac{OA}{OB} = \frac{\sqrt{2}\,a}{a} = \sqrt{2},$$

$$\csc 45° = \frac{OA}{BA} = \frac{\sqrt{2}\,a}{a} = \sqrt{2}.$$

23. Functions of an angle of 60°. Let OAC be an equilateral triangle. Then is it also equiangular. From the vertex A draw AB perpendicular to OC. Then in the right triangle OAB, angle $O = 60°$, and angle $OAB = 30°$. Also, $OB = \frac{1}{2}\,OC = \frac{1}{2}\,OA$.

Let $OB = a$. Then $OA = 2\,a$, and $AB = \sqrt{OA^2 - OB^2}$

$$= \sqrt{4\,a^2 - a^2} = \sqrt{3\,a^2} = a\sqrt{3}.$$

The trigonometric functions of 60° can now be found as follows:

$$\sin 60° = \frac{BA}{OA} = \frac{a\sqrt{3}}{2\,a} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3},$$

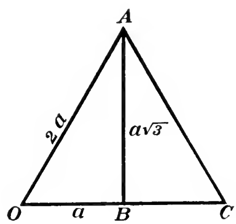
$$\cos 60° = \frac{OB}{OA} = \frac{a}{2\,a} = \frac{1}{2},$$

$$\tan 60° = \frac{BA}{OB} = \frac{a\sqrt{3}}{a} = \sqrt{3},$$

$$\cot 60° = \frac{OB}{BA} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3},$$

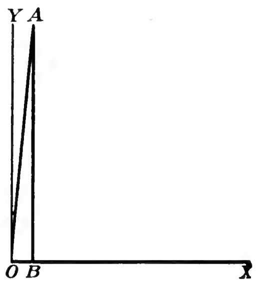
$$\sec 60° = \frac{OA}{OB} = \frac{2\,a}{a} = 2,$$

$$\csc 60° = \frac{OA}{BA} = \frac{2\,a}{a\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$$



24. Functions of an angle of 90°. Let the angle AOB (p. 34) be very nearly a right angle. Then the angle A is very small, and B the foot of the perpendicular from A to OB is very near the vertex O. When the angle O approaches a right angle, AB

will approach coincidence with AO, and B will approach coincidence with O. Hence,



$$\sin 90^\circ = \frac{BA}{OA} = \frac{OA}{OA} = 1,$$

$$\cos 90^\circ = \frac{OB}{OA} = \frac{0}{OA} = 0,$$

$$\tan 90^\circ = \frac{BA}{OB} = \frac{OA}{0} = \infty,$$

$$\cot 90^\circ = \frac{OB}{BA} = \frac{0}{OA} = 0,$$

$$\sec 90^\circ = \frac{OA}{OB} = \frac{OA}{0} = \infty,$$

$$\csc 90^\circ = \frac{OA}{BA} = \frac{OA}{OA} = 1.$$

The real meaning of these equations is that, as the angle approaches 90° as a limit, the sine of the angle approaches 1 as a limit, the cosine approaches 0 as a limit, the tangent approaches ∞ as a limit, etc. It is, however, customary to say $\sin 90^\circ = 1$, $\cos 90^\circ = 0$, $\tan 90^\circ = \infty$, etc.

A more complete discussion of the values of the trigonometric functions for limiting cases such as the above is given later. See Art. 41, p. 59.

25. In the following table are collected the results obtained in the last five sections. These results are exceedingly important, and the student should become thoroughly familiar with them before proceeding further.

	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cosine	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞
cotangent	∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0
secant	1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	∞
cosecant	∞	2	$\sqrt{2}$	$\frac{2}{3}\sqrt{3}$	1

It is necessary to commit to memory only one half of this table. The remainder can be obtained at any time by means of the relations which were found in Art. 17, of which the following is a condensed statement: *Any trigonometric function of an acute angle is equal to the corresponding co-function of its complement.*

EXERCISE VI

Verify the following:

1. $\cos 0° + \sin 30° + \sin 90° = 2\frac{1}{2}$.
2. $\cos 0° \cos 60° + \sin 0° \sin 60° + \sin 30° = 1$.
3. $\tan^2 30° + \sec^2 30° = 1\frac{2}{3}$.
4. $\cos^2 60° + \cos^2 45° + \cos^2 30° = \frac{3}{2}$.
5. $\sin 60° \cos 30° + \cos 60° \sin 30° = 1$.
6. $\sin^2 30° \tan^2 45° + \sec^2 60° \sin^2 90° = 4\frac{1}{4}$.
7. $(\sin 30° + \cos 60°)(\sec 45° + \csc 45°) = 2\sqrt{2}$.
8. $\sin 30° \sin 45° \sin 60° \tan 60° = \frac{3}{8}\sqrt{2}$.
9. $\cot 30° \tan 60° \sin 45° \cos 45° = \frac{3}{2}$.
10. $\tan^2 45° + \sin^2 30° - \cos^2 30° - \frac{3}{4}\tan^2 30° = \frac{1}{4}$.

Prove the following identities:

11. $\sin A \cos (90° - A) \sec (90° - A) \equiv \sin A$.
12. $\cos A \cos (90° - A) \sin (90° - A) \csc A \equiv \cos^2 A$.
13. $\tan (90° - A) \cot (90° - A) \tan A$
$$\equiv \cos (90° - A) \csc (90° - A).$$
14. $\dfrac{\cos (90° - A)}{\csc (90° - A)} \cdot \dfrac{\cot (90° - A)}{\sin A} \equiv \sin A$.
15. $\cos^2 A \sec^2 (90° - A) \tan^2 A \cot^2 (90° - A) \equiv \tan^2 A$.
16. $\dfrac{\tan^2 (90° - A)}{\sin^2 A} \cdot \dfrac{\cos^2 A}{\cot^2 (90° - A)} \cdot \dfrac{\csc^2 (90° - A)}{\sec^2 (90° - A)} \equiv \cot^4 A$.
17. $\dfrac{\cos (90° - A)}{\text{covers } A} \cdot \dfrac{1 - \cos (90° - A)}{\sin (90 - A)} \equiv \tan A$.
18. $\sec^2 A \equiv 1 + \cos^2 (90° - A) \csc^2 (90° - A)$.
19. $\csc^2 A \equiv 1 + \sin^2 (90° - A) \sec^2 (90° - A)$.
20. $\dfrac{\cot^2 (90° - A)}{\csc^2 (90° - A)} \cdot \dfrac{\tan^2 (90° - A)}{\sin^2 (90° - A)} \equiv 1$.

CHAPTER IV

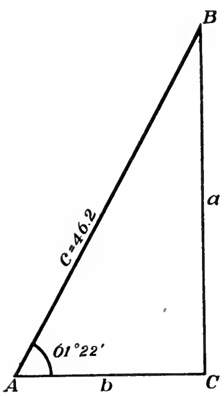
THE RIGHT TRIANGLE

26. In order to solve a right triangle, two parts besides the right angle must be given, of which at least one must be a side. The known parts may be:

1. An acute angle and the hypotenuse.
2. An acute angle and the opposite leg.
3. An acute angle and the adjacent leg.
4. The hypotenuse and either leg.
5. The two legs.

27. In the preceding sections we have seen that the trigonometric functions are pure numbers; and in the case of the angles 0°, 30°, 45°, 60°, and 90°, the values of these functions have been ascertained. From a trigonometric table the values of the functions of any angle can be found; and by the aid of these values the solution of any triangle can be effected.

The method for each case arising under right triangles is illustrated by the following examples:



CASE 1

Given $A = 61° \, 22'$, $c = 46.2$; find B, a, b.

(1) $B = 90° - 61° \, 22' = 28° \, 38'$.

(2) $\sin A = \frac{a}{c}$. $\therefore a = c \sin A$

$= 46.2 \times 0.8777.$

$\therefore a = 40.54.$

(3) $\cos A = \frac{b}{c}$. $\therefore b = c \cos A$

$= 46.2 \times 0.4792.$

$\therefore b = 22.14.$

Case 2

Given $A = 31° 17'$, $a = 321$; find B, c, b.

(1) $B = 90° - 31° 17' = 58° 43'$.

(2) $\sin A = \frac{a}{c}$. $\therefore c = \frac{a}{\sin A}$

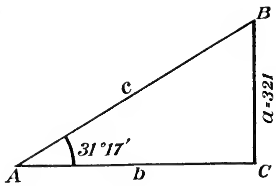
$= \frac{321}{0.5193} = 618.14.$

$\therefore c = 618.14.$

(3) $\tan A = \frac{a}{b}$. $\therefore b = \frac{a}{\tan A}$

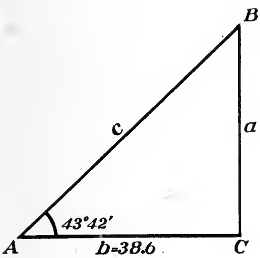
$= \frac{321}{0.6076}.$

$\therefore b = 528.31.$



Case 3

Given $A = 43° 42'$, $b = 38.6$; find B, a, c.



(1) $B = 90° - 43° 42' = 46° 18'$.

(2) $\tan A = \frac{a}{b}$. $\therefore a = b \tan A$

$= 38.6 \times 0.9556.$

$\therefore a = 36.89.$

(3) $\cos A = \frac{b}{c}$. $\therefore c = \frac{b}{\cos A}$

$= \frac{38.6}{0.7230}.$

$\therefore c = 53.39.$

Case 4

Given $a = 36.4$, $c = 48.4$; find A, B, b.

(1) $\sin A = \frac{a}{c}$

$= \frac{36.4}{48.4}$

$= 0.7521$, nearly.

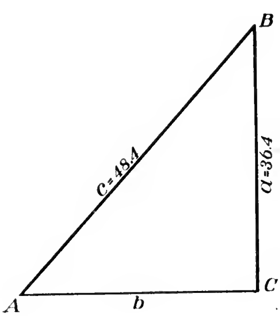
$\therefore A = 48° 46'.$

(2) $B = 90° - 48° 46' = 41° 14'$.

(3) $\tan A = \frac{a}{b}$. $\therefore b = \frac{a}{\tan A}$

$= \frac{36.4}{1.141}.$

$\therefore b = 31.9.$



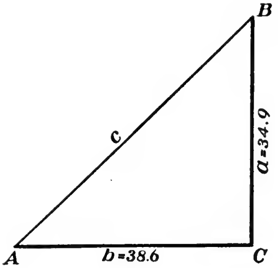
The value of b could also be found directly by means of the familiar geometric relation

$$c^2 = a^2 + b^2,$$

from which we have $$b = \sqrt{c^2 - a^2}.$$

Case 5

Given $a = 34.9$, $b = 38.6$; find A, B, c.



(1) $\tan A = \frac{a}{b}$

$= \frac{34.9}{38.6}$

$= 0.9041.$

$\therefore A = 42° 7'.$

(2) $B = 90° - 42° 7' = 47° 53'.$

(3) $\sin A = \frac{a}{c}. \quad \therefore c = \frac{a}{\sin A}$

$= \frac{34.9}{0.6706}.$

$\therefore c = 52.04.$

The value of c can also be found directly by means of the relation

$$c^2 = \sqrt{a^2 + b^2}.$$

From the methods of solution illustrated in the examples given in Art. 27, we deduce the following general rule:

Rule for the solution of right triangles. *From the equation* $A + B = 90°$, *and from the equations that define the functions of an acute angle of a right triangle, select an equation in which the required part is the only unknown quantity. From this equation find an expression for the required part, and compute the value of this part from the expression thus obtained.*

If a and c, or b and c, have values that differ but little from each other, the methods here given will yield inaccurate results. In such cases the method of Art. 101, p. 144, should be employed.

The student will find it advantageous to check his results in all cases, to avoid numerical errors as far as possible. Any method of checking can be employed that involves a process of solution different from the one used in first obtaining the required part.

EXERCISE VII

In the following examples, use the first two parts as the given parts, and solve for the three remaining parts :

1. $A = 21° 19'$, $c = 18$. $B = 68° 41'$, $a = 6.5$, $b = 16.8$.
2. $B = 40° 44'$, $c = 31$. $A = 49° 16'$, $b = 20.2$, $a = 23.5$.
3. $A = 71° 38'$, $c = 5.4$. $B = 18° 22'$, $a = 5.12$, $b = 1.7$.
4. $B = 13° 14'$, $c = 92$. $A = 72° 46'$, $b = 21$, $a = 89.6$.
5. $A = 63° 11'$, $a = 12$. $B = 26° 49'$, $b = 6.1$, $c = 13.4$.
6. $B = 43° 52'$, $b = 70$. $A = 46° 8'$, $a = 72.8$, $c = 101$.
7. $A = 19° 36'$, $b = 42$. $B = 70° 24'$, $a = 15$, $c = 44.6$.
8. $B = 56° 17'$, $a = 9$. $A = 33° 43'$, $b = 13.5$, $c = 16.2$.
9. $a = 12.6$, $c = 26$. $A = 28° 59'$, $B = 61° 1'$, $b = 22.7$.
10. $b = 42.6$, $c = 46$. $A = 22° 10'$, $B = 67° 50'$, $a = 17.4$.

Solution by Logarithms

28. Problems in the solution of triangles can usually be performed quite as expeditiously by the use of logarithms as by the use of the actual values of the trigonometric functions, and in many cases the amount of labor is very greatly reduced by the use of logarithms.

The method of solution by logarithms in the different cases that arise in connection with right triangles is illustrated by the following problems:

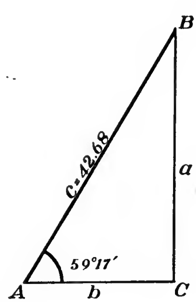
Case 1

Given $A = 59° 17'$, $c = 42.68$; find B, a, b,

(1) $B = 90° - 59° 17' = 30° 43'$.

(2) $\sin A = \frac{a}{c}$. $\therefore a = c \sin A$.

$$\begin{aligned} \log a &= \log c + \log \sin A. \\ \log c &= 1.63022 \\ \log \sin A &= 9.93435 - 10 \\ \log a &= 1.56457 \\ \therefore a &= 36.69. \end{aligned}$$

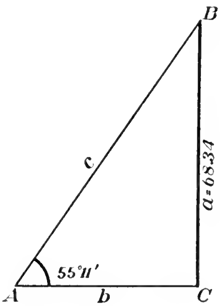


(3) $\cos A = \frac{b}{c}$; $b = c \cos A$.

$$\begin{aligned} \log c &= 1.63022 \\ \log \cos A &= 9.70824 - 10 \\ \log b &= 1.33846 \\ \therefore b &= 21.8. \end{aligned}$$

Case 2

Given $A = 55° 11'$, $a = 68.34$; find B, b, c.



(1) $B = 90° - 55°, 11' = 34° 49'$.

(2) $\tan A = \frac{a}{b}$. $\therefore b = \frac{a}{\tan A}$.

$\log b = \log a + \text{colog} \tan A$.

$$\begin{aligned} \log a &= 1.83467 \\ \text{colog} \tan A &= 9.84227 - 10 \\ \hline \log b &= 1.67694 \\ \therefore b &= 47.527. \end{aligned}$$

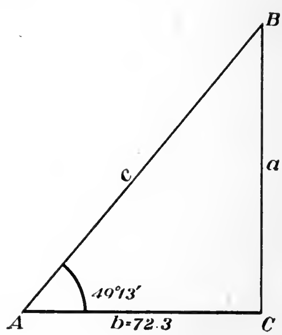
(3) $\sin A = \frac{a}{c}$. $\therefore c = \frac{a}{\sin A}$.

$\log c = \log a + \text{colog} \sin A$.

$$\begin{aligned} \log a &= 1.83467 \\ \text{colog} \sin A &= 0.08567 \\ \hline \log c &= 1.92034 \\ \therefore c &= 83.242. \end{aligned}$$

Case 3

Given $A = 49° 13'$, $b = 72.3$; find B, a, c.



(1) $B = 90° - 49° 13' = 40° 47'$.

(2) $\tan A = \frac{a}{b}$. $\therefore a = b \tan A$.

$\log a = \log b + \log \tan A$.

$$\begin{aligned} \log b &= 1.85914 \\ \log \tan A &= 10.06416 - 10 \\ \hline \log a &= 1.92330 - 10 \\ a &= 83.81. \end{aligned}$$

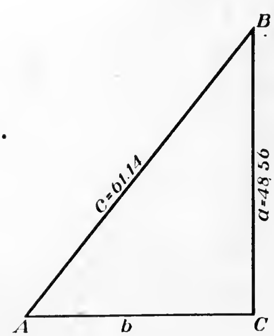
(3) $\cos A = \frac{b}{c}$. $\therefore c = \frac{b}{\cos A}$.

$\log c = \log b + \text{colog} \cos A$.

$$\begin{aligned} \log b &= 1.85914. \\ \text{colog} \cos A &= 0.18495 \\ \hline \log c &= 2.04409 \\ \therefore c &= 110.68 \end{aligned}$$

Case 4

Given $c = 61.14$, $a = 48.56$; find A, B, b.



(1) $\sin A = \frac{a}{c}$.

$\log \sin A = \log a + \text{colog}\, c$.

$$\begin{aligned} \log a &= 1.68628 \\ \text{colog}\, c &= 8.21367 - 10 \\ \hline \log \sin A &= 9.89995 - 10 \\ \therefore A &= 52° 35'. \end{aligned}$$

(2) $\cos A = \frac{b}{c}$. $\therefore b = c \cos A$.

$$\log b = \log c + \log \cos A.$$
$$\log c = 1.78633.$$
$$\log \cos A = 9.78362 - 10$$
$$\log b = 1.56995$$
$$b = 37.149.$$

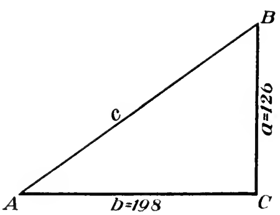
(3) $\tan B = \frac{b}{a}$.

$$\log \tan B = \log b + \text{colog}\, a.$$
$$\log b = 1.56995$$
$$\text{colog}\, a = 8.31372 - 10$$
$$\log \tan B = 9.88367 - 10$$
$$\therefore B = 37° 25'.$$

Case 5

Given $a = 126$, $b = 198$; find A, B, c.

(1) $\tan A = \frac{a}{b}$.



$$\log \tan A = \log a + \text{colog}\, b.$$
$$\log a = 2.10037$$
$$\text{colog}\, b = 7.70333 - 10$$
$$\log \tan A = 9.80370 - 10$$
$$\therefore A = 32° 28'.$$

(2) $\tan B = \frac{b}{a}$.

$$\log \tan B = \log b + \text{colog}\, a.$$
$$\log b = 2.29667$$
$$\text{colog}\, a = 7.89963 - 10$$
$$\log \tan B = 10.19630 - 10$$
$$\therefore B = 57° 32'.$$

(3) $\sin A = \frac{a}{c}$.

$$c = \frac{a}{\sin A},$$
$$\log c = \log a + \text{colog} \sin A.$$
$$\log a = 2.10037$$
$$\text{colog} \sin A = 0.27018$$
$$\log c = 2.37055$$
$$\therefore c = 234.72.$$

NOTE. In the last two cases the angle B might have been found directly by subtracting A from 90°. It is, however, better to determine the value of the second angle independently, as a means of checking the work.

12

AREA OF THE RIGHT TRIANGLE

29. The area of any triangle is equal to one half the product of the base and the altitude. In the case of the right triangle either of the legs can be regarded as the base and the other as the altitude. Hence the area of a right triangle can be found when any two parts are known, provided one or both the known parts are sides, by computing, if necessary, the legs of the triangle, and then taking one half their product. That is,

If a, b, denote the legs of a right triangle, and Δ the area, then

$$\Delta = \tfrac{1}{2}\, ab. \tag{1}$$

Ex. 1. In the right triangle ABC, given $A = 36°\ 14'$, $a = 26.8$; to find the area.

First find $\log b$ by the method of Case 2, p. 40. Then we have

$$\begin{aligned} \log \Delta &= \log a + \log b + \text{colog } 2. \\ \log a &= 1.42813 \\ \log b &= 1.56315 \\ \text{colog } 2 &= \underline{9.69897 - 10} \\ \log \Delta &= 2.69025 \\ \therefore \Delta &= 490.06. \end{aligned}$$

Ex. 2. In the right triangle ABC, given $A = 40°\ 23$, $c = 39.6$; to find the area.

First find $\log a$ and $\log b$ as in Case 1, p. 39. Then we have

$$\begin{aligned} \log \Delta &= \log a + \log b + \text{colog } 2. \\ \log a &= 1.40921 \\ \log b &= 1.47950 \\ \text{colog } 2 &= \underline{9.69897 - 10} \\ \log \Delta &= 2.58768 \\ \therefore \Delta &= 386.97. \end{aligned}$$

EXERCISE VIII

Solve the following right triangles, finding the angles to the nearest minute:

1. Given $A = 34°\ 10'$, $a = 21$;
   find $B = 55°\ 50'$, $b = 30.939$, $c = 37.39$.

2. Given $B = 50°\ 12'$, $a = 65$;
   find $A = 39°\ 48'$, $b = 78.15$, $c = 101.55$.

3. Given $B = 47° 15'$, $c = 54.39$;
find $A = 42° 45'$, $a = 36.92$, $b = 39.94$.

4. Given $A = 31° 25'$, $c = 45.62$;
find $B = 58° 35'$, $b = 38.93$, $a = 23.78$.

5. Given $A = 29° 17'$, $c = 31.68$;
find $B = 60° 43'$, $a = 15.495$, $b = 27.63$.

6. Given $A = 49° 17'$, $c = 36.48$;
find $B = 40° 43'$, $a = 27.65$, $b = 23.796$.

7. Given $A = 41° 9'$, $b = 156$;
find $B = 48° 51'$, $a = 136.33$, $c = 207.17$.

8. Given $B = 59° 11'$, $b = 221$;
find $A = 30° 49'$, $a = 131.83$, $c = 257.33$.

9. Given $B = 62° 55'$, $c = 92.4$;
find $A = 27° 5'$, $a = 42.068$, $b = 82.268$.

10. Given $A = 29° 31'$, $a = 290.6$;
find $B = 60° 29'$, $b = 513.29$, $c = 589.85$.

11. Given $B = 45° 20'$, $a = 41.46$;
find $A = 44° 40'$, $b = 41.946$, $c = 58.979$.

12. Given $a = 20.08$, $c = 28.26$;
find $A = 45° 17'$, $B = 44° 43'$, $b = 19.885$.

13. Given $B = 55° 13'$, $a = 72.96$;
find $A = 34° 47'$, $b = 105.04$, $c = 127.89$.

14. Given $B = 51° 19'$, $b = 106.8$;
find $A = 38° 41'$, $a = 85.512$, $c = 136.81$.

15. Given $B = 59° 49'$, $a = 254.36$;
find $A = 30° 11'$, $b = 437.33$, $c = 505.92$.

16. Given $A = 51° 50'$, $b = 6.813$;
find $B = 38° 10'$, $a = 8.668$, $c = 11.025$.

17. Given $B = 57° 46'$, $b = 0.0688$;
find $A = 32° 14'$, $a = 0.04338$, $c = 0.08134$.

18. Given $b = 963.3$, $c = 1465$;
find $A = 48° 53'$, $B = 41° 7'$, $a = 1103.7$.

19. Given $a = 691$, $c = 877.62$;
find $A = 51° 56'$, $B = 38° 4'$, $b = 541.05$.

20. Given $a = 62.36$, $b = 33.823$;
find $A = 61° 32'$, $B = 28° 28'$, $c = 70.96$.

In the following examples find the required angles to the nearest second:

21. Given $A = 41° 38' 20''$, $b = 262.38$;
find $B = 48° 21' 40''$, $a = 233.27$, $c = 351.08$.

22. Given $A = 71° 14' 12''$, $c = 129.3$;
find $B = 18° 45' 48''$, $a = 122.43$, $b = 41.6$.

23. Given $A = 41° 17' 30''$, $a = 29.41$;
find $B = 48° 42' 30''$, $b = 33.486$, $c = 44.568$.

24. Given $B = 61° 12' 15''$, $c = 382.6$;
find $A = 28° 47' 45''$, $a = 184.29$, $b = 335.29$.

25. Given $b = 1426$, $c = 2291.2$;
find $A = 51° 30' 38''$, $B = 38° 29' 22''$, $a = 1793.38$.

26. Given $B = 54° 2' 28''$, $a = 49.628$;
find $A = 35° 57' 32''$, $b = 68.41$, $c = 84.514$.

27. Given $a = 35.421$, $b = 18.168$;
find $A = 62° 50' 46''$, $B = 27° 9' 14''$, $c = 39.81$.

28. Given $a = 39.313$, $b = 19.852$;
find $A = 63° 12' 26''$, $B = 26° 47' 34''$, $c = 44.036$.

29. Given $a = 126.43$, $b = 131.52$;
find $A = 43° 52' 9''$, $B = 46° 7' 51''$, $c = 182.44$.

30. Given $a = 476.32$, $c = 812.36$;
find $A = 35° 53' 53''$, $B = 54° 6' 7''$, $b = 658.05$.

31. Given $A = 68° 17' 22''$, $c = 269.4$;
find $B = 21° 42' 38''$, $a = 250.29$, $b = 99.658$.

In the following ten examples find the area of the triangle in each case, having given:

32. $a = 10$, $b = 12$.

33. $a = 268$, $b = 316$.

34. $a = 3$, $c = 5$.

35. $b = 20.7844$, $c = 24$.

36. $A = 35°$, $a = 16$.

37. $A = 42° 27'$, $b = 50$.

38. $A = 54° 24'$, $c = 90$.

39. $B = 39° 55'$, $a = 294$.

40. $B = 66° 36'$, $b = 48$.

41. $B = 70° 52'$, $c = 582$.

42. Find the value of $\triangle$ in terms of a and c.

43. Find the value of $\triangle$ in terms of a and A.

44. Find the value of $\triangle$ in terms of a and B.

45. Find the value of $\triangle$ in terms of c and A.

46. Given $\triangle = 72$, $a = 9$; find A.

47. Given $\triangle = 72$, $b = 9$; find A.

48. Given $\triangle = 250$, $A = 40°$; find a.

49. Given $\triangle = 250$, $B = 29° 30'$; find a.

50. Given $\triangle = 254.2$, $c = 32$; find B.

51. The hypotenuse of a right triangle is 28 and one of the legs is 13. Find the angle opposite the given leg.

52. The legs of a right triangle are 36 and 39, respectively. Find the angle opposite the shorter leg.

53. The tangent of one of the acute angles of a right triangle is $\frac{9}{21}$. Find the angle.

54. The cotangent of one of the acute angles of a right triangle is $\frac{14}{11}$. What is the angle?

55. One of the acute angles of a right triangle is 49° 38′ and the adjacent leg is 68.42. Find the hypotenuse and the other leg.

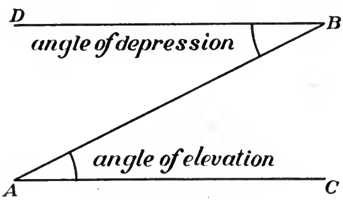
56. The legs of a right triangle are 41625.3 and 11362.7, respectively. Find the larger angle.

57. The hypotenuse of a right triangle is 262.46 and one of the acute angles is 28° 15′ 42″. Find the opposite leg.

58. The legs of a right triangle are 515.38 and 221.34, respectively. Find the hypotenuse.

59. One of the acute angles of a right triangle is 46° 21′ and the adjacent leg is 26.38. Find by natural functions the other leg and the hypotenuse.

Angle of elevation and angle of depression. The angle of elevation of an object above the point of observation is the angle between a line from the eye of the observer to the object and a horizontal line in the same vertical plane. The angle of depression of an object below the point of observation is the angle between a line from the eye of the observer to the object and a horizontal line in the same vertical plane.



In the figure, BAC is the angle of elevation of the point B above the point A; and DBA is the angle of depression of the point A below the point B.

60. The angle of elevation of the top of a tower 80 ft. high is 41° 49′. What is the distance of the point of observation from the foot of the tower?

61. At a distance of 31.15 ft. from the foot of a vertical cliff the angle of elevation of the top of the cliff is 56° 18′. What is the height of the cliff?

62. From the top of a monument the angle of depression of a point on the ground, on the same level as the foot of the monument, is 43° 41′. The point is found by measurement to be 128.29 ft. distant from the foot of the monument. What is the height of the monument?

63. From the top of a hill 304 ft. 9 in. in height the angle of depression of an object on the ground is 40° 37′. What is the distance of the object from a point directly below the point of observation and on the same level with the object?

64. What is the height of a tree that casts a shadow 42.6 ft. long when the angle of elevation of the sun is 60° 11′?

65. What must be the length of a ladder set at an angle of 71° 14′ with the ground to reach a window 21.88 ft. high?

66. To find the width of a river a point P is selected on one bank, and a distance of 138.2 ft. is measured parallel to the course of the river from the given point P to a point Q. Directly opposite Q, on the other side of the river, is the point S, and the angle SPQ is found to be 66° 11′. What is the width of the river?

67. A guy rope 49.11 ft. long is attached to the top of a mast, and makes an angle of 50° 56′ with the level of the ground. What is the height of the mast?

68. The top of a flag pole, broken by the wind, falls so that it touches the ground at a distance of 19.73 ft. from the foot of the pole, and is inclined to the ground at an angle of 65° 40′. What is the height of the portion that remains standing, and what was the total height of the pole?

69. What is the angle of elevation of an inclined plane that rises 26 ft. in a horizontal distance of 31.9 ft.?

70. A man walking on a level plain toward a tower observes that at a certain point the angle of elevation of the top of the tower is 30°; on walking 300 ft. directly toward the tower the angle of elevation of the top is found to be 60°. What is the height of the tower?

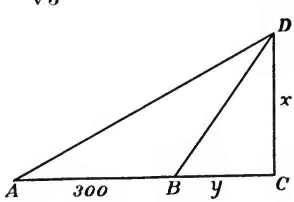
SOLUTION. Let x = the height of the tower and y = the distance from the second point of observation to the foot of the tower.

From the triangle ACD $\dfrac{x}{300+y} = \tan 30° = \dfrac{1}{\sqrt{3}}$,

$\therefore\ y = \sqrt{3}\,x - 300;$

from the triangle BCD $\dfrac{x}{y} = \tan 60° = \sqrt{3}$,

$y = \dfrac{x}{\sqrt{3}};$



equating these values of y, we have

$$\sqrt{3}\,x - 300 = \frac{x}{\sqrt{3}},$$

$$2\,x = 300 \times 1.732,$$

$$x = 259.8.$$

68, 70, 71, § 30, 31

NOTE. In solving problem 70 natural functions have been employed. On p. 156 a method will be given by means of which problems of this kind can be solved by the use of logarithms. In the following problems it is recommended that natural functions be employed.

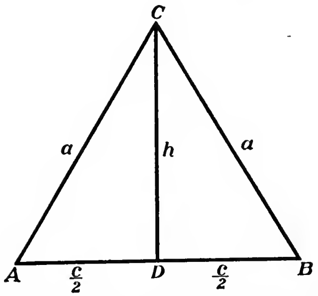
71. At a point on a level plain the angle of elevation of the top of a church spire is 45°, and at a point 50 ft. nearer, and in the same straight line with the first point and the church, the corresponding angle of elevation is 60°. What is the height of the spire?

72. From the top of a cliff 150 ft. high the angles of depression of the top and bottom of a tower are 30° and 60°, respectively. What is the height of the tower?

73. The angles of elevation of the top of a tower, taken at two points 268 ft. apart and in the same straight line with the tower, are 21° 14′ and 53° 46′, respectively. What is the height of the tower?

74. At the foot of a mountain the angle of elevation of the summit is 45°; one mile up the slope of the mountain, which rises at an inclination of 30°, the angle of elevation of the summit is 60°. What is the height of the mountain?

75. At a certain point south of a tower the angle of elevation of the top of the tower is 60°, and at a point 300 ft. east of the point the corresponding angle of elevation is 30°. What is the height of the tower?



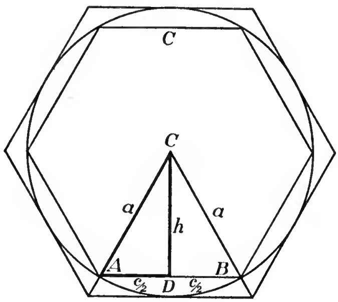
30. The isosceles triangle. The perpendicular from the vertex, C, of an isosceles triangle to the base divides the triangle into two equal right triangles.

Any two parts of either of these right triangles being given, one or both of which are sides, the right triangle can be completely determined. Therefore the isosceles triangle also can be completely determined.

Denoting the base of the isosceles triangle by c, and the altitude by h, the area, Δ, is given by the formula

$$\Delta = \tfrac{1}{2}\, ch. \qquad (1)$$

31. The regular polygon. A regular polygon is divided into equal isosceles triangles by lines drawn from the center to the vertices of the polygon. Each of the isosceles triangles is divided into two equal right triangles by the apothem of the polygon.



Any side of either of these right triangles being given, the polygon can be completely determined if the number of sides is known.

For the angles at the center of the polygon can be found when the number of sides, n, is known, by dividing 360° by n. Taking one half of this angle as one of the acute angles of the right triangle, and combining it with the given side, we have at our disposal two parts of a right triangle, one of which is a side. The remaining parts can then be found by the methods already given for the solution of right triangles.

Denoting the perimeter of the polygon by p and the apothem by h, the area of the polygon can be found by the following formula:

$$\triangle = \tfrac{1}{2}\, ph. \qquad (1)$$

It should be remembered that the legs of the isosceles triangles are radii of the circumscribed circle, and the apothem is the radius of the inscribed circle of the polygon.

EXERCISE IX

Solve the following isosceles triangles, finding the part indicated in each case:

1.	Given $c = 83.2$,	$h = 56.9$;	find C.
2.	Given $c = 92.56$,	$h = 59.72$;	find C.
3.	Given $c = 252.64$,	$C = 62°\ 28'\ 36''$;	find a.
4.	Given $C = 142°\ 27'\ 44''$,	$a = 92.452$;	find c.
5.	Given $C = 102°\ 44'\ 42''$,	$h = 92.96$;	find a.
6.	Given $c = 85.32$,	$h = 49.84$;	find A.
7.	Given $c = 136.48$,	$h = 60.51$;	find a.
8.	Given $h = 1426.3$,	$a = 2291.2$;	find A.

9. Find the value of Δ in terms of a and C.

10. Find the value of Δ in terms of a and A.

11. Find the value of Δ in terms of h and A.

Solve the following regular polygons, having given:

12. $n = 10$, $c = 3$.

13. $n = 8$, $h = 2$.

14. $n = 6$, $c = 12$.

15. $n = 20$, $a = 10$.

16. What is the area of a regular octagon formed by cutting away the corners of a square whose side is 6?

17. What is the area of a circle inscribed in an equilateral triangle whose side is 20?

18. What is the area of a regular polygon of 18 sides if the radius of the circumscribed circle is 2?

19. One of the diagonals of a regular pentagon is 12.15. What is the area of the pentagon?

20. Compute the area of a regular heptagon if the length of one of its sides is 13.88.

21. The radius of the circumscribed circle of a regular dodecagon is 27. What is the area?

CHAPTER V

THE APPLICATION OF ALGEBRAIC SIGNS TO TRIGONOMETRY

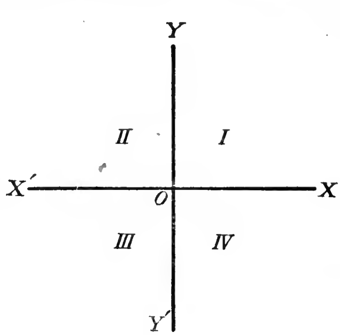
32. In the preceding work no attempt has been made to apply the definitions of any of the trigonometric functions to any except positive acute angles.

These definitions will now be extended so as to apply to negative as well as to positive angles, and to angles of any magnitude whatever.

33. The coördinate axes. The location of a point or a line lying in a given plane is often described by referring it to two intersecting straight lines in that plane, called **coördinate axes.** These lines are usually drawn perpendicular to each other.

Let the two lines XX' and YY' intersect at right angles. Then the plane of these lines is divided into four quadrants, designated as the first, second, third, and fourth quadrants, respectively. These quadrants are numbered as indicated in the figure.

34. Coördinates of a point in a plane. The location of any point in the plane determined by the axes XX' and YY' is described by means of its perpendicular distances from these axes.



The distance of a point from YY' measured along a line parallel to XX' is called the **abscissa** of the point; and the distance of a point from XX', measured on a line parallel to YY' is called the **ordinate** of the point.

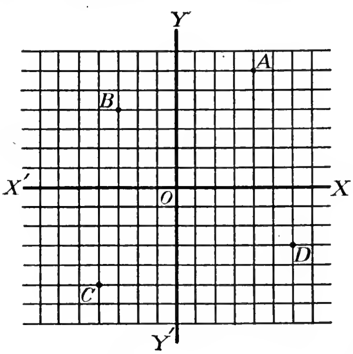
The abscissa of a point is usually designated by the letter x, and the ordinate by the letter y. These two distances, taken together, are called the **coördinates** of the point.

The line XX' is called the axis of abscissas, and the line YY' is called the axis of ordinates. These axes are, for the sake of brevity, often called the x-axis and the y-axis, respectively. Their point of intersection, O, is called the **origin.**

Any abscissa measured to the right of YY' is considered positive, and any abscissa measured to the left of YY' is considered negative.

Any ordinate measured above XX' is considered positive, and any ordinate measured below XX' is considered negative.

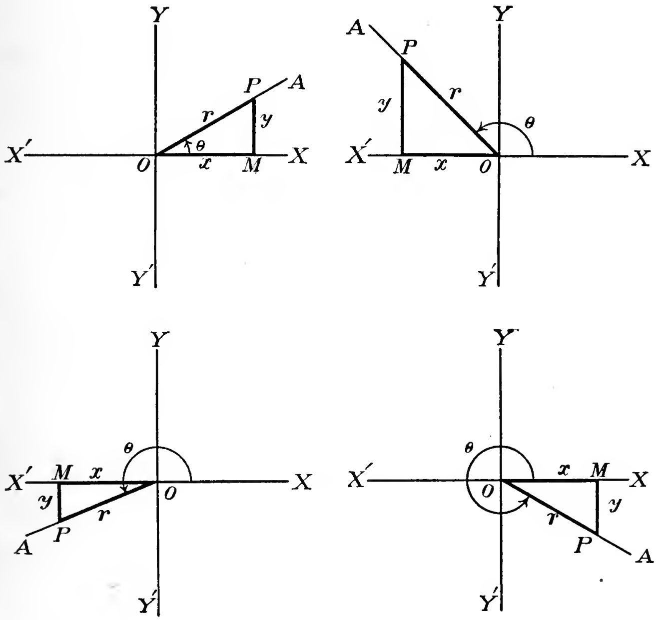
The coördinates of a point determine its position completely. For example, if the point A is 4 units from YY' and 6 units from XX', its position can be located as follows: measure off on XX' a distance equal to 4 units, and through the point thus found draw a line parallel to YY'. Also, measure off on YY' a distance equal to 6 units, and through the point thus determined, draw a line parallel to XX'. The intersection, A, of these two lines is the required point. The abscissa of A is 4, and its ordinate is 6, and this point, whose location is given by means of its coördinates, is called the point (4, 6).



The point B, located in a similar manner, has for its coördinates $x = -3$ and $y = 4$; and this point B is called the point $(-3, 4)$. The point C is called the point $(-4, -5)$; and the point D is called the point $(6, -3)$. In a similar manner we can locate any other point (a, b), where a and b are any real quantities whatever, either positive or negative.

35. Trigonometric functions of any angle. Let the line OA (p. 53) start from OX and revolve in a positive direction until it occupies a position in any one of the four quadrants. From any point P in the revolving line draw a perpendicular PM to the axis of abscissas, XX'. In each of the four figures we have

$OM = x$ and $MP = y$. Let the distance $OP = r$. The trigonometric functions of the angle XOA, which may be represented by θ, are then, for all positions of OA, defined as follows:



$$\sin\theta = \frac{\text{ordinate}}{\text{revolving line}} = \frac{y}{r},$$

$$\cos\theta = \frac{\text{abscissa}}{\text{revolving line}} = \frac{x}{r},$$

$$\tan\theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x},$$

$$\cot\theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y},$$

$$\sec\theta = \frac{\text{revolving line}}{\text{abscissa}} = \frac{r}{x},$$

$$\csc\theta = \frac{\text{revolving line}}{\text{ordinate}} = \frac{r}{y},$$

The functions vers θ and covers θ are defined in a manner similar to that employed in the case of the right triangle, as follows:

$$\text{vers}\,\theta = 1 - \cos\theta,$$

$$\text{covers}\,\theta = 1 - \sin\theta.$$

Note. In the case of cot 0°, csc 0°, tan 90°, sec 90°, cot 180°, csc 180°, tan 270°, sec 270°, cot 360° and csc 360°, these definitions fail. For, taking as an illustration the tangent of 90°, we have in that case a fraction whose numerator is r and whose denominator is 0. The value of tan 90° is, then, if we attempt to use the above definition, given by this fraction whose numerator is r, and whose denominator is 0. But there is no such thing as division by 0, hence, according to the definition given, the symbol tan 90° has no meaning. This and other similar cases will be discussed later. (See pp. 57–63.)

36. In a manner precisely similar to that employed in Art. 16 it can be proved that, for any value whatever of θ the following relations are true:

$$\sin^2\theta + \cos^2\theta = 1; \qquad (1)$$

$$\sec^2\theta = 1 + \tan^2\theta; \qquad (2)$$

$$\csc^2\theta = 1 + \cot^2\theta. \qquad (3)$$

Also, from the definitions of the functions, the following relations are immediately derived:

$$\sin\theta = \frac{1}{\csc\theta}, \quad \therefore \sin\theta\csc\theta = 1, \qquad (4)$$

$$\cos\theta = \frac{1}{\sec\theta}, \quad \therefore \cos\theta\sec\theta = 1, \qquad (5)$$

$$\tan\theta = \frac{1}{\cot\theta}, \quad \therefore \tan\theta\cot\theta = 1. \qquad (6)$$

Also, since,

$$\cos\theta = \frac{x}{r}, \quad \therefore x = r\cos\theta, \qquad (7)$$

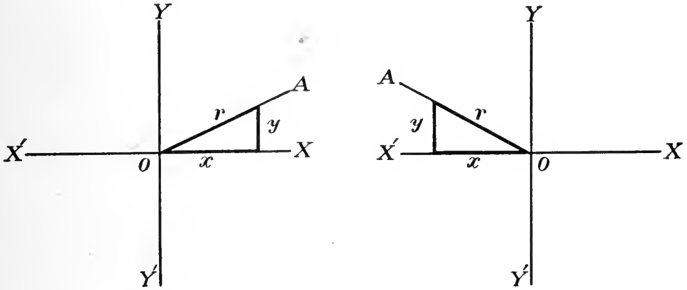
$$\sin\theta = \frac{y}{r}, \quad \therefore y = r\sin\theta, \qquad (8)$$

$$\tan\theta = \frac{y}{x}, \quad \therefore y = x\tan\theta. \qquad (9)$$

37. Signs of the trigonometric functions. In dealing with the functions of an acute angle of a right triangle (Art. 14, p. 20), no attention was paid to the question of positive or negative signs. All lines employed in that connection were considered positive; hence the value of each of the functions was considered positive. But in dealing with the general angle we have to consider both positive and negative lines, and as a result the signs of the functions undergo certain changes as the revolving line passes from quadrant to quadrant.

First Quadrant. Assume that the revolving line is always positive, and let it occupy any position in the first quadrant.

In this position both x and y are positive; hence, since r is also positive, both numerator and denominator are positive in the case of each of the functions. Therefore all the trigonometric functions are positive for the angle in the first quadrant.



Second Quadrant. Let the revolving line occupy any position in the second quadrant. In this case x is negative and y is positive; and we have the following results:

The sine is a fraction whose numerator and denominator are both positive; therefore the sine of an angle in the second quadrant is positive.

The cosine is a fraction whose numerator is negative and whose denominator is positive; therefore the cosine of an angle in the second quadrant is negative.

The tangent is a fraction whose numerator is positive and whose denominator is negative; therefore the tangent of an angle in the second quadrant is negative.

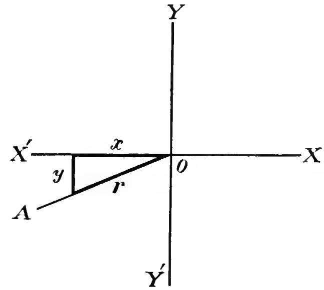
The cotangent is a fraction whose numerator is negative and whose denominator is positive; therefore the cotangent of an angle in the second quadrant is negative.

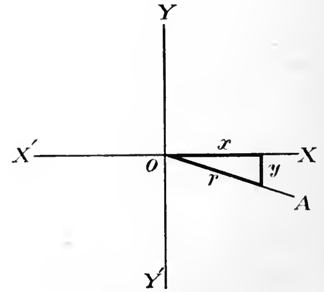
The secant is a fraction whose numerator is positive and whose denominator is negative; therefore the secant of an angle in the second quadrant is negative.

The cosecant is a fraction whose numerator and denominator are both positive; therefore the cosecant of an angle in the second quadrant is positive.

Third Quadrant. Let the revolving line occupy any position in the third quadrant. In this case both x and y are negative; therefore the following results can at once be obtained:

The sine is negative.
The cosine is negative.
The tangent is positive.
The cotangent is positive.
The secant is negative.
The cosecant is negative.





Fourth Quadrant. Let the revolving line occupy any position in the fourth quadrant. In this case x is positive and y is negative; therefore the following results can at once be obtained:

The sine is negative.
The cosine is positive.
The tangent is negative.
The cotangent is negative.
The secant is positive.
The cosecant is negative.

The above results are conveniently grouped together by means of the following table:

		Y		
sine	+		sine	+
cosine	−		cosine	+
tangent	−		tangent	+
cotangent	−		cotangent	+
secant	−		secant	+
cosecant	+		cosecant	+
X′				X
sine	−		sine	−
cosine	−		cosine	+
tangent	+		tangent	−
cotangent	+		cotangent	−
secant	−		secant	+
cosecant	−		cosecant	−
		Y′		

From the definitions of the versed sine and of the coversed sine it follows that these two functions are always positive.

38. Changes in sign and magnitude of the trigonometric functions as the angle increases from 0° to 360°.

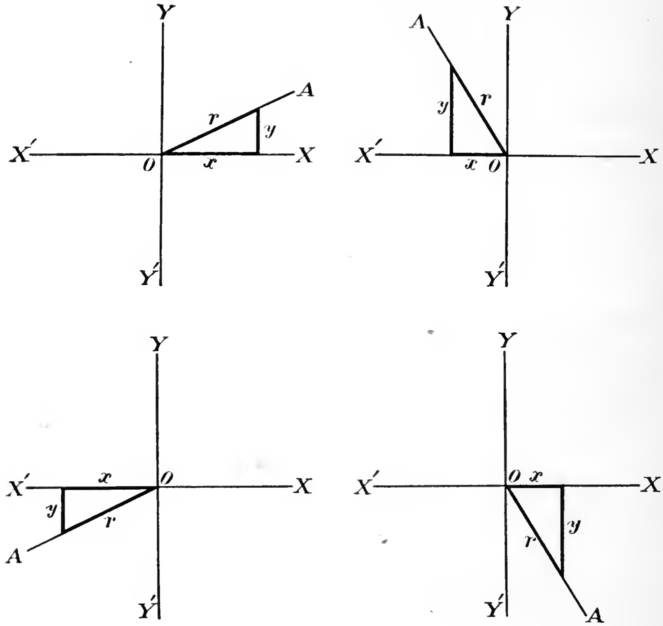
As before, we assume for the revolving line a constant length, r. As the revolving line starts from its initial position we have $x = r$, and $y = 0$. As the angle θ, which is generated by the revolution of this line, increases from 0° to 90°, y increases and x decreases; and when OA coincides with OY, we have $x = 0$, and $y = r$. Hence, as the angle increases from 0° to 90°, x decreases from r to 0, and y increases from 0 to r.

As the angle increases from 90° to 180°, x decreases — increases numerically — from 0 to $-r$ and y decreases from r to 0.

As the angle increases from 180° to 270°, x increases — decreases numerically — from $-r$ to 0 and y decreases — increases numerically — from 0 to $-r$.

As the angle increases from 270° to 360°, x increases from 0 to r and y increases — decreases numerically — from $-r$ to 0.

Inasmuch as all changes in sign and magnitude among the trigonometric functions are directly dependent on the changes just noted, the following results are now obtained without difficulty.



39. Sine. As the angle increases from 0° to 90° the numerator of the fraction that expresses the value of the sine increases from 0 to r, and the denominator r remains constant. Hence the sine increases from 0 to 1. As the angle increases still further, the numerator begins to decrease, the denominator still remaining constant, and at 180° the numerator becomes 0. Hence as the angle increases from 90° to 180° the sine decreases from 1 to 0. As the revolving line enters the third quadrant, y becomes negative and continues to decrease algebraically, becoming $-r$ when the angle equals 270°. Hence in the third quadrant the sine is negative, and as the angle increases from 180° to 270° the sine decreases from 0 to -1. In the fourth quadrant y continues negative; but as the angle increases y increases algebraically, and when the revolving line reaches its original position, y again becomes 0. Hence as the angle increases from 270° to 360° the sine is negative, and increases from -1 to 0.

Collecting the above results for the sake of convenience we have the following statement :

In the first quadrant the sine increases from 0 to 1; in the second it decreases from 1 to 0; in the third it decreases from 0 to -1; in the fourth it increases from -1 to 0.

40. Cosine. In a manner similar to that employed in the case of the sine, the following results are obtained:

As the angle increases from 0° to 90° the cosine decreases from $\frac{r}{r}$ to $\frac{0}{r}$, *i.e.* from 1 to 0. As the angle increases from 90° to 180° the cosine decreases — increases numerically — from $\frac{0}{r}$ to $\frac{-r}{r}$, *i.e.* from 0 to -1. As the angle increases from 180° to 270° the cosine increases — decreases numerically — from $\frac{-r}{r}$ to $\frac{0}{r}$, *i.e.* from -1 to 0. As the angle increases from 270° to 360° the cosine increases from $\frac{0}{r}$ to $\frac{r}{r}$, *i.e.* from 0 to 1.

41. Tangent. The value of the tangent is the value of the fraction $\frac{y}{x}$. When the angle is very small, the numerator of this fraction is very small, and the denominator is very nearly equal to r. Hence the tangent of the angle is very small; or, as it is commonly expressed, when the angle equals 0°, the tangent of the angle is also equal to 0.

As the angle increases the numerator y increases and the denominator x decreases. Hence the tangent of the angle increases. When the angle is nearly 90°, the numerator is very nearly equal to r; and as the angle approaches 90° the value of the numerator continually increases, approaching r as its limit. At the same time the value of the denominator continually decreases, approaching 0 as its limit. Hence, as θ approaches 90° the value of $\tan\theta$ can be made to exceed any finite number previously assigned, no matter how great that number may be. This is usually expressed by saying that when the angle is equal to 90°, the tangent of the angle is equal to infinity. Hence,

In the first quadrant the tangent increases from $\frac{0}{r}$ to $\frac{r}{0}$, *i.e.* from 0 to ∞.

In the second quadrant the denominator x becomes negative while the numerator y remains positive. Hence the tangent

12

of an angle in the second quadrant is negative. When the angle is but little greater than 90°, the numerator is very nearly equal to r and the denominator is very small, and negative. Therefore, as the revolving line enters the second quadrant, the numerical value of the tangent can be taken to be greater than any negative finite limit previously assigned. That is, when the angle is in the second quadrant and differs from 90° by an amount that is less than any finite number assigned in advance, no matter how small that number may be, the tangent of the angle is negative and is numerically greater than any finite limit assigned in advance. To express this we shall say that $\tan 90° = -\infty$. It is thus seen that $\tan 90°$ will be called equal to either $+\infty$ or $-\infty$ according as the angle is approaching the limit 90° from the positive direction, or as the revolving line is leaving the position at which the angle equals 90° and is just entering the second quadrant. As the angle increases, the numerator decreases and the denominator, which is negative, increases numerically. Hence, the tangent decreases numerically — increases algebraically — and when the angle becomes equal to 180°, the tangent becomes equal to 0. Hence,

In the second quadrant the tangent increases from $\frac{r}{0}$ to $\frac{0}{-r}$, *i.e.* from $-\infty$ to 0.

In the third quadrant both numerator and denominator are negative. Hence the tangent is positive. The numerator increases numerically from 0 to $-r$, and the denominator decreases numerically from $-r$ to 0. Hence,

In the third quadrant the tangent increases from $\frac{0}{-r}$ to $\frac{-r}{0}$, *i.e.* from 0 to ∞.

In the fourth quadrant the numerator is negative and the denominator is positive. Hence the tangent is negative. The numerator decreases numerically from $-r$ to 0, and the denominator increases from 0 to r. Hence,

In the fourth quadrant the tangent increases from $\frac{-r}{0}$ to $\frac{0}{r}$, *i.e.* from $-\infty$ to 0.

The same restriction is to be observed with respect to the value of $\tan 270°$ as was noted in connection with $\tan 90°$. That is, if the angle is in the third quadrant and is approaching 270° as its limit, the tangent of the angle can be made to exceed in magnitude any finite positive limit previously assigned. If it is in the fourth quadrant, the tangent is negative and can be made to exceed in numerical magnitude any finite limit previously assigned. For this reason it is customary to say that $\tan 270° = \pm\infty$.

42. Cotangent. The value of the cotangent is the value of the fraction $\frac{x}{y}$. When the angle is very small, the numerator is nearly equal to r and the denominator is nearly equal to 0. Hence the value of the cotangent of 0° is infinity. Then, letting the angle increase, and reasoning in the same manner as in the case of the tangent, we obtain the following results:

In the first quadrant the cotangent is positive and decreases from $\frac{r}{0}$ to $\frac{0}{r}$, *i.e.* from ∞ to 0.

In the second quadrant the cotangent is negative and decreases from $\frac{0}{r}$ to $\frac{-r}{0}$, *i.e.* from 0 to $-\infty$.

In the third quadrant the cotangent is positive and decreases from $\frac{-r}{0}$ to $\frac{0}{-r}$, *i.e.* from ∞ to 0.

In the fourth quadrant the cotangent is negative and decreases from $\frac{0}{-r}$ to $\frac{r}{0}$, *i.e.* from 0 to $-\infty$.

Remarks similar to those made in connection with tan 90° and tan 270° apply to cot 0°, cot 180°, and cot 360°.

43. Secant. The value of the secant is the value of the fraction $\frac{r}{x}$. The numerator remains constant for all positions of the revolving line, while the denominator varies. When the angle is very small, the numerator and the denominator are approximately equal. Hence the secant of 0° is equal to unity. As the angle increases the denominator x decreases, thus causing the value of the secant to increase. When the angle is nearly equal to 90°, the denominator is nearly equal to 0, and approaches 0 as its limit. Therefore the secant can be

11
14

made to exceed any finite limit previously assigned. We shall express this by saying that sec $90° = \infty$. Hence,

As the angle increases from 0° to 90° the secant increases from $\frac{r}{r}$ to $\frac{r}{0}$, *i.e.* from $+1$ to $+\infty$.

When the revolving line enters the second quadrant the denominator x becomes negative and begins to increase numerically — decrease algebraically — becoming equal to $-r$ when the angle becomes 180°. Hence, beginning with a negative value numerically greater than any finite limit assigned in advance, the secant increases — decreases numerically — until it reaches the value -1. Hence,

As the angle increases from 90° to 180° the secant increases from $\frac{r}{0}$ to $\frac{r}{-r}$, *i.e.* from $-\infty$ to -1.

In the third quadrant the denominator continues negative, but begins to decrease — increase numerically — as soon as the revolving line enters the quadrant. At 270° the denominator becomes 0. Hence,

As the angle increases from 180° to 270° the secant decreases from $\frac{r}{-r}$ to $\frac{r}{0}$, *i.e.* from -1 to $-\infty$.

In the fourth quadrant the denominator again becomes positive, and increases from 0 to r as the angle increases from 270° to 360°, returning to its original value when the revolving line completes one entire revolution. Hence,

As the angle increases from 270° to 360° the secant decreases from $\frac{r}{0}$ to $\frac{r}{r}$, *i.e.* from ∞ to 1.

The same restriction is to be observed with respect to the value of sec 270° as was noted in connection with sec 90°. That is, if the angle is in the third quadrant and is approaching 270° as its limit, the secant of the angle can be made to exceed in numerical magnitude any finite negative limit assigned in advance. If the angle is in the fourth quadrant, the secant is positive, and can be made to exceed in magnitude any finite positive limit assigned in advance. We shall express this by saying that sec $270° = \pm\infty$.

44. Cosecant. The value of the cosecant is the value of the fraction $\frac{r}{y}$. Remembering that the numerator remains constant, and tracing out the changes in sign and magnitude of the denominator, as in the case of the secant, we obtain the following results:

As the angle increases from 0° to 90° the cosecant decreases from $\frac{r}{0}$ to $\frac{r}{r}$, *i.e.* from ∞ to 1.

As the angle increases from 90° to 180° the cosecant increases from $\frac{r}{r}$ to $\frac{r}{0}$, *i.e.* from 1 to ∞.

As the angle increases from 180° to 270° the cosecant increases from $\frac{r}{0}$ to $\frac{r}{-r}$, *i.e.* from $-\infty$ to -1.

As the angle increases from 270° to 360° the cosecant decreases from $\frac{r}{-r}$ to $\frac{r}{0}$, *i.e.* from -1 to $-\infty$.

Remarks similar to those made in connection with sec 90° and sec 270° apply to csc 0°, csc 180°, and csc 360°.

The changes that take place in the sign and magnitude of the different trigonometric functions are conveniently grouped together in the following table:

Y

Second Quadrant		First Quadrant	
sine	decreases from 1 to 0	sine	increases from 0 to 1
cosine	decreases from 0 to -1	cosine	decreases from 1 to 0
tangent	increases from $-\infty$ to 0	tangent	increases from 0 to ∞
cotangent	decreases from 0 to $-\infty$	cotangent	decreases from ∞ to 0
secant	increases from $-\infty$ to -1	secant	increases from 1 to ∞
cosecant	increases from 1 to ∞	cosecant	decreases from ∞ to 1

X' X

Third Quadrant		Fourth Quadrant	
sine	decreases from 0 to -1	sine	increases from -1 to 0
cosine	increases from -1 to 0	cosine	increases from 0 to 1
tangent	increases from 0 to ∞	tangent	increases from $-\infty$ to 0
cotangent	decreases from ∞ to 0	cotangent	decreases from 0 to $-\infty$
secant	decreases from -1 to $-\infty$	secant	decreases from ∞ to 1
cosecant	increases from $-\infty$ to -1	cosecant	decreases from -1 to $-\infty$

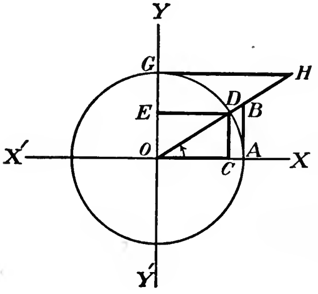
Y'

45. After the changes in sign and magnitude have been obtained for the first three functions, the corresponding changes for the last three can be found by remembering that the cotangent, secant, and cosecant are the reciprocals of the tangent, the cosine, and the sine respectively. The student should verify the above results by obtaining them in this manner also.

In connection with the general definitions of the trigonometric functions given on p. 53, it was noted that these definitions failed in the case of certain functions for certain values of the angle. These cases have been explained in some detail in Arts. 41–44, and we now have definitions of the tangent, the cotangent, the secant, and the cosecant of any angle from 0° to 360° inclusive; and hence, by the usual considerations, Arts. 50–57, definitions of these functions for any angle whatever.

In order that the relations between tan 90° and cot 90°, tan 270° and cot 270°, sec 90° and cos 90°, etc., may be the same as that between the same functions in the case of other angles we shall say that $\frac{1}{\infty} = 0$, and $\frac{1}{0} = \infty$. But the student is cautioned that "∞" is not a number in the usual sense of the word, and that these two equations are not to be taken literally. They are used merely for the sake of expressing concisely the result of a definite limiting process, a process much more complicated than that of ordinary division.

46. Geometrical representation of the trigonometric functions. The trigonometric functions are pure numbers, the value in each case being a ratio between two given magnitudes. These magnitudes are represented by lines, and if the length of the revolving line is properly chosen, it is possible to represent the values of the functions themselves by lines.



Let the revolving line be the radius of a circle, and let its value be assumed to be unity.

The sine of the angle AOB is $\frac{CD}{OD}$. But since $OD=1$, we may say

$$\sin\theta = \frac{CD}{OD} = \frac{CD}{1} = CD.$$

Similarly,

$$\cos\theta = \frac{OC}{OD} = \frac{OC}{1} = OC,$$

$$\tan\theta = \frac{CD}{OC} = \frac{AB}{OA} = \frac{AB}{1} = AB,$$

$$\cot\theta = \frac{OC}{CD} = \frac{GH}{OG} = \frac{GH}{1} = GH,$$

$$\sec\theta = \frac{OD}{OC} = \frac{OB}{OA} = \frac{OB}{1} = OB,$$

$$\csc\theta = \frac{OD}{CD} = \frac{OH}{OG} = \frac{OH}{1} = OH,$$

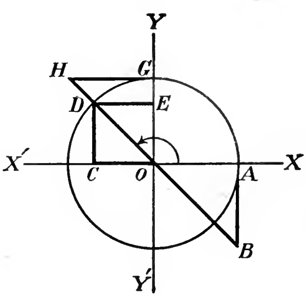
$$\text{vers}\,\theta = 1 - \cos\theta = OA - OC = AC,$$

$$\text{covers}\,\theta = 1 - \sin\theta = OG - OE = GE.$$

For an angle of the second quadrant the so-called "line values" of the trigonometric functions are obtained as follows:

$$\sin\theta = \frac{CD}{OD} = \frac{CD}{1} = CD,$$

$$\cos\theta = \frac{OC}{OD} = \frac{OC}{1} = OC,$$



$$\tan\theta = \frac{CD}{OC} = \frac{AB}{OA} = \frac{AB}{1} = AB,$$

$$\cot\theta = \frac{OC}{CD} = \frac{GH}{OG} = \frac{GH}{1} = GH,$$

$$\sec\theta = \frac{OD}{OC} = \frac{OB}{OA} = \frac{OB}{1} = OB,$$

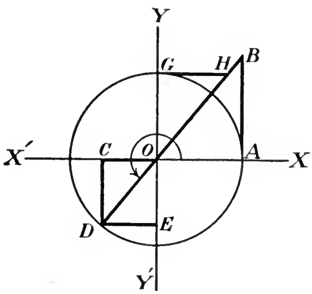
$$\csc\theta = \frac{OD}{CD} = \frac{OH}{OG} = \frac{OH}{1} = OH,$$

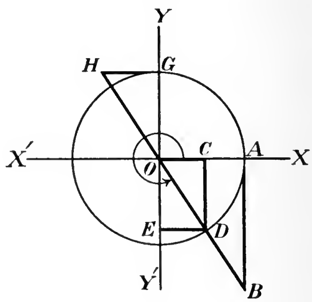
$$\text{vers}\,\theta = 1 - \cos\theta = OA - OC = OA + CO = CA,$$

$$\text{covers}\,\theta = 1 - \sin\theta = OG - OE = EG.$$

The change in sign when OC is replaced by CO in obtaining the value of the versed sine should be noted carefully.

For angles of the third and fourth quadrants the line values are obtained in a manner similar to that employed in connection with angles of the first two quadrants. The figures are lettered so that the following values hold for both:





$$\sin\theta = \frac{CD}{OD} = \frac{CD}{1} = CD,$$

$$\cos\theta = \frac{OC}{OD} = \frac{OC}{1} = OC,$$

$$\tan\theta = \frac{CD}{OC} = \frac{AB}{OA} = \frac{AB}{1} = AB,$$

$$\cot\theta = \frac{OC}{CD} = \frac{GH}{OG} = \frac{GH}{1} = GH,$$

$$\sec\theta = \frac{OD}{OC} = \frac{OB}{OA} = \frac{OB}{1} = OB,$$

$$\csc\theta = \frac{OD}{CD} = \frac{OH}{OG} = \frac{OH}{1} = OH,$$

$$\text{vers}\,\theta = 1 - \cos\theta = OA - OC = CA,$$

$$\text{covers}\,\theta = 1 - \sin\theta = OG - OE = EG.$$

The signs of the trigonometric functions when used as lines are, of course, the same as when they are used as ratios. It will be noticed that when the line that represents the sine extends upward from the axis of abscissas, or horizontal diameter, the sine is positive; when it extends downward, the sine is negative. The cosine is positive when the line that represents its value extends toward the right from the origin, negative when it extends toward the left. The tangent is positive when its line extends upward from the axis of abscissas, or horizontal diameter, negative when it extends downward. The cotangent is positive when its line extends toward the right from the axis of ordinates, or vertical diameter, negative when it extends toward the left. The secant and the cosecant are positive when their respective lines extend in the same direction from the origin as the revolving line, negative when they extend in an opposite direction. The versed sine is considered as extending toward the right from the foot of the sine, and the coversed sine upward from the foot of the perpendicular dropped from the extremity of the revolving line to the vertical diameter. Both are always positive.

The trigonometric functions were originally used as lines; and the numerical value was, in each case, the length of the line in terms of the revolving line, or the radius of the circle, taken as a unit. There are certain advantages connected with the use of these line values, but for general purposes the ratios are so much more convenient than the line values that they have now come into almost universal use.

47. Limiting values of the trigonometric functions. In discussing the variation in the values of the different functions the following limits were found. In the case of the sine the positive limit was 1, and the negative limit was -1. For the cosine also these limits were $+1$ and -1 respectively. For the tangent and the cotangent the limits were $+\infty$ and $-\infty$. For the secant and the cosecant it was found that the positive values that these functions could take were comprehended between $+1$ and $+\infty$, and the negative values between -1 and $-\infty$. Hence, we can make the following definite statement respecting the limits between which the different functions can vary:

The sine can take any value between $+1$ and -1.
The cosine can take any value between $+1$ and -1.
The tangent can take any value between $+\infty$ and $-\infty$.
The cotangent can take any value between $+\infty$ and $-\infty$.

The secant can take any value between $+1$ and $+\infty$, and between -1 and $-\infty$.

The cosecant can take any value between $+1$ and $+\infty$, and between -1 and $-\infty$.

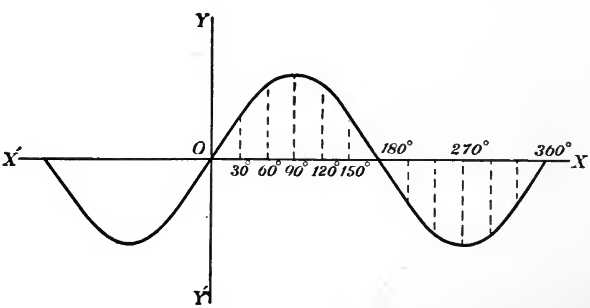
From the definitions of the versed sine and the coversed sine it follows that each of these functions can take any value between 0 and $+2$.

48. Graphs of the trigonometric functions. The graphs of the trigonometric functions can be plotted in the ordinary manner if the values of the angles are taken as ordinates and the corresponding values of the functions as abscissas.

Sine. For the sine we form the following table of values from the equation

$$y = \sin x.$$

In this table the values of the sine are, for convenience in plotting, given decimally, instead of in the ordinary common fractions.



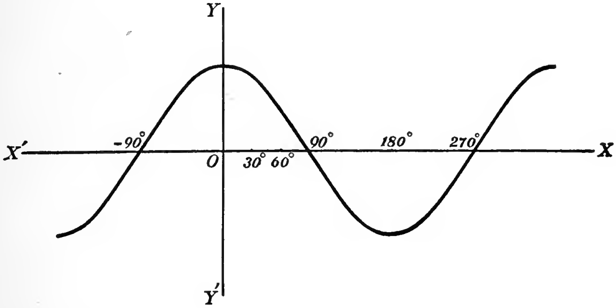
x	y
0°	0
30°	.5
45°	.71
60°	.87
90°	1
120°	.87
135°	.71
150°	.5
180°	0
225°	− .71
270°	− 1
315°	− .71
360°	0
415°	.71
450°	1
495°	.71
etc.	etc.

Continuing this table, and plotting the points thus determined, we find that the graph is a curve consisting of an infinite number of waves like those in the figure. By using negative values of the angle we obtain similar waves at the left of the origin. The curve is called the sine curve, or sinusoid.

Cosine. The graph of the equation

$$y = \cos x$$

is found in a similar manner. Forming a table of values, and plotting the points determined by these values, we find that the cosine curve has the following form.

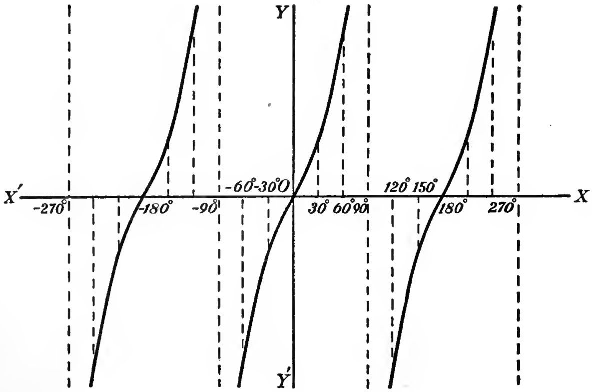


Tangent. The table of values for x and y formed from the equation

$$y = \tan x$$

is as follows.

x	y
0°	0
30°	.58
45°	1
60°	1.73
90°	∞
120°	– 1.73
135°	– 1
150°	– .58
180°	0
210°	.58
225°	1
240°	1.73
270°	∞
300°	– 1.73
315°	– 1
330°	–.58
360°	0
390°	.58
etc.	etc.

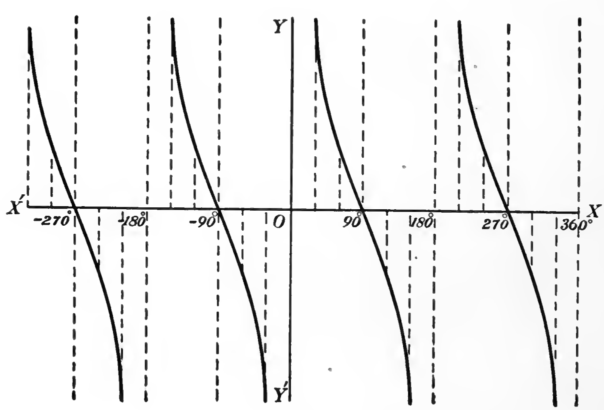


Continuing the table, and plotting the points determined by the values thus found, we obtain the tangent curve, which consists of an infinite number of branches, each like one of those in the figure. Negative values of the angle give an infinite number of like branches at the left of the origin.

Cotangent. The graph of the equation

$$y = \cot x$$

is similar to that of $y = \tan x$, except that the points where the different branches cross the x-axis are 90° to the right of those where the tangent curve branches cross, and the curvature is toward the right instead of toward the left. The form of the graph is shown in the following figure.



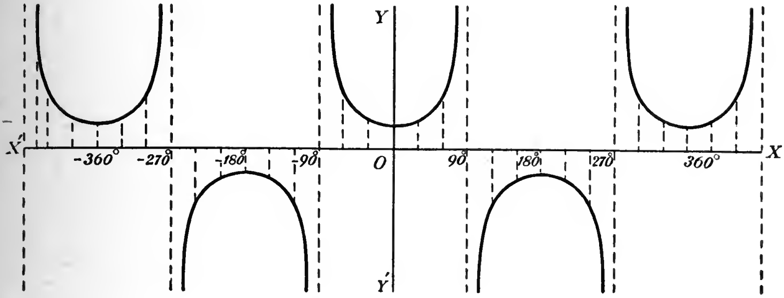
x	y
0°	∞
30°	1.73
45°	1
60°	.58
90°	0
120°	– .58
135°	– 1
150°	– 1.73
180°	– ∞
210°	1.73
225°	1
240°	.58
270°	0
300°	– .58
315°	– 1
330°	– 1.73
360°	∞
390°	1.73
etc.	etc.

Secant. The table of values for the equation

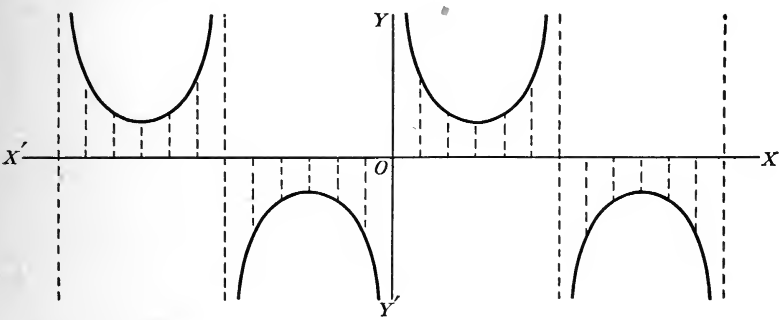
$$y = \sec x$$

can readily be found if it is remembered that $\sec x$ is the reciprocal of $\cos x$. The graph has the form shown in the first figure on p. 71.

Cosecant. The graph of the cosecant is similar in form to that of the secant, but the relative position of the various



branches with respect to the y-axis is different. The graph is shown in the following figure.



49. Periods of the trigonometric functions. In considering the changes in value through which the functions pass as the angle increases, it is seen that the sine, for example, takes all its possible values, in both increasing and decreasing order of change, while the angle is increasing from 0° to 360°. As the angle increases from 360° to 720° the values of the sine which were obtained in the first 360° are repeated, this repetition of values occurring in the original order. The same cycle of values will again occur in the next 360°, and so on, for each complete revolution of the generating, or revolving line. The angle formed by the generating line while this regular recurrence of values takes place is called the **period** of the sine; and in accordance with this result we may say that

The period of the sine is 360°, or 2π.

A similar course of reasoning shows us that 360° is also the period of the cosine, of the secant, and of the cosecant.

The values of the tangent repeat themselves completely with each increase of 180° in the angle. Hence,

The period of the tangent is 180°, or π.

The period of the cotangent is the same as the period of the tangent.

EXERCISE X

1. Trace the changes in sign and magnitude of $\sin\theta$ as θ varies from $-\frac{\pi}{2}$ to $-\frac{3\pi}{2}$; from $-270°$ to $-450°$.

2. Trace the changes in sign and magnitude of $\cos A$ as A varies from $-\pi$ to -2π.

3. Trace the changes in sign and magnitude of $\tan A$ as A varies from $-180°$ to $-540°$.

4. Trace the changes in sign and magnitude of $\sec A$ as A varies from $-90°$ to $-270°$.

Find the value of each of the following:

5. $\sin\theta + \cos\theta$ when $\theta = 60°$.

6. $\sin^2\theta + 2\cos\theta$ when $\theta = 45°$.

7. $\sin A + \tan A$ when $A = 135°$.

8. $\sin 60° + \tan 240°$.

9. $\cos 0° \cos 30° + \tan 135° \cot 315°$.

10. $\cos 0° \tan 60° - \sec^2 30° \cot 225°$.

11. $2\sin 90° \sec^2 30° + \cos 180° \tan 315°$.

12. $2\sec^2\pi \cos 0° + 3\sin^3\frac{3\pi}{2} - \csc\frac{\pi}{2}$.

13. $\cos\pi \tan\frac{\pi}{4} - \sec^2\frac{11\pi}{6}\tan^2\frac{3\pi}{4}$.

14. For which of the following values of θ is $\sin\theta - \cos\theta$ positive and for which is it negative?

$\theta = 0°$; $\theta = 60°$; $\theta = 120°$; $\theta = 210°$; $\theta = 240°$; $\theta = 300°$; $\theta = 330°$.

all

14 11 12 13

15. For which of the following values of θ is $\sin\theta + \cos\theta$ positive and for which is it negative?

$\theta = 135°$; $\theta = 210°$; $\theta = 300°$; $\theta = 315°$; $\theta = 330°$.

16. Prove that $\sec^6\theta - \tan^6\theta = 3\sec^2\theta\tan^2\theta + 1$.

17. If $\cos\theta = \frac{a^2 - b^2}{a^2 + b^2}$, find $\sin\theta$ and $\tan\theta$.

18. If $\tan\theta = \frac{2a^2 + 2a}{2a + 1}$, find $\cos\theta$ and $\sin\theta$.

19. Prove the equation $\sin\theta = x + \frac{1}{x}$ impossible for all real values of x.

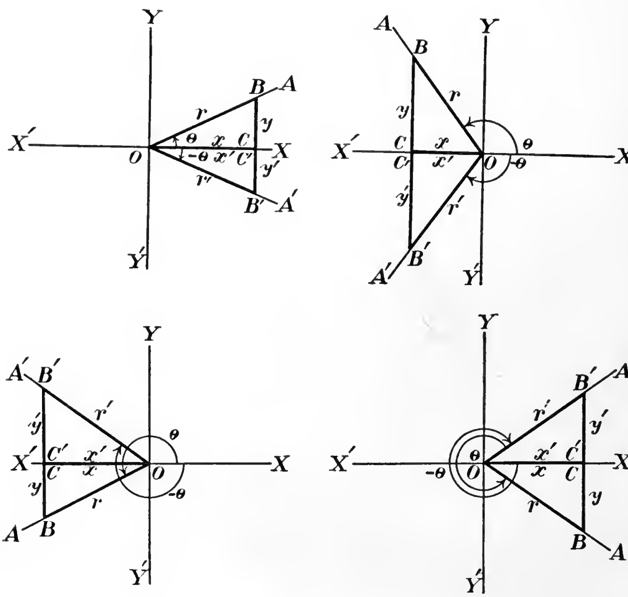
20. Prove the equation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ impossible unless $x = y$.

CHAPTER VI

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

50. Functions of an angle $-\theta$ in terms of functions of θ. Let the revolving line OA generate an angle θ, of any magnitude. The final position of OA is, then, in any one of the four quadrants, as shown in the figures. Also, let the line OA' generate an angle $-\theta$, equal in magnitude to the positive angle θ, generated by OA.

Take $OB = OB'$, and from B and B' draw perpendiculars BC, $B'C'$, to XX'. Then are the triangles OBC, $OB'C'$, equal geometrically, since they are right triangles having the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other. Hence the points C, C', coincide, $BC = B'C'$, and $OC = O'C'$.



For convenience, let $OB = r$, $OB' = r'$, $BC = y$, $B'C' = y'$, $OC = x$, $OC' = x'$; then for each of the four figures we have

$$\sin(-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\sin\theta,$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = \frac{-y}{x} = -\tan\theta,$$

$$\cot(-\theta) = \frac{x'}{y'} = \frac{x}{-y} = -\cot\theta,$$

$$\sec(-\theta) = \frac{r'}{x'} = \frac{r}{x} = \sec\theta,$$

$$\csc(-\theta) = \frac{r'}{y'} = \frac{r}{-y} = -\csc\theta.$$

EXAMPLES.

1. $\sin(-30°) = -\sin 30° = -\frac{1}{2}$,

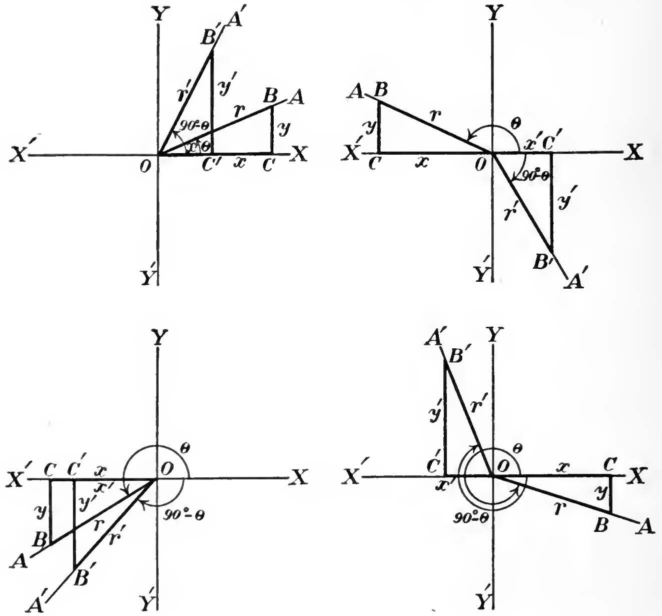
2. $\cos(-45°) = \cos 45° = \frac{\sqrt{2}}{2}$,

3. $\tan(-60°) = -\tan 60° = -\sqrt{3}$.

51. Functions of an angle $90° - \theta$ in terms of functions of θ. Let the revolving line OA (p. 76) generate an angle θ, of any magnitude, and at the same time let OA' generate an angle whose magnitude is $90° - \theta$.

As before, take $OB = OB'$, and from B, B', draw perpendiculars BC, $B'C'$, to XX'. The triangles OBC, $OB'C'$, are, in each of the four figures, equal geometrically. The proof should be supplied by the student.

With the same notation as in the previous figures we have, considering only the actual lengths of the lines, and paying no attention to positive and negative signs, $r = r'$, $y = x'$, $x = y'$.



The following equations then hold true for all possible cases:

$$\sin (90^\circ - \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos \theta,$$

$$\cos (90^\circ - \theta) = \frac{x'}{r'} = \frac{y}{r} = \sin \theta,$$

$$\tan (90^\circ - \theta) = \frac{y'}{x'} = \frac{x}{y} = \cot \theta,$$

$$\cot (90^\circ - \theta) = \frac{x'}{y'} = \frac{y}{x} = \tan \theta,$$

$$\sec (90^\circ - \theta) = \frac{r'}{x'} = \frac{r}{y} = \csc \theta,$$

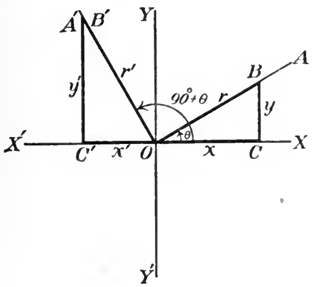
$$\csc (90^\circ - \theta) = \frac{r'}{y'} = \frac{r}{x} = \sec \theta.$$

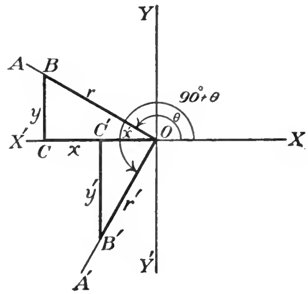
NOTE. For the special case that occurs when θ is an acute angle, these relations were established independently in connection with the definitions of the functions of an acute angle of a right triangle (Art. 17, p. 23).

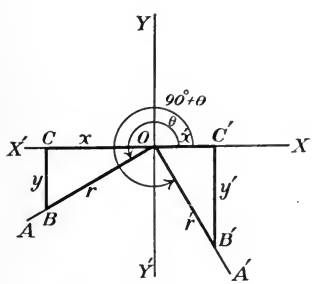
52. Functions of an angle $90° + \theta$ in terms of functions of θ. Let the revolving line OA generate an angle θ, of any magnitude, and at the same time let OA' generate an angle whose magnitude is $90° + \theta$.

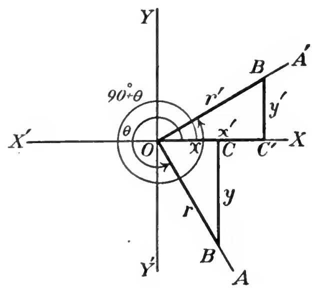
As in each of the previous cases, take $OB = OB'$, and from B, B', draw perpendiculars BC, $B'C'$, to XX'. The triangles OBC, $OB'C'$ are, in each of the four figures, equal geometrically. The proof should be supplied by the student.

With the notation used in the previous cases we have, considering only the actual lengths of the lines, and paying no attention to positive and negative signs, $r = r'$, $x = y'$, $y = x'$. If positive and negative signs are taken into account, these equations become $r = r'$, $x = y'$, $y = -x'$.









The following equations then hold true for all possible cases:

$$\sin(90° + \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\cos(90° + \theta) = \frac{x'}{r'} = \frac{-y}{r} = -\sin\theta,$$

$$\tan(90° + \theta) = \frac{y'}{x'} = \frac{x}{-y} = -\cot\theta,$$

$$\cot(90° + \theta) = \frac{x'}{y'} = \frac{-y}{x} = -\tan\theta,$$

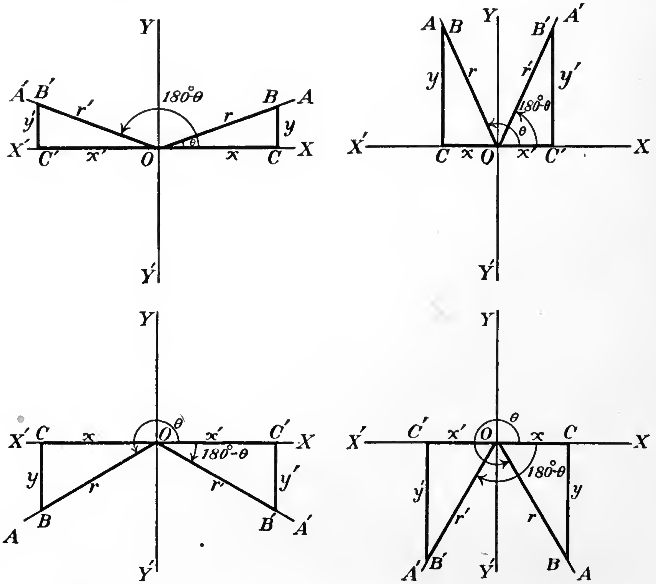
$$\sec(90° + \theta) = \frac{r'}{x'} = \frac{r}{-y} = -\csc\theta,$$

$$\csc(90° + \theta) = \frac{r'}{y'} = \frac{r}{x} = \sec\theta.$$

EXAMPLES.

1. $\sin(90° + 30°) = \cos 30° = \frac{1}{2}\sqrt{3},$
2. $\cos(90° + 45°) = -\sin 45° = -\frac{1}{2}\sqrt{2},$
3. $\tan(90° + 60°) = -\cot 60° = -\frac{1}{3}\sqrt{3},$
4. $\cot(90° + 120°) = -\tan 120° = -(-\sqrt{3}) = \sqrt{3},$
5. $\sec(90° + 135°) = -\csc 135° = -\sqrt{2},$
6. $\csc(90° + 150°) = \sec 150° = -\frac{2}{3}\sqrt{3}.$

53. Functions of an angle 180° − θ in terms of functions of θ. Let the revolving line OA generate an angle θ, of any magnitude, and at the same time let OA' generate an angle whose magnitude is $180° - \theta$.



As in the previous cases, take $OB = OB'$, and from B, B', draw perpendiculars BC, $B'C'$, to XX'. The triangles OBC, $OB'C'$, are, in each of the four figures, equal geometrically. The student should supply the proof.

With the notation used in the previous cases we have, considering only the actual lengths of the lines, and paying no attention to positive and negative signs, $r = r'$, $x = x'$, $y = y'$. If positive and negative signs are taken into account, the second equation becomes $x = -x'$.

The following equations then hold true for all possible cases:

$$\sin(180° - \theta) = \frac{y'}{r'} = \frac{y}{r} = \sin\theta,$$

$$\cos(180° - \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\cos\theta,$$

$$\tan(180° - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\tan\theta,$$

$$\cot(180° - \theta) = \frac{x'}{y'} = \frac{-x}{y} = -\cot\theta,$$

$$\sec(180° - \theta) = \frac{r'}{x'} = \frac{r}{-x} = -\sec\theta,$$

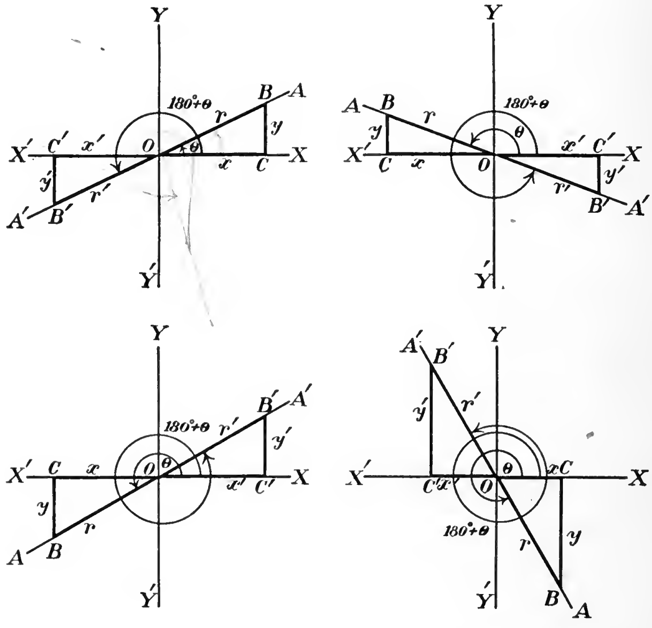
$$\csc(180° - \theta) = \frac{r'}{y'} = \frac{r}{y} = \csc\theta.$$

EXAMPLES.

1. $\sin(180° - 30°) = \sin 30° = \frac{1}{2}.$
2. $\cos(180° - 60°) = -\cos 60° = -\frac{1}{2},$
3. $\tan(180° - 45°) = -\tan 45° = -1,$
4. $\cot(180° - 120°) = -\cot 120° = -\left(-\frac{\sqrt{3}}{3}\right) = \frac{1}{3}\sqrt{3},$
5. $\sec(180° - 135°) = -\sec 135° = -(-\sqrt{2}) = \sqrt{2},$
6. $\csc(180° - 150°) = \csc 150° = 2.$

54. Functions of an angle $180° + \theta$ in terms of functions of θ. Let the revolving line OA generate an angle θ, of any magnitude, and at the same time let OA' generate an angle whose magnitude is $180° + \theta$.

As in the cases already considered, take $OB = OB'$, and from B, B', draw perpendiculars BC, $B'C'$, to XX'. The triangles OBC, $OB'C'$, are, in each of the four figures, equal geometrically. The student should supply the proof.



With the notation used in the previous cases we have, considering only the actual lengths of the lines, and paying no attention to positive and negative signs, $r = r'$, $x = x'$, $y = y'$. If positive and negative signs are taken into account, the last two equations become $x = -x'$, $y = -y'$ respectively.

The following equations then hold true for all possible cases:

$$\sin(180° + \theta) = \frac{y'}{r'} = \frac{-y}{r} = -\sin\theta,$$

$$\cos(180° + \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\cos\theta,$$

$$\tan(180° + \theta) = \frac{y'}{x'} = \frac{-y}{-x} = \tan\theta,$$

$$\cot(180° + \theta) = \frac{x'}{y'} = \frac{-x}{-y} = \cot\theta,$$

$$\sec(180° + \theta) = \frac{r'}{x'} = \frac{r}{-y} = -\sec\theta,$$

$$\csc(180° + \theta) = \frac{r'}{y'} = \frac{r}{-x} = -\csc\theta.$$

EXAMPLES.

1. $\sin(180° + 30°) = -\sin 30° = -\frac{1}{2}$,
2. $\cos(180° + 45°) = -\cos 45° = -\frac{1}{2}\sqrt{2}$,
3. $\tan(180° + 60°) = \tan 60° = \sqrt{3}$,
4. $\cot(180° + 120°) = \cot 120° = -\frac{1}{3}\sqrt{3}$,
5. $\sec(180° + 135°) = -\sec 135° = -(-\sqrt{2}) = \sqrt{2}$,
6. $\csc(180° + 150°) = -\csc 150° = -2$.

55. In a manner precisely similar to that employed in the preceding sections, we can determine the functions of an angle $270° - \theta$ in terms of functions of θ. The figures for each quadrant should be constructed by the student, and the values obtained, as in the cases which have just been considered.

These relations, true for all values of θ, are as follows:

$$\sin(270° - \theta) = -\cos\theta,$$
$$\cos(270° - \theta) = -\sin\theta,$$
$$\tan(270° - \theta) = \cot\theta,$$
$$\cot(270° - \theta) = \tan\theta,$$
$$\sec(270° - \theta) = -\csc\theta,$$
$$\csc(270° - \theta) = -\sec\theta.$$

56. The corresponding values of functions of an angle $270° + \theta$ in terms of functions of θ can also be obtained in a manner similar to that employed in the cases already discussed (Art. 50–54). These values are as follows:

$$\sin(270° + \theta) = -\cos\theta,$$
$$\cos(270° + \theta) = \sin\theta,$$

12
13

$$\tan(270° + \theta) = -\cot\theta,$$
$$\cot(270° + \theta) = -\tan\theta,$$
$$\sec(270° + \theta) = \csc\theta,$$
$$\csc(270° + \theta) = -\sec\theta.$$

EXAMPLES.

$$\sin(270° - 210°) = -\cos 210° = -(-\tfrac{1}{2}\sqrt{3}) = \tfrac{1}{2}\sqrt{3},$$
$$\cos(270° - 150°) = -\sin 150° = -\tfrac{1}{2},$$
$$\tan(270° + 135°) = -\cot 135° = -(-1) = 1,$$
$$\cot(270° - 240° = \tan 240° = \sqrt{3},$$
$$\sec(270° + 30°) = \csc 30° = 2,$$
$$\csc(270° + 60°) = -\sec 60° = -2.$$

57. Functions of an angle 360° + θ in terms of functions of θ. When the revolving line has generated an angle $360° + \theta$, its position is the same as that occupied after it has generated the angle θ. Hence,

The functions of an angle $360° + \theta$ are the same as the corresponding functions of θ.

Also, since the revolving line returns to its initial position after any number of complete revolutions, in either a positive or negative direction, it follows that, when n is any positive or negative integer or zero,

Functions of an angle $n \times 360° + \theta$ are equal to the corresponding functions of θ.

In a similar manner it may be shown that the functions of $n \times 360° - \theta$ are equal to the corresponding functions of $-\theta$.

58. By means of the equations contained in Arts. 50–57, pp. 74–82, the functions of any angle can be found in terms of functions of an angle less than 90°.

For example,
$$\sin 2151° = \sin(5 \times 360° + 351°)$$
$$= \sin 351°$$
$$= \sin(270° + 81°)$$
$$= -\cos 81°.$$

Similarly,
$$\cos(-2058°) = \cos 2058°$$
$$= \cos(5 \times 360° + 258°)$$
$$= \cos 258°$$
$$= \cos(270° - 12°)$$
$$= -\sin 12°.$$

By reductions of this kind it is easy to find the values of functions of any large angle, either positive or negative. Multiples of 360° should first be subtracted, and the remainder of the reduction performed by the theorems of this chapter.

59. The following table contains the values of the functions of the angles between 0° and 360° which are of most frequent occurrence in elementary mathematics.

	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°
sine	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1
cosine	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{3}$	-1	0
tangent	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	$\pm\infty$	$-\sqrt{3}$	-1	$-\frac{1}{3}\sqrt{3}$	0	$\pm\infty$
cotangent	$\pm\infty$	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0	$-\frac{1}{3}\sqrt{3}$	-1	$-\sqrt{3}$	$\pm\infty$	0
secant	1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	$\pm\infty$	-2	$-\sqrt{2}$	$-\frac{2}{3}\sqrt{3}$	-1	$\pm\infty$
cosecant	$\pm\infty$	2	$\sqrt{2}$	$\frac{2}{3}\sqrt{3}$	1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	$\pm\infty$	-1

NOTE. In the above table the double sign, which is used wherever the value ∞ occurs, signifies that either the positive or the negative value is obtained according as the revolving line approaches the given position from the one or from the other side. For example, $\tan 90° = +\infty$ if the revolving line approaches the positive portion of the y-axis from the right, *i.e.* through positive rotation; $\tan 90° = -\infty$ if the revolving line approaches the same position from the left, *i.e.* through negative rotation.

EXERCISE XI

Prove that

1. $\sin 210° \tan 225° + \cos 300° \cot 315° = -1.$
2. $\cos 240° \cos 120° - \sin 120° \cos 150° = 1.$
3. $\tan 120° \cot 150° + \sec 120° \csc 150° = -1.$
4. $\tan 675° \sec 540° + \cot 495° \csc 450° = 0.$

5. For what values of A between 0° and 360° are $\sin A$ and $\cos A$ equal? For what values are $\tan A$ and $\cot A$ equal?

6. What sign has $\sin A + \cos A$ for the following angles? $A=120°$; $A=135°$; $A=150°$; $A=300°$; $A=315°$; $A=690°$; $A=\frac{6\pi}{7}$.

7. What sign has $\sin A - \cos A$ for each of the following angles? $A=210°$; $A=225°$; $A=240°$; $A=300°$; $A=625°$; $A=780°$; $A=\frac{5\pi}{3}$.

8. What sign has $\tan A - \cot A$ for each of the following angles? $A=60°$; $A=120°$; $A=135°$; $A=150°$; $A=210°$; $A=225°$; $A=\frac{11\pi}{6}$.

9. Find all the angles less than 360° that satisfy the following relations:

(*a*) $\sin\theta = -\frac{\sqrt{3}}{2}$; (*b*) $\cos\theta = -\frac{\sqrt{3}}{2}$; (*c*) $\tan\theta = -1$; (*d*) $\cot\theta = \sqrt{3}$.

10. Prove $\sec(A - 180°) = -\sec A$.

11. Prove $\cot(A - 270°) = -\tan A$.

12. Prove $\cos A + \cos(90° + A) + \cos(180° - A) - \cos(270° - A) = 0$.

13. Prove

$$\frac{\tan(180° + A)}{\tan(180° - A)} \cdot \frac{\cot(270° - A)}{\cot(270° + A)} \cdot \frac{\sec(360° - A)}{\csc(360° + A)} = \tan A.$$

14. Find the value of

$$\frac{\sin(-A)}{\cos(90°+A)} + \frac{\cos(-A)}{\sin(90°+A)} + \frac{\tan(-A)}{\cot(90°+A)}.$$

15. Express in simplest form

$$\frac{\cos(180° - A)}{\sin(180° + A)} \cdot \frac{\tan(270° - A)}{\cot(270° + A)} \cdot \sec A.$$

CHAPTER VII

GENERAL EXPRESSION FOR ALL ANGLES HAVING A GIVEN TRIGONOMETRIC FUNCTION

60. From the definitions of the trigonometric functions it is evident that a given angle can have but one sine, one cosine, one tangent, etc.

But the converse statement is not true. A given sine may belong to any one of an infinite number of angles. The same is true of the cosine, the tangent, or of any of the other trigonometric functions. This has already been alluded to incidentally (Arts. 50–57, pp. 74–82). Expressions will now be found for all angles that have a given sine, a given cosine, a given tangent, etc.

61. When the revolving line has made one complete revolution in either direction, it has generated an angle of $\pm 2\pi$ radians; when it has made two complete revolutions, it has generated an angle of $\pm 4\pi$ radians; and, in general, when it has made three, four, five, etc., revolutions, it has generated an angle of $\pm 6\pi$, $\pm 8\pi$, $\pm 10\pi$, etc., radians.

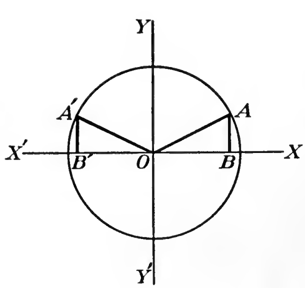
These statements may be combined into a single expression by means of the following statement:

When the revolving line has made any number of complete revolutions in either direction, it has generated an angle of $2n\pi$ radians, where n is some positive or negative integer or zero.

62. General expression for all angles that have the same sine. Let XOA (p. 86) be any convenient angle, α, and let XOA' be equal to $\pi - \alpha$. By Art. 53, the sine of XOA = the sine of XOA'; or $\sin\alpha = \sin(\pi - \alpha)$. Also, $\pi - \alpha$ is the only other angle between 0° and 360°, or between 0 and 2π whose sine is equal to the sine of α. But (Art. 61) any angle whose initial line coincides with OX and whose terminal line also coincides with OX is represented by the expression $2n\pi$. Hence, all angles whose initial lines coincide with OX and whose terminal lines coincide with OA are represented by the expression $2n\pi + \alpha$.

Any angle whose initial line coincides with OX and whose terminal line coincides with OX' is represented by the expression $2n\pi + \pi$, or $(2n + 1)\pi$. Hence, all angles whose initial lines coincide with OX and whose terminal lines coincide with OA' are represented by the expression $(2n + 1)\pi - \alpha$. These two expressions, $2n\pi + \alpha$ and $(2n + 1)\pi - \alpha$, are both included in the more general expression $n\pi + (-1)^n\alpha$; that is, α is to be added to any even multiple of π, and subtracted from any odd multiple of π. This will be understood if successive values are substituted for n, and the resulting positions of the terminal line noted. This is conveniently done by means of the following table:

If	$n = 0,$	$n\pi + (-1)^n\alpha = \alpha,$
	$n = 1,$	$n\pi + (-1)^n\alpha = \pi - \alpha,$
	$n = 2,$	$n\pi + (-1)^n\alpha = 2\pi + \alpha,$
	$n = 3,$	$n\pi + (-1)^n\alpha = 3\pi - \alpha,$
	$n = 4,$	$n\pi + (-1)^n\alpha = 4\pi + \alpha,$
	$n = 5,$	$n\pi + (-1)^n\alpha = 5\pi - \alpha,$
	$n = 6,$	$n\pi + (-1)^n\alpha = 6\pi + \alpha,$
	$n = 7,$	$n\pi + (-1)^n\alpha = 7\pi - \alpha,$
	$n = 8,$	$n\pi + (-1)^n\alpha = 8\pi + \alpha,$
	$n = 9,$	$n\pi + (-1)^n\alpha = 9\pi - \alpha,$
	. . . .	. . . .
	$n = -1,$	$n\pi + (-1)^n\alpha = -\pi - \alpha,$
	$n = -2,$	$n\pi + (-1)^n\alpha = -2\pi + \alpha.$
	. . . .	. . . .

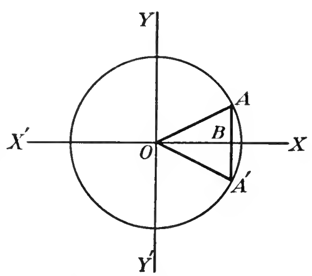


In this table we observe that whenever n is an even number, the expression $(-1)^n = +1$, and the angle that the revolving line has then generated is (Art. 61, p. 85) a certain number of complete revolutions plus the angle α. If n is an odd number, the expression $(-1)^n = -1$, and the angle that the revolving line has generated is a certain number of complete revolutions plus a half revolution, minus the angle α. That is,

The expression $n\pi + (-1)^n \alpha$ *is a general expression for all angles that have the same sine as the angle* α.

63. In this connection it should be noted that, when n is any positive or negative integer or zero, $2n$ is, by definition, an even number, and $2n+1$ is an odd number.

64. General expression for all angles that have the same cosine. The cosine of the angle $360° - \alpha$, or $2\pi - \alpha$, is equal to the cosine of the angle α; and $2\pi - \alpha$ is the only angle between 0 and 2π that has the same cosine as the angle α.



But, reasoning in the same manner as in Art. 62, all angles whose initial lines coincide with OX and whose terminal lines coincide with OA are included in the expression $2n\pi + \alpha$; and all angles whose initial lines coincide with OX and whose terminal lines coincide with OA' are included in the expression $2n\pi - \alpha$. Hence,

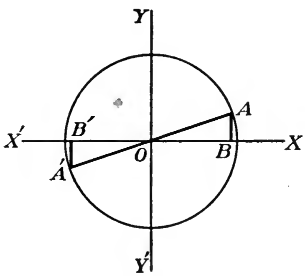
The expression $2n\pi \pm \alpha$ *is a general expression for all angles that have the same cosine as the angle* α.

65. General expression for all angles that have the same tangent. The tangent of $180° + \alpha$, or $\pi + \alpha$, is equal to the tangent of α; and $\pi + \alpha$ is the only angle between 0 and 2π that has the same tangent as the angle α (see fig. p. 88).

All angles whose initial lines coincide with OX and whose terminal lines coincide with OA are included in the general expression $2n\pi + \alpha$, and all angles whose initial lines coincide with OX and whose terminal lines coincide with OA' are included in the general expression $(2n+1)\pi + \alpha$. But, since $2n$ signifies only even integers, and $2n+1$ only odd integers, while n includes all integers, both even and odd, the two expressions, $2n\pi + \alpha$ and $(2n+1)\pi + \alpha$, can be combined as follows:

The expression $n\pi + \alpha$ *is a general expression for all angles that have the same tangent as the angle* α.

66. Since $\cot\alpha$ is the reciprocal of $\tan\alpha$, the general expression for all angles that have the same cotangent as the angle α is $n\pi + \alpha$.



Since $\sec\alpha$ is the reciprocal of $\cos\alpha$, the general expression for all angles that have the same secant as the angle α is $2n\pi \pm \alpha$.

Since $\csc\alpha$ is the reciprocal of $\sin\alpha$, the general expression for all angles that have the same cosecant as the angle α is $n\pi + (-1)^n\alpha$.

67. In the following examples, and in practical work generally, the smallest positive value of α is taken. This is done simply for convenience, the results just obtained being perfectly general.

EXERCISE XII

1. What is the general expression for all angles whose sine is $\frac{1}{2}$?

The smallest positive angle whose sine equals $\frac{1}{2}$ is 30°, or $\frac{\pi}{6}$.

$$\therefore\ \theta = \frac{\pi}{6} \text{ is the smallest positive angle,}$$

and (Art. 62) $\theta = n\pi + (-1)^n\frac{\pi}{6}$ is the general expression for all angles whose sine is $\frac{1}{2}$.

2. What is the general expression for all angles whose tangent is $\sqrt{3}$?

The smallest positive angle whose tangent is $\sqrt{3}$ is 60°, or $\frac{\pi}{3}$.

$$\therefore\ \theta = \frac{\pi}{3} \text{ is the smallest positive angle,}$$

and (Art. 65) $\theta = n\pi + \frac{\pi}{3}$ is the general expression for all angles whose tangent is $\sqrt{3}$.

3. What is the general expression for all angles whose cosine is $-\frac{1}{2}$, and whose tangent is $-\sqrt{3}$?

The only angles between 0° and 360° whose cosine is $-\frac{1}{2}$ are 120° and 240°.
The only angles between 0° and 360° whose tangent is $-\sqrt{3}$ are 120° and 300 .
The only angle that satisfies both these conditions is 120°, or $\frac{2}{3}\pi$.

$$\therefore\ \theta = 2n\pi + \tfrac{2}{3}\pi.$$

Another general expression for the same angles is $(2n+1)\pi - \frac{1}{3}\pi$.

Find the general value of θ which satisfies each of the following equations:

4. $\sin\theta = \frac{1}{2}\sqrt{2}$.

5. $\sin\theta = 1$.

6. $\sin\theta = -\frac{1}{2}\sqrt{3}$.

7. $\sin\theta = -\frac{1}{2}$.

8. $\cos\theta = \frac{1}{2}\sqrt{3}$.

9. $\cos\theta = -\frac{1}{2}\sqrt{2}$.

10. $\cos\theta = 0$.

11. $\cos\theta = -1$.

12. $\tan\theta = 1$.

13. $\tan\theta = -\frac{1}{3}\sqrt{3}$.

14. $\sin^2\theta = \frac{1}{4}$.

15. $\cos^2\theta = \frac{1}{2}$.

16. $3\tan^2\theta = 1$.

17. $3\sec^2\theta = 4$.

18. $\cot^2\theta = 1$.

19. $\tan^2\theta = 2\sin^2\theta$.

20. $2\tan^2\theta = \sec^2\theta$.

21. What is the general value of θ that satisfies both of the following equations?

$$\sin\theta = \frac{1}{2}\sqrt{3}, \text{ and } \cos\theta = \frac{1}{2}.$$

22. What is the general value of θ that satisfies both of the following equations?

$$\sin\theta = -\frac{1}{2}, \text{ and } \cos\theta = -\frac{1}{2}\sqrt{3}.$$

In the following five examples, show that the same angles are indicated by both the given expressions.

23. $n\pi + \frac{\pi}{2}$, and $2n\pi \pm \frac{\pi}{2}$.

24. $n\pi + (-1)^n\frac{\pi}{3}$, and $2n\pi + \frac{\pi}{2} \pm \frac{\pi}{6}$.

25. $n\pi - \frac{\pi}{6}$, and $n\pi + \frac{5}{6}\pi$.

26. $n\pi + \frac{\pi}{3}$, and $-n\pi + \frac{\pi}{3}$.

27. $(4n + 3)\frac{\pi}{2}$, and $2n\pi - \frac{\pi}{2}$.

68. An equation involving trigonometric functions of an unknown angle is called a **trigonometric equation.**

The solution of a trigonometric equation involves the determination of all angles that satisfy the equation.

In solving a trigonometric equation, the smallest positive angle that satisfies it should first be determined, and then the general value should be found for all angles that satisfy it. This has been illustrated in the examples of Exercise XII, and will be still further shown in those of the following set.

EXERCISE XIII

1. Solve the equation $\cos^2\theta + 2\sin^2\theta = \frac{5}{4}$.

This may be written

$$\cos^2\theta + 2 - 2\cos^2\theta = \tfrac{5}{4}.$$

$$\therefore \cos^2\theta = \tfrac{3}{4}.$$

$$\cos\theta = \pm\tfrac{1}{2}\sqrt{3}.$$

The smallest positive angle whose cosine is $\frac{1}{2}\sqrt{3}$ is 30°, or $\frac{\pi}{6}$.

Therefore, using the positive result, $\theta = 2n\pi \pm \frac{\pi}{6}$

Also, the smallest positive angle whose cosine is $-\frac{1}{2}\sqrt{3}$ is 150°, or $\frac{5}{6}\pi$.

Therefore, using the negative result, $\theta = 2n\pi \pm \frac{5}{6}\pi$, or $(2n+1)\pi \pm \frac{\pi}{6}$.

These two sets of values may be combined in the single expression $n\pi \pm \frac{\pi}{6}$.

$$\therefore \theta = n\pi \pm \frac{\pi}{6}.$$

2. Solve the equation $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$.

This may be written

$$2 - 2\sin^2\theta - \sqrt{3}\sin\theta + 1 = 0.$$

$$2\sin^2\theta + \sqrt{3}\sin\theta - 3 = 0.$$

Factoring,

$$(\sin\theta + \sqrt{3})(2\sin\theta - \sqrt{3}) = 0.$$

$$\therefore \sin\theta = -\sqrt{3}, \text{ or } \sin\theta = \tfrac{1}{2}\sqrt{3}.$$

The sine of an angle cannot be numerically greater than 1; therefore, the first equation gives no solution.

The smallest positive angle that satisfies the equation

$$\sin\theta = \tfrac{1}{2}\sqrt{3},$$

is

$$\theta = 60°, \text{ or } \frac{\pi}{3},$$

and (Art. 62) the general expression for the value of all angles that have the same sine as 60° is

$$\theta = n\pi + (-1)^n\frac{\pi}{3}.$$

Therefore, the most general expression for all angles that satisfy the original equation is $n\pi + (-1)^n\frac{\pi}{3}$.

3. Solve the equation $\tan 4\theta = \cot 3\theta$.

This may be written $\tan 4\theta = \tan\left(\frac{\pi}{2} - 3\theta\right)$, by Art. 51,

$$= \tan\left(n\pi + \frac{\pi}{2} - 3\theta\right), \text{ by Art. 65.}$$

$$\therefore 4\theta = n\pi + \frac{\pi}{2} - 3\theta,$$

$$7\theta = n\pi + \frac{\pi}{2}.$$

$$\theta = \frac{1}{7}\left(n\pi + \frac{\pi}{2}\right).$$

Solve the following equations, finding the general value of θ in each case:

4. $2\sin^2\theta - \cos\theta = 1$.

5. $\tan^2\theta + \sec\theta = 1$.

6. $\cot^2\theta - \csc\theta = 1$.

7. $\cos^2\theta - \sin\theta = \frac{1}{4}$.

8. $2\sin^2\theta + 3\cos\theta = 0$.

9. $2\cos^2\theta + \cos\theta = 1$.

10. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$.

11. $3\sin^2\theta - 2\sin\theta = 1$.

12. $\sec^2\theta + \tan^2\theta = 3$.

13. $\csc^2\theta - \cot\theta = 3$.

14. $\tan^2\theta + \cot^2\theta = 2$.

15. $\sin 5\theta = \sin 2\theta$.

16. $\sin 3\theta = \sin 9\theta$.

17. $\cos 6\theta = \cos 2\theta$.

18. $\cos 4\theta = \cos 5\theta$.

19. $\cos m\theta = \cos n\theta$.

20. $\cos 4\theta = \sin 2\theta$.

21. $\sin 4\theta = \cos 2\theta$.

22. $\tan 2\theta = \tan 3\theta$.

23. $\cot 5\theta = \cot 2\theta$.

24. $\tan 4\theta = \cot 5\theta$.

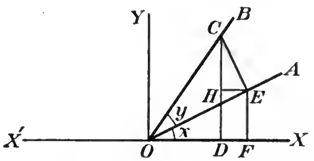
25. $\tan m\theta + \cot n\theta = 0$.

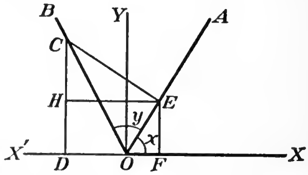
26. $\tan 2\theta \tan\theta = 1$.

CHAPTER VIII

RELATIONS BETWEEN THE TRIGONOMETRIC FUNCTIONS OF TWO OR MORE ANGLES

69. Sine and cosine of the sum of two angles. Let x and y be acute angles, and let $x + y$ be either acute or obtuse. In both figures the lettering is so arranged that the following demonstrations apply to either case.





From C, any point in OB, draw $CD \perp XX'$, and $CE \perp OA$; and from E draw $EH \parallel XX'$ and $EF \perp XX'$.

Since $$\angle x = \angle OEH = 90° - \angle HEC = \angle HCE,$$

$$\therefore \angle x = \angle HCE.$$

Then we have
$$\sin(x+y) = \frac{DC}{OC}$$
$$= \frac{DH + HC}{OC}$$
$$= \frac{FE}{OC} + \frac{HC}{OC}$$
$$= \frac{FE}{OE}\frac{OE}{OC} + \frac{HC}{CE}\frac{CE}{OC}$$
$$= \sin x \cos y + \cos \angle HCE \sin y.$$

$$\therefore \mathbf{\sin(x + y) = \sin x \cos y + \cos x \sin y.} \qquad (1)$$

Also,
$$\cos(x+y) = \frac{OD}{OC}$$
$$= \frac{OF - DF}{OC}$$
$$= \frac{OF}{OC} - \frac{HE}{OC}$$
$$= \frac{OF}{OE}\frac{OE}{OC} - \frac{HE}{CE}\frac{CE}{OC}$$
$$= \cos x \cos y - \sin \angle HCE \sin y.$$

$$\therefore \mathbf{\cos(x + y) = \cos x \cos y - \sin x \sin y.} \qquad (2)$$

70. The above proofs are given only for the case when both x and y are acute.

To prove the formulas true for all values of x and y we proceed as follows:

Let x and y be acute angles, and let $x_1 = 90° + x$; then we have (Art. 52),

$$\sin x_1 = \cos x, \text{ and } \cos x_1 = -\sin x. \qquad (1)$$

Then,
$$\sin (x_1 + y) = \sin (90° + x + y)$$
$$= \cos (x + y), \qquad \text{(Art. 52)} \quad (2)$$

where x and y are both acute angles.

But (Art. 69, p. 92) when x and y are both acute angles,

$$\cos (x + y) = \cos x \cos y - \sin x \sin y.$$

Substituting in this equation the values given in (1) and (2), we have

$$\mathbf{\sin (x_1 + y) = \sin x_1 \cos y + \cos x_1 \sin y.} \qquad \text{Q.E.D.}$$

In like manner,
$$\cos (x_1 + y) = \cos (90° + x + y)$$
$$= - \sin (x + y), \qquad \text{(Art. 52)} \quad (3)$$

where x and y are both acute angles.

But (Art. 69, p. 92) when x and y are both acute angles,

$$- \sin (x + y) = - \sin x \cos y - \cos x \sin y.$$

Substituting in this equation the values given in (1) and (3), we have

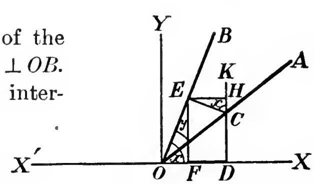
$$\mathbf{\cos (x_1 + y) = \cos x_1 \cos y - \sin x_1 \sin y.} \qquad \text{Q.E.D.}$$

Formulas (1) and (2) (Art. 69, p. 92) have now been proved for the case when x is obtuse and y is acute.

Letting $y_1 = (90° + y)$, and proceeding in the same manner, we can establish these formulas for the case when both angles are obtuse.

Then, letting $x_2 = 90° + x_1$, $y_2 = 90° + y_1$, $x_3 = 90° + x_2$, etc., and proceeding in a precisely similar manner, we can establish the formulas for all possible values of x and y.

71. Sine and cosine of the difference of two angles. Let x and y be two acute angles, placed as represented in the figure. It is here assumed that $x > y$.



From C, any point in the final position of the generating line OA, draw $CD \perp OX$ and $CE \perp OB$. Prolong DC, and from E draw $EH \parallel OX$, intersecting DC produced in H.

Since

$$\angle x = \angle BEH = 90° - \angle HEC = \angle ECH,$$
$$\therefore \angle x = \angle ECH.$$

Then,
$$\sin(x-y) = \frac{CD}{OC}$$
$$= \frac{FE - HC}{OC}$$
$$= \frac{FE}{OE}\frac{OE}{OC} - \frac{HC}{EC}\frac{EC}{OC}$$
$$= \sin x \cos y - \cos \angle ECH \sin y.$$
$$\therefore \sin(x-y) = \sin x \cos y - \cos x \sin y. \qquad (1)$$

In like manner,
$$\cos(x-y) = \frac{OD}{OC}$$
$$= \frac{OF + FD}{OC}$$
$$= \frac{OF}{OC} + \frac{EH}{OC}$$
$$= \frac{OF}{OE}\frac{OE}{OC} + \frac{EH}{EC}\frac{EC}{OC}$$
$$= \cos x \cos y + \sin \angle ECH \sin y.$$
$$\therefore \cos(x-y) = \cos x \cos y + \sin x \sin y. \qquad (2)$$

These proofs have been given on the assumption that $x > y$. To prove that they are true when $x < y$, we proceed as follows:

$$\sin(x-y) = \sin[-(y-x)]$$
$$= -\sin(y-x), \qquad \text{(Art. 50, p. 75)}$$
$$= -\sin y \cos x + \cos y \sin x,$$

or, rearranging the terms and the factors in each term,

$$\sin(x-y) = \sin x \cos y - \cos x \sin y. \qquad \text{Q. E. D.} \quad (3)$$

In like manner,
$$\cos(x-y) = \cos[-(y-x)]$$
$$= \cos(y-x) \qquad \text{(Art. 50)}$$
$$= \cos y \cos x + \sin y \sin x,$$

or, rearranging the factors in each term,

$$\cos(x-y) = \cos x \cos y + \sin x \sin y. \qquad \text{Q. E. D.} \quad (4)$$

72. The formulas of Art. 71 have now been proved for all cases when x and y are both acute angles. To prove that they are true for all possible values of x and y, we proceed as follows:

Let x and y be acute angles, and let $x_1 = 90° + x$. Then,

$$\sin x_1 = \cos x, \text{ and } \cos x_1 = -\sin x. \qquad (1)$$

Then we have
$$\sin(x_1 - y) = \sin(90° + x - y)$$
$$= \cos(x-y). \qquad \text{(Art. 52)} \quad (2)$$

But since x and y are acute angles,

$$\cos(x - y) = \cos x \cos y + \sin x \sin y. \tag{3}$$

Substituting in (3) the values given in (1) and (2), we have

$$\mathbf{\sin(x_1 - y) = \sin x_1 \cos y - \cos x_1 \sin y.} \quad \text{Q. E. D.} \tag{4}$$

In like manner,

$$\cos(x_1 - y) = \cos(90° + x - y) = -\sin(x - y). \tag{5}$$

But since x and y are acute angles,

$$-\sin(x - y) = -(\sin x \cos y - \cos x \sin y).$$

(Art. 71, p. 94) (6)

Substituting in (6) the values given in (1) and (5), we have

$$\mathbf{\cos(x_1 - y) = \cos x_1 \cos y + \sin x_1 \sin y.} \quad \text{Q. E. D.} \tag{7}$$

Formulas (1) and (2) (Art. 71, p. 94) have now been proved for the case when x is an obtuse angle and y is an acute angle.

Letting $y_1 = 90° + y$, and proceeding as before, we can establish these formulas for the case when both angles are obtuse.

Then, letting $x_2 = 90° + x_1$, $y_2 = 90° + y_1$, $x_3 = 90° + x_2$, etc., and proceeding in a precisely similar manner, we can establish the formulas for all possible values of x and y.

EXERCISE XIV

1. Find the value of sin 75°.

$$\begin{aligned}\sin 75° &= \sin(45° + 30°)\\ &= \sin 45° \cos 30° + \cos 45° \sin 30°\\ &= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}.\end{aligned}$$

2. Find the value of sin 15°.

$$\begin{aligned}\sin 15° &= \sin(45° - 30°)\\ &= \sin 45° \cos 30° - \cos 45° \sin 30°\\ &= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}.\end{aligned}$$

3. Find the value of $\cos 105°$.

$$\begin{aligned}\cos 105° &= \cos(60° + 45°)\\ &= \cos 60° \cos 45° - \sin 60° \sin 45°\\ &= \frac{1}{2}\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}.\end{aligned}$$

4. If $\sin \alpha = \frac{3}{5}$ and $\sin \beta = \frac{12}{13}$, find $\sin(\alpha - \beta)$.

5. If $\sin \alpha = \frac{3}{5}$ and $\cos \beta = \frac{12}{13}$, find $\cos(\alpha + \beta)$.

6. If $\cos \alpha = \frac{40}{41}$, and $\cos \beta = \frac{4}{5}$, find $\cos(\alpha - \beta)$.

Prove that

7. $\sin(60° + \theta) - \sin \theta = \sin(60° - \theta)$.

8. $\sin 105° + \cos 105° = \cos 45°$.

9. $\sin 75° - \sin 15° = \cos 105° + \cos 15°$.

10. $\sin(45° - \theta)\cos(45° - \phi) - \cos(45° - \theta)\sin(45° - \phi)$
$= \sin(\phi - \theta)$.

HINT. Let $x = 45° - \theta$ and $y = 45° - \phi$. Then compare with (1), Art. 66. The converse application of the x-y formulas, as illustrated by this example, is of frequent occurrence.

11. $\sin(45° + \theta)\cos(45° - \phi) + \cos(45° + \theta)\sin(45° - \phi)$
$= \cos(\theta - \phi)$.

12. $\cos(45° - \theta)\cos(45° + \theta) - \sin(45° - \theta)\sin(45° + \theta) = 0$.

13. $\cos(30° + \alpha)\cos(30° - \alpha) + \sin(30° + \alpha)\sin(30° - \alpha)$
$= \cos 2\alpha$.

14. $\cos \alpha \cos(\beta - \alpha) - \sin \alpha \sin(\beta - \alpha) = \cos \beta$.

15. $\sin(n+1)\alpha \sin(n-1)\alpha + \cos(n+1)\alpha \cos(n-1)\alpha$
$= \cos 2\alpha$.

16. $\sin(n+1)\alpha \sin(n+2)\alpha + \cos(n+1)\alpha \cos(n+2)\alpha$
$= \cos \alpha$.

17. $\sin(\alpha - \beta + 15)\cos(\beta - \alpha + 15)$
$- \cos(\alpha - \beta + 15)\sin(\beta - \alpha + 15) = \sin(2\alpha - 2\beta)$.

The following examples are of especial importance, and are often used as standard formulas.

18. $$\sin 75^\circ = \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

19. $$\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

20. $$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

21. $$\sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x.$$

73. Tangent of the sum and of the difference of two angles. For all values of x and y we have (Art. 69)

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

and

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

$$\therefore \quad \tan(x+y) = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Dividing both numerator and denominator by $\cos x \cos y$, we have

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}}.$$

$$\therefore \quad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \qquad (1)$$

In like manner,

$$\tan(x-y) = \frac{\sin(x-y)}{\cos(x-y)}$$

$$= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}$$

$$= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} + \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin x \sin y}{\cos x \cos y}}.$$

$$\therefore \quad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}. \qquad (2)$$

74. Cotangent of the sum and of the difference of two angles. For all values of x and y we have

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)}.$$

Expanding $\cos(x+y)$ and $\sin(x+y)$, dividing both numerator and denominator by $\sin x \sin y$, and reducing, we have

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}. \tag{1}$$

In a similar manner it can be proved that

$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}. \tag{2}$$

75. Formulas (1) and (2), Art. 69, (1) and (2), Art. 71, (1) and (2), Art. 73, and (1) and (2), Art. 74, are often referred to as the **addition and subtraction formulas.** The addition formulas are sometimes known as the $x+y$ formulas, and the subtraction formulas as the $x-y$ formulas. When reference is made to both groups together, the general expression, "the x-y formulas," is often employed.

76. From the formulas for the functions of the sum of two angles the formulas for the functions of the sum of three angles are at once obtained, as follows:

$$\begin{aligned}\sin(x+y+z) &= \sin[(x+y)+z] \\ &= \sin(x+y)\cos z + \cos(x+y)\sin z \\ &= (\sin x \cos y + \cos x \sin y)\cos z \\ &\qquad + (\cos x \cos y - \sin x \sin y)\sin z.\end{aligned}$$

$$\begin{aligned}\therefore \sin(x+y+z) &= \sin x \cos y \cos z + \cos x \sin y \cos z \\ &\quad + \cos x \cos y \sin z - \sin x \sin y \sin z.\end{aligned} \tag{1}$$

In like manner it can be proved that

$$\begin{aligned}\cos(x+y+z) &= \cos x \cos y \cos z - \cos x \sin y \sin z \\ &\quad - \sin x \cos y \sin z - \sin x \sin y \cos z,\end{aligned} \tag{2}$$

and that

$$\begin{aligned}\tan(x+y+z) &= \frac{\sin(x+y+z)}{\cos(x+y+z)} \\ &= \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan x \tan z - \tan y \tan z}.\end{aligned} \tag{3}$$

EXERCISE XV

1. If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$, find $\tan(\alpha + \beta)$.

2. If $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{40}{41}$, find $\tan(\beta - \alpha)$.

3. If $\tan\alpha = \frac{4}{3}$ and $\cot\beta = \frac{5}{12}$, find $\cot(\alpha + \beta)$.

4. If $\tan\alpha = \frac{4}{3}$ and $\beta = 45°$, find $\tan(\alpha + \beta)$.

5. If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$, find $\tan(2\alpha + \beta)$.

6. If $\tan\alpha = \dfrac{n}{n+1}$ and $\tan\beta = \dfrac{1}{2n+1}$, find $\tan(\alpha + \beta)$.

7. If $\tan\alpha = \frac{5}{6}$ and $\tan\beta = \frac{1}{11}$, prove that $\alpha + \beta = 45°$.

The next four examples are of especial importance, and are often used as standard formulas.

8. $\tan(45° + x) = \dfrac{1 + \tan x}{1 - \tan x}$.

9. $\tan(45° - x) = \dfrac{1 - \tan x}{1 + \tan x}$.

10. $\tan 15° = \cot 75° = 2 - \sqrt{3}$.

11. $\tan 75° = \cot 15° = 2 + \sqrt{3}$.

12. $\cot\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{\cot\theta - 1}{\cot\theta + 1}$.

13. $\cot\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{\cot\theta + 1}{\cot\theta - 1}$.

14. $\tan\left(\dfrac{\pi}{4} + \theta\right)\tan\left(\dfrac{3\pi}{4} + \theta\right) = -1$.

15. Prove the identity $\cos(\alpha + \beta)\cos\beta + \sin(\alpha + \beta)\sin\beta = \cos\alpha$.

HINT. Let $\alpha + \beta = x$ and $\beta = y$. Then compare with (2), Art. 69. Many of the remaining examples can be worked without difficulty by applying the addition or subtraction formulas directly.

16. $\sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha = \sin 3\alpha$.

17. $\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha = \sin 2\alpha$.

18. $\cos 3\alpha\cos 2\alpha - \sin 3\alpha\sin 2\alpha = \cos 5\alpha$.

19. $\dfrac{\sin 2\alpha}{\sec\alpha} + \dfrac{\cos 2\alpha}{\csc\alpha} = \sin 3\alpha$.

20. $\sin(60° + \alpha)\cos(30° + \alpha) - \cos(60° + \alpha)\sin(30° + \alpha) = \frac{1}{2}$.

21. $\dfrac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha\tan\alpha} = \tan 3\alpha$.

22. $\dfrac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)} = \tan 2\alpha$.

23. $\dfrac{\tan \alpha - \tan(\alpha - \beta)}{1 + \tan \alpha \tan(\alpha - \beta)} = \tan \beta.$

24. $\dfrac{\cot 3\alpha \cot 2\alpha + 1}{\cot 2\alpha - \cot 3\alpha} = \cot \alpha.$

25. $\tan 2\theta - \tan \theta = \tan \theta \sec 2\theta.$

26. $\sec 2\theta = 1 + \tan 2\theta \tan \theta.$

27. $\csc 2\theta = \cot \theta - \cot 2\theta.$

28. $\dfrac{\tan 3\theta - \tan 2\theta}{1 + \tan 3\theta \tan 2\theta} = \dfrac{\tan 4\theta - \tan 3\theta}{1 + \tan 4\theta \tan 3\theta}.$

29. $\tan(45° + \theta) - \tan(45° - \theta) = \dfrac{4 \tan \theta}{1 - \tan^2 \theta}.$

30. $\dfrac{\sin(x + y)}{\cos(x - y)} = \dfrac{\cot x + \cot y}{1 + \cot x \cot y}.$

77. The algebraic sum of two sines or of two cosines in the form of a product. For all values of x and y we have (Arts. 69 and 71)

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

and

$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

Adding and subtracting, we have

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y, \qquad (1)$$

and

$$\sin(x + y) - \sin(x - y) = 2 \cos x \sin y. \qquad (2)$$

Also (Arts. 69 and 71),

$$\cos(x + y) = \cos x \cos y - \sin x \sin y,$$

and

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

Adding and subtracting, as before, we have

$$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y, \qquad (3)$$

and

$$\cos(x + y) - \cos(x - y) = -2 \sin x \sin y. \qquad (4)$$

Let

$$x + y = u, \text{ and } x - y = v.$$

Solving these two equations for x and y,

$$x = \frac{u + v}{2}, \text{ and } y = \frac{u - v}{2}.$$

Substituting these values of x and y in (1), (2), (3), and (4), we have

$$\sin u + \sin v = 2 \sin \frac{u + v}{2} \cos \frac{u - v}{2}; \qquad (5)$$

$$\sin u - \sin v = 2 \cos \frac{u + v}{2} \sin \frac{v - u}{2}; \qquad (6)$$

$$\cos u + \cos v = 2 \cos \frac{u + v}{2} \cos \frac{u - v}{2}; \qquad (7)$$

$$\cos u - \cos v = -2 \sin \frac{u + v}{2} \sin \frac{u - v}{2}. \qquad (8)$$

These formulas are among the most important of all the formulas of trigonometry. The student should commit them carefully to memory, and become perfectly familiar with their application. They will sometimes be referred to as the *u-v* formulas.

As illustrations of the manner in which certain expressions can be simplified by the application of one or more of these processes, the following examples are given:

1. $$\sin 70° - \sin 10° = 2\cos\frac{70° + 10°}{2}\sin\frac{70° - 10°}{2}$$
$$= 2\cos 40° \sin 30°$$
$$= \cos 40°.$$

2. $$\frac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \frac{2\cos\frac{75° + 15°}{2}\sin\frac{75° - 15°}{2}}{2\cos\frac{75° + 15°}{2}\cos\frac{75° - 15°}{2}}$$
$$= \frac{2\cos 45° \sin 30°}{2\cos 45° \cos 30°}$$
$$= \tan 30°$$
$$= \tfrac{1}{3}\sqrt{3} = 0.57735.$$

3. $$\frac{(\sin 6\theta + \sin 2\theta)(\cos 2\theta - \cos 4\theta)}{(\sin 5\theta + \sin\theta)(\cos 3\theta - \cos 5\theta)}$$
$$= \frac{(2\sin 4\theta\cos 2\theta)(2\sin 3\theta\sin\theta)}{(2\sin 3\theta\cos 2\theta)(2\sin 4\theta\sin\theta)} = 1.$$

EXERCISE XVI

Prove the following relations:

1. $\sin 70° + \sin 50° = \sqrt{3}\cos 10°.$

2. $\dfrac{\sin 8\theta - \sin 6\theta}{\cos 8\theta + \cos 6\theta} = \tan\theta.$ **3.** $\dfrac{\sin 2\theta + \sin 6\theta}{\cos 2\theta + \cos 6\theta} = \tan 4\theta.$

4. $\dfrac{\sin 5\theta - \sin\theta}{\sin 6\theta - \sin 2\theta} = \cos 3\theta\sec 4\theta.$

5. $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \tan(A + B)\cot(A - B).$

6. $\dfrac{\sin\theta+\sin 2\theta}{\cos\theta-\cos 2\theta}=\cot\dfrac{\theta}{2}.$ 7. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{(A+B)}{2}.$

8. $\dfrac{\sin A-\sin B}{\cos B-\cos A}=\cot\dfrac{(A+B)}{2}.$

9. $\sin(A+B)+\cos(A-B)=2\sin(45^\circ+B)\cos(45^\circ-A).$

10. $\dfrac{\cos 5A-\cos 3A}{\sin 5A-\sin 3A}+\dfrac{\cos 2A-\cos 4A}{\sin 4A-\sin 2A}=-\dfrac{\sin A}{\cos 4A\cos 3A}.$

11. $\sin(60^\circ+A)-\sin(60^\circ-A)=\sin A.$

12. $\cos(30^\circ-\theta)+\cos(30^\circ+\theta)=\sqrt{3}\cos\theta.$

13. $\cos\left(\dfrac{\pi}{4}+\theta\right)-\cos\left(\dfrac{\pi}{4}-\theta\right)=-\sqrt{2}\sin\theta.$

14. $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\tan 4\theta.$

15. $\dfrac{\sin\theta-\sin 5\theta+\sin 9\theta-\sin 13\theta}{\cos\theta-\cos 5\theta-\cos 9\theta+\cos 13\theta}=\cot 4\theta.$

16. $\dfrac{\sin x+\sin y}{\sin x-\sin y}=\tan\dfrac{x+y}{2}\cot\dfrac{x-y}{2}.$

17. $\dfrac{\cos x+\cos y}{\cos x-\cos y}=\cot\dfrac{x+y}{2}\cot\dfrac{y-x}{2}.$

18. $\cos 3\theta+\cos 5\theta+\cos 7\theta+\cos 15\theta=4\cos 4\theta\cos 5\theta\cos 6\theta.$

19. $\dfrac{\cos(2A-3B)+\cos 3B}{\sin(2A-3B)+\sin 3B}=\cot A.$

20. $\sin 50^\circ+\sin 10^\circ-\sin 70^\circ=0.$

21. $\dfrac{\cos(A-3B)-\cos(3A+B)}{\sin(3A+B)+\sin(A-3B)}=\tan(A+2B).$

22. $\sin 80^\circ+\sin 70^\circ-\sin 10^\circ-\sin 20^\circ=+\sin 40^\circ+\sin 50^\circ.$

23. $\cos x+\cos 2x+\cos 4x+\cos 5x=4\cos\dfrac{x}{2}\cos\dfrac{3x}{2}\cos 3x.$

24. $\sin(\alpha+\beta+\gamma)+\sin(\alpha-\beta-\gamma)+\sin(\alpha+\beta-\gamma)$
$+\sin(\alpha-\beta+\gamma)=4\sin\alpha\cos\beta\cos\gamma.$

25. $\sin 2\alpha+\sin 2\beta+\sin 2\gamma-\sin 2(\alpha+\beta+\gamma)$
$=4\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta).$

26. $\dfrac{\cos\theta+2\cos 3\theta+\cos 5\theta}{\cos 3\theta+2\cos 5\theta+\cos 7\theta}=\dfrac{\cos 3\theta}{\cos 5\theta}.$

27. $\dfrac{\sin 3\theta+2\sin 5\theta+\sin 7\theta}{\sin\theta+2\sin 3\theta+\sin 5\theta}=\sin 5\theta\csc 3\theta.$

28. $\dfrac{\sin(A+B)-2\sin A+\sin(A-B)}{\cos(A+B)-2\cos A+\cos(A-B)}=\tan A.$

29.

$$\frac{\cos(x+y+z)+\cos(-x+y+z)+\cos(x-y+z)+\cos(x+y-z)}{\sin(x+y+z)+\sin(-x+y+z)-\sin(x-y+z)+\sin(x+y-z)}=\cot y.$$

30. $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ=0.$

78. The product of two sines, of two cosines, or of a sine and a cosine expressed in the form of an algebraic sum.

In (1), (2), (3), and (4), Art. 77, the *u-v* formulas are expressed in a form which is quite as important as that already considered, and which is so convenient, and of such frequent application that the formulas are here reproduced in that form. Using the left for the right and the right for the left members, they are

$$2\sin x\cos y=\sin(x+y)+\sin(x-y); \quad (1)$$

$$2\cos x\sin y=\sin(x+y)-\sin(x-y); \quad (2)$$

$$2\cos x\cos y=\cos(x+y)+\cos(x-y); \quad (3)$$

$$-2\sin x\sin y=\cos(x+y)-\cos(x-y). \quad (4)$$

These formulas are the converse of the *u-v* formulas, and may be conveniently referred to by that name. The two groups taken together are useful in solving problems and in performing investigations which, without them, could be handled only with the greatest difficulty.

EXERCISE XVII

1. Express in the form of a sum or difference $2 \sin 6\theta \sin 4\theta$.

$$2 \sin 6\theta \sin 4\theta = -(\cos(6\theta + 4\theta) - \cos(6\theta - 4\theta))$$
$$= -(\cos 10\theta - \cos 2\theta)$$
$$= \cos 2\theta - \cos 10\theta.$$

2. Express in the form of a sum or difference $\cos(A - 2B)$ $\sin(A + 2B)$.

$$\cos(A - 2B)\sin(A + 2B) = \tfrac{1}{2}(\sin 2A - \sin(-4B))$$
$$= \tfrac{1}{2}(\sin 2A + \sin 4B).$$

3. Find the value of $2 \sin 75° \sin 15°$.

$$2 \sin 75° \sin 15° = \cos(75° - 15°) - \cos(75° + 15°)$$
$$= \cos 60° - \cos 90°$$
$$= \tfrac{1}{2} - 0$$
$$= \tfrac{1}{2}.$$

Express as a sum or difference the following:

4. $2 \sin 6\theta \cos 2\theta$.

5. $2 \cos 4\theta \sin 2\theta$.

6. $\cos \dfrac{\theta}{2} \sin \dfrac{3\theta}{2}$.

7. $\sin \dfrac{5\theta}{2} \cos \dfrac{7\theta}{2}$.

8. $\cos \dfrac{\theta}{4} \cos \dfrac{3\theta}{4}$.

9. $2 \sin(2A + B)\cos(A - B)$.

10. $2 \cos 3A \cos(A - 2B)$.

11. $\sin(60° + \theta)\cos(60° - \theta)$.

Prove the following identities:

12. $\cos(120° + \theta)\cos(120° - \theta) = \tfrac{1}{4}(2 \cos 2\theta - 1)$.

13. $\cos(30° - \theta)\cos(60° - \theta) = \tfrac{1}{2}(2 \sin 2\theta + \sqrt{3})$.

14. $\sin(120° - \theta)\cos(60° + \theta) = \tfrac{1}{2}(\sin 60° - 2\theta)$.

15. $\sin(\theta + 45°)\sin(\theta - 45°) = -\tfrac{1}{2}\cos 2\theta$.

16. $\cos 3\theta \sin 2\theta - \cos 4\theta \sin \theta = \cos 2\theta \sin \theta$.

17. $\sin 3\theta \sin 6\theta + \sin \theta \sin 2\theta = \sin 4\theta \sin 5\theta$.

18. $\sin 2\theta \cos \theta + \sin 6\theta \cos \theta = \sin 3\theta \cos 2\theta + \sin 5\theta \cos 2\theta$.

19. $\cos(40° - \theta)\cos(40° + \theta) + \cos(50° + \theta)\cos(50° - \theta) =$ $\cos 2\theta$.

20. $\sin A \cos(A+B) - \cos A \sin(A-B) = \cos 2A \sin B.$

21. $2\cos\frac{3\pi}{11}\cos\frac{4\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{10\pi}{11} = 0.$

22. $4\sin A \sin B \sin C = \sin(B+C-A) + \sin(C+A-B)$
$+ \sin(A+B-C) - \sin(A+B+C).$

23. $\dfrac{\cos 3A \sin 2A - \cos 4A \sin A}{\cos 5A \cos 2A - \cos 4A \cos 3A} = -\cot 2A.$

24. $4\sin\theta \sin(60^\circ+\theta)\sin(60^\circ-\theta) = \sin 3\theta.$

25. $4\cos\theta\cos\left(\frac{2\pi}{3}+\theta\right)\cos\left(\frac{2\pi}{3}-\theta\right) = \cos 3\theta.$

26. $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{1}{8}\sqrt{3}.$

27. $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}.$

CHAPTER IX

FUNCTIONS OF MULTIPLE AND SUBMULTIPLE ANGLES

79. Functions of an angle in terms of functions of half the angle. If in the addition formulas, Arts. 69, 71, 73, and 74, we put $x = y$, we have

$$\sin(x + x) = \sin x \cos x + \cos x \sin x,$$

$$\cos(x + x) = \cos x \cos x - \sin x \sin x,$$

$$\tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x},$$

and
$$\cot(x + x) = \frac{\cot x \cot x - 1}{\cot x + \cot x};$$

i.e.
$$\sin 2x = 2 \sin x \cos x; \quad (1)$$

$$\cos 2x = \cos^2 x - \sin^2 x; \quad (2)$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}; \quad (3)$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}. \quad (4)$$

In these formulas $2x$ may have any value whatever; or, in other words, $2x$ is any angle whatever.

Hence, these formulas are to be regarded as formulas for expressing the values of functions of an angle in terms of functions of half the angle. They may also, of course, be regarded as formulas for expressing the functions of twice an angle in terms of functions of the angle itself.

80. If we let $2x = \theta$, we have the formulas in the following useful form:

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}; \quad (1)$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \qquad (2)$$

$$= 1 - 2\sin^2\frac{\theta}{2}$$

$$= 2\cos^2\frac{\theta}{2} - 1.$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}; \qquad (3)$$

$$\cot\theta = \frac{\cot^2\frac{\theta}{2} - 1}{2\cot\frac{\theta}{2}}.$$

81. Functions of an angle $3x$ in terms of functions of x.

If in the addition formulas we put $y = 2x$, we obtain expressions for the value of functions of $3x$ in terms of functions of x, as follows:

$$\begin{aligned}\sin(x + 2x) &= \sin x\cos 2x + \cos x\sin 2x\\ &= \sin x(\cos^2 x - \sin^2 x) + \cos x \cdot 2\sin x\cos x\\ &= \sin x(1 - 2\sin^2 x) + 2\sin x(1 - \sin^2 x)\\ &= \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x.\end{aligned}$$

$$\therefore\ \sin 3x = 3\sin x - 4\sin^3 x. \qquad (1)$$

In like manner,

$$\begin{aligned}\cos(x + 2x) &= \cos x\cos 2x - \sin x\sin 2x\\ &= \cos x(\cos^2 x - \sin^2 x) - \sin x \cdot 2\sin x\cos x\\ &= \cos x(2\cos^2 x - 1) - 2(1 - \cos^2 x)\cos x\\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x.\end{aligned}$$

$$\therefore\ \cos 3x = 4\cos^3 x - 3\cos x. \qquad (2)$$

Also, $$\tan 3x = \frac{\tan x + \tan 2x}{1 - \tan x\tan 2x} = \frac{\tan x + \dfrac{2\tan x}{1 - \tan^2 x}}{1 - \tan x\dfrac{2\tan x}{1 - \tan^2 x}}.$$

$$\therefore\ \tan 3x = \frac{3\tan x \pm \tan^3 x}{1 - 3\tan^2 x}. \qquad (3)$$

In a similar manner it is possible to obtain formulas for the functions of higher multiples of x in terms of functions of x.

82. Functions of an angle expressed in terms of functions of twice the angle.

Since $\cos 2x = 1 - 2\sin^2 x,$

we have $2\sin^2 x = 1 - \cos 2x.$

$$\therefore \sin x = \pm\sqrt{\frac{1-\cos 2x}{2}}. \quad (1)$$

Also, $\cos 2x = 2\cos^2 x - 1,$

$2\cos^2 x = 1 + \cos 2x.$

$$\therefore \cos x = \pm\sqrt{\frac{1+\cos 2x}{2}}. \quad (2)$$

Dividing (1) by (2) we have

$$\tan x = \pm\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}. \quad (3)$$

These formulas are often given in the following form, where $x = \frac{\theta}{2}.$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}. \quad (4)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}. \quad (5)$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}. \quad (6)$$

In this form they are to be regarded as formulas for expressing the values of functions of a half-angle in terms of functions of the angle itself.

The magnitude of the angle determines which of the two signs preceding the radical is to be employed.

EXERCISE XVIII

1. If $\sin\theta = \frac{1}{3}$, find $\sin 2\theta$ and $\sin 3\theta$.
2. If $\sin\theta = \frac{1}{4}$, find $\cos 2\theta$ and $\cos 3\theta$.
3. If $\cos\theta = \frac{3}{5}$, find $\sin 2\theta$ and $\cos 3\theta$.
4. If $\tan\theta = \frac{1}{2}$, find $\tan 2\theta$ and $\tan 3\theta$.
5. If $\tan\theta = \frac{1}{3}$, find $\sin 2\theta$ and $\tan 3\theta$.

Prove the following identities:

6. $\cos^4\theta - \sin^4\theta = \cos 2\theta.$

7. $\tan\theta + \cot\theta = 2\csc 2\theta.$

8. $\cot\theta - \tan\theta = \dfrac{2}{\tan 2\theta}.$

9. $\dfrac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta} = \cos 2\theta.$

10. $\sec 2\theta = \dfrac{\cot^2\theta + 1}{\cot^2\theta - 1}.$

The next six equations are especially important, and may be regarded as standard formulas.

11. $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta.$

12. $\tan\theta = \dfrac{\sin 2\theta}{1 + \cos 2\theta}.$

13. $\cot\theta = \dfrac{\sin 2\theta}{1 - \cos 2\theta}.$

14. $\tan\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{\sin\theta}.$

15. $\cot\dfrac{\theta}{2} = \dfrac{1 + \cos\theta}{\sin\theta}.$

16. $\left(\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right)^2 = 1 - \sin\theta.$

17. $2\cos^2\dfrac{\theta}{2} = \dfrac{1 + \sec\theta}{\sec\theta}.$

18. $\cos 2\theta = \dfrac{2 - \sec^2\theta}{\sec^2\theta}.$

19. $\sin^2\dfrac{\theta}{2} = \dfrac{\sec\theta - 1}{2\sec\theta}.$

20. $\dfrac{\cos\theta}{1 - \sin\theta} = \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right).$

21. $\dfrac{\sin(A+B)}{\cos(A-B)} = \dfrac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}.$

22. $\dfrac{\cos 2\theta}{1 + \sin 2\theta} = \tan(45° - \theta).$

23. $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2.$

24. $\dfrac{1 - \cos A + \cos B - \cos(A+B)}{1 + \cos A - \cos B - \cos(A+B)} = \tan\dfrac{A}{2}\tan\dfrac{B}{2}.$

25. $\tan(45° + \theta) + \tan(45° - \theta) = \dfrac{2}{\cos 2\theta}.$

26. $\tan 2\theta - \sec\theta\sin\theta = \tan\theta\sec 2\theta.$

27. $\dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cos\alpha - \sin\beta\cos\beta} = \tan(\alpha + \beta).$

28. $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} - \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\tan 2\theta.$

29. $\dfrac{\cos(\theta + 15^\circ)}{\sin(\theta + 15^\circ)} - \dfrac{\sin(\theta - 15^\circ)}{\cos(\theta - 15^\circ)} = \dfrac{4\cos 2\theta}{1 + 2\sin 2\theta}.$

30. $\dfrac{\cos 2\theta + \cos\theta + 1}{\sin 2\theta + \sin\theta} = \cot\theta.$

31. $\dfrac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \tan\theta.$

32. $\dfrac{-\tan\theta + 1}{\tan\theta + 1} = \dfrac{1 - \sin 2\theta}{\cos 2\theta}.$

33. $\dfrac{\sin 2\theta}{1 - \cos 2\theta} \cdot \dfrac{1 - \cos\theta}{\cos\theta} = \tan\dfrac{\theta}{2}.$

34. $\dfrac{\sin(n+1)\theta + \sin(n-1)\theta + 2\sin n\theta}{\cos(n-1)\theta - \cos(n+1)\theta} = \cot\dfrac{\theta}{2}.$

35. $\dfrac{\cos 3\theta + \sin 3\theta}{\cos\theta - \sin\theta} = 1 + 2\sin 2\theta.$

36. $\sin 6\theta + \sin 4\theta - \sin 2\theta = 4\sin 2\theta\cos\theta\cos 3\theta.$

37. $(\sec 2\theta + 1)\sqrt{\sec^2\theta - 1} = \tan 2\theta.$

38. $4\cos\theta\cos(60^\circ - \theta)\cos(60^\circ + \theta) = \cos 3\theta.$

39. $16\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ = 1.$

40. $\tan(45^\circ + \theta) = \sqrt{\dfrac{1 + \sin 2\theta}{1 - \sin 2\theta}}.$

41. $\dfrac{\sin(n+1)\theta - \sin(n-1)\theta}{\cos(n+1)\theta + \cos(n-1)\theta + 2\cos n\theta} = \tan\dfrac{\theta}{2}.$

42. $\cos^2(n+1)\theta - \cos^2 n\theta = -\sin(2n+1)\theta\sin\theta.$

83. Identities that are true for angles whose sum is 180° or 90°. When three angles are involved whose sum is either 90° or 180°, many relations are found to exist that do not hold true for angles in general.

For, if $A + B + C = 180^\circ$, we have (Art. 53, p. 78),

$$\sin(A+B) = \sin C,\ \cos(A+B) = -\cos C,\ \tan(A+B) = -\tan C,$$

and similar relations hold between functions of the sum of any two of the given angles, and the corresponding functions of the third angle, since the sum of any two is the supplement of the third.

Also, if $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°$, the sum of any two of these angles is the complement of the third. Therefore,

$$\sin\left(\frac{A}{2}+\frac{B}{2}\right)=\cos\frac{C}{2},\ \cos\left(\frac{A}{2}+\frac{B}{2}\right)=\sin\frac{C}{2},\ \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\frac{C}{2},$$

and similar relations hold between functions of the sum of any two of the angles and the corresponding co-functions of the third.

Ex. 1. If $A+B+C=180°$, prove that

$$\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C.$$

$$\begin{aligned}\text{Left member} &= 2\sin(A+B)\cos(A-B)-2\sin C\cos C\\ &= 2\sin C\cos(A-B)+2\sin C\cos(A+B)\\ &= 2\sin C\,[\cos(A+B)+\cos(A-B)]\\ &= 2\sin C\,(2\cos A\cos B)\\ &= 4\cos A\cos B\sin C.\end{aligned}$$

Ex. 2. If $A+B+C=180°$, prove that

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$\begin{aligned}\text{Left member} &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}\\ &= 1+2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}\\ &= 1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)\\ &= 1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)\\ &= 1+2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)\\ &= 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\end{aligned}$$

Ex. 3. If $A + B + C = 180°$, prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Since $A + B = 180° - C$, $\tan (A + B) = - \tan C$;

i.e.
$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = - \tan C.$$

Clearing of fractions, $\tan A + \tan B = - \tan C + \tan A \tan B \tan C.$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

EXERCISE XIX

If $A + B + C = 180°$, prove that

1. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

2. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$

3. $\cos 2A - \cos 2B + \cos 2C = 1 - 4 \sin A \cos B \sin C.$

4. $\sin 2A - \sin 2B - \sin 2C = -4 \sin A \cos B \cos C.$

5. $\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$

6. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$

7. $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$

8. $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C.$

9. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C.$

10. $\dfrac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{B}{2}.$

11. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C} = \cot A \cot B.$

12. $\dfrac{1 + \cos A - \cos B + \cos C}{1 + \cos A + \cos B - \cos C} = \tan \frac{B}{2} \cot \frac{C}{2}.$

13. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$

$\cot B(\cot A + \cot C) + \cot A \cot C = \cot B\left[\frac{\cot A \cot C - 1}{\cot(A+C)}\right] + \cot A \cot$

$= -\cot A \cot C + 1 - \cot A \cot C = 1.$

Page 98.

14. $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1.$

15. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$

16. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

17. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

18. $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{A+B}{4}\cos\frac{B+C}{4}\cos\frac{C+A}{4}.$

19. $\sin(A + B - C) + \sin(B + C - A) + \sin(C + A - B)$

$$= 4\sin A \sin B \sin C.$$

20. $\sin(A + 2B) + \sin(B + 2C) + \sin(C + 2A)$

$$= 4\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}.$$

CHAPTER X

INVERSE TRIGONOMETRIC FUNCTIONS

84. If $\sin\theta = a$, where a is any known quantity, θ may have any one of an infinite number of values. The symbol "$\sin^{-1} a$" is used to denote the angle whose sine is a, and is, accordingly, read "the angle whose sine is a." It is sometimes called the *inverse sine*, or the *anti sine* of a.

To illustrate the use of this notation, let us take the equation

$$\cos\theta = \tfrac{1}{2}. \tag{1}$$

We know from this that θ may equal 60°, 300°, 420°, 660°, ⋯. To state the fact that θ may equal any one of these angles, we employ the equation

$$\theta = \cos^{-1}\tfrac{1}{2}, \tag{2}$$

which is read "$\theta =$ the angle whose cosine is $\frac{1}{2}$."
We are then to understand that (1) and (2) are inverse statements, the former asserting that the cosine of some angle, θ, is equal to $\frac{1}{2}$, and the latter asserting that θ is the angle whose cosine is $\frac{1}{2}$.

From (2) we also understand that 60°, 300°, 420°, ⋯, are angles that satisfy the equation, since the cosine of each of these angles is $\frac{1}{2}$. In other words, (2) is satisfied by any of the angles (Art. 64, p. 87) included in the general expression

$$2n\pi \pm \frac{\pi}{3}.$$

Similarly, if $$\tan\theta = 1,$$
then the equation $$\theta = \tan^{-1} 1$$

asserts that θ may equal 45°, 225°, 405°, ⋯. That is, θ may have any one of the values represented by the expression

$$n\pi + \frac{\pi}{4}.$$

It is strongly urged that the student become familiar at the outset with the idea that the expressions $\sin^{-1} 1$, $\cos^{-1}\frac{1}{2}$, $\tan^{-1}\sqrt{3}$, etc., are single symbols, and denote angles. They represent angles just as definitely as do the symbols θ, ϕ, A, B, x, y, etc., which are used so frequently for that purpose. The only point to be noted is, that the angle which is represented in this manner is described by means of one of its trigonometric functions.

85. Angles expressed by the symbols $\sin^{-1} a$, $\cos^{-1}\frac{1}{2}\sqrt{3}$, $\tan^{-1} 1$, etc., are called **inverse trigonometric functions,** or **inverse circular functions.**

Since a central angle has the same magnitude in degrees as the intercepted arc, these functions are used to represent arcs as well as angles. The notation arc sin a, arc cos $\frac{1}{2}$, arc tan $\frac{1}{3}\sqrt{3}$, etc., is often used instead of $\sin^{-1} a$, $\cos^{-1}\frac{1}{2}$, $\tan^{-1}\frac{1}{3}\sqrt{3}$, etc.

In using the notation here adopted, the student should note that the symbol -1 is not an algebraic exponent. That is,

$$\sin^{-1} a \text{ is not the same as } (\sin a)^{-1}.$$

The former expression denotes the angle whose sine is a, and the latter denotes $\dfrac{1}{\sin a}$, or $\csc a$.

86. The smallest numerical value of an angle whose sine, cosine, tangent, etc., have given values, is called the **principal value** of the angle.

Thus, the principal values of

$$\sin^{-1}\frac{1}{2}, \quad \cos^{-1}\left(-\frac{1}{2}\right), \quad \tan^{-1}(-1), \quad \cot^{-1}\frac{\sqrt{3}}{3},$$

are $\quad 30°, \quad \pm 120°, \quad -45°, \quad 60°.$

In a case like the second, where two values are given, which are numerically equal but have opposite signs, the positive value is usually understood. Thus, the principal value of $\cos^{-1}(-\frac{1}{2})$ is usually considered to be 120°.

To avoid ambiguity, it will be understood that, when any of these symbols are employed, the principal values of the angles are referred to.

If a is positive, the principal values of all the inverse functions except $\text{vers}^{-1} a$ and $\text{covers}^{-1} a$ lie between 0° and 90°. The principal value of $\text{vers}^{-1} a$ lies between 0° and 180°, and the principal value of $\text{covers}^{-1} a$ lies between 0° and 90°, or between 180° and 270°.

If a is negative, the principal values of $\sin^{-1} a$ and $\csc^{-1} a$ lie between 0° and − 90°, or between 180° and 270°. The principal values of $\cos^{-1} a$ and $\sec^{-1} a$ lie between 90° and 180°, or between −90° and −180°. The principal values of $\tan^{-1} a$ and $\cot^{-1} a$ lie between 90° and 180°. As stated above, the positive values of these angles are usually employed. Since vers θ and covers θ are always positive, $\text{vers}^{-1} a$ and $\text{covers}^{-1} a$ are impossible when a is negative.

87. Ex. 1. Prove that $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$. (1)

Let $\sin^{-1}\frac{3}{5} = \alpha,\quad \cos^{-1}\frac{12}{13} = \beta,\quad \cos^{-1}\frac{33}{65} = \gamma.$

Then, $\sin\alpha = \frac{3}{5},\quad \cos\beta = \frac{12}{13},\quad \cos\gamma = \frac{33}{65}.$

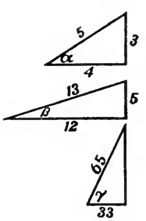
We are to prove that $\alpha + \beta = \gamma$. (2)

This can be done by proving that any function of $\alpha + \beta$ is equal to the same function of γ, since, if two sines, two cosines, two tangents, ⋯, are equal, the principal values of the angles are also equal.

In this case we select the cosines; and we are now to prove that

$$\cos(\alpha + \beta) = \cos\gamma. \qquad (3)$$

Expanding, $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos\gamma.$ (4)



The values of $\cos\beta$, $\sin\alpha$, and $\cos\gamma$ are already known; and, obtaining the values of $\cos\alpha$ and $\sin\beta$ from the figures in the margin, and substituting in (4), we have

$$\frac{4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{5}{13} = \frac{33}{65},$$

$$\frac{48}{65} - \frac{15}{65} = \frac{33}{65}.$$

$$\therefore \cos(\alpha + \beta) = \cos\gamma.$$

$$\therefore \alpha + \beta = \gamma.$$

Ex. 2. Prove that $\cos^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$.

Let $\cos^{-1}\frac{3}{5} = \alpha,\quad \sin^{-1}\frac{5}{13} = \beta,\quad \sin^{-1}\frac{16}{65} = \gamma.$

Then, $\cos\alpha = \frac{3}{5},\quad \sin\beta = \frac{5}{13},\quad \sin\gamma = \frac{16}{65}.$

We are to prove that $\alpha + \beta + \gamma = \frac{\pi}{2}$,

or, $$\alpha + \beta = \frac{\pi}{2} - \gamma.$$

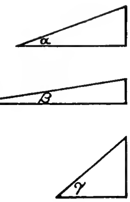
Selecting in this case the sines, we proceed as follows:

$$\sin(\alpha + \beta) = \sin\left(\frac{\pi}{2} - \gamma\right)$$

$$= \cos\gamma.$$

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \cos\gamma.$$

Substituting numerical values, we have



$$\tfrac{4}{5}\cdot\tfrac{12}{13}+\tfrac{3}{5}\cdot\tfrac{5}{13}=\tfrac{63}{65},$$

$$\tfrac{48}{65}+\tfrac{15}{65}=\tfrac{63}{65}.$$

$$\therefore\ \alpha+\beta=\frac{\pi}{2}-\gamma.$$

$$\alpha+\beta+\gamma=\frac{\pi}{2}.$$

Ex. 3. Prove that

$$2\sin^{-1}\frac{1}{\sqrt{10}}+\tan^{-1}\frac{1}{7}-\cos^{-1}\frac{1}{\sqrt{2}}=0.$$

Let $\sin^{-1}\frac{1}{\sqrt{10}}=\alpha,\quad \tan^{-1}\frac{1}{7}=\beta,\quad \cos^{-1}\frac{1}{\sqrt{2}}=\gamma.$

Then, $\sin\alpha=\frac{1}{\sqrt{10}},\quad \tan\beta=\frac{1}{7},\quad \cos\gamma=\frac{1}{\sqrt{2}}.$

We are to prove that $2\alpha+\beta-\gamma=0,$

or, $2\alpha+\beta=\gamma.$

Selecting the tangents as convenient functions to deal with in this case, we proceed as follows:

$$\tan(2\alpha+\beta)=\tan\gamma$$

$$\text{The left member}=\frac{\tan 2\alpha+\tan\beta}{1-\tan 2\alpha\tan\beta}.$$

$$=\frac{\dfrac{2\tan\alpha}{1-\tan^2\alpha}+\tan\beta}{1-\dfrac{2\tan\alpha}{1-\tan^2\alpha}\tan\beta}=\frac{\dfrac{\frac{2}{3}}{1-\frac{1}{9}}+\frac{1}{7}}{1-\dfrac{\frac{2}{3}}{1-\frac{1}{9}}\cdot\frac{1}{7}}$$

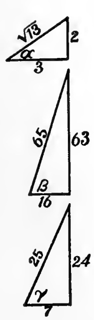
$$=\tfrac{150}{150}=1.$$

But $\tan\gamma=1.$

$$\therefore\ \tan(2\alpha+\beta)=\tan\gamma.$$

$$\therefore\quad 2\alpha+\beta=\gamma,$$

$$2\alpha+\beta-\gamma=0.$$



Ex. 4. Prove that

$$2\sin^{-1}\frac{2}{\sqrt{13}}+\cos^{-1}\frac{16}{65}+\frac{1}{2}\tan^{-1}\frac{24}{7}=\pi.$$

Let $\sin^{-1}\frac{2}{\sqrt{13}}=\alpha,\quad \cos^{-1}\frac{16}{65}=\beta,\quad \tan^{-1}\frac{24}{7}=\gamma.$

Then, $\sin\alpha=\frac{2}{\sqrt{13}},\quad \cos\beta=\frac{16}{65},\quad \tan\gamma=\frac{24}{7}.$

We are to prove that

$$2\alpha+\beta+\tfrac{1}{2}\gamma=\pi,$$

or, $2\alpha+\beta=\pi-\tfrac{1}{2}\gamma.$

Selecting the sines as the most convenient functions with which to work in this case, we proceed as follows:

$$\sin(2\alpha+\beta)=\sin(\pi-\tfrac{1}{2}\gamma).$$

$$\sin 2\alpha\cos\beta+\cos 2\alpha\sin\beta=\sin\tfrac{1}{2}\gamma. \qquad (1)$$

All the functions of α, β, and γ can be determined at once from the proper figures; and the values of $\sin 2\alpha$, $\cos 2\beta$, and $\sin\frac{1}{2}\gamma$ must be computed.

$$\sin 2\alpha=2\sin\alpha\cos\alpha=2\cdot\frac{2}{\sqrt{13}}\cdot\frac{3}{\sqrt{13}}=\frac{12}{13}.$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=\frac{9}{13}-\frac{4}{13}=\frac{5}{13}.$$

$$\sin\frac{1}{2}\gamma=\sqrt{\frac{1-\cos\gamma}{2}}=\sqrt{\frac{1-\frac{7}{25}}{2}}=\frac{3}{5}.$$

Substituting in (1), we have

$$\tfrac{12}{13}\cdot\tfrac{66}{65}+\tfrac{5}{13}\cdot\tfrac{63}{65}=\tfrac{3}{5},$$

$$\frac{192+315}{845}=\frac{3}{5},$$

$$\tfrac{3}{5}=\tfrac{3}{5}.$$

$$\therefore\quad 2\alpha+\beta=\pi-\tfrac{1}{2}\gamma.$$

$$2\alpha+\beta+\tfrac{1}{2}\gamma=\pi.$$

EXERCISE XX

Prove that

1. $\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{15}{17}$.
2. $\sin^{-1}\frac{5}{13}=\tan^{-1}\frac{5}{12}$.
3. $\cos^{-1}\frac{63}{65}=\csc^{-1}\frac{65}{16}$.
4. $\sin^{-1}\dfrac{3}{5}=2\sin^{-1}\dfrac{1}{\sqrt{10}}$.
5. $\tan^{-1}\frac{4}{3}-\tan^{-1}\frac{1}{7}=\tan^{-1}1$.
6. $\sin^{-1}\frac{16}{65}+\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{3}{5}$.
7. $\sin^{-1}\frac{3}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$.
8. $\tan^{-1}\frac{2}{11}+\cot^{-1}\frac{24}{7}=\tan^{-1}\frac{1}{2}$.
9. $\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{3}=\sin^{-1}\dfrac{1}{\sqrt{2}}$.
10. $\sin^{-1}\dfrac{1}{\sqrt{5}}+\tan^{-1}\dfrac{1}{3}=\cos^{-1}\dfrac{1}{\sqrt{2}}$.
11. $\cot^{-1}\frac{84}{13}+\cot^{-1}\frac{15}{8}=\cot^{-1}\frac{4}{3}$.
12. $2\tan^{-1}\frac{2}{3}=\cot^{-1}\frac{5}{12}$.
13. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{32}{43}$.

14. $\tan^{-1}\frac{2}{9}+\cot^{-1}4=\frac{1}{2}\cos^{-1}\frac{3}{5}$.

15. $\sin^{-1}\dfrac{3}{5}+\cot^{-1}\dfrac{5}{3}-\tan^{-1}\dfrac{8}{19}=\dfrac{\pi}{4}$.

16. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{5}{6}+\tan^{-1}\frac{1}{11}$.

17. $\tan^{-1}\frac{4}{5}=\frac{1}{2}\cot^{-1}\frac{9}{40}$.

18. $2\cos^{-1}\frac{12}{13}=\tan^{-1}\frac{120}{119}$.

19. $\cos^{-1}x=2\cos^{-1}\sqrt{\dfrac{1+x}{2}}$.

20. $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}$.

21. $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$.

22. $\tan^{-1}\dfrac{x}{y}-\tan^{-1}\dfrac{x-y}{x+y}=\dfrac{\pi}{4}$.

23. $\sin^{-1}a+\cos^{-1}b=\cos^{-1}(b\sqrt{1-a^2}-a\sqrt{1-b^2})$.

24. $\tan^{-1}\dfrac{a-b}{1+ab}+\tan^{-1}\dfrac{b-c}{1+bc}+\tan^{-1}\dfrac{c-a}{1+ca}=0$.

25. $\sin(2\sin^{-1}a)=2a\sqrt{1-a^2}$.

26. $\sin\left(\cos^{-1}\dfrac{4}{5}\right)=\tan\left(\sin^{-1}\dfrac{3}{\sqrt{34}}\right)$.

27. $\sin(\sin^{-1}a+\sin^{-1}b)=a\sqrt{1-b^2}+b\sqrt{1-a^2}$.

28. $\tan(\tan^{-1}a+\tan^{-1}b)=\dfrac{a+b}{1-ab}$.

29. $\tan(2\tan^{-1}a)=\dfrac{2a}{1-a}$.

30. $\cos(2\tan^{-1}\frac{1}{7})=\sin(4\tan^{-1}\frac{1}{3})$.

88. Solution of equations expressed in the inverse notation. The method of solution of equations that are expressed in terms of inverse functions is best illustrated by means of examples.

Ex. 1. Solve the equation

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\tfrac{8}{31}.$$

Let $\tan^{-1}(x+1)=\alpha,\quad \tan^{-1}(x-1)=\beta,\quad \tan^{-1}\frac{8}{31}=\gamma.$

Then, $\tan\alpha=x+1,\quad \tan\beta=x-1,\quad \tan\gamma=\frac{8}{31}.$

To find what values of x will satisfy the equation

$$\alpha+\beta=\gamma$$

when $\tan\alpha$, $\tan\beta$, and $\tan\gamma$ have the above values, we proceed as follows:

$$\tan(\alpha+\beta)=\tan\gamma.$$

Then the left member $$=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$=\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}$$

$$=\frac{2x}{2-x^2}.$$

Equating this to $\tan\gamma$, we have

$$\frac{2x}{2-x^2}=\frac{8}{31},$$

$$62x=16-8x^2.$$

Solving, we have $$x=\tfrac{1}{4}, \text{ or } -8.$$

The second value is inadmissible as long as we use the principal value of the angles. Therefore, $$x=\tfrac{1}{4}.$$

Ex. 2. Solve the equation

$$\tan^{-1}x+\tan^{-1}(1-x)=2\tan^{-1}\sqrt{x-x^2}.$$

Let $\tan^{-1}x=\alpha$, $\tan^{-1}(1-x)=\beta$, $\tan^{-1}\sqrt{x-x^2}=\gamma$.

Then, $\tan\alpha=x$, $\tan\beta=1-x$, $\tan\gamma=\sqrt{x-x^2}$.

To find what values of x will satisfy the equation we proceed as follows:

$$\tan(\alpha+\beta)=\tan 2\gamma,$$

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{2\tan\gamma}{1-\tan^2\gamma},$$

$$\frac{x+(1-x)}{1-x(1-x)}=\frac{2\sqrt{x-x^2}}{1-x+x^2},$$

$$1=2\sqrt{x-x^2}.$$

Solving, $$x=\tfrac{1}{2}.$$

EXERCISE XXI

Solve the following equations:

1. $\sin^{-1}x=\cos^{-1}x.$
2. $\sin^{-1}x=\cos^{-1}(-x).$
3. $\tan^{-1}x=\cot^{-1}x.$
4. $\tan^{-1}x=\cot^{-1}(-x).$
5. $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}.$
6. $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}.$
7. $\tan^{-1}x+2\tan^{-1}\frac{1}{x}=\frac{2\pi}{3}.$
8. $\cot^{-1}x+\cot^{-1}2x=\frac{3\pi}{4}.$

9. $\tan^{-1}(x+1) - \cot^{-1}\dfrac{1}{x-1} = \tan^{-1}\dfrac{1}{2}.$

10. $\tan^{-1} 2x + \tan^{-1} 3x = \dfrac{3\pi}{4}.$

11. $\cos^{-1} x - \cos^{-1}\sqrt{1-x^2} = \cos^{-1} x\sqrt{3}.$

12. $\sin^{-1}(3x-2) + \cos^{-1} x = \cos^{-1}\sqrt{1-x^2}.$

13. $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}.$

14. $2\tan^{-1}\dfrac{1}{2} + \sin^{-1}\dfrac{4}{5} = \sin^{-1}\dfrac{1}{x}.$

15. $\sin(\cot^{-1}\tfrac{1}{2}) = \tan(\cos^{-1}\sqrt{x}).$

16. $\tan(\cos^{-1} x) = \sin(\cot^{-1}\tfrac{1}{2}).$

17. $\sin^{-1}\dfrac{1}{x} = \sin^{-1}\dfrac{1}{a} + \sin^{-1}\dfrac{1}{b}.$

18. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{2}{\sin x}\right) = \tan^{-1}(\cos x).$

19. $\csc^{-1} x = \sec^{-1}\dfrac{a}{\sqrt{a^2-1}} + \cos^{-1}\dfrac{\sqrt{b^2-1}}{b}.$

20. $\cos^{-1}\dfrac{x^2-1}{x^2+1} + \tan^{-1}\dfrac{2x}{x^2-1} = \dfrac{2\pi}{3}.$

21. $\tan^{-1}\dfrac{x+1}{x-1} + \tan\dfrac{x-1}{x} = \tan^{-1}(-7).$

CHAPTER XI

THE GENERAL SOLUTION OF TRIGONOMETRIC EQUATIONS

89. A **trigonometric equation** is an equation in which the unknown quantity or quantities appear in the form of trigonometric functions.

These equations have been used with the utmost freedom in previous chapters, though no formal definition has been given until the present time. They have been used in many different ways, involving one or more functions, one or more angles, and one or more values of the given angles in any single equation.

At first the only angles used were acute angles, and an equation was understood to involve functions of an acute angle only. Then the idea was introduced of an angle unrestricted in magnitude; and after this had been done, all results were freed from the restraints which had previously been imposed by the fact that we were dealing with acute angles only.

A large class of the equations with which we have previously been concerned consist of trigonometric identities, that is, equations in which both sides had the same value for all possible values of the angles employed, though the form might be different.

Examples of these are the formulas that have been proved from time to time, as, $\sin^2\theta + \cos^2\theta = 1$; $\sin(x+y) = \sin x \cos y + \cos x \sin y$; etc. Equations of this kind are true for all possible values of the angle or angles involved.

But trigonometric equations are, of course, not ordinarily true for all values of the angles involved. For example, if we consider the equation

$$\cos\theta = \tfrac{1}{2},$$

we see at once that we can assign but two values of θ between 0° and 360° that satisfy this equation. In other words,

$\cos\theta = \frac{1}{2}$ is true only for $\theta = 60°$ and $\theta = 300°$, as long as θ is restricted to values between $0°$ and $360°$. If angles of unrestricted magnitude are allowed, $\cos\theta = \frac{1}{2}$ is satisfied by all values of θ that are included in the general expression

$$\theta = 2n\pi \pm \frac{\pi}{3},$$

and by no other values.

In like manner, the equation

$$\tan\theta = \tfrac{1}{3}\sqrt{3}$$

is satisfied by all values of θ that are given by the general expression

$$\theta = n\pi + \frac{\pi}{6},$$

and by no other values;

$$\sin\theta = \tfrac{1}{2}\sqrt{2}$$

by those values of θ that are given by the expression

$$\theta = n\pi + (-1)^n \frac{\pi}{4},$$

and by no other values; and so on for other examples that might be given. In all these illustrations it is to be understood that n is any positive or negative integer or zero.

The **solution** of an equation is the determination of the value of the angle or angles that satisfy the equation. In Art. 67, p. 88, a method of solution was given by means of which some of the simpler forms of trigonometric equations could be treated. But at that time only a limited number of the formulas of transformation were at our disposal. Hence, the number of classes of equations that could be handled was necessarily quite limited.

The methods of reduction and transformation that are now available make it possible to solve many classes of equations that were formerly quite out of our reach, and also to simplify some of the methods previously employed. The present chapter will illustrate some of the simpler cases of this kind.

This work should be looked upon as an extension of that given in Art. 68, p. 90.

90. Solution of equations of the form

$$a \cos \theta + b \sin \theta = c. \tag{1}$$

A simple method of solving equations of this form is furnished by the introduction of what are termed **auxiliary angles,** as follows:

Assume a right triangle whose legs are a, b, and designate by ϕ the angle lying opposite the leg b. The hypotenuse of this right triangle is $\sqrt{a^2+b^2}$, and we now have

$$\cos \phi = \frac{a}{\sqrt{a^2+b^2}}, \text{ and } \sin \phi = \frac{b}{\sqrt{a^2+b^2}}.$$

Dividing each member of the original equation by $\sqrt{a^2+b^2}$, we have

$$\frac{a}{\sqrt{a^2+b^2}} \cos \theta + \frac{b}{\sqrt{a^2+b^2}} \sin \theta = \frac{c}{\sqrt{a^2+b^2}}. \tag{2}$$

Substituting $\cos \phi$ and $\sin \phi$ for their respective values in this equation we have

$$\cos \phi \cos \theta + \sin \phi \sin \theta = \frac{c}{\sqrt{a^2+b^2}},$$

or,

$$\cos (\theta - \phi) = \frac{c}{\sqrt{a^2+b^2}}.$$

Since a, b, and c are known, $\cos (\theta - \phi)$ is known, and $\theta - \phi$ can at once be found from the tables. Calling this angle α, for convenience we have

$$\cos (\theta - \phi) = \cos \alpha.$$

$$\therefore \quad \theta - \phi = 2 n\pi \pm \alpha, \text{ Art. 64, p. 87.}$$

$$\theta = 2 n\pi + \phi \pm \alpha.$$

The cosine of an angle can never be numerically greater than unity. Hence, in dealing with the equation $\cos(\theta - \phi) = \frac{c}{\sqrt{a^2+b^2}}$ it is to be remembered that we must have $c \overline{\leqq} \sqrt{a^2+b^2}$. If $c > \sqrt{a^2+b^2}$, there is no real value of $\theta - \phi$ which will satisfy the equation.

Ex. 1. Solve the equation $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$.

Dividing both sides of the equation by $\sqrt{3+1}$, *i.e.* by 2, we have

$$\tfrac{1}{2}\sqrt{3}\cos\theta + \tfrac{1}{2}\sin\theta = \tfrac{1}{2}\sqrt{2}.$$

In this case we have $a = \sqrt{3}$, $b = 1$, and $\sqrt{a^2 + b^2} = 2$. Hence, the auxiliary angle ϕ is equal to 30°. The original equation then becomes

$$\cos 30^\circ \cos\theta + \sin 30^\circ \sin\theta = \tfrac{1}{2}\sqrt{2}.$$

$$\cos(\theta - 30^\circ) = \tfrac{1}{2}\sqrt{2}.$$

But $\frac{1}{2}\sqrt{2}$ is the cosine of 45°. Hence, we write

$$\cos(\theta - 30^\circ) = \cos 45^\circ.$$

$$\therefore\ \theta - 30^\circ = 2n\pi \pm \frac{\pi}{4},$$

$$\theta = 2n\pi + \frac{\pi}{6} \pm \frac{\pi}{4}.$$

Ex. 2. Solve the equation $5\cos\theta + 2\sin\theta = 4$.

In this problem we have $a = 5$, and $b = 2$. Dividing both sides of the equation by $\sqrt{a^2 + b^2}$, we have

$$\frac{5}{\sqrt{29}}\cos\theta + \frac{2}{\sqrt{29}}\sin\theta = \frac{4}{\sqrt{29}}. \qquad (1)$$

In the preceding example we were able to find the value of ϕ from the familiar coefficients $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$, which we already knew were the cosine and sine respectively of 30°. But in this example we have unfamiliar values to consider.

From the figure on the margin of the page we see that ϕ is an angle whose cotangent is $\frac{2}{5}$. Turning to the tables, we find that the value of ϕ is 68° 12′; and (1) can now be written

$$\cos 68^\circ 12' \cos\theta + \sin 68^\circ 12' \sin\theta = \frac{4}{\sqrt{29}}.$$

Letting α equal the angle whose cosine is $\frac{4}{\sqrt{29}}$ this becomes

$$\cos(\theta - 68^\circ 12') = \cos\alpha.$$

Reducing the value of $\frac{4}{\sqrt{29}}$ to a decimal, we find it to be 0.7428; and, consulting the tables, we find that the angle whose cosine is 0.7428 is 42° 2′. Therefore,

$$\cos(\theta - 68^\circ 12') = \cos 42^\circ 2',$$

$$\theta - 68^\circ 12' = 2n\pi \pm 42^\circ 2',$$

$$\theta = 2n\pi + 68^\circ 12' \pm 42^\circ 2'.$$

Note. Each of the foregoing examples could have been solved by replacing either $\sin\theta$ or $\cos\theta$ by its value in terms of the other, then obtaining the value of the single function involved, and finally obtaining the value of θ from the value of this function. But the process just explained is much simpler and better.

91. Solution of equations involving multiple angles. The simplest forms of equations involving multiple angles have already been considered (Art. 68, p. 90). But these, and also many other forms of equations in which multiple angles appear, are more conveniently treated by means of the various reduction formulas that are now available.

The following problems will illustrate some of the methods of most frequent application.

Ex. 1. Solve the equation $\sin 3\theta + \sin 7\theta = \sin 5\theta$.

By (5), Art. 77, p. 100, we have

$$2 \sin 5\theta \cos 2\theta = \sin 5\theta.$$

$$\therefore \sin 5\theta = 0, \text{ or } \cos 2\theta = \tfrac{1}{2}.$$

If $\sin 5\theta = 0$, then $5\theta = n\pi$.

$$\therefore \theta = \frac{n\pi}{5}.$$

If $\cos 2\theta = \frac{1}{2}$, then $2\theta = 2n\pi \pm \frac{\pi}{3}$.

$$\therefore \theta = n\pi \pm \frac{\pi}{6}.$$

Therefore, the general values of θ that satisfy the equation

$$\sin 3\theta + \sin 7\theta = \sin 5\theta,$$

are $\theta = \frac{n\pi}{5}$, and $\theta = n\pi \pm \frac{\pi}{6}$.

Ex. 2. Solve the equation $\cos 4\theta - \cos 6\theta - \sin 2\theta = 0$.

Applying the proper reduction formulas, we have

$$2 \sin 5\theta \sin \theta - 2 \sin \theta \cos \theta = 0.$$

$$\therefore \sin \theta (\sin 5\theta - \cos \theta) = 0.$$

From the first factor we have

$$\sin \theta = 0.$$

$$\therefore \theta = n\pi. \qquad (1)$$

From the second factor we have

$$\cos \theta = \sin 5\theta$$

$$= \cos\left(\frac{\pi}{2} - 5\theta\right).$$

$$\therefore \theta = 2n\pi \pm \left(\frac{\pi}{2} - 5\theta\right).$$

Using the positive sign, we have

$$6\theta = 2n\pi + \frac{\pi}{2}.$$

$$\therefore \theta = \frac{n\pi}{3} + \frac{\pi}{12}. \qquad (2)$$

Using the negative sign, we have

$$-4\theta = 2n\pi - \frac{\pi}{2},$$

$$4\theta = 2n\pi + \frac{\pi}{2},$$

[the sign of n being left unchanged because n denotes all negative as well as all positive integers]

$$\theta = \frac{n\pi}{2} + \frac{\pi}{8}. \qquad (3)$$

Collecting the values given in (1), (2), and (3), we have as the general values of θ that satisfy the given equation

$$\theta = n\pi,\ \theta = \frac{n\pi}{3} + \frac{\pi}{12},\ \text{and } \theta = \frac{n\pi}{2} + \frac{\pi}{8}.$$

Ex. 3. Solve the equation $\cos 2x = \cos x + \sin x$.

Expressing $\cos 2x$ in terms of functions of x we have

$$\cos^2 x - \sin^2 x = \cos x + \sin x;$$

$$(\cos x + \sin x)(\cos x - \sin x) = \cos x + \sin x.$$

$$\therefore \cos x + \sin x = 0, \qquad (1)$$

or,

$$\cos x - \sin x = 1. \qquad (2)$$

From (1) we have $\tan x = -1.$

$$\therefore x = n\pi - \frac{\pi}{4}.$$

From (2) we have

$$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}},$$

or,

$$\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x = \cos\frac{\pi}{4},$$

$$\cos\left(x + \frac{\pi}{4}\right) = \cos\frac{\pi}{4}.$$

$$\therefore x + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}.$$

$$x = 2n\pi, \text{ or } x = 2n\pi - \frac{\pi}{2}.$$

EXERCISE XXII

Solve the following equations:

1. $\cos x - \sqrt{3}\sin x = 1.$
2. $\sin x - \sqrt{3}\cos x = 1.$
3. $\sin\theta + \sqrt{3}\cos\theta = \sqrt{2}.$
4. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}.$
5. $\sin\theta + \cos\theta = \sqrt{2}.$
6. $\cos\alpha - \sin\alpha = \frac{1}{2}\sqrt{2}.$
7. $\cos\alpha + \sin\alpha = -\sqrt{2}.$
8. $\sin m\theta + \sin n\theta = 0.$
9. $\cos m\theta + \cos n\theta = 0.$
10. $3\sin x + 2\cos x = 2.$
11. $6\cos\theta - 3\sin\theta = 3.$
12. $4\cos\theta - 3\sin\theta = 5.$

13. $\sin 7x - \sin 4x + \sin x = 0.$

14. $\sin 5x - \sin 3x + \sin x = 0.$

15. $\sin 7x - \sin x = \sin 3x.$

16. $\sin 4x - \sin 2x = \cos 3x.$

17. $\cos\theta + \cos 2\theta + \cos 3\theta = 0.$

18. $\sin\theta + \sin 2\theta + \sin 3\theta = 0.$

19. $\cos 7\theta - \cos\theta = -\sin 4\theta.$

20. $\cos 2\theta - \cos\theta - \sin 2\theta + \sin\theta = 0.$

21. $\sin 4\theta - \sin 3\theta + \sin 2\theta - \sin\theta = 0.$

22. $\cos 7\theta + \cos 5\theta + \cos 3\theta + \cos\theta = 0.$

23. $2\cos 2\theta = \cos 3\theta + \sin\theta.$

24. $\cos k\partial - \cos(k-2)\theta = \sin\theta.$

25. $\sin 5\theta\cos\theta - \sin 6\theta\cos 2\theta = 0.$

26. $\sin\frac{k+1}{2}\theta - \sin\frac{k-1}{2}\theta = \sin\theta.$

27. $\cos 3\theta + 2\cos\theta = 0.$

28. $\cos 2\theta + \sin 3\theta = 0.$

29. $\cos 5\theta + \cos\theta = \sqrt{2}\cos 3\theta.$

30. $\sin\theta + \sqrt{3}\cos 4\theta = \sin 7\theta.$

31. $\cos 2\theta - \cos^2\theta = 0.$

32. $\cos 3\theta + 8\cos^3\theta = 0.$

33. $\sin 3\theta - 8\sin^3\theta = 0.$

34. $\sin 2\theta + 3\sin\theta = 0.$

35. $\csc\theta - \cot\theta = \sqrt{3}.$

36. $\cot\theta - \tan\theta = 2.$

37. $\sec\theta - \csc\theta = 2\sqrt{2}.$

38. $\cot 2\theta - \cot\theta = -2.$

39. $\sec 4\theta - \sec 2\theta = 2.$

40. $\sec\theta + \csc\theta = 2\sqrt{2}.$

41. $\tan 3\theta + \tan 2\theta + \tan\theta = 0.$

42. $\tan 3\theta - \tan 2\theta - \tan\theta = 0.$

43. $\tan 3\theta + \tan\theta = 2\tan 2\theta.$

44. $\sin 5\theta\cos\theta - \sin 4\theta\cos 2\theta = 0.$

92. Changes in sign and magnitude of the expression $a\cos x + b\sin x$. In connection with the solution of equations of the form

$$a\cos x + b\sin x = 0,$$

it is often useful to trace the changes in sign and magnitude of the left member of the equation as x increases from 0° to 360°.

The simplest case occurs when $a=1$ and $b=1$; in which case we have simply $\sin x + \cos x$ to examine. Proceeding as in Art. 90 we have

$$\cos x + \sin x = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]$$

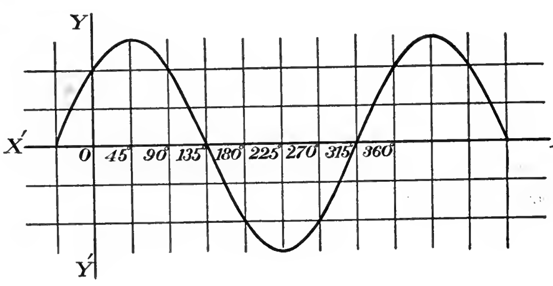
$$= \sqrt{2}(\sin x \cos 45° + \cos x \sin 45°)$$

$$= \sqrt{2}\sin(x + 45°).$$

For convenience we replace $\cos x + \sin x$ by y, and then, forming the equation $y = \sqrt{2}\sin(x+45°)$, we form the following table of values.

Plotting the graph by the method explained in Art. 48, we have the following result.

x	y
0°	1
45°	$\sqrt{2}$
90°	1
135°	0
180°	-1
225°	$-\sqrt{2}$
270°	-1
315°	0
360°	1



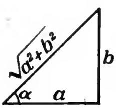
Since the greatest value that the sine of any angle can have is 1, the maximum value of this expression occurs when $\sin(x+45°)=1$, *i.e.* when $x=45°$. This gives $\sqrt{2}$ as the maximum value of the expression $\sin x + \cos x$.

In like manner, the minimum value of the expression is $-\sqrt{2}$, which corresponds to the angle $x = 225°$.

If the table of values is extended, and the graph is plotted for values of x greater than 360°, the values of y, *i.e.* of $\cos x + \sin x$, will be repeated in their original order; that is, $\cos x + \sin x$ is a periodic function with a period of 360°. (See Art. 49, p. 71.)

93. When a or b, or both a and b, are different from unity, the process is slightly modified, as follows:

$$a \cos x + b \sin x = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right)$$
$$= \sqrt{a^2+b^2}\,(\cos x \cos \alpha + \sin x \sin \alpha)$$
$$= \sqrt{a^2+b^2}\cos(x-\alpha).$$



Here, as is readily seen from the figure on the margin of the page, it has been assumed that α is the angle whose cosine is $\frac{a}{\sqrt{a^2+b^2}}$ and whose sine is $\frac{b}{\sqrt{a^2+b^2}}$. When a and b are known, α can be found, as in Art. 90, p. 124.

The table of values can then be obtained and the graph constructed, as in the preceding case.

Since $\cos(x-\alpha)$ has 1 for its maximum value and -1 for its minimum value, the expression $a \cos x + b \sin x$ has $\sqrt{a^2+b^2}$ for its maximum value and $-\sqrt{a^2-b^2}$ for its minimum value.

NOTE. In computing the table of values for the purpose of constructing the graph, the values of y can always be obtained directly from the expression as it is originally given, without any reduction whatever. This is sometimes preferable; and in certain cases, as for example the functions given in Examples 7, 9, 10, and 11 in the following set, it is easier to compute the values directly than to compute them after transforming the expression.

EXERCISE XXIII

Trace the changes in sign and magnitude of the following expressions as x increases from 0° to 360°. Find the period and construct the graph in each case.

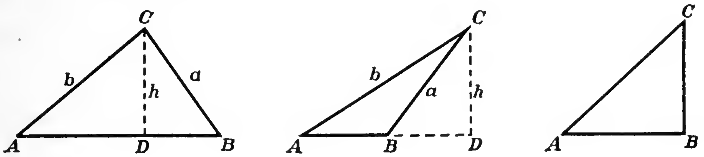
1. $\sin x - \cos x$.
2. $\sqrt{3}\sin x + \cos x$.
3. $\sin x + \sqrt{3}\cos x$.
4. $\sqrt{3}\sin x - \cos x$.
5. $\sin x + \sqrt{3}\cos x$.
6. $2\sin x + 3\cos x$.
7. $\cos 2\theta$.
8. $\sin\theta\cos\theta$.
9. $\cos 3\theta$.
10. $\sin 3\theta$.
11. $\tan 2\theta$.
12. $\dfrac{\sin 2\theta - \sin\theta}{\cos 2\theta + \cos\theta}$.

CHAPTER XII

THE OBLIQUE TRIANGLE

94. The law of sines. Let A, B, C denote the angles of a triangle, and a, b, c respectively the sides opposite.

From any vertex, as C, draw CD perpendicular to AB, meeting AB, or AB produced, in D.



From the first figure we have

$$\frac{h}{b} = \sin A.$$

$$\therefore\ h = b \sin A.$$

Also,
$$\frac{h}{a} = \sin B.$$

$$\therefore\ h = a \sin B.$$

Equating these values of h we have

$$b \sin A = a \sin B.$$

From the second figure we have

$$\frac{h}{b} = \sin A.$$

$$\therefore\ h = b \sin A.$$

Also,
$$\frac{h}{a} = \sin (180° - B) = \sin B,$$

whence as before,
$$b \sin A = a \sin B,$$

Therefore in either case we have the same result,

$$b \sin A = a \sin B;$$

i.e.
$$\frac{a}{b} = \frac{\sin A}{\sin B}. \qquad (1)$$

In like manner drawing perpendiculars from the vertices A and B to the opposite sides respectively we can prove that

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \qquad (2)$$

and
$$\frac{a}{c} = \frac{\sin A}{\sin C}. \qquad (3)$$

The results obtained in (1), (2), and (3) enable us to state the law of sines as follows:

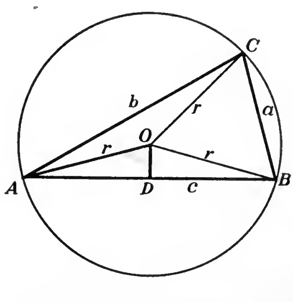
The sides of a triangle are proportional to the sines of the opposite angles.

Equations (1), (2), and (3) are often combined and written in the following manner:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad (4)$$

95. The geometric meaning of each of the three ratios just stated will be understood from the following:

Let ABC be any triangle, and let a circle be circumscribed about the triangle. From the center O to the vertices of the triangle draw the radii OA, OB, OC, respectively, and also draw OD perpendicular to AB.



By geometry

$$\angle AOB = 2\angle C.$$

$$\therefore \angle AOD = \angle C.$$

From this we have

$$AD = r \sin \angle AOD,$$
$$= r \sin C.$$
$$\therefore c = 2r \sin C.$$

In like manner it can be proved that

$$a = 2r \sin A,$$

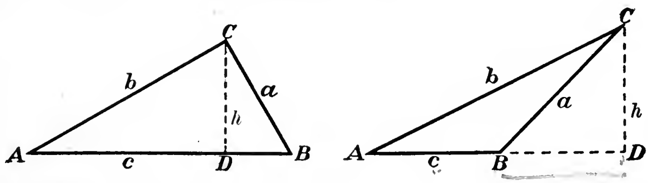
and
$$b = 2r \sin B.$$

Equating the values of $2r$ obtained from these three equations we have

$$2r = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$ That is,

The ratio of any side of a triangle to the sine of the opposite angle is equal to the diameter of the circumscribed circle.

96. The law of cosines. Let ABC be any triangle, and let CD, the perpendicular from the vertex C to the opposite side, meet AB, produced if necessary, in D.



From the first figure we have

$$\begin{aligned} a^2 &= h^2 + \overline{BD}^2 \\ &= h^2 + (c - AD)^2 \\ &= h^2 + c^2 - 2c \cdot AD + \overline{AD}^2 \\ &= (h^2 + AD^2) + c^2 - 2c \cdot AD \\ &= b^2 + c^2 - 2c \cdot b\cos A. \end{aligned}$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From the second figure we have

$$\begin{aligned} a^2 &= h^2 + BD^2 \\ &= h^2 + (AD - c)^2 \\ &= h^2 + AD^2 - 2c \cdot AD + c^2 \\ &= b^2 + c^2 - 2c \cdot b\cos A. \end{aligned}$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}. \qquad (1)$$

Therefore, the same result is obtained for both triangles.

In like manner, drawing perpendiculars from A and B to the opposite sides respectively, we can prove that

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}, \qquad (2)$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}. \qquad (3)$$

Equations (1), (2), and (3) are often useful when expressed in the following form:

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A, \\ b^2 &= c^2 + a^2 - 2ca\cos B, \\ c^2 &= a^2 + b^2 - 2ab\cos C. \end{aligned}\right\} \qquad (4)$$

The law of cosines can now be stated as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice their product into the cosine of the included angle.

97. The law of tangents. We have already proved that, in any triangle, $\frac{a}{b} = \frac{\sin A}{\sin B}$.

Therefore, considering this equation as a proportion, and taking the four quantities by division and composition,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$$

$$= \cot\frac{A+B}{2}\tan\frac{A-B}{2}.$$

$$\therefore \quad \frac{a-b}{a+b} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}. \qquad (1)$$

In like manner it can be proved that

$$\frac{a-c}{a+c} = \frac{\tan\frac{A-C}{2}}{\tan\frac{A+C}{2}}, \qquad (2)$$

and

$$\frac{b-c}{b+c} = \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}. \qquad (3)$$

The law of tangents can now be stated as follows:

The difference of two sides of a triangle is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

NOTE. In using the formulas of this section it is better to let the greater side and the greater angle precede the smaller in all cases. The formulas are true, whichever order is used; but if the smaller side and the smaller angle precede the greater side and the greater angle respectively, negative numbers are introduced, and if logarithms are to be employed, these numbers should be avoided whenever it is possible to do so.

98. The given parts. In the solution of oblique plane triangles four cases occur. In each case three parts are given, as follows:

1. One side and two angles.
2. Two sides and the angle opposite one of them.
3. Two sides and the included angle.
4. Three sides.

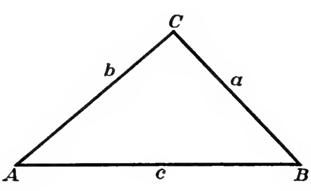
The formulas developed in Arts. 94–97 are sufficient for the solution of every possible case that can arise. These cases will now be considered separately.

99. CASE 1. **Given one side and two angles.** Let the given angles be A and B, and the given side a. The formulas for solution are as follows:

1. $C = 180° - (A + B)$.

2. $\dfrac{b}{a} = \dfrac{\sin B}{\sin A}$, $\quad \therefore b = \dfrac{a \sin B}{\sin A}$.

3. $\dfrac{c}{a} = \dfrac{\sin C}{\sin A}$, $\quad \therefore c = \dfrac{a \sin C}{\sin A}$.



Ex. 1. Given $a = 467$, $A = 56° 28'$, $B = 69° 14'$; find the remaining parts.

The work may be conveniently arranged as follows:

$$C = 180° - (A + B) = 54° 18'.$$

(1) By natural functions.

$$b = a \times \sin B \div \sin A = 467 \times 0.9350 \div 0.8336 = 523.8.$$

$$c = a \times \sin C \div \sin A = 467 \times 0.8121 \div 0.8336 = 454.95.$$

(2) By logarithms.

$$\log b = \log a + \log \sin B - \log \sin A$$
$$= \log a + \log \sin B + \operatorname{colog} \sin A.$$
$$\log c = \log a + \log \sin C - \log \sin A$$
$$= \log a + \log \sin C + \operatorname{colog} \sin A.$$

$\log a =$	2.66932	$\log a =$	2.66932
$\log \sin B =$	9.97083 − 10	$\log \sin C =$	9.90960 − 10
$\operatorname{colog} \sin A =$	0.07906	$\operatorname{colog} \sin A =$	0.07906
	2.71921		2.65798
$b =$	523.85	$c =$	454.97

NOTE. To insure accuracy the student should check all results by solving each problem by a second method, entirely independent of the first; or by the same method, using a different combination of parts. In the case under consideration it is usually sufficient to employ the same method, *i.e.* the law of sines, combining the parts in a manner different from that employed in the first place. For example, after c has been found we can solve again for b by the formula $b = \frac{c \sin B}{\sin C}$, as follows:

$$\log c = 2.65798$$
$$\log \sin B = 9.97083 - 10$$
$$\operatorname{colog} \sin C = 0.09040$$
$$\log b = 2.71921 \qquad b = 523.85 \textit{ check.}$$

EXERCISE XXIV

Solve the following triangles:

1. Given $a = 438.3$, $A = 43° \, 50' \, 24''$, $B = 69° \, 30' \, 12''$.
   Ans. $C = 66° \, 39' \, 24''$, $b = 592.74$, $c = 580.$[illegible].

2. Given $b = 421$, $A = 31° \, 12'$, $B = 36° \, 20'$.
   Ans. $C = 112° \, 28'$, $a = 368.08$, $c = 656.63$.

3. Given $a = 412$, $A = 58° \, 14'$, $B = 65° \, 37'$.
   Ans. $C = 56° \, 9'$, $b = 441.37$, $c = 402.45$.

4. Given $b = 81.5$, $B = 43° \, 44' \, 18''$, $C = 75° \, 2' \, 42''$.
   Ans. $A = 61° \, 13'$, $a = 103.32$, $c = 113.89$.

5. Given $c = 77.93$, $B = 22° \, 15' \, 20''$, $C = 41° \, 50' \, 30''$.
   Ans. $A = 115° \, 53' \, 10''$, $a = 105.07$, $b = 44.23$.

6. Given $c = 6.98$, $A = 25° \, 7' \, 10''$, $C = 36° \, 12' \, 24''$.
   Ans. $B = 118° \, 40' \, 30''$, $A = 5.016$, $b = 10.37$.

7. Given $a = 928.4$, $A = 61° 17' 15''$, $C = 58° 18' 40''$.
Ans. $B = 60° 24' 5''$, $c = 900.78$, $b = 920.45$.

8. Given $a = 328.4$, $A = 29° 41' 12''$, $B = 37° 50' 24''$.
Ans. $C = 112° 28' 24''$, $b = 406.77$, $c = 612.73$.

9. Given $A = 64° 35'$, $C = 73° 49'$, $a = 213.47$.
Ans. $B = 41° 36'$, $b = 156.92$, $c = 226.98$.

10. Given $A = 41° 23' 47''$, $B = 124° 49'$, $b = 65.536$.
Ans. $C = 13° 47' 13''$, $a = 52.788$, $c = 19.023$.

11. Two points, A and B, are separated by a body of water. In order to find the distance between them a line AC is measured 612.3 ft. in length, and the angles BAC, ACB are measured and are found to be 49° 17′ and 68° 11′ respectively. What is the distance from A to B?

12. It is desired to find the distance of a lighthouse A to each of two stations B, C, situated on shore, and in the same horizontal plane with the base of the lighthouse. BC is $2\frac{1}{4}$ miles, $\angle ABC$ is 39° 38′, and $\angle ACB$ is 74° 56′. Find AB and AC.

13. The angles of elevation of a balloon that has ascended vertically between two stations one mile apart on a level plain, and in the same vertical plane with the balloon, are 29° 41′ and 37° 17′ respectively. What is the distance of the balloon from each station, and what is its vertical height above the plain?

14. Solve the preceding problem on the supposition that both the stations are on the same side of the balloon.

15. To find the width of a stream a line AB, 351 ft. long, is measured on one side, parallel to the bank. On the opposite side of the stream a stake C is set, and the angles CAB, CBA, are observed and are found to be 38° 17′ and 31° 29′ respectively. What is the width of the stream?

16. From the top and bottom of a column the angles of elevation of the top of a tower 236 ft. high standing on the same horizontal plane are 44° 27′ and 61° 31′ respectively. What is the height of the column?

17. When the altitude of the sun is 49° 52′, a pole standing on the slope of a hill inclined 16° 55′ to the level of the plain casts a shadow directly down the hill a distance of 45 ft. 8 in. What is the height of the pole?

18. An observer in a balloon measures the angle of depression of an object on the ground and finds it to be 63° 58′. After ascending vertically 582 ft. he finds the angle of depression of the same object 74° 49′. What was the height of the balloon at the time of the first observation?

19. From a ship the bearings of two objects were found to be N.N.W. and N.E. by N., respectively. After sailing due east 10 miles the two objects were in a line bearing W.N.W. How far apart were the objects?

NOTE. For an explanation of the term "bearing," and for instruction in reading angles by means of the compass, see p. 176.

20. From a ship a lighthouse bears N. 21° 12′ E. After the ship has sailed S. 25° 12′ E. $2\frac{1}{4}$ miles the lighthouse bears due north. Find the distance of the lighthouse from the last point of observation.

100. CASE 2. **Given two sides and the angle opposite one of them.** Let the given parts be the sides a and b, and the angle A. The required parts can be found in the following manner:

By the law of sines

$$\frac{\sin B}{\sin A}=\frac{b}{a}, \qquad \therefore \sin B=\frac{b \sin A}{a}. \tag{1}$$

From this equation the angle B can be found.

Then, $$C=180°-(A+B).$$

Also, $$\frac{c}{a}=\frac{\sin C}{\sin A}, \qquad \therefore c=\frac{a \sin C}{\sin A}. \tag{2}$$

In solving for the angle opposite the second side, in this case the angle B, it is to be noted that two solutions are possible, since the sines of supplementary angles are equal (Art. 53, p. 79).

The following considerations will determine the number of solutions for any given set of conditions.

If $a > b$, then $A > B$, and B is necessarily an acute angle, since a triangle can have but one obtuse angle. Therefore there is one and only one solution.

If $a = b$, then $A = B$, and both A and B are acute angles. Therefore there is one and only one solution, an isosceles triangle.

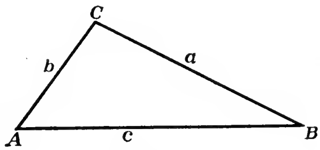


FIG. 1.
One solution, $a > b$

If $a < b$, then $A < B$, and A is an acute angle. In this case B may be either acute or obtuse, and there will be two solutions if $a > CD$, the perpendicular drawn from the vertex C to AB, produced if necessary. That is, either of the two triangles AB_1C, AB_2C, will satisfy the given conditions. But the perpendicular $CD = b \sin A$. Therefore, if A is acute and $a < b$, and

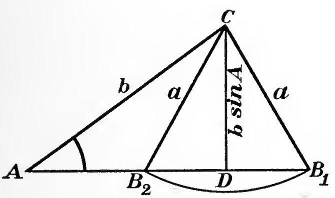


FIG. 2.
Two solutions, $a > b \sin A$

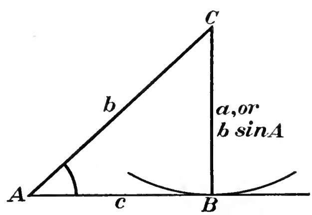


FIG. 3.
One solution, $a = b \sin A$

if $a > b \sin A$, there are two solutions. The angles AB_1C, AB_2C, are supplementary, since $\angle AB_1C = \angle B_1B_2C$. Both angles are given by the formula

$$\sin B = \frac{b \sin A}{a}.$$

If $a = b \sin A$, that is, if a is equal to the perpendicular CD, *there is but one solution, a right triangle.* This is also seen from the fact that when $a = b \sin A$, the value of $\sin B$ reduces to unity. This gives $B = 90°$.

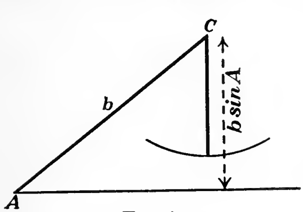


FIG. 4.
No solution, $a < b \sin A$

If $a < b \sin A$, that is, if a is less than the perpendicular CD, *there is no solution, and the triangle is impossible.* This is also seen from the fact that when $a < b \sin A$, the fraction $\frac{b \sin A}{a}$ is greater than unity. But this fraction is in all cases equal to sin B; and as the sine of an angle can never exceed unity the triangle is therefore impossible.

These results may be summarized as follows:

Two solutions.

A acute, $a < b$, and $a > b \sin A$.

One solution.

(*a*) A obtuse and $a > b$.

(*b*) A acute and $a = b \sin A$.

(*c*) A acute and $a > b$.

No solution.

(*a*) A acute and $a < b \sin A$.

(*b*) A obtuse and $a = b$ or $a < b$.

To determine the number of solutions, first note whether A is acute or obtuse. Then, on examining the different cases just studied, it is seen that there can never be more than one solution *unless A is acute and the side opposite A is less than the side adjacent.* In this case there may be two solutions, one solution, or no solution.

The comparison between a and $b \sin A$ is often most conveniently made by means of the natural value of sin A. In many cases the computation can be performed mentally; for all that is now desired is to determine whether a is less than, equal to, or greater than $b \sin A$.

If logarithms are used, we compute log sin B. The results are as follows.

(*a*) $\log \sin B > 0$, no solution.

(*b*) $\log \sin B = 0$, one solution, a right triangle.

(*c*) $\log \sin B < 0$, one solution if $a > b$, and two solutions if $a < b$ and A is acute.

The student should bear in mind that the given parts are not necessarily a, b, and A; they may be any two sides and the angle opposite one of them. If other parts are given than those mentioned above, the proper modifications should be made in the formulas for determining the number of solutions.

Ex. **1.** Given $a = 26$, $b = 72$, $A = 30°$; find the remaining parts.

Since $\sin A = \frac{1}{2}$, we have $b \sin A = 36$. Hence, the triangle is impossible as $a < 36$.

Ex. 2. Given $a=88$, $b=103$, $A=120°$; find the remaining parts.

Here A is obtuse and $a<b$; therefore the triangle is impossible.

Ex. 3. Given $a=738.4$, $b=1185.7$, $A=79°\ 38'\ 40''$; find the remaining parts.

Solving by logarithms we proceed as follows:

$$\sin B=\frac{b\sin A}{a},$$

$$\begin{aligned}\log b &= 3.07397\\ \log\sin A &= 9.99287-10\\ \text{colog}\, a &= 7.13171-10\\ \hline \log\sin B &= 10.19855-10\end{aligned}$$

Since $\log\sin B>0$, there is no solution.

Ex. 4. Given $a=28.2$, $c=45.65$, $A=38°\ 7'\ 7.5''$; find the remaining parts.

Proceeding as in Ex. 3 we have

$$\sin C=\frac{c\sin A}{a},$$

$$\begin{aligned}\log c &= 1.65944\\ \log\sin A &= 9.79081-10\\ \text{colog}\, a &= 8.54975-10\\ \hline \log\sin C &= 10.00000-10\end{aligned}$$

$\therefore C=90°$, and the triangle is a right triangle.

Solving for B and b by the usual methods employed in the case of right triangles (Arts. 26 and 27, pp. 36–38), we find $B=51°\,50'\,52.5''$, $b=35.998$.

Ex. 5. Given $a=67.53$, $b=56.82$, $A=77°\ 14'\ 19''$; find the remaining parts.

Here $a>b$ and A is acute; therefore there is but one solution. The unknown parts are found in the following manner:

$$\begin{aligned}\log b &= 1.75450\\ \log\sin A &= 9.98914-10\\ \text{colog}\, a &= 8.17050-10\\ \hline \log\sin B &= 9.91414-10\\ \therefore B &= 55°\,8'\,47''.\end{aligned}$$

$$\begin{aligned}C &= 180°-(A+B)\\ &= 46°\,36'\,54''\end{aligned}$$

$$\begin{aligned}\log b &= 1.75450\\ \log\sin C &= 9.86843-10\\ \text{colog}\sin A &= 0.08586\\ \hline \log c &= 1.70879\\ \therefore c &= 51.143.\end{aligned}$$

Check:

$$\begin{aligned}\log a &= 1.82950\\ \log\sin C &= 9.86843-10\\ \text{colog}\sin A &= 0.01086\\ \hline \log c &= 1.70879\\ c &= 51.143\end{aligned}$$

Ex. 6. Given $a = 168.32$, $b = 221.46$, $A = 33° 39' 16''$; find the remaining parts.

In this case the simplest method of finding the number of solutions is to obtain the value of $b \sin A$ by multiplying the value of b, 221.46, by the natural value of $\sin A$, and comparing the result with 168.32, the value of a. The sine of $33° 39' 16''$ is approximately 0.55. Hence, it is seen at a glance that $b \sin A$ is a trifle over one half of 221.46; that is, much less than a Hence, since A is acute and $a < b$, there are two solutions.

The work of computation, exhibited in compact form, is as follows:

$$\begin{array}{rl} \log b &= 2.34529 \\ \log \sin A &= 9.74365 - 10 \\ \text{colog } a &= \underline{7.77387 - 10} \\ \log \sin B &= 9.86281 - 10 \\ \therefore B_1 &= 46° 48' 50'', \\ B_2 &= 133° 11' 10''. \\ \therefore C &= 99° 31' 54'', \text{ or, } 13° 9' 34''. \end{array}
\qquad
\begin{array}{rl|l} \log a &= 2.22613 & 2.22613 \\ \log \sin C &= 9.99396 - 10 & 9.35729 - 10 \\ \text{colog } \sin A &= \underline{0.25635} & \underline{0.25635} \\ \log c &= 2.47644 & 1.83977 \\ \therefore c_1 &= 299.53, & c_2 = 69.147. \end{array}$$

NOTE. The method of checking results is the same as that used in connection with Case 1. In Ex. 5 above the check work for c is given. After a little practice this work can be performed with great rapidity. Every result obtained by the student should be subjected to a check of some kind.

EXERCISE XXV

1. Determine the number of solutions in each of the following cases:

(1)	$a = 30$,	$b = 60$,	$A = 30°$.
(2)	$a = 20$,	$b = 60$,	$A = 30°$.
(3)	$a = 40$,	$b = 60$,	$A = 30°$.
(4)	$a = 750$,	$b = 638$,	$A = 69° 30'$.
(5)	$a = 38.8$,	$b = 45.5$,	$A = 60°$.
(6)	$a = 226$,	$b = 196$,	$A = 123° 40'$.

2. Given $a = 109.68$, $c = 467$, $A = 13° 35'$;
find $C = 90°$, $B = 76° 25'$, $b = 453.94$.

3. Given $a = 392$, $b = 124$, $A = 36° 41' 42''$;
find $B = 10° 53' 45''$ $C = 132° 24' 33''$ $c = 484.37$.

4. Given $a = 168.2$, $b = 218.6$, $A = 34° 22' 50''$;
find $B_1 = 47° 12' 49''$, $C_1 = 98° 24' 21''$, $c_1 = 294.67$,
$B_2 = 132° 47' 11''$; $C_2 = 12° 49' 59''$, $c_2 = 66.16$.

5. Given $b=8472.2$, $c=3211.7$, $C=16°\,33'\,45''$;
find $A_1=114°\,40'\,42''$, $B_1=48°\,45'\,33''$, $a_1=10238$.
$A_2=32°\,11'\,48''$, $B_2=131°\,14'\,27''$, $a_2=6003.4$.

6. Given $a=506$, $b=432$, $A=36°\,7'\,12''$;
find $B=30°\,13'$, $C=113°\,39'\,48''$, $c=786.22$.

7. Given $a=36.27$, $b=23.96$, $B=30°\,26'\,14''$;
find $A_1=50°\,4'\,24''$, $C_1=99°\,29'\,22''$, $c_1=46.65$,
$A_2=129°\,55'\,36''$, $C_2=19°\,38'\,10''$, $c_2=15.894$.

8. Given $a=283.4$, $b=268.5$, $A=60°\,40'\,26''$;
find $B=55°\,41'\,23''$, $C=63°\,38'\,11$, $c=291.25$.

9. Given $a=158$, $b=179$, $A=21°\,17'\,22''$;
find $B_1=24°\,17'\,18''$, $C_1=134°\,25'\,20''$, $a_1=310.8$,
$B_2=155°\,42'\,42''$, $C_2=2°\,59'\,56''$, $a_2=22.767$.

10. Given $a=628.2$, $b=234.4$, $A=119°\,40'\,40''$;
find $B=18°\,54'\,58''$, $C=41°\,24'\,22''$, $c=478.22$.

11. Given $a=86.14$, $b=97.82$, $A=38°\,15'\,14''$;
find $B_1=44°\,40'\,42''$, $C_1=97°\,4'\,4''$, $c_1=138.07$,
$B_2=135°\,19'\,18''$, $C_2=6°\,25'\,28''$, $c_2=15.57$.

12. Given $a=158$, $b=179$, $A=21°\,17'\,22''$;
find $B=24°\,17'\,18''$, $C=134°\,25'\,20''$, $c=310.8$,
$B'=155°\,42'\,42''$, $C'=2°\,59'\,56''$, $c'=22.77$.

13. Given $a=36.38$, $b=23.92$, $A=39°\,2'\,14''$;
find $B=24°\,27'\,49''$, $C=116°\,29'\,57''$, $c=51.69$.

14. Given $a=0.09593$, $b=0.16864$, $B=125°\,33'$;
find $A=27°\,34'\,12''$, $C=26°\,52'\,48''$, $c=0.09375$.

15. Given $a=354.16$, $b=433.86$, $A=36°\,1'\,4''$;
find $B_1=46°\,5'\,5''$, $C_1=97°\,53'\,51''$, $c_1=596.57$,
$B_2=133°\,54'\,55''$, $C_2=10°\,4'\,1''$, $c_2=105.26$.

16. Given $a=25.675$, $b=50.139$, $B=68° 4' 14''$;
find $A=28° 21' 42''$, $C=83° 34' 4''$, $c=53.709$.

17. Given $a=542.99$, $b=310.71$, $A=122° 49' 17''$;
find $B=28° 44' 34''$, $C=28° 26' 9''$, $c=307.66$.

18. Given $a=346.66$, $c=412.33$, $A=24° 19' 51''$;
find $B_1=126° 19' 31''$, $C_1=29° 20' 38''$, $b_1=677.87$,
$B_2=5° 0' 47''$, $C_2=150° 39' 22''$, $b_2=73.524$.

19. Given $a=56.82$, $b=67.53$, $B=77° 14' 19''$;
find $A=55° 8' 47''$, $C=47° 36' 54''$, $c=51.14$.

101. Given two sides and their included angle.

First method. When one angle C is given, the remaining angles can be found by the law of tangents (Art. 97, p. 134), which can be expressed in the following manner:

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\tan\frac{A+B}{2}.$$

The angle $\frac{A+B}{2}=90°-\frac{C}{2}$. Hence, its value is known, and the value of $\frac{A-B}{2}$ can be obtained from the above equation. The values of A and B can then be found as follows:

$$\frac{A+B}{2}+\frac{A-B}{2}=A,$$

and

$$\frac{A+B}{2}-\frac{A-B}{2}=B.$$

The remaining side c can now be found by the law of sines in either of the two following ways:

$$c=\frac{a\sin C}{\sin A}, \text{ or } c=\frac{b\sin C}{\sin B}.$$

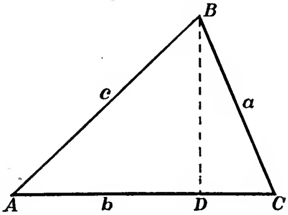
Second method. The third side c can be found directly by the law of cosines (Art. 96, p. 133), as follows:

$$c=\sqrt{a^2+b^2-2ab\cos C};$$

and the angles A and B can then be found by the law of sines, as follows:

$$\sin A = \frac{a \sin C}{c},\quad \sin B = \frac{b \sin C}{c}.$$

Third method. In the triangle ABC let the given parts be a, b, C. From the vertex B draw BD perpendicular to AC.



Then, $$BD = a \sin C,$$

and $$DC = a \cos C.$$

Now $$\tan A = \frac{BD}{AD}$$

$$= \frac{BD}{AC - DC}.$$

Substituting in this equation the values of BD and DC, we have

$$\tan A = \frac{a \sin C}{b - a \cos C}.$$

In like manner, drawing a perpendicular from A to the side BC it can be proved that

$$\tan B = \frac{b \sin C}{a - b \cos C}.$$

The third side can now be found by the law of sines, as under the first method.

NOTE. The first method is the best for use when all the unknown parts are desired. If only the third side is desired, the second method can be used to advantage. The second and third methods are not suitable for computation by means of logarithms.

Ex. 1. Given $a = 138.65$, $b = 226.19$, $C = 59° 12' 54''$; find the remaining parts.

$$\begin{aligned} b - a &= 78.54 \\ b + a &= 364.84 \\ B + A &= 120° 47' 6'' \\ \frac{B+A}{2} &= 60° 23' 33'' \\ \frac{B-A}{2} &= 22° 53' 31'' \\ A &= 37° 30' 2'' \\ B &= 83° 17' 4'' \end{aligned}$$

$$\begin{aligned} \log(b - a) &= 1.94221 \\ \log\tan\frac{B+A}{2} &= 10.24546 - 10 \\ \operatorname{colog}(b + a) &= 7.43790 - 10 \\ \log\tan\frac{B-A}{2} &= 9.62557 - 10 \\ \frac{B-A}{2} &= 22° 53' 31'' \end{aligned}$$

$$
\begin{aligned}
\log a &= 2.14192\\
\log \sin C &= 9.93494-10\\
\operatorname{colog} \sin A &= \underline{10.21554-10}\\
\log c &= 2.29240\\
c &= 196.06
\end{aligned}
$$

Check:

$$
\begin{aligned}
\log b &= 2.35447\\
\log \sin C &= 9.93494-10\\
\operatorname{colog} \sin B &= \underline{0.00299}\\
\log c &= 2.29240\\
c &= 196.06
\end{aligned}
$$

NOTE. In the solution of this problem b precedes a since $b > a$. (Compare Art. 97, p. 134.) In finding c we use A rather than B, because B is so near $90°$ that any solution obtained by means of its sine is likely to be inaccurate.

NOTE. In Ex. 1 the check solution gives a result exactly equal to that obtained by the original solution. In the work near the top of p. 136 the check solution also gave a result exactly equal to that obtained in the original solution. In general, however, the check solution may be expected to differ slightly from the original.

Ex. 2. Given $a = 7$, $c = 9$, $B = 60°$; find the third side b.

In this problem the second method furnishes the solution with the smallest amount of labor.

$$b^2 = a^2 + c^2 - 2ac \cos B,$$

$$b = \sqrt{49 + 81 - 2 \cdot 7 \cdot 9 \cdot \tfrac{1}{2}} = \sqrt{67}.$$

$$\therefore\ b = 8.1854.$$

EXERCISE XXVI

1. Given $a = 426$, $b = 582$, $C = 52° 18'$;
find $A = 46° 21' 16''$, $B = 81° 20' 44''$, $c = 465.8$.

2. Given $b = 123$, $c = 211$, $A = 115° 22'$;
find $B = 41° 46' 45''$, $C = 22° 51' 15''$, $a = 286.16$.

3. Given $a = 121.6$, $c = 192.2$, $B = 114° 42'$;
find $A = 24° 26' 49''$, $C = 40° 51' 11''$, $b = 266.94$.

4. Given $a = 619$, $b = 515$, $C = 39° 17'$;
find $A = 84° 46' 10''$, $B = 55° 56' 50''$, $c = 393.56$.

5. Given $b = 35.218$, $c = 54.176$, $A = 32° 48' 14''$;
find $B = 37° 49' 35''$, $C = 109° 22' 11''$, $a = 31.112$.

6. Given $a = 46.792$, $c = 61.234$, $B = 45° 29' 16''$;
find $A = 49° 34' 5''$, $C = 84° 56' 39''$, $b = 43.836$.

7. Given $b = 718.01$, $c = 228.88$, $A = 68° 55' 2''$;
find $B = 92° 30' 47''$, $C = 18° 34' 11''$, $a = 670.61$.

8. Given $b = 2478.1$, $c = 5134.8$, $A = 137° 8' 49''$;
find $B = 13° 37' 43.5''$, $C = 29° 13' 27.5''$, $a = 7152.5$.

9. Given $a = 4.1203$, $b = 4.9538$, $C = 65° 38' 52''$;
find $A = 49° 4' 18''$, $B = 65° 16' 50''$, $c = 4.9683$.

10. Given $a = 0.59217$, $b = 0.21833$, $C = 41° 37' 4''$;
find $A = 119° 42' 18''$, $B = 18° 40' 38''$, $c = 0.4528$.

11. Two objects A and B are separated by a body of water. In order to find the distance between them a third point C is chosen from which each of these points is visible, and the following measurements are made: $CA = 2560$ ft., $CB = 3120$ ft., and $\angle ACB = 105° 35'$. Find the distance from A to B.

12. If two sides of a triangle are 68.6 ft. and 92.2 ft. respectively and the included angle is 112° 42′, what is the third side?

13. Find the distance between two points A, B, which are separated by a marsh, when the distances of these points from a third point C are 4214 ft. and 6932 ft. respectively, and the angle ACB is 51° 11.

14. In an isosceles triangle each of the equal sides is 9 and the included angle is 60°. Find the third side.

15. In an isosceles triangle each of the equal sides is 9 and the included angle is 120°. Find the third side.

16. There are two points, A, B, on the bank of a river, but owing to a curve in its course it is impossible to measure the distance between them directly. A third point C is chosen such that the distances $AC = 1460$ ft. and $BC = 1680$ ft. can be measured, and the angle ACB is found to be 68° 42′ 30″. What is the distance from A to B?

17. In a given triangle two of the sides are 6 and 9 respectively, and the included angle is 38°. What is the third side?

18. The diagonals of a parallelogram are 8 and 10 respectively, and they intersect at an angle of 60°. What are the sides of the parallelogram?

19. If two sides of a triangle are 1468 and 2136 respectively and the included angle is $72° 21' 14''$, what are the values of the other angles?

20. There are two points, A, B, so situated that they are not visible from each other, and there is no other point from which both can be seen. To find the distance from A to B two other points C, D, are selected so that A and D are visible from C, and B and C are visible from D; and the following measurements are made: $CD = 826.5$ ft., $\angle ACD = 121° 12'$, $\angle BCD = 58° 55'$, $\angle ADC = 49° 12'$, $\angle ADB = 62° 38'$. What is the distance from A to B?

102. Given the three sides a, b, c. When the three sides of a triangle are given, the angles can be found directly from the formulas proved in Art. 96, p. 133.

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}; \tag{1}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca}; \tag{2}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}. \tag{3}$$

In order to obtain a form suitable for computation by means of logarithms we proceed as follows:

Let the sum of the sides of the triangle $a + b + c = 2\,s$. Then we have

$$a + b - c = 2\,(s - c),$$
$$b + c - a = 2\,(s - a),$$
$$c + a - b = 2\,(s - b).$$

Then,

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2\,bc}$$
$$= \frac{2\,bc - (b^2 + c^2 - a^2)}{2\,bc}$$
$$= \frac{a^2 - (b - c)^2}{2\,bc}$$
$$= \frac{(a + b - c)(a - b + c)}{2\,bc}$$
$$= \frac{2\,(s - b)(s - c)}{bc}.$$

Also (Art. 82, p. 108), $1 - \cos A = 2\sin^2\frac{A}{2}$.

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}. \qquad (4)$$

Note. Since $A < 180°$, being one of the angles of a triangle, $\frac{A}{2} < 90°$; therefore $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, and $\tan\frac{A}{2}$ are positive. Hence the radical in (4), and the corresponding expressions in (5) and (6) below, are always positive.

In like manner,
$$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2\,bc}$$
$$= \frac{2\,bc + b^2 + c^2 - a^2}{2\,bc}$$
$$= \frac{(b+c)^2 - a^2}{2\,bc}$$
$$= \frac{(b+c-a)(b+c+a)}{2\,bc}$$
$$= \frac{2\,s(s-a)}{bc}.$$

Also (Art. 82, p. 108), $1 + \cos A = 2\cos^2\frac{A}{2}$.

$$\therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \qquad (5)$$

Dividing (4) by (5), we have

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \qquad (6)$$

In like manner it can be proved that

$$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \qquad \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \qquad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Any one of the three formulas just given can be used in finding the angle required. If the half angle is very small, the cosine formula will not give a result as accurate as either the sine formula or the tangent formula, since the cosines of angles that are very small differ but little from each other; and for a similar reason the sine formula should not be used when the half angle is near 90°. In general the tangent formula is better than either of the others.

To insure as great a degree of accuracy as possible, it is better to solve for all the angles rather than solve for two angles and then subtract their sum from 180°. If each angle is computed separately and their sum is found to be within two or three seconds of 180°, the work of solution is probably correct.

If all the angles are to be computed, the following variation of the tangent formula may be found useful.

$$\tan\frac{A}{2}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$$

$$=\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Putting $$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}=r, \qquad (7)$$

we have $$\tan\frac{A}{2}=\frac{r}{s-a}. \qquad (8)$$

In like manner, $$\tan\frac{B}{2}=\frac{r}{s-b}; \qquad (9)$$

$$\tan\frac{C}{2}=\frac{r}{s-c}. \qquad (10)$$

Ex. **1.** Given $a=79.3$, $b=94.2$, $c=66.9$; find all the angles.

The work of solving for A and B is as follows:

$a=79.3$	$s-a=40.9$
$b=94.2$	$s-b=26$
$c=66.9$	$s-c=53.3$
$2s=240.4$	$s=120.2$
$s=120.2$	

$$\begin{aligned}
\log(s-b) &= 1.41497\\
\log(s-c) &= 1.72673\\
\text{colog}\,(s-a) &= 8.38828-10\\
\text{colog}\, s &= \underline{7.92010-10}\\
&\ 2)\underline{19.45008-20}\\
\log\tan\frac{A}{2} &= 9.72504-10\\
\therefore\ \frac{A}{2} &= 27^\circ\, 57'\, 56''.\\
A &= 55^\circ\, 55'\, 52''.
\end{aligned}
\qquad
\begin{aligned}
\log(s-c) &= 1.72673\\
\log(s-a) &= 1.61172\\
\text{colog}\,(s-b) &= 8.58503-10\\
\text{colog}\, s &= \underline{7.92010-10}\\
&\ 2)\underline{19.84358-20}\\
\log\tan\frac{B}{2} &= 9.92179-10\\
\therefore\ \frac{B}{2} &= 39^\circ\, 52'\, 6.9''.\\
B &= 79^\circ\, 44'\, 13.8''.\\
A+B &= 135^\circ\, 40'\, 5.8''.\\
\therefore\ C &= 44^\circ\, 19'\, 54.2''.
\end{aligned}$$

If the value of C is found by logarithms in the same manner as were the values of A and B, it will be found to be $44^\circ\, 19'\, 56.8''$, which is $2.6''$ larger than the value found by subtracting the sum of A and B from 180°. The sum of the three angles, when all are found independently, is $180^\circ\, 0'\, 2.6''$. The sum of the three angles determined in this manner is rarely equal to exactly 180°. This is due to the fact that logarithmic computation is almost always slightly inexact. It is customary in practical work to divide the error among the three angles according to the probable amount for each angle.

Ex. 2. Solve the preceding example by the use of formulas (8), (9), and (10).

In solving by this method it is best to find all the logarithms at the outset, and then to subtract the logarithms of $s-a$, $s-b$, $s-c$, respectively, from the logarithm of r. A compact arrangement of the work can be secured by following the model below.

$$\begin{aligned}
a &= 79.3\\
b &= 94.2\\
c &= \underline{66.9}\\
2s &= 240.4\\
s &= \underline{120.2}\\
s-a &= 40.9\\
s-b &= 26\\
s-c &= \underline{53.3}\\
s &= 120.2\ \textit{Check.}
\end{aligned}
\qquad
\begin{aligned}
\log(s-a) &= 1.61172\\
\log(s-b) &= 1.41497\\
\log(s-c) &= 1.72673\\
\text{colog}\, s &= \underline{7.92010-10}\\
\log r^2 &= 2.67352\\
\log r &= 1.33676
\end{aligned}
\qquad
\begin{aligned}
\log\tan\frac{A}{2} &= 9.72504-10\\
\log\tan\frac{B}{2} &= 9.92179-10\\
\log\tan\frac{C}{2} &= 9.61003-10\\
\frac{A}{2} &= 27^\circ\, 57'\, 56''\\
\frac{B}{2} &= 39^\circ\, 52'\, 6.9''\\
\frac{C}{2} &= 22^\circ\, 9'\, 58.4''\\
A &= 55^\circ\, 55'\, 52''\\
B &= 79^\circ\, 44'\, 13.8''\\
C &= \underline{44^\circ\, 19'\, 56.8''}\\
\textit{Check.}\quad A+B+C &= 180^\circ\, 0'\, 2.6''
\end{aligned}$$

EXERCISE XXVII

1. Given $a = 56$, $b = 58$, $c = 64$;
find $A = 54° 22' 43''$, $B = 57° 20' 32''$, $C = 68° 16' 44''$.

2. Given $a = 54$, $b = 52$, $c = 68$;
find $A = 51° 24' 3.8''$, $B = 48° 48' 52.8''$, $C = 79° 47' 7.6''$.

3. Given $a = 35$, $b = 41$, $c = 47$;
find $A = 46° 15' 5''$, $B = 57° 48' 16''$, $C = 75° 56' 41.5''$.

4. Given $a = 73$, $b = 82$, $c = 91$;
find $A = 49° 34' 58''$, $B = 58° 46' 58''$, $C = 71° 38' 4''$.

5. Given $a = 47$, $b = 51$, $c = 58$;
find $A = 50° 35' 18''$, $B = 56° 58' 4''$, $C = 72° 26' 40''$.

6. Given $a = 286$, $b = 321$, $c = 463$;
find $A = 37° 33' 57''$, $B = 43° 10' 46''$, $C = 99° 15' 23''$.

7. Given $a = 138$, $b = 246$, $c = 321$;
find $A = 23° 47' 23''$, $B = 45° 58' 41''$, $C = 110° 14'$

8. Given $a = 196$, $b = 211$, $c = 173$;
find $A = 60° 25' 31''$, $B = 69° 26'$, $C = 50° 8' 36''$.

9. Given $a = 48.3$, $b = 53.2$, $c = 62.7$;
find $A = 48° 24' 24''$, $B = 55° 27' 44''$, $C = 76° 7' 55''$.

10. Given $a = 226.4$, $b = 431.6$, $c = 316.8$;
find $A = 30° 35' 53''$, $B = 103° 58' 55''$ $C = 45° 25' 8''$.

11. Given $a = 262.43$, $b = 514.36$, $c = 556.25$;
find $A = 50° 59' 48''$, $B = 59° 48' 44''$, $C = 69° 11' 32''$.

12. Given $a = 2243.8$, $b = 2469.2$, $c = 3125.6$;
find $A = 45° 26' 3''$, $B = 51° 37' 42''$, $C = 82° 56' 19''$.

13. Given $a = 25617$, $b = 34178$, $c = 23194$;
find $A = 48° 31' 56''$, $B = 88° 44' 34''$, $C = 42° 43' 30''$.

14. Given $a = 0.34177$, $b = 0.45623$, $c = 0.58216$;
find $A = 35° 54' 30''$, $B = 51° 31' 34''$, $C = 92° 33' 56''$.

15. Given $a = 11.682$, $b = 14.468$, $c = 20.386$;
find $A = 34° 6' 13''$, $B = 43° 58' 47''$, $C = 101° 54' 58''$.

16. Given $a = 1.9141$, $b = 1.8365$, $c = 1.2854$;
find $A = 73° 14' 32,''$ $B = 66° 44' 22''$, $C = 40° 1' 5''$.

17. The sides of a triangle are respectively 36.92, 31.84, 26.14. Find the smallest angle of the triangle.

18. The sides of a triangle are in the ratio of 29 : 21 : 38. Find the medium angle.

19. The sides of a triangle are to each other as 3 : 4 : 5. Find all the angles.

20. In a given triangle $a = 8$, $b = 8$, $c = 8$. Find all the angles.

21. Three cities are respectively 22.6, 21.4, 19.6 miles apart. If the curvature of the earth is disregarded, what angles are made by the lines joining the cities?

22. In discussing the solution of a triangle when two sides and the angle opposite one of them are given, it was noted that two solutions were possible when an angle was found by means of its sine. Why does not a similar ambiguity exist when an angle is found by means of formula (4), p. 149?

23. The sides of a triangle are $a = 7$, $b = 8$, $c = 5$. Find the angle A.

24. The sides of a triangle are $a = 7$, $b = 5$, $c = 3$. Find the angle A.

25. An object 16.2 ft. in length is so situated that one end is $17\frac{1}{2}$ ft. and the other is 11.9 ft. from the eye of an observer. What angle does the object subtend at the eye?

103. Area of a triangle. In geometry it was proved that the area of a triangle ($\triangle$) can be found by either of the following formulas:

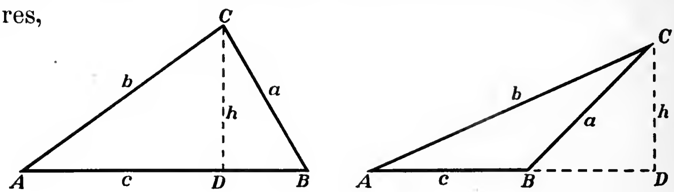
$$\triangle = \tfrac{1}{2} \text{ base} \times \text{altitude},$$

or,

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

The work of finding the area of a triangle can be greatly simplified by trigonometry, as will be seen from the following section.

104. Case 1. **Given two sides and the included angle.** The area of any triangle is equal to one half the product of the base and the altitude. Therefore, using either of the following figures,



$$\triangle = \tfrac{1}{2} c\, CD.$$

But

$$CD = a \sin B.$$

$$\therefore \quad \triangle = \tfrac{1}{2} ac \sin B. \tag{1}$$

In like manner it can be proved that

$$\triangle = \tfrac{1}{2} bc \sin A, \tag{2}$$

and

$$\triangle = \tfrac{1}{2} ab \sin C. \tag{3}$$

Case 2. **Given a side and the two adjacent angles.** By the law of sines (Art. 94, p. 131),

$$a : c = \sin A : \sin C.$$

$$c = \frac{a \sin C}{\sin A}.$$

Substituting this value of c in (1), Case 1, we have.

$$\triangle = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

But since $A + B + C = 180°$, $\sin A = \sin (B + C)$;

$$\therefore \triangle = \frac{a^2 \sin B \sin C}{2 \sin (B + C)}. \tag{4}$$

CASE 3. **Given the three sides.** In Art. 80, p. 106, it was proved that

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}.$$

But (Art. 102, p. 149),

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

and

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Substituting these values in the above equation, we have

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Substituting this value of $\sin A$ in (2), we have

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}. \qquad (5)$$

CASE 4. **Given three sides and the radius of the circumscribed circle.** By Art. 95, p. 132, we have

$$\sin A = \frac{a}{2r},$$

where r is the radius of the circumscribed circle. Substituting this value in (2), we have

$$\triangle = \frac{abc}{4r}. \qquad (6)$$

CASE 5. **Given three sides and the radius of the inscribed circle.**

Let r be the radius of the inscribed circle. The triangle can be divided into three triangles whose bases are a, b, c, respectively, and whose common altitude is r. Then

$$\triangle = \tfrac{1}{2} r(a+b+c). \qquad (7)$$

April 3 — 153–11, 147–9, 152–11
area

CHAPTER XIII

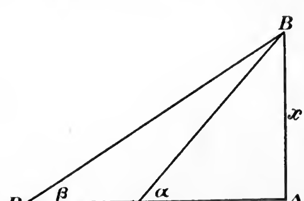
MISCELLANEOUS PROBLEMS IN HEIGHTS AND DISTANCES

105. In this chapter certain problems will be considered that are frequently met in land surveying, railroad work, etc. The degree of accuracy required in practical problems of this kind can only be known after the nature of the special problem under consideration is known. Hence, in the examples that are here considered no attempt is made to conform to the ordinary practice of field surveyors. In many classes of problems that they are called upon to solve a sufficient degree of accuracy is secured if the angles are measured to single minutes and the computations are performed by means of four-place tables of logarithms; while in others the measurements are made with the greatest possible accuracy and the computations are performed with the aid of eight-, ten-, or twelve-place tables. For this reason it is quite impracticable for an elementary text-book in trigonometry to attempt to conform to field usage.

The tables used in the solution of the problems in this chapter are five-place tables.

106. The height of an object by means of observations made at distant points.

Let AB represent the height of an object, and let C, D, be two points of observation on the same level with A, so situated that A, C, D, are in the same straight line. Let the angle of elevation of B at C be α, and at D be β, and let $DC=a$. Then from the triangle ABC



$$\frac{x}{BC} = \sin\alpha, \qquad (1)$$

and from the triangle DCB

$$\frac{BC}{a} = \frac{\sin\beta}{\sin(\alpha - \beta)}.$$

$$\therefore BC = \frac{a\sin\beta}{\sin(\alpha-\beta)}. \tag{2}$$

Substituting this value of BC in (1), and reducing we have

$$x = \frac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)},$$

a formula which gives the value of x in a form suitable for computation either by logarithms or by natural functions.

Ex. 1. What is the height of a tower when the angles of elevation of the top of the tower from two points 250 ft. apart on the ground and in the same straight line with the foot of the tower are 30° and 60° respectively?

Here $a = 250$, $\alpha = 60°$, and $\beta = 30°$. Therefore

$$x = \frac{250 \sin 60° \sin 30°}{\sin 30°}$$

$$= 250 \cdot \tfrac{1}{2}\sqrt{3} = 216.5 \text{ ft.}$$

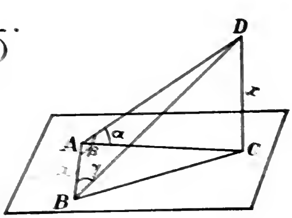
107. If the height of an object is to be determined, and no two points can be found that are in the same straight line, and at the same time conveniently situated for observation, the following method is often employed:

From A measure AB in any convenient direction. Let the angle of elevation of the top of the object D, measured at A, be α, and let the distance AB be a. At A and B measure the angles $DAB = \beta$, and $DBA = \gamma$, respectively. Then in the triangle ADB

$$\angle ADB = 180° - (\beta + \gamma).$$

Therefore,

$$\frac{AD}{a} = \frac{\sin\gamma}{\sin(180° - (\beta+\gamma))} = \frac{\sin\gamma}{\sin(\beta+\gamma)}.$$



Using the value of AD obtained from this equation, we have

$$x = AD\sin\alpha = \frac{a\sin\alpha\sin\gamma}{\sin(\beta+\gamma)}.$$

MISCELLANEOUS EXAMPLES

THE RIGHT TRIANGLE

1. The angle of elevation of the top of a vertical cliff 426.28 ft. high, taken from a point on the same level as the foot of the cliff, is 59° 51′ 14″. What is the distance from the foot of the cliff to the point where the observation was taken?

2. A pole 36 ft. high casts a shadow 39 ft. long. What is the angle of elevation of the top of the pole, measured at the extremity of the shadow?

3. The height of a room is 12.62 ft. and its length is 14.44 ft. What is the angle of elevation of one of the upper corners of the room taken at the lower corner on the same side?

4. What is the elevation of the sun when a tree 31.6 ft. high casts a shadow 42.9 ft. in length?

5. What angle does a ladder 25.2 ft. long make with the ground when it just reaches the sill of a window 18.95 ft. above the level on which the foot of the ladder rests?

6. The angle of depression of a point on the ground, measured from the top of a building 49.27 ft. high, is 34° 6′ 36″. What is the distance from the foot of the building to the given point?

7. The length of the diagonal of a rectangular field is 247.39 ft., and the angle between the diagonal and the shorter side of the field is 29° 40′ 36″. What is the width of the field?

8. A path measuring 256.4 ft. in length leads diagonally across a rectangular plot of ground, making with one of the sides an angle of 61° 12′ 52″. What is the length of the side?

9. The angle of elevation of a balloon measured at a certain point is 71° 14′ 12″, and from this point to a point directly below the balloon the horizontal distance is 415.9 ft. What is the height of the balloon and its distance from the point of observation?

10. A kite is fastened to a string 483.2 ft. long, and the string makes an angle of 63° 19′ 28″ with the level of the ground. What is the vertical height of the kite above the ground, no allowance being made for the sagging of the string?

11. To ascertain the width of a river a distance AB is measured along one of the banks 262.38 ft. Directly across the river from B is a point C, and the angle BAC is found upon measurement to be 41° 38′ 20″. Required the width of the river.

12. Two forces, of 198.52 lb. and 393.13 lb. respectively, are acting at right angles to each other. What is the resultant of the two forces, and what is the angle which the direction of each force makes with the resultant?

13. What is the radius of the parallel passing through a point on the earth's surface whose latitude is 43° 15′, the radius of the earth being reckoned as 3956 mi.?

14. The angle of elevation of the top of a hill viewed from a certain point is 29° 17′, and from a point 362.4 ft. nearer, measured directly toward the hill, the angle of elevation is 48° 12′. Required the height of the hill.

15. From the top of a mountain the angles of depression of two milestones 2 mi. apart and in the same vertical plane with the top of the mountain are 10° 14′ 42″ and 5° 38′ 46″ respectively. What is the height of the mountain?

16. A flagstaff which is broken at a certain distance above the ground falls so that its tip touches the ground at a distance of 13.5 ft. from the foot of the portion which remains standing. The length of the part broken over is 35.1 ft. What was the total height of the staff before it was broken over?

17. If the angle of depression of the visible horizon, observed from the top of a mountain 3 mi. in height, is 2° 13′ 59″, what is the diameter of the earth?

18 A ladder 30 ft. long when set in a certain position between two buildings will reach a point 20 ft. from the ground on one of the buildings, and on being turned over without having the position of its foot changed it reaches a point on the other building 15 ft. from the ground. What is the angle between the two positions of the ladder? (Solve by natural functions.)

19. A lighthouse 50 ft. high stands on the top of a rock. The angle of elevation of the top of the rock and of the top of the lighthouse, measured from the deck of a vessel, are $6° 5'$ and $6° 58''$ respectively. What is the height of the rock, and the distance from the vessel to the foot of the rock? (Solve by natural functions.)

20. At any point on the earth's surface a line is drawn tangent to the surface at that point. If the earth is considered a sphere whose diameter is 7912.4 mi., how far from the surface will the line be at the end of 1 mi.?

21. A building 50 ft. high stands at the foot of a hill, and from the top of the hill the angles of depression of the top and of the bottom of the building are $45° 15'$ and $47° 12'$ respectively. What is the height of the hill?

22. The angles of a triangle are $1:2:3$, and the perpendicular from the greatest angle to the side opposite is 15 ft. Required the sides of the triangle.

23. A bridge of five equal spans crosses a river, each span measuring 100 ft. from center to center. From a boat moored in line with one of the middle piers the length of the bridge subtends a right angle. What is the distance from the boat to the bridge? (Solve by natural functions.)

24. An observer on a vessel at anchor sees another vessel due north of him; at the end of fifteen minutes it bears E., and half an hour later it bears S.E. How long after it is first seen is it nearest the observer, and in what direction is it sailing, its course being supposed to be in a straight line from the time of the first to the time of the last observation? (Solve by natural functions.)

25. A statue on a column subtends the same angle at distances of 27 and of 33 ft. from the column. If the tangent of the angle equals $\frac{1}{10}$, what is the height of the statue? (Solve by natural functions.)

26. A tower 145 ft. high stands on an elevation 75 ft. high. At what point in the plain on which the elevation stands must an observation be made in order that the tower and the height of the elevation may subtend equal angles? (Solve by natural functions.)

27. A flagstaff 50 ft. high stands in the center of a plot of ground in the form of an equilateral triangle. Each side of the triangle subtends at the top of the staff an angle of 60°. What is the length of one of the sides of the triangle? (Solve by natural functions.)

28. A tower stands on the slope of a hill that has an inclination of 15° to the level of the plain. At a point 80 ft. farther up the hill it is found that the tower subtends an angle of 30°. Prove that the tower is $40(\sqrt{6}-\sqrt{2})$ ft. in height.

29. At a distance of 300 ft. from the foot of a tower the angle of elevation is one third as great as it is at a distance of 60 ft. What is the height of the tower?

THE OBLIQUE TRIANGLE

30. The angles of elevation of a balloon measured at the same instant at two points on level ground and in the same vertical plane as the balloon are 41° 56′ and 28° 14′ respectively. The two points from which the angles are measured are 3462 ft. apart and on the same side of the balloon. Required its height at the time of observation.

31. The angle of depression of an object viewed from the top of a tower is 50° 12′ 56″, and the angle of depression of a second object 250 ft. farther away, and in a straight line with the first object and the foot of the tower is 31° 19′ 54″. What is the height of the tower?

32. The angles of depression of two objects on a level plain, viewed from an elevation in the same vertical plane with the objects, are 48° 12′ and 29° 17′ respectively, and the distance between the two points is 362.4 ft. Required the height of the point of observation.

33. The sides of a triangular plot of ground are 138 ft., 246 ft., and 321 ft. respectively. What is the greatest angle formed by the sides?

34. Two objects are separated by a building, and it is required to find the distance between them. At a third point, distant 268 ft. and 315 ft. respectively from the given objects, the angle subtended by the line connecting the objects is measured and is found to be 108° 17′. What is the distance between the objects?

35. What is the distance between two points A, B, when the distances from these points to a third point C are 6282 ft. and 2344 ft. respectively, and the angle BAC is 119° 40′ 40″? Is more than one solution possible? Why? (See Art. 100, p. 138.)

36. The distance between two points A, B, cannot be obtained directly by the use of the chain or tape because of an intervening body of water. A third point C is chosen from which both A and B are visible, and the following measurements are then made: $AC = 3101.8$ ft., $\angle CAB = 51°\ 28'$, $\angle ABC = 70°\ 37'\ 33''$. What is the required distance?

37. In a system of triangulation the sides of a triangle connecting the stations on the tops of three hills have been computed and have been found to be 54,692.73 ft., 61,284.39 ft., and 42,798.64 ft. respectively. What are the values of the angles of this triangle as computed from the sides?

38. An observation station A is set up in a field along one side of which runs a straight, level road. Two points of observation on the road, B, C, one fourth of a mile apart, are chosen, on opposite sides of the first station and the angles ABC, ACB, are measured and found to be 46° 20′ 28″ and 38° 24′ 48″ respectively. What is the distance from the station A to the road?

39. The distances from a point on shore to two buoys are known to be 1286 ft. and 2466 ft. respectively, and the angle subtended at that point by the line connecting the buoys is 42° 14′ 16″. What is the distance between the buoys?

40. A tripod is set up on a rock, and to find the distance from the tripod to the shore a line 8500 ft. in length is measured along the shore, and at each extremity of the line the angle is measured which subtends the line connecting the tripod with the other end of the line. The angles are found to be 46° 28′ and 43° 32′ respectively. Find the distance from the tripod to the line of measurement along the shore.

41. Two vessels lying at anchor 1 mi. apart are observed from a third vessel sailing east to be in a straight line due north. After sailing an hour and a half one of the vessels bears N.W. and the other W.N.W. Find the rate at which the vessel is sailing.

42. The distance between two points A, B, is to be determined, where B is accessible and A is not. A kite is sent up and made fast, and its position C is determined to be 517.3 yd. vertically above D, which is on the same level with A and B. The following angles are then measured: $ACB = 13° 15' 15''$, $CAD = 21° 9' 18''$, $DBC = 23° 15' 34''$. What is the distance from A to B?

43. Two forces, of 410 lb. and 320 lb. respectively, are acting at an angle of 51° 37′. Required the direction and intensity of the resultant.

44. A kite A has been sent up and is fastened to the ground at a point C. The kite has drifted a certain distance and now stands directly above a point B, which is on the same level as C, but is separated from it by obstacles which render direct measurement impracticable; and the height of the kite is desired. To ascertain this a line is measured from C to a point D, 4262.4 ft. in length, and the following angles are measured: $ACB = 31° 17' 14''$, $ACD = 66° 14' 52''$, $CDA = 52° 51' 38''$. Required the vertical height of the kite above the point B. (See Art. 107.)

45. Two rocks are to be charted. To ascertain the distance between them the angles of elevation of a point at the top of a cliff 527.4 ft. high are taken and are found to be 21° 8′ 16″ and 23° 14′ 20″ respectively, and the angle subtended by the line connecting the rocks, measured at a point at the top of the cliff, is 16° 3′ 30″. Required the distance between the rocks.

46. A balloon, A, is sighted at the same instant from two points, B, C, which are on the same level, and are 262.4 ft. apart. The angle of elevation of the balloon at B is 41° 15′ 24″, $\angle ABC = 62° \; 48' \; 14''$, $\angle ACB = 59° \; 14' \; 21''$. What is the height of the balloon at the instant of observation?

47. A tower stands on the slope of a hill which makes an angle of 16° with the horizon. At a distance of 95 ft. from the foot of the tower, measured directly up the side of the hill, the height of the tower subtends an angle of 38°. What is the height of the tower?

48. A tree stands E.S.E. of an observer, and at noon the extremity of the shadow of the tree is directly N.E. of the position in which he is standing. The length of the shadow is 60 ft., and the angle of elevation of the top of the tree viewed from the position of the observer is 45°. What is the height of the tree? (Solve by natural functions.)

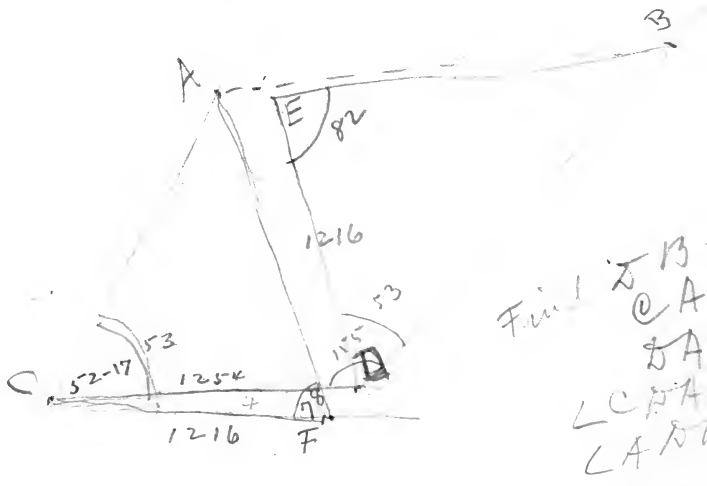
49. It is required to find the distance between two points, A, B, neither of which is accessible. For that purpose a base line, CD, 4968 ft. long, is measured, and the following angles are observed: $ACD = 108° \; 14'$, $BCD = 41° \; 15'$, $BDC = 115° \; 21'$, $ADC = 39° \; 42'$. What is the distance from A to B?

50. Two points are so situated that it is not possible to measure directly from one to the other, but observations can be taken at either point. Two other points, C, D, are chosen, 5226 ft. apart, and the following angles are measured: $ACB = 15° \; 18' \; 24''$, $DAC = 21° \; 12' \; 46''$, $DBC = 23° \; 18' \; 42''$, $ADC = BDC = 90°$. What is the distance from A to B?

51. To find the distance between two inaccessible points, A, B, two other points, C, D, are chosen, so situated that from either of them the three other points can be seen; and the following measurements are then made: $CD = 826.5$ ft., $\angle ACD = 121° \; 12'$, $\angle BCD = 58° \; 55'$, $\angle ADC = 49° \; 12'$, $\angle ADB = 62° \; 38'$. What is the distance from A to B?

52. Two points, A, B, are so situated that only one point, C, can be found which is conveniently situated for observation, from which both can be seen. A fourth point, D, is found from which A and C can be seen, and a fifth point, E, from which B and C can be seen. The following measurements are taken, from which it is required that the distance from A to B shall be computed: $CD = 6428.72$ ft., $CE = 5872.54$ ft., $\angle ACB = 66° \ 14'$, $\angle BCE = 41° \ 17'$, $\angle CEB = 117° \ 42'$, $\angle ACD = 69° \ 38'$, $\angle CDA = 64° \ 21'$.

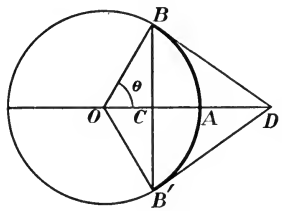
53. Two points, A, B, are so situated that no point can be found from which both can be seen. Two other points, C, D, are found, so placed that A and D can be seen from C and B from D, and also two additional points, E, F, so placed that A and C can be seen from F, and B and D from E. The following data can now be obtained for the determination of the distance from A to B: $CD = 1254$ ft., $CF = 1216$ ft., $DE = 1216$ ft., $\angle AFC = 78° \ 14' \ 15''$, $\angle FCA = 53° \ 51' \ 40''$, $\angle ACD = 52° \ 17' \ 18''$, $\angle CDB = 155° \ 24' \ 20''$, $\angle BDE = 53° \ 49' \ 8''$, $\angle DEB = 82° \ 57'$. What is the length of the line AB?



CHAPTER XIV

FUNCTIONS OF VERY SMALL ANGLES — HYPERBOLIC FUNCTIONS — TRIGONOMETRIC ELIMINATION

108. Trigonometric functions of very small angles. Let AOB be any angle less than 90°. With O as a center and any radius OA describe a circle.



Draw BC perpendicular to OA, and produce it to intersect the circle in B'.

Draw tangents to the circle at B, B'. These tangents will, by geometry, intersect OA produced in the same point D. Then

$$\text{chord } BB' < \text{arc } BB' < BD + B'D.$$

Dividing by 2, $\quad CB < \text{arc } AB < DB.$

$$\therefore \frac{CB}{OB} < \frac{\text{arc } AB}{OB} < \frac{BD}{OB}.$$

But $\frac{CB}{OB} = \sin\theta$, $\frac{BD}{OB} = \tan\theta$, and $\frac{\text{arc } AB}{OB} =$ the circular measure of the angle θ, or of the arc AB (Art. 13, p. 16). Therefore,

$$\sin\theta < \theta < \tan\theta.$$

This important result may be expressed as follows:

When $\theta < 90°$, $\sin\theta$, θ, and $\tan\theta$ are in the ascending order of magnitude.

109. Dividing the inequality just obtained by $\sin\theta$, we have

$$1 < \frac{\theta}{\sin\theta} < \sec\theta,$$

or,

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Therefore, $\frac{\sin\theta}{\theta}$ lies between 1 and $\cos\theta$ for all values of θ between 0 and $\frac{\pi}{2}$.

But as θ approaches 0 as its limit, $\cos\theta$ approaches 1 as its limit; and at the same time $\frac{1}{\cos\theta}$ approaches 1 as its limit.

Therefore, when θ is very small, and is approaching 0 as its limit, $\frac{\sin\theta}{\theta}$ lies between 1 and a quantity that may be made to differ from 1 by a quantity ϵ which may be made as small as we please; and as θ approaches 0 as its limit, ϵ also approaches 0 as its limit.

In other words, when θ approaches 0 as its limit, $\frac{\sin\theta}{\theta}$ approaches 1 as its limit. This fact is often expressed by the statement that when θ is very small, $\sin\theta = \theta$, approximately.

In like manner it can be shown that as θ approaches 0 as its limit, $\tan\theta$ will also approach the limit 0; that is, when θ is very small, $\tan\theta = \theta$ approximately.

From the above it follows also that when θ is very small, $\sin\theta = \tan\theta$, approximately.

In this discussion it should be remembered that θ is expressed in circular measure; *i.e.* θ is the number of radians in the angle or arc under consideration.

EXERCISE XXVIII

1. Find the sine and the cosine of 1′.

Let x be the circular measure of 1′.

Then, $$x = \frac{2\pi}{360 \times 60} = \frac{3.14159+}{10800} = 0.000290888+.$$

Therefore, since $$x > \sin x > 0, \quad \text{(Art. 108)}$$
sin 1′ lies between 0 and 0.000290889.

Also, $$\cos 1' = \sqrt{1 - \sin^2 1'}$$
$$> \sqrt{1 - (0.000290888)^2}$$
$$> 0.9999999.$$
$$\therefore \cos 1' = 0.9999999+. \quad (1)$$

But (Art. 108, p. 166), $$\sin x > x \cos x.$$
$$\therefore \sin 1' > 0.000290888 \times 0.9999999$$
$$< 0.000290887. \quad (2)$$

Therefore, $\sin 1'$ lies between (1) and (2); *i.e.*

$$\sin 1' = 0.00029088^{+},$$

and the next decimal place is either 7 or 8.

Find approximately the values of the following:

2. $\sin 10'$. **4.** $\sin 7'$. **6.** $\cos 15'$.

3. $\cos 10'$. **5.** $\sin 15'$. **7.** $\sin 8''$.

HYPERBOLIC FUNCTIONS

110. In the differential calculus it is proved that the following equations are true for all values of x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots; \tag{1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots; \tag{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \tag{3}$$

where $e = 2.7182818 \cdots$ is the base of the natural system of logarithms. In (1) and (2) x is the value of the angle or arc expressed in radians.

If in (3) x is replaced by ix, where $i = \sqrt{-1}$, we have

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right). \tag{4}$$

The series in the first parenthesis is the same as the right member of (2), and that in the second parenthesis is the same as the right member of (1). Hence, replacing these series by their values, we have equation (4) in the following form:

$$e^{ix} = \cos x + i \sin x. \tag{5}$$

In a precisely similar manner it may be shown that

$$e^{-ix} = \cos x - i \sin x. \tag{6}$$

Adding (5) and (6), and dividing by 2, we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}. \tag{7}$$

Subtracting (6) from (5) and dividing by $2i$, and the corresponding value for $\sin x$ is obtained:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{8}$$

These equations give the values of the sine and the cosine of any angle whatever in exponential form.

111. If in (5) and (6) of the preceding section we replace x by ix, the following equations are obtained:

$$e^{-x} = \cos ix + i \sin ix; \tag{1}$$

$$e^{x} = \cos ix - i \sin ix. \tag{2}$$

By addition and subtraction we obtain from these the results below:

$$\cos ix = \frac{e^{x} + e^{-x}}{2}; \tag{3}$$

$$\sin ix = \frac{i(e^{x} - e^{-x})}{2}. \tag{4}$$

It will be noticed that the exponential functions which occur in the right-hand members of (3) and (4) possess a striking similarity to those which appear in (7) and (8) of the preceding section. It has been found convenient to make use of this similarity, and, corresponding to the exponential values of $\sin x$ and $\cos x$ given in those equations, to give the following definitions:

$\frac{e^{x} + e^{-x}}{2}$ is called the hyperbolic cosine of x,

and $\frac{e^{x} - e^{-x}}{2}$ is called the hyperbolic sine of x.

These functions are written in abbreviated form $\cosh x$ and and $\sinh x$ respectively. Accordingly we have

$$\cosh x = \frac{e^{x} + e^{-x}}{2} = \cos ix; \tag{5}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} = -i \sin ix. \tag{6}$$

The name *hyperbolic* is applied to these functions because they bear to the equilateral hyperbola a relation analogous to that which $\sin x$ and $\cos x$ bear to the circle. (Art. 46, p. 64.)

The other hyperbolic functions are defined as follows:

$$\tanh x = \frac{\sinh x}{\cosh x}; \tag{7}$$

$$\coth x = \frac{\cosh x}{\sinh x}; \tag{8}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}; \tag{9}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}. \tag{10}$$

112. Ex. 1. Prove the relation $\sinh 0 = 0$.

By (6), Art. 111, we have

$$\sinh 0 = \frac{e^0 - e^0}{2} = \frac{0}{2} = 0.$$

Ex. 2. Prove the relation

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

By definition

$$\begin{aligned}\sinh(x+y) &= -i(\sin(ix+iy)) \\ &= -i(\sin ix \cos iy + \cos ix \sin iy) \\ &= -i(i \sinh x \cosh y + i \cosh x \sinh y) \\ &= \sinh x \cosh y + \cosh x \sinh y.\end{aligned}$$

Ex. 3. Prove the relation

$$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}.$$

By definition

$$\begin{aligned}\sinh x + \sinh y &= -i(\sin ix + \sin iy) \\ &= -i\left(2 \sin \frac{i(x+y)}{2} \cos \frac{i(x-y)}{2}\right) \\ &= 2\left(-i \sin \frac{i(x+y)}{2} \cos \frac{i(x-y)}{2}\right) \\ &= 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}.\end{aligned}$$

EXERCISE XXIX

Prove the following identities:

1. $\cosh 0 = 1.$
2. $\sinh \frac{\pi i}{2} = i.$
3. $\cosh \frac{\pi i}{2} = 0.$
4. $\sinh \pi i = 0.$
5. $\cosh \pi i = -1.$
6. $\sinh 2n\pi = 0.$
7. $\cosh 2n\pi = 1.$
8. $\tanh 0 = 0.$
9. $\sin(-ix) = -\sin ix.$
10. $\cos(-ix) = \cos ix.$
11. $\tan ix = i \tanh x.$
12. $\sinh(-x) = -\sinh x.$
13. $\cosh(-x) = \cosh x.$
14. $\coth(-x) = -\coth x.$
15. $\text{sech}(-x) = \text{sech}\, x.$
16. $\text{csch}(-x) = -\text{csch}\, x.$
17. $\cosh^2 x - \sinh^2 x = 1.$
18. $\text{sech}^2 x + \tanh^2 x = 1.$
19. $\text{csch}^2 x - \coth^2 x = -1.$
20. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.$
21. $\sinh 2x = 2 \sinh x \cosh x.$
22. $\cosh 2x = \cosh^2 x + \sinh^2 x.$
23. $\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}.$
24. $\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2}.$
25. $\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}.$

113. The notation for inverse hyperbolic functions is the same as for inverse circular functions (Art. 84, p. 114).

If $$y = \sinh x,$$

then, $$x = \sinh^{-1} y.$$

But by (6), p. 169, $$y = \frac{e^x - e^{-x}}{2}.$$

Solving this equation for x, we have

$$x = \log(y + \sqrt{y^2 + 1}). \qquad (1)$$

$$\therefore \sinh^{-1} y = \log(y + \sqrt{y^2 + 1}).$$

In like manner, $\cosh^{-1}y = \log(y + \sqrt{y^2 - 1})$; (2)

$$\tanh^{-1}y = \frac{1}{2}\log\frac{1+y}{1-y}; \quad (3)$$

$$\coth^{-1}y = \tanh^{-1}\frac{1}{y} = \frac{1}{2}\log\frac{y+1}{y-1}; \quad (4)$$

$$\operatorname{sech}^{-1}y = \cosh^{-1}\frac{1}{y} = \log\frac{1+\sqrt{1-y^2}}{y}; \quad (5)$$

$$\operatorname{csch}^{-1}y = \sinh^{-1}\frac{1}{y} = \log\frac{1+\sqrt{1+y^2}}{y}. \quad (6)$$

EXERCISE XXX

Prove the following relations:

1. $\tanh^{-1}\dfrac{2x}{1+x^2} = 2\tanh^{-1}x.$

2. $\sinh^{-1}2x = 2\sinh^{-1}x\cosh^{-1}x.$

3. $\sinh^{-1}x = \cosh^{-1}\sqrt{1+x^2}.$

4. $\sinh^{-1}x = \tanh^{-1}\dfrac{x}{\sqrt{1+x^2}}.$

5. $\tanh^{-1}x + \tanh^{-1}y = \tanh^{-1}\dfrac{x+y}{1+xy}.$

ELIMINATION

114. It often happens that two or more equations are given that contain trigonometric functions of some angle, or perhaps of more than one angle. From these equations a single equation is to be obtained from which all trigonometric functions have been eliminated.

In theory the required elimination can always be performed, but in practice this often involves processes that are somewhat complicated; and the desired results are obtained with a greater or less degree of difficulty.

No general rule for work of this kind can be given; and the process is best illustrated by a few examples.

115. Ex. 1. Find the values of r and θ from the equations

$$r \sin \theta = a; \qquad (1)$$
$$r \cos \theta = b. \qquad (2)$$

Squaring and adding,

$$r^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2,$$
$$r^2 = a^2 + b^2,$$
$$r = \sqrt{a^2 + b^2}.$$

Also, dividing (1) by (2),

$$\tan\theta = \frac{a}{b},$$
$$\theta = \tan^{-1}\frac{a}{b}.$$

Ex. 2. Find the equation of relation between a and b if

$$\sin^3\theta = a, \quad \text{and} \quad \cos^3\theta = b.$$

From the values here given we have

$$\sin\theta = a^{\frac{1}{3}}, \quad \text{and} \quad \cos\theta = b^{\frac{1}{3}}.$$

But for all values of θ, $\sin^2\theta + \cos^2\theta = 1$.

Therefore, substituting, $a^{\frac{2}{3}} + b^{\frac{2}{3}} = 1$,
which is the equation desired.

Ex. 3. Eliminate θ from the equations,

$$a\cos\theta + b\sin\theta = c,$$
$$d\cos\theta + e\sin\theta = f.$$

Solving by any of the ordinary methods of elimination,

$$\sin\theta = \frac{cd - af}{bd - ae},$$
$$\cos\theta = \frac{bf - ce}{bd - ae}.$$

Substituting these values of $\sin\theta$ and $\cos\theta$ in

$$\sin^2\theta + \cos^2\theta = 1,$$

and reducing, we have

$$(bf - ce)^2 + (cd - af)^2 = (bd - ae)^2.$$

Ex. 4. Eliminate θ from the equations

$$\cot\theta + \tan\theta = a; \qquad (1)$$
$$\sec\theta - \cos\theta = b. \qquad (2)$$

From (1) $\qquad a = \dfrac{1}{\tan\theta} + \tan\theta = \dfrac{1 + \tan^2\theta}{\tan\theta}.$

$$\therefore a = \frac{\sec^2\theta}{\tan\theta}. \qquad (3)$$

From (2) $$b = \sec\theta - \frac{1}{\sec\theta} = \frac{\sec^2\theta - 1}{\sec\theta}.$$

$$\therefore\ b = \frac{\tan^2\theta}{\sec\theta}. \qquad (4)$$

From (3) and (4) $$a^2b = \sec^3\theta, \text{ and } ab^2 = \tan^3\theta.$$

But $$\sec^2\theta - \tan^2\theta = 1.$$

$$\therefore\ (a^2b)^{\frac{2}{3}} - (ab^2)^{\frac{2}{3}} = 1,$$

or, $$a^{\frac{4}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}b^{\frac{4}{3}} = 1.$$

Ex. 5. Eliminate θ from the equations

$$\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = \cos 2\theta; \qquad (1)$$

$$\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 2\sin 2\theta. \qquad (2)$$

Multiplying (1) by $\cos\theta$ and (2) by $\sin\theta$ and adding the resulting equations, we obtain

$$\frac{x}{a} = \cos\theta\cos 2\theta + 2\sin 2\theta\sin\theta$$
$$= \cos\theta\cos 2\theta + \sin\theta\sin 2\theta + \sin\theta\sin 2\theta$$
$$= \cos\theta + 2\sin^2\theta\cos\theta. \qquad (3)$$

In like manner, multiplying (1) by $\sin\theta$ and (2) by $\cos\theta$ and subtracting, we obtain

$$\frac{y}{b} = 2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta$$
$$= \sin\theta + 2\sin\theta\cos^2\theta. \qquad (4)$$

Adding (3) and (4),

$$\frac{x}{a} + \frac{y}{b} = \cos\theta + \sin\theta + 2\sin\theta\cos\theta(\cos\theta + \sin\theta)$$
$$= (\cos\theta + \sin\theta)(1 + 2\sin\theta\cos\theta)$$
$$= (\cos\theta + \sin\theta)(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta)$$
$$= (\cos\theta + \sin\theta)^3.$$

$$\therefore\ \cos\theta + \sin\theta = \left(\frac{x}{a} + \frac{y}{b}\right)^{\frac{1}{3}}. \qquad (5)$$

By subtracting (4) from (3) and reducing the result, we find that

$$\cos\theta - \sin\theta = \left(\frac{x}{a} - \frac{y}{b}\right)^{\frac{1}{3}}. \qquad (6)$$

Squaring (5) and (6) and adding the results, we obtain the following, which is the desired equation:

$$2 = \left(\frac{x}{a} + \frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{x}{a} - \frac{y}{b}\right)^{\frac{2}{3}}.$$

Ex. 6. From the following simultaneous equations, find the values of r, ϕ, θ:

$$r \sin\theta \cos\phi = a; \qquad (1)$$
$$r \cos\theta \cos\phi = b; \qquad (2)$$
$$r \sin\phi = c. \qquad (3)$$

Dividing (1) by (2), $\tan\theta = \frac{a}{b}$. $\therefore \theta = \tan^{-1}\frac{a}{b}$. (4)

Squaring (1) and (2) and adding,

$$r^2 \cos^2\phi = a^2 + b^2. \qquad (5)$$

Taking the square root of (5), and then dividing (3) by this result,

$$\tan\phi = \frac{c}{\sqrt{a^2+b^2}}. \quad \therefore \phi = \tan^{-1}\frac{c}{\sqrt{a^2+b^2}}. \qquad (6)$$

Squaring (3) and adding the result to (5),

$$r^2 = a^2 + b^2 + c^2,$$
$$r = \sqrt{a^2+b^2+c^2}.$$

EXERCISE XXXI

1. Find r and θ if $r \sin\theta = 1.25$ and $r\cos\theta = 2.165$.

Eliminate θ from the equations following:

2. $a\cos\theta + b\sin\theta = c$, and $b\cos\theta - a\sin\theta = d$.

3. $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$, and $\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta = 1$.

4. $a\sec\theta - b\tan\theta = c$, and $d\sec\theta + c\tan\theta = b$.

5. $a\cos 2\theta = b\sin\theta$, and $c\sin 2\theta = d\cos\theta$.

6. $\cos\theta + \sin\theta = a$, and $\cos 2\theta = b$.

7. $\sin\theta + \cos\theta = a$, and $\tan\theta + \cot\theta = b$.

8. $\cot\theta + \cos\theta = a$, and $\cot\theta - \cos\theta = b$.

9. $\sin\theta - \cos\theta = a$, and $\csc\theta - \sin\theta = b$.

10. $\sin\theta + \cos\theta\sin 2\theta = a$, and $\cos\theta + \sin\theta\sin 2\theta = b$.

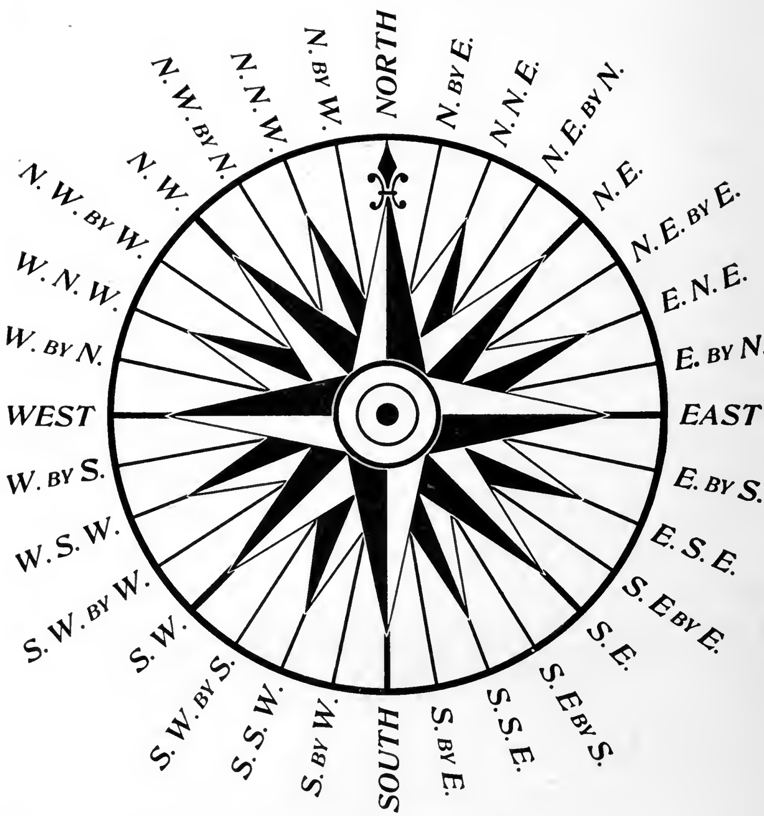
11. $\sec\theta - \cos\theta = a$, and $\csc\theta - \sin\theta = b$.

Eliminate θ and ϕ from the following equations:

12. $\tan\theta + \tan\phi = a$, $\cot\theta + \cot\phi = b$, and $\theta + \phi = \alpha$.

13. $\sin\theta + \sin\phi = a$, $\cos\theta + \cos\phi = b$, and $\theta - \phi = \alpha$.

14. $a\cos^2\theta + b\sin^2\theta = c\cos^2\phi$, $a\sin^2\theta + b\cos^2\theta = d\sin^2\phi$, and $c\tan^2\theta - d\tan^2\phi = 0$.



Mariner's Compass

SPHERICAL TRIGONOMETRY

CHAPTER XV

GENERAL THEOREMS AND FORMULAS

116. Spherical trigonometry is that branch of trigonometry which treats of the solution of spherical triangles.

117. The following definitions and theorems are to be found in works on solid geometry. For a discussion of the definitions and for proofs of the theorems the student is referred to any text-book on that subject.

DEFINITIONS AND THEOREMS

1. The curve of intersection of a plane and a sphere **is a circle.**

2. A **great circle** is a circle formed by a plane that passes through the center of the sphere.

3. A **small circle** is a circle formed by a plane that intersects the sphere without passing through its center.

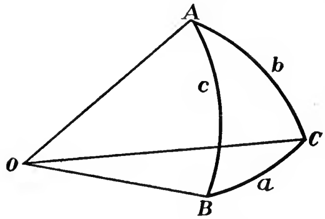
4. Through any two points on the surface of a sphere one and only one great circle can be passed, unless these points are at opposite extremities of a diameter of the sphere.

5. A **spherical angle** is the angle between two arcs of great circles. It is equal to the angle between the tangents to the two circles drawn at their point of intersection; it is also equal in angular magnitude to the dihedral angle formed by the planes of the two great circles.

6. A **spherical polygon** is a portion of the surface of the sphere bounded by three or more arcs of great circles.

7. A **spherical triangle** is a spherical polygon of three sides.

118. Let ABC be any spherical triangle, and O the center of the sphere on whose surface the triangle is drawn. The vertices are represented geometrically by the letters A, B, C, and the same letters are used to designate the angles lying at these vertices respectively. The sides opposite these angles are designated by the corresponding letters a, b, c. Since O is the center of the sphere, $OA = OB = OC$, each being a radius of the same sphere. Also, the arcs a, b, c, are the measures of the central angles BOC, AOC, AOB, respectively.



THEOREMS. The following theorems on spherical triangles were proved in solid geometry.

I. *The sum of any two sides of a spherical triangle is greater than the third side.**

II. *In any spherical triangle the greatest side is opposite the greatest angle, and conversely. Also, equal sides are opposite equal angles.*

III. *Any angle of a spherical triangle is less than* 180°.

IV. *The sum of the angles of a spherical triangle is greater than* 180° *and less than* 540°; *i.e.* $180° < A + B + C < 540°$.

V. *Any side of a spherical triangle is less than* 180°.

VI. *The sum of the sides of a spherical traingle is less than* 360°; *i.e.* $a + b + c < 360°$.

VII. *The difference of any two angles of a spherical triangle has the same sign as the difference of the corresponding opposite sides; e.g.* $A - B$ *and* $a - b$ *are of the same sign.*

VIII. *If from the vertices of a spherical triangle as poles, arcs of great circles are drawn, a second spherical triangle will be formed which is called the* **polar** *of the first triangle.*

Let ABC be any spherical triangle, and a', b', c' be arcs of great circles drawn with A, B, C, respectively as poles. If these arcs are extended and the great circles are fully drawn, the surface of the sphere is divided into

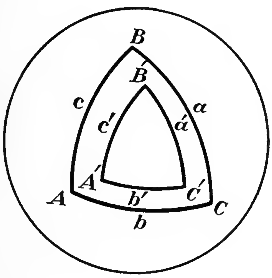
* Three great circles intersect on the surface of a sphere in such a way as to form eight triangles; and one of these triangles always satisfies the theorems of this section. Only such triangles are considered in this work.

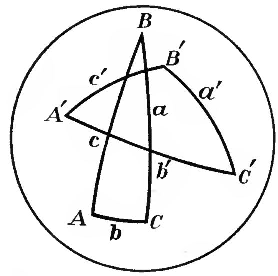
eight spherical triangles. That triangle $A'B'C'$ is called the **polar** of ABC which is so situated that A and A' lie on the same side of BC; B and B' on the same side of AC; C and C' on the same side of AB.

Any angle of a spherical triangle is the supplement of the side opposite in its polar.

If ABC is the polar of $A'B'C'$, then conversely $A'B'C'$ is the polar of ABC.

Let ABC and $A'B'C'$ be two polar triangles, and let a, b, c, and a', b', c', be the sides opposite the like-named angles in the two triangles respectively.





Then, $A = 180° - a'$, $A' = 180° - a$,

$B = 180° - b'$, $B' = 180° - b$,

$C = 180° - c'$. $C' = 180° - c$.

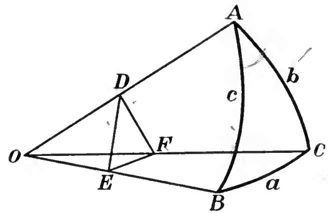
Spherical triangles are called *isosceles*, *equilateral*, *equiangular*, *right*, and *oblique* under the same conditions as are the corresponding plane triangles.

It is to be remembered, however, that a spherical triangle may have one, two, or three right angles. If it contains two right angles, it is called a *bi-rectangular* spherical triangle; and if it contains three right angles, it is a *tri-rectangular* spherical triangle.

NOTE. The length of a side of a spherical triangle, expressed in linear measure, can not be determined until the radius of the sphere is known.

119. FUNDAMENTAL THEOREM. *To express a side of a spherical triangle in terms of the other two sides and of the angle opposite:*

Let ABC be a spherical triangle and O the center of the sphere.



From D, any point in the radius OA, draw DE, DF, perpendicular to OA, in the planes OAB, OAC, respectively. Connect EF.

Then is the plane angle EDF equal to the angle A (Art. 117, p. 177).

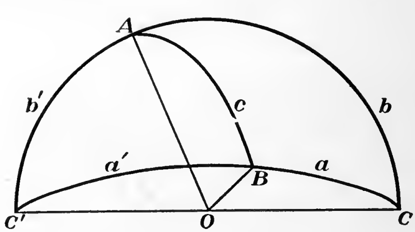
In the plane triangles DEF, OEF, we have (Art. 96, p. 133)
$$EF^2 = DE^2 + DF^2 - 2\,DE \cdot DF \cos A,$$
and
$$EF^2 = OE^2 + OF^2 - 2\,OE \cdot OF \cos a.$$

Equating these values of EF^2 and transposing, we have
$$OE^2 - DE^2 + OF^2 - DF^2 + 2\,DE \cdot DF \cos A - 2\,OE \cdot OF \cos a = 0.$$

Substituting OD^2 for $OE^2 - DE^2$, and also for $OF^2 - DF^2$, this becomes
$$2\,OD^2 + 2\,DE \cdot DF \cos A - 2\,OE \cdot OF \cos a = 0.$$

Dividing by $2\,OE \cdot OF$,
$$\frac{OD}{OE} \cdot \frac{OD}{OF} + \frac{DE}{OE} \cdot \frac{DF}{OF} \cos A - \cos a = 0;$$
i.e.
$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad (1)$$

120. An examination of the figure which accompanies the demonstration in the preceding article shows that the implied supposition is there made that both b and c are less than 90°, but that no restriction is placed on a.



In order to establish the truth of the theorem for all values of a and b we proceed as follows:

Let b be greater than 90°. Produce the arcs CA and CB until they intersect again in C'.

Since $AC > 90°$, we have $AC' < 90°$. Therefore in the triangle ABC',
$$AC' < 90°,$$
and, by hypothesis,
$$AB < 90°,$$
while BC' is unrestricted.

Applying (1), Art. 119, to the triangle ABC', we have

$$\cos a' = \cos b' \cos c + \sin b' \sin c \cos \angle C'AB. \qquad (1)$$

But (Art. 53, p. 78), $\cos a' = -\cos a,$

$$\cos b' = -\cos b,$$

and $\cos \angle C'AB = -\cos A.$

Substituting these values in (1), we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

In like manner it can be shown that the theorem remains true if a and b are both greater than 90°. Hence, it is true for all spherical triangles which come within the scope of our work.

Also, by drawing the perpendiculars DE, DF, from some point in the radius OB in the planes BOC, BOA, respectively, in the figure of Art. 119, we can obtain a corresponding formula for expressing the value of $\cos b$; and by drawing these perpendiculars from some point in the radius OC, in the planes COA, COB, respectively, a similar formula for the value of $\cos c$. Therefore,

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A, \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B, \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} \qquad (2)$$

The above are relations involving the sides and one of the angles of a spherical triangle.

From these equations the following are at once derived:

$$\left.\begin{aligned} \cos A &= \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \\ \cos B &= \frac{\cos b - \cos c \cos a}{\sin c \sin a}, \\ \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b}. \end{aligned}\right\} \qquad (3)$$

These relations express the values of the cosines of the angles of a spherical triangle in terms of the sides of the triangles.

121. After the first of the three formulas in (2) or in (3) in the preceding article has been obtained the others can be derived from it by a cyclic interchange of the letters a, b, c, replacing at the same time A by B, and B by C.

122. The law of sines. From plane trigonometry we have the relation

$$\sin^2 A = 1 - \cos^2 A.$$

Replacing $\cos^2 A$ by its value from (3) in the preceding section,

$$\sin^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}.$$

Expanding, reducing, and rearranging terms, we have

$$\sin^2 A = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Dividing both sides of the equation by $\sin^2 a$ and extracting the square root, we obtain

$$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}. \quad (1)$$

In a precisely similar manner it can be proved that $\frac{\sin B}{\sin b}$ and also that $\frac{\sin C}{\sin c}$ have the same value. Therefore, since each of these ratios has the same value, they are equal to each other.

$$\therefore \quad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}, \quad (2)$$

which is the law of sines. It may be stated in words as follows:

The sines of the sides of a spherical triangle are to each other as the sines of the opposite angles.

An inspection of (1) shows that a cyclic interchange of the letters a, b, c, and A, B, C, leaves the right member of the equation unchanged, while the left member is changed into $\frac{\sin B}{\sin b}$ and $\frac{\sin C}{\sin c}$ successively. Hence, after (1) has been proved, (2) can be established by cyclic interchange of letters.

123. To derive a relation involving the angles and one of the sides of a spherical triangle.

Let $A'B'C'$ and ABC be two spherical triangles polar to each other. Then (Art. 118, p. 179),

$$\left.\begin{aligned} a' &= 180^\circ - A, \\ b' &= 180^\circ - B, \\ c' &= 180^\circ - C. \end{aligned}\right\} \qquad (1)$$

By (1), Art. 119, p. 180,

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'. \qquad (2)$$

But, by (1),

$$\begin{aligned} \cos a' &= -\cos A, & \sin b' &= \sin B, \\ \cos b' &= -\cos B, & \sin c' &= \sin C, \\ \cos c' &= -\cos C. & \cos A' &= -\cos a. \end{aligned}$$

Substituting these values in (2), we have

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a.$$

In like manner we can obtain corresponding values for $\cos B$ and for $\cos C$.

Therefore,

$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b, \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c. \end{aligned}\right\} \qquad (3)$$

From these equations the following are at once derived:

$$\left.\begin{aligned} \cos a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \\ \cos b &= \frac{\cos B + \cos C \cos A}{\sin C \sin A}, \\ \cos c &= \frac{\cos C + \cos A \cos B}{\sin A \sin B}. \end{aligned}\right\} \qquad (4)$$

124. To derive a relation involving two angles and the sides of a spherical triangle.

Resuming (1), Art. 119, p. 180, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Substituting in this equation the value of $\cos c$ obtained from (2), Art. 120, p. 181,

$$\cos a = \cos b\,(\cos a \cos b + \sin a \sin b \cos C) + \sin b \sin c \cos A$$
$$= \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin c \cos A.$$
$$\cos a\,(1 - \cos^2 b) = \sin a \sin b \cos b \cos C + \sin b \sin c \cos A.$$

Substituting for $1 - \cos^2 b$ its value, $\sin^2 b$, and dividing both sides of the equation by $\sin b$, we obtain the desired relation,

$$\cos a \sin b = \sin a \cos b \cos C + \sin c \cos A. \qquad (1)$$

In like manner we can obtain corresponding expressions for the value of $\cos a \sin c$, of $\cos b \sin c$, etc. Therefore,

$$\left.\begin{aligned}
\cos a \sin b &= \sin a \cos b \cos C + \sin c \cos A,\\
\cos a \sin c &= \sin a \cos c \cos B + \sin b \cos A,\\
\cos b \sin a &= \sin b \cos a \cos C + \sin c \cos B,\\
\cos b \sin c &= \sin b \cos c \cos A + \sin a \cos B,\\
\cos c \sin b &= \sin c \cos b \cos A + \sin a \cos C,\\
\cos c \sin a &= \sin c \cos a \cos B + \sin b \cos C,
\end{aligned}\right\} \qquad (2)$$

125. To derive a relation involving two sides and the angles of a spherical triangle.

Resuming the first equation under (3), Art. 123, p. 183, we have

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Substituting in this equation the value of $\cos C$ obtained from the third equation of the same set,

$$\cos A = -\cos B\,(-\cos A \cos B + \sin A \sin B \cos c)$$
$$+ \sin B \sin C \cos a$$
$$= \cos A \cos^2 B - \sin A \sin B \cos B \cos c + \sin B \sin C \cos a.$$

Transposing and factoring,

$$\cos A\,(1 - \cos^2 B) = -\sin A \sin B \cos B \cos c + \sin B \sin C \cos a.$$

Replacing $1-\cos^2 B$ by its value, $\sin^2 B$, and dividing both sides of the equation by $\sin B$, we obtain the desired relation,

$$\cos A \sin B = \cos a \sin C - \cos c \sin A \cos B. \qquad (1)$$

In like manner we can obtain corresponding expressions for the value of $\cos A \sin C$, $\cos B \sin A$, etc. Therefore,

$$\left.\begin{aligned}
\cos A \sin B &= \cos a \sin C - \cos c \cos B \sin A,\\
\cos A \sin C &= \cos a \sin B - \cos b \cos C \sin A,\\
\cos C \sin B &= \cos c \sin A - \cos a \cos B \sin C,\\
\cos C \sin A &= \cos c \sin B - \cos b \cos A \sin C,\\
\cos B \sin A &= \cos b \sin C - \cos c \cos A \sin B,\\
\cos B \sin C &= \cos b \sin A - \cos a \cos C \sin B.
\end{aligned}\right\} \qquad (2)$$

126. From the formulas in Art. 124 a group of important relations is derived, as follows:

From the first of the six formulas there given we have

$$\cos a \sin b = \sin a \cos b \cos C + \sin c \cos A.$$

Dividing both sides of the equation by $\sin a$,

$$\frac{\cos a}{\sin a} \sin b = \cos b \cos C + \frac{\sin c}{\sin a} \cos A.$$

Replacing $\frac{\sin c}{\sin a}$ by its equal $\frac{\sin C}{\sin A}$, this becomes

$$\cot a \sin b = \cos b \cos C + \sin C \frac{\cos A}{\sin A}.$$

$$\therefore \cot a \sin b = \cos b \cos C + \sin C \cot A. \qquad (1)$$

In like manner we can obtain corresponding expressions for the value of $\cot a \sin c$, $\cot b \sin a$, etc. Therefore,

$$\left.\begin{aligned}
\cot a \sin b &= \cos b \cos C + \sin C \cot A,\\
\cot a \sin c &= \cos c \cos B + \sin B \cot A,\\
\cot b \sin c &= \cos c \cos A + \sin A \cot B,\\
\cot b \sin a &= \cos a \cos C + \sin C \cot B,\\
\cot c \sin a &= \cos a \cos B + \sin B \cot C,\\
\cot c \sin b &= \cos b \cos A + \sin A \cot C.
\end{aligned}\right\} \qquad (2)$$

127. The values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, $\tan\frac{A}{2}$, etc., in terms of the sides of the triangle.

From (3), Art. 120, p. 181,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

From this we have

$$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c},$$

by Art. 68,

$$= \frac{\cos(b-c) - \cos a}{\sin b \sin c}.$$

Dividing by 2, and applying (8), Art. 77, p. 100,

$$\frac{1-\cos A}{2} = \frac{\sin\frac{a+b-c}{2}\sin\frac{a-b+c}{2}}{\sin b \sin c}.$$

Putting $a + b + b = 2s$, and replacing $\frac{1-\cos A}{2}$ by its equal $\sin^2\frac{A}{2}$ (Art. 82, p. 108), we have

$$\sin^2\frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}.$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}. \qquad (1)$$

In like manner,

$$1 + \cos A = \frac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c},$$

$$\frac{1+\cos A}{2} = \frac{\cos a - \cos(b+c)}{2 \sin b \sin c},$$

$$\cos^2\frac{A}{2} = \frac{\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}}{\sin b \sin c},$$

$$\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}. \qquad (2)$$

Dividing (1) by (2), we have

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}. \qquad (3)$$

Since any angle of a spherical triangle is less than 180°, all the functions of the half angles are positive; *i.e.* $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, $\tan\frac{A}{2}$, are all positive. Therefore the signs of the radical expressions in (1), (2), and (3) are positive.

Since s, a, b, c, $s-a$, $s-b$, $s-c$ are severally less than 180° and positive, the values obtained in (1), (2), and (3) are real.

128. The values of $\sin\frac{a}{2}$, $\cos\frac{a}{2}$, $\tan\frac{a}{2}$, etc., in terms of the angles of the triangle.

From (4), Art. 123, p. 183,

$$\cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C}.$$

Therefore, $$1-\cos a=\frac{\sin B\sin C-\cos B\cos C-\cos A}{\sin B\sin C}.$$

$$=\frac{-\cos(B+C)-\cos A}{\sin B\sin C},$$

and $$1+\cos a=\frac{\sin B\sin C+\cos B\cos C+\cos A}{\sin B\sin C}$$

$$=\frac{\cos(B-C)+\cos A}{\sin B\sin C}.$$

Putting $A+B+C=2S$, and proceeding as in the last section, we obtain

$$\sin\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}; \qquad (1)$$

$$\cos\frac{a}{2}=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}; \qquad (2)$$

$$\tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}. \qquad (3)$$

Since any side of a spherical triangle is less than 180°, all the functions of the half sides are positive; *i.e.* $\sin\frac{a}{2}$, $\cos\frac{a}{2}$, $\tan\frac{a}{2}$, are all positive. Therefore the signs of the radical expressions in (1), (2), and (3) are positive.

To prove that these expressions are real we proceed as follows:

Let $A'B'C'$ be the polar triangle of ABC, and let a', b', c', be the sides of $A'B'C'$ which lie opposite the angles A, B, C, respectively of the original triangle.

Then, since a', b', c', are supplements of A, B, C, respectively, and since $a' < b' + c'$, we have

$$180° - A < (180° - B) + (180° - C).$$

Transposing, $$B + C - A < 180°;$$

i.e. $$S - A < 90°.$$

Therefore, $\cos(S - A)$ is positive.

Also, since $A + B + C$ lies between 360° and 540°, S lies between 180° and 270°. Hence $\cos S$ is negative; *i.e.* $-\cos S$ is positive.

Therefore the radical expressions in (1), (2), and (3) are real.

129. Gauss's equations. From Art. 69, p. 92,

$$\cos\left(\frac{A}{2} + \frac{B}{2}\right) = \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}.$$

Substituting in this equation the values of $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$, and corresponding values for $\cos\frac{B}{2}$ and $\sin\frac{B}{2}$ (Art. 127, p. 186), we have

$$\cos\left(\frac{A}{2} + \frac{B}{2}\right) = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \cdot \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}}$$

$$- \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \cdot \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}}$$

$$= \frac{\sin s - \sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin b \sin a}}.$$

But by Art. 77, p. 100, and Art. 80, p. 106,

$$\frac{\sin s - \sin (s - c)}{\sin c} = \frac{2 \cos \frac{2s - c}{2} \sin \frac{c}{2}}{2 \sin \frac{c}{2} \cos \frac{c}{2}} \tag{1}$$

$$= \frac{\cos \frac{a+b}{2}}{\cos \frac{c}{2}},$$

and by Art. 127, p. 186,

$$\sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}} = \sin \frac{C}{2}.$$

Substituting these values in (1), and reducing, we have

$$\cos \frac{A+B}{2} = \frac{\cos \frac{a+b}{2}}{\cos \frac{c}{2}} \sin \frac{C}{2}. \tag{2}$$

In like manner corresponding values can be obtained for $\sin \frac{A+B}{2}$, $\sin \frac{A-B}{2}$, and $\cos \frac{A-B}{2}$. These four relations, which are commonly known as Gauss's Equations, are as follows:

$$\cos \frac{A+B}{2} = \frac{\cos \frac{a+b}{2}}{\cos \frac{c}{2}} \sin \frac{C}{2}; \tag{3}$$

$$\sin \frac{A+B}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{c}{2}} \cos \frac{C}{2}; \tag{4}$$

$$\cos \frac{A-B}{2} = \frac{\sin \frac{a+b}{2}}{\sin \frac{c}{2}} \sin \frac{C}{2}; \tag{5}$$

$$\sin \frac{A-B}{2} = \frac{\sin \frac{a-b}{2}}{\sin \frac{c}{2}} \cos \frac{C}{2}. \tag{6}$$

130. Napier's analogies. From Gauss's Equations the following are derived. The method of derivation is obvious, and the work is left as an exercise for the student.

$$\tan\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2}; \tag{1}$$

$$\tan\frac{A-B}{2}=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2}; \tag{2}$$

$$\tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}; \tag{3}$$

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}. \tag{4}$$

131. Special formulas for the solution of spherical right triangles. If one of the angles of the triangle, as C, is a right angle, the following special formulas are derived from those established in the preceding sections:

From (2), Art. 120, p. 181,

$$\cos c=\cos a\cos b+\sin a\sin b\cos C. \tag{1}$$

But, since $C=90°$, $\cos C=0$. Therefore the second term of the right member becomes zero. Therefore,

$$\cos c=\cos a\cos b.$$

In a manner similar to that just employed, the following formulas are derived for the special case when C is a right angle.

From (2), Art. 122, p. 182, formulas for finding either of the oblique angles when the hypotenuse and the opposite leg are given.

$$\left.\begin{aligned}\sin A&=\frac{\sin a}{\sin c},\\ \text{and}\qquad \sin B&=\frac{\sin b}{\sin c}.\end{aligned}\right\} \tag{2}$$

From (3), Art. 123, p. 183, formulas for finding either of the oblique angles when the opposite leg and the other oblique angle are given.

$$\left.\begin{aligned}\cos A &= \cos a \sin B,\\ \cos B &= \cos b \sin A.\end{aligned}\right\} \quad (3)$$

From (2) and (3) are derived the following formulas for finding an oblique angle when the hypotenuse and the adjacent leg are given.

$$\left.\begin{aligned}\cos A &= \tan b \cot c,\\ \cos B &= \tan a \cot c.\end{aligned}\right\} \quad (4)$$

From (2), Art. 126, p. 185, formulas for finding the oblique angles when the legs are given.

$$\left.\begin{aligned}\tan A &= \frac{\tan a}{\sin b},\\ \tan B &= \frac{\tan b}{\sin a}.\end{aligned}\right\} \quad (5)$$

From (3), Art. 123, p. 183, formulas for finding the legs when the two oblique angles are given.

$$\left.\begin{aligned}\cos a &= \frac{\cos A}{\sin B},\\ \cos b &= \frac{\cos B}{\sin A}.\end{aligned}\right\} \quad (6)$$

Multiplying together the two formulas just obtained, and replacing the left member of the product, $\cos a \cos b$, by its value given in (1), we have the following formula for finding the hypotenuse when the two oblique angles are given:

$$\cos c = \cot A \cot B. \quad (7)$$

132. Napier's rules. The formulas of the last section are sufficient for the solution of every possible case that can arise under spherical right triangles. But it is often better to solve the various cases that arise under right triangles by two convenient and simple rules devised by Napier, the inventor of logarithms.

These rules are constructed by supposing that a right triangle has five parts. These parts, which are usually called Napier's parts, are

(1) The two legs.

(2) The complement of the hypotenuse.

(3) The complements of the two oblique angles.

The right angle is not considered, and plays no part whatever in the solution of a triangle by this method.

Any one of the five parts may be regarded as the middle part. The two parts immediately adjacent to this are called the adjacent parts, and the other two are called the opposite parts.

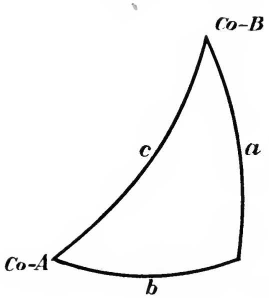
Napier's rules for the solution of spherical right triangles are as follows:

1. The sine of the middle part is equal to the product of the **tan**gents of the **ad**jacent parts.

2. The sine of the middle part is equal to the product of the **co**sines of the **op**posite parts.

The similarity of the vowel sounds in the syllables *tan-*, *ad-* and *co-*, *op-* renders it easy to remember these rules, and also to distinguish them from each other.

To test the correctness of these rules, assume any three parts as the given parts. For example, let the given parts be a, b, and *co-A*. In this case b is the middle part, and a, *co-A*, are to be considered adjacent parts. Hence we have



$$\sin b = \tan a \tan (co\text{-}A)$$
$$= \tan a \cot A.$$

This is the same as the first of the two formulas under (5), Art. 131, p. 191, which has already been proved to be true.

As another illustration, let the given parts be a, *co A*, *co-B*. Here *co-A* is the middle part, and a, *co-B* are to be considered opposite parts. Hence

$$\sin (co\text{-}A) = \cos a \cos (co\text{-}B),$$
$$\cos A = \cos a \sin B.$$

This is the same as the first of the two formulas under (3), Art. 130, p. 190, which has already been proved to be true.

In like manner Napier's rules as applied to any other group of three parts will be found to reduce to one of the formulas already proved.

133. Definition. Two angles, or an angle and a side, are said to be of the **same species** when both are greater or both are less than 90°; they are said to be of **opposite species** when one is greater and the other is less than 90°.

In any right triangle if a and b are of the same species, the hypotenuse c is less than 90°; if a and b are of opposite species, c is greater than 90°.

This follows from (1), Art. 130, p. 190. For if a and b are both greater or both less than 90°, the product $\cos a \cos b$ is positive. Therefore $\cos c$ is positive; therefore c is less than 90°.

But if a and b are of opposite species, the product $\cos a \cos b$ is negative. Therefore $\cos c$ is negative; therefore c is greater than 90°.

EXERCISE XXXII

1. Prove that in any right triangle a leg and the angle opposite are of the same species.

2. By the aid of Napier's rules derive the formulas in (6), Art. 131, p. 190.

3. If the hypotenuse of a right triangle is equal to 90°, what must be the values of a and b? Why?

4. Prove $\tan^2 \frac{B}{2} = \frac{\sin(c-a)}{\sin(c+a)}$.

5. Prove $\tan^2 \frac{C}{2} = \frac{\cos(A+B)}{\cos(B-A)}$.

6. If $a = 90°$ and $b = 90°$, what must be the values of the remaining parts of the right triangle?

7. In a right triangle a side and the hypotenuse are of the same or of opposite species according as the included angle is less or greater than 90°.

(2)

CHAPTER XVI

SOLUTION OF SPHERICAL TRIANGLES

134. A spherical triangle is determined when any three of its parts are known. That is, when any three parts are given, the remaining parts can be computed.

In the solution of spherical triangles we have six cases to consider, as follows: having given,

(1) The three sides.
(2) Two sides and the included angle.
(3) Two sides and the angle opposite one of them.
(4) Two angles and the side opposite one of them.
(5) Two angles and the included side.
(6) The three angles.

135. The right triangle. We proceed first to the consideration of the right triangle. We shall suppose that C is the right angle; and here, as in Plane Trigonometry, only two parts are known in addition to the right angle.

136. Ambiguous cases. Whenever a solution is obtained by means of the sine or the cosecant, the solution is ambiguous, because, both sine and cosecant being positive in the second quadrant as well as in the first, a given value of either of these functions is, in general, satisfied by two angles, one in the first and the other in the second quadrant.

Hence, whenever a required part of a spherical triangle is found by means of the sine or the cosecant, it is necessary to test the result, and to determine whether or not both solutions are admissible.

When a solution is found by means of the cosine, tangent, cotangent, or secant, there is no ambiguity, since each of these functions is positive in the first quadrant and negative in the second quadrant.

For this reason it is of great importance that the student should note carefully the sign of each of the functions that appear in an equation.

137. Case 1. **Given two legs, *a* and *b*; to find *c*, *A*, *B*.** The formulas for solution are contained in (1) and (5), Art. 131, p. 190, or are obtained directly from Napier's Rules, and are as follows:

$$\cos c = \cos a \cos b; \tag{1}$$

$$\tan A = \frac{\tan a}{\sin b}; \tag{2}$$

$$\tan B = \frac{\tan b}{\sin a}. \tag{3}$$

For a check formula use $\cos c = \cot A \cot B$.

Ex. 1. Given $a = 46° 50'$, $b = 31° 15'$; find c, A, B.

$$\begin{aligned} \log \cos a &= 9.83513 - 10 \\ \log \cos b &= 9.93192 - 10 \\ \log \cos c &= 9.76705 - 10. \\ \therefore c &= 54° 12' 25''. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 10.02781 - 10 \\ \text{colog} \sin b &= 10.28502 - 10 \\ \log \tan A &= 10.31283 - 10. \\ A &= 64° 3' 9''. \end{aligned}$$

$$\begin{aligned} \log \tan b &= 9.78306 - 10 \\ \log \sin a &= 9.86295 - 10 \\ \log \tan B &= 9.92011 - 10. \\ B &= 39° 45' 32''. \end{aligned}$$

Since c is obtained by means of its cosine and A and B by means of their tangents, there is no ambiguity respecting the result. Both a and b are in the first quadrant; therefore $\cos a$ and $\cos b$ are positive. It follows from this that the right member of (1) is positive when applied to this particular problem; therefore $\cos c$ is positive, and consequently c is in the first quadrant.

In like manner it can be shown that A and B are in the first quadrant.

When only one solution exists that will satisfy the conditions of a problem, the solution is said to be **unique**.

Ex. 2. Given $a = 38° 44' 40''$, $b = 42° 26' 28''$;

find $c = 54° 51' 37''$, $A = 49° 56' 12''$, $B = 55° 36' 44''$.

138. Case 2. **Given the hypotenuse c, and one of the legs a; to find b, A, B.** The formulas for solution are (Art. 131, p. 190)

$$\cos b = \frac{\cos c}{\cos a},$$

$$\sin A = \frac{\sin a}{\sin c},$$

$$\cos B = \frac{\tan a}{\tan c}.$$

For a check formula use

$$\cos B = \cos b \sin A \text{ (Art. 131, p. 191).}$$

The solutions for b and B, being obtained in each case by means of a cosine, are unique.

The solution for A, being obtained by means of its sine, is apparently ambiguous. But by Art. 133, p. 193, a and A are of the same species. Hence, as a is given, the species of A becomes known at once, and the ambiguity disappears.

Ex. 1. Given $c = 54° 36' 30''$, $a = 23° 17' 40''$;
find $b = 50° 54' 30''$, $A = 29° 1' 5''$, $B = 72° 11' 20''$.

Ex. 2. Given $c = 98° 15' 12''$, $a = 133° 40' 24''$;
find $b = 78° 0' 7''$, $A = 133° 2' 30''$, $B = 81° 15' 40''$.

139. Case 3. **Given one of the legs a and the opposite angle A; to find b, c, B.** The formulas for solution are as follows, (Art. 131, p. 190):

$$\sin c = \frac{\sin a}{\sin A},$$

$$\sin b = \frac{\tan a}{\tan A},$$

$$\sin B = \frac{\cos A}{\cos a}.$$

For a check formula use $\sin b = \dfrac{\tan a}{\tan A}$. (Art. 131, p. 191)

The solution is ambiguous, being obtained in each case by means of a sine. The different cases that may arise are as follows:

(1) If $a = A$, then $\sin a = \sin A$, $\tan a = \tan A$, and $\cos a = \cos A$; therefore $\sin c = 1$, $\sin b = 1$, and $\sin B = 1$. Hence the solution is unique.

(2) If c and a are of the same species, then $B < 90°$; therefore $b < 90°$ (Ex. 7, p. 193).

(3) If c and a are of opposite species, then $B > 90°$; therefore $b > 90°$ (Ex. 7, p. 193).

After c has been computed b and B may be found, if other formulas than those given above are desired, by the following (Art. 131, p. 191):

$$\cos b = \frac{\cos c}{\cos a},$$

$$\cos B = \frac{\tan a}{\tan c}.$$

These formulas give unique solutions for b and B, but for obtaining c it is necessary to make use of the sine. As any given value of the sine is satisfied by two supplementary values of the angle, this case often gives two solutions.

Ex. 1. Given $a = 70° 55' 50''$, $A = 82° 58' 6''$;

Find $c_1 = 72° 13' 45''$, $b_1 = 20° 54' 18''$, $B_1 = 22° 0' 19''$.

or, $c_2 = 107° 46' 15''$, $b_2 = 159° 5' 42''$, $B_2 = 157° 59' 41''$.

Ex. 2. Given $a = 76° 59' 59''$, $A = 39° 50' 56''$.

The triangle is impossible. Why?

140. Case 4. **Given one of the legs a and the adjacent angle B; to find c, b, A.** The formulas for solution are (Art. 131, p. 191)

$$\tan c = \frac{\tan a}{\cos B},$$

$$\cos A = \cos a \sin B,$$

$$\tan b = \sin a \tan B.$$

For a check formula use

$$\cos A = \frac{\tan b}{\tan c}. \qquad \text{(Art. 131, p. 191)}$$

The solution is unique. Why?

Ex. 1. Given $a = 21° 5' 15''$, $B = 39° 8' 10''$;

find $c = 26° 26' 6''$, $A = 53° 55' 13''$, $b = 16° 19' 5''$.

Ex. 2. Given $a = 59° 27' 32''$, $B = 36° 24' 25''$;

find $c = 64° 35' 56''$, $b = 32° 25' 17''$, $A = 72° 26' 47''$.

141. CASE 5. **Given the hypotenuse c and one of the oblique angles A; to find a, b, B.** The formulas for solution are (Art. 131, p. 190)

$$\sin a = \sin c \sin A,$$

$$\tan b = \tan c \cos A,$$

$$\cot B = \cos c \tan A.$$

For a check formula use

$$\sin a = \tan b \cot B \text{ (Art. 131, p. 191).}$$

The solution for a, being obtained by means of its sine, is apparently ambiguous. But since A is given, and since a and A are of the same species, the proper value of a can always be determined. Hence the solution is unique.

Ex. 1. Given $c = 117° 39' 48''$, $A = 127° 20' 25''$;

find $a = 135° 14' 18''$, $b = 49° 9' 58''$, $B = 58° 40' 37''$.

Ex. 2. Given $c = 68° 50' 31''$, $A = 55° 11' 17''$;

find $a = 49° 58'$, $b = 55° 51' 53''$, $B = 62° 33' 58''$.

142. CASE 6. **Given the two oblique angles A, B; to find a, b, c.** The formulas for solution are (Art. 131, p. 191)

$$\cos a = \frac{\cos A}{\sin B},$$

$$\cos b = \frac{\cos B}{\sin A},$$

$$\cos c = \cot A \cot B.$$

For a check formula use

$$\cos c = \cos a \cos b \text{ (Art. 131, p. 190).}$$

The solution is unique.

Ex. 1. Given $A=63° 25' 32''$, $B=136° 1' 27''$;
find $a=49° 53' 16''$, $b=143° 34' 30''$, $c=121° 13' 34''$.

Ex. 2. Given $A=119° 20' 11''$, $B=114° 7' 35''$;
find $a=122° 28' 6''$, $b=117° 57' 42''$, $c=75° 25' 16''$.

143. The isosceles spherical triangle. An isosceles spherical triangle can always be solved by means of the formulas employed in the solution of spherical right triangles; for, by passing an arc of a great circle through the vertex and the middle point of the side opposite, the isosceles triangle can always be divided into two symmetrical right triangles.

EXERCISE XXXIII

1. In a right spherical triangle given $c=20° 50' 52''$, $a=15° 12' 44''$; find b, A, B.

2. In a right spherical triangle given $a=75° 28' 24''$, $b=33° 37' 8''$; find c, A, B.

3. In a right spherical triangle given $a=66° 9' 9''$, $A=155° 49' 46''$; find b, c, B.

4. In a right spherical triangle given $a=122° 5'$, $B=125° 40'$; find b, c, A.

5. In a right spherical triangle given $c=115° 35' 4''$, $A=57° 29'$; find a, b, B.

6. In a right spherical triangle given $A=45° 23' 8''$, $B=58° 17'$; find a, b, c.

7. In a right spherical triangle given $c=80° 28' 44''$, $A=33° 20' 24''$; find a, b, B.

8. In a right spherical triangle given $c=139° 42'$, $a=21° 47' 46''$; find b, A, B.

9. In a right spherical triangle given $a=110° 38'$, $B=153° 55' 40''$; find b, c, A.

10. In a right spherical triangle given $a=112° 49'$, $A=100° 27'$; find b, c, B.

11. In a right spherical triangle given $a = 55° 52'$, $b = 34° 46' 42''$; find c, A, B.

12. In a right spherical triangle given $A = 54° 20'$, $B = 64° 49' 51''$; find a, b, c.

13. In a right spherical triangle if $a = b$, prove that $\cos^2 a = \cos c$.

14. In a right spherical triangle prove that

$$\sin b = \cos c \tan a \tan B.$$

15. In a right spherical triangle prove that

$$\sin^2 A + \sin^2 B = 1 + \sin^2 a \sin^2 B.$$

16. In a right spherical triangle prove that

$$\sin (b + c) = 2 \cos^2 \frac{A}{2} \cos b \sin c.$$

THE OBLIQUE SPHERICAL TRIANGLE

144. In solving oblique spherical triangles we have six cases to consider, as follows:

CASE 1. **Given the three sides a, b, c; to find A, B, C.** The formulas for solution are (Art. 127, p. 186).

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}; \tag{1}$$

$$\tan \frac{B}{2} = \sqrt{\frac{\sin (s - c) \sin (s - a)}{\sin s \sin (s - b)}}; \tag{2}$$

$$\tan \frac{C}{2} = \sqrt{\frac{\sin (s - a) \sin (s - b)}{\sin s \sin (s - c)}}. \tag{3}$$

The corresponding formulas for the sines or for the cosines of the half angles may be employed (Art. 127, p. 186), but in general the tangent formulas are to be preferred.

If all the angles are to be found, a saving of labor can be effected in the following manner.

Multiply both numerator and denominator of the fraction under the radical sign in (1) by sin $(s-a)$. Then let

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

and we may write

$$\tan\frac{A}{2} = \frac{\tan r}{\sin(s-a)}.$$

Making the corresponding changes in (2) and (3), we have the three equations:

$$\tan\frac{A}{2} = \frac{\tan r}{\sin(s-a)},$$

$$\tan\frac{B}{2} = \frac{\tan r}{\sin(s-b)},$$

$$\tan\frac{C}{2} = \frac{\tan r}{\sin(s-c)}.$$

If these formulas are employed, it will be found that the work of solution can be more compactly arranged and more conveniently carried out than by the use of any other method.

Ex. 1. Given $a = 51° 43' 18''$, $b = 38° 2' 20''$, $c = 75° 11' 30''$; find A.

$$\begin{aligned} a &= 51° 43' 18'' \\ b &= 38° \ \ 2' 20'' \\ c &= 75° 11' 30'' \\ \hline 2s &= 164° 57' 8'' \\ s &= 82° 28' 34'' \\ \hline s-a &= 30° 45' 16'' \\ s-b &= 44° 26' 14'' \\ s-c &= \ \ 7° 17' \ \ 4'' \\ \hline s &= 82° 28' 34'' \quad \textit{Check.} \end{aligned}$$

$$\begin{aligned} \log\sin(s-b) &= 9.84518-10 \\ \log\sin(s-c) &= 9.10311-10 \\ \text{colog}\sin s &= 0.00375 \\ \text{colog}\sin(s-a) &= 0.29127 \\ \hline & 2)\overline{19.24331-20} \\ \log\tan\frac{A}{2} &= \ \ 9.62166-10 \\ \frac{A}{2} &= 22° 42' 27.4'' \\ A &= 45° 24' 55'' \end{aligned}$$

Ex. 2. Given $a = 125° 40' 14''$, $b = 53° 56' 12''$, $c = 98° 51' 16''$; find A, B, C.

$$
\begin{aligned}
a &= 125° \; 40' \; 14'' \\
b &= \;\; 53° \; 56' \; 12'' \\
c &= \;\; 98° \; 51' \; 16'' \\
2s &= 278° \; 27' \; 42'' \\
s &= 139° \; 13' \; 51'' \\
s - a &= 13° \; 33' \; 37'' \\
s - b &= 85° \; 17' \; 39'' \\
s - c &= 40° \; 22' \; 35'' \\
\log \sin (s - a) &= 9.37008 - 10 \\
\log \sin (s - b) &= 9.99854 - 10 \\
\log \sin (s - c) &= 9.81145 - 10 \\
\text{colog} \sin s &= 0.12071 \\
\log \tan^2 r &= 19.30078 - 20 \\
\log \tan r &= 9.65039 - 10
\end{aligned}
$$

$$
\begin{aligned}
\log \tan \frac{A}{2} &= 0.28031 \\
\log \tan \frac{B}{2} &= 9.65185 - 10 \\
\log \tan \frac{C}{2} &= 9.83894 - 10 \\
\frac{A}{2} &= 62° \; 19' \; 33'' \\
\frac{B}{2} &= 24° \;\; 9' \; 38'' \\
\frac{C}{2} &= 34° \; 36' \; 40'' \\
A &= 124° \; 39' \; 6'' \\
B &= 48° \; 19' \; 16'' \\
C &= 69° \; 13' \; 20''
\end{aligned}
$$

EXERCISE XXXIV

1. In a spherical triangle given $a = 119° 22' 27''$, $b = 60° 44' 40''$, $c = 108° 37' 3''$; find A, B, C.

2. In a spherical triangle given $a = 53° 42'$, $b = 118° 39' 28''$, $c = 130° 38' 20''$; find A, B, C.

3. In a spherical triangle given $a = 129° 17' 36''$, $b = 109° 29' 18''$, $c = 83° 14'$; find the largest angle.

4. In a spherical triangle given $a = 22° 56' 46''$, $b = 60° 47'$, $c = 69° 49' 32''$; find B and C.

145. Case 2. **Given two sides a, b, and the included angle, C; to find A, B, c.** The angles A, B, may be found by the first two of Napier's Analogies (Art. 130, p. 190):

$$\tan \frac{A + B}{2} = \frac{\cos \frac{a - b}{2}}{\cos \frac{a + b}{2}} \cot \frac{C}{2},$$

$$\tan \frac{A - B}{2} = \frac{\sin \frac{a - b}{2}}{\sin \frac{a + b}{2}} \cot \frac{C}{2}.$$

From the values of $\frac{A-B}{2}$ and $\frac{A+B}{2}$ obtained from these equations the values of A and B can at once be found.

The value of c can then be obtained from any one of Gauss's Equations (Art. 129, p. 189); for example,

$$\cos\frac{c}{2} = \frac{\cos\frac{a+b}{2}}{\cos\frac{A+B}{2}}\sin\frac{C}{2}.$$

The solution is unique.

EXERCISE XXXV

1. In a spherical triangle given $a = 85° 54' 16''$, $b = 125° 7' 27''$, $C = 52° 6' 26''$; find A, B, c.

2. In a spherical triangle given $a = 119° 32' 30''$, $b = 86° 31' 35''$, $C = 49° 40' 22''$; find A, B, c.

3. In a spherical triangle given $b = 61° 23' 18''$, $c = 48° 30' 6''$, $A = 60° 53' 24''$; find B, C, a.

4. In a spherical triangle given $a = 72° 40' 40''$, $c = 110° 33' 38''$, $B = 53° 50' 20''$; find A, C, b.

5. In a spherical triangle given $b = 68° 20' 25''$, $c = 52° 18' 15''$, $A = 117° 12' 20''$; find B, C, a.

146. Case 3. **Given two sides a, b, and the angle opposite one of them A; to find B, C, c.** The value of B can be found by means of the law of sines (Art. 122, p. 182), from which we have

$$\sin B = \frac{\sin A}{\sin a}\sin b. \tag{1}$$

After B has been determined C and c can be found by the use of the first and the third of Napier's Analogies.

$$\cot\frac{C}{2} = \frac{\cos\frac{a+b}{2}}{\cos\frac{a-b}{2}}\tan\frac{A+B}{2}; \tag{2}$$

$$\tan\frac{c}{2} = \frac{\cos\frac{A+B}{2}}{\cos\frac{A-B}{2}}\tan\frac{a+b}{2}. \tag{3}$$

Since B is determined by means of its sine, the solution is ambiguous.

The following tests may be conveniently employed to determine the number of solutions.

If $\sin A \sin b > \sin a$, there is no solution; for in that case $\sin B > 1$, which is impossible.

If $\sin A \sin b < \sin a$, (1) is satisfied by two supplementary values of B. But $\frac{A+B}{2}$ and $\frac{a+b}{2}$ are necessarily of the same species. Therefore, if both these values of B satisfy this condition, there are two solutions; if not, there is but one.

NOTE. To make use of the test just given it is necessary that we first solve for B. There are several methods of testing for the number of solutions without first finding B, but it is not thought best to include any of them in this work. For a full explanation of them the student is referred to more extended treatises on the subject of Spherical Trigonometry.

Ex. 1. Given $a = 56° 30'$, $b = 31° 20'$, $A = 105° 11' 10''$; find B, C, c.

Since in this case $\sin A \sin b < \sin a$, there may be either one or two solutions. To test for the number of solutions we find the possible values of B.

$$\log \sin A = 9.98456 - 10$$

$$\log \sin b = 9.71602 - 10$$

$$\text{colog} \sin a = 0.07889$$

$$\log \sin B = \overline{9.77947 - 10}$$

$$B = 37° 0' 3'',$$

or,

$$B = 142° 59' 57''.$$

We have from data given, $\frac{a+b}{2} < 90°$.

$$\therefore \frac{A+B}{2} < 90°.$$

This shows that only the smaller of the two values of B is admissible.

Therefore there is but one solution.

The work of solution may be compactly and conveniently arranged as follows:

$$a+b=87^\circ\ 50' \qquad \frac{a+b}{2}=43^\circ\ 35'$$

$$a-b=25^\circ\ 10' \qquad \frac{a-b}{2}=12^\circ\ 35$$

$$A+B=142^\circ\ 11'\ 13'' \qquad \frac{A+B}{2}=71^\circ\ 5'\ 36.5''$$

$$A-B=68^\circ\ 11'\ 7'' \qquad \frac{A-B}{2}=34^\circ\ 5'\ 33.5''$$

$$\log\sin\frac{A+B}{2}=9.97591-10 \qquad \log\sin\frac{a+b}{2}=9.84112-10$$

$$\text{colog}\sin\frac{A-B}{2}=0.25140 \qquad \text{colog}\sin\frac{a-b}{2}=0.66182$$

$$\log\tan\frac{a-b}{2}=\underline{9.34874-10} \qquad \log\tan\frac{A-B}{2}=\underline{9.83053-10}$$

$$\log\tan\frac{c}{2}=9.57605-10 \qquad \log\cot\frac{C}{2}=0.33347$$

$$\frac{c}{2}=20^\circ\ 38'\ 38'' \qquad \frac{C}{2}=24^\circ\ 53'\ 31''$$

$$c=41^\circ\ 17'\ 16'' \qquad C=49^\circ\ 47'\ 2''$$

EXERCISE XXXVI

1. In a spherical triangle given $a=71^\circ\ 14'$, $b=122^\circ\ 27'\ 18''$, $A=77^\circ\ 23'\ 24''$; find B, C, c.

2. In a spherical triangle given $a=80^\circ\ 5'\ 16''$, $b=82^\circ\ 4'$, $A=83^\circ\ 34'\ 12''$; find B, C, c.

3. In a spherical triangle given $a=151^\circ\ 22'\ 30''$, $b=133^\circ\ 31'\ 25''$, $A=143^\circ\ 32'\ 28''$; find B, C, c.

4. In a spherical triangle given $a=30^\circ\ 38'$, $b=31^\circ\ 29'\ 45''$, $A=87^\circ\ 53'\ 20''$; find the remaining parts.

147 CASE 4. **Given two angles A, B, and the side opposite one of them a; to find C, b, c.** As in the preceding case one of the parts, in this case b, can be found by means of the law of sines, from which we have (Art. 122, p. 182)

$$\sin b=\frac{\sin B}{\sin A}\sin a. \qquad (1)$$

The values of c and C can then be found by means of the fourth and the second of Napier's Analogies:

$$\tan\frac{c}{2}=\frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}}\tan\frac{a-b}{2}; \quad (2)$$

$$\cot\frac{C}{2}=\frac{\sin\frac{a+b}{2}}{\sin\frac{a-b}{2}}\tan\frac{A-B}{2}. \quad (3)$$

The solution is ambiguous, the value of b being determined by means of its sine.

If $\sin B\sin a > \sin A$, there is no solution; for in that case $\sin b > 1$, which is impossible.

If $\sin B \sin a < \sin A$, (1) is satisfied by two supplementary values of b. To ascertain whether or not both these values are admissible we proceed in a manner similar to that employed in the last case. If both values of b satisfy the condition imposed by the fact that $\frac{A+B}{2}$ and $\frac{a+b}{2}$ are of the same species, there are two solutions; otherwise there is but one.

NOTE. The number of solutions can always be determined by forming the polar of the given triangle and then determining by the tests under Case 3 the number of solutions of that triangle. The number of solutions of the given triangle is always the same as the number of solutions of its polar.

Ex. 1. Given $A = 29° 43' 12''$, $B = 45° 4' 18''$, $a = 36° 19' 32''$; find b, c, C.

In this case $\sin B \sin a < \sin A$; therefore there may be either one or two solutions. Solving for b, we proceed as follows:

$$\begin{aligned} \log\sin B &= 9.85003 - 10 \\ \log\sin a &= 9.77260 - 10 \\ \text{colog}\sin A &= \underline{0.30473} \\ \log\sin b &= 9.92736 - 10 \\ b &= 57° 48' 38'', \end{aligned}$$

or, $b = 122° 13' 22''$.

We have from data given, $\frac{A+B}{2} < 90°$.

$$\therefore \frac{a+b}{2} < 90°.$$

Both of the values of b just found satisfy this condition. Hence, there are two solutions. The values of C and c can now be found in the ordinary manner, both values of b being employed.

EXERCISE XXXVII

1. In a spherical triangle given $A = 109° 20' 10''$, $B = 134° 16' 24''$, $a = 148° 48' 40''$; find b, c, C.

2. In a spherical triangle given $A = 113° 30'$, $B = 125° 31' 34''$, $a = 66° 44' 40''$; find b, c, C.

3. In a spherical triangle given $A = 28° 35' 5''$, $B = 47° 51' 15''$, $a = 38° 41' 32''$; find b, c, C.

4. In a spherical triangle given $A = 24° 30'$, $B = 38° 15'$, $a = 65° 22'$; find b, c, C.

148. CASE 5. **Given a side c and the two adjacent angles A, B; to find a, b, C.** The third and fourth of Napier's Analogies may be used for determining the values of a and b (Art. 130, p. 190):

$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}.$$

$$\tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}.$$

From these formulas the values of a and b can be obtained. The value of C can then be found by means of the first of Napier's Analogies:

$$\tan\frac{C}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{A+B}{2}.$$

The solution is unique.

Ex. 1. Given $A = 108° \ 28' \ 55''$, $B = 38° \ 11' \ 27''$, $c = 52° \ 29'$; find a, b, C.

$$\frac{A-B}{2} = 35° \ 8' \ 44'' \qquad \log \sin \frac{A-B}{2} = 9.76016 - 10$$

$$\frac{A+B}{2} = 73° \ 20' \ 11'' \qquad \log \tan \frac{c}{2} = 9.69282 - 10$$

$$\frac{c}{2} = 26° \ 14' \ 30'' \qquad \text{colog} \sin \frac{A+B}{2} = 0.01863$$

$$\log \tan \frac{a-b}{2} = 9.47161 - 10$$

$$\frac{a-b}{2} = 16° \ 30' \ 1.3''$$

$$\log \cos \frac{A-B}{2} = 9.91259 - 10$$

$$\log \tan \frac{c}{2} = 9.69282 - 10$$

$$\text{colog} \cos \frac{A+B}{2} = 0.54250$$

$$\log \tan \frac{a+b}{2} = 10.14791 - 10$$

$$\frac{a+b}{2} = 54° \ 34' \ 24.4''$$

$$\frac{a-b}{2} = 16° \ 30' \ 1.3''$$

$$a = 71° \ 4' \ 26''$$

$$b = 38° \ 4' \ 23''$$

$$\log \cos \frac{a-b}{2} = 9.98174 - 10$$

$$\log \cot \frac{A+B}{2} = 9.47599 - 10$$

$$\text{colog} \cos \frac{a+b}{2} = 0.23682$$

$$\log \tan \frac{C}{2} = 9.69455 - 10$$

$$\frac{C}{2} = 26° \ 19' \ 56''$$

$$C = 52° \ 39' \ 52''$$

EXERCISE XXXVIII

1. In a spherical triangle given $A = 126° \ 40' \ 50''$, $B = 81° \ 45' \ 42''$, $c = 51° \ 56' \ 12''$; find a, b, C.

2. In a spherical triangle given $B = 27° \ 27' \ 36''$, $C = 40° \ 44' \ 20''$, $a = 155° \ 16'$; find b, c, A.

3. In a spherical triangle given $A = 127° \ 19' \ 38''$, $C = 108° \ 41' \ 30''$, $b = 125° \ 22' \ 32''$; find a, c, B.

4. In a spherical triangle given $A = 154° \ 20' \ 42''$, $B = 79° \ 16' \ 22''$, $c = 85° \ 24' \ 28''$; find a, b, C.

149. Case 6. **Given the three angles A, B, C; to find the three sides a, b, c.** Any of the three groups of formulas in Art. 128, p. 187, can be used. The formulas for the tangents are recommended in preference to those for the sines or for the cosines.

$$\tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}; \tag{1}$$

$$\tan\frac{b}{2}=\sqrt{\frac{-\cos S\cos(S-B)}{\cos(S-C)\cos(S-A)}}; \tag{2}$$

$$\tan\frac{c}{2}=\sqrt{\frac{-\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}}. \tag{3}$$

If all three of the sides are to be found, it is convenient to proceed in a manner similar to that employed in Art. 144, p. 200, where three sides were given and three angles were to be found.

Multiplying both numerator and denominator of the fraction under the radical sign in (1) by $\cos(S-A)$ we have

$$\tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos^2(S-A)}{\cos(S-A)\cos(S-B)\cos(S-C)}}$$

Putting $\tan R=\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}$,

we may write

$$\tan\frac{a}{2}=\tan R\cos(S-A).$$

Making the corresponding changes in (2) and (3), we have the three equations

$$\tan\frac{a}{2}=\tan R\cos(S-A),$$

$$\tan\frac{b}{2}=\tan R\cos(S-B),$$

$$\tan\frac{c}{2}=\tan R\cos(S-C).$$

The solution is unique.

Ex. 1. Given $A = 221°$, $B = 128°$, $C = 153°$; to find a.

The formula for $\tan\frac{a}{2}$, with the algebraic sign of each factor written above it for convenience, is as follows:

$$\tan\frac{a}{2} = \sqrt{\frac{-\overset{-}{\cos S}\,\overset{+}{\cos(S-A)}}{\overset{-}{\cos(S-B)}\,\overset{-}{\cos(S-C)}}}.$$

$A = 221°$	$\log\cos S = 9.51264 - 10$
$B = 128°$	$\log\cos(S-A) = 9.93753 - 10$
$C = 153°$	$\text{colog}\cos(S-B) = 0.26389$
$2S = 502°$	$\text{colog}\cos(S-C) = 0.85644$
$S = 251°$	$2)\underline{20.57050 - 20}$
$S - A = 30°$	$\log\tan\frac{a}{2} = 10.28525 - 10$
$S - B = 123°$	$\frac{a}{2} = 62°\ 35'\ 35''$
$S - C = 98°$	$a = 125°\ 11'\ 10''$

The result is real (Art. 128, p. 187), the four negative signs under the radical producing a positive quantity.

Ex. 2. Given $A = 21°\ 26'\ 20''$, $B = 56°\ 46'\ 28''$, $C = 115°\ 23'\ 4''$; find a, b, c.

Proceeding by the second method, we first find the value of $\log\tan R$. The following is suggested as a convenient arrangement of the work:

$$\tan R = \sqrt{\frac{-\overset{-}{\cos S}}{\overset{+}{\cos(S-A)}\,\overset{+}{\cos(S-B)}\,\overset{+}{\cos(S-C)}}}.$$

$A = 21°\ 26'\ 20''$	$\log\tan\frac{a}{2} = 9.30865 - 10$
$B = 56°\ 46'\ 28''$	
$C = 115°\ 23'\ 4''$	$\log\tan\frac{b}{2} = 9.79007 - 10$
$2S = 193°\ 35'\ 52''$	
$S = 96°\ 47'\ 56''$	$\log\tan\frac{c}{2} = 9.87055 - 10$
$S - A = 75°\ 21'\ 36''$	
$S - B = 40°\ \ 1'\ 28''$	$\frac{a}{2} = 11°\ 30'\ 17.5''$
$S - C = -19°\ 24'\ 52''$	
$\log\cos S = 9.07330 - 10$	$\frac{b}{2} = 31°\ 39'\ 43''$
$\text{colog}\cos(S-A) = 0.59732$	
$\text{colog}\cos(S-B) = 0.11590$	$\frac{c}{2} = 36°\ 35'\ 4.6''$
$\text{colog}\cos(S-C) = \underline{0.02542}$	$a = 23°\ \ 0'\ 35''$
$\log\tan^2 R = 9.81194 - 10$	$b = 63°\ 19'\ 26''$
$\log\tan R = 9.90597 - 10$	$c \doteq 73°\ 10'\ \ 9''$

EXERCISE XXXIX

1. In a spherical triangle given $A = 121° 40' 24''$, $B = 60° 12' 22''$, $C = 105° 40'$; find a, b, c.

2. In a spherical triangle given $A = 58° 20' 27''$, $B = 84° 30' 30''$, $C = 61° 35' 10''$; find a, b, c.

3. In a spherical triangle given $A = 105° 14' 4''$, $B = 55° 31' 24''$, $C = 88° 51' 6''$; find a, b, c.

4. In a spherical triangle given $A = 171° 49' 33''$, $B = 5° 15' 23''$, $C = 9° 18' 28''$; find a, b, c.

THE AREA OF A SPHERICAL TRIANGLE

150. In considering the problem of finding the area of a spherical triangle we have two principal cases to consider.

I. **Given the three angles A, B, C.**

Let r = radius of sphere.

E = spherical excess $= A + B + C - 180°$.

$\triangle$ = area of triangle.

It is proved in geometry that the area of a spherical triangle is to the area of the surface of the sphere as its spherical excess, in degrees, is to 720°. Hence, we have

$$\triangle : 4\pi r^2 = E : 720°.$$

$$\therefore \triangle = \frac{E}{180°}\pi r^2.$$

II. **Given the three sides a, b, c.**

The problem is to express the value of E in terms of the sides.

(1) Cagnoli's Theorem.

$$\sin\frac{E}{2} = \sin\frac{A + B + C - \pi}{2}$$

$$= \sin\frac{A+B}{2}\cos\frac{C-\pi}{2} + \cos\frac{A+B}{2}\sin\frac{C-\pi}{2}$$

$$= \sin\frac{A+B}{2}\sin\frac{C}{2} - \cos\frac{A+B}{2}\cos\frac{C}{2}$$

$$= \frac{\sin \frac{C}{2} \cos \frac{C}{2}}{\cos \frac{c}{2}} \left(\cos \frac{a-b}{2} - \cos \frac{a+b}{2} \right) \quad \text{(Art. 130, p. 190)}$$

$$= \frac{\sin \frac{C}{2} \cos \frac{C}{2} \left(2 \sin \frac{a}{2} \sin \frac{b}{2} \right)}{\cos \frac{c}{2}} \quad \text{(Art. 77, p. 100)}$$

$$= \frac{\sin \frac{a}{2} \sin \frac{b}{2}}{\cos \frac{c}{2}} \cdot \frac{2\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}{\sin a \sin b}. \quad \text{(Art. 127, p. 186)}$$

Replacing $\sin a$ and $\sin b$ by their values (Art. 80, p. 106) and canceling, we have

$$\sin \frac{E}{2} = \frac{\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}{2 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}.$$

(2). L'Huilier's Theorem. This theorem, which expresses the value of E by means of its tangent, is derived as follows:

$$\tan \frac{E}{4} = \frac{\sin \frac{E}{4}}{\cos \frac{E}{4}} = \frac{\sin \frac{A+B+C-\pi}{4}}{\cos \frac{A+B+C-\pi}{4}}$$

$$= \frac{\sin \frac{A+B+C-\pi}{4}}{\cos \frac{A+B+C-\pi}{4}} \cdot \frac{\cos \frac{A+B+\pi-C}{4}}{\cos \frac{A+B+\pi-C}{4}}$$

$$= \frac{\sin \frac{A+B}{2} - \sin \frac{\pi - C}{2}}{\cos \frac{A+B}{2} + \cos \frac{\pi - C}{2}} \quad \text{(Art. 77, p. 100)}$$

$$=\frac{\sin\frac{A+B}{2}-\cos\frac{C}{2}}{\cos\frac{A+B}{2}+\sin\frac{C}{2}}$$

$$=\frac{\cos\frac{a-b}{2}-\cos\frac{c}{2}}{\cos\frac{a+b}{2}+\cos\frac{c}{2}}\cdot\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}} \qquad \text{(Art. 129, p. 189)}$$

$$=\frac{\sin\frac{s-b}{2}\sin\frac{s-a}{2}}{\cos\frac{s}{2}\cos\frac{s-c}{2}}\cot\frac{C}{2}. \qquad \text{(Art. 77, p. 100)}$$

$$=\sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}}. \qquad \text{(Art. 127, p. 186)}$$

(3) All other cases may be solved by first finding the three sides or the three angles, and then applying the proper formula.

ANSWERS

PLANE TRIGONOMETRY

Exercise I. Pages 11, 12

1. $\frac{1}{3}$.
2. $1\frac{1}{3}$.
3. 0.7581+.
4. 1.2737+.
5. 2.5419−.
6. 3.5693+.
7. 40°, 60°, 80°.
17. 5°, 25°, 150°.
18. 30°, 360°, 21600°.

Exercise II. Pages 14–16

3. 30°.
4. 120°.
5. 36°.
6. 54°.
7. 270°.
8. 150°.
9. 540°.
10. 2700°.
11. $\frac{\pi}{4}$.
12. $\frac{2\pi}{3}$.
13. $\frac{3\pi}{4}$.
14. $\frac{5\pi}{4}$.
15. $\frac{121\pi}{360}$.
16. $\frac{463\pi}{720}$.
17. $\frac{8599\pi}{5400}$.
18. $\frac{20533\pi}{5400}$.
19. $\frac{n\pi}{180}$.
20. $\frac{A\pi}{180}$.
21. $\frac{1}{2}$.
22. $\frac{13021\pi}{30000}$.
23. 27°, 63°.
24. 52°, 56°, 72°.
25. $\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15}$.
26. 30°, 60°, 90°.
27. 4, 6.
28. $\frac{3\pi}{5}, \frac{5\pi}{7}, \frac{7\pi}{9}$.
29. $\frac{1}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, -\frac{1}{2}$.
30. $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$.
31. 150°, $\frac{5\pi}{6}$; 82° 30′, $\frac{11\pi}{24}$; 135°, $\frac{3\pi}{4}$.
32. $5\frac{10}{11}$ minutes past four; $54\frac{6}{11}$ minutes past four.

Exercise III. Pages 17–19

5. 1.77.
6. 28° 7′ 30″.
7. 0.265 sec.
8. 40 yd.
9. 2° 8′ 52.8″.
10. 861,031 mi. (approximately).
11. 3962.95.
12. 14° 19′ 26.2″.
13. 1.047 radians.
14. 51.56.
15. 102 ft. (approximately).
16. 5 : 4.
17. 3.1416.
18. $\frac{\pi}{3}, \frac{4}{9}\pi, \frac{5}{9}\pi, \frac{2}{3}\pi$.
19. 3.1416.
20. 0.000097+.
21. 65° 24′ 30.4″.
22. $98\frac{2}{11}$.
23. 1 mi. 908 ft. nearly.
24. 7 mi. 1237.2 ft.
25. 18° and 58°.
26. 19.099′.
27. $\frac{60\,a\pi}{10800}$.
28. 0.00004848.

Exercise V. Pages 29, 30

7. $\frac{4}{11}\sqrt{7}$.
8. $\frac{50}{61}$.
9. $\frac{5}{4}$, $\frac{4}{3}$.
10. $\frac{3}{20}\sqrt{10}$, $\frac{7}{3}$.
11. $\frac{61}{60}$, $\frac{60}{61}$.
12. $\frac{5}{3}$, $\frac{4}{5}$.
13. $\frac{6}{61}\sqrt{61}$, $\frac{5}{61}\sqrt{61}$.
14. $\frac{2}{5}\sqrt{6}$, $2\sqrt{6}$.
15. $\frac{11}{60}$, $\frac{1}{61}$.
16. $\frac{3}{8}\sqrt{7}$, $\frac{1}{21}\sqrt{7}$.
17. $\frac{5}{9}$, $\frac{9}{28}\sqrt{14}$.
18. $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, etc.; $\sin B = \frac{15}{17}$, $\cos B = \frac{8}{17}$, etc.
19. $\sin A = \frac{x^2 - y^2}{x^2 + y^2}$, $\cos A = \frac{2xy}{x^2 + y^2}$, etc.; $\sin B = \frac{2xy}{x^2 + y^2}$, $\cos B = \frac{x^2 - y^2}{x^2 + y^2}$, etc.
20. $\frac{3}{4}$.
21. $\frac{1}{3}$.
22. $\frac{5}{3}$.
23. $\frac{15}{17}$.

Exercise VIII. Pages 42–48

32. 60.
33. 45188.
34. 6.
35. 124.71.
36. 182.8.
37. 1143.4.
38. 1916.64.
39. 36157.5.
40. 498.51.
41. 52444.44.
42. $\frac{1}{2} a\sqrt{c^2 - a^2}$.
43. $\frac{1}{2} a^2 \cot A$.
44. $\frac{1}{2} a^2 \tan B$.
45. $\frac{1}{2} c^2 \sin A \cos A$.
46. 29° 22′.
47. 60° 38′.
48. 20.48.
49. 33.64.
50. 41° 36′.
51. 24° 54′ 16″.
52. 42° 42′ 34″.
53. 23° 11′ 55″.
54. 38° 9′ 25″.
55. 80.49, 105.64.
56. 74° 43′ 54″.
57. 124.27.
58. 560.88.
59. 25.165, 36.458.
60. 89.44.
61. 46.71.
62. 122.53.
63. 355.34.
64. 74.335.
65. 42.838.
66. 313.1.
67. 38.13.
68. 43.63.
69. 39° 11′.
71. 118.3.
72. 100.
73. 145.58.

Exercise IX. Pages 49, 50

1. 64° 20′ 26″.
2. 75° 32′ 50″.
3. 243.57.
4. 175.068.
5. 148.91′.
6. ~~30° 17′.~~ 49° 26′
7. 91.204.
8. 3° 34′ 8″.
9. $na^2 \sin\frac{C}{2}\cos\frac{C}{2}$.
10. $na^2 \sin A \cos A$.
11. $nh^2 \cot A$.
12. $\Delta = 69.24$.
13. $\Delta = 1325.46$.
14. $\Delta = 3741.18$.
15. $\Delta = 309.01$.
16. $\Delta = 29.82$.
17. $\Delta = 104.71$.
18. $\Delta = 12.312$.
19. $\Delta = 115.92$.
20. $\Delta = 700.616$.
21. $\Delta = 2186.95$.

Exercise X. Pages 72, 73

5. $\frac{\sqrt{3} + 1}{2}$.
6. $\frac{1 + 2\sqrt{2}}{2}$.
7. $\frac{\sqrt{2} - 2}{2}$.
8. $\frac{3\sqrt{3}}{2}$.
9. $\frac{\sqrt{3} + 2}{2}$.
10. $\frac{3\sqrt{3} - 4}{3}$.
11. $\frac{11}{3}$.
12. -2.
13. $-\frac{7}{3}$.

14. Positive for 60°, 120°, 210°, 330°; negative for 0°, 240°, 300°.

15. Positive for 330°; negative for 210°, 300°; zero for 135°.

17. $\frac{2ab}{a^2+b^2}, \frac{2ab}{a^2-b^2}.$

18. $\frac{2a+1}{2a^2+2a+1}, \frac{2a^2+2a}{2a^2+2a+1}.$

Exercise XI. Pages 83, 84

5. 45° and 225°; 45°, 135°, 225°, 315°.

6. Positive for 120° and 690°; negative for 150°, 300, and $\frac{6\pi}{7}$; zero for 135° and 315°.

7. Positive for 210° and 780°; negative for 240°, 300°, 625° and $\frac{5\pi}{3}$; zero for 225°.

8. Positive for 60°, 150°, and $\frac{11\pi}{6}$; negative for 120° and 210°; zero for 135° and 225°.

9. (*a*) 240° and 300°; (*b*) 210° and 330°; (*c*) 135° and 315°; (*d*) 30° and 210°.

14. 3.

15. $-\cot^2 A \csc A$.

Exercise XII. Pages 88, 89

4. $\theta = n\pi + (-1)^n \frac{\pi}{4}.$

5. $\theta = 2n\pi + \frac{\pi}{2}.$

6. $\theta = n\pi - (-1)^n \frac{\pi}{3}.$

7. $\theta = n\pi - (-1)^n \frac{\pi}{6}.$

8. $\theta = 2n\pi \pm \frac{\pi}{6}.$

9. $\theta = (2n+1)\pi \pm \frac{\pi}{4}.$

10. $\theta = 2n\pi \pm \frac{\pi}{2}.$

11. $\theta = (2n+1)\pi.$

12. $\theta = n\pi + \frac{\pi}{4}.$

13. $\theta = n\pi - \frac{\pi}{6}.$

14. $\theta = n\pi \pm \frac{\pi}{6}.$

15. $\theta = n\pi \pm \frac{\pi}{4}.$

16. $\theta = n\pi \pm \frac{\pi}{6}.$

17. $\theta = n\pi \pm \frac{\pi}{6}.$

18. $\theta = n\pi \pm \frac{\pi}{4}.$

19. $\theta = n\pi$ or $n\pi \pm \frac{\pi}{4}.$

20. $\theta = n\pi \pm \frac{\pi}{4}.$

21. $\theta = 2n\pi + \frac{\pi}{3}.$

22. $\theta = (2n+1)\pi + \frac{\pi}{6}.$

Exercise XIII. Pages 90, 91

4. $2n\pi \pm \frac{\pi}{3}$, or $(2n+1)\pi$.

5. $2n\pi$, or $2n\pi \pm \frac{\pi}{3}$.

6. $n\pi + (-1)^n \frac{\pi}{6}$, or $n\pi + (-1)^n \frac{3\pi}{2}$.

7. $n\pi + (-1)^n \frac{\pi}{6}$.

8. $2n\pi \pm \frac{2\pi}{3}$.

9. $2n\pi \pm \frac{\pi}{3}$, or $(2n+1)\pi$.

10. $2n\pi \pm \frac{\pi}{3}$.

11. $2n\pi + \frac{\pi}{2}$, or $\sin\theta = -\frac{1}{3}$.

12. $n\pi \pm \frac{\pi}{4}$.

13. $n\pi + \frac{\pi}{4}$, or $\cot\theta = -2$.

14. $n\pi \pm \frac{\pi}{4}$.

15. $(2n+1)\frac{\pi}{7}$, or $\frac{2n\pi}{3}$.

16. $\frac{n\pi}{6} + \frac{\pi}{12}$, or $\frac{n\pi}{3}$.

17. $\frac{n\pi}{4}$, or $\frac{n\pi}{2}$.

18. $2n\pi$, or $\frac{2n\pi}{9}$.

19. $\frac{2r\pi}{m-n}$, or $\frac{2r\pi}{m+n}$.

20. $n\pi - \frac{\pi}{4}$, or $\frac{n\pi}{3} + \frac{\pi}{12}$.

21. $n\pi + \frac{\pi}{4}$, or $\frac{n\pi}{3} + \frac{\pi}{12}$.

22. $n\pi$.

23. $\frac{n\pi}{3}$.

24. $\frac{n\pi}{9} + \frac{\pi}{18}$.

25. $\frac{(2r+1)\pi}{2(m-n)}$.

26. $(2n+1)\frac{\pi}{6}$.

Exercise XIV. Pages 95–97

4. $-\frac{33}{65}$. 5. $\frac{33}{65}$. 6. $\frac{157}{205}$.

Exercise XV. Pages 99, 100

1. 1. 2. $\frac{37}{284}$. 3. $-\frac{33}{56}$. 4. -4. 5. 3. 6. 1.

Exercise XVIII. Pages 108–110

1. $\frac{4\sqrt{2}}{9}$, $\frac{23}{27}$. 2. $\frac{7}{8}$, $\frac{3\sqrt{15}}{16}$. 3. $\frac{24}{25}$, $-\frac{117}{125}$. 4. $\frac{4}{3}$, $6\frac{1}{2}$. 5. $\frac{3\sqrt{10}}{5}$, $1\frac{5}{9}$.

Exercise XXI. Pages 120, 121

1. $\pm\frac{1}{2}\sqrt{2}$.
2. $\pm\frac{1}{2}\sqrt{2}$.
3. ± 1.
4. x imaginary.
5. 13.
6. $\frac{\sqrt{3}}{2\sqrt{7}}$.
7. $\sqrt{3}$.
8. $\frac{-3 \pm \sqrt{17}}{4}$.
9. $\frac{1}{2}$.
10. 1 or $-\frac{1}{6}$.
11. 0 or $\pm\frac{1}{2}$.
12. 1 or $\frac{1}{2}$.
13. $\pm\frac{1}{2}\sqrt{2}$.
14. $\frac{25}{24}$.
15. $\frac{5}{9}$.
16. $\frac{1}{3}\sqrt{5}$.
17. $\frac{ab}{\sqrt{a^2-1}+\sqrt{b^2-1}}$.
18. $n\pi$ or $n\pi + \frac{\pi}{4}$.
19. $\frac{ab}{\sqrt{a^2-1}+\sqrt{b^2-1}}$.
20. $\sqrt{3}$.
21. 2.

Exercise XXII. Page 127, 128

1. $2n\pi$, or $2n\pi - \frac{2\pi}{3}$.

2. $2n\pi + \frac{\pi}{2}$, or $(2n+1)\pi + \frac{\pi}{6}$.

3. $2n\pi + \frac{5\pi}{12}$, or $2n\pi - \frac{\pi}{12}$.

4. $n\pi + \frac{\pi}{6} + (-1)^n\frac{\pi}{4}$.

5. $2n\pi + \frac{\pi}{4}$.

6. $2n\pi + \frac{\pi}{12}$, or $2n\pi - \frac{7\pi}{12}$.

7. $2n\pi + \frac{5\pi}{4}$, or $2n\pi - \frac{3\pi}{4}$.

8. $\frac{2k\pi}{m+n}$, or $\frac{(2k+1)\pi}{m-n}$.

9. $\frac{(2k+1)\pi}{m \pm n}$.

10. $2n\pi$, or $2n\pi + 112° 38'$.

11. $n\pi + (-1)^n 36° 52'$, or $2n\pi - \frac{\pi}{2}$.

12. $2n\pi - 36° 52'$.

13. $\frac{n\pi}{4}$, or $\frac{2n\pi}{3} \pm \frac{\pi}{9}$.

14. $\frac{n\pi}{3}$, or $n\pi \pm \frac{\pi}{6}$.

15. $\frac{n\pi}{3}$, or $\frac{n\pi}{2} \pm \frac{\pi}{12}$.

16. $\frac{2n\pi}{3} \pm \frac{\pi}{6}$, or $n\pi + (-1)^n\frac{\pi}{6}$.

17. $\frac{n\pi}{2} + \frac{\pi}{4}$, or $2n\pi \pm \frac{2\pi}{3}$.

18. $\frac{n\pi}{2}$, or $2n\pi \pm \frac{2\pi}{3}$.

19. $\frac{n\pi}{4}$, or $\frac{n\pi}{3} + (-1)^n\frac{\pi}{18}$.

20. $2n\pi$, or $(4n+3)\frac{\pi}{6}$.

21. $2n\pi$, $(2n+1)\frac{\pi}{2}$, or $(2n+1)\frac{\pi}{5}$.

22. $(2n+1)\frac{\pi}{2}$, $(2n+1)\frac{\pi}{4}$, or $(2n+1)\frac{\pi}{8}$.

23. $2n\pi$, or $n\pi \pm \frac{\pi}{4}$.

24. $n\pi$, or $\frac{1}{n-1}\left(n\pi - (-1)^n\frac{\pi}{6}\right)$.

25. $n\pi$, or $(2n+1)\frac{\pi}{14}$.

26. $2n\pi$, or $\frac{4n\pi}{n+1}$, or $\frac{4n\pi}{n-1}$.

27. $n\pi \pm \frac{\pi}{3}$, or $2n\pi \pm \frac{\pi}{2}$.

28. $2n\pi - \frac{\pi}{2}$, or $\frac{2n\pi}{5} - \frac{\pi}{10}$.

29. $n\pi \pm \frac{\pi}{8}$, or $(2n+1)\frac{\pi}{6}$.

30. $(2n+1)\frac{\pi}{8}$, or $\frac{n\pi}{3} + (-1)^n\frac{\pi}{9}$.

31. $n\pi$.

32. $(2n+1)\frac{\pi}{2}$, or $n\pi \pm \frac{\pi}{3}$.

33. $n\pi$, or $n\pi \pm \frac{\pi}{6}$.

34. $n\pi$.

35. $2n\pi$, or $2n\pi + \frac{2\pi}{3}$.

36. $\frac{n\pi}{2} + \frac{\pi}{8}$.

37. $n\pi - \frac{\pi}{4}$, or $\frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$.

38. $\frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$.

39. $(2n+1)\frac{\pi}{10}$, or $(2n+1)\frac{\pi}{2}$.

40. $n\pi + \frac{\pi}{4}$, or $\frac{n\pi}{2} - (-1)^n\frac{\pi}{12}$.

41. $\frac{n\pi}{2}$, or $n\pi \pm \frac{\pi}{3}$.

42. $n\pi$, or $n\pi \pm \frac{\pi}{3}$.

43. $n\pi$, or $\frac{n\pi}{2}$.

44. $n\pi$, or $\frac{n\pi}{3} + \frac{\pi}{6}$.

Exercise XXIV. Pages 136–138

11. 640.65 ft.

12. $AC = 8332.2$ ft., $AB = 12163.53$ ft.

13. Distances 2841.2 ft., 3475.46 ft. Height 1721.08 ft.

14. Distances 11975.68 ft., 24182.77 ft. Height 19769.54 ft.

15. 121.04 ft. **16.** 171.15 ft. **17.** 110.39 ft. **19.** 4.588 mi. **20.** 4.506 mi.

Exercise XXVI. Pages 146–148

11. 4536.4 ft. **13.** 5402.6 ft. **15.** 15.6. **17.** 5.65.

12. 134.49 ft. **14.** 9. **16.** 1781.2 ft. **18.** 4.58 : 9.81.

19. $A = 39° 46' 0.4''$, $B = 68° 2' 45.6''$. **20.** 4494.3 ft.

Exercise XXVII. Pages 152, 153

17. 43° 55′ 13″. **20.** 60°, 60°, 60°.

18. 49° 8′ 46″. **21.** 66° 44′ 2″, 60° 26′ 53″, 52° 49′ 9″.

19. 30°, 60°, 90°. **23.** 60°. **24.** 120°. **25.** 73° 44′.

Miscellaneous Examples. Pages 158–165

1. 247.56 ft. **3.** 41° 9′ 7″. **5.** 48° 45′ 44″. **7.** 122.48 ft.

2. 42° 42′ 34″. **4.** 36° 22′ 21″. **6.** 72.75 ft. **8.** 123.47 ft.

9. Height = 1224.3 ft.; distance = 1292.9 ft. **10.** 431.78 ft.

11. 233.27 ft. **12.** 440.36 lb.; 63° 12′ 26″, 26° 47′ 34″.

13. 2881.46 mi. **15.** 2304.52 ft. **17.** 7912.8 mi.

14. 407.61 ft. **16.** 67.5 ft. **18.** 108° 11′.

19. Height = 350.67 ft., distance = 3205.15 ft. **20.** 8.0076 in.

21. 746 ft. **22.** 17.32, 30, 34.64. **23.** 244.95.

24. $\tan^{-1}\frac{2}{3}$; $\frac{9}{52}$ of an hour. **25.** 6 ft.

26. 136.13 ft. from the foot of the tower. **27.** 61.24 ft.

29. 109.9 ft. **31.** 308.66 ft. **33.** 110° 16′. **35.** 4782.2 ft.

30. 4621.1 ft. **32.** 407.61 ft. **34.** 473.3 ft. **36.** 2785.6 ft.

37. 60° 20′ 8″, 76° 49′ 18″, 42° 50′ 29″. **38.** 595.84 ft.

39. 1743.36 ft. **40.** 4244.4 ft. **41.** ~~9.1 mi. an hour.~~ **42.** 383.37 yd.

43. Resultant = 658.36 lb.; angle bet. resultant and greater force 22° 23′ 43″.

44. 2019.62 ft. **47.** 63.08. **50.** 3883 ft. **52.** 13451.52 ft.

45. 410.35 ft. **48.** 45.92 ft. **51.** 4494.3 ft. **53.** 1949.77 ft.

46. 178.88 ft. **49.** 10520.49 ft.

Exercise XXVIII. Pages 167, 168

2. 0.0029089. **4.** 0.002036. **6.** 0.99999.

3. 0.9999958. **5.** 0.004363. **7.** 0.00003878.

Exercise XXXI. Page 175

1. $r = 2.5,\ \theta = 2.165.$
2. $a^2 + b^2 = c^2 + d^2.$
3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$
4. $a^2 + d^2 = b^2 + c^2.$
5. $2ac^2 - ad^2 = bcd.$
6. $a^4 - 2a^2 + b^2 = 0.$
7. $a^2b - b = 2.$
8. $(a^2 - b^2)^2 - 16ab = 0.$
9. $(a^2 + 1)^2 + 2b(a^2 + 1)(a + b) - 4(a + b)^2 = 0.$
10. $(a + b)^{\frac{2}{3}} + (a - b)^{\frac{2}{3}} = 2.$
11. $a^{\frac{2}{3}} + b^{\frac{2}{3}} - (ab)^{-\frac{2}{3}} = 0.$
12. $(a - b)\tan\alpha + ab = 0.$
13. $a^2 - b^2 - 2\cos\alpha - 2 = 0.$
14. $(a + b)(c + d) - 2cd = 0.$

SPHERICAL TRIGONOMETRY

Exercise XXXIII. Pages 199, 200

1. $b = 14° 25' 20''$, $A = 47° 30' 46''$, $B = 44° 25' 26''$.
2. $c = 77° 56' 37''$, $A = 81° 50' 9''$, $B = 34° 28' 58''$.
3. Impossible.
4. $c = 69° 55' 18''$, $b = 130° 15' 58''$, $A = 115° 33' 51''$.
5. $a = 49° 30' 54''$, $b = 131° 41' 29''$, $B = 124° 6' 53''$.
6. $a = 34° 20' 53''$, $b = 42° 23' 40''$, $c = 52° 25' 39''$.
7. $a = 80° 28' 44''$, $b = 78° 38' 54''$, $B = 83° 47' 23''$.
8. $b = 145° 13' 27''$, $A = 35° 2' 7''$, $B = 118° 8' 2''$.
9. $b = 155° 23' 47''$, $c = 71° 18' 48''$, $A = 98° 54' 34''$.
10. $b_1 = 153° 59' 53''$, $c_1 = 69° 36'$, $B_1 = 152° 6' 47''$.
    $b_2 = 26° 0' 7''$, $c_2 = 110° 24'$, $B_2 = 27° 53' 13''$.
11. $c = 62° 33' 19''$, $A = 68° 51' 35''$, $B = 39° 59' 48''$.
12. $a = 49° 53' 28''$, $b = 58° 26'$, $c = 70° 17' 27''$.

Exercise XXXIV. Page 202

1. $A = 113° 51' 22''$, $B = 66° 17' 20''$, $C = 96° 0' 18''$.
2. $A = 65° 10'$, $B = 98° 50' 37''$, $C = 125° 17' 48''$.
3. $A = 129° 22' 58''$, $B = 109° 41' 38''$, $C = 97° 21' 36''$.
4. $A = 23° 16' 48''$, $B = 62° 13' 34''$, $C = 107° 54' 18''$.

Exercise XXXV. Page 203

1. $A = 117° 33' 50''$, $B = 46° 37' 46''$, $c = 62° 36' 45''$.
2. $A = 116° 0' 7''$, $B = 51° 34' 15''$, $c = 57° 51' 26''$.
3. $B = 101° 4' 47''$, $C = 40° 8' 22''$, $a = 57° 31' 43''$.
4. $A = 35° 18' 32''$, $C = 126° 39' 6''$, $b = 77° 10' 36''$.
5. $B = 69° 28' 26''$, $C = 42° 13' 34''$, $a = 76° 17' 36''$.

Exercise XXXVI. Page 205

1. $B = 119° 34' 43''$, $C = 96° 55' 26''$, $c = 105° 36' 14''$.
2. $B = 63° 1' 40''$, $C = 84° 50' 28''$, $c = 80° 51' 28''$.
3. $B_1 = 63° 55' 10''$. $C_1 = 33° 51' 5''$, $c_1 = 26° 41' 4''$.
   $B_2 = 116° 4' 50''$, $C_2 = 92° 8' 32''$, $c_2 = 53° 40' 8''$.
4. Impossible.

Exercise XXXVII. Page 207

1. $b = 156° 51' 40''$,	$c = 30° 57' 43''$,	$C = 69° 37' 20''$.
2. $b = 125° 22' 40''$,	$c = 155° 48' 12''$,	$C = 155° 50' 58''$.
3. $b_1 = 75° 38' 40''$,	$c_1 = 102° 0' 42''$,	$C_1 = 48° 27' 53''$.
4. $b_2 = 104° 21' 20''$,	$c_2 = 134° 30' 27''$,	$C_2 = 146° 55' 13''$.

Exercise XXXVIII. Page 208

1. $a = 129° 29' 29''$,	$b = 107° 45' 45''$,	$C = 54° 54' 16''$.
2. $b = 36° 23' 38''$,	$c = 122° 53' 23''$,	$A = 161° 1' 28''$.
3. $a = 123° 21' 30''$,	$c = 84° 15' 24''$,	$B = 129° 4' 47''$.
4. $a = 153° 51' 21''$,	$b = 89° 26'$,	$C = 78° 21' 23''$.

Exercise XXXIX. Page 211

1. $a = 142° 5' 25''$,	$b = 38° 47' 39''$,	$c = 135° 57' 44''$.
2. $a = 49° 20' 39''$,	$b = 62° 31' 13''$,	$c = 51° 37' 5''$.
3. $a = 107° 45' 46''$,	$b = 54° 27' 19''$,	$c = 99° 18' 46''$.
4. $a = 118° 52' 50''$,	$b = 34° 20' 45''$,	$c = 84° 53' 32''$.

FIVE-PLACE LOGARITHMIC AND TRIGONOMETRIC TABLES

BASED ON THE TABLES OF F. G. GAUSS

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CONANT TRIG. TABLES.
W. P. I

INTRODUCTION

1. A **logarithm** is the exponent by which a number a must be affected in order that the result shall be a given number m. That is, if $a^x = m$, then x is called the logarithm of m to the base a. The above equation written in logarithmic form is $\log_a m = x$.

Any positive number except 1 may be used as the base of a system of logarithms. In practical work involving numerical computation 10 is the base that is universally employed.

All computations by means of logarithms are based on the following theorems :

2. *The logarithm of a product is equal to the sum of the logarithms of the factors.*

PROOF. Let m and n be any two positive numbers, and let x and y be their logarithms respectively. Then

$$m \cdot n = 10^x \cdot 10^y = 10^{x+y}.$$

$$\therefore \log(mn) = x + y = \log m + \log n.$$

3. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

PROOF.
$$\frac{m}{n} = \frac{10^x}{10^y} = 10^{x-y}.$$

$$\therefore \log \frac{m}{n} = x - y = \log m - \log n.$$

4. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the index of the power.*

PROOF.
$$m^y = (10^x)^y = 10^{xy}.$$

$$\therefore \log m^y = xy = y \log m.$$

5. *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

PROOF.
$$\sqrt[y]{m} = \sqrt[y]{10^x} = 10^{\frac{x}{y}}.$$

$$\therefore \log \sqrt[y]{m} = \frac{x}{y} = \frac{\log m}{y}.$$

6. The logarithm of any integral power or root of 10 is an integral number. The logarithms of all other positive numbers are fractions.

Negative numbers have no logarithms. If any logarithmic computation is to be performed which involves negative numbers, the problem should be solved as though the numbers were all positive; and the algebraic sign of the result should then be determined by the usual methods of algebra.

7. The logarithm of a number consists of two parts, an integral part and a decimal. The integral part is called the **characteristic**, and the decimal part the **mantissa**. As logarithms are usually printed the mantissa is always positive. The characteristic may be positive, negative, or zero. The characteristic of the logarithm of any number may be found by one of the following rules:

I. *The characteristic of the logarithm of a number greater than one is positive, and is one less than the number of digits in the integral part of the number.*

II. *The characteristic of the logarithm of a decimal fraction is negative, and is numerically one greater than the number of ciphers immediately after the decimal point.*

For example, the characteristic of the logarithm of 3286 is 3; of 294645 is 5; of 0.0241 is -2; of 0.000649 is -4.

For the sake of convenience a negative characteristic is often changed in form by adding to it and subtracting from it the number 10. For example, if the characteristic of a logarithm is -2, and the mantissa is .38416, the logarithm may be written $8.38416 - 10$. If the characteristic is -1 and the mantissa is .74925, the logarithm may be written $9.74925 - 10$. If the negative forms of the characteristics are retained, the above logarithms are written $\bar{2}.38416$ and $\bar{1}.74925$ respectively. When it is remembered that the mantissas are positive, the reason for writing the negative sign of a characteristic above instead of before it will be readily understood.

In all work connected with the logarithms in the following tables the characteristics, when negative, are to be understood as being increased and diminished by 10.

TABLE I

Directions for finding the logarithm of a number.

8. When the number is between 1 and 100.

The entire logarithm, including both characteristic and mantissa, is given on p. 9.

9. Numbers containing one or two significant figures.

The mantissa is found on p. 9. It is the same for all numbers containing the same significant figures arranged in the same order, no matter where the decimal point is placed.

The characteristic is found by means of the rules given above. For example,

$\log 53 = 1.72428$, $\log .53 = 9.72428 - 10$,
$\log 5.3 = 0.72428$, $\log .053 = 8.72428 - 10$.

10. Numbers containing three significant figures.

The number, no attention being paid to the decimal point, is found at the left of the page in the column headed No. The mantissa is found on a line with the number, and in the column headed 0. The characteristic is found as before, by one or the other of the rules on p. 4.

For example,

$\log 763 = 2.88252$, $\log .0763 = 8.88252 - 10$,
$\log 76.3 = 1.88252$, $\log .00763 = 7.88252 - 10$.

11. Numbers containing four significant figures.

The first three significant figures are found in the column headed No., and the fourth is at the top of the page. On a line with the first three figures, and in the column headed by the fourth figure, the mantissa is found. The characteristic is determined as in the previous cases.

For example,

$\log 296300 = 5.47173$, $\log .2963 = 9.47173 - 10$,
$\log 29.63 = 1.47173$, $\log .0002963 = 6.47173 - 10$.

12. Numbers containing more than four significant figures.

Let the number whose logarithm is required be 61487. Since the number lies between 61480 and 61490, the logarithm of the required number lies between the logarithms of those numbers, *i.e.* between 4.78873 and 4.78880.

Now $\log 61490 = 4.78880$
and $\log 61480 = 4.78873$
giving a difference of $.00007$

Hence, we see that an increase of 10 in the number produces an increase of .00007 in the logarithm. But the actual increase we have to consider in the number is 7. Now if an increase of 10 in the number produces an increase of .00007 in the logarithm, an increase of 7 in the number will produce an increase of $\frac{7}{10}$ of .00007, or .000049. Calling this correction .00005, we have

$$\begin{aligned} \log 61480 &= 4.78873 \\ \text{correction} &= \underline{.00005} \\ \therefore \log 61487 &= 4.78878 \end{aligned}$$

It is here assumed that an increase in the number is accompanied by a proportional increase in the logarithm of the number. This is not true; but in obtaining logarithms from a table, that assumption is always made. If greater accuracy is desired, it will be necessary to use tables containing a greater number of figures.

Directions for finding the number corresponding to a given logarithm.

13. Logarithms whose mantissas are found in the table.

When the exact mantissa of a logarithm is found in the table, the first three significant figures of the number corresponding to the logarithm are found in the column headed No., and on a line with the given mantissa. The fourth significant figure is at the top of the column in which the given mantissa is found.

For example,

2.68529 is the logarithm of 484.5. See p. 17.
9.68529 – 10 is the logarithm of 0.4845.
7.68529 – 10 is the logarithm of 0.004845.
5.68529 is the logarithm of 484500.

14. Logarithms whose mantissas are not found in the table.

When the exact mantissa of the given logarithm is not found in the table, the first four significant figures of the number corresponding to the logarithm are the same as the first four significant figures of the number corresponding to the next smaller logarithm. The remaining figures are found by interpolation, as illustrated in the following.

To find the number corresponding to the logarithm 3.44127.

Number corresponding to 3.44138 is 2763 See p. 13.
Number corresponding to $\underline{3.44122}$ is $\underline{2762}$
.00016 1

Thus we see that an increase of .00016 in the logarithm corresponds to an increase of 1 in the number. But the given logarithm, 3.44127, is .00005 greater than the logarithm of the number 2762.

Therefore, the increase in the required number is $\frac{.00005}{.00016}$, or, more simply, $\frac{5}{16}$ of 1. This gives .31 as the required increase. Hence 2762.31 is the number whose logarithm is 3.44127.

Similarly,

78.565 is the number whose logarithm is 1.89523.

58317.5 is the number whose logarithm is 4.76580.

.17532 is the number whose logarithm is 9.24383 − 10.

15. Cologarithms.

The cologarithm of a number is the logarithm of the reciprocal of that number.

Since the reciprocal of a number is unity divided by that number, and since the logarithm of unity is 0, it follows that the cologarithm of a number is found by subtracting the logarithm of the number from 0, or from 10 − 10.

For example,

$$\text{colog } 256 = \log \tfrac{1}{256} = \log 1 - \log 256 = 0 - 2.40824 = -2.40824.$$

To avoid the use of negative logarithms the above work is performed, and the value of the above result is expressed as follows:

$$\begin{aligned} \log 1 &= 10.00000 - 10 \\ \log 256 &= \underline{\ \ 2.40824 \qquad\ } \\ \therefore \text{colog } 256 &= \ \ 7.59176 - 10. \end{aligned}$$

From the definition of a cologarithm it follows that the effect of subtracting the logarithm of a number is the same as that of adding its cologarithm. For example, finding the logarithm of $\frac{293}{478}$ by each of the two possible methods, we have:

By Logarithms		By Cologarithms	
log 293 =	12.46687 − 10	log 293 =	2.46687
log 478 =	2.67943	colog 478 =	7.32057 − 10
Subtracting,	9.78744 − 10	Adding,	9.78744 − 10

The result is the same in both cases.

TABLE III

This table contains the logarithmic sine and tangent for every second from 0′ to 3′, and the logarithmic cosine and cotangent for every second from 89° 57′ to 90°; the logarithmic sine, cosine, and tangent for every ten seconds from 0° to 2°, and the logarithmic sine, cosine, and cotangent for every ten seconds from 88° to 90°; and the logarithmic sine, cosine, tangent, and cotangent for every minute from 1° to 89°.

16. The logarithmic sine, cosine, tangent, or cotangent of an angle less than 90°.

If the angle is less than 45°, use the column having the name of the proper function at the top, and the column of minutes at the left of the page; if the angle is between 45° and 90°, use the column having the name of the proper function at the bottom, and the column of minutes at the right of the page.

To illustrate the use of this table, let us find the logarithm of sin 26° 28′ 12″.

By p. 48, $\log \sin 26° \, 28' = 9.64902 - 10.$

The difference between log sin 26° 28′ and log sin 26° 29′ is .00025. Increasing the former by $\frac{12}{60}$ of this difference, or .00005, we have

$$\log \sin 26° \, 28' \, 12'' = 9.64907 - 10.$$

As a further illustration, find log tan 71° 38′ 19″.

By p. 44, $\log \tan 71° \, 38' = 10.47885 - 10.$

Increasing this by $\frac{13}{60}$ of .00042, *i.e.* by .00013, we have

$$\log \tan 71° \, 38' \, 19'' = 10.47898 - 10.$$

If the logarithm of a cosine or of a cotangent is to be found, the correction for seconds must be *subtracted*, since these functions decrease as the angle increases from 0° to 90°.

17. The angle corresponding to a logarithmic sine, cosine, tangent, or cotangent.

Find the angle whose log tan = 9.65647 – 10.

The next smaller logarithmic tangent is (p. 47) 9.65636 – 10, which corresponds to an angle of 24° 23′. The difference between this logarithm and the log tan 24° 24′ is .00033, and the difference between log tan 24° 23′ and the given logarithm is .00011. Therefore, the angle corresponding to the next smaller logarithm, *i.e.* 24° 23′, must be increased by $\frac{11}{33}$ of 60″, *i.e.* by 20″. Hence,

$$9.65647 - 10 = \log \tan 24° \, 23' \, 20''.$$

In the case of the logarithm of the cosine or of the cotangent we work from the next *larger* logarithm to the next smaller, instead of from the smaller to the larger as in the case of the sine and the tangent.

TABLE IV

This table contains the numerical or natural values of the sine, cosine, tangent, and cotangent for every minute from 0° to 90°.

TABLE I

THE COMMON OR BRIGGS

LOGARITHMS

OF THE NATURAL NUMBERS

FROM 1 TO 10000

1-100

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
0	———	**20**	1. 30 103	**40**	1. 60 206	**60**	1. 77 815	**80**	1. 90 309
1	0. 00 000	21	1. 32 222	41	1. 61 278	61	1. 78 533	81	1. 90 849
2	0. 30 103	22	1. 34 242	42	1. 62 325	62	1. 79 239	82	1. 91 381
3	0. 47 712	23	1. 36 173	43	1. 63 347	63	1. 79 934	83	1. 91 908
4	0. 60 206	24	1. 38 021	44	1. 64 345	64	1. 80 618	84	1. 92 428
5	0. 69 897	25	1. 39 794	45	1. 65 321	65	1. 81 291	85	1. 92 942
6	0. 77 815	26	1. 41 497	46	1. 66 276	66	1. 81 954	86	1. 93 450
7	0. 84 510	27	1. 43 136	47	1. 67 210	67	1. 82 607	87	1. 93 952
8	0. 90 309	28	1. 44 716	48	1. 68 124	68	1. 83 251	88	1. 94 448
9	0. 95 424	29	1. 46 240	49	1. 69 020	69	1. 83 885	89	1. 94 939
10	1. 00 000	**30**	1. 47 712	**50**	1. 69 897	**70**	1. 84 510	**90**	1. 95 424
11	1. 04 139	31	1. 49 136	51	1. 70 757	71	1. 85 126	91	1. 95 904
12	1. 07 918	32	1. 50 515	52	1. 71 600	72	1. 85 733	92	1. 96 379
13	1. 11 394	33	1. 51 851	53	1. 72 428	73	1. 86 332	93	1. 96 848
14	1. 14 613	34	1. 53 148	54	1. 73 239	74	1. 86 923	94	1. 97 313
15	1. 17 609	35	1. 54 407	55	1. 74 036	75	1. 87 506	95	1. 97 772
16	1. 20 412	36	1. 55 630	56	1. 74 819	76	1. 88 081	96	1. 98 227
17	1. 23 045	37	1. 56 820	57	1. 75 587	77	1. 88 649	97	1. 98 677
18	1. 25 527	38	1. 57 978	58	1. 76 343	78	1. 89 209	98	1. 99 123
19	1. 27 875	39	1. 59 106	59	1. 77 085	79	1. 89 763	99	1. 99 564
20	1. 30 103	**40**	1. 60 206	**60**	1. 77 815	**80**	1. 90 309	**100**	2. 00 000

No.	0	1	2	3	4	5	6	7	8	9
100	00 000	00 043	00 087	00 130	00 173	00 217	00 260	00 303	00 346	00 389
101	00 432	00 475	00 518	00 561	00 604	00 647	00 689	00 732	00 775	00 817
102	00 860	00 903	00 945	00 988	01 030	01 072	01 115	01 157	01 199	01 242
103	01 284	01 326	01 368	01 410	01 452	01 494	01 536	01 578	01 620	01 662
104	01 703	01 745	01 787	01 828	01 870	01 912	01 953	01 995	02 036	02 078
105	02 119	02 160	02 202	02 243	02 284	02 325	02 366	02 407	02 449	02 490
106	02 531	02 572	02 612	02 653	02 694	02 735	02 776	02 816	02 857	02 898
107	02 938	02 979	03 019	03 060	03 100	03 141	03 181	03 222	03 262	03 302
108	03 342	03 383	03 423	03 463	03 503	03 543	03 583	03 623	03 663	03 703
109	03 743	03 782	03 822	03 862	03 902	03 941	03 981	04 021	04 060	04 100
110	04 139	04 179	04 218	04 258	04 297	04 336	04 376	04 415	04 454	04 493
111	04 532	04 571	04 610	04 650	04 689	04 727	04 766	04 805	04 844	04 883
112	04 922	04 961	04 999	05 038	05 077	05 115	05 154	05 192	05 231	05 269
113	05 308	05 346	05 385	05 423	05 461	05 500	05 538	05 576	05 614	05 652
114	05 690	05 729	05 767	05 805	05 843	05 881	05 918	05 956	05 994	06 032
115	06 070	06 108	06 145	06 183	06 221	06 258	06 296	06 333	06 371	06 408
116	06 446	06 483	06 521	06 558	06 595	06 633	06 670	06 707	06 744	06 781
117	06 819	06 856	06 893	06 930	06 967	07 004	07 041	07 078	07 115	07 151
118	07 188	07 225	07 262	07 298	07 335	07 372	07 408	07 445	07 482	07 518
119	07 555	07 591	07 628	07 664	07 700	07 737	07 773	07 809	07 846	07 882
120	07 918	07 954	07 990	08 027	08 063	08 099	08 135	08 171	08 207	08 243
121	08 279	08 314	08 350	08 386	08 422	08 458	08 493	08 529	08 565	08 600
122	08 636	08 672	08 707	08 743	08 778	08 814	08 849	08 884	08 920	08 955
123	08 991	09 026	09 061	09 096	09 132	09 167	09 202	09 237	09 272	09 307
124	09 342	09 377	09 412	09 447	09 482	09 517	09 552	09 587	09 621	09 656
125	09 691	09 726	09 760	09 795	09 830	09 864	09 899	09 934	09 968	10 003
126	10 037	10 072	10 106	10 140	10 175	10 209	10 243	10 278	10 312	10 346
127	10 380	10 415	10 449	10 483	10 517	10 551	10 585	10 619	10 653	10 687
128	10 721	10 755	10 789	10 823	10 857	10 890	10 924	10 958	10 992	11 025
129	11 059	11 093	11 126	11 160	11 193	11 227	11 261	11 294	11 327	11 361
130	11 394	11 428	11 461	11 494	11 528	11 561	11 594	11 628	11 661	11 694
131	11 727	11 760	11 793	11 826	11 860	11 893	11 926	11 959	11 992	12 024
132	12 057	12 090	12 123	12 156	12 189	12 222	12 254	12 287	12 320	12 352
133	12 385	12 418	12 450	12 483	12 516	12 548	12 581	12 613	12 646	12 678
134	12 710	12 743	12 775	12 808	12 840	12 872	12 905	12 937	12 969	13 001
135	13 033	13 066	13 098	13 130	13 162	13 194	13 226	13 258	13 290	13 322
136	13 354	13 386	13 418	13 450	13 481	13 513	13 545	13 577	13 609	13 640
137	13 672	13 704	13 735	13 767	13 799	13 830	13 862	13 893	13 925	13 956
138	13 988	14 019	14 051	14 082	14 114	14 145	14 176	14 208	14 239	14 270
139	14 301	14 333	14 364	14 395	14 426	14 457	14 489	14 520	14 551	14 5[illegible]
140	14 613	14 644	14 675	14 706	14 737	14 768	14 799	14 829	14 860	14 891
141	14 922	14 953	14 983	15 014	15 045	15 076	15 106	15 137	15 168	15 198
142	15 229	15 259	15 290	15 320	15 351	15 381	15 412	15 442	15 473	15 503
143	15 534	15 564	15 594	15 625	15 655	15 685	15 715	15 746	15 776	15 806
144	15 836	15 866	15 897	15 927	15 957	15 987	16 017	16 047	16 077	16 107
145	16 137	16 167	16 197	16 227	16 256	16 286	16 316	16 346	16 376	16 406
146	16 435	16 465	16 495	16 524	16 554	16 584	16 613	16 643	16 673	16 702
147	16 732	16 761	16 791	16 820	16 850	16 879	16 909	16 938	16 967	16 997
148	17 026	17 056	17 085	17 114	17 143	17 173	17 202	17 231	17 260	17 289
149	17 319	17 348	17 377	17 406	17 435	17 464	17 493	17 522	17 551	17 580
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
150	17 609	17 638	17 667	17 696	17 725	17 754	17 782	17 811	17 840	17 869
151	17 898	17 926	17 955	17 984	18 013	18 041	18 070	18 099	18 127	18 156
152	18 184	18 213	18 241	18 270	18 298	18 327	18 355	18 384	18 412	18 441
153	18 469	18 498	18 526	18 554	18 583	18 611	18 639	18 667	18 696	18 724
154	18 752	18 780	18 808	18 837	18 865	18 893	18 921	18 949	18 977	19 005
155	19 033	19 061	19 089	19 117	19 145	19 173	19 201	19 229	19 257	19 285
156	19 312	19 340	19 368	19 396	19 424	19 451	19 479	19 507	19 535	19 562
157	19 590	19 618	19 645	19 673	19 700	19 728	19 756	19 783	19 811	19 838
158	19 866	19 893	19 921	19 948	19 976	20 003	20 030	20 058	20 085	20 112
159	20 140	20 167	20 194	20 222	20 249	20 276	20 303	20 330	20 358	20 385
160	20 412	20 439	20 466	20 493	20 520	20 548	20 575	20 602	20 629	20 656
161	20 683	20 710	20 737	20 763	20 790	20 817	20 844	20 871	20 898	20 925
162	20 952	20 978	21 005	21 032	21 059	21 085	21 112	21 139	21 165	21 192
163	21 219	21 245	21 272	21 299	21 325	21 352	21 378	21 405	21 431	21 458
164	21 484	21 511	21 537	21 564	21 590	21 617	21 643	21 669	21 696	21 722
165	21 748	21 775	21 801	21 827	21 854	21 880	21 906	21 932	21 958	21 985
166	22 011	22 037	22 063	22 089	22 115	22 141	22 167	22 194	22 220	22 246
167	22 272	22 298	22 324	22 350	22 376	22 401	22 427	22 453	22 479	22 505
168	22 531	22 557	22 583	22 608	22 634	22 660	22 686	22 712	22 737	22 763
169	22 789	22 814	22 840	22 866	22 891	22 917	22 943	22 968	22 994	23 019
170	23 045	23 070	23 096	23 121	23 147	23 172	23 198	23 223	23 249	23 274
171	23 300	23 325	23 350	23 376	23 401	23 426	23 452	23 477	23 502	23 528
172	23 553	23 578	23 603	23 629	23 654	23 679	23 704	23 729	23 754	23 779
173	23 805	23 830	23 855	23 880	23 905	23 930	23 955	23 980	24 005	24 030
174	24 055	24 080	24 105	24 130	24 155	24 180	24 204	24 229	24 254	24 279
175	24 304	24 329	24 353	24 378	24 403	24 428	24 452	24 477	24 502	24 527
176	24 551	24 576	24 601	24 625	24 650	24 674	24 699	24 724	24 748	24 773
177	24 797	24 822	24 846	24 871	24 895	24 920	24 944	24 969	24 993	25 018
178	25 042	25 066	25 091	25 115	25 139	25 164	25 188	25 212	25 237	25 261
179	25 285	25 310	25 334	25 358	25 382	25 406	25 431	25 455	25 479	25 503
180	25 527	25 551	25 575	25 600	25 624	25 648	25 672	25 696	25 720	25 744
181	25 768	25 792	25 816	25 840	25 864	25 888	25 912	25 935	25 959	25 983
182	26 007	26 031	26 055	26 079	26 102	26 126	26 150	26 174	26 198	26 221
183	26 245	26 269	26 293	26 316	26 340	26 364	26 387	26 411	26 435	26 458
184	26 482	26 505	26 529	26 553	26 576	26 600	26 623	26 647	26 670	26 694
185	26 717	26 741	26 764	26 788	26 811	26 834	26 858	26 881	26 905	26 928
186	26 951	26 975	26 998	27 021	27 045	27 068	27 091	27 114	27 138	27 161
[illegible]	27 184	27 207	27 231	27 254	27 277	27 300	27 323	27 346	27 370	27 393
[illegible]	7 416	27 439	27 462	27 485	27 508	27 531	27 554	27 577	27 600	27 623
[illegible]	7 646	27 669	27 692	27 715	27 738	27 761	27 784	27 807	27 830	27 852
[illegible]	27 875	27 898	27 921	27 944	27 967	27 989	28 012	28 035	28 058	28 081
191	28 103	28 126	28 149	28 171	28 194	28 217	28 240	28 262	28 285	28 307
192	28 330	28 353	28 375	28 398	28 421	28 443	28 466	28 488	28 511	28 533
193	28 556	28 578	28 601	28 623	28 646	28 668	28 691	28 713	28 735	28 758
194	28 780	28 803	28 825	28 847	28 870	28 892	28 914	28 937	28 959	28 981
195	29 003	29 026	29 048	29 070	29 092	29 115	29 137	29 159	29 181	29 203
196	29 226	29 248	29 270	29 292	29 314	29 336	29 358	29 380	29 403	29 425
197	29 447	29 469	29 491	29 513	29 535	29 557	29 579	29 601	29 623	29 645
198	29 667	29 688	29 710	29 732	29 754	29 776	29 798	29 820	29 842	29 863
199	29 885	29 907	29 929	29 951	29 973	29 994	30 016	30 038	30 060	30 081
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
200	30 103	30 125	30 146	30 168	30 190	30 211	30 233	30 255	30 276	30 298
201	30 320	30 341	30 363	30 384	30 406	30 428	30 449	30 471	30 492	30 514
202	30 535	30 557	30 578	30 600	30 621	30 643	30 664	30 685	30 707	30 728
203	30 750	30 771	30 792	30 814	30 835	30 856	30 878	30 899	30 920	30 942
204	30 963	30 984	31 006	31 027	31 048	31 069	31 091	31 112	31 133	31 154
205	31 175	31 197	31 218	31 239	31 260	31 281	31 302	31 323	31 345	31 366
206	31 387	31 408	31 429	31 450	31 471	31 492	31 513	31 534	31 555	31 576
207	31 597	31 618	31 639	31 660	31 681	31 702	31 723	31 744	31 765	31 785
208	31 806	31 827	31 848	31 869	31 890	31 911	31 931	31 952	31 973	31 994
209	32 015	32 035	32 056	32 077	32 098	32 118	32 139	32 160	32 181	32 201
210	32 222	32 243	32 263	32 284	32 305	32 325	32 346	32 366	32 387	32 408
211	32 428	32 449	32 469	32 490	32 510	32 531	32 552	32 572	32 593	32 613
212	32 634	32 654	32 675	32 695	32 715	32 736	32 756	32 777	32 797	32 818
213	32 838	32 858	32 879	32 899	32 919	32 940	32 960	32 980	33 001	33 021
214	33 041	33 062	33 082	33 102	33 122	33 143	33 163	33 183	33 203	33 224
215	33 244	33 264	33 284	33 304	33 325	33 345	33 365	33 385	33 405	33 425
216	33 445	33 465	33 486	33 506	33 526	33 546	33 566	33 586	33 606	33 626
217	33 646	33 666	33 686	33 706	33 726	33 746	33 766	33 786	33 806	33 826
218	33 846	33 866	33 885	33 905	33 925	33 945	33 965	33 985	34 005	34 025
219	34 044	34 064	34 084	34 104	34 124	34 143	34 163	34 183	34 203	34 223
220	34 242	34 262	34 282	34 301	34 321	34 341	34 361	34 380	34 400	34 420
221	34 439	34 459	34 479	34 498	34 518	34 537	34 557	34 577	34 596	34 616
222	34 635	34 655	34 674	34 694	34 713	34 733	34 753	34 772	34 792	34 811
223	34 830	34 850	34 869	34 889	34 908	34 928	34 947	34 967	34 986	35 005
224	35 025	35 044	35 064	35 083	35 102	35 122	35 141	35 160	35 180	35 199
225	35 218	35 238	35 257	35 276	35 295	35 315	35 334	35 353	35 372	35 392
226	35 411	35 430	35 449	35 468	35 488	35 507	35 526	35 545	35 564	35 583
227	35 603	35 622	35 641	35 660	35 679	35 698	35 717	35 736	35 755	35 774
228	35 793	35 813	35 832	35 851	35 870	35 889	35 908	35 927	35 946	35 965
229	35 984	36 003	36 021	36 040	36 059	36 078	36 097	36 116	36 135	36 154
230	36 173	36 192	36 211	36 229	36 248	36 267	36 286	36 305	36 324	36 342
231	36 361	36 380	36 399	36 418	36 436	36 455	36 474	36 493	36 511	36 530
232	36 549	36 568	36 586	36 605	36 624	36 642	36 661	36 680	36 698	36 717
233	36 736	36 754	36 773	36 791	36 810	36 829	36 847	36 866	36 884	36 903
234	36 922	36 940	36 959	36 977	36 996	37 014	37 033	37 051	37 070	37 088
235	37 107	37 125	37 144	37 162	37 181	37 199	37 218	37 236	37 254	37 273
236	37 291	37 310	37 328	37 346	37 365	37 383	37 401	37 420	37 438	37 457
237	37 475	37 493	37 511	37 530	37 548	37 566	37 585	37 603	37 621	[illegible]
238	37 658	37 676	37 694	37 712	37 731	37 749	37 767	37 785	37 803	[illegible]
239	37 840	37 858	37 876	37 894	37 912	37 931	37 949	37 967	37 985	[illegible]
240	38 021	38 039	38 057	38 075	38 093	38 112	38 130	38 148	38 166	[illegible]
241	38 202	38 220	38 238	38 256	38 274	38 292	38 310	38 328	38 346	38 364
242	38 382	38 399	38 417	38 435	38 453	38 471	38 489	38 507	38 525	38 543
243	38 561	38 578	38 596	38 614	38 632	38 650	38 668	38 686	38 703	38 721
244	38 739	38 757	38 775	38 792	38 810	38 828	38 846	38 863	38 881	38 899
245	38 917	38 934	38 952	38 970	38 987	39 005	39 023	39 041	39 058	39 076
246	39 094	39 111	39 129	39 146	39 164	39 182	39 199	39 217	39 235	39 252
247	39 270	39 287	39 305	39 322	39 340	39 358	39 375	39 393	39 410	39 428
248	39 445	39 463	39 480	39 498	39 515	39 533	39 550	39 568	39 585	39 602
249	39 620	39 637	39 655	39 672	39 690	39 707	39 724	39 742	39 759	39 777
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

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250	39 794	39 811	39 829	39 846	39 863	39 881	39 898	39 915	39 933	39 950
251	39 967	39 985	40 002	40 019	40 037	40 054	40 071	40 088	40 106	40 123
252	40 140	40 157	40 175	40 192	40 209	40 226	40 243	40 261	40 278	40 295
253	40 312	40 329	40 346	40 364	40 381	40 398	40 415	40 432	40 449	40 466
254	40 483	40 500	40 518	40 535	40 552	40 569	40 586	40 603	40 620	40 637
255	40 654	40 671	40 688	40 705	40 722	40 739	40 756	40 773	40 790	40 807
256	40 824	40 841	40 858	40 875	40 892	40 909	40 926	40 943	40 960	40 976
257	40 993	41 010	41 027	41 044	41 061	41 078	41 095	41 111	41 128	41 145
258	41 162	41 179	41 196	41 212	41 229	41 246	41 263	41 280	41 296	41 313
259	41 330	41 347	41 363	41 380	41 397	41 414	41 430	41 447	41 464	41 481
260	41 497	41 514	41 531	41 547	41 564	41 581	41 597	41 614	41 631	41 647
261	41 664	41 681	41 697	41 714	41 731	41 747	41 764	41 780	41 797	41 814
262	41 830	41 847	41 863	41 880	41 896	41 913	41 929	41 946	41 963	41 979
263	41 996	42 012	42 029	42 045	42 062	42 078	42 095	42 111	42 127	42 144
264	42 160	42 177	42 193	42 210	42 226	42 243	42 259	42 275	42 292	42 308
265	42 325	42 341	42 357	42 374	42 390	42 406	42 423	42 439	42 455	42 472
266	42 488	42 504	42 521	42 537	42 553	42 570	42 586	42 602	42 619	42 635
267	42 651	42 667	42 684	42 700	42 716	42 732	42 749	42 765	42 781	42 797
268	42 813	42 830	42 846	42 862	42 878	42 894	42 911	42 927	42 943	42 959
269	42 975	42 991	43 008	43 024	43 040	43 056	43 072	43 088	43 104	43 120
270	43 136	43 152	43 169	43 185	43 201	43 217	43 233	43 249	43 265	43 281
271	43 297	43 313	43 329	43 345	43 361	43 377	43 393	43 409	43 425	43 441
272	43 457	43 473	43 489	43 505	43 521	43 537	43 553	43 569	43 584	43 600
273	43 616	43 632	43 648	43 664	43 680	43 696	43 712	43 727	43 743	43 759
274	43 775	43 791	43 807	43 823	43 838	43 854	43 870	43 886	43 902	43 917
275	43 933	43 949	43 965	43 981	43 996	44 012	44 028	44 044	44 059	44 075
276	44 091	44 107	44 122	44 138	44 154	44 170	44 185	44 201	44 217	44 232
277	44 248	44 264	44 279	44 295	44 311	44 326	44 342	44 358	44 373	44 389
278	44 404	44 420	44 436	44 451	44 467	44 483	44 498	44 514	44 529	44 545
279	44 560	44 576	44 592	44 607	44 623	44 638	44 654	44 669	44 685	44 700
280	44 716	44 731	44 747	44 762	44 778	44 793	44 809	44 824	44 840	44 855
281	44 871	44 886	44 902	44 917	44 932	44 948	44 963	44 979	44 994	45 010
282	45 025	45 040	45 056	45 071	45 086	45 102	45 117	45 133	45 148	45 163
283	45 179	45 194	45 209	45 225	45 240	45 255	45 271	45 286	45 301	45 317
284	45 332	45 347	45 362	45 378	45 393	45 408	45 423	45 439	45 454	45 469
285	45 484	45 500	45 515	45 530	45 545	45 561	45 576	45 591	45 606	45 621
286	45 637	45 652	45 667	45 682	45 697	45 712	45 728	45 743	45 758	45 773
287	45 788	45 803	45 818	45 834	45 849	45 864	45 879	45 894	45 909	45 924
288	45 939	45 954	45 969	45 984	46 000	46 015	46 030	46 045	46 060	46 075
289	46 090	46 105	46 120	46 135	46 150	46 165	46 180	46 195	46 210	46 225
290	46 240	46 255	46 270	46 285	46 300	46 315	46 330	46 345	46 359	46 374
291	46 389	46 404	46 419	46 434	46 449	46 464	46 479	46 494	46 509	46 523
292	46 538	46 553	46 568	46 583	46 598	46 613	46 627	46 642	46 657	46 672
293	46 687	46 702	46 716	46 731	46 746	46 761	46 776	46 790	46 805	46 820
294	46 835	46 850	46 864	46 879	46 894	46 909	46 923	46 938	46 953	46 967
295	46 982	46 997	47 012	47 026	47 041	47 056	47 070	47 085	47 100	47 114
296	47 129	47 144	47 159	47 173	47 188	47 202	47 217	47 232	47 246	47 261
297	47 276	47 290	47 305	47 319	47 334	47 349	47 363	47 378	47 392	47 407
298	47 422	47 436	47 451	47 465	47 480	47 494	47 509	47 524	47 538	47 553
299	47 567	47 582	47 596	47 611	47 625	47 640	47 654	47 669	47 683	47 698

No. 0 1 2 3 4 5 6 7 8 9

No.	0	1	2	3	4	5	6	7	8	9
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301	47 857	47 871	47 885	47 900	47 914	47 929	47 943	47 958	47 972	47 986
302	48 001	48 015	48 029	48 044	48 058	48 073	48 087	48 101	48 116	48 130
303	48 144	48 159	48 173	48 187	48 202	48 216	48 230	48 244	48 259	48 273
304	48 287	48 302	48 316	48 330	48 344	48 359	48 373	48 387	48 401	48 416
305	48 430	48 444	48 458	48 473	48 487	48 501	48 515	48 530	48 544	48 558
306	48 572	48 586	48 601	48 615	48 629	48 643	48 657	48 671	48 686	48 700
307	48 714	48 728	48 742	48 756	48 770	48 785	48 799	48 813	48 827	48 841
308	48 855	48 869	48 883	48 897	48 911	48 926	48 940	48 954	48 968	48 982
309	48 996	49 010	49 024	49 038	49 052	49 066	49 080	49 094	49 108	49 122
310	49 136	49 150	49 164	49 178	49 192	49 206	49 220	49 234	49 248	49 262
311	49 276	49 290	49 304	49 318	49 332	49 346	49 360	49 374	49 388	49 402
312	49 415	49 429	49 443	49 457	49 471	49 485	49 499	49 513	49 527	49 541
313	49 554	49 568	49 582	49 596	49 610	49 624	49 638	49 651	49 665	49 679
314	49 693	49 707	49 721	49 734	49 748	49 762	49 776	49 790	49 803	49 817
315	49 831	49 845	49 859	49 872	49 886	49 900	49 914	49 927	49 941	49 955
316	49 969	49 982	49 996	50 010	50 024	50 037	50 051	50 065	50 079	50 092
317	50 106	50 120	50 133	50 147	50 161	50 174	50 188	50 202	50 215	50 229
318	50 243	50 256	50 270	50 284	50 297	50 311	50 325	50 338	50 352	50 365
319	50 379	50 393	50 406	50 420	50 433	50 447	50 461	50 474	50 488	50 501
320	50 515	50 529	50 542	50 556	50 569	50 583	50 596	50 610	50 623	50 637
321	50 651	50 664	50 678	50 691	50 705	50 718	50 732	50 745	50 759	50 772
322	50 786	50 799	50 813	50 826	50 840	50 853	50 866	50 880	50 893	50 907
323	50 920	50 934	50 947	50 961	50 974	50 987	51 001	51 014	51 028	51 041
324	51 055	51 068	51 081	51 095	51 108	51 121	51 135	51 148	51 162	51 175
325	51 188	51 202	51 215	51 228	51 242	51 255	51 268	51 282	51 295	51 308
326	51 322	51 335	51 348	51 362	51 375	51 388	51 402	51 415	51 428	51 441
327	51 455	51 468	51 481	51 495	51 508	51 521	51 534	51 548	51 561	51 574
328	51 587	51 601	51 614	51 627	51 640	51 654	51 667	51 680	51 693	51 706
329	51 720	51 733	51 746	51 759	51 772	51 786	51 799	51 812	51 825	51 838
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333	52 244	52 257	52 270	52 284	52 297	52 310	52 323	52 336	52 349	52 362
334	52 375	52 388	52 401	52 414	52 427	52 440	52 453	52 466	52 479	52 492
335	52 504	52 517	52 530	52 543	52 556	52 569	52 582	52 595	52 608	52 621
336	52 634	52 647	52 660	52 673	52 686	52 699	52 711	52 724	52 737	52 750
337	52 763	52 776	52 789	52 802	52 815	52 827	52 840	52 853	52 866	52 879
338	52 892	52 905	52 917	52 930	52 943	52 956	52 969	52 982	52 994	53 007
339	53 020	53 033	53 046	53 058	53 071	53 084	53 097	53 110	53 122	53 135
340	53 148	53 161	53 173	53 186	53 199	53 212	53 224	53 237	53 250	53 263
341	53 275	53 288	53 301	53 314	53 326	53 339	53 352	53 364	53 377	53 390
342	53 403	53 415	53 428	53 441	53 453	53 466	53 479	53 491	53 504	53 517
343	53 529	53 542	53 555	53 567	53 580	53 593	53 605	53 618	53 631	53 643
344	53 656	53 668	53 681	53 694	53 706	53 719	53 732	53 744	53 757	53 769
345	53 782	53 794	53 807	53 820	53 832	53 845	53 857	53 870	53 882	53 895
346	53 908	53 920	53 933	53 945	53 958	53 970	53 983	53 995	54 008	54 020
347	54 033	54 045	54 058	54 070	54 083	54 095	54 108	54 120	54 133	54 145
348	54 158	54 170	54 183	54 195	54 208	54 220	54 233	54 245	54 258	54 270
349	54 283	54 295	54 307	54 320	54 332	54 345	54 357	54 370	54 382	54 394
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

No.	0	1	2	3	4	5	6	7	8	9
350	54 407	54 419	54 432	54 444	54 456	54 469	54 481	54 494	54 506	54 518
351	54 531	54 543	54 555	54 568	54 580	54 593	54 605	54 617	54 630	54 642
352	54 654	54 667	54 679	54 691	54 704	54 716	54 728	54 741	54 753	54 765
353	54 777	54 790	54 802	54 814	54 827	54 839	54 851	54 864	54 876	54 888
354	54 900	54 913	54 925	54 937	54 949	54 962	54 974	54 986	54 998	55 011
355	55 023	55 035	55 047	55 060	55 072	55 084	55 096	55 108	55 121	55 133
356	55 145	55 157	55 169	55 182	55 194	55 206	55 218	55 230	55 242	55 255
357	55 267	55 279	55 291	55 303	55 315	55 328	55 340	55 352	55 364	55 376
358	55 388	55 400	55 413	55 425	55 437	55 449	55 461	55 473	55 485	55 497
359	55 509	55 522	55 534	55 546	55 558	55 570	55 582	55 594	55 606	55 618
360	55 630	55 642	55 654	55 666	55 678	55 691	55 703	55 715	55 727	55 739
361	55 751	55 763	55 775	55 787	55 799	55 811	55 823	55 835	55 847	55 859
362	55 871	55 883	55 895	55 907	55 919	55 931	55 943	55 955	55 967	55 979
363	55 991	56 003	56 015	56 027	56 038	56 050	56 062	56 074	56 086	56 098
364	56 110	56 122	56 134	56 146	56 158	56 170	56 182	56 194	56 205	56 217
365	56 229	56 241	56 253	56 265	56 277	56 289	56 301	56 312	56 324	56 336
366	56 348	56 360	56 372	56 384	56 396	56 407	56 419	56 431	56 443	56 455
367	56 467	56 478	56 490	56 502	56 514	56 526	56 538	56 549	56 561	56 573
368	56 585	56 597	56 608	56 620	56 632	56 644	56 656	56 667	56 679	56 691
369	56 703	56 714	56 726	56 738	56 750	56 761	56 773	56 785	56 797	56 808
370	56 820	56 832	56 844	56 855	56 867	56 879	56 891	56 902	56 914	56 926
371	56 937	56 949	56 961	56 972	56 984	56 996	57 008	57 019	57 031	57 043
372	57 054	57 066	57 078	57 089	57 101	57 113	57 124	57 136	57 148	57 159
373	57 171	57 183	57 194	57 206	57 217	57 229	57 241	57 252	57 264	57 276
374	57 287	57 299	57 310	57 322	57 334	57 345	57 357	57 368	57 380	57 392
375	57 403	57 415	57 426	57 438	57 449	57 461	57 473	57 484	57 496	57 507
376	57 519	57 530	57 542	57 553	57 565	57 576	57 588	57 600	57 611	57 623
377	57 634	57 646	57 657	57 669	57 680	57 692	57 703	57 715	57 726	57 738
378	57 749	57 761	57 772	57 784	57 795	57 807	57 818	57 830	57 841	57 852
379	57 864	57 875	57 887	57 898	57 910	57 921	57 933	57 944	57 955	57 967
380	57 978	57 990	58 001	58 013	58 024	58 035	58 047	58 058	58 070	58 081
381	58 092	58 104	58 115	58 127	58 138	58 149	58 161	58 172	58 184	58 195
382	58 206	58 218	58 229	58 240	58 252	58 263	58 274	58 286	58 297	58 309
383	58 320	58 331	58 343	58 354	58 365	58 377	58 388	58 399	58 410	58 422
384	58 433	58 444	58 456	58 467	58 478	58 490	58 501	58 512	58 524	58 535
385	58 546	58 557	58 569	58 580	58 591	58 602	58 614	58 625	58 636	58 647
386	58 659	58 670	58 681	58 692	58 704	58 715	58 726	58 737	58 749	58 760
387	58 771	58 782	58 794	58 805	58 816	58 827	58 838	58 850	58 861	58 872
388	58 883	58 894	58 906	58 917	58 928	58 939	58 950	58 961	58 973	58 984
389	58 995	59 006	59 017	59 028	59 040	59 051	59 062	59 073	59 084	59 095
390	59 106	59 118	59 129	59 140	59 151	59 162	59 173	59 184	59 195	59 207
391	59 218	59 229	59 240	59 251	59 262	59 273	59 284	59 295	59 306	59 318
392	59 329	59 340	59 351	59 362	59 373	59 384	59 395	59 406	59 417	59 428
393	59 439	59 450	59 461	59 472	59 483	59 494	59 506	59 517	59 528	59 539
394	59 550	59 561	59 572	59 583	59 594	59 605	59 616	59 627	59 638	59 649
395	59 660	59 671	59 682	59 693	59 704	59 715	59 726	59 737	59 748	59 759
396	59 770	59 780	59 791	59 802	59 813	59 824	59 835	59 846	59 857	59 868
397	59 879	59 890	59 901	59 912	59 923	59 934	59 945	59 956	59 966	59 977
398	59 988	59 999	60 010	60 021	60 032	60 043	60 054	60 065	60 076	60 086
399	60 097	60 108	60 119	60 130	60 141	60 152	60 163	60 173	60 184	60 195
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
400	60 206	60 217	60 228	60 239	60 249	60 260	60 271	60 282	60 293	60 304
401	60 314	60 325	60 336	60 347	60 358	60 369	60 379	60 390	60 401	60 412
402	60 423	60 433	60 444	60 455	60 466	60 477	60 487	60 498	60 509	60 520
403	60 531	60 541	60 552	60 563	60 574	60 584	60 595	60 606	60 617	60 627
404	60 638	60 649	60 660	60 670	60 681	60 692	60 703	60 713	60 724	60 735
405	60 746	60 756	60 767	60 778	60 788	60 799	60 810	60 821	60 831	60 842
406	60 853	60 863	60 874	60 885	60 895	60 906	60 917	60 927	60 938	60 949
407	60 959	60 970	60 981	60 991	61 002	61 013	61 023	61 034	61 045	61 055
408	61 066	61 077	61 087	61 098	61 109	61 119	61 130	61 140	61 151	61 162
409	61 172	61 183	61 194	61 204	61 215	61 225	61 236	61 247	61 257	61 268
410	61 278	61 289	61 300	61 310	61 321	61 331	61 342	61 352	61 363	61 374
411	61 384	61 395	61 405	61 416	61 426	61 437	61 448	61 458	61 469	61 479
412	61 490	61 500	61 511	61 521	61 532	61 542	61 553	61 563	61 574	61 584
413	61 595	61 606	61 616	61 627	61 637	61 648	61 658	61 669	61 679	61 690
414	61 700	61 711	61 721	61 731	61 742	61 752	61 763	61 773	61 784	61 794
415	61 805	61 815	61 826	61 836	61 847	61 857	61 868	61 878	61 888	61 899
416	61 909	61 920	61 930	61 941	61 951	61 962	61 972	61 982	61 993	62 003
417	62 014	62 024	62 034	62 045	62 055	62 066	62 076	62 086	62 097	62 107
418	62 118	62 128	62 138	62 149	62 159	62 170	62 180	62 190	62 201	62 211
419	62 221	62 232	62 242	62 252	62 263	62 273	62 284	62 294	62 304	62 315
420	62 325	62 335	62 346	62 356	62 366	62 377	62 387	62 397	62 408	62 418
421	62 428	62 439	62 449	62 459	62 469	62 480	62 490	62 500	62 511	62 521
422	62 531	62 542	62 552	62 562	62 572	62 583	62 593	62 603	62 613	62 624
423	62 634	62 644	62 655	62 665	62 675	62 685	62 696	62 706	62 716	62 726
424	62 737	62 747	62 757	62 767	62 778	62 788	62 798	62 808	62 818	62 829
425	62 839	62 849	62 859	62 870	62 880	62 890	62 900	62 910	62 921	62 931
426	62 941	62 951	62 961	62 972	62 982	62 992	63 002	63 012	63 022	63 033
427	63 043	63 053	63 063	63 073	63 083	63 094	63 104	63 114	63 124	63 134
428	63 144	63 155	63 165	63 175	63 185	63 195	63 205	63 215	63 225	63 236
429	63 246	63 256	63 266	63 276	63 286	63 296	63 306	63 317	63 327	63 337
430	63 347	63 357	63 367	63 377	63 387	63 397	63 407	63 417	63 428	63 438
431	63 448	63 458	63 468	63 478	63 488	63 498	63 508	63 518	63 528	63 538
432	63 548	63 558	63 568	63 579	63 589	63 599	63 609	63 619	63 629	63 639
433	63 649	63 659	63 669	63 679	63 689	63 699	63 709	63 719	63 729	63 739
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437	64 048	64 058	64 068	64 078	64 088	64 098	64 108	64 118	64 128	64 137
438	64 147	64 157	64 167	64 177	64 187	64 197	64 207	64 217	64 227	64 237
439	64 246	64 256	64 266	64 276	64 286	64 296	64 306	64 316	64 326	64 335
440	64 345	64 355	64 365	64 375	64 385	64 395	64 404	64 414	64 424	64 434
441	64 444	64 454	64 464	64 473	64 483	64 493	64 503	64 513	64 523	64 532
442	64 542	64 552	64 562	64 572	64 582	64 591	64 601	64 611	64 621	64 631
443	64 640	64 650	64 660	64 670	64 680	64 689	64 699	64 709	64 719	64 729
444	64 738	64 748	64 758	64 768	64 777	64 787	64 797	64 807	64 816	64 826
445	64 836	64 846	64 856	64 865	64 875	64 885	64 895	64 904	64 914	64 924
446	64 933	64 943	64 953	64 963	64 972	64 982	64 992	65 002	65 011	65 021
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448	65 128	65 137	65 147	65 157	65 167	65 176	65 186	65 196	65 205	65 215
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No.	0	1	2	3	4	5	6	7	8	9

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451	65 418	65 427	65 437	65 447	65 456	65 466	65 475	65 485	65 495	65 504
452	65 514	65 523	65 533	65 543	65 552	65 562	65 571	65 581	65 591	65 600
453	65 610	65 619	65 629	65 639	65 648	65 658	65 667	65 677	65 686	65 696
454	65 706	65 715	65 725	65 734	65 744	65 753	65 763	65 772	65 782	65 792
455	65 801	65 811	65 820	65 830	65 839	65 849	65 858	65 868	65 877	65 887
456	65 896	65 906	65 916	65 925	65 935	65 944	65 954	65 963	65 973	65 982
457	65 992	66 001	66 011	66 020	66 030	66 039	66 049	66 058	66 068	66 077
458	66 087	66 096	66 106	66 115	66 124	66 134	66 143	66 153	66 162	66 172
459	66 181	66 191	66 200	66 210	66 219	66 229	66 238	66 247	66 257	66 266
460	66 276	66 285	66 295	66 304	66 314	66 323	66 332	66 342	66 351	66 361
461	66 370	66 380	66 389	66 398	66 408	66 417	66 427	66 436	66 445	66 455
462	66 464	66 474	66 483	66 492	66 502	66 511	66 521	66 530	66 539	66 549
463	66 558	66 567	66 577	66 586	66 596	66 605	66 614	66 624	66 633	66 642
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466	66 839	66 848	66 857	66 867	66 876	66 885	66 894	66 904	66 913	66 922
467	66 932	66 941	66 950	66 960	66 969	66 978	66 987	66 997	67 006	67 015
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469	67 117	67 127	67 136	67 145	67 154	67 164	67 173	67 182	67 191	67 201
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472	67 394	67 403	67 413	67 422	67 431	67 440	67 449	67 459	67 468	67 477
473	67 486	67 495	67 504	67 514	67 523	67 532	67 541	67 550	67 560	67 569
474	67 578	67 587	67 596	67 605	67 614	67 624	67 633	67 642	67 651	67 660
475	67 669	67 679	67 688	67 697	67 706	67 715	67 724	67 733	67 742	67 752
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481	68 215	68 224	68 233	68 242	68 251	68 260	68 269	68 278	68 287	68 296
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484	68 485	68 494	68 502	68 511	68 520	68 529	68 538	68 547	68 556	68 565
485	68 574	68 583	68 592	68 601	68 610	68 619	68 628	68 637	68 646	68 655
486	68 664	68 673	68 681	68 690	68 699	68 708	68 717	68 726	68 735	68 744
487	68 753	68 762	68 771	68 780	68 789	68 797	68 806	68 815	68 824	68 833
488	68 842	68 851	68 860	68 869	68 878	68 886	68 895	68 904	68 913	68 922
489	68 931	68 940	68 949	68 958	68 966	68 975	68 984	68 993	69 002	69 011
490	69 020	69 028	69 037	69 046	69 055	69 064	69 073	69 082	69 090	69 099
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492	69 197	69 205	69 214	69 223	69 232	69 241	69 249	69 258	69 267	69 276
493	69 285	69 294	69 302	69 311	69 320	69 329	69 338	69 346	69 355	69 364
494	69 373	69 381	69 390	69 399	69 408	69 417	69 425	69 434	69 443	69 452
495	69 461	69 469	69 478	69 487	69 496	69 504	69 513	69 522	69 531	69 539
496	69 548	69 557	69 566	69 574	69 583	69 592	69 601	69 609	69 618	69 627
497	69 636	69 644	69 653	69 662	69 671	69 679	69 688	69 697	69 705	69 714
498	69 723	69 732	69 740	69 749	69 758	69 767	69 775	69 784	69 793	69 801
499	69 810	69 819	69 827	69 836	69 845	69 854	69 862	69 871	69 880	69 888
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

No.	0	1	2	3	4	5	6	7	8	9
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501	69 984	69 992	70 001	70 010	70 018	70 027	70 036	70 044	70 053	70 062
502	70 070	70 079	70 088	70 096	70 105	70 114	70 122	70 131	70 140	70 148
503	70 157	70 165	70 174	70 183	70 191	70 200	70 209	70 217	70 226	70 234
504	70 243	70 252	70 260	70 269	70 278	70 286	70 295	70 303	70 312	70 321
505	70 329	70 338	70 346	70 355	70 364	70 372	70 381	70 389	70 398	70 406
506	70 415	70 424	70 432	70 441	70 449	70 458	70 467	70 475	70 484	70 492
507	70 501	70 509	70 518	70 526	70 535	70 544	70 552	70 561	70 569	70 578
508	70 586	70 595	70 603	70 612	70 621	70 629	70 638	70 646	70 655	70 663
509	70 672	70 680	70 689	70 697	70 706	70 714	70 723	70 731	70 740	70 749
510	70 757	70 766	70 774	70 783	70 791	70 800	70 808	70 817	70 825	70 834
511	70 842	70 851	70 859	70 868	70 876	70 885	70 893	70 902	70 910	70 919
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514	71 096	71 105	71 113	71 122	71 130	71 139	71 147	71 155	71 164	71 172
515	71 181	71 189	71 198	71 206	71 214	71 223	71 231	71 240	71 248	71 257
516	71 265	71 273	71 282	71 290	71 299	71 307	71 315	71 324	71 332	71 341
517	71 349	71 357	71 366	71 374	71 383	71 391	71 399	71 408	71 416	71 425
518	71 433	71 441	71 450	71 458	71 466	71 475	71 483	71 492	71 500	71 508
519	71 517	71 525	71 533	71 542	71 550	71 559	71 567	71 575	71 584	71 592
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525	72 016	72 024	72 032	72 041	72 049	72 057	72 066	72 074	72 082	72 090
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527	72 181	72 189	72 198	72 206	72 214	72 222	72 230	72 239	72 247	72 255
528	72 263	72 272	72 280	72 288	72 296	72 304	72 313	72 321	72 329	72 337
529	72 346	72 354	72 362	72 370	72 378	72 387	72 395	72 403	72 411	72 419
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531	72 509	72 518	72 526	72 534	72 542	72 550	72 558	72 567	72 575	72 583
532	72 591	72 599	72 607	72 616	72 624	72 632	72 640	72 648	72 656	72 665
533	72 673	72 681	72 689	72 697	72 705	72 713	72 722	72 730	72 738	72 746
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536	72 916	72 925	72 933	72 941	72 949	72 957	72 965	72 973	72 981	72 989
537	72 997	73 006	73 014	73 022	73 030	73 038	73 046	73 054	73 062	73 070
538	73 078	73 086	73 094	73 102	73 111	73 119	73 127	73 135	73 143	73 151
539	73 159	73 167	73 175	73 183	73 191	73 199	73 207	73 215	73 223	73 231
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543	73 480	73 488	73 496	73 504	73 512	73 520	73 528	73 536	73 544	73 552
544	73 560	73 568	73 576	73 584	73 592	73 600	73 608	73 616	73 624	73 632
545	73 640	73 648	73 656	73 664	73 672	73 679	73 687	73 695	73 703	73 711
546	73 719	73 727	73 735	73 743	73 751	73 759	73 767	73 775	73 783	73 791
547	73 799	73 807	73 815	73 823	73 830	73 838	73 846	73 854	73 862	73 870
548	73 878	73 886	73 894	73 902	73 910	73 918	73 926	73 933	73 941	73 949
549	73 957	73 965	73 973	73 981	73 989	73 997	74 005	74 013	74 020	74 028
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

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551	74 115	74 123	74 131	74 139	74 147	74 155	74 162	74 170	74 178	74 186
552	74 194	74 202	74 210	74 218	74 225	74 233	74 241	74 249	74 257	74 265
553	74 273	74 280	74 288	74 296	74 304	74 312	74 320	74 327	74 335	74 343
554	74 351	74 359	74 367	74 374	74 382	74 390	74 398	74 406	74 414	74 421
555	74 429	74 437	74 445	74 453	74 461	74 468	74 476	74 484	74 492	74 500
556	74 507	74 515	74 523	74 531	74 539	74 547	74 554	74 562	74 570	74 578
557	74 586	74 593	74 601	74 609	74 617	74 624	74 632	74 640	74 648	74 656
558	74 663	74 671	74 679	74 687	74 695	74 702	74 710	74 718	74 726	74 733
559	74 741	74 749	74 757	74 764	74 772	74 780	74 788	74 796	74 803	74 811
560	74 819	74 827	74 834	74 842	74 850	74 858	74 865	74 873	74 881	74 889
561	74 896	74 904	74 912	74 920	74 927	74 935	74 943	74 950	74 958	74 966
562	74 974	74 981	74 989	74 997	75 005	75 012	75 020	75 028	75 035	75 043
563	75 051	75 059	75 066	75 074	75 082	75 089	75 097	75 105	75 113	75 120
564	75 128	75 136	75 143	75 151	75 159	75 166	75 174	75 182	75 189	75 197
565	75 205	75 213	75 220	75 228	75 236	75 243	75 251	75 259	75 266	75 274
566	75 282	75 289	75 297	75 305	75 312	75 320	75 328	75 335	75 343	75 351
567	75 358	75 366	75 374	75 381	75 389	75 397	75 404	75 412	75 420	75 427
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569	75 511	75 519	75 526	75 534	75 542	75 549	75 557	75 565	75 572	75 580
570	75 587	75 595	75 603	75 610	75 618	75 626	75 633	75 641	75 648	75 656
571	75 664	75 671	75 679	75 686	75 694	75 702	75 709	75 717	75 724	75 732
572	75 740	75 747	75 755	75 762	75 770	75 778	75 785	75 793	75 800	75 808
573	75 815	75 823	75 831	75 838	75 846	75 853	75 861	75 868	75 876	75 884
574	75 891	75 899	75 906	75 914	75 921	75 929	75 937	75 944	75 952	75 959
575	75 967	75 974	75 982	75 989	75 997	76 005	76 012	76 020	76 027	76 035
576	76 042	76 050	76 057	76 065	76 072	76 080	76 087	76 095	76 103	76 110
577	76 118	76 125	76 133	76 140	76 148	76 155	76 163	76 170	76 178	76 185
578	76 193	76 200	76 208	76 215	76 223	76 230	76 238	76 245	76 253	76 260
579	76 268	76 275	76 283	76 290	76 298	76 305	76 313	76 320	76 328	76 335
580	76 343	76 350	76 358	76 365	76 373	76 380	76 388	76 395	76 403	76 410
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582	76 492	76 500	76 507	76 515	76 522	76 530	76 537	76 545	76 552	76 559
583	76 567	76 574	76 582	76 589	76 597	76 604	76 612	76 619	76 626	76 634
584	76 641	76 649	76 656	76 664	76 671	76 678	76 686	76 693	76 701	76 708
585	76 716	76 723	76 730	76 738	76 745	76 753	76 760	76 768	76 775	76 782
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591	77 159	77 166	77 173	77 181	77 188	77 195	77 203	77 210	77 217	77 225
592	77 232	77 240	77 247	77 254	77 262	77 269	77 276	77 283	77 291	77 298
593	77 305	77 313	77 320	77 327	77 335	77 342	77 349	77 357	77 364	77 371
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596	77 525	77 532	77 539	77 546	77 554	77 561	77 568	77 576	77 583	77 590
597	77 597	77 605	77 612	77 619	77 627	77 634	77 641	77 648	77 656	77 663
598	77 670	77 677	77 685	77 692	77 699	77 706	77 714	77 721	77 728	77 735
599	77 743	77 750	77 757	77 764	77 772	77 779	77 786	77 793	77 801	77 808
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
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601	77 887	77 895	77 902	77 909	77 916	77 924	77 931	77 938	77 945	77 952
602	77 960	77 967	77 974	77 981	77 988	77 996	78 003	78 010	78 017	78 025
603	78 032	78 039	78 046	78 053	78 061	78 068	78 075	78 082	78 089	78 097
604	78 104	78 111	78 118	78 125	78 132	78 140	78 147	78 154	78 161	78 168
605	78 176	78 183	78 190	78 197	78 204	78 211	78 219	78 226	78 233	78 240
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611	78 604	78 611	78 618	78 625	78 633	78 640	78 647	78 654	78 661	78 668
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614	78 817	78 824	78 831	78 838	78 845	78 852	78 859	78 866	78 873	78 880
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616	78 958	78 965	78 972	78 979	78 986	78 993	79 000	79 007	79 014	79 021
617	79 029	79 036	79 043	79 050	79 057	79 064	79 071	79 078	79 085	79 092
618	79 099	79 106	79 113	79 120	79 127	79 134	79 141	79 148	79 155	79 162
619	79 169	79 176	79 183	79 190	79 197	79 204	79 211	79 218	79 225	79 232
620	79 239	79 246	79 253	79 260	79 267	79 274	79 281	79 288	79 295	79 302
621	79 309	79 316	79 323	79 330	79 337	79 344	79 351	79 358	79 365	79 372
622	79 379	79 386	79 393	79 400	79 407	79 414	79 421	79 428	79 435	79 442
623	79 449	79 456	79 463	79 470	79 477	79 484	79 491	79 498	79 505	79 511
624	79 518	79 525	79 532	79 539	79 546	79 553	79 560	79 567	79 574	79 581
625	79 588	79 595	79 602	79 609	79 616	79 623	79 630	79 637	79 644	79 650
626	79 657	79 664	79 671	79 678	79 685	79 692	79 699	79 706	79 713	79 720
627	79 727	79 734	79 741	79 748	79 754	79 761	79 768	79 775	79 782	79 789
628	79 796	79 803	79 810	79 817	79 824	79 831	79 837	79 844	79 851	79 858
629	79 865	79 872	79 879	79 886	79 893	79 900	79 906	79 913	79 920	79 927
630	79 934	79 941	79 948	79 955	79 962	79 969	79 975	79 982	79 989	79 996
631	80 003	80 010	80 017	80 024	80 030	80 037	80 044	80 051	80 058	80 065
632	80 072	80 079	80 085	80 092	80 099	80 106	80 113	80 120	80 127	80 134
633	80 140	80 147	80 154	80 161	80 168	80 175	80 182	80 188	80 195	80 202
634	80 209	80 216	80 223	80 229	80 236	80 243	80 250	80 257	80 264	80 271
635	80 277	80 284	80 291	80 298	80 305	80 312	80 318	80 325	80 332	80 339
636	80 346	80 353	80 359	80 366	80 373	80 380	80 387	80 393	80 400	80 407
637	80 414	80 421	80 428	80 434	80 441	80 448	80 455	80 462	80 468	80 475
638	80 482	80 489	80 496	80 502	80 509	80 516	80 523	80 530	80 536	80 543
639	80 550	80 557	80 564	80 570	80 577	80 584	80 591	80 598	80 604	80 611
640	80 618	80 625	80 632	80 638	80 645	80 652	80 659	80 665	80 672	80 679
641	80 686	80 693	80 699	80 706	80 713	80 720	80 726	80 733	80 740	80 747
642	80 754	80 760	80 767	80 774	80 781	80 787	80 794	80 801	80 808	80 814
643	80 821	80 828	80 835	80 841	80 848	80 855	80 862	80 868	80 875	80 882
644	80 889	80 895	80 902	80 909	80 916	80 922	80 929	80 936	80 943	80 949
645	80 956	80 963	80 969	80 976	80 983	80 990	80 996	81 003	81 010	81 017
646	81 023	81 030	81 037	81 043	81 050	81 057	81 064	81 070	81 077	81 084
647	81 090	81 097	81 104	81 111	81 117	81 124	81 131	81 137	81 144	81 151
648	81 158	81 164	81 171	81 178	81 184	81 191	81 198	81 204	81 211	81 218
649	81 224	81 231	81 238	81 245	81 251	81 258	81 265	81 271	81 278	81 285
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650	81 291	81 298	81 305	81 311	81 318	81 325	81 331	81 338	81 345	81 351
651	81 358	81 365	81 371	81 378	81 385	81 391	81 398	81 405	81 411	81 418
652	81 425	81 431	81 438	81 445	81 451	81 458	81 465	81 471	81 478	81 485
653	81 491	81 498	81 505	81 511	81 518	81 525	81 531	81 538	81 544	81 551
654	81 558	81 564	81 571	81 578	81 584	81 591	81 598	81 604	81 611	81 617
655	81 624	81 631	81 637	81 644	81 651	81 657	81 664	81 671	81 677	81 684
656	81 690	81 697	81 704	81 710	81 717	81 723	81 730	81 737	81 743	81 750
657	81 757	81 763	81 770	81 776	81 783	81 790	81 796	81 803	81 809	81 816
658	81 823	81 829	81 836	81 842	81 849	81 856	81 862	81 869	81 875	81 882
659	81 889	81 895	81 902	81 908	81 915	81 921	81 928	81 935	81 941	81 948
660	81 954	81 961	81 968	81 974	81 981	81 987	81 994	82 000	82 007	82 014
661	82 020	82 027	82 033	82 040	82 046	82 053	82 060	82 066	82 073	82 079
662	82 086	82 092	82 099	82 105	82 112	82 119	82 125	82 132	82 138	82 145
663	82 151	82 158	82 164	82 171	82 178	82 184	82 191	82 197	82 204	82 210
664	82 217	82 223	82 230	82 236	82 243	82 249	82 256	82 263	82 269	82 276
665	82 282	82 289	82 295	82 302	82 308	82 315	82 321	82 328	82 334	82 341
666	82 347	82 354	82 360	82 367	82 373	82 380	82 387	82 393	82 400	82 406
667	82 413	82 419	82 426	82 432	82 439	82 445	82 452	82 458	82 465	82 471
668	82 478	82 484	82 491	82 497	82 504	82 510	82 517	82 523	82 530	82 536
669	82 543	82 549	82 556	82 562	82 569	82 575	82 582	82 588	82 595	82 601
670	82 607	82 614	82 620	82 627	82 633	82 640	82 646	82 653	82 659	82 666
671	82 672	82 679	82 685	82 692	82 698	82 705	82 711	82 718	82 724	82 730
672	82 737	82 743	82 750	82 756	82 763	82 769	82 776	82 782	82 789	82 795
673	82 802	82 808	82 814	82 821	82 827	82 834	82 840	82 847	82 853	82 860
674	82 866	82 872	82 879	82 885	82 892	82 898	82 905	82 911	82 918	82 924
675	82 930	82 937	82 943	82 950	82 956	82 963	82 969	82 975	82 982	82 988
676	82 995	83 001	83 008	83 014	83 020	83 027	83 033	83 040	83 046	83 052
677	83 059	83 065	83 072	83 078	83 085	83 091	83 097	83 104	83 110	83 117
678	83 123	83 129	83 136	83 142	83 149	83 155	83 161	83 168	83 174	83 181
679	83 187	83 193	83 200	83 206	83 213	83 219	83 225	83 232	83 238	83 245
680	83 251	83 257	83 264	83 270	83 276	83 283	83 289	83 296	83 302	83 308
681	83 315	83 321	83 327	83 334	83 340	83 347	83 353	83 359	83 366	83 372
682	83 378	83 385	83 391	83 398	83 404	83 410	83 417	83 423	83 429	83 436
683	83 442	83 448	83 455	83 461	83 467	83 474	83 480	83 487	83 493	83 499
684	83 506	83 512	83 518	83 525	83 531	83 537	83 544	83 550	83 556	83 563
685	83 569	83 575	83 582	83 588	83 594	83 601	83 607	83 613	83 620	83 626
686	83 632	83 639	83 645	83 651	83 658	83 664	83 670	83 677	83 683	83 689
687	83 696	83 702	83 708	83 715	83 721	83 727	83 734	83 740	83 746	83 753
688	83 759	83 765	83 771	83 778	83 784	83 790	83 797	83 803	83 809	83 816
689	83 822	83 828	83 835	83 841	83 847	83 853	83 860	83 866	83 872	83 879
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691	83 948	83 954	83 960	83 967	83 973	83 979	83 985	83 992	83 998	84 004
692	84 011	84 017	84 023	84 029	84 036	84 042	84 048	84 055	84 061	84 067
693	84 073	84 080	84 086	84 092	84 098	84 105	84 111	84 117	84 123	84 130
694	84 136	84 142	84 148	84 155	84 161	84 167	84 173	84 180	84 186	84 192
695	84 198	84 205	84 211	84 217	84 223	84 230	84 236	84 242	84 248	84 255
696	84 261	84 267	84 273	84 280	84 286	84 292	84 298	84 305	84 311	84 317
697	84 323	84 330	84 336	84 342	84 348	84 354	84 361	84 367	84 373	84 379
698	84 386	84 392	84 398	84 404	84 410	84 417	84 423	84 429	84 435	84 442
699	84 448	84 454	84 460	84 466	84 473	84 479	84 485	84 491	84 497	84 504
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
700	84 510	84 516	84 522	84 528	84 535	84 541	84 547	84 553	84 559	84 566
701	84 572	84 578	84 584	84 590	84 597	84 603	84 609	84 615	84 621	84 628
702	84 634	84 640	84 646	84 652	84 658	84 665	84 671	84 677	84 683	84 689
703	84 696	84 702	84 708	84 714	84 720	84 726	84 733	84 739	84 745	84 751
704	84 757	84 763	84 770	84 776	84 782	84 788	84 794	84 800	84 807	84 813
705	84 819	84 825	84 831	84 837	84 844	84 850	84 856	84 862	84 868	84 874
706	84 880	84 887	84 893	84 899	84 905	84 911	84 917	84 924	84 930	84 936
707	84 942	84 948	84 954	84 960	84 967	84 973	84 979	84 985	84 991	84 997
708	85 003	85 009	85 016	85 022	85 028	85 034	85 040	85 046	85 052	85 058
709	85 065	85 071	85 077	85 083	85 089	85 095	85 101	85 107	85 114	85 120
710	85 126	85 132	85 138	85 144	85 150	85 156	85 163	85 169	85 175	85 181
711	85 187	85 193	85 199	85 205	85 211	85 217	85 224	85 230	85 236	85 242
712	85 248	85 254	85 260	85 266	85 272	85 278	85 285	85 291	85 297	85 303
713	85 309	85 315	85 321	85 327	85 333	85 339	85 345	85 352	85 358	85 364
714	85 370	85 376	85 382	85 388	85 394	85 400	85 406	85 412	85 418	85 425
715	85 431	85 437	85 443	85 449	85 455	85 461	85 467	85 473	85 479	85 485
716	85 491	85 497	85 503	85 509	85 516	85 522	85 528	85 534	85 540	85 546
717	85 552	85 558	85 564	85 570	85 576	85 582	85 588	85 594	85 600	85 606
718	85 612	85 618	85 625	85 631	85 637	85 643	85 649	85 655	85 661	85 667
719	85 673	85 679	85 685	85 691	85 697	85 703	85 709	85 715	85 721	85 727
720	85 733	85 739	85 745	85 751	85 757	85 763	85 769	85 775	85 781	85 788
721	85 794	85 800	85 806	85 812	85 818	85 824	85 830	85 836	85 842	85 848
722	85 854	85 860	85 866	85 872	85 878	85 884	85 890	85 896	85 902	85 908
723	85 914	85 920	85 926	85 932	85 938	85 944	85 950	85 956	85 962	85 968
724	85 974	85 980	85 986	85 992	85 998	86 004	86 010	86 016	86 022	86 028
725	86 034	86 040	86 046	86 052	86 058	86 064	86 070	86 076	86 082	86 088
726	86 094	86 100	86 106	86 112	86 118	86 124	86 130	86 136	86 141	86 147
727	86 153	86 159	86 165	86 171	86 177	86 183	86 189	86 195	86 201	86 207
728	86 213	86 219	86 225	86 231	86 237	86 243	86 249	86 255	86 261	86 267
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731	86 392	86 398	86 404	86 410	86 415	86 421	86 427	86 433	86 439	86 445
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734	86 570	86 576	86 581	86 587	86 593	86 599	86 605	86 611	86 617	86 623
735	86 629	86 635	86 641	86 646	86 652	86 658	86 664	86 670	86 676	86 682
736	86 688	86 694	86 700	86 705	86 711	86 717	86 723	86 729	86 735	86 741
737	86 747	86 753	86 759	86 764	86 770	86 776	86 782	86 788	86 794	86 800
738	86 806	86 812	86 817	86 823	86 829	86 835	86 841	86 847	86 853	86 859
739	86 864	86 870	86 876	86 882	86 888	86 894	86 900	86 906	86 911	86 917
740	86 923	86 929	86 935	86 941	86 947	86 953	86 958	86 964	86 970	86 976
741	86 982	86 988	86 994	86 999	87 005	87 011	87 017	87 023	87 029	87 035
742	87 040	87 046	87 052	87 058	87 064	87 070	87 075	87 081	87 087	87 093
743	87 099	87 105	87 111	87 116	87 122	87 128	87 134	87 140	87 146	87 151
744	87 157	87 163	87 169	87 175	87 181	87 186	87 192	87 198	87 204	87 210
745	87 216	87 221	87 227	87 233	87 239	87 245	87 251	87 256	87 262	87 268
746	87 274	87 280	87 286	87 291	87 297	87 303	87 309	87 315	87 320	87 326
747	87 332	87 338	87 344	87 349	87 355	87 361	87 367	87 373	87 379	87 384
748	87 390	87 396	87 402	87 408	87 413	87 419	87 425	87 431	87 437	87 442
749	87 448	87 454	87 460	87 466	87 471	87 477	87 483	87 489	87 495	87 500
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
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751	87 564	87 570	87 576	87 581	87 587	87 593	87 599	87 604	87 610	87 616
752	87 622	87 628	87 633	87 639	87 645	87 651	87 656	87 662	87 668	87 674
753	87 679	87 685	87 691	87 697	87 703	87 708	87 714	87 720	87 726	87 731
754	87 737	87 743	87 749	87 754	87 760	87 766	87 772	87 777	87 783	87 789
755	87 795	87 800	87 806	87 812	87 818	87 823	87 829	87 835	87 841	87 846
756	87 852	87 858	87 864	87 869	87 875	87 881	87 887	87 892	87 898	87 904
757	87 910	87 915	87 921	87 927	87 933	87 938	87 944	87 950	87 955	87 961
758	87 967	87 973	87 978	87 984	87 990	87 996	88 001	88 007	88 013	88 018
759	88 024	88 030	88 036	88 041	88 047	88 053	88 058	88 064	88 070	88 076
760	88 081	88 087	88 093	88 098	88 104	88 110	88 116	88 121	88 127	88 133
761	88 138	88 144	88 150	88 156	88 161	88 167	88 173	88 178	88 184	88 190
762	88 195	88 201	88 207	88 213	88 218	88 224	88 230	88 235	88 241	88 247
763	88 252	88 258	88 264	88 270	88 275	88 281	88 287	88 292	88 298	88 304
764	88 309	88 315	88 321	88 326	88 332	88 338	88 343	88 349	88 355	88 360
765	88 366	88 372	88 377	88 383	88 389	88 395	88 400	88 406	88 412	88 417
766	88 423	88 429	88 434	88 440	88 446	88 451	88 457	88 463	88 468	88 474
767	88 480	88 485	88 491	88 497	88 502	88 508	88 513	88 519	88 525	88 530
768	88 536	88 542	88 547	88 553	88 559	88 564	88 570	88 576	88 581	88 587
769	88 593	88 598	88 604	88 610	88 615	88 621	88 627	88 632	88 638	88 643
770	88 649	88 655	88 660	88 666	88 672	88 677	88 683	88 689	88 694	88 700
771	88 705	88 711	88 717	88 722	88 728	88 734	88 739	88 745	88 750	88 756
772	88 762	88 767	88 773	88 779	88 784	88 790	88 795	88 801	88 807	88 812
773	88 818	88 824	88 829	88 835	88 840	88 846	88 852	88 857	88 863	88 868
774	88 874	88 880	88 885	88 891	88 897	88 902	88 908	88 913	88 919	88 925
775	88 930	88 936	88 941	88 947	88 953	88 958	88 964	88 969	88 975	88 981
776	88 986	88 992	88 997	89 003	89 009	89 014	89 020	89 025	89 031	89 037
777	89 042	89 048	89 053	89 059	89 064	89 070	89 076	89 081	89 087	89 092
778	89 098	89 104	89 109	89 115	89 120	89 126	89 131	89 137	89 143	89 148
779	89 154	89 159	89 165	89 170	89 176	89 182	89 187	89 193	89 198	89 204
780	89 209	89 215	89 221	89 226	89 232	89 237	89 243	89 248	89 254	89 260
781	89 265	89 271	89 276	89 282	89 287	89 293	89 298	89 304	89 310	89 315
782	89 321	89 326	89 332	89 337	89 343	89 348	89 354	89 360	89 365	89 371
783	89 376	89 382	89 387	89 393	89 398	89 404	89 409	89 415	89 421	89 426
784	89 432	89 437	89 443	89 448	89 454	89 459	89 465	89 470	89 476	89 481
785	89 487	89 492	89 498	89 504	89 509	89 515	89 520	89 526	89 531	89 537
786	89 542	89 548	89 553	89 559	89 564	89 570	89 575	89 581	89 586	89 592
787	89 597	89 603	89 609	89 614	89 620	89 625	89 631	89 636	89 642	89 647
788	89 653	89 658	89 664	89 669	89 675	89 680	89 686	89 691	89 697	89 702
789	89 708	89 713	89 719	89 724	89 730	89 735	89 741	89 746	89 752	89 757
790	89 763	89 768	89 774	89 779	89 785	89 790	89 796	89 801	89 807	89 812
791	89 818	89 823	89 829	89 834	89 840	89 845	89 851	89 856	89 862	89 867
792	89 873	89 878	89 883	89 889	89 894	89 900	89 905	89 911	89 916	89 922
793	89 927	89 933	89 938	89 944	89 949	89 955	89 960	89 966	89 971	89 977
794	89 982	89 988	89 993	89 998	90 004	90 009	90 015	90 020	90 026	90 031
795	90 037	90 042	90 048	90 053	90 059	90 064	90 069	90 075	90 080	90 086
796	90 091	90 097	90 102	90 108	90 113	90 119	90 124	90 129	90 135	90 140
797	90 146	90 151	90 157	90 162	90 168	90 173	90 179	90 184	90 189	90 195
798	90 200	90 206	90 211	90 217	90 222	90 227	90 233	90 238	90 244	90 249
799	90 255	90 260	90 266	90 271	90 276	90 282	90 287	90 293	90 298	90 304
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

No.	0	1	2	3	4	5	6	7	8	9
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358
801	90 363	90 369	90 374	90 380	90 385	90 390	90 396	90 401	90 407	90 412
802	90 417	90 423	90 428	90 434	90 439	90 445	90 450	90 455	90 461	90 466
803	90 472	90 477	90 482	90 488	90 493	90 499	90 504	90 509	90 515	90 520
804	90 526	90 531	90 536	90 542	90 547	90 553	90 558	90 563	90 569	90 574
805	90 580	90 585	90 590	90 596	90 601	90 607	90 612	90 617	90 623	90 628
806	90 634	90 639	90 644	90 650	90 655	90 660	90 666	90 671	90 677	90 682
807	90 687	90 693	90 698	90 703	90 709	90 714	90 720	90 725	90 730	90 736
808	90 741	90 747	90 752	90 757	90 763	90 768	90 773	90 779	90 784	90 789
809	90 795	90 800	90 806	90 811	90 816	90 822	90 827	90 832	90 838	90 843
810	90 849	90 854	90 859	90 865	90 870	90 875	90 881	90 886	90 891	90 897
811	90 902	90 907	90 913	90 918	90 924	90 929	90 934	90 940	90 945	90 950
812	90 956	90 961	90 966	90 972	90 977	90 982	90 988	90 993	90 998	91 004
813	91 009	91 014	91 020	91 025	91 030	91 036	91 041	91 046	91 052	91 057
814	91 062	91 068	91 073	91 078	91 084	91 089	91 094	91 100	91 105	91 110
815	91 116	91 121	91 126	91 132	91 137	91 142	91 148	91 153	91 158	91 164
816	91 169	91 174	91 180	91 185	91 190	91 196	91 201	91 206	91 212	91 217
817	91 222	91 228	91 233	91 238	91 243	91 249	91 254	91 259	91 265	91 270
818	91 275	91 281	91 286	91 291	91 297	91 302	91 307	91 312	91 318	91 323
819	91 328	91 334	91 339	91 344	91 350	91 355	91 360	91 365	91 371	91 376
820	91 381	91 387	91 392	91 397	91 403	91 408	91 413	91 418	91 424	91 429
821	91 434	91 440	91 445	91 450	91 455	91 461	91 466	91 471	91 477	91 482
822	91 487	91 492	91 498	91 503	91 508	91 514	91 519	91 524	91 529	91 535
823	91 540	91 545	91 551	91 556	91 561	91 566	91 572	91 577	91 582	91 587
824	91 593	91 598	91 603	91 609	91 614	91 619	91 624	91 630	91 635	91 640
825	91 645	91 651	91 656	91 661	91 666	91 672	91 677	91 682	91 687	91 693
826	91 698	91 703	91 709	91 714	91 719	91 724	91 730	91 735	91 740	91 745
827	91 751	91 756	91 761	91 766	91 772	91 777	91 782	91 787	91 793	91 798
828	91 803	91 808	91 814	91 819	91 824	91 829	91 834	91 840	91 845	91 850
829	91 855	91 861	91 866	91 871	91 876	91 882	91 887	91 892	91 897	91 903
830	91 908	91 913	91 918	91 924	91 929	91 934	91 939	91 944	91 950	91 955
831	91 960	91 965	91 971	91 976	91 981	91 986	91 991	91 997	92 002	92 007
832	92 012	92 018	92 023	92 028	92 033	92 038	92 044	92 049	92 054	92 059
833	92 065	92 070	92 075	92 080	92 085	92 091	92 096	92 101	92 106	92 111
834	92 117	92 122	92 127	92 132	92 137	92 143	92 148	92 153	92 158	92 163
835	92 169	92 174	92 179	92 184	92 189	92 195	92 200	92 205	92 210	92 215
836	92 221	92 226	92 231	92 236	92 241	92 247	92 252	92 257	92 262	92 267
837	92 273	92 278	92 283	92 288	92 293	92 298	92 304	92 309	92 314	92 319
838	92 324	92 330	92 335	92 340	92 345	92 350	92 355	92 361	92 366	92 371
839	92 376	92 381	92 387	92 392	92 397	92 402	92 407	92 412	92 418	92 423
840	92 428	92 433	92 438	92 443	92 449	92 454	92 459	92 464	92 469	92 474
841	92 480	92 485	92 490	92 495	92 500	92 505	92 511	92 516	92 521	92 526
842	92 531	92 536	92 542	92 547	92 552	92 557	92 562	92 567	92 572	92 578
843	92 583	92 588	92 593	92 598	92 603	92 609	92 614	92 619	92 624	92 629
844	92 634	92 639	92 645	92 650	92 655	92 660	92 665	92 670	92 675	92 681
845	92 686	92 691	92 696	92 701	92 706	92 711	92 716	92 722	92 727	92 732
846	92 737	92 742	92 747	92 752	92 758	92 763	92 768	92 773	92 778	92 783
847	92 788	92 793	92 799	92 804	92 809	92 814	92 819	92 824	92 829	92 834
848	92 840	92 845	92 850	92 855	92 860	92 865	92 870	92 875	92 881	92 886
849	92 891	92 896	92 901	92 906	92 911	92 916	92 921	92 927	92 932	92 937
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
850	92 942	92 947	92 952	92 957	92 962	92 967	92 973	92 978	92 983	92 988
851	92 993	92 998	93 003	93 008	93 013	93 018	93 024	93 029	93 034	93 039
852	93 044	93 049	93 054	93 059	93 064	93 069	93 075	93 080	93 085	93 090
853	93 095	93 100	93 105	93 110	93 115	93 120	93 125	93 131	93 136	93 141
854	93 146	93 151	93 156	93 161	93 166	93 171	93 176	93 181	93 186	93 192
855	93 197	93 202	93 207	93 212	93 217	93 222	93 227	93 232	93 237	93 242
856	93 247	93 252	93 258	93 263	93 268	93 273	93 278	93 283	93 288	93 293
857	93 298	93 303	93 308	93 313	93 318	93 323	93 328	93 334	93 339	93 344
858	93 349	93 354	93 359	93 364	93 369	93 374	93 379	93 384	93 389	93 394
859	93 399	93 404	93 409	93 414	93 420	93 425	93 430	93 435	93 440	93 445
860	93 450	93 455	93 460	93 465	93 470	93 475	93 480	93 485	93 490	93 495
861	93 500	93 505	93 510	93 515	93 520	93 526	93 531	93 536	93 541	93 546
862	93 551	93 556	93 561	93 566	93 571	93 576	93 581	93 586	93 591	93 596
863	93 601	93 606	93 611	93 616	93 621	93 626	93 631	93 636	93 641	93 646
864	93 651	93 656	93 661	93 666	93 671	93 676	93 682	93 687	93 692	93 697
865	93 702	93 707	93 712	93 717	93 722	93 727	93 732	93 737	93 742	93 747
866	93 752	93 757	93 762	93 767	93 772	93 777	93 782	93 787	93 792	93 797
867	93 802	93 807	93 812	93 817	93 822	93 827	93 832	93 837	93 842	93 847
868	93 852	93 857	93 862	93 867	93 872	93 877	93 882	93 887	93 892	93 897
869	93 902	93 907	93 912	93 917	93 922	93 927	93 932	93 937	93 942	93 947
870	93 952	93 957	93 962	93 967	93 972	93 977	93 982	93 987	93 992	93 997
871	94 002	94 007	94 012	94 017	94 022	94 027	94 032	94 037	94 042	94 047
872	94 052	94 057	94 062	94 067	94 072	94 077	94 082	94 086	94 091	94 096
873	94 101	94 106	94 111	94 116	94 121	94 126	94 131	94 136	94 141	94 146
874	94 151	94 156	94 161	94 166	94 171	94 176	94 181	94 186	94 191	94 196
875	94 201	94 206	94 211	94 216	94 221	94 226	94 231	94 236	94 240	94 245
876	94 250	94 255	94 260	94 265	94 270	94 275	94 280	94 285	94 290	94 295
877	94 300	94 305	94 310	94 315	94 320	94 325	94 330	94 335	94 340	94 345
878	94 349	94 354	94 359	94 364	94 369	94 374	94 379	94 384	94 389	94 394
879	94 399	94 404	94 409	94 414	94 419	94 424	94 429	94 433	94 438	94 443
880	94 448	94 453	94 458	94 463	94 468	94 473	94 478	94 483	94 488	94 493
881	94 498	94 503	94 507	94 512	94 517	94 522	94 527	94 532	94 537	94 542
882	94 547	94 552	94 557	94 562	94 567	94 571	94 576	94 581	94 586	94 591
883	94 596	94 601	94 606	94 611	94 616	94 621	94 626	94 630	94 635	94 640
884	94 645	94 650	94 655	94 660	94 665	94 670	94 675	94 680	94 685	94 689
885	94 694	94 699	94 704	94 709	94 714	94 719	94 724	94 729	94 734	94 738
886	94 743	94 748	94 753	94 758	94 763	94 768	94 773	94 778	94 783	94 787
887	94 792	94 797	94 802	94 807	94 812	94 817	94 822	94 827	94 832	94 836
888	94 841	94 846	94 851	94 856	94 861	94 866	94 871	94 876	94 880	94 885
889	94 890	94 895	94 900	94 905	94 910	94 915	94 919	94 924	94 929	94 934
890	94 939	94 944	94 949	94 954	94 959	94 963	94 968	94 973	94 978	94 983
891	94 988	94 993	94 998	95 002	95 007	95 012	95 017	95 022	95 027	95 032
892	95 036	95 041	95 046	95 051	95 056	95 061	95 066	95 071	95 075	95 080
893	95 085	95 090	95 095	95 100	95 105	95 109	95 114	95 119	95 124	95 129
894	95 134	95 139	95 143	95 148	95 153	95 158	95 163	95 168	95 173	95 177
895	95 182	95 187	95 192	95 197	95 202	95 207	95 211	95 216	95 221	95 226
896	95 231	95 236	95 240	95 245	95 250	95 255	95 260	95 265	95 270	95 274
897	95 279	95 284	95 289	95 294	95 299	95 303	95 308	95 313	95 318	95 323
898	95 328	95 332	95 337	95 342	95 347	95 352	95 357	95 361	95 366	95 371
899	95 376	95 381	95 386	95 390	95 395	95 400	95 405	95 410	95 415	95 419
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

No.	0	1	2	3	4	5	6	7	8	9
900	95 424	95 429	95 434	95 439	95 444	95 448	95 453	95 458	95 463	95 468
901	95 472	95 477	95 482	95 487	95 492	95 497	95 501	95 506	95 511	95 516
902	95 521	95 525	95 530	95 535	95 540	95 545	95 550	95 554	95 559	95 564
903	95 569	95 574	95 578	95 583	95 588	95 593	95 598	95 602	95 607	95 612
904	95 617	95 622	95 626	95 631	95 636	95 641	95 646	95 650	95 655	95 660
905	95 665	95 670	95 674	95 679	95 684	95 689	95 694	95 698	95 703	95 708
906	95 713	95 718	95 722	95 727	95 732	95 737	95 742	95 746	95 751	95 756
907	95 761	95 766	95 770	95 775	95 780	95 785	95 789	95 794	95 799	95 804
908	95 809	95 813	95 818	95 823	95 828	95 832	95 837	95 842	95 847	95 852
909	95 856	95 861	95 866	95 871	95 875	95 880	95 885	95 890	95 895	95 899
910	95 904	95 909	95 914	95 918	95 923	95 928	95 933	95 938	95 942	95 947
911	95 952	95 957	95 961	95 966	95 971	95 976	95 980	95 985	95 990	95 995
912	95 999	96 004	96 009	96 014	96 019	96 023	96 028	96 033	96 038	96 042
913	96 047	96 052	96 057	96 061	96 066	96 071	96 076	96 080	96 085	96 090
914	96 095	96 099	96 104	96 109	96 114	96 118	96 123	96 128	96 133	96 137
915	96 142	96 147	96 152	96 156	96 161	96 166	96 171	96 175	96 180	96 185
916	96 190	96 194	96 199	96 204	96 209	96 213	96 218	96 223	96 227	96 232
917	96 237	96 242	96 246	96 251	96 256	96 261	96 265	96 270	96 275	96 280
918	96 284	96 289	96 294	96 298	96 303	96 308	96 313	96 317	96 322	96 327
919	96 332	96 336	96 341	96 346	96 350	96 355	96 360	96 365	96 369	96 374
920	96 379	96 384	96 388	96 393	96 398	96 402	96 407	96 412	96 417	96 421
921	96 426	96 431	96 435	96 440	96 445	96 450	96 454	96 459	96 464	96 468
922	96 473	96 478	96 483	96 487	96 492	96 497	96 501	96 506	96 511	96 515
923	96 520	96 525	96 530	96 534	96 539	96 544	96 548	96 553	96 558	96 562
924	96 567	96 572	96 577	96 581	96 586	96 591	96 595	96 600	96 605	96 609
925	96 614	96 619	96 624	96 628	96 633	96 638	96 642	96 647	96 652	96 656
926	96 661	96 666	96 670	96 675	96 680	96 685	96 689	96 694	96 699	96 703
927	96 708	96 713	96 717	96 722	96 727	96 731	96 736	96 741	96 745	96 750
928	96 755	96 759	96 764	96 769	96 774	96 778	96 783	96 788	96 792	96 797
929	96 802	96 806	96 811	96 816	96 820	96 825	96 830	96 834	96 839	96 844
930	96 848	96 853	96 858	96 862	96 867	96 872	96 876	96 881	96 886	96 890
931	96 895	96 900	96 904	96 909	96 914	96 918	96 923	96 928	96 932	96 937
932	96 942	96 946	96 951	96 956	96 960	96 965	96 970	96 974	96 979	96 984
933	96 988	96 993	96 997	97 002	97 007	97 011	97 016	97 021	97 025	97 030
934	97 035	97 039	97 044	97 049	97 053	97 058	97 063	97 067	97 072	97 077
935	97 081	97 086	97 090	97 095	97 100	97 104	97 109	97 114	97 118	97 123
936	97 128	97 132	97 137	97 142	97 146	97 151	97 155	97 160	97 165	97 169
937	97 174	97 179	97 183	97 188	97 192	97 197	97 202	97 206	97 211	97 216
938	97 220	97 225	97 230	97 234	97 239	97 243	97 248	97 253	97 257	97 262
939	97 267	97 271	97 276	97 280	97 285	97 290	97 294	97 299	97 304	97 308
940	97 313	97 317	97 322	97 327	97 331	97 336	97 340	97 345	97 350	97 354
941	97 359	97 364	97 368	97 373	97 377	97 382	97 387	97 391	97 396	97 400
942	97 405	97 410	97 414	97 419	97 424	97 428	97 433	97 437	97 442	97 447
943	97 451	97 456	97 460	97 465	97 470	97 474	97 479	97 483	97 488	97 493
944	97 497	97 502	97 506	97 511	97 516	97 520	97 525	97 529	97 534	97 539
945	97 543	97 548	97 552	97 557	97 562	97 566	97 571	97 575	97 580	97 585
946	97 589	97 594	97 598	97 603	97 607	97 612	97 617	97 621	97 626	97 630
947	97 635	97 640	97 644	97 649	97 653	97 658	97 663	97 667	97 672	97 676
948	97 681	97 685	97 690	97 695	97 699	97 704	97 708	97 713	97 717	97 722
949	97 727	97 731	97 736	97 740	97 745	97 749	97 754	97 759	97 763	97 768

No. 0 1 2 3 4 5 6 7 8 9

No.	0	1	2	3	4	5	6	7	8	9
950	97 772	97 777	97 782	97 786	97 791	97 795	97 800	97 804	97 809	97 813
951	97 818	97 823	97 827	97 832	97 836	97 841	97 845	97 850	97 855	97 859
952	97 864	97 868	97 873	97 877	97 882	97 886	97 891	97 896	97 900	97 905
953	97 909	97 914	97 918	97 923	97 928	97 932	97 937	97 941	97 946	97 950
954	97 955	97 959	97 964	97 968	97 973	97 978	97 982	97 987	97 991	97 996
955	98 000	98 005	98 009	98 014	98 019	98 023	98 028	98 032	98 037	98 041
956	98 046	98 050	98 055	98 059	98 064	98 068	98 073	98 078	98 082	98 087
957	98 091	98 096	98 100	98 105	98 109	98 114	98 118	98 123	98 127	98 132
958	98 137	98 141	98 146	98 150	98 155	98 159	98 164	98 168	98 173	98 177
959	98 182	98 186	98 191	98 195	98 200	98 204	98 209	98 214	98 218	98 223
960	98 227	98 232	98 236	98 241	98 245	98 250	98 254	98 259	98 263	98 268
961	98 272	98 277	98 281	98 286	98 290	98 295	98 299	98 304	98 308	98 313
962	98 318	98 322	98 327	98 331	98 336	98 340	98 345	98 349	98 354	98 358
963	98 363	98 367	98 372	98 376	98 381	98 385	98 390	98 394	98 399	98 403
964	98 408	98 412	98 417	98 421	98 426	98 430	98 435	98 439	98 444	98 448
965	98 453	98 457	98 462	98 466	98 471	98 475	98 480	98 484	98 489	98 493
966	98 498	98 502	98 507	98 511	98 516	98 520	98 525	98 529	98 534	98 538
967	98 543	98 547	98 552	98 556	98 561	98 565	98 570	98 574	98 579	98 583
968	98 588	98 592	98 597	98 601	98 605	98 610	98 614	98 619	98 623	98 628
969	98 632	98 637	98 641	98 646	98 650	98 655	98 659	98 664	98 668	98 673
970	98 677	98 682	98 686	98 691	98 695	98 700	98 704	98 709	98 713	98 717
971	98 722	98 726	98 731	98 735	98 740	98 744	98 749	98 753	98 758	98 762
972	98 767	98 771	98 776	98 780	98 784	98 789	98 793	98 798	98 802	98 807
973	98 811	98 816	98 820	98 825	98 829	98 834	98 838	98 843	98 847	98 851
974	98 856	98 860	98 865	98 869	98 874	98 878	98 883	98 887	98 892	98 896
975	98 900	98 905	98 909	98 914	98 918	98 923	98 927	98 932	98 936	98 941
976	98 945	98 949	98 954	98 958	98 963	98 967	98 972	98 976	98 981	98 985
977	98 989	98 994	98 998	99 003	99 007	99 012	99 016	99 021	99 025	99 029
978	99 034	99 038	99 043	99 047	99 052	99 056	99 061	99 065	99 069	99 074
979	99 078	99 083	99 087	99 092	99 096	99 100	99 105	99 109	99 114	99 118
980	99 123	99 127	99 131	99 136	99 140	99 145	99 149	99 154	99 158	99 162
981	99 167	99 171	99 176	99 180	99 185	99 189	99 193	99 198	99 202	99 207
982	99 211	99 216	99 220	99 224	99 229	99 233	99 238	99 242	99 247	99 251
983	99 255	99 260	99 264	99 269	99 273	99 277	99 282	99 286	99 291	99 295
984	99 300	99 304	99 308	99 313	99 317	99 322	99 326	99 330	99 335	99 339
985	99 344	99 348	99 352	99 357	99 361	99 366	99 370	99 374	99 379	99 383
986	99 388	99 392	99 396	99 401	99 405	99 410	99 414	99 419	99 423	99 427
987	99 432	99 436	99 441	99 445	99 449	99 454	99 458	99 463	99 467	99 471
988	99 476	99 480	99 484	99 489	99 493	99 498	99 502	99 506	99 511	99 515
989	99 520	99 524	99 528	99 533	99 537	99 542	99 546	99 550	99 555	99 559
990	99 564	99 568	99 572	99 577	99 581	99 585	99 590	99 594	99 599	99 603
991	99 607	99 612	99 616	99 621	99 625	99 629	99 634	99 638	99 642	99 647
992	99 651	99 656	99 660	99 664	99 669	99 673	99 677	99 682	99 686	99 691
993	99 695	99 699	99 704	99 708	99 712	99 717	99 721	99 726	99 730	99 734
994	99 739	99 743	99 747	99 752	99 756	99 760	99 765	99 769	99 774	99 778
995	99 782	99 787	99 791	99 795	99 800	99 804	99 808	99 813	99 817	99 822
996	99 826	99 830	99 835	99 839	99 843	99 848	99 852	99 856	99 861	99 865
997	99 870	99 874	99 878	99 883	99 887	99 891	99 896	99 900	99 904	99 909
998	99 913	99 917	99 922	99 926	99 930	99 935	99 939	99 944	99 948	99 952
999	99 957	99 961	99 965	99 970	99 974	99 978	99 983	99 987	99 991	99 996
1000	00 000	00 004	00 009	00 013	00 017	00 022	00 026	00 030	00 035	00 039
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

		Log
Circumference of the Circle in Degrees	= 360	2.55 630 250
Circumference of the Circle in Minutes	= 21 600	4.33 445 375
Circumference of the Circle in Seconds	= 1 296 000	6.11 260 500
If the radius = 1, the semi-circumference is $\pi = 3.14\,159\,265\,358\,979\,323\,846\,264\,338\,328$		0.49 714 987

Also	Log		Log
		$\pi^2 = 9.86\,960\,440$	0.99 429 975
$2\pi = 6.28\,318\,531$	0.79 817 987	$\frac{1}{\pi^2} = 0.10\,132\,118$	9.00 570 025 − 10
$4\pi = 12.56\,637\,061$	1.09 920 986		
$\frac{\pi}{2} = 1.57\,079\,633$	0.19 611 988	$\sqrt{\pi} = 1.77\,245\,385$	0.24 857 494
$\frac{\pi}{3} = 1.04\,719\,755$	0.02 002 862	$\frac{1}{\sqrt{\pi}} = 0.56\,418\,958$	9.75 142 506 − 10
$\frac{4\pi}{3} = 4.18\,879\,020$	0.62 208 861	$\sqrt{\frac{3}{\pi}} = 0.97\,720\,502$	9.98 998 569 − 10
$\frac{\pi}{4} = 0.78\,539\,816$	9.89 508 988 − 10	$\sqrt{\frac{4}{\pi}} = 1.12\,837\,917$	0.05 245 506
$\frac{\pi}{6} = 0.52\,359\,878$	9.71 899 862 − 10	$\sqrt[3]{\pi} = 1.46\,459\,189$	0.16 571 662
$\frac{1}{\pi} = 0.31\,830\,989$	9.50 285 013 − 10	$\frac{1}{\sqrt[3]{\pi}} = 0.68\,278\,406$	9.83 428 338 − 10
$\frac{1}{2\pi} = 0.15\,915\,494$	9.20 182 013 − 10	$\sqrt[3]{\pi^2} = 2.14\,502\,940$	0.33 143 325
$\frac{3}{\pi} = 0.95\,492\,966$	9.97 997 138 − 10	$\sqrt[3]{\frac{3}{4\pi}} = 0.62\,035\,049$	9.79 263 713 − 10
$\frac{4}{\pi} = 1.27\,323\,954$	0.10 491 012		
$\frac{3}{4\pi} = 0.23\,873\,241$	9.37 791 139 − 10	$\sqrt[3]{\frac{\pi}{6}} = 0.80\,599\,598$	9.90 633 287 − 10

	Log
Angle θ, whose arc is equal to the radius r, is	
in degrees, $\theta^\circ = \frac{180}{\pi} = 57.29\,577\,951^\circ$	1.75 812 263
in minutes, $\theta' = \frac{10\,800}{\pi} = 3\,437.74\,677'$	3.53 627 388
in seconds, $\theta'' = \frac{648\,000}{\pi} = 206\,264.806''$	5.31 442 513
Angle 2θ, whose arc is equal to twice the radius, $2r$, is	
in degrees, $2\theta^\circ = \frac{360}{\pi} = 114.59\,155\,903^\circ$	2.05 915 263
in minutes, $2\theta' = \frac{21\,600}{\pi} = 6\,875.49\,354'$	3.83 730 388
in seconds, $2\theta'' = \frac{1\,296\,000}{\pi} = 412\,529.612''$	5.61 545 513
If the radius $r = 1$, the length of the arc is:	
for 1 degree $= \frac{1}{\theta^\circ} = \frac{\pi}{180} = 0.01\,745\,329$	8.24 187 737 − 10
for 1 minute $= \frac{1}{\theta'} = \frac{\pi}{10\,800} = 0.00\,029\,089$	6.46 372 612 − 10
for 1 second $= \frac{1}{\theta''} = \frac{\pi}{648\,000} = 0.00\,000\,485$	4.68 557 487 − 10
for ½ degree $= \frac{1}{2\theta^\circ} = \frac{\pi}{360} = 0.00\,872\,665$	7.94 084 737 − 10
for ½ minute $= \frac{1}{2\theta'} = \frac{\pi}{21\,600} = 0.00\,014\,544$	6.16 269 612 − 10
for ½ second $= \frac{1}{2\theta''} = \frac{\pi}{1\,296\,000} = 0.00\,000\,242$	4.38 454 487 − 10
Sin 1″, when the radius $r = 1$, is . . . $= 0.00\,000\,485$	4.68 557 487 − 10

TABLE III

LOGARITHMS

OF THE

TRIGONOMETRIC FUNCTIONS

From 0° 0′ to 0° 3′, and from 89° 57′ to 90°, for every second
From 0° to 2°, and from 88° to 90°, for every ten seconds
From 1° to 89°, for every minute

NOTE. — The characteristic of every logarithm in the following table is too large by 10. Therefore, – 10 should be written after every logarithm.

L sin and L tan 0° **L sin and L tan**

″	0′	1′	2′	″	″	0′	1′	2′	″
0	——	6.46 373	6.76 476	**60**	**30**	6.16 270	6.63 982	6.86 167	**30**
1	4.68 557	6.47 090	6.76 836	59	31	6.17 694	6.64 462	6.86 455	29
2	4.98 660	6.47 797	6.77 193	58	32	6.19 072	6.64 936	6.86 742	28
3	5.16 270	6.48 492	6.77 548	57	33	6.20 409	6.65 406	6.87 027	27
4	5.28 763	6.49 175	6.77 900	56	34	6.21 705	6.65 870	6.87 310	26
5	5.38 454	6.49 849	6.78 248	55	35	6.22 964	6.66 330	6.87 591	25
6	5.46 373	6.50 512	6.78 595	54	36	6.24 188	6.66 785	6.87 870	24
7	5.53 067	6.51 165	6.78 938	53	37	6.25 378	6.67 235	6.88 147	23
8	5.58 866	6.51 808	6.79 278	52	38	6.26 536	6.67 680	6.88 423	22
9	5.63 982	6.52 442	6.79 616	51	39	6.27 664	6.68 121	6.88 697	21
10	5.68 557	6.53 067	6.79 952	**50**	**40**	6.28 763	6.68 557	6.88 969	**20**
11	5.72 697	6 53 683	6.80 285	49	41	6.29 836	6.68 990	6.89 240	19
12	5.76 476	6.54 291	6.80 615	48	42	6.30 882	6.69 418	6.89 509	18
13	5.79 952	6.54 890	6.80 943	47	43	6.31 904	6.69 841	6.89 776	17
14	5.83 170	6.55 481	6.81 268	46	44	6.32 903	6.70 261	6.90 042	16
15	5.86 167	6.56 064	6.81 591	45	45	6.33 879	6.70 676	6.90 306	15
16	5.88 969	6.56 639	6.81 911	44	46	6.34 833	6.71 088	6.90 568	14
17	5.91 602	6.57 207	6.82 230	43	47	6.35 767	6.71 496	6.90 829	13
18	5.94 085	6.57 767	6.82 545	42	48	6.36 682	6.71 900	6.91 088	12
19	5.96 433	6.58 320	6.82 859	41	49	6.37 577	6.72 300	6.91 346	11
20	5.98 660	6.58 866	6.83 170	**40**	**50**	6.38 454	6.72 697	6.91 602	**10**
21	6.00 779	6.59 406	6.83 479	39	51	6.39 315	6.73 090	6.91 857	9
22	6.02 800	6.59 939	6.83 786	38	52	6.40 158	6.73 479	6.92 110	8
23	6.04 730	6.60 465	6.84 091	37	53	6.40 985	6.73 865	6.92 362	7
24	6.06 579	6.60 985	6.84 394	36	54	6.41 797	6.74 248	6.92 612	6
25	6.08 351	6.61 499	6.84 694	35	55	6.42 594	6.74 627	6.92 861	5
26	6.10 055	6.62 007	6.84 993	34	56	6.43 376	6.75 003	6.93 109	4
27	6.11 694	6.62 509	6.85 289	33	57	6.44 145	6.75 376	6.93 355	3
28	6.13 273	6.63 006	6.85 584	32	58	6.44 900	6.75 746	6.93 599	2
29	6.14 797	6.63 496	6.85 876	31	59	6.45 643	6.76 112	6.93 843	1
30	6.16 270	6.63 982	6.86 167	**30**	**60**	6.46 373	6.76 476	6.94 085	**0**
″	**59′**	**58′**	**57′**	″	″	**59′**	**58′**	**57′**	″

L cos and L cot 89° **L cos and L cot**

0°

′	″	L sin	L tan	L cos	″	′	′	″	L sin	L tan	L cos	″	′
0	0	——	——	10.00 000	0	**60**	**10**	0	7.46 373	7.46 373	10.00 000	0	**50**
	10	5.68 557	5.68 557	10.00 000	50			10	7.47 090	7.47 091	10.00 000	50	
	20	5.98 660	5.98 660	10.00 000	40			20	7.47 797	7.47 797	10.00 000	40	
	30	6.16 270	6.16 270	10.00 000	30			30	7.48 491	7.48 492	10.00 000	30	
	40	6.28 763	6.28 763	10.00 000	20			40	7.49 175	7.49 175	10.00 000	20	
	50	6.38 454	6.38 454	10.00 000	10			50	7.49 849	7.49 849	10.00 000	10	
1	0	6.46 373	6.46 373	10.00 000	0	**59**	**11**	0	7.50 512	7.50 512	10.00 000	0	**49**
	10	6.53 067	6.53 067	10.00 000	50			10	7.51 165	7.51 165	10.00 000	50	
	20	6.58 866	6.58 866	10.00 000	40			20	7.51 808	7.51 809	10.00 000	40	
	30	6.63 982	6.63 982	10.00 000	30			30	7.52 442	7.52 443	10.00 000	30	
	40	6.68 557	6.68 557	10.00 000	20			40	7.53 067	7.53 067	10.00 000	20	
	50	6.72 697	6.72 697	10.00 000	10			50	7.53 683	7.53 683	10.00 000	10	
2	0	6.76 476	6.76 476	10.00 000	0	**58**	**12**	0	7.54 291	7.54 291	10.00 000	0	**48**
	10	6.79 952	6.79 952	10.00 000	50			10	7.54 890	7.54 890	10.00 000	50	
	20	6.83 170	6.83 170	10.00 000	40			20	7.55 481	7.55 481	10.00 000	40	
	30	6.86 167	6.86 167	10.00 000	30			30	7.56 064	7.56 064	10.00 000	30	
	40	6.88 969	6.88 969	10.00 000	20			40	7.56 639	7.56 639	10.00 000	20	
	50	6.91 602	6.91 602	10.00 000	10			50	7.57 206	7.57 206	10.00 000	10	
3	0	6.94 085	6.94 085	10.00 000	0	**57**	**13**	0	7.57 767	7.57 767	10.00 000	0	**47**
	10	6.96 433	6.96 433	10.00 000	50			10	7.58 320	7.58 320	10.00 000	50	
	20	6.98 660	6.98 661	10.00 000	40			20	7.58 866	7.58 867	10.00 000	40	
	30	7.00 779	7.00 779	10.00 000	30			30	7.59 406	7.59 406	10.00 000	30	
	40	7.02 800	7.02 800	10.00 000	20			40	7.59 939	7.59 939	10.00 000	20	
	50	7.04 730	7.04 730	10.00 000	10			50	7.60 465	7.60 466	10.00 000	10	
4	0	7.06 579	7.06 579	10.00 000	0	**56**	**14**	0	7.60 985	7.60 986	10.00 000	0	**46**
	10	7.08 351	7.08 352	10.00 000	50			10	7.61 499	7.61 500	10.00 000	50	
	20	7.10 055	7.10 055	10.00 000	40			20	7.62 007	7.62 008	10.00 000	40	
	30	7.11 694	7.11 694	10.00 000	30			30	7.62 509	7.62 510	10.00 000	30	
	40	7.13 273	7.13 273	10.00 000	20			40	7.63 006	7.63 006	10.00 000	20	
	50	7.14 797	7.14 797	10.00 000	10			50	7.63 496	7.63 497	10.00 000	10	
5	0	7.16 270	7.16 270	10.00 000	0	**55**	**15**	0	7.63 982	7.63 982	10.00 000	0	**45**
	10	7.17 694	7.17 694	10.00 000	50			10	7.64 461	7.64 462	10.00 000	50	
	20	7.19 072	7.19 073	10.00 000	40			20	7.64 936	7.64 937	10.00 000	40	
	30	7.20 409	7.20 409	10.00 000	30			30	7.65 406	7.65 406	10.00 000	30	
	40	7.21 705	7.21 705	10.00 000	20			40	7.65 870	7.65 871	10.00 000	20	
	50	7.22 964	7.22 964	10.00 000	10			50	7.66 330	7.66 330	10.00 000	10	
6	0	7.24 188	7.24 188	10.00 000	0	**54**	**16**	0	7.66 784	7.66 785	10.00 000	0	**44**
	10	7.25 378	7.25 378	10.00 000	50			10	7.67 235	7.67 235	10.00 000	50	
	20	7.26 536	7.26 536	10.00 000	40			20	7.67 680	7.67 680	10.00 000	40	
	30	7.27 664	7.27 664	10.00 000	30			30	7.68 121	7.68 121	10.00 000	30	
	40	7.28 763	7.28 764	10.00 000	20			40	7.68 557	7.68 558	9.99 999	20	
	50	7.29 836	7.29 836	10.00 000	10			50	7.68 989	7.68 990	9.99 999	10	
7	0	7.30 882	7.30 882	10.00 000	0	**53**	**17**	0	7.69 417	7.69 418	9.99 999	0	**43**
	10	7.31 904	7.31 904	10.00 000	50			10	7.69 841	7.69 842	9.99 999	50	
	20	7.32 903	7.32 903	10.00 000	40			20	7.70 261	7.70 261	9.99 999	40	
	30	7.33 879	7.33 879	10.00 000	30			30	7.70 676	7.70 677	9.99 999	30	
	40	7.34 833	7.34 833	10.00 000	20			40	7.71 088	7.71 088	9.99 999	20	
	50	7.35 767	7.35 767	10.00 000	10			50	7.71 496	7.71 496	9.99 999	10	
8	0	7.36 682	7.36 682	10.00 000	0	**52**	**18**	0	7.71 900	7.71 900	9.99 999	0	**42**
	10	7.37 577	7.37 577	10.00 000	50			10	7.72 300	7.72 301	9.99 999	50	
	20	7.38 454	7.38 455	10.00 000	40			20	7.72 697	7.72 697	9.99 999	40	
	30	7.39 314	7.39 315	10.00 000	30			30	7.73 090	7.73 090	9.99 999	30	
	40	7.40 158	7.40 158	10.00 000	20			40	7.73 479	7.73 480	9.99 999	20	
	50	7.40 985	7.40 985	10.00 000	10			50	7.73 865	7.73 866	9.99 999	10	
9	0	7.41 797	7.41 797	10.00 000	0	**51**	**19**	0	7.74 248	7.74 248	9.99 999	0	**41**
	10	7.42 594	7.42 594	10.00 000	50			10	7.74 627	7.74 628	9.99 999	50	
	20	7.43 376	7.43 376	10.00 000	40			20	7.75 003	7.75 004	9.99 999	40	
	30	7.44 145	7.44 145	10.00 000	30			30	7.75 376	7.75 377	9.99 999	30	
	40	7.44 900	7.44 900	10.00 000	20			40	7.75 745	7.75 746	9.99 999	20	
	50	7.45 643	7.45 643	10.00 000	10			50	7.76 112	7.76 113	9.99 999	10	
10	0	7.46 373	7.46 373	10.00 000	0	**50**	**20**	0	7.76 475	7.76 476	9.99 999	0	**40**
′	″	L cos	L cot	L sin	″	′	′	″	L cos	L cot	L sin	″	′

89°

′	″	L sin	L tan	L cos	″	′	′	″	L sin	L tan	L cos	″	′
20	0	7.76 475	7.76 476	9.99 999	0	**40**	**30**	0	7.94 084	7.94 086	9.99 998	0	**30**
	10	7.76 836	7.76 837	9.99 999	50			10	7.94 325	7.94 326	9.99 998	50	
	20	7.77 193	7.77 194	9.99 999	40			20	7.94 564	7.94 566	9.99 998	40	
	30	7.77 548	7.77 549	9.99 999	30			30	7.94 802	7.94 804	9.99 998	30	
	40	7.77 899	7.77 900	9.99 999	20			40	7.95 039	7.95 040	9.99 998	20	
	50	7.78 248	7.78 249	9.99 999	10			50	7.95 274	7.95 276	9.99 998	10	
21	0	7.78 594	7.78 595	9.99 999	0	**39**	**31**	0	7.95 508	7.95 510	9.99 998	0	**29**
	10	7.78 938	7.78 938	9.99 999	50			10	7.95 741	7.95 743	9.99 998	50	
	20	7.79 278	7.79 279	9.99 999	40			20	7.95 973	7.95 974	9.99 998	40	
	30	7.79 616	7.79 617	9.99 999	30			30	7.96 203	7.96 205	9.99 998	30	
	40	7.79 952	7.79 952	9.99 999	20			40	7.96 432	7.96 434	9.99 998	20	
	50	7.80 284	7.80 285	9.99 999	10			50	7.96 660	7.96 662	9.99 998	10	
22	0	7.80 615	7.80 615	9.99 999	0	**38**	**32**	0	7.96 887	7.96 889	9.99 998	0	**28**
	10	7.80 942	7.80 943	9.99 999	50			10	7.97 113	7.97 114	9.99 998	50	
	20	7.81 268	7.81 269	9.99 999	40			20	7.97 337	7.97 339	9.99 998	40	
	30	7.81 591	7.81 591	9.99 999	30			30	7.97 560	7.97 562	9.99 998	30	
	40	7.81 911	7.81 912	9.99 999	20			40	7.97 782	7.97 784	9.99 998	20	
	50	7.82 229	7.82 230	9.99 999	10			50	7.98 003	7.98 005	9.99 998	10	
23	0	7.82 545	7.82 546	9.99 999	0	**37**	**33**	0	7.98 223	7.98 225	9.99 998	0	**27**
	10	7.82 859	7.82 860	9.99 999	50			10	7.98 442	7.98 444	9.99 998	50	
	20	7.83 170	7.83 171	9.99 999	40			20	7.98 660	7.98 662	9.99 998	40	
	30	7.83 479	7.83 480	9.99 999	30			30	7.98 876	7.98 878	9.99 998	30	
	40	7.83 786	7.83 787	9.99 999	20			40	7.99 092	7.99 094	9.99 998	20	
	50	7.84 091	7.84 092	9.99 999	10			50	7.99 306	7.99 308	9.99 998	10	
24	0	7.84 393	7.84 394	9.99 999	0	**36**	**34**	0	7.99 520	7.99 522	9.99 998	0	**26**
	10	7.84 694	7.84 695	9.99 999	50			10	7.99 732	7.99 734	9.99 998	50	
	20	7.84 992	7.84 993	9.99 999	40			20	7.99 943	7.99 946	9.99 998	40	
	30	7.85 289	7.85 290	9.99 999	30			30	8.00 154	8.00 156	9.99 998	30	
	40	7.85 583	7.85 584	9.99 999	20			40	8.00 363	8.00 365	9.99 998	20	
	50	7.85 876	7.85 877	9.99 999	10			50	8.00 571	8.00 574	9.99 998	10	
25	0	7.86 166	7.86 167	9.99 999	0	**35**	**35**	0	8.00 779	8.00 781	9.99 998	0	**25**
	10	7.86 455	7.86 456	9.99 999	50			10	8.00 985	8.00 987	9.99 998	50	
	20	7.86 741	7.86 743	9.99 999	40			20	8.01 190	8.01 193	9.99 998	40	
	30	7.87 026	7.87 027	9.99 999	30			30	8.01 395	8.01 397	9.99 998	30	
	40	7.87 309	7.87 310	9.99 999	20			40	8.01 598	8.01 600	9.99 998	20	
	50	7.87 590	7.87 591	9.99 999	10			50	8.01 801	8.01 803	9.99 998	10	
26	0	7.87 870	7.87 871	9.99 999	0	**34**	**36**	0	8.02 002	8.02 004	9.99 998	0	**24**
	10	7.88 147	7.88 148	9.99 999	50			10	8.02 203	8.02 205	9.99 998	50	
	20	7.88 423	7.88 424	9.99 999	40			20	8.02 402	8.02 405	9.99 998	40	
	30	7.88 697	7.88 698	9.99 999	30			30	8.02 601	8.02 604	9.99 998	30	
	40	7.88 969	7.88 970	9.99 999	20			40	8.02 799	8.02 801	9.99 998	20	
	50	7.89 240	7.89 241	9.99 999	10			50	8.02 996	8.02 998	9.99 998	10	
27	0	7.89 509	7.89 510	9.99 999	0	**33**	**37**	0	8.03 192	8.03 194	9.99 997	0	**23**
	10	7.89 776	7.89 777	9.99 999	50			10	8.03 387	8.03 390	9.99 997	50	
	20	7.90 041	7.90 043	9.99 999	40			20	8.03 581	8.03 584	9.99 997	40	
	30	7.90 305	7.90 307	9.99 999	30			30	8.03 775	8.03 777	9.99 997	30	
	40	7.90 568	7.90 569	9.99 999	20			40	8.03 967	8.03 970	9.99 997	20	
	50	7.90 829	7.90 830	9.99 999	10			50	8.04 159	8.04 162	9.99 997	10	
28	0	7.91 088	7.91 089	9.99 999	0	**32**	**38**	0	8.04 350	8.04 353	9.99 997	0	**22**
	10	7.91 346	7.91 347	9.99 999	50			10	8.04 540	8.04 543	9.99 997	50	
	20	7.91 602	7.91 603	9.99 999	40			20	8.04 729	8.04 732	9.99 997	40	
	30	7.91 857	7.91 858	9.99 999	30			30	8.04 918	8.04 921	9.99 997	30	
	40	7.92 110	7.92 111	9.99 998	20			40	8.05 105	8.05 108	9.99 997	20	
	50	7.92 362	7.92 363	9.99 998	10			50	8.05 292	8.05 295	9.99 997	10	
29	0	7.92 612	7.92 613	9.99 998	0	**31**	**39**	0	8.05 478	8.05 481	9.99 997	0	**21**
	10	7.92 861	7.92 862	9.99 998	50			10	8.05 663	8.05 666	9.99 997	50	
	20	7.93 108	7.93 110	9.99 998	40			20	8.05 848	8.05 851	9.99 997	40	
	30	7.93 354	7.93 356	9.99 998	30			30	8.06 031	8.06 034	9.99 997	30	
	40	7.93 599	7.93 601	9.99 998	20			40	8.06 214	8.06 217	9.99 997	20	
	50	7.93 842	7.93 844	9.99 998	10			50	8.06 396	8.06 399	9.99 997	10	
30	0	7.94 084	7.94 086	9.99 998	0	**30**	**40**	0	8.06 578	8.06 581	9.99 997	0	**20**
′	″	L cos	L cot	L sin	″	′	′	″	L cos	L cot	L sin	″	′

′	″	L sin	L tan	L cos	″	′	′	″	L sin	L tan	L cos	″	′
40	0	8.06 578	8.06 581	9.99 997	0	**20**	**50**	0	8.16 268	8.16 273	9.99 995	0	**10**
	10	8.06 758	8.06 761	9.99 997	50			10	8.16 413	8.16 417	9.99 995	50	
	20	8.06 938	8.06 941	9.99 997	40			20	8.16 557	8.16 561	9.99 995	40	
	30	8.07 117	8.07 120	9.99 997	30			30	8.16 700	8.16 705	9.99 995	30	
	40	8.07 295	8.07 298	9.99 997	20			40	8.16 843	8.16 848	9.99 995	20	
	50	8.07 473	8.07 476	9.99 997	10			50	8.16 986	8.16 991	9.99 995	10	
41	0	8.07 650	8.07 653	9.99 997	0	**19**	**51**	0	8.17 128	8.17 133	9.99 995	0	**9**
	10	8.07 826	8.07 829	9.99 997	50			10	8.17 270	8.17 275	9.99 995	50	
	20	8.08 002	8.08 005	9.99 997	40			20	8.17 411	8.17 416	9.99 995	40	
	30	8.08 176	8.08 180	9.99 997	30			30	8.17 552	8.17 557	9.99 995	30	
	40	8.08 350	8.08 354	9.99 997	20			40	8.17 692	8.17 697	9.99 995	20	
	50	8.08 524	8.08 527	9.99 997	10			50	8.17 832	8.17 837	9.99 995	10	
42	0	8.08 696	8.08 700	9.99 997	0	**18**	**52**	0	8.17 971	8.17 976	9.99 995	0	**8**
	10	8.08 868	8.08 872	9.99 997	50			10	8.18 110	8.18 115	9.96 995	50	
	20	8.09 040	8.09 043	9.99 997	40			20	8.18 249	8.18 254	9.99 995	40	
	30	8.09 210	8.09 214	9.99 997	30			30	8.18 387	8.18 392	9.99 995	30	
	40	8.09 380	8.09 384	9.99 997	20			40	8.18 524	8.18 530	9.99 995	20	
	50	8.09 550	8.09 553	9.99 997	10			50	8.18 662	8.18 667	9.99 995	10	
43	0	8.09 718	8.09 722	9.99 997	0	**17**	**53**	0	8.18 798	8.18 804	9.99 995	0	**7**
	10	8.09 886	8.09 890	9.99 997	50			10	8.18 935	8.18 940	9.99 995	50	
	20	8.10 054	8.10 057	9.99 997	40			20	8.19 071	8.19 076	9.99 995	40	
	30	8.10 220	8.10 224	9.99 997	30			30	8.19 206	8.19 211	9.99 995	30	
	40	8.10 386	8.10 390	9.99 996	20			40	8.19 341	8.19 347	9.99 995	20	
	50	8.10 552	8.10 555	9.99 996	10			50	8.19 476	8.19 481	9.99 995	10	
44	0	8.10 717	8.10 720	9.99 996	0	**16**	**54**	0	8.19 610	8.19 616	9.99 995	0	**6**
	10	8.10 881	8.10 884	9.99 996	50			10	8.19 744	8.19 749	9.99 995	50	
	20	8.11 044	8.11 048	9.99 996	40			20	8.19 877	8.19 883	9.99 995	40	
	30	8 11 207	8.11 211	9.99 996	30			30	8.20 010	8.20 016	9-99 995	30	
	40	8.11 370	8.11 373	9.99 996	20			40	8.20 143	8.20 149	9.99 995	20	
	50	8.11 531	8.11 535	9.99 996	10			50	8.20 275	8.20 281	9.99 994	10	
45	0	8.11 693	8.11 696	9.99 996	0	**15**	**55**	0	8.20 407	8.20 413	9.99 994	0	**5**
	10	8.11 853	8.11 857	9.99 996	50			10	8.20 538	8.20 544	9.99 994	50	
	20	8.12 013	8.12 017	9.99 996	40			20	8.20 669	8 20 675	9.99 994	40	
	30	8.12 172	8.12 176	9.99 996	30			30	8.20 800	8.20 806	9.99 994	30	
	40	8.12 331	8 12 335	9.99 996	20			40	8.20 930	8.20 936	9.99 994	20	
	50	8.12 489	8.12 493	9.99 996	10			50	8.21 060	8.21 066	9.99 994	10	
46	0	8.12 647	8.12 651	9.99 996	0	**14**	**56**	0	8.21 189	8.21 195	9.99 994	0	**4**
	10	8.12 804	8.12 808	9.99 996	50			10	8.21 319	8.21 324	9.99 994	50	
	20	8.12 961	8.12 965	9.99 996	40			20	8.21 447	8.21 453	9.99 994	40	
	30	8.13 117	8.13 121	9.99 996	30			30	8.21 576	8.21 581	9.99 994	30	
	40	8.13 272	8.13 276	9.99 996	20			40	8.21 703	8.21 709	9.99 994	20	
	50	8.13 427	8.13 431	9.99 996	10			50	8.21 831	8.21 837	9.99 994	10	
47	0	8.13 581	8.13 585	9.99 996	0	**13**	**57**	0	8.21 958	8.21 964	9.99 994	0	**3**
	10	8.13 735	8.13 739	9.99 996	50			10	8.22 085	8.22 091	9.99 994	50	
	20	8.13 888	8.13 892	9.99 996	40			20	8.22 211	8.22 217	9.99 994	40	
	30	8.14 041	8.14 045	9.99 996	30			30	8 22 337	8.22 343	9.99 994	30	
	40	8.14 193	8.14 197	9.99 996	20			40	8 22 463	8.22 469	9.99 994	20	
	50	8.14 344	8.14 348	9.99 996	10			50	8.22 588	8.22 595	9.99 994	10	
48	0	8.14 495	8.14 500	9.99 996	0	**12**	**58**	0	8.22 713	8.22 720	9.99 994	0	**2**
	10	8.14 646	8.14 650	9.99 996	50			10	8.22 838	8.22 844	9.99 994	50	
	20	8.14 796	8.14 800	9.99 996	40			20	8.22 962	8.22 968	9.99 994	40	
	30	8 14 945	8.14 950	9.99 996	30			30	8.23 086	8.23 092	9.99 994	30	
	40	8.15 094	8.15 099	9.99 996	20			40	8.23 210	8.23 216	9.99 994	20	
	50	8.15 243	8.15 247	9.99 996	10			50	8.23 333	8.23 339	9.99 994	10	
49	0	8.15 391	8.15 395	9.99 996	0	**11**	**59**	0	8.23 456	8.23 462	9.99 994	0	**1**
	10	8.15 538	8.15 543	9.99 996	50			10	8.23 578	8.23 585	9.99 994	50	
	20	8.15 685	8.15 690	9.99 996	40			20	8.23 700	8.23 707	9.99 994	40	
	30	8.15 832	8.15 836	9.99 995	30			30	8.23 822	8.23 829	9.99 993	30	
	40	8.15 978	8.15 982	9.99 995	20			40	8.23 944	8.23 950	9.99 993	20	
	50	8.16 123	8.16 128	9.99 995	10			50	8.24 065	8.24 071	9.99 993	10	
50	0	8.16 268	8.16 273	9.99 995	0	**10**	**60**	0	8.24 186	8.24 192	9.99 993	0	**0**
′	″	L cos	L cot	L sin	″	′	′	″	L cos	L cot	L sin	″	′

′	″	L sin	L tan	L cos	″	′	′	″	L sin	L tan	L cos	″	′
0	0	8.24 186	8.24 192	9.99 993	0	**60**	**10**	0	8.30 879	8.30 888	9.99 991	0	**50**
	10	8.24 306	8.24 313	9.99 993	50			10	8.30 983	8.30 992	9.99 991	50	
	20	8.24 426	8.24 433	9.99 993	40			20	8.31 086	8.31 095	9.99 991	40	
	30	8.24 546	8.24 553	9.99 993	30			30	8.31 188	8.31 198	9.99 991	30	
	40	8.24 665	8.24 672	9.99 993	20			40	8.31 291	8.31 300	9.99 991	20	
	50	8.24 785	8.24 791	9.99 993	10			50	8.31 393	8.31 403	9.99 991	10	
1	0	8.24 903	8.24 910	9.99 993	0	**59**	**11**	0	8.31 495	8.31 505	9.99 991	0	**49**
	10	8.25 022	8.25 029	9.99 993	50			10	8.31 597	8.31 606	9.99 991	50	
	20	8.25 140	8.25 147	9.99 993	40			20	8.31 699	8.31 708	9.99 991	40	
	30	8.25 258	8.25 265	9.99 993	30			30	8.31 800	8.31 809	9.99 991	30	
	40	8.25 375	8.25 382	9.99 993	20			40	8.31 901	8.31 911	9.99 991	20	
	50	8.25 493	8.25 500	9.99 993	10			50	8.32 002	8.32 012	9.99 991	10	
2	0	8.25 609	8.25 616	9.99 993	0	**58**	**12**	0	8.32 103	8.32 112	9.99 990	0	**48**
	10	8.25 726	8.25 733	9.99 993	50			10	8.32 203	8.32 213	9.99 990	50	
	20	8.25 842	8.25 849	9.99 993	40			20	8.32 303	8.32 313	9.99 990	40	
	30	8.25 958	8.25 965	9.99 993	30			30	8.32 403	8.32 413	9.99 990	30	
	40	8.26 074	8.26 081	9.99 993	20			40	8.32 503	8.32 513	9.99 990	20	
	50	8.26 189	8.26 196	9.99 993	10			50	8.32 602	8.32 612	9.99 990	10	
3	0	8.26 304	8.26 312	9.99 993	0	**57**	**13**	0	8.32 702	8.32 711	9.99 990	0	**47**
	10	8.26 419	8 26 426	9.99 993	50			10	8.32 801	8.32 811	9.99 990	50	
	20	8.26 553	8.26 541	9.99 993	40			20	8.32 899	8.32 909	9.99 990	40	
	30	8.26 648	8.26 655	9.99 993	30			30	8.32 998	8.33 008	9.99 990	30	
	40	8.26 761	8.26 769	9.99 993	20			40	8.33 096	8.33 106	9.99 990	20	
	50	8.26 875	8.26 882	9.99 993	10			50	8.33 195	8.33 205	9.99 990	10	
4	0	8.26 988	8.26 996	9.99 992	0	**56**	**14**	0	8.33 292	8.33 302	9.99 990	0	**46**
	10	8.27 101	8.27 109	9.99 992	50			10	8.33 390	8.33 400	9.99 990	50	
	20	8.27 214	8.27 221	9.99 992	40			20	8.33 488	8.33 498	9.99 990	40	
	30	8.27 326	8.27 334	9.99 992	30			30	8.33 585	8.33 595	9.99 990	30	
	40	8.27 438	8.27 446	9.99 992	20			40	8.33 682	8.33 692	9.99 990	20	
	50	8.27 550	8.27 558	9.99 992	10			50	8.33 779	8.33 789	9.99 990	10	
5	0	8.27 661	8.27 669	9.99 992	0	**55**	**15**	0	8.33 875	8.33 886	9.99 990	0	**45**
	10	8.27 773	8.27 780	9.99 992	50			10	8.33 972	8.33 982	9.99 990	50	
	20	8.27 883	8.27 891	9.99 992	40			20	8.34 086	8.34 078	9.99 990	40	
	30	8.27 994	8.28 002	9.99 992	30			30	8.34 164	8.34 174	9.99 990	30	
	40	8.28 104	8 28 112	9.99 992	20			40	8.34 260	8.34 270	9.99 989	20	
	50	8.28 215	8.28 223	9.99 992	10			50	8.34 355	8.34 366	9.99 989	10	
6	0	8.28 324	8.28 332	9.99 992	0	**54**	**16**	0	8.34 450	8.34 461	9.99 989	0	**44**
	10	8.28 434	8.28 442	9.99 992	50			10	8.34 546	8.34 556	9.99 989	50	
	20	8.28 543	8.28 551	9.99 992	40			20	8.34 640	8.34 651	9.99 989	40	
	30	8.28 652	8.28 660	9.99 992	30			30	8.34 735	8.34 746	9.99 989	30	
	40	8.28 761	8.28 769	9.99 992	20			40	8.34 830	8.34 840	9.99 989	20	
	50	8.28 869	8.28 877	9.99 992	10			50	8.34 924	8.34 935	9.99 989	10	
7	0	8.28 977	8.28 986	9.99 992	0	**53**	**17**	0	8.35 018	8.35 029	9.99 989	0	**43**
	10	8.29 085	8.29 094	9.99 992	50			10	8.35 112	8.35 123	9.99 989	50	
	20	8.29 093	8.29 201	9.99 992	40			20	8.35 206	8.35 217	9.99 989	40	
	30	8.29 300	8.29 309	9.99 992	30			30	8.35 299	8.35 310	9.99 989	30	
	40	8.29 407	8.29 416	9.99 992	20			40	8.35 392	8.35 403	9.99 989	20	
	50	8.29 514	8.29 523	9.99 992	10			50	8.35 485	8.35 497	9.99 989	10	
8	0	8.29 621	8.29 629	9.99 992	0	**52**	**18**	0	8.35 578	8.35 590	9.99 989	0	**42**
	10	8.29 727	8.29 736	9.99 991	50			10	8.35 671	8.35 682	9.99 989	50	
	20	8.29 833	8.29 842	9.99 991	40			20	8.35 764	8.35 775	9.99 989	40	
	30	8.29 939	8.29 947	9.99 991	30			30	8.35 856	8.35 867	9.99 989	30	
	40	8.30 044	8.30 053	9.99 991	20			40	8.35 948	8.35 959	9.99 989	20	
	50	8.30 150	8.30 158	9.99 991	10			50	8.36 040	8.36 051	9.99 989	10	
9	0	8.30 255	8.30 263	9.99 991	0	**51**	**19**	0	8.36 131	8.36 143	9.99 989	0	**41**
	10	8.30 359	8.30 368	9.99 991	50			10	8.36 223	8.36 235	9.99 988	50	
	20	8.30 464	8.30 473	9.99 991	40			20	8.36 314	8.36 326	9.99 988	40	
	30	8.30 568	8.30 577	9.99 991	30			30	8.36 405	8.36 417	9.99 988	30	
	40	8.30 672	8.30 681	9.99 991	20			40	8 36 496	8.36 508	9.99 988	20	
	50	8.30 776	8.30 785	9.99 991	10			50	8.36 587	8.36 599	9.99 988	10	
10	0	8.30 879	8.30 888	9.99 991	0	**50**	**20**	0	8.36 678	8.36 689	9.99 988	0	**40**
′	″	L cos	L cot	L sin	″	′	′	″	L cos	L cot	L sin	″	′

1°

′	″	L sin	L tan	L cos	″	′	′	″	L sin	L tan	L cos	″	′
20	0	8.36 678	8.36 689	9.99 988	0	**40**	**30**	0	8.41 792	8.41 807	9.99 985	0	**30**
	10	8.36 768	8.36 780	9.99 988	50			10	8.41 872	8.41 887	9.99 985	50	
	20	8.36 858	8.36 870	9.99 988	40			20	8.41 952	8.41 967	9.99 985	40	
	30	8.36 948	8.36 960	9.99 988	30			30	8.42 032	8.42 048	9.99 985	30	
	40	8.37 038	8.37 050	9.99 988	20			40	8.42 112	8.42 127	9.99 985	20	
	50	8.37 128	8.37 140	9.99 988	10			50	8.42 192	8.42 207	9.99 985	10	
21	0	8.37 217	8.37 229	9.99 988	0	**39**	**31**	0	8.42 272	8.42 287	9.99 985	0	**29**
	10	8.37 306	8.37 318	9.99 988	50			10	8.42 351	8.42 366	9.99 985	50	
	20	8.37 395	8.37 408	9.99 988	40			20	8.42 430	8.42 446	9.99 985	40	
	30	8.37 484	8.37 497	9.99 988	30			30	8.42 510	8.42 525	9.99 985	30	
	40	8.37 573	8.37 585	9.99 988	20			40	8.42 589	8.42 406	9.99 985	20	
	50	8.37 662	8.37 674	9.99 988	10			50	8.42 667	8.42 683	9.99 985	10	
22	0	8.37 750	8.37 762	9.99 988	0	**38**	**32**	0	8.42 746	8.42 762	9.99 984	0	**28**
	10	8.37 838	8.37 850	9.99 988	50			10	8.42 825	8.42 840	9.96 984	50	
	20	8.37 926	8.37 938	9.99 988	40			20	8.42 903	8.42 919	9.99 984	40	
	30	8.38 014	8.38 026	9.99 987	30			30	8.42 982	8.42 997	9.99 984	30	
	40	8.38 101	8.38 114	9.99 987	20			40	8.43 060	8.43 075	9.99 984	20	
	50	8.38 189	8.38 202	9.99 987	10			50	8.43 138	8.43 154	9.99 984	10	
23	0	8.38 276	8.38 289	9.99 987	0	**37**	**33**	0	8.43 216	8.43 232	9.99 984	0	**27**
	10	8.38 363	8.38 376	9.99 987	50			10	8.43 293	8.43 309	9.99 984	50	
	20	8.38 450	8.38 463	9.99 987	40			20	8.43 371	8.43 387	9.99 984	40	
	30	8.38 537	8.38 550	9.99 987	30			30	8.43 448	8.43 464	9.99 984	30	
	40	8.38 624	8 38 636	9.99 987	20			40	8.43 526	8.43 542	9.99 984	20	
	50	8.38 710	8.38 723	9.99 987	10			50	8.43 603	8.43 619	9.99 984	10	
24	0	8.38 796	8.38 809	9.99 987	0	**36**	**34**	0	8.43 680	8.43 696	9.99 984	0	**26**
	10	8.38 882	8.38 895	9.99 987	50			10	8.43 757	8.43 773	9.99 984	50	
	20	8.38 968	8.38 981	9.99 987	40			20	8.43 834	8.43 850	9.99 984	40	
	30	8.39 054	8.39 067	9.99 987	30			30	8.43 910	8.43 927	9.99 984	30	
	40	8.39 139	8.39 153	9.99 987	20			40	8.43 987	8.44 003	9.99 984	20	
	50	8.39 225	8.39 238	9.99 987	10			50	8.44 063	8.44 080	9.99 983	10	
25	0	8.39 310	8.39 323	9.99 987	0	**35**	**35**	0	8.44 139	8.44 156	9.99 983	0	**25**
	10	8.39 395	8.39 408	9.99 987	50			10	8.44 216	8.44 232	9.99 983	50	
	20	8.39 480	8.39 493	9.99 987	40			20	8.44 292	8.44 308	9.99 983	40	
	30	8.39 565	8.39 587	9.99 987	30			30	8.44 367	8.44 384	9.99 983	30	
	40	8.39 649	8.39 663	9.99 987	20			40	8.44 443	8.44 460	9.99 983	20	
	50	8.39 734	8.39 747	9.99 986	10			50	8.44 519	8.44 536	9.99 983	10	
26	0	8.39 818	8.39 832	9.99 986	0	**34**	**36**	0	8.44 594	8.44 611	9.99 983	0	**24**
	10	8.39 902	8.39 916	9.99 986	50			10	8.44 669	8.44 686	9.99 983	50	
	20	8.39 986	8.40 000	9.99 986	40			20	8.44 745	8.44 762	9.99 983	40	
	30	8.40 070	8.40 083	9.99 986	30			30	8.44 820	8.44 837	9.99 983	30	
	40	8.40 153	8.40 167	9.99 986	20			40	8.44 895	8.44 912	9.99 983	20	
	50	8.40 237	8.40 251	9.99 986	10			50	8.44 969	8.44 987	9.99 983	10	
27	0	8.40 320	8.40 334	9.99 986	0	**33**	**37**	0	8.45 044	8.45 061	9.99 983	0	**23**
	10	8.40 403	8.40 417	9.99 986	50			10	8.45 119	8.45 136	9.99 983	50	
	20	8.40 486	8.40 500	9.99 986	40			20	8.45 193	8.45 210	9.99 983	40	
	30	8.40 569	8.40 583	9.99 986	30			30	8.45 267	8.45 285	9.99 983	30	
	40	8.40 651	8.40 665	9.99 986	20			40	8.45 341	8.45 359	9.99 982	20	
	50	8.40 734	8.40 748	9.99 986	10			50	8.45 415	8.45 433	9.99 982	10	
28	0	8.40 816	8.40 830	9.99 986	0	**32**	**38**	0	8.45 489	8.45 507	9.99 982	0	**22**
	10	8.40 898	8.40 913	9.99 986	50			10	8.45 563	8.45 581	9.99 982	50	
	20	8.40 980	8.40 995	9.99 986	40			20	8.45 637	8.45 655	9.99 982	40	
	30	8.41 062	8.41 077	9.99 986	30			30	8.45 710	8.45 728	9.99 982	30	
	40	8 41 144	8.41 158	9.99 986	20			40	8.45 784	8.45 802	9.99 982	20	
	50	8.41 225	8.41 240	9.99 986	10			50	8.45 857	8.45 875	9.99 982	10	
29	0	8.41 307	8.41 321	9.99 985	0	**31**	**39**	0	8.45 930	8.45 948	9.99 982	0	**21**
	10	8.41 388	8.41 403	9.99 985	50			10	8.46 003	8.46 021	9.99 982	50	
	20	8.41 469	8.41 484	9.99 985	40			20	8.46 076	8.46 094	9.99 982	40	
	30	8.41 550	8.41 565	9.99 985	30			30	8.46 149	8.46 167	9.99 982	30	
	40	8.41 631	8.41 646	9.99 985	20			40	8.46 222	8.46 240	9.99 982	20	
	50	8.41 711	8.41 726	9.99 985	10			50	8.46 294	8.46 312	9.99 982	10	
30	0	8.41 792	8.41 807	9.99 985	0	**30**	**40**	0	8.46 366	8.46 385	9.99 932	0	**20**
′	″	L cos	L cot	L sin	″	′	′	″	L cos	L cot	L sin	″	′

88°

′	″	L sin	L tan	L cos	″	′	′	″	L sin	L tan	L cos	″	′
40	0	8.46 366	8.46 385	9.99 982	0	**20**	**50**	0	8.50 504	8.50 527	9.99 978	0	**10**
	10	8.46 439	8.46 457	9.99 982	50			10	8.50 570	8.50 593	9.99 978	50	
	20	8.46 511	8.46 529	9.99 982	40			20	8.50 636	8.50 658	9.99 978	40	
	30	8.46 583	8.46 602	9.99 981	30			30	8.50 701	8.50 724	9.99 978	30	
	40	8.46 655	8.46 674	9.99 981	20			40	8.50 767	8.50 789	9.99 977	20	
	50	8.46 727	8.46 745	9.99 981	10			50	8.50 832	8.50 855	9.99 977	10	
41	0	8.46 799	8.46 817	9.99 981	0	**19**	**51**	0	8.50 897	8.50 920	9.99 977	0	**9**
	10	8.46 870	8.46 889	9.99 981	50			10	8.50 963	8.50 985	9.99 977	50	
	20	8.46 942	8.46 960	9.99 981	40			20	8.51 028	8.51 050	9.99 977	40	
	30	8.47 013	8.47 032	9.99 981	30			30	8.51 092	8.51 015	9.99 977	30	
	40	8.47 084	8.47 103	9.99 981	20			40	8.51 157	8.51 180	9.99 977	20	
	50	8.47 155	8.47 174	9.99 981	10			50	8.51 222	8.51 245	9.99 977	10	
42	0	8.47 226	8.47 245	9.99 981	0	**18**	**52**	0	8.51 287	8.51 310	9.99 977	0	**8**
	10	8.47 297	8.47 316	9.99 981	50			10	8.51 351	8.51 374	9.99 977	50	
	20	8.47 368	8.47 387	9.99 981	40			20	8.51 416	8.51 439	9.99 977	40	
	30	8.47 439	8.47 458	9.99 981	30			30	8.51 480	8.51 503	9.99 977	30	
	40	8.47 509	8.47 528	9.99 981	20			40	8.51 544	8.51 568	9.99 977	20	
	50	8.47 580	8.47 599	9.99 981	10			50	8.51 609	8.51 632	9.99 977	10	
43	0	8.47 650	8.47 669	9.99 981	0	**17**	**53**	0	8.51 673	8.51 696	9.99 977	0	**7**
	10	8.47 720	8.47 740	9.99 980	50			10	8.51 737	8.51 760	9.99 976	50	
	20	8.47 790	8.47 810	9.99 980	40			20	8.51 801	8.51 824	9.99 976	40	
	30	8.47 860	8.47 880	9.99 980	30			30	8.51 864	8.51 888	9.99 976	30	
	40	8.47 930	8.47 950	9.99 980	20			40	8.51 928	8.51 952	9.99 976	20	
	50	8.48 000	8.48 020	9.99 980	10			50	8.51 992	8.52 015	9.99 976	10	
44	0	8.48 096	8.48 090	9.99 980	0	**16**	**54**	0	8.52 055	8.52 079	9.99 976	0	**6**
	10	8.48 139	8.48 159	9.99 980	50			10	8.52 119	8.52 143	9.99 976	50	
	20	8.48 208	8.48 228	9.99 980	40			20	8.52 182	8.52 206	9.99 976	40	
	30	8.48 278	8.48 298	9.99 980	30			30	8.52 245	8.52 269	9.99 976	30	
	40	8.48 347	8.48 367	9.99 980	20			40	8.52 308	8.52 332	9.99 976	20	
	50	8.48 416	8.48 436	9.99 980	10			50	8.52 371	8.52 396	9.99 976	10	
45	0	8.48 485	8.48 505	9.99 980	0	**15**	**55**	0	8.52 434	8.52 459	9.99 976	0	**5**
	10	8.48 554	8.48 574	9.99 980	50			10	8.52 497	8.52 522	9.99 976	50	
	20	8.48 622	8.48 643	9.99 980	40			20	8.52 560	8.52 584	9.99 976	40	
	30	8.48 691	8.48 711	9.99 980	30			30	8.52 623	8.52 647	9.99 975	30	
	40	8.48 760	8.48 780	9.99 979	20			40	8.52 685	8.52 710	9.99 975	20	
	50	8.48 828	8.48 849	9.99 979	10			50	8.52 748	8.52 772	9.99 975	10	
46	0	8.48 896	8.48 917	9.99 979	0	**14**	**56**	0	8.52 810	8.52 835	9.99 975	0	**4**
	10	8.48 965	8.48 985	9.99 979	50			10	8.52 872	8.52 897	9.99 975	50	
	20	8.49 033	8.49 053	9.99 979	40			20	8.52 935	8.52 960	9.99 975	40	
	30	8.49 101	8.49 121	9.99 979	30			30	8.52 997	8.53 022	9.99 975	30	
	40	8.49 169	8.49 189	9.99 979	20			40	8.53 059	8.53 084	9.99 975	20	
	50	8.49 236	8.49 257	9.99 979	10			50	8.53 121	8.53 146	9.99 975	10	
47	0	8.49 304	8.49 325	9.99 979	0	**13**	**57**	0	8.53 183	8.53 208	9.99 975	0	**3**
	10	8.49 372	8.49 393	9.99 979	50			10	8.53 245	8.53 270	9.99 975	50	
	20	8.49 439	8.49 460	9.99 979	40			20	8.53 306	8.53 332	9.99 975	40	
	30	8.49 506	8.49 528	9.99 979	30			30	8.53 368	8.53 393	9.99 975	30	
	40	8.49 574	8.49 595	9.99 979	20			40	8.53 429	8.53 455	9.99 975	20	
	50	8.49 641	8.49 662	9.99 979	10			50	8.53 491	8.53 516	9.99 974	10	
48	0	8.49 708	8.49 729	9.99 979	0	**12**	**58**	0	8.53 552	8.53 578	9.99 974	0	**2**
	10	8.49 775	8.49 796	9.99 979	50			10	8.53 614	8.53 639	9.99 974	50	
	20	8.49 842	8.49 863	9.99 978	40			20	8.53 675	8.53 700	9.99 974	40	
	30	8.49 908	8.49 930	9.99 978	30			30	8.53 736	8.53 762	9.99 974	30	
	40	8.49 975	8.49 997	9.99 978	20			40	8.53 797	8.53 823	9.99 974	20	
	50	8.50 042	8.50 063	9.99 978	10			50	8.53 858	8.53 884	9.99 974	10	
49	0	8.50 108	8.50 130	9.99 978	0	**11**	**59**	0	8.53 919	8.53 945	9.99 974	0	**1**
	10	8.50 174	8.50 196	9.99 978	50			10	8.53 979	8.54 005	9.99 974	50	
	20	8.50 241	8.50 263	9.99 978	40			20	8.54 040	8.54 066	9.99 974	40	
	30	8.50 307	8.50 329	9.99 978	30			30	8.54 101	8.54 127	9.99 974	30	
	40	8.50 373	8.50 395	9.99 978	20			40	8.54 161	8.54 187	9.99 974	20	
	50	8.50 439	8.50 461	9.99 978	10			50	8.54 222	8.54 248	9.99 974	10	
50	0	8.50 504	8.50 527	9.99 978	0	**10**	**60**	0	8.54 282	8.54 308	9.99 974	0	**0**
′	″	L cos	L cot	L sin	″	′	′	″	L cos	L cot	L sin	″	′

1°

′	8 L sin	8 L tan	11 L cot	9 L cos	′
0	.24 186	.24 192	.75 808	.99 993	**60**
1	.24 903	.24 910	.75 090	.99 993	59
2	.25 609	.25 616	.74 384	.99 993	58
3	.26 304	.26 312	.73 688	.99 993	57
4	.26 988	.26 996	.73 004	.99 992	56
5	.27 661	.27 669	.72 331	.99 992	55
6	.28 324	.28 332	.71 668	.99 992	54
7	.28 977	.28 986	.71 014	.99 992	53
8	.29 621	.29 629	.70 371	.99 992	52
9	.30 255	.30 263	.69 737	.99 991	51
10	.30 879	.30 888	.69 112	.99 991	**50**
11	.31 495	.31 505	.68 495	.99 991	49
12	.32 103	.32 112	.67 888	.99 990	48
13	.32 702	.32 711	.67 289	.99 990	47
14	.33 292	.33 302	.66 698	.99 990	46
15	.33 875	.33 886	.66 114	.99 990	45
16	.34 450	.34 461	.65 539	.99 989	44
17	.35 018	.35 029	.64 971	.99 989	43
18	.35 578	.35 590	.64 410	.99 989	42
19	.36 131	.36 143	.63 857	.99 989	41
20	.36 678	.36 689	.63 311	.99 988	**40**
21	.37 217	.37 229	.62 771	.99 988	39
22	.37 750	.37 762	.62 238	.99 988	38
23	.38 276	.38 289	.61 711	.99 987	37
24	.38 796	.38 809	.61 191	.99 987	36
25	.39 310	.39 323	.60 677	.99 987	35
26	.39 818	.39 832	.60 168	.99 986	34
27	.40 320	.40 334	.59 666	.99 986	33
28	.40 816	.40 830	.59 170	.99 986	32
29	.41 307	.41 321	.58 679	.99 985	31
30	.41 792	.41 807	.58 193	.99 985	**30**
31	.42 272	.42 287	.57 713	.99 985	29
32	.42 746	.42 762	.57 238	.99 984	28
33	.43 216	.43 232	.56 768	.99 984	27
34	.43 680	.43 696	.56 304	.99 984	26
35	.44 139	.44 156	.55 844	.99 983	25
36	.44 594	.44 611	.55 389	.99 983	24
37	.45 044	.45 061	.54 939	.99 983	23
38	.45 489	.45 507	.54 493	.99 982	22
39	.45 930	.45 948	.54 052	.99 982	21
40	.46 366	.46 385	.53 615	.99 982	**20**
41	.46 799	.46 817	.53 183	.99 981	19
42	.47 226	.47 245	.52 755	.99 981	18
43	.47 650	.47 669	.52 331	.99 981	17
44	.48 069	.48 089	.51 911	.99 980	16
45	.48 485	.48 505	.51 495	.99 980	15
46	.48 896	.48 917	.51 083	.99 979	14
47	.49 304	.49 325	.50 675	.99 979	13
48	.49 708	.49 729	.50 271	.99 979	12
49	.50 108	.50 130	.49 870	.99 978	11
50	.50 504	.50 527	.49 473	.99 978	**10**
51	.50 897	.50 920	.49 080	.99 977	9
52	.51 287	.51 310	.48 690	.99 977	8
53	.51 673	.51 696	.48 304	.99 977	7
54	.52 055	.52 079	.47 921	.99 976	6
55	.52 434	.52 459	.47 541	.99 976	5
56	.52 810	.52 835	.47 165	.99 975	4
57	.53 183	.53 208	.46 792	.99 975	3
58	.53 552	.53 578	.46 422	.99 974	2
59	.53 919	.53 945	.46 055	.99 974	1
60	.54 282	.54 308	.45 692	.99 974	**0**
′	8 L cos	8 L cot	11 L tan	9 L sin	′

88°

2°

′	8 L sin	8 L tan	11 L cot	9 L cos	′
0	.54 282	.54 308	.45 692	.99 974	**60**
1	.54 642	.54 669	.45 331	.99 973	59
2	.54 999	.55 027	.44 973	.99 973	58
3	.55 354	.55 382	.44 618	.99 972	57
4	.55 705	.55 734	.44 266	.99 972	56
5	.56 054	.56 083	.43 917	.99 971	55
6	.56 400	.56 429	.43 571	.99 971	54
7	.56 743	.56 773	.43 227	.99 970	53
8	.57 084	.57 114	.42 886	.99 970	52
9	.57 421	.57 452	.42 548	.99 969	51
10	.57 757	.57 788	.42 212	.99 969	**50**
11	.58 089	.58 121	.41 879	.99 968	49
12	.58 419	.58 451	.41 549	.99 968	48
13	.58 747	.58 779	.41 221	.99 967	47
14	.59 072	.59 105	.40 895	.99 967	46
15	.59 395	.59 428	.40 572	.99 967	45
16	.59 715	.59 749	.40 251	.99 966	44
17	.60 033	.60 068	.39 932	.99 966	43
18	.60 349	.60 384	.39 616	.99 965	42
19	.60 662	.60 698	.39 302	.99 964	41
20	.60 973	.61 009	.38 991	.99 964	**40**
21	.61 282	.61 319	.38 681	.99 963	39
22	.61 589	.61 626	.38 374	.99 963	38
23	.61 894	.61 931	.38 069	.99 962	37
24	.62 196	.62 234	.37 766	.99 962	36
25	.62 497	.62 535	.37 465	.99 961	35
26	.62 795	.62 834	.37 166	.99 961	34
27	.63 091	.63 131	.36 869	.99 960	33
28	.63 385	.63 426	.36 574	.99 960	32
29	.63 678	.63 718	.36 282	.99 959	31
30	.63 968	.64 009	.35 991	.99 959	**30**
31	.64 256	.64 298	.35 702	.99 958	29
32	.64 543	.64 585	.35 415	.99 958	28
33	.64 827	.64 870	.35 130	.99 957	27
34	.65 110	.65 154	.34 846	.99 956	26
35	.65 391	.65 435	.34 565	.99 956	25
36	.65 670	.65 715	.34 285	.99 955	24
37	.65 947	.65 993	.34 007	.99 955	23
38	.66 223	.66 269	.33 731	.99 954	22
39	.66 497	.66 543	.33 457	.99 954	21
40	.66 769	.66 816	.33 184	.99 953	**20**
41	.67 039	.67 087	.32 913	.99 952	19
42	.67 308	.67 356	.32 644	.99 952	18
43	.67 575	.67 624	.32 376	.99 951	17
44	.67 841	.67 890	.32 110	.99 951	16
45	.68 104	.68 154	.31 846	.99 950	15
46	.68 367	.68 417	.31 583	.99 949	14
47	.68 627	.68 678	.31 322	.99 949	13
48	.68 886	.68 938	.31 062	.99 948	12
49	.69 144	.69 196	.30 804	.99 948	11
50	.69 400	.69 453	.30 547	.99 947	**10**
51	.69 654	.69 708	.30 292	.99 946	9
52	.69 907	.69 962	.30 038	.99 946	8
53	.70 159	.70 214	.29 786	.99 945	7
54	.70 409	.70 465	.29 535	.99 944	6
55	.70 658	.70 714	.29 286	.99 944	5
56	.70 905	.70 962	.29 038	.99 943	4
57	.71 151	.71 208	.28 792	.99 942	3
58	.71 395	.71 453	.28 547	.99 942	2
59	.71 638	.71 697	.28 303	.99 941	1
60	.71 880	.71 940	.28 060	.99 940	**0**
′	8 L cos	8 L cot	11 L tan	9 L sin	′

87°

3°

′	8 L sin	8 L tan	11 L cot	9 L cos	′
0	.71 880	.71 940	.28 060	.99 940	**60**
1	.72 120	.72 181	.27 819	.99 940	59
2	.72 359	.72 420	.27 580	.99 939	58
3	.72 597	.72 659	.27 341	.99 938	57
4	.72 834	.72 896	.27 104	.99 938	56
5	.73 069	.73 132	.26 868	.99 937	55
6	.73 303	.73 366	.26 634	.99 936	54
7	.73 535	.73 600	.26 400	.99 936	53
8	.73 767	.73 832	.26 168	.99 935	52
9	.73 997	.74 063	.25 937	.99 934	51
10	.74 226	.74 292	.25 708	.99 934	**50**
11	.74 454	.74 521	.25 479	.99 933	49
12	.74 680	.74 748	.25 252	.99 932	48
13	.74 906	.74 974	.25 026	.99 932	47
14	.75 130	.75 199	.24 801	.99 931	46
15	.75 353	.75 423	.24 577	.99 930	45
16	.75 575	.75 645	.24 355	.99 929	44
17	.75 795	.75 867	.24 133	.99 929	43
18	.76 015	.76 087	.23 913	.99 928	42
19	.76 234	.76 306	.23 694	.99 927	41
20	.76 451	.76 525	.23 475	.99 926	**40**
21	.76 667	.76 742	.23 258	.99 926	39
22	.76 883	.76 958	.23 042	.99 925	38
23	.77 097	.77 173	.22 827	.99 924	37
24	.77 310	.77 387	.22 613	.99 923	36
25	.77 522	.77 600	.22 400	.99 923	35
26	.77 733	.77 811	.22 189	.99 922	34
27	.77 943	.78 022	.21 978	.99 921	33
28	.78 152	.78 232	.21 768	.99 920	32
29	.78 360	.78 441	.21 559	.99 920	31
30	.78 568	.78 649	.21 351	.99 919	**30**
31	.78 774	.78 855	.21 145	.99 918	29
32	.78 979	.79 061	.20 939	.99 917	28
33	.79 183	.79 266	.20 734	.99 917	27
34	.79 386	.79 470	.20 530	.99 916	26
35	.79 588	.79 673	.20 327	.99 915	25
36	.79 789	.79 875	.20 125	.99 914	24
37	.79 990	.80 076	.19 924	.99 913	23
38	.80 189	.80 277	.19 723	.99 913	22
39	.80 388	.80 476	.19 524	.99 912	21
40	.80 585	.80 674	.19 326	.99 911	**20**
41	.80 782	.80 872	.19 128	.99 910	19
42	.80 978	.81 068	.18 932	.99 909	18
43	.81 173	.81 264	.18 736	.99 909	17
44	.81 367	.81 459	.18 541	.99 908	16
45	.81 560	.81 653	.18 347	.99 907	15
46	.81 752	.81 846	.18 154	.99 906	14
47	.81 944	.82 038	.17 962	.99 905	13
48	.82 134	.82 230	.17 770	.99 904	12
49	.82 324	.82 420	.17 580	.99 904	11
50	.82 513	.82 610	.17 390	.99 903	**10**
51	.82 701	.82 799	.17 201	.99 902	9
52	.82 888	.82 987	.17 013	.99 901	8
53	.83 075	.83 175	.16 825	.99 900	7
54	.83 261	.83 361	.16 639	.99 899	6
55	.83 446	.83 547	.16 453	.99 898	5
56	.83 630	.83 732	.16 268	.99 898	4
57	.83 813	.83 916	.16 084	.99 897	3
58	.83 996	.84 100	.15 900	.99 896	2
59	.84 177	.84 282	.15 718	.99 895	1
60	.84 358	.84 464	.15 536	.99 894	**0**
′	8 L cos	8 L cot	11 L tan	9 L sin	′

86°

4°

′	8 L sin	8 L tan	11 L cot	9 L cos	′
0	.84 358	.84 464	.15 536	.99 894	**60**
1	.84 539	.84 646	.15 354	.99 893	59
2	.84 718	.84 826	.15 174	.99 892	58
3	.84 897	.85 006	.14 994	.99 891	57
4	.85 075	.85 185	.14 815	.99 891	56
5	.85 252	.85 363	.14 637	.99 890	55
6	.85 429	.85 540	.14 460	.99 889	54
7	.85 605	.85 717	.14 283	.99 888	53
8	.85 780	.85 893	.14 107	.99 887	52
9	.85 955	.86 069	.13 931	.99 886	51
10	.86 128	.86 243	.13 757	.99 885	**50**
11	.86 301	.86 417	.13 583	.99 884	49
12	.86 474	.86 591	.13 409	.99 883	48
13	.86 645	.86 763	.13 237	.99 882	47
14	.86 816	.86 935	.13 065	.99 881	46
15	.86 987	.87 106	.12 894	.99 880	45
16	.87 156	.87 277	.12 723	.99 879	44
17	.87 325	.87 447	.12 553	.99 879	43
18	.87 494	.87 616	.12 384	.99 878	42
19	.87 661	.87 785	.12 215	.99 877	41
20	.87 829	.87 953	.12 047	.99 876	**40**
21	.87 995	.88 120	.11 880	.99 875	39
22	.88 161	.88 287	.11 713	.99 874	38
23	.88 326	.88 453	.11 547	.99 873	37
24	.88 490	.88 618	.11 382	.99 872	36
25	.88 654	.88 783	.11 217	.99 871	35
26	.88 817	.88 948	.11 052	.99 870	34
27	.88 980	.89 111	.10 889	.99 869	33
28	.89 142	.89 274	.10 726	.99 868	32
29	.89 304	.89 437	.10 563	.99 867	31
30	.89 464	.89 598	.10 402	.99 866	**30**
31	.89 625	.89 760	.10 240	.99 865	29
32	.89 784	.89 920	.10 080	.99 864	28
33	.89 943	.90 080	.09 920	.99 863	27
34	.90 102	.90 240	.09 760	.99 862	26
35	.90 260	.90 399	.09 601	.99 861	25
36	.90 417	.90 557	.09 443	.99 860	24
37	.90 574	.90 715	.09 285	.99 859	23
38	.90 730	.90 872	.09 128	.99 858	22
39	.90 885	.91 029	.08 971	.99 857	21
40	.91 040	.91 185	.08 815	.99 856	**20**
41	.91 195	.91 340	.08 660	.99 855	19
42	.91 349	.91 495	.08 505	.99 854	18
43	.91 502	.91 650	.08 350	.99 853	17
44	.91 655	.91 803	.08 197	.99 852	16
45	.91 807	.91 957	.08 043	.99 851	15
46	.91 959	.92 110	.07 890	.99 850	14
47	.92 110	.92 262	.07 738	.99 848	13
48	.92 261	.92 414	.07 586	.99 847	12
49	.92 411	.92 565	.07 435	.99 846	11
50	.92 561	.92 716	.07 284	.99 845	**10**
51	.92 710	.92 866	.07 134	.99 844	9
52	.92 859	.93 016	.06 984	.99 843	8
53	.93 007	.93 165	.06 835	.99 842	7
54	.93 154	.93 313	.06 687	.99 841	6
55	.93 301	.93 462	.06 538	.99 840	5
56	.93 448	.93 609	.06 391	.99 839	4
57	.93 594	.93 756	.06 244	.99 838	3
58	.93 740	.93 903	.06 097	.99 837	2
59	.93 885	.94 049	.05 951	.99 836	1
60	.94 030	.94 195	.05 805	.99 834	**0**
′	8 L cos	8 L cot	11 L tan	9 L sin	′

85°

5°

′	8 L sin	8 L tan	11 L cot	9 L cos	′
0	.94 030	.94 195	.05 805	.99 834	**60**
1	.94 174	.94 340	.05 660	.99 833	59
2	.94 317	.94 485	.05 515	.99 832	58
3	.94 461	.94 630	.05 370	.99 831	57
4	.94 603	.94 773	.05 227	.99 830	56
5	.94 746	.94 917	.05 083	.99 829	55
6	.94 887	.95 060	.04 940	.99 828	54
7	.95 029	.95 202	.04 798	.99 827	53
8	.95 170	.95 344	.04 656	.99 825	52
9	.95 310	.95 486	.04 514	.99 824	51
10	.95 450	.95 627	.04 373	.99 823	**50**
11	.95 589	.95 767	.04 233	.99 822	49
12	.95 728	.95 908	.04 092	.99 821	48
13	.95 867	.96 047	.03 953	.99 820	47
14	.96 005	.96 187	.03 813	.99 819	46
15	.96 143	.96 325	.03 675	.99 817	45
16	.96 280	.96 464	.03 536	.99 816	44
17	.96 417	.96 602	.03 398	.99 815	43
18	.96 553	.96 739	.03 261	.99 814	42
19	.96 689	.96 877	.03 123	.99 813	41
20	.96 825	.97 013	.02 987	.99 812	**40**
21	.96 960	.97 150	.02 850	.99 810	39
22	.97 095	.97 285	.02 715	.99 809	38
23	.97 229	.97 421	.02 579	.99 808	37
24	.97 363	.97 556	.02 444	.99 807	36
25	.97 496	.97 691	.02 309	.99 806	35
26	.97 629	.97 825	.02 175	.99 804	34
27	.97 762	.97 959	.02 041	.99 803	33
28	.97 894	.98 092	.01 908	.99 802	32
29	.98 026	.98 225	.01 775	.99 801	31
30	.98 157	.98 358	.01 642	.99 800	**30**
31	.98 288	.98 490	.01 510	.99 798	29
32	.98 419	.98 622	.01 378	.99 797	28
33	.98 549	.98 753	.01 247	.99 796	27
34	.98 679	.98 884	.01 116	.99 795	26
35	.98 808	.99 015	.00 985	.99 793	25
36	.98 937	.99 145	.00 855	.99 792	24
37	.99 066	.99 275	.00 725	.99 791	23
38	.99 194	.99 405	.00 595	.99 790	22
39	.99 322	.99 534	.00 466	.99 788	21
40	.99 450	.99 662	.00 338	.99 787	**20**
41	.99 577	.99 791	.00 209	.99 786	19
42	.99 704	.99 919	.00 081	.99 785	18
43	.99 830	.00 046	.99 954	.99 783	17
44	.99 956	.00 174	.99 826	.99 782	16
45	.00 082	.00 301	.99 699	.99 781	15
46	.00 207	.00 427	.99 573	.99 780	14
47	.00 332	.00 553	.99 447	.99 778	13
48	.00 456	.00 679	.99 321	.99 777	12
49	.00 581	.00 805	.99 195	.99 776	11
50	.00 704	.00 930	.99 070	.99 775	**10**
51	.00 828	.01 055	.98 945	.99 773	9
52	.00 951	.01 179	.98 821	.99 772	8
53	.01 074	.01 303	.98 697	.99 771	7
54	.01 196	.01 427	.98 573	.99 769	6
55	.01 318	.01 550	.98 450	.99 768	5
56	.01 440	.01 673	.98 327	.99 767	4
57	.01 561	.01 796	.98 204	.99 765	3
58	.01 682	.01 918	.98 082	.99 764	2
59	.01 803	.02 040	.97 960	.99 763	1
60	.01 923	.02 162	.97 838	.99 761	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

84°

6°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.01 923	.02 162	.97 838	.99 761	**60**
1	.02 043	.02 283	.97 717	.99 760	59
2	.02 163	.02 404	.97 596	.99 759	58
3	.02 283	.02 525	.97 475	.99 757	57
4	.02 402	.02 645	.97 355	.99 756	56
5	.02 520	.02 766	.97 234	.99 755	55
6	.02 639	.02 885	.97 115	.99 753	54
7	.02 757	.03 005	.96 995	.99 752	53
8	.02 874	.03 124	.96 876	.99 751	52
9	.02 992	.03 242	.96 758	.99 749	51
10	.03 109	.03 361	.96 639	.99 748	**50**
11	.03 226	.03 479	.96 521	.99 747	49
12	.03 342	.03 597	.96 403	.99 745	48
13	.03 458	.03 714	.96 286	.99 744	47
14	.03 574	.03 832	.96 168	.99 742	46
15	.03 690	.03 948	.96 052	.99 741	45
16	.03 805	.04 065	.95 935	.99 740	44
17	.03 920	.04 181	.95 819	.99 738	43
18	.04 034	.04 297	.95 703	.99 737	42
19	.04 149	.04 413	.95 587	.99 736	41
20	.04 262	.04 528	.95 472	.99 734	**40**
21	.04 376	.04 643	.95 357	.99 733	39
22	.04 490	.04 758	.95 242	.99 731	38
23	.04 603	.04 873	.95 127	.99 730	37
24	.04 715	.04 987	.95 013	.99 728	36
25	.04 828	.05 101	.94 899	.99 727	35
26	.04 940	.05 214	.94 786	.99 726	34
27	.05 052	.05 328	.94 672	.99 724	33
28	.05 164	.05 441	.94 559	.99 723	32
29	.05 275	.05 553	.94 447	.99 721	31
30	.05 386	.05 666	.94 334	.99 720	**30**
31	.05 497	.05 778	.94 222	.99 718	29
32	.05 607	.05 890	.94 110	.99 717	28
33	.05 717	.06 002	.93 998	.99 716	27
34	.05 827	.06 113	.93 887	.99 714	26
35	.05 937	.06 224	.93 776	.99 713	25
36	.06 046	.06 335	.93 665	.99 711	24
37	.06 155	.06 445	.93 555	.99 710	23
38	.06 264	.06 556	.93 444	.99 708	22
39	.06 372	.06 666	.93 334	.99 707	21
40	.06 481	.06 775	.93 225	.99 705	**20**
41	.06 589	.06 885	.93 115	.99 704	19
42	.06 696	.06 994	.93 006	.99 702	18
43	.06 804	.07 103	.92 897	.99 701	17
44	.06 911	.07 211	.92 789	.99 699	16
45	.07 018	.07 320	.92 680	.99 698	15
46	.07 124	.07 428	.92 572	.99 696	14
47	.07 231	.07 536	.92 464	.99 695	13
48	.07 337	.07 643	.92 357	.99 693	12
49	.07 442	.07 751	.92 249	.99 692	11
50	.07 548	.07 858	.92 142	.99 690	**10**
51	.07 653	.07 964	.92 036	.99 689	9
52	.07 758	.08 071	.91 929	.99 687	8
53	.07 863	.08 177	.91 823	.99 686	7
54	.07 968	.08 283	.91 717	.99 684	6
55	.08 072	.08 389	.91 611	.99 683	5
56	.08 176	.08 495	.91 505	.99 681	4
57	.08 280	.08 600	.91 400	.99 680	3
58	.08 383	.08 705	.91 295	.99 678	2
59	.08 486	.08 810	.91 190	.99 677	1
60	.08 589	.08 914	.91 086	.99 675	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

83°

7°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.08 589	.08 914	.91 086	.99 675	**60**
1	.08 692	.09 019	.90 981	.99 674	59
2	.08 795	.09 123	.90 877	.99 672	58
3	.08 897	.09 227	.90 773	.99 670	57
4	.08 999	.09 330	.90 670	.99 669	56
5	.09 101	.09 434	.90 566	.99 667	55
6	.09 202	.09 537	.90 463	.99 666	54
7	.09 304	.09 640	.90 360	.99 664	53
8	.09 405	.09 742	.90 258	.99 663	52
9	.09 506	.09 845	.90 155	.99 661	51
10	.09 606	.09 947	.90 053	.99 659	**50**
11	.09 707	.10 049	.89 951	.99 658	49
12	.09 807	.10 150	.89 850	.99 656	48
13	.09 907	.10 252	.89 748	.99 655	47
14	.10 006	.10 353	.89 647	.99 653	46
15	.10 106	.10 454	.89 546	.99 651	45
16	.10 205	.10 555	.89 445	.99 650	44
17	.10 304	.10 656	.89 344	.99 648	43
18	.10 402	.10 756	.89 244	.99 647	42
19	.10 501	.10 856	.89 144	.99 645	41
20	.10 599	.10 956	.89 044	.99 643	**40**
21	.10 697	.11 056	.88 944	.99 642	39
22	.10 795	.11 155	.88 845	.99 640	38
23	.10 893	.11 254	.88 746	.99 638	37
24	.10 990	.11 353	.88 647	.99 637	36
25	.11 087	.11 452	.88 548	.99 635	35
26	.11 184	.11 551	.88 449	.99 633	34
27	.11 281	.11 649	.88 351	.99 632	33
28	.11 377	.11 747	.88 253	.99 630	32
29	.11 474	.11 845	.88 155	.99 629	31
30	.11 570	.11 943	.88 057	.99 627	**30**
31	.11 666	.12 040	.87 960	.99 625	29
32	.11 761	.12 138	.87 862	.99 624	28
33	.11 857	.12 235	.87 765	.99 622	27
34	.11 952	.12 332	.87 668	.99 620	26
35	.12 047	.12 428	.87 572	.99 618	25
36	.12 142	.12 525	.87 475	.99 617	24
37	.12 236	.12 621	.87 379	.99 615	23
38	.12 331	.12 717	.87 283	.99 613	22
39	.12 425	.12 813	.87 187	.99 612	21
40	.12 519	.12 909	.87 091	.99 610	**20**
41	.12 612	.13 004	.86 996	.99 608	19
42	.12 706	.13 099	.86 901	.99 607	18
43	.12 799	.13 194	.86 806	.99 605	17
44	.12 892	.13 289	.86 711	.99 603	16
45	.12 985	.13 384	.86 616	.99 601	15
46	.13 078	.13 478	.86 522	.99 600	14
47	.13 171	.13 573	.86 427	.99 598	13
48	.13 263	.13 667	.86 333	.99 596	12
49	.13 355	.13 761	.86 239	.99 595	11
50	.13 447	.13 854	.86 146	.99 593	**10**
51	.13 539	.13 948	.86 052	.99 591	9
52	.13 630	.14 041	.85 959	.99 589	8
53	.13 722	.14 134	.85 866	.99 588	7
54	.13 813	.14 227	.85 773	.99 586	6
55	.13 904	.14 320	.85 680	.99 584	5
56	.13 994	.14 412	.85 588	.99 582	4
57	.14 085	.14 504	.85 496	.99 581	3
58	.14 175	.14 597	.85 403	.99 579	2
59	.14 266	.14 688	.85 312	.99 577	1
60	.14 356	.14 780	.85 220	.99 575	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

82°

8°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.14 356	.14 780	.85 220	.99 575	**60**
1	.14 445	.14 872	.85 128	.99 574	59
2	.14 535	.14 963	.85 037	.99 572	58
3	.14 624	.15 054	.84 946	.99 570	57
4	.14 714	.15 145	.84 855	.99 568	56
5	.14 803	.15 236	.84 764	.99 566	55
6	.14 891	.15 327	.84 673	.99 565	54
7	.14 980	.15 417	.84 583	.99 563	53
8	.15 069	.15 508	.84 492	.99 561	52
9	.15 157	.15 598	.84 402	.99 559	51
10	.15 245	.15 688	.84 312	.99 557	**50**
11	.15 333	.15 777	.84 223	.99 556	49
12	.15 421	.15 867	.84 133	.99 554	48
13	.15 508	.15 956	.84 044	.99 552	47
14	.15 596	.16 046	.83 954	.99 550	46
15	.15 683	.16 135	.83 865	.99 548	45
16	.15 770	.16 224	.83 776	.99 546	44
17	.15 857	.16 312	.83 688	.99 545	43
18	.15 944	.16 401	.83 599	.99 543	42
19	.16 030	.16 489	.83 511	.99 541	41
20	.16 116	.16 577	.83 423	.99 539	**40**
21	.16 203	.16 665	.83 335	.99 537	39
22	.16 289	.16 753	.83 247	.99 535	38
23	.16 374	.16 841	.83 159	.99 533	37
24	.16 460	.16 928	.83 072	.99 532	36
25	.16 545	.17 016	.82 984	.99 530	35
26	.16 631	.17 103	.82 897	.99 528	34
27	.16 716	.17 190	.82 810	.99 526	33
28	.16 801	.17 277	.82 723	.99 524	32
29	.16 886	.17 363	.82 637	.99 522	31
30	.16 970	.17 450	.82 550	.99 520	**30**
31	.17 055	.17 536	.82 464	.99 518	29
32	.17 139	.17 622	.82 378	.99 517	28
33	.17 223	.17 708	.82 292	.99 515	27
34	.17 307	.17 794	.82 206	.99 513	26
35	.17 391	.17 880	.82 120	.99 511	25
36	.17 474	.17 965	.82 035	.99 509	24
37	.17 558	.18 051	.81 949	.99 507	23
38	.17 641	.18 136	.81 864	.99 505	22
39	.17 724	.18 221	.81 779	.99 503	21
40	.17 807	.18 306	.81 694	.99 501	**20**
41	.17 890	.18 391	.81 609	.99 499	19
42	.17 973	.18 475	.81 525	.99 497	18
43	.18 055	.18 560	.81 440	.99 495	17
44	.18 137	.18 644	.81 356	.99 494	16
45	.18 220	.18 728	.81 272	.99 492	15
46	.18 302	.18 812	.81 188	.99 490	14
47	.18 383	.18 896	.81 104	.99 488	13
48	.18 465	.18 979	.81 021	.99 486	12
49	.18 547	.19 063	.80 937	.99 484	11
50	.18 628	.19 146	.80 854	.99 482	**10**
51	.18 709	.19 229	.80 771	.99 480	9
52	.18 790	.19 312	.80 688	.99 478	8
53	.18 871	.19 395	.80 605	.99 476	7
54	.18 952	.19 478	.80 522	.99 474	6
55	.19 033	.19 561	.80 439	.99 472	5
56	.19 113	.19 643	.80 357	.99 470	4
57	.19 193	.19 725	.80 275	.99 468	3
58	.19 273	.19 807	.80 193	.99 466	2
59	.19 353	.19 889	.80 111	.99 464	1
60	.19 433	.19 971	.80 029	.99 462	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

81°

9°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.19 433	.19 971	.80 029	.99 462	**60**
1	.19 513	.20 053	.79 947	.99 460	59
2	.19 592	.20 134	.79 866	.99 458	58
3	.19 672	.20 216	.79 784	.99 456	57
4	.19 751	.20 297	.79 703	.99 454	56
5	.19 830	.20 378	.79 622	.99 452	55
6	.19 909	.20 459	.79 541	.99 450	54
7	.19 988	.20 540	.79 460	.99 448	53
8	.20 067	.20 621	.79 379	.99 446	52
9	.20 145	.20 701	.79 299	.99 444	51
10	.20 223	.20 782	.79 218	.99 442	**50**
11	.20 302	.20 862	.79 138	.99 440	49
12	.20 380	.20 942	.79 058	.99 438	48
13	.20 458	.21 022	.78 978	.99 436	47
14	.20 535	.21 102	.78 898	.99 434	46
15	.20 613	.21 182	.78 818	.99 432	45
16	.20 691	.21 261	.78 739	.99 429	44
17	.20 768	.21 341	.78 659	.99 427	43
18	.20 845	.21 420	.78 580	.99 425	42
19	.20 922	.21 499	.78 501	.99 423	41
20	.20 999	.21 578	.78 422	.99 421	**40**
21	.21 076	.21 657	.78 343	.99 419	39
22	.21 153	.21 736	.78 264	.99 417	38
23	.21 229	.21 814	.78 186	.99 415	37
24	.21 306	.21 893	.78 107	.99 413	36
25	.21 382	.21 971	.78 029	.99 411	35
26	.21 458	.22 049	.77 951	.99 409	34
27	.21 534	.22 127	.77 873	.99 407	33
28	.21 610	.22 205	.77 795	.99 404	32
29	.21 685	.22 283	.77 717	.99 402	31
30	.21 761	.22 361	.77 639	.99 400	**30**
31	.21 836	.22 438	.77 562	.99 398	29
32	.21 912	.22 516	.77 484	.99 396	28
33	.21 987	.22 593	.77 407	.99 394	27
34	.22 062	.22 670	.77 330	.99 392	26
35	.22 137	.22 747	.77 253	.99 390	25
36	.22 211	.22 824	.77 176	.99 388	24
37	.22 286	.22 901	.77 099	.99 385	23
38	.22 361	.22 977	.77 023	.99 383	22
39	.22 435	.23 054	.76 946	.99 381	21
40	.22 509	.23 130	.76 870	.99 379	**20**
41	.22 583	.23 206	.76 794	.99 377	19
42	.22 657	.23 283	.76 717	.99 375	18
43	.22 731	.23 359	.76 641	.99 372	17
44	.22 805	.23 435	.76 565	.99 370	16
45	.22 878	.23 510	.76 490	.99 368	15
46	.22 952	.23 586	.76 414	.99 366	14
47	.23 025	.23 661	.76 339	.99 364	13
48	.23 098	.23 737	.76 263	.99 362	12
49	.23 171	.23 812	.76 188	.99 359	11
50	.23 244	.23 887	.76 113	.99 357	**10**
51	.23 317	.23 962	.76 038	.99 355	9
52	.23 390	.24 037	.75 963	.99 353	8
53	.23 462	.24 112	.75 888	.99 351	7
54	.23 535	.24 186	.75 814	.99 348	6
55	.23 607	.24 261	.75 739	.99 346	5
56	.23 679	.24 335	.75 665	.99 344	4
57	.23 752	.24 410	.75 590	.99 342	3
58	.23 823	.24 484	.75 516	.99 340	2
59	.23 895	.24 558	.75 442	.99 337	1
60	.23 967	.24 632	.75 368	.99 335	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

80°

10°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.23 967	.24 632	.75 368	.99 335	**60**
1	.24 039	.24 706	.75 294	.99 333	59
2	.24 110	.24 779	.75 221	.99 331	58
3	.24 181	.24 853	.75 147	.99 328	57
4	.24 253	.24 926	.75 074	.99 326	56
5	.24 324	.25 000	.75 000	.99 324	55
6	.24 395	.25 073	.74 927	.99 322	54
7	.24 466	.25 146	.74 854	.99 319	53
8	.24 536	.25 219	.74 781	.99 317	52
9	.24 607	.25 292	.74 708	.99 315	51
10	.24 677	.25 365	.74 635	.99 313	**50**
11	.24 748	.25 437	.74 563	.99 310	49
12	.24 818	.25 510	.74 490	.99 308	48
13	.24 888	.25 582	.74 418	.99 306	47
14	.24 958	.25 655	.74 345	.99 304	46
15	.25 028	.25 727	.74 273	.99 301	45
16	.25 098	.25 799	.74 201	.99 299	44
17	.25 168	.25 871	.74 129	.99 297	43
18	.25 237	.25 943	.74 057	.99 294	42
19	.25 307	.26 015	.73 985	.99 292	41
20	.25 376	.26 086	.73 914	.99 290	**40**
21	.25 445	.26 158	.73 842	.99 288	39
22	.25 514	.26 229	.73 771	.99 285	38
23	.25 583	.26 301	.73 699	.99 283	37
24	.25 652	.26 372	.73 628	.99 281	36
25	.25 721	.26 443	.73 557	.99 278	35
26	.25 790	.26 514	.73 486	.99 276	34
27	.25 858	.26 585	.73 415	.99 274	33
28	.25 927	.26 655	.73 345	.99 271	32
29	.25 995	.26 726	.73 274	.99 269	31
30	.26 063	.26 797	.73 203	.99 267	**30**
31	.26 131	.26 867	.73 133	.99 264	29
32	.26 199	.26 937	.73 063	.99 262	28
33	.26 267	.27 008	.72 992	.99 260	27
34	.26 335	.27 078	.72 922	.99 257	26
35	.26 403	.27 148	.72 852	.99 255	25
36	.26 470	.27 218	.72 782	.99 252	24
37	.26 538	.27 288	.72 712	.99 250	23
38	.26 605	.27 357	.72 643	.99 248	22
39	.26 672	.27 427	.72 573	.99 245	21
40	.26 739	.27 496	.72 504	.99 243	**20**
41	.26 806	.27 566	.72 434	.99 241	19
42	.26 873	.27 635	.72 365	.99 238	18
43	.26 940	.27 704	.72 296	.99 236	17
44	.27 007	.27 773	.72 227	.99 233	16
45	.27 073	.27 842	.72 158	.99 231	15
46	.27 140	.27 911	.72 089	.99 229	14
47	.27 206	.27 980	.72 020	.99 226	13
48	.27 273	.28 049	.71 951	.99 224	12
49	.27 339	.28 117	.71 883	.99 221	11
50	.27 405	.28 186	.71 814	.99 219	**10**
51	.27 471	.28 254	.71 746	.99 217	9
52	.27 537	.28 323	.71 677	.99 214	8
53	.27 602	.28 391	.71 609	.99 212	7
54	.27 668	.28 459	.71 541	.99 209	6
55	.27 734	.28 527	.71 473	.99 207	5
56	.27 799	.28 595	.71 405	.99 204	4
57	.27 864	.28 662	.71 338	.99 202	3
58	.27 930	.28 730	.71 270	.99 200	2
59	.27 995	.28 798	.71 202	.99 197	1
60	.28 060	.28 865	.71 135	.99 195	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

79°

11°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.28 060	.28 865	.71 135	.99 195	**60**
1	.28 125	.28 933	.71 067	.99 192	59
2	.28 190	.29 000	.71 000	.99 190	58
3	.28 254	.29 067	.70 933	.99 187	57
4	.28 319	.29 134	.70 866	.99 185	56
5	.28 384	.29 201	.70 799	.99 182	55
6	.28 448	.29 268	.70 732	.99 180	54
7	.28 512	.29 335	.70 665	.99 177	53
8	.28 577	.29 402	.70 598	.99 175	52
9	.28 641	.29 468	.70 532	.99 172	51
10	.28 705	.29 535	.70 465	.99 170	**50**
11	.28 769	.29 601	.70 399	.99 167	49
12	.28 833	.29 668	.70 332	.99 165	48
13	.28 896	.29 734	.70 266	.99 162	47
14	.28 960	.29 800	.70 200	.99 160	46
15	.29 024	.29 866	.70 134	.99 157	45
16	.29 087	.29 932	.70 068	.99 155	44
17	.29 150	.29 998	.70 002	.99 152	43
18	.29 214	.30 064	.69 936	.99 150	42
19	.29 277	.30 130	.69 870	.99 147	41
20	.29 340	.30 195	.69 805	.99 145	**40**
21	.29 403	.30 261	.69 739	.99 142	39
22	.29 466	.30 326	.69 674	.99 140	38
23	.29 529	.30 391	.69 609	.99 137	37
24	.29 591	.30 457	.69 543	.99 135	36
25	.29 654	.30 522	.69 478	.99 132	35
26	.29 716	.30 587	.69 413	.99 130	34
27	.29 779	.30 652	.69 348	.99 127	33
28	.29 841	.30 717	.69 283	.99 124	32
29	.29 903	.30 782	.69 218	.99 122	31
30	.29 966	.30 846	.69 154	.99 119	**30**
31	.30 028	.30 911	.69 089	.99 117	29
32	.30 090	.30 975	.69 025	.99 114	28
33	.30 151	.31 040	.68 960	.99 112	27
34	.30 213	.31 104	.68 896	.99 109	26
35	.30 275	.31 168	.68 832	.99 106	25
36	.30 336	.31 233	.68 767	.99 104	24
37	.30 398	.31 297	.68 703	.99 101	23
38	.30 459	.31 361	.68 639	.99 099	22
39	.30 521	.31 425	.68 575	.99 096	21
40	.30 582	.31 489	.68 511	.99 093	**20**
41	.30 643	.31 552	.68 448	.99 091	19
42	.30 704	.31 616	.68 384	.99 088	18
43	.30 765	.31 679	.68 321	.99 086	17
44	.30 826	.31 743	.68 257	.99 083	16
45	.30 887	.31 806	.68 194	.99 080	15
46	.30 947	.31 870	.68 130	.99 078	14
47	.31 008	.31 933	.68 067	.99 075	13
48	.31 068	.31 996	.68 004	.99 072	12
49	.31 129	.32 059	.67 941	.99 070	11
50	.31 189	.32 122	.67 878	.99 067	**10**
51	.31 250	.32 185	.67 815	.99 064	9
52	.31 310	.32 248	.67 752	.99 062	8
53	.31 370	.32 311	.67 689	.99 059	7
54	.31 430	.32 373	.67 627	.99 056	6
55	.31 490	.32 436	.67 564	.99 054	5
56	.31 549	.32 498	.67 502	.99 051	4
57	.31 609	.32 561	.67 439	.99 048	3
58	.31 669	.32 623	.67 377	.99 046	2
59	.31 728	.32 685	.67 315	.99 043	1
60	.31 788	.32 747	.67 253	.99 040	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

78°

12°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.31 788	.32 747	.67 253	.99 040	**60**
1	.31 847	.32 810	.67 190	.99 038	59
2	.31 907	.32 872	.67 128	.99 035	58
3	.31 966	.32 933	.67 067	.99 032	57
4	.32 025	.32 995	.67 005	.99 030	56
5	.32 084	.33 057	.66 943	.99 027	55
6	.32 143	.33 119	.66 881	.99 024	54
7	.32 202	.33 180	.66 820	.99 022	53
8	.32 261	.33 242	.66 758	.99 019	52
9	.32 319	.33 303	.66 697	.99 016	51
10	.32 378	.33 365	.66 635	.99 013	**50**
11	.32 437	.33 426	.66 574	.99 011	49
12	.32 495	.33 487	.66 513	.99 008	48
13	.32 553	.33 548	.66 452	.99 005	47
14	.32 612	.33 609	.66 391	.99 002	46
15	.32 670	.33 670	.66 330	.99 000	45
16	.32 728	.33 731	.66 269	.98 997	44
17	.32 786	.33 792	.66 208	.98 994	43
18	.32 844	.33 853	.66 147	.98 991	42
19	.32 902	.33 913	.66 087	.98 989	41
20	.32 960	.33 974	.66 026	.98 986	**40**
21	.33 018	.34 034	.65 966	.98 983	39
22	.33 075	.34 095	.65 905	.98 980	38
23	.33 133	.34 155	.65 845	.98 978	37
24	.33 190	.34 215	.65 785	.98 975	36
25	.33 248	.34 276	.65 724	.98 972	35
26	.33 305	.34 336	.65 664	.98 969	34
27	.33 362	.34 396	.65 604	.98 967	33
28	.33 420	.34 456	.65 544	.98 964	32
29	.33 477	.34 516	.65 484	.98 961	31
30	.33 534	.34 576	.65 424	.98 958	**30**
31	.33 591	.34 635	.65 365	.98 955	29
32	.33 647	.34 695	.65 305	.98 953	28
33	.33 704	.34 755	.65 245	.98 950	27
34	.33 761	.34 814	.65 186	.98 947	26
35	.33 818	.34 874	.65 126	.98 944	25
36	.33 874	.34 933	.65 067	.98 941	24
37	.33 931	.34 992	.65 008	.98 938	23
38	.33 987	.35 051	.64 949	.98 936	22
39	.34 043	.35 111	.64 889	.98 933	21
40	.34 100	.35 170	.64 830	.98 930	**20**
41	.34 156	.35 229	.64 771	.98 927	19
42	.34 212	.35 288	.64 712	.98 924	18
43	.34 268	.35 347	.64 653	.98 921	17
44	.34 324	.35 405	.64 595	.98 919	16
45	.34 380	.35 464	.64 536	.98 916	15
46	.34 436	.35 523	.64 477	.98 913	14
47	.34 491	.35 581	.64 419	.98 910	13
48	.34 547	.35 640	.64 360	.98 907	12
49	.34 602	.35 698	.64 302	.98 904	11
50	.34 658	.35 757	.64 243	.98 901	**10**
51	.34 713	.35 815	.64 185	.98 898	9
52	.34 769	.35 873	.64 127	.98 896	8
53	.34 824	.35 931	.64 069	.98 893	7
54	.34 879	.35 989	.64 011	.98 890	6
55	.34 934	.36 047	.63 953	.98 887	5
56	.34 989	.36 105	.63 895	.98 884	4
57	.35 044	.36 163	.63 837	.98 881	3
58	.35 099	.36 221	.63 779	.98 878	2
59	.35 154	.36 279	.63 721	.98 875	1
60	.35 209	.36 336	.63 664	.98 872	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

77°

13°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.35 209	.36 336	.63 664	.98 872	**60**
1	.35 263	.36 394	.63 606	.98 869	59
2	.35 318	.36 452	.63 548	.98 867	58
3	.35 373	.36 509	.63 491	.98 864	57
4	.35 427	.36 566	.63 434	.98 861	56
5	.35 481	.36 624	.63 376	.98 858	55
6	.35 536	.36 681	.63 319	.98 855	54
7	.35 590	.36 738	.63 262	.98 852	53
8	.35 644	.36 795	.63 205	.98 849	52
9	.35 698	.36 852	.63 148	.98 846	51
10	.35 752	.36 909	.63 091	.98 843	**50**
11	.35 806	.36 966	.63 034	.98 840	49
12	.35 860	.37 023	.62 977	.98 837	48
13	.35 914	.37 080	.62 920	.98 834	47
14	.35 968	.37 137	.62 863	.98 831	46
15	.36 022	.37 193	.62 807	.98 828	45
16	.36 075	.37 250	.62 750	.98 825	44
17	.36 129	.37 306	.62 694	.98 822	43
18	.36 182	.37 363	.62 637	.98 819	42
19	.36 236	.37 419	.62 581	.98 816	41
20	.36 289	.37 476	.62 524	.98 813	**40**
21	.36 342	.37 532	.62 468	.98 810	39
22	.36 395	.37 588	.62 412	.98 807	38
23	.36 449	.37 644	.62 356	.98 804	37
24	.36 502	.37 700	.62 300	.98 801	36
25	.36 555	.37 756	.62 244	.98 798	35
26	.36 608	.37 812	.62 188	.98 795	34
27	.36 660	.37 868	.62 132	.98 792	33
28	.36 713	.37 924	.62 076	.98 789	32
29	.36 766	.37 980	.62 020	.98 786	31
30	.36 819	.38 035	.61 965	.98 783	**30**
31	.36 871	.38 091	.61 909	.98 780	29
32	.36 924	.38 147	.61 853	.98 777	28
33	.36 976	.38 202	.61 798	.98 774	27
34	.37 028	.38 257	.61 743	.98 771	26
35	.37 081	.38 313	.61 687	.98 768	25
36	.37 133	.38 368	.61 632	.98 765	24
37	.37 185	.38 423	.61 577	.98 762	23
38	.37 237	.38 479	.61 521	.98 759	22
39	.37 289	.38 534	.61 466	.98 756	21
40	.37 341	.38 589	.61 411	.98 753	**20**
41	.37 393	.38 644	.61 356	.98 750	19
42	.37 445	.38 699	.61 301	.98 746	18
43	.37 497	.38 754	.61 246	.98 743	17
44	.37 549	.38 808	.61 192	.98 740	16
45	.37 600	.38 863	.61 137	.98 737	15
46	.37 652	.38 918	.61 082	.98 734	14
47	.37 703	.38 972	.61 028	.98 731	13
48	.37 755	.39 027	.60 973	.98 728	12
49	.37 806	.39 082	.60 918	.98 725	11
50	.37 858	.39 136	.60 864	.98 722	**10**
51	.37 909	.39 190	.60 810	.98 719	9
52	.37 960	.39 245	.60 755	.98 715	8
53	.38 011	.39 299	.60 701	.98 712	7
54	.38 062	.39 353	.60 647	.98 709	6
55	.38 113	.39 407	.60 593	.98 706	5
56	.38 164	.39 461	.60 539	.98 703	4
57	.38 215	.39 515	.60 485	.98 700	3
58	.38 266	.39 569	.60 431	.98 697	2
59	.38 317	.39 623	.60 377	.98 694	1
60	.38 368	.39 677	.60 323	.98 690	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

76°

14°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.38 368	.39 677	.60 323	.98 690	**60**
1	.38 418	.39 731	.60 269	.98 687	59
2	.38 469	.39 785	.60 215	.98 684	58
3	.38 519	.39 838	.60 162	.98 681	57
4	.38 570	.39 892	.60 108	.98 678	56
5	.38 620	.39 945	.60 055	.98 675	55
6	.38 670	.39 999	.60 001	.98 671	54
7	.38 721	.40 052	.59 948	.98 668	53
8	.38 771	.40 106	.59 894	.98 665	52
9	.38 821	.40 159	.59 841	.98 662	51
10	.38 871	.40 212	.59 788	.98 659	**50**
11	.38 921	.40 266	.59 734	.98 656	49
12	.38 971	.40 319	.59 681	.98 652	48
13	.39 021	.40 372	.59 628	.98 649	47
14	.39 071	.40 425	.59 575	.98 646	46
15	.39 121	.40 478	.59 522	.98 643	45
16	.39 170	.40 531	.59 469	.98 640	44
17	.39 220	.40 584	.59 416	.98 636	43
18	.39 270	.40 636	.59 364	.98 633	42
19	.39 319	.40 689	.59 311	.98 630	41
20	.39 369	.40 742	.59 258	.98 627	**40**
21	.39 418	.40 795	.59 205	.98 623	39
22	.39 467	.40 847	.59 153	.98 620	38
23	.39 517	.40 900	.59 100	.98 617	37
24	.39 566	.40 952	.59 048	.98 614	36
25	.39 615	.41 005	.58 995	.98 610	35
26	.39 664	.41 057	.58 943	.98 607	34
27	.39 713	.41 109	.58 891	.98 604	33
28	.39 762	.41 161	.58 839	.98 601	32
29	.39 811	.41 214	.58 786	.98 597	31
30	.39 860	.41 266	.58 734	.98 594	**30**
31	.39 909	.41 318	.58 682	.98 591	29
32	.39 958	.41 370	.58 630	.98 588	28
33	.40 006	.41 422	.58 578	.98 584	27
34	.40 055	.41 474	.58 526	.98 581	26
35	.40 103	.41 526	.58 474	.98 578	25
36	.40 152	.41 578	.58 422	.98 574	24
37	.40 200	.41 629	.58 371	.98 571	23
38	.40 249	.41 681	.58 319	.98 568	22
39	.40 297	.41 733	.58 267	.98 565	21
40	.40 346	.41 784	.58 216	.98 561	**20**
41	.40 394	.41 836	.58 164	.98 558	19
42	.40 442	.41 887	.58 113	.98 555	18
43	.40 490	.41 939	.58 061	.98 551	17
44	.40 538	.41 990	.58 010	.98 548	16
45	.40 586	.42 041	.57 959	.98 545	15
46	.40 634	.42 093	.57 907	.98 541	14
47	.40 682	.42 144	.57 856	.98 538	13
48	.40 730	.42 195	.57 805	.98 535	12
49	.40 778	.42 246	.57 754	.98 531	11
50	.40 825	.42 297	.57 703	.98 528	**10**
51	.40 873	.42 348	.57 652	.98 525	9
52	.40 921	.42 399	.57 601	.98 521	8
53	.40 968	.42 450	.57 550	.98 518	7
54	.41 016	.42 501	.57 499	.98 515	6
55	.41 063	.42 552	.57 448	.98 511	5
56	.41 111	.42 603	.57 397	.98 508	4
57	.41 158	.42 653	.57 347	.98 505	3
58	.41 205	.42 704	.57 296	.98 501	2
59	.41 252	.42 755	.57 245	.98 498	1
60	.41 300	.42 805	.57 195	.98 494	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

75°

15°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.41 300	.42 805	.57 195	.98 494	**60**
1	.41 347	.42 856	.57 144	.98 491	59
2	.41 394	.42 906	.57 094	.98 488	58
3	.41 441	.42 957	.57 043	.98 484	57
4	.41 488	.43 007	.56 993	.98 481	56
5	.41 535	.43 057	.56 943	.98 477	55
6	.41 582	.43 108	.56 892	.98 474	54
7	.41 628	.43 158	.56 842	.98 471	53
8	.41 675	.43 208	.56 792	.98 467	52
9	.41 722	.43 258	.56 742	.98 464	51
10	.41 768	.43 308	.56 692	.98 460	**50**
11	.41 815	.43 358	.56 642	.98 457	49
12	.41 861	.43 408	.56 592	.98 453	48
13	.41 908	.43 458	.56 542	.98 450	47
14	.41 954	.43 508	.56 492	.98 447	46
15	.42 001	.43 558	.56 442	.98 443	45
16	.42 047	.43 607	.56 393	.98 440	44
17	.42 093	.43 657	.56 343	.98 436	43
18	.42 140	.43 707	.56 293	.98 433	42
19	.42 186	.43 756	.56 244	.98 429	41
20	.42 232	.43 806	.56 194	.98 426	**40**
21	.42 278	.43 855	.56 145	.98 422	39
22	.42 324	.43 905	.56 095	.98 419	38
23	.42 370	.43 954	.56 046	.98 415	37
24	.42 416	.44 004	.55 996	.98 412	36
25	.42 461	.44 053	.55 947	.98 409	35
26	.42 507	.44 102	.55 898	.98 405	34
27	.42 553	.44 151	.55 849	.98 402	33
28	.42 599	.44 201	.55 799	.98 398	32
29	.42 644	.44 250	.55 750	.98 395	31
30	.42 690	.44 299	.55 701	.98 391	**30**
31	.42 735	.44 348	.55 652	.98 388	29
32	.42 781	.44 397	.55 603	.98 384	28
33	.42 826	.44 446	.55 554	.98 381	27
34	.42 872	.44 495	.55 505	.98 377	26
35	.42 917	.44 544	.55 456	.98 373	25
36	.42 962	.44 592	.55 408	.98 370	24
37	.43 008	.44 641	.55 359	.98 366	23
38	.43 053	.44 690	.55 310	.98 363	22
39	.43 098	.44 738	.55 262	.98 359	21
40	.43 143	.44 787	.55 213	.98 356	**20**
41	.43 188	.44 836	.55 164	.98 352	19
42	.43 233	.44 884	.55 116	.98 349	18
43	.43 278	.44 933	.55 067	.98 345	17
44	.43 323	.44 981	.55 019	.98 342	16
45	.43 367	.45 029	.54 971	.98 338	15
46	.43 412	.45 078	.54 922	.98 334	14
47	.43 457	.45 126	.54 874	.98 331	13
48	.43 502	.45 174	.54 826	.98 327	12
49	.43 546	.45 222	.54 778	.98 324	11
50	.43 591	.45 271	.54 729	.98 320	**10**
51	.43 635	.45 319	.54 681	.98 317	9
52	.43 680	.45 367	.54 633	.98 313	8
53	.43 724	.45 415	.54 585	.98 309	7
54	.43 769	.45 463	.54 537	.98 306	6
55	.43 813	.45 511	.54 489	.98 302	5
56	.43 857	.45 559	.54 441	.98 299	4
57	.43 901	.45 606	.54 394	.98 295	3
58	.43 946	.45 654	.54 346	.98 291	2
59	.43 990	.45 702	.54 298	.98 288	1
60	.44 034	.45 750	.54 250	.98 284	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

74°

16°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.44 034	.45 750	.54 250	.98 284	**60**
1	.44 078	.45 797	.54 203	.98 281	59
2	.44 122	.45 845	.54 155	.98 277	58
3	.44 166	.45 892	.54 108	.98 273	57
4	.44 210	.45 940	.54 060	.98 270	56
5	.44 253	.45 987	.54 013	.98 266	55
6	.44 297	.46 035	.53 965	.98 262	54
7	.44 341	.46 082	.53 918	.98 259	53
8	.44 385	.46 130	.53 870	.98 255	52
9	.44 428	.46 177	.53 823	.98 251	51
10	.44 472	.46 224	.53 776	.98 248	**50**
11	.44 516	.46 271	.53 729	.98 244	49
12	.44 559	.46 319	.53 681	.98 240	48
13	.44 602	.46 366	.53 634	.98 237	47
14	.44 646	.46 413	.53 587	.98 233	46
15	.44 689	.46 460	.53 540	.98 229	45
16	.44 733	.46 507	.53 493	.98 226	44
17	.44 776	.46 554	.53 446	.98 222	43
18	.44 819	.46 601	.53 399	.98 218	42
19	.44 862	.46 648	.53 352	.98 215	41
20	.44 905	.46 694	.53 306	.98 211	**40**
21	.44 948	.46 741	.53 259	.98 207	39
22	.44 992	.46 788	.53 212	.98 204	38
23	.45 035	.46 835	.53 165	.98 200	37
24	.45 077	.46 881	.53 119	.98 196	36
25	.45 120	.46 928	.53 072	.98 192	35
26	.45 163	.46 975	.53 025	.98 189	34
27	.45 206	.47 021	.52 979	.98 185	33
28	.45 249	.47 068	.52 932	.98 181	32
29	.45 292	.47 114	.52 886	.98 177	31
30	.45 334	.47 160	.52 840	.98 174	**30**
31	.45 377	.47 207	.52 793	.98 170	29
32	.45 419	.47 253	.52 747	.98 166	28
33	.45 462	.47 299	.52 701	.98 162	27
34	.45 504	.47 346	.52 654	.98 159	26
35	.45 547	.47 392	.52 608	.98 155	25
36	.45 589	.47 438	.52 562	.98 151	24
37	.45 632	.47 484	.52 516	.98 147	23
38	.45 674	.47 530	.52 470	.98 144	22
39	.45 716	.47 576	.52 424	.98 140	21
40	.45 758	.47 622	.52 378	.98 136	**20**
41	.45 801	.47 668	.52 332	.98 132	19
42	.45 843	.47 714	.52 286	.98 129	18
43	.45 885	.47 760	.52 240	.98 125	17
44	.45 927	.47 806	.52 194	.98 121	16
45	.45 969	.47 852	.52 148	.98 117	15
46	.46 011	.47 897	.52 103	.98 113	14
47	.46 053	.47 943	.52 057	.98 110	13
48	.46 095	.47 989	.52 011	.98 106	12
49	.46 136	.48 035	.51 965	.98 102	11
50	.46 178	.48 080	.51 920	.98 098	**10**
51	.46 220	.48 126	.51 874	.98 094	9
52	.46 262	.48 171	.51 829	.98 090	8
53	.46 303	.48 217	.51 783	.98 087	7
54	.46 345	.48 262	.51 738	.98 083	6
55	.46 386	.48 307	.51 693	.98 079	5
56	.46 428	.48 353	.51 647	.98 075	4
57	.46 469	.48 398	.51 602	.98 071	3
58	.46 511	.48 443	.51 557	.98 067	2
59	.46 552	.48 489	.51 511	.98 063	1
60	.46 594	.48 534	.51 466	.98 060	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

73°

17°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.46 594	.48 534	.51 466	.98 060	**60**
1	.46 635	.48 579	.51 421	.98 056	59
2	.46 676	.48 624	.51 376	.98 052	58
3	.46 717	.48 669	.51 331	.98 048	57
4	.46 758	.48 714	.51 286	.98 044	56
5	.46 800	.48 759	.51 241	.98 040	55
6	.46 841	.48 804	.51 196	.98 036	54
7	.46 882	.48 849	.51 151	.98 032	53
8	.46 923	.48 894	.51 106	.98 029	52
9	.46 964	.48 939	.51 061	.98 025	51
10	.47 005	.48 984	.51 016	.98 021	**50**
11	.47 045	.49 029	.50 971	.98 017	49
12	.47 086	.49 073	.50 927	.98 013	48
13	.47 127	.49 118	.50 882	.98 009	47
14	.47 168	.49 163	.50 837	.98 005	46
15	.47 209	.49 207	.50 793	.98 001	45
16	.47 249	.49 252	.50 748	.97 997	44
17	.47 290	.49 296	.50 704	.97 993	43
18	.47 330	.49 341	.50 659	.97 989	42
19	.47 371	.49 385	.50 615	.97 986	41
20	.47 411	.49 430	.50 570	.97 982	**40**
21	.47 452	.49 474	.50 526	.97 978	39
22	.47 492	.49 519	.50 481	.97 974	38
23	.47 533	.49 563	.50 437	.97 970	37
24	.47 573	.49 607	.50 393	.97 966	36
25	.47 613	.49 652	.50 348	.97 962	35
26	.47 654	.49 696	.50 304	.97 958	34
27	.47 694	.49 740	.50 260	.97 954	33
28	.47 734	.49 784	.50 216	.97 950	32
29	.47 774	.49 828	.50 172	.97 946	31
30	.47 814	.49 872	.50 128	.97 942	**30**
31	.47 854	.49 916	.50 084	.97 938	29
32	.47 894	.49 960	.50 040	.97 934	28
33	.47 934	.50 004	.49 996	.97 930	27
34	.47 974	.50 048	.49 952	.97 926	26
35	.48 014	.50 092	.49 908	.97 922	25
36	.48 054	.50 136	.49 864	.97 918	24
37	.48 094	.50 180	.49 820	.97 914	23
38	.48 133	.50 223	.49 777	.97 910	22
39	.48 173	.50 267	.49 733	.97 906	21
40	.48 213	.50 311	.49 689	.97 902	**20**
41	.48 252	.50 355	.49 645	.97 898	19
42	.48 292	.50 398	.49 602	.97 894	18
43	.48 332	.50 442	.49 558	.97 890	17
44	.48 371	.50 485	.49 515	.97 886	16
45	.48 411	.50 529	.49 471	.97 882	15
46	.48 450	.50 572	.49 428	.97 878	14
47	.48 490	.50 616	.49 384	.97 874	13
48	.48 529	.50 659	.49 341	.97 870	12
49	.48 568	.50 703	.49 297	.97 866	11
50	.48 607	.50 746	.49 254	.97 861	**10**
51	.48 647	.50 789	.49 211	.97 857	9
52	.48 686	.50 833	.49 167	.97 853	8
53	.48 725	.50 876	.49 124	.97 849	7
54	.48 764	.50 919	.49 081	.97 845	6
55	.48 803	.50 962	.49 038	.97 841	5
56	.48 842	.51 005	.48 995	.97 837	4
57	.48 881	.51 048	.48 952	.97 833	3
58	.48 920	.51 092	.48 908	.97 829	2
59	.48 959	.51 135	.48 865	.97 825	1
60	.48 998	.51 178	.48 822	.97 821	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

72°

18°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.48 998	.51 178	.48 822	.97 821	**60**
1	.49 037	.51 221	.48 779	.97 817	59
2	.49 076	.51 264	.48 736	.97 812	58
3	.49 115	.51 306	.48 694	.97 808	57
4	.49 153	.51 349	.48 651	.97 804	56
5	.49 192	.51 392	.48 608	.97 800	55
6	.49 231	.51 435	.48 565	.97 796	54
7	.49 269	.51 478	.48 522	.97 792	53
8	.49 308	.51 520	.48 480	.97 788	52
9	.49 347	.51 563	.48 437	.97 784	51
10	.49 385	.51 606	.48 394	.97 779	**50**
11	.49 424	.51 648	.48 352	.97 775	49
12	.49 462	.51 691	.48 309	.97 771	48
13	.49 500	.51 734	.48 266	.97 767	47
14	.49 539	.51 776	.48 224	.97 763	46
15	.49 577	.51 819	.48 181	.97 759	45
16	.49 615	.51 861	.48 139	.97 754	44
17	.49 654	.51 903	.48 097	.97 750	43
18	.49 692	.51 946	.48 054	.97 746	42
19	.49 730	.51 988	.48 012	.97 742	41
20	.49 768	.52 031	.47 969	.97 738	**40**
21	.49 806	.52 073	.47 927	.97 734	39
22	.49 844	.52 115	.47 885	.97 729	38
23	.49 882	.52 157	.47 843	.97 725	37
24	.49 920	.52 200	.47 800	.97 721	36
25	.49 958	.52 242	.47 758	.97 717	35
26	.49 996	.52 284	.47 716	.97 713	34
27	.50 034	.52 326	.47 674	.97 708	33
28	.50 072	.52 368	.47 632	.97 704	32
29	.50 110	.52 410	.47 590	.97 700	31
30	.50 148	.52 452	.47 548	.97 696	**30**
31	.50 185	.52 494	.47 506	.97 691	29
32	.50 223	.52 536	.47 464	.97 687	28
33	.50 261	.52 578	.47 422	.97 683	27
34	.50 298	.52 620	.47 380	.97 679	26
35	.50 336	.52 661	.47 339	.97 674	25
36	.50 374	.52 703	.47 297	.97 670	24
37	.50 411	.52 745	.47 255	.97 666	23
38	.50 449	.52 787	.47 213	.97 662	22
39	.50 486	.52 829	.47 171	.97 657	21
40	.50 523	.52 870	.47 130	.97 653	**20**
41	.50 561	.52 912	.47 088	.97 649	19
42	.50 598	.52 953	.47 047	.97 645	18
43	.50 635	.52 995	.47 005	.97 540	17
44	.50 673	.53 037	.46 963	.97 636	16
45	.50 710	.53 078	.46 922	.97 632	15
46	.50 747	.53 120	.46 880	.97 628	14
47	.50 784	.53 161	.46 839	.97 623	13
48	.50 821	.53 202	.46 798	.97 619	12
49	.50 858	.53 244	.46 756	.97 615	11
50	.50 896	.53 285	.46 715	.97 610	**10**
51	.50 933	.53 327	.46 673	.97 606	9
52	.50 970	.53 368	.46 632	.97 602	8
53	.51 007	.53 409	.46 591	.97 597	7
54	.51 043	.53 450	.46 550	.97 593	6
55	.51 080	.53 492	.46 508	.97 589	5
56	.51 117	.53 533	.46 467	.97 584	4
57	.51 154	.53 574	.46 426	.97 580	3
58	.51 191	.53 615	.46 385	.97 576	2
59	.51 227	.53 656	.46 344	.97 571	1
60	.51 264	.53 697	.46 303	.97 567	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

71°

19°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.51 264	.53 697	.46 303	.97 567	**60**
1	.51 301	.53 738	.46 262	.97 563	59
2	.51 338	.53 779	.46 221	.97 558	58
3	.51 374	.53 820	.46 180	.97 554	57
4	.51 411	.53 861	.46 139	.97 550	56
5	.51 447	.53 902	.46 098	.97 545	55
6	.51 484	.53 943	.46 057	.97 541	54
7	.51 520	.53 984	.46 016	.97 536	53
8	.51 557	.54 025	.45 975	.97 532	52
9	.51 593	.54 065	.45 935	.97 528	51
10	.51 629	.54 106	.45 894	.97 523	**50**
11	.51 666	.54 147	.45 853	.97 519	49
12	.51 702	.54 187	.45 813	.97 515	48
13	.51 738	.54 228	.45 772	.97 510	47
14	.51 774	.54 269	.45 731	.97 506	46
15	.51 811	.54 309	.45 691	.97 501	45
16	.51 847	.54 350	.45 650	.97 497	44
17	.51 883	.54 390	.45 610	.97 492	43
18	.51 919	.54 431	.45 569	.97 488	42
19	.51 955	.54 471	.45 529	.97 484	41
20	.51 991	.54 512	.45 488	.97 479	**40**
21	.52 027	.54 552	.45 448	.97 475	39
22	.52 063	.54 593	.45 407	.97 470	38
23	.52 099	.54 633	.45 367	.97 466	37
24	.52 135	.54 673	.45 327	.97 461	36
25	.52 171	.54 714	.45 286	.97 457	35
26	.52 207	.54 754	.45 246	.97 453	34
27	.52 242	.54 794	.45 206	.97 448	33
28	.52 278	.54 835	.45 165	.97 444	32
29	.52 314	.54 875	.45 125	.97 439	31
30	.52 350	.54 915	.45 085	.97 435	**30**
31	.52 385	.54 955	.45 045	.97 430	29
32	.52 421	.54 995	.45 005	.97 426	28
33	.52 456	.55 035	.44 965	.97 421	27
34	.52 492	.55 075	.44 925	.97 417	26
35	.52 527	.55 115	.44 885	.97 412	25
36	.52 563	.55 155	.44 845	.97 408	24
37	.52 598	.55 195	.44 805	.97 403	23
38	.52 634	.55 235	.44 765	.97 399	22
39	.52 669	.55 275	.44 725	.97 394	21
40	.52 705	.55 315	.44 685	.97 390	**20**
41	.52 740	.55 355	.44 645	.97 385	19
42	.52 775	.55 395	.44 605	.97 381	18
43	.52 811	.55 434	.44 566	.97 376	17
44	.52 846	.55 474	.44 526	.97 372	16
45	.52 881	.55 514	.44 486	.97 367	15
46	.52 916	.55 554	.44 446	.97 363	14
47	.52 951	.55 593	.44 407	.97 358	13
48	.52 986	.55 633	.44 367	.97 353	12
49	.53 021	.55 673	.44 327	.97 349	11
50	.53 056	.55 712	.44 288	.97 344	**10**
51	.53 092	.55 752	.44 248	.97 340	9
52	.53 126	.55 791	.44 209	.97 335	8
53	.53 161	.55 831	.44 169	.97 331	7
54	.53 196	.55 870	.44 130	.97 326	6
55	.53 231	.55 910	.44 090	.97 322	5
56	.53 266	.55 949	.44 051	.97 317	4
57	.53 301	.55 989	.44 011	.97 312	3
58	.53 336	.56 028	.43 972	.97 308	2
59	.53 370	.56 067	.43 933	.97 303	1
60	.53 405	.56 107	.43 893	.97 299	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

70°

20°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.53 405	.56 107	.43 893	.97 299	**60**
1	.53 440	.56 146	.43 854	.97 294	59
2	.53 475	.56 185	.43 815	.97 289	58
3	.53 509	.56 224	.43 776	.97 285	57
4	.53 544	.56 264	.43 736	.97 280	56
5	.53 578	.56 303	.43 697	.97 276	55
6	.53 613	.56 342	.43 658	.97 271	54
7	.53 647	.56 381	.43 619	.97 266	53
8	.53 682	.56 420	.43 580	.97 262	52
9	.53 716	.56 459	.43 541	.97 257	51
10	.53 751	.56 498	.43 502	.97 252	**50**
11	.53 785	.56 537	.43 463	.97 248	49
12	.53 819	.56 576	.43 424	.97 243	48
13	.53 854	.56 615	.43 385	.97 238	47
14	.53 888	.56 654	.43 346	.97 234	46
15	.53 922	.56 693	.43 307	.97 229	45
16	.53 957	.56 732	.43 268	.97 224	44
17	.53 991	.56 771	.43 229	.97 220	43
18	.54 025	.56 810	.43 190	.97 215	42
19	.54 059	.56 849	.43 151	.97 210	41
20	.54 093	.56 887	.43 113	.97 206	**40**
21	.54 127	.56 926	.43 074	.97 201	39
22	.54 161	.56 965	.43 035	.97 196	38
23	.54 195	.57 004	.42 996	.97 192	37
24	.54 229	.57 042	.42 958	.97 187	36
25	.54 263	.57 081	.42 919	.97 182	35
26	.54 297	.57 120	.42 880	.97 178	34
27	.54 331	.57 158	.42 842	.97 173	33
28	.54 365	.57 197	.42 803	.97 168	32
29	.54 399	.57 235	.42 765	.97 163	31
30	.54 433	.57 274	.42 726	.97 159	**30**
31	.54 466	.57 312	.42 688	.97 154	29
32	.54 500	.57 351	.42 649	.97 149	28
33	.54 534	.57 389	.42 611	.97 145	27
34	.54 567	.57 428	.42 572	.97 140	26
35	.54 601	.57 466	.42 534	.97 135	25
36	.54 635	.57 504	.42 496	.97 130	24
37	.54 668	.57 543	.42 457	.97 126	23
38	.54 702	.57 581	.42 419	.97 121	22
39	.54 735	.57 619	.42 381	.97 116	21
40	.54 769	.57 658	.42 342	.97 111	**20**
41	.54 802	.57 696	.42 304	.97 107	19
42	.54 836	.57 734	.42 266	.97 102	18
43	.54 869	.57 772	.42 228	.97 097	17
44	.54 903	.57 810	.42 190	.97 092	16
45	.54 936	.57 849	.42 151	.97 087	15
46	.54 969	.57 887	.42 113	.97 083	14
47	.55 003	.57 925	.42 075	.97 078	13
48	.55 036	.57 963	.42 037	.97 073	12
49	.55 069	.58 001	.41 999	.97 068	11
50	.55 102	.58 039	.41 961	.97 063	**10**
51	.55 136	.58 077	.41 923	.97 059	9
52	.55 169	.58 115	.41 885	.97 054	8
53	.55 202	.58 153	.41 847	.97 049	7
54	.55 235	.58 191	.41 809	.97 044	6
55	.55 268	.58 229	.41 771	.97 039	5
56	.55 301	.58 267	.41 733	.97 035	4
57	.55 334	.58 304	.41 696	.97 030	3
58	.55 367	.58 342	.41 658	.97 025	2
59	.55 400	.58 380	.41 620	.97 020	1
60	.55 433	.58 418	.41 582	.97 015	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

69°

21°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.55 433	.58 418	.41 582	.97 015	**60**
1	.55 466	.58 455	.41 545	.97 010	59
2	.55 499	.58 493	.41 507	.97 005	58
3	.55 532	.58 531	.41 469	.97 001	57
4	.55 564	.58 569	.41 431	.96 996	56
5	.55 597	.58 606	.41 394	.96 991	55
6	.55 630	.58 644	.41 356	.96 986	54
7	.55 663	.58 681	.41 319	.96 981	53
8	.55 695	.58 719	.41 281	.96 976	52
9	.55 728	.58 757	.41 243	.96 971	51
10	.55 761	.58 794	.41 206	.96 966	**50**
11	.55 793	.58 832	.41 168	.96 962	49
12	.55 826	.58 869	.41 131	.96 957	48
13	.55 858	.58 907	.41 093	.96 952	47
14	.55 891	.58 944	.41 056	.96 947	46
15	.55 923	.58 981	.41 019	.96 942	45
16	.55 956	.59 019	.40 981	.96 937	44
17	.55 988	.59 056	.40 944	.96 932	43
18	.56 021	.59 094	.40 906	.96 927	42
19	.56 053	.59 131	.40 869	.96 922	41
20	.56 085	.59 168	.40 832	.96 917	**40**
21	.56 118	.59 205	.40 795	.96 912	39
22	.56 150	.59 243	.40 757	.96 907	38
23	.56 182	.59 280	.40 720	.96 903	37
24	.56 215	.59 317	.40 683	.96 898	36
25	.56 247	.59 354	.40 646	.96 893	35
26	.56 279	.59 391	.40 609	.96 888	34
27	.56 311	.59 429	.40 571	.96 883	33
28	.56 343	.59 466	.40 534	.96 878	32
29	.56 375	.59 503	.40 497	.96 873	31
30	.56 408	.59 540	.40 460	.96 868	**30**
31	.56 440	.59 577	.40 423	.96 863	29
32	.56 472	.59 614	.40 386	.96 858	28
33	.56 504	.59 651	.40 349	.96 853	27
34	.56 536	.59 688	.40 312	.96 848	26
35	.56 568	.59 725	.40 275	.96 843	25
36	.56 599	.59 762	.40 238	.96 838	24
37	.56 631	.59 799	.40 201	.96 833	23
38	.56 663	.59 835	.40 165	.96 828	22
39	.56 695	.59 872	.40 128	.96 823	21
40	.56 727	.59 909	.40 091	.96 818	**20**
41	.56 759	.59 946	.40 054	.96 813	19
42	.56 790	.59 983	.40 017	.96 808	18
43	.56 822	.60 019	.39 981	.96 803	17
44	.56 854	.60 056	.39 944	.96 798	16
45	.56 886	.60 093	.39 907	.96 793	15
46	.56 917	.60 130	.39 870	.96 788	14
47	.56 949	.60 166	.39 834	.96 783	13
48	.56 980	.60 203	.39 797	.96 778	12
49	.57 012	.60 240	.39 760	.96 772	11
50	.57 044	.60 276	.39 724	.96 767	**10**
51	.57 075	.60 313	.39 687	.96 762	9
52	.57 107	.60 349	.39 651	.96 757	8
53	.57 138	.60 386	.39 614	.96 752	7
54	.57 169	.60 422	.39 578	.96 747	6
55	.57 201	.60 459	.39 541	.96 742	5
56	.57 232	.60 495	.39 505	.96 737	4
57	.57 264	.60 532	.39 468	.96 732	3
58	.57 295	.60 568	.39 432	.96 727	2
59	.57 326	.60 605	.39 395	.96 722	1
60	.57 358	.60 641	.39 359	.96 717	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

68°

22°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.57 358	.60 641	.39 359	.96 717	**60**
1	.57 389	.60 677	.39 323	.96 711	59
2	.57 420	.60 714	.39 286	.96 706	58
3	.57 451	.60 750	.39 250	.96 701	57
4	.57 482	.60 786	.39 214	.96 696	56
5	.57 514	.60 823	.39 177	.96 691	55
6	.57 545	.60 859	.39 141	.96 686	54
7	.57 576	.60 895	.39 105	.96 681	53
8	.57 607	.60 931	.39 069	.96 676	52
9	.57 638	.60 967	.39 033	.96 670	51
10	.57 669	.61 004	.38 996	.96 665	**50**
11	.57 700	.61 040	.38 960	.96 660	49
12	.57 731	.61 076	.38 924	.96 655	48
13	.57 762	.61 112	.38 888	.96 650	47
14	.57 793	.61 148	.38 852	.96 645	46
15	.57 824	.61 184	.38 816	.96 640	45
16	.57 855	.61 220	.38 780	.96 634	44
17	.57 885	.61 256	.38 744	.96 629	43
18	.57 916	.61 292	.38 708	.96 624	42
19	.57 947	.61 328	.38 672	.96 619	41
20	.57 978	.61 364	.38 636	.96 614	**40**
21	.58 008	.61 400	.38 600	.96 608	39
22	.58 039	.61 436	.38 564	.96 603	38
23	.58 070	.61 472	.38 528	.96 598	37
24	.58 101	.61 508	.38 492	.96 593	36
25	.58 131	.61 544	.38 456	.96 588	35
26	.58 162	.61 579	.38 421	.96 582	34
27	.58 192	.61 615	.38 385	.96 577	33
28	.58 223	.61 651	.38 349	.96 572	32
29	.58 253	.61 687	.38 313	.96 567	31
30	.58 284	.61 722	.38 278	.96 562	**30**
31	.58 314	.61 758	.38 242	.96 556	29
32	.58 345	.61 794	.38 206	.96 551	28
33	.58 375	.61 830	.38 170	.96 546	27
34	.58 406	.61 865	.38 135	.96 541	26
35	.58 436	.61 901	.38 099	.96 535	25
36	.58 467	.61 936	.38 064	.96 530	24
37	.58 497	.61 972	.38 028	.96 525	23
38	.58 527	.62 008	.37 992	.96 520	22
39	.58 557	.62 043	.37 957	.96 514	21
40	.58 588	.62 079	.37 921	.96 509	**20**
41	.58 618	.62 114	.37 886	.96 504	19
42	.58 648	.62 150	.37 850	.96 498	18
43	.58 678	.62 185	.37 815	.96 493	17
44	.58 709	.62 221	.37 779	.96 488	16
45	.58 739	.62 256	.37 744	.96 483	15
46	.58 769	.62 292	.37 708	.96 477	14
47	.58 799	.62 327	.37 673	.96 472	13
48	.58 829	.62 362	.37 638	.96 467	12
49	.58 859	.62 398	.37 602	.96 461	11
50	.58 889	.62 433	.37 567	.96 456	**10**
51	.58 919	.62 468	.37 532	.96 451	9
52	.58 949	.62 504	.37 496	.96 445	8
53	.58 979	.62 539	.37 461	.96 440	7
54	.59 009	.62 574	.37 426	.96 435	6
55	.59 039	.62 609	.37 391	.96 429	5
56	.59 069	.62 645	.37 355	.96 424	4
57	.59 098	.62 680	.37 320	.96 419	3
58	.59 128	.62 715	.37 285	.96 413	2
59	.59 158	.62 750	.37 250	.96 408	1
60	.59 188	.62 785	.37 215	.96 403	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

67°

23°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.59 188	.62 785	.37 215	.96 403	**60**
1	.59 218	.62 820	.37 180	.96 397	59
2	.59 247	.62 855	.37 145	.96 392	58
3	.59 277	.62 890	.37 110	.96 387	57
4	.59 307	.62 926	.37 074	.96 381	56
5	.59 336	.62 961	.37 039	.96 376	55
6	.59 366	.62 996	.37 004	.96 370	54
7	.59 396	.63 031	.36 969	.96 365	53
8	.59 425	.63 066	.36 934	.96 360	52
9	.59 455	.63 101	.36 899	.96 354	51
10	.59 484	.63 135	.36 865	.96 349	**50**
11	.59 514	.63 170	.36 830	.96 343	49
12	.59 543	.63 205	.36 795	.96 338	48
13	.59 573	.63 240	.36 760	.96 333	47
14	.59 602	.63 275	.36 725	.96 327	46
15	.59 632	.63 310	.36 690	.96 322	45
16	.59 661	.63 345	.36 655	.96 316	44
17	.59 690	.63 379	.36 621	.96 311	43
18	.59 720	.63 414	.36 586	.96 305	42
19	.59 749	.63 449	.36 551	.96 300	41
20	.59 778	.63 484	.36 516	.96 294	**40**
21	.59 808	.63 519	.36 481	.96 289	39
22	.59 837	.63 553	.36 447	.96 284	38
23	.59 866	.63 588	.36 412	.96 278	37
24	.59 895	.63 623	.36 377	.96 273	36
25	.59 924	.63 657	.36 343	.96 267	35
26	.59 954	.63 692	.36 308	.96 262	34
27	.59 983	.63 726	.36 274	.96 256	33
28	.60 012	.63 761	.36 239	.96 251	32
29	.60 041	.63 796	.36 204	.96 245	31
30	.60 070	.63 830	.36 170	.96 240	**30**
31	.60 099	.63 865	.36 135	.96 234	29
32	.60 128	.63 899	.36 101	.96 229	28
33	.60 157	.63 934	.36 066	.96 223	27
34	.60 186	.63 968	.36 032	.96 218	26
35	.60 215	.64 003	.35 997	.96 212	25
36	.60 244	.64 037	.35 963	.96 207	24
37	.60 273	.64 072	.35 928	.96 201	23
38	.60 302	.64 106	.35 894	.96 196	22
39	.60 331	.64 140	.35 860	.96 190	21
40	.60 359	.64 175	.35 825	.96 185	**20**
41	.60 388	.64 209	.35 791	.96 179	19
42	.60 417	.64 243	.35 757	.96 174	18
43	.60 446	.64 278	.35 722	.96 168	17
44	.60 474	.64 312	.35 688	.96 162	16
45	.60 503	.64 346	.35 654	.96 157	15
46	.60 532	.64 381	.35 619	.96 151	14
47	.60 561	.64 415	.35 585	.96 146	13
48	.60 589	.64 449	.35 551	.96 140	12
49	.60 618	.64 483	.35 517	.96 135	11
50	.60 646	.64 517	.35 483	.96 129	**10**
51	.60 675	.64 552	.35 448	.96 123	9
52	.60 704	.64 586	.35 414	.96 118	8
53	.60 732	.64 620	.35 380	.96 112	7
54	.60 761	.64 654	.35 346	.96 107	6
55	.60 789	.64 688	.35 312	.96 101	5
56	.60 818	.64 722	.35 278	.96 095	4
57	.60 846	.64 756	.35 244	.96 090	3
58	.60 875	.64 790	.35 210	.96 084	2
59	.60 903	.64 824	.35 176	.96 079	1
60	.60 931	.64 858	.35 142	.96 073	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

66°

24°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.60 931	.64 858	.35 142	.96 073	**60**
1	.60 960	.64 892	.35 108	.96 067	59
2	.60 988	.64 926	.35 074	.96 062	58
3	.61 016	.64 960	.35 040	.96 056	57
4	.61 045	.64 994	.35 006	.96 050	56
5	.61 073	.65 028	.34 972	.96 045	55
6	.61 101	.65 062	.34 938	.96 039	54
7	.61 129	.65 096	.34 904	.96 034	53
8	.61 158	.65 130	.34 870	.96 028	52
9	.61 186	.65 164	.34 836	.96 022	51
10	.61 214	.65 197	.34 803	.96 017	**50**
11	.61 242	.65 231	.34 769	.96 011	49
12	.61 270	.65 265	.34 735	.96 005	48
13	.61 298	.65 299	.34 701	.96 000	47
14	.61 326	.65 333	.34 667	.95 994	46
15	.61 354	.65 366	.34 634	.95 988	45
16	.61 382	.65 400	.34 600	.95 982	44
17	.61 411	.65 434	.34 566	.95 977	43
18	.61 438	.65 467	.34 533	.95 971	42
19	.61 466	.65 501	.34 499	.95 965	41
20	.61 494	.65 535	.34 465	.95 960	**40**
21	.61 522	.65 568	.34 432	.95 954	39
22	.61 550	.65 602	.34 398	.95 948	38
23	.61 578	.65 636	.34 364	.95 942	37
24	.61 606	.65 669	.34 331	.95 937	36
25	.61 634	.65 703	.34 297	.95 931	35
26	.61 662	.65 736	.34 264	.95 925	34
27	.61 689	.65 770	.34 230	.95 920	33
28	.61 717	.65 803	.34 197	.95 914	32
29	.61 745	.65 837	.34 163	.95 908	31
30	.61 773	.65 870	.34 130	.95 902	**30**
31	.61 800	.65 904	.34 096	.95 897	29
32	.61 828	.65 937	.34 063	.95 891	28
33	.61 856	.65 971	.34 029	.95 885	27
34	.61 883	.66 004	.33 996	.95 879	26
35	.61 911	.66 038	.33 962	.95 873	25
36	.61 939	.66 071	.33 929	.95 868	24
37	.61 966	.66 104	.33 896	.95 862	23
38	.61 994	.66 138	.33 862	.95 856	22
39	.62 021	.66 171	.33 829	.95 850	21
40	.62 049	.66 204	.33 796	.95 844	**20**
41	.62 076	.66 238	.33 762	.95 839	19
42	.62 104	.66 271	.33 729	.95 833	18
43	.62 131	.66 304	.33 696	.95 827	17
44	.62 159	.66 337	.33 663	.95 821	16
45	.62 186	.66 371	.33 629	.95 815	15
46	.62 214	.66 404	.33 596	.95 810	14
47	.62 241	.66 437	.33 563	.95 804	13
48	.62 268	.66 470	.33 530	.95 798	12
49	.62 296	.66 503	.33 497	.95 792	11
50	.62 323	.66 537	.33 463	.95 786	**10**
51	.62 350	.66 570	.33 430	.95 780	9
52	.62 377	.66 603	.33 397	.95 775	8
53	.62 405	.66 636	.33 364	.95 769	7
54	.62 432	.66 669	.33 331	.95 763	6
55	.62 459	.66 702	.33 298	.95 757	5
56	.62 486	.66 735	.33 265	.95 751	4
57	.62 513	.66 768	.33 232	.95 745	3
58	.62 541	.66 801	.33 199	.95 739	2
59	.62 568	.66 834	.33 166	.95 733	1
60	.62 595	.66 867	.33 133	.95 728	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

65°

25°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.62 595	.66 867	.33 133	.95 728	**60**
1	.62 622	.66 900	.33 100	.95 722	59
2	.62 649	.66 933	.33 067	.95 716	58
3	.62 676	.66 966	.33 034	.95 710	57
4	.62 703	.66 999	.33 001	.95 704	56
5	.62 730	.67 032	.32 968	.95 698	55
6	.62 757	.67 065	.32 935	.95 692	54
7	.62 784	.67 098	.32 902	.95 686	53
8	.62 811	.67 131	.32 869	.95 680	52
9	.62 838	.67 163	.32 837	.95 674	51
10	.62 865	.67 196	.32 804	.95 668	**50**
11	.62 892	.67 229	.32 771	.95 663	49
12	.62 918	.67 262	.32 738	.95 657	48
13	.62 945	.67 295	.32 705	.95 651	47
14	.62 972	.67 327	.32 673	.95 645	46
15	.62 999	.67 360	.32 640	.95 639	45
16	.63 026	.67 393	.32 607	.95 633	44
17	.63 052	.67 426	.32 574	.95 627	43
18	.63 079	.67 458	.32 542	.95 621	42
19	.63 106	.67 491	.32 509	.95 615	41
20	.63 133	.67 524	.32 476	.95 609	**40**
21	.63 159	.67 556	.32 444	.95 603	39
22	.63 186	.67 589	.32 411	.95 597	38
23	.63 213	.67 622	.32 378	.95 591	37
24	.63 239	.67 654	.32 346	.95 585	36
25	.63 266	.67 687	.32 313	.95 579	35
26	.63 292	.67 719	.32 281	.95 573	34
27	.63 319	.67 752	.32 248	.95 567	33
28	.63 345	.67 785	.32 215	.95 561	32
29	.63 372	.67 817	.32 183	.95 555	31
30	.63 398	.67 850	.32 150	.95 549	**30**
31	.63 425	.67 882	.32 118	.95 543	29
32	.63 451	.67 915	.32 085	.95 537	28
33	.63 478	.67 947	.32 053	.95 531	27
34	.63 504	.67 980	.32 020	.95 525	26
35	.63 531	.68 012	.31 988	.95 519	25
36	.63 557	.68 044	.31 956	.95 513	24
37	.63 583	.68 077	.31 923	.95 507	23
38	.63 610	.68 109	.31 891	.95 500	22
39	.63 636	.68 142	.31 858	.95 494	21
40	.63 662	.68 174	.31 826	.95 488	**20**
41	.63 689	.68 206	.31 794	.95 482	19
42	.63 715	.68 239	.31 761	.95 476	18
43	.63 741	.68 271	.31 729	.95 470	17
44	.63 767	.68 303	.31 697	.95 464	16
45	.63 794	.68 336	.31 664	.95 458	15
46	.63 820	.68 368	.31 632	.95 452	14
47	.63 846	.68 400	.31 600	.95 446	13
48	.63 872	.68 432	.31 568	.95 440	12
49	.63 898	.68 465	.31 535	.95 434	11
50	.63 924	.68 497	.31 503	.95 427	**10**
51	.63 950	.68 529	.31 471	.95 421	9
52	.63 976	.68 561	.31 439	.95 415	8
53	.64 002	.68 593	.31 407	.95 409	7
54	.64 028	.68 626	.31 374	.95 403	6
55	.64 054	.68 658	.31 342	.95 397	5
56	.64 080	.68 690	.31 310	.95 391	4
57	.64 106	.68 722	.31 278	.95 384	3
58	.64 132	.68 754	.31 246	.95 378	2
59	.64 158	.68 786	.31 214	.95 372	1
60	.64 184	.68 818	.31 182	.95 366	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

64°

26°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.64 184	.68 818	.31 182	.95 366	**60**
1	.64 210	.68 850	.31 150	.95 360	59
2	.64 236	.68 882	.31 118	.95 354	58
3	.64 262	.68 914	.31 086	.95 348	57
4	.64 288	.68 946	.31 054	.95 341	56
5	.64 313	.68 978	.31 022	.95 335	55
6	.64 339	.69 010	.30 990	.95 329	54
7	.64 365	.69 042	.30 958	.95 323	53
8	.64 391	.69 074	.30 926	.95 317	52
9	.64 417	.69 106	.30 894	.95 310	51
10	.64 442	.69 138	.30 862	.95 304	**50**
11	.64 468	.69 170	.30 830	.95 298	49
12	.64 494	.69 202	.30 798	.95 292	48
13	.64 519	.69 234	.30 766	.95 286	47
14	.64 545	.69 266	.30 734	.95 279	46
15	.64 571	.69 298	.30 702	.95 273	45
16	.64 596	.69 329	.30 671	.95 267	44
17	.64 622	.69 361	.30 639	.95 261	43
18	.64 647	.69 393	.30 607	.95 254	42
19	.64 673	.69 425	.30 575	.95 248	41
20	.64 698	.69 457	.30 543	.95 242	**40**
21	.64 724	.69 488	.30 512	.95 236	39
22	.64 749	.69 520	.30 480	.95 229	38
23	.64 775	.69 552	.30 448	.95 223	37
24	.64 800	.69 584	.30 416	.95 217	36
25	.64 826	.69 615	.30 385	.95 211	35
26	.64 851	.69 647	.30 353	.95 204	34
27	.64 877	.69 679	.30 321	.95 198	33
28	.64 902	.69 710	.30 290	.95 192	32
29	.64 927	.69 742	.30 258	.95 185	31
30	.64 953	.69 774	.30 226	.95 179	**30**
31	.64 978	.69 805	.30 195	.95 173	29
32	.65 003	.69 837	.30 163	.95 167	28
33	.65 029	.69 868	.30 132	.95 160	27
34	.65 054	.69 900	.30 100	.95 154	26
35	.65 079	.69 932	.30 068	.95 148	25
36	.65 104	.69 963	.30 037	.95 141	24
37	.65 130	.69 995	.30 005	.95 135	23
38	.65 155	.70 026	.29 974	.95 129	22
39	.65 180	.70 058	.29 942	.95 122	21
40	.65 205	.70 089	.29 911	.95 116	**20**
41	.65 230	.70 121	.29 879	.95 110	19
42	.65 255	.70 152	.29 848	.95 103	18
43	.65 281	.70 184	.29 816	.95 097	17
44	.65 306	.70 215	.29 785	.95 090	16
45	.65 331	.70 247	.29 753	.95 084	15
46	.65 356	.70 278	.29 722	.95 078	14
47	.65 381	.70 309	.29 691	.95 071	13
48	.65 406	.70 341	.29 659	.95 065	12
49	.65 431	.70 372	.29 628	.95 059	11
50	.65 456	.70 404	.29 596	.95 052	**10**
51	.65 481	.70 435	.29 565	.95 046	9
52	.65 506	.70 466	.29 534	.95 039	8
53	.65 531	.70 498	.29 502	.95 033	7
54	.65 556	.70 529	.29 471	.95 027	6
55	.65 580	.70 560	.29 440	.95 020	5
56	.65 605	.70 592	.29 408	.95 014	4
57	.65 630	.70 623	.29 377	.95 007	3
58	.65 655	.70 654	.29 346	.95 001	2
59	.65 680	.70 685	.29 315	.94 995	1
60	.65 705	.70 717	.29 283	.94 988	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

63°

27°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.65 705	.70 717	.29 283	.94 988	**60**
1	.65 729	.70 748	.29 252	.94 982	59
2	.65 754	.70 779	.29 221	.94 975	58
3	.65 779	.70 810	.29 190	.94 969	57
4	.65 804	.70 841	.29 159	.94 962	56
5	.65 828	.70 873	.29 127	.94 956	55
6	.65 853	.70 904	.29 096	.94 949	54
7	.65 878	.70 935	.29 065	.94 943	53
8	.65 902	.70 966	.29 034	.94 936	52
9	.65 927	.70 997	.29 003	.94 930	51
10	.65 952	.71 028	.28 972	.94 923	**50**
11	.65 976	.71 059	.28 941	.94 917	49
12	.66 001	.71 090	.28 910	.94 911	48
13	.66 025	.71 121	.28 879	.94 904	47
14	.66 050	.71 153	.28 847	.94 898	46
15	.66 075	.71 184	.28 816	.94 891	45
16	.66 099	.71 215	.28 785	.94 885	44
17	.66 124	.71 246	.28 754	.94 878	43
18	.66 148	.71 277	.28 723	.94 871	42
19	.66 173	.71 308	.28 692	.94 865	41
20	.66 197	.71 339	.28 661	.94 858	**40**
21	.66 221	.71 370	.28 630	.94 852	39
22	.66 246	.71 401	.28 599	.94 845	38
23	.66 270	.71 431	.28 569	.94 839	37
24	.66 295	.71 462	.28 538	.94 832	36
25	.66 319	.71 493	.28 507	.94 826	35
26	.66 343	.71 524	.28 476	.94 819	34
27	.66 368	.71 555	.28 445	.94 813	33
28	.66 392	.71 586	.28 414	.94 806	32
29	.66 416	.71 617	.28 383	.94 799	31
30	.66 441	.71 648	.28 352	.94 793	**30**
31	.66 465	.71 679	.28 321	.94 786	29
32	.66 489	.71 709	.28 291	.94 780	28
33	.66 513	.71 740	.28 260	.94 773	27
34	.66 537	.71 771	.28 229	.94 767	26
35	.66 562	.71 802	.28 198	.94 760	25
36	.66 586	.71 833	.28 167	.94 753	24
37	.66 610	.71 863	.28 137	.94 747	23
38	.66 634	.71 894	.28 106	.94 740	22
39	.66 658	.71 925	.28 075	.94 734	21
40	.66 682	.71 955	.28 045	.94 727	**20**
41	.66 706	.71 986	.28 014	.94 720	19
42	.66 731	.72 017	.27 983	.94 714	18
43	.66 755	.72 048	.27 952	.94 707	17
44	.66 779	.72 078	.27 922	.94 700	16
45	.66 803	.72 109	.27 891	.94 694	15
46	.66 827	.72 140	.27 860	.94 687	14
47	.66 851	.72 170	.27 830	.94 680	13
48	.66 875	.72 201	.27 799	.94 674	12
49	.66 899	.72 231	.27 769	.94 667	11
50	.66 922	.72 262	.27 738	.94 660	**10**
51	.66 946	.72 293	.27 707	.94 654	9
52	.66 970	.72 323	.27 677	.94 647	8
53	.66 994	.72 354	.27 646	.94 640	7
54	.67 018	.72 384	.27 616	.94 634	6
55	.67 042	.72 415	.27 585	.94 627	5
56	.67 066	.72 445	.27 555	.94 620	4
57	.67 090	.72 476	.27 524	.94 614	3
58	.67 113	.72 506	.27 494	.94 607	2
59	.67 137	.72 537	.27 463	.94 600	1
60	.67 161	.72 567	.27 433	.94 593	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

62°

28°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.67 161	.72 567	.27 433	.94 593	**60**
1	.67 185	.72 598	.27 402	.94 587	59
2	.67 208	.72 628	.27 372	.94 580	58
3	.67 232	.72 659	.27 341	.94 573	57
4	.67 256	.72 689	.27 311	.94 567	56
5	.67 280	.72 720	.27 280	.94 560	55
6	.67 303	.72 750	.27 250	.94 553	54
7	.67 327	.72 780	.27 220	.94 546	53
8	.67 350	.72 811	.27 189	.94 540	52
9	.67 374	.72 841	.27 159	.94 533	51
10	.67 398	.72 872	.27 128	.94 526	**50**
11	.67 421	.72 902	.27 098	.94 519	49
12	.67 445	.72 932	.27 068	.94 513	48
13	.67 468	.72 963	.27 037	.94 506	47
14	.67 492	.72 993	.27 007	.94 499	46
15	.67 515	.73 023	.26 977	.94 492	45
16	.67 539	.73 054	.26 946	.94 485	44
17	.67 562	.73 084	.26 916	.94 479	43
18	.67 586	.73 114	.26 886	.94 472	42
19	.67 609	.73 144	.26 856	.94 465	41
20	.67 633	.73 175	.26 825	.94 458	**40**
21	.67 656	.73 205	.26 795	.94 451	39
22	.67 680	.73 235	.26 765	.94 445	38
23	.67 703	.73 265	.26 735	.94 438	37
24	.67 726	.73 295	.26 705	.94 431	36
25	.67 750	.73 326	.26 674	.94 424	35
26	.67 773	.73 356	.26 644	.94 417	34
27	.67 796	.73 386	.26 614	.94 410	33
28	.67 820	.73 416	.26 584	.94 404	32
29	.67 843	.73 446	.26 554	.94 397	31
30	.67 866	.73 476	.26 524	.94 390	**30**
31	.67 890	.73 507	.26 493	.94 383	29
32	.67 913	.73 537	.26 463	.94 376	28
33	.67 936	.73 567	.26 433	.94 369	27
34	.67 959	.73 597	.26 403	.94 362	26
35	.67 982	.73 627	.26 373	.94 355	25
36	.68 006	.73 657	.26 343	.94 349	24
37	.68 029	.73 687	.26 313	.94 342	23
38	.68 052	.73 717	.26 283	.94 335	22
39	.68 075	.73 747	.26 253	.94 328	21
40	.68 098	.73 777	.26 223	.94 321	**20**
41	.68 121	.73 807	.26 193	.94 314	19
42	.68 144	.73 837	.26 163	.94 307	18
43	.68 167	.73 867	.26 133	.94 300	17
44	.68 190	.73 897	.26 103	.94 293	16
45	.68 213	.73 927	.26 073	.94 286	15
46	.68 237	.73 957	.26 043	.94 279	14
47	.68 260	.73 987	.26 013	.94 273	13
48	.68 283	.74 017	.25 983	.94 266	12
49	.68 305	.74 047	.25 953	.94 259	11
50	.68 328	.74 077	.25 923	.94 252	**10**
51	.68 351	.74 107	.25 893	.94 245	9
52	.68 374	.74 137	.25 863	.94 238	8
53	.68 397	.74 166	.25 834	.94 231	7
54	.68 420	.74 196	.25 804	.94 224	6
55	.68 443	.74 226	.25 774	.94 217	5
56	.68 466	.74 256	.25 744	.94 210	4
57	.68 489	.74 286	.25 714	.94 203	3
58	.68 512	.74 316	.25 684	.94 196	2
59	.68 534	.74 345	.25 655	.94 189	1
60	.68 557	.74 375	.25 625	.94 182	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

61°

29°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.68 557	.74 375	.25 625	.94 182	**60**
1	.68 580	.74 405	.25 595	.94 175	59
2	.68 603	.74 435	.25 565	.94 168	58
3	.68 625	.74 465	.25 535	.94 161	57
4	.68 648	.74 494	.25 506	.94 154	56
5	.68 671	.74 524	.25 476	.94 147	55
6	.68 694	.74 554	.25 446	.94 140	54
7	.68 716	.74 583	.25 417	.94 133	53
8	.68 739	.74 613	.25 387	.94 126	52
9	.68 762	.74 643	.25 357	.94 119	51
10	.68 784	.74 673	.25 327	.94 112	**50**
11	.68 807	.74 702	.25 298	.94 105	49
12	.68 829	.74 732	.25 268	.94 098	48
13	.68 852	.74 762	.25 238	.94 090	47
14	.68 875	.74 791	.25 209	.94 083	46
15	.68 897	.74 821	.25 179	.94 076	45
16	.68 920	.74 851	.25 149	.94 069	44
17	.68 942	.74 880	.25 120	.94 062	43
18	.68 965	.74 910	.25 090	.94 055	42
19	.68 987	.74 939	.25 061	.94 048	41
20	.69 010	.74 969	.25 031	.94 041	**40**
21	.69 032	.74 998	.25 002	.94 034	39
22	.69 055	.75 028	.24 972	.94 027	38
23	.69 077	.75 058	.24 942	.94 020	37
24	.69 100	.75 087	.24 913	.94 012	36
25	.69 122	.75 117	.24 883	.94 005	35
26	.69 144	.75 146	.24 854	.93 998	34
27	.69 167	.75 176	.24 824	.93 991	33
28	.69 189	.75 205	.24 795	.93 984	32
29	.69 212	.75 235	.24 765	.93 977	31
30	.69 234	.75 264	.24 736	.93 970	**30**
31	.69 256	.75 294	.24 706	.93 963	29
32	.69 279	.75 323	.24 677	.93 955	28
33	.69 301	.75 353	.24 647	.93 948	27
34	.69 323	.75 382	.24 618	.93 941	26
35	.69 345	.75 411	.24 589	.93 934	25
36	.69 368	.75 441	.24 559	.93 927	24
37	.69 390	.75 470	.24 530	.93 920	23
38	.69 412	.75 500	.24 500	.93 912	22
39	.69 434	.75 529	.24 471	.93 905	21
40	.69 456	.75 558	.24 442	.93 898	**20**
41	.69 479	.75 588	.24 412	.93 891	19
42	.69 501	.75 617	.24 383	.93 884	18
43	.69 523	.75 647	.24 353	.93 876	17
44	.69 545	.75 676	.24 324	.93 869	16
45	.69 567	.75 705	.24 295	.93 862	15
46	.69 589	.75 735	.24 265	.93 855	14
47	.69 611	.75 764	.24 236	.93 847	13
48	.69 633	.75 793	.24 207	.93 840	12
49	.69 655	.75 822	.24 178	.93 833	11
50	.69 677	.75 852	.24 148	.93 826	**10**
51	.69 699	.75 881	.24 119	.93 819	9
52	.69 721	.75 910	.24 090	.93 811	8
53	.69 743	.75 939	.24 061	.93 804	7
54	.69 765	.75 969	.24 031	.93 797	6
55	.69 787	.75 998	.24 002	.93 789	5
56	.69 809	.76 027	.23 973	.93 782	4
57	.69 831	.76 056	.23 944	.93 775	3
58	.69 853	.76 086	.23 914	.93 768	2
59	.69 875	.76 115	.23 885	.93 760	1
60	.69 897	.76 144	.23 856	.93 753	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

60°

30°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.69 897	.76 144	.23 856	.93 753	**60**
1	.69 919	.76 173	.23 827	.93 746	59
2	.69 941	.76 202	.23 798	.93 738	58
3	.69 963	.76 231	.23 769	.93 731	57
4	.69 984	.76 261	.23 739	.93 724	56
5	.70 006	.76 290	.23 710	.93 717	55
6	.70 028	.76 319	.23 681	.93 709	54
7	.70 050	.76 348	.23 652	.93 702	53
8	.70 072	.76 377	.23 623	.93 695	52
9	.70 093	.76 406	.23 594	.93 687	51
10	.70 115	.76 435	.23 565	.93 680	**50**
11	.70 137	.76 464	.23 536	.93 673	49
12	.70 159	.76 493	.23 507	.93 665	48
13	.70 180	.76 522	.23 478	.93 658	47
14	.70 202	.76 551	.23 449	.93 650	46
15	.70 224	.76 580	.23 420	.93 643	45
16	.70 245	.76 609	.23 391	.93 636	44
17	.70 267	.76 639	.23 361	.93 628	43
18	.70 288	.76 668	.23 332	.93 621	42
19	.70 310	.76 697	.23 303	.93 614	41
20	.70 332	.76 725	.23 275	.93 606	**40**
21	.70 353	.76 754	.23 246	.93 599	39
22	.70 375	.76 783	.23 217	.93 591	38
23	.70 396	.76 812	.23 188	.93 584	37
24	.70 418	.76 841	.23 159	.93 577	36
25	.70 439	.76 870	.23 130	.93 569	35
26	.70 461	.76 899	.23 101	.93 562	34
27	.70 482	.76 928	.23 072	.93 554	33
28	.70 504	.76 957	.23 043	.93 547	32
29	.70 525	.76 986	.23 014	.93 539	31
30	.70 547	.77 015	.22 985	.93 532	**30**
31	.70 568	.77 044	.22 956	.93 525	29
32	.70 590	.77 073	.22 927	.93 517	28
33	.70 611	.77 101	.22 899	.93 510	27
34	.70 633	.77 130	.22 870	.93 502	26
35	.70 654	.77 159	.22 841	.93 495	25
36	.70 675	.77 188	.22 812	.93 487	24
37	.70 697	.77 217	.22 783	.93 480	23
38	.70 718	.77 246	.22 754	.93 472	22
39	.70 739	.77 274	.22 726	.93 465	21
40	.70 761	.77 303	.22 697	.93 457	**20**
41	.70 782	.77 332	.22 668	.93 450	19
42	.70 803	.77 361	.22 639	.93 442	18
43	.70 824	.77 390	.22 610	.93 435	17
44	.70 846	.77 418	.22 582	.93 427	16
45	.70 867	.77 447	.22 553	.93 420	15
46	.70 888	.77 476	.22 524	.93 412	14
47	.70 909	.77 505	.22 495	.93 405	13
48	.70 931	.77 533	.22 467	.93 397	12
49	.70 952	.77 562	.22 438	.93 390	11
50	.70 973	.77 591	.22 409	.93 382	**10**
51	.70 994	.77 619	.22 381	.93 375	9
52	.71 015	.77 648	.22 352	.93 367	8
53	.71 036	.77 677	.22 323	.93 360	7
54	.71 058	.77 706	.22 294	.93 352	6
55	.71 079	.77 734	.22 266	.93 344	5
56	.71 100	.77 763	.22 237	.93 337	4
57	.71 121	.77 791	.22 209	.93 329	3
58	.71 142	.77 820	.22 180	.93 322	2
59	.71 163	.77 849	.22 151	.93 314	1
60	.71 184	.77 877	.22 123	.93 307	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

59°

31°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.71 184	.77 877	.22 123	.93 307	**60**
1	.71 205	.77 906	.22 094	.93 299	59
2	.71 226	.77 935	.22 065	.93 291	58
3	.71 247	.77 963	.22 037	.93 284	57
4	.71 268	.77 992	.22 008	.93 276	56
5	.71 289	.78 020	.21 980	.93 269	55
6	.71 310	.78 049	.21 951	.93 261	54
7	.71 331	.78 077	.21 923	.93 253	53
8	.71 352	.78 106	.21 894	.93 246	52
9	.71 373	.78 135	.21 865	.93 238	51
10	.71 393	.78 163	.21 837	.93 230	**50**
11	.71 414	.78 192	.21 808	.93 223	49
12	.71 435	.78 220	.21 780	.93 215	48
13	.71 456	.78 249	.21 751	.93 207	47
14	.71 477	.78 277	.21 723	.93 200	46
15	.71 498	.78 306	.21 694	.93 192	45
16	.71 519	.78 334	.21 666	.93 184	44
17	.71 539	.78 363	.21 637	.93 177	43
18	.71 560	.78 391	.21 609	.93 169	42
19	.71 581	.78 419	.21 581	.93 161	41
20	.71 602	.78 448	.21 552	.93 154	**40**
21	.71 622	.78 476	.21 524	.93 146	39
22	.71 643	.78 505	.21 495	.93 138	38
23	.71 664	.78 533	.21 467	.93 131	37
24	.71 685	.78 562	.21 438	.93 123	36
25	.71 705	.78 590	.21 410	.93 115	35
26	.71 726	.78 618	.21 382	.93 108	34
27	.71 747	.78 647	.21 353	.93 100	33
28	.71 767	.78 675	.21 325	.93 092	32
29	.71 788	.78 704	.21 296	.93 084	31
30	.71 809	.78 732	.21 268	.93 077	**30**
31	.71 829	.78 760	.21 240	.93 069	29
32	.71 850	.78 789	.21 211	.93 061	28
33	.71 870	.78 817	.21 183	.93 053	27
34	.71 891	.78 845	.21 155	.93 046	26
35	.71 911	.78 874	.21 126	.93 038	25
36	.71 932	.78 902	.21 098	.93 030	24
37	.71 952	.78 930	.21 070	.93 022	23
38	.71 973	.78 959	.21 041	.93 014	22
39	.71 994	.78 987	.21 013	.93 007	21
40	.72 014	.79 015	.20 985	.92 999	**20**
41	.72 034	.79 043	.20 957	.92 991	19
42	.72 055	.79 072	.20 928	.92 983	18
43	.72 075	.79 100	.20 900	.92 976	17
44	.72 096	.79 128	.20 872	.92 968	16
45	.72 116	.79 156	.20 844	.92 960	15
46	.72 137	.79 185	.20 815	.92 952	14
47	.72 157	.79 213	.20 787	.92 944	13
48	.72 177	.79 241	.20 759	.92 936	12
49	.72 198	.79 269	.20 731	.92 929	11
50	.72 218	.79 297	.20 703	.92 921	**10**
51	.72 238	.79 326	.20 674	.92 913	9
52	.72 259	.79 354	.20 646	.92 905	8
53	.72 279	.79 382	.20 618	.92 897	7
54	.72 299	.79 410	.20 590	.92 889	6
55	.72 320	.79 438	.20 562	.92 881	5
56	.72 340	.79 466	.20 534	.92 874	4
57	.72 360	.79 495	.20 505	.92 866	3
58	.72 381	.79 523	.20 477	.92 858	2
59	.72 401	.79 551	.20 449	.92 850	1
60	.72 421	.79 579	.20 421	.92 842	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

58°

32°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.72 421	.79 579	.20 421	.92 842	**60**
1	.72 441	.79 607	.20 393	.92 834	59
2	.72 461	.79 635	.20 365	.92 826	58
3	.72 482	.79 663	.20 337	.92 818	57
4	.72 502	.79 691	.20 309	.92 810	56
5	.72 522	.79 719	.20 281	.92 803	55
6	.72 542	.79 747	.20 253	.92 795	54
7	.72 562	.79 776	.20 224	.92 787	53
8	.72 582	.79 804	.20 196	.92 779	52
9	.72 602	.79 832	.20 168	.92 771	51
10	.72 622	.79 860	.20 140	.92 763	**50**
11	.72 643	.79 888	.20 112	.92 755	49
12	.72 663	.79 916	.20 084	.92 747	48
13	.72 683	.79 944	.20 056	.92 739	47
14	.72 703	.79 972	.20 028	.92 731	46
15	.72 723	.80 000	.20 000	.92 723	45
16	.72 743	.80 028	.19 972	.92 715	44
17	.72 763	.80 056	.19 944	.92 707	43
18	.72 783	.80 084	.19 916	.92 699	42
19	.72 803	.80 112	.19 888	.92 691	41
20	.72 823	.80 140	.19 860	.92 683	**40**
21	.72 843	.80 168	.19 832	.92 675	39
22	.72 863	.80 195	.19 805	.92 667	38
23	.72 883	.80 223	.19 777	.92 659	37
24	.72 902	.80 251	.19 749	.92 651	36
25	.72 922	.80 279	.19 721	.92 643	35
26	.72 942	.80 307	.19 693	.92 635	34
27	.72 962	.80 335	.19 665	.92 627	33
28	.72 982	.80 363	.19 637	.92 619	32
29	.73 002	.80 391	.19 609	.92 611	31
30	.73 022	.80 419	.19 581	.92 603	**30**
31	.73 041	.80 447	.19 553	.92 595	29
32	.73 061	.80 474	.19 526	.92 587	28
33	.73 081	.80 502	.19 498	.92 579	27
34	.73 101	.80 530	.19 470	.92 571	26
35	.73 121	.80 558	.19 442	.92 563	25
36	.73 140	.80 586	.19 414	.92 555	24
37	.73 160	.80 614	.19 386	.92 546	23
38	.73 180	.80 642	.19 358	.92 538	22
39	.73 200	.80 669	.19 331	.92 530	21
40	.73 219	.80 697	.19 303	.92 522	**20**
41	.73 239	.80 725	.19 275	.92 514	19
42	.73 259	.80 753	.19 247	.92 506	18
43	.73 278	.80 781	.19 219	.92 498	17
44	.73 298	.80 808	.19 192	.92 490	16
45	.73 318	.80 836	.19 164	.92 482	15
46	.73 337	.80 864	.19 136	.92 473	14
47	.73 357	.80 892	.19 108	.92 465	13
48	.73 377	.80 919	.19 081	.92 457	12
49	.73 396	.80 947	.19 053	.92 449	11
50	.73 416	.80 975	.19 025	.92 441	**10**
51	.73 435	.81 003	.18 997	.92 433	9
52	.73 455	.81 030	.18 970	.92 425	8
53	.73 474	.81 058	.18 942	.92 416	7
54	.73 494	.81 086	.18 914	.92 408	6
55	.73 513	.81 113	.18 887	.92 400	5
56	.73 533	.81 141	.18 859	.92 392	4
57	.73 552	.81 169	.18 831	.92 384	3
58	.73 572	.81 196	.18 804	.92 376	2
59	.73 591	.81 224	.18 776	.92 367	1
60	.73 611	.81 252	.18 748	.92 359	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

57°

33°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.73 611	.81 252	.18 748	.92 359	**60**
1	.73 630	.81 279	.18 721	.92 351	59
2	.73 650	.81 307	.18 693	.92 343	58
3	.73 669	.81 335	.18 665	.92 335	57
4	.73 689	.81 362	.18 638	.92 326	56
5	.73 708	.81 390	.18 610	.92 318	55
6	.73 727	.81 418	.18 582	.92 310	54
7	.73 747	.81 445	.18 555	.92 302	53
8	.73 766	.81 473	.18 527	.92 293	52
9	.73 785	.81 500	.18 500	.92 285	51
10	.73 805	.81 528	.18 472	.92 277	**50**
11	.73 824	.81 556	.18 444	.92 269	49
12	.73 843	.81 583	.18 417	.92 260	48
13	.73 863	.81 611	.18 389	.92 252	47
14	.73 882	.81 638	.18 362	.92 244	46
15	.73 901	.81 666	.18 334	.92 235	45
16	.73 921	.81 693	.18 307	.92 227	44
17	.73 940	.81 721	.18 279	.92 219	43
18	.73 959	.81 748	.18 252	.92 211	42
19	.73 978	.81 776	.18 224	.92 202	41
20	.73 997	.81 803	.18 197	.92 194	**40**
21	.74 017	.81 831	.18 169	.92 186	39
22	.74 036	.81 858	.18 142	.92 177	38
23	.74 055	.81 886	.18 114	.92 169	37
24	.74 074	.81 913	.18 087	.92 161	36
25	.74 093	.81 941	.18 059	.92 152	35
26	.74 113	.81 968	.18 032	.92 144	34
27	.74 132	.81 996	.18 004	.92 136	33
28	.74 151	.82 023	.17 977	.92 127	32
29	.74 170	.82 051	.17 949	.92 119	31
30	.74 189	.82 078	.17 922	.92 111	**30**
31	.74 208	.82 106	.17 894	.92 102	29
32	.74 227	.82 133	.17 867	.92 094	28
33	.74 246	.82 161	.17 839	.92 086	27
34	.74 265	.82 188	.17 812	.92 077	26
35	.74 284	.82 215	.17 785	.92 069	25
36	.74 303	.82 243	.17 757	.92 060	24
37	.74 322	.82 270	.17 730	.92 052	23
38	.74 341	.82 298	.17 702	.92 044	22
39	.74 360	.82 325	.17 675	.92 035	21
40	.74 379	.82 352	.17 648	.92 027	**20**
41	.74 398	.82 380	.17 620	.92 018	19
42	.74 417	.82 407	.17 593	.92 010	18
43	.74 436	.82 435	.17 565	.92 002	17
44	.74 455	.82 462	.17 538	.91 993	16
45	.74 474	.82 489	.17 511	.91 985	15
46	.74 493	.82 517	.17 483	.91 976	14
47	.74 512	.82 544	.17 456	.91 968	13
48	.74 531	.82 571	.17 429	.91 959	12
49	.74 549	.82 599	.17 401	.91 951	11
50	.74 568	.82 626	.17 374	.91 942	**10**
51	.74 587	.82 653	.17 347	.91 934	9
52	.74 606	.82 681	.17 319	.91 925	8
53	.74 625	.82 708	.17 292	.91 917	7
54	.74 644	.82 735	.17 265	.91 908	6
55	.74 662	.82 762	.17 238	.91 900	5
56	.74 681	.82 790	.17 210	.91 891	4
57	.74 700	.82 817	.17 183	.91 883	3
58	.74 719	.82 844	.17 156	.91 874	2
59	.74 737	.82 871	.17 129	.91 866	1
60	.74 756	.82 899	.17 101	.91 857	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

56°

34°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.74 756	.82 899	.17 101	.91 857	**60**
1	.74 775	.82 926	.17 074	.91 849	59
2	.74 794	.82 953	.17 047	.91 840	58
3	.74 812	.82 980	.17 020	.91 832	57
4	.74 831	.83 008	.16 992	.91 823	56
5	.74 850	.83 035	.16 965	.91 815	55
6	.74 868	.83 062	.16 938	.91 806	54
7	.74 887	.83 089	.16 911	.91 798	53
8	.74 906	.83 117	.16 883	.91 789	52
9	.74 924	.83 144	.16 856	.91 781	51
10	.74 943	.83 171	.16 829	.91 772	**50**
11	.74 961	.83 198	.16 802	.91 763	49
12	.74 980	.83 225	.16 775	.91 755	48
13	.74 999	.83 252	.16 748	.91 746	47
14	.75 017	.83 280	.16 720	.91 738	46
15	.75 036	.83 307	.16 693	.91 729	45
16	.75 054	.83 334	.16 666	.91 720	44
17	.75 073	.83 361	.16 639	.91 712	43
18	.75 091	.83 388	.16 612	.91 703	42
19	.75 110	.83 415	.16 585	.91 695	41
20	.75 128	.83 442	.16 558	.91 686	**40**
21	.75 147	.83 470	.16 530	.91 677	39
22	.75 165	.83 497	.16 503	.91 669	38
23	.75 184	.83 524	.16 476	.91 660	37
24	.75 202	.83 551	.16 449	.91 651	36
25	.75 221	.83 578	.16 422	.91 643	35
26	.75 239	.83 605	.16 395	.91 634	34
27	.75 258	.83 632	.16 368	.91 625	33
28	.75 276	.83 659	.16 341	.91 617	32
29	.75 294	.83 686	.16 314	.91 608	31
30	.75 313	.83 713	.16 287	.91 599	**30**
31	.75 331	.83 740	.16 260	.91 591	29
32	.75 350	.83 768	.16 232	.91 582	28
33	.75 368	.83 795	.16 205	.91 573	27
34	.75 386	.83 822	.16 178	.91 565	26
35	.75 405	.83 849	.16 151	.91 556	25
36	.75 423	.83 876	.16 124	.91 547	24
37	.75 441	.83 903	.16 097	.91 538	23
38	.75 459	.83 930	.16 070	.91 530	22
39	.75 478	.83 957	.16 043	.91 521	21
40	.75 496	.83 984	.16 016	.91 512	**20**
41	.75 514	.84 011	.15 989	.91 504	19
42	.75 533	.84 038	.15 962	.91 495	18
43	.75 551	.84 065	.15 935	.91 486	17
44	.75 569	.84 092	.15 908	.91 477	16
45	.75 587	.84 119	.15 881	.91 469	15
46	.75 605	.84 146	.15 854	.91 460	14
47	.75 624	.84 173	.15 827	.91 451	13
48	.75 642	.84 200	.15 800	.91 442	12
49	.75 660	.84 227	.15 773	.91 433	11
50	.75 678	.84 254	.15 746	.91 425	**10**
51	.75 696	.84 280	.15 720	.91 416	9
52	.75 714	.84 307	.15 693	.91 407	8
53	.75 733	.84 334	.15 666	.91 398	7
54	.75 751	.84 361	.15 639	.91 389	6
55	.75 769	.84 388	.15 612	.91 381	5
56	.75 787	.84 415	.15 585	.91 372	4
57	.75 805	.84 442	.15 558	.91 363	3
58	.75 823	.84 469	.15 531	.91 354	2
59	.75 841	.84 496	.15 504	.91 345	1
60	.75 859	.84 523	.15 477	.91 336	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

55°

35°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.75 859	.84 523	.15 477	.91 336	**60**
1	.75 877	.84 550	.15 450	.91 328	59
2	.75 895	.84 576	.15 424	.91 319	58
3	.75 913	.84 603	.15 397	.91 310	57
4	.75 931	.84 630	.15 370	.91 301	56
5	.75 949	.84 657	.15 343	.91 292	55
6	.75 967	.84 684	.15 316	.91 283	54
7	.75 985	.84 711	.15 289	.91 274	53
8	.76 003	.84 738	.15 262	.91 266	52
9	.76 021	.84 764	.15 236	.91 257	51
10	.76 039	.84 791	.15 209	.91 248	**50**
11	.76 057	.84 818	.15 182	.91 239	49
12	.76 075	.84 845	.15 155	.91 230	48
13	.76 093	.84 872	.15 128	.91 221	47
14	.76 111	.84 899	.15 101	.91 212	46
15	.76 129	.84 925	.15 075	.91 203	45
16	.76 146	.84 952	.15 048	.91 194	44
17	.76 164	.84 979	.15 021	.91 185	43
18	.76 182	.85 006	.14 994	.91 176	42
19	.76 200	.85 033	.14 967	.91 167	41
20	.76 218	.85 059	.14 941	.91 158	**40**
21	.76 236	.85 086	.14 914	.91 149	39
22	.76 253	.85 113	.14 887	.91 141	38
23	.76 271	.85 140	.14 860	.91 132	37
24	.76 289	.85 166	.14 834	.91 123	36
25	.76 307	.85 193	.14 807	.91 114	35
26	.76 324	.85 220	.14 780	.91 105	34
27	.76 342	.85 247	.14 753	.91 096	33
28	.76 360	.85 273	.14 727	.91 087	32
29	.76 378	.85 300	.14 700	.91 078	31
30	.76 395	.85 327	.14 673	.91 069	**30**
31	.76 413	.85 354	.14 646	.91 060	29
32	.76 431	.85 380	.14 620	.91 051	28
33	.76 448	.85 407	.14 593	.91 042	27
34	.76 466	.85 434	.14 566	.91 033	26
35	.76 484	.85 460	.14 540	.91 023	25
36	.76 501	.85 487	.14 513	.91 014	24
37	.76 519	.85 514	.14 486	.91 005	23
38	.76 537	.85 540	.14 460	.90 996	22
39	.76 554	.85 567	.14 433	.90 987	21
40	.76 572	.85 594	.14 406	.90 978	**20**
41	.76 590	.85 620	.14 380	.90 969	19
42	.76 607	.85 647	.14 353	.90 960	18
43	.76 625	.85 674	.14 326	.90 951	17
44	.76 642	.85 700	.14 300	.90 942	16
45	.76 660	.85 727	.14 273	.90 933	15
46	.76 677	.85 754	.14 246	.90 924	14
47	.76 695	.85 780	.14 220	.90 915	13
48	.76 712	.85 807	.14 193	.90 906	12
49	.76 730	.85 834	.14 166	.90 896	11
50	.76 747	.85 860	.14 140	.90 887	**10**
51	.76 765	.85 887	.14 113	.90 878	9
52	.76 782	.85 913	.14 087	.90 869	8
53	.76 800	.85 940	.14 060	.90 860	7
54	.76 817	.85 967	.14 033	.90 851	6
55	.76 835	.85 993	.14 007	.90 842	5
56	.76 852	.86 020	.13 980	.90 832	4
57	.76 870	.86 046	.13 954	.90 823	3
58	.76 887	.86 073	.13 927	.90 814	2
59	.76 904	.86 100	.13 900	.90 805	1
60	.76 922	.86 126	.13 874	.90 796	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

54°

36°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.76 922	.86 126	.13 874	.90 796	**60**
1	.76 939	.86 153	.13 847	.90 787	59
2	.76 957	.86 179	.13 821	.90 777	58
3	.76 974	.86 206	.13 794	.90 768	57
4	.76 991	.86 232	.13 768	.90 759	56
5	.77 009	.86 259	.13 741	.90 750	55
6	.77 026	.86 285	.13 715	.90 741	54
7	.77 043	.86 312	.13 688	.90 731	53
8	.77 061	.86 338	.13 662	.90 722	52
9	.77 078	.86 365	.13 635	.90 713	51
10	.77 095	.86 392	.13 608	.90 704	**50**
11	.77 112	.86 418	.13 582	.90 694	49
12	.77 130	.86 445	.13 555	.90 685	48
13	.77 147	.86 471	.13 529	.90 676	47
14	.77 164	.86 498	.13 502	.90 667	46
15	.77 181	.86 524	.13 476	.90 657	45
16	.77 199	.86 551	.13 449	.90 648	44
17	.77 216	.86 577	.13 423	.90 639	43
18	.77 233	.86 603	.13 397	.90 630	42
19	.77 250	.86 630	.13 370	.90 620	41
20	.77 268	.86 656	.13 344	.90 611	**40**
21	.77 285	.86 683	.13 317	.90 602	39
22	.77 302	.86 709	.13 291	.90 592	38
23	.77 319	.86 736	.13 264	.90 583	37
24	.77 336	.86 762	.13 238	.90 574	36
25	.77 353	.86 789	.13 211	.90 565	35
26	.77 370	.86 815	.13 185	.90 555	34
27	.77 387	.86 842	.13 158	.90 546	33
28	.77 405	.86 868	.13 132	.90 537	32
29	.77 422	.86 894	.13 106	.90 527	31
30	.77 439	.86 921	.13 079	.90 518	**30**
31	.77 456	.86 947	.13 053	.90 509	29
32	.77 473	.86 974	.13 026	.90 499	28
33	.77 490	.87 000	.13 000	.90 490	27
34	.77 507	.87 027	.12 973	.90 480	26
35	.77 524	.87 053	.12 947	.90 471	25
36	.77 541	.87 079	.12 921	.90 462	24
37	.77 558	.87 106	.12 894	.90 452	23
38	.77 575	.87 132	.12 868	.90 443	22
39	.77 592	.87 158	.12 842	.90 434	21
40	.77 609	.87 185	.12 815	.90 424	**20**
41	.77 626	.87 211	.12 789	.90 415	19
42	.77 643	.87 238	.12 762	.90 405	18
43	.77 660	.87 264	.12 736	.90 396	17
44	.77 677	.87 290	.12 710	.90 386	16
45	.77 694	.87 317	.12 683	.90 377	15
46	.77 711	.87 343	.12 657	.90 368	14
47	.77 728	.87 369	.12 631	.90 358	13
48	.77 744	.87 396	.12 604	.90 349	12
49	.77 761	.87 422	.12 578	.90 339	11
50	.77 778	.87 448	.12 552	.90 330	**10**
51	.77 795	.87 475	.12 525	.90 320	9
52	.77 812	.87 501	.12 499	.90 311	8
53	.77 829	.87 527	.12 473	.90 301	7
54	.77 846	.87 554	.12 446	.90 292	6
55	.77 862	.87 580	.12 420	.90 282	5
56	.77 879	.87 606	.12 394	.90 273	4
57	.77 896	.87 633	.12 367	.90 263	3
58	.77 913	.87 659	.12 341	.90 254	2
59	.77 930	.87 685	.12 315	.90 244	1
60	.77 946	.87 711	.12 289	.90 235	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

53°

37°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.77 946	.87 711	.12 289	.90 235	**60**
1	.77 963	.87 738	.12 262	.90 225	59
2	.77 980	.87 764	.12 236	.90 216	58
3	.77 997	.87 790	.12 210	.90 206	57
4	.78 013	.87 817	.12 183	.90 197	56
5	.78 030	.87 843	.12 157	.90 187	55
6	.78 047	.87 869	.12 131	.90 178	54
7	.78 063	.87 895	.12 105	.90 168	53
8	.78 080	.87 922	.12 078	.90 159	52
9	.78 097	.87 948	.12 052	.90 149	51
10	.78 113	.87 974	.12 026	.90 139	**50**
11	.78 130	.88 000	.12 000	.90 130	49
12	.78 147	.88 027	.11 973	.90 120	48
13	.78 163	.88 053	.11 947	.90 111	47
14	.78 180	.88 079	.11 921	.90 101	46
15	.78 197	.88 105	.11 895	.90 091	45
16	.78 213	.88 131	.11 869	.90 082	44
17	.78 230	.88 158	.11 842	.90 072	43
18	.78 246	.88 184	.11 816	.90 063	42
19	.78 263	.88 210	.11 790	.90 053	41
20	.78 280	.88 236	.11 764	.90 043	**40**
21	.78 296	.88 262	.11 738	.90 034	39
22	.78 313	.88 289	.11 711	.90 024	38
23	.78 329	.88 315	.11 685	.90 014	37
24	.78 346	.88 341	.11 659	.90 005	36
25	.78 362	.88 367	.11 633	.89 995	35
26	.78 379	.88 393	.11 607	.89 985	34
27	.78 395	.88 420	.11 580	.89 976	33
28	.78 412	.88 446	.11 554	.89 966	32
29	.78 428	.88 472	.11 528	.89 956	31
30	.78 445	.88 498	.11 502	.89 947	**30**
31	.78 461	.88 524	.11 476	.89 937	29
32	.78 478	.88 550	.11 450	.89 927	28
33	.78 494	.88 577	.11 423	.89 918	27
34	.78 510	.88 603	.11 397	.89 908	26
35	.78 527	.88 629	.11 371	.89 898	25
36	.78 543	.88 655	.11 345	.89 888	24
37	.78 560	.88 681	.11 319	.89 879	23
38	.78 576	.88 707	.11 293	.89 869	22
39	.78 592	.88 733	.11 267	.89 859	21
40	.78 609	.88 759	.11 241	.89 849	**20**
41	.78 625	.88 786	.11 214	.89 840	19
42	.78 642	.88 812	.11 188	.89 830	18
43	.78 658	.88 838	.11 162	.89 820	17
44	.78 674	.88 864	.11 136	.89 810	16
45	.78 691	.88 890	.11 110	.89 801	15
46	.78 707	.88 916	.11 084	.89 791	14
47	.78 723	.88 942	.11 058	.89 781	13
48	.78 739	.88 968	.11 032	.89 771	12
49	.78 756	.88 994	.11 006	.89 761	11
50	.78 772	.89 020	.10 980	.89 752	**10**
51	.78 788	.89 046	.10 954	.89 742	9
52	.78 805	.89 073	.10 927	.89 732	8
53	.78 821	.89 099	.10 901	.89 722	7
54	.78 837	.89 125	.10 875	.89 712	6
55	.78 853	.89 151	.10 849	.89 702	5
56	.78 869	.89 177	.10 823	.88 693	4
57	.78 886	.89 203	.10 797	.89 683	3
58	.78 902	.89 229	.10 771	.89 673	2
59	.78 918	.89 255	.10 745	.89 663	1
60	.78 934	.89 281	.10 719	.89 653	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

52°

38°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.78 934	.89 281	.10 719	.89 653	**60**
1	.78 950	.89 307	.10 693	.89 643	59
2	.78 967	.89 333	.10 667	.89 633	58
3	.78 983	.89 359	.10 641	.89 624	57
4	.78 999	.89 385	.10 615	.89 614	56
5	.79 015	.89 411	.10 589	.89 604	55
6	.79 031	.89 437	.10 563	.89 594	54
7	.79 047	.89 463	.10 537	.89 584	53
8	.79 063	.89 489	.10 511	.89 574	52
9	.79 079	.89 515	.10 485	.89 564	51
10	.79 095	.89 541	.10 459	.89 554	**50**
11	.79 111	.89 567	.10 433	.89 544	49
12	.79 128	.89 593	.10 407	.89 534	48
13	.79 144	.89 619	.10 381	.89 524	47
14	.79 160	.89 645	.10 355	.89 514	46
15	.79 176	.89 671	.10 329	.89 504	45
16	.79 192	.89 697	.10 303	.89 495	44
17	.79 208	.89 723	.10 277	.89 485	43
18	.79 224	.89 749	.10 251	.89 475	42
19	.79 240	.89 775	.10 225	.89 465	41
20	.79 256	.89 801	.10 199	.89 455	**40**
21	.79 272	.89 827	.10 173	.89 445	39
22	.79 288	.89 853	.10 147	.89 435	38
23	.79 304	.89 879	.10 121	.89 425	37
24	.79 319	.89 905	.10 095	.89 415	36
25	.79 335	.89 931	.10 069	.89 405	35
26	.79 351	.89 957	.10 043	.89 395	34
27	.79 367	.89 983	.10 017	.89 385	33
28	.79 383	.90 009	.09 991	.89 375	32
29	.79 399	.90 035	.09 965	.89 364	31
30	.79 415	.90 061	.09 939	.89 354	**30**
31	.79 431	.90 086	.09 914	.89 344	29
32	.79 447	.90 112	.09 888	.89 334	28
33	.79 463	.90 138	.09 862	.89 324	27
34	.79 478	.90 164	.09 836	.89 314	26
35	.79 494	.90 190	.09 810	.89 304	25
36	.79 510	.90 216	.09 784	.89 294	24
37	.79 526	.90 242	.09 758	.89 284	23
38	.79 542	.90 268	.09 732	.89 274	22
39	.79 558	.90 294	.09 706	.89 264	21
40	.79 573	.90 320	.09 680	.89 254	**20**
41	.79 589	.90 346	.09 654	.89 244	19
42	.79 605	.90 371	.09 629	.89 233	18
43	.79 621	.90 397	.09 603	.89 223	17
44	.79 636	.90 423	.09 577	.89 213	16
45	.79 652	.90 449	.09 551	.89 203	15
46	.79 668	.90 475	.09 525	.89 193	14
47	.79 684	.90 501	.09 499	.89 183	13
48	.79 699	.90 527	.09 473	.89 173	12
49	.79 715	.90 553	.09 447	.89 162	11
50	.79 731	.90 578	.09 422	.89 152	**10**
51	.79 746	.90 604	.09 396	.89 142	9
52	.79 762	.90 630	.09 370	.89 132	8
53	.79 778	.90 656	.09 344	.89 122	7
54	.79 793	.90 682	.09 318	.89 112	6
55	.79 809	.90 708	.09 292	.89 101	5
56	.79 825	.90 734	.09 266	.89 091	4
57	.79 840	.90 759	.09 241	.89 081	3
58	.79 856	.90 785	.09 215	.89 071	2
59	.79 872	.90 811	.09 189	.89 060	1
60	.79 887	.90 837	.09 163	.89 050	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

51°

39°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.79 887	.90 837	.09 163	.89 050	**60**
1	.79 903	.90 863	.09 137	.89 040	59
2	.79 918	.90 889	.09 111	.89 030	58
3	.79 934	.90 914	.09 086	.89 020	57
4	.79 950	.90 940	.09 060	.89 009	56
5	.79 965	.90 966	.09 034	.88 999	55
6	.79 981	.90 992	.09 008	.88 989	54
7	.79 996	.91 018	.08 982	.88 978	53
8	.80 012	.91 043	.08 957	.88 968	52
9	.80 027	.91 069	.08 931	.88 958	51
10	.80 043	.91 095	.08 905	.88 948	**50**
11	.80 058	.91 121	.08 879	.88 937	49
12	.80 074	.91 147	.08 853	.88 927	48
13	.80 089	.91 172	.08 828	.88 917	47
14	.80 105	.91 198	.08 802	.88 906	46
15	.80 120	.91 224	.08 776	.88 896	45
16	.80 136	.91 250	.08 750	.88 886	44
17	.80 151	.91 276	.08 724	.88 875	43
18	.80 166	.91 301	.08 699	.88 865	42
19	.80 182	.91 327	.08 673	.88 855	41
20	.80 197	.91 353	.08 647	.88 844	**40**
21	.80 213	.91 379	.08 621	.88 834	39
22	.80 228	.91 404	.08 596	.88 824	38
23	.80 244	.91 430	.08 570	.88 813	37
24	.80 259	.91 456	.08 544	.88 803	36
25	.80 274	.91 482	.08 518	.88 793	35
26	.80 290	.91 507	.08 493	.88 782	34
27	.80 305	.91 533	.08 467	.88 772	33
28	.80 320	.91 559	.08 441	.88 761	32
29	.80 336	.91 585	.08 415	.88 751	31
30	.80 351	.91 610	.08 390	.88 741	**30**
31	.80 366	.91 636	.08 364	.88 730	29
32	.80 382	.91 662	.08 338	.88 720	28
33	.80 397	.91 688	.08 312	.88 709	27
34	.80 412	.91 713	.08 287	.88 699	26
35	.80 428	.91 739	.08 261	.88 688	25
36	.80 443	.91 765	.08 235	.88 678	24
37	.80 458	.91 791	.08 209	.88 668	23
38	.80 473	.91 816	.08 184	.88 657	22
39	.80 489	.91 842	.08 158	.88 647	21
40	.80 504	.91 868	.08 132	.88 636	**20**
41	.80 519	.91 893	.08 107	.88 626	19
42	.80 534	.91 919	.08 081	.88 615	18
43	.80 550	.91 945	.08 055	.88 605	17
44	.80 565	.91 971	.08 029	.88 594	16
45	.80 580	.91 996	.08 004	.88 584	15
46	.80 595	.92 022	.07 978	.88 573	14
47	.80 610	.92 048	.07 952	.88 563	13
48	.80 625	.92 073	.07 927	.88 552	12
49	.80 641	.92 099	.07 901	.88 542	11
50	.80 656	.92 125	.07 875	.88 531	**10**
51	.80 671	.92 150	.07 850	.88 521	9
52	.80 686	.92 176	.07 824	.88 510	8
53	.80 701	.92 202	.07 798	.88 499	7
54	.80 716	.92 227	.07 773	.88 489	6
55	.80 731	.92 253	.07 747	.88 478	5
56	.80 746	.92 279	.07 721	.88 468	4
57	.80 762	.92 304	.07 696	.88 457	3
58	.80 777	.92 330	.07 670	.88 447	2
59	.80 792	.92 356	.07 644	.88 436	1
60	.80 807	.92 381	.07 619	.88 425	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

50°

40°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.80 807	.92 381	.07 619	.88 425	**60**
1	.80 822	.92 407	.07 593	.88 415	59
2	.80 837	.92 433	.07 567	.88 404	58
3	.80 852	.92 458	.07 542	.88 394	57
4	.80 867	.92 484	.07 516	.88 383	56
5	.80 882	.92 510	.07 490	.88 372	55
6	.80 897	.92 535	.07 465	.88 362	54
7	.80 912	.92 561	.07 439	.88 351	53
8	.80 927	.92 587	.07 413	.88 340	52
9	.80 942	.92 612	.07 388	.88 330	51
10	.80 957	.92 638	.07 362	.88 319	**50**
11	.80 972	.92 663	.07 337	.88 308	49
12	.80 987	.92 689	.07 311	.88 298	48
13	.81 002	.92 715	.07 285	.88 287	47
14	.81 017	.92 740	.07 260	.88 276	46
15	.81 032	.92 766	.07 234	.88 266	45
16	.81 047	.92 792	.07 208	.88 255	44
17	.81 061	.92 817	.07 183	.88 244	43
18	.81 076	.92 843	.07 157	.88 234	42
19	.81 091	.92 868	.07 132	.88 223	41
20	.81 106	.92 894	.07 106	.88 212	**40**
21	.81 121	.92 920	.07 080	.88 201	39
22	.81 136	.92 945	.07 055	.88 191	38
23	.81 151	.92 971	.07 029	.88 180	37
24	.81 166	.92 996	.07 004	.88 169	36
25	.81 180	.93 022	.06 978	.88 158	35
26	.81 195	.93 048	.06 952	.88 148	34
27	.81 210	.93 073	.06 927	.88 137	33
28	.81 225	.93 099	.06 901	.88 126	32
29	.81 240	.93 124	.06 876	.88 115	31
30	.81 254	.93 150	.06 850	.88 105	**30**
31	.81 269	.93 175	.06 825	.88 094	29
32	.81 284	.93 201	.06 799	.88 083	28
33	.81 299	.93 227	.06 773	.88 072	27
34	.81 314	.93 252	.06 748	.88 061	26
35	.81 328	.93 278	.06 722	.88 051	25
36	.81 343	.93 303	.06 697	.88 040	24
37	.81 358	.93 329	.06 671	.88 029	23
38	.81 372	.93 354	.06 646	.88 018	22
39	.81 387	.93 380	.06 620	.88 007	21
40	.81 402	.93 406	.06 594	.87 996	**20**
41	.81 417	.93 431	.06 569	.87 985	19
42	.81 431	.93 457	.06 543	.87 975	18
43	.81 446	.93 482	.06 518	.87 964	17
44	.81 461	.93 508	.06 492	.87 953	16
45	.81 475	.93 533	.06 467	.87 942	15
46	.81 490	.93 559	.06 441	.87 931	14
47	.81 505	.93 584	.06 416	.87 920	13
48	.81 519	.93 610	.06 390	.87 909	12
49	.81 534	.93 636	.06 364	.87 898	11
50	.81 549	.93 661	.06 339	.87 887	**10**
51	.81 563	.93 687	.06 313	.87 877	9
52	.81 578	.93 712	.06 288	.87 866	8
53	.81 592	.93 738	.06 262	.87 855	7
54	.81 607	.93 763	.06 237	.87 844	6
55	.81 622	.93 789	.06 211	.87 833	5
56	.81 636	.93 814	.06 186	.87 822	4
57	.81 651	.93 840	.06 160	.87 811	3
58	.81 665	.93 865	.06 135	.87 800	2
59	.81 680	.93 891	.06 109	.87 789	1
60	.81 694	.93 916	.06 084	.87 778	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

49°

41°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.81 694	.93 916	.06 084	.87 778	**60**
1	.81 709	.93 942	.06 058	.87 767	59
2	.81 723	.93 967	.06 033	.87 756	58
3	.81 738	.93 993	.06 007	.87 745	57
4	.81 752	.94 018	.05 982	.87 734	56
5	.81 767	.94 044	.05 956	.87 723	55
6	.81 781	.94 069	.05 931	.87 712	54
7	.81 796	.94 095	.05 905	.87 701	53
8	.81 810	.94 120	.05 880	.87 690	52
9	.81 825	.94 146	.05 854	.87 679	51
10	.81 839	.94 171	.05 829	.87 668	**50**
11	.81 854	.94 197	.05 803	.87 657	49
12	.81 868	.94 222	.05 778	.87 646	48
13	.81 882	.94 248	.05 752	.87 635	47
14	.81 897	.94 273	.05 727	.87 624	46
15	.81 911	.94 299	.05 701	.87 613	45
16	.81 926	.94 324	.05 676	.87 601	44
17	.81 940	.94 350	.05 650	.87 590	43
18	.81 955	.94 375	.05 625	.87 579	42
19	.81 969	.94 401	.05 599	.87 568	41
20	.81 983	.94 426	.05 574	.87 557	**40**
21	.81 998	.94 452	.05 548	.87 546	39
22	.82 012	.94 477	.05 523	.87 535	38
23	.82 026	.94 503	.05 497	.87 524	37
24	.82 041	.94 528	.05 472	.87 513	36
25	.82 055	.94 554	.05 446	.87 501	35
26	.82 069	.94 579	.05 421	.87 490	34
27	.82 084	.94 604	.05 396	.87 479	33
28	.82 098	.94 630	.05 370	.87 468	32
29	.82 112	.94 655	.05 345	.87 457	31
30	.82 126	.94 681	.05 319	.87 446	**30**
31	.82 141	.94 706	.05 294	.87 434	29
32	.82 155	.94 732	.05 268	.87 423	28
33	.82 169	.94 757	.05 243	.87 412	27
34	.82 184	.94 783	.05 217	.87 401	26
35	.82 198	.94 808	.05 192	.87 390	25
36	.82 212	.94 834	.05 166	.87 378	24
37	.82 226	.94 859	.05 141	.87 367	23
38	.82 240	.94 884	.05 116	.87 356	22
39	.82 255	.94 910	.05 090	.87 345	21
40	.82 269	.94 935	.05 065	.87 334	**20**
41	.82 283	.94 961	.05 039	.87 322	19
42	.82 297	.94 986	.05 014	.87 311	18
43	.82 311	.95 012	.04 988	.87 300	17
44	.82 326	.95 037	.04 963	.87 288	16
45	.82 340	.95 062	.04 938	.87 277	15
46	.82 354	.95 088	.04 912	.87 266	14
47	.82 368	.95 113	.04 887	.87 255	13
48	.82 382	.95 139	.04 861	.87 243	12
49	.82 396	.95 164	.04 836	.87 232	11
50	.82 410	.95 190	.04 810	.87 221	**10**
51	.82 424	.95 215	.04 785	.87 209	9
52	.82 439	.95 240	.04 760	.87 198	8
53	.82 453	.95 266	.04 734	.87 187	7
54	.82 467	.95 291	.04 709	.87 175	6
55	.82 481	.95 317	.04 683	.87 164	5
56	.82 495	.95 342	.04 658	.87 153	4
57	.82 509	.95 368	.04 632	.87 141	3
58	.82 523	.95 393	.04 607	.87 130	2
59	.82 537	.95 418	.04 582	.87 119	1
60	.82 551	.95 444	.04 556	.87 107	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

48°

42°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.82 551	.95 444	.04 556	.87 107	**60**
1	.82 565	.95 469	.04 531	.87 096	59
2	.82 579	.95 495	.04 505	.87 085	58
3	.82 593	.95 520	.04 480	.87 073	57
4	.82 607	.95 545	.04 455	.87 062	56
5	.82 621	.95 571	.04 429	.87 050	55
6	.82 635	.95 596	.04 404	.87 039	54
7	.82 649	.95 622	.04 378	.87 028	53
8	.82 663	.95 647	.04 353	.87 016	52
9	.82 677	.95 672	.04 328	.87 005	51
10	.82 691	.95 698	.04 302	.86 993	**50**
11	.82 705	.95 723	.04 277	.86 982	49
12	.82 719	.95 748	.04 252	.86 970	48
13	.82 733	.95 774	.04 226	.86 959	47
14	.82 747	.95 799	.04 201	.86 947	46
15	.82 761	.95 825	.04 175	.86 936	45
16	.82 775	.95 850	.04 150	.86 924	44
17	.82 788	.95 875	.04 125	.86 913	43
18	.82 802	.95 901	.04 099	.86 902	42
19	.82 816	.95 926	.04 074	.86 890	41
20	.82 830	.95 952	.04 048	.86 879	**40**
21	.82 844	.95 977	.04 023	.86 867	39
22	.82 858	.96 002	.03 998	.86 855	38
23	.82 872	.96 028	.03 972	.86 844	37
24	.82 885	.96 053	.03 947	.86 832	36
25	.82 899	.96 078	.03 922	.86 821	35
26	.82 913	.96 104	.03 896	.86 809	34
27	.82 927	.96 129	.03 871	.86 798	33
28	.82 941	.96 155	.03 845	.86 786	32
29	.82 955	.96 180	.03 820	.86 775	31
30	.82 968	.96 205	.03 795	.86 763	**30**
31	.82 982	.96 231	.03 769	.86 752	29
32	.82 996	.96 256	.03 744	.86 740	28
33	.83 010	.96 281	.03 719	.86 728	27
34	.83 023	.96 307	.03 693	.86 717	26
35	.83 037	.96 332	.03 668	.86 705	25
36	.83 051	.96 357	.03 643	.86 694	24
37	.83 065	.96 383	.03 617	.86 682	23
38	.83 078	.96 408	.03 592	.86 670	22
39	.83 092	.96 433	.03 567	.86 659	21
40	.83 106	.96 459	.03 541	.86 647	**20**
41	.83 120	.96 484	.03 516	.86 635	19
42	.83 133	.96 510	.03 490	.86 624	18
43	.83 147	.96 535	.03 465	.86 612	17
44	.83 161	.96 560	.03 440	.86 600	16
45	.83 174	.96 586	.03 414	.86 589	15
46	.83 188	.96 611	.03 389	.86 577	14
47	.83 202	.96 636	.03 364	.86 565	13
48	.83 215	.96 662	.03 338	.86 554	12
49	.83 229	.96 687	.03 313	.86 542	11
50	.83 242	.96 712	.03 288	.86 530	**10**
51	.83 256	.96 738	.03 262	.86 518	9
52	.83 270	.96 763	.03 237	.86 507	8
53	.83 283	.96 788	.03 212	.86 495	7
54	.83 297	.96 814	.03 186	.86 483	6
55	.83 310	.96 839	.03 161	.86 472	5
56	.83 324	.96 864	.03 136	.86 460	4
57	.83 338	.96 890	.03 110	.86 448	3
58	.83 351	.96 915	.03 085	.86 436	2
59	.83 365	.96 940	.03 060	.86 425	1
60	.83 378	.96 966	.03 034	.86 413	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

47°

43°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.83 378	.96 966	.03 034	.86 413	**60**
1	.83 392	.96 991	.03 009	.86 401	59
2	.83 405	.97 016	.02 984	.86 389	58
3	.83 419	.97 042	.02 958	.86 377	57
4	.83 432	.97 067	.02 933	.86 366	56
5	.83 446	.97 092	.02 908	.86 354	55
6	.83 459	.97 118	.02 882	.86 342	54
7	.83 473	.97 143	.02 857	.86 330	53
8	.83 486	.97 168	.02 832	.86 318	52
9	.83 500	.97 193	.02 807	.86 306	51
10	.83 513	.97 219	.02 781	.86 295	**50**
11	.83 527	.97 244	.02 756	.86 283	49
12	.83 540	.97 269	.02 731	.86 771	48
13	.83 554	.97 295	.02 705	.86 259	47
14	.83 567	.97 320	.02 680	.86 247	46
15	.83 581	.97 345	.02 655	.86 235	45
16	.83 594	.97 371	.02 629	.86 223	44
17	.83 608	.97 396	.02 604	.86 211	43
18	.83 621	.97 421	.02 579	.86 200	42
19	.83 634	.97 447	.02 553	.86 188	41
20	.83 648	.97 472	.02 528	.86 176	**40**
21	.83 661	.97 497	.02 503	.86 164	39
22	.83 674	.97 523	.02 477	.86 152	38
23	.83 688	.97 548	.02 452	.86 140	37
24	.83 701	.97 573	.02 427	.86 128	36
25	.83 715	.97 598	.02 402	.86 116	35
26	.83 728	.97 624	.02 376	.86 104	34
27	.83 741	.97 649	.02 351	.86 092	33
28	.83 755	.97 674	.02 326	.86 080	32
29	.83 768	.97 700	.02 300	.86 068	31
30	.83 781	.97 725	.02 275	.86 056	**30**
31	.83 795	.97 750	.02 250	.86 044	29
32	.83 808	.97 776	.02 224	.86 032	28
33	.83 821	.97 801	.02 199	.86 020	27
34	.83 834	.97 826	.02 174	.86 008	26
35	.83 848	.97 851	.02 149	.85 996	25
36	.83 861	.97 877	.02 123	.85 984	24
37	.83 874	.97 902	.02 098	.85 972	23
38	.83 887	.97 927	.02 073	.85 960	22
39	.83 901	.97 953	.02 047	.85 948	21
40	.83 914	.97 978	.02 022	.85 936	**20**
41	.83 927	.98 003	.01 997	.85 924	19
42	.83 940	.98 029	.01 971	.85 912	18
43	.88 954	.98 054	.01 946	.85 900	17
44	.83 967	.98 079	.01 921	.85 888	16
45	.83 980	.98 104	.01 896	.85 876	15
46	.83 993	.98 130	.01 870	.85 864	14
47	.84 006	.98 155	.01 845	.85 851	13
48	.84 020	.98 180	.01 820	.85 839	12
49	.84 033	.98 206	.01 794	.85 827	11
50	.84 046	.98 231	.01 769	.85 815	**10**
51	.84 059	.98 256	.01 744	.85 803	9
52	.84 072	.98 281	.01 719	.85 791	8
53	.84 085	.98 307	.01 693	.85 779	7
54	.84 098	.98 332	.01 668	.85 766	6
55	.84 112	.98 357	.01 643	.85 754	5
56	.84 125	.98 383	.01 617	.85 742	4
57	.84 138	.98 408	.01 592	.85 730	3
58	.84 151	.98 433	.01 567	.85 718	2
59	.84 164	.98 458	.01 542	.85 706	1
60	.84 177	.98 484	.01 516	.85 693	**0**
′	9 L cos	9 L cot	10 L tan	9 L sin	′

46°

44°

′	9 L sin	9 L tan	10 L cot	9 L cos	′
0	.84 177	.98 484	.01 516	.85 693	**60**
1	.84 190	.98 509	.01 491	.85 681	59
2	.84 203	.98 534	.01 466	.85 669	58
3	.84 216	.98 560	.01 440	.85 657	57
4	.84 229	.98 585	.01 415	.85 645	56
5	.84 242	.98 610	.01 390	.85 632	55
6	.84 255	.98 635	.01 365	.85 620	54
7	.84 269	.98 661	.01 339	.85 608	53
8	.84 282	.98 686	.01 314	.85 596	52
9	.84 295	.98 711	.01 289	.85 583	51
10	.84 308	.98 737	.01 263	.85 571	**50**
11	.84 321	.98 762	.01 238	.85 559	49
12	.84 334	.98 787	.01 213	.85 547	48
13	.84 347	.98 812	.01 188	.85 534	47
14	.84 360	.98 838	.01 162	.85 522	46
15	.84 373	.98 863	.01 137	.85 510	45
16	.84 385	.98 888	.01 112	.85 497	44
17	.84 398	.98 913	.01 087	.85 485	43
18	.84 411	.98 939	.01 061	.85 473	42
19	.84 424	.98 964	.01 036	.85 460	41
20	.84 437	.98 989	.01 011	.85 448	**40**
21	.84 450	.99 015	.00 985	.85 436	39
22	.84 463	.99 040	.00 960	.85 423	38
23	.84 476	.99 065	.00 935	.85 411	37
24	.84 489	.99 090	.00 910	.85 399	36
25	.84 502	.99 116	.00 884	.85 386	35
26	.84 515	.99 141	.00 859	.85 374	34
27	.84 528	.99 166	.00 834	.85 361	33
28	.84 540	.99 191	.00 809	.85 349	32
29	.84 553	.99 217	.00 783	.85 337	31
30	.84 566	.99 242	.00 758	.85 324	**30**
31	.84 579	.99 267	.00 733	.85 312	29
32	.84 592	.99 293	.00 707	.85 299	28
33	.84 605	.99 318	.00 682	.85 287	27
34	.84 618	.99 343	.00 657	.85 274	26
35	.84 630	.99 368	.00 632	.85 262	25
36	.84 643	.99 394	.00 606	.85 250	24
37	.84 656	.99 419	.00 581	.85 237	23
38	.84 669	.99 444	.00 556	.85 225	22
39	.84 682	.99 469	.00 531	.85 212	21
40	.84 694	.99 495	.00 505	.85 200	**20**
41	.84 707	.99 520	.00 480	.85 187	19
42	.84 720	.99 545	.00 455	.85 175	18
43	.84 733	.99 570	.00 430	.85 162	17
44	.84 745	.99 596	.00 404	.85 150	16
45	.84 758	.99 621	.00 379	.85 137	15
46	.84 771	.99 646	.00 354	.85 125	14
47	.84 784	.99 672	.00 328	.85 112	13
48	.84 796	.99 697	.00 303	.85 100	12
49	.84 809	.99 722	.00 278	.85 087	11
50	.84 822	.99 747	.00 253	.85 074	**10**
51	.84 835	.99 773	.00 227	.85 062	9
52	.84 847	.99 798	.00 202	.85 049	8
53	.84 860	.99 823	.00 177	.85 037	7
54	.84 873	.99 848	.00 152	.85 024	6
55	.84 885	.99 874	.00 126	.85 012	5
56	.84 898	.99 899	.00 101	.84 999	4
57	.84 911	.99 924	.00 076	.84 986	3
58	.84 923	.99 949	.00 051	.84 974	2
59	.84 936	.99 975	.00 025	.84 961	1
60	.84 949	.00 000	.00 000	.84 949	**0**
′	9 L cos	10 L cot	10 L tan	9 L sin	′

45°

	0°				
′	sin	tan	cot	cos	′
0	.00 000	.00 000	∞	1.0000	**60**
1	.00 029	.00 029	3437.7	1.0000	59
2	.00 058	.00 058	1718.9	1.0000	58
3	.00 087	.00 087	1145.9	1.0000	57
4	.00 116	.00 116	859.44	1.0000	56
5	.00 145	.00 145	687.55	1.0000	55
6	.00 175	.00 175	572.96	1.0000	54
7	.00 204	.00 204	491.11	1.0000	53
8	.00 233	.00 233	429.72	1.0000	52
9	.00 262	.00 262	381.97	1.0000	51
10	.00 291	.00 291	343.77	1.0000	**50**
11	.00 320	.00 320	312.52	.99 999	49
12	.00 349	.00 349	286.48	.99 999	48
13	.00 378	.00 378	264.44	.99 999	47
14	.00 407	.00 407	245.55	.99 999	46
15	.00 436	.00 436	229.18	.99 999	45
16	.00 465	.00 465	214.86	.99 999	44
17	.00 495	.00 495	202.22	.99 999	43
18	.00 524	.00 524	190.98	.99 999	42
19	.00 553	.00 553	180.93	.99 998	41
20	.00 582	.00 582	171.89	.99 998	**40**
21	.00 611	.00 611	163.70	.99 998	39
22	.00 640	.00 640	156.26	.99 998	38
23	.00 669	.00 669	149.47	.99 998	37
24	.00 698	.00 698	143.24	.99 998	36
25	.00 727	.00 727	137.51	.99 997	35
26	.00 756	.00 756	132.22	.99 997	34
27	.00 785	.00 785	127.32	.99 997	33
28	.00 814	.00 815	122.77	.99 997	32
29	.00 844	.00 844	118.54	.99 996	31
30	.00 873	.00 873	114.59	.99 996	**30**
31	.00 902	.00 902	110.89	.99 996	29
32	.00 931	.00 931	107.43	.99 996	28
33	.00 960	.00 960	104.17	.99 995	27
34	.00 989	.00 989	101.11	.99 995	26
35	.01 018	.01 018	98.218	.99 995	25
36	.01 047	.01 047	95.489	.99 995	24
37	.01 076	.01 076	92.908	.99 994	23
38	.01 105	.01 105	90.463	.99 994	22
39	.01 134	.01 135	88.144	.99 994	21
40	.01 164	.01 164	85.940	.99 993	**20**
41	.01 193	.01 193	83.844	.99 993	19
42	.01 222	.01 222	81.847	.99 993	18
43	.01 251	.01 251	79.943	.99 992	17
44	.01 280	.01 280	78.126	.99 992	16
45	.01 309	.01 309	76.390	.99 991	15
46	.01 338	.01 338	74.729	.99 991	14
47	.01 367	.01 367	73.139	.99 991	13
48	.01 396	.01 396	71.615	.99 990	12
49	.01 425	.01 425	70.153	.99 990	11
50	.01 454	.01 455	68.750	.99 989	**10**
51	.01 483	.01 484	67.402	.99 989	9
52	.01 513	.01 513	66.105	.99 989	8
53	.01 542	.01 542	64.858	.99 988	7
54	.01 571	.01 571	63.657	.99 988	6
55	.01 600	.01 600	62.499	.99 987	5
56	.01 629	.01 629	61.383	.99 987	4
57	.01 658	.01 658	60.306	.99 986	3
58	.01 687	.01 687	59.266	.99 986	2
59	.01 716	.01 716	58.261	.99 985	1
60	.01 745	.01 746	57.290	.99 985	**0**
′	cos	cot	tan	sin	′
	89°				

	1°				
′	sin	tan	cot	cos	′
0	.01 745	.01 746	57.290	.99 985	**60**
1	.01 774	.01 775	56.351	.99 984	59
2	.01 803	.01 804	55.442	.99 984	58
3	.01 832	.01 833	54.561	.99 983	57
4	.01 862	.01 862	53.709	.99 983	56
5	.01 891	.01 891	52.882	.99 982	55
6	.01 920	.01 920	52.081	.99 982	54
7	.01 949	.01 949	51.303	.99 981	53
8	.01 978	.01 978	50.549	.99 980	52
9	.02 007	.02 007	49.816	.99 980	51
10	.02 036	.02 036	49.104	.99 979	**50**
11	.02 065	.02 066	48.412	.99 979	49
12	.02 094	.02 095	47.740	.99 978	48
13	.02 123	.02 124	47.085	.99 977	47
14	.02 152	.02 153	46.449	.99 977	46
15	.02 181	.02 182	45.829	.99 976	45
16	.02 211	.02 211	45.226	.99 976	44
17	.02 240	.02 240	44.639	.99 975	43
18	.02 269	.02 269	44.066	.99 974	42
19	.02 298	.02 298	43.508	.99 974	41
20	.02 327	.02 328	42.964	.99 973	**40**
21	.02 356	.02 357	42.433	.99 972	39
22	.02 385	.02 386	41.916	.99 972	38
23	.02 414	.02 415	41.411	.99 971	37
24	.02 443	.02 444	40.917	.99 970	36
25	.02 472	.02 473	40.436	.99 969	35
26	.02 501	.02 502	39.965	.99 969	34
27	.02 530	.02 531	39.506	.99 968	33
28	.02 560	.02 560	39.057	.99 967	32
29	.02 589	.02 589	38.618	.99 966	31
30	.02 618	.02 619	38.188	.99 966	**30**
31	.02 647	.02 648	37.769	.99 965	29
32	.02 676	.02 677	37.358	.99 964	28
33	.02 705	.02 706	36.956	.99 963	27
34	.02 734	.02 735	36.563	.99 963	26
35	.02 763	.02 764	36.178	.99 962	25
36	.02 792	.02 793	35.801	.99 961	24
37	.02 821	.02 822	35.431	.99 960	23
38	.02 850	.02 851	35.070	.99 959	22
39	.02 879	.02 881	34.715	.99 959	21
40	.02 908	.02 910	34.368	.99 958	**20**
41	.02 938	.02 939	34.027	.99 957	19
42	.02 967	.02 968	33.694	.99 956	18
43	.02 996	.02 997	33.366	.99 955	17
44	.03 025	.03 026	33.045	.99 954	16
45	.03 054	.03 055	32.730	.99 953	15
46	.03 083	.03 084	32.421	.99 952	14
47	.03 112	.03 114	32.118	.99 952	13
48	.03 141	.03 143	31.821	.99 951	12
49	.03 170	.03 172	31.528	.99 950	11
50	.03 199	.03 201	31.242	.99 949	**10**
51	.03 228	.03 230	30.960	.99 948	9
52	.03 257	.03 259	30.683	.99 947	8
53	.03 286	.03 288	30.412	.99 946	7
54	.03 316	.03 317	30.145	.99 945	6
55	.03 345	.03 346	29.882	.99 944	5
56	.03 374	.03 376	29.624	.99 943	4
57	.03 403	.03 405	29.371	.99 942	3
58	.03 432	.03 434	29.122	.99 941	2
59	.03 461	.03 463	28.877	.99 940	1
60	.03 490	.03 492	28.636	.99 939	**0**
′	cos	cot	tan	sin	′
	88°				

	2°				
′	sin	tan	cot	cos	′
0	.03 490	.03 492	28.636	.99 939	**60**
1	.03 519	.03 521	28.399	.99 938	59
2	.03 548	.03 550	28.166	.99 937	58
3	.03 577	.03 579	27.937	.99 936	57
4	.03 606	.03 609	27.712	.99 935	56
5	.03 635	.03 638	27.490	.99 934	55
6	.03 664	.03 667	27.271	.99 933	54
7	.03 693	.03 696	27.057	.99 932	53
8	.03 723	.03 725	26.845	.99 931	52
9	.03 752	.03 754	26.637	.99 930	51
10	.03 781	.03 783	26.432	.99 929	**50**
11	.03 810	.03 812	26.230	.99 927	49
12	.03 839	.03 842	26.031	.99 926	48
13	.03 868	.03 871	25.835	.99 925	47
14	.03 897	.03 900	25.642	.99 924	46
15	.03 926	.03 929	25.452	.99 923	45
16	.03 955	.03 958	25.264	.99 922	44
17	.03 984	.03 987	25.080	.99 921	43
18	.04 013	.04 016	24.898	.99 919	42
19	.04 042	.04 046	24.719	.99 918	41
20	.04 071	.04 075	24.542	.99 917	**40**
21	.04 100	.04 104	24.368	.99 916	39
22	.04 129	.04 133	24.196	.99 915	38
23	.04 159	.04 162	24.026	.99 913	37
24	.04 188	.04 191	23.859	.99 912	36
25	.04 217	.04 220	23.695	.99 911	35
26	.04 246	.04 250	23.532	.99 910	34
27	.04 275	.04 279	23.372	.99 909	33
28	.04 304	.04 308	23.214	.99 907	32
29	.04 333	.04 337	23.058	.99 906	31
30	.04 362	.04 366	22.904	.99 905	**30**
31	.04 391	.04 395	22.752	.99 904	29
32	.04 420	.04 424	22.602	.99 902	28
33	.04 449	.04 454	22.454	.99 901	27
34	.04 478	.04 483	22.308	.99 900	26
35	.04 507	.04 512	22.164	.99 898	25
36	.04 536	.04 541	22.022	.99 897	24
37	.04 565	.04 570	21.881	.99 896	23
38	.04 594	.04 599	21.743	.99 894	22
39	.04 623	.04 628	21.606	.99 893	21
40	.04 653	.04 658	21.470	.99 892	**20**
41	.04 682	.04 687	21.337	.99 890	19
42	.04 711	.04 716	21.205	.99 889	18
43	.04 740	.04 745	21.075	.99 888	17
44	.04 769	.04 774	20.946	.99 886	16
45	.04 798	.04 803	20.819	.99 885	15
46	.04 827	.04 833	20.693	.99 883	14
47	.04 856	.04 862	20.569	.99 882	13
48	.04 885	.04 891	20.446	.99 881	12
49	.04 914	.04 920	20.325	.99 879	11
50	.04 943	.04 949	20.206	.99 878	**10**
51	.04 972	.04 978	20.087	.99 876	9
52	.05 001	.05 007	19.970	.99 875	8
53	.05 030	.05 037	19.855	.99 873	7
54	.05 059	.05 066	19.740	.99 872	6
55	.05 088	.05 095	19.627	.99 870	5
56	.05 117	.05 124	19.516	.99 869	4
57	.05 146	.05 153	19.405	.99 867	3
58	.05 175	.05 182	19.296	.99 866	2
59	.05 205	.05 212	19.188	.99 864	1
60	.05 234	.05 241	19.081	.99 863	**0**
′	cos	cot	tan	sin	′
	87°				

	3°				
′	sin	tan	cot	cos	′
0	.05 234	.05 241	19.081	.99 863	**60**
1	.05 263	.05 270	18.976	.99 861	59
2	.05 292	.05 299	18.871	.99 860	58
3	.05 321	.05 328	18.768	.99 858	57
4	.05 350	.05 357	18.666	.99 857	56
5	.05 379	.05 387	18.564	.99 855	55
6	.05 408	.05 416	18.464	.99 854	54
7	.05 437	.05 445	18.366	.99 852	53
8	.05 466	.05 474	18.268	.99 851	52
9	.05 495	.05 503	18.171	.99 849	51
10	.05 524	.05 533	18.075	.99 847	**50**
11	.05 553	.05 562	17.980	.99 846	49
12	.05 582	.05 591	17.886	.99 844	48
13	.05 611	.05 620	17.793	.99 842	47
14	.05 640	.05 649	17.702	.99 841	46
15	.05 669	.05 678	17.611	.99 839	45
16	.05 698	.05 708	17.521	.99 838	44
17	.05 727	.05 737	17.431	.99 836	43
18	.05 756	.05 766	17.343	.99 834	42
19	.05 785	.05 795	17.256	.99 833	41
20	.05 814	.05 824	17.169	.99 831	**40**
21	.05 844	.05 854	17.084	.99 829	39
22	.05 873	.05 883	16.999	.99 827	38
23	.05 902	.05 912	16.915	.99 826	37
24	.05 931	.05 941	16.832	.99 824	36
25	.05 960	.05 970	16.750	.99 822	35
26	.05 989	.05 999	16.668	.99 821	34
27	.06 018	.06 029	16.587	.99 819	33
28	.06 047	.06 058	16.507	.99 817	32
29	.06 076	.06 087	16.428	.99 815	31
30	.06 105	.06 116	16.350	.99 813	**30**
31	.06 134	.06 145	16.272	.99 812	29
32	.06 163	.06 175	16.195	.99 810	28
33	.06 192	.06 204	16.119	.99 808	27
34	.06 221	.06 233	16.043	.99 806	26
35	.06 250	.06 262	15.969	.99 804	25
36	.06 279	.06 291	15.895	.99 803	24
37	.06 308	.06 321	15.821	.99 801	23
38	.06 337	.06 350	15.748	.99 799	22
39	.06 366	.06 379	15.676	.99 797	21
40	.06 395	.06 408	15.605	.99 795	**20**
41	.06 424	.06 438	15.534	.99 793	19
42	.06 453	.06 467	15.464	.99 792	18
43	.06 482	.06 496	15.394	.99 790	17
44	.06 511	.06 525	15.325	.99 788	16
45	.06 540	.06 554	15.257	.99 786	15
46	.06 569	.06 584	15.189	.99 784	14
47	.06 598	.06 613	15.122	.99 782	13
48	.06 627	.06 642	15.056	.99 780	12
49	.06 656	.06 671	14.990	.99 778	11
50	.06 685	.06 700	14.924	.99 776	**10**
51	.06 714	.06 730	14.860	.99 774	9
52	.06 743	.06 759	14.795	.99 772	8
53	.06 773	.06 788	14.732	.99 770	7
54	.06 802	.06 817	14.669	.99 768	6
55	.06 831	.06 847	14.606	.99 766	5
56	.06 860	.06 876	14.544	.99 764	4
57	.06 889	.06 905	14.482	.99 762	3
58	.06 918	.06 934	14.421	.99 760	2
59	.06 947	.06 963	14.361	.99 758	1
60	.06 976	.06 993	14.301	.99 756	**0**
′	cos	cot	tan	sin	′
	86°				

′	4° sin	tan	cot	cos	′
0	.06 976	.06 993	14.301	.99 756	60
1	.07 005	.07 022	14.241	.99 754	59
2	.07 034	.07 051	14.182	.99 752	58
3	.07 063	.07 080	14.124	.99 750	57
4	.07 092	.07 110	14.065	.99 748	56
5	.07 121	.07 139	14.008	.99 746	55
6	.07 150	.07 168	13.951	.99 744	54
7	.07 179	.07 197	13.894	.99 742	53
8	.07 208	.07 227	13.838	.99 740	52
9	.07 237	.07 256	13.782	.99 738	51
10	.07 266	.07 285	13.727	.99 736	50
11	.07 295	.07 314	13.672	.99 734	49
12	.07 324	.07 344	13.617	.99 731	48
13	.07 353	.07 373	13.563	.99 729	47
14	.07 382	.07 402	13.510	.99 727	46
15	.07 411	.07 431	13.457	.99 725	45
16	.07 440	.07 461	13.404	.99 723	44
17	.07 469	.07 490	13.352	.99 721	43
18	.07 498	.07 519	13.300	.99 719	42
19	.07 527	.07 548	13.248	.99 716	41
20	.07 556	.07 578	13.197	.99 714	40
21	.07 585	.07 607	13.146	.99 712	39
22	.07 614	.07 636	13.096	.99 710	38
23	.07 643	.07 665	13.046	.99 708	37
24	.07 672	.07 695	12.996	.99 705	36
25	.07 701	.07 724	12.947	.99 703	35
26	.07 730	.07 753	12.898	.99 701	34
27	.07 759	.07 782	12.850	.99 699	33
28	.07 788	.07 812	12 801	.99 696	32
29	.07 817	.07 841	12.754	.99 694	31
30	.07 846	.07 870	12.706	.99 692	30
31	.07 875	.07 899	12.659	.99 689	29
32	.07 904	.07 929	12.612	.99 687	28
33	.07 933	.07 958	12.566	.99 685	27
34	.07 962	.07 987	12.520	.99 683	26
35	.07 991	.08 017	12.474	.99 680	25
36	.08 020	.08 046	12.429	.99 678	24
37	.08 049	.08 075	12.384	.99 676	23
38	.08 078	.08 104	12.339	.99 673	22
39	.08 107	.08 134	12.295	.99 671	21
40	.08 136	.08 163	12.251	.99 668	20
41	.08 165	.08 192	12.207	.99 666	19
42	.08 194	.08 221	12.163	.99 664	18
43	.08 223	.08 251	12.120	.99 661	17
44	.08 252	.08 280	12.077	.99 659	16
45	.08 281	.08 309	12.035	.99 657	15
46	.08 310	.08 339	11.992	.99 654	14
47	.08 339	.08 368	11.950	.99 652	13
48	.08 368	.08 397	11.909	.99 649	12
49	.08 397	.08 427	11.867	.99 647	11
50	.08 426	.08 456	11.826	.99 644	10
51	.08 455	.08 485	11.785	.99 642	9
52	.08 484	.08 514	11.745	.99 639	8
53	.08 513	.08 544	11.705	.99 637	7
54	.08 542	.08 573	11.664	.99 635	6
55	.08 571	.08 602	11.625	.99 632	5
56	.08 600	.08 632	11.585	.99 630	4
57	.08 629	.08 661	11.546	.99 627	3
58	.08 658	.08 690	11.507	.99 625	2
59	.08 687	.08 720	11.468	.99 622	1
60	.08 716	.08 749	11.430	.99 619	0
′	cos	cot	tan	sin 85°	′

′	5° sin	tan	cot	cos	′
0	.08 716	.08 749	11.430	.99 619	60
1	.08 745	.08 778	11.392	.99 617	59
2	.08 774	.08 807	11.354	.99 614	58
3	.08 803	.08 837	11.316	.99 612	57
4	.08 831	.08 866	11.279	.99 609	56
5	.08 860	.08 895	11.242	.99 607	55
6	.08 889	.08 925	11.205	.99 604	54
7	.08 918	.08 954	11.168	.99 602	53
8	.08 947	.08 983	11.132	.99 599	52
9	.08 976	.09 013	11.095	.99 596	51
10	.09 005	.09 042	11.059	.99 594	50
11	.09 034	.09 071	11.024	.99 591	49
12	.09 063	.09 101	10.988	.99 588	48
13	.09 092	.09 130	10.953	.99 586	47
14	.09 121	.09 159	10.918	.99 583	46
15	.09 150	.09 189	10.883	.99 580	45
16	.09 179	.09 218	10.848	.99 578	44
17	.09 208	.09 247	10.814	.99 575	43
18	.09 237	.09 277	10.780	.99 572	42
19	.09 266	.09 306	10.746	.99 570	41
20	.09 295	.09 335	10.712	.99 567	40
21	.09 324	.09 365	10.678	.99 564	39
22	.09 353	.09 394	10.645	.99 562	38
23	.09 382	.09 423	10.612	.99 559	37
24	.09 411	.09 453	10.579	.99 556	36
25	.09 440	.09 482	10.546	.99 553	35
26	.09 469	.09 511	10.514	.99 551	34
27	.09 498	.09 541	10.481	.99 548	33
28	.09 527	.09 570	10.449	.99 545	32
29	.09 556	.09 600	10.417	.99 542	31
30	.09 585	.09 629	10.385	.99 540	30
31	.09 614	.09 658	10.354	.99 537	29
32	.09 642	.09 688	10.322	.99 534	28
33	.09 671	.09 717	10.291	.99 531	27
34	.09 700	.09 746	10.260	.99 528	26
35	.09 729	.09 776	10.229	.99 526	25
36	.09 758	.09 805	10.199	.99 523	24
37	.09 787	.09 834	10.168	.99 520	23
38	.09 816	.09 864	10.138	.99 517	22
39	.09 845	.09 893	10.108	.99 514	21
40	.09 874	.09 923	10.078	.99 511	20
41	.09 903	.09 952	10.048	.99 508	19
42	.09 932	.09 981	10.019	.99 506	18
43	.09 961	.10 011	9.9893	.99 503	17
44	.09 990	.10 040	9.9601	.99 500	16
45	.10 019	.10 069	9.9310	.99 497	15
46	.10 048	.10 099	9.9021	.99 494	14
47	.10 077	.10 128	9.8734	.99 491	13
48	.10 106	.10 158	9.8448	.99 488	12
49	.10 135	.10 187	9.8164	.99 485	11
50	.10 164	.10 216	9.7882	.99 482	10
51	.10 192	.10 246	9.7601	.99 479	9
52	.10 221	.10 275	9.7322	.99 476	8
53	.10 250	.10 305	9.7044	.99 473	7
54	.10 279	.10 334	9.6768	.99 470	6
55	.10 308	.10 363	9.6493	.99 467	5
56	.10 337	.10 393	9.6220	.99 464	4
57	.10 366	.10 422	9.5949	.99 461	3
58	.10 395	.10 452	9.5679	.99 458	2
59	.10 424	.10 481	9.5411	.99 455	1
60	.10 453	.10 510	9.5144	.99 452	0
′	cos	cot	tan	sin 84°	′

6°					
′	sin	tan	cot	cos	′
0	.10 453	.10 510	9.5144	.99 452	**60**
1	.10 482	.10 540	9.4878	.99 449	59
2	.10 511	.10 569	9.4614	.99 446	58
3	.10 540	.10 599	9.4352	.99 443	57
4	.10 569	.10 628	9.4090	.99 440	56
5	.10 597	.10 657	9.3831	.99 437	55
6	.10 626	.10 687	9.3572	.99 434	54
7	.10 655	.10 716	9.3315	.99 431	53
8	.10 684	.10 746	9.3060	.99 428	52
9	.10 713	.10 775	9.2806	.99 424	51
10	.10 742	.10 805	9.2553	.99 421	**50**
11	.10 771	.10 834	9.2302	.99 418	49
12	.10 800	.10 863	9.2052	.99 415	48
13	.10 829	.10 893	9.1803	.99 412	47
14	.10 858	.10 922	9.1555	.99 409	46
15	.10 887	.10 952	9.1309	.99 406	45
16	.10 916	.10 981	9.1065	.99 402	44
17	.10 945	.11 011	9.0821	.99 399	43
18	.10 973	.11 040	9.0579	.99 396	42
19	.11 002	.11 070	9.0338	.99 393	41
20	.11 031	.11 099	9.0098	.99 390	**40**
21	.11 060	.11 128	8.9860	.99 386	39
22	.11 089	.11 158	8.9623	.99 383	38
23	.11 118	.11 187	8.9387	.99 380	37
24	.11 147	.11 217	8.9152	.99 377	36
25	.11 176	.11 246	8.8919	.99 374	35
26	.11 205	.11 276	8.8686	.99 370	34
27	.11 234	.11 305	8.8455	.99 367	33
28	.11 263	.11 335	8.8225	.99 364	32
29	.11 291	.11 364	8.7996	.99 360	31
30	.11 320	.11 394	8.7769	.99 357	**30**
31	.11 349	.11 423	8.7542	.99 354	29
32	.11 378	.11 452	8.7317	.99 351	28
33	.11 407	.11 482	8.7093	.99 347	27
34	.11 436	.11 511	8.6870	.99 344	26
35	.11 465	.11 541	8.6648	.99 341	25
36	.11 494	.11 570	8.6427	.99 337	24
37	.11 523	.11 600	8.6208	.99 334	23
38	.11 552	.11 629	8.5989	.99 331	22
39	.11 580	.11 659	8.5772	.99 327	21
40	.11 609	.11 688	8.5555	.99 324	**20**
41	.11 638	.11 718	8.5340	.99 320	19
42	.11 667	.11 747	8.5126	.99 317	18
43	.11 696	.11 777	8.4913	.99 314	17
44	.11 725	.11 806	8.4701	.99 310	16
45	.11 754	.11 836	8.4490	.99 307	15
46	.11 783	.11 865	8.4280	.99 303	14
47	.11 812	.11 895	8.4071	.99 300	13
48	.11 840	.11 924	8.3863	.99 297	12
49	.11 869	.11 954	8.3656	.99 293	11
50	.11 898	.11 983	8.3450	.99 290	**10**
51	.11 927	.12 013	8.3245	.99 286	9
52	.11 956	.12 042	8.3041	.99 283	8
53	.11 985	.12 072	8.2838	.99 279	7
54	.12 014	.12 101	8.2636	.99 276	6
55	.12 043	.12 131	8.2434	.99 272	5
56	.12 071	.12 160	8.2234	.99 269	4
57	.12 100	.12 190	8.2035	.99 265	3
58	.12 129	.12 219	8.1837	.99 262	2
59	.12 158	.12 249	8.1640	.99 258	1
60	.12 187	.12 278	8.1443	.99 255	**0**
′	cos	cot	tan	sin	′
83°					

7°					
′	sin	tan	cot	cos	′
0	.12 187	.12 278	8.1443	.99 255	**60**
1	.12 216	.12 308	8.1248	.99 251	59
2	.12 245	.12 338	8.1054	.99 248	58
3	.12 274	.12 367	8.0860	.99 244	57
4	.12 302	.12 397	8.0667	.99 240	56
5	.12 331	.12 426	8.0476	.99 237	55
6	.12 360	.12 456	8.0285	.99 233	54
7	.12 389	.12 485	8.0095	.99 230	53
8	.12 418	.12 515	7.9906	.99 226	52
9	.12 447	.12 544	7.9718	.99 222	51
10	.12 476	.12 574	7.9530	.99 219	**50**
11	.12 504	.12 603	7.9344	.99 215	49
12	.12 533	.12 633	7.9158	.99 211	48
13	.12 562	.12 662	7.8973	.99 208	47
14	.12 591	.12 692	7.8789	.99 204	46
15	.12 620	.12 722	7.8606	.99 200	45
16	.12 649	.12 751	7.8424	.99 197	44
17	.12 678	.12 781	7.8243	.99 193	43
18	.12 706	.12 810	7.8062	.99 189	42
19	.12 735	.12 840	7.7882	.99 186	41
20	.12 764	.12 869	7.7704	.99 182	**40**
21	.12 793	.12 899	7.7525	.99 178	39
22	.12 822	.12 929	7.7348	.99 175	38
23	.12 851	.12 958	7.7171	.99 171	37
24	.12 880	.12 988	7.6996	.99 167	36
25	.12 908	.13 017	7.6821	.99 163	35
26	.12 937	.13 047	7.6647	.99 160	34
27	.12 966	.13 076	7.6473	.99 156	33
28	.12 995	.13 106	7.6301	.99 152	32
29	.13 024	.13 136	7.6129	.99 148	31
30	.13 053	.13 165	7.5958	.99 144	**30**
31	.13 081	.13 195	7.5787	.99 141	29
32	.13 110	.13 224	7.5618	.99 137	28
33	.13 139	.13 254	7.5449	.99 133	27
34	.13 168	.13 284	7.5281	.99 129	26
35	.13 197	.13 313	7.5113	.99 125	25
36	.13 226	.13 343	7.4947	.99 122	24
37	.13 254	.13 372	7.4781	.99 118	23
38	.13 283	.13 402	7.4615	.99 114	22
39	.13 312	.13 432	7.4451	.99 110	21
40	.13 341	.13 461	7.4287	.99 106	**20**
41	.13 370	.13 491	7.4124	.99 102	19
42	.13 399	.13 521	7.3962	.99 098	18
43	.13 427	.13 550	7.3800	.99 094	17
44	.13 456	.13 580	7.3639	.99 091	16
45	.13 485	.13 609	7.3479	.99 087	15
46	.13 514	.13 639	7.3319	.99 083	14
47	.13 543	.13 669	7.3160	.99 079	13
48	.13 572	.13 698	7.3002	.99 075	12
49	.13 600	.13 728	7.2844	.99 071	11
50	.13 629	.13 758	7.2687	.99 067	**10**
51	.13 658	.13 787	7.2531	.99 063	9
52	.13 687	.13 817	7.2375	.99 059	8
53	.13 716	.13 846	7.2220	.99 055	7
54	.13 744	.13 876	7.2066	.99 051	6
55	.13 773	.13 906	7.1912	.99 047	5
56	.13 802	.13 935	7.1759	.99 043	4
57	.13 831	.13 965	7.1607	.99 039	3
58	.13 860	.13 995	7.1455	.99 035	2
59	.13 889	.14 024	7.1304	.99 031	1
60	.13 917	.14 054	7.1154	.99 027	**0**
′	cos	cot	tan	sin	′
82°					

	8°				
′	sin	tan	cot	cos	′
0	.13 917	.14 054	7.1154	.99 027	**60**
1	.13 946	.14 084	7.1004	.99 023	59
2	.13 975	.14 113	7.0855	.99 019	58
3	.14 004	.14 143	7.0706	.99 015	57
4	.14 033	.14 173	7.0558	.99 011	56
5	.14 061	.14 202	7.0410	.99 006	55
6	.14 090	.14 232	7.0264	.99 002	54
7	.14 119	.14 262	7.0117	.98 998	53
8	.14 148	.14 291	6.9972	.98 994	52
9	.14 177	.14 321	6.9827	.98 990	51
10	.14 205	.14 351	6.9682	.98 986	**50**
11	.14 234	.14 381	6.9538	.98 982	49
12	.14 263	.14 410	6.9395	.98 978	48
13	.14 292	.14 440	6.9252	.98 973	47
14	.14 320	.14 470	6.9110	.98 969	46
15	.14 349	.14 499	6.8969	.98 965	45
16	.14 378	.14 529	6.8828	.98 961	44
17	.14 407	.14 559	6.8687	.98 957	43
18	.14 436	.14 588	6.8548	.98 953	42
19	.14 464	.14 618	6.8408	.98 948	41
20	.14 493	.14 648	6.8269	.98 944	**40**
21	.14 522	.14 678	6.8131	.98 940	39
22	.14 551	.14 707	6.7994	.98 936	38
23	.14 580	.14 737	6.7856	.98 931	37
24	.14 608	.14 767	6.7720	.98 927	36
25	.14 637	.14 796	6.7584	.98 923	35
26	.14 666	.14 826	6.7448	.98 919	34
27	.14 695	.14 856	6.7313	.98 914	33
28	.14 723	.14 886	6.7179	.98 910	32
29	.14 752	.14 915	6.7045	.98 906	31
30	.14 781	.14 945	6.6912	.98 902	**30**
31	.14 810	.14 975	6.6779	.98 897	29
32	.14 838	.15 005	6.6646	.98 893	28
33	.14 867	.15 034	6.6514	.98 889	27
34	.14 896	.15 064	6.6383	.98 884	26
35	.14 925	.15 094	6.6252	.98 880	25
36	.14 954	.15 124	6.6122	.98 876	24
37	.14 982	.15 153	6.5992	.98 871	23
38	.15 011	.15 183	6.5863	.98 867	22
39	.15 040	.15 213	6.5734	.98 863	21
40	.15 069	.15 243	6.5606	.98 858	**20**
41	.15 097	.15 272	6.5478	.98 854	19
42	.15 126	.15 302	6.5350	.98 849	18
43	.15 155	.15 332	6.5223	.98 845	17
44	.15 184	.15 362	6.5097	.98 841	16
45	.15 212	.15 391	6.4971	.98 836	15
46	.15 241	.15 421	6.4846	.98 832	14
47	.15 270	.15 451	6.4721	.98 827	13
48	.15 299	.15 481	6.4596	.98 823	12
49	.15 327	.15 511	6.4472	.98 818	11
50	.15 356	.15 540	6.4348	.98 814	**10**
51	.15 385	.15 570	6.4225	.98 809	9
52	.15 414	.15 600	6.4103	.98 805	8
53	.15 442	.15 630	6.3980	.98 800	7
54	.15 471	.15 660	6.3859	.98 796	6
55	.15 500	.15 689	6.3737	.98 791	5
56	.15 529	.15 719	6.3617	.98 787	4
57	.15 557	.15 749	6.3496	.98 782	3
58	.15 586	.15 779	6.3376	.98 778	2
59	.15 615	.15 809	6.3257	.98 773	1
60	.15 643	.15 838	6.3138	.98 769	**0**
′	cos	cot	tan	sin	′
	81°				

	9°				
′	sin	tan	cot	cos	′
0	.15 643	.15 838	6.3138	.98 769	**60**
1	.15 672	.15 868	6.3019	.98 764	59
2	.15 701	.15 898	6.2901	.98 760	58
3	.15 730	.15 928	6.2783	.98 755	57
4	.15 758	.15 958	6.2666	.98 751	56
5	.15 787	.15 988	6.2549	.98 746	55
6	.15 816	.16 017	6.2432	.98 741	54
7	.15 845	.16 047	6.2316	.98 737	53
8	.15 873	.16 077	6.2200	.98 732	52
9	.15 902	.16 107	6.2085	.98 728	51
10	.15 931	.16 137	6.1970	.98 723	**50**
11	.15 959	.16 167	6.1856	.98 718	49
12	.15 988	.16 196	6.1742	.98 714	48
13	.16 017	.16 226	6.1628	.98 709	47
14	.16 046	.16 256	6.1515	.98 704	46
15	.16 074	.16 286	6.1402	.98 700	45
16	.16 103	.16 316	6.1290	.98 695	44
17	.16 132	.16 346	6.1178	.98 690	43
18	.16 160	.16 376	6.1066	.98 686	42
19	.16 189	.16 405	6.0955	.98 681	41
20	.16 218	.16 435	6.0844	.98 676	**40**
21	.16 246	.16 465	6.0734	.98 671	39
22	.16 275	.16 495	6.0624	.98 667	38
23	.16 304	.16 525	6.0514	.98 662	37
24	.16 333	.16 555	6.0405	.98 657	36
25	.16 361	.16 585	6.0296	.98 652	35
26	.16 390	.16 615	6.0188	.98 648	34
27	.16 419	.16 645	6.0080	.98 643	33
28	.16 447	.16 674	5.9972	.98 638	32
29	.16 476	.16 704	5.9865	.98 633	31
30	.16 505	.16 734	5.9758	.98 629	**30**
31	.16 533	.16 764	5.9651	.98 624	29
32	.16 562	.16 794	5.9545	.98 619	28
33	.16 591	.16 824	5.9439	.98 614	27
34	.16 620	.16 854	5.9333	.98 609	26
35	.16 648	.16 884	5.9228	.98 604	25
36	.16 677	.16 914	5.9124	.98 600	24
37	.16 706	.16 944	5.9019	.98 595	23
38	.16 734	.16 974	5.8915	.98 590	22
39	.16 763	.17 004	5.8811	.98 585	21
40	.16 792	.17 033	5.8708	.98 580	**20**
41	.16 820	.17 063	5.8605	.98 575	19
42	.16 849	.17 093	5.8502	.98 570	18
43	.16 878	.17 123	5.8400	.98 565	17
44	.16 906	.17 153	5.8298	.98 561	16
45	.16 935	.17 183	5.8197	.98 556	15
46	.16 964	.17 213	5.8095	.98 551	14
47	.16 992	.17 243	5.7994	.98 546	13
48	.17 021	.17 273	5.7894	.98 541	12
49	.17 050	.17 303	5.7794	.98 536	11
50	.17 078	.17 333	5.7694	.98 531	**10**
51	.17 107	.17 363	5.7594	.98 526	9
52	.17 136	.17 393	5.7495	.98 521	8
53	.17 164	.17 423	5.7396	.98 516	7
54	.17 193	.17 453	5.7297	.98 511	6
55	.17 222	.17 483	5.7199	.98 506	5
56	.17 250	.17 513	5.7101	.98 501	4
57	.17 279	.17 543	5.7004	.98 496	3
58	.17 308	.17 573	5.6906	.98 491	2
59	.17 336	.17 603	5.6809	.98 486	1
60	.17 365	.17 633	5.6713	.98 481	**0**
′	cos	cot	tan	sin	′
	80°				

′	10° sin	tan	cot	cos	′
0	.17 365	.17 633	5.6713	.98 481	**60**
1	.17 393	.17 663	5.6617	.98 476	59
2	.17 422	.17 693	5.6521	.98 471	58
3	.17 451	.17 723	5.6425	.98 466	57
4	.17 479	.17 753	5.6329	.98 461	56
5	.17 508	.17 783	5.6234	.98 455	55
6	.17 537	.17 813	5.6140	.98 450	54
7	.17 565	.17 843	5.6045	.98 445	53
8	.17 594	.17 873	5.5951	.98 440	52
9	.17 623	.17 903	5.5857	.98 435	51
10	.17 651	.17 933	5.5764	.98 430	**50**
11	.17 680	.17 963	5.5671	.98 425	49
12	.17 708	.17 993	5.5578	.98 420	48
13	.17 737	.18 023	5.5485	.98 414	47
14	.17 766	.18 053	5.5393	.98 409	46
15	.17 794	.18 083	5.5301	.98 404	45
16	.17 823	.18 113	5.5209	.98 399	44
17	.17 852	.18 143	5.5118	.98 394	43
18	.17 880	.18 173	5.5026	.98 389	42
19	.17 909	.18 203	5.4936	.98 383	41
20	.17 937	.18 233	5.4845	.98 378	**40**
21	.17 966	.18 263	5.4755	.98 373	39
22	.17 995	.18 293	5.4665	.98 368	38
23	.18 023	.18 323	5.4575	.98 362	37
24	.18 052	.18 353	5.4486	.98 357	36
25	.18 081	.18 384	5.4397	.98 352	35
26	.18 109	.18 414	5.4308	.98 347	34
27	.18 138	.18 444	5.4219	.98 341	33
28	.18 166	.18 474	5.4131	.98 336	32
29	.18 195	.18 504	5.4043	.98 331	31
30	.18 224	.18 534	5.3955	.98 325	**30**
31	.18 252	.18 564	5.3868	.98 320	29
32	.18 281	.18 594	5.3781	.98 315	28
33	.18 309	.18 624	5.3694	.98 310	27
34	.18 338	.18 654	5.3607	.98 304	26
35	.18 367	.18 684	5.3521	.98 299	25
36	.18 395	.18 714	5.3435	.98 294	24
37	.18 424	.18 745	5.3349	.98 288	23
38	.18 452	.18 775	5.3263	.98 283	22
39	.18 481	.18 805	5.3178	.98 277	21
40	.18 509	.18 835	5.3093	.98 272	**20**
41	.18 538	.18 865	5.3008	.98 267	19
42	.18 567	.18 895	5.2924	.98 261	18
43	.18 595	.18 925	5.2839	.98 256	17
44	.18 624	.18 955	5.2755	.98 250	16
45	.18 652	.18 986	5.2672	.98 245	15
46	.18 681	.19 016	5.2588	.98 240	14
47	.13 710	.19 046	5.2505	.98 234	13
48	.18 738	.19 076	5.2422	.98 229	12
49	.18 767	.19 106	5.2339	.98 223	11
50	.18 795	.19 136	5.2257	.98 218	**10**
51	.18 824	.19 166	5.2174	.98 212	9
52	.18 852	.19 197	5.2092	.98 207	8
53	.18 881	.19 227	5.2011	.98 201	7
54	.18 910	.19 257	5.1929	.98 196	6
55	.18 938	.19 287	5.1848	.98 190	5
56	.18 967	.19 317	5.1767	.98 185	4
57	.18 995	.19 347	5.1686	.98 179	3
58	.19 024	.19 378	5.1606	.98 174	2
59	.19 052	.19 408	5.1526	.98 168	1
60	.19 081	.19 438	5.1446	.98 163	**0**
′	cos	cot	tan	sin 79°	′

′	11° sin	tan	cot	cos	′
0	.19 081	.19 438	5.1446	.98 163	**60**
1	.19 109	.19 468	5.1366	.98 157	59
2	.19 138	.19 498	5.1286	.98 152	58
3	.19 167	.19 529	5.1207	.98 146	57
4	.19 195	.19 559	5.1128	.98 140	56
5	.19 224	.19 589	5.1049	.98 135	55
6	.19 252	.19 619	5.0970	.98 129	54
7	.19 281	.19 649	5.0892	.98 124	53
8	.19 309	.19 680	5.0814	.98 118	52
9	.19 338	.19 710	5.0736	.98 112	51
10	.19 366	.19 740	5.0658	.98 107	**50**
11	.19 395	.19 770	5.0581	.98 101	49
12	.19 423	.19 801	5.0504	.98 096	48
13	.19 452	.19 831	5.0427	.98 090	47
14	.19 481	.19 861	5.0350	.98 084	46
15	.19 509	.19 891	5.0273	.98 079	45
16	.19 538	.19 921	5.0197	.98 073	44
17	.19 566	.19 952	5.0121	.98 067	43
18	.19 595	.19 982	5.0045	.98 061	42
19	.19 623	.20 012	4.9969	.98 056	41
20	.19 652	.20 042	4.9894	.98 050	**40**
21	.19 680	.20 073	4.9819	.98 044	39
22	.19 709	.20 103	4.9744	.98 039	38
23	.19 737	.20 133	4.9669	.98 033	37
24	.19 766	.20 164	4.9594	.98 027	36
25	.19 794	.20 194	4.9520	.98 021	35
26	.19 823	.20 224	4.9446	.98 016	34
27	.19 851	.20 254	4.9372	.98 010	33
28	.19 880	.20 285	4.9298	.98 004	32
29	.19 908	.20 315	4.9225	.97 998	31
30	.19 937	.20 345	4.9152	.97 992	**30**
31	.19 965	.20 376	4.9078	.97 987	29
32	.19 994	.20 406	4.9006	.97 981	28
33	.20 022	.20 436	4.8933	.97 975	27
34	.20 051	.20 466	4.8860	.97 969	26
35	.20 079	.20 497	4.8788	.97 963	25
36	.20 108	.20 527	4.8716	.97 958	24
37	.20 136	.20 557	4.8644	.97 952	23
38	.20 165	.20 588	4.8573	.97 946	22
39	.20 193	.20 618	4.8501	.97 940	21
40	.20 222	.20 648	4.8430	.97 934	**20**
41	.20 250	.20 679	4.8359	.97 928	19
42	.20 279	.20 709	4.8288	.97 922	18
43	.20 307	.20 739	4.8218	.97 916	17
44	.20 336	.20 770	4.8147	.97 910	16
45	.20 364	.20 800	4.8077	.97 905	15
46	.20 393	.20 830	4.8007	.97 899	14
47	.20 421	.20 861	4.7937	.97 893	13
48	.20 450	.20 891	4.7867	.97 887	12
49	.20 478	.20 921	4.7798	.97 881	11
50	.20 507	.20 952	4.7729	.97 875	**10**
51	.20 535	.20 982	4.7659	.97 869	9
52	.20 563	.21 013	4.7591	.97 863	8
53	.20 592	.21 043	4.7522	.97 857	7
54	.20 620	.21 073	4.7453	.97 851	6
55	.20 649	.21 104	4.7385	.97 845	5
56	.20 677	.21 134	4.7317	.97 839	4
57	.20 706	.21 164	4.7249	.97 833	3
58	.20 734	.21 195	4.7181	.97 827	2
59	.20 763	.21 225	4.7114	.97 821	1
60	.20 791	.21 256	4.7046	.97 815	**0**
′	cos	cot	tan	sin 78°	′

	12°				
′	sin	tan	cot	cos	′
0	.20 791	.21 256	4.7046	.97 815	**60**
1	.20 820	.21 286	4.6979	.97 809	59
2	.20 848	.21 316	4.6912	.97 803	58
3	.20 877	.21 347	4.6845	.97 797	57
4	.20 905	.21 377	4.6779	.97 791	56
5	.20 933	.21 408	4.6712	.97 784	55
6	.20 962	.21 438	4.6646	.97 778	54
7	.20 990	.21 469	4.6580	.97 772	53
8	.21 019	.21 499	4.6514	.97 766	52
9	.21 047	.21 529	4.6448	.97 760	51
10	.21 076	.21 560	4.6382	.97 754	**50**
11	.21 104	.21 590	4.6317	.97 748	49
12	.21 132	.21 621	4.6252	.97 742	48
13	.21 161	.21 651	4.6187	.97 735	47
14	.21 189	.21 682	4.6122	.97 729	46
15	.21 218	.21 712	4.6057	.97 723	45
16	.21 246	.21 743	4.5993	.97 717	44
17	.21 275	.21 773	4.5928	.97 711	43
18	.21 303	.21 804	4.5864	.97 705	42
19	.21 331	.21 834	4.5800	.97 698	41
20	.21 360	.21 864	4.5736	.97 692	**40**
21	.21 388	.21 895	4.5673	.97 686	39
22	.21 417	.21 925	4.5609	.97 680	38
23	.21 445	.21 956	4.5546	.97 673	37
24	.21 474	.21 986	4.5483	.97 667	36
25	.21 502	.22 017	4.5420	.97 661	35
26	.21 530	.22 047	4.5357	.97 655	34
27	.21 559	.22 078	4.5294	.97 648	33
28	.21 587	.22 108	4.5232	.97 642	32
29	.21 616	.22 139	4.5169	.97 636	31
30	.21 644	.22 169	4.5107	.97 630	**30**
31	.21 672	.22 200	4.5045	.97 623	29
32	.21 701	.22 231	4.4983	.97 617	28
33	.21 729	.22 261	4.4922	.97 611	27
34	.21 758	.22 292	4.4860	.97 604	26
35	.21 786	.22 322	4.4799	.97 598	25
36	.21 814	.22 353	4.4737	.97 592	24
37	.21 843	.22 383	4.4676	.97 585	23
38	.21 871	.22 414	4.4615	.97 579	22
39	.21 899	.22 444	4.4555	.97 573	21
40	.21 928	.22 475	4.4494	.97 566	**20**
41	.21 956	.22 505	4.4434	.97 560	19
42	.21 985	.22 536	4.4373	.97 553	18
43	.22 013	.22 567	4.4313	.97 547	17
44	.22 041	.22 597	4.4253	.97 541	16
45	.22 070	.22 628	4.4194	.97 534	15
46	.22 098	.22 658	4.4134	.97 528	14
47	.22 126	.22 689	4.4075	.97 521	13
48	.22 155	.22 719	4.4015	.97 515	12
49	.22 183	.22 750	4.3956	.97 508	11
50	.22 212	.22 781	4.3897	.97 502	**10**
51	.22 240	.22 811	4.3838	.97 496	9
52	.22 268	.22 842	4.3779	.97 489	8
53	.22 297	.22 872	4.3721	.97 483	7
54	.22 325	.22 903	4.3662	.97 476	6
55	.22 353	.22 934	4.3604	.97 470	5
56	.22 382	.22 964	4.3546	.97 463	4
57	.22 410	.22 995	4.3488	.97 457	3
58	.22 438	.23 026	4.3430	.97 450	2
59	.22 467	.23 056	4.3372	.97 444	1
60	.22 495	.23 087	4.3315	.97 437	**0**
′	cos	cot	tan	sin	′
	77°				

	13°				
′	sin	tan	cot	cos	′
0	.22 495	.23 087	4.3315	.97 437	**60**
1	.22 523	.23 117	4.3257	.97 430	59
2	.22 552	.23 148	4.3200	.97 424	58
3	.22 580	.23 179	4.3143	.97 417	57
4	.22 608	.23 209	4.3086	.97 411	56
5	.22 637	.23 240	4.3029	.97 404	55
6	.22 665	.23 271	4.2972	.97 398	54
7	.22 693	.23 301	4.2916	.97 391	53
8	.22 722	.23 332	4.2859	.97 384	52
9	.22 750	.23 363	4.2803	.97 378	51
10	.22 778	.23 393	4.2747	.97 371	**50**
11	.22 807	.23 424	4.2691	.97 365	49
12	.22 835	.23 455	4.2635	.97 358	48
13	.22 863	.23 485	4.2580	.97 351	47
14	.22 892	.23 516	4.2524	.97 345	46
15	.22 920	.23 547	4.2468	.97 338	45
16	.22 948	.23 578	4.2413	.97 331	44
17	.22 977	.23 608	4.2358	.97 325	43
18	.23 005	.23 639	4.2303	.97 318	42
19	.23 033	.23 670	4.2248	.97 311	41
20	.23 062	.23 700	4.2193	.97 304	**40**
21	.23 090	.23 731	4.2139	.97 298	39
22	.23 118	.23 762	4.2084	.97 291	38
23	.23 146	.23 793	4.2030	.97 284	37
24	.23 175	.23 823	4.1976	.97 278	36
25	.23 203	.23 854	4.1922	.97 271	35
26	.23 231	.23 885	4.1868	.97 264	34
27	.23 260	.23 916	4.1814	.97 257	33
28	.23 288	.23 946	4.1760	.97 251	32
29	.23 316	.23 977	4.1706	.97 244	31
30	.23 345	.24 008	4.1653	.97 237	**30**
31	.23 373	.24 039	4.1600	.97 230	29
32	.23 401	.24 069	4.1547	.97 223	28
33	.23 429	.24 100	4.1493	.97 217	27
34	.23 458	.24 131	4.1441	.97 210	26
35	.23 486	.24 162	4.1388	.97 203	25
36	.23 514	.24 193	4.1335	.97 196	24
37	.23 542	.24 223	4.1282	.97 189	23
38	.23 571	.24 254	4.1230	.97 182	22
39	.23 599	.24 285	4.1178	.97 176	21
40	.23 627	.24 316	4.1126	.97 169	**20**
41	.23 656	.24 347	4.1074	.97 162	19
42	.23 684	.24 377	4.1022	.97 155	18
43	.23 712	.24 408	4.0970	.97 148	17
44	.23 740	.24 439	4.0918	.97 141	16
45	.23 769	.24 470	4.0867	.97 134	15
46	.23 797	.24 501	4.0815	.97 127	14
47	.23 825	.24 532	4.0764	.97 120	13
48	.23 853	.24 562	4.0713	.97 113	12
49	.23 882	.24 593	4.0662	.97 106	11
50	.23 910	.24 624	4.0611	.97 100	**10**
51	.23 938	.24 655	4.0560	.97 093	9
52	.23 966	.24 686	4.0509	.97 086	8
53	.23 995	.24 717	4.0459	.97 079	7
54	.24 023	.24 747	4.0408	.97 072	6
55	.24 051	.24 778	4.0358	.97 065	5
56	.24 079	.24 809	4.0308	.97 058	4
57	.24 108	.24 840	4.0257	.97 051	3
58	.24 136	.24 871	4.0207	.97 044	2
59	.24 164	.24 902	4.0158	.97 037	1
60	.24 192	.24 933	4.0108	.97 030	**0**
′	cos	cot	tan	sin	′
	76°				

14°

′	sin	tan	cot	cos	′
0	.24 192	.24 933	4.0108	.97 030	**60**
1	.24 220	.24 964	4.0058	.97 023	59
2	.24 249	.24 995	4.0009	.97 015	58
3	.24 277	.25 026	3.9959	.97 008	57
4	.24 305	.25 056	3.9910	.97 001	56
5	.24 333	.25 087	3.9861	.96 994	55
6	.24 362	.25 118	3.9812	.96 987	54
7	.24 390	.25 149	3.9763	.96 980	53
8	.24 418	.25 180	3.9714	.96 973	52
9	.24 446	.25 211	3.9665	.96 966	51
10	.24 474	.25 242	3.9617	.96 959	**50**
11	.24 503	.25 273	3.9568	.96 952	49
12	.24 531	.25 304	3.9520	.96 945	48
13	.24 559	.25 335	3.9471	.96 937	47
14	.24 587	.25 366	3.9423	.96 930	46
15	.24 615	.25 397	3.9375	.96 923	45
16	.24 644	.25 428	3.9327	.96 916	44
17	.24 672	.25 459	3.9279	.96 909	43
18	.24 700	.25 490	3.9232	.96 902	42
19	.24 728	.25 521	3.9184	.96 894	41
20	.24 756	.25 552	3.9136	.96 887	**40**
21	.24 784	.25 583	3.9089	.96 880	39
22	.24 813	.25 614	3.9042	.96 873	38
23	.24 841	.25 645	3.8995	.96 866	37
24	.24 869	.25 676	3.8947	.96 858	36
25	.24 897	.25 707	3.8900	.96 851	35
26	.24 925	.25 738	3.8854	.96 844	34
27	.24 954	.25 769	3.8807	.96 837	33
28	.24 982	.25 800	3.8760	.96 829	32
29	.25 010	.25 831	3.8714	.96 822	31
30	.25 038	.25 862	3.8667	.96 815	**30**
31	.25 066	.25 893	3.8621	.96 807	29
32	.25 094	.25 924	3.8575	.96 800	28
33	.25 122	.25 955	3.8528	.96 793	27
34	.25 151	.25 986	3.8482	.96 786	26
35	.25 179	.26 017	3.8436	.96 778	25
36	.25 207	.26 048	3.8391	.96 771	24
37	.25 235	.26 079	3.8345	.96 764	23
38	.25 263	.26 110	3.8299	.96 756	22
39	.25 291	.26 141	3.8254	.96 749	21
40	.25 320	.26 172	3.8208	.96 742	**20**
41	.25 348	.26 203	3.8163	.96 734	19
42	.25 376	.26 235	3.8118	.96 727	18
43	.25 404	.26 266	3.8073	.96 719	17
44	.25 432	.26 297	3.8028	.96 712	16
45	.25 460	.26 328	3.7983	.96 705	15
46	.25 488	.26 359	3.7938	.96 697	14
47	.25 516	.26 390	3.7893	.96 690	13
48	.25 545	.26 421	3.7848	.96 682	12
49	.25 573	.26 452	3.7804	.96 675	11
50	.25 601	.26 483	3.7760	.96 667	**10**
51	.25 629	.26 515	3.7715	.96 660	9
52	.25 657	.26 546	3.7671	.96 653	8
53	.25 685	.26 577	3.7627	.96 645	7
54	.25 713	.26 608	3.7583	.96 638	6
55	.25 741	.26 639	3.7539	.96 630	5
56	.25 769	.26 670	3.7495	.96 623	4
57	.25 798	.26 701	3.7451	.96 615	3
58	.25 826	.26 733	3.7408	.96 608	2
59	.25 854	.26 764	3.7364	.96 600	1
60	.25 882	.26 795	3.7321	.96 593	**0**
′	cos	cot	tan	sin	′

75°

15°

′	sin	tan	cot	cos	′
0	.25 882	.26 795	3.7321	.96 593	**60**
1	.25 910	.26 826	3.7277	.96 585	59
2	.25 938	.26 857	3.7234	.96 578	58
3	.25 966	.26 888	3.7191	.96 570	57
4	.25 994	.26 920	3.7148	.96 562	56
5	.26 022	.26 951	3.7105	.96 555	55
6	.26 050	.26 982	3.7062	.96 547	54
7	.26 079	.27 013	3.7019	.96 540	53
8	.26 107	.27 044	3.6976	.96 532	52
9	.26 135	.27 076	3.6933	.96 524	51
10	.26 163	.27 107	3.6891	.96 517	**50**
11	.26 191	.27 138	3.6848	.96 509	49
12	.26 219	.27 169	3.6806	.96 502	48
13	.26 247	.27 201	3.6764	.96 494	47
14	.26 275	.27 232	3.6722	.96 486	46
15	.26 303	.27 263	3.6680	.96 479	45
16	.26 331	.27 294	3.6638	.96 471	44
17	.26 359	.27 326	3.6596	.96 463	43
18	.26 387	.27 357	3.6554	.96 456	42
19	.26 415	.27 388	3.6512	.96 448	41
20	.26 443	.27 419	3.6470	.96 440	**40**
21	.26 471	.27 451	3.6429	.96 433	39
22	.26 500	.27 482	3.6387	.96 425	38
23	.26 528	.27 513	3.6346	.96 417	37
24	.26 556	.27 545	3.6305	.96 410	36
25	.26 584	.27 576	3.6264	.96 402	35
26	.26 612	.27 607	3.6222	.96 394	34
27	.26 640	.27 638	3.6181	.96 386	33
28	.26 668	.27 670	3.6140	.96 379	32
29	.26 696	.27 701	3.6100	.96 371	31
30	.26 724	.27 732	3.6059	.96 363	**30**
31	.26 752	.27 764	3.6018	.96 355	29
32	.26 780	.27 795	3.5978	.96 347	28
33	.26 808	.27 826	3.5937	.96 340	27
34	.26 836	.27 858	3.5897	.96 332	26
35	.26 864	.27 889	3.5856	.96 324	25
36	.26 892	.27 921	3.5816	.96 316	24
37	.26 920	.27 952	3.5776	.96 308	23
38	.26 948	.27 983	3.5736	.96 301	22
39	.26 976	.28 015	3.5696	.96 293	21
40	.27 004	.28 046	3.5656	.96 285	**20**
41	.27 032	.28 077	3.5616	.96 277	19
42	.27 060	.28 109	3.5576	.96 269	18
43	.27 088	.28 140	3.5536	.96 261	17
44	.27 116	.28 172	3.5497	.96 253	16
45	.27 144	.28 203	3.5457	.96 246	15
46	.27 172	.28 234	3.5418	.96 238	14
47	.27 200	.28 266	3.5379	.96 230	13
48	.27 228	.28 297	3.5339	.96 222	12
49	.27 256	.28 329	3.5300	.96 214	11
50	.27 284	.28 360	3.5261	.96 206	**10**
51	.27 312	.28 391	3.5222	.96 198	9
52	.27 340	.28 423	3.5183	.96 190	8
53	.27 368	.28 454	3.5144	.96 182	7
54	.27 396	.28 486	3.5105	.96 174	6
55	.27 424	.28 517	3.5067	.96 166	5
56	.27 452	.28 549	3.5028	.96 158	4
57	.27 480	.28 580	3.4989	.96 150	3
58	.27 508	.28 612	3.4951	.96 142	2
59	.27 536	.28 643	3.4912	.96 134	1
60	.27 564	.28 675	3.4874	.96 126	**0**
′	cos	cot	tan	sin	′

74°

	16°				
′	sin	tan	cot	cos	′
0	.27 564	.28 675	3.4874	.96 126	**60**
1	.27 592	.28 706	3.4836	.96 118	59
2	.27 620	.28 738	3.4798	.96 110	58
3	.27 648	.28 769	3.4760	.96 102	57
4	.27 676	.28 801	3.4722	.96 094	56
5	.27 704	.28 832	3.4684	.96 086	55
6	.27 731	.28 864	3.4646	.96 078	54
7	.27 759	.28 895	3.4608	.96 970	53
8	.27 787	.28 927	3.4570	.96 062	52
9	.27 815	.28 958	3.4533	.96 954	51
10	.27 843	.28 990	3.4495	.96 046	**50**
11	.27 871	.29 021	3.4458	.96 037	49
12	.27 899	.29 053	3.4420	.96 029	48
13	.27 927	.29 084	3.4383	.96 021	47
14	.27 955	.29 116	3.4346	.96 013	46
15	.27 983	.29 147	3.4308	.96 005	45
16	.28 011	.29 179	3.4271	.95 997	44
17	.28 039	.29 210	3.4234	.95 989	43
18	.28 067	.29 242	3.4197	.95 981	42
19	.28 095	.29 274	3.4160	.95 972	41
20	.28 123	.29 305	3.4124	.95 964	**40**
21	.28 150	.29 337	3.4087	.95 956	39
22	.28 178	.29 368	3.4050	.95 948	38
23	.28 206	.29 400	3.4014	.95 940	37
24	.28 234	.29 432	3.3977	.95 931	36
25	.28 262	.29 463	3.3941	.95 923	35
26	.28 290	.29 495	3.3904	.95 915	34
27	.28 318	.29 526	3.3868	.95 907	33
28	.28 346	.29 558	3.3832	.95 898	32
29	.28 374	.29 590	3.3796	.95 890	31
30	.28 402	.29 621	3.3759	.95 882	**30**
31	.28 429	.29 653	3.3723	.95 874	29
32	.28 457	.29 685	3.3687	.95 865	28
33	.28 485	.29 716	3.3652	.95 857	27
34	.28 513	.29 748	3.3616	.95 849	26
35	.28 541	.29 780	3.3580	.95 841	25
36	.28 569	.29 811	3.3544	.95 832	24
37	.28 597	.29 843	3.3509	.95 824	23
38	.28 625	.29 875	3.3473	.95 816	22
39	.28 652	.29 906	3.3438	.95 807	21
40	.28 680	.29 938	3.3402	.95 799	**20**
41	.28 708	.29 970	3.3367	.95 791	19
42	.28 736	.30 001	3.3332	.95 782	18
43	.28 764	.30 033	3.3297	.95 774	17
44	.28 792	.30 065	3.3261	.95 766	16
45	.28 820	.30 097	3.3226	.95 757	15
46	.28 847	.30 128	3.3191	.95 749	14
47	.28 875	.30 160	3.3156	.95 740	13
48	.28 903	.30 192	3.3122	.95 732	12
49	.28 931	.30 224	3.3087	.95 724	11
50	.28 959	.30 255	3.3052	.95 715	**10**
51	.28 987	.30 287	3.3017	.95 707	9
52	.29 015	.30 319	3.2983	.95 698	8
53	.29 042	.30 351	3.2948	.95 690	7
54	.29 070	.30 382	3.2914	.95 681	6
55	.29 098	.30 414	3.2879	.95 673	5
56	.29 126	.30 446	3.2845	.95 664	4
57	.29 154	.30 478	3.2811	.95 656	3
58	.29 182	.30 509	3.2777	.95 647	2
59	.29 209	.30 541	3.2743	.95 639	1
60	.29 237	.30 573	3.2709	.95 630	**0**
′	cos	cot	tan	sin	′
	73°				

	17°				
′	sin	tan	cot	cos	′
0	.29 237	.30 573	3.2709	.95 630	**60**
1	.29 265	.30 605	3.2675	.95 622	59
2	.29 293	.30 637	3.2641	.95 613	58
3	.29 321	.30 669	3.2607	.95 605	57
4	.29 348	.30 700	3.2573	.95 596	56
5	.29 376	.30 732	3.2539	.95 588	55
6	.29 404	.30 764	3.2506	.95 579	54
7	.29 432	.30 796	3.2472	.95 571	53
8	.29 460	.30 828	3.2438	.95 562	52
9	.29 487	.30 860	3.2405	.95 554	51
10	.29 515	.30 891	3.2371	.95 545	**50**
11	.29 543	.30 923	3.2338	.95 536	49
12	.29 571	.30 955	3.2305	.95 528	48
13	.29 599	.30 987	3.2272	.95 519	47
14	.29 626	.31 019	3.2238	.95 511	46
15	.29 654	.31 051	3.2205	.95 502	45
16	.29 682	.31 083	3.2172	.95 493	44
17	.29 710	.31 115	3.2139	.95 485	43
18	.29 737	.31 147	3.2106	.95 476	42
19	.29 765	.31 178	3.2073	.95 467	41
20	.29 793	.31 210	3.2041	.95 459	**40**
21	.29 821	.31 242	3.2008	.95 450	39
22	.29 849	.31 274	3.1975	.95 441	38
23	.29 876	.31 306	3.1943	.95 433	37
24	.29 904	.31 338	3.1910	.95 424	36
25	.29 932	.31 370	3.1878	.95 415	35
26	.29 960	.31 402	3.1845	.95 407	34
27	.29 987	.31 434	3.1813	.95 398	33
28	.30 015	.31 466	3.1780	.95 389	32
29	.30 043	.31 498	3.1748	.95 380	31
30	.30 071	.31 530	3.1716	.95 372	**30**
31	.30 098	.31 562	3.1684	.95 363	29
32	.30 126	.31 594	3.1652	.95 354	28
33	.30 154	.31 626	3.1620	.95 345	27
34	.30 182	.31 658	3.1588	.95 337	26
35	.30 209	.31 690	3.1556	.95 328	25
36	.30 237	.31 722	3.1524	.95 319	24
37	.30 265	.31 754	3.1492	.95 310	23
38	.30 292	.31 786	3.1460	.95 301	22
39	.30 320	.31 818	3.1429	.95 293	21
40	.30 348	.31 850	3.1397	.95 284	**20**
41	.30 376	.31 882	3.1366	.95 275	19
42	.30 403	.31 914	3.1334	.95 266	18
43	.30 431	.31 946	3.1303	.95 257	17
44	.30 459	.31 978	3.1271	.95 248	16
45	.30 486	.32 010	3.1240	.95 240	15
46	.30 514	.32 042	3.1209	.95 231	14
47	.30 542	.32 074	3.1178	.95 222	13
48	.30 570	.32 106	3.1146	.95 213	12
49	.30 597	.32 139	3.1115	.95 204	11
50	.30 625	.32 171	3.1084	.95 195	**10**
51	.30 653	.32 203	3.1053	.95 186	9
52	.30 680	.32 235	3.1022	.95 177	8
53	.30 708	.32 267	3.0991	.95 168	7
54	.30 736	.32 299	3.0961	.95 159	6
55	.30 763	.32 331	3.0930	.95 150	5
56	.30 791	.32 363	3.0899	.95 142	4
57	.30 819	.32 396	3.0868	.95 133	3
58	.30 846	.32 428	3.0838	.95 124	2
59	.30 874	.32 460	3.0807	.95 115	1
60	.30 902	.32 492	3.0777	.95 106	**0**
′	cos	cot	tan	sin	′
	72°				

′	18° sin	tan	cot	cos	′
0	.30 902	.32 492	3.0777	.95 106	**60**
1	.30 929	.32 524	3.0746	.95 097	59
2	.30 957	.32 556	3.0716	.95 088	58
3	.30 985	.32 588	3.0686	.95 079	57
4	.31 012	.32 621	3.0655	.95 070	56
5	.31 040	.32 653	3.0625	.95 061	55
6	.31 068	.32 685	3.0595	.95 052	54
7	.31 095	.32 717	3.0565	.95 043	53
8	.31 123	.32 749	3.0535	.95 033	52
9	.31 151	.32 782	3.0505	.95 024	51
10	.31 178	.32 814	3.0475	.95 015	**50**
11	.31 206	.32 846	3.0445	.95 006	49
12	.31 233	.32 878	3.0415	.94 997	48
13	.31 261	.32 911	3.0385	.94 988	47
14	.31 289	.32 943	3.0356	.94 979	46
15	.31 316	.32 975	3.0326	.94 970	45
16	.31 344	.33 007	3.0296	.94 961	44
17	.31 372	.33 040	3.0267	.94 952	43
18	.31 399	.33 072	3.0237	.94 943	42
19	.31 427	.33 104	3.0208	.94 933	41
20	.31 454	.33 136	3.0178	.94 924	**40**
21	.31 482	.33 169	3.0149	.94 915	39
22	.31 510	.33 201	3.0120	.94 906	38
23	.31 537	.33 233	3.0090	.94 897	37
24	.31 565	.33 266	3.0061	.94 888	36
25	.31 593	.33 298	3.0032	.94 878	35
26	.31 620	.33 330	3.0003	.94 869	34
27	.31 648	.33 363	2.9974	.94 860	33
28	.31 675	.33 395	2.9945	.94 851	32
29	.31 703	.33 427	2.9916	.94 842	31
30	.31 730	.33 460	2.9887	.94 832	**30**
31	.31 758	.33 492	2.9858	.94 823	29
32	.31 786	.33 524	2.9829	.94 814	28
33	.31 813	.33 557	2.9800	.94 805	27
34	.31 841	.33 589	2.9772	.94 795	26
35	.31 868	.33 621	2.9743	.94 786	25
36	.31 896	.33 654	2.9714	.94 777	24
37	.31 923	.33 686	2.9686	.94 768	23
38	.31 951	.33 718	2.9657	.94 758	22
39	.31 979	.33 751	2.9629	.94 749	21
40	.32 006	.33 783	2.9600	.94 740	**20**
41	.32 034	.33 816	2.9572	.94 730	19
42	.32 061	.33 848	2.9544	.94 721	18
43	.32 089	.33 881	2.9515	.94 712	17
44	.32 116	.33 913	2.9487	.94 702	16
45	.32 144	.33 945	2.9459	.94 693	15
46	.32 171	.33 978	2.9431	.94 684	14
47	.32 199	.34 010	2.9403	.94 674	13
48	.32 227	.34 043	2.9375	.94 665	12
49	.32 254	.34 075	2.9347	.94 656	11
50	.32 282	.34 108	2.9319	.94 646	**10**
51	.32 309	.34 140	2.9291	.94 637	9
52	.32 337	.34 173	2.9263	.94 627	8
53	.32 364	.34 205	2.9235	.94 618	7
54	.32 392	.34 238	2.9208	.94 609	6
55	.32 419	.34 270	2.9180	.94 599	5
56	.32 447	.34 303	2.9152	.94 590	4
57	.32 474	.34 335	2.9125	.94 580	3
58	.32 502	.34 368	2.9097	.94 571	2
59	.32 529	.34 400	2.9070	.94 561	1
60	.32 557	.34 433	2.9042	.94 552	**0**
′	cos	cot	tan	sin 71°	′

′	19° sin	tan	cot	cos	′
0	.32 557	.34 433	2.9042	.94 552	**60**
1	.32 584	.34 465	2.9015	.94 542	59
2	.32 612	.34 498	2.8987	.94 533	58
3	.32 639	.34 530	2.8960	.94 523	57
4	.32 667	.34 563	2.8933	.94 514	56
5	.32 694	.34 596	2.8905	.94 504	55
6	.32 722	.34 628	2.8878	.94 495	54
7	.32 749	.34 661	2.8851	.94 485	53
8	.32 777	.34 693	2.8824	.94 476	52
9	.32 804	.34 726	2.8797	.94 466	51
10	.32 832	.34 758	2.8770	.94 457	**50**
11	.32 859	.34 791	2.8743	.94 447	49
12	.32 887	.34 824	2.8716	.94 438	48
13	.32 914	.34 856	2.8689	.94 428	47
14	.32 942	.34 889	2.8662	.94 418	46
15	.32 969	.34 922	2.8636	.94 409	45
16	.32 997	.34 954	2.8609	.94 399	44
17	.33 024	.34 987	2.8582	.94 390	43
18	.33 051	.35 020	2.8556	.94 380	42
19	.33 079	.35 052	2.8529	.94 370	41
20	.33 106	.35 085	2.8502	.94 361	**40**
21	.33 134	.35 118	2.8476	.94 351	39
22	.33 161	.35 150	2.8449	.94 342	38
23	.33 189	.35 183	2.8423	.94 332	37
24	.33 216	.35 216	2.8397	.94 322	36
25	.33 244	.35 248	2.8370	.94 313	35
26	.33 271	.35 281	2.8344	.94 303	34
27	.33 298	.35 314	2.8318	.94 293	33
28	.33 326	.35 346	2.8291	.94 284	32
29	.33 353	.35 379	2.8265	.94 274	31
30	.33 381	.35 412	2.8239	.94 264	**30**
31	.33 408	.35 445	2.8213	.94 254	29
32	.33 436	.35 477	2.8187	.94 245	28
33	.33 463	.35 510	2.8161	.94 235	27
34	.33 490	.35 543	2.8135	.94 225	26
35	.33 518	.35 576	2.8109	.94 215	25
36	.33 545	.35 608	2.8083	.94 206	24
37	.33 573	.35 641	2.8057	.94 196	23
38	.33 600	.35 674	2.8032	.94 186	22
39	.33 627	.35 707	2.8006	.94 176	21
40	.33 655	.35 740	2.7980	.94 167	**20**
41	.33 682	.35 772	2.7955	.94 157	19
42	.33 710	.35 805	2.7929	.94 147	18
43	.33 737	.35 838	2.7903	.94 137	17
44	.33 764	.35 871	2.7878	.94 127	16
45	.33 792	.35 904	2.7852	.94 118	15
46	.33 819	.35 937	2.7827	.94 108	14
47	.33 846	.35 969	2.7801	.94 098	13
48	.33 874	.36 002	2.7776	.94 088	12
49	.33 901	.36 035	2.7751	.94 078	11
50	.33 929	.36 068	2.7725	.94 068	**10**
51	.33 956	.36 101	2.7700	.94 058	9
52	.33 983	.36 134	2.7675	.94 049	8
53	.34 011	.36 167	2.7650	.94 039	7
54	.34 038	.36 199	2.7625	.94 029	6
55	.34 065	.36 232	2.7600	.94 019	5
56	.34 093	.36 265	2.7575	.94 009	4
57	.34 120	.36 298	2.7550	.93 999	3
58	.34 147	.36 331	2.7525	.93 989	2
59	.34 175	.36 364	2.7500	.93 979	1
60	.34 202	.36 397	2.7475	.93 969	**0**
′	cos	cot	tan	sin 70°	′

20°					
′	sin	tan	cot	cos	′
0	.34 202	.36 397	2.7475	.93 969	**60**
1	.34 229	.36 430	2.7450	.93 959	59
2	.34 257	.36 463	2.7425	.93 949	58
3	.34 284	.36 496	2.7400	.93 939	57
4	.34 311	.36 529	2.7376	.93 929	56
5	.34 339	.36 562	2.7351	.93 919	55
6	.34 366	.36 595	2.7326	.93 909	54
7	.34 393	.36 628	2.7302	.93 899	53
8	.34 421	.36 661	2.7277	.93 889	52
9	.34 448	.36 694	2.7253	.93 879	51
10	.34 475	.36 727	2.7228	.93 869	**50**
11	.34 503	.36 760	2.7204	.93 859	49
12	.34 530	.36 793	2.7179	.93 849	48
13	.34 557	.36 826	2.7155	.93 839	47
14	.34 584	.36 859	2.7130	.93 829	46
15	.34 612	.36 892	2.7106	.93 819	45
16	.34 639	.36 925	2.7082	.93 809	44
17	.34 666	.36 958	2.7058	.93 799	43
18	.34 694	.36 991	2.7034	.93 789	42
19	.34 721	.37 024	2.7009	.93 779	41
20	.34 748	.37 057	2.6985	.93 769	**40**
21	.34 775	.37 090	2.6961	.93 759	39
22	.34 803	.37 123	2.6937	.93 748	38
23	.34 830	.37 157	2.6913	.93 738	37
24	.34 857	.37 190	2.6889	.93 728	36
25	.34 884	.37 223	2.6865	.93 718	35
26	.34 912	.37 256	2.6841	.93 708	34
27	.34 939	.37 289	2.6818	.93 698	33
28	.34 966	.37 322	2.6794	.93 688	32
29	.34 993	.37 355	2.6770	.93 677	31
30	.35 021	.37 388	2.6746	.93 667	**30**
31	.35 048	.37 422	2.6723	.93 657	29
32	.35 075	.37 455	2.6699	.93 647	28
33	.35 102	.37 488	2.6675	.93 637	27
34	.35 130	.37 521	2.6652	.93 626	26
35	.35 157	.37 554	2.6628	.93 616	25
36	.35 184	.37 588	2.6605	.93 606	24
37	.35 211	.37 621	2.6581	.93 596	23
38	.35 239	.37 654	2.6558	.93 585	22
39	.35 266	.37 687	2.6534	.93 575	21
40	.35 293	.37 720	2.6511	.93 565	**20**
41	.35 320	.37 754	2.6488	.93 555	19
42	.35 347	.37 787	2.6464	.93 544	18
43	.35 375	.37 820	2.6441	.93 534	17
44	.35 402	.37 853	2.6418	.93 524	16
45	.35 429	.37 887	2.6395	.93 514	15
46	.35 456	.37 920	2.6371	.93 503	14
47	.35 484	.37 953	2.6348	.93 493	13
48	.35 511	.37 986	2.6325	.93 483	12
49	.35 538	.38 020	2.6302	.93 472	11
50	.35 565	.38 053	2.6279	.93 462	**10**
51	.35 592	.38 086	2.6256	.93 452	9
52	.35 619	.38 120	2.6233	.93 441	8
53	.35 647	.38 153	2.6210	.93 431	7
54	.35 674	.38 186	2.6187	.93 420	6
55	.35 701	.38 220	2.6165	.93 410	5
56	.35 728	.38 253	2.6142	.93 400	4
57	.35 755	.38 286	2.6119	.93 389	3
58	.35 782	.38 320	2.6096	.93 379	2
59	.35 810	.38 353	2.6074	.93 368	1
60	.35 837	.38 386	2.6051	.93 358	**0**
′	cos	cot	tan	sin	′
69°					

21°					
′	sin	tan	cot	cos	′
0	.35 837	.38 386	2.6051	.93 358	**60**
1	.35 864	.38 420	2.6028	.93 348	59
2	.35 891	.38 453	2.6006	.93 337	58
3	.35 918	.38 487	2.5983	.93 327	57
4	.35 945	.38 520	2.5961	.93 316	56
5	.35 973	.38 553	2.5938	.93 306	55
6	.36 000	.38 587	2.5916	.93 295	54
7	.36 027	.38 620	2.5893	.93 285	53
8	.36 054	.38 654	2.5871	.93 274	52
9	.36 081	.38 687	2.5848	.93 264	51
10	.36 108	.38 721	2.5826	.93 253	**50**
11	.36 135	.38 754	2.5804	.93 243	49
12	.36 162	.38 787	2.5782	.93 232	48
13	.36 190	.38 821	2.5759	.93 222	47
14	.36 217	.38 854	2.5737	.93 211	46
15	.36 244	.38 888	2.5715	.93 201	45
16	.36 271	.38 921	2.5693	.93 190	44
17	.36 298	.38 955	2.5671	.93 180	43
18	.36 325	.38 988	2.5649	.93 169	42
19	.36 352	.39 022	2.5627	.93 159	41
20	.36 379	.39 055	2.5605	.93 148	**40**
21	.36 406	.39 089	2.5583	.93 137	39
22	.36 434	.39 122	2.5561	.93 127	38
23	.36 461	.39 156	2.5539	.93 116	37
24	.36 488	.39 190	2.5517	.93 106	36
25	.36 515	.39 223	2.5495	.93 095	35
26	.36 542	.39 257	2.5473	.93 084	34
27	.36 569	.39 290	2.5452	.93 074	33
28	.36 596	.39 324	2.5430	.93 063	32
29	.36 623	.39 357	2.5408	.93 052	31
30	.36 650	.39 391	2.5386	.93 042	**30**
31	.36 677	.39 425	2.5365	.93 031	29
32	.36 704	.39 458	2.5343	.93 020	28
33	.36 731	.39 492	2.5322	.93 010	27
34	.36 758	.39 526	2.5300	.92 999	26
35	.36 785	.39 559	2.5279	.92 988	25
36	.36 812	.39 593	2.5257	.92 978	24
37	.36 839	.39 626	2.5236	.92 967	23
38	.36 867	.39 660	2.5214	.92 956	22
39	.36 894	.39 694	2.5193	.92 945	21
40	.36 921	.39 727	2.5172	.92 935	**20**
41	.36 948	.39 761	2.5150	.92 924	19
42	.36 975	.39 795	2.5129	.92 913	18
43	.37 002	.39 829	2.5108	.92 902	17
44	.37 029	.39 862	2.5086	.92 892	16
45	.37 056	.39 896	2.5065	.92 881	15
46	.37 083	.39 930	2.5044	.92 870	14
47	.37 110	.39 963	2.5023	.92 859	13
48	.37 137	.39 997	2.5002	.92 849	12
49	.37 164	.40 031	2.4981	.92 838	11
50	.37 191	.40 065	2.4960	.92 827	**10**
51	.37 218	.40 098	2.4939	.92 816	9
52	.37 245	.40 132	2.4918	.92 805	8
53	.37 272	.40 166	2.4897	.92 794	7
54	.37 299	.40 200	2.4876	.92 784	6
55	.37 326	.40 234	2.4855	.92 773	5
56	.37 353	.40 267	2.4834	.92 762	4
57	.37 380	.40 301	2.4813	.92 751	3
58	.37 407	.40 335	2.4792	.92 740	2
59	.37 434	.40 369	2.4772	.92 729	1
60	.37 461	.40 403	2.4751	.92 718	**0**
′	cos	cot	tan	sin	′
68°					

	22°				
′	sin	tan	cot	cos	′
0	.37 461	.40 403	2.4751	.92 718	**60**
1	.37 488	.40 436	2.4730	.92 707	59
2	.37 515	.40 470	2.4709	.92 697	58
3	.37 542	.40 504	2.4689	.92 686	57
4	.37 569	.40 538	2.4668	.92 675	56
5	.37 595	.40 572	2.4648	.92 664	55
6	.37 622	.40 606	2.4627	.92 653	54
7	.37 649	.40 640	2.4606	.92 642	53
8	.37 676	.40 674	2.4586	.92 631	52
9	.37 703	.40 707	2.4566	.92 620	51
10	.37 730	.40 741	2.4545	.92 609	**50**
11	.37 757	.40 775	2.4525	.92 598	49
12	.37 784	.40 809	2.4504	.92 587	48
13	.37 811	.40 843	2.4484	.92 576	47
14	.37 838	.40 877	2.4464	.92 565	46
15	.37 865	.40 911	2.4443	.92 554	45
16	.37 892	.40 945	2.4423	.92 543	44
17	.37 919	.40 979	2.4403	.92 532	43
18	.37 946	.41 013	2.4383	.92 521	42
19	.37 973	.41 047	2.4362	.92 510	41
20	.37 999	.41 081	2.4342	.92 499	**40**
21	.38 026	.41 115	2.4322	.92 488	39
22	.38 053	.41 149	2.4302	.92 477	38
23	.38 080	.41 183	2.4282	.92 466	37
24	.38 107	.41 217	2.4262	.92 455	36
25	.38 134	.41 251	2.4242	.92 444	35
26	.38 161	.41 285	2.4222	.92 432	34
27	.38 188	.41 319	2.4202	.92 421	33
28	.38 215	.41 353	2.4182	.92 410	32
29	.38 241	.41 387	2.4162	.92 399	31
30	.38 268	.41 421	2.4142	.92 388	**30**
31	.38 295	.41 455	2.4122	.92 377	29
32	.38 322	.41 490	2.4102	.92 366	28
33	.38 349	.41 524	2.4083	.92 355	27
34	.38 376	.41 558	2.4063	.92 343	26
35	.38 403	.41 592	2.4043	.92 332	25
36	.38 430	.41 626	2.4023	.92 321	24
37	.38 456	.41 660	2.4004	.92 310	23
38	.38 483	.41 694	2.3984	.92 299	22
39	.38 510	.41 728	2.3964	.92 287	21
40	.38 537	.41 763	2.3945	.92 276	**20**
41	.38 564	.41 797	2.3925	.92 265	19
42	.38 591	.41 831	2.3906	.92 254	18
43	.38 617	.41 865	2.3886	.92 243	17
44	.38 644	.41 899	2.3867	.92 231	16
45	.38 671	.41 933	2.3847	.92 220	15
46	.38 698	.41 968	2.3828	.92 209	14
47	.38 725	.42 002	2.3808	.92 198	13
48	.38 752	.42 036	2.3789	.92 186	12
49	.38 778	.42 070	2.3770	.92 175	11
50	.38 805	.42 105	2.3750	.92 164	**10**
51	.38 832	.42 139	2.3731	.92 152	9
52	.38 859	.42 173	2.3712	.92 141	8
53	.38 886	.42 207	2.3693	.92 130	7
54	.38 912	.42 242	2.3673	.92 119	6
55	.38 939	.42 276	2.3654	.92 107	5
56	.38 966	.42 310	2.3635	.92 096	4
57	.38 993	.42 345	2.3616	.92 085	3
58	.39 020	.42 379	2.3597	.92 073	2
59	.39 046	.42 413	2.3578	.92 062	1
60	.39 073	.42 447	2.3559	.92 050	**0**
′	cos	cot	tan	sin	′
	67°				

	23°				
′	sin	tan	cot	cos	′
0	.39 073	.42 447	2.3559	.92 050	**60**
1	.39 100	.42 482	2.3539	.92 039	59
2	.39 127	.42 516	2.3520	.92 028	58
3	.39 153	.42 551	2.3501	.92 016	57
4	.39 180	.42 585	2.3483	.92 005	56
5	.39 207	.42 619	2.3464	.91 994	55
6	.39 234	.42 654	2.3445	.91 982	54
7	.39 260	.42 688	2.3426	.91 971	53
8	.39 287	.42 722	2.3407	.91 959	52
9	.39 314	.42 757	2.3388	.91 948	51
10	.39 341	.42 791	2.3369	.91 936	**50**
11	.39 367	.42 826	2.3351	.91 925	49
12	.39 394	.42 860	2.3332	.91 914	48
13	.39 421	.42 894	2.3313	.91 902	47
14	.39 448	.42 929	2.3294	.91 891	46
15	.39 474	.42 963	2.3276	.91 879	45
16	.39 501	.42 998	2.3257	.91 868	44
17	.39 528	.43 032	2.3238	.91 856	43
18	.39 555	.43 067	2.3220	.91 845	42
19	.39 581	.43 101	2.3201	.91 833	41
20	.39 608	.43 136	2.3183	.91 822	**40**
21	.39 635	.43 170	2.3164	.91 810	39
22	.39 661	.43 205	2.3146	.91 799	38
23	.39 688	.43 239	2.3127	.91 787	37
24	.39 715	.43 274	2.3109	.91 775	36
25	.39 741	.43 308	2.3090	.91 764	35
26	.39 768	.43 343	2.3072	.91 752	34
27	.39 795	.43 378	2.3053	.91 741	33
28	.39 822	.43 412	2.3035	.91 729	32
29	.39 848	.43 447	2.3017	.91 718	31
30	.39 875	.43 481	2.2998	.91 706	**30**
31	.39 902	.43 516	2.2980	.91 694	29
32	.39 928	.43 550	2.2962	.91 683	28
33	.39 955	.43 585	2.2944	.91 671	27
34	.39 982	.43 620	2.2925	.91 660	26
35	.40 008	.43 654	2.2907	.91 648	25
36	.40 035	.43 689	2.2889	.91 636	24
37	.40 062	.43 724	2.2871	.91 625	23
38	.40 088	.43 758	2.2853	.91 613	22
39	.40 115	.43 793	2.2835	.91 601	21
40	.40 141	.43 828	2.2817	.91 590	**20**
41	.40 168	.43 862	2.2799	.91 578	19
42	.40 195	.43 897	2.2781	.91 566	18
43	.40 221	.43 932	2.2763	.91 555	17
44	.40 248	.43 966	2.2745	.91 543	16
45	.40 275	.44 001	2.2727	.91 531	15
46	.40 301	.44 036	2.2709	.91 519	14
47	.40 328	.44 071	2.2691	.91 508	13
48	.40 355	.44 105	2.2673	.91 496	12
49	.40 381	.44 140	2.2655	.91 484	11
50	.40 408	.44 175	2.2637	.91 472	**10**
51	.40 434	.44 210	2.2620	.91 461	9
52	.40 461	.44 244	2.2602	.91 449	8
53	.40 488	.44 279	2.2584	.91 437	7
54	.40 514	.44 314	2.2566	.91 425	6
55	.40 541	.44 349	2.2549	.91 414	5
56	.40 567	.44 384	2.2531	.91 402	4
57	.40 594	.44 418	2.2513	.91 390	3
58	.40 621	.44 453	2.2496	.91 378	2
59	.40 647	.44 488	2.2478	.91 366	1
60	.40 674	.44 523	2.2460	.91 355	**0**
′	cos	cot	tan	sin	′
	66°				

	24°				
′	sin	tan	cot	cos	′
0	.40 674	.44 523	2.2460	.91 355	**60**
1	.40 700	.44 558	2.2443	.91 343	59
2	.40 727	.44 593	2.2425	.91 331	58
3	.40 753	.44 627	2.2408	.91 319	57
4	.40 780	.44 662	2.2390	.91 307	56
5	.40 806	.44 697	2.2373	.91 295	55
6	.40 833	.44 732	2.2355	.91 283	54
7	.40 860	.44 767	2.2338	.91 272	53
8	.40 886	.44 802	2.2320	.91 260	52
9	.40 913	.44 837	2.2303	.91 248	51
10	.40 939	.44 872	2.2286	.91 236	**50**
11	.40 966	.44 907	2.2268	.91 224	49
12	.40 992	.44 942	2.2251	.91 212	48
13	.41 019	.44 977	2.2234	.91 200	47
14	.41 045	.45 012	2.2216	.91 188	46
15	.41 072	.45 047	2.2199	.91 176	45
16	.41 098	.45 082	2.2182	.91 164	44
17	.41 125	.45 117	2.2165	.91 152	43
18	.41 151	.45 152	2.2148	.91 140	42
19	.41 178	.45 187	2.2130	.91 128	41
20	.41 204	.45 222	2.2113	.91 116	**40**
21	.41 231	.45 257	2.2096	.91 104	39
22	.41 257	.45 292	2.2079	.91 092	38
23	.41 284	.45 327	2.2062	.91 080	37
24	.41 310	.45 362	2.2045	.91 068	36
25	.41 337	.45 397	2.2028	.91 056	35
26	.41 363	.45 432	2.2011	.91 044	34
27	.41 390	.45 467	2.1994	.91 032	33
28	.41 416	.45 502	2.1977	.91 020	32
29	.41 443	.45 538	2.1960	.91 008	31
30	.41 469	.45 573	2.1943	.90 996	**30**
31	.41 496	.45 608	2.1926	.90 984	29
32	.41 522	.45 643	2.1909	.90 972	28
33	.41 549	.45 678	2.1892	.90 960	27
34	.41 575	.45 713	2.1876	.90 948	26
35	.41 602	.45 748	2.1859	.90 936	25
36	.41 628	.45 784	2.1842	.90 924	24
37	.41 655	.45 819	2.1825	.90 911	23
38	.41 681	.45 854	2.1808	.90 899	22
39	.41 707	.45 889	2.1792	.90 887	21
40	.41 734	.45 924	2.1775	.90 875	**20**
41	.41 760	.45 960	2.1758	.90 863	19
42	.41 787	.45 995	2.1742	.90 851	18
43	.41 813	.46 030	2.1725	.90 839	17
44	.41 840	.46 065	2.1708	.90 826	16
45	.41 866	.46 101	2.1692	.90 814	15
46	.41 892	.46 136	2.1675	.90 802	14
47	.41 919	.46 171	2.1659	.90 790	13
48	.41 945	.46 206	2.1642	.90 778	12
49	.41 972	.46 242	2.1625	.90 766	11
50	.41 998	.46 277	2.1609	.90 753	**10**
51	.42 024	.46 312	2.1592	.90 741	9
52	.42 051	.46 348	2.1576	.90 729	8
53	.42 077	.46 383	2.1560	.90 717	7
54	.42 104	.46 418	2.1543	.90 704	6
55	.42 130	.46 454	2.1527	.90 692	5
56	.42 156	.46 489	2.1510	.90 680	4
57	.42 183	.46 525	2.1494	.90 668	3
58	.42 209	.46 560	2.1478	.90 655	2
59	.42 235	.46 595	2.1461	.90 643	1
60	.42 262	.46 631	2.1445	.90 631	**0**
′	cos	cot	tan	sin	′
	65°				

	25°				
′	sin	tan	cot	cos	′
0	.42 262	.46 631	2.1445	.90 631	**60**
1	.42 288	.46 666	2.1429	.90 618	59
2	.42 315	.46 702	2.1413	.90 606	58
3	.42 341	.46 737	2.1396	.90 594	57
4	.42 367	.46 772	2.1380	.90 582	56
5	.42 394	.46 808	2.1364	.90 569	55
6	.42 420	.46 843	2.1348	.90 557	54
7	.42 446	.46 879	2.1332	.90 545	53
8	.42 473	.46 914	2.1315	.90 532	52
9	.42 499	.46 950	2.1299	.90 520	51
10	.42 525	.46 985	2.1283	.90 507	**50**
11	.42 552	.47 021	2.1267	.90 495	49
12	.42 578	.47 056	2.1251	.90 483	48
13	.42 604	.47 092	2.1235	.90 470	47
14	.42 631	.47 128	2.1219	.90 458	46
15	.42 657	.47 163	2.1203	.90 446	45
16	.42 683	.47 199	2.1187	.90 433	44
17	.42 709	.47 234	2.1171	.90 421	43
18	.42 736	.47 270	2.1155	.90 408	42
19	.42 762	.47 305	2.1139	.90 396	41
20	.42 788	.47 341	2.1123	.90 383	**40**
21	.42 815	.47 377	2.1107	.90 371	39
22	.42 841	.47 412	2.1092	.90 358	38
23	.42 867	.47 448	2.1076	.90 346	37
24	.42 894	.47 483	2.1060	.90 334	36
25	.42 920	.47 519	2.1044	.90 321	35
26	.42 946	.47 555	2.1028	.90 309	34
27	.42 972	.47 590	2.1013	.90 296	33
28	.42 999	.47 626	2.0997	.90 284	32
29	.43 025	.47 662	2.0981	.90 271	31
30	.43 051	.47 698	2.0965	.90 259	**30**
31	.43 077	.47 733	2.0950	.90 246	29
32	.43 104	.47 769	2.0934	.90 233	28
33	.43 130	.47 805	2.0918	.90 221	27
34	.43 156	.47 840	2.0903	.90 208	26
35	.43 182	.47 876	2.0887	.90 196	25
36	.43 209	.47 912	2.0872	.90 183	24
37	.43 235	.47 948	2.0856	.90 171	23
38	.43 261	.47 984	2.0840	.90 158	22
39	.43 287	.48 019	2.0825	.90 146	21
40	.43 313	.48 055	2.0809	.90 133	**20**
41	.43 340	.48 091	2.0794	.90 120	19
42	.43 366	.48 127	2.0778	.90 108	18
43	.43 392	.48 163	2.0763	.90 095	17
44	.43 418	.48 198	2.0748	.90 082	16
45	.43 445	.48 234	2.0732	.90 070	15
46	.43 471	.48 270	2.0717	.90 057	14
47	.43 497	.48 306	2.0701	.90 045	13
48	.43 523	.48 342	2.0686	.90 032	12
49	.43 549	.48 378	2.0671	.90 019	11
50	.43 575	.48 414	2.0655	.90 007	**10**
51	.43 602	.48 450	2.0640	.89 994	9
52	.43 628	.48 486	2.0625	.89 981	8
53	.43 654	.48 521	2.0609	.89 968	7
54	.43 680	.48 557	2.0594	.89 956	6
55	.43 706	.48 593	2.0579	.89 943	5
56	.43 733	.48 629	2.0564	.89 930	4
57	.43 759	.48 665	2.0549	.89 918	3
58	.43 785	.48 701	2.0533	.89 905	2
59	.43 811	.48 737	2.0518	.89 892	1
60	.43 837	.48 773	2.0503	.89 879	**0**
′	cos	cot	tan	sin	′
	64°				

	26°				
′	sin	tan	cot	cos	′
0	.43 837	.48 773	2.0503	.89 879	**60**
1	.43 863	.48 809	2.0488	.89 867	59
2	.43 889	.48 845	2.0473	.89 854	58
3	.43 916	.48 881	2.0458	.89 841	57
4	.43 942	.48 917	2.0443	.89 828	56
5	.43 968	.48 953	2.0428	.89 816	55
6	.43 994	.48 989	2.0413	.89 803	54
7	.44 020	.49 026	2.0398	.89 790	53
8	.44 046	.49 062	2.0383	.89 777	52
9	.44 072	.49 098	2.0368	.89 764	51
10	.44 098	.49 134	2.0353	.89 752	**50**
11	.44 124	.49 170	2.0338	.89 739	49
12	.44 151	.49 206	2.0323	.89 726	48
13	.44 177	.49 242	2.0308	.89 713	47
14	.44 203	.49 278	2.0293	.89 700	46
15	.44 229	.49 315	2.0278	.89 687	45
16	.44 255	.49 351	2.0263	.89 674	44
17	.44 281	.49 387	2.0248	.89 662	43
18	.44 307	.49 423	2.0233	.89 649	42
19	.44 333	.49 459	2.0219	.89 636	41
20	.44 359	.49 495	2.0204	.89 623	**40**
21	.44 385	.49 532	2.0189	.89 610	39
22	.44 411	.49 568	2.0174	.89 597	38
23	.44 437	.49 604	2.0160	.89 584	37
24	.44 464	.49 640	2.0145	.89 571	36
25	.44 490	.49 677	2.0130	.89 558	35
26	.44 516	.49 713	2.0115	.89 545	34
27	.44 542	.49 749	2.0101	.89 532	33
28	.44 568	.49 786	2.0086	.89 519	32
29	.44 594	.49 822	2.0072	.89 506	31
30	.44 620	.49 858	2.0057	.89 493	**30**
31	.44 646	.49 894	2.0042	.89 480	29
32	.44 672	.49 931	2.0028	.89 467	28
33	.44 698	.49 967	2.0013	.89 454	27
34	.44 724	.50 004	1.9999	.89 441	26
35	.44 750	.50 040	1.9984	.89 428	25
36	.44 776	.50 076	1.9970	.89 415	24
37	.44 802	.50 113	1.9955	.89 402	23
38	.44 828	.50 149	1.9941	.89 389	22
39	.44 854	.50 185	1.9926	.89 376	21
40	.44 880	.50 222	1.9912	.89 363	**20**
41	.44 906	.50 258	1.9897	.89 350	19
42	.44 932	.50 295	1.9883	.89 337	18
43	.44 958	.50 331	1.9868	.89 324	17
44	.44 984	.50 368	1.9854	.89 311	16
45	.45 010	.50 404	1.9840	.89 298	15
46	.45 036	.50 441	1.9825	.89 285	14
47	.45 062	.50 477	1.9811	.89 272	13
48	.45 088	.50 514	1.9797	.89 259	12
49	.45 114	.50 550	1.9782	.89 245	11
50	.45 140	.50 587	1.9768	.89 232	**10**
51	.45 166	.50 623	1.9754	.89 219	9
52	.45 192	.50 660	1.9740	.89 206	8
53	.45 218	.50 696	1.9725	.89 193	7
54	.45 243	.50 733	1.9711	.89 180	6
55	.45 269	.50 769	1.9697	.89 167	5
56	.45 295	.50 806	1.9683	.89 153	4
57	.45 321	.50 843	1.9669	.89 140	3
58	.45 347	.50 879	1.9654	.89 127	2
59	.45 373	.50 916	1.9640	.89 114	1
60	.45 399	.50 953	1.9626	.89 101	**0**
′	cos	cot	tan	sin	′
	63°				

	27°				
′	sin	tan	cot	cos	′
0	.45 399	.50 953	1.9626	.89 101	**60**
1	.45 425	.50 989	1.9612	.89 087	59
2	.45 451	.51 026	1.9598	.89 074	58
3	.45 477	.51 063	1.9584	.89 061	57
4	.45 503	.51 099	1.9570	.89 048	56
5	.45 529	.51 136	1.9556	.89 035	55
6	.45 554	.51 173	1.9542	.89 021	54
7	.45 580	.51 209	1.9528	.89 008	53
8	.45 606	.51 246	1.9514	.88 995	52
9	.45 632	.51 283	1.9500	.88 981	51
10	.45 658	.51 319	1.9486	.88 968	**50**
11	.45 684	.51 356	1.9472	.88 955	49
12	.45 710	.51 393	1.9458	.88 942	48
13	.45 736	.51 430	1.9444	.88 928	47
14	.45 762	.51 467	1.9430	.88 915	46
15	.45 787	.51 503	1.9416	.88 902	45
16	.45 813	.51 540	1.9402	.88 888	44
17	.45 839	.51 577	1.9388	.88 875	43
18	.45 865	.51 614	1.9375	.88 862	42
19	.45 891	.51 651	1.9361	.88 848	41
20	.45 917	.51 688	1.9347	.88 835	**40**
21	.45 942	.51 724	1.9333	.88 822	39
22	.45 968	.51 761	1.9319	.88 808	38
23	.45 994	.51 798	1.9306	.88 795	37
24	.46 020	.51 835	1.9292	.88 782	36
25	.46 046	.51 872	1.9278	.88 768	35
26	.46 072	.51 909	1.9265	.88 755	34
27	.46 097	.51 946	1.9251	.88 741	33
28	.46 123	.51 983	1.9237	.88 728	32
29	.46 149	.52 020	1.9223	.88 715	31
30	.46 175	.52 057	1.9210	.88 701	**30**
31	.46 201	.52 094	1.9196	.88 688	29
32	.46 226	.52 131	1.9183	.88 674	28
33	.46 252	.52 168	1.9169	.88 661	27
34	.46 278	.52 205	1.9155	.88 647	26
35	.46 304	.52 242	1.9142	.88 634	25
36	.46 330	.52 279	1.9128	.88 620	24
37	.46 355	.52 316	1.9115	.88 607	23
38	.46 381	.52 353	1.9101	.88 593	22
39	.46 407	.52 390	1.9088	.88 580	21
40	.46 433	.52 427	1.9074	.88 566	**20**
41	.46 458	.52 464	1.9061	.88 553	19
42	.46 484	.52 501	1.9047	.88 539	18
43	.46 510	.52 538	1.9034	.88 526	17
44	.46 536	.52 575	1.9020	.88 512	16
45	.46 561	.52 613	1.9007	.88 499	15
46	.46 587	.52 650	1.8993	.88 485	14
47	.46 613	.52 687	1.8980	.88 472	13
48	.46 639	.52 724	1.8967	.88 458	12
49	.46 664	.52 761	1.8953	.88 445	11
50	.46 690	.52 798	1.8940	.88 431	**10**
51	.46 716	.52 836	1.8927	.88 417	9
52	.46 742	.52 873	1.8913	.88 404	8
53	.46 767	.52 910	1.8900	.88 390	7
54	.46 793	.52 947	1.8887	.88 377	6
55	.46 819	.52 985	1.8873	.88 363	5
56	.46 844	.53 022	1.8860	.88 349	4
57	.46 870	.53 059	1.8847	.88 336	3
58	.46 896	.53 096	1.8834	.88 322	2
59	.46 921	.53 134	1.8820	.88 308	1
60	.46 947	.53 171	1.8807	.88 295	**0**
′	cos	cot	tan	sin	′
	62°				

28°

′	sin	tan	cot	cos	′
0	.46 947	.53 171	1.8807	.88 295	**60**
1	.46 973	.53 208	1.8794	.88 281	59
2	.46 999	.53 246	1.8781	.88 267	58
3	.47 024	.53 283	1.8768	.88 254	57
4	.47 050	.53 320	1.8755	.88 240	56
5	.47 076	.53 358	1.8741	.88 226	55
6	.47 101	.53 395	1.8728	.88 213	54
7	.47 127	.53 432	1.8715	.88 199	53
8	.47 153	.53 470	1.8702	.88 185	52
9	.47 178	.53 507	1.8689	.88 172	51
10	.47 204	.53 545	1.8676	.88 158	**50**
11	.47 229	.53 582	1.8663	.88 144	49
12	.47 255	.53 620	1.8650	.88 130	48
13	.47 281	.53 657	1.8637	.88 117	47
14	.47 306	.53 694	1.8624	.88 103	46
15	.47 332	.53 732	1.8611	.88 089	45
16	.47 358	.53 769	1.8598	.88 075	44
17	.47 383	.53 807	1.8585	.88 062	43
18	.47 409	.53 844	1.8572	.88 048	42
19	.47 434	.53 882	1.8559	.88 034	41
20	.47 460	.53 920	1.8546	.88 020	**40**
21	.47 486	.53 957	1.8533	.88 006	39
22	.47 511	.53 995	1.8520	.87 993	38
23	.47 537	.54 032	1.8507	.87 979	37
24	.47 562	.54 070	1.8495	.87 965	36
25	.47 588	.54 107	1.8482	.87 951	35
26	.47 614	.54 145	1.8469	.87 937	34
27	.47 639	.54 183	1.8456	.87 923	33
28	.47 665	.54 220	1.8443	.87 909	32
29	.47 690	.54 258	1.8430	.87 896	31
30	.47 716	.54 296	1.8418	.87 882	**30**
31	.47 741	.54 333	1.8405	.87 868	29
32	.47 767	.54 371	1.8392	.87 854	28
33	.47 793	.54 409	1.8379	.87 840	27
34	.47 818	.54 446	1.8367	.87 826	26
35	.47 844	.54 484	1.8354	.87 812	25
36	.47 869	.54 522	1.8341	.87 798	24
37	.47 895	.54 560	1.8329	.87 784	23
38	.47 920	.54 597	1.8316	.87 770	22
39	.47 946	.54 635	1.8303	.87 756	21
40	.47 971	.54 673	1.8291	.87 743	**20**
41	.47 997	.54 711	1.8278	.87 729	19
42	.48 022	.54 748	1.8265	.87 715	18
43	.48 048	.54 786	1.8253	.87 701	17
44	.48 073	.54 824	1.8240	.87 687	16
45	.48 099	.54 862	1.8228	.87 673	15
46	.48 124	.54 900	1.8215	.87 659	14
47	.48 150	.54 938	1.8202	.87 645	13
48	.48 175	.54 975	1.8190	.87 631	12
49	.48 201	.55 013	1.8177	.87 617	11
50	.48 226	.55 051	1.8165	.87 603	**10**
51	.48 252	.55 089	1.8152	.87 589	9
52	.48 277	.55 127	1.8140	.87 575	8
53	.48 303	.55 165	1.8127	.87 561	7
54	.48 328	.55 203	1.8115	.87 546	6
55	.48 354	.55 241	1.8103	.87 532	5
56	.48 379	.55 279	1.8090	.87 518	4
57	.48 405	.55 317	1.8078	.87 504	3
58	.48 430	.55 355	1.8065	.87 490	2
59	.48 456	.55 393	1.8053	.87 476	1
60	.48 481	.55 431	1.8040	.87 462	**0**
′	cos	cot	tan	sin	′

61°

29°

′	sin	tan	cot	cos	′
0	.48 481	.55 431	1.8040	.87 462	**60**
1	.48 506	.55 469	1.8028	.87 448	59
2	.48 532	.55 507	1.8016	.87 434	58
3	.48 557	.55 545	1.8003	.87 420	57
4	.48 583	.55 583	1.7991	.87 406	56
5	.48 608	.55 621	1.7979	.87 391	55
6	.48 634	.55 659	1.7966	.87 377	54
7	.48 659	.55 697	1.7954	.87 363	53
8	.48 684	.55 736	1.7942	.87 349	52
9	.48 710	.55 774	1.7930	.87 335	51
10	.48 735	.55 812	1.7917	.87 321	**50**
11	.48 761	.55 850	1.7905	.87 306	49
12	.48 786	.55 888	1.7893	.87 292	48
13	.48 811	.55 926	1.7881	.87 278	47
14	.48 837	.55 964	1.7868	.87 264	46
15	.48 862	.56 003	1.7856	.87 250	45
16	.48 888	.56 041	1.7844	.87 235	44
17	.48 913	.56 079	1.7832	.87 221	43
18	.48 938	.56 117	1.7820	.87 207	42
19	.48 964	.56 156	1.7808	.87 193	41
20	.48 989	.56 194	1.7796	.87 178	**40**
21	.49 014	.56 232	1.7783	.87 164	39
22	.49 040	.56 270	1.7771	.87 150	38
23	.49 065	.56 309	1.7759	.87 136	37
24	.49 090	.56 347	1.7747	.87 121	36
25	.49 116	.56 385	1.7735	.87 107	35
26	.49 141	.56 424	1.7723	.87 093	34
27	.49 166	.56 462	1.7711	.87 079	33
28	.49 192	.56 501	1.7699	.87 064	32
29	.49 217	.56 539	1.7687	.87 050	31
30	.49 242	.56 577	1.7675	.87 036	**30**
31	.49 268	.56 616	1.7663	.87 021	29
32	.49 293	.56 654	1.7651	.87 007	28
33	.49 318	.56 693	1.7639	.86 993	27
34	.49 344	.56 731	1.7627	.86 978	26
35	.49 369	.56 769	1.7615	.86 964	25
36	.49 394	.56 808	1.7603	.86 949	24
37	.49 419	.56 846	1.7591	.86 935	23
38	.49 445	.56 885	1.7579	.86 921	22
39	.49 470	.56 923	1.7567	.86 906	21
40	.49 495	.56 962	1.7556	.86 892	**20**
41	.49 521	.57 000	1.7544	.86 878	19
42	.49 546	.57 039	1.7532	.86 863	18
43	.49 571	.57 078	1.7520	.86 849	17
44	.49 596	.57 116	1.7508	.86 834	16
45	.49 622	.57 155	1.7496	.86 820	15
46	.49 647	.57 193	1.7485	.86 805	14
47	.49 672	.57 232	1.7473	.86 791	13
48	.49 697	.57 271	1.7461	.86 777	12
49	.49 723	.57 309	1.7449	.86 762	11
50	.49 748	.57 348	1.7437	.86 748	**10**
51	.49 773	.57 386	1.7426	.86 733	9
52	.49 798	.57 425	1.7414	.86 719	8
53	.49 824	.57 464	1.7402	.86 704	7
54	.49 849	.57 503	1.7391	.86 690	6
55	.49 874	.57 541	1.7379	.86 675	5
56	.49 899	.57 580	1.7367	.86 661	4
57	.49 924	.57 619	1.7355	.86 646	3
58	.49 950	.57 657	1.7344	.86 632	2
59	.49 975	.57 696	1.7332	.86 617	1
60	.50 000	.57 735	1.7321	.86 603	**0**
′	cos	cot	tan	sin	′

60°

30°

′	sin	tan	cot	cos	′
0	.50 000	.57 735	1.7321	.86 603	**60**
1	.50 025	.57 774	1.7309	.86 588	59
2	.50 050	.57 813	1.7297	.86 573	58
3	.50 076	.57 851	1.7286	.86 559	57
4	.50 101	.57 890	1.7274	.86 544	56
5	.50 126	.57 929	1.7262	.86 530	55
6	.50 151	.57 968	1.7251	.86 515	54
7	.50 176	.58 007	1.7239	.86 501	53
8	.50 201	.58 046	1.7228	.86 486	52
9	.50 227	.58 085	1.7216	.86 471	51
10	.50 252	.58 124	1.7205	.86 457	**50**
11	.50 277	.58 162	1.7193	.86 442	49
12	.50 302	.58 201	1.7182	.86 427	48
13	.50 327	.58 240	1.7170	.86 413	47
14	.50 352	.58 279	1.7159	.86 398	46
15	.50 377	.58 318	1.7147	.86 384	45
16	.50 403	.58 357	1.7136	.86 369	44
17	.50 428	.58 396	1.7124	.86 354	43
18	.50 453	.58 435	1.7113	.86 340	42
19	.50 478	.58 474	1.7102	.86 325	41
20	.50 503	.58 513	1.7090	.86 310	**40**
21	.50 528	.58 552	1.7079	.86 295	39
22	.50 553	.58 591	1.7067	.86 281	38
23	.50 578	.58 631	1.7056	.86 266	37
24	.50 603	.58 670	1.7045	.86 251	36
25	.50 628	.58 709	1.7033	.86 237	35
26	.50 654	.58 748	1.7022	.86 222	34
27	.50 679	.58 787	1.7011	.86 207	33
28	.50 704	.58 826	1.6999	.86 192	32
29	.50 729	.58 865	1.6988	.86 178	31
30	.50 754	.58 905	1.6977	.86 163	**30**
31	.50 779	.58 944	1.6965	.86 148	29
32	.50 804	.58 983	1.6954	.86 133	28
33	.50 829	.59 022	1.6943	.86 119	27
34	.50 854	.59 061	1.6932	.86 104	26
35	.50 879	.59 101	1.6920	.86 089	25
36	.50 904	.59 140	1.6909	.86 074	24
37	.50 929	.59 179	1.6898	.86 059	23
38	.50 954	.59 218	1.6887	.86 045	22
39	.50 979	.59 258	1.6875	.86 030	21
40	.51 004	.59 297	1.6864	.86 015	**20**
41	.51 029	.59 336	1.6853	.86 000	19
42	.51 054	.59 376	1.6842	.85 985	18
43	.51 079	.59 415	1.6831	.85 970	17
44	.51 104	.59 454	1.6820	.85 956	16
45	.51 129	.59 494	1.6808	.85 941	15
46	.51 154	.59 533	1.6797	.85 926	14
47	.51 179	.59 573	1.6786	.85 911	13
48	.51 204	.59 612	1.6775	.85 896	12
49	.51 229	.59 651	1.6764	.85 881	11
50	.51 254	.59 691	1.6753	.85 866	**10**
51	.51 279	.59 730	1.6742	.85 851	9
52	.51 304	.59 770	1.6731	.85 836	8
53	.51 329	.59 809	1.6720	.85 821	7
54	.51 354	.59 849	1.6709	.85 806	6
55	.51 379	.59 888	1.6698	.85 792	5
56	.51 404	.59 928	1.6687	.85 777	4
57	.51 429	.59 967	1.6676	.85 762	3
58	.51 454	.60 007	1.6665	.85 747	2
59	.51 479	.60 046	1.6654	.85 732	1
60	.51 504	.60 086	1.6643	.85 717	**0**
′	cos	cot	tan	sin	′

59°

31°

′	sin	tan	cot	cos	′
0	.51 504	.60 086	1.6643	.85 717	**60**
1	.51 529	.60 126	1.6632	.85 702	59
2	.51 554	.60 165	1.6621	.85 687	58
3	.51 579	.60 205	1.6610	.85 672	57
4	.51 604	.60 245	1.6599	.85 657	56
5	.51 628	.60 284	1.6588	.85 642	55
6	.51 653	.60 324	1.6577	.85 627	54
7	.51 678	.60 364	1.6566	.85 612	53
8	.51 703	.60 403	1.6555	.85 597	52
9	.51 728	.60 443	1.6545	.85 582	51
10	.51 753	.60 483	1.6534	.85 567	**50**
11	.51 778	.60 522	1.6523	.85 551	49
12	.51 803	.60 562	1.6512	.85 536	48
13	.51 828	.60 602	1.6501	.85 521	47
14	.51 852	.60 642	1.6490	.85 506	46
15	.51 877	.60 681	1.6479	.85 491	45
16	.51 902	.60 721	1.6469	.85 476	44
17	.51 927	.60 761	1.6458	.85 461	43
18	.51 952	.60 801	1.6447	.85 446	42
19	.51 977	.60 841	1.6436	.85 431	41
20	.52 002	.60 881	1.6426	.85 416	**40**
21	.52 026	.60 921	1.6415	.85 401	39
22	.52 051	.60 960	1.6404	.85 385	38
23	.52 076	.61 000	1.6393	.85 370	37
24	.52 101	.61 040	1.6383	.85 355	36
25	.52 126	.61 080	1.6372	.85 340	35
26	.52 151	.61 120	1.6361	.85 325	34
27	.52 175	.61 160	1.6351	.85 310	33
28	.52 200	.61 200	1.6340	.85 294	32
29	.52 225	.61 240	1.6329	.85 279	31
30	.52 250	.61 280	1.6319	.85 264	**30**
31	.52 275	.61 320	1.6308	.85 249	29
32	.52 299	.61 360	1.6297	.85 234	28
33	.52 324	.61 400	1.6287	.85 218	27
34	.52 349	.61 440	1.6276	.85 203	26
35	.52 374	.61 480	1.6265	.85 188	25
36	.52 399	.61 520	1.6255	.85 173	24
37	.52 423	.61 561	1.6244	.85 157	23
38	.52 448	.61 601	1.6234	.85 142	22
39	.52 473	.61 641	1.6223	.85 127	21
40	.52 498	.61 681	1.6212	.85 112	**20**
41	.52 522	.61 721	1.6202	.85 096	19
42	.57 547	.61 761	1.6191	.85 081	18
43	.52 572	.61 801	1.6181	.85 066	17
44	.52 597	.61 842	1.6170	.85 051	16
45	.52 621	.61 882	1.6160	.85 035	15
46	.52 646	.61 922	1.6149	.85 020	14
47	.52 671	.61 962	1.6139	.85 005	13
48	.52 696	.62 003	1.6128	.84 989	12
49	.52 720	.62 043	1.6118	.84 974	11
50	.52 745	.62 083	1.6107	.84 959	**10**
51	.52 770	.62 124	1.6097	.84 943	9
52	.52 794	.62 164	1.6087	.84 928	8
53	.52 819	.62 204	1.6076	.84 913	7
54	.52 844	.62 245	1.6066	.84 897	6
55	.52 869	.62 285	1.6055	.84 882	5
56	.52 893	.62 325	1.6045	.84 866	4
57	.52 918	.62 366	1.6034	.84 851	3
58	.52 943	.62 406	1.6024	.84 836	2
59	.52 967	.62 446	1.6014	.84 820	1
60	.52 992	.62 487	1.6003	.84 805	**0**
′	cos	cot	tan	sin	′

58°

′	sin	tan	cot	cos	′
		32°			
0	.52 992	.62 487	1.6003	.84 805	60
1	.53 017	.62 527	1.5993	.84 789	59
2	.53 041	.62 568	1.5983	.84 774	58
3	.53 066	.62 608	1.5972	.84 759	57
4	.53 091	.62 649	1.5962	.84 743	56
5	.53 115	.62 689	1.5952	.84 728	55
6	.53 140	.62 730	1.5941	.84 712	54
7	.53 164	.62 770	1.5931	.84 697	53
8	.53 189	.62 811	1.5921	.84 681	52
9	.53 214	.62 852	1.5911	.84 666	51
10	.53 238	.62 892	1.5900	.84 650	50
11	.53 263	.62 933	1.5890	.84 635	49
12	.53 288	.62 973	1.5880	.84 619	48
13	.53 312	.63 014	1.5869	.84 604	47
14	.53 337	.63 055	1.5859	.84 588	46
15	.53 361	.63 095	1.5849	.84 573	45
16	.53 386	.63 136	1.5839	.84 557	44
17	.53 411	.63 177	1.5829	.84 542	43
18	.53 435	.63 217	1.5818	.84 526	42
19	.53 460	.63 258	1.5808	.84 511	41
20	.53 484	.63 299	1.5798	.84 495	40
21	.53 509	.63 340	1.5788	.84 480	39
22	.53 534	.63 380	1.5778	.84 464	38
23	.53 558	.63 421	1.5768	.84 448	37
24	.53 583	.63 462	1.5757	.84 433	36
25	.53 607	.63 503	1.5747	.84 417	35
26	.53 632	.63 544	1.5737	.84 402	34
27	.53 656	.63 584	1.5727	.84 386	33
28	.53 681	.63 625	1.5717	.84 370	32
29	.53 705	.63 666	1.5707	.84 355	31
30	.53 730	.63 707	1.5697	.84 339	30
31	.53 754	.63 748	1.5687	.84 324	29
32	.53 779	.63 789	1.5677	.84 308	28
33	.53 804	.63 830	1.5667	.84 292	27
34	.53 828	.63 871	1.5657	.84 277	26
35	.53 853	.63 912	1.5647	.84 261	25
36	.53 877	.63 953	1.5637	.84 245	24
37	.53 902	.63 994	1.5627	.84 230	23
38	.53 926	.64 035	1.5617	.84 214	22
39	.53 951	.64 076	1.5607	.84 198	21
40	.53 975	.64 117	1.5597	.84 182	20
41	.54 000	.64 158	1.5587	.84 167	19
42	.54 024	.64 199	1.5577	.84 151	18
43	.54 049	.64 240	1.5567	.84 135	17
44	.54 073	.64 281	1.5557	.84 120	16
45	.54 097	.64 322	1.5547	.84 104	15
46	.54 122	.64 363	1.5537	.84 088	14
47	.54 146	.64 404	1.5527	.84 072	13
48	.54 171	.64 446	1.5517	.84 057	12
49	.54 195	.64 487	1.5507	.84 041	11
50	.54 220	.64 528	1.5497	.84 025	10
51	.54 244	.64 569	1.5487	.84 009	9
52	.54 269	.64 610	1.5477	.83 994	8
53	.54 293	.64 652	1.5468	.83 978	7
54	.54 317	.64 693	1.5458	.83 962	6
55	.54 342	.64 734	1.5448	.83 946	5
56	.54 366	.64 775	1.5438	.83 930	4
57	.54 391	.64 817	1.5428	.83 915	3
58	.54 415	.64 858	1.5418	.83 899	2
59	.54 440	.64 899	1.5408	.83 883	1
60	.54 464	.64 941	1.5399	.83 867	0
′	cos	cot	tan	sin	′
		57°			

′	sin	tan	cot	cos	′
		33°			
0	.54 464	.64 941	1.5399	.83 867	60
1	.54 488	.64 982	1.5389	.83 851	59
2	.54 513	.65 024	1.5379	.83 835	58
3	.54 537	.65 065	1.5369	.83 819	57
4	.54 561	.65 106	1.5359	.83 804	56
5	.54 586	.65 148	1.5350	.83 788	55
6	.54 610	.65 189	1.5340	.83 772	54
7	.54 635	.65 231	1.5330	.83 756	53
8	.54 659	.65 272	1.5320	.83 740	52
9	.54 683	.65 314	1.5311	.83 724	51
10	.54 708	.65 355	1.5301	.83 708	50
11	.54 732	.65 397	1.5291	.83 692	49
12	.54 756	.65 438	1.5282	.83 676	48
13	.54 781	.65 480	1.5272	.83 660	47
14	.54 805	.65 521	1.5262	.83 645	46
15	.54 829	.65 563	1.5253	.83 629	45
16	.54 854	.65 604	1.5243	.83 613	44
17	.54 878	.65 646	1.5233	.83 597	43
18	.54 902	.65 688	1.5224	.83 581	42
19	.54 927	.65 729	1.5214	.83 565	41
20	.54 951	.65 771	1.5204	.83 549	40
21	.54 975	.65 813	1.5195	.83 533	39
22	.54 999	.65 854	1.5185	.83 517	38
23	.55 024	.65 896	1.5175	.83 501	37
24	.55 048	.65 938	1.5166	.83 485	36
25	.55 072	.65 980	1.5156	.83 469	35
26	.55 097	.66 021	1.5147	.83 453	34
27	.55 121	.66 063	1.5137	.83 437	33
28	.55 145	.66 105	1.5127	.83 421	32
29	.55 169	.66 147	1.5118	.83 405	31
30	.55 194	.66 189	1.5108	.83 389	30
31	.55 218	.66 230	1.5099	.83 373	29
32	.55 242	.66 272	1.5089	.83 356	28
33	.55 266	.66 314	1.5080	.83 340	27
34	.55 291	.66 356	1.5070	.83 324	26
35	.55 315	.66 398	1.5061	.83 308	25
36	.55 339	.66 440	1.5051	.83 292	24
37	.55 363	.66 482	1.5042	.83 276	23
38	.55 388	.66 524	1.5032	.83 260	22
39	.55 412	.66 566	1.5023	.83 244	21
40	.55 436	.66 608	1.5013	.83 228	20
41	.55 460	.66 650	1.5004	.83 212	19
42	.55 484	.66 692	1.4994	.83 195	18
43	.55 509	.66 734	1.4985	.83 179	17
44	.55 533	.66 776	1.4975	.83 163	16
45	.55 557	.66 818	1.4966	.83 147	15
46	.55 581	.66 860	1.4957	.83 131	14
47	.55 605	.66 902	1.4947	.83 115	13
48	.55 630	.66 944	1.4938	.83 098	12
49	.55 654	.66 986	1.4928	.83 082	11
50	.55 678	.67 028	1.4919	.83 066	10
51	.55 702	.67 071	1.4910	.83 050	9
52	.55 726	.67 113	1.4900	.83 034	8
53	.55 750	.67 155	1.4891	.83 017	7
54	.55 775	.67 197	1.4882	.83 001	6
55	.55 799	.67 239	1.4872	.82 985	5
56	.55 823	.67 282	1.4863	.82 969	4
57	.55 847	.67 324	1.4854	.82 953	3
58	.55 871	.67 366	1.4844	.82 936	2
59	.55 895	.67 409	1.4835	.82 920	1
60	.55 919	.67 451	1.4826	.82 904	0
′	cos	cot	tan	sin	′
		56°			

′	34° sin	tan	cot	cos	′
0	.55 919	.67 451	1.4826	.82 904	60
1	.55 943	.67 493	1.4816	.82 887	59
2	.55 968	.67 536	1.4807	.82 871	58
3	.55 992	.67 578	1.4798	.82 855	57
4	.56 016	.67 620	1.4788	.82 839	56
5	.56 040	.67 663	1.4779	.82 822	55
6	.56 064	.67 705	1.4770	.82 806	54
7	.56 088	.67 748	1.4761	.82 790	53
8	.56 112	.67 790	1.4751	.82 773	52
9	.56 136	.67 832	1.4742	.82 757	51
10	.56 160	.67 875	1.4733	.82 741	50
11	.56 184	.67 917	1.4724	.82 724	49
12	.56 208	.67 960	1.4715	.82 708	48
13	.56 232	.68 002	1.4705	.82 692	47
14	.56 256	.68 045	1.4696	.82 675	46
15	.56 280	.68 088	1.4687	.82 659	45
16	.56 305	.68 130	1.4678	.82 643	44
17	.56 329	.68 173	1.4669	.82 626	43
18	.56 353	.68 215	1.4659	.82 610	42
19	.56 377	.68 258	1.4650	.82 593	41
20	.56 401	.68 301	1.4641	.82 577	40
21	.56 425	.68 343	1.4632	.82 561	39
22	.56 449	.68 386	1.4623	.82 544	38
23	.56 473	.68 429	1.4614	.82 528	37
24	.56 497	.68 471	1.4605	.82 511	36
25	.56 521	.68 514	1.4596	.82 495	35
26	.56 545	.68 557	1.4586	.82 478	34
27	.56 569	.68 600	1.4577	.82 462	33
28	.56 593	.68 642	1.4568	.82 446	32
29	.56 617	.68 685	1.4559	.82 429	31
30	.56 641	.68 728	1.4550	.82 413	30
31	.56 665	.68 771	1.4541	.82 396	29
32	.56 689	.68 814	1.4532	.82 380	28
33	.56 713	.68 857	1.4523	.82 363	27
34	.56 736	.68 900	1.4514	.82 347	26
35	.56 760	.68 942	1.4505	.82 330	25
36	.56 784	.68 985	1.4496	.82 314	24
37	.56 808	.69 028	1.4487	.82 297	23
38	.56 832	.69 071	1.4478	.82 281	22
39	.56 856	.69 114	1.4469	.82 264	21
40	.56 880	.69 157	1.4460	.82 248	20
41	.56 904	.69 200	1.4451	.82 231	19
42	.56 928	.69 243	1.4442	.82 214	18
43	.56 952	.69 286	1.4433	.82 198	17
44	.56 976	.69 329	1.4424	.82 181	16
45	.57 000	.69 372	1.4415	.82 165	15
46	.57 024	.69 416	1.4406	.82 148	14
47	.57 047	.69 459	1.4397	.82 132	13
48	.57 071	.69 502	1.4388	.82 115	12
49	.57 095	.69 545	1.4379	.82 098	11
50	.57 119	.69 588	1.4370	.82 082	10
51	.57 143	.69 631	1.4361	.82 065	9
52	.57 167	.69 675	1.4352	.82 048	8
53	.57 191	.69 718	1.4344	.82 032	7
54	.57 215	.69 761	1.4335	.82 015	6
55	.57 238	.69 804	1.4326	.81 999	5
56	.57 262	.69 847	1.4317	.81 982	4
57	.57 286	.69 891	1.4308	.81 965	3
58	.57 310	.69 934	1.4299	.81 949	2
59	.57 334	.69 977	1.4290	.81 932	1
60	.57 358	.70 021	1.4281	.81 915	0
′	cos	cot	tan	sin	′
	55°				

′	35° sin	tan	cot	cos	′
0	.57 358	.70 021	1.4281	.81 915	60
1	.57 381	.70 064	1.4273	.81 899	59
2	.57 405	.70 107	1.4264	.81 882	58
3	.57 429	.70 151	1.4255	.81 865	57
4	.57 453	.70 194	1.4246	.81 848	56
5	.57 477	.70 238	1.4237	.81 832	55
6	.57 501	.70 281	1.4229	.81 815	54
7	.57 524	.70 325	1.4220	.81 798	53
8	.57 548	.70 368	1.4211	.81 782	52
9	.57 572	.70 412	1.4202	.81 765	51
10	.57 596	.70 455	1.4193	.81 748	50
11	.57 619	.70 499	1.4185	.81 731	49
12	.57 643	.70 542	1.4176	.81 714	48
13	.57 667	.70 586	1.4167	.81 698	47
14	.57 691	.70 629	1.4158	.81 681	46
15	.57 715	.70 673	1.4150	.81 664	45
16	.57 738	.70 717	1.4141	.81 647	44
17	.57 762	.70 760	1.4132	.81 631	43
18	.57 786	.70 804	1.4124	.81 614	42
19	.57 810	.70 848	1.4115	.81 597	41
20	.57 833	.70 891	1.4106	.81 580	40
21	.57 857	.70 935	1.4097	.81 563	39
22	.57 881	.70 979	1.4089	.81 546	38
23	.57 904	.71 023	1.4080	.81 530	37
24	.57 928	.71 066	1.4071	.81 513	36
25	.57 952	.71 110	1.4063	.81 496	35
26	.57 976	.71 154	1.4054	.81 479	34
27	.57 999	.71 198	1.4045	.81 462	33
28	.58 023	.71 242	1.4037	.81 445	32
29	.58 047	.71 285	1.4028	.81 428	31
30	.58 070	.71 329	1.4019	.81 412	30
31	.58 094	.71 373	1.4011	.81 395	29
32	.58 118	.71 417	1.4002	.81 378	28
33	.58 141	.71 461	1.3994	.81 361	27
34	.58 165	.71 505	1.3985	.81 344	26
35	.58 189	.71 549	1.3976	.81 327	25
36	.58 212	.71 593	1.3968	.81 310	24
37	.58 236	.71 637	1.3959	.81 293	23
38	.58 260	.71 681	1.3951	.81 276	22
39	.58 283	.71 725	1.3942	.81 259	21
40	.58 307	.71 769	1.3934	.81 242	20
41	.58 330	.71 813	1.3925	.81 225	19
42	.58 354	.71 857	1.3916	.81 208	18
43	.58 378	.71 901	1.3908	.81 191	17
44	.58 401	.71 946	1.3899	.81 174	16
45	.58 425	.71 990	1.3891	.81 157	15
46	.58 449	.72 034	1.3882	.81 140	14
47	.58 472	.72 078	1.3874	.81 123	13
48	.58 496	.72 122	1.3865	.81 106	12
49	.58 519	.72 167	1.3857	.81 089	11
50	.58 543	.72 211	1.3848	.81 072	10
51	.58 567	.72 255	1.3840	.81 055	9
52	.58 590	.72 299	1.3831	.81 038	8
53	.58 614	.72 344	1.3823	.81 021	7
54	.58 637	.72 388	1.3814	.81 004	6
55	.58 661	.72 432	1.3806	.80 987	5
56	.58 684	.72 477	1.3798	.80 970	4
57	.58 708	.72 521	1.3789	.80 953	3
58	.58 731	.72 565	1.3781	.80 936	2
59	.58 755	.72 610	1.3772	.80 919	1
60	.58 779	.72 654	1.3764	.80 902	0
′	cos	cot	tan	sin	′
	54°				

36°

′	sin	tan	cot	cos	′
0	.58 779	.72 654	1.3764	.80 902	**60**
1	.58 802	.72 699	1.3755	.80 885	59
2	.58 826	.72 743	1.3747	.80 867	58
3	.58 849	.72 788	1.3739	.80 850	57
4	.58 873	.72 832	1.3730	.80 833	56
5	.58 896	.72 877	1.3722	.80 816	55
6	.58 920	.72 921	1.3713	.80 799	54
7	.58 943	.72 966	1.3705	.80 782	53
8	.58 967	.73 010	1.3697	.80 765	52
9	.58 990	.73 055	1.3688	.80 748	51
10	.59 014	.73 100	1.3680	.80 730	**50**
11	.59 037	.73 144	1.3672	.80 713	49
12	.59 061	.73 189	1.3663	.80 696	48
13	.59 084	.73 234	1.3655	.80 679	47
14	.59 108	.73 278	1.3647	.80 662	46
15	.59 131	.73 323	1.3638	.80 644	45
16	.59 154	.73 368	1.3630	.80 627	44
17	.59 178	.73 413	1.3622	.80 610	43
18	.59 201	.73 457	1.3613	.80 593	42
19	.59 225	.73 502	1.3605	.80 576	41
20	.59 248	.73 547	1.3597	.80 558	**40**
21	.59 272	.73 592	1.3588	.80 541	39
22	.59 295	.73 637	1.3580	.80 524	38
23	.59 318	.73 681	1.3572	.80 507	37
24	.59 342	.73 726	1.3564	.80 489	36
25	.59 365	.73 771	1.3555	.80 472	35
26	.59 389	.73 816	1.3547	.80 455	34
27	.59 412	.73 861	1.3539	.80 438	33
28	.59 436	.73 906	1.3531	.80 420	32
29	.59 459	.73 951	1.3522	.80 403	31
30	.59 482	.73 996	1.3514	.80 386	**30**
31	.59 506	.74 041	1.3506	.80 368	29
32	.59 529	.74 086	1.3498	.80 351	28
33	.59 552	.74 131	1.3490	.80 334	27
34	.59 576	.74 176	1.3481	.80 316	26
35	.59 599	.74 221	1.3473	.80 299	25
36	.59 622	.74 267	1.3465	.80 282	24
37	.59 646	.74 312	1.3457	.80 264	23
38	.59 669	.74 357	1.3449	.80 247	22
39	.59 693	.74 402	1.3440	.80 230	21
40	.59 716	.74 447	1.3432	.80 212	**20**
41	.59 739	.74 492	1.3424	.80 195	19
42	.59 763	.74 538	1.3416	.80 178	18
43	.59 786	.74 583	1.3408	.80 160	17
44	.59 809	.74 628	1.3400	.80 143	16
45	.59 832	.74 674	1.3392	.80 125	15
46	.59 856	.74 719	1.3384	.80 108	14
47	.59 879	.74 764	1.3375	.80 091	13
48	.59 902	.74 810	1.3367	.80 073	12
49	.59 926	.74 855	1.3359	.80 056	11
50	.59 949	.74 900	1.3351	.80 038	**10**
51	.59 972	.74 946	1.3343	.80 021	9
52	.59 995	.74 991	1.3335	.80 003	8
53	.60 019	.75 037	1.3327	.79 986	7
54	.60 042	.75 082	1.3319	.79 968	6
55	.60 065	.75 128	1.3311	.79 951	5
56	.60 089	.75 173	1.3303	.79 934	4
57	.60 112	.75 219	1.3295	.79 916	3
58	.60 135	.75 264	1.3287	.79 899	2
59	.60 158	.75 310	1.3278	.79 881	1
60	.60 182	.75 355	1.3270	.79 864	**0**
′	cos	cot	tan	sin	′

53°

37°

′	sin	tan	cot	cos	′
0	.60 182	.75 355	1.3270	.79 864	**60**
1	.60 205	.75 401	1.3262	.79 846	59
2	.60 228	.75 447	1.3254	.79 829	58
3	.60 251	.75 492	1.3246	.79 811	57
4	.60 274	.75 538	1.3238	.79 793	56
5	.60 298	.75 584	1.3230	.79 776	55
6	.60 321	.75 629	1.3222	.79 758	54
7	.60 344	.75 675	1.3214	.79 741	53
8	.60 367	.75 721	1.3206	.79 723	52
9	.60 390	.75 767	1.3198	.79 706	51
10	.60 414	.75 812	1.3190	.79 688	**50**
11	.60 437	.75 858	1.3182	.79 671	49
12	.60 460	.75 904	1.3175	.79 653	48
13	.60 483	.75 950	1.3167	.79 635	47
14	.60 506	.75 996	1.3159	.79 618	46
15	.60 529	.76 042	1.3151	.79 600	45
16	.60 553	.76 088	1.3143	.79 583	44
17	.60 576	.76 134	1.3135	.79 565	43
18	.60 599	.76 180	1.3127	.79 547	42
19	.60 622	.76 226	1.3119	.79 530	41
20	.60 645	.76 272	1.3111	.79 512	**40**
21	.60 668	.76 318	1.3103	.79 494	39
22	.60 691	.76 364	1.3095	.79 477	38
23	.60 714	.76 410	1.3087	.79 459	37
24	.60 738	.76 456	1.3079	.79 441	36
25	.60 761	.76 502	1.3072	.79 424	35
26	.60 784	.76 548	1.3064	.79 406	34
27	.60 807	.76 594	1.3056	.79 388	33
28	.60 830	.76 640	1.3048	.79 371	32
29	.60 853	.76 686	1.3040	.79 353	31
30	.60 876	.76 733	1.3032	.79 335	**30**
31	.60 899	.76 779	1.3024	.79 318	29
32	.60 922	.76 825	1.3017	.79 300	28
33	.60 945	.76 871	1.3009	.79 282	27
34	.60 968	.76 918	1.3001	.79 264	26
35	.60 991	.76 964	1.2993	.79 247	25
36	.61 015	.77 010	1.2985	.79 229	24
37	.61 038	.77 057	1.2977	.79 211	23
38	.61 061	.77 103	1.2970	.79 193	22
39	.61 084	.77 149	1.2962	.79 176	21
40	.61 107	.77 196	1.2954	.79 158	**20**
41	.61 130	.77 242	1.2946	.79 140	19
42	.61 153	.77 289	1.2938	.79 122	18
43	.61 176	.77 335	1.2931	.79 105	17
44	.61 199	.77 382	1.2923	.79 087	16
45	.61 222	.77 428	1.2915	.79 069	15
46	.61 245	.77 475	1.2907	.79 051	14
47	.61 268	.77 521	1.2900	.79 033	13
48	.61 291	.77 568	1.2892	.79 016	12
49	.61 314	.77 615	1.2884	.78 998	11
50	.61 337	.77 661	1.2876	.78 980	**10**
51	.61 360	.77 708	1.2869	.78 962	9
52	.61 383	.77 754	1.2861	.78 944	8
53	.61 406	.77 801	1.2853	.78 926	7
54	.61 429	.77 848	1.2846	.78 908	6
55	.61 451	.77 895	1.2838	.78 891	5
56	.61 474	.77 941	1.2830	.78 873	4
57	.61 497	.77 988	1.2822	.78 855	3
58	.61 520	.78 035	1.2815	.78 837	2
59	.61 543	.78 082	1.2807	.78 819	1
60	.61 566	.78 129	1.2799	.78 801	**0**
′	cos	cot	tan	sin	′

52°

	38°				
′	sin	tan	cot	cos	′
0	.61 566	.78 129	1.2799	.78 801	**60**
1	.61 589	.78 175	1.2792	.78 783	59
2	.61 612	.78 222	1.2784	.78 765	58
3	.61 635	.78 269	1.2776	.78 747	57
4	.61 658	.78 316	1.2769	.78 729	56
5	.61 681	.78 363	1.2761	.78 711	55
6	.61 704	.78 410	1.2753	.78 694	54
7	.61 726	.78 457	1.2746	.78 676	53
8	.61 749	.78 504	1.2738	.78 658	52
9	.61 772	.78 551	1.2731	.78 640	51
10	.61 795	.78 598	1.2723	.78 622	**50**
11	.61 818	.78 645	1.2715	.78 604	49
12	.61 841	.78 692	1.2708	.78 586	48
13	.61 864	.78 739	1.2700	.78 568	47
14	.61 887	.78 786	1.2693	.78 550	46
15	.61 909	.78 834	1.2685	.78 532	45
16	.61 932	.78 881	1.2677	.78 514	44
17	.61 955	.78 928	1.2670	.78 496	43
18	.61 978	.78 975	1.2662	.78 478	42
19	.62 001	.79 022	1.2655	.78 460	41
20	.62 024	.79 070	1.2647	.78 442	**40**
21	.62 046	.79 117	1.2640	.78 424	39
22	.62 069	.79 164	1.2632	.78 405	38
23	.62 092	.79 212	1.2624	.78 387	37
24	.62 115	.79 259	1.2617	.78 369	36
25	.62 138	.79 306	1.2609	.78 351	35
26	.62 160	.79 354	1.2602	.78 333	34
27	.62 183	.79 401	1.2594	.78 315	33
28	.62 206	.79 449	1.2587	.78 297	32
29	.62 229	.79 496	1.2579	.78 279	31
30	.62 251	.79 544	1.2572	.78 261	**30**
31	.62 274	.79 591	1.2564	.78 243	29
32	.62 297	.79 639	1.2557	.78 225	28
33	.62 320	.79 686	1.2549	.78 206	27
34	.62 342	.79 734	1.2542	.78 188	26
35	.62 365	.79 781	1.2534	.78 170	25
36	.62 388	.79 829	1.2527	.78 152	24
37	.62 411	.79 877	1.2519	.78 134	23
38	.62 433	.79 924	1.2512	.78 116	22
39	.62 456	.79 972	1.2504	.78 098	21
40	.62 479	.80 020	1.2497	.78 079	**20**
41	.62 502	.80 067	1.2489	.78 061	19
42	.62 524	.80 115	1.2482	.78 043	18
43	.62 547	.80 163	1.2475	.78 025	17
44	.62 570	.80 211	1.2467	.78 007	16
45	.62 592	.80 258	1.2460	.77 988	15
46	.62 615	.80 306	1.2452	.77 970	14
47	.62 638	.80 354	1.2445	.77 952	13
48	.62 660	.80 402	1.2437	.77 934	12
49	.62 683	.80 450	1.2430	.77 916	11
50	.62 706	.80 498	1.2423	.77 897	**10**
51	.62 728	.80 546	1.2415	.77 879	9
52	.62 751	.80 594	1.2408	.77 861	8
53	.62 774	.80 642	1.2401	.77 843	7
54	.62 796	.80 690	1.2393	.77 824	6
55	.62 819	.80 738	1.2386	.77 806	5
56	.62 842	.80 786	1.2378	.77 788	4
57	.62 864	.80 834	1.2371	.77 769	3
58	.62 887	.80 882	1.2364	.77 751	2
59	.62 909	.80 930	1.2356	.77 733	1
60	.62 932	.80 978	1.2349	.77 715	**0**
′	cos	cot	tan	sin	′
	51°				

	39°				
′	sin	tan	cot	cos	′
0	.62 932	.80 978	1.2349	.77 715	**60**
1	.62 955	.81 027	1.2342	.77 696	59
2	.62 977	.81 075	1.2334	.77 678	58
3	.63 000	.81 123	1.2327	.77 660	57
4	.63 022	.81 171	1.2320	.77 641	56
5	.63 045	.81 220	1.2312	.77 623	55
6	.63 068	.81 268	1.2305	.77 605	54
7	.63 090	.81 316	1.2298	.77 586	53
8	.63 113	.81 364	1.2290	.77 568	52
9	.63 135	.81 413	1.2283	.77 550	51
10	.63 158	.81 461	1.2276	.77 531	**50**
11	.63 180	.81 510	1.2268	.77 513	49
12	.63 203	.81 558	1.2261	.77 494	48
13	.63 225	.81 606	1.2254	.77 476	47
14	.63 248	.81 655	1.2247	.77 458	46
15	.63 271	.81 703	1.2239	.77 439	45
16	.63 293	.81 752	1.2232	.77 421	44
17	.63 316	.81 800	1.2225	.77 402	43
18	.63 338	.81 849	1.2218	.77 384	42
19	.63 361	.81 898	1.2210	.77 366	41
20	.63 383	.81 946	1.2203	.77 347	**40**
21	.63 406	.81 995	1.2196	.77 329	39
22	.63 428	.82 044	1.2189	.77 310	38
23	.63 451	.82 092	1.2181	.77 292	37
24	.63 473	.82 141	1.2174	.77 273	36
25	.63 496	.82 190	1.2167	.77 255	35
26	.63 518	.82 238	1.2160	.77 236	34
27	.63 540	.82 287	1.2153	.77 218	33
28	.63 563	.82 336	1.2145	.77 199	32
29	.63 585	.82 385	1.2138	.77 181	31
30	.63 608	.82 434	1.2131	.77 162	**30**
31	.63 630	.82 483	1.2124	.77 144	29
32	.63 653	.82 531	1.2117	.77 125	28
33	.63 675	.82 580	1.2109	.77 107	27
34	.63 698	.82 629	1.2102	.77 088	26
35	.63 720	.82 678	1.2095	.77 070	25
36	.63 742	.82 727	1.2088	.77 051	24
37	.63 765	.82 776	1.2081	.77 033	23
38	.63 787	.82 825	1.2074	.77 014	22
39	.63 810	.82 874	1.2066	.76 996	21
40	.63 832	.82 923	1.2059	.76 977	**20**
41	.63 854	.82 972	1.2052	.76 959	19
42	.63 877	.83 022	1.2045	.76 940	18
43	.63 899	.83 071	1.2038	.76 921	17
44	.63 922	.83 120	1.2031	.76 903	16
45	.63 944	.83 169	1.2024	.76 884	15
46	.63 966	.83 218	1.2017	.76 866	14
47	.63 989	.83 268	1.2009	.76 847	13
48	.64 011	.83 317	1.2002	.76 828	12
49	.64 033	.83 366	1.1995	.76 810	11
50	.64 056	.83 415	1.1988	.76 791	**10**
51	.64 078	.83 465	1.1981	.76 772	9
52	.64 100	.83 514	1.1974	.76 754	8
53	.64 123	.83 564	1.1967	.76 735	7
54	.64 145	.83 613	1.1960	.76 717	6
55	.64 167	.83 662	1.1953	.76 698	5
56	.64 190	.83 712	1.1946	.76 679	4
57	.64 212	.83 761	1.1939	.76 661	3
58	.64 234	.83 811	1.1932	.76 642	2
59	.64 256	.83 860	1.1925	.76 623	1
60	.64 279	.83 910	1.1918	.76 604	**0**
′	cos	cot	tan	sin	′
	50°				

40°					
′	sin	tan	cot	cos	′
0	.64 279	.83 910	1.1918	.76 604	**60**
1	.64 301	.83 960	1.1910	.76 586	59
2	.64 323	.84 009	1.1903	.76 567	58
3	.64 346	.84 059	1.1896	.76 548	57
4	.64 368	.84 108	1.1889	.76 530	56
5	.64 390	.84 158	1.1882	.76 511	55
6	.64 412	.84 208	1.1875	.76 492	54
7	.64 435	.84 258	1.1868	.76 473	53
8	.64 457	.84 307	1.1861	.76 455	52
9	.64 479	.84 357	1.1854	.76 436	51
10	.64 501	.84 407	1.1847	.76 417	**50**
11	.64 524	.84 457	1.1840	.76 398	49
12	.64 546	.84 507	1.1833	.76 380	48
13	.64 568	.84 556	1.1826	.76 361	47
14	.64 590	.84 606	1.1819	.76 342	46
15	.64 612	.84 656	1.1812	.76 323	45
16	.64 635	.84 706	1.1806	.76 304	44
17	.64 657	.84 756	1.1799	.76 286	43
18	.64 679	.84 806	1.1792	.76 267	42
19	.64 701	.84 856	1.1785	.76 248	41
20	.64 723	.84 906	1.1778	.76 229	**40**
21	.64 746	.84 956	1.1771	.76 210	39
22	.64 768	.85 006	1.1764	.76 192	38
23	.64 790	.85 057	1.1757	.76 173	37
24	.64 812	.85 107	1.1750	.76 154	36
25	.64 834	.85 157	1.1743	.76 135	35
26	.64 856	.85 207	1.1736	.76 116	34
27	.64 878	.85 257	1.1729	.76 097	33
28	.64 901	.85 308	1.1722	.76 078	32
29	.64 923	.85 358	1.1715	.76 059	31
30	.64 945	.85 408	1.1708	.76 041	**30**
31	.64 967	.85 458	1.1702	.76 022	29
32	.64 989	.85 509	1.1695	.76 003	28
33	.65 011	.85 559	1.1688	.75 984	27
34	.65 033	.85 609	1.1681	.75 965	26
35	.65 055	.85 660	1.1674	.75 946	25
36	.65 077	.85 710	1.1667	.75 927	24
37	.65 100	.85 761	1.1660	.75 908	23
38	.65 122	.85 811	1.1653	.75 889	22
39	.65 144	.85 862	1.1647	.75 870	21
40	.65 166	.85 912	1.1640	.75 851	**20**
41	.65 188	.85 963	1.1633	.75 832	19
42	.65 210	.86 014	1.1626	.75 813	18
43	.65 232	.86 064	1.1619	.75 794	17
44	.65 254	.86 115	1.1612	.75 775	16
45	.65 276	.86 166	1.1606	.75 756	15
46	.65 298	.86 216	1.1599	.75 738	14
47	.65 320	.86 267	1.1592	.75 719	13
48	.65 342	.86 318	1.1585	.75 700	12
49	.65 364	.86 368	1.1578	.75 680	11
50	.65 386	.86 419	1.1571	.75 661	**10**
51	.65 408	.86 470	1.1565	.75 642	9
52	.65 430	.86 521	1.1558	.75 623	8
53	.65 452	.86 572	1.1551	.75 604	7
54	.65 474	.86 623	1.1544	.75 585	6
55	.65 496	.86 674	1.1538	.75 566	5
56	.65 518	.86 725	1.1531	.75 547	4
57	.65 540	.86 776	1.1524	.75 528	3
58	.65 562	.86 827	1.1517	.75 509	2
59	.65 584	.86 878	1.1510	.75 490	1
60	.65 606	.86 929	1.1504	.75 471	**0**
′	cos	cot	tan	sin	′
	49°				

41°					
′	sin	tan	cot	cos	′
0	.65 606	.86 929	1.1504	.75 471	**60**
1	.65 628	.86 980	1.1497	.75 452	59
2	.65 650	.87 031	1.1490	.75 433	58
3	.65 672	.87 082	1.1483	.75 414	57
4	.65 694	.87 133	1.1477	.75 395	56
5	.65 716	.87 184	1.1470	.75 375	55
6	.65 738	.87 236	1.1463	.75 356	54
7	.65 759	.87 287	1.1456	.75 337	53
8	.65 781	.87 338	1.1450	.75 318	52
9	.65 803	.87 389	1.1443	.75 299	51
10	.65 825	.87 441	1.1436	.75 280	**50**
11	.65 847	.87 492	1.1430	.75 261	49
12	.65 869	.87 543	1.1423	.75 241	48
13	.65 891	.87 595	1.1416	.75 222	47
14	.65 913	.87 646	1.1410	.75 203	46
15	.65 935	.87 698	1.1403	.75 184	45
16	.65 956	.87 749	1.1396	.75 165	44
17	.65 978	.87 801	1.1389	.75 146	43
18	.66 000	.87 852	1.1383	.75 126	42
19	.66 022	.87 904	1.1376	.75 107	41
20	.66 044	.87 955	1.1369	.75 088	**40**
21	.66 066	.88 007	1.1363	.75 069	39
22	.66 088	.88 059	1.1356	.75 050	38
23	.66 109	.88 110	1.1349	.75 030	37
24	.66 131	.88 162	1.1343	.75 011	36
25	.66 153	.88 214	1.1336	.74 992	35
26	.66 175	.88 265	1.1329	.74 973	34
27	.66 197	.88 317	1.1323	.74 953	33
28	.66 218	.88 369	1.1316	.74 934	32
29	.66 240	.88 421	1.1310	.74 915	31
30	.66 262	.88 473	1.1303	.74 896	**30**
31	.66 284	.88 524	1.1296	.74 876	29
32	.66 306	.88 576	1.1290	.74 857	28
33	.66 327	.88 628	1.1283	.74 838	27
34	.66 349	.88 680	1.1276	.74 818	26
35	.66 371	.88 732	1.1270	.74 799	25
36	.66 393	.88 784	1.1263	.74 780	24
37	.66 414	.88 836	1.1257	.74 760	23
38	.66 436	.88 888	1.1250	.74 741	22
39	.66 458	.88 940	1.1243	.74 722	21
40	.66 480	.88 992	1.1237	.74 703	**20**
41	.66 501	.89 045	1.1230	.74 683	19
42	.66 523	.89 097	1.1224	.74 664	18
43	.66 545	.89 149	1.1217	.74 644	17
44	.66 566	.89 201	1.1211	.74 625	16
45	.66 588	.89 253	1.1204	.74 606	15
46	.66 610	.89 306	1.1197	.74 586	14
47	.66 632	.89 358	1.1191	.74 567	13
48	.66 653	.89 410	1.1184	.74 548	12
49	.66 675	.89 463	1.1178	.74 528	11
50	.66 697	.89 515	1.1171	.74 509	**10**
51	.66 718	.89 567	1.1165	.74 489	9
52	.66 740	.89 620	1.1158	.74 470	8
53	.66 762	.89 672	1.1152	.74 451	7
54	.66 783	.89 725	1.1145	.74 431	6
55	.66 805	.89 777	1.1139	.74 412	5
56	.66 827	.89 830	1.1132	.74 392	4
57	.66 848	.89 883	1.1126	.74 373	3
58	.66 870	.89 935	1.1119	.74 353	2
59	.66 891	.89 988	1.1113	.74 334	1
60	.66 913	.90 040	1.1106	.74 314	**0**
′	cos	cot	tan	sin	′
	48°				

′	42° sin	tan	cot	cos	′
0	.66 913	.90 040	1.1106	.74 314	**60**
1	.66 935	.90 093	1.1100	.74 295	59
2	.66 956	.90 146	1.1093	.74 276	58
3	.66 978	.90 199	1.1087	.74 256	57
4	.66 999	.90 251	1.1080	.74 237	56
5	.67 021	.90 304	1.1074	.74 217	55
6	.67 043	.90 357	1.1067	.74 198	54
7	.67 064	.90 410	1.1061	.74 178	53
8	.67 086	.90 463	1.1054	.74 159	52
9	.67 107	.90 516	1.1048	.74 139	51
10	.67 129	.90 569	1.1041	.74 120	**50**
11	.67 151	.90 621	1.1035	.74 100	49
12	.67 172	.90 674	1.1028	.74 080	48
13	.67 194	.90 727	1.1022	.74 061	47
14	.67 215	.90 781	1.1016	.74 041	46
15	.67 237	.90 834	1.1009	.74 022	45
16	.67 258	.90 887	1.1003	.74 002	44
17	.67 280	.90 940	1.0996	.73 983	43
18	.67 301	.90 993	1.0990	.73 963	42
19	.67 323	.91 046	1.0983	.73 944	41
20	.67 344	.91 099	1.0977	.73 924	**40**
21	.67 366	.91 153	1.0971	.73 904	39
22	.67 387	.91 206	1.0964	.73 885	38
23	.67 409	.91 259	1.0958	.73 865	37
24	.67 430	.91 313	1.0951	.73 846	36
25	.67 452	.91 366	1.0945	.73 826	35
26	.67 473	.91 419	1.0939	.73 806	34
27	.67 495	.91 473	1.0932	.73 787	33
28	.67 516	.91 526	1.0926	.73 767	32
29	.67 538	.91 580	1.0919	.73 747	31
30	.67 559	.91 633	1.0913	.73 728	**30**
31	.67 580	.91 687	1.0907	.73 708	29
32	.67 602	.91 740	1.0900	.73 688	28
33	.67 623	.91 794	1.0894	.73 669	27
34	.67 645	.91 847	1.0888	.73 649	26
35	.67 666	.91 901	1.0881	.73 629	25
36	.67 688	.91 955	1.0875	.73 610	24
37	.67 709	.92 008	1.0869	.73 590	23
38	.67 730	.92 062	1.0862	.73 570	22
39	.67 752	.92 116	1.0856	.73 551	21
40	.67 773	.92 170	1.0850	.73 531	**20**
41	.67 795	.92 224	1.0843	.73 511	19
42	.67 816	.92 277	1.0837	.73 491	18
43	.67 837	.92 331	1.0831	.73 472	17
44	.67 859	.92 385	1.0824	.73 452	16
45	.67 880	.92 439	1.0818	.73 432	15
46	.67 901	.92 493	1.0812	.73 413	14
47	.67 923	.92 547	1.0805	.73 393	13
48	.67 944	.92 601	1.0799	.73 373	12
49	.67 965	.92 655	1.0793	.73 353	11
50	.67 987	.92 709	1.0786	.73 333	**10**
51	.68 008	.92 763	1.0780	.73 314	9
52	.68 029	.92 817	1.0774	.73 294	8
53	.68 051	.92 872	1.0768	.73 274	7
54	.68 072	.92 926	1.0761	.73 254	6
55	.68 093	.92 980	1.0755	.73 234	5
56	.68 115	.93 034	1.0749	.73 215	4
57	.68 136	.93 088	1.0742	.73 195	3
58	.68 157	.93 143	1.0736	.73 175	2
59	.68 179	.93 197	1.0730	.73 155	1
60	.68 200	.93 252	1.0724	.73 135	**0**
′	cos	cot	tan	sin 47°	′

′	43° sin	tan	cot	cos	′
0	.68 200	.93 252	1.0724	.73 135	**60**
1	.68 221	.93 306	1.0717	.73 116	59
2	.68 242	.93 360	1.0711	.73 096	58
3	.68 264	.93 415	1.0705	.73 076	57
4	.68 285	.93 469	1.0699	.73 056	56
5	.68 306	.93 524	1.0692	.73 036	55
6	.68 327	.93 578	1.0686	.73 016	54
7	.68 349	.93 633	1.0680	.72 996	53
8	.68 370	.93 688	1.0674	.72 976	52
9	.68 391	.93 742	1.0668	.72 957	51
10	.68 412	.93 797	1.0661	.72 937	**50**
11	.68 434	.93 852	1.0655	.72 917	49
12	.68 455	.93 906	1.0649	.72 897	48
13	.68 476	.93 961	1.0643	.72 877	47
14	.68 497	.94 016	1.0637	.72 857	46
15	.68 518	.94 071	1.0630	.72 837	45
16	.68 539	.94 125	1.0624	.72 817	44
17	.68 561	.94 180	1.0618	.72 797	43
18	.68 582	.94 235	1.0612	.72 777	42
19	.68 603	.94 290	1.0606	.72 757	41
20	.68 624	.94 345	1.0599	.72 737	**40**
21	.68 645	.94 400	1.0593	.72 717	39
22	.68 666	.94 455	1.0587	.72 697	38
23	.68 688	.94 510	1.0581	.72 677	37
24	.68 709	.94 565	1.0575	.72 657	36
25	.68 730	.94 620	1.0569	.72 637	35
26	.68 751	.94 676	1.0562	.72 617	34
27	.68 772	.94 731	1.0556	.72 597	33
28	.68 793	.94 786	1.0550	.72 577	32
29	.68 814	.94 841	1.0544	.72 557	31
30	.68 835	.94 896	1.0538	.72 537	**30**
31	.68 857	.94 952	1.0532	.72 517	29
32	.68 878	.95 007	1.0526	.72 497	28
33	.68 899	.95 062	1.0519	.72 477	27
34	.68 920	.95 118	1.0513	.72 457	26
35	.68 941	.95 173	1.0507	.72 437	25
36	.68 962	.95 229	1.0501	.72 417	24
37	.68 983	.95 284	1.0495	.72 397	23
38	.69 004	.95 340	1.0489	.72 377	22
39	.69 025	.95 395	1.0483	.72 357	21
40	.69 046	.95 451	1.0477	.72 337	**20**
41	.69 067	.95 506	1.0470	.72 317	19
42	.69 088	.95 562	1.0464	.72 297	18
43	.69 109	.95 618	1.0458	.72 277	17
44	.69 130	.95 673	1.0452	.72 257	16
45	.69 151	.95 729	1.0446	.72 236	15
46	.69 172	.95 785	1.0440	.72 216	14
47	.69 193	.95 841	1.0434	.72 196	13
48	.69 214	.95 897	1.0428	.72 176	12
49	.69 235	.95 952	1.0422	.72 156	11
50	.69 256	.96 008	1.0416	.72 136	**10**
51	.69 277	.96 064	1.0410	.72 116	9
52	.69 298	.96 120	1.0404	.72 095	8
53	.69 319	.96 176	1.0398	.72 075	7
54	.69 340	.96 232	1.0392	.72 055	6
55	.69 361	.96 288	1.0385	.72 035	5
56	.69 382	.96 344	1.0379	.72 015	4
57	.69 403	.96 400	1.0373	.71 995	3
58	.69 424	.96 457	1.0367	.71 974	2
59	.69 445	.96 513	1.0361	.71 954	1
60	.69 466	.96 569	1.0355	.71 934	**0**
′	cos	cot	tan	sin 46°	′

44°					
′	sin	tan	cot	cos	′
0	.69 466	.96 569	1.0355	.71 934	**60**
1	.69 487	.96 625	1.0349	.71 914	59
2	.69 508	.96 681	1.0343	.71 894	58
3	.69 529	.96 738	1.0337	.71 873	57
4	.69 549	.96 794	1.0331	.71 853	56
5	.69 570	.96 850	1.0325	.71 833	55
6	.69 591	.96 907	1.0319	.71 813	54
7	.69 612	.96 963	1.0313	.71 792	53
8	.69 633	.97 020	1.0307	.71 772	52
9	.69 654	.97 076	1.0301	.71 752	51
10	.69 675	.97 133	1.0295	.71 732	**50**
11	.69 696	.97 189	1.0289	.71 711	49
12	.69 717	.97 246	1.0283	.71 691	48
13	.69 737	.97 302	1.0277	.71 671	47
14	.69 758	.97 359	1.0271	.71 650	46
15	.69 779	.97 416	1.0265	.71 630	45
16	.69 800	.97 472	1.0259	.71 610	44
17	.69 821	.97 529	1.0253	.71 590	43
18	.69 842	.97 586	1.0247	.71 569	42
19	.69 862	.97 643	1.0241	.71 549	41
20	.69 883	.97 700	1.0235	.71 529	**40**
21	.69 904	.97 756	1.0230	.71 508	39
22	.69 925	.97 813	1.0224	.71 488	38
23	.69 946	.97 870	1.0218	.71 468	37
24	.69 966	.97 927	1.0212	.71 447	36
25	.69 987	.97 984	1.0206	.71 427	35
26	.70 008	.98 041	1.0200	.71 407	34
27	.70 029	.98 098	1.0194	.71 386	33
28	.70 049	.98 155	1.0188	.71 366	32
29	.70 070	.98 213	1.0182	.71 345	31
30	.70 091	.98 270	1.0176	.71 325	**30**
31	.70 112	.98 327	1.0170	.71 305	29
32	.70 132	.98 384	1.0164	.71 284	28
33	.70 153	.98 441	1.0158	.71 264	27
34	.70 174	.98 499	1.0152	.71 243	26
35	.70 195	.98 556	1.0147	.71 223	25
36	.70 215	.98 613	1.0141	.71 203	24
37	.70 236	.98 671	1.0135	.71 182	23
38	.70 257	.98 728	1.0129	.71 162	22
39	.70 277	.98 786	1.0123	.71 141	21
40	.70 298	.98 843	1.0117	.71 121	**20**
41	.70 319	.98 901	1.0111	.71 100	19
42	.70 339	.98 958	1.0105	.71 080	18
43	.70 360	.99 016	1.0099	.71 059	17
44	.70 381	.99 073	1.0094	.71 039	16
45	.70 401	.99 131	1.0088	.71 019	15
46	.70 422	.99 189	1.0082	.70 998	14
47	.70 443	.99 247	1.0076	.70 978	13
48	.70 463	.99 304	1.0070	.70 957	12
49	.70 484	.99 362	1.0064	.70 937	11
50	.70 505	.99 420	1.0058	.70 916	**10**
51	.70 525	.99 478	1.0052	.70 896	9
52	.70 546	.99 536	1.0047	.70 875	8
53	.70 567	.99 594	1.0041	.70 855	7
54	.70 587	.99 652	1.0035	.70 834	6
55	.70 608	.99 710	1.0029	.70 813	5
56	.70 628	.99 768	1.0023	.70 793	4
57	.70 649	.99 826	1.0017	.70 772	3
58	.70 670	.99 884	1.0012	.70 752	2
59	.70 690	.99 942	1.0006	.70 731	1
60	.70 711	1.0000	1.0000	.70 711	**0**
′	cos	cot	tan	sin	′
45°					

Page 153 Ex 14
" 147 Ex 8
" 144 Ex. 18
"

Page 25 Even; Page 34 sin cos
" 30, Ex. 13, 15, 17, 19, 21, 23

Page 34 Derive the values in the table
" 35 even
" 47 Ex. 65, 66, 67
" 49 Ex. 5, 6
" 50 Ex. 14, 21
74. 79 Revi
" 82 work all Exs.
" 84 " "

Pg 92 - 96, on P. 96 - 97

Pg 99 Even Ex.
100
102 Ex. 8, 10, 12, 14
104 " 18, 16, 14, 12
106
108 Form 1, 2, 3, 4, 5, 6

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118 Odd
120 1-5
131
133
134
139
148 tan

P. 154 Derive
P. 157
Derive De Moivre
sin 5x, cos 5x

p. 102, 4. 14 — 23

p. 104, 2 4. 7 — 15

Brown 0
Keating 0
0

Cowan
Jones