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Florian Cajori

PLANE AND SPHERICAL TRIGONOMETRY

BY

ELMER A. LYMAN

MICHIGAN STATE NORMAL COLLEGE

AND

EDWIN C. GODDARD

UNIVERSITY OF MICHIGAN

ALLYN AND BACON

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PREFACE.

Many American text-books on trigonometry treat the solution of triangles quite fully; English text-books elaborate analytical trigonometry; but no book available seems to meet both needs adequately. To do that is the first aim of the present work, in the preparation of which nearly everything has been worked out and tested by the authors in their classes.

The work entered upon, other features demanded attention. For some unaccountable reason nearly all books, in proving the formulæ for functions of $\alpha \pm \beta$, treat the same line as both positive and negative, thus vitiating the proof; and proofs given for acute angles are (without further discussion) supposed to apply to all angles, or it is suggested that the student can draw other figures and show that the formulæ hold in all cases. As a matter of fact the average student cannot show anything of the kind; and if he could, the proof would still apply only to combinations of conditions the same as those in the figures actually drawn. These difficulties are avoided by so wording the proofs that the language applies to figures involving any angles, and to avoid drawing the indefinite number of figures necessary fully to establish the formulæ geometrically, the general case is proved algebraically (see page 58).

Inverse functions are introduced early, and used constantly. Wherever computations are introduced they are made by means of logarithms. The average student, using logarithms for a short time and only at the end of the subject, straightway forgets what manner of things they are. It is hoped, by dint of much practice, extended over as long a time as possible, to give the student a command of logarithms that will stay. The fundamental formulæ of trigonometry must be memorized. There is no substitute for this. For this purpose oral work is introduced, and there are frequent lists of review problems involving all principles and formulæ previously developed. These lists serve the

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For the usual course in plane trigonometry Chapters I–VII, omitting Arts. 26, 27, contain enough. Articles marked * (as Art. * 26) may be omitted unless the teacher finds time for them without neglecting the rest of the work. Classes that can accomplish more will find a most interesting field opened in the other chapters. More problems are provided than any student is expected to solve, in order that different selections may be assigned to different students, or to classes in different years. *Do not assign work too fast. Make sure the student has memorized and can use each preceding formula, before taking up new ones.*

No complete acknowledgment of help received could here be made. The authors are under obligation to many for general hints, and to several who, after going over the proof with care, have given valuable suggestions. The standard works of Levett and Davison, Hobson, Henrici and Treutlein, and others have been freely consulted, and while many of the problems have been prepared by the authors in their class-room work, they have not hesitated to take, from such standard collections as writers generally have drawn upon, any problems that seemed better adapted than others to the work. Quality has not been knowingly sacrificed to originality. Corrections and suggestions will be gladly received at any time.

E. A. L., Ypsilanti.
E. C. G., Ann Arbor.

October, 1900.

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PLANE TRIGONOMETRY.

CHAPTER I.

ANGLES—MEASUREMENT OF ANGLES.

1. Angles. It is difficult, if not impossible, to define an angle. This difficulty may be avoided by telling how it is formed. *If a line revolve about one of its points, an angle is generated*, the magnitude of the angle depending on the amount of the rotation.

Thus, if one side of the angle θ, as OR, be originally in the position OX, and be revolved about the point O to the position in the figure, the angle XOR is generated. OX is called the *initial* line, and any position of OR the *terminal* line of the angle formed. The angle θ is considered *positive if generated by a counter-clockwise rotation of OR*, and hence *negative if generated by a clockwise rotation.* The magnitude of θ depends on the amount of rotation of OR, and since the amount of such rotation may be unlimited, there is no limit to the possible magnitude of angles, for, evidently, the revolving line may reach the position OR by rotation through an acute angle θ, and, likewise, by rotation through once, twice, $\cdots$, n times 360°, plus the acute angle θ. So that XOR may mean the acute angle θ, $\theta + 360°$, $\theta + 720°$, $\cdots$, $\theta + n \cdot 360°$.

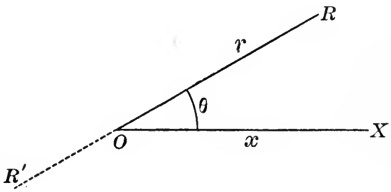


FIG. 1.

In reading an angle, read first the initial line, then the terminal line. Thus in the figure the acute angle XOR, or xr, is a positive angle, and ROX, or rx, an equal negative angle.

Ex. **1.** Show that if the initial lines for $\frac{1}{2}$, $\frac{9}{2}$, $\frac{25}{2}$, $-\frac{7}{2}$, right angles are the same, the terminal lines may coincide.

2. Name four other angles having the same initial and terminal lines as $\frac{1}{3}$ of a right angle; as $\frac{5}{4}$ of a right angle; as $\frac{3}{2}$ of a right angle.

2. Rectangular axes. Any plane surface may be divided by two perpendicular straight lines XX' and YY' into four portions, or *quadrants*.

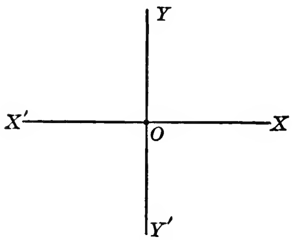


FIG. 2.

XX' is known as the *x-axis*, YY' as the *y-axis*, and the two together are called *axes of reference*. Their intersection O is the *origin*, and the four portions of the plane surface, XOY, YOX', $X'OY'$, $Y'OX$, are called respectively the *first*, *second*, *third*, and *fourth quadrants*. The position of any point in the plane is determined when we know its *distances* and *directions* from the axes.

3. Any direction may be considered positive. Then the opposite direction must be negative. Thus, if AB represents any positive line, BA is an equal negative line. Mathematicians usually consider *lines measured in the same direction as* OX *or* OY (Fig. 2) *as positive. Then lines measured in the same direction as* OX' *or* OY' *must be negative.*

The distance of any point from the y-axis is called the *abscissa*, its distance from the x-axis the *ordinate*, of that point; the two together are the *coördinates* of the point, usually denoted by the letters x and y respectively, and written (x, y).

When taken with their proper signs, the coördinates define completely the position of the point. Thus, if the point P is $+a$ units from YY', and $+b$ units from XX', any convenient unit of length being chosen, the position of P is known. For we have only to measure a distance ON equal to a units along OX, and then from N measure a distance b units parallel to OY, and we arrive at the position of the point P, (a, b). In like manner we may locate P', $(-a, b)$, in the second quadrant, P'', $(-a, -b)$, in the third quadrant, and P''', $(a, -b)$, in the fourth quadrant.

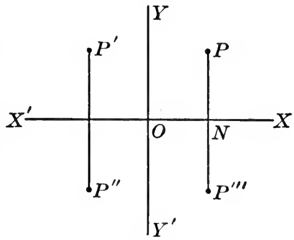


FIG. 3.

Ex. Locate $(2, -2)$; $(0, 0)$; $(-8, -7)$; $(0, 5)$; $(-2, 0)$; $(2, 2)$; (m, n).

4. If OX is the initial line, θ is said to be an *angle of the first*, *second*, *third*, or *fourth quadrant*, according as its terminal line is in the first, second, third, or fourth quadrant. It is clear that as OR rotates its *quality* is in no way affected, and hence it is *in all positions considered positive*, and its extension through O, OR', negative.

The student should notice that the initial line may take any position and revolve in either direction. While it is customary to consider the counter-clockwise rotation as forming a positive angle, yet the conditions of a figure may be such that a positive angle may be generated by a clockwise rotation. Thus the angle XOR in each figure may be traced as a positive angle by revolving the initial line OX to the position OR. No confusion can result if the fact is clear that when an angle is read XOR, OX is considered a positive line revolving to the position OR. OX' and OR' then are negative lines in whatever directions drawn. These conceptions are mere matters of agreement, and the agreement may be determined in a particular case by the conditions of the problem quite as well as by such general agreements of mathematicians as those referred to in Arts. 3 and 4 above.

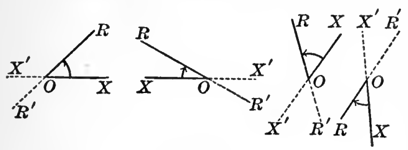


FIG. 4.

5. Measurement. All measurements are made in terms of some fixed standard adopted as a unit. This unit must be of the same kind as the quantity measured. Thus, length is measured in terms of a unit length, surface in terms of a unit surface, weight in terms of a unit weight, value in terms of a unit value, an angle in terms of a unit angle.

The measure of a given quantity is the number of times it contains the unit selected.

Thus the area of a given surface in square feet is the number of times it contains the unit surface 1 sq. ft.; the length of a road in miles, the number of times it contains the unit length 1 mi.; the weight of a cargo of iron ore in tons, the number of times it contains the unit weight 1 ton; the value of an estate, the number of times it contains the unit value $1.

The same quantity may have different measures, according to the unit chosen. So the measure of 80 acres, when the unit surface is 1 acre, is 80, when the unit surface is 1 sq. rd., is 12,800, when the unit surface is 1 sq. yd., is 387,200. What is its measure in square feet?

6. The essentials of a good unit of measure are:

1. *That it be invariable*, *i.e.* under all conditions bearing the same ratio to equal magnitudes.

2. *That it be convenient* for practical or theoretical purposes.

3. *That it be of the same kind as the quantity measured.*

7. Two systems of measuring angles are in use, the *sexagesimal* and the *circular*.

The *sexagesimal* system is used in most practical applications. The right angle, the unit of measure in geometry, though it is invariable, as a measure is too large for convenience. Accordingly it is divided into 90 equal parts, called *degrees*. The degree is divided into 60 *minutes*, and the minute into 60 *seconds*. Degrees, minutes, seconds, are indicated by the marks $^\circ \, ' \, ''$, as $36^\circ \, 20' \, 15''$.

The division of a right angle into hundredths, with subdivisions into hundredths, would be more convenient. The French have proposed such a *centesimal* system, dividing the right angle into 100 grades, the grade into 100 minutes, and the minute into 100 seconds, marked ^{g} $^{\backslash}$ $^{\backslash\backslash}$, as 50^{g} $70^{\backslash}$ $28^{\backslash\backslash}$. The great labor involved in changing mathematical tables, instruments, and records of observation to the new system has prevented its adoption.

8. The *circular* system is important in theoretical considerations. It is based on the fact that for a given angle the ratio of the length of its arc to the length of the radius of that arc is constant, *i.e.* for a fixed angle the ratio *arc : radius* is the same no matter what the length of the radius. In the figure, for the angle θ,

$$\frac{OA}{AA'} = \frac{OB}{BB'} = \frac{OC}{CC'} = \cdots.$$

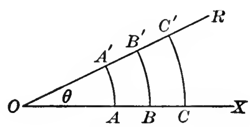


FIG. 5.

That this ratio of arc to radius for a fixed angle is constant follows from the established geometrical principles:

1. The circumference of any circle is 2π times its radius.
2. Angles at the centre are in the same ratio as their arcs.

The Radian. It follows that an angle whose arc is equal in length to the radius is a constant angle for all circles, since in four right angles, or the perigon, there are always 2π such angles. *This constant angle, whose arc is equal in length to the radius, is taken as the unit angle of circular measure, and is called the radian.* From the definition we have

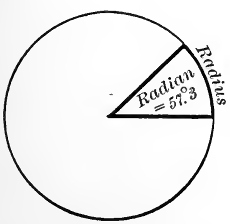


FIG. 6.

$$4 \text{ right angles} = 360° = 2\pi \text{ radians},$$

$$2 \text{ right angles} = 180° = \pi \text{ radians},$$

$$1 \text{ right angle} = 90° = \frac{\pi}{2} \text{ radians}.$$

π is a numerical quantity, 3.14159+, and not an angle. When we speak of 180° as π, 90° as $\frac{\pi}{2}$, etc., we always mean π *radians*, $\frac{\pi}{2}$ *radians*, etc.

9. To change from one system of measurement to the other we use the relation,

$$2\,\pi \text{ radians} = 360°.$$

$$\therefore\ 1 \text{ radian} = \frac{180°}{\pi} = 57°.2958-;$$

i.e. the radian is 57°.3, *approximately.*

Ex. **1.** Express in radians 75° 30′.

$$75°\,30' = 75°.5;\ 1 \text{ radian} = 57°.3.$$

$$\therefore\ 75°\,30' = \frac{75.5}{57.3} = 1.317 \text{ radians.}$$

2. Express in degree measure 3.6 radians.

$$1 \text{ radian} = 57°.3.$$

$$\therefore\ 3.6 \text{ radians} = 3.6 \times 57°.3 = 206°\,16'\,48''.$$

EXAMPLES.

1. Construct, approximately, the following angles: 50°, – 20°, 90°, 179°, – 135°, 400°, – 380°, 1140°, $\frac{\pi}{4}$ radians, $\frac{\pi}{3}$ radians, $-\frac{\pi}{6}$ radians, $3\,\pi$ radians, $-\frac{3\,\pi}{4}$ radians, $\frac{12\,\pi}{5}$ radians. Of which quadrant is each angle?

2. What is the measure of:

(*a*) $\frac{5}{4}$ of a right angle, when 30° is the unit of measure?
(*b*) an acre, when a square whose side is 10 rds. is the unit?
(*c*) *m* miles, when *y* yards is the unit?

3. What is the unit of measure, when the measure of $2\frac{1}{2}$ miles is 50?

4. The Michigan Central R.R. is 535 miles long, and the Ann Arbor R.R. is 292 miles long. Express the length of the first in terms of the second as a unit.

5. What will be the measure of the radian when the right angle is taken for the unit? Of the right angle when the radian is the unit?

6. In which quadrant is 45°? 10°? – 60°? 145°? 1145°? – 725°? Express each in right angles; in radians.

7. Express in sexagesimal measure

$$\frac{\pi}{3},\ \frac{\pi}{12},\ 1,\ 6.28,\ \frac{1}{\pi},\ \frac{7\,\pi}{3},\ -\frac{4\,\pi}{3}, \text{ radians.}$$

8. Express in each system an interior angle of a regular hexagon; an exterior angle.

9. Find the distance in miles between two places on the earth's equator which are $11° 15'$ apart. (The earth's radius is about 3963 miles.)

10. Find the length of an arc which subtends an angle of 4 radians at the centre of a circle of radius 12 ft. 3 in.

11. An arc 15 yds. long contains 3 radians. Find the radius of the circle.

12. Show that the hour and minute hands of a watch turn through angles of $30'$ and $6°$ respectively per minute; also find in degrees and in radians the angle turned through by the minute hand in 3 hrs. 20 mins.

13. Find the number of seconds in an arc of 1 mile on the equator; also the length in miles of an arc of $1'$ (1 knot).

14. Find to three decimal places the radius of a circle in which the arc of $71° 36' 3''.6$ is 15 in. long.

15. Find the ratio of $\frac{\pi}{6}$ to $5°$.

16. What is the shortest distance measured on the earth's surface from the equator to Ann Arbor, latitude $+ 42° 16' 48''$?

17. The difference of two angles is $10°$, and the circular measure of their sum is 2. Find the circular measure of each angle.

18. A water wheel of radius 6 ft. makes 30 revolutions per minute. Find the number of miles per hour travelled by a point on the rim.

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

10. Trigonometry, as the word indicates, was originally concerned with the measurement of triangles. It now includes the analytical treatment of certain *functions of angles*, as well as the solution of triangles by means of certain relations between the functions of the angles of those triangles.

11. Function. If one quantity depends upon another for its value, the first is called a *function* of the second. It always follows that the second quantity is also a function of the first; and, in general, functions are so related that if one is constant the other is constant, and if either varies in value, the other varies. This relation may be extended to any number of mutually dependent quantities.

Illustration. If a train moves at a rate of 30 miles per hour, the distance travelled is a function of the rate and time, the time is a function of the rate and distance, and the rate is a function of the time and distance.

Again, the circumference of a circle is a function of the radius, and the radius of the circumference, for so long as either is constant the other is constant, and if either changes in value, the other changes, since circumference and radius are connected by the relation $C = 2\pi R$.

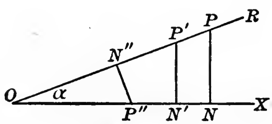


Fig. 7.

Once more, in the right triangle NOP, the ratio of any two sides is a function of the angle α, because all the right triangles of which α is one angle are similar, *i.e.* the ratio of two corresponding sides is constant so long as α is constant, and varies if α varies.

Thus, the ratios

$$\frac{NP}{OP}=\frac{N'P'}{OP'}=\frac{N''P''}{OP''}$$

and
$$\frac{ON}{NP}=\frac{ON'}{N'P'}=\frac{ON''}{N''P''}, \text{ etc.},$$

depend on α for their values, *i.e.* are functions of α.

12. The trigonometric functions. In trigonometry six *functions of angles* are usually employed, called the *trigonometric functions.*

By definition *these functions are the six ratios between the sides of the triangle of reference of the given angle. The triangle of reference is formed by drawing from some point in the initial line, or the initial line produced, a perpendicular to that line meeting the terminal line of the angle.*

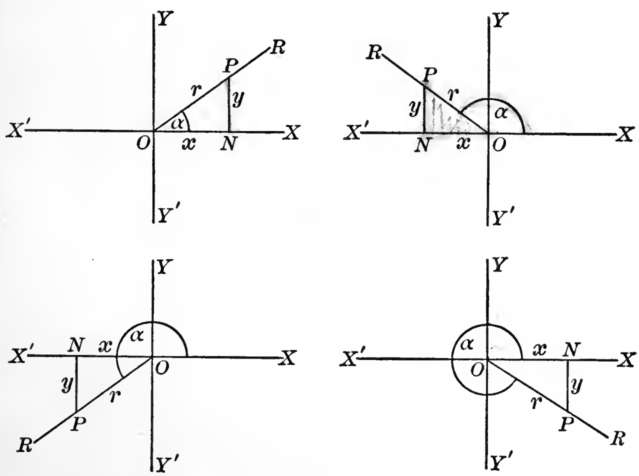


FIG. 8.

Let α be an angle of any quadrant. Each triangle of reference of α, NOP, is formed by drawing a perpendicular to OX, or OX produced, meeting the terminal line OR in P.

If α is greater than 360°, its triangle of reference would not differ from one of the above triangles.

It is perhaps worthy of notice that the *triangle of reference* might be defined to be the triangle formed by drawing a perpendicular to either side of the angle, or that side produced, meeting the other side or the other side produced. In the figure, NOP is in all cases the triangle of reference of α. The principles of the following pages are the same no matter which of the triangles is considered the triangle of reference. It will, however, be as well, and perhaps clearer, to use the triangle defined under Fig. 8, and we shall always draw the triangle as there described.

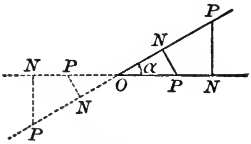


FIG. 9.

13. The trigonometric functions of α (Fig. 8) are called the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, and *cosecant* of α. These are abbreviated in writing to sin α, cos α, tan α, cot α, sec α, csc α, and are defined as follows:

$$\sin\alpha = \frac{\text{perp.}}{\text{hyp.}} = \frac{y}{r}, \quad \text{whence} \quad y = r \sin\alpha;$$

$$\cos\alpha = \frac{\text{base}}{\text{hyp.}} = \frac{x}{r}, \quad \text{whence} \quad x = r \cos\alpha;$$

$$\tan\alpha = \frac{\text{perp.}}{\text{base}} = \frac{y}{x}, \quad \text{whence} \quad y = x \tan\alpha;$$

$$\cot\alpha = \frac{\text{base}}{\text{perp.}} = \frac{x}{y}, \quad \text{whence} \quad x = y \cot\alpha;$$

$$\sec\alpha = \frac{\text{hyp.}}{\text{base}} = \frac{r}{x}, \quad \text{whence} \quad r = x \sec\alpha;$$

$$\csc\alpha = \frac{\text{hyp.}}{\text{perp.}} = \frac{r}{y}, \quad \text{whence} \quad r = y \csc\alpha.$$

$1 - \cos\alpha$ and $1 - \sin\alpha$, called *versed-sine* α and *coversed-sine* α, respectively, are sometimes used.

Ex. **1**. Write the trigonometric functions of β, NPO (Fig. 8), and compare with those of α above.

The meaning of the prefix *co* in cosine, cotangent, and cosecant appears from the relations of Ex. 1. For the *sine of an angle* equals the *cosine*, *i.e.* the *complement-sine*, of the *complement of that angle;* the *tangent* *of an angle* equals the *cotangent of its complementary angle*, and the *secant of an angle* equals the *cosecant of its complementary angle.*

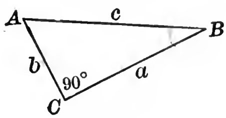


FIG. 10.

2. Express each side of triangle ABC in terms of another side, and some function of an angle in all possible ways, as $a = b \tan A$, etc.

14. Constancy of the trigonometric functions. It is important to notice why these ratios are *functions of the angle*, *i.e.* are the same for equal angles and different for unequal angles. This is shown by the principles of similar triangles.

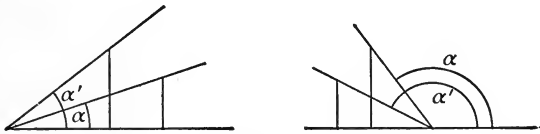


FIG. 11.

In each figure show that in all possible triangles of reference for α the ratios are the same, but in the triangles of reference for α and α', respectively, the ratios are different.

The student must notice that $\sin\alpha$ is a *single symbol.* It is the *name of a number*, or *fraction*, belonging to the angle α; and if it be at any time convenient, we may denote $\sin\alpha$ by a *single letter*, such as o, or x. Also, $\sin^2\alpha$ is an abbreviation for $(\sin\alpha)^2$, *i.e.* for $(\sin\alpha)\times(\sin\alpha)$. Such abbreviations are used because they are convenient. Lock, *Elementary Trigonometry.*

15. Fundamental relations. From the definitions of Art. 13 the following reciprocal relations are apparent:

$$\sin\alpha = \frac{1}{\csc\alpha}, \qquad \csc\alpha = \frac{1}{\sin\alpha},$$

$$\cos\alpha = \frac{1}{\sec\alpha}, \qquad \sec\alpha = \frac{1}{\cos\alpha},$$

$$\tan\alpha = \frac{1}{\cot\alpha}, \qquad \cot\alpha = \frac{1}{\tan\alpha}.$$

Also from the definitions,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}.$$

From the right triangle *NOP*, page 9,

$$y^2 + x^2 = r^2;$$

whence (1) $$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1,$$

(2) $$\frac{y^2}{x^2} + 1 = \frac{r^2}{x^2},$$

(3) $$1 + \frac{x^2}{y^2} = \frac{r^2}{y^2}.$$

From (1) $\sin^2 \alpha + \cos^2 \alpha = 1$; $\sin \alpha = \sqrt{1 - \cos^2 \alpha}$; $\cos \alpha = ?$

(2) $\tan^2 \alpha + 1 = \sec^2 \alpha$; $\tan \alpha = \sqrt{\sec^2 \alpha - 1}$; $\sec \alpha = ?$

(3) $1 + \cot^2 \alpha = \csc^2 \alpha$; $\cot \alpha = \sqrt{\csc^2 \alpha - 1}$; $\csc \alpha = ?$

The foregoing definitions and fundamental relations are of the highest importance, and must be mastered at once. The student of trigonometry is helpless without perfect familiarity with them.

These relations are true for all values of α, positive or negative, but the signs of the functions are not in all cases positive, as appears from the fact that in the triangles of reference in Fig. 8 x and y are sometimes negative. The equations $\sin \alpha = \pm \sqrt{1 - \cos^2 \alpha}$, $\tan \alpha = \pm \sqrt{\sec^2 \alpha - 1}$, $\cot \alpha = \pm \sqrt{\csc^2 \alpha - 1}$, have the double sign $\pm$. Which sign is to be used in a given case depends on the quadrant in which α lies.

16. The relations of Art. 15 enable us to express any function in terms of any other, or when one function is given, to find all the others.

Ex. **1.** To express the other functions in terms of tangent:

$$\sin \alpha = \frac{1}{\csc \alpha} = \frac{1}{\sqrt{1 + \cot^2 \alpha}} = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}; \quad \cot \alpha = \frac{1}{\tan \alpha};$$

$$\cos \alpha = \frac{1}{\sec \alpha} = \frac{1}{\sqrt{1 + \tan^2 \alpha}}; \quad \sec \alpha = \sqrt{1 + \tan^2 \alpha};$$

$$\tan \alpha = \tan \alpha; \quad \csc \alpha = \frac{\sqrt{1 + \tan^2 \alpha}}{\tan \alpha}.$$

In like manner determine the relations to complete the following table:

	sin α	cos α	tan α	cot α	sec α	csc α
sin α			$\dfrac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$			
cos α			$\dfrac{1}{\sqrt{1+\tan^2\alpha}}$			
tan α			$\tan\alpha$			
cot α			$\dfrac{1}{\tan\alpha}$			
sec α			$\sqrt{1+\tan^2\alpha}$			
csc α			$\dfrac{\sqrt{1+\tan^2\alpha}}{\tan\alpha}$			

2. Given $\sin\alpha = \frac{3}{4}$; find the other functions.

$$\cos\alpha = \sqrt{1-\tfrac{9}{16}} = \tfrac{1}{4}\sqrt{7};\quad \tan\alpha = \frac{\frac{3}{4}}{\frac{1}{4}\sqrt{7}} = \frac{3}{\sqrt{7}} = \tfrac{3}{7}\sqrt{7};$$

$$\cot\alpha = \frac{1}{\frac{3}{7}\sqrt{7}} = \tfrac{1}{3}\sqrt{7};\quad \sec\alpha = \frac{1}{\frac{1}{4}\sqrt{7}} = \frac{4}{\sqrt{7}} = \tfrac{4}{7}\sqrt{7};\quad \csc\alpha = \frac{1}{\frac{3}{4}} = \frac{4}{3}.$$

3. Given $\tan\phi + \cot\phi = 2$; find $\sin\phi$.

$$\tan\phi + \frac{1}{\tan\phi} = 2,\quad \tan^2\phi - 2\tan\phi + 1 = 0,\quad \tan\phi = 1.$$

$$\therefore\ \sin\phi = \frac{\tan\phi}{\sqrt{1+\tan^2\phi}} = \tfrac{1}{2}\sqrt{2}.$$

Or, expressing in terms of sine directly, $\dfrac{\sin\phi}{\cos\phi} + \dfrac{\cos\phi}{\sin\phi} = 2,$

$$\sin^2\phi + \cos^2\phi = 2\sin\phi\cos\phi,\quad \sin^2\phi - 2\sin\phi\cos\phi + \cos^2\phi = 0;$$

whence $\quad \sin\phi - \cos\phi = 0,\ \sin\phi = \cos\phi.\quad \therefore\ \sin\phi = \tfrac{1}{2}\sqrt{2}.$

4. Prove $\sec^4 x - \sec^2 x = \tan^2 x + \tan^4 x$.

$$\sec^4 x - \sec^2 x = \sec^2 x(\sec^2 x - 1) = (1+\tan^2 x)\tan^2 x = \tan^2 x + \tan^4 x.$$

5. Prove $\sin^6 y + \cos^6 y = 1 - 3\sin^2 y\cos^2 y$.

$$\sin^6 y + \cos^6 y = (\sin^2 y + \cos^2 y)(\sin^4 y - \sin^2 y\cos^2 y + \cos^4 y)$$
$$= (\sin^2 y + \cos^2 y)^2 - 3\sin^2 y\cos^2 y = 1 - 3\sin^2 y\cos^2 y.$$

6. Prove $\dfrac{\tan z}{1-\cot z}+\dfrac{\cot z}{1-\tan z}=\sec z\csc z+1.$

$$\frac{\tan z}{1-\cot z}+\frac{\cot z}{1-\tan z}=\frac{\dfrac{\sin z}{\cos z}}{1-\dfrac{\cos z}{\sin z}}+\frac{\dfrac{\cos z}{\sin z}}{1-\dfrac{\sin z}{\cos z}}$$

$$=\frac{\sin^2 z}{\cos z(\sin z-\cos z)}+\frac{\cos^2 z}{\sin z(\cos z-\sin z)}$$

$$=\frac{\sin^3 z-\cos^3 z}{\sin z\cos z(\sin z-\cos z)}=\frac{\sin^2 z+\sin z\cos z+\cos^2 z}{\sin z\cos z}$$

$$=\frac{1+\sin z\cos z}{\sin z\cos z}=\frac{1}{\sin z\cos z}+1=\sec z\csc z+1.$$

In solving problems like 3, 4, 5, and 6 above, it is usually safe, if no other step suggests itself, to express all other functions of one member in terms of sine and cosine. The resulting expression may then be reduced by the principles of algebra to the expression in the other member of the equation. For further suggestions as to the solution of trigonometric equations and identities see page 66.

EXAMPLES.

1. Find the values of all the functions of α, if $\sin\alpha=\frac{2}{3}$; if $\tan\alpha=\frac{3}{4}$; if $\sec\alpha=2$; if $\cos\alpha=\frac{1}{3}\sqrt{3}$; if $\cot\alpha=\frac{1}{7}$; if $\csc\alpha=\sqrt{2}$.

2. Compute the functions of each acute angle in the right triangles whose sides are: (1) 3, 4, 5; (2) 8, 15, 17; (3) 480, 31, 481; (4) a, b, c; (5) $\dfrac{2xy}{x-y}$, $\dfrac{x^2+y^2}{x-y}$, $x+y$.

3. If $\cos\alpha=\frac{8}{17}$, find the value of $\dfrac{\sin\alpha+\tan\alpha}{\cos\alpha-\cot\alpha}$.

4. If $2\cos\alpha=2-\sin\alpha$, find $\tan\alpha$.

5. If $\sec^2\alpha\csc^2\alpha-4=0$, find $\cot\alpha$.

6. Solve for $\sin\beta$ in $13\sin\beta+5\cos^2\beta=11$.

Prove

7. $\sin^4\phi-\cos^4\phi=1-2\cos^2\phi$.

8. $(\sin\alpha+\cos\alpha)(\sin\alpha-\cos\alpha)=2\sin^2\alpha-1$.

9. $(\sec\alpha+\tan\alpha)(\sec\alpha-\tan\alpha)=1$.

10. $\cos^2\beta(\sec^2\beta-2\sin^2\beta)=\cos^4\beta+\sin^4\beta$.

11. $\tan v+\sec v=\dfrac{\cos v}{1-\sin v}$.

12. $\dfrac{\sin w}{1-\cos w}=\dfrac{1+\cos w}{\sin w}$.

13. $(\sec\theta+1)(1-\cos\theta)=\tan^2\theta\cos\theta$.

14. $\sin^4 t - \sin^2 t = \cos^4 t - \cos^2 t.$

15. $\dfrac{\sin\beta}{1-\sin\beta} + \dfrac{1+\sin\beta}{\sin\beta} = \sec^2\beta\,(\csc\beta + 1).$

16. $(\tan A + \cot A)^2 = \sec^2 A \csc^2 A.$

17. $\sec^2 x - \sin^2 x = \tan^2 x + \cos^2 x.$

In the triangle ABC, right angled at C,

18. Given $\cos A = \frac{8}{17}$, $BC = 45$, find $\tan B$, and AB.

19. If $\cos A = \dfrac{m^2 - n^2}{m^2 + n^2}$, and $AB = m^2 + n^2$, find AC and BC.

20. If $AC = m + n$, $BC = m - n$, find $\sin A$, $\cos B$.

21. In examples 18, 19, 20, above, prove $\sin^2 A + \cos^2 A = 1$; $1 + \tan^2 A = \sec^2 A$.

17. Functions of certain angles. The trigonometric functions are numerical quantities which may be determined for any angle. In general these values are taken from tables prepared for the purpose, but the principles already studied enable us to calculate the functions of the following angles.

18. Functions of 0°. If α be a very small angle, the value of y is very small, and decreases as α diminishes. Clearly, when α approaches 0° as a limit, y likewise approaches 0, and x approaches r, so that when $\alpha = 0°$,

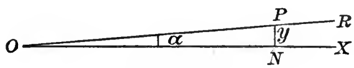


FIG. 12.

$$y = 0, \text{ and } x = r.$$

$$\therefore\ \sin 0° = \frac{y}{r} = 0, \qquad \cot 0° = \frac{1}{\tan 0°} = \infty,$$

$$\cos 0° = \frac{x}{r} = 1, \qquad \sec 0° = \frac{1}{\cos 0°} = 1,$$

$$\tan 0° = \frac{y}{x} = 0, \qquad \csc 0° = \frac{1}{\sin 0°} = \infty.$$

In the figure of Art. 18, by diminishing α it is clear that we can make y as small as we please, and by making α small enough, we can make the *value of y less than any assignable quantity, however small, so that $\sin\alpha$ approaches as a limit* 0. This is what we mean when we say $\sin 0° = 0$. In like manner, it is evident that, by sufficiently diminishing α we can make *$\cot\alpha$ greater than any assignable* quantity. This we express by saying $\cot 0° = \infty$.

19. Functions of 30°. Let NOP be the triangle of reference for an angle of 30°. Make triangle $NOP' = NOP$. Then POP' is an equilateral triangle (why?), and ON bisects PP'. Hence

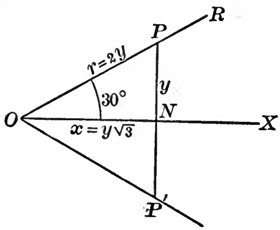


FIG. 13.

$$PP' = r = 2y.$$

Also $x = \sqrt{r^2 - y^2} = \sqrt{3y^2} = y\sqrt{3}.$

$$\therefore \; \sin 30° = \frac{y}{r} = \frac{y}{2y} = \frac{1}{2}, \qquad \csc 30° = 2,$$

$$\cos 30° = \frac{x}{r} = \frac{y\sqrt{3}}{2y} = \tfrac{1}{2}\sqrt{3}, \qquad \sec 30° = \tfrac{2}{3}\sqrt{3},$$

$$\tan 30° = \frac{y}{x} = \frac{y}{y\sqrt{3}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}, \qquad \cot 30° = \sqrt{3}.$$

20. Functions of 45°. Let NOP be the triangle of reference. If angle $NOP = 45°$, $OPN = 45°$.

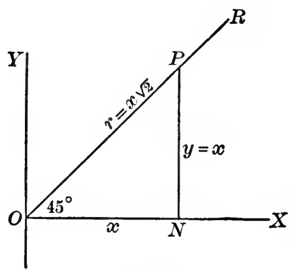


FIG. 14.

$$\therefore \; y = x, \text{ and } r = \sqrt{x^2 + y^2} = \sqrt{2x^2} = x\sqrt{2}.$$

Then $$\sin 45° = \frac{y}{r} = \frac{x}{x\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

$$\cos 45° = \frac{x}{r} = \frac{x}{x\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

$$\tan 45° = \frac{y}{x} = \frac{x}{x} = 1.$$

Find *cot* 45°, *sec* 45°, *csc* 45°.

21. Functions of 60°. The functions of 60° may be computed by means of the figure, or they may be written from the functions of the complement, or 30°. Let the student in both ways show that

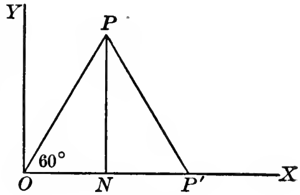


FIG. 15.

$$sin\ 60° = \tfrac{1}{2}\sqrt{3}, \quad cos\ 60° = \tfrac{1}{2},$$
$$tan\ 60° = \sqrt{3}.$$

Compute also the other functions of 60°.

22. Functions of 90°. If α be an angle very near 90°, the value of x is very small, and decreases as α increases toward 90°. Clearly when α approaches 90° as a limit, x approaches 0, and y approaches r, so that when

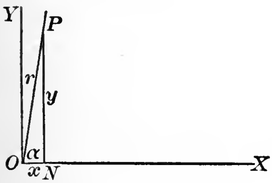


FIG. 16.

$$\alpha = 90°, \quad x = 0, \quad y = r.$$
$$\therefore sin\ 90° = 1, \ cos\ 90° = 0, \ tan\ 90° = \infty.$$

Compute the other functions. Also find the functions of 90° from those of its complement, 0°.

23. It is of great convenience to the student to remember the functions of these angles. *They are easily found by recalling the relative values of the sides of the triangles of reference for the respective angles,* or the values of the other functions may readily be computed by means of the fundamental relations, if the values of the sine and cosine are remembered, as follows:

α	0°	30°	45°	60°	90°
sine	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
cosine	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$

ORAL WORK.

1. Which is greater, $\sin 45°$ or $\frac{1}{2}\sin 90°$? $\sin 60°$ or $2\sin 30°$?

2. From the functions of 60°, find those of 30°; from the functions of 90°, those of 0°. Why are the functions of 45° equal to the co-functions of 45°?

3. Given $\sin A = \frac{1}{3}$, find $\cos A$; $\tan A$.

4. Show that $\sin B \csc B = 1$; $\cos C \sec C = 1$; $\cot x \tan x = 1$.

5. Show that $\sec^2\theta - \tan^2\theta = \csc^2\theta - \cot^2\theta = \sin^2\theta + \cos^2\theta$.

6. Show that $\tan 30° \tan 60° = \cot 60° \cot 30° = \tan 45°$.

7. Show that $\tan 60° \sin^2 45° = \cos 30° \sin 90°$.

8. Show that $\cos\alpha \tan\alpha = \sin\alpha$; $\sin\beta \cot\beta = \cos\beta$.

9. Show that $\dfrac{1-\tan^2 30°}{1+\tan^2 30°} = \cos 60° = \frac{1}{2}\cos 0°$.

10. Show that $(\tan y + \cot y)\sin y \cos y = 1$.

EXAMPLES.

1. Show that $\sin 30° \cos 60° + \cos 30° \sin 60° = \sin 90°$.

2. Show that $\cos 60° \cos 30° + \sin 60° \sin 30° = \cos 30°$.

3. Show that $\sin 45° \cos 0° - \cos 45° \sin 0° = \cos 45°$.

4. Show that $\cos^2 45° - \sin^2 45° = \cos 90°$.

5. Show that $\dfrac{\tan 45° + \tan 0°}{1 - \tan 45° \tan 0°} = \tan 45°$.

If $A = 60°$, verify

6. $\sin\dfrac{A}{2} = \sqrt{\dfrac{1-\cos A}{2}}$.

7. $\tan\dfrac{A}{2} = \sqrt{\dfrac{1-\cos A}{1+\cos A}}$.

8. $\cos A = 2\cos^2\dfrac{A}{2} - 1 = 1 - 2\sin^2\dfrac{A}{2}$.

If $\alpha = 0°$, $\beta = 30°$, $\gamma = 45°$, $\delta = 60°$, $\epsilon = 90°$, find the values of

9. $\sin\beta + \cos\delta$.

10. $\cos\beta + \tan\delta$.

11. $\sin\beta\cos\delta + \cos\beta\sin\delta - \sin\epsilon$.

12. $(\sin\beta + \sin\epsilon)(\cos\alpha + \cos\delta) - 4\sin\alpha(\cos\gamma + \sin\epsilon)$.

24. Variations in the trigonometric functions.

Signs. Thus far no account has been taken of the *signs of the functions.* By the definitions it appears that these depend on the signs of x, y, and r. Now r is always positive, and from the figures it is seen that x is positive in the first

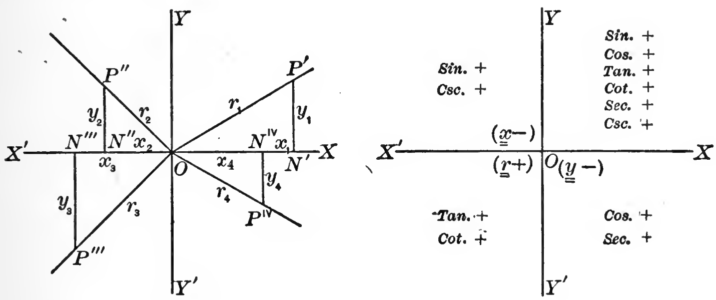


FIG. 17.

and fourth quadrants, and y is positive in the first and second. Hence

For an angle in the *first quadrant* all functions are *positive*, since x, y, r are *positive.*

In the *second quadrant* x alone is *negative*, so that those functions whose ratios involve x, viz. *cosine*, *tangent*, *cotangent*, *secant*, are *negative;* the others, *sine* and *cosecant*, are *positive.*

In the *third quadrant* x and y are both *negative*, so that those functions involving r, viz. *sine*, *cosine*, *secant*, *cosecant*, are *negative;* the others, *tangent* and *cotangent*, are *positiue.*

In the *fourth quadrant* y is *negative*, so that *sine*, *tangent*, *cotangent*, *cosecant* are *negative*, and *cosine* and *secant*, *positive.*

Values. In the triangle of reference of any angle, the hypotenuse r is never less than x or y. Then if r be taken of any fixed length, as the angle varies, the base and perpendicular of the triangle of reference may each vary in length from 0 to r. Hence the ratios $\frac{x}{r}$ and $\frac{y}{r}$ can never be greater than 1, nor if x and y are negative, less than -1; and $\frac{r}{x}$, $\frac{r}{y}$

cannot have values between + 1 and − 1. But the ratios $\frac{y}{x}$ and $\frac{x}{y}$ may vary without limit, *i.e.* from $+\infty$ to $-\infty$. Therefore the possible values of the functions of an angle are:

sine and cosine between + 1 and − 1,

i.e. *sine and cosine cannot be numerically greater than* 1;

tangent and cotangent between $+\infty$ and $-\infty$,

i.e. *tangent and cotangent may have any real value;*

secant and cosecant between $+\infty$ and + 1, and − 1 and $-\infty$,

i.e. *secant and cosecant may have any real values, except values between* + 1 and − 1.

These limits are indicated in the following figures. The student should carefully verify.

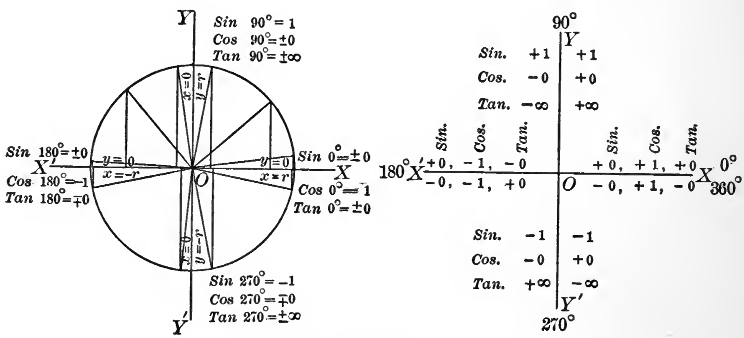


FIG. 18.

25. In tracing the changes in the values of the functions as α changes from 0° to 360°, consider the revolving line r as of fixed length. Then x and y may have any length between 0 and r.

Sine. At 0°, $\sin\alpha = \frac{y}{r} = \frac{0}{r} = 0$. As α increases through the first quadrant, y increases from 0 to r, whence $\frac{y}{r}$ increases from 0 to 1. In passing to 180° $\sin\alpha$ decreases from 1 to 0, since y decreases from r to 0. As α passes through 180°, y changes sign, and in the third quadrant decreases to negative r, so that sin α decreases from 0 to -1. In the fourth quadrant y increases from negative r to 0, and hence sin α increases from -1 to 0.

Cosine depends on changing values of x. Show that, as α increases from 0° to 360°, cos α varies in the four quadrants as follows: 1 to 0, 0 to -1, -1 to 0, 0 to 1.

Tangent depends on changing values of both y and x.

$$\text{At } 0°,\ y = 0,\ x = r, \qquad \text{at } 180°,\ y = 0,\ x = -r,$$

$$\text{at } 90°,\ x = 0,\ y = r, \qquad \text{at } 270°,\ x = 0,\ y = -r.$$

Hence $\tan 0° = \dfrac{y}{x} = \dfrac{0}{r} = 0.$ As α passes to 90°, y increases to r, and x decreases to 0, so that tan α increases from 0 to ∞. As α passes through 90°, x changes sign, so that tan α changes from positive to negative by passing through ∞. In the second quadrant x decreases to negative r, y to 0, and tan α passes from $-\infty$ to 0. As α passes through 180°, tan α changes from minus to plus by passing through 0, because at 180° y changes to minus. In the third quadrant tan α passes from 0 to ∞, changing sign at 270° by passing through ∞, because at 270° x changes to plus. In the fourth quadrant tan α passes from $-\infty$ to 0.

Cotangent. In like manner show that cot α passes through the values ∞ to 0, 0 to $-\infty$, ∞ to 0, 0 to $-\infty$, as α passes from 0° to 360°.

Secant depends on x for its value. Noting the change in x as under cosine, we see that secant passes from 1 to ∞, $-\infty$ to -1, -1 to $-\infty$, ∞ to 1.

Cosecant passes through the values ∞ to 1, 1 to ∞, $-\infty$ to -1, -1 to $-\infty$.

The student should trace the changes in each function fully, as has been done for sine and tangent, giving the reasons at each step.

α	0° to 90°	90° to 180°	180° to 270°	270° to 360°
sin	0 to 1	1 to 0	-0 to -1	-1 to -0
cos	1 to 0	-0 to -1	-1 to -0	0 to 1
tan	0 to ∞	$-\infty$ to -0	0 to ∞	$-\infty$ to -0
cot	∞ to 0	-0 to $-\infty$	∞ to 0	-0 to $-\infty$
sec	1 to ∞	$-\infty$ to -1	-1 to $-\infty$	∞ to 1
csc	∞ to 1	1 to ∞	$-\infty$ to -1	-1 to $-\infty$

* **26. Graphic representation of functions.** These variations are clearly brought out by graphic representations of the functions. Two cases will be considered: I, when α is a constant angle; II, when α is a variable angle.

I. *When α is a constant angle.*

The trigonometric functions are ratios, pure numbers. By so choosing the triangle of reference that the denominator of the ratio is a side of unit length, the side forming the numerator of that ratio will be a geometrical representation of the value of that function, *e.g.* if in Fig. 19 $r = 1$, then $\sin\alpha = \frac{y}{r} = \frac{y}{1} = y$. This may be done by making α a central angle in a circle of radius 1, and drawing triangles of reference as follows:

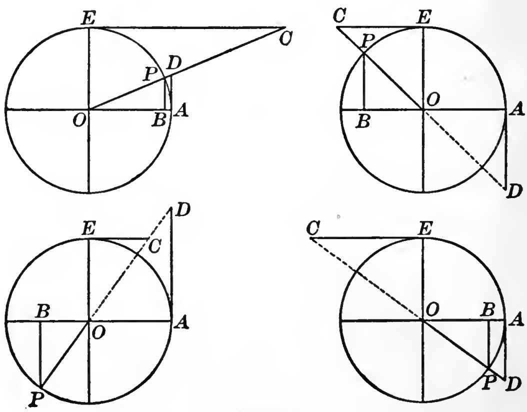


FIG. 19.

In all the figures $AOP = \alpha$, and

$$\sin\alpha = \frac{BP}{OP} = \frac{BP}{1} = BP,$$

$$\cos\alpha = \frac{OB}{OP} = \frac{OB}{1} = OB,$$

$$\tan\alpha = \frac{BP}{OB} = \frac{AD}{OA} = \frac{AD}{1} = AD,$$

$$\cot\alpha = \frac{OA}{AD} = \frac{EC}{OE} = \frac{EC}{1} = EC,$$

$$\sec\alpha = \frac{OP}{OB} = \frac{OD}{OA} = \frac{OD}{1} = OD,$$

$$\csc\alpha = \frac{OP}{BP} = \frac{OC}{OE} = \frac{OC}{1} = OC.$$

It appears then that, by taking a radius 1,

sine is represented by the perpendicular to the initial line, drawn from that line to the terminus of the arc subtending the given angle;

cosine is represented by the line from the vertex of the angle to the foot of the sine;

tangent is represented by the geometrical tangent drawn from the origin of the arc to the terminal line, produced if necessary;

cotangent is represented by the geometrical tangent drawn from a point 90° from the origin of the arc to the terminal line, produced if necessary;

secant is represented by the terminal line, or the terminal line produced, from the origin to its intersection with the tangent line;

cosecant is represented by the terminal line, or the terminal line produced, from the origin to its intersection with the cotangent line.

These lines are not the functions, but in triangles drawn as explained their lengths are equal to the numerical values of the functions, and in this sense the lines may be said to *represent* the functions. It will be noticed also that their directions indicate the *signs* of the functions. Let the student by means of these representations verify the results of Arts. 24 and 25.

II. *When α is a variable angle.*

Take XX' and YY' as axes of reference, and let angle units be measured along the x-axis, and values of the functions parallel to the y-axis, as in Art. 3. We may write corresponding values of the angle and the functions thus:

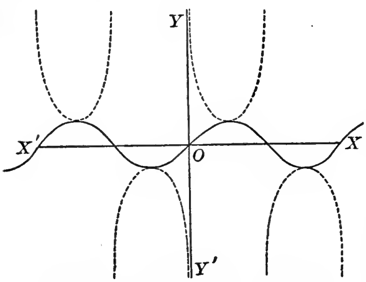
$$\alpha = 0°,\ 30°,\ 45°,\ 60°,\ 90°,\ 120°,\ 135°,\ 150°,\ 180°,\ 210°,\ 225°,$$

$$\sin\alpha = 0,\ \tfrac{1}{2},\ \tfrac{1}{2}\sqrt{2},\ \tfrac{1}{2}\sqrt{3},\ 1,\ \tfrac{1}{2}\sqrt{3},\ \tfrac{1}{2}\sqrt{2},\ \tfrac{1}{2},\ 0,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\sqrt{2},$$

$$\alpha = 240°,\ 270°,\ 300°,\ 315°,\ 330°,\ 360°,\ -30°,\ -45°,\ -60°,\ -90°, \text{ etc.},$$

$$\sin\alpha = -\tfrac{1}{2}\sqrt{3},\ -1,\ -\tfrac{1}{2}\sqrt{3},\ -\tfrac{1}{2}\sqrt{2},\ -\tfrac{1}{2},\ 0,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\sqrt{2},\ -\tfrac{1}{2}\sqrt{3},\ -1, \text{ etc.}$$

These values will be sufficient to determine the form of the curve representing the function. By taking angles between those above, and computing the values of the function, as given in mathematical tables, the form of the curve can be determined to any required degree of accuracy. Reducing the above fractions to decimals, it will be convenient to make the y-units large in comparison with the x-units. In the figure one x-unit represents 15°, and one y-unit 0.25.



Curves of Sine and Cosecant.

Sine ————

Cosecant - - - - - -

FIG. 20.

Measuring the angle values along the x-axis, and from these points of division measuring the corresponding values of $\sin\alpha$ parallel to the y-axis, as in Art. 3, we have, approximately,

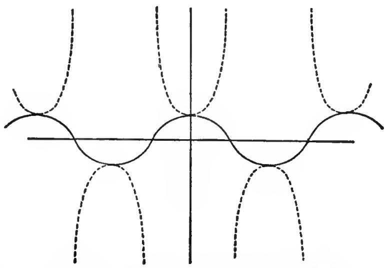
$$OX_1 = 30° = 2 \text{ units}, \qquad OX_2 = 45° = 3 \text{ units},$$
$$X_1Y_1 = \tfrac{1}{2} = 2 \text{ units}, \qquad X_2Y_2 = 0.71 = 2.84 \text{ units},$$

$$OX_3 = 60° = 4 \text{ units, etc.},$$
$$X_3Y_3 = 0.86 = 3.44 \text{ units, etc.}$$

We have now only to draw through the points Y_1, Y_2, Y_3, etc., thus determined, a continuous curve, and we have the *sine-curve* or *sinusoid*.

The dotted curve in the figure is the *cosecant curve*. Let the student compute values, as above, and draw the curve.

In like manner draw the *cosine* and *secant* curves, as follows:



Curves of Cosine and Secant.
Cosine ———
Secant ---------

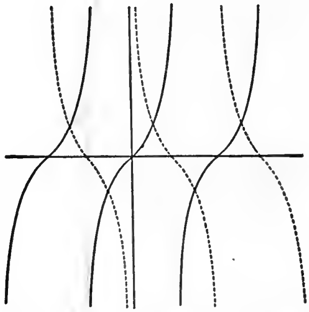
FIG. 21.

Tangent curve. Compute values for the angle α and for $\tan \alpha$, as before:

$$\alpha = 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°,$$
$$\tan \alpha = 0, \tfrac{1}{3}\sqrt{3}, 1, \sqrt{3}, \pm\infty, -\sqrt{3}, -1, -\tfrac{1}{3}\sqrt{3}, 0, \tfrac{1}{3}\sqrt{3}, 1, \sqrt{3}, \pm\infty,$$

$$\alpha = -30°, -45°, -60°, -90°, \text{ etc.},$$
$$\tan \alpha = -\tfrac{1}{3}\sqrt{3}, -1, -\sqrt{3}, \pm\infty, \text{ etc.}$$

Then lay off the values of α and of $\tan \alpha$ along the x, and parallel to the y-axis, respectively. It will be noted that, as α approaches 90°, tan α increases to ∞, and when α passes 90°, tan α is negative. Hence the value is measured parallel



Curves of Tangent and Cotangent.
Tangent ———
Cotangent-----------

FIG. 22.

to the y-axis downward, thus giving a discontinuous curve, as in the figure.

* **27**. The following principles are illustrated by the curves:

1. The sine and cosine are continuous for varying values of the angle, and lie within the limits $+1$ and -1. Sine changes sign as the angle passes through 180°, 360°, ⋯, $n\,180°$, while cosine changes sign as the angle passes through 90°, 270°, ⋯, $(2n+1)\ 90°$. Tangent and cotangent are discontinuous, the one as the angle approaches 90°, 270°, ⋯, $(2n+1)\ 90°$, the other as the angle approaches 180°, 360°, ⋯, $n\,180°$, and each changes sign as the angle passes through these values. The limiting values of tangent and cotangent are $+\infty$ and $-\infty$.

2. A line parallel to the y-axis cuts any of the curves in but one point, showing that for any value of α there is but one value of any function of α. But a line parallel to the x-axis cuts any of the curves in an indefinite number of points, if at all, showing that for any value of the function there are an indefinite number of values, if any, of α.

3. The curves afford an excellent illustration of the variations in sign and value of the functions, as α varies from 0 to 360°, as discussed in Art. 25. Let the student trace these changes.

4. From the curves it is evident that the functions are *periodic*, *i.e.* each increase of the angle through 360° in the case of the sine and cosine, or through 180° in the case of the tangent and cotangent, produces a portion of the curve like that produced by the first variation of the angle within those limits.

5. The difference in rapidity of change of the functions at different values of α is important, and reference will be made to this in computations of triangles. (See Art. 64, Case III.) A glance at the curves shows that sine is changing in value rapidly at 0°, 180°, etc., while near 90°, 270°, etc., the rate of change is slow. But cosine has a slow rate of change at 0°, 180°, etc., and a rapid rate at 90°, 270°, etc. Tangent and cotangent change rapidly throughout.

Ex. Let the student discuss secant and cosecant curves.

ORAL WORK.

1. Express in radians 180°, 120°, 45°; in degrees, $\frac{11}{7}$ radians, 2π, $\frac{2}{3}\pi$, $\frac{3}{2}\pi$.

2. If $\frac{1}{3}$ of a right angle be the unit, what is the measure of $\frac{1}{2}$ of a right angle? of 90°? of 135°?

3. Which is greater, $\cos 30°$ or $\frac{1}{2}\cos 60°$? $\tan\frac{\pi}{6}$ or $\cot\frac{\pi}{3}$? $\sin\frac{\pi}{4}$ or $\cos\frac{\pi}{4}$?

4. Express $\sin\alpha$ in terms of $\sec\alpha$; of $\tan\alpha$; $\tan\alpha$ in terms of $\cos\alpha$; of $\sec\alpha$.

5. Given $\sin\alpha = \frac{3}{5}$, find $\tan\alpha$. If $\tan\alpha = 1$, find $\sin\alpha$, $\csc\alpha$, $\cot\alpha$; also $\tan 2\alpha$, $\sin 2\alpha$, $\cos 2\alpha$.

6. If $\cos\alpha = \frac{1}{2}$, find $\sin\frac{\alpha}{2}$, $\tan\frac{\alpha}{2}$.

7. In what quadrant is angle t, if both $\sin t$ and $\cos t$ are minus? if $\sin t$ is plus and $\cos t$ minus? if $\tan t$ and $\cot t$ are both minus? if $\sin t$ and $\csc t$ are of the same sign? Why?

8. Of the numbers 3, $\frac{4}{5}$, -5, $-\frac{1}{3}$, a, $-b$, ∞, 0, which may be a value of $\sin p$? of $\sec p$? of $\tan p$? Why?

EXAMPLES.

1. If $\sin 26° 40' = 0.44880$, find, correct to 0.00001, the cosine and tangent.

2. If $\tan \alpha = \sqrt{3}$, and $\cot \beta = \frac{1}{3}\sqrt{3}$, find $\sin \alpha \cos \beta - \cos \alpha \sin \beta$.

3. Evaluate $\dfrac{\sin 30° \cot 30° - \cos 60° \tan 60°}{\sin 90° \cos 0°}$.

Prove the identities:

4. $\tan A(1 - \cot^2 A) + \cot A(1 - \tan^2 A) = 0$.

5. $(\sin A + \sec A)^2 + (\cos A + \csc A)^2 = (1 + \sec A \csc A)^2$.

6. $\sin^2 x \cos x \csc x - \cos^3 x \csc x \sin^2 x + \cos^4 x \sec x \sin x = \sin^3 x \cos x + \cos^3 x \sin x$.

7. $\tan^2 w + \cot^2 w = \sec^2 w \csc^2 w - 2$.

8. $\sec^2 v + \cos^2 v = 2 + \tan^2 v \sin^2 v$.

9. $\cos^2 t + 1 = 2\cos^3 t \sec t + \sin^2 t$.

10. $\csc^2 t - \sec^2 t = \cos^2 t \csc^2 t - \sin^2 t \sec^2 t$.

11. The sine of an angle is $\dfrac{m^2 - n^2}{m^2 + n^2}$; find the other functions.

12. If $\tan A + \sin A = m$, $\tan A - \sin A = n$, prove $m^2 - n^2 = 4\sqrt{mn}$.

Solve for one function of the angle involved the equations:

13. $\sin \theta + 2\cos \theta = 1$.

14. $\dfrac{\cos \alpha}{\tan \alpha} = \dfrac{3}{2}$.

15. $\sqrt{3}\csc^2 \theta = 4\cot \theta$.

16. $2\sin^2 x + \cos x - 1 = 0$.

17. $\sec^2 x - 7\tan x - 9 = 0$.

18. $3\csc y + 10\cot y - 35 = 0$.

19. $\sin^2 v - \frac{7}{3}\cos v - 1 = 0$.

20. $a\sec^2 w + b\tan w + c - a = 0$.

21. If $\dfrac{\sin A}{\sin B} = \sqrt{2}$, $\dfrac{\tan A}{\tan B} = \sqrt{3}$, find A and B.

22. Find to five decimal places the arc which subtends the angle of 1° at the centre of a circle whose radius is 4000 miles.

23. If $\csc A = \frac{2}{3}\sqrt{3}$, find the other functions, when A lies between $\dfrac{\pi}{2}$ and π.

24. In each of two triangles the angles are in G. P. The least angle of one of them is three times the least angle of the other, and the sum of the greatest angles is 240°. Find the circular measure of each of the angles.

CHAPTER III.

FUNCTIONS OF ANY ANGLE—INVERSE FUNCTIONS.

28. By an examination of the figure of Art. 24 it is seen that all the fundamental relations between the functions hold true for any value of α. The table of Art. 16 expresses the functions of α, whatever be its magnitude, in terms of each of the other functions of that angle if the $\pm$ sign be prefixed to the radicals.

The definitions of the trigonometric functions (Art. 12) apply to angles of any size and sign, but it is always possible to express the functions of any angle in terms of the functions of a *positive acute* angle.

The functions of any angle θ, greater than 360°, are the same as those of $\theta \pm n \cdot 360°$, since θ and $\theta \pm n \cdot 360°$ have the same triangle of reference. Thus the functions of 390°, or of 750°, are the same as the functions of $390° - 360°$, or of $750° - 2 \cdot 360°$, *i.e.* of 30°, as is at once seen by drawing a figure. So also the functions of $-315°$, or of $-675°$ are the same as those of $-315° + 360°$, or of $-675° + 2 \cdot 360°$, *i.e.* of 45°.

For functions of angles less than 360° the relations of this chapter are important.

29. *To find the relations of the functions of* $-\theta$, $90° \pm \theta$, $180° \pm \theta$, *and* $270° \pm \theta$ *to the functions of* θ, θ *being any angle.*

Four sets of figures are drawn, I for θ an acute angle, II for θ obtuse, III for θ an angle of the third quadrant, and IV for θ an angle of the fourth quadrant.

In every case generate the angles forming the compound angles separately, *i.e.* turn the revolving line first through

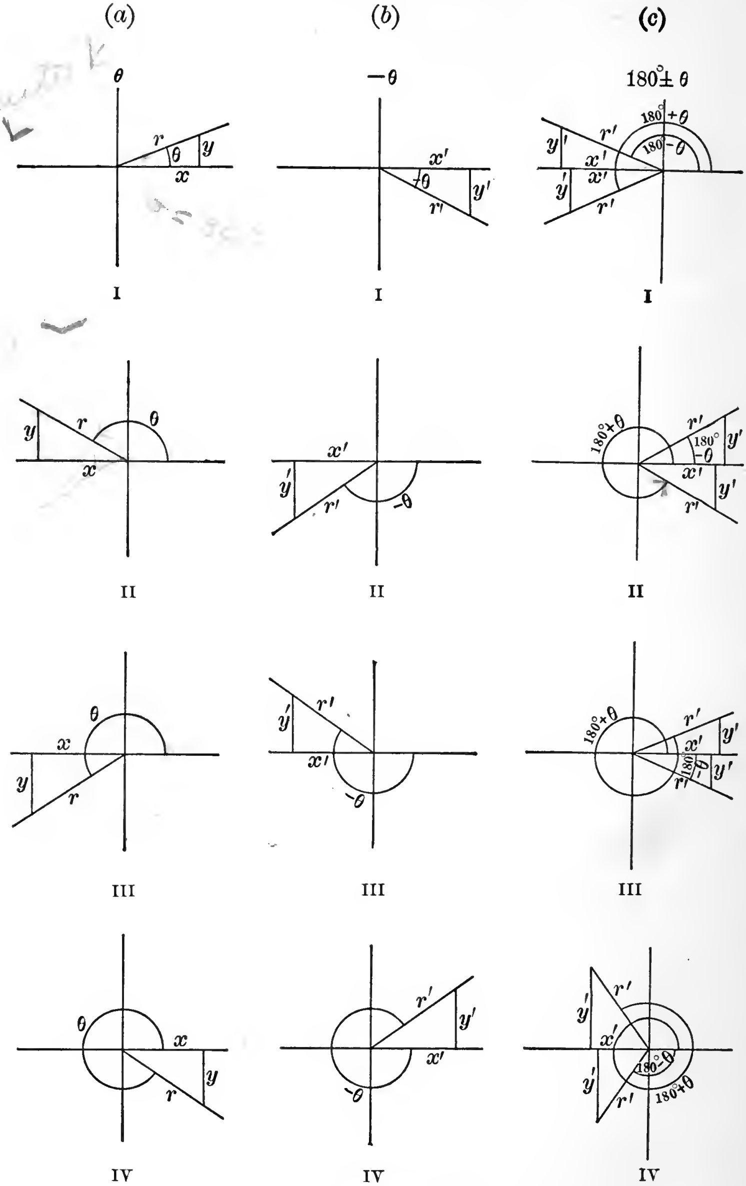


Fig. 23.

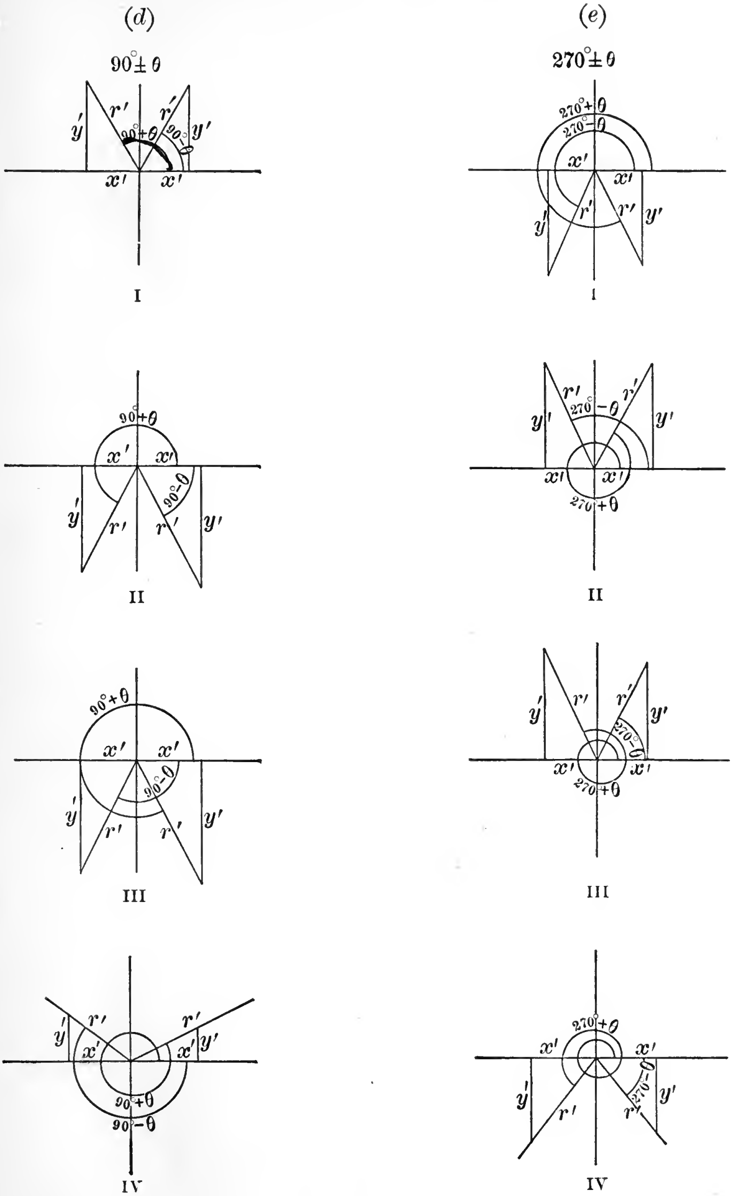


Fig. 23.

0°, 90°, 180°, or 270°, and then from this position through θ, or $-\theta$, as the case may be. Form the triangles of reference for (*a*) the angle θ, (*b*) $-\theta$, (*c*) $180° \pm \theta$, (*d*) $90° \pm \theta$, (*e*) $270° \pm \theta$.

The triangles of reference (*a*), (*b*), (*c*), (*d*), and (*e*), in each of the four sets of figures, I, II, III, IV, are similar, being mutually equiangular, since all have a right angle and one acute angle equal each to each. Hence the sides x, y, r of the triangles (*a*) are homologous to x', y', r' of the corresponding triangles (*b*) and (*c*), but to y', x', r', of the corresponding triangles (*d*) and (*e*). For the sides x of triangle (*a*) and x' of the triangles (*b*) and (*c*) are opposite equal angles, and hence are homologous, but the sides y' are opposite this same angle in triangles (*d*) and (*e*), and therefore sides y' of (*d*) and (*e*) are homologous to x of (*a*).

Attending to the signs of x and x', y and y' in the similar triangles (*a*) and (*b*),

$$\sin(-\theta) = \frac{y'}{r'} = -\frac{y}{r} = -\sin\theta,$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta.$$

Also in the similar triangles (*a*) and (*c*),

$$\sin(180° - \theta) = \frac{y'}{r'} = \frac{y}{r} = \sin\theta,$$

$$\cos(180° - \theta) = \frac{x'}{r'} = -\frac{x}{r} = -\cos\theta,$$

$$\tan(180° - \theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta.$$

In like manner show that

$$\sin(180° + \theta) = -\sin\theta,$$

$$\cos(180° + \theta) = -\cos\theta,$$

$$\tan(180° + \theta) = \tan\theta.$$

Again, in the similar triangles (*a*) and (*d*),

$$\sin(90° + \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\cos(90° + \theta) = \frac{x'}{r'} = -\frac{y}{r} = -\sin\theta,$$

$$\tan(90° + \theta) = \frac{y'}{x'} = -\frac{x}{y} = -\cot\theta.$$

Show that

$$\sin(90° - \theta) = \cos\theta,$$

$$\cos(90° - \theta) = \sin\theta,$$

$$\tan(90° - \theta) = \cot\theta.$$

Finally, from the similar triangles (*a*) and (*e*), show that

$$\sin(270° \pm \theta) = -\cos\theta,$$

$$\cos(270° \pm \theta) = \pm\sin\theta,$$

$$\tan(270° \pm \theta) = \mp\cot\theta.$$

From the reciprocal relations the student can at once write the corresponding relations for secant, cosecant, and cotangent.

30. Since in each of the four cases x', y' of triangles (*b*) and (*c*) are homologous to x, y of triangle (*a*), while x', y' of the triangles (*d*) and (*e*) are homologous to y, x of triangle (*a*), we may express the relations of the last article thus:

The functions of $\begin{cases} \pm\theta \\ 180° \pm \theta \end{cases}$ *correspond to the same functions of* θ*, while those of* $\begin{cases} 90° \pm \theta \\ 270° \pm \theta \end{cases}$ *correspond to the co-functions of* θ*, due attention being paid to the signs.*

The student can readily determine the sign in any given case, whether θ be acute or obtuse, by considering in what quadrant the compound angle, $90° \pm \theta$, $180° \pm \theta$, etc., would lie if θ were an acute angle, and prefixing to the corresponding functions of θ the signs of the respective functions for an angle in that quadrant. Thus $90° + \theta$, if θ be acute, is an angle of the second quadrant, so that sine and cosecant are plus, the other functions minus. It will be seen that $\sin(90° + \theta) = +\cos\theta$, $\cos(90° + \theta) = -\sin\theta$, etc., and this will be true whatever be the magnitude of θ. It will assist in fixing in the memory these important relations to notice that when in the compound angle θ is measured from the y-axis, as in $90° \pm \theta$, $270° \pm \theta$, the functions of one angle correspond to the co-functions of the other, but when in the compound angle θ is measured from the x-axis, as in $\pm\theta$, $180° \pm \theta$, then the functions of one angle correspond to the same functions of the other.

These relations, as has been noted in Art. 28, can be extended to angles greater than 360°, and it may be stated generally that

$$\text{function } \theta = \pm \text{ function } (2n \cdot 90° \pm \theta),$$

$$\text{function } \theta = \pm \text{ co-function } [(2n+1)\, 90° \pm \theta].$$

Computation tables contain angles less than 90° only. The chief utility of the above relations will be the reduction of functions of angles greater than 90° to functions of acute angles. Thus, to find $\tan 130° 20'$, look in the tables for $\cot 40° 20'$, or for $\tan 49° 40'$. Why?

Ex. **1.** What angles less than 360° have the same numerical cosine as 20°?

$$\cos 20° = -\cos(180° \pm 20°) = \cos(360° - 20°).$$

$\therefore$ 200°, 160°, 340° have the same cosine numerically as 20°.

2. Find the functions of 135°; of 210°.

$$\sin 135° = \sin(90° + 45°) = \cos 45° = \tfrac{1}{2}\sqrt{2},$$

$$\cos 135° = \cos(180° - 45°) = -\cos 45° = -\tfrac{1}{2}\sqrt{2}, \text{ etc.}$$

$$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\tfrac{1}{2}.$$

Let the student give the other functions for each angle.

ORAL WORK.

1. Determine the sine and tangent of each of the following angles: 30°, 120°, – 30°, – 60°, $\frac{5}{6}\pi$, $2\frac{2}{3}\pi$, – 135°, – π.

2. Which is the greater, sin 30° or sin(– 30°)? tan 135° or tan 45°? cos 60° or cos(– 60°)? sin 22° 30′ or cos 67° 30′?

3. What positive angle has the same tangent as $\frac{\pi}{3}$? the same sine as 50°?

4. If $\tan\theta = -1$, find $\sin\theta$.

5. Find sin 510°, cos(– 60°), tan 150°.

6. Reduce in two ways to functions of a positive acute angle, cos 122° tan 140° 30′, sin(– 60°).

7. Find all positive values of x, less than 360°, satisfying the following equations: $\cos x = \cos 45°$, $\sin 2x = \sin 10°$, $\tan 3x = \tan 60°$, $\sin x = \sin 30°$, $\tan x = \tan 135°$.

8. What angles are determined when (*a*) sine and cosine are +? (*b*) cotangent and sine are –? (*c*) sine + and cosine –? (*d*) cosine – and cotangent +?

INVERSE FUNCTIONS.

31. That a is the sine of an angle θ may be expressed in two ways, viz., $\sin\theta = a$, or, inversely, $\theta = \sin^{-1} a$, the latter being read, *θ equals an angle whose sine is a*, or, more briefly, *θ is the anti-sine of a.*

The notation $\sin^{-1} a$, $\cos^{-1} a$, $\tan^{-1} a$, etc., is not a fortunate one, but is so generally accepted that a change is not probable. The symbol may have been suggested from the fact that if $ax = b$, then $x = a^{-1}b$, whence, by analogy, if $\sin\theta = a$, $\theta = \sin^{-1} a$. But the likeness is an analogy only, for there is no similarity in *meaning*. $\sin^{-1} a$ *is an angle* θ, where $\sin\theta = a$, and is entirely different from $(\sin a)^{-1} = \frac{1}{\sin a}$. In Europe the symbols arc sin a, arc cos a, etc., are employed.

32. Principal value. We have found that in $\sin\theta = a$, for any value of θ, a can have but one value; but in $\theta = \sin^{-1} a$, for any value of a there are an indefinite number of values of θ (Art. 27, 2).

Thus, when $\sin\theta = a$, if $a = \frac{1}{2}$, θ may be 30°, 150°, 390°, 510°, – 330°, etc., or, in general, $n\pi + (-1)^n 30°$.

In the solution of problems involving inverse functions, the numerically least of these angles, called the *principal value*, is always used; *i.e.* we understand that $\sin^{-1} a$, $\tan^{-1} a$, are angles between $+90°$ and $-90°$, while the limits of $\cos^{-1} a$ are $0°$ and $180°$.

Thus, $\sin^{-1}\frac{1}{2} = 30°$, $\sin^{-1}(-\frac{1}{2}) = -30°$, $\cos^{-1}\frac{1}{2} = 60°$, $\cos^{-1}(-\frac{1}{2}) = 120°$.

ORAL WORK.

How many degrees in each of the following angles? How many radians?

1. $\cos^{-1}\frac{\sqrt{3}}{2}$?
2. $\tan^{-1} 1$?
3. $\cot^{-1}(-\sqrt{3})$?
4. $\sin^{-1}(-\frac{1}{2}\sqrt{2})$?
5. $\cos^{-1}(-\frac{1}{2}\sqrt{2})$?
6. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$?
7. $\tan^{-1}\sqrt{3}$?
8. $\cos^{-1} 0$?
9. $\sin^{-1} 1$?
10. $\tan^{-1} 0$?
11. $\tan^{-1}(-1)$?
12. $\sin^{-1}(-1)$?

Find the values of the functions:

13. $\sin(\tan^{-1}\frac{1}{3}\sqrt{3})$.
14. $\tan(\cos^{-1} 1)$.
15. $\tan(\cot^{-1}[-\infty])$.
16. $\cos(\tan^{-1}\infty)$.
17. $\sin(\sin^{-1}\frac{1}{2}\sqrt{2})$.
18. $\tan(\tan^{-1} x)$.
19. $\cos(\sin^{-1} 0)$.
20. $\sin(\cos^{-1}[-1])$.
21. $\cos(\cot^{-1}\sqrt{3})$.
22. $\tan(\sin^{-1}[-1])$.
23. $\sin(\tan^{-1}[-1])$.

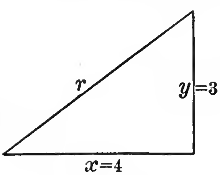


FIG. 24.

Ex. 1. Construct $\cot^{-1}\frac{4}{3}$.

Construct the right triangle xyr, so that $x = 4$, $y = 3$, whence angle $xr = \cot^{-1}\frac{4}{3}$.

2. Find $\cos(\tan^{-1}\frac{8}{15})$.

Let $\theta = \tan^{-1}\frac{8}{15}$, whence

$$\tan\theta = \tfrac{8}{15}, \text{ and } \cos\theta = \tfrac{15}{17}.$$

$$\therefore \cos\theta = \cos(\tan^{-1}\tfrac{8}{15}) = \tfrac{15}{17}.$$

3. If $\theta = \csc^{-1} a$, prove $\theta = \cos^{-1}\frac{\sqrt{a^2-1}}{a}$.

$$\csc\theta = a; \ \therefore \sin\theta = \frac{1}{a},$$

and
$$\cos\theta = \sqrt{1 - \frac{1}{a^2}} = \frac{\sqrt{a^2-1}}{a}, \text{ or } \theta = \cos^{-1}\frac{\sqrt{a^2-1}}{a}.$$

EXAMPLES.

1. Construct $\sin^{-1}\frac{2}{3}$, $\tan^{-1}\frac{5}{12}$, $\cos^{-1}(-\frac{1}{2})$.

2. Find $\tan(\sin^{-1}\frac{5}{13})$, $\sin(\tan^{-1}\frac{5}{12})$.

3. If $\theta = \sin^{-1} a$, prove $\theta = \tan^{-1}\dfrac{a}{\sqrt{1-a^2}}$.

4. Show that $\sin^{-1} a = 90° - \cos^{-1} a$.

5. Prove $\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3} = \dfrac{\pi}{2}$.

6. Prove $\tan^{-1}\left(\sin\dfrac{\pi}{2}\right) = \cos^{-1}\frac{1}{2}\sqrt{2}$.

7. What angles, less than 360°, have the same tangent numerically as 10°?

8. Given $\tan 143° 22' = -0.74357$; find, correct to 0.00001, sine and cosine.

9. If $\cot^2(90° + \beta) + \csc(90° - \beta) - 1 = 0$, find $\tan\beta$.

10. Find all positive values of x, less than 360°, when $\sin x = \sin 22° 30'$; when $\tan 2x = \tan 60°$.

11. When is $\sin x = \dfrac{a^2 + b^2}{2ab}$ possible, and when impossible? a = b

12. Verify $\sin^{-1}\frac{1}{2} + \cos^{-1}\dfrac{\sqrt{3}}{2} + \tan^{-1}\sqrt{3} = \sin^{-1}\dfrac{\sqrt{3}}{2}$.

13. What values of x will satisfy $\sin^{-1}(x^2 - x) = 30°$?

14. If $\tan^2\theta - \sec^2\alpha = 1$, prove $\sec\theta + \tan^3\theta\csc\theta = (3 + \tan^2\alpha)^{\frac{3}{2}}$.

15. Prove $\sin A(1 + \tan A) + \cos A(1 + \cot A) = \sec A + \csc A$.

16. Solve the simultaneous equations:

$$\sin^{-1}(2x + 3y) = 30° \text{ and } 3x + 2y = 2.$$

17. Verify (a) $\tan 60° = \sqrt{\dfrac{1 - \cos 120°}{1 + \cos 120°}}$.

(b) $\cos 60° = \dfrac{1 - \tan^2 30°}{1 + \tan^2 30°}$.

(c) $2\sin^2 60° = 1 - \cos 120°$.

18. Show that the cosine of the complement of $\dfrac{\pi}{6}$ equals the sine of the supplement of $\dfrac{\pi}{6}$.

37 1 9 38 3 Even Page 71

REVIEW.

Before leaving a problem the student should review and master all principles involved.

1. Construct $\cos^{-1}\frac{8}{17}$; $\sin^{-1}(-\frac{3}{4})$; $\tan^{-1}2$.

2. Find $\cos(\sin^{-1}\frac{3}{5})$; $\tan(\cos^{-1}[-\frac{1}{2}])$.

3. Prove $\cot^{-1}a = \cos^{-1}\dfrac{a}{\sqrt{1+a^2}}$.

4. Given $\alpha = \cot^{-1}\frac{3}{5}$, find $\tan\alpha + \sin(90^\circ + \alpha)$.

5. Find $\tan\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\dfrac{\sqrt{3}}{2}\right)$.

6. State the fundamental relations between the trigonometric functions in terms of the inverse functions. Thus, Page 11

$$\sin^{-1}a = \csc^{-1}\frac{1}{a},\quad \sin^{-1}a = \cos^{-1}\sqrt{1-a^2}, \text{ etc.}$$

7. Find all the angles, less than 360°, whose cosine equals sin 120°.

8. Given $\cot^{-1}2.8449$, find the sine and cosine of the angle, correct to 0.0001.

9. If $\tan^2(180^\circ - \theta) - \sec(180^\circ + \theta) = 5$, find $\cos\theta$.

10. If $\sin\theta = \frac{2}{3}$, find $\dfrac{\tan^2\theta + \cos^2\theta}{\tan^2\theta - \cos^2\theta}$.

11. Is $\sin x - 2\cos x + 3\sin x - 6 = 0$ a possible equation?

12. Verify (*a*) $\sin 60^\circ = \dfrac{2\tan 30^\circ}{1+\tan^2 30^\circ}$.

(*b*) $2\cos^2 60^\circ = 1 + \cos 120^\circ$.

(*c*) $\cos 60^\circ - \cos 90^\circ = 2\cos^2 30^\circ - 2\cos^2 45^\circ$.

13. If $\sin x = \dfrac{a(a+2b)}{a^2+2ab+2b^2}$, find $\sec x$ and $\tan x$.

14. Prove $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} + \dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta} = 2\csc\theta$.

15. Prove

$$\cos 45^\circ + \cos 135^\circ + \cos 30^\circ + \cos 150^\circ - \cos 210^\circ + \cos 270^\circ = \sin 60^\circ.$$

16. If $\tan\theta = \dfrac{b}{\sqrt{a^2-b^2}}$, prove that

$$\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta) - \sec\theta = \frac{a}{b}.$$

17. Solve $\sin^2 x + \sin^2(x+90^\circ) + \sin^2(x+180^\circ) = 1$.

18. Given $\cos^2 \alpha = m \sin \alpha - n$, find $\sin \alpha$.

19. If $\sin^2 \beta = \frac{3}{2 \sec \beta}$, find β.

20. Given $\tan 238° = 1.6$, find $\sin 148°$.

21. Prove $\tan^{-1} m + \cot^{-1} m = 90°$. angles being acute.

22. Find $\sin (\sin^{-1} p + \cos^{-1} p)$.

23. Solve $\cot^2 \theta (2 \csc \theta - 3) + 3 (\csc \theta - 1) = 0$.

24. Prove $\sin^2 \alpha \sec^2 \beta + \tan^2 \beta \cos^2 \alpha = \sin^2 \alpha + \tan^2 \beta$.

25. Prove $\cos^6 V + \sin^6 V = 1 - 3 \sin^2 V + 3 \sin^4 V$.

26. What values of A satisfy $\sin 2A = \cos 3A$?

27. If $\tan C = \frac{\sqrt{1 - m^2}}{m}$, and $\tan D = \sqrt{\frac{1 - \cos C}{1 + \cos C}}$, find $\tan D$ in terms of m.

28. If $\sin x - \cos x + 4 \cos^2 x = 2$, find $\tan x$; $\sec x$.

29. Does the value of $\sec x$, derived from $\sec^2 x = \frac{1 - 2 \cos^2 x}{1 - \cos^2 x}$, give a possible value of x?

30. Prove

$$[\cot (90° - A) - \tan (90° + A)] [\sin (180° - A) \sin (90° + A)] = 1.$$

31. Prove $(1 + \sin A)^2 [\cot A + 2 \sec A (1 - \csc A)] + \csc A \cos^3 A = 0$.

32. Given $\sin x = m \sin y$, and $\tan x = n \tan y$, find $\cos x$ and $\cos y$.

33. Given $\cot 201° = 2.6$, find $\cos 111°$.

34. Find the value of

$$\cos^{-1} \tfrac{1}{2} + \sin^{-1} \tfrac{1}{2}\sqrt{2} + \csc^{-1}(-1) + \tan^{-1} 1 - 2 \cot^{-1} \sqrt{3}.$$

35. Solve $2 \cos^2 \theta + 11 \sin \theta - 7 = 0$.

36. Prove

$$\cos^2 B + \cos^2 (B + 90°) + \cos^2 (B + 180°) + \cos^2 (B + 270°) = 2.$$

CHAPTER IV.

COMPUTATION TABLES.

33. Natural functions. It has been noted that the trigonometric functions of angles are *numbers*, but the values were found for only a few angles, viz. 0°, 30°, 45°, 60°, 90°, etc. In computations, however, it is necessary to know the values of the functions of any angle, and tables have been prepared giving the numerical values of the functions of all angles between 0° and 90° to every minute. In these tables the functions of any given angle, and conversely the angle corresponding to any given function, can be found to any required degree of accuracy; *e.g.* by looking in the tables we find $\sin 24° 26' = 0.41363$, and also $1.6415 = \tan 58° 39'$. These numbers are called the *natural functions*, as distinguished from their logarithms, which are called the *logarithmic functions* of the angles.

Ex. **1.** Find from the tables of natural functions:

$\sin 35° 14'$; $\cos 54° 46'$; $\tan 78° 29'$; $\cos 112° 58'$; $\sin 135°$.

2. Find the angles less than 180° corresponding to:

$\sin^{-1} 0.37865$; $\cos^{-1} 0.37865$; $\tan^{-1} 0.58670$; $\cos^{-1} 0.00291$; $\sin^{-1} 0.99999$.

34. Logarithms. The arithmetical processes of multiplication, division, involution, and evolution, are greatly abridged by the use of tables of logarithms of numbers and of the trigonometric ratios, which are numbers. The principles involved are illustrated in the following table:

Write in parallel columns a geometrical progression having the ratio 2, and an arithmetical progression having the difference 1, as follows:

G. P.	A. P.
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
2048	11
4096	12
8192	13
16384	14
32768	15
65536	16
131072	17
262144	18
524288	19
1048576	20

It will be perceived that the numbers in the second column are the indices of the powers of 2 producing the corresponding numbers in the first column, thus: $2^6 = 64$, $2^{11} = 2048$, $2^{18} = 262144$, etc. The use of such a table will be illustrated by examples.

Ex. **1.** Multiply 8192 by 128.

From the table, $8192 = 2^{13}$, $128 = 2^7$. Then by actual multiplication, $8192 \times 128 = 1048576$, or by the law of indices, $2^{13} \times 2^7 = 2^{20} = 1048576$ (from table).

Notice that the simple operation of addition is substituted for multiplication by adding the numbers in the second column opposite the given factors in the first column. This sum corresponds to the number in the first column which is the required product.

2. Divide 16384 by 512.

$16384 \div 512 = 32$, which corresponds to the result obtained by use of the table, or $2^{14} \div 2^9 = 2^5 = 32$. The operation of subtraction takes the place of division.

3. Find $\sqrt[6]{262144}$.

$$\sqrt[6]{262144} = \sqrt[6]{2^{18}} = 2^{\frac{18}{6}} = 2^3 = 8.$$

In the table, 262144 is opposite 18. $18 \div 6 = 3$, which is opposite 8, the required root; *i.e.* simple division takes the place of the tedious process of evolution.

4. Cube 64.

5. Multiply 256 by 4096.

6. Find $\sqrt[5]{32768}$.

7. Divide 1048576 by 32768.

35. The above table can be made as complete as desired by continually inserting between successive numbers in the first column the geometrical mean, and between the opposite numbers in the second, the arithmetical mean, but in practice logarithms are computed by other methods. The numbers in the second column are called the *logarithms* of the numbers opposite in the first column. 2 is called the *base* of this system, so that the *logarithm of a number is the exponent by which the base is affected to produce the number.*

Thus, the logarithm of 512 to the base 2 is 9, since $2^9 = 512$.

Logarithms were invented by a Scotchman, John Napier, early in the seventeenth century, but his method of constructing tables was different from the above. See *Encyc. Brit.*, art. "*Logarithms*," for an exceedingly interesting account. De Morgan says that by the aid of logarithms the labor of computing has been reduced for the mathematician to about one-tenth part of the previous expense of time and labor, while Laplace has said that John Napier, by the invention of logarithms, lengthened the life of the astronomer by one-half.

Columns similar to those above might be formed with any other number as base. For practical purposes, however, 10 is always taken as the base of the system, called the *common system*, in distinction from the *natural system*, of which the base is 2.71828 ⋯, the value of the exponential series (*Higher Algebra*). The natural system is used in theoretical discussions. It follows that *common logarithms* are *indices*, *positive or negative*, *of the powers of 10.*

Thus, $10^3 = 1000$; *i.e.* $\log 1000 = 3$;

$$10^{-2} = \frac{1}{10^2} = 0.01; \text{ i.e. } \log 0.01 = -2.$$

36. Characteristic and mantissa. Clearly most numbers are not integral powers of 10. Thus 300 is more than the second and less than the third power of 10, so that

$$\log 300 = 2 \text{ plus a decimal.}$$

Evidently the logarithms of numbers generally consist of an integral and a decimal part, called respectively the *characteristic* and the *mantissa* of the logarithms.

37. Characteristic law. The characteristic of the logarithm of a number is *independent* of the digits composing the number, but *depends* on the position of the decimal point, and *is found by counting the number of places the first significant figure in the number is removed from the units' place, being positive or negative according as the first significant* *figure is at the left or the right of units' place.* This follows from the fact that common logarithms are indices of powers of 10, and that 10^n, n being a positive integer, contains $n+1$ places, while 10^{-n} contains $n-1$ zeros at the right of units' place. Thus in 146.043 the first significant figure is two places at the left of units' place; the characteristic of log 146.043 is therefore 2. In 0.00379 the first significant digit is three places at the right of units' place, and the characteristic of log 0.00379 is -3.

To avoid the use of negative characteristics, such characteristics are increased by 10, and -10 is written after the logarithm. Thus, instead of $\log 0.00811 = \bar{3}.90902$, write $7.90902 - 10$. In practice the -10 is generally not written, but it must always be remembered and accounted for in the result.

Ex. Determine the characteristic of the logarithm of:

1; 46; 0.009; 14796.4; 230.001; $10^5 \times 76$; 0.525; 1.03; 0.000426.

38. Mantissa law. The mantissa of the logarithm of a number is *independent* of the position of the decimal point, but *depends* on the digits composing the number, *is always positive*, and *is found* in the tables.

For, moving the decimal point multiplies or divides a number by an integral power of 10, *i.e.* adds to or subtracts from the logarithm an integer, and hence does not affect the mantissa. Thus,

$$\log 225.67 = \log 225.67,$$
$$\log 2256.7 = \log 225.67 \times 10^1 = \log 225.67 + 1,$$
$$\log 22567.0 = \log 225.67 \times 10^2 = \log 225.67 + 2,$$
$$\log 22.567 = \log 225.67 \times 10^{-1} = \log 225.67 + (-1),$$
$$\log 0.22567 = \log 225.67 \times 10^{-3} = \log 225.67 + (-3),$$

so that the mantissæ of the logarithms of all numbers composed of the digits 22567 in that order are the same, .35347. Moving the decimal point affects the characteristic only. *The student must remember that the mantissa is always positive.*

Log 0.0022567 is never written $-3+.35347$, but $\bar{3}.35347$, the minus sign being written above to indicate that the characteristic alone is negative. In computations negative characteristics are avoided by adding and subtracting 10, as has been explained.

39. We may now define the *logarithm of a number as the index of the power to which a fixed number, called the base, must be raised to produce the given number.*

Thus, $a^x = b$, and $x = \log_a b$ (where $\log_a b$ is read logarithm of b to the base a) are equivalent expressions. The relation between base, logarithm, and number is always

$$(\text{base})^{\log} = \text{number}.$$

To illustrate: $\log_2 8 = 3$ is the same as $2^3 = 8$; $\log_3 81 = 4$ and $3^4 = 81$ are equivalent expressions; and so are $\log_{10} 1000 = 3$ and $10^3 = 1000$, and $\log_{10} 0.001 = -3$ and $10^{-3} = 0.001$.

Find the value of:

$$\log_4 64;\ \log_5 125;\ \log_3 243;\ \log_a (a)^{\frac{4}{3}};\ \log_{27} 3;\ \log_x 1.$$

40. From the definition it follows that the laws of indices apply to logarithms, and we have:

I. *The logarithm of a product equals the sum of the logarithms of the factors.*

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

III. *The logarithm of a power equals the index of the power times the logarithm of the number.*

IV. *The logarithm of a root equals the logarithm of the number divided by the index of the root.*

For if $a^x = n$ and $a^y = m$,

then $n \times m = a^{x+y}, \quad \therefore \log nm = x + y = \log n + \log m;$

and $n \div m = a^{x-y}, \quad \therefore \log \dfrac{n}{m} = x - y = \log n - \log m;$

also $n^r = (a^x)^r = a^{rx}, \quad \therefore \log n^r = rx = r \times \log n;$

finally, $\sqrt[r]{n} = \sqrt[r]{a^x} = a^{\frac{x}{r}}, \quad \therefore \log \sqrt[r]{n} = \dfrac{x}{r} = \dfrac{1}{r} \log n.$

EXAMPLES.

Given log 2 = 0.30103, log 3 = 0.47712, log 5 = 0.69897, find:

1. $\log 4$.
2. $\log 6$.
3. $\log 10$.
4. $\log 9$.
5. $\log 25$.
6. $\log \sqrt{3}$.
7. $\log 15^3$.
8. $\log \frac{8}{3}$.
9. $\log 15 \times 9$.
10. $\log \sqrt{\frac{72}{25}}$.
11. $\log \sqrt{\frac{9^2 \times 5^3}{2^4 \times 10}}$.

USE OF TABLES.

41. To find the logarithm of a number.

First. Find the characteristic, as in Art. 37.

Second. Find the mantissa in the tables, thus:

(*a*) When the number consists of not more than four figures.

In the column N of the tables find the first three figures, and in the row N the fourth figure of the number. The mantissa of the logarithm will be found in the row opposite the first three figures and in the column of the fourth figure.

Illustration. Find log 42.38.

The characteristic is 1. (Why?)

In the table in column N find the figures 423, and on the same page in row N the figure 8. The last three figures of the mantissa, 716, lie at the intersection of column 8 and row 423. To make the tables more compact the first two figures of the mantissa, 62, are printed in column 0 only. Then log 42.38 = 1.62716.

Find
$$\log 0.8734 = \bar{1}.94121,$$
$$\log 3.5 = \log 3.500 = 0.54407,$$
$$\log 36350 = 4.56050.$$

(*b*) When the number consists of more than four figures.

Find the mantissa of the logarithm of the number composed of the first four figures as above. To correct for the remaining figures we *interpolate* by means of the *principle of proportional parts*, according to which *it is assumed that, for differences small as compared with the numbers, the differences* *between several numbers are proportional to the differences between their logarithms.*

The theorem is only approximately correct, but its use leads to results accurate enough for ordinary computations.

Ex. **1.** To find log 89.4562.

As above, mantissa of $\log 894500 = 0.95158$,

mantissa of $\log 894600 = 0.95163$,

$\therefore \log 894600 - \log 894500 = 0.00005$, *called the tabular difference.*

Let $\log 894562 - \log 894500 = x$ hundred-thousandths.

Now, by the principle of proportional parts,

$$\frac{\log 894562 - \log 894500}{\log 894600 - \log 894500} = \frac{894562 - 894500}{894600 - 894500},$$

or $\frac{x}{5} = \frac{62}{100}$, whence $x = .62$ of $5 = 3.1$

$$\therefore \log 89.4562 = 1.95158 + 0.00003 = 1.95161,$$

all figures after the fifth place being rejected in five-place tables. If, however, the sixth place be 5 or more, it is the practice to add 1 to the figure in the fifth place. Thus, if $x = 0.0000456$, we should call it 0.00005, and add 5 to the mantissa.

2. Find log 537.0643.

To interpolate we have $x : 9 = 643 : 1000$, *i.e.* $x = 5.787$;

$$\therefore \log 537.0643 = 2.72997 + 0.00006.$$

3. Find $\log 0.0168342 = \bar{2}.22619$.

4. Find $\log 39642.7 = 4.59816$.

42. To find the number corresponding to a given logarithm.

The characteristic of the logarithm determines the position of the decimal point (Art. 37).

(*a*) If the mantissa is in the tables, the required number is found at once.

Ex. **1.** Find $\log^{-1} 1.94621$ (read, the number whose logarithm is 1.94621).

The mantissa is found in the tables at the intersection of row 883 and column 5.

$$\therefore \log^{-1} 1.94621 = 88.35,$$

the characteristic 1 showing that there are two integral places.

(*b*) If the exact mantissa of the given logarithm is not in the tables, the first four figures of the corresponding number are found, and to these are annexed figures found by interpolating by means of the principle of proportional parts, as follows:

Find the two successive mantissæ between which the given mantissa lies. Then, by the principle of proportional parts, the amount to be added to the four figures already found is such a part of 1 as the difference between the successive mantissæ is of the difference between the smaller of them and the given mantissa.

2. Find $\log^{-1} 1.43764$.

$$\begin{array}{rr} \text{Mantissa of } \log 2740 = & 0.43775 \\ \text{of } \log 2739 = & 0.43759 \\ \hline \text{Differences} \quad 1 & 16 \end{array}$$

$$\begin{array}{rr} \text{Mantissa of log required number} = & 0.43764 \\ \text{of } \log 2739 = & 0.43759 \\ \hline \text{Differences} \quad x & 5 \end{array}$$

By p. p. $\quad x : 1 = 5 : 16$ and $x = \frac{5}{16} = 0.3125$.

Annexing these figures, $\log^{-1} 1.43764 = 27.3931+$.

3. Find $\log^{-1} \bar{1}.48762$.

The differences in logarithms are 14, 6.

$$\therefore x = \frac{6}{14} = .428+,$$

$$\text{and } \log^{-1} \bar{1}.48762 = 0.307343+.$$

4. Find $\log 891.59$; $\log 0.023$; $\log \frac{1}{2}$; $\log 0.1867$; $\log \sqrt{2}$.

5. Find $\log^{-1} 2.21042$; $\log^{-1} 0.55115$; $\log^{-1} \bar{1}.89003$.

43. Logarithms of trigonometric functions. These might be found by first taking from the tables the natural functions of the given angle, and then the logarithms of these numbers. It is more expeditious, however, to use tables showing directly the logarithms of the functions of angles less than 90° to every minute. Functions of angles greater than 90° are reduced to functions of angles less than 90° by the formulæ of Art. 29. To make the work correct for seconds, or any fractional part of a minute, interpolation is necessary by the principle of proportional parts, thus :

Ex. 1. Find $\log \sin 28° 32' 21''$.

In the table of logarithms of trigonometric functions, find 28° at the top of the page, and in the minute column at the left find 32′. Then under log sin column find $\log \sin 28° 32' = 9.67913 - 10$

$$\log \sin 28° 33 = 9.67936 - 10$$

Differences 1′ 23

By p. p. $x : 23 = 21'' : 60''$, *i.e.* $x = \frac{21}{60} \times 23 = 8.05.$

$$\therefore \log \sin 28° 32' 21'' = 9.67913 + 0.00008 - 10$$
$$= 9.67921 - 10.$$

Whenever functions of angles are less than unity, *i.e.* are decimals (as sine and cosine always are, except when equal to unity, and as tangent is for angles less than 45°), the characteristic of the logarithm will be negative, and, accordingly, 10 is always added in the tables, and it must be remembered that 10 is to be subtracted. Thus, in the example above, the characteristic of the logarithm is not 9, but $\bar{1}$, and the logarithm is not 9.67913, as written in the tables, but 9.67913 − 10.

2. Find $\log \cos 67° 27' 50''$.

In the table of logarithms at the foot of the page, find 67°, and in the minute column at the right, 27′. Then computing the difference as above, $x = 25$.

But it must be noted that cosine decreases as the angle increases toward 90°. Hence, $\log \cos 67° 27' 50''$ is less than $\log \cos 67° 27'$, *i.e.* the difference 25 must be subtracted, so that

$$\log \cos 67° 27' 50'' = 9.58375 - 0.00025 - 10$$
$$= 9.58350 - 10.$$

44. To find the angle when the logarithm is given, find the successive logarithms between which the given logarithm lies, compute by the principle of proportional parts the seconds, and add them to the less of the two angles corresponding to the successive logarithms. This will not necessarily be the angle corresponding to the less of the two logarithms; for, as has been seen, the number, and, therefore, the logarithm, may decrease as the angle increases.

Ex. 1. Find the angle whose log tan is 9.88091.

$$\begin{aligned} \log\tan 37°\,14' &= 9.88079-10 \\ \log\tan 37°\,15' &= 9.88105-10 \end{aligned}$$

Differences $\quad 60'' \qquad 26$

$$\begin{aligned} \log\tan 37°\,14' &= 9.88079-10 \\ \log\tan \text{angle required} &= 9.88091-10 \end{aligned}$$

Differences $\quad x'' \qquad 12$

$\therefore\ x:60=12:26$, or $x''=\frac{12}{26}\times 60''=28''$, approximately, and the angle is $37°\,14'\,28''$.

2. Find the angle whose $\log\cos = 9.82348$.

We find $x=\frac{6}{14}\times 60''=26''$, and the angle is $48°\,14'\,26''$.

3. Show that $\log\cos\ 25°\,31'\,20''=9.95541$;

$\log\sin 110°\,25'\,20''=9.97181$;

$\log\tan\ 49°\,52'\,10''=0.07417$.

4. Show that the angle whose log tan is 9.92501 is $40°\,4'\,39''$; whose log sin is 9.88365 is $49°\,54'\,18''$; whose log cos is 9.50828 is $71°\,11'\,49''$.

45. Cologarithms. In examples involving multiplications and divisions it is more convenient, if n is any divisor, to add $\log\frac{1}{n}$ than to subtract $\log n$. The logarithm of $\frac{1}{n}$ is called the cologarithm of n. Since

$$\log\frac{1}{n}=\log 1-\log n=0-\log n,$$

it follows that $\operatorname{colog} n=-\log n$, *i.e.* $\log n$ subtracted from zero. To avoid negative results, add and subtract 10.

Ex. 1. Find colog 2963.

$$\begin{aligned} \log 1 &= 10.00000-10 \\ \log 2963 &= \ \ 3.47173 \\ \therefore\ \operatorname{colog} 2963 &= \ \ 6.52827-10 \end{aligned}$$

2. Find colog tan 16° 17′.

$$\begin{aligned} \log 1 &= 10.00000-10 \\ \log\tan 16°\,17' &= \ \ 9.46554-10 \\ \therefore\ \operatorname{colog}\tan 16°\,17' &= \ \ 0.53446 \end{aligned}$$

By means of the definitions of the trigonometric functions, the parts of a right triangle may be computed if any two parts, one of them being a side, are given. Thus,

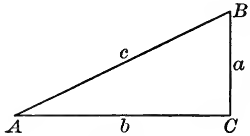


FIG. 25.

given a and A in the rt. triangle ABC.

Then $c = a \div \sin A,\ b = a \div \tan A,$

and $B = 90° - A.$

Again, if a and b are given, then

$\tan A = \frac{a}{b},\ c = a \div \sin A,$ and $B = 90° - A.$

3. Given $c = 25.643,\ B = 37°\ 25'\ 20''$, compute the other parts.

$$A = 90° - 37°\,25'\,20'' = 52°\ 34'\ 40''.$$

$$\begin{array}{rlcrl} a &= c \cos B. & \qquad & b &= a \tan B. \\ \log c &= 1.40897 & & \log a &= 1.30889 \\ \log \cos B &= 9.89992 & & \log \tan B &= 9.88376 \\ \hline \log a &= 1.30889 & & \log b &= 1.19265 \\ \therefore\ a &= 20.365. & & \therefore\ b &= 15.583. \end{array}$$

Check: $c^2 = a^2 + b^2 = 20.365^2 + 15.583^2 = 657.57 = 25.643^2.$

4. Given $b = 0.356,\ B = 63°\ 28'\ 40''$, compute the other parts.

$$A = 26°\ 31'\ 20''.$$

$$\begin{array}{rlcrl} c &= \dfrac{b}{\sin B}. & \qquad & a &= \dfrac{b}{\tan B}. \\ \log b &= 9.55145 & & \log b &= 9.55145 \\ \text{colog} \sin B &= 0.04829 & & \text{colog} \tan B &= 9.69816 \\ \hline \log c &= 9.59974 & & \log a &= 9.24961 \\ c &= 0.3979 & & a &= 0.1777 \end{array}$$

Check: $c^2 - a^2 = 0.1583 - 0.03157 = 0.12673 = b^2.$

EXAMPLES.

Compute the other parts:

1. Given $a = 9.325,\ A = 43°\ 22'\ 35''$.

2. Given $c = 240.32,\ a = 174.6$.

3. Given $B = 76°\ 14'\ 23'',\ a = 147.53$.

4. Given $a = 2789.42,\ b = 4632.19$.

5. Given $c = 0.0213,\ A = 23°\ 14''$.

6. Given $b = 2,\ c = 3$.

CHAPTER V.

APPLICATIONS.

46. Many problems in measurements of heights and distances may be solved by applying the preceding principles. By means of instruments certain distances and angles may be measured, and from the data thus determined other distances and angles computed. The most common instruments are the *chain*, the *transit*, and the *compass.*

The *chain* is used to measure distances. Two kinds are in use, the *engineer's chain* and the *Gunter's chain.* They each contain 100 links, each link in the engineer's chain being 12 inches long, and in the Gunter's 7.92 inches.

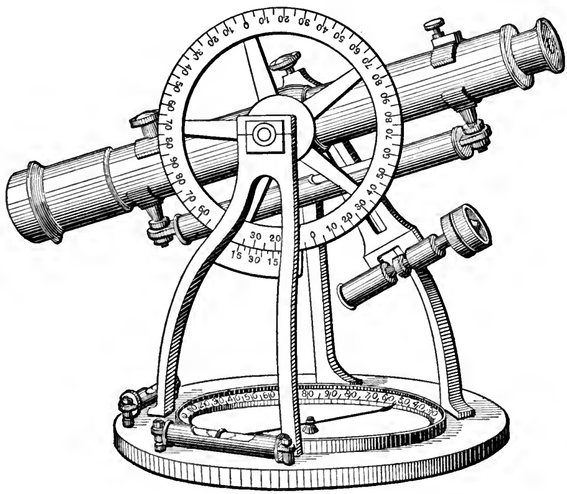


Fig. 26.

The *transit* is the instrument most used to measure horizontal angles, and with certain attachments to measure vertical angles. The figure shows the form of the instrument.

The *mariner's compass* is used to determine the directions, or *bearings*, of objects at sea. Each quadrant is divided into 8 parts, making the 32 points of the compass, so that each point contains $11° 15'$.

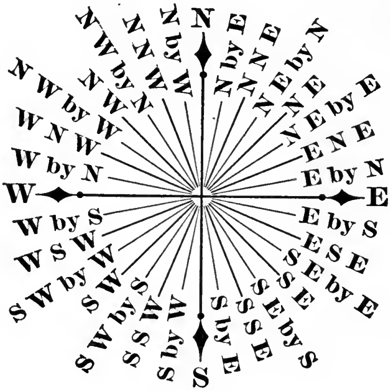


Fig. 27.

47. The angle between the horizontal plane and the line of vision from the eye to the object is called the *angle of elevation*, or of *depression*, according as the object is above or below the observer.

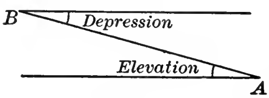


Fig. 28.

It is evident that the elevation angle of B, as seen from A, is equal to the depression angle of A, as seen from B, so that in the solution of examples the two angles are interchangeable.

PROBLEMS.

48. Some of the more common problems met with in practice are illustrated by the following:

To find the height of an object when the foot is accessible.

The distance BC, and the elevation angle B are measured, and x is determined from the relation $x = BC \tan B$.

Fig. 29.

Ex. 1. The elevation angle of a cliff measured from a point 300 ft. from its base is found to be 30°. How high is the cliff?

$$BC = 300,\ B = 30°.$$

Then $$x = 300 \cdot \tan 30° = 300 \cdot \tfrac{1}{3}\sqrt{3} = 100\sqrt{3}.$$

2. From a point 175 ft. from the foot of a tree the elevation of the top is found to be 27° 19′. Find the height of the tree.

The problem may be solved by the use of natural functions, or of logarithms. The work should be arranged for the solution before the tables are opened. Let the student complete.

$$BC = 175. \quad B = 27° \, 19'.$$

Then $x = BC \tan B.$

$\log BC =$

$\log \tan B =$

$\log x =$

$\therefore x = 90.39.$

Or by natural functions,

$BC = 175$

$\tan B = 0.5165$

$\therefore x = 90.3875.$

To find the height of an object when the foot is inaccessible.

Measure BB', θ and θ'.

Then $$x = \frac{BC}{\cot\theta} = \frac{BB' + B'C}{\cot\theta}.$$

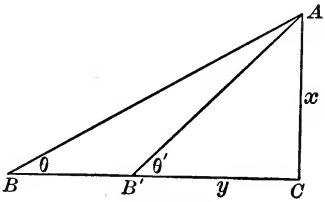


FIG. 30.

But $B'C = x \cot\theta'$, whence substituting,

$$x = \frac{BB'}{\cot\theta - \cot\theta'},$$

which is best solved by the use of the natural functions of θ and θ'.

3. Measured from a certain point at its base the elevation of the peak of a mountain is 60°. At a distance of one mile directly from this point the elevation is 30°. Find the height of the mountain.

$$BB' = 5280 \text{ ft.},\quad \theta = 30°,\quad \theta' = 60°.$$

$$x = \frac{y + 5280}{\cot 30°}. \quad \text{But } y = x \cot 60°.$$

$$\therefore x = \frac{5280}{\cot 30° - \cot 60°} = 4572.48 \text{ ft.}$$

In surveying it is often necessary to make measurements across a stream or other obstacle too wide to be spanned by a single chain.

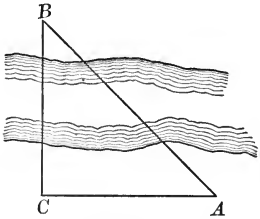


FIG. 31.

To find the distance from C to a point B on the opposite side of a stream.

At C measure a right angle, and take CA a convenient distance. Measure angle A, then

$$BC = CA \cdot \tan A.$$

4. Find CB when angle $A = 47° 16'$, and $CA = 250$ ft.

5. From a point due south of a kite its elevation is found to be $42° 30'$; from a point 20 yds. due west from this point the elevation is $36° 24'$. How high is the kite above the ground?

$$AB = x \cdot \cot 42° 30',$$

$$AC = x \cdot \cot 36° 24',$$

$$AC^2 - AB^2 = BC^2 = 400.$$

$$\therefore x^2 (\cot^2 36° 24' - \cot^2 42° 30') = 400,$$

whence

$$x^2 = \frac{400}{.6489}, \text{ and } x = \frac{20}{.805} = 24.84 \text{ yds.}$$

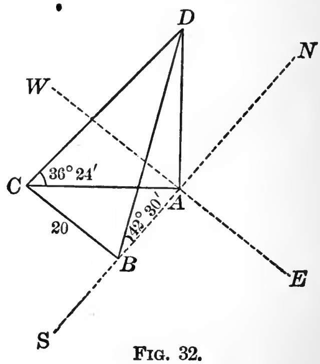


FIG. 32.

EXAMPLES.

1. What is the altitude of the sun when a tree 71.5 ft. high casts a shadow 37.75 ft. long?

2. What is the height of a balloon directly over Ann Arbor when its elevation at Ypsilanti, 8 miles away, is 10° 15′?

3. The Washington monument is 555 ft. high. How far apart are two observers who, from points due east, see the top of the monument at elevations of 23° 20′ and 47° 30′, respectively?

4. A mountain peak is observed from the base and top of a tower 200 ft. high. The elevation angles being 25° 30′ and 23° 15′, respectively, compute the height of the mountain above the base of the tower.

5. From a point in the street between two buildings the elevation angles of the tops of the buildings are 30° and 60°. On moving across the street 20 ft. toward the first building the elevation angles are found to be each 45°. Find the width of the street and the height of each building.

6. From the peak of a mountain two towns are observed due south. The first is seen at a depression of 48° 40′, and the second, 8 miles farther away and in the same horizontal plane, at a depression of 20° 50′. What is the height of the mountain above the plane?

7. A building 145 ft. long is observed from a point directly in front of one corner. The length of the building subtends $\tan^{-1} 3$, and the height $\tan^{-1} 2$. Find the height.

8. An inaccessible object is observed to lie due N.E. After the observer has moved S.E. 2 miles, the object lies N.N.E. Find the distance of the object from each point of observation.

9. Assuming the earth to be a sphere with a radius of 3963 miles, find the height of a lighthouse just visible from a point 15 miles distant at sea.

10. The angle of elevation of a tower 120 ft. high due north of an observer was 35°; what will be its angle of elevation from a point due west from the first point of observation 250 ft.? Also the distance of the observer from the base of the tower in each position?

11. A railway 5 miles long has a uniform grade of 2° 30′; find the rise per mile. What is the grade when the road rises 70 ft. in one mile?

(The grade depends on the tangent of the angle.)

12. The foot of a ladder is in the street at a point 30 ft. from the line of a building, and just reaches a window $22\frac{1}{2}$ ft. above the ground. By turning the ladder over it just reaches a window 36 ft. above the ground on the other side of the street. Find the breadth of the street.

13. From a point 200 ft. from the base of the Forefathers' monument at Plymouth, the base and summit of the statue of Faith are at an elevation of 12° 40′ 48″ and 22° 2′ 53″, respectively; find the height of the statue and of the pedestal on which it stands.

14. At a distance of 100 ft. measured in a horizontal plane from the foot of a tower, a flagstaff standing on the top of the tower subtends an angle of 8°, while the tower subtends an angle of 42° 20′. Find the length of the flagstaff.

15. The length of a string attached to a kite is 300 ft. The kite's elevation is 56° 6′. Find the height of the kite.

16. From two rocks at sea level, 50 ft. apart, the top of a cliff is observed in the same vertical plane with the rocks. The angles of elevation of the cliff from the two rocks are 24° 40′ and 32° 30′. What is the height of the cliff above the sea?

CHAPTER VI.

GENERAL FORMULÆ — TRIGONOMETRIC EQUATIONS AND IDENTITIES.

49. Thus far functions of single angles only have been considered. Relations will now be developed to express functions of angles which are sums, differences, multiples, or sub-multiples of single angles in terms of the functions of the single angles from which they are formed.

First it will be shown that,

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta,$$

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}.$$

The following cases must be considered:

1. α, β, $\alpha + \beta$ acute angles.
2. α, β, acute, but $\alpha + \beta$ an obtuse angle.
3. Either α, or β, or both, of any magnitude, positive or negative.

The figures apply to cases 1 and 2.

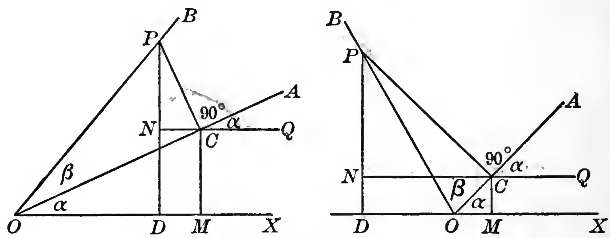


Fig. 33.

Let the terminal line revolve through the angle α, and then through the angle β, to the position OB, so that angle

$XOB = \alpha + \beta$. Through any point P in OB draw perpendiculars to the sides of α, DP and CP, and through C draw a perpendicular and a parallel to OX, MC and NC.

Then the angle $QCA = \alpha$ (why?), and CNP is the triangle of reference for angle $QCP = 90° + \alpha$.

CNP is sometimes treated as the triangle of reference for angle CPN. The fallacy of this appears when we develop $\cos(\alpha + \beta)$, in which PC would be treated as both plus and minus.

Now $\sin(\alpha+\beta) = \sin XOB = \dfrac{DP}{OP} = \dfrac{MC}{OP} + \dfrac{NP}{OP}$,

or expressing in trigonometric ratios,

$$= \frac{MC}{OC}\cdot\frac{OC}{OP} + \frac{NP}{CP}\cdot\frac{CP}{OP}$$

$$= \sin\alpha\cos\beta + \sin(90° + \alpha)\sin\beta.$$

Hence, since $\sin(90° + \alpha) = \cos\alpha$, we have

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

In like manner

$$\cos(\alpha+\beta) = \cos XOB = \frac{OD}{OP} = \frac{OM}{OP} + \frac{CN}{OP},$$

or expressing in trigonometric ratios,

$$= \frac{OM}{OC}\cdot\frac{OC}{OP} + \frac{CN}{CP}\cdot\frac{CP}{OP}$$

$$= \cos\alpha\cos\beta + \cos(90° + \alpha)\sin\beta.$$

And since $\cos(90° + \alpha) = -\sin\alpha$, we have

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

It will be noted that the wording of the demonstration applies to both figures, the only difference being that when $\alpha+\beta$ is obtuse OD is negative. CN is negative in each figure.

50. In the case, when α, or β, or both, are of any magnitude, positive or negative, figures may be constructed as before described by *drawing through any point in the terminal line of β a perpendicular to each side of α, and through the foot of the perpendicular on the terminal line of α a perpendicular and a parallel to the initial line of α.* Noting negative lines, the demonstrations already given will be found to apply for all values of α and β.

To make the proof complete by this method would require an unlimited number of figures, *e.g.* we might take α obtuse, both α and β obtuse, either or both greater than 180°, or than 360°, or negative angles, etc.

Instead of this, however, the generality of the proposition is more readily shown algebraically, as follows:

Let $\alpha' = 90° + \alpha$ be any obtuse angle, and α, β, acute angles.

Then

$$\begin{aligned}\sin(\alpha' + \beta) &= \sin(90° + \alpha + \beta) = \cos(\alpha + \beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \sin(90° + \alpha)\cos\beta + \cos(90° + \alpha)\sin\beta \text{(why?)}\\ &= \sin\alpha'\cos\beta + \cos\alpha'\sin\beta.\end{aligned}$$

In like manner, considering any obtuse angle $\beta' = 90° + \beta$, it can be shown that

$$\sin(\alpha' + \beta') = \sin\alpha'\cos\beta' + \cos\alpha'\sin\beta'.$$

Show that $\cos(\alpha' + \beta') = \cos\alpha'\cos\beta' - \sin\alpha'\sin\beta'$.

By further substitutions, *e.g.* $\alpha'' = 90° \pm \alpha'$, $\beta'' = 90° \pm \beta'$, etc., it is clear that the above relations hold for all values, positive or negative, of the angles α and β.

Since α and β may have any values, we may put $-\beta$ for β, and $\sin(\alpha + [-\beta])$

$$\begin{aligned}= \sin(\alpha - \beta) &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \text{ (why?)}.\end{aligned}$$

Also $$\begin{aligned}\cos(\alpha - \beta) &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

Finally,

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} \pm \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \mp \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

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ORAL WORK.

By the above formulæ develop:

1. $\sin(2A + 3B)$.

2. $\cos(90° - B)$.

3. $\tan(45° + \phi)$.

4. $\sin 2A = \sin(A + A)$.

5. $\cos 2\theta$.

6. $\tan(180° + C)$.

7. $\sin 90° = \sin(45° + 45°)$.

8. $\cos 90°$.

9. $\tan 90°$.

10. $\sin(90° + \beta + \gamma)$.

11. $\cos(270° - m - n)$.

12. $\tan(90° + m + n)$.

Ex. **1.** Find sin 75°.

$$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{1 + \sqrt{3}}{2\sqrt{2}} = 0.9659.$$

2. Find tan 15°.

$$\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3} = 0.2679.$$

3. Prove $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = 2.$

Combining, $\dfrac{\sin 3A \cos A - \cos 3A \sin A}{\sin A \cos A} = \dfrac{\sin(3A - A)}{\sin A \cos A}$

$$= \frac{\sin 2A}{\sin A \cos A} = \frac{\sin(A + A)}{\sin A \cos A} = \frac{\sin A \cos A + \cos A \sin A}{\sin A \cos A} = 2.$$

4. Prove $\tan^{-1} a + \tan^{-1} b = \tan^{-1} \dfrac{a + b}{1 - ab}.$

Let $\alpha = \tan^{-1} a,\ \beta = \tan^{-1} b,\ \gamma = \tan^{-1} \dfrac{a + b}{1 - ab}.$

Hence, $\tan\alpha = a,\ \tan\beta = b,\ \tan\gamma = \dfrac{a + b}{1 - ab}.$

Then $\alpha + \beta = \gamma$, and hence $\tan(\alpha + \beta) = \tan\gamma$.

Expanding, $\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \tan\gamma.$

Substituting, $\dfrac{a + b}{1 - ab} = \dfrac{a + b}{1 - ab}.$

EXAMPLES.

1. Find $\cos 15°$, $\tan 75°$.

2. Prove $\cot(\alpha \pm \beta) = \dfrac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$.

3. Prove geometrically $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$,
and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,

given
(*a*) α acute, β obtuse;
(*b*) α, β, obtuse;
(*c*) α, β, either, or both, negative angles.

4. Prove geometrically $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

Verify the formula by assigning values to α and β, and finding the values of the functions from the tables of natural tangents.

5. Prove $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$.

6. Show that $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.

7. Given $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{3}{4}$, find $\sin(\alpha+\beta)$

8. Given $\sin 280° = s$, find $\sin 170°$.

9. If $\alpha = 67° 22'$, $\beta = 128° 40'$, by use of the tables of natural functions verify the formulæ on page 56.

10. Prove $\tan^{-1}\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}} = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{a}$.

11. Prove $\tan^{-1}\dfrac{2x-b}{b\sqrt{3}} + \tan^{-1}\dfrac{2b-x}{x\sqrt{3}} = \tan^{-1}\sqrt{3}$.

12. Prove $\sec^{-1}\dfrac{a}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a}$.

13. If $\alpha+\beta=\omega$, prove $\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\omega = \sin^2\omega$.

14. Solve $\frac{1}{2}\sin\theta = 1 - \cos\theta$.

15. Prove $\sin(A+B)\cos A - \cos(A+B)\sin A = \sin B$.

16. Prove $\cos(A+B)\cos(A-B) + \sin(A+B)\sin(A-B) = \cos 2B$.

17. Prove $\sin(2\alpha-\beta)\cos(\alpha-2\beta)$
$- \cos(2\alpha-\beta)\sin(\alpha-2\beta) = \sin(\alpha+\beta)$.

18. Prove $\sin(n-1)\alpha\cos(n+1)\alpha + \cos(n-1)\alpha\sin(n+1)\alpha = \sin 2n\alpha$.

19. Prove $\sin(135° - \theta) + \cos(135° + \theta) = 0$.

20. Prove $1 - \tan^2\alpha\tan^2\beta = \dfrac{\cos^2\beta - \sin^2\alpha}{\cos^2\alpha\cos^2\beta}$.

21. Prove $\dfrac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha\tan\beta$.

22. $\tan^2\left(\dfrac{\pi}{4} - \alpha\right) = \dfrac{1 - 2\sin\alpha\cos\alpha}{1 + 2\sin\alpha\cos\alpha}$.

51. The following formulæ are very important and should be carefully memorized. They enable us to change sums and differences to products, *i.e.* to displace terms by factors.

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2},$$

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}.$$

Since $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$,

and $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$,

then $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$, (1)

and $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$. (2)

Also since $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,

and $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$,

then $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$, (3)

and $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$. (4)

Put $\alpha + \beta = \theta$

and $\alpha - \beta = \phi$

$$2\alpha = \theta + \phi, \text{ and } \alpha = \frac{\theta+\phi}{2},$$

$$2\beta = \theta - \phi, \text{ and } \beta = \frac{\theta-\phi}{2}.$$

Substituting in (1), (2), (3), (4), we have the above formulæ.

EXAMPLES.

1. Prove $\dfrac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta} = \tan\dfrac{3\theta}{2}$.

By formulæ of last article the first member becomes

$$\frac{2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}}{2\cos\frac{3\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{3\theta}{2}.$$

2. Prove $\dfrac{\sin\alpha + 2\sin 3\alpha + \sin 5\alpha}{\sin 3\alpha + 2\sin 5\alpha + \sin 7\alpha} = \dfrac{\sin 3\alpha}{\sin 5\alpha}$.

$$\frac{(\sin\alpha + \sin 5\alpha) + 2\sin 3\alpha}{(\sin 3\alpha + \sin 7\alpha) + 2\sin 5\alpha} = \frac{2\sin 3\alpha\cos 2\alpha + 2\sin 3\alpha}{2\sin 5\alpha\cos 2\alpha + 2\sin 5\alpha}$$

$$= \frac{(\cos 2\alpha + 1)\sin 3\alpha}{(\cos 2\alpha + 1)\sin 5\alpha} = \frac{\sin 3\alpha}{\sin 5\alpha}.$$

3. Prove $\dfrac{\sin(4A - 2B) + \sin(4B - 2A)}{\cos(4A - 2B) + \cos(4B - 2A)} = \tan(A + B)$.

$$\frac{2\sin\frac{4A - 2B + 4B - 2A}{2}\cos\frac{4A - 2B - 4B + 2A}{2}}{2\cos\frac{4A - 2B + 4B - 2A}{2}\cos\frac{4A - 2B - 4B + 2A}{2}}$$

$$= \frac{\sin(A + B)}{\cos(A + B)} = \tan(A + B).$$

4. Prove $\sin 50° - \sin 70° + \sin 10° = 0$.

$$2\cos\frac{50° + 70°}{2}\sin\frac{50° - 70°}{2} = 2\cos 60°\sin(-10°) = -\sin 10°.$$

5. Prove $\dfrac{\cos 2\alpha\cos 3\alpha - \cos 2\alpha\cos 7\alpha + \cos\alpha\cos 10\alpha}{\sin 4\alpha\sin 3\alpha - \sin 2\alpha\sin 5\alpha + \sin 4\alpha\sin 7\alpha} = \cot 6\alpha\cot 5\alpha$.

By (3) and (4), p. 61,

$$\frac{\cos 5\alpha + \cos\alpha - \cos 9\alpha - \cos 5\alpha + \cos 11\alpha + \cos 9\alpha}{\cos\alpha - \cos 7\alpha - \cos 3\alpha + \cos 7\alpha + \cos 3\alpha - \cos 11\alpha}$$

$$= \frac{\cos\alpha + \cos 11\alpha}{\cos\alpha - \cos 11\alpha} = \frac{2\cos 6\alpha\cos 5\alpha}{2\sin 6\alpha\sin 5\alpha} = \cot 6\alpha\cot 5\alpha.$$

ORAL WORK.

By the formulæ of Art. 51 transform:

6. $\cos 5\alpha + \cos\alpha$.

7. $\cos\alpha - \cos 5\alpha$.

8. $2\sin 3\theta\cos\theta$.

9. $\sin 2\alpha - \sin 4\alpha$.

10. $\cos 9\theta \cos 2\theta$.

11. $\sin\theta + \sin\frac{\theta}{2}$.

12. $\sin 75° \sin 15°$.

13. $\cos 7p - \cos 2p$.

14. $\cos(2p + 3q)\sin(2p - 3q)$.

15. $\sin\frac{3t}{2} - \sin\frac{t}{2}$.

16. $\cos(30° + 2\phi)\sin(30° - \phi)$.

17. $\sin(2r + s) + \sin(2r - s)$.

18. $\cos(2\beta - \alpha) - \cos 3\alpha$.

19. $\sin 36° + \sin 54°$.

20. $\cos 60° + \cos 20°$.

21. $\sin 30° + \cos 30°$.

Prove:

22. $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \tan\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$.

23. $\dfrac{\cos\alpha + \cos\beta}{\cos\beta - \cos\alpha} = \cot\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$.

24. $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan\dfrac{x+y}{2}$.

25. $\dfrac{\sin x - \sin y}{\cos x - \cos y} = -\cot\dfrac{x+y}{2}$.

26. $\cos 55° + \sin 25° = \sin 85°$.

Simplify:

27. $\dfrac{\sin B + \sin 2B + \sin 3B}{\cos B + \cos 2B + \cos 3B}$.

28. $\dfrac{\sin C - \sin 4C + \sin 7C - \sin 10C}{\cos C - \cos 4C + \cos 7C - \cos 10C}$.

52. *Functions of an angle in terms of those of the half angle.*

If in $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, $\alpha = \beta$,

then $\sin(\alpha + \alpha) = \sin 2\alpha = 2\sin\alpha\cos\alpha$.

In like manner

$$\cos(\alpha + \alpha) = \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= 2\cos^2\alpha - 1$$
$$= 1 - 2\sin^2\alpha;$$

and
$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$

ORAL WORK.

Ex. Express in terms of functions of half the given angles:

1. $\sin 4\alpha$.
2. $\cos 3p$.
3. $\tan 5t$.
4. $\cos x$.
5. $\sin\frac{\beta}{2}$.
6. $\sin(2p - q)$.
7. $\cos(30° + 2\phi)$.
8. $\sin(x + y)$.

9. From the functions of 30° find those of 60°; from the functions of 45°, those of 90°.

53. *Functions of an angle in terms of those of twice the angle.*

By Art. 52, $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2} - 1.$

$\therefore 2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha,$ and $2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha.$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}; \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}.$$

$$\therefore \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

Explain the significance of the ± sign before the radicals.

Express in terms of the double angle the functions of 120°; 50°; 90°, with proper signs prefixed.

Ex. 1. Express in terms of functions of twice the given angles each of the functions in Examples 1–8 above.

2. From the functions of 45° find those of 22° 30′; from the functions of 36°, those of 18° (see tables of natural functions).

3. Find the corresponding functions of twice and of half each of the following angles, and verify results by the tables of natural functions:

Given $\sin 26° 42' = 0.4493,$

$\tan 62° 24' = 1.9128,$

$\cos 21° 34' = 0.9300.$

4. Prove $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{x}{2}.$

5. $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}.$

6. If A, B, C are angles of a triangle, prove

$$\sin A + \sin C + \sin B = 4 \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

7. If $\cos^2 \alpha + \cos^2 2\alpha + \cos^2 3\alpha = 1$, then

$$\cos \alpha \cos 2\alpha \cos 3\alpha = 0.$$

8. Prove $\cot A - \cot 2A = \csc 2A$.

9. Prove $\dfrac{\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)} = \left[\dfrac{1 - \tan\frac{\phi}{2}}{1 + \tan\frac{\phi}{2}}\right]^2.$

10. $\dfrac{\tan \alpha}{\tan(\alpha + \phi)} = 1 - \dfrac{2 \sin \phi}{\sin(2\alpha + \phi) + \sin \phi}.$

11. If $y = \tan^{-1} \dfrac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}$, prove $x^2 = \sin 2y$.

12. Prove $\tan^{-1} \dfrac{\sqrt{1 + x^2} - 1}{x} + \tan^{-1} \dfrac{2x}{1 - x^2} = \dfrac{5}{2} \tan^{-1} x.$

13. If $y = \sin^{-1} \dfrac{x}{\sqrt{1 + x^2}}$, prove $x = \tan y$.

14. Prove $\cos^2 \alpha + \cos^2 \beta - 1 = \cos(\alpha + \beta) \cos(\alpha - \beta)$.

15. Prove $\sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2} = 2 \sin \dfrac{\alpha - \beta}{2}.$

16. Prove $\sin^{-1} \sqrt{\dfrac{x}{a + x}} = \tan^{-1} \sqrt{\dfrac{x}{a}} = \dfrac{1}{2} \cos^{-1} \dfrac{a - x}{a + x}.$

17. Prove $\cos^2 \theta - \cos^2 \phi = \sin(\phi + \theta) \sin(\phi - \theta)$.

18. Prove $\tan A + \tan(A + 120°) + \tan(A - 120°) = 3 \tan 3A$.

19. Prove $\tan \alpha - \tan \dfrac{\alpha}{2} = \tan \dfrac{\alpha}{2} \sec \alpha.$

20. $3 \tan^{-1} a = \tan^{-1} \dfrac{3a - a^3}{1 - 3a^2}.$

21. $\cos^2 3A (\tan^2 3A - \tan^2 A) = 8 \sin^2 A \cos 2A$.

22. $1 + \cos 2(A - B) \cos 2B = \cos^2 A + \cos^2 (A - 2B)$.

23. $\cot^2\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right) = \dfrac{2 \csc 2\theta - \sec \theta}{2 \csc 2\theta + \sec \theta}.$

TRIGONOMETRIC EQUATIONS AND IDENTITIES.

54. Identities. It was shown in Chapter I that

$$\sin^2\theta + \cos^2\theta = 1$$

is true for all values of θ, and in Chapter VI, that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

is true for all values of α and β. It may be shown that

$$\frac{\sin 2A}{1 + \cos 2A} = \tan A$$

is true for all values of A, thus:

$$\frac{\sin 2A}{1 + \cos 2A} = \frac{2\sin A\cos A}{1 + 2\cos^2 A - 1} \quad \text{(by trigonometric transformation)}$$

$$= \frac{\sin A}{\cos A} \quad \text{(by algebraic transformation)}$$

$$= \tan A \quad \text{(by trigonometric transformation).}$$

Such expressions are called *trigonometric identities. They are true for all values of the angles involved.*

55. Equations. The expression

$$2\cos^2\alpha - 3\cos\alpha + 1 = 0$$

is true for but two values of $\cos\alpha$, viz. $\cos\alpha = \frac{1}{2}$ and 1, *i.e.* the expression is true for $\alpha = 0°, 60°, 300°$, and for no other positive angles less than 360°. Such expressions are called *trigonometric equations. They are true only for particular values of the angles involved.*

56. Method of attack. The transformations necessary at any step in the proof of identities, or the solution of equations, are either *trigonometric*, or *algebraic; i.e.* in proving an identity, or solving an equation, the student must choose at each step to apply either some principles of algebra, or some trigonometric relations. If at any step no algebraic operation seems advantageous, then usually the expression should be simplified by endeavoring to state the *different functions* involved in terms of a *single function* of the angle, or if there are *multiple angles*, to reduce all to functions of a *single angle.*

$$\text{Transformations}\begin{cases}\text{Algebraic}\\[2ex] \text{Trigonometric, to change to a}\begin{cases}\text{Single function}\\ \text{Single angle}\end{cases}\end{cases}$$

No other transformations are needed, and the student will be greatly assisted by remembering that the ready solution of a trigonometric problem consists in wisely choosing at each step between the possible algebraic and trigonometric transformations. Problems involving trigonometric functions will in general be simplified by expressing them entirely in terms of sine and cosine.

EXAMPLES.

1. Prove $$\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2.$$

By algebra, $$\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = \frac{\sin 3A\cos A - \cos 3A\sin A}{\sin A\cos A}$$

by trigonometry, $$= \frac{\sin(3A - A)}{\sin A\cos A} = \frac{\sin 2A}{\sin A\cos A}$$

$$= \frac{2\sin A\cos A}{\sin A\cos A} = 2.$$

Or, by trigonometry,

$$\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = \frac{3\sin A - 4\sin^3 A}{\sin A} - \frac{4\cos^3 A - 3\cos A}{\cos A}$$

by algebra, $$= 3 - 4\sin^2 A - 4\cos^2 A + 3$$

$$= 6 - 4(\sin^2 A + \cos^2 A) = 2.$$

2. Prove $$\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\tan 8\theta}{\tan 2\theta}.$$

No algebraic operation simplifies. Two trigonometric changes are needed. 1. To change the functions to a single function, sine or cosine. 2. To change the angles to a single angle, $8A$, $4A$, or $2A$.

By trigonometry and algebra,

$$\frac{\dfrac{1-\cos 8\theta}{\cos 8\theta}}{\dfrac{1-\cos 4\theta}{\cos 4\theta}} = \frac{\dfrac{\sin 8\theta}{\cos 8\theta}}{\dfrac{\sin 2\theta}{\cos 2\theta}};$$

by algebra, $$\frac{\cos 4\theta(1-\cos 8\theta)}{1-\cos 4\theta} = \frac{\sin 8\theta \cos 2\theta}{\sin 2\theta};$$

by trigonometry,

$$\frac{\cos 4\theta(1-1+2\sin^2 4\theta)}{1-1+2\sin^2 2\theta} = \frac{2\sin 4\theta \cos 4\theta \cos 2\theta}{\sin 2\theta};$$

by algebra, $$\frac{\sin 4\theta}{\sin 2\theta} = 2\cos 2\theta;$$

and $$\sin 4\theta = 2\sin 2\theta \cos 2\theta,$$

which is a trigonometric identity.

3. Solve $2\cos^2\theta + 3\sin\theta = 0$.

By trigonometry, $2(1-\sin^2\theta) + 3\sin\theta = 0$,

a quadratic equation in $\sin\theta$.

By algebra, $$2\sin^2\theta - 3\sin\theta - 2 = 0,$$

and $$(\sin\theta - 2)(2\sin\theta + 1) = 0.$$

$$\therefore \sin\theta = 2, \text{ or } -\tfrac{1}{2}. \quad \text{Verify.}$$

The value 2 must be rejected. Why?

$\therefore \theta = 210°$, and $330°$ are the only positive values less than $360°$ that satisfy the equation.

4. Solve $\sec\theta - \tan\theta = 2$.

Here $\tan\theta = -0.75$, $\therefore$ from the tables of natural functions,

$$\theta = 143° 7' 48'', \text{ or } 323° 7' 48''.$$

Find $\sec\theta$, and verify.

5. Solve $2\sin\theta\sin 3\theta - \sin^2 2\theta = 0$.

By trigonometry, $\cos 2\theta - \cos 4\theta - \sin^2 2\theta = 0$,

also $$\cos 2\theta - \cos^2 2\theta + \sin^2 2\theta - \sin^2 2\theta = 0.$$

By algebra, $$\cos 2\theta(1-\cos 2\theta) = 0.$$

$$\therefore \cos 2\theta = 0 \text{ or } 1,$$

and $$2\theta = 90°, 270°, 0°, \text{ or } 360°,$$

whence $$\theta = 45°, 135°, 0°, \text{ or } 180°. \quad \textbf{Verify.}$$

Or, by trigonometry,

$$2 \sin \theta(3 \sin \theta - 4 \sin^3 \theta) - 4 \sin^2 \theta \cos^2 \theta = 0;$$

by trigonometry and algebra,

$$6 \sin^2 \theta - 8 \sin^4 \theta - 4 \sin^2 \theta + 4 \sin^4 \theta = 0;$$

by algebra, $$2 \sin^2 \theta - 4 \sin^4 \theta = 0,$$

and $$2 \sin^2 \theta(1 - 2 \sin^2 \theta) = 0.$$

$$\therefore \sin \theta = 0, \text{ or } \pm\sqrt{\tfrac{1}{2}},$$

and $$\theta = 0°, 180°, 45°, 135°, 225°, \text{ or } 315°.$$

The last two values do not appear in the first solution, because only angles less than 360° are considered, and the solution there gave values of 2θ, which in the last two cases would be 450° and 630°.

Solve: **6.** $\tan \theta = \cot \theta$. **8.** $2 \cos 2\theta - 2 \sin \theta = 1$.

7. $\sin^2 \theta + \cos \theta = 1$. **9.** $\sin 2\theta \cos \theta = \sin \theta$.

Prove: **10.** $2 \cot 2A = \cot A - \tan A$.

11. $\cos 2x + \cos 2y = 2 \cos(x + y) \cos(x - y)$.

12. $(\cos \alpha + \sin \alpha)^2 = 1 + \sin 2\alpha$.

57. Simultaneous trigonometric equations.

13. Solve $$\cos(x + y) + \cos(x - y) = 2,$$

$$\sin\frac{x}{2} + \sin\frac{y}{2} = 0.$$

By trigonometry,

$$\cos x \cos y - \sin x \sin y + \cos x \cos y + \sin x \sin y = 2,$$

so that $$\cos x \cos y = 1;$$

also, $$\sqrt{\frac{1 - \cos x}{2}} + \sqrt{\frac{1 - \cos y}{2}} = 0,$$

and $$\therefore \cos x = \cos y.$$

Substituting, $$\cos^2 x = 1,$$

$$\cos x = \pm 1.$$

$$\therefore x = 0°, \text{ or } 180°,$$

and $$y = x = 0°, \text{ or } 180°.$$ Verify.

14. Solve for R and F.

$$W - F\sin i - R\cos i = 0,$$
$$W + F\cos i - R\sin i = 0.$$

To eliminate F,

$$W\cos i - F\sin i\cos i - R\cos^2 i = 0,$$
$$W\sin i + F\cos i\sin i - R\sin^2 i = 0.$$

Adding, $$W(\sin i + \cos i) - R(\sin^2 i + \cos^2 i) = 0.$$

$$\therefore\ R = W(\sin i + \cos i).$$

Substituting, $W - F\sin i - W(\sin i + \cos i)\cos i = 0$

$$\therefore\ F = \frac{W - W(\sin i + \cos i)\cos i}{\sin i}.$$

If $W = 3$ tons, and $i = 22°\,30'$, compute F and R.

$$R = 3(0.3827 + 0.9239) = 3.9198.$$

$$F = \frac{3 - 3(0.3827 + 0.9239)0.9239}{0.3827} = -1.624.$$

Solve:

15. $472\cot\theta - 263\cot\phi = 490,\ 307\cot\theta - 379\cot\phi = 0.$

16. $\sin 2x + 1 = \cos x + 2\sin x.$

17. $\cos^2\theta + \sin\theta = 1.$

18. If $2h(\cos^2\theta - \sin^2\theta) - 2a\sin\theta\cos\theta + 2b\sin\theta\cos\theta = 0$, prove $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}.$

Prove:

19. $\tan y = (1 + \sec y)\tan\frac{y}{2}.$

20. $2\cot^{-1}x = \csc^{-1}\frac{1+x^2}{2x}.$

21. $\sin(\phi + 45°) + \sin(\phi + 135°) = \sqrt{2}\cos\phi.$

22. $\frac{\cos v + \cos 3v}{\cos 3v + \cos 5v} = \frac{1}{2\cos 2v - \sec 2v}.$

23. $\cos 3x - \sin 3x = (\cos x + \sin x)(1 - 2\sin 2x).$

Solve:

24. $\sin 2\theta + \sin\theta = \cos 2\theta + \cos\theta.$

25. $4\cos(\theta + 60°) - \sqrt{2} = \sqrt{6} - 4\cos(\theta + 30°).$

26. $\cot 2\theta = \tan\theta - 1.$

27. $\cos\theta + \cos 2\theta + \cos 3\theta = 0.$

28. $\sin 2x + \sqrt{3}\cos 2x = 1.$

29. $3\tan^2 p + 8\cos^2 p = 7.$

30. Determine for what relative values of P and W the following equation is true:

$$\cos^2\frac{y}{2} - \frac{P}{W}\cos\frac{y}{2} - \frac{1}{2} = 0.$$

31. Compute N from the equation $N + \frac{W}{3}\cos\alpha - \frac{W}{3}\sin\alpha - W\cos\alpha = 0$, when $W = 2000$ pounds and α satisfies the equation $2\sin\alpha = 1 + \cos\alpha$.

32. $\sin\theta - \tan\phi(\cos\theta + \sin\theta) = \cos\theta,\ \sin\theta - \tan\phi\cos\theta = 1.$

Prove:

33. $\cot(t + 15°) - \tan(t - 15°) = \dfrac{4\cos 2t}{2\sin 2t + 1}.$

34. $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13} = \sin^{-1}\frac{16}{65}.$

35. $\tan\left(\dfrac{\pi}{4} + \dfrac{\omega}{2}\right) = \sqrt{\dfrac{1 + \sin\omega}{1 - \sin\omega}}.$

36. $2\sin^{-1}\frac{1}{2} = \cos^{-1}\frac{1}{2}.$

37. If $\sin A$ is a geometric mean between $\sin B$ and $\cos B$, prove $\cos 2A = 2\sin(45 - B)\cos(45 + B)$.

38. Prove $\sin(\alpha + \beta + \gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma.$

Also find $\cos(\alpha + \beta + \gamma)$.

39. Prove $\tan(\alpha + \beta + \gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}.$

If α, β, and γ are angles of a triangle, prove

40. $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma.$

41. $\cot\dfrac{\alpha}{2} + \cot\dfrac{\beta}{2} + \cot\dfrac{\gamma}{2} = \cot\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}\cot\dfrac{\gamma}{2}.$

If $\alpha + \beta + \gamma = 90°$, prove

42. $\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1.$

Prove:

43. $\sin n\alpha = 2\sin(n-1)\alpha\cos\alpha - \sin(n-2)\alpha.$

44. $\cos n\alpha = 2\cos(n-1)\alpha\cos\alpha - \cos(n-2)\alpha.$

45. $\tan n\alpha = \dfrac{\tan(n-1)\alpha + \tan\alpha}{1 - \tan(n-1)\alpha\tan\alpha}.$

CHAPTER VII.

TRIANGLES.

58. In geometry it has been shown that a triangle is determined, except in the ambiguous case, if there are given any three independent parts, as follows:

I. Two angles and a side.

II. Two sides and an angle,

(*a*) the angle being included by the given sides,

(*b*) the angle being opposite one of the given sides (ambiguous case).

III. Three sides.

The angles of a triangle are not three *independent* parts, since they are connected by the relation $A + B + C = 180°$.

The three angles of a triangle will be designated A, B, C, the sides opposite, a, b, c.

But the principles of geometry do not enable us to *compute* the unknown parts. This is accomplished by the following laws of trigonometry:

I. *Law of Sines*, $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

II. *Law of Tangents*, $$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{a - b}{a + b}, \text{ etc.}$$

III. *Law of Cosines*, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \text{ etc.}$$

59. Law of Sines. *In any triangle the sides are proportional to the sines of the angles opposite.*

Let ABC be any triangle, p the perpendicular from B on b. In I (Fig. 34), C is an *acute*, in II, an *obtuse*, in III, a *right* angle. The demonstration applies to each triangle, but in II, $\sin ACB = \sin DCB$ (why?); in III, $\sin C = 1$ (why?).

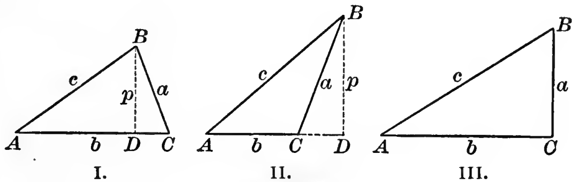


FIG. 34.

Now $\qquad \sin A = \frac{p}{c}, \qquad \therefore\ p = c \sin A.$

$\qquad \sin C = \frac{p}{a}, \qquad \therefore\ p = a \sin C.$

Equating values of p, $\qquad c \sin A = a \sin C,$

or,
$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

By dropping a perpendicular from A, or C, on a, or c, show that
$$\frac{\sin B}{b} = \frac{\sin C}{c}, \text{ or } \frac{\sin A}{a} = \frac{\sin B}{b},$$

whence
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

60. Law of Tangents. *The tangent of half the difference of two angles of a triangle is to the tangent of half their sum, as the difference of the sides opposite is to their sum.*

By Art. 59, $\qquad \frac{a}{b} = \frac{\sin A}{\sin B}.$

By composition and division,
$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}$$
$$= \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)};$$

or,
$$\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{a-b}{a+b}.$$

61. Law of Cosines. *The cosine of any angle of a triangle is equal to the quotient of the sum of the squares of the adjacent sides less the square of the opposite side, divided by twice the product of the adjacent sides.*

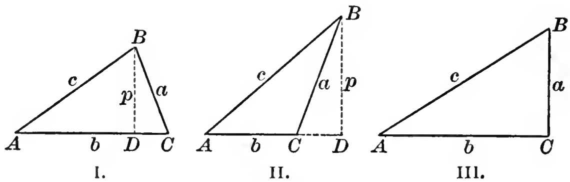


FIG. 34.

In each figure $$a^2 = p^2 + DC^2$$
$$= c^2 - AD^2 + (b - AD)^2$$

(in Fig. 34, II, DC is negative; in III, zero)

$$= c^2 - AD^2 + b^2 - 2\,b \cdot AD + AD^2$$
$$= b^2 + c^2 - 2\,b \cdot AD.$$

But

$AD = c \cos A, \quad \therefore\; a^2 = b^2 + c^2 - 2\,bc \cos A;$

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

Prove that $$\cos B = \frac{a^2 + c^2 - b^2}{2\,ac},$$

and $$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

62. Though these formulæ may be used for the solution of the triangle, they are not adapted to the use of logarithms (why?). Hence we derive the following:

Since $\cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$,

we have

$$2\cos^2\frac{A}{2} = 1 + \cos A, \text{ and } 2\sin^2\frac{A}{2} = 1 - \cos A.$$

From the latter

$$2\sin^2\frac{A}{2}=1-\frac{b^2+c^2-a^2}{2\,bc}=\frac{2\,bc-b^2-c^2+a^2}{2\,bc}$$

$$=\frac{a^2-(b-c)^2}{2\,bc}=\frac{(a-b+c)(a+b-c)}{2\,bc}.$$

Let $a+b+c=2\,s$, then $a+b-c=a+b+c-2\,c=2\,s-2\,c$;

i.e. $$a+b-c=2(s-c).$$

In like manner, $$a-b+c=2(s-b).$$

$$-a+b+c=2(s-a).$$

Substituting, $$2\sin^2\frac{A}{2}=\frac{2(s-b)\cdot 2(s-c)}{2\,bc}.$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Show that $$\sin\frac{B}{2}=?$$

also $$\sin\frac{C}{2}=?$$

From $$2\cos^2\frac{A}{2}=1+\cos A,$$

show that $$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

also $$\cos\frac{B}{2}=?$$

and $$\cos\frac{C}{2}=?$$

Also derive the formulæ

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\tan\frac{B}{2}=?$$

$$\tan\frac{C}{2}=?$$

63. Area of the triangle. In the figures of Art. 59 the area of the triangle $ABC = \Delta = \frac{1}{2}pb$.

But $\quad p = c \sin A. \quad \therefore \ \Delta = \frac{1}{2} bc \sin A.$ (i)

Again, by law of sines, $b = \dfrac{c \sin B}{\sin C}$.

Substituting,
$$\Delta = \frac{c^2 \sin A \sin B}{2 \sin C}$$
$$= \frac{c^2 \sin A \sin B}{2 \sin(A+B)} \text{ (why?).} \qquad \text{(ii)}$$

Finally, since $\sin A = 2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$, we have from (i)

$$\Delta = \tfrac{1}{2} bc \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2} = bc\sqrt{\frac{s(s-a)(s-b)(s-c)}{bc \cdot bc}}$$

or
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}. \qquad \text{(iii)}$$

Find Δ; (1) Given $a = 10$, $b = 12$, $C = 45°$.

(2) Given $a = 4$, $b = 5$, $c = 6$.

(3) Given $a = 2$, $B = 45°$, $C = 60°$.

SOLUTION OF TRIANGLES.

64. For the solution of triangles we have the following formulæ, which should be carefully memorized:

I. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

II. $\tan \frac{1}{2}(A - B) = \dfrac{a-b}{a+b} \tan \frac{1}{2}(A + B)$.

III. $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, or $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$,

or $\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$. $= \dfrac{\sqrt{(s-a)(s-b)(s-c) \div s}}{s-a}$

IV. $\Delta = \frac{1}{2} bc \sin A = \dfrac{c^2 \sin A \sin B}{2 \sin(A+B)} = \sqrt{s(s-a)(s-b)(s-c)}$.

Which of the above formulæ shall be used in the solution of a given triangle must be determined by examining the parts known, as will appear in Art. 69. It is always possible to express each of the unknown parts in terms of three known parts.

In solving triangles such as Case I, Art. 58, the law of sines applies; for, if the given side is not opposite either given angle, the third angle of the triangle is found from the relation $A + B + C = 180°$, and then three of the four quantities in $\frac{\sin A}{a} = \frac{\sin B}{b}$ being known, the solution gives the fourth.

In Case II (b) the law of sines applies, but in II (a) two only of the four quantities in $\frac{\sin A}{a} = \frac{\sin B}{b}$ are known. Therefore, we resort to the formula

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B),$$

in which all the factors of the second member are known.

In Case III, $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ is clearly applicable, and is *preferred* to the formulæ for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$; for, first, it is more accurate since tangent varies in magnitude from 0 to ∞, while sine and cosine lie between 0 and 1. (See Art. 27, 5.)

Let the student satisfy himself on this point by finding, correct to seconds, the angle whose logarithmic sine is 9.99992, and whose logarithmic tangent is 1.71668. Does the first determine the angle? Does the second?

And, second, it is more convenient, since in the complete solution of the triangle by $\sin \frac{A}{2}$ *six* logarithms must be taken from the table, by $\cos \frac{A}{2}$ *seven*, and by $\tan \frac{A}{2}$ but *four*.

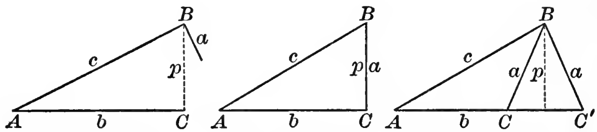
The right triangle may be solved as a special case by the law of sines, since $\sin C = 1$.

65. Ambiguous case. In geometry it was proved that a triangle having two sides and an angle opposite one of them of given magnitude is not always determined. The marks of the undetermined or ambiguous triangle are:

1. *The parts given are two sides and an angle opposite one.*
2. *The given angle is acute.*
3. *The side opposite this angle is less than the other given side.*

When these marks are all present, the number of solutions must be tested in one of two ways:

(*a*) From the figure it is apparent that there will be *no solution* when the side opposite is less than the perpendicular p; *one solution* when side a equals p; and *two solutions* when a is greater than p.



No Solution. *One Solution.* *Two Solutions.*

FIG. 35.

And since $\sin A = \frac{p}{c}$, it follows that there will be *no solution*, *one solution*, *two solutions*, according as $\sin A \gtreqless \frac{a}{c}$.

(*b*) A good test is found in solving by means of logarithms; and there will be *no solutions*, *one solution*, *two solutions*, according as log sin C proves to be *impossible*, *zero*, *possible*, *i.e.* as log sin C is positive, zero, or negative. This results from the fact that sine cannot be greater than unity, whence log sine must have a negative characteristic, or be zero.

66. In computations *time* and *accuracy* assume more than usual importance. *Time* will be saved by an orderly arrangement of the formulæ for the complete solution, before opening the book of logarithms, thus:

Given A, B, a. Solve completely.

$$C = 180° - (A+B),\quad b = \frac{a \sin B}{\sin A},\quad c = \frac{a \sin C}{\sin A},\quad \Delta = \tfrac{1}{2}\, ab \sin C.$$

180°	$\log a =$	$\log a =$
$A + B =$ ______	$\log \sin B =$	$\log \sin C =$
$\therefore C =$	$\text{colog} \sin A =$ ______	$\text{colog} \sin A =$ ______
	$\log b =$	$\log c =$
	$\therefore b =$	$\therefore c =$

	Check:
$\log a =$	$\log (s - b) =$
$\log b =$	$\log (s - c) =$
$\log \sin C =$	$\text{colog}\, s =$
$\text{colog}\, 2 =$ ______	$\text{colog}\, (s - a) =$ ______
$\log \Delta =$	2) ______
$\therefore \Delta =$	$\log \tan \frac{A}{2} =$
	$\therefore A =$

67. *Accuracy* must be secured by checks on the work at every step; *e.g.* in adding columns of logarithms, first add up, and then check by adding down. Too much care cannot be given to verification in the simple operations of addition, subtraction, multiplication, and division. A final check should be made by using other formulæ involving the parts in a different way, as in the check above. As far as possible the parts originally given should be used throughout in the solution, so that an error in computing one part may not affect later computations.

68. The formulæ should always be *solved for the unknown part before using*, and it should be noted whether the solution gives one value, or more than one, for each part; *e.g.* the same value of $\sin B$ belongs to two supplementary angles, one or both of which may be possible, as in the ambiguous case.

69. Write formulæ for the complete solution of the following triangles, showing whether you find no solution, one solution, two or more solutions, in each case, with reasons for your conclusion:

	a	b	c	A	B	C
1.				81° 26′ 28″	44° 11′ 20″	54° 22′ 12″
2.		78.54		63° 18′ 20″		41° 30′ 18″
3.		135.82	26.89	53° 28′ 30″		
4.	0.75	0.85	0.95			
5.	243		562			36° 15′ 40″
6.		38.75	25.92			63° 50′ 10″
7.	0.058			78° 15′	33° 46′	
8.	2986		1493			30°
9.		48	50		26° 15′	

MODEL SOLUTIONS.

1. Given $a = 0.785$, $b = 0.85$, $c = 0.633$. Solve completely.

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\ \tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}},\ \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Check: $A + B + C = 180°$. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

$$\begin{aligned} a &= 0.735 \\ b &= 0.85 \\ c &= 0.633 \\ 2)&\underline{2.268} \\ s &= 1.134 \\ s-a &= 0.349 \\ s-b &= 0.284 \\ s-c &= 0.501 \end{aligned}$$

$$\begin{aligned} \log(s-b) &= 9.45332 \\ \log(s-c) &= 9.69984 \\ \text{colog}\, s &= 9.94539 \\ \text{colog}\,(s-a) &= 0.45717 \\ 2)&\underline{19.55572} \\ \log\tan\tfrac{1}{2}A &= 9.77786 \\ \tfrac{1}{2}A &= 30°\,56'\,49'' \\ A &= 61°\,53'\,38'' \end{aligned}$$

$$\begin{aligned} \log(s-a) &= 9.54283 \\ \log(s-c) &= 9.69984 \\ \text{colog}\, s &= 9.94539 \\ \text{colog}\,(s-b) &= 0.54668 \\ 2)&\underline{19.73474} \\ \log\tan\tfrac{1}{2}B &= 9.86737 \\ \tfrac{1}{2}B &= 36°\,23'\,2'' \\ B &= 72°\,46'\,4'' \end{aligned}$$

Check:

$$\begin{aligned} A &= 61°\,53'\,38'' \\ B &= 72°\,46'\,\ \,4'' \\ C &= 45°\,20'\,20'' \\ &\overline{180°\ \ 0'\ \ 2''} \end{aligned}$$

$$\begin{aligned} \log(s-a) &= 9.54283 \\ \log(s-b) &= 9.45332 \\ \text{colog}\, s &= 9.94539 \\ \text{colog}\,(s-c) &= 0.30016 \\ 2)&\underline{19.24170} \\ \log\tan\tfrac{1}{2}C &= 9.62085 \\ \tfrac{1}{2}C &= 22°\,40'\,10'' \\ C &= 45°\,20'\,20'' \end{aligned}$$

$$\begin{aligned} \log s &= 0.05461 \\ \log(s-a) &= 9.54283 \\ \log(s-b) &= 9.45332 \\ \log(s-c) &= 9.69984 \\ 2)&\underline{18.75060} \\ \log\Delta &= 9.37530 \\ \Delta &= 0.2373 \end{aligned}$$

Solve: (1) Given $a = 30$, $b = 40$, $c = 50$.

(2) Given $a = 2159$, $b = 1431.6$, $c = 914.8$.

(3) Given $a = 78.54$, $b = 32.56$, $c = 48.9$.

2. Given $A = 57°\,23'\,12''$, $C = 68°\,15'\,30''$, $c = 832.56$. Solve completely.

$$a = \frac{c \sin A}{\sin C}, \quad b = \frac{c \sin B}{\sin C}, \quad \Delta = \tfrac{1}{2} bc \sin A.$$

$B = 180° - (A + C)$
$= 54°\,21'\,18''$.

Check: $\tan \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

$\log c = 2.92042$	$\log c = 2.92042$	$\log b = 2.86236$
$\log \sin A = 9.92548$	$\log \sin B = 9.90990$	$\log c = 2.92042$
$\text{colog} \sin C = 0.03204$	$\text{colog} \sin C = 0.03204$	$\log \sin A = 9.92548$
$\log a = 2.87794$	$\log b = 2.86236$	$\log 2\Delta = 5.70826$
$a = 754.98$	$b = 728.38$	$\Delta = \frac{510811}{2} = 255405.5$

Check: $a = 754.98$	$s - a = 402.98$	$\log(s-b) = 2.63304$
$b = 728.38$	$s - b = 429.58$	$\log(s-c) = 2.51242$
$c = 832.56$	$s - c = 325.40$	$\text{colog}\, s = 6.93634$
$2)2315.92$		$\text{colog}(s-a) = 7.39471$
$s = 1157.96$		$2)19.47651$
		$\log \tan \frac{1}{2} A = 9.73826$
		$\frac{1}{2} A = 28°\,41'\,38''$
		$A = 57°\,23'\,16''$

Solve:

(1) Given $a = 215.73$, $B = 92°\,15'$, $C = 28°\,14'$.

(2) Given $b = 0.827$, $A = 78°\,14'\,20''$, $B = 63°\,42'\,30''$.

(3) Given $b = 7.54$, $c = 6.93$, $B = 54°\,28'\,40''$.

3. Given $a = 25.384$, $c = 52.925$, $B = 28°\,32'\,20''$. Solve completely.

(Why not use the same formulæ as in Example 1, or 2?)

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \tan \frac{C+A}{2}, \quad b = \frac{c \sin B}{\sin C}, \quad \Delta = \tfrac{1}{2} ac \sin B.$$

$180° - B = C + A = 151°\,27'\,40''$.
$\therefore \frac{1}{2}(C + A) = 75°\,43'\,50''$.

Check: $b = \dfrac{a \sin B}{\sin A}$.

$c = 52.925$	$\log(c-a) = 1.43998$	$\therefore \frac{1}{2}(C-A) = 54°\,7'38''$
$a = 25.384$	$\text{colog}(c+a) = 8.10619$	$\frac{1}{2}(C+A) = 75°43'50''$
$c+a = 78.309$	$\log \tan \frac{1}{2}(C+A) = 0.59460$	adding, $C = 129°51'28''$
$c-a = 27.541$	$\log \tan \frac{1}{2}(C-A) = 0.14077$	subtracting, $A = 21°36'12''$

$\log c = 1.72366$		$\log a = 1.40456$
$\log \sin B = 9.67921$		$\log c = 1.72366$
$\text{colog} \sin C = 0.11484$	*Check:* $\log a = 1.40456$	$\log \sin B = 9.67921$
$\log b = 1.51771$	$\log \sin B = 9.67921$	$\log 2\Delta = 2.70743$
	$\text{colog} \sin A = 0.43395$	
$b = 32.939$	$\log b = 1.51772$	$\Delta = \frac{509.83}{2} = 254.965$

Solve: (1) Given $a = 0.325$, $c = 0.426$, $B = 48° 50' 10''$.

(2) Given $b = 4291$, $c = 3194$, $A = 73° 24' 50''$.

(3) Given $b = 5.38$, $c = 12.45$, $A = 62° 14' 40''$.

4. *Ambiguous cases.* Since the required angle is found in terms of its sine, and since $\sin \alpha = \sin (180° - \alpha)$, it follows that there may be two values of α, one in the first, and the other in the second quadrant, their sum being 180°. In the following examples the student should note that all the marks of the ambiguous case are present. The solutions will show the treatment of the ambiguous triangle having no solution, one solution, two solutions.

(*a*) Given $b = 70$, $c = 40$, $C = 47° 32' 10''$. Solve. Why ambiguous?

$$\sin B = \frac{b \sin C}{c}.$$

$$\begin{aligned} \log b &= 1.84510 \\ \log \sin C &= 9.86788 \\ \text{colog } c &= 8.39794 \\ \hline \log \sin B &= 0.11092 \end{aligned}$$

$\therefore$ B is impossible, and there is no solution. Why? Show the same by $\sin C > \frac{c}{b}$.

(*b*) Given $a = 1.5$, $c = 1.7$, $A = 61° 55' 38''$. Solve.

$$\sin C = \frac{c \sin A}{a}.$$

$$\begin{aligned} \log c &= 0.23045 \\ \log \sin A &= 9.94564 \\ \text{colog } a &= 9.82391 \\ \hline \log \sin C &= 0.00000 \\ C &= 90° \end{aligned}$$

and there is one solution. Why? Show the same by $\sin A = \frac{a}{c}$. Solve for the remaining parts and check the work.

(c) Given $a = 0.235$, $b = 0.189$, $B = 36° 28' 20''$. Solve.

$$\sin A = \frac{a \sin B}{b}, \qquad c = \frac{b \sin C}{\sin B}.$$

$$\begin{aligned} \log a &= 9.37107 \\ \log \sin B &= 9.77411 \\ \text{colog } b &= 0.72354 \\ \hline \log \sin A &= 9.86872 \\ A &= 47° 39' 25'' \\ &\text{or } 132° 20' 35''. \end{aligned}
\qquad
\begin{aligned} \log b &= 9.27646 & & 9.27646 \\ \log \sin C &= 9.99772 & \text{or } & 9.28774 \\ \text{colog } \sin B &= 0.22589 & & 0.22589 \\ \hline \log c &= 9.50007 & \text{or } & 8.79009 \\ c &= 0.31628 & \text{or } & 0.06167 \end{aligned}$$

$\therefore \; C = 95° 52' 15''$ or $11° 11' 5''$.

Solve for Δ, and check. Show the same by $\sin B < \frac{b}{a}$.

Solve:

(1) Given $b = 216.4$, $c = 593.2$, $B = 98° 15'$.

(2) Given $a = 22$, $b = 75$, $B = 32° 20'$.

(3) Given $a = 0.353$, $c = 0.295$, $A = 46° 15' 20''$.

(4) Given $a = 293.445$, $b = 450$, $A = 40° 42'$.

(5) Given $b = 531.03$, $c = 629.20$, $B = 34° 28' 16''$.

Solve completely, given:

	a	b	c	A	B	C
1.	50	60				78° 27′ 47″
2.		10	11			93° 35′
3.	4	5	6			
4.			10	109° 28′ 16″	38° 56′ 54″	
5.	40	51		49° 28′ 32″		
6.	352.25	513.27	482.68			
7.	0.573	0.394		112° 4′		
8.	107.087			56° 15′	48° 35′	
9.			$\sqrt{2}$	117°	45°	
10.	197.63	246.35		34° 27′		
11.	4090	3850	3811			
12.	3795				73° 15′ 15″	42° 18′ 30″
13.		234.7	185.4	84° 36′		
14.		26.234	22.6925		49° 8′ 24″	
15.	273	136		72° 25′ 13″		

APPLICATIONS.

70. Measurements of heights and distances often lead to the solution of oblique triangles. With this exception, the methods of Chapter V apply, as will be illustrated in the following problems.

The *bearing* of a line is the angle it makes with a north and south line, as determined by the magnetic needle of the mariner's compass. If the bearing does not correspond to any of the points of the compass, it is usual to express it thus: N. 40° W., meaning that the line bears from N. 40° toward W.

EXAMPLES.

1. When the altitude of the sun is 48°, a pole standing on a slope inclined to the horizon at an angle of 15° casts a shadow directly down the slope 44.3 ft. How high is the pole?

2. A tree standing on a mountain side rising at an angle of 18° 30′ breaks 32 ft. from the foot. The top strikes down the slope of the mountain 28 ft. from the foot of the tree. Find the height of the tree.

3. From one corner of a triangular lot the other corners are found to be 120 ft. E. by N., and 150 ft. S. by W. Find the area of the lot, and the length of the fence required to enclose it.

4. A surveyor observed two inaccessible headlands, A and B. A was W. by N. and B, N.E. He went 20 miles N., when they were S.W. and S. by E. How far was A from B?

5. The bearings of two objects from a ship were N. by W. and N.E. by N. After sailing E. 11 miles, they were in the same line W.N.W. Find the distance between them.

6. From the top and bottom of a vertical column the elevation angles of the summit of a tower 225 ft. high and standing on the same horizontal plane are 45° and 55°. Find the height of the column.

7. An observer in a balloon 1 mile high observes the depression angle of an object on the ground to be 35° 20′. After ascending vertically and uniformly for 10 mins., he observes the depression angle of the same object to be 55° 40′. Find the rate of ascent of the balloon in miles per hour.

8. A statue 10 ft. high standing on a column subtends, at a point 100 ft. from the base of the column and in the same horizontal plane, the same angle as that subtended by a man 6 ft. high, standing at the foot of the column. Find the height of the column.

9. From a balloon at an elevation of 4 miles the dip of the horizon is 2° 33′ 40″. Required the earth's radius.

10. Two ships sail from Boston, one S.E. 50 miles, the other N.E. by E. 60 miles. Find the bearing and distance of the second ship from the first.

11. The sides of a valley are two parallel ridges sloping at an angle of 30°. A man walks 200 yds. up one slope and observes the angle of elevation of the other ridge to be 15°. Show that the height of the observed ridge is 273.2 yds.

12. To determine the height of a mountain, a north and south base line 1000 yds. long is measured; from one end of the base line the summit bears E. 10° N., and is at an altitude of 13° 14′. From the other end it bears E. 46° 30′ N. Find the height of the mountain.

13. The shadow of a cloud at noon is cast on a spot 1600 ft. due west of an observer. At the same instant he finds that the cloud is at an elevation of 23° in a direction W. 14° S. Find the height of the cloud and the altitude of the sun.

14. From the base of a mountain the elevation of its summit is 54° 20′. From a point 3000 ft. toward the summit up a plane rising at an angle of 25° 30′ the elevation angle is 68° 42′. Find the height of the mountain.

15. From two observations on the same meridian, and 92° 14′ apart, the zenith angles of the moon are observed to be 44° 54′ 21″ and 48° 42′ 57″. Calling the earth's radius 3956.2 miles, find the distance to the moon.

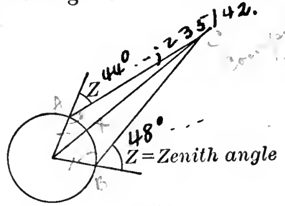


FIG. 36.

16. The distances from a point to three objects are 1130, 1850, 1456, and the angles subtended by the distances between the three objects are respectively 102° 10′, 142°, and 115° 50′. Find the distances between the three objects.

17. From a ship A running N.E. 6 mi. an hour direct to a port distant 35 miles, another ship B is seen steering toward the same port, its bearing from A being E.S.E., and distance 12 miles. After keeping on their courses $1\frac{1}{2}$ hrs., B is seen to bear from A due E. Find B's course and rate of sailing.

18. From the mast of a ship 64 ft. high the light of a lighthouse is just visible when 30 miles distant. Find the height of the lighthouse, the earth's radius being 3956.2 miles.

19. From a ship two lighthouses are observed due N.E. After sailing 20 miles E. by S., the lighthouses bear N.N.W. and N. by E. Find the distance between the lighthouses.

20. A lighthouse is seen N. 20° E. from a vessel sailing S. 25° E. A mile further on it appears due N. Determine its distance at the last observation.

EXAMPLES FOR REVIEW.

In connection with each problem the student should review all principles involved. The following list of problems will then furnish a thorough review of the book. In solving equations, find all values of the unknown angle less than 360° that satisfy the equation.

1. If $\tan\alpha = \frac{1}{7}$, $\tan\beta = \frac{1}{2}$, show that $\tan(\beta - 2\alpha) = \frac{2}{11}$.

2. Prove $\tan\alpha + \cot\alpha = 2\csc 2\alpha$.

3. From the identities $\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1$, and $2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A$,

prove $$2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A},$$

and $$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A}.$$

4. Remove the ambiguous signs in Ex. 3 when A is in turn an angle of each quadrant.

5. A wall 20 feet high bears S. 59° 5′ E.; find the width of its shadow on a horizontal plane when the sun is due S. and at an altitude of 60°.

6. Solve $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x$.

7. Prove $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$.

8. If $A = 60°$, $B = 45°$, $C = 30°$, evaluate

$$\frac{\tan A + \tan B + \tan C}{\tan A\tan B + \tan B\tan C + \tan C\tan A}.$$

9. Prove $\dfrac{\cos(A+B)\cos C}{\cos(A+C)\cos B} = \dfrac{1-\tan A\tan B}{1-\tan A\tan C}$.

10. Solve completely the triangle whose known parts are $b = 2.35$, $c = 1.96$, $C = 38° 45'.4$.

11. Find the functions of 18°, 36°, 54°, 72°.

Let $x = 18°$. Then $2x = 36°$, $3x = 54°$, and $2x + 3x = 90°$.

12. If $\cot\alpha = \frac{p}{q}$, find the value of

$$\sin\alpha + \cos\alpha + \tan\alpha + \cot\alpha + \sec\alpha + \csc\alpha.$$

13. Prove $\dfrac{\sin 3\alpha \sin 2\beta - \sin 3\beta \sin 2\alpha}{\sin 2\alpha \sin \beta - \sin 2\beta \sin \alpha} = 1 + 4\cos\alpha\cos\beta.$

14. From a ship sailing due N., two lighthouses bear N.E. and N.N.E., respectively; after sailing 20 miles they are observed to bear due E. Find the distance between the lighthouses.

15. Solve $1 - 2\sin x = \sin 3x.$

16. Prove $\sin^{-1}\sqrt{\dfrac{a}{a+b}} = \tan^{-1}\sqrt{\dfrac{a}{b}}.$

17. If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, then $\cos\theta + \sin\theta = \sqrt{2}\cos\theta.$

18. Solve completely the triangle ABC, given $a = 0.256$, $b = 0.387$, $C = 102° 20'.5$.

19. Prove $\tan(30° + \alpha)\tan(30° - \alpha) = \dfrac{2\cos 2\alpha - 1}{2\cos 2\alpha + 1}.$

20. Solve $\tan(45° - \theta) + \tan(45° + \theta) = 4.$

21. Prove $\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha - \sin^2\beta.$

22. Prove $\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta = \cos^2\alpha - \sin^2\beta.$

23. A man standing due S. of a water tower 150 feet high finds its elevation to be 72° 30′; he walks due W. to A street, where the elevation is 44° 50′; proceeding in the same direction one block to B street, he finds the elevation to be 22° 30′. What is the length of the block between A and B streets?

24. Prove $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}.$

25. If $P = 60°$, $Q = 45°$, $R = 30°$, evaluate

$$\frac{\sin P\cos Q + \tan P\cos Q}{\sin P\cos P + \cot P\cot R}.$$

26. If $\cos(90° + \alpha) = -\frac{3}{4}$, evaluate $3\cos 2\alpha + 4\sin 2\alpha.$

27. If $\sin B + \sin C = m$, $\cos B + \cos C = n$, show that $\tan\dfrac{B+C}{2} = \dfrac{m}{n}.$

28. Show that $\sin 2\beta$ can never be greater than $2\sin\beta$.

29. Prove $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} = \tan^{-1}\frac{56}{33}.$

30. Solve $\cot^{-1}x + \sin^{-1}\dfrac{1}{5}\sqrt{5} = \dfrac{\pi}{4}.$

31. Solve $\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x.$

32. A man standing between two towers, 200 feet from the base of the higher, which is 90 feet high, observes their altitudes to be the same; 70 feet nearer the shorter tower he finds the altitude of one is twice that of the other. Find the height of the shorter tower, and his original distance from it.

33. Solve $\cos 3\beta + 8\cos^3\beta = 0$.

34. Solve $\cot m - \tan(180^\circ + m) = \sec m + \sec(90^\circ - m)$.

35. Solve $\dfrac{1 - \tan t}{1 + \tan t} = 2\cos 2t$.

36. Prove $\cot A + \cot B = \dfrac{\sin(A + B)}{\sin A \sin B}$.

37. Prove $\cot P - \cot Q = -\dfrac{\sin(P - Q)}{\sin P \sin Q}$.

38. In the triangle ABC prove

$$a = b\sin C + c\sin B,$$
$$b = c\sin A + a\sin C,$$
$$c = a\sin B + b\sin A.$$

39. Solve completely the triangle, given

$$a = 927.56,\ b = 648.25,\ c = 738.42.$$

40. Prove $\cos^2\alpha - \sin(30^\circ + \alpha)\sin(30^\circ - \alpha) = \frac{3}{4}$.

41. Prove $\tan 3x \tan x = \dfrac{\cos 2x - \cos 4x}{\cos 2x + \cos 4x}$.

42. Simplify $\cos(270^\circ + \alpha) + \sin(180^\circ + \alpha) + \cos(90^\circ + \alpha)$.

43. Simplify $\tan(270^\circ - \theta) - \tan(90^\circ + \theta) + \tan(270^\circ + \theta)$.

44. Solve $\cos 3\phi - \cos 2\phi + \cos\phi = 0$.

45. Solve $\cos A + \cos 3A + \cos 5A + \cos 7A = 0$.

46. The topmast of a yacht from a point on the deck subtends the same angle α, that the part below it does. Show that if the topmast be a feet high, the length of the part below it is $a\cos 2\alpha$.

47. A horizontal line AB is measured 400 yards long. From a point in AB a balloon ascends vertically till its elevation angles at A and B are $64^\circ 15'$ and $48^\circ 20'$, respectively. Find the height of the balloon.

48. If $\cos\phi = n\sin\alpha$, and $\cot\phi = \dfrac{\sin\alpha}{\tan\beta}$, prove $\cos\beta = \dfrac{n}{\sqrt{1 + n^2\cos^2\alpha}}$.

49. Find $\cos 3\alpha$, when $\tan 2\alpha = -\frac{3}{4}$.

50. Solve completely the triangle, given $a = 0.296$, $B = 28^\circ 47'.3$, $C = 84^\circ 25'$.

51. Evaluate $\sin 300^\circ + \cos 240^\circ + \tan 225^\circ$.

52. Evaluate $\sec\dfrac{2\pi}{3} - \csc\dfrac{5\pi}{3} + \tan\dfrac{4\pi}{3}$.

53. If
$$\tan\theta = \frac{\sin\alpha\cos\gamma - \sin\beta\sin\gamma}{\cos\alpha\cos\gamma - \cos\beta\sin\gamma}$$
and
$$\tan\phi = \frac{\sin\alpha\sin\gamma - \sin\beta\cos\gamma}{\cos\alpha\sin\gamma - \cos\beta\cos\gamma},$$
show that $\tan(\theta + \phi) = \tan(\alpha + \beta)$.

54. If $\tan 466^\circ\, 15'\, 38'' = -\frac{24}{7}$, find the sine and cosine of $233^\circ\, 7'\, 49''$.

55. Prove $\dfrac{\csc\alpha - \cot\alpha}{\sec\alpha + \tan\alpha} = \dfrac{\sec\alpha - \tan\alpha}{\csc\alpha + \cot\alpha}$.

56. Prove $\dfrac{\cos(\alpha - 3\beta) - \cos(3\alpha - \beta)}{\sin 2\alpha + \sin 2\beta} = 2\sin(\alpha - \beta)$.

57. Prove $\sin 80^\circ = \sin 40^\circ + \sin 20^\circ$.

58. Prove $\cos 20^\circ = \cos 40^\circ + \cos 80^\circ$.

59. Prove $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}$.

60. From the deck of a ship a rock bears N.N.W. After the ship has sailed 10 miles E.N.E., the rock bears due W. Find its distance from the ship at each observation.

61. Find the length of an arc of 80° in a circle of 4 feet radius.

62. Given $\tan\theta = \frac{4}{3}$, $\tan\phi = \frac{5}{12}$, evaluate $\sin(\theta + \phi) + \cos(\theta - \phi)$.

63. If $\tan\theta = 2\tan\phi$, show that $\sin(\theta + \phi) = 3\sin(\theta - \phi)$.

64. Prove $\cos(\alpha+\beta)\cos(\alpha-\beta) + \sin(\alpha+\beta)\sin(\alpha-\beta) = \dfrac{1-\tan^2\beta}{1+\tan^2\beta}$.

65. Solve $4\cos 2\theta + 3\cos\theta = 1$.

66. Solve $3\sin\alpha = 2\sin(60^\circ - \alpha)$.

67. Prove $(\sin\alpha - \csc\alpha)^2 - (\tan\alpha - \cot\alpha)^2 + (\cos\alpha - \sec\alpha)^2 = 1$.

68. Prove $2(\sin^6\alpha + \cos^6\alpha) + 1 = 3(\sin^4\alpha + \cos^4\alpha)$.

69. Prove $\csc 2\beta + \cot 4\beta = \cot\beta - \csc 4\beta$.

70. If $\tan p = \dfrac{5}{12}$, $\cos 2q = \dfrac{527}{625}$, then $\csc\dfrac{p - q}{2} = 5\sqrt{13}$.

71. Solve completely the triangle, given
$$a = 0.0654,\quad b = 0.092,\quad B = 38^\circ\, 40'.4.$$

72. Solve completely the triangle, given
$$b = 10,\quad c = 26,\quad B = 22^\circ\, 37'.$$

73. A railway train is travelling along a curve of $\frac{1}{3}$ mile radius at the rate of 25 miles per hour. Through what angle (in circular measure) will it turn in half a minute?

74. Express the following angles in circular measure:

$$63^\circ, \quad 4^\circ\,30', \quad 6^\circ\,12'\,36''.$$

75. Express the following angles in sexagesimal measure:

$$\frac{\pi}{6}, \quad \frac{3\pi}{8}, \quad \frac{17\pi}{64}.$$

76. If A, B, C are angles of a triangle, prove

$$\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}.$$

77. Prove $\sin 2x + \sin 2y + \sin 2z = 4 \sin x \sin y \sin z$, when x, y, z are the angles of a triangle.

78. Prove $\sec\alpha = 1 + \tan\alpha \tan\frac{\alpha}{2}$.

79. Prove $\sin^2(\alpha + \beta) - \sin^2(\alpha - \beta) = \sin 2\alpha \sin 2\beta$.

80. Prove $\cos^2(\alpha + \beta) - \sin^2(\alpha - \beta) = \cos 2\alpha \cos 2\beta$.

81. Prove $\dfrac{\sin 19p + \sin 17p}{\sin 10p + \sin 8p} = 2 \cos 9p$.

82. Consider with reference to their ambiguity the triangles whose known parts are:

(a) $a = 2743$, $b = 6452$, $B = 43^\circ\,15'$;

(b) $a = 0.3854$, $c = 0.2942$, $C = 38^\circ\,20'$;

(c) $b = 5$, $c = 53$, $B = 15^\circ\,22'$;

(d) $a = 20$, $b = 90$, $A = 63^\circ\,28'.5$.

83. From a ship at sea a lighthouse is observed to bear S.E. After the ship sailed N.E. 6 miles the bearing of the lighthouse is S. 27° 30′ E. Find the distance of the lighthouse at each time of observation.

84. Prove $\dfrac{\sin(\theta + 3\phi) + \sin(3\theta + \phi)}{\sin 2\theta + \sin 2\phi} = 2\cos(\theta + \phi)$.

85. Prove $\cos 15^\circ - \sin 15^\circ = \dfrac{1}{\sqrt{2}}$.

86. Show that $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$
$= \cos^2\beta - \sin^2\alpha$.

87. Show that $\tan(\alpha + 45^\circ)\tan(\alpha - 45^\circ) = \dfrac{2\sin^2\alpha - 1}{2\cos^2\alpha - 1}$.

88. Solve $\sin(x + y)\sin(x - y) = \frac{1}{2}$, $\cos(x + y)\cos(x - y) = 0$.

89. Prove $\dfrac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha} = \tan\dfrac{\alpha}{2}$.

90. Prove $\tan 2\theta + \sec 2\theta = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$.

91. If $\tan\phi = \dfrac{b}{a}$, then $a\cos 2\phi + b\sin 2\phi = a$.

92. Prove $\sin^{-1}\dfrac{1}{\sqrt{5}} + \cot^{-1} 3 = \dfrac{\pi}{4}$.

93. Solve $\cos A + \cos 7A = \cos 4A$.

94. Two sides of a triangle, including an acute angle, are 5 and 7, the area is 14; find the other side.

95. Show that $\dfrac{3\cos 3\theta - 2\cos\theta - \cos 5\theta}{\sin 5\theta - 3\sin 3\theta + 4\sin\theta} = \tan 2\theta$.

96. A regular pyramid stands on a square base one side of which is 173.6 feet. This side makes an angle of 67° with one edge. What is the height of the pyramid?

97. From points directly opposite on the banks of a river 500 yards wide the mast of a ship lying between them is observed to be at an elevation of 10° 28′.4 and 12° 14′.5, respectively. Find the height of the mast.

98. Show that $(\sin 60° - \sin 45°)(\cos 30° + \cos 45°) = \sin^2 30°$.

99. Find x if $\sin^{-1} x + \sin^{-1}\dfrac{x}{2} = \dfrac{\pi}{4}$.

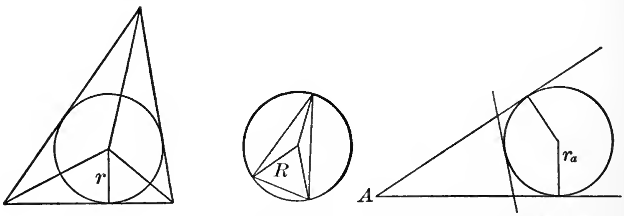
100. Trace the changes in sign and value of $\sin\alpha + \cos\alpha$ as α changes from 0° to 360°.

CHAPTER VIII.

MISCELLANEOUS PROPOSITIONS.

71. The circle inscribed in a given triangle is often called the *incircle* of the triangle, its centre the *incentre*, and its radius is denoted by r. The incentre is the point of intersection of the three bisectors of the angles of the triangle (geometry).

The circle circumscribed about a triangle is called the *circumcircle*, its centre the *circumcentre*, and its radius R. The circumcentre is the point of intersection of perpendiculars erected at the middle points of the three sides of the triangle (geometry).



Incircle. Circumcircle. Escribed circle opposite A.

FIG. 37.

The circle which touches any side of a triangle and the other two sides produced is called the *escribed* circle; its radius is denoted by r_a, r_b, or r_c, according as the escribed circle is opposite angle A, B, or C.

Again, the altitudes from the vertices of a triangle meet in a point called the *orthocentre* of the triangle.

Finally, the medians of a triangle meet in a point called the *centroid*, which is two-thirds of the length of the median from the vertex of the angle from which that median is drawn (geometry).

Certain properties of the above will now be considered.

72. To find the radius of the incircle.

Let Δ, Δ', Δ'', Δ''' represent the areas of triangles ABC, COB, AOC, BOA, respectively.

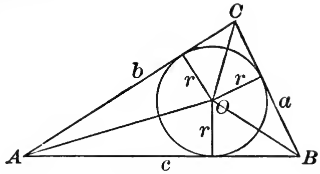


FIG. 38.

Then

$$\Delta = \Delta' + \Delta'' + \Delta'''$$

$$= \tfrac{1}{2}(a + b + c)r = sr.$$

And since $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, (Art. 63)

$$\therefore\ r = \frac{\Delta}{s} = \frac{\sqrt{(s-a)(s-b)(s-c)}}{s}.$$

COR. To express the angles in terms of r and the sides, divide each member of the above equation by $s - a$.

Then $\dfrac{r}{s-a} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \tan\tfrac{1}{2}A.$ (Art. 62)

In like manner $\tan\tfrac{1}{2}B = \dfrac{r}{s-b}$; $\tan\tfrac{1}{2}C = \dfrac{r}{s-c}$.

To find the radius of the circumcircle.

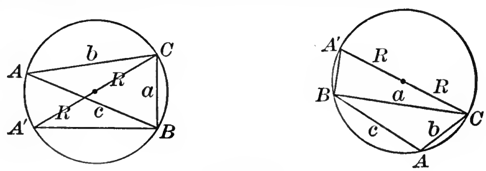


FIG. 39.

In the figure ABC is the given triangle, and $A'C$ a diameter of the circumcircle. Then, angle $A = A'$, or $180° - A'$.

$$\therefore\ \sin A = \sin A'.$$

Since $A'BC$ is a right angle,

$$\sin A' = \frac{BC}{A'C} = \frac{a}{2R}.$$

$$\therefore\ R = \frac{\tfrac{1}{2}a}{\sin A}.$$

COR. 1. As above, $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, which is another proof of the "law of sines."

COR. 2. From $R = \frac{a}{2 \sin A}$, we have

$$R = \frac{abc}{2\,bc \sin A} = \frac{abc}{4\,\Delta}, \text{ where } \Delta = \text{area } ABC.$$

74. To find the radii of the escribed circles.

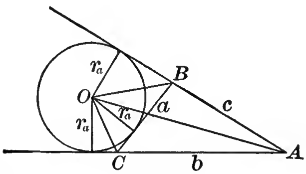


FIG. 40.

Represent areas ABC, BOA, AOC, BOC, by Δ, Δ', Δ'', Δ''', respectively. Then r_a is the altitude of each of the triangles BOA, AOC, BOC.

Now
$$\Delta = \Delta' + \Delta'' - \Delta'''$$
$$= \tfrac{1}{2} r_a c + \tfrac{1}{2} r_a b - \tfrac{1}{2} r_a a$$
$$= \tfrac{1}{2} r_a (c + b - a) = r_a (s - a).$$
$$\therefore\ r_a = \frac{\Delta}{s - a}.$$

In like manner, $r_b = \frac{\Delta}{s - b}$; $r_c = \frac{\Delta}{s - c}$.

75. The orthocentre.

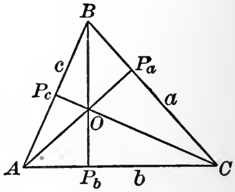


FIG. 41.

Denote the perpendiculars on the sides a, b, c, by AP_a, BP_b, CP_c, and let it be required to find the distances from their intersection O to the sides of the triangle, and also to the vertices.

$$OP_b = AP_b \tan CAO.$$

But $AP_b = c \cos A$, and $CAO = 90° - C$.

$$\therefore\ OP_b = c \cos A \cot C = \frac{c}{\sin C} \cos A \cos C.$$
$$= 2R \cos A \cos C. \qquad \text{(Art. 73, Cor. 1)}$$

In like manner, $OP_c = 2R \cos B \cos A,$

$$OP_a = 2R \cos C \cos B.$$

Again, the distances from the orthocentre to the vertices are,

$$OA = \frac{AP_b}{\cos CAO} = \frac{c \cos A}{\sin C}$$

$$= 2R \cos A.$$

Also, $$OB = 2R \cos B,$$

and $$OC = 2R \cos C.$$

76. Centroid and medians.

The lengths of the medians may be computed as follows:

In the figure the medians to the sides a, b, c, are AM_a, BM_b, CM_c, meeting in the centroid O.

Now, by the law of cosines, from the triangle BM_bC,

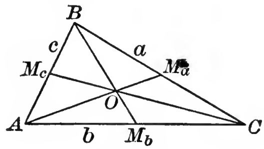


FIG. 42.

$$BM_b^2 = a^2 + M_bC^2 - 2a \cdot M_bC \cdot \cos C$$

$$= a^2 + \frac{b^2}{4} - ab \cos C.$$

But, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab},$

$$\therefore\ BM_b^2 = a^2 + \frac{b^2}{4} - \frac{a^2 + b^2 - c^2}{2} = \frac{2a^2 + 2c^2 - b^2}{4},$$

whence, $BM_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} = \frac{1}{2}\sqrt{a^2 + c^2 + 2ac \cos B}$

since $$\frac{a^2 + c^2 - b^2}{2ac} = \cos B.$$

In like manner,

$$CM_c = \tfrac{1}{2}\sqrt{2b^2 + 2a^2 - c^2} = \tfrac{1}{2}\sqrt{b^2 + a^2 + 2ba \cos C},$$

and $$AM_a = \tfrac{1}{2}\sqrt{2c^2 + 2b^2 - a^2} = \tfrac{1}{2}\sqrt{c^2 + b^2 + 2cb \cos A}.$$

EXAMPLES.

1. In the triangle, $a = 25$, $b = 35$, $c = 45$, find R, r, r_a.

2. Given $a = 0.354$, $b = 0.548$, $C = 28° 34' 20''$, find the distances to C and B, from the circumcentre, the incentre, the centroid, and the orthocentre.

3. In the ambiguous triangle show that the circumcircles of the two triangles, when there are two solutions, are equal.

4. Prove that $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$.

5. In any triangle prove $\Delta = \sqrt{r\, r_a r_b r_c}$.

6. Prove that the product of the distances of the incentre from the vertices of the triangle is $4\, r^2 R$.

7. Prove that the area of all triangles of given perimeter that can be circumscribed about a given circle is constant.

8. Prove that the area of the triangle ABC is $Rr(\sin A + \sin B + \sin C)$.

CHAPTER IX.

SERIES — DE MOIVRE'S THEOREM — HYPERBOLIC FUNCTIONS.

77. First consider some series by means of which logarithms of numbers and the natural functions of angles may be computed. For this purpose the following series is important:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

It may be derived as follows:

By the binomial theorem,

$$\left(1+\frac{1}{n}\right)^{nx}$$

$$= 1 + nx \cdot \frac{1}{n} + \frac{nx(nx-1)}{2!} \cdot \frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{3!} \cdot \frac{1}{n^3} + \cdots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \cdots,$$

and if n increase without limit,

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

This is called the *exponential series*, and is represented by e^x, so that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots.$$

It is shown in higher algebra that this equation holds for all values of x; whence, if $x = 1$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots.$$

This value of e is taken as the base of the natural or Naperian system of logarithms.

This value e, however, is not the base of the system of logarithms computed by Napier, but its reciprocal instead. The natural logarithm is used in the theoretical treatment of logarithms, and, as will presently appear, it is customary to compute the common logarithm by first finding the natural, and then multiplying it by a constant multiplier called the modulus, Art. 82; *i.e.* in the Naperian system the modulus is taken as 1, and the base is computed. In the common system the base 10 is chosen and the modulus computed.

78. From the exponential series the value of e may be computed to any required degree of accuracy.

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots.$$

$$1 + 1 + \frac{1}{\underline{|2}} = 2.5$$

$$\frac{1}{\underline{|3}} = 0.1666666666$$

$$\frac{1}{\underline{|4}} = 0.0416666666$$

$$\frac{1}{\underline{|5}} = 0.0083333333$$

$$\frac{1}{\underline{|6}} = 0.0013888888$$

$$\frac{1}{\underline{|7}} = 0.0001984126$$

$$\frac{1}{\underline{|8}} = 0.0000248015$$

$$\frac{1}{\underline{|9}} = 0.0000027557$$

$$\frac{1}{\underline{|10}} = 0.0000002755$$

. . . .

Adding, $e = 2.7182818$, correct to 7 decimal places.

79. To expand a^x in ascending powers of x.

$$e^z = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots + \frac{z^r}{\lfloor r} + \cdots.$$

Let $a^x = e^z$, then $z = \log_e a^x = x \cdot \log_e a.$ (Arts. 35, 40)

Substituting

$$a^x = 1 + x \cdot \log_e a + \frac{x^2 (\log_e a)^2}{\lfloor 2} + \frac{x^3 (\log_e a)^3}{\lfloor 3} + \cdots.$$

Now put $1 + a$ for a, and

$$(1+a)^x = 1 + x \cdot \log_e (1+a)$$
$$+ \frac{x^2 [\log_e (1+a)]^2}{\lfloor 2} + \frac{x^3 [\log_e (1+a)]^3}{\lfloor 3} + \cdots.$$

But by the binomial theorem,

$$(1+a)^x = 1 + xa + \frac{x(x-1)}{\lfloor 2} a^2 + \frac{x(x-1)(x-2)}{\lfloor 3} a^3 + \cdots.$$

Equating coefficients of x in the second members of the above equations,

$$\log_e (1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \cdots;$$

or writing x for a,

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

In this form the series is of little practical use, since it converges very slowly, and only when x is between $+1$ and -1 (higher algebra).

Put $-x$ for x, and

$$\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots;$$

$\therefore \log_e (1+x) - \log_e (1-x)$

$$= \log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

Finally, put $\frac{1}{2n+1}$ for x, and

$$\log_e \frac{n+1}{n} = \log_e (n+1) - \log_e n$$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots\right\},$$

$$\therefore \log_e (n+1) = \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots\right\},$$

a series which is rapidly convergent.

80. From this series a table of logarithms to the base e may be computed.

To find $\log_e 2$ put $n = 1$. Then, since $\log_e 1 = 0$, the series becomes

$$\log_e 2 = \log_e 1 + 2\left\{\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} + \frac{1}{11 \cdot 3^{11}} + \frac{1}{13 \cdot 3^{13}} + \cdots\right\} = 0.693147.$$

The computations may be arranged thus:

3	2.00000000	
9	.66666667	= .66666667
9	.07407407 ÷ 3	= .02469136
9	.00823045 ÷ 5	= .00164609
9	.00091449 ÷ 7	= .00013064
9	.00010161 ÷ 9	= .00001129
9	.00001129 ÷ 11	= .00000103
	.00000125 ÷ 13	= .00000009
		.69314717

whence $\log_e 2 = 0.693147$, correct to 6 decimal places.

To find $\log_e 3$, put $n = 2$, and

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} + \cdots\right).$$

log$_e$ 5 = 1.6094379
" 6 = 1.7917595
" 11 = 2.3978953

5	2.00000000
25	.40000000 $\qquad$ = .40000000
25	.01600000 ÷ 3 = .00533333
25	.00640000 ÷ 5 = .00012800
25	.00025600 ÷ 7 = .00000366
	.00000102 ÷ 9 = .00000011

$$.40546510$$
$$\log_e 2 = .69314717$$
$$\therefore \log_e 3 = 1.098612,$$

correct to 6 decimal places.

$\log_e 4 = 2 \times \log_e 2$, $\log_e 6 = \log_e 3 + \log_e 2$, etc. (Why?) The logarithms of prime numbers may be computed as above by giving proper values to n.

81. Having computed the logarithms of numbers to base e, the logarithms to any other base may be computed by means of the following relation:

Let $\qquad \log_a n = x$; then $a^x = n$.

Also, $\qquad \log_b n = y$; then $b^y = n$,

$$\therefore a^x = b^y.$$

Hence, $\qquad \log_a(a^x) = \log_a(b^y)$,

and $\qquad \therefore x = y \log_a b$.

It follows that $\qquad \log_a n = \log_b n \cdot \log_a b$;

whence $\qquad \mathbf{\log_b n = \log_a n \cdot \frac{1}{\log_a b}}.$

This factor $\frac{1}{\log_a b}$ is called the *modulus* of the system of logarithms to base b. Using it as a multiplier, logarithms of numbers to base b are computed at once from the logarithms of the same numbers to any other base a.

82. To compute the common logarithms.

Common logarithms are computed from the Naperian by use of the modulus $\frac{1}{\log_e 10}$; *i.e.*

$$\log_{10} n = \log_e n \cdot \frac{1}{\log_e 10}.$$

By Art. 80, $\log_e 10$ can be found, and

$$\frac{1}{log_e 10} = .434294, \textit{ the modulus of the common system.}$$

Ex. Compute the common logarithms of:

2, 3, 4, 6, 5, 10, 15, 216, 3375.

COMPLEX NUMBERS.

83. In algebra it is shown that the general expression for complex numbers is $a + bi$, where a represents all the real terms of the expression, b the coefficients of all the imaginary terms, and i is so defined that $i^2 = -1$; whence

$$i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \text{ etc.}$$

The laws of operation in algebra are found to apply to complex numbers. Moreover, it is further shown that if two complex numbers are equal, the real terms are equal, and the imaginary terms are equal; *i.e.* if

$$a + bi = c + di,$$

then $$a = c \text{ and } b = d.$$

Finally, the complex number may be graphically represented as follows:

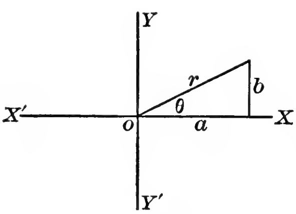


FIG. 43.

The real number is measured along OX, a units; the imaginary parallel to OY, b units. The line r is a graphic representation of $a + bi$.

Since $\qquad a = r\cos\theta$ and $b = r\sin\theta$,

$$\therefore\ a + bi = r(\cos\theta + i\sin\theta).$$

The properties of complex numbers are best developed by using this trigonometric form. If r be taken as unity, then $\cos\theta + i\sin\theta$ represents any complex number.

84. De Moivre's Theorem. To prove that, for any value of n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

I. When n is a positive integer.

By multiplication,

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$
$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

In like manner,

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$
$$= \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma);$$

and finally,

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)\cdots \text{ to } n \text{ factors}$$
$$= \cos(\alpha+\beta+\gamma+\cdots) + i\sin(\alpha+\beta+\gamma+\cdots).$$

Now let $\alpha = \beta = \gamma = \cdots$, and the above becomes

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

II. When n is a negative integer.

Let $n = -m$; then

$$(\cos\alpha + i\sin\alpha)^n = (\cos\alpha + i\sin\alpha)^{-m}$$
$$= \frac{1}{(\cos\alpha + i\sin\alpha)^m} = \frac{1}{\cos m\alpha + i\sin m\alpha}$$
$$= \frac{\cos m\alpha - i\sin m\alpha}{(\cos m\alpha + i\sin m\alpha)(\cos m\alpha - i\sin m\alpha)}$$
$$= \frac{\cos m\alpha - i\sin m\alpha}{\cos^2 m\alpha + \sin^2 m\alpha}$$
$$= \cos m\alpha - i\sin m\alpha = \cos(-m)\alpha + i\sin(-m)\alpha.$$

Substituting n for $-m$, the equation becomes

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

III. When n is a fraction, positive or negative.

Let $n = \frac{p}{q}$, p and q being any integers.

Now

$$\left(\cos\frac{\alpha}{q} + i\sin\frac{\alpha}{q}\right)^q = \cos q\cdot\frac{\alpha}{q} + i\sin q\cdot\frac{\alpha}{q} = \cos\alpha + i\sin\alpha \quad \text{(by I)}.$$

Then $$\left(\cos\frac{\alpha}{q} + i\sin\frac{\alpha}{q}\right) = (\cos\alpha + i\sin\alpha)^{\frac{1}{q}}.$$

Raising each member to the power p,

$$(\cos\alpha + i\sin\alpha)^{\frac{p}{q}} = \left(\cos\frac{\alpha}{q} + i\sin\frac{\alpha}{q}\right)^p = \cos\frac{p}{q}\alpha + i\sin\frac{p}{q}\alpha.$$

COMPUTATIONS OF NATURAL FUNCTIONS.

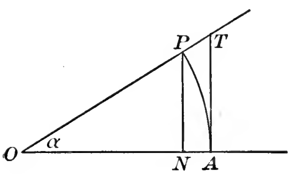


FIG. 44.

85. The radian measure of an acute angle is greater than its sine and less than its tangent, *i.e.*

$$\sin\alpha < \alpha < \tan\alpha.$$

Let α be the circular, or radian, measure of any acute angle AOP.

Then, in the figure,

$$\text{area of sector } OAP < \text{area of triangle } OAT,$$

$$\textit{i.e. } \tfrac{1}{2} OA \cdot \text{arc } AP < \tfrac{1}{2} OA \cdot AT.$$

$$\therefore \text{ arc } AP < AT.$$

Now, since $$NP < \text{arc } AP,$$

$$\frac{NP}{OP} < \frac{\text{arc } AP}{OP} < \frac{AT}{OP}.$$

But $$\frac{\text{arc } AP}{OP} = \text{circular measure of } AOP = \alpha;$$

whence $$\sin\alpha < \alpha < \tan\alpha.$$

86. Since $$\sin\alpha < \alpha < \tan\alpha,$$

$$1 < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha}.$$

Hence, however small α may be, $\frac{\alpha}{\sin\alpha}$ lies between 1 and $\frac{1}{\cos\alpha}$. When α approaches 0, $\cos\alpha$ approaches unity. Therefore, by diminishing α sufficiently, we may make $\frac{\alpha}{\sin\alpha}$ differ from unity by an amount less than any assignable quantity. This we express by saying that when α approaches 0, $\frac{\alpha}{\sin\alpha}$ approaches unity as a limit, *i.e.* $\frac{\alpha}{\sin\alpha}=1$, approximately. Multiplying by $\cos\alpha$ ($=1$, nearly), we have $\frac{\alpha}{\tan\alpha}=1$, approximately. Whence, if α approaches 0, $\tan\alpha=\sin\alpha=\alpha$, approximately.

87. Sine and cosine series.

$$\cos n\alpha + i\sin n\alpha = (\cos\alpha + i\sin\alpha)^n, \text{ (De Moivre's Theorem).}$$

Expanding the second member by the binomial formula, it becomes,

$$\cos^n\alpha + n\cos^{n-1}\alpha\cdot i\sin\alpha + \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\alpha\,.\,i^2\sin^2\alpha$$
$$+\frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\alpha\cdot i^3\sin^3\alpha$$
$$+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\alpha\cdot i^4\sin^4\alpha+\cdots.$$

Substituting the values of i^2, i^3, i^4, etc., we have

$$\cos n\alpha + i\sin n\alpha = \cos^n\alpha - \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\alpha\sin^2\alpha$$
$$+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\alpha\sin^4\alpha - \cdots$$
$$+i\left(n\cos^{n-1}\alpha\sin\alpha - \frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\alpha\sin^3\alpha+\cdots\right).$$

Equating the real and imaginary parts in the two members,

$$cos\, n\alpha = cos^n\, \alpha - \frac{n(n-1)}{\underline{|2}} cos^{n-2}\, \alpha\, sin^2\, \alpha$$

$$+ \frac{n(n-1)(n-2)(n-3)}{\underline{|4}} cos^{n-4}\, \alpha\, sin^4\, \alpha - \cdots,$$

and $$sin\, n\alpha = n\, cos^{n-1}\, \alpha - \frac{n(n-1)(n-2)}{\underline{|3}} cos^{n-3}\, \alpha\, sin^3\, \alpha + \cdots.$$

Ex. **1.** Find $\cos 3\alpha$; $\sin 3\alpha$.

In the above put $n = 3$, and $\cos 3\alpha = \cos^3 \alpha - 3 \cos \alpha \sin^2 \alpha$

$$= 4 \cos^3 \alpha - 3 \cos \alpha;$$

also $$\sin^3 \alpha = 3 \cos^2 \alpha \sin \alpha - \sin^3 \alpha$$

$$= 3 \sin \alpha - 4 \sin^3 \alpha.$$

2. Find $\sin 4\alpha$; $\cos 4\alpha$; $\sin 5\alpha$; $\cos 5\alpha$.

It will be noticed that in the series for *cos nα* and *sin nα* the terms are alternately positive and negative, and that the series continues till there is a zero factor in the numerator.

88. If now in the above series we let $n\alpha = \theta$, then

$$\cos\theta = \cos^n \alpha - \frac{\frac{\theta}{\alpha}\left(\frac{\theta}{\alpha} - 1\right)}{\underline{|2}} \cos^{n-2} \alpha \sin^2 \alpha$$

$$\frac{\frac{\theta}{\alpha}\left(\frac{\theta}{\alpha} - 1\right)\left(\frac{\theta}{\alpha} - 2\right)\left(\frac{\theta}{\alpha} - 3\right)}{\underline{|4}} \cos^{n-4} \alpha \sin^4 \alpha - \cdots$$

$$= \cos^n \alpha - \frac{\theta(\theta - \alpha)}{\underline{|2}} \cos^{n-2} \alpha \left(\frac{\sin \alpha}{\alpha}\right)^2$$

$$+ \frac{\theta(\theta - \alpha)(\theta - 2\alpha)(\theta - 3\alpha)}{\underline{|4}} \cos^{n-4} \alpha \left(\frac{\sin \alpha}{\alpha}\right)^4 - \cdots.$$

If now θ remain constant, and α decrease without limit, then will n become indefinitely great, and $\frac{\sin \alpha}{\alpha}$ and every power thereof, and $\cos\alpha$ and every power of $\cos\alpha$ will approach unity as a limit, so that

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \cdots.$$

Similarly,
$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \cdots.$$

By algebra it is shown that these series are convergent for all values of θ. By their use we can compute values of $\sin\theta$ and $\cos\theta$ to any required degree of accuracy.

Show from the above that $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\,\theta^5}{15} + \cdots.$

Ex. **1.** Compute the value of $\sin 1°$, correct to 5 places.

In $\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \cdots$, make θ the radian measure of

$$1° = \frac{\pi}{180} = 0.01745 +.$$

Then,
$$\theta = 0.01745 +$$
$$\frac{\theta^3}{\lfloor 3} = 0.0000008.$$
$$\therefore \sin\theta = 0.01745 +.$$

The terms of the series after the first do not affect the fifth place, so that the value is given by the first term, an illustration of the fact that, if α is small, $\sin\alpha = \alpha$, approximately. Compare the value of $\tan 1°$.

2. Show that $\sin 10° = 0.17365$; $\cos 10° = 0.98481$; $\sin 15° = 0.25882$; $\cos 60° = 0.50000$.

3. Find the sine and cosine of $18°\,30'$; $22°\,15'$; $67°\,45'$.

It is unnecessary to compute the functions beyond 30°, for since

$$\sin(30° + \theta) + \sin(30° - \theta) = \cos\theta \text{ (why?)},$$
$$\therefore \sin(30° + \theta) = \cos\theta - \sin(30° - \theta).$$

So, also,
$$\cos(30° + \theta) = \cos(30° - \theta) - \sin\theta.$$

Giving θ proper values the functions of any angle from 30° to 45° are determined at once from the functions of angles less than 30°.

Thus,
$$\sin 31° = \cos 1° - \sin 29°;$$
$$\cos 31° = \cos 29° - \sin 1°.$$

4. Find sine and cosine of 40°; of 50°.

89. The following are sometimes useful in applied mathematics:

Ex. **1.** To find the sum of a series of sines of angles in A. P., such as

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+[n-1]\beta).$$

$$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right),$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{3\beta}{2}\right),$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{3\beta}{2}\right) - \cos\left(\alpha+\frac{5\beta}{2}\right),$$

. . . .

$$2\sin(\alpha+[n-1]\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{2n-3}{2}\beta\right) - \cos\left(\alpha+\frac{2n-1}{2}\beta\right).$$

Adding

$$2\{\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+[n-1]\beta)\}\sin\frac{\beta}{2}$$

$$= \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$

$$= 2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n}{2}\beta.$$

$$\therefore \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+[n-1]\beta)$$

$$= \frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n}{2}\beta}{\sin\frac{\beta}{2}}.$$

Similarly it can be shown that

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+[n-1]\beta)$$

$$= \frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n}{2}\beta}{\sin\frac{\beta}{2}}.$$

90. The series $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots + \frac{x^r}{\lfloor r} + \cdots$ is proved in higher algebra to be true for all values of x, real or imaginary. Then if $x = i\theta$,

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{\lfloor 2} + \frac{i^3\theta^3}{\lfloor 3} + \cdots + \frac{i^r\theta^r}{\lfloor r} + \cdots$$

$$= 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \cdots + i\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \cdots\right).$$

$$\therefore\ e^{i\theta} = \cos\theta + i\sin\theta \text{ (Art. 87).}$$

In like manner, $e^{-i\theta} = \cos\theta - i\sin\theta.$

Adding, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$;

subtracting, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

HYPERBOLIC FUNCTIONS.

91. Since $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ are true for all values of θ, let $\theta = i\theta$.

Then, $\sin(i\theta) = \dfrac{e^{-\theta} - e^{\theta}}{2i} = i\dfrac{e^{\theta} - e^{-\theta}}{2} = i\sinh\theta,$

and $\cos(i\theta) = \dfrac{e^{\theta} + e^{-\theta}}{2} = \cosh\theta,$

so that $\tan(i\theta) = \dfrac{\sin(i\theta)}{\cos(i\theta)} = \dfrac{i\sinh\theta}{\cosh\theta} = i\tanh\theta,$

where *sinh* θ, *cosh* θ, *tanh* θ, are called the *hyperbolic sine, cosine*, and *tangent* of θ. The hyperbolic cotangent, secant, and cosecant of θ are obtained from the hyperbolic sine, cosine, and tangent, just as the corresponding circular functions, cotangent, secant, and cosecant, are obtained from tangent, cosine, and sine. The hyperbolic functions have the same geometric relations to the rectangular hyperbola that the circular functions have to the circle, hence the name hyperbolic functions.

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}, \qquad \therefore \operatorname{csch}\theta = \frac{2}{e^\theta - e^{-\theta}};$$

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}, \qquad \therefore \operatorname{sech}\theta = \frac{2}{e^\theta + e^{-\theta}};$$

$$\tanh\theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}, \qquad \therefore \coth\theta = \frac{e^\theta + e^{-\theta}}{e^\theta - e^{-\theta}}.$$

92. From the relations of Art. 91 it appears that to any relation between the circular functions there corresponds a relation between the hyperbolic functions.

Since $$\cos^2(i\theta) + \sin^2(i\theta) = 1,$$

$$\cosh^2\theta + i^2\sinh^2\theta = 1,$$

or $$\cosh^2\theta - \sinh^2\theta = 1.$$

This may also be derived thus:

$$\cosh^2\theta - \sinh^2\theta = \left(\frac{e^\theta + e^{-\theta}}{2}\right)^2 - \left(\frac{e^\theta - e^{-\theta}}{2}\right)^2$$

$$= \frac{e^{2\theta} + 2 + e^{-2\theta} - e^{2\theta} + 2 - e^{-2\theta}}{4} = 1.$$

Also since

$$\sin(i\alpha + i\beta) = \sin(i\alpha)\cos(i\beta) + \cos(i\alpha)\sin(i\beta),$$

$$\therefore i\sinh(\alpha + \beta) = i\sinh\alpha\cosh\beta + \cosh\alpha \cdot i\sinh\beta,$$

and $$\sinh(\alpha + \beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta.$$

Let the student verify this relation from the exponential values of sinh and cosh.

EXAMPLES.

Prove

1. $\cosh(\alpha + \beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta.$
2. $\cosh(\alpha + \beta) - \cosh(\alpha - \beta) = 2\sinh\alpha\sinh\beta.$
3. $\cosh 2\theta = 1 + 2\sinh^2\theta = 2\cosh^2\theta - 1.$
4. $\sinh 2\alpha = 2\sinh\alpha\cosh\alpha.$

5. $\cosh\frac{\theta}{2}=\sqrt{\frac{1+\cosh\theta}{2}}$; $\sinh\frac{\theta}{2}=\sqrt{\frac{\cosh\theta-1}{2}}$.

6. $\sinh 3\theta = 3\sinh\theta + 4\sinh^3\theta$.

7. $\sinh\theta + \sinh\phi = 2\sinh\frac{\theta+\phi}{2}\cosh\frac{\theta-\phi}{2}$.

8. $\sinh\alpha+\sinh(\alpha+\beta)+\sinh(\alpha+2\beta)+\cdots+\sinh(\alpha+[n-1]\beta)$

$$=\frac{\sinh\left(\alpha+\frac{n-1}{2}\beta\right)\sinh\frac{n}{2}\beta}{\sinh\frac{\beta}{2}}$$

9. $\tanh(\theta+\phi)=\frac{\tanh\theta+\tanh\phi}{1+\tanh\theta\tanh\phi}$.

10. $\sinh^{-1}x=\cosh^{-1}\sqrt{1+x^2}=\tanh^{-1}\frac{x}{\sqrt{1+x^2}}$.

11. $\cosh(\alpha+\beta)\cosh(\alpha-\beta)=\cosh^2\alpha+\sinh^2\beta=\cosh^2\beta+\sinh^2\alpha$.

12. $2\cosh n\alpha\cosh\alpha=\cosh(n+1)\alpha+\cosh(n-1)\alpha$.

13. $\cosh\alpha=\frac{1}{2}(e^{\alpha}+e^{-\alpha})=1+\frac{\alpha^2}{\underline{|2}}+\frac{\alpha^4}{\underline{|4}}+\cdots$.

14. $\sinh\alpha=\frac{1}{2}(e^{\alpha}-e^{-\alpha})=\alpha+\frac{\alpha^3}{\underline{|3}}+\frac{\alpha^5}{\underline{|5}}+\cdots$.

15. $\tanh^{-1}a+\tanh^{-1}b=\tanh^{-1}\frac{a+b}{1+ab}$.

SPHERICAL TRIGONOMETRY.

CHAPTER X.

SPHERICAL TRIANGLES.

93. Spherical trigonometry is concerned chiefly with the solution of spherical triangles. Its applications are for the most part in geodesy and astronomy.

The following definitions and theorems of geometry are for convenience of reference stated here.

A *great circle* is a plane section of a sphere passing through the centre. Other plane sections are *small circles.*

The shortest distance between two points on a sphere is measured on the arc of a great circle, less than 180°, which joins them.

A *spherical triangle* is any portion of the surface of a sphere bounded by three arcs of great circles. We shall consider only triangles whose sides are arcs not greater than 180° in length.

The *polar triangle* of any spherical triangle is the triangle whose sides are drawn with the vertices of the first triangle as poles. If ABC is the polar of $A'B'C'$, then $A'B'C'$ is the polar of ABC.

In any spherical triangle,

The sum of two sides $>$ *the third side.*

The greatest side is opposite the greatest angle, and conversely.

Each angle $< 180°$; *the sum of the angles* $> 180°$, *and* $< 540°$.

Each side $< 180°$; *the sum of the sides* $< 360°$.

The sides of a spherical triangle are the supplements of the angles opposite in the polar triangle, and conversely.

If two angles are equal the sides opposite are equal, and conversely.

The sides of a spherical triangle subtend angles at the centre of the sphere which contain the same number of angle degrees as the arc does of arc degrees; *i.e.* an angle at the centre and its arc have the same measure numerically.

The arc does not measure the angle for they have not the same unit of measurement, but we say they have the same numerical measure; *i.e.* the arc contains the unit arc as many times as the angle contains the unit angle.

The angles of a spherical triangle are said to be measured by the plane angle included by tangents to the sides of the angle at their intersection. They have therefore the same numerical measure as the dihedral angle between the planes of the arcs.

In the figure the following have the same numerical measure:

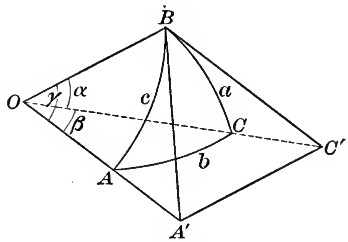


FIG. 45.

arc a and angle α;

arc b and angle β;

arc c and angle γ;

plane angle $A'BC'$;

spherical angle B and dihedral angle $A–BO–C$;

spherical angle C and dihedral angle $B–CO–A$;

spherical angle A and dihedral angle $C–AO–B$.

$A'C'B$ and $C'A'B$ have not the same measure as spherical angles C and A, for BA', $A'C'$, $C'B$ are not perpendicular to OA or OC.

94. In plane trigonometry the trigonometric functions were treated as functions of the angles. But since an angle and its subtending arc vary together and have the same numerical measure, it is clear that the trigonometric ratios are functions of the arcs, and may be so considered. All the relations between the functions are the same whether we consider them with reference to the angle or the arc, so that all the identities of plane trigonometry are true for the functions of the arcs.

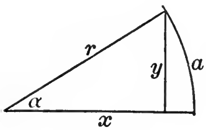


FIG. 46.

Thus in the figure we may write,

$$\sin \alpha = \frac{y}{r} \text{ or } \sin a = \frac{y}{r};$$

$$\sin^2 \alpha + \cos^2 \alpha = 1, \text{ or } \sin^2 a + \cos^2 a = 1;$$

$$\cos 2\alpha = 2\cos^2 \alpha - 1, \text{ or } \cos 2a = 2\cos^2 a - 1.$$

GENERAL FORMULÆ FOR SPHERICAL TRIANGLES.

95. The solutions of spherical triangles may be effected by formulæ now to be developed:

First it will be shown that in any spherical triangle

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

The following cases must be considered:

I. Both b and $c < 90°$.

II. $b > 90°$, $c < 90°$.

III. Both b and $c > 90°$.

IV. Either b or $c = 90°$.

V. $b = c = 90°$.

The figure applies to Case I.

Let ABC be a spherical triangle, a, b, c its sides, and O the centre of the sphere.

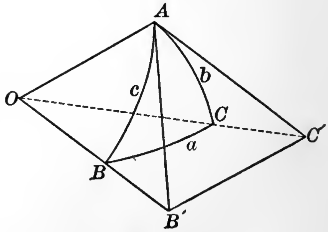


FIG. 47.

Draw AC' and AB' tangent to the sides b, c at A. (The same result would be obtained by drawing AB', AC' perpendicular to OA at any point to meet OB, OC.) Since these tangents lie in the planes of the circles to which they are drawn, they will meet OC and OB in C' and B', and the angle $C'AB'$ will be the measure of the angle A of the spherical triangle ABC. Since OAB', OAC' are right angles, AOB', AOC' must be acute, and hence sides c, b are each $< 90°$.

In the triangles $C'AB'$ and $C'OB'$,

$$C'B'^2 = AC'^2 + AB'^2 - 2\,AC' \cdot AB' \cos C'AB',$$

and $$B'C'^2 = OC'^2 + OB'^2 - 2\,OC' \cdot OB' \cos C'OB'.$$

Subtracting and noting that

$$\cos C'AB' = \cos A \text{ and } \cos C'OB' = \cos a,$$

we have

$$0 = OC'^2 - AC'^2 + OB'^2 - AB'^2 + 2\,AC' \cdot AB' \cos A - 2\,OC' \cdot OB' \cos a.$$

But $OC'^2 - AC'^2 = OA^2$ and $OB'^2 - AB'^2 = OA^2$.

Hence, $$0 = OA^2 + AC' \cdot AB' \cos A - OC' \cdot OB' \cos a;$$

or $$\cos a = \frac{OA}{OC'} \cdot \frac{OA}{OB'} + \frac{AC'}{OC'} \cdot \frac{AB'}{OB'} \cos A.$$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Similarly,

$$\cos b = \cos a \cos c + \sin a \sin c \cos B,$$

and $$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

These formulæ are important, and should be carefully memorized.

II. $b > 90°$; $c < 90°$.

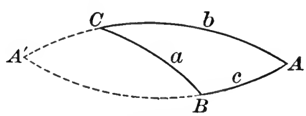


FIG. 48.

In the triangle ABC, let $b > 90°$ and $c < 90°$. Complete the lune $BACA'$. Then in the triangle $A'CB$ the sides a and $A'C$ are both less than $90°$, and by (I)

$$\cos A'B = \cos A'C \cos a + \sin A'C \sin a \cos A'CB.$$

But $A'B = 180° - c$, $A'C = 180° - b$, and $A'CB = 180° - C$.

$$\therefore \cos(180° - c) = \cos(180° - b)\cos a + \sin(180° - b)\sin a \cos(180° - C);$$

or $$-\cos c = (-\cos b)\cos a + \sin b \sin a(-\cos C),$$

and $$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

A similar proof will apply in case $c > 90°$, $b < 90°$.

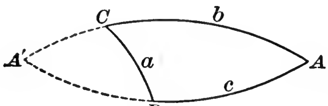


FIG. 49.

III. Both b and $c > 90°$.

In the triangle ABC, let both b and $c > 90°$. Complete the lune $ABA'C$. Then since $A'C$ and $A'B$ are both $< 90°$,

$$\cos a = \cos A'C \cos A'B + \sin A'C \sin A'B \cos A'.$$

But $A' = A$, $A'C = 180° - b$, $A'B = 180° - c$.

$$\therefore \cos a = \cos(180° - b)\cos(180° - c) + \sin(180° - b)\sin(180° - c)\cos A;$$

or $$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Cases IV and V are left to the student as exercises.

96. Since the angles of the polar triangle are the supplements of the sides opposite in the first triangle, we have

$$a' = 180° - A, \quad b' = 180° - B,$$

$$c' = 180° - C, \quad A' = 180° - a.$$

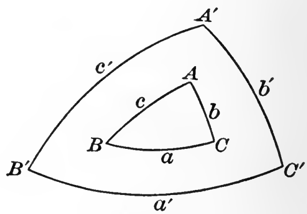


FIG. 50.

Substituting in

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A',$$

we have

$$-\cos(180° - A) = \cos(180° - B)\cos(180° - C) + \sin(180° - B)\sin(180° - C)\cos(180° - a);$$

or $$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a).$$

Changing signs,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Similarly, $$\cos B = -\cos A \cos C + \sin A \sin C \cos b,$$

and $$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

97. In any spherical triangle to prove $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$.

Since $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

$$\therefore \sin^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Hence,

$$\sin A = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c - 2 \cos a \cos b \cos c}}{\sin b \sin c}$$

and $$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c - 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

By a similar process, $\dfrac{\sin B}{\sin b}$ and $\dfrac{\sin C}{\sin c}$ will be found equal to the same expression.

$$\therefore \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

98. Expressions for sine, cosine, and tangent of half an angle in terms of functions of the sides.

We have $2\sin^2\frac{A}{2} = 1 - \cos A$

$$= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos(b - c) - \cos a}{\sin b \sin c}.$$

Then $2\sin^2\frac{A}{2} = \frac{2\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin b \sin c}$ (Art. 51)

$$= \frac{2\sin(s-b)\sin(s-c)}{\sin b \sin c},$$

when $2s = a + b + c.$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}.$$

Similarly, $\sin\frac{B}{2} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin a \sin c}},$

and $\sin\frac{C}{2} = \sqrt{\frac{\sin(s-b)\sin(s-a)}{\sin a \sin b}}.$

Also from the relation

$$2\cos^2\frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

we have $\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$

Also, $\cos\frac{B}{2} = \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}},$

and $\cos\frac{C}{2} = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}.$

From the above,

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

Also,
$$\tan\frac{B}{2}=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}},$$

and
$$\tan\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}.$$

Compare the formulæ thus far derived with the corresponding formulæ for solving plane triangles. The similarity in forms will assist in memorizing the formulæ for solving spherical triangles.

99. From the formulæ of Art. 96, the student can easily prove the following relations:

$$\sin\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}},$$

where
$$2S=A+B+C.$$

$$\sin\frac{b}{2}=?,$$

$$\sin\frac{c}{2}=?.$$

$$\cos\frac{a}{2}=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}},$$

$$\cos\frac{b}{2}=?,$$

$$\cos\frac{c}{2}=?.$$

$$\tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}},$$

$$\tan\frac{b}{2}=?,$$

$$\tan\frac{c}{2}=?.$$

100. Napier's Analogies.

Since
$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}=\frac{\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}}{\sqrt{\dfrac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}}}$$

$$=\sqrt{\frac{\sin^2(s-b)}{\sin^2(s-a)}}=\frac{\sin(s-b)}{\sin(s-a)};$$

by composition and division,

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}}=\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)},$$

$$\frac{\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\dfrac{\sin\frac{B}{2}}{\cos\frac{B}{2}}}{\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}-\dfrac{\sin\frac{B}{2}}{\cos\frac{B}{2}}}=\frac{\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(2s-a-b)\sin\frac{1}{2}(a-b)},$$
(Art. 51)

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(2s-a-b)}{\tan\frac{1}{2}(a-b)}$$

$$=\frac{\tan\frac{c}{2}}{\tan\frac{1}{2}(a-b)},\quad \text{since } 2s-a-b=c.$$

$$\therefore \tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

To find an expression for $\tan\frac{1}{2}(A-B)$ we have only to consider the polar triangle, and by substituting $180°-A$ for a, etc., $180°-a$ for A, etc., we have the following relations:

$$\tfrac{1}{2}(a-b)=\tfrac{1}{2}(180°-A-180°+B)=-\tfrac{1}{2}(A-B);$$

also, $\frac{1}{2}(A-B) = -\frac{1}{2}(a-b)$;

$$\tfrac{1}{2}(A+B) = \tfrac{1}{2}(180° - a + 180° - b) = 180° - \tfrac{1}{2}(a+b);$$

and $$\frac{c}{2} = 90° - \frac{C}{2}.$$

The formula then becomes, applying Art. 29,

$$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

Formulæ for $\tan\frac{1}{2}(a+b)$, $\tan\frac{1}{2}(A+B)$ are derived as follows:

Since

$$\tan\frac{A}{2}\cdot\tan\frac{B}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\cdot\sin(s-a)}}\cdot\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\cdot\sin(s-b)}},$$

$$\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin(s-c)}{\sin s}.$$

By composition and division,

$$\frac{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}} = \frac{\sin s + \sin(s-c)}{\sin s - \sin(s-c)},$$

whence $$\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{c}{2}}, \qquad \text{(Art. 51)}$$

since $$2s - c = a + b,$$

or, $$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

The value of $\tan \frac{1}{2}(A + B)$ is derived by substituting in terms of the corresponding elements of the polar triangle.

$$\frac{\cos \frac{1}{2}(a - b)}{-\cos \frac{1}{2}(a + b)} = \frac{-\tan \frac{1}{2}(A + B)}{\cot \frac{C}{2}},$$

$$\therefore \tan \frac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

Similar relations among the other elements of the triangle may be derived, or they may be written from the above by proper changes of A, B, C, a, b, c in the formulæ. The student should write them out as exercises.

101. Delambre's Analogies.

Since $$\sin \frac{1}{2}(A + B) = \sin \frac{A}{2} \cos \frac{B}{2} + \cos \frac{A}{2} \sin \frac{B}{2},$$

then

$$\sin \frac{1}{2}(A + B) = \frac{\sin (s - b) + \sin (s - a)}{\sin c} \cdot \sqrt{\frac{\sin s \cdot \sin (s - c)}{\sin a \cdot \sin b}}.$$

(Art. 98)

Hence, $$\frac{\sin \frac{1}{2}(A + B)}{\cos \frac{C}{2}} = \frac{\sin (s - b) + \sin (s - a)}{\sin c}$$

$$= \frac{2 \sin \frac{c}{2} \cos \frac{1}{2}(a - b)}{2 \sin \frac{c}{2} \cos \frac{c}{2}}, \quad \text{(Art. 51)}$$

and $$\sin \frac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{c}{2}} \cos \frac{C}{2};$$

In like manner derive

$$\sin \frac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{c}{2}} \cos \frac{C}{2};$$

$$\cos \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a+b)}{\cos \frac{c}{2}} \sin \frac{C}{2};$$

$$\cos \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a+b)}{\sin \frac{c}{2}} \sin \frac{C}{2}.$$

These formulæ are often called Gauss's Formulæ, but they were first discovered by Delambre in 1807. Afterwards Gauss, independently, discovered them, and published them in his *Theoria Motus*.

102. Formulæ for solving *right spherical triangles* are derived from the foregoing by putting $C = 90°$, whence $\sin C = 1$, $\cos C = 0$.

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \qquad \text{(Art. 95)}$$

becomes

$$\cos c = \cos a \cos b. \qquad (1)$$

Substituting the value of $\cos a$ from (1), and simplifying,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \qquad \text{(Art. 95)}$$

becomes

$$\cos A = \frac{\tan b}{\tan c}. \qquad (2)$$

Again,

$$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c} \qquad \text{(Art. 97)}$$

in the right triangle is

$$\sin A = \frac{\sin a}{\sin c}. \qquad (3)$$

Dividing (3) by (2),

$$\tan A = \frac{\sin a \cos b}{\cos c \sin b} = \frac{\sin a \cos a \cos b}{\cos c \cos a \sin b} = \frac{\sin a}{\cos a \sin b},$$

since

$$\cos a \cos b = \cos c.$$

$$\therefore \tan A \sin b = \tan a. \qquad (4)$$

From (4) $\tan a = \tan A \sin b,$

also, $\tan b = \tan B \sin a.$

Multiplying, $\tan a \tan b = \tan A \tan B \sin a \sin b,$

or, $$\cot A \cot B = \cos a \cos b = \cos c. \qquad (5)$$

From (2) and (3), by division,

$$\frac{\cos A}{\sin B} = \frac{\dfrac{\tan b}{\tan c}}{\dfrac{\sin b}{\sin c}} = \frac{\cos c}{\cos b} = \cos a.$$

$$\therefore \cos A = \cos a \sin B. \qquad (6)$$

Let the student write formulæ (2), (3), (4), (6) for B. It will be noticed that (1) and (5) give values for c only, while (2), (3), (4), (6) apply only to A and B.

103. Formulæ (1)–(6) are sufficient for the solution of right spherical triangles if any two parts besides the right angle are given. They are easily remembered by comparison with corresponding formulæ in plane trigonometry. Two rules, invented by Napier, and called *Napier's Rules of Circular Parts*, include all the formulæ of Art. 102.

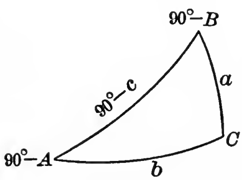


FIG. 51.

Omitting C, and taking the complements of A, c, and B, the parts of the triangle taken in order are a, b, $90° - A$, $90° - c$, $90° - B$. These are called the *circular parts* of the triangle.

Any one of the five parts may be selected as the *middle part*, the two parts next to it are called the *adjacent parts*, and the remaining two the *opposite parts*. Thus, if a be taken as the middle part, $90° - B$ and b are the adjacent parts, and $90° - c$, $90 - A$ the opposite parts.

Napier's Two Rules are as follows:

The sine of the middle part equals the product of the tangents of the adjacent parts.

The sine of the middle part equals the product of the cosines of the opposite parts.

It will aid the memory somewhat to notice that **i** occurs in sine and middle, **a** in tangent and adjacent, and **o** in cosine and opposite, these words being associated in the rules.

The value of the above rules is frequently questioned, most computers preferring to associate the formulæ with the corresponding formulæ of plane trigonometry.

These rules may be proved by taking each of the parts as the middle part, and showing that the formulæ derived from the rules reduce to one of the six formulæ of Art. 102.

Then, if b is the middle part, by the rules,

$$\sin b = \tan a \tan(90° - A) = \tan a \cot A, \text{ or } \tan A = \frac{\tan a}{\sin b},$$

$$\sin b = \cos(90° - c)\cos(90° - B) = \sin c \sin B,$$

or

$$\sin B = \frac{\sin b}{\sin c},$$

results which agree with (4) and (3), Art. 102. If any other part be taken as the middle part, the rules will be found to hold.

104. Area of the spherical triangle.

If r = radius of the sphere,

E = spherical excess of the triangle $= A + B + C - 180°$,

Δ = area of spherical triangle, then by geometry

$$\Delta = Er^2 \times \frac{\pi}{180}.$$

If the three angles are not known, E may be computed by one of the following methods, and Δ found as above.

Cagnoli's Method.

$$\sin\frac{E}{2} = \sin\tfrac{1}{2}(A + B + C - 180^\circ)$$

$$= \sin\tfrac{1}{2}(A + B)\sin\frac{C}{2} - \cos\tfrac{1}{2}(A + B)\cos\frac{C}{2}$$

$$= \left[\cos\tfrac{1}{2}(a - b) - \cos\tfrac{1}{2}(a + b)\right]\frac{\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{c}{2}} \qquad \text{(Art. 101)}$$

$$= \frac{2\sin\frac{a}{2}\sin\frac{b}{2}}{\cos\frac{c}{2}} \cdot \frac{\sqrt{\sin s\sin(s - a)\sin(s - b)\sin(s - c)}}{\sin a\sin b}.$$

(Arts. 51, 98)

$$\sin\frac{E}{2} = \frac{\sqrt{\sin s\sin(s - a)\sin(s - b)\sin(s - c)}}{2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}.$$

Lhuilier's Method.

$$\tan\frac{E}{4} = \frac{\sin\frac{1}{4}(A + B + C - 180^\circ)}{\cos\frac{1}{4}(A + B + C - 180^\circ)}.$$

Now, multiply each term of the fraction by

$$2\cos\tfrac{1}{4}(A + B - C + 180^\circ),$$

and by Art. 51, (1) and (3), the equation becomes

$$\tan\frac{E}{4} = \frac{\sin\frac{1}{2}(A + B) - \cos\frac{C}{2}}{\cos\frac{1}{2}(A + B) + \sin\frac{C}{2}}$$

$$= \frac{\left[\cos\frac{1}{2}(a - b) - \cos\frac{c}{2}\right]\cos\frac{C}{2}}{\left[\cos\frac{1}{2}(a + b) + \cos\frac{c}{2}\right]\sin\frac{C}{2}} \qquad \text{(Art. 101)}$$

$$= \frac{\sin\frac{1}{2}(s - b)\sin\frac{1}{2}(s - a)}{\cos\frac{s}{2}\cos\frac{1}{2}(s - c)} \cdot \sqrt{\frac{\sin s\sin(s - c)}{\sin(s - a)\sin(s - b)}}.$$

(Art. 51)

By Art. 52, introducing the coefficient under the radical,

$$\tan\frac{E}{4}=\sqrt{\tan\frac{s}{2}\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}.$$

If two sides and the included angle are given, E may be determined as follows:

$$\cos\frac{E}{2}=\cos\tfrac{1}{2}(A+B+C-180^\circ)$$

$$=\cos\tfrac{1}{2}(A+B)\sin\frac{C}{2}+\sin\tfrac{1}{2}(A+B)\cos\frac{C}{2}$$

$$=\cos\tfrac{1}{2}(a+b)\sin^2\frac{C}{2}+\cos\tfrac{1}{2}(a-b)\cos^2\frac{C}{2}\quad\text{(Art. 101)}$$

$$=\frac{\cos\frac{a}{2}\cos\frac{b}{2}+\sin\frac{a}{2}\sin\frac{b}{2}\cos C}{\cos\frac{c}{2}}.$$

But $$\sin\frac{E}{2}=\frac{\sin\frac{a}{2}\sin\frac{b}{2}\cdot 2\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{c}{2}}.\quad\text{(Cagnoli's Method)}$$

Dividing this equation by the above,

$$\tan\frac{E}{2}=\frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin C}{\cos\frac{a}{2}\cos\frac{b}{2}+\sin\frac{a}{2}\sin\frac{b}{2}\cos C}.$$

This formula is not suitable for logarithmic computations. Usually it is better to compute the angles by Napier's Analogies, and solve by $\Delta=Er^2\times\frac{\pi}{180}$.

EXAMPLES.

1. Show that $\cos a = \cos b \cos c + \sin b \sin c \cos A$ becomes $\sec A = 1 + \sec a$, when $a = b = c$.

2. If $a + b + c = \pi$, prove

$$(a)\quad \cos a = \tan\frac{B}{2}\tan\frac{C}{2}.$$

$$(b)\quad \cos^2\frac{A}{2} = \frac{\cos a}{\sin b \sin c}.$$

$$(c)\quad \sin^2\frac{A}{2} = \cot b \cot c.$$

$$(d)\quad \cos A + \cos B + \cos C = 1.$$

$$(e)\quad \sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1.$$

3. Prove $$\frac{\sin\frac{E}{2}\cos\frac{1}{2}(A - E)}{\sin\frac{A}{2}} = \frac{\sin\frac{S}{2}\sin\frac{1}{2}(s - a)}{\cos\frac{a}{2}}.$$ (Art. 104)

4. Show that $\cos a \sin b = \sin a \cos b \cos C + \sin c \cos A$.

CHAPTER XI.

SOLUTION OF SPHERICAL TRIANGLES.

105. According to the principles of spherical geometry any three parts are sufficient to *determine* a spherical triangle; the other parts are *computed*, if any three are given, by the formulæ of trigonometry. The known parts may be:

I. Three sides, or three angles.

II. Two sides and the included angle, or two angles and the included side.

III. Two sides and an angle opposite one, or two angles and a side opposite one.

It will appear that, as in plane geometry, III may be ambiguous.

The *signs* of the functions in the formulæ are important since the cosines and tangents of arcs and angles greater than 90° are negative; whether the part sought is greater or less than 90° is therefore determined by the sign of the function in terms of which it is found unless this function be sine. In this case the result is ambiguous, since $\sin \alpha$ and $\sin(180° - \alpha)$ have the same sign and value. Thus if the solution gives $\log \sin a = 9.56504$, we may have either $a = 21° \, 33'$, or $158° \, 27'$. The conditions of the problem must determine which values apply to the triangle in question.

The negative signs, when they occur, will be indicated thus:

$$\log \cos 115° \, 20' = 9.63135^{-},$$

indicating, not that the logarithm is negative, but that in the final result account must be made of the fact that $\cos 115° \, 20'$ is negative.

106. *Formulæ for the solution of triangles.*

I. $$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}.$$

II. $$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

III. $$\tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}.$$

IV. $$\tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

V. $$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

VI. $$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

VII. $$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

VIII. $$\Delta=Er^2\frac{\pi}{180},$$

where E is determined by

$$\tan\frac{E}{4}=\sqrt{\tan\frac{s}{2}\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}.$$

Right triangles may be solved as special cases of oblique triangles, or by the following :

(1) $\cos c=\cos a\cos b$.

(2) $\cos A=\dfrac{\tan b}{\tan c}$.

(3) $\sin A=\dfrac{\sin a}{\sin c}$.

(4) $\tan A\sin b=\tan a$.

(5) $\cot A\cot B=\cos c$.

(6) $\cos A=\cos a\sin B$.

The formula to be used in any case may be determined by applying Napier's Rule of Circular Parts.

107. In solving a triangle the student should select formulæ in which all parts save one are known, and solve for that one (see page 77). Referring to Arts. 105 and 106, it will appear that solutions are effected as follows :

Case I by formulæ II, or III, check by I.

Case II by formulæ VI, VII, I, or IV, V, I, check by IV or VI.

Case III by formulæ I, IV, or I, VI, check by VI or IV.

MODEL SOLUTIONS.

108. **1.** Given $a = 46° 24'$, $b = 67° 14'$, $c = 81° 12'$. Solve.

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}},\quad \tan\frac{B}{2} = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}},$$

$$\tan\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}.\quad \textit{Check: } \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

Arrange and solve as in Example 1, page 80.

Ans. $A = 46° 13'.5$, $B =$, $C =$.

Solve: (1) $A = 96° 45'$, $B = 108° 30'$, $C = 116° 15'$.

(Use formulæ III in the same manner as in Example 1.)

(2) $a = 108° 14'$, $b = 75° 29'$, $c = 56° 37'$.

(3) $A = 57° 50'$, $B = 98° 20'$, $C = 63° 40'$.

2. Given $b = 113° 3'$, $c = 82° 39'$, $A = 138° 50'$. Solve.

$$\tan\tfrac{1}{2}(B+C) = \frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}\cot\frac{A}{2},\quad \tan\tfrac{1}{2}(B-C) = \frac{\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b+c)}\cot\frac{A}{2},$$

$$\tfrac{1}{2}(B+C) \pm \tfrac{1}{2}(B-C) = B, \text{ or } C,\qquad \sin a = \frac{\sin A \sin b}{\sin B}.$$

$$\textit{Check: } \tan\frac{a}{2} = \frac{\tan\frac{1}{2}(b-c)\sin\frac{1}{2}(B+C)}{\sin\frac{1}{2}(B-C)}.$$

$$\begin{array}{lll}
b = 113°\ 3' & \log\cos {}^1(b-c) = 9.98453 & \log\sin\frac{1}{2}(b-c) = 9.41861 \\
c = \underline{82° 39'} & \text{colog}\cos\frac{1}{2}(b+c) = 0.86461^{-} & \text{colog}\sin\frac{1}{2}(b+c) = 0.00409 \\
\frac{1}{2}(b+c) = 97° 51' & \log\cot\frac{A}{2} = \underline{9.57466} & \log\cot\frac{A}{2} = \underline{9.57466} \\
\frac{1}{2}(b-c) = 15° 12' & & \\
\frac{1}{2}A = 69° 25' & \log\tan\frac{1}{2}(B+C) = 0.42380^{-} & \log\tan\frac{1}{2}(B-C) = 8.99736 \\
 & \frac{1}{2}(B+C) = 110° 39' & \frac{1}{2}(B-C) = 5° 40'.6 \\
 & \frac{1}{2}(B-C) = \underline{\ \ 5° 40'.6} & \\
 & \therefore B = 116° 19'.6 & \\
 & \text{and } C = 104° 58'.4 &
\end{array}$$

	Check:
$\log \sin A = 9.81839$	$\log \tan \frac{1}{2}(b-c) = 9.43408$
$\log \sin b = 9.96387$	$\log \sin \frac{1}{2}(B+C) = 9.97116$
$\text{cologs in } B = 0.04756$	$\text{colog sin } \frac{1}{2}(B-C) = 1.00474$
$\log \sin a = 9.82982$	$\log \tan \frac{a}{2} = 0.40998$
$a = 137° 29'$	$a = 137° 29'$

Notice that $\tan \frac{1}{2}(B+C)$ is —. Hence, $\frac{1}{2}(B+C)$ is greater than 90°, *i.e.* 110° 39′.

Solve: (1) $A = 68° 40'$, $B = 56° 20'$, $c = 84° 30'$.
(Use formulæ IV, V, I. Compare Example 2.)

(2) $a = 102° 22'$, $b = 78° 17'$, $C = 125° 28'$.

(3) $A = 130° 5'$, $B = 32° 26'$, $c = 51° 6'$.

109. Ambiguous cases. By the principles of geometry the spherical triangle is not necessarily determined by two sides and an angle opposite, nor by two angles and a side opposite. The triangle may be ambiguous. By geometrical principles it is shown that the marks of the ambiguous spherical triangle are:

1. The parts given are two angles and the side opposite one, or two sides and the angle opposite one.

2. The side, or angle, opposite differs from 90° more than the other given side, or angle.

3. Both sides, or angles, given are either greater than 90°, or less than 90°.

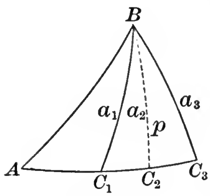


FIG. 52.

In the right triangle ABC_2,

$$\sin a = \sin A \sin c. \quad \text{(formula (3))}$$

Therefore there will be no solution, one solution, or two solutions, according as $\sin a \lesseqgtr \sin A \sin c$, *i.e.* according as $a \lesseqgtr$ the perpendicular p. (See Art. 65.)

But the most expeditious means of determining the ambiguity is found in the solution of the triangle. The use of formula I gives the solution in terms of sine, so that it is to be expected that two values of the part sought may be possible; and whether the triangle be ambiguous or not, there must be some means of determining which of the two angles, α and $180° - \alpha$, that have the same sine is to be used. If there are two solutions, both values are used.

This is determined in the further solution of the triangle by formula VI, which may be written

$$\tan\frac{b}{2} = \frac{\cos\frac{1}{2}(A+C)\tan\frac{1}{2}(a+c)}{\cos\frac{1}{2}(A-C)}.$$

Now $\frac{b}{2} < 90°$, whence $\tan\frac{b}{2}$ is $+$. Then if for both values of C, found by the sine formula, the second member is $+$, there are two solutions; if the second member is $-$ for either value of C, there is but one solution; while if both values of C make the second member $-$, there is no solution. The various cases will be illustrated by problems.

3. Given $a = 62° 15'.4$, $b = 103° 18'.8$, $A = 53° 42'.6$. Solve.

$$\sin B = \frac{\sin b \sin A}{\sin a}, \qquad \tan\frac{c}{2} = \frac{\cos\frac{1}{2}(A+B)\tan\frac{1}{2}(a+b)}{\cos\frac{1}{2}(A-B)},$$

$$\sin C = \frac{\sin c \sin A}{\sin a}. \qquad \textit{Check: } \cot\frac{C}{2} = \frac{\tan\frac{1}{2}(A-B)\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}.$$

Solving the first formula gives

$$\log \sin B = 9.94756,$$

whence

$$B_1 = 62° 24'.4,$$

$$B_2 = 117° 35'.6.$$

For each of the values B_1 and B_2,

$$\frac{\cos\frac{1}{2}(A+B)\tan\frac{1}{2}(a+b)}{\cos\frac{1}{2}(A-B)}$$

is $+$ and therefore equal to $\tan\frac{c}{2}$. Hence there are two solutions. Find

$$c = 153° \ 9'.6, \text{ or } 70° 25'.4$$

and

$$C = 155° 43'.2, \text{ or } 59° \ 6'.2$$

4. Given $a = 46° 45'.5$, $A = 73° 11'.3$, $B = 61° 18'.2$. Solve.

$$\sin b = \frac{\sin a \sin B}{\sin A}, \qquad \cot\frac{C}{2} = \frac{\tan\frac{1}{2}(A-B)\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)},$$

$$\sin c = \frac{\sin a \sin C}{\sin A}. \qquad \textit{Check: } \tan\frac{c}{2} = \frac{\tan\frac{1}{2}(a-b)\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}.$$

Solving for b gives $\log \sin b = 9.82446,$

whence $b_1 = 41° 52'.5,$

and $b_2 = 138° 7'.5.$

For the value b_1 the fraction

$$\frac{\tan \frac{1}{2}(A - B) \cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)}$$

is +, but for b_2 $\cos \frac{1}{2}(a + b)$ is −, making the fraction −, and hence it can not equal $\cot \frac{C}{2}$, which is +. There is then but one solution. Find

$$C = 60° 42'.7, \quad c = 41° 35'.1.$$

5. Given $a = 162° 30'$, $A = 49° 50'$, $B = 57° 52'$. Solve.

Solving gives $\log \sin b = 9.52274,$

whence $b_1 = 19° 27'.9,$

$b_2 = 160° 32'.1.$

For both values, b_1 and b_2, $\cos \frac{1}{2}(a + b)$ is −. Therefore,

$$\frac{\tan \frac{1}{2}(A - B) \cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)}$$

is − and not equal to $\cot \frac{C}{2}$. Hence the triangle is impossible.

Solve, testing for the number of solutions:

(1) $b = 106° 24'.5,$ $c = 40° 20',$ $C = 38° 45'.6.$

(2) $a = 80° 50,$ $A = 131° 40',$ $B = 65° 25'.$

(3) $a = 60° 31'.4,$ $b = 147° 32'.1,$ $B = 143° 50'.$

(4) $a = 55° 30',$ $c = 139° 5',$ $A = 43° 25'.$

RIGHT TRIANGLES.

110. Right triangles are a special case of oblique triangles, but are usually solved by formulæ (1) to (6), Art. 106. Students should have no difficulty in applying these.

Computers generally question the utility of Napier's Rules of Circular Parts. For those who prefer the rules a problem will be solved by their use.

6. Given $c = 86° 51'$, $B = 18° 3'.5$, $C = 90°$.

The parts sought are a, b, A, and it is immaterial which is computed first. a and A are adjacent to c and B, while b is the middle part of c and B. Then by Napier's first rule

$$\sin(90° - B) = \tan(90° - c)\tan a;$$

or $$\tan a = \frac{\cos B}{\cot c} = \cos B \tan c,$$

which is formula (2).

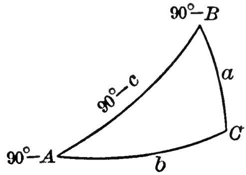


FIG. 53.

By the same rule

$$\sin(90° - c) = \tan(90° - A)\tan(90° - B),$$

or $$\cot A = \frac{\cos c}{\cot B} = \cos c \tan B, \qquad \text{formula (5).}$$

Finally by the second rule

$$\sin b = \cos(90° - c)\cos(90° - B) = \sin c \sin B, \qquad \text{formula (3).}$$

The solutions give $a = 86° 41'.2$, $b = 18° 1'.8$, $A = 88° 58'.4$. Verify.

111. Species. Two angles or sides of a spherical triangle are said to be of the *same species* if they are both less, or both greater, than 90°. They are of *opposite species* when one is greater and the other less than 90°. Since the sides and angles of a spherical triangle may, any or all, be less or greater than 90°, it is necessary in solutions to determine whether each part is more or less than 90°. The directions already given are sufficient in oblique triangles. In right triangles the sign of the function will determine if the solution gives the result in terms of cosine or tangent, but not if the result is found in terms of sine. Thus in Example 6, above, we have $\log \sin b = 9.49068$, whence $b = 18° 1'.8$, or $161° 58'.2$. By formula (4) $\sin b = \frac{\tan a}{\tan A}$. Now $\sin b$ is always +, therefore, $\tan a$ and $\tan A$ must be of the same sign, whence *in any right spherical triangle an oblique angle and its opposite side must be of the same species.*

Again by formula (1) $\cos c = \cos a \cos b$. Now $\cos c$ is + or – according as c is less or greater than 90°. If then $c < 90°$, $\cos a$ and $\cos b$ are of the same sign, but if $c > 90°$, $\cos a$ and $\cos b$ are of opposite sign. Therefore, *if the* *hypotenuse of a right spherical triangle is less than* 90°, *the other sides, and hence the angles opposite, are of the same species; but if the hypotenuse be greater than* 90°, *the other sides, and the angles opposite, are of opposite species.*

112. Ambiguous right triangles.

When the parts given are a side adjacent to the right angle, and the angle opposite this side, the triangle is ambiguous, for solving for the hypotenuse by formula (3) gives

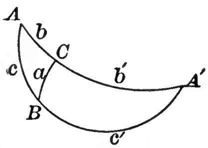


FIG. 54.

$$\sin c = \frac{\sin a}{\sin A},$$

from which there result two values of c. By the last rule of species it follows that to the values of c, one $<90°$, the other $>90°$, there will correspond two values for b, one of the same species as a, the other of opposite species.

Clearly $\sin c \gtreqless 1$, according as $\sin a \gtreqless \sin A$, and hence there will be no solution, one solution, or two solutions, according as $\sin a \gtreqless \sin A$.

Solve the spherical triangles, right angled at C, given:

(1) $b = 73° 21'.4, \quad c = 84° 48'.7.$

(2) $c = 54° 28', \quad B = 128° 12'.6.$

(3) $b = 45° 42', \quad B = 135° 42'.$

(4) $a = 108° 22'.3, \quad b = 120° 14'.5.$

(5) $a = 70° 50', \quad A = 170° 40'.$

(6) $b = 32° 8'.4, \quad B = 46° 2'.8.$

(7) $b = 34° 28', \quad c = 62° 50'.$

(8) $c = 102° 35', \quad B = 17° 45'.$

(9) $a = 92° 16', \quad c = 57° 35'.$

FIVE-PLACE

LOGARITHMIC AND TRIGONOMETRIC

TABLES

ADAPTED FROM GAUSS'S TABLES

BY

ELMER A. LYMAN

MICHIGAN STATE NORMAL COLLEGE

AND

EDWIN C. GODDARD

UNIVERSITY OF MICHIGAN

ALLYN AND BACON

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TABLE I.

THE COMMON LOGARITHMS OF NUMBERS FROM 1 TO 10009.

N.	L. 0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	44	43	42
1	4,4	4,3	4,2
2	8,8	8,6	8,4
3	13,2	12,9	12,6
4	17,6	17,2	16,8
5	22,0	21,5	21,0
6	26,4	25,8	25,2
7	30,8	30,1	29,4
8	35,2	34,4	33,6
9	39,6	38,7	37,8

	41	40	39
1	4,1	4,0	3,9
2	8,2	8,0	7,8
3	12,3	12,0	11,7
4	16,4	16,0	15,6
5	20,5	20,0	19,5
6	24,6	24,0	23,4
7	28,7	28,0	27,3
8	32,8	32,0	31,2
9	36,9	36,0	35,1

	38	37	36
1	3,8	3,7	3,6
2	7,6	7,4	7,2
3	11,4	11,1	10,8
4	15,2	14,8	14,4
5	19,0	18,5	18,0
6	22,8	22,2	21,6
7	26,6	25,9	25,2
8	30,4	29,6	28,8
9	34,2	33,3	32,4

	35	34	33
1	3,5	3,4	3,3
2	7,0	6,8	6,6
3	10,5	10,2	9,9
4	14,0	13,6	13,2
5	17,5	17,0	16,5
6	21,0	20,4	19,8
7	24,5	23,8	23,1
8	28,0	27,2	26,4
9	31,5	30,6	29,7

	32	31	30
1	3,2	3,1	3,0
2	6,4	6,2	6,0
3	9,6	9,3	9,0
4	12,8	12,4	12,0
5	16,0	15,5	15,0
6	19,2	18,6	18,0
7	22,4	21,7	21,0
8	25,6	24,8	24,0
9	28,8	27,9	27,0

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	29	28	27	26	25	24	23	22	21
1	2,9	2,8	2,7	2,6	2,5	2,4	2,3	2,2	2,1
2	5,8	5,6	5,4	5,2	5,0	4,8	4,6	4,4	4,2
3	8,7	8,4	8,1	7,8	7,5	7,2	6,9	6,6	6,3
4	11,6	11,2	10,8	10,4	10,0	9,6	9,2	8,8	8,4
5	14,5	14,0	13,5	13,0	12,5	12,0	11,5	11,0	10,5
6	17,4	16,8	16,2	15,6	15,0	14,4	13,8	13,2	12,6
7	20,3	19,6	18,9	18,2	17,5	16,8	16,1	15,4	14,7
8	23,2	22,4	21,6	20,8	20,0	19,2	18,4	17,6	16,8
9	26,1	25,2	24,3	23,4	22,5	21,6	20,7	19,8	18,9

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	22	21
1	2,2	2,1
2	4,4	4,2
3	6,6	6,3
4	8,8	8,4
5	11,0	10,5
6	13,2	12,6
7	15,4	14,7
8	17,6	16,8
9	19,8	18,9

	20
1	2,0
2	4,0
3	6,0
4	8,0
5	10,0
6	12,0
7	14,0
8	16,0
9	18,0

	19
1	1,9
2	3,8
3	5,7
4	7,6
5	9,5
6	11,4
7	13,3
8	15,2
9	17,1

	18
1	1,8
2	3,6
3	5,4
4	7,2
5	9,0
6	10,8
7	12,6
8	14,4
9	16,2

	17
1	1,7
2	3,4
3	5,1
4	6,8
5	8,5
6	10,2
7	11,9
8	13,6
9	15,3

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842

| N. | L. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

P. P.

	18	17	16	15	14
1	1,8	1,7	1,6	1,5	1,4
2	3,6	3,4	3,2	3,0	2,8
3	5,4	5,1	4,8	4,5	4,2
4	7,2	6,8	6,4	6,0	5,6
5	9,0	8,5	8,0	7,5	7,0
6	10,8	10,2	9,6	9,0	8,4
7	12,6	11,9	11,2	10,5	9,8
8	14,4	13,6	12,8	12,0	11,2
9	16,2	15,3	14,4	13,5	12,6

N.	L. 0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N.	**L. 0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

P. P.

	15	14	13	12
1	1,5	1,4	1,3	1,2
2	3,0	2,8	2,6	2,4
3	4,5	4,2	3,9	3,6
4	6,0	5,6	5,2	4,8
5	7,5	7,0	6,5	6,0
6	9,0	8,4	7,8	7,2
7	10,5	9,8	9,1	8,4
8	12,0	11,2	10,4	9,6
9	13,5	12,6	11,7	10,8

N.	L. 0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	13
1	1,3
2	2,6
3	3,9
4	5,2
5	6,5
6	7,8
7	9,1
8	10,4
9	11,7

	12
1	1,2
2	2,4
3	3,6
4	4,8
5	6,0
6	7,2
7	8,4
8	9,6
9	10,8

	11
1	1,1
2	2,2
3	3,3
4	4,4
5	5,5
6	6,6
7	7,7
8	8,8
9	9,9

	10
1	1,0
2	2,0
3	3,0
4	4,0
5	5,0
6	6,0
7	7,0
8	8,0
9	9,0

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	11	10	9
1	1,1	1,0	0,9
2	2,2	2,0	1,8
3	3,3	3,0	2,7
4	4,4	4,0	3,6
5	5,5	5,0	4,5
6	6,6	6,0	5,4
7	7,7	7,0	6,3
8	8,8	8,0	7,2
9	9,9	9,0	8,1

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
450	65 321	331	341	350	360	369	379	389	398	408	
451	418	427	437	447	456	466	475	485	495	504	
452	514	523	533	543	552	562	571	581	591	600	
453	610	619	629	639	648	658	667	677	686	696	
454	706	715	725	734	744	753	763	772	782	792	
455	801	811	820	830	839	849	858	868	877	887	
456	896	906	916	925	935	944	954	963	973	982	**10**
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077	1 | 1,0
458	66 087	096	106	115	124	134	143	153	162	172	2 | 2,0
459	181	191	200	210	219	229	238	247	257	266	3 | 3,0
460	276	285	295	304	314	323	332	342	351	361	4 | 4,0
461	370	380	389	398	408	417	427	436	445	455	5 | 5,0
462	464	474	483	492	502	511	521	530	539	549	6 | 6,0
463	558	567	577	586	596	605	614	624	633	642	7 | 7,0
464	652	661	671	680	689	699	708	717	727	736	8 | 8,0
465	745	755	764	773	783	792	801	811	820	829	9 | 9,0
466	839	848	857	867	876	885	894	904	913	922	
467	932	941	950	960	969	978	987	997	*006	*015	
468	67 025	034	043	052	062	071	080	089	099	108	
469	117	127	136	145	154	164	173	182	191	201	
470	210	219	228	237	247	256	265	274	284	293	
471	302	311	321	330	339	348	357	367	376	385	**9**
472	394	403	413	422	431	440	449	459	468	477	1 | 0,9
473	486	495	504	514	523	532	541	550	560	569	2 | 1,8
474	578	587	596	605	614	624	633	642	651	660	3 | 2,7
475	669	679	688	697	706	715	724	733	742	752	4 | 3,6
476	761	770	779	788	797	806	815	825	834	843	5 | 4,5
477	852	861	870	879	888	897	906	916	925	934	6 | 5,4
478	943	952	961	970	979	988	997	*006	*015	*024	7 | 6,3
479	68 034	043	052	061	070	079	088	097	106	115	8 | 7,2
480	124	133	142	151	160	169	178	187	196	205	9 | 8,1
481	215	224	233	242	251	260	269	278	287	296	
482	305	314	323	332	341	350	359	368	377	386	
483	395	404	413	422	431	440	449	458	467	476	
484	485	494	502	511	520	529	538	547	556	565	
485	574	583	592	601	610	619	628	637	646	655	
486	664	673	681	690	699	708	717	726	735	744	**8**
487	753	762	771	780	789	797	806	815	824	833	1 | 0,8
488	842	851	860	869	878	886	895	904	913	922	2 | 1,6
489	931	940	949	958	966	975	984	993	*002	*011	3 | 2,4
490	69 020	028	037	046	055	064	073	082	090	099	4 | 3,2
491	108	117	126	135	144	152	161	170	179	188	5 | 4,0
492	197	205	214	223	232	241	249	258	267	276	6 | 4,8
493	285	294	302	311	320	329	338	346	355	364	7 | 5,6
494	373	381	390	399	408	417	425	434	443	452	8 | 6,4
495	461	469	478	487	496	504	513	522	531	539	9 | 7,2
496	548	557	566	574	583	592	601	609	618	627	
497	636	644	653	662	671	679	688	697	705	714	
498	723	732	740	749	758	767	775	784	793	801	
499	810	819	827	836	845	854	862	871	880	888	
500	897	906	914	923	932	940	949	958	966	975	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	248	257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
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P. P.

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P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
550	74 036	044	052	060	068	076	084	092	099	107	
551	115	123	131	139	147	155	162	170	178	186	
552	194	202	210	218	225	233	241	249	257	265	
553	273	280	288	296	304	312	320	327	335	343	
554	351	359	367	374	382	390	398	406	414	421	
555	429	437	445	453	461	468	476	484	492	500	
556	507	515	523	531	539	547	554	562	570	578	
557	586	593	601	609	617	624	632	640	648	656	
558	663	671	679	687	695	702	710	718	726	733	
559	741	749	757	764	772	780	788	796	803	811	
560	819	827	834	842	850	858	865	873	881	889	
561	896	904	912	920	927	935	943	950	958	966	**8**
562	974	981	989	997	*005	*012	*020	*028	*035	*043	1 | 0,8
563	75 051	059	066	074	082	089	097	105	113	120	2 | 1,6
564	128	136	143	151	159	166	174	182	189	197	3 | 2,4
565	205	213	220	228	236	243	251	259	266	274	4 | 3,2
566	282	289	297	305	312	320	328	335	343	351	5 | 4,0
567	358	366	374	381	389	397	404	412	420	427	6 | 4,8
568	435	442	450	458	465	473	481	488	496	504	7 | 5,6
569	511	519	526	534	542	549	557	565	572	580	8 | 6,4
570	587	595	603	610	618	626	633	641	648	656	9 | 7,2
571	664	671	679	686	694	702	709	717	724	732	
572	740	747	755	762	770	778	785	793	800	808	
573	815	823	831	838	846	853	861	868	876	884	
574	891	899	906	914	921	929	937	944	952	959	
575	967	974	982	989	997	*005	*012	*020	*027	*035	
576	76 042	050	057	065	072	080	087	095	103	110	
577	118	125	133	140	148	155	163	170	178	185	
578	193	200	208	215	223	230	238	245	253	260	
579	268	275	283	290	298	305	313	320	328	335	
580	343	350	358	365	373	380	388	395	403	410	
581	418	425	433	440	448	455	462	470	477	485	**7**
582	492	500	507	515	522	530	537	545	552	559	1 | 0,7
583	567	574	582	589	597	604	612	619	626	634	2 | 1,4
584	641	649	656	664	671	678	686	693	701	708	3 | 2,1
585	716	723	730	738	745	753	760	768	775	782	4 | 2,8
586	790	797	805	812	819	827	834	842	849	856	5 | 3,5
587	864	871	879	886	893	901	908	916	923	930	6 | 4,2
588	938	945	953	960	967	975	982	989	997	*004	7 | 4,9
589	77 012	019	026	034	041	048	056	063	070	078	8 | 5,6
590	085	093	100	107	115	122	129	137	144	151	9 | 6,3
591	159	166	173	181	188	195	203	210	217	225	
592	232	240	247	254	262	269	276	283	291	298	
593	305	313	320	327	335	342	349	357	364	371	
594	379	386	393	401	408	415	422	430	437	444	
595	452	459	466	474	481	488	495	503	510	517	
596	525	532	539	546	554	561	568	576	583	590	
597	597	605	612	619	627	634	641	648	656	663	
598	670	677	685	692	699	706	714	721	728	735	
599	743	750	757	764	772	779	786	793	801	808	
600	815	822	830	837	844	851	859	866	873	880	
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N.	L. 0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
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P. P.

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P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
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P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
700	84 510	516	522	528	535	541	547	553	559	566	
701	572	578	584	590	597	603	609	615	621	628	
702	634	640	646	652	658	665	671	677	683	689	
703	696	702	708	714	720	726	733	739	745	751	
704	757	763	770	776	782	788	794	800	807	813	
705	819	825	831	837	844	850	856	862	868	874	**7**
706	880	887	893	899	905	911	917	924	930	936	1 | 0,7
707	942	948	954	960	967	973	979	985	991	997	2 | 1,4
708	85 003	009	016	022	028	034	040	046	052	058	3 | 2,1
709	065	071	077	083	089	095	101	107	114	120	4 | 2,8
710	126	132	138	144	150	156	163	169	175	181	5 | 3,5
711	187	193	199	205	211	217	224	230	236	242	6 | 4,2
712	248	254	260	266	272	278	285	291	297	303	7 | 4,9
713	309	315	321	327	333	339	345	352	358	364	8 | 5,6
714	370	376	382	388	394	400	406	412	418	425	9 | 6,3
715	431	437	443	449	455	461	467	473	479	485	
716	491	497	503	509	516	522	528	534	540	546	
717	552	558	564	570	576	582	588	594	600	606	
718	612	618	625	631	637	643	649	655	661	667	
719	673	679	685	691	697	703	709	715	721	727	
720	733	739	745	751	757	763	769	775	781	788	**6**
721	794	800	806	812	818	824	830	836	842	848	1 | 0,6
722	854	860	866	872	878	884	890	896	902	908	2 | 1,2
723	914	920	926	932	938	944	950	956	962	968	3 | 1,8
724	974	980	986	992	998	*004	*010	*016	*022	*028	4 | 2,4
725	86 034	040	046	052	058	064	070	076	082	088	5 | 3,0
726	094	100	106	112	118	124	130	136	141	147	6 | 3,6
727	153	159	165	171	177	183	189	195	201	207	7 | 4,2
728	213	219	225	231	237	243	249	255	261	267	8 | 4,8
729	273	279	285	291	297	303	308	314	320	326	9 | 5,4
730	332	338	344	350	356	362	368	374	380	386	
731	392	398	404	410	415	421	427	433	439	445	
732	451	457	463	469	475	481	487	493	499	504	
733	510	516	522	528	534	540	546	552	558	564	
734	570	576	581	587	593	599	605	611	617	623	
735	629	635	641	646	652	658	664	670	676	682	**5**
736	688	694	700	705	711	717	723	729	735	741	1 | 0,5
737	747	753	759	764	770	776	782	788	794	800	2 | 1,0
738	806	812	817	823	829	835	841	847	853	859	3 | 1,5
739	864	870	876	882	888	894	900	906	911	917	4 | 2,0
740	923	929	935	941	947	953	958	964	970	976	5 | 2,5
741	982	988	994	999	*005	*011	*017	*023	*029	*035	6 | 3,0
742	87 040	046	052	058	064	070	075	081	087	093	7 | 3,5
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744	157	163	169	175	181	186	192	198	204	210	9 | 4,5
745	216	221	227	233	239	245	251	256	262	268	
746	274	280	286	291	297	303	309	315	320	326	
747	332	338	344	349	355	361	367	373	379	384	
748	390	396	402	408	413	419	425	431	437	442	
749	448	454	460	466	471	477	483	489	495	500	
750	506	512	518	523	529	535	541	547	552	558	
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750	87 506	512	518	523	529	535	541	547	552	558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89 042	048	053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	487	492	498	504	509	515	520	526	531	537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
795	90 037	042	048	053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
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P. P.

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P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
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801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	580	585	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	*004
813	91 009	014	020	025	030	036	041	046	052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	735	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	845	850
829	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	950	955
831	960	965	971	976	981	986	991	997	*002	*007
832	92 012	018	023	028	033	038	044	049	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	195	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	6
1	0,6
2	1,2
3	1,8
4	2,4
5	3,0
6	3,6
7	4,2
8	4,8
9	5,4

	5
1	0,5
2	1,0
3	1,5
4	2,0
5	2,5
6	3,0
7	3,5
8	4,0
9	4,5

N.	L. 0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468

N. L. 0 1 2 3 4 5 6 7 8 9 P. P.

P. P.

	6	5	4
1	0,6	0,5	0,4
2	1,2	1,0	0,8
3	1,8	1,5	1,2
4	2,4	2,0	1,6
5	3,0	2,5	2,0
6	3,6	3,0	2,4
7	4,2	3,5	2,8
8	4,8	4,0	3,2
9	5,4	4,5	3,6

N.	L. 0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035	039	044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350	354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	543	548	552	557	562	566	571	575	580	585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	5
1	0,5
2	1,0
3	1,5
4	2,0
5	2,5
6	3,0
7	3,5
8	4,0
9	4,5

	4
1	0,4
2	0,8
3	1,2
4	1,6
5	2,0
6	2,4
7	2,8
8	3,2
9	3,6

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
950	97 772	777	782	786	791	795	800	804	809	813	
951	818	823	827	832	836	841	845	850	855	859	
952	864	868	873	877	882	886	891	896	900	905	
953	909	914	918	923	928	932	937	941	946	950	
954	955	959	964	968	973	978	982	987	991	996	
955	98 000	005	009	014	019	023	028	032	037	041	
956	046	050	055	059	064	068	073	078	082	087	
957	091	096	100	105	109	114	118	123	127	132	
958	137	141	146	150	155	159	164	168	173	177	
959	182	186	191	195	200	204	209	214	218	223	
960	227	232	236	241	245	250	254	259	263	268	
961	272	277	281	286	290	295	299	304	308	313	**5**
962	318	322	327	331	336	340	345	349	354	358	1 | 0,5
963	363	367	372	376	381	385	390	394	399	403	2 | 1,0
964	408	412	417	421	426	430	435	439	444	448	3 | 1,5
965	453	457	462	466	471	475	480	484	489	493	4 | 2,0
966	498	502	507	511	516	520	525	529	534	538	5 | 2,5
967	543	547	552	556	561	565	570	574	579	583	6 | 3,0
968	588	592	597	601	605	610	614	619	623	628	7 | 3,5
969	632	637	641	646	650	655	659	664	668	673	8 | 4,0
970	677	682	686	691	695	700	704	709	713	717	9 | 4,5
971	722	726	731	735	740	744	749	753	758	762	
972	767	771	776	780	784	789	793	798	802	807	
973	811	816	820	825	829	834	838	843	847	851	
974	856	860	865	869	874	878	883	887	892	896	
975	900	905	909	914	918	923	927	932	936	941	
976	945	949	954	958	963	967	972	976	981	985	
977	989	994	998	*003	*007	*012	*016	*021	*025	*029	
978	99 034	038	043	047	052	056	061	065	069	074	
979	078	083	087	092	096	100	105	109	114	118	
980	123	127	131	136	140	145	149	154	158	162	
981	167	171	176	180	185	189	193	198	202	207	**4**
982	211	216	220	224	229	233	238	242	247	251	1 | 0,4
983	255	260	264	269	273	277	282	286	291	295	2 | 0,8
984	300	304	308	313	317	322	326	330	335	339	3 | 1,2
985	344	348	352	357	361	366	370	374	379	383	4 | 1,6
986	388	392	396	401	405	410	414	419	423	427	5 | 2,0
987	432	436	441	445	449	454	458	463	467	471	6 | 2,4
988	476	480	484	489	493	498	502	506	511	515	7 | 2,8
989	520	524	528	533	537	542	546	550	555	559	8 | 3,2
990	564	568	572	577	581	585	590	594	599	603	9 | 3,6
991	607	612	616	621	625	629	634	638	642	647	
992	651	656	660	664	669	673	677	682	686	691	
993	695	699	704	708	712	717	721	726	730	734	
994	739	743	747	752	756	760	765	769	774	778	
995	782	787	791	795	800	804	808	813	817	822	
996	826	830	835	839	843	848	852	856	861	865	
997	870	874	878	883	887	891	896	900	904	909	
998	913	917	922	926	930	935	939	944	948	952	
999	957	961	965	970	974	978	983	987	991	996	
1000	00 000	004	009	013	017	022	026	030	035	039	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

NOTES ON TABLES I AND II.

The logarithms of numbers are in general incommensurable. In these tables they are given correct to five places of decimals. If the sixth place is 5 or more, the next larger number is used in the fifth place. Thus log $8102 = 3.908549+$; in five-place tables this is written $3.9085\bar{5}$, the dash above the 5 showing that the logarithm is less than given.

So log $8133 = 3.910251-$; in five-place tables this is written $3.9102\dot{5}$, the dot above the 5 showing that the logarithm is more than given.

In the natural functions of the angles (Table II) all numbers are decimals for sine and cosine (why ?), and for tangent and cotangent, except where the decimal point is used to indicate that part of the number is integral. When no decimal point is printed in the tables it is to be understood. When the natural function is a pure decimal the characteristic of the logarithm is negative. Accordingly, in the tables 10 is added, and in the result this must be allowed for. Thus

$$\text{nat. sin } 44° 20' = 0.69883, \quad \log \sin 44° 20' = \bar{1}.84437,$$

or, as printed in the tables, 9.84437, which means $9.84437 - 10$.

TABLE II.

THE LOGARITHMIC AND NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS OF ANGLES FROM 0° TO 90°.

0°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c. d.	Log. Cot	Nat.	
0	00000	—		10000	0.00000	00000	—		—	∞	60
1	029	6.46373	30103	000	0.00000	029	6.46373	30103	3.53627	3437.7	59
2	058	6.76476	17609	000	0.00000	058	6.76470	17609	3.23524	1718.9	58
3	087	6.94085	12494	000	0.00000	087	6.94085	12494	3.05915	1145.9	57
4	116	7.06579	9691	000	0.00000	116	7.06579	9691	2.93421	859.44	56
5	00145	7.16270	7918	10000	0.00000	00145	7.16270	7918	2.83730	687.55	55
6	175	7.24188	6694	000	0.00000	175	7.24188	6694	2.75812	572.96	54
7	204	7.30882	5800	000	0.00000	204	7.30882	5800	2.69118	491.11	53
8	233	7.36682	5115	000	0.00000	233	7.36682	5115	2.63318	429.72	52
9	262	7.41797	4576	000	0.00000	262	7.41797	4576	2.58203	381.97	51
10	00291	7.46373	4139	10000	0.00000	00291	7.46373	4139	2.53627	343.77	50
11	320	7.50512	3779	99999	0.00000	320	7.50512	3779	2.49488	312.52	49
12	349	7.54291	3476	999	0.00000	349	7.54291	3476	2.45709	286.48	48
13	378	7.57767	3218	999	0.00000	378	7.57767	3219	2.42233	264.44	47
14	407	7.60985	2997	999	0.00000	407	7.60986	2996	2.39014	245.55	46
15	00436	7.63982	2802	99999	0.00000	00436	7.63982	2803	2.36018	229.18	45
16	465	7.66784	2633	999	0.00000	465	7.66785	2633	2.33215	214.86	44
17	495	7.69417	2483	999	9.99999	495	7.69418	2482	2.30582	202.22	43
18	524	7.71900	2348	999	9.99999	524	7.71900	2348	2.28100	190.98	42
19	553	7.74248	2227	998	9.99999	553	7.74248	2228	2.25752	180.93	41
20	00582	7.76475	2119	99998	9.99999	00582	7.76476	2119	2.23524	171.89	40
21	611	7.78594	2021	998	9.99999	611	7.78595	2020	2.21405	163.70	39
22	640	7.80615	1930	998	9.99999	640	7.80615	1931	2.19385	156.26	38
23	669	7.82545	1848	998	9.99999	669	7.82546	1848	2.17454	149.47	37
24	698	7.84393	1773	998	9.99999	698	7.84394	1773	2.15606	143.24	36
25	00727	7.86100	1704	99997	9.99999	00727	7.86167	1704	2.13833	137.51	35
26	756	7.87870	1639	997	9.99999	756	7.87871	1639	2.12129	132.22	34
27	785	7.89509	1579	997	9.99999	785	7.89510	1579	2.10490	127.32	33
28	814	7.91088	1524	997	9.99999	815	7.91089	1524	2.08911	122.77	32
29	844	7.92612	1472	996	9.99998	844	7.92613	1473	2.07387	118.54	31
30	00873	7.94034	1424	99996	9.99998	00873	7.94086	1424	2.05914	114.59	30
31	902	7.95508	1379	996	9.99998	902	7.95510	1379	2.04490	110.89	29
32	931	7.96887	1336	996	9.99998	931	7.96889	1336	2.03111	107.43	28
33	960	7.98223	1297	995	9.99998	960	7.98225	1297	2.01775	104.17	27
34	989	7.99520	1259	995	9.99998	989	7.99522	1259	2.00478	101.11	26
35	01018	8.00779	1223	99995	9.99998	01018	8.00781	1223	1.99219	98.218	25
36	047	8.02002	1190	995	9.99998	047	8.02004	1190	1.97996	95.489	24
37	076	8.03192	1158	994	9.99997	076	8.03194	1159	1.96806	92.908	23
38	105	8.04350	1128	994	9.99997	105	8.04353	1128	1.95647	90.463	22
39	134	8.05478	1100	994	9.99997	135	8.05481	1100	1.94519	88.144	21
40	01164	8.06578	1072	99993	9.99997	01164	8.06581	1072	1.93419	85.940	20
41	193	8.07650	1046	993	9.99997	193	8.07653	1047	1.92347	83.844	19
42	222	8.08696	1022	993	9.99997	222	8.08700	1022	1.91300	81.847	18
43	251	8.09718	999	992	9.99997	251	8.09722	998	1.90278	79.943	17
44	280	8.10717	976	992	9.99996	280	8.10720	976	1.89280	78.126	16
45	01309	8.11693	954	99991	9.99996	01309	8.11696	955	1.88304	76.390	15
46	338	8.12647	934	991	9.99996	338	8.12651	934	1.87349	74.729	14
47	367	8.13581	914	991	9.99996	367	8.13585	915	1.86415	73.139	13
48	396	8.14495	896	990	9.99996	396	8.14500	895	1.85500	71.615	12
49	425	8.15391	877	990	9.99996	425	8.15395	878	1.84605	70.153	11
50	01454	8.16268	860	99989	9.99995	01455	8.16273	860	1.83727	68.750	10
51	483	8.17128	843	989	9.99995	484	8.17133	843	1.82867	67.402	9
52	513	8.17971	827	989	9.99995	513	8.17976	828	1.82024	66.105	8
53	542	8.18798	812	988	9.99995	542	8.18804	812	1.81196	64.858	7
54	571	8.19610	797	988	9.99995	571	8.19616	797	1.80384	63.657	6
55	01600	8.20407	782	99987	9.99994	01600	8.20413	782	1.79587	62.499	5
56	629	8.21189	769	987	9.99994	629	8.21195	769	1.78805	61.383	4
57	658	8.21958	755	986	9.99994	658	8.21964	756	1.78036	60.306	3
58	687	8.22713	743	986	9.99994	687	8.22720	742	1.77280	59.266	2
59	716	8.23456	730	985	9.99994	716	8.23462	730	1.76538	58.261	1
60	745	8.24186		985	9.99993	746	8.24192		1.75808	57.290	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c. d.	Log. Tan	Nat.	′

89°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	01745	8.24186	717	99985	9.99993	01746	8.24192	718	1.75808	57.290	60
1	774	8.24903	706	984	9.99993	775	8.24910	706	1.75090	56.351	59
2	803	8.25609	695	984	9.99993	804	8.25616	696	1.74384	55.442	58
3	832	8.26304	684	983	9.99993	833	8.26312	684	1.73688	54.561	57
4	862	8.26988	673	983	9.99992	862	8.26996	673	1.73004	53.709	56
5	01891	8.27661	663	99982	9.99992	01891	8.27669	663	1.72331	52.882	55
6	920	8.28324	653	982	9.99992	920	8.28332	654	1.71668	.081	54
7	949	8.28977	644	981	9.99992	949	8.28986	643	1.71014	51.303	53
8	978	8.29621	634	980	9.99992	978	8.29629	634	1.70371	50.549	52
9	02007	8.30255	624	980	9.99991	02007	8.30263	625	1.69737	49.816	51
10	02036	8.30879	616	99979	9.99991	02036	8.30888	617	1.69112	49.104	50
11	065	8.31495	608	979	9.99991	066	8.31505	607	1.68495	48.412	49
12	094	8.32103	599	978	9.99990	095	8.32112	599	1.67888	47.740	48
13	123	8.32702	590	977	9.99990	124	8.32711	591	1.67289	.085	47
14	152	8.33292	583	977	9.99990	153	8.33302	584	1.66698	46.449	46
15	02181	8.33875	575	99976	9.99990	02182	8.33886	575	1.66114	45.829	45
16	211	8.34450	568	976	9.99989	211	8.34461	568	1.65539	.226	44
17	240	8.35018	560	975	9.99989	240	8.35029	561	1.64971	44.639	43
18	269	8.35578	553	974	9.99989	269	8.35590	553	1.64410	.066	42
19	298	8.36131	547	974	9.99989	298	8.36143	546	1.63857	43.508	41
20	02327	8.36678	539	99973	9.99988	02328	8.36689	540	1.63311	42.964	40
21	356	8.37217	533	972	9.99988	357	8.37229	533	1.62771	.433	39
22	385	8.37750	526	972	9.99988	386	8.37762	527	1.62238	41.916	38
23	414	8.38276	520	971	9.99987	415	8.38289	520	1.61711	.411	37
24	443	8.38796	514	970	9.99987	444	8.38809	514	1.61191	40.917	36
25	02472	8.39310	508	99969	9.99987	02473	8.39323	509	1.60677	40.436	35
26	501	8.39818	502	969	9.99986	502	8.39832	502	1.60168	39.965	34
27	530	8.40320	496	968	9.99986	531	8.40334	496	1.59666	.506	33
28	560	8.40816	491	967	9.99986	560	8.40830	491	1.59170	.057	32
29	589	8.41307	485	966	9.99985	589	8.41321	486	1.58679	38.618	31
30	02618	8.41792	480	99966	9.99985	02619	8.41807	480	1.58193	38.188	30
31	647	8.42272	474	965	9.99985	648	8.42287	475	1.57713	37.769	29
32	676	8.42746	470	964	9.99984	677	8.42762	470	1.57238	.358	28
33	705	8.43216	464	963	9.99984	706	8.43232	464	1.56768	36.956	27
34	734	8.43680	459	963	9.99984	735	8.43696	460	1.56304	.563	26
35	02763	8.44139	455	99962	9.99983	02764	8.44156	455	1.55844	36.178	25
36	792	8.44594	450	961	9.99983	793	8.44611	450	1.55389	35.801	24
37	821	8.45044	445	960	9.99983	822	8.45061	446	1.54939	.431	23
38	850	8.45489	441	959	9.99982	851	8.45507	441	1.54493	.070	22
39	879	8.45930	436	959	9.99982	881	8.45948	437	1.54052	34.715	21
40	02908	8.46366	433	99958	9.99982	02910	8.46385	432	1.53615	34.368	20
41	938	8.46799	427	957	9.99981	939	8.46817	428	1.53183	.027	19
42	967	8.47226	424	956	9.99981	968	8.47245	424	1.52755	33.694	18
43	996	8.47650	419	955	9.99981	997	8.47669	420	1.52331	.366	17
44	03025	8.48069	416	954	9.99980	03026	8.48089	416	1.51911	.045	16
45	03054	8.48485	411	99953	9.99980	03055	8.48505	412	1.51495	32.730	15
46	083	8.48896	408	952	9.99979	084	8.48917	408	1.51083	.421	14
47	112	8.49304	404	952	9.99979	114	8.49325	404	1.50675	.118	13
48	141	8.49708	400	951	9.99979	143	8.49729	401	1.50271	31.821	12
49	170	8.50108	396	950	9.99978	172	8.50130	397	1.49870	.528	11
50	03199	8.50504	393	99949	9.99978	03201	8.50527	393	1.49473	31.242	10
51	228	8.50897	390	948	9.99977	230	8.50920	390	1.49080	30.960	9
52	257	8.51287	386	947	9.99977	259	8.51310	386	1.48690	.683	8
53	286	8.51673	382	946	9.99977	288	8.51696	383	1.48304	.412	7
54	316	8.52055	379	945	9.99976	317	8.52079	380	1.47921	.145	6
55	03345	8.52434	376	99944	9.99976	03346	8.52459	376	1.47541	29.882	5
56	374	8.52810	373	943	9.99975	376	8.52835	373	1.47165	.624	4
57	403	8.53183	369	942	9.99975	405	8.53208	370	1.46792	.371	3
58	432	8.53552	367	941	9.99974	434	8.53578	367	1.46422	.122	2
59	461	8.53919	363	940	9.99974	463	8.53945	363	1.46055	28.877	1
60	490	8.54282		939	9.99974	492	8.54308		1.45692	.636	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Sin Log.	d.	Nat. Cos	Cos Log.	Nat. Tan	Tan Log.	c.d.	Log. Cot	Cot Nat.	
0	03490	8.54282	360	99939	9.99974	03492	8.54308	361	1.45692	28.636	60
1	519	8.54642	357	938	9.99973	521	8.54669	358	1.45331	.399	59
2	548	8.54999	355	937	9.99973	550	8.55027	355	1.44973	.166	58
3	577	8.55354	351	936	9.99972	579	8.55382	352	1.44618	27.937	57
4	606	8.55705	349	935	9.99972	609	8.55734	349	1.44266	.712	56
5	03635	8.56054	346	99934	9.99971	03638	8.56083	346	1.43917	27.490	55
6	664	8.56400	343	933	9.99971	667	8.56429	344	1.43571	.271	54
7	693	8.56743	341	932	9.99970	696	8.56773	341	1.43227	.057	53
8	723	8.57084	337	931	9.99970	725	8.57114	338	1.42886	26.845	52
9	752	8.57421	336	930	9.99969	754	8.57452	336	1.42548	.637	51
10	03781	8.57757	332	99929	9.99969	03783	8.57788	333	1.42212	26.432	50
11	810	8.58089	330	927	9.99968	812	8.58121	330	1.41879	.230	49
12	839	8.58419	328	926	9.99968	842	8.58451	328	1.41549	.031	48
13	868	8.58747	325	925	9.99967	871	8.58779	326	1.41221	25.835	47
14	897	8.59072	323	924	9.99967	900	8.59105	323	1.40895	.642	46
15	03926	8.59395	320	99923	9.99967	03929	8.59428	321	1.40572	25.452	45
16	955	8.59715	318	922	9.99966	958	8.59749	319	1.40251	.264	44
17	984	8.60033	316	921	9.99966	987	8.60068	316	1.39932	.080	43
18	04013	8.60349	313	919	9.99965	04016	8.60384	314	1.39616	24.898	42
19	042	8.60662	311	918	9.99964	046	8.60698	311	1.39302	.719	41
20	04071	8.60973	309	99917	9.99964	04075	8.61009	310	1.38991	24.542	40
21	100	8.61282	307	916	9.99963	104	8.61319	307	1.38681	.368	39
22	129	8.61589	305	915	9.99963	133	8.61626	305	1.38374	.196	38
23	159	8.61894	302	913	9.99962	162	8.61931	303	1.38069	.026	37
24	188	8.62196	301	912	9.99962	191	8.62234	301	1.37766	23.859	36
25	04217	8.62497	298	99911	9.99961	04220	8.62535	299	1.37465	23.695	35
26	246	8.62795	296	910	9.99961	250	8.62834	297	1.37166	.532	34
27	275	8.63091	294	909	9.99960	279	8.63131	295	1.36869	.372	33
28	304	8.63385	293	907	9.99960	308	8.63426	292	1.36574	.214	32
29	333	8.63678	290	906	9.99959	337	8.63718	291	1.36282	.058	31
30	04362	8.63968	288	99905	9.99959	04366	8.64009	289	1.35991	22.904	30
31	391	8.64256	287	904	9.99958	395	8.64298	287	1.35702	.752	29
32	420	8.64543	284	902	9.99958	424	8.64585	285	1.35415	.602	28
33	449	8.64827	283	901	9.99957	454	8.64870	284	1.35130	.454	27
34	478	8.65110	281	900	9.99956	483	8.65154	281	1.34846	.308	26
35	04507	8.65391	279	99898	9.99956	04512	8.65435	280	1.34565	22.164	25
36	536	8.65670	277	897	9.99955	541	8.65715	278	1.34285	.022	24
37	565	8.65947	276	896	9.99955	570	8.65993	276	1.34007	21.881	23
38	594	8.66223	274	894	9.99954	599	8.66269	274	1.33731	.743	22
39	623	8.66497	272	893	9.99954	628	8.66543	273	1.33457	.606	21
40	04653	8.66769	270	99892	9.99953	04658	8.66816	271	1.33184	21.470	20
41	682	8.67039	269	890	9.99952	687	8.67087	269	1.32913	.337	19
42	711	8.67308	267	889	9.99952	716	8.67356	268	1.32644	.205	18
43	740	8.67575	266	888	9.99951	745	8.67624	266	1.32376	.075	17
44	769	8.67841	263	886	9.99951	774	8.67890	264	1.32110	20.946	16
45	04798	8.68104	263	99885	9.99950	04803	8.68154	263	1.31846	20.819	15
46	827	8.68367	260	883	9.99949	833	8.68417	261	1.31583	.693	14
47	856	8.68627	259	882	9.99949	862	8.68678	260	1.31322	.569	13
48	885	8.68886	258	881	9.99948	891	8.68938	258	1.31062	.446	12
49	914	8.69144	256	879	9.99948	920	8.69196	257	1.30804	.325	11
50	04943	8.69400	254	99878	9.99947	04949	8.69453	255	1.30547	20.206	10
51	972	8.69654	253	876	9.99946	978	8.69708	254	1.30292	.087	9
52	05001	8.69907	252	875	9.99946	05007	8.69962	252	1.30038	19.970	8
53	030	8.70159	250	873	9.99945	037	8.70214	251	1.29786	.855	7
54	059	8.70409	249	872	9.99944	066	8.70465	249	1.29535	.740	6
55	05088	8.70658	247	99870	9.99944	05095	8.70714	248	1.29286	19.627	5
56	117	8.70905	246	869	9.99943	124	8.70962	246	1.29038	.516	4
57	146	8.71151	244	867	9.99942	153	8.71208	245	1.28792	.405	3
58	175	8.71395	243	866	9.99942	182	8.71453	244	1.28547	.296	2
59	205	8.71638	242	864	9.99941	212	8.71697	243	1.28303	.188	1
60	234	8.71880		863	9.99940	241	8.71940		1.28060	.081	0
	Nat. Cos	Cos Log.	d.	Nat. Sin	Sin Log.	Nat. Cot	Cot Log.	c.d.	Log. Tan	Tan Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	05234	8.71880	240	99863	9.99940	05341	8.71940	241	1.28060	19.081	**60**
1	263	8.72120	239	861	9.99940	270	8.72181	239	1.27819	18.976	59
2	292	8.72359	238	860	9.99939	299	8.72420	239	1.27580	.871	58
3	321	8.72597	237	858	9.99938	328	8.72659	237	1.27341	.768	57
4	350	8.72834	235	857	9.99938	357	8.72896	236	1.27104	.666	56
5	05379	8.73069	234	99855	9.99937	05387	8.73132	234	1.26868	18.564	**55**
6	408	8.73303	232	854	9.99936	416	8.73366	234	1.26634	.464	54
7	437	8.73535	232	852	9.99936	445	8.73600	232	1.26400	.366	53
8	466	8.73767	230	851	9.99935	474	8.73832	231	1.26168	.268	52
9	495	8.73997	229	849	9.99934	503	8.74063	229	1.25937	.171	51
10	05524	8.74226	228	99847	9.99934	05533	8.74292	229	1.25708	18.075	**50**
11	553	8.74454	226	846	9.99933	562	8.74521	227	1.25479	17.980	49
12	582	8.74680	226	844	9.99932	591	8.74748	226	1.25252	.886	48
13	611	8.74906	224	842	9.99932	620	8.74974	225	1.25026	.793	47
14	640	8.75130	223	841	9.99931	649	8.75199	224	1.24801	.702	46
15	05669	8.75353	222	99839	9.99930	05678	8.75423	222	1.24577	17.611	**45**
16	698	8.75575	220	838	9.99929	708	8.75645	222	1.24355	.521	44
17	727	8.75795	220	836	9.99929	737	8.75867	220	1.24133	.431	43
18	756	8.76015	219	834	9.99928	766	8.76087	219	1.23913	.343	42
19	785	8.76234	217	833	9.99927	795	8.76306	219	1.23694	.256	41
20	05814	8.76451	216	99831	9.99926	05824	8.76525	217	1.23475	17.169	**40**
21	844	8.76667	216	829	9.99926	854	8.76742	216	1.23258	.084	39
22	873	8.76883	214	827	9.99925	883	8.76958	215	1.23042	16.999	38
23	902	8.77097	213	826	9.99924	912	8.77173	214	1.22827	.915	37
24	931	8.77310	212	824	9.99923	941	8.77387	213	1.22613	.832	36
25	05960	8.77522	211	99822	9.99923	05970	8.77600	211	1.22400	16.750	**35**
26	989	8.77733	210	821	9.99922	999	8.77811	211	1.22189	.668	34
27	06018	8.77943	209	819	9.99921	06029	8.78022	210	1.21978	.587	33
28	047	8.78152	208	817	9.99920	058	8.78232	209	1.21768	.507	32
29	076	8.78360	208	815	9.99920	087	8.78441	208	1.21559	.428	31
30	06105	8.78568	206	99813	9.99919	06116	8.78649	206	1.21351	16.350	**30**
31	134	8.78774	205	812	9.99918	145	8.78855	206	1.21145	.272	29
32	163	8.78979	204	810	9.99917	175	8.79061	205	1.20939	.195	28
33	192	8.79183	203	808	9.99917	204	8.79266	204	1.20734	.119	27
34	221	8.79386	202	806	9.99916	233	8.79470	203	1.20530	.043	26
35	06250	8.79588	201	99804	9.99915	06262	8.79673	202	1.20327	15.969	**25**
36	279	8.79789	201	803	9.99914	291	8.79875	201	1.20125	.895	24
37	308	8.79990	199	801	9.99913	321	8.80076	201	1.19924	.821	23
38	337	8.80189	199	799	9.99913	350	8.80277	199	1.19723	.748	22
39	366	8.80388	197	797	9.99912	379	8.80476	198	1.19524	.676	21
40	06395	8.80585	197	99795	9.99911	06408	8.80674	198	1.19326	15.605	**20**
41	424	8.80782	196	793	9.99910	438	8.80872	196	1.19128	.534	19
42	453	8.80978	195	792	9.99909	467	8.81068	196	1.18932	.464	18
43	482	8.81173	194	790	9.99909	496	8.81264	195	1.18736	.394	17
44	511	8.81367	193	788	9.99908	525	8.81459	194	1.18541	.325	16
45	06540	8.81560	192	99786	9.99907	06554	8.81653	193	1.18347	15.257	**15**
46	569	8.81752	192	784	9.99906	584	8.81846	192	1.18154	.189	14
47	598	8.81944	190	782	9.99905	613	8.82038	192	1.17962	.122	13
48	627	8.82134	190	780	9.99904	642	8.82230	190	1.17770	.056	12
49	656	8.82324	189	778	9.99904	671	8.82420	190	1.17580	14.990	11
50	06685	8.82513	188	99776	9.99903	06700	8.82610	189	1.17390	14.924	**10**
51	714	8.82701	187	774	9.99902	730	8.82799	188	1.17201	.860	9
52	743	8.82888	187	772	9.99901	759	8.82987	188	1.17013	.795	8
53	773	8.83075	186	770	9.99900	788	8.83175	186	1.16825	.732	7
54	802	8.83261	185	768	9.99899	817	8.83361	186	1.16639	.669	6
55	06831	8.83446	184	99766	9.99898	06847	8.83547	185	1.16453	14.606	**5**
56	860	8.83630	183	764	9.99898	876	8.83732	184	1.16268	.544	4
57	889	8.83813	183	762	9.99897	905	8.83916	184	1.16084	.482	3
58	918	8.83996	181	760	9.99896	934	8.84100	182	1.15900	.421	2
59	947	8.84177	181	758	9.99895	963	8.84282	182	1.15718	.361	1
60	976	8.84358		756	9.99894	993	8.84464		1.15536	.301	**0**
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

4°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	06976	8.84358	181	99756	9.99894	06993	8.84464	182	1.15536	14.301	60
1	07005	8.84539	179	754	9.99893	07022	8.84646	180	1.15354	.241	59
2	034	8.84718	179	752	9.99892	051	8.84826	180	1.15174	.182	58
3	063	8.84897	178	750	9.99891	080	8.85006	179	1.14994	.124	57
4	092	8.85075	177	748	9.99891	110	8.85185	178	1.14815	.065	56
5	07121	8.85252	177	99746	9.99890	07139	8.85363	177	1.14637	14.008	55
6	150	8.85429	176	744	9.99889	168	8.85540	177	1.14460	13.951	54
7	179	8.85605	175	742	9.99888	197	8.85717	176	1.14283	.894	53
8	208	8.85780	175	740	9.99887	227	8.85893	176	1.14107	.838	52
9	237	8.85955	173	738	9.99886	256	8.86069	174	1.13931	.782	51
10	07266	8.86128	173	99736	9.99885	07285	8.86243	174	1.13757	13.727	50
11	295	8.86301	173	734	9.99884	314	8.86417	174	1.13583	.672	49
12	324	8.86474	171	731	9.99883	344	8.86591	172	1.13409	.617	48
13	353	8.86645	171	729	9.99882	373	8.86763	172	1.13237	.563	47
14	382	8.86816	171	727	9.99881	402	8.86935	171	1.13065	.510	46
15	07411	8.86987	169	99725	9.99880	07431	8.87106	171	1.12894	13.457	45
16	440	8.87156	169	723	9.99879	461	8.87277	170	1.12723	.404	44
17	469	8.87325	169	721	9.99879	490	8.87447	169	1.12553	.352	43
18	498	8.87494	167	719	9.99878	519	8.87616	169	1.12384	.300	42
19	527	8.87661	168	716	9.99877	548	8.87785	168	1.12215	.248	41
20	07556	8.87829	166	99714	9.99876	07578	8.87953	167	1.12047	13.197	40
21	585	8.87995	166	712	9.99875	607	8.88120	167	1.11880	.146	39
22	614	8.88161	165	710	9.99874	636	8.88287	166	1.11713	.096	38
23	643	8.88326	164	708	9.99873	665	8.88453	165	1.11547	.046	37
24	672	8.88490	164	705	9.99872	695	8.88618	165	1.11382	12.996	36
25	07701	8.88654	163	99703	9.99871	07724	8.88783	165	1.11217	12.947	35
26	730	8.88817	163	701	9.99870	753	8.88948	163	1.11052	.898	34
27	759	8.88980	162	699	9.99869	782	8.89111	163	1.10889	.850	33
28	788	8.89142	162	696	9.99868	812	8.89274	163	1.10726	.801	32
29	817	8.89304	160	694	9.99867	841	8.89437	161	1.10563	.754	31
30	07846	8.89464	161	99692	9.99866	07870	8.89598	162	1.10402	12.706	30
31	875	8.89625	159	689	9.99865	899	8.89760	160	1.10240	.659	29
32	904	8.89784	159	687	9.99864	929	8.89920	160	1.10080	.612	28
33	933	8.89943	159	685	9.99863	958	8.90080	160	1.09920	.566	27
34	962	8.90102	158	683	9.99862	987	8.90240	159	1.09760	.520	26
35	07991	8.90260	157	99680	9.99861	08017	8.90399	158	1.09601	12.474	25
36	08020	8.90417	157	678	9.99860	046	8.90557	158	1.09443	.429	24
37	049	8.90574	156	676	9.99859	075	8.90715	157	1.09285	.384	23
38	078	8.90730	155	673	9.99858	104	8.90872	157	1.09128	.339	22
39	107	8.90885	155	671	9.99857	134	8.91029	156	1.08971	.295	21
40	08136	8.91040	155	99668	9.99856	08163	8.91185	155	1.08815	12.251	20
41	165	8.91195	154	666	9.99855	192	8.91340	155	1.08660	.207	19
42	194	8.91349	153	664	9.99854	221	8.91495	155	1.08505	.163	18
43	223	8.91502	153	661	9.99853	251	8.91650	153	1.08350	.120	17
44	252	8.91655	152	659	9.99852	280	8.91803	154	1.08197	.077	16
45	08281	8.91807	152	99657	9.99851	08309	8.91957	153	1.08043	12.035	15
46	310	8.91959	151	654	9.99850	339	8.92110	152	1.07890	11.992	14
47	339	8.92110	151	652	9.99848	368	8.92262	152	1.07738	.950	13
48	368	8.92261	150	649	9.99847	397	8.92414	151	1.07586	.909	12
49	397	8.92411	150	647	9.99846	427	8.92565	151	1.07435	.867	11
50	08426	8.92561	149	99644	9.99845	08456	8.92716	150	1.07284	11.826	10
51	455	8.92710	149	642	9.99844	485	8.92866	150	1.07134	.785	9
52	484	8.92859	148	639	9.99843	514	8.93016	149	1.06984	.745	8
53	513	8.93007	147	637	9.99842	544	8.93165	148	1.06835	.705	7
54	542	8.93154	147	635	9.99841	573	8.93313	149	1.06687	.664	6
55	08571	8.93301	147	99632	9.99840	08602	8.93462	147	1.06538	11.625	5
56	600	8.93448	146	630	9.99839	632	8.93609	147	1.06391	.585	4
57	629	8.93594	146	627	9.99838	661	8.93756	147	1.06244	.546	3
58	658	8.93740	145	625	9.99837	690	8.93903	146	1.06097	.507	2
59	687	8.93885	145	622	9.99836	720	8.94049	146	1.05951	.468	1
60	716	8.94030		619	9.99834	749	8.94195		1.05805	.430	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

85°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	08716	8.94030	144	99619	9.99834	08749	8.94195	145	1.05805	11.430	60
1	745	8.94174	143	617	9.99833	778	8.94340	145	1.05660	.392	59
2	774	8.94317	144	614	9.99832	807	8.94485	145	1.05515	.354	58
3	803	8.94461	142	612	9.99831	837	8.94630	143	1.05370	.316	57
4	831	8.94603	143	609	9.99830	866	8.94773	144	1.05227	.279	56
5	08860	8.94746	141	99607	9.99829	08895	8.94917	143	1.05083	11.242	55
6	889	8.94887	142	604	9.99828	925	8.95060	142	1.04940	.205	54
7	918	8.95029	141	602	9.99827	954	8.95202	142	1.04798	.168	53
8	947	8.95170	140	599	9.99825	983	8.95344	142	1.04656	.132	52
9	976	8.95310	140	596	9.99824	09013	8.95486	141	1.04514	.095	51
10	09005	8.95450	139	99594	9.99823	09042	8.95627	140	1.04373	11.059	50
11	034	8.95589	139	591	9.99822	071	8.95767	141	1.04233	.024	49
12	063	8.95728	139	588	9.99821	101	8.95908	139	1.04092	10.988	48
13	092	8.95867	138	586	9.99820	130	8.96047	140	1.03953	.953	47
14	121	8.96005	138	583	9.99819	159	8.96187	138	1.03813	.918	46
15	09150	8.96143	137	99580	9.99817	09189	8.96325	139	1.03675	10.883	45
16	179	8.96280	137	578	9.99816	218	8.96464	138	1.03536	.848	44
17	208	8.96417	136	575	9.99815	247	8.96602	137	1.03398	.814	43
18	237	8.96553	136	572	9.99814	277	8.96739	138	1.03261	.780	42
19	266	8.96689	136	570	9.99813	306	8.96877	136	1.03123	.746	41
20	09295	8.96825	135	99567	9.99812	09335	8.97013	137	1.02987	10.712	40
21	324	8.96960	135	564	9.99810	365	8.97150	135	1.02850	.678	39
22	353	8.97095	134	562	9.99809	394	8.97285	136	1.02715	.645	38
23	382	8.97229	134	559	9.99808	423	8.97421	135	1.02579	.612	37
24	411	8.97363	133	556	9.99807	453	8.97556	135	1.02444	.579	36
25	09440	8.97496	133	99553	9.99806	09482	8.97691	134	1.02309	10.546	35
26	469	8.97629	133	551	9.99804	511	8.97825	134	1.02175	.514	34
27	498	8.97762	132	548	9.99803	541	8.97959	133	1.02041	.481	33
28	527	8.97894	132	545	9.99802	570	8.98092	133	1.01908	.449	32
29	556	8.98026	131	542	9.99801	600	8.98225	133	1.01775	.417	31
30	09585	8.98157	131	99540	9.99800	09629	8.98358	132	1.01642	10.385	30
31	614	8.98288	131	537	9.99798	658	8.98490	132	1.01510	.354	29
32	642	8.98419	130	534	9.99797	688	8.98622	131	1.01378	.322	28
33	671	8.98549	130	531	9.99796	717	8.98753	131	1.01247	.291	27
34	700	8.98679	129	528	9.99795	746	8.98884	131	1.01116	.260	26
35	09729	8.98808	129	99526	9.99793	09776	8.99015	130	1.00985	10.229	25
36	758	8.98937	129	523	9.99792	805	8.99145	130	1.00855	.199	24
37	787	8.99066	128	520	9.99791	834	8.99275	130	1.00725	.168	23
38	816	8.99194	128	517	9.99790	864	8.99405	129	1.00595	.138	22
39	845	8.99322	128	514	9.99788	893	8.99534	128	1.00466	.108	21
40	09874	8.99450	127	99511	9.99787	09923	8.99662	129	1.00338	10.078	20
41	903	8.99577	127	508	9.99786	952	8.99791	128	1.00209	.048	19
42	932	8.99704	126	506	9.99785	981	8.99919	127	1.00081	.019	18
43	961	8.99830	126	503	9.99783	10011	9.00046	128	0.99954	9.9893	17
44	990	8.99956	126	500	9.99782	040	9.00174	127	0.99826	601	16
45	10019	9.00082	125	99497	9.99781	10069	9.00301	126	0.99699	9.9310	15
46	048	9.00207	125	494	9.99780	099	9.00427	126	0.99573	021	14
47	077	9.00332	124	491	9.99778	128	9.00553	126	0.99447	9.8734	13
48	106	9.00456	125	488	9.99777	158	9.00679	126	0.99321	448	12
49	135	9.00581	123	485	9.99776	187	9.00805	125	0.99195	164	11
50	10164	9.00704	124	99482	9.99775	10216	9.00930	125	0.99070	9.7882	10
51	192	9.00828	123	479	9.99773	246	9.01055	124	0.98945	601	9
52	221	9.00951	123	476	9.99772	275	9.01179	124	0.98821	322	8
53	250	9.01074	122	473	9.99771	305	9.01303	124	0.98697	044	7
54	279	9.01196	122	470	9.99769	334	9.01427	123	0.98573	9.6768	6
55	10308	9.01318	122	99467	9.99768	10363	9.01550	123	0.98450	9.6493	5
56	337	9.01440	121	464	9.99767	393	9.01673	123	0.98327	220	4
57	366	9.01561	121	461	9.99765	422	9.01796	122	0.98204	9.5949	3
58	395	9.01682	121	458	9.99764	452	9.01918	122	0.98082	679	2
59	424	9.01803	120	455	9.99763	481	9.02040	122	0.97960	411	1
60	453	9.01923		452	9.99761	510	9.02162		0.97838	144	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	10453	9.01923	120	99452	9.99761	10510	9.02162	121	0.97838	9.5144	60
1	482	9.02043	120	449	9.99760	540	9.02283	121	0.97717	9.4878	59
2	511	9.02163	120	446	9.99759	569	9.02404	121	0.97596	614	58
3	540	9.02283	119	443	9.99757	599	9.02525	120	0.97475	352	57
4	569	9.02402	118	440	9.99756	628	9.02645	121	0.97355	090	56
5	10597	9.02520	119	99437	9.99755	10657	9.02766	119	0.97234	9.3831	55
6	626	9.02639	118	434	9.99753	687	9.02885	120	0.97115	572	54
7	655	9.02757	117	431	9.99752	716	9.03005	119	0.96995	315	53
8	684	9.02874	118	428	9.99751	746	9.03124	118	0.96876	060	52
9	713	9.02992	117	424	9.99749	775	9.03242	119	0.96758	9.2806	51
10	10742	9.03109	117	99421	9.99748	10805	9.03361	118	0.96639	9.2553	50
11	771	9.03226	116	418	9.99747	834	9.03479	118	0.96521	302	49
12	800	9.03342	116	415	9.99745	863	9.03597	117	0.96403	052	48
13	829	9.03458	116	412	9.99744	893	9.03714	118	0.96286	9.1803	47
14	858	9.03574	116	409	9.99742	922	9.03832	116	0.96168	555	46
15	10887	9.03690	115	99406	9.99741	10952	9.03948	117	0.96052	9.1309	45
16	916	9.03805	115	402	9.99740	981	9.04065	116	0.95935	065	44
17	945	9.03920	114	399	9.99738	11011	9.04181	116	0.95819	9.0821	43
18	973	9.04034	115	396	9.99737	040	9.04297	116	0.95703	579	42
19	11002	9.04149	113	393	9.99736	070	9.04413	115	0.95587	338	41
20	11031	9.04262	114	99390	9.99734	11099	9.04528	115	0.95472	9.0098	40
21	060	9.04376	114	386	9.99733	128	9.04643	115	0.95357	8.9860	39
22	089	9.04490	113	383	9.99731	158	9.04758	115	0.95242	623	38
23	118	9.04603	112	380	9.99730	187	9.04873	114	0.95127	387	37
24	147	9.04715	113	377	9.99728	217	9.04987	114	0.95013	152	36
25	11176	9.04828	112	99374	9.99727	11246	9.05101	113	0.94899	8.8919	35
26	205	9.04940	112	370	9.99726	276	9.05214	114	0.94786	686	34
27	234	9.05052	112	367	9.99724	305	9.05328	113	0.94672	455	33
28	263	9.05164	111	364	9.99723	335	9.05441	112	0.94559	225	32
29	291	9.05275	111	360	9.99721	364	9.05553	113	0.94447	8.7996	31
30	11320	9.05386	111	99357	9.99720	11394	9.05666	112	0.94334	8.7769	30
31	349	9.05497	110	354	9.99718	423	9.05778	112	0.94222	542	29
32	378	9.05607	110	351	9.99717	452	9.05890	112	0.94110	317	28
33	407	9.05717	110	347	9.99716	482	9.06002	111	0.93998	093	27
34	436	9.05827	110	344	9.99714	511	9.06113	111	0.93887	8.6870	26
35	11465	9.05937	109	99341	9.99713	11541	9.06224	111	0.93776	8.6648	25
36	494	9.06046	109	337	9.99711	570	9.06335	110	0.93665	427	24
37	523	9.06155	109	334	9.99710	600	9.06445	111	0.93555	208	23
38	552	9.06264	108	331	9.99708	629	9.06556	110	0.93444	8.5989	22
39	580	9.06372	109	327	9.99707	659	9.06666	109	0.93334	772	21
40	11609	9.06481	108	99324	9.99705	11688	9.06775	110	0.93225	8.5555	20
41	638	9.06589	107	320	9.99704	718	9.06885	109	0.93115	340	19
42	667	9.06696	108	317	9.99702	747	9.06994	109	0.93006	126	18
43	696	9.06804	107	314	9.99701	777	9.07103	108	0.92897	8.4913	17
44	725	9.06911	107	310	9.99699	806	9.07211	109	0.92789	701	16
45	11754	9.07018	106	99307	9.99698	11836	9.07320	108	0.92680	8.4490	15
46	783	9.07124	107	303	9.99696	865	9.07428	108	0.92572	280	14
47	812	9.07231	106	300	9.99695	895	9.07536	107	0.92464	071	13
48	840	9.07337	105	297	9.99693	924	9.07643	108	0.92357	8.3863	12
49	869	9.07442	106	293	9.99692	954	9.07751	107	0.92249	656	11
50	11898	9.07548	105	99290	9.99690	11983	9.07858	106	0.92142	8.3450	10
51	927	9.07653	105	286	9.99689	12013	9.07964	107	0.92036	245	9
52	956	9.07758	105	283	9.99687	042	9.08071	106	0.91929	041	8
53	985	9.07863	105	279	9.99686	072	9.08177	106	0.91823	8.2838	7
54	12014	9.07968	104	276	9.99684	101	9.08283	106	0.91717	636	6
55	12043	9.08072	104	99272	9.99683	12131	9.08389	106	0.91611	8.2434	5
56	071	9.08176	104	269	9.99681	160	9.08495	105	0.91505	234	4
57	100	9.08280	103	265	9.99680	190	9.08600	105	0.91400	035	3
58	129	9.08383	103	262	9.99678	219	9.08705	105	0.91295	8.1837	2
59	158	9.08486	103	258	9.99677	249	9.08810	104	0.91190	640	1
60	187	9.08589		255	9.99675	278	9.08914		0.91086	443	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

'	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	12187	9.08589	103	99255	9.99675	12278	9.08914	105	0.91086	8.1443	60
1	216	9.08692	103	251	9.99674	308	9.09019	104	0.90981	248	59
2	245	9.08795	102	248	9.99672	338	9.09123	104	0.90877	054	58
3	274	9.08897	102	244	9.99670	367	9.09227	103	0.90773	8.0860	57
4	302	9.08999	102	240	9.99669	397	9.09330	104	0.90670	667	56
5	12331	9.09101	101	99237	9.99667	12426	9.09434	103	0.90566	8.0476	55
6	360	9.09202	102	233	9.99666	456	9.09537	103	0.90463	285	54
7	389	9.09304	101	230	9.99664	485	9.09640	102	0.90360	095	53
8	418	9.09405	101	226	9.99663	515	9.09742	103	0.90258	7.9906	52
9	447	9.09506	100	222	9.99661	544	9.09845	102	0.90155	718	51
10	12476	9.09606	101	99219	9.99659	12574	9.09947	102	0.90053	7.9530	50
11	504	9.09707	100	215	9.99658	603	9.10049	101	0.89951	344	49
12	533	9.09807	100	211	9.99656	633	9.10150	102	0.89850	158	48
13	562	9.09907	99	208	9.99655	662	9.10252	101	0.89748	7.8973	47
14	591	9.10006	100	204	9.99653	692	9.10353	101	0.89647	789	46
15	12620	9.10106	99	99200	9.99651	12722	9.10454	101	0.89546	7.8606	45
16	649	9.10205	99	197	9.99650	751	9.10555	101	0.89445	424	44
17	678	9.10304	98	193	9.99648	781	9.10656	100	0.89344	243	43
18	706	9.10402	99	189	9.99647	810	9.10756	100	0.89244	062	42
19	735	9.10501	98	186	9.99645	840	9.10856	100	0.89144	7.7882	41
20	12764	9.10599	98	99182	9.99643	12869	9.10956	100	0.89044	7.7704	40
21	793	9.10697	98	178	9.99642	899	9.11056	99	0.88944	525	39
22	822	9.10795	98	175	9.99640	929	9.11155	99	0.88845	348	38
23	851	9.10893	97	171	9.99638	958	9.11254	99	0.88746	171	37
24	880	9.10990	97	167	9.99637	988	9.11353	99	0.88647	7.6996	36
25	12908	9.11087	97	99163	9.99635	13017	9.11452	99	0.88548	7.6821	35
26	937	9.11184	97	160	9.99633	047	9.11551	98	0.88449	647	34
27	966	9.11281	96	156	9.99632	076	9.11649	98	0.88351	473	33
28	995	9.11377	97	152	9.99630	106	9.11747	98	0.88253	301	32
29	13024	9.11474	96	148	9.99629	136	9.11845	98	0.88155	129	31
30	13053	9.11570	96	99144	9.99627	13165	9.11943	97	0.88057	7.5958	30
31	081	9.11666	95	141	9.99625	195	9.12040	98	0.87960	787	29
32	110	9.11761	96	137	9.99624	224	9.12138	97	0.87862	618	28
33	139	9.11857	95	133	9.99622	254	9.12235	97	0.87765	449	27
34	168	9.11952	95	129	9.99620	284	9.12332	96	0.87668	281	26
35	13197	9.12047	95	99125	9.99618	13313	9.12428	97	0.87572	7.5113	25
36	226	9.12142	94	122	9.99617	343	9.12525	96	0.87475	7.4947	24
37	254	9.12236	95	118	9.99615	372	9.12621	96	0.87379	781	23
38	283	9.12331	94	114	9.99613	402	9.12717	96	0.87283	615	22
39	312	9.12425	94	110	9.99612	432	9.12813	96	0.87187	451	21
40	13341	9.12519	93	99106	9.99610	13461	9.12909	95	0.87091	7.4287	20
41	370	9.12612	94	102	9.99608	491	9.13004	95	0.86996	124	19
42	399	9.12706	93	098	9.99607	521	9.13099	95	0.86901	7.3962	18
43	427	9.12799	93	094	9.99605	550	9.13194	95	0.86806	800	17
44	456	9.12892	93	091	9.99603	580	9.13289	95	0.86711	639	16
45	13485	9.12985	93	99087	9.99601	13609	9.13384	94	0.86616	7.3479	15
46	514	9.13078	93	083	9.99600	639	9.13478	95	0.86522	319	14
47	543	9.13171	92	079	9.99598	669	9.13573	94	0.86427	160	13
48	572	9.13263	92	075	9.99596	698	9.13667	94	0.86333	002	12
49	600	9.13355	92	071	9.99595	728	9.13761	93	0.86239	7.2844	11
50	13629	9.13447	92	99067	9.99593	13758	9.13854	94	0.86146	7.2687	10
51	658	9.13539	91	063	9.99591	787	9.13948	93	0.86052	531	9
52	687	9.13630	92	059	9.99589	817	9.14041	93	0.85959	375	8
53	716	9.13722	91	055	9.99588	846	9.14134	93	0.85866	220	7
54	744	9.13813	91	051	9.99586	876	9.14227	93	0.85773	066	6
55	13773	9.13904	90	99047	9.99584	13906	9.14320	92	0.85680	7.1912	5
56	802	9.13994	91	043	9.99582	935	9.14412	92	0.85588	759	4
57	831	9.14085	90	039	9.99581	965	9.14504	93	0.85496	607	3
58	860	9.14175	91	035	9.99579	995	9.14597	91	0.85403	455	2
59	889	9.14266	90	031	9.99577	14024	9.14688	92	0.85312	304	1
60	917	9.14356		027	9.99575	054	9.14780		0.85220	154	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	'

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	13917	9.14356	89	99027	9.99575	14054	9.14780	92	0.85220	7.1154	60
1	946	9.14445	90	023	9.99574	084	9.14872	91	0.85128	004	59
2	975	9.14535	89	019	9.99572	113	9.14963	91	0.85037	7.0855	58
3	14004	9.14624	90	015	9.99570	143	9.15054	91	0.84946	706	57
4	033	9.14714	89	011	9.99568	173	9.15145	91	0.84855	558	56
5	14061	9.14803	88	99006	9.99566	14202	9.15236	91	0.84764	7.0410	55
6	090	9.14891	89	002	9.99565	232	9.15327	90	0.84673	264	54
7	119	9.14980	89	98998	9.99563	262	9.15417	91	0.84583	117	53
8	148	9.15069	88	994	9.99561	291	9.15508	90	0.84492	6.9972	52
9	177	9.15157	88	990	9.99559	321	9.15598	90	0.84402	827	51
10	14205	9.15245	88	98986	9.99557	14351	9.15688	89	0.84312	6.9682	50
11	234	9.15333	88	982	9.99556	381	9.15777	90	0.84223	538	49
12	263	9.15421	87	978	9.99554	410	9.15867	89	0.84133	395	48
13	292	9.15508	88	973	9.99552	440	9.15956	90	0.84044	252	47
14	320	9.15596	87	969	9.99550	470	9.16046	89	0.83954	110	46
15	14349	9.15683	87	98965	9.99548	14499	9.16135	89	0.83865	6.8969	45
16	378	9.15770	87	961	9.99546	529	9.16224	88	0.83776	828	44
17	407	9.15857	87	957	9.99545	559	9.16312	89	0.83688	687	43
18	436	9.15944	86	953	9.99543	588	9.16401	88	0.83599	548	42
19	464	9.16030	86	948	9.99541	618	9.16489	88	0.83511	408	41
20	14493	9.16116	87	98944	9.99539	14648	9.16577	88	0.83423	6.8269	40
21	522	9.16203	86	940	9.99537	678	9.16665	88	0.83335	131	39
22	551	9.16289	85	936	9.99535	707	9.16753	88	0.83247	6.7994	38
23	580	9.16374	86	931	9.99533	737	9.16841	87	0.83159	856	37
24	608	9.16460	85	927	9.99532	767	9.16928	88	0.83072	720	36
25	14637	9.16545	86	98923	9.99530	14796	9.17016	87	0.82984	6.7584	35
26	666	9.16631	85	919	9.99528	826	9.17103	87	0.82897	448	34
27	695	9.16716	85	914	9.99526	856	9.17190	87	0.82810	313	33
28	723	9.16801	85	910	9.99524	886	9.17277	86	0.82723	179	32
29	752	9.16886	84	906	9.99522	915	9.17363	87	0.82637	045	31
30	14781	9.16970	85	98902	9.99520	14945	9.17450	86	0.82550	6.6912	30
31	810	9.17055	84	897	9.99518	975	9.17536	86	0.82464	779	29
32	838	9.17139	84	893	9.99517	15005	9.17622	86	0.82378	646	28
33	867	9.17223	84	889	9.99515	034	9.17708	86	0.82292	514	27
34	896	9.17307	84	884	9.99513	064	9.17794	86	0.82206	383	26
35	14925	9.17391	83	98880	9.99511	15094	9.17880	85	0.82120	6.6252	25
36	954	9.17474	84	876	9.99509	124	9.17965	86	0.82035	122	24
37	982	9.17558	83	871	9.99507	153	9.18051	85	0.81949	6.5992	23
38	15011	9.17641	83	867	9.99505	183	9.18136	85	0.81864	863	22
39	040	9.17724	83	863	9.99503	213	9.18221	85	0.81779	734	21
40	15069	9.17807	83	98858	9.99501	15243	9.18306	85	0.81694	6.5606	20
41	097	9.17890	83	854	9.99499	272	9.18391	84	0.81609	478	19
42	126	9.17973	82	849	9.99497	302	9.18475	85	0.81525	350	18
43	155	9.18055	82	845	9.99495	332	9.18560	84	0.81440	223	17
44	184	9.18137	83	841	9.99494	362	9.18644	84	0.81356	097	16
45	15212	9.18220	82	98836	9.99492	15391	9.18728	84	0.81272	6.4971	15
46	241	9.18302	81	832	9.99490	421	9.18812	84	0.81188	846	14
47	270	9.18383	82	827	9.99488	451	9.18896	83	0.81104	721	13
48	299	9.18465	82	823	9.99486	481	9.18979	84	0.81021	596	12
49	327	9.18547	81	818	9.99484	511	9.19063	83	0.80937	472	11
50	15356	9.18628	81	98814	9.99482	15540	9.19146	83	0.80854	6.4348	10
51	385	9.18709	81	809	9.99480	570	9.19229	83	0.80771	225	9
52	414	9.18790	81	805	9.99478	600	9.19312	83	0.80688	103	8
53	442	9.18871	81	800	9.99476	630	9.19395	83	0.80605	6.3980	7
54	471	9.18952	81	796	9.99474	660	9.19478	83	0.80522	859	6
55	15500	9.19033	80	98791	9.99472	15689	9.19561	82	0.80439	6.3737	5
56	529	9.19113	80	787	9.99470	719	9.19643	82	0.80357	617	4
57	557	9.19193	80	782	9.99468	749	9.19725	82	0.80275	496	3
58	586	9.19273	80	778	9.99466	779	9.19807	82	0.80193	376	2
59	615	9.19353	80	773	9.99464	809	9.19889	82	0.80111	257	1
60	643	9.19433		769	9.99462	838	9.19971		0.80029	138	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	15643	9.19433	80	98769	9.99462	15838	9.19971	82	0.80029	6.3138	60
1	672	9.19513	79	764	9.99460	868	9.20053	81	0.79947	019	59
2	701	9.19592	80	760	9.99458	898	9.20134	82	0.79866	6.2901	58
3	730	9.19672	79	755	9.99456	928	9.20216	81	0.79784	783	57
4	758	9.19751	79	751	9.99454	958	9.20297	81	0.79703	666	56
5	15787	9.19830	79	98746	9.99452	15988	9.20378	81	0.79622	6.2549	55
6	816	9.19909	79	741	9.99450	16017	9.20459	81	0.79541	432	54
7	845	9.19988	79	737	9.99448	047	9.20540	81	0.79460	316	53
8	873	9.20067	78	732	9.99446	077	9.20621	80	0.79379	200	52
9	902	9.20145	78	728	9.99444	107	9.20701	81	0.79299	085	51
10	15931	9.20223	79	98723	9.99442	16137	9.20782	80	0.79218	6.1970	50
11	959	9.20302	78	718	9.99440	167	9.20862	80	0.79138	856	49
12	988	9.20380	78	714	9.99438	196	9.20942	80	0.79058	742	48
13	16017	9.20458	77	709	9.99436	226	9.21022	80	0.78978	628	47
14	046	9.20535	78	704	9.99434	256	9.21102	80	0.78898	515	46
15	16074	9.20613	78	98700	9.99432	16286	9.21182	79	0.78818	6.1402	45
16	103	9.20691	77	695	9.99429	316	9.21261	80	0.78739	290	44
17	132	9.20768	77	690	9.99427	346	9.21341	79	0.78659	178	43
18	160	9.20845	77	686	9.99425	376	9.21420	79	0.78580	066	42
19	189	9.20922	77	681	9.99423	405	9.21499	79	0.78501	6.0955	41
20	16218	9.20999	77	98676	9.99421	16435	9.21578	79	0.78422	6.0844	40
21	246	9.21076	77	671	9.99419	465	9.21657	79	0.78343	734	39
22	275	9.21153	76	667	9.99417	495	9.21736	78	0.78264	624	38
23	304	9.21229	77	662	9.99415	525	9.21814	79	0.78186	514	37
24	333	9.21306	76	657	9.99413	555	9.21893	78	0.78107	405	36
25	16361	9.21382	76	98652	9.99411	16585	9.21971	78	0.78029	6.0296	35
26	390	9.21458	76	648	9.99409	615	9.22049	78	0.77951	188	34
27	419	9.21534	76	643	9.99407	645	9.22127	78	0.77873	080	33
28	447	9.21610	75	638	9.99404	674	9.22205	78	0.77795	5.9972	32
29	476	9.21685	76	633	9.99402	704	9.22283	78	0.77717	865	31
30	16505	9.21761	75	98629	9.99400	16734	9.22361	77	0.77639	5.9758	30
31	533	9.21836	76	624	9.99398	764	9.22438	78	0.77562	651	29
32	562	9.21912	75	619	9.99396	794	9.22516	77	0.77484	545	28
33	591	9.21987	75	614	9.99394	824	9.22593	77	0.77407	439	27
34	620	9.22062	75	609	9.99392	854	9.22670	77	0.77330	333	26
35	16648	9.22137	74	98604	9.99390	16884	9.22747	77	0.77253	5.9228	25
36	677	9.22211	75	600	9.99388	914	9.22824	77	0.77176	124	24
37	706	9.22286	75	595	9.99385	944	9.22901	76	0.77099	019	23
38	734	9.22361	74	590	9.99383	974	9.22977	77	0.77023	5.8915	22
39	763	9.22435	74	585	9.99381	17004	9.23054	76	0.76946	811	21
40	16792	9.22509	74	98580	9.99379	17033	9.23130	76	0.76870	5.8708	20
41	820	9.22583	74	575	9.99377	063	9.23206	77	0.76794	605	19
42	849	9.22657	74	570	9.99375	093	9.23283	76	0.76717	502	18
43	878	9.22731	74	565	9.99372	123	9.23359	76	0.76641	400	17
44	906	9.22805	73	561	9.99370	153	9.23435	75	0.76565	298	16
45	16935	9.22878	74	98556	9.99368	17183	9.23510	76	0.76490	5.8197	15
46	964	9.22952	73	551	9.99366	213	9.23586	75	0.76414	095	14
47	992	9.23025	73	546	9.99364	243	9.23661	76	0.76339	5.7994	13
48	17021	9.23098	73	541	9.99362	273	9.23737	75	0.76263	894	12
49	050	9.23171	73	536	9.99359	303	9.23812	75	0.76188	794	11
50	17078	9.23244	73	98531	9.99357	17333	9.23887	75	0.76113	5.7694	10
51	107	9.23317	73	526	9.99355	363	9.23962	75	0.76038	594	9
52	136	9.23390	72	521	9.99353	393	9.24037	75	0.75963	495	8
53	164	9.23462	73	516	9.99351	423	9.24112	74	0.75888	396	7
54	193	9.23535	72	511	9.99348	453	9.24186	75	0.75814	297	6
55	17222	9.23607	72	98506	9.99346	17483	9.24261	74	0.75739	5.7199	5
56	250	9.23679	73	501	9.99344	513	9.24335	75	0.75665	101	4
57	279	9.23752	71	496	9.99342	543	9.24410	74	0.75590	004	3
58	308	9.23823	72	491	9.99340	573	9.24484	74	0.75516	5.6906	2
59	336	9.23895	72	486	9.99337	603	9.24558	74	0.75442	809	1
60	365	9.23967		481	9.99335	633	9.24632		0.75368	713	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

10°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	17365	9.23967	72	98481	9.99335	2	17633	9.24632	74	0.75368	5.6713	60
1	393	9.24039	71	476	9.99333	2	663	9.24706	73	0.75294	617	59
2	422	9.24110	71	471	9.99331	3	693	9.24779	74	0.75221	521	58
3	451	9.24181	72	466	9.99328	2	723	9.24853	73	0.75147	425	57
4	479	9.24253	71	461	9.99326	2	753	9.24926	74	0.75074	329	56
5	17508	9.24324	71	98455	9.99324	2	17783	9.25000	73	0.75000	5.6234	55
6	537	9.24395	71	450	9.99322	3	813	9.25073	73	0.74927	140	54
7	565	9.24466	70	445	9.99319	2	843	9.25146	73	0.74854	045	53
8	594	9.24536	71	440	9.99317	2	873	9.25219	73	0.74781	5.5951	52
9	623	9.24607	70	435	9.99315	2	903	9.25292	73	0.74708	857	51
10	17651	9.24677	71	98430	9.99313	3	17933	9.25365	72	0.74635	5.5764	50
11	680	9.24748	70	425	9.99310	2	963	9.25437	73	0.74563	671	49
12	708	9.24818	70	420	9.99308	2	993	9.25510	72	0.74490	578	48
13	737	9.24888	70	414	9.99306	2	18023	9.25582	73	0.74418	485	47
14	766	9.24958	70	409	9.99304	3	053	9.25655	72	0.74345	393	46
15	17794	9.25028	70	98404	9.99301	2	18083	9.25727	72	0.74273	5.5301	45
16	823	9.25098	70	399	9.99299	2	113	9.25799	72	0.74201	209	44
17	852	9.25168	69	394	9.99297	3	143	9.25871	72	0.74129	118	43
18	880	9.25237	70	389	9.99294	2	173	9.25943	72	0.74057	026	42
19	909	9.25307	69	383	9.99292	2	203	9.26015	71	0.73985	5.4936	41
20	17937	9.25376	69	98378	9.99290	2	18233	9.26086	72	0.73914	5.4845	40
21	966	9.25445	69	373	9.99288	3	263	9.26158	71	0.73842	755	39
22	995	9.25514	69	368	9.99285	2	293	9.26229	72	0.73771	665	38
23	18023	9.25583	69	362	9.99283	2	323	9.26301	71	0.73699	575	37
24	052	9.25652	69	357	9.99281	3	353	9.26372	71	0.73628	486	36
25	18081	9.25721	69	98352	9.99278	2	18384	9.26443	71	0.73557	5.4397	35
26	109	9.25790	68	347	9.99276	2	414	9.26514	71	0.73486	308	34
27	138	9.25858	69	341	9.99274	3	444	9.26585	70	0.73415	219	33
28	166	9.25927	68	336	9.99271	2	474	9.26655	71	0.73345	131	32
29	195	9.25995	68	331	9.99269	2	504	9.26726	71	0.73274	043	31
30	18224	9.26063	68	98325	9.99267	3	18534	9.26797	70	0.73203	5.3955	30
31	252	9.26131	68	320	9.99264	2	564	9.26867	70	0.73133	868	29
32	281	9.26199	68	315	9.99262	2	594	9.26937	71	0.73063	781	28
33	309	9.26267	68	310	9.99260	3	624	9.27008	70	0.72992	694	27
34	338	9.26335	68	304	9.99257	2	654	9.27078	70	0.72922	607	26
35	18367	9.26403	67	98299	9.99255	3	18684	9.27148	70	0.72852	5.3521	25
36	395	9.26470	68	294	9.99252	2	714	9.27218	70	0.72782	435	24
37	424	9.26538	67	288	9.99250	2	745	9.27288	69	0.72712	349	23
38	452	9.26605	67	283	9.99248	3	775	9.27357	70	0.72643	263	22
39	481	9.26672	67	277	9.99245	2	805	9.27427	69	0.72573	178	21
40	18509	9.26739	67	98272	9.99243	2	18835	9.27496	70	0.72504	5.3093	20
41	538	9.26806	67	267	9.99241	3	865	9.27566	69	0.72434	008	19
42	567	9.26873	67	261	9.99238	2	895	9.27635	69	0.72365	5.2924	18
43	595	9.26940	67	256	9.99236	3	925	9.27704	69	0.72296	839	17
44	624	9.27007	66	250	9.99233	2	955	9.27773	69	0.72227	755	16
45	18652	9.27073	67	98245	9.99231	2	18986	9.27842	69	0.72158	5.2672	15
46	681	9.27140	66	240	9.99229	3	19016	9.27911	69	0.72089	588	14
47	710	9.27206	67	234	9.99226	2	046	9.27980	69	0.72020	505	13
48	738	9.27273	66	229	9.99224	3	076	9.28049	68	0.71951	422	12
49	767	9.27339	66	223	9.99221	2	106	9.28117	69	0.71883	339	11
50	18795	9.27405	66	98218	9.99219	2	19136	9.28186	68	0.71814	5.2257	10
51	824	9.27471	66	212	9.99217	3	166	9.28254	69	0.71746	174	9
52	852	9.27537	65	207	9.99214	2	197	9.28323	68	0.71677	092	8
53	881	9.27602	66	201	9.99212	3	227	9.28391	68	0.71609	011	7
54	910	9.27668	66	196	9.99209	2	257	9.28459	68	0.71541	5.1929	6
55	18938	9.27734	65	98190	9.99207	3	19287	9.28527	68	9.71473	5.1848	5
56	967	9.27799	65	185	9.99204	2	317	9.28595	67	0.71405	767	4
57	995	9.27864	66	179	9.99202	2	347	9.28662	68	0.71338	686	3
58	19024	9.27930	65	174	9.99200	3	378	9.28730	68	0.71270	606	2
59	052	9.27995	65	168	9.99197	2	408	9.28798	67	0.71202	526	1
60	081	9.28060		163	9.99195		438	9.28865		0.71135	446	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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11°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	19081	9.28060	65	98163	9.99195	3	19438	9.28865	68	0.71135	5.1446	60
1	109	9.28125	65	157	9.99192	2	468	9.28933	67	0.71067	366	59
2	138	9.28190	64	152	9.99190	3	498	9.29000	67	0.71000	286	58
3	167	9.28254	65	146	9.99187	2	529	9.29067	67	0.70933	207	57
4	195	9.28319	65	140	9.99185	3	559	9.29134	67	0.70866	128	56
5	19224	9.28384	64	98135	9.99182	2	19589	9.29201	67	0.70799	5.1049	55
6	252	9.28448	64	129	9.99180	3	619	9.29268	67	0.70732	5.0970	54
7	281	9.28512	65	124	9.99177	2	649	9.29335	67	0.70665	892	53
8	309	9.28577	64	118	9.99175	3	680	9.29402	66	0.70598	814	52
9	338	9.28641	64	112	9.99172	2	710	9.29468	67	0.70532	736	51
10	19366	9.28705	64	98107	9.99170	3	19740	9.29535	66	0.70465	5.0658	50
11	395	9.28769	64	101	9.99167	2	770	9.29601	67	0.70399	581	49
12	423	9.28833	63	096	9.99165	3	801	9.29668	66	0.70332	504	48
13	452	9.28896	64	090	9.99162	2	831	9.29734	66	0.70266	427	47
14	481	9.28960	64	084	9.99160	3	861	9.29800	66	0.70200	350	46
15	19509	9.29024	63	98079	9.99157	2	19891	9.29866	66	0.70134	5.0273	45
16	538	9.29087	63	073	9.99155	3	921	9.29932	66	0.70068	197	44
17	566	9.29150	64	067	9.99152	2	952	9.29998	66	0.70002	121	43
18	595	9.29214	63	061	9.99150	3	982	9.30064	66	0.69936	045	42
19	623	9.29277	63	056	9.99147	2	20012	9.30130	65	0.69870	4.9969	41
20	19652	9.29340	63	98050	9.99145	3	20042	9.30195	66	0.69805	4.9894	40
21	680	9.29403	63	044	9.99142	2	073	9.30261	65	0.69739	819	39
22	709	9.29466	63	039	9.99140	3	103	9.30326	65	0.69674	744	38
23	737	9.29529	62	033	9.99137	2	133	9.30391	66	0.69609	669	37
24	766	9.29591	63	027	9.99135	3	164	9.30457	65	0.69543	594	36
25	19794	9.29654	62	98021	9.99132	2	20194	9.30522	65	0.69478	4.9520	35
26	823	9.29716	63	016	9.99130	3	224	9.30587	65	0.69413	446	34
27	851	9.29779	62	010	9.99127	3	254	9.30652	65	0.69348	372	33
28	880	9.29841	62	004	9.99124	2	285	9.30717	65	0.69283	298	32
29	908	9.29903	63	97998	9.99122	3	315	9.30782	64	0.69218	225	31
30	19937	9.29966	62	97992	9.99119	2	20345	9.30846	65	0.69154	4.9152	30
31	965	9.30028	62	987	9.99117	3	376	9.30911	64	0.69089	078	29
32	994	9.30090	61	981	9.99114	2	406	9.30975	65	0.69025	006	28
33	20022	9.30151	62	975	9.99112	3	436	9.31040	64	0.68960	4.8933	27
34	051	9.30213	62	969	9.99109	3	466	9.31104	64	0.68896	860	26
35	20079	9.30275	61	97963	9.99106	2	20497	9.31168	65	0.68832	4.8788	25
36	108	9.30336	62	958	9.99104	3	527	9.31233	64	0.68767	716	24
37	136	9.30398	61	952	9.99101	2	557	9.31297	64	0.68703	644	23
38	165	9.30459	62	946	9.99099	3	588	9.31361	64	0.68639	573	22
39	193	9.30521	61	940	9.99096	3	618	9.31425	64	0.68575	501	21
40	20222	9.30582	61	97934	9.99093	2	20648	9.31489	63	0.68511	4.8430	20
41	250	9.30643	61	928	9.99091	3	679	9.31552	64	0.68448	359	19
42	279	9.30704	61	922	9.99088	2	709	9.31616	63	0.68384	288	18
43	307	9.30765	61	916	9.99086	3	739	9.31679	64	0.68321	218	17
44	336	9.30826	61	910	9.99083	3	770	9.31743	63	0.68257	147	16
45	20364	9.30887	60	97905	9.99080	2	20800	9.31806	64	0.68194	4.8077	15
46	393	9.30947	61	899	9.99078	3	830	9.31870	63	0.68130	007	14
47	421	9.31008	60	893	9.99075	3	861	9.31933	63	0.68067	4.7937	13
48	450	9.31068	61	887	9.99072	2	891	9.31996	63	0.68004	867	12
49	478	9.31129	60	881	9.99070	3	921	9.32059	63	0.67941	798	11
50	20507	9.31189	61	97875	9.99067	3	20952	9.32122	63	0.67878	4.7729	10
51	535	9.31250	60	869	9.99064	2	982	9.32185	63	0.67815	659	9
52	563	9.31310	60	863	9.99062	3	21013	9.32248	63	0.67752	591	8
53	592	9.31370	60	857	9.99059	3	043	9.32311	62	0.67689	522	7
54	620	9.31430	60	851	9.99056	2	073	9.32373	63	0.67627	453	6
55	20649	9.31490	59	97845	9.99054	3	21104	9.32436	62	0.67564	4.7385	5
56	677	9.31549	60	839	9.99051	3	134	9.32498	63	0.67502	317	4
57	706	9.31609	60	833	9.99048	2	164	9.32561	62	0.67439	249	3
58	734	9.31669	59	827	9.99046	3	195	9.32623	62	0.67377	181	2
59	763	9.31728	60	821	9.99043	3	225	9.32685	62	0.67315	114	1
60	791	9.31788		815	9.99040		256	9.32747		0.67253	046	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	20791	9.31788	59	97815	9.99040	2	21256	9.32747	63	0.67253	4.7046	60
1	820	9.31847	60	809	9.99038	3	286	9.32810	62	0.67190	4.6979	59
2	848	9.31907	59	803	9.99035	3	316	9.32872	61	0.67128	912	58
3	877	9.31966	59	797	9.99032	2	347	9.32933	62	0.67067	845	57
4	905	9.32025	59	791	9.99030	3	377	9.32995	62	0.67005	779	56
5	20933	9.32084	59	97784	9.99027	3	21408	9.33057	62	0.66943	4.6712	55
6	962	9.32143	59	778	9.99024	2	438	9.33119	61	0.66881	646	54
7	990	9.32202	59	772	9.99022	3	469	9.33180	62	0.66820	580	53
8	21019	9.32261	58	766	9.99019	3	499	9.33242	61	0.66758	514	52
9	047	9.32319	59	760	9.99016	3	529	9.33303	62	0.66697	448	51
10	21076	9.32378	59	97754	9.99013	2	21560	9.33365	61	0.66635	4.6382	50
11	104	9.32437	58	748	9.99011	3	590	9.33426	61	0.66574	317	49
12	132	9.32495	58	742	9.99008	3	621	9.33487	61	0.66513	252	48
13	161	9.32553	59	735	9.99005	3	651	9.33548	61	0.66452	187	47
14	189	9.32612	58	729	9.99002	2	682	9.33609	61	0.66391	122	46
15	21218	9.32670	58	97723	9.99000	3	21712	9.33670	61	0.66330	4.6057	45
16	246	9.32728	58	717	9.98997	3	743	9.33731	61	0.66269	4.5993	44
17	275	9.32786	58	711	9.98994	3	773	9.33792	61	0.66208	928	43
18	303	9.32844	58	705	9.98991	2	804	9.33853	60	0.66147	864	42
19	331	9.32902	58	698	9.98989	3	834	9.33913	61	0.66087	800	41
20	21360	9.32960	58	97692	9.98986	3	21864	9.33974	60	0.66026	4.5736	40
21	388	9.33018	57	686	9.98983	3	895	9.34034	61	0.65966	673	39
22	417	9.33075	58	680	9.98980	2	925	9.34095	60	0.65905	609	38
23	445	9.33133	57	673	9.98978	3	956	9.34155	60	0.65845	546	37
24	474	9.33190	58	667	9.98975	3	986	9.34215	61	0.65785	483	36
25	21502	9.33248	57	97661	9.98972	3	22017	9.34276	60	0.65724	4.5420	35
26	530	9.33305	57	655	9.98969	2	047	9.34336	60	0.65664	357	34
27	559	9.33362	58	648	9.98967	3	078	9.34396	60	0.65604	294	33
28	587	9.33420	57	642	9.98964	3	108	9.34456	60	0.65544	232	32
29	616	9.33477	57	636	9.98961	3	139	9.34516	60	0.65484	169	31
30	21644	9.33534	57	97630	9.98958	3	22169	9.34576	59	0.65424	4.5107	30
31	672	9.33591	56	623	9.98955	2	200	9.34635	60	0.65365	045	29
32	701	9.33647	57	617	9.98953	3	231	9.34695	60	0.65305	4.4983	28
33	729	9.33704	57	611	9.98950	3	261	9.34755	59	0.65245	922	27
34	758	9.33761	57	604	9.98947	3	292	9.34814	60	0.65186	860	26
35	21786	9.33818	56	97598	9.98944	3	22322	9.34874	59	0.65126	4.4799	25
36	814	9.33874	57	592	9.98941	3	353	9.34933	59	0.65067	737	24
37	843	9.33931	56	585	9.98938	2	383	9.34992	59	0.65008	676	23
38	871	9.33987	56	579	9.98936	3	414	9.35051	60	0.64949	615	22
39	899	9.34043	57	573	9.98933	3	444	9.35111	59	0.64889	555	21
40	21928	9.34100	56	97566	9.98930	3	22475	9.35170	59	0.64830	4.4494	20
41	956	9.34156	56	560	9.98927	3	505	9.35229	59	0.64771	434	19
42	985	9.34212	56	553	9.98924	3	536	9.35288	59	0.64712	373	18
43	22013	9.34268	56	547	9.98921	2	567	9.35347	58	0.64653	313	17
44	041	9.34324	56	541	9.98919	3	597	9.35405	59	0.64595	253	16
45	22070	9.34380	56	97534	9.98916	3	22628	9.35464	59	0.64536	4.4194	15
46	098	9.34436	55	528	9.98913	3	658	9.35523	58	0.64477	134	14
47	126	9.34491	56	521	9.98910	3	689	9.35581	59	0.64419	075	13
48	155	9.34547	55	515	9.98907	3	719	9.35640	58	0.64360	015	12
49	183	9.34602	56	508	9.98904	3	750	9.35698	59	0.64302	4.3956	11
50	22212	9.34658	55	97502	9.98901	3	22781	9.35757	58	0.64243	4.3897	10
51	240	9.34713	56	496	9.98898	2	811	9.35815	58	0.64185	838	9
52	268	9.34769	55	489	9.98896	3	842	9.35873	58	0.64127	779	8
53	297	9.34824	55	483	9.98893	3	872	9.35931	58	0.64069	721	7
54	325	9.34879	55	476	9.98890	3	903	9.35989	58	0.64011	662	6
55	22353	9.34934	55	97470	9.98887	3	22934	9.36047	58	0.63953	4.3604	5
56	382	9.34989	55	463	9.98884	3	964	9.36105	58	0.63895	546	4
57	410	9.35044	55	457	9.98881	3	995	9.36163	58	0.63837	488	3
58	438	9.35099	55	450	9.98878	3	23026	9.36221	58	0.63779	430	2
59	467	9.35154	55	444	9.98875	3	056	9.36279	57	0.63721	372	1
60	495	9.35209		437	9.98872		087	9.36336		0.63664	315	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	22495	9.35209	54	97437	9.98872	3	23087	9.36336	58	0.63664	4.3315	60
1	523	9.35263	55	430	9.98869	2	117	9.36394	58	0.63606	257	59
2	552	9.35318	55	424	9.98867	3	148	9.36452	57	0.63548	200	58
3	580	9.35373	54	417	9.98864	3	179	9.36509	57	0.63491	143	57
4	608	9.35427	54	411	9.98861	3	209	9.36566	58	0.63434	086	56
5	22637	9.35481	55	97404	9.98858	3	23240	9.36624	57	0.63376	4.3029	55
6	665	9.35536	54	398	9.98855	3	271	9.36681	57	0.63319	4.2972	54
7	693	9.35590	54	391	9.98852	3	301	9.36738	57	0.63262	916	53
8	722	9.35644	54	384	9.98849	3	332	9.36795	57	0.63205	859	52
9	750	9.35698	54	378	9.98846	3	363	9.36852	57	0.63148	803	51
10	22778	9.35752	54	97371	9.98843	3	23393	9.36909	57	0.63091	4.2747	50
11	807	9.35806	54	365	9.98840	3	424	9.36966	57	0.63034	691	49
12	835	9.35860	54	358	9.98837	3	455	9.37023	57	0.62977	635	48
13	863	9.35914	54	351	9.98834	3	485	9.37080	57	0.62920	580	47
14	892	9.35968	54	345	9.98831	3	516	9.37137	56	0.62863	524	46
15	22920	9.36022	53	97338	9.98828	3	23547	9.37193	57	0.62807	4.2468	45
16	948	9.36075	54	331	9.98825	3	578	9.37250	56	0.62750	413	44
17	977	9.36129	53	325	9.98822	3	608	9.37306	57	0.62694	358	43
18	23005	9.36182	54	318	9.98819	3	639	9.37363	56	0.62637	303	42
19	033	9.36236	53	311	9.98816	3	670	9.37419	57	0.62581	248	41
20	23062	9.36289	53	97304	9.98813	3	23700	9.37476	56	0.62524	4.2193	40
21	090	9.36342	53	298	9.98810	3	731	9.37532	56	0.62468	139	39
22	118	9.36395	54	291	9.98807	3	762	9.37588	56	0.62412	084	38
23	146	9.36449	53	284	9.98804	3	793	9.37644	56	0.62356	030	37
24	175	9.36502	53	278	9.98801	3	823	9.37700	56	0.62300	4.1976	36
25	23203	9.36555	53	97271	9.98798	3	23854	9.37756	56	0.62244	4.1922	35
26	231	9.36608	52	264	9.98795	3	885	9.37812	56	0.62188	868	34
27	260	9.36660	53	257	9.98792	3	916	9.37868	56	0.62132	814	33
28	288	9.36713	53	251	9.98789	3	946	9.37924	56	0.62076	760	32
29	316	9.36766	53	244	9.98786	3	977	9.37980	55	0.62020	706	31
30	23345	9.36819	52	97237	9.98783	3	24008	9.38035	56	0.61965	4.1653	30
31	373	9.36871	53	230	9.98780	3	039	9.38091	56	0.61909	600	29
32	401	9.36924	52	223	9.98777	3	069	9.38147	55	0.61853	547	28
33	429	9.36976	52	217	9.98774	3	100	9.38202	55	0.61798	493	27
34	458	9.37028	53	210	9.98771	3	131	9.38257	56	0.61743	441	26
35	23486	9.37081	52	97203	9.98768	3	24162	9.38313	55	0.61687	4.1388	25
36	514	9.37133	52	196	9.98765	3	193	9.38368	55	0.61632	335	24
37	542	9.37185	52	189	9.98762	3	223	9.38423	56	0.61577	282	23
38	571	9.37237	52	182	9.98759	3	254	9.38479	55	0.61521	230	22
39	599	9.37289	52	176	9.98756	3	285	9.38534	55	0.61466	178	21
40	23627	9.37341	52	97169	9.98753	3	24316	9.38589	55	0.61411	4.1126	20
41	656	9.37393	52	162	9.98750	4	347	9.38644	55	0.61356	074	19
42	684	9.37445	52	155	9.98746	3	377	9.38699	55	0.61301	022	18
43	712	9.37497	52	148	9.98743	3	408	9.38754	54	0.61246	4.0970	17
44	740	9.37549	51	141	9.98740	3	439	9.38808	55	0.61192	918	16
45	23769	9.37600	52	97134	9.98737	3	24470	9.38863	55	0.61137	4.0867	15
46	797	9.37652	51	127	9.98734	3	501	9.38918	54	0.61082	815	14
47	825	9.37703	52	120	9.98731	3	532	9.38972	55	0.61028	764	13
48	853	9.37755	51	113	9.98728	3	562	9.39027	55	0.60973	713	12
49	882	9.37806	52	106	9.98725	3	593	9.39082	54	0.60918	662	11
50	23910	9.37858	51	97100	9.98722	3	24624	9.39136	54	0.60864	4.0611	10
51	938	9.37909	51	093	9.98719	4	655	9.39190	55	0.60810	560	9
52	966	9.37960	51	086	9.98715	3	686	9.39245	54	0.60755	509	8
53	995	9.38011	51	079	9.98712	3	717	9.39299	54	0.60701	459	7
54	24023	9.38062	51	072	9.98709	3	747	9.39353	54	0.60647	408	6
55	24051	9.38113	51	97065	9.98706	3	24778	9.39407	54	0.60593	4.0358	5
56	079	9.38164	51	058	9.98703	3	809	9.39461	54	0.60539	308	4
57	108	9.38215	51	051	9.98700	3	840	9.39515	54	0.60485	257	3
58	136	9.38266	51	044	9.98697	3	871	9.39569	54	0.60431	207	2
59	164	9.38317	51	037	9.98694	4	902	9.39623	54	0.60377	158	1
60	192	9.38368		030	9.98690		933	9.39677		0.60323	108	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	24192	9.38368	50	97030	9.98690	3	24933	9.39677	54	0.60323	4.0108	60
1	220	9.38418	51	023	9.98687	3	964	9.39731	54	0.60269	058	59
2	249	9.38469	50	015	9.98684	3	995	9.39785	53	0.60215	009	58
3	277	9.38519	51	008	9.98681	3	25026	9.39838	54	0.60162	3.9959	57
4	305	9.38570	50	001	9.98678	3	056	9.39892	53	0.60108	910	56
5	24333	9.38620	50	96994	9.98675	4	25087	9.39945	54	0.60055	3.9861	55
6	362	9.38670	51	987	9.98671	3	118	9.39999	53	0.60001	812	54
7	390	9.38721	50	980	9.98668	3	149	9.40052	54	0.59948	763	53
8	418	9.38771	50	973	9.98665	3	180	9.40106	53	0.59894	714	52
9	446	9.38821	50	966	9.98662	3	211	9.40159	53	0.59841	665	51
10	24474	9.38871	50	96959	9.98659	3	25242	9.40212	54	0.59788	3.9617	50
11	503	9.38921	50	952	9.98656	4	273	9.40266	53	0.59734	568	49
12	531	9.38971	50	945	9.98652	3	304	9.40319	53	0.59681	520	48
13	559	9.39021	50	937	9.98649	3	335	9.40372	53	0.59628	471	47
14	587	9.39071	50	930	9.98646	3	366	9.40425	53	0.59575	423	46
15	24615	9.39121	49	96923	9.98643	3	25397	9.40478	53	0.59522	3.9375	45
16	644	9.39170	50	916	9.98640	4	428	9.40531	53	0.59469	327	44
17	672	9.39220	50	909	9.98636	3	459	9.40584	52	0.59416	279	43
18	700	9.39270	49	902	9.98633	3	490	9.40636	53	0.59364	232	42
19	728	9.39319	50	894	9.98630	3	521	9.40689	53	0.59311	184	41
20	24756	9.39369	49	96887	9.98627	4	25552	9.40742	53	0.59258	3.9136	40
21	784	9.39418	49	880	9.98623	3	583	9.40795	52	0.59205	089	39
22	813	9.39467	50	873	9.98620	3	614	9.40847	53	0.59153	042	38
23	841	9.39517	49	866	9.98617	3	645	9.40900	52	0.59100	3.8995	37
24	869	9.39566	49	858	9.98614	4	676	9.40952	53	0.59048	947	36
25	24897	9.39615	49	96851	9.98610	3	25707	9.41005	52	0.58995	3.8900	35
26	925	9.39664	49	844	9.98607	3	738	9.41057	52	0.58943	854	34
27	954	9.39713	49	837	9.98604	3	769	9.41109	52	0.58891	807	33
28	982	9.39762	49	829	9.98601	4	800	9.41161	53	0.58839	760	32
29	25010	9.39811	49	822	9.98597	3	831	9.41214	52	0.58786	714	31
30	25038	9.39860	49	96815	9.98594	3	25862	9.41266	52	0.58734	3.8667	30
31	066	9.39909	49	807	9.98591	3	893	9.41318	52	0.58682	621	29
32	094	9.39958	48	800	9.98588	4	924	9.41370	52	0.58630	575	28
33	122	9.40006	49	793	9.98584	3	955	9.41422	52	0.58578	528	27
34	151	9.40055	48	786	9.98581	3	986	9.41474	52	0.58526	482	26
35	25179	9.40103	49	96778	9.98578	4	26017	9.41526	52	0.58474	3.8436	25
36	207	9.40152	48	771	9.98574	3	048	9.41578	51	0.58422	391	24
37	235	9.40200	49	764	9.98571	3	079	9.41629	52	0.58371	345	23
38	263	9.40249	48	756	9.98568	3	110	9.41681	52	0.58319	299	22
39	291	9.40297	49	749	9.98565	4	141	9.41733	51	0.58267	254	21
40	25320	9.40346	48	96742	9.98561	3	26172	9.41784	52	0.58216	3.8208	20
41	348	9.40394	48	734	9.98558	3	203	9.41836	51	0.58164	163	19
42	376	9.40442	48	727	9.98555	4	235	9.41887	52	0.58113	118	18
43	404	9.40490	48	719	9.98551	3	266	9.41939	51	0.58061	073	17
44	432	9.40538	48	712	9.98548	3	297	9.41990	51	0.58010	028	16
45	25460	9.40586	48	96705	9.98545	4	26328	9.42041	52	0.57959	3.7983	15
46	488	9.40634	48	697	9.98541	3	359	9.42093	51	0.57907	938	14
47	516	9.40682	48	690	9.98538	3	390	9.42144	51	0.57856	893	13
48	545	9.40730	48	682	9.98535	4	421	9.42195	51	0.57805	848	12
49	573	9.40778	47	675	9.98531	3	452	9.42246	51	0.57754	804	11
50	25601	9.40825	48	96667	9.98528	3	26483	9.42297	51	0.57703	3.7760	10
51	629	9.40873	48	660	9.98525	4	515	9.42348	51	0.57652	715	9
52	657	9.40921	47	653	9.98521	3	546	9.42399	51	0.57601	671	8
53	685	9.40968	48	645	9.98518	3	577	9.42450	51	0.57550	627	7
54	713	9.41016	47	638	9.98515	4	608	9.42501	51	0.57499	583	6
55	25741	9.41063	48	96630	9.98511	3	26639	9.42552	51	0.57448	3.7539	5
56	769	9.41111	47	623	9.98508	3	670	9.42603	50	0.57397	495	4
57	798	9.41158	47	615	9.98505	4	701	9.42653	51	0.57347	451	3
58	826	9.41205	47	608	9.98501	3	733	9.42704	51	0.57296	408	2
59	854	9.41252	48	600	9.98498	4	764	9.42755	50	0.57245	364	1
60	882	9.41300		593	9.98494		795	9.42805		0.57195	321	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

15°

′	Nat. Sin	Log. Sin	d.	Nat. Cos	Log. Cos	d.	Nat. Tan	Log. Tan	c.d.	Log. Cot	Nat. Cot	
0	25882	9.41300	47	96593	9.98494	3	26795	9.42805	51	0.57195	3.7321	60
1	910	9.41347	47	585	9.98491	3	826	9.42856	50	0.57144	277	59
2	938	9.41394	47	578	9.98488	4	857	9.42906	51	0.57094	234	58
3	966	9.41441	47	570	9.98484	3	888	9.42957	50	0.57043	191	57
4	994	9.41488	47	562	9.98481	4	920	9.43007	50	0.56993	148	56
5	26022	9.41535	47	96555	9.98477	3	26951	9.43057	51	0.56943	3.7105	55
6	050	9.41582	46	547	9.98474	3	982	9.43108	50	0.56892	062	54
7	079	9.41628	47	540	9.98471	4	27013	9.43158	50	0.56842	019	53
8	107	9.41675	47	532	9.98467	3	044	9.43208	50	0.56792	3.6976	52
9	135	9.41722	46	524	9.98464	4	076	9.43258	50	0.56742	933	51
10	26163	9.41768	47	96517	9.98460	3	27107	9.43308	50	0.56692	3.6891	50
11	191	9.41815	46	509	9.98457	4	138	9.43358	50	0.56642	848	49
12	219	9.41861	47	502	9.98453	3	169	9.43408	50	0.56592	806	48
13	247	9.41908	46	494	9.98450	3	201	9.43458	50	0.56542	764	47
14	275	9.41954	47	486	9.98447	4	232	9.43508	50	0.56492	722	46
15	26303	9.42001	46	96479	9.98443	3	27263	9.43558	49	0.56442	3.6680	45
16	331	9.42047	46	471	9.98440	4	294	9.43607	50	0.56393	638	44
17	359	9.42093	47	463	9.98436	3	326	9.43657	50	0.56343	596	43
18	387	9.42140	46	456	9.98433	4	357	9.43707	49	0.56293	554	42
19	415	9.42186	46	448	9.98429	3	388	9.43756	50	0.56244	512	41
20	26443	9.42232	46	96440	9.98426	4	27419	9.43806	49	0.56194	3.6470	40
21	471	9.42278	46	433	9.98422	3	451	9.43855	50	0.56145	429	39
22	500	9.42324	46	425	9.98419	4	482	9.43905	49	0.56095	387	38
23	528	9.42370	46	417	9.98415	3	513	9.43954	50	0.56046	346	37
24	556	9.42416	45	410	9.98412	3	545	9.44004	49	0.55996	305	36
25	26584	9.42461	46	96402	9.98409	4	27576	9.44053	49	0.55947	3.6264	35
26	612	9.42507	46	394	9.98405	3	607	9.44102	49	0.55898	222	34
27	640	9.42553	46	386	9.98402	4	638	9.44151	50	0.55849	181	33
28	668	9.42599	45	379	9.98398	3	670	9.44201	49	0.55799	140	32
29	696	9.42644	46	371	9.98395	4	701	9.44250	49	0.55750	100	31
30	26724	9.42690	45	96363	9.98391	3	27732	9.44299	49	0.55701	3.6059	30
31	752	9.42735	46	355	9.98388	4	764	9.44348	49	0.55652	018	29
32	780	9.42781	45	347	9.98384	3	795	9.44397	49	0.55603	3.5978	28
33	808	9.42826	46	340	9.98381	4	826	9.44446	49	0.55554	937	27
34	836	9.42872	45	332	9.98377	4	858	9.44495	49	0.55505	897	26
35	26864	9.42917	45	96324	9.98373	3	27889	9.44544	48	0.55456	3.5856	25
36	892	9.42962	46	316	9.98370	4	921	9.44592	49	0.55408	816	24
37	920	9.43008	45	308	9.98366	3	952	9.44641	49	0.55359	776	23
38	948	9.43053	45	301	9.98363	4	983	9.44690	48	0.55310	736	22
39	976	9.43098	45	293	9.98359	3	28015	9.44738	49	0.55262	696	21
40	27004	9.43143	45	96285	9.98356	4	28046	9.44787	49	0.55213	3.5656	20
41	032	9.43188	45	277	9.98352	3	077	9.44836	48	0.55164	616	19
42	060	9.43233	45	269	9.98349	4	109	9.44884	49	0.55116	576	18
43	088	9.43278	45	261	9.98345	3	140	9.44933	48	0.55067	536	17
44	116	9.43323	44	253	9.98342	4	172	9.44981	48	0.55019	497	16
45	27144	9.43367	45	96246	9.98338	4	28203	9.45029	49	0.54971	3.5457	15
46	172	9.43412	45	238	9.98334	3	234	9.45078	48	0.54922	418	14
47	200	9.43457	45	230	9.98331	4	266	9.45126	48	0.54874	379	13
48	228	9.43502	44	222	9.98327	3	297	9.45174	48	0.54826	339	12
49	256	9.43546	45	214	9.98324	4	329	9.45222	49	0.54778	300	11
50	27284	9.43591	44	96206	9.98320	3	28360	9.45271	48	0.54729	3.5261	10
51	312	9.43635	45	198	9.98317	4	391	9.45319	48	0.54681	222	9
52	340	9.43680	44	190	9.98313	4	423	9.45367	48	0.54633	183	8
53	368	9.43724	45	182	9.98309	3	454	9.45415	48	0.54585	144	7
54	396	9.43769	44	174	9.98306	4	486	9.45463	48	0.54537	105	6
55	27424	9.43813	44	96166	9.98302	3	28517	9.45511	48	0.54489	3.5067	5
56	452	9.43857	44	158	9.98299	4	549	9.45559	47	0.54441	028	4
57	480	9.43901	45	150	9.98295	4	580	9.45606	48	0.54394	3.4989	3
58	508	9.43946	44	142	9.98291	3	612	9.45654	48	0.54346	951	2
59	536	9.43990	44	134	9.98288	4	643	9.45702	48	0.54298	912	1
60	564	9.44034		126	9.98284		675	9.45750		0.54250	874	0
	Nat. Cos	Log. Cos	d.	Nat. Sin	Log. Sin	d.	Nat. Cot	Log. Cot	c.d.	Log. Tan	Nat. Tan	′

74°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	27564	9.44034	44	96126	9.98284	3	28675	9.45750	47	0.54250	3.4874	60
1	592	9.44078	44	118	9.98281	4	706	9.45797	48	0.54203	836	59
2	620	9.44122	44	110	9.98277	4	738	9.45845	47	0.54155	798	58
3	648	9.44166	44	102	9.98273	3	769	9.45892	48	0.54108	760	57
4	676	9.44210	43	094	9.98270	4	801	9.45940	47	0.54060	722	56
5	27704	9.44253	44	96086	9.98266	4	28832	9.45987	48	0.54013	3.4684	55
6	731	9.44297	44	078	9.98262	3	864	9.46035	47	0.53965	646	54
7	759	9.44341	44	070	9.98259	4	895	9.46082	48	0.53918	608	53
8	787	9.44385	43	062	9.98255	4	927	9.46130	47	0.53870	570	52
9	815	9.44428	44	054	9.98251	3	958	9.46177	47	0.53823	533	51
10	27843	9.44472	44	96046	9.98248	4	28990	9.46224	47	0.53776	3.4495	50
11	871	9.44516	43	037	9.98244	4	29021	9.46271	48	0.53729	458	49
12	899	9.44559	43	029	9.98240	3	053	9.46319	47	0.53681	420	48
13	927	9.44602	44	021	9.98237	4	084	9.46366	47	0.53634	383	47
14	955	9.44646	43	013	9.98233	4	116	9.46413	47	0.53587	346	46
15	27983	9.44689	44	96005	9.98229	3	29147	9.46460	47	0.53540	3.4308	45
16	28011	9.44733	43	95997	9.98226	4	179	9.46507	47	0.53493	271	44
17	039	9.44776	43	989	9.98222	4	210	9.46554	47	0.53446	234	43
18	067	9.44819	43	981	9.98218	3	242	9.46601	47	0.53399	197	42
19	095	9.44862	43	972	9.98215	4	274	9.46648	46	0.53352	160	41
20	28123	9.44905	43	95964	9.98211	4	29305	9.46694	47	0.53306	3.4124	40
21	150	9.44948	44	956	9.98207	3	337	9.46741	47	0.53259	087	39
22	178	9.44992	43	948	9.98204	4	368	9.46788	47	0.53212	050	38
23	206	9.45035	42	940	9.98200	4	400	9.46835	46	0.53165	014	37
24	234	9.45077	43	931	9.98196	4	432	9.46881	47	0.53119	3.3977	36
25	28262	9.45120	43	95923	9.98192	3	29463	9.46928	47	0.53072	3.3941	35
26	290	9.45163	43	915	9.98189	4	495	9.46975	46	0.53025	904	34
27	318	9.45206	43	907	9.98185	4	526	9.47021	47	0.52979	868	33
28	346	9.45249	43	898	9.98181	4	558	9.47068	46	0.52932	832	32
29	374	9.45292	42	890	9.98177	3	590	9.47114	46	0.52886	796	31
30	28402	9.45334	43	95882	9.98174	4	29621	9.47160	47	0.52840	3.3759	30
31	429	9.45377	42	874	9.98170	4	653	9.47207	46	0.52793	723	29
32	457	9.45419	43	865	9.98166	4	685	9.47253	46	0.52747	687	28
33	485	9.45462	42	857	9.98162	3	716	9.47299	47	0.52701	652	27
34	513	9.45504	43	849	9.98159	4	748	9.47346	46	0.52654	616	26
35	28541	9.45547	42	95841	9.98155	4	29780	9.47392	46	0.52608	3.3580	25
36	569	9.45589	43	832	9.98151	4	811	9.47438	46	0.52562	544	24
37	597	9.45632	42	824	9.98147	3	843	9.47484	46	0.52516	509	23
38	625	9.45674	42	816	9.98144	4	875	9.47530	46	0.52470	473	22
39	652	9.45716	42	807	9.98140	4	906	9.47576	46	0.52424	438	21
40	28680	9.45758	43	95799	9.98136	4	29938	9.47622	46	0.52378	3.3402	20
41	708	9.45801	42	791	9.98132	3	970	9.47668	46	0.52332	367	19
42	736	9.45843	42	782	9.98129	4	30001	9.47714	46	0.52286	332	18
43	764	9.45885	42	774	9.98125	4	033	9.47760	46	0.52240	297	17
44	792	9.45927	42	766	9.98121	4	065	9.47806	46	0.52194	261	16
45	28820	9.45969	42	95757	9.98117	4	30097	9.47852	45	0.52148	3.3226	15
46	847	9.46011	42	749	9.98113	3	128	9.47897	46	0.52103	191	14
47	875	9.46053	42	740	9.98110	4	160	9.47943	46	0.52057	156	13
48	903	9.46095	41	732	9.98106	4	192	9.47989	46	0.52011	122	12
49	931	9.46136	42	724	9.98102	4	224	9.48035	45	0.51965	087	11
50	28959	9.46178	42	95715	9.98098	4	30255	9.48080	46	0.51920	3.3052	10
51	987	9.46220	42	707	9.98094	4	287	9.48126	45	0.51874	017	9
52	29015	9.46262	41	698	9.98090	3	319	9.48171	46	0.51829	3.2983	8
53	042	9.46303	42	690	9.98087	4	351	9.48217	45	0.51783	948	7
54	070	9.46345	41	681	9.98083	4	382	9.48262	45	0.51738	914	6
55	29098	9.46386	42	95673	9.98079	4	30414	9.48307	46	0.51693	3.2879	5
56	126	9.46428	41	664	9.98075	4	446	9.48353	45	0.51647	845	4
57	154	9.46469	42	656	9.98071	4	478	9.48398	45	0.51602	811	3
58	182	9.46511	41	647	9.98067	4	509	9.48443	46	0.51557	777	2
59	209	9.46552	42	639	9.98063	3	541	9.48489	45	0.51511	743	1
60	237	9.46594		630	9.98060		573	9.48534		0.51466	709	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

17°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	29237	9.46594	41	95630	9.98060	4	30573	9.48534	45	0.51466	3.2709	60
1	265	9.46635	41	622	9.98056	4	605	9.48579	45	0.51421	675	59
2	293	9.46676	41	613	9.98052	4	637	9.48624	45	0.51376	641	58
3	321	9.46717	41	605	9.98048	4	669	9.48669	45	0.51331	607	57
4	348	9.46758	42	596	9.98044	4	700	9.48714	45	0.51286	573	56
5	29376	9.46800	41	95588	9.98040	4	30732	9.48759	45	0.51241	3.2539	55
6	404	9.46841	41	579	9.98036	4	764	9.48804	45	0.51196	506	54
7	432	9.46882	41	571	9.98032	3	796	9.48849	45	0.51151	472	53
8	460	9.46923	41	562	9.98029	4	828	9.48894	45	0.51106	438	52
9	487	9.46964	41	554	9.98025	4	860	9.48939	45	0.51061	405	51
10	29515	9.47005	40	95545	9.98021	4	30891	9.48984	45	0.51016	3.2371	50
11	543	9.47045	41	536	9.98017	4	923	9.49029	44	0.50971	338	49
12	571	9.47086	41	528	9.98013	4	955	9.49073	45	0.50927	305	48
13	599	9.47127	41	519	9.98009	4	987	9.49118	45	0.50882	272	47
14	626	9.47168	41	511	9.98005	4	31019	9.49163	44	0.50837	238	46
15	29654	9.47209	40	95502	9.98001	4	31051	9.49207	45	0.50793	3.2205	45
16	682	9.47249	41	493	9.97997	4	083	9.49252	44	0.50748	172	44
17	710	9.47290	40	485	9.97993	4	115	9.49296	45	0.50704	139	43
18	737	9.47330	41	476	9.97989	3	147	9.49341	44	0.50659	106	42
19	765	9.47371	40	467	9.97986	4	178	9.49385	45	0.50615	073	41
20	29793	9.47411	41	95459	9.97982	4	31210	9.49430	44	0.50570	3.2041	40
21	821	9.47452	40	450	9.97978	4	242	9.49474	45	0.50526	008	39
22	849	9.47492	41	441	9.97974	4	274	9.49519	44	0.50481	3.1975	38
23	876	9.47533	40	433	9.97970	4	306	9.49563	44	0.50437	943	37
24	904	9.47573	40	424	9.97966	4	338	9.49607	45	0.50393	910	36
25	29932	9.47613	41	95415	9.97962	4	31370	9.49652	44	0.50348	3.1878	35
26	960	9.47654	40	407	9.97958	4	402	9.49696	44	0.50304	845	34
27	987	9.47694	40	398	9.97954	4	434	9.49740	44	0.50260	813	33
28	30015	9.47734	40	389	9.97950	4	466	9.49784	44	0.50216	780	32
29	043	9.47774	40	380	9.97946	4	498	9.49828	44	0.50172	748	31
30	30071	9.47814	40	95372	9.97942	4	31530	9.49872	44	0.50128	3.1716	30
31	098	9.47854	40	363	9.97938	4	562	9.49916	44	0.50084	684	29
32	126	9.47894	40	354	9.97934	4	594	9.49960	44	0.50040	652	28
33	154	9.47934	40	345	9.97930	4	626	9.50004	44	0.49996	620	27
34	182	9.47974	40	337	9.97926	4	658	9.50048	44	0.49952	588	26
35	30209	9.48014	40	95328	9.97922	4	31690	9.50092	44	0.49908	3.1556	25
36	237	9.48054	40	319	9.97918	4	722	9.50136	44	0.49864	524	24
37	265	9.48094	39	310	9.97914	4	754	9.50180	43	0.49820	492	23
38	292	9.48133	40	301	9.97910	4	786	9.50223	44	0.49777	460	22
39	320	9.48173	40	293	9.97906	4	818	9.50267	44	0.49733	429	21
40	30348	9.48213	39	95284	9.97902	4	31850	9.50311	44	0.49689	3.1397	20
41	376	9.48252	40	275	9.97898	4	882	9.50355	43	0.49645	366	19
42	403	9.48292	40	266	9.97894	4	914	9.50398	44	0.49602	334	18
43	431	9.48332	39	257	9.97890	4	946	9.50442	43	0.49558	303	17
44	459	9.48371	40	248	9.97886	4	978	9.50485	44	0.49515	271	16
45	30486	9.48411	39	95240	9.97882	4	32010	9.50529	43	0.49471	3.1240	15
46	514	9.48450	40	231	9.97878	4	042	9.50572	44	0.49428	209	14
47	542	9.48490	39	222	9.97874	4	074	9.50616	43	0.49384	178	13
48	570	9.48529	39	213	9.97870	4	106	9.50659	44	0.49341	146	12
49	597	9.48568	39	204	9.97866	5	139	9.50703	43	0.49297	115	11
50	30625	9.48607	40	95195	9.97861	4	32171	9.50746	43	0.49254	3.1084	10
51	653	9.48647	39	186	9.97857	4	203	9.50789	44	0.49211	053	9
52	680	9.48686	39	177	9.97853	4	235	9.50833	43	0.49167	022	8
53	708	9.48725	39	168	9.97849	4	267	9.50876	43	0.49124	3.0991	7
54	736	9.48764	39	159	9.97845	4	299	9.50919	43	0.49081	961	6
55	30763	9.48803	39	95150	9.97841	4	32331	9.50962	43	0.49038	3.0930	5
56	791	9.48842	39	142	9.97837	4	363	9.51005	43	0.48995	899	4
57	819	9.48881	39	133	9.97833	4	396	9.51048	44	0.48952	868	3
58	846	9.48920	39	124	9.97829	4	428	9.51092	43	0.48908	838	2
59	874	9.48959	39	115	9.97825	4	460	9.51135	43	0.48865	807	1
60	902	9.48998		106	9.97821		492	9.51178		0.48822	777	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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18°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	30902	9.48998	39	95106	9.97821	4	32492	9.51178	43	0.48822	3.0777	60
1	929	9.49037	39	097	9.97817	5	524	9.51221	43	0.48779	746	59
2	957	9.49076	39	088	9.97812	4	556	9.51264	42	0.48736	716	58
3	985	9.49115	38	079	9.97808	4	588	9.51306	43	0.48694	686	57
4	31012	9.49153	39	070	9.97804	4	621	9.51349	43	0.48651	655	56
5	31040	9.49192	39	95061	9.97800	4	32653	9.51392	43	0.48608	3.0625	55
6	068	9.49231	38	052	9.97796	4	685	9.51435	43	0.48565	595	54
7	095	9.49269	39	043	9.97792	4	717	9.51478	42	0.48522	565	53
8	123	9.49308	39	033	9.97788	4	749	9.51520	43	0.48480	535	52
9	151	9.49347	38	024	9.97784	5	782	9.51563	43	0.48437	505	51
10	31178	9.49385	39	95015	9.97779	4	32814	9.51606	42	0.48394	3.0475	50
11	206	9.49424	38	006	9.97775	4	846	9.51648	43	0.48352	445	49
12	233	9.49462	38	94997	9.97771	4	878	9.51691	43	0.48309	415	48
13	261	9.49500	39	988	9.97767	4	911	9.51734	42	0.48266	385	47
14	289	9.49539	38	979	9.97763	4	943	9.51776	43	0.48224	356	46
15	31316	9.49577	38	94970	9.97759	5	32975	9.51819	42	0.48181	3.0326	45
16	344	9.49615	39	961	9.97754	4	33007	9.51861	42	0.48139	296	44
17	372	9.49654	38	952	9.97750	4	040	9.51903	43	0.48097	267	43
18	399	9.49692	38	943	9.97746	4	072	9.51946	42	0.48054	237	42
19	427	9.49730	38	933	9.97742	4	104	9.51988	43	0.48012	208	41
20	31454	9.49768	38	94924	9.97738	4	33136	9.52031	42	0.47969	3.0178	40
21	482	9.49806	38	915	9.97734	5	169	9.52073	42	0.47927	149	39
22	510	9.49844	38	906	9.97729	4	201	9.52115	42	0.47885	120	38
23	537	9.49882	38	897	9.97725	4	233	9.52157	43	0.47843	090	37
24	565	9.49920	38	888	9.97721	4	266	9.52200	42	0.47800	061	36
25	31593	9.49958	38	94878	9.97717	4	33298	9.52242	42	0.47758	3.0032	35
26	620	9.49996	38	869	9.97713	5	330	9.52284	42	0.47716	003	34
27	648	9.50034	38	860	9.97708	4	363	9.52326	42	0.47674	2.9974	33
28	675	9.50072	38	851	9.97704	4	395	9.52368	42	0.47632	945	32
29	703	9.50110	38	842	9.97700	4	427	9.52410	42	0.47590	916	31
30	31730	9.50148	37	94832	9.97696	5	33460	9.52452	42	0.47548	2.9887	30
31	758	9.50185	38	823	9.97691	4	492	9.52494	42	0.47506	858	29
32	786	9.50223	38	814	9.97687	4	524	9.52536	42	0.47464	829	28
33	813	9.50261	37	805	9.97683	4	557	9.52578	42	0.47422	800	27
34	841	9.50298	38	795	9.97679	5	589	9.52620	41	0.47380	772	26
35	31868	9.50336	38	94786	9.97674	4	33621	9.52661	42	0.47339	2.9743	25
36	896	9.50374	37	777	9.97670	4	654	9.52703	42	0.47297	714	24
37	923	9.50411	38	768	9.97666	4	686	9.52745	42	0.47255	686	23
38	951	9.50449	37	758	9.97662	5	718	9.52787	42	0.47213	657	22
39	979	9.50486	37	749	9.97657	4	751	9.52829	41	0.47171	629	21
40	32006	9.50523	38	94740	9.97653	4	33783	9.52870	42	0.47130	2.9600	20
41	034	9.50561	37	730	9.97649	4	816	9.52912	41	0.47088	572	19
42	061	9.50598	37	721	9.97645	5	848	9.52953	42	0.47047	544	18
43	089	9.50635	38	712	9.97640	4	881	9.52995	42	0.47005	515	17
44	116	9.50673	37	702	9.97636	4	913	9.53037	41	0.46963	487	16
45	32144	9.50710	37	94693	9.97632	4	33945	9.53078	42	0.46922	2.9459	15
46	171	9.50747	37	684	9.97628	5	978	9.53120	41	0.46880	431	14
47	199	9.50784	37	674	9.97623	4	34010	9.53161	41	0.46839	403	13
48	227	9.50821	37	665	9.97619	4	043	9.53202	42	0.46798	375	12
49	254	9.50858	38	656	9.97615	5	075	9.53244	41	0.46756	347	11
50	32282	9.50896	37	94646	9.97610	4	34108	9.53285	42	0.46715	2.9319	10
51	309	9.50933	37	637	9.97606	4	140	9.53327	41	0.46673	291	9
52	337	9.50970	37	627	9.97602	5	173	9.53368	41	0.46632	263	8
53	364	9.51007	36	618	9.97597	4	205	9.53409	41	0.46591	235	7
54	392	9.51043	37	609	9.97593	4	238	9.53450	42	0.46550	208	6
55	32419	9.51080	37	94599	9.97589	5	34270	9.53492	41	0.46508	2.9180	5
56	447	9.51117	37	590	9.97584	4	303	9.53533	41	0.46467	152	4
57	474	9.51154	37	580	9.97580	4	335	9.53574	41	0.46426	125	3
58	502	9.51191	36	571	9.97576	5	368	9.53615	41	0.46385	097	2
59	529	9.51227	37	561	9.97571	4	400	9.53656	41	0.46344	070	1
60	557	9.51264		552	9.97567		433	9.53697		0.46303	042	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	32557	9.51264	37	94552	9.97567	4	34433	9.53697	41	0.46303	2.9042	60
1	584	9.51301	37	542	9.97563	5	465	9.53738	41	0.46262	015	59
2	612	9.51338	36	533	9.97558	4	498	9.53779	41	0.46221	2.8987	58
3	639	9.51374	37	523	9.97554	4	530	9.53820	41	0.46180	960	57
4	667	9.51411	36	514	9.97550	5	563	9.53861	41	0.46139	933	56
5	32694	9.51447	37	94504	9.97545	4	34596	9.53902	41	0.46098	2.8905	55
6	722	9.51484	36	495	9.97541	5	628	9.53943	41	0.46057	878	54
7	749	9.51520	37	485	9.97536	4	661	9.53984	41	0.46016	851	53
8	777	9.51557	36	476	9.97532	4	693	9.54025	40	0.45975	824	52
9	804	9.51593	36	466	9.97528	5	726	9.54065	41	0.45935	797	51
10	32832	9.51629	37	94457	9.97523	4	34758	9.54106	41	0.45894	2.8770	50
11	859	9.51666	36	447	9.97519	4	791	9.54147	40	0.45853	743	49
12	887	9.51702	36	438	9.97515	5	824	9.54187	41	0.45813	716	48
13	914	9.51738	36	428	9.97510	4	856	9.54228	41	0.45772	689	47
14	942	9.51774	37	418	9.97506	5	889	9.54269	40	0.45731	662	46
15	32969	9.51811	36	94409	9.97501	4	34922	9.54309	41	0.45691	2.8636	45
16	997	9.51847	36	399	9.97497	5	954	9.54350	40	0.45650	609	44
17	33024	9.51883	36	390	9.97492	4	987	9.54390	41	0.45610	582	43
18	051	9.51919	36	380	9.97488	4	35020	9.54431	40	0.45569	556	42
19	079	9.51955	36	370	9.97484	5	052	9.54471	41	0.45529	529	41
20	33106	9.51991	36	94361	9.97479	4	35085	9.54512	40	0.45488	2.8502	40
21	134	9.52027	36	351	9.97475	5	118	9.54552	41	0.45448	476	39
22	161	9.52063	36	342	9.97470	4	150	9.54593	40	0.45407	449	38
23	189	9.52099	36	332	9.97466	5	183	9.54633	40	0.45367	423	37
24	216	9.52135	36	322	9.97461	4	216	9.54673	41	0.45327	397	36
25	33244	9.52171	36	94313	9.97457	4	35248	9.54714	40	0.45286	2.8370	35
26	271	9.52207	35	303	9.97453	5	281	9.54754	40	0.45246	344	34
27	298	9.52242	36	293	9.97448	4	314	9.54794	41	0.45206	318	33
28	326	9.52278	36	284	9.97444	5	346	9.54835	40	0.45165	291	32
29	353	9.52314	36	274	9.97439	4	379	9.54875	40	0.45125	265	31
30	33381	9.52350	35	94264	9.97435	5	35412	9.54915	40	0.45085	2.8239	30
31	408	9.52385	36	254	9.97430	4	445	9.54955	40	0.45045	213	29
32	436	9.52421	35	245	9.97426	5	477	9.54995	40	0.45005	187	28
33	463	9.52456	36	235	9.97421	4	510	9.55035	40	0.44965	161	27
34	490	9.52492	35	225	9.97417	5	543	9.55075	40	0.44925	135	26
35	33518	9.52527	36	94215	9.97412	4	35576	9.55115	40	0.44885	2.8109	25
36	545	9.52563	35	206	9.97408	5	608	9.55155	40	0.44845	083	24
37	573	9.52598	36	196	9.97403	4	641	9.55195	40	0.44805	057	23
38	600	9.52634	35	186	9.97399	5	674	9.55235	40	0.44765	032	22
39	627	9.52669	36	176	9.97394	4	707	9.55275	40	0.44725	006	21
40	33655	9.52705	35	94167	9.97390	5	35740	9.55315	40	0.44685	2.7980	20
41	682	9.52740	35	157	9.97385	4	772	9.55355	40	0.44645	955	19
42	710	9.52775	36	147	9.97381	5	805	9.55395	39	0.44605	929	18
43	737	9.52811	35	137	9.97376	4	838	9.55434	40	0.44566	903	17
44	764	9.52846	35	127	9.97372	5	871	9.55474	40	0.44526	878	16
45	33792	9.52881	35	94118	9.97367	4	35904	9.55514	40	0.44486	2.7852	15
46	819	9.52916	35	108	9.97363	5	937	9.55554	39	0.44446	827	14
47	846	9.52951	35	098	9.97358	5	969	9.55593	40	0.44407	801	13
48	874	9.52986	35	088	9.97353	4	36002	9.55633	40	0.44367	776	12
49	901	9.53021	35	078	9.97349	5	035	9.55673	39	0.44327	751	11
50	33929	9.53056	36	94068	9.97344	4	36068	9.55712	40	0.44288	2.7725	10
51	956	9.53092	34	058	9.97340	5	101	9.55752	39	0.44248	700	9
52	983	9.53126	35	049	9.97335	4	134	9.55791	40	0.44209	675	8
53	34011	9.53161	35	039	9.97331	5	167	9.55831	39	0.44169	650	7
54	038	9.53196	35	029	9.97326	4	199	9.55870	40	0.44130	625	6
55	34065	9.53231	35	94019	9.97322	5	36232	9.55910	39	0.44090	2.7600	5
56	093	9.53266	35	009	9.97317	5	265	9.55949	40	0.44051	575	4
57	120	9.53301	35	93999	9.97312	4	298	9.55989	39	0.44011	550	3
58	147	9.53336	34	989	9.97308	5	331	9.56028	39	0.43972	525	2
59	175	9.53370	35	979	9.97303	4	364	9.56067	40	0.43933	500	1
60	202	9.53405		969	9.97299		397	9.56107		0.43893	475	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

70°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	34202	9.53405	35	93969	9.97299	5	36397	9.56107	39	0.43893	2.7475	60
1	229	9.53440	35	959	9.97294	5	430	9.56146	39	0.43854	450	59
2	257	9.53475	34	949	9.97289	4	463	9.56185	39	0.43815	425	58
3	284	9.53509	35	939	9.97285	5	496	9.56224	40	0.43776	400	57
4	311	9.53544	34	929	9.97280	4	529	9.56264	39	0.43736	376	56
5	34339	9.53578	35	93919	9.97276	5	36562	9.56303	39	0.43697	2.7351	55
6	366	9.53613	34	909	9.97271	5	595	9.56342	39	0.43658	326	54
7	393	9.53647	35	899	9.97266	4	628	9.56381	39	0.43619	302	53
8	421	9.53682	34	889	9.97262	5	661	9.56420	39	0.43580	277	52
9	448	9.53716	35	879	9.97257	5	694	9.56459	39	0.43541	253	51
10	34475	9.53751	34	93869	9.97252	4	36727	9.56498	39	0.43502	2.7228	50
11	503	9.53785	34	859	9.97248	5	760	9.56537	39	0.43463	204	49
12	530	9.53819	35	849	9.97243	5	793	9.56576	39	0.43424	179	48
13	557	9.53854	34	839	9.97238	4	826	9.56615	39	0.43385	155	47
14	584	9.53888	34	829	9.97234	5	859	9.56654	39	0.43346	130	46
15	34612	9.53922	35	93819	9.97229	5	36892	9.56693	39	0.43307	2.7106	45
16	639	9.53957	34	809	9.97224	4	925	9.56732	39	0.43268	082	44
17	666	9.53991	34	799	9.97220	5	958	9.56771	39	0.43229	058	43
18	694	9.54025	34	789	9.97215	5	991	9.56810	39	0.43190	034	42
19	721	9.54059	34	779	9.97210	4	37024	9.56849	38	0.43151	009	41
20	34748	9.54093	34	93769	9.97206	5	37057	9.56887	39	0.43113	2.6985	40
21	775	9.54127	34	759	9.97201	5	090	9.56926	39	0.43074	961	39
22	803	9.54161	34	748	9.97196	4	123	9.56965	39	0.43035	937	38
23	830	9.54195	34	738	9.97192	5	157	9.57004	38	0.42996	913	37
24	857	9.54229	34	728	9.97187	5	190	9.57042	39	0.42958	889	36
25	34884	9.54263	34	93718	9.97182	4	37223	9.57081	39	0.42919	2.6865	35
26	912	9.54297	34	708	9.97178	5	256	9.57120	38	0.42880	841	34
27	939	9.54331	34	698	9.97173	5	289	9.57158	39	0.42842	818	33
28	966	9.54365	34	688	9.97168	5	322	9.57197	38	0.42803	794	32
29	993	9.54399	34	677	9.97163	4	355	9.57235	39	0.42765	770	31
30	35021	9.54433	33	93667	9.97159	5	37388	9.57274	38	0.42726	2.6746	30
31	048	9.54466	34	657	9.97154	5	422	9.57312	39	0.42688	723	29
32	075	9.54500	34	647	9.97149	4	455	9.57351	38	0.42649	699	28
33	102	9.54534	33	637	9.97145	5	488	9.57389	39	0.42611	675	27
34	130	9.54567	34	626	9.97140	5	521	9.57428	38	0.42572	652	26
35	35157	9.54601	34	93616	9.97135	5	37554	9.57466	38	0.42534	2.6628	25
36	184	9.54635	33	606	9.97130	4	588	9.57504	39	0.42496	605	24
37	211	9.54668	34	596	9.97126	5	621	9.57543	38	0.42457	581	23
38	239	9.54702	33	585	9.97121	5	654	9.57581	38	0.42419	558	22
39	266	9.54735	34	575	9.97116	5	687	9.57619	39	0.42381	534	21
40	35293	9.54769	33	93565	9.97111	4	37720	9.57658	38	0.42342	2.6511	20
41	320	9.54802	34	555	9.97107	5	754	9.57696	38	0.42304	488	19
42	347	9.54836	33	544	9.97102	5	787	9.57734	38	0.42266	464	18
43	375	9.54869	34	534	9.97097	5	820	9.57772	38	0.42228	441	17
44	402	9.54903	33	524	9.97092	5	853	9.57810	39	0.42190	418	16
45	35429	9.54936	33	93514	9.97087	4	37887	9.57849	38	0.42151	2.6395	15
46	456	9.54969	34	503	9.97083	5	920	9.57887	38	0.42113	371	14
47	484	9.55003	33	493	9.97078	5	953	9.57925	38	0.42075	348	13
48	511	9.55036	33	483	9.97073	5	986	9.57963	38	0.42037	325	12
49	538	9.55069	33	472	9.97068	5	38020	9.58001	38	0.41999	302	11
50	35565	9.55102	34	93462	9.97063	4	38053	9.58039	38	0.41961	2.6279	10
51	592	9.55136	33	452	9.97059	5	086	9.58077	38	0.41923	256	9
52	619	9.55169	33	441	9.97054	5	120	9.58115	38	0.41885	233	8
53	647	9.55202	33	431	9.97049	5	153	9.58153	38	0.41847	210	7
54	674	9.55235	33	420	9.97044	5	186	9.58191	38	0.41809	187	6
55	36701	9.55268	33	93410	9.97039	4	38220	9.58229	38	0.41771	2.6165	5
56	728	9.55301	33	400	9.97035	5	253	9.58267	37	0.41733	142	4
57	755	9.55334	33	389	9.97030	5	286	9.58304	38	0.41696	119	3
58	782	9.55367	33	379	9.97025	5	320	9.58342	38	0.41658	096	2
59	810	9.55400	33	368	9.97020	5	353	9.58380	38	0.41620	074	1
60	837	9.55433		358	9.97015		386	9.58418		0.41582	051	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	35837	9.55433	33	93358	9.97015	5	38386	9.58418	37	0.41582	2.6051	60
1	864	9.55466	33	348	9.97010	5	420	9.58455	38	0.41545	028	59
2	891	9.55499	33	337	9.97005	4	453	9.58493	38	0.41507	006	58
3	918	9.55532	32	327	9.97001	5	487	9.58531	38	0.41469	2.5983	57
4	945	9.55564	33	316	9.96996	5	520	9.58569	37	0.41431	961	56
5	35973	9.55597	33	93306	9.96991	5	38553	9.58606	38	0.41394	2.5938	55
6	36000	9.55630	33	295	9.96986	5	587	9.58644	37	0.41356	916	54
7	027	9.55663	32	285	9.96981	5	620	9.58681	38	0.41319	893	53
8	054	9.55695	33	274	9.96976	5	654	9.58719	38	0.41281	871	52
9	081	9.55728	33	264	9.96971	5	687	9.58757	37	0.41243	848	51
10	36108	9.55761	32	93253	9.96966	4	38721	9.58794	38	0.41206	2.5826	50
11	135	9.55793	33	243	9.96962	5	754	9.58832	37	0.41168	804	49
12	162	9.55826	32	232	9.96957	5	787	9.58869	38	0.41131	782	48
13	190	9.55858	33	222	9.96952	5	821	9.58907	37	0.41093	759	47
14	217	9.55891	32	211	9.96947	5	854	9.58944	37	0.41056	737	46
15	36244	9.55923	33	93201	9.96942	5	38888	9.58981	38	0.41019	2.5715	45
16	271	9.55956	32	190	9.96937	5	921	9.59019	37	0.40981	693	44
17	298	9.55988	33	180	9.96932	5	955	9.59056	38	0.40944	671	43
18	325	9.56021	32	169	9.96927	5	988	9.59094	37	0.40906	649	42
19	352	9.56053	32	159	9.96922	5	39022	9.59131	37	0.40869	627	41
20	36379	9.56085	33	93148	9.96917	5	39055	9.59168	37	0.40832	2.5605	40
21	406	9.56118	32	137	9.96912	5	089	9.59205	38	0.40795	583	39
22	434	9.56150	32	127	9.96907	4	122	9.59243	37	0.40757	561	38
23	461	9.56182	33	116	9.96903	5	156	9.59280	37	0.40720	539	37
24	488	9.56215	32	106	9.96898	5	190	9.59317	37	0.40683	517	36
25	36515	9.56247	32	93095	9.96893	5	39223	9.59354	37	0.40646	2.5495	35
26	542	9.56279	32	084	9.96888	5	257	9.59391	38	0.40609	473	34
27	569	9.56311	32	074	9.96883	5	290	9.59429	37	0.40571	452	33
28	596	9.56343	32	063	9.96878	5	324	9.59466	37	0.40534	430	32
29	623	9.56375	33	052	9.96873	5	357	9.59503	37	0.40497	408	31
30	36650	9.56408	32	93042	9.96868	5	39391	9.59540	37	0.40460	2.5386	30
31	677	9.56440	32	031	9.96863	5	425	9.59577	37	0.40423	365	29
32	704	9.56472	32	020	9.96858	5	458	9.59614	37	0.40386	343	28
33	731	9.56504	32	010	9.96853	5	492	9.59651	37	0.40349	322	27
34	758	9.56536	32	92999	9.96848	5	526	9.59688	37	0.40312	300	26
35	36785	9.56568	31	92988	9.96843	5	39559	9.59725	37	0.40275	2.5279	25
36	812	9.56599	32	978	9.96838	5	593	9.59762	37	0.40238	257	24
37	839	9.56631	32	967	9.96833	5	626	9.59799	36	0.40201	236	23
38	867	9.56663	32	956	9.96828	5	660	9.59835	37	0.40165	214	22
39	894	9.56695	32	945	9.96823	5	694	9.59872	37	0.40128	193	21
40	36921	9.56727	32	92935	9.96818	5	39727	9.59909	37	0.40091	2.5172	20
41	948	9.56759	31	924	9.96813	5	761	9.59946	37	0.40054	150	19
42	975	9.56790	32	913	9.96808	5	795	9.59983	36	0.40017	129	18
43	37002	9.56822	32	902	9.96803	5	829	9.60019	37	0.39981	108	17
44	029	9.56854	32	892	9.96798	5	862	9.60056	37	0.39944	086	16
45	37056	9.56886	31	92881	9.96793	5	39896	9.60093	37	0.39907	2.5065	15
46	083	9.56917	32	870	9.96788	5	930	9.60130	36	0.39870	044	14
47	110	9.56949	31	859	9.96783	5	963	9.60166	37	0.39834	023	13
48	137	9.56980	32	849	9.96778	6	997	9.60203	37	0.39797	002	12
49	164	9.57012	32	838	9.96772	5	40031	9.60240	36	0.39760	2.4981	11
50	37191	9.57044	31	92827	9.96767	5	40065	9.60276	37	0.39724	2.4960	10
51	218	9.57075	32	816	9.96762	5	098	9.60313	36	0.39687	939	9
52	245	9.57107	31	805	9.96757	5	132	9.60349	37	0.39651	918	8
53	272	9.57138	31	794	9.96752	5	166	9.60386	36	0.39614	897	7
54	299	9.57169	32	784	9.96747	5	200	9.60422	37	0.39578	876	6
55	37326	9.57201	31	92773	9.96742	5	40234	9.60459	36	0.39541	2.4855	5
56	353	9.57232	32	762	9.96737	5	267	9.60495	37	0.39505	834	4
57	380	9.57264	31	751	9.96732	5	301	9.60532	36	0.39468	813	3
58	407	9.57295	31	740	9.96727	5	335	9.60568	37	0.39432	792	2
59	434	9.57326	32	729	9.96722	5	369	9.60605	36	0.39395	772	1
60	461	9.57358		718	9.96717		403	9.60641		0.39359	751	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	37461	9.57358	31	92718	9.96717	6	40403	9.60641	36	0.39359	2.4751	60
1	488	9.57389	31	707	9.96711	5	436	9.60677	37	0.39323	730	59
2	515	9.57420	31	697	9.96706	5	470	9.60714	36	0.39286	709	58
3	542	9.57451	31	686	9.96701	5	504	9.60750	36	0.39250	689	57
4	569	9.57482	32	675	9.96696	5	538	9.60786	37	0.39214	668	56
5	37595	9.57514	31	92664	9.96691	5	40572	9.60823	36	0.39177	2.4648	55
6	622	9.57545	31	653	9.96686	5	606	9.60859	36	0.39141	627	54
7	649	9.57576	31	642	9.96681	5	640	9.60895	36	0.39105	606	53
8	676	9.57607	31	631	9.96676	6	674	9.60931	36	0.39069	586	52
9	703	9.57638	31	620	9.96670	5	707	9.60967	37	0.39033	566	51
10	37730	9.57669	31	92609	9.96665	5	40741	9.61004	36	0.38996	2.4545	50
11	757	9.57700	31	598	9.96660	5	775	9.61040	36	0.38960	525	49
12	784	9.57731	31	587	9.96655	5	809	9.61076	36	0.38924	504	48
13	811	9.57762	31	576	9.96650	5	843	9.61112	36	0.38888	484	47
14	838	9.57793	31	565	9.96645	5	877	9.61148	36	0.38852	464	46
15	37865	9.57824	31	92554	9.96640	6	40911	9.61184	36	0.38816	2.4443	45
16	892	9.57855	30	543	9.96634	5	945	9.61220	36	0.38780	423	44
17	919	9.57885	31	532	9.96629	5	979	9.61256	36	0.38744	403	43
18	946	9.57916	31	521	9.96624	5	41013	9.61292	36	0.38708	383	42
19	973	9.57947	31	510	9.96619	5	047	9.61328	36	0.38672	362	41
20	37999	9.57978	30	92499	9.96614	6	41081	9.61364	36	0.38636	2.4342	40
21	38026	9.58008	31	488	9.96608	5	115	9.61400	36	0.38600	322	39
22	053	9.58039	31	477	9.96603	5	149	9.61436	36	0.38564	302	38
23	080	9.58070	31	466	9.96598	5	183	9.61472	36	0.38528	282	37
24	107	9.58101	30	455	9.96593	5	217	9.61508	36	0.38492	262	36
25	38134	9.58131	31	92444	9.96588	6	41251	9.61544	35	0.38456	2.4242	35
26	161	9.58162	30	432	9.96582	5	285	9.61579	36	0.38421	222	34
27	188	9.58192	31	421	9.96577	5	319	9.61615	36	0.38385	202	33
28	215	9.58223	30	410	9.96572	5	353	9.61651	36	0.38349	182	32
29	241	9.58253	31	399	9.96567	5	387	9.61687	35	0.38313	162	31
30	38268	9.58284	30	92388	9.96562	6	41421	9.61722	36	0.38278	2.4142	30
31	295	9.58314	31	377	9.96556	5	455	9.61758	36	0.38242	122	29
32	322	9.58345	30	366	9.96551	5	490	9.61794	36	0.38206	102	28
33	349	9.58375	31	355	9.96546	5	524	9.61830	35	0.38170	083	27
34	376	9.58406	30	343	9.96541	6	558	9.61865	36	0.38135	063	26
35	38403	9.58436	31	92332	9.96535	5	41592	9.61901	35	0.38099	2.4043	25
36	430	9.58467	30	321	9.96530	5	626	9.61936	36	0.38064	023	24
37	456	9.58497	30	310	9.96525	5	660	9.61972	36	0.38028	004	23
38	483	9.58527	30	299	9.96520	6	694	9.62008	35	0.37992	2.3984	22
39	510	9.58557	31	287	9.96514	5	728	9.62043	36	0.37957	964	21
40	38537	9.58588	30	92276	9.96509	5	41763	9.62079	35	0.37921	2.3945	20
41	564	9.58618	30	265	9.96504	6	797	9.62114	36	0.37886	925	19
42	591	9.58648	30	254	9.96498	5	831	9.62150	35	0.37850	906	18
43	617	9.58678	31	243	9.96493	5	865	9.62185	36	0.37815	886	17
44	644	9.58709	30	231	9.96488	5	899	9.62221	35	0.37779	867	16
45	38671	9.58739	30	92220	9.96483	6	41933	9.62256	36	0.37744	2.3847	15
46	698	9.58769	30	209	9.96477	5	968	9.62292	35	0.37708	828	14
47	725	9.58799	30	198	9.96472	5	42002	9.62327	35	0.37673	808	13
48	752	9.58829	30	186	9.96467	6	036	9.62362	36	0.37638	789	12
49	778	9.58859	30	175	9.96461	5	070	9.62398	35	0.37602	770	11
50	38805	9.58889	30	92164	9.96456	5	42105	9.62433	35	0.37567	2.3750	10
51	832	9.58919	30	152	9.96451	6	139	9.62468	36	0.37532	731	9
52	859	9.58949	30	141	9.96445	5	173	9.62504	35	0.37496	712	8
53	886	9.58979	30	130	9.96440	5	207	9.62539	35	0.37461	693	7
54	912	9.59009	30	119	9.96435	6	242	9.62574	35	0.37426	673	6
55	38939	9.59039	30	92107	9.96429	5	42276	9.62609	36	0.37391	2.3654	5
56	966	9.59069	29	096	9.96424	5	310	9.62645	35	0.37355	635	4
57	993	9.59098	30	085	9.96419	6	345	9.62680	35	0.37320	616	3
58	39020	9.59128	30	073	9.96413	5	379	9.62715	35	0.37285	597	2
59	046	9.59158	30	062	9.96408	5	413	9.62750	35	0.37250	578	1
60	073	9.59188		050	9.96403		447	9.62785		0.37215	559	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat Tan	Log.	c.d.	Log. Cot	Nat.	
0	39073	9.59188	30	92050	9.96403	6	42447	9.62785	35	0.37215	2.3559	60
1	100	9.59218	29	039	9.96397	5	482	9.62820	35	0.37180	539	59
2	127	9.59247	30	028	9.96392	5	516	9.62855	35	0.37145	520	58
3	153	9.59277	30	016	9.96387	6	551	9.62890	36	0.37110	501	57
4	180	9.59307	29	005	9.96381	5	585	9.62926	35	0.37074	483	56
5	39207	9.59336	30	91994	9.96376	6	42619	9.62961	35	0.37039	2.3464	55
6	234	9.59366	30	982	9.96370	5	654	9.62996	35	0.37004	445	54
7	260	9.59396	29	971	9.96365	5	688	9.63031	35	0.36969	426	53
8	287	9.59425	30	959	9.96360	6	722	9.63066	35	0.36934	407	52
9	314	9.59455	29	948	9.96354	5	757	9.63101	34	0.36899	388	51
10	39341	9.59484	30	91936	9.96349	6	42791	9.63135	35	0.36865	2.3369	50
11	367	9.59514	29	925	9.96343	5	826	9.63170	35	0.36830	351	49
12	394	9.59543	30	914	9.96338	5	860	9.63205	35	0.36795	332	48
13	421	9.59573	29	902	9.96333	6	894	9.63240	35	0.36760	313	47
14	448	9.59602	30	891	9.96327	5	929	9.63275	35	0.36725	294	46
15	39474	9.59632	29	91879	9.96322	6	42963	9.63310	35	0.36690	2.3276	45
16	501	9.59661	29	868	9.96316	5	998	9.63345	34	0.36655	257	44
17	528	9.59690	30	856	9.96311	6	43032	9.63379	35	0.36621	238	43
18	555	9.59720	29	845	9.96305	5	067	9.63414	35	0.36586	220	42
19	581	9.59749	29	833	9.96300	6	101	9.63449	35	0.36551	201	41
20	39608	9.59778	30	91822	9.96294	5	43136	9.63484	35	0.36516	2.3183	40
21	635	9.59808	29	810	9.96289	5	170	9.63519	34	0.36481	164	39
22	661	9.59837	29	799	9.96284	6	205	9.63553	35	0.36447	146	38
23	688	9.59866	29	787	9.96278	5	239	9.63588	35	0.36412	127	37
24	715	9.59895	29	775	9.96273	6	274	9.63623	34	0.36377	109	36
25	39741	9.59924	30	91764	9.96267	5	43308	9.63657	35	0.36343	2.3090	35
26	768	9.59954	29	752	9.96262	6	343	9.63692	34	0.36308	072	34
27	795	9.59983	29	741	9.96256	5	378	9.63726	35	0.36274	053	33
28	822	9.60012	29	729	9.96251	6	412	9.63761	35	0.36239	035	32
29	848	9.60041	29	718	9.96245	5	447	9.63796	34	0.36204	017	31
30	39875	9.60070	29	91706	9.96240	6	43481	9.63830	35	0.36170	2.2998	30
31	902	9.60099	29	694	9.96234	5	516	9.63865	34	0.36135	980	29
32	928	9.60128	29	683	9.96229	6	550	9.63899	35	0.36101	962	28
33	955	9.60157	29	671	9.96223	5	585	9.63934	34	0.36066	944	27
34	982	9.60186	29	660	9.96218	6	620	9.63968	35	0.36032	925	26
35	40008	9.60215	29	91648	9.96212	5	43654	9.64003	34	0.35997	2.2907	25
36	035	9.60244	29	636	9.96207	6	689	9.64037	35	0.35963	889	24
37	062	9.60273	29	625	9.96201	5	724	9.64072	34	0.35928	871	23
38	088	9.60302	29	613	9.96196	6	758	9.64106	34	0.35894	853	22
39	115	9.60331	28	601	9.96190	5	793	9.64140	35	0.35860	835	21
40	40141	9.60359	29	91590	9.96185	6	43828	9.64175	34	0.35825	2.2817	20
41	168	9.60388	29	578	9.96179	5	862	9.64209	34	0.35791	799	19
42	195	9.60417	29	566	9.96174	6	897	9.64243	35	0.35757	781	18
43	221	9.60446	28	555	9.96168	6	932	9.64278	34	0.35722	763	17
44	248	9.60474	29	543	9.96162	5	966	9.64312	34	0.35688	745	16
45	40275	9.60503	29	91531	9.96157	6	44001	9.64346	35	0.35654	2.2727	15
46	301	9.60532	29	519	9.96151	5	036	9.64381	34	0.35619	709	14
47	328	9.60561	28	508	9.96146	6	071	9.64415	34	0.35585	691	13
48	355	9.60589	29	496	9.96140	5	105	9.64449	34	0.35551	673	12
49	381	9.60618	28	484	9.96135	6	140	9.64483	34	0.35517	655	11
50	40408	9.60646	29	91472	9.96129	6	44175	9.64517	35	0.35483	2.2637	10
51	434	9.60675	29	461	9.96123	5	210	9.64552	34	0.35448	620	9
52	461	9.60704	28	449	9.96118	6	244	9.64586	34	0.35414	602	8
53	488	9.60732	29	437	9.96112	5	279	9.64620	34	0.35380	584	7
54	514	9.60761	28	425	9.96107	6	314	9.64654	34	0.35346	566	6
55	40541	9.60789	29	91414	9.96101	6	44349	9.64688	34	0.35312	2.2549	5
56	567	9.60818	28	402	9.96095	5	384	9.64722	34	0.35278	531	4
57	594	9.60846	29	390	9.96090	6	418	9.64756	34	0.35244	513	3
58	621	9.60875	28	378	9.96084	5	453	9.64790	34	0.35210	496	2
59	647	9.60903	28	366	9.96079	6	488	9.64824	34	0.35176	478	1
60	674	9.60931		355	9.96073		523	9.64858		0.35142	460	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	40674	9.60931	29	91355	9.96073	6	44523	9.64858	34	0.35142	2.2460	60
1	700	9.60960	28	343	9.96067	5	558	9.64892	34	0.35108	443	59
2	727	9.60988	28	331	9.96062	6	593	9.64926	34	0.35074	425	58
3	753	9.61016	29	319	9.96056	6	627	9.64960	34	0.35040	408	57
4	780	9.61045	28	307	9.96050	5	662	9.64994	34	0.35006	390	56
5	40806	9.61073	28	91295	9.96045	6	44697	9.65028	34	0.34972	2.2373	55
6	833	9.61101	28	283	9.96039	5	732	9.65062	34	0.34938	355	54
7	860	9.61129	29	272	9.96034	6	767	9.65096	34	0.34904	338	53
8	886	9.61158	28	260	9.96028	6	802	9.65130	34	0.34870	320	52
9	913	9.61186	28	248	9.96022	5	837	9.65164	33	0.34836	303	51
10	40939	9.61214	28	91236	9.96017	6	44872	9.65197	34	0.34803	2.2286	50
11	966	9.61242	28	224	9.96011	6	907	9.65231	34	0.34769	268	49
12	992	9.61270	28	212	9.96005	5	942	9.65265	34	0.34735	251	48
13	41019	9.61298	28	200	9.96000	6	977	9.65299	34	0.34701	234	47
14	045	9.61326	28	188	9.95994	6	45012	9.65333	33	0.34667	216	46
15	41072	9.61354	28	91176	9.95988	6	45047	9.65366	34	0.34634	2.2199	45
16	098	9.61382	29	164	9.95982	5	082	9.65400	34	0.34600	182	44
17	125	9.61411	27	152	9.95977	6	117	9.65434	33	0.34566	165	43
18	151	9.61438	28	140	9.95971	6	152	9.65467	34	0.34533	148	42
19	178	9.61466	28	128	9.95965	5	187	9.65501	34	0.34499	130	41
20	41204	9.61494	28	91116	9.95960	6	45222	9.65535	33	0.34465	2.2113	40
21	231	9.61522	28	104	9.95954	6	257	9.65568	34	0.34432	096	39
22	257	9.61550	28	092	9.95948	6	292	9.65602	34	0.34398	079	38
23	284	9.61578	28	080	9.95942	5	327	9.65636	33	0.34364	062	37
24	310	9.61606	28	068	9.95937	6	362	9.65669	34	0.34331	045	36
25	41337	9.61634	28	91056	9.95931	6	45397	9.65703	33	0.34297	2.2028	35
26	363	9.61662	27	044	9.95925	5	432	9.65736	34	0.34264	011	34
27	390	9.61689	28	032	9.95920	6	467	9.65770	33	0.34230	2.1994	33
28	416	9.61717	28	020	9.95914	6	502	9.65803	34	0.34197	977	32
29	443	9.61745	28	008	9.95908	6	538	9.65837	33	0.34163	960	31
30	41469	9.61773	27	90996	9.95902	5	45573	9.65870	34	0.34130	2.1943	30
31	496	9.61800	28	984	9.95897	6	608	9.65904	33	0.34096	926	29
32	522	9.61828	28	972	9.95891	6	643	9.65937	34	0.34063	909	28
33	549	9.61856	27	960	9.95885	6	678	9.65971	33	0.34029	892	27
34	575	9.61883	28	948	9.95879	6	713	9.66004	34	0.33996	876	26
35	41602	9.61911	28	90936	9.95873	5	45748	9.66038	33	0.33962	2.1859	25
36	628	9.61939	27	924	9.95868	6	784	9.66071	33	0.33929	842	24
37	655	9.61966	28	911	9.95862	6	819	9.66104	34	0.33896	825	23
38	681	9.61994	27	899	9.95856	6	854	9.66138	33	0.33862	808	22
39	707	9.62021	28	887	9.95850	6	889	9.66171	33	0.33829	792	21
40	41734	9.62049	27	90875	9.95844	5	45924	9.66204	34	0.33796	2.1775	20
41	760	9.62076	28	863	9.95839	6	960	9.66238	33	0.33762	758	19
42	787	9.62104	27	851	9.95833	6	995	9.66271	33	0.33729	742	18
43	813	9.62131	28	839	9.95827	6	46030	9.66304	33	0.33696	725	17
44	840	9.62159	27	826	9.95821	6	065	9.66337	34	0.33663	708	16
45	41866	9.62186	28	90814	9.95815	5	46101	9.66371	33	0.33629	2.1692	15
46	892	9.62214	27	802	9.95810	6	136	9.66404	33	0.33596	675	14
47	919	9.62241	27	790	9.95804	6	171	9.66437	33	0.33563	659	13
48	945	9.62268	28	778	9.95798	6	206	9.66470	33	0.33530	642	12
49	972	9.62296	27	766	9.95792	6	242	9.66503	34	0.33497	625	11
50	41998	9.62323	27	90753	9.95786	6	46277	9.66537	33	0.33463	2.1609	10
51	42024	9.62350	27	741	9.95780	5	312	9.66570	33	0.33430	592	9
52	051	9.62377	28	729	9.95775	6	348	9.66603	33	0.33397	576	8
53	077	9.62405	27	717	9.95769	6	383	9.66636	33	0.33364	560	7
54	104	9.62432	27	704	9.95763	6	418	9.66669	33	0.33331	543	6
55	42130	9.62459	27	90692	9.95757	6	46454	9.66702	33	0.33298	2.1527	5
56	156	9.62486	27	680	9.95751	6	489	9.66735	33	0.33265	510	4
57	183	9.62513	28	668	9.95745	6	525	9.66768	33	0.33232	494	3
58	209	9.62541	27	655	9.95739	6	560	9.66801	33	0.33199	478	2
59	235	9.62568	27	643	9.95733	5	595	9.66834	33	0.33166	461	1
60	262	9.62595		631	9.95728		631	9.66867		0.33133	445	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	42262	9.62595	27	90631	9.95728	6	46631	9.66867	33	0.33133	2.1445	60
1	288	9.62622	27	618	9.95722	6	666	9.66900	33	0.33100	429	59
2	315	9.62649	27	606	9.95716	6	702	9.66933	33	0.33067	413	58
3	341	9.62676	27	594	9.95710	6	737	9.66966	33	0.33034	396	57
4	367	9.62703	27	582	9.95704	6	772	9.66999	33	0.33001	380	56
5	42394	9.62730	27	90569	9.95698	6	46808	9.67032	33	0.32968	2.1364	55
6	420	9.62757	27	557	9.95692	6	843	9.67065	33	0.32935	348	54
7	446	9.62784	27	545	9.95686	6	879	9.67098	33	0.32902	332	53
8	473	9.62811	27	532	9.95680	6	914	9.67131	32	0.32869	315	52
9	499	9.62838	27	520	9.95674	6	950	9.67163	33	0.32837	299	51
10	42525	9.62865	27	90507	9.95668	5	46985	9.67196	33	0.32804	2.1283	50
11	552	9.62892	26	495	9.95663	6	47021	9.67229	33	0.32771	267	49
12	578	9.62918	27	483	9.95657	6	056	9.67262	33	0.32738	251	48
13	604	9.62945	27	470	9.95651	6	092	9.67295	32	0.32705	235	47
14	631	9.62972	27	458	9.95645	6	128	9.67327	33	0.32673	219	46
15	42657	9.62999	27	90446	9.95639	6	47163	9.67360	33	0.32640	2.1203	45
16	683	9.63026	26	433	9.95633	6	199	9.67393	33	0.32607	187	44
17	709	9.63052	27	421	9.95627	6	234	9.67426	32	0.32574	171	43
18	736	9.63079	27	408	9.95621	6	270	9.67458	33	0.32542	155	42
19	762	9.63106	27	396	9.95615	6	305	9.67491	33	0.32509	139	41
20	42788	9.63133	26	90383	9.95609	6	47341	9.67524	32	0.32476	2.1123	40
21	815	9.63159	27	371	9.95603	6	377	9.67556	33	0.32444	107	39
22	841	9.63186	27	358	9.95597	6	412	9.67589	33	0.32411	092	38
23	867	9.63213	26	346	9.95591	6	448	9.67622	32	0.32378	076	37
24	894	9.63239	27	334	9.95585	6	483	9.67654	33	0.32346	060	36
25	42920	9.63266	26	90321	9.95579	6	47519	9.67687	32	0.32313	2.1044	35
26	946	9.63292	27	309	9.95573	6	555	9.67719	33	0.32281	028	34
27	972	9.63319	26	296	9.95567	6	590	9.67752	33	0.32248	013	33
28	999	9.63345	27	284	9.95561	6	626	9.67785	32	0.32215	2.0997	32
29	43025	9.63372	26	271	9.95555	6	662	9.67817	33	0.32183	981	31
30	43051	9.63398	27	90259	9.95549	6	47698	9.67850	32	0.32150	2.0965	30
31	077	9.63425	26	246	9.95543	6	733	9.67882	33	0.32118	950	29
32	104	9.63451	27	233	9.95537	6	769	9.67915	32	0.32085	934	28
33	130	9.63478	26	221	9.95531	6	805	9.67947	33	0.32053	918	27
34	156	9.63504	27	208	9.95525	6	840	9.67980	32	0.32020	903	26
35	43182	9.63531	26	90196	9.95519	6	47876	9.68012	32	0.31988	2.0887	25
36	209	9.63557	26	183	9.95513	6	912	9.68044	33	0.31956	872	24
37	235	9.63583	27	171	9.95507	7	948	9.68077	32	0.31923	856	23
38	261	9.63610	26	158	9.95500	6	984	9.68109	33	0.31891	840	22
39	287	9.63636	26	146	9.95494	6	48019	9.68142	32	0.31858	825	21
40	43313	9.63662	27	90133	9.95488	6	48055	9.68174	32	0.31826	2.0809	20
41	340	9.63689	26	120	9.95482	6	091	9.68206	33	0.31794	794	19
42	366	9.63715	26	108	9.95476	6	127	9.68239	32	0.31761	778	18
43	392	9.63741	26	095	9.95470	6	163	9.68271	32	0.31729	763	17
44	418	9.63767	27	082	9.95464	6	198	9.68303	33	0.31697	748	16
45	43445	9.63794	26	90070	9.95458	6	48234	9.68336	32	0.31664	2.0732	15
46	471	9.63820	26	057	9.95452	6	270	9.68368	32	0.31632	717	14
47	497	9.63846	26	045	9.95446	6	306	9.68400	32	0.31600	701	13
48	523	9.63872	26	032	9.95440	6	342	9.68432	33	0.31568	686	12
49	549	9.63898	26	019	9.95434	7	378	9.68465	32	0.31535	671	11
50	43575	9.63924	26	90007	9.95427	6	48414	9.68497	32	0.31503	2.0655	10
51	602	9.63950	26	89994	9.95421	6	450	9.68529	32	0.31471	640	9
52	628	9.63976	26	981	9.95415	6	486	9.68561	32	0.31439	625	8
53	654	9.64002	26	968	9.95409	6	521	9.68593	33	0.31407	609	7
54	680	9.64028	26	956	9.95403	6	557	9.68626	32	0.31374	594	6
55	43706	9.64054	26	89943	9.95397	6	48593	9.68658	32	0.31342	2.0579	5
56	733	9.64080	26	930	9.95391	7	629	9.68690	32	0.31310	564	4
57	759	9.64106	26	918	9.95384	6	665	9.68722	32	0.31278	549	3
58	785	9.64132	26	905	9.95378	6	701	9.68754	32	0.31246	533	2
59	811	9.64158	26	892	9.95372	6	737	9.68786	32	0.31214	518	1
60	837	9.64184		879	9.95366		773	9.68818		0.31182	503	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	43837	9.64184	26	89879	9.95366	6	48773	9.68818	32	0.31182	2.0503	60
1	863	9.64210	26	867	9.95360	6	809	9.68850	32	0.31150	488	59
2	889	9.64236	26	854	9.95354	6	845	9.68882	32	0.31118	473	58
3	916	9.64262	26	841	9.95348	7	881	9.68914	32	0.31086	458	57
4	942	9.64288	25	828	9.95341	6	917	9.68946	32	0.31054	443	56
5	43968	9.64313	26	89816	9.95335	6	48953	9.68978	32	0.31022	2.0428	55
6	994	9.64339	26	803	9.95329	6	989	9.69010	32	0.30990	413	54
7	44020	9.64365	26	790	9.95323	6	49026	9.69042	32	0.30958	398	53
8	046	9.64391	26	777	9.95317	7	062	9.69074	32	0.30926	383	52
9	072	9.64417	25	764	9.95310	6	098	9.69106	32	0.30894	368	51
10	44098	9.64442	26	89752	9.95304	6	49134	9.69138	32	0.30862	2.0353	50
11	124	9.64468	26	739	9.95298	6	170	9.69170	32	0.30830	338	49
12	151	9.64494	25	726	9.95292	6	206	9.69202	32	0.30798	323	48
13	177	9.64519	26	713	9.95286	7	242	9.69234	32	0.30766	308	47
14	203	9.64545	26	700	9.95279	6	278	9.69266	32	0.30734	293	46
15	44229	9.64571	25	89687	9.95273	6	49315	9.69298	31	0.30702	2.0278	45
16	255	9.64596	26	674	9.95267	6	351	9.69329	32	0.30671	263	44
17	281	9.64622	25	662	9.95261	7	387	9.69361	32	0.30639	248	43
18	307	9.64647	26	649	9.95254	6	423	9.69393	32	0.30607	233	42
19	333	9.64673	25	636	9.95248	6	459	9.69425	32	0.30575	219	41
20	44359	9.64698	26	89623	9.95242	6	49495	9.69457	31	0.30543	2.0204	40
21	385	9.64724	25	610	9.95236	7	532	9.69488	32	0.30512	189	39
22	411	9.64749	26	597	9.95229	6	568	9.69520	32	0.30480	174	38
23	437	9.64775	25	584	9.95223	6	604	9.69552	32	0.30448	160	37
24	464	9.64800	26	571	9.95217	6	640	9.69584	31	0.30416	145	36
25	44490	9.64826	25	89558	9.95211	7	49677	9.69615	32	0.30385	2.0130	35
26	516	9.64851	26	545	9.95204	6	713	9.69647	32	0.30353	115	34
27	542	9.64877	25	532	9.95198	6	749	9.69679	31	0.30321	101	33
28	568	9.64902	25	519	9.95192	7	786	9.69710	32	0.30290	086	32
29	594	9.64927	26	506	9.95185	6	822	9.69742	32	0.30258	072	31
30	44620	9.64953	25	89493	9.95179	6	49858	9.69774	31	0.30226	2.0057	30
31	646	9.64978	25	480	9.95173	6	894	9.69805	32	0.30195	042	29
32	672	9.65003	26	467	9.95167	7	931	9.69837	31	0.30163	028	28
33	698	9.65029	25	454	9.95160	6	967	9.69868	32	0.30132	013	27
34	724	9.65054	25	441	9.95154	6	50004	9.69900	32	0.30100	1.9999	26
35	44750	9.65079	25	89428	9.95148	7	50040	9.69932	31	0.30068	1.9984	25
36	776	9.65104	26	415	9.95141	6	076	9.69963	32	0.30037	970	24
37	802	9.65130	25	402	9.95135	6	113	9.69995	31	0.30005	955	23
38	828	9.65155	25	389	9.95129	7	149	9.70026	32	0.29974	941	22
39	854	9.65180	25	376	9.95122	6	185	9.70058	31	0.29942	926	21
40	44880	9.65205	25	89363	9.95116	6	50222	9.70089	32	0.29911	1.9912	20
41	906	9.65230	25	350	9.95110	7	258	9.70121	31	0.29879	897	19
42	932	9.65255	26	337	9.95103	6	295	9.70152	32	0.29848	883	18
43	958	9.65281	25	324	9.95097	7	331	9.70184	31	0.29816	868	17
44	984	9.65306	25	311	9.95090	6	368	9.70215	32	0.29785	854	16
45	45010	9.65331	25	89298	9.95084	6	50404	9.70247	31	0.29753	1.9840	15
46	036	9.65356	25	285	9.95078	7	441	9.70278	31	0.29722	825	14
47	062	9.65381	25	272	9.95071	6	477	9.70309	32	0.29691	811	13
48	088	9.65406	25	259	9.95065	6	514	9.70341	31	0.29659	797	12
49	114	9.65431	25	245	9.95059	7	550	9.70372	32	0.29628	782	11
50	45140	9.65456	25	89232	9.95052	6	50587	9.70404	31	0.29596	1.9768	10
51	166	9.65481	25	219	9.95046	7	623	9.70435	31	0.29565	754	9
52	192	9.65506	25	206	9.95039	6	660	9.70466	32	0.29534	740	8
53	218	9.65531	25	193	9.95033	6	696	9.70498	31	0.29502	725	7
54	243	9.65556	24	180	9.95027	7	733	9.70529	31	0.29471	711	6
55	45269	9.65580	25	89167	9.95020	6	50769	9.70560	32	0.29440	1.9697	5
56	295	9.65605	25	153	9.95014	7	806	9.70592	31	0.29408	683	4
57	321	9.65630	25	140	9.95007	6	843	9.70623	31	0.29377	669	3
58	347	9.65655	25	127	9.95001	6	879	9.70654	31	0.29346	654	2
59	373	9.65680	25	114	9.94995	7	916	9.70685	32	0.29315	640	1
60	399	9.65705		101	9.94988		953	9.70717		0.29283	626	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat Tan	Log.	c.d.	Log. Cot	Nat.	
0	45399	9.65705	24	89101	9.94988	6	50953	9.70717	31	0.29283	1.9626	60
1	425	9.65729	25	087	9.94982	7	989	9.70748	31	0.29252	612	59
2	451	9.65754	25	074	9.94975	6	51026	9.70779	31	0.29221	598	58
3	477	9.65779	25	061	9.94969	7	063	9.70810	31	0.29190	584	57
4	503	9.65804	24	048	9.94962	6	099	9.70841	32	0.29159	570	56
5	45529	9.65828	25	89035	9.94956	7	51136	9.70873	31	0.29127	1.9556	55
6	554	9.65853	25	021	9.94949	6	173	9.70904	31	0.29096	542	54
7	580	9.65878	24	008	9.94943	7	209	9.70935	31	0.29065	528	53
8	606	9.65902	25	88995	9.94936	6	246	9.70966	31	0.29034	514	52
9	632	9.65927	25	981	9.94930	7	283	9.70997	31	0.29003	500	51
10	45658	9.65952	24	88968	9.94923	6	51319	9.71028	31	0.28972	1.9486	50
11	684	9.65976	25	955	9.94917	6	356	9.71059	31	0.28941	472	49
12	710	9.66001	24	942	9.94911	7	393	9.71090	31	0.28910	458	48
13	736	9.66025	25	928	9.94904	6	430	9.71121	32	0.28879	444	47
14	762	9.66050	25	915	9.94898	7	467	9.71153	31	0.28847	430	46
15	45787	9.66075	24	88902	9.94891	6	51503	9.71184	31	0.28816	1.9416	45
16	813	9.66099	25	888	9.94885	7	540	9.71215	31	0.28785	402	44
17	839	9.66124	24	875	9.94878	7	577	9.71246	31	0.28754	388	43
18	865	9.66148	25	862	9.94871	6	614	9.71277	31	0.28723	375	42
19	891	9.66173	24	848	9.94865	7	651	9.71308	31	0.28692	361	41
20	45917	9.66197	24	88835	9.94858	6	51688	9.71339	31	0.28661	1.9347	40
21	942	9.66221	25	822	9.94852	7	724	9.71370	31	0.28630	333	39
22	968	9.66246	24	808	9.94845	6	761	9.71401	30	0.28599	319	38
23	994	9.66270	25	795	9.94839	7	798	9.71431	31	0.28569	306	37
24	46020	9.66295	24	782	9.94832	6	835	9.71462	31	0.28538	292	36
25	46046	9.66319	24	88768	9.94826	7	51872	9.71493	31	0.28507	1.9278	35
26	072	9.66343	25	755	9.94819	6	909	9.71524	31	0.28476	265	34
27	097	9.66368	24	741	9.94813	7	946	9.71555	31	0.28445	251	33
28	123	9.66392	24	728	9.94806	7	983	9.71586	31	0.28414	237	32
29	149	9.66416	25	715	9.94799	6	52020	9.71617	31	0.28383	223	31
30	46175	9.66441	24	88701	9.94793	7	52057	9.71648	31	0.28352	1.9210	30
31	201	9.66465	24	688	9.94786	6	094	9.71679	30	0.28321	196	29
32	226	9.66489	24	674	9.94780	7	131	9.71709	31	0.28291	183	28
33	252	9.66513	24	661	9.94773	6	168	9.71740	31	0.28260	169	27
34	278	9.66537	25	647	9.94767	7	205	9.71771	31	0.28229	155	26
35	46304	9.66562	24	88634	9.94760	7	52242	9.71802	31	0.28198	1.9142	25
36	330	9.66586	24	620	9.94753	6	279	9.71833	30	0.28167	128	24
37	355	9.66610	24	607	9.94747	7	316	9.71863	31	0.28137	115	23
38	381	9.66634	24	593	9.94740	6	353	9.71894	31	0.28106	101	22
39	407	9.66658	24	580	9.94734	7	390	9.71925	30	0.28075	088	21
40	46433	9.66682	24	88566	9.94727	7	52427	9.71955	31	0.28045	1.9074	20
41	458	9.66706	25	553	9.94720	6	464	9.71986	31	0.28014	061	19
42	484	9.66731	24	539	9.94714	7	501	9.72017	31	0.27983	047	18
43	510	9.66755	24	526	9.94707	7	538	9.72048	30	0.27952	034	17
44	536	9.66779	24	512	9.94700	6	575	9.72078	31	0.27922	020	16
45	46561	9.66803	24	88499	9.94694	7	52613	9.72109	31	0.27891	1.9007	15
46	587	9.66827	24	485	9.94687	7	650	9.72140	30	0.27860	1.8993	14
47	613	9.66851	24	472	9.94680	6	687	9.72170	31	0.27830	980	13
48	639	9.66875	24	458	9.94674	7	724	9.72201	30	0.27799	967	12
49	664	9.66899	23	445	9.94667	7	761	9.72231	31	0.27769	953	11
50	46690	9.66922	24	88431	9.94660	6	52798	9.72262	31	0.27738	1.8940	10
51	716	9.66946	24	417	9.94654	7	836	9.72293	30	0.27707	927	9
52	742	9.66970	24	404	9.94647	7	873	9.72323	31	0.27677	913	8
53	767	9.66994	24	390	9.94640	6	910	9.72354	30	0.27646	900	7
54	793	9.67018	24	377	9.94634	7	947	9.72384	31	0.27616	887	6
55	46819	9.67042	24	88363	9.94627	7	52985	9.72415	30	0.27585	1.8873	5
56	844	9.67066	24	349	9.94620	6	53022	9.72445	31	0.27555	860	4
57	870	9.67090	23	336	9.94614	7	059	9.72476	30	0.27524	847	3
58	896	9.67113	24	322	9.94607	7	096	9.72506	31	0.27494	834	2
59	921	9.67137	24	308	9.94600	7	134	9.72537	30	0.27463	820	1
60	947	9.67161		295	9.94593		171	9.72567		0.27433	807	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	46947	9.67161	24	88295	9.94593	6	53171	9.72567	31	0.27433	1.8807	60
1	973	9.67185	23	281	9.94587	7	208	9.72598	30	0.27402	794	59
2	999	9.67208	24	267	9.94580	7	246	9.72628	31	0.27372	781	58
3	47024	9.67232	24	254	9.94573	6	283	9.72659	30	0.27341	768	57
4	050	9.67256	24	240	9.94567	7	320	9.72689	31	0.27311	755	56
5	47076	9.67280	23	88226	9.94560	7	53358	9.72720	30	0.27280	1.8741	55
6	101	9.67303	24	213	9.94553	7	395	9.72750	30	0.27250	728	54
7	127	9.67327	23	199	9.94546	6	432	9.72780	31	0.27220	715	53
8	153	9.67350	24	185	9.94540	7	470	9.72811	30	0.27189	702	52
9	178	9.67374	24	172	9.94533	7	507	9.72841	31	0.27159	689	51
10	47204	9.67398	23	88158	9.94526	7	53545	9.72872	30	0.27128	1.8676	50
11	229	9.67421	24	144	9.94519	6	582	9.72902	30	0.27098	663	49
12	255	9.67445	23	130	9.94513	7	620	9.72932	31	0.27068	650	48
13	281	9.67468	24	117	9.94506	7	657	9.72963	30	0.27037	637	47
14	306	9.67492	23	103	9.94499	7	694	9.72993	30	0.27007	624	46
15	47332	9.67515	24	88089	9.94492	7	53732	9.73023	31	0.26977	1.8611	45
16	358	9.67539	23	075	9.94485	6	769	9.73054	30	0.26946	598	44
17	383	9.67562	24	062	9.94479	7	807	9.83084	30	0.26916	585	43
18	409	9.67586	23	048	9.94472	7	844	9.73114	30	0.26886	572	42
19	434	9.67609	24	034	9.94465	7	882	9.73144	31	0.26856	559	41
20	47460	9.67633	23	88020	9.94458	7	53920	9.73175	30	0.26825	1.8546	40
21	486	9.67656	24	006	9.94451	6	957	9.73205	30	0.26795	533	39
22	511	9.67680	23	87993	9.94445	7	995	9.73235	30	0.26765	520	38
23	537	9.67703	23	979	9.94438	7	54032	9.73265	30	0.26735	507	37
24	562	9.67726	24	965	9.94431	7	070	9.73295	31	0.26705	495	36
25	47588	9.67750	23	87951	9.94424	7	54107	9.73326	30	0.26674	1.8482	35
26	614	9.67773	23	937	9.94417	7	145	9.73356	30	0.26644	469	34
27	639	9.67796	24	923	9.94410	6	183	9.73386	30	0.26614	456	33
28	665	9.67820	23	909	9.94404	7	220	9.73416	30	0.26584	443	32
29	690	9.67843	23	896	9.94397	7	258	9.73446	30	0.26554	430	31
30	47716	9.67866	24	87882	9.94390	7	54296	9.73476	31	0.26524	1.8418	30
31	741	9.67890	23	868	9.94383	7	333	9.73507	30	0.26493	405	29
32	767	9.67913	23	854	9.94376	7	371	9.73537	30	0.26463	392	28
33	793	9.67936	23	840	9.94369	7	409	9.73567	30	0.26433	379	27
34	818	9.67959	23	826	9.94362	7	446	9.73597	30	0.26403	367	26
35	47844	9.67982	24	87812	9.94355	6	54484	9.73627	30	0.26373	1.8354	25
36	869	9.68006	23	798	9.94349	7	522	9.73657	30	0.26343	341	24
37	895	9.68029	23	784	9.94342	7	560	9.73687	30	0.26313	329	23
38	920	9.68052	23	770	9.94335	7	597	9.73717	30	0.26283	316	22
39	946	9.68075	23	756	9.94328	7	635	9.73747	30	0.26253	303	21
40	47971	9.68098	23	87743	9.94321	7	54673	9.73777	30	0.26223	1.8291	20
41	997	9.68121	23	729	9.94314	7	711	9.73807	30	0.26193	278	19
42	48022	9.68144	23	715	9.94307	7	748	9.73837	30	0.26163	265	18
43	048	9.68167	23	701	9.94300	7	786	9.73867	30	0.26133	253	17
44	073	9.68190	23	687	9.94293	7	824	9.73897	30	0.26103	240	16
45	48099	9.68213	24	87673	9.94286	7	54862	9.73927	30	0.26073	1.8228	15
46	124	9.68237	23	659	9.94279	6	900	9.73957	30	0.26043	215	14
47	150	9.68260	23	645	9.94273	7	938	9.73987	30	0.26013	202	13
48	175	9.68283	22	631	9.94266	7	975	9.74017	30	0.25983	190	12
49	201	9.68305	23	617	9.94259	7	55013	9.74047	30	0.25953	177	11
50	48226	9.68328	23	87603	9.94252	7	55051	9.74077	30	0.25923	1.8165	10
51	252	9.68351	23	589	9.94245	7	089	9.74107	30	0.25893	152	9
52	277	9.68374	23	575	9.94238	7	127	9.74137	29	0.25863	140	8
53	303	9.68397	23	561	9.94231	7	165	9.74166	30	0.25834	127	7
54	328	9.68420	23	546	9.94224	7	203	9.74196	30	0.25804	115	6
55	48354	9.68443	23	87532	9.94217	7	55241	9.74226	30	0.25774	1.8103	5
56	379	9.68466	23	518	9.94210	7	279	9.74256	30	0.25744	090	4
57	405	9.68489	23	504	9.94203	7	317	9.74286	30	0.25714	078	3
58	430	9.68512	22	490	9.94196	7	355	9.74316	29	0.25684	065	2
59	456	9.68534	23	476	9.94189	7	393	9.74345	30	0.25655	053	1
60	481	9.68557		462	9.94182		431	9.74375		0.25625	040	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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29°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	48481	9.68557	23	87462	9.94182	7	55431	9.74375	30	0.25625	1.8040	60
1	506	9.68580	23	448	9.94175	7	469	9.74405	30	0.25595	028	59
2	532	9.68603	22	434	9.94168	7	507	9.74435	30	0.25565	016	58
3	557	9.68625	23	420	9.94161	7	545	9.74465	29	0.25535	003	57
4	583	9.68648	23	406	9.94154	7	583	9.74494	30	0.25506	1.7991	56
5	48608	9.68671	23	87391	9.94147	7	55621	9.74524	30	0.25476	1.7979	55
6	634	9.68694	22	377	9.94140	7	659	9.74554	29	0.25446	966	54
7	659	9.68716	23	363	9.94133	7	697	9.74583	30	0.25417	954	53
8	684	9.68739	23	349	9.94126	7	736	9.74613	30	0.25387	942	52
9	710	9.68762	22	335	9.94119	7	774	9.74643	30	0.25357	930	51
10	48735	9.68784	23	87321	9.94112	7	55812	9.74673	29	0.25327	1.7917	50
11	761	9.68807	22	306	9.94105	7	850	9.74702	30	0.25298	905	49
12	786	9.68829	23	292	9.94098	8	888	9.74732	30	0.25268	893	48
13	811	9.68852	23	278	9.94090	7	926	9.74762	29	0.25238	881	47
14	837	9.68875	22	264	9.94083	7	964	9.74791	30	0.25209	868	46
15	48862	9.68897	23	87250	9.94076	7	56003	9.74821	30	0.25179	1.7856	45
16	888	9.68920	22	235	9.94069	7	041	9.74851	29	0.25149	844	44
17	913	9.68942	23	221	9.94062	7	079	9.74880	30	0.25120	832	43
18	938	9.68965	22	207	9.94055	7	117	9.74910	29	0.25090	820	42
19	964	9.68987	23	193	9.94048	7	156	9.74939	30	0.25061	808	41
20	48989	9.69010	22	87178	9.94041	7	56194	9.74969	29	0.25031	1.7796	40
21	49014	9.69032	23	164	9.94034	7	232	9.74998	30	0.25002	783	39
22	040	9.69055	22	150	9.94027	7	270	9.75028	30	0.24972	771	38
23	065	9.69077	23	136	9.94020	8	309	9.75058	29	0.24942	759	37
24	090	9.69100	22	121	9.94012	7	347	9.75087	30	0.24913	747	36
25	49116	9.69122	22	87107	9.94005	7	56385	9.75117	29	0.24883	1.7735	35
26	141	9.69144	23	093	9.93998	7	424	9.75146	30	0.24854	723	34
27	166	9.69167	22	079	9.93991	7	462	9.75176	29	0.24824	711	33
28	192	9.69189	23	064	9.93984	7	501	9.75205	30	0.24795	699	32
29	217	9.69212	22	050	9.93977	7	539	9.75235	29	0.24765	687	31
30	49242	9.69234	22	87036	9.93970	7	56577	9.75264	30	0.24736	1.7675	30
31	268	9.69256	23	021	9.93963	8	616	9.75294	29	0.24706	663	29
32	293	9.69279	22	007	9.93955	7	654	9.75323	30	0.24677	651	28
33	318	9.69301	22	86993	9.93948	7	693	9.75353	29	0.24647	639	27
34	344	9.69323	22	978	9.93941	7	731	9.75382	29	0.24618	627	26
35	49369	9.69345	23	86964	9.93934	7	56769	9.75411	30	0.24589	1.7615	25
36	394	9.69368	22	949	9.93927	7	808	9.75441	29	0.24559	603	24
37	419	9.69390	22	935	9.93920	8	846	9.75470	30	0.24530	591	23
38	445	9.69412	22	921	9.93912	7	885	9.75500	29	0.24500	579	22
39	470	9.69434	22	906	9.93905	7	923	9.75529	29	0.24471	567	21
40	49495	9.69456	23	86892	9.93898	7	56962	9.75558	30	0.24442	1.7556	20
41	521	9.69479	22	878	9.93891	7	57000	9.75588	29	0.24412	544	19
42	546	9.69501	22	863	9.93884	8	039	9.75617	30	0.24383	532	18
43	571	9.69523	22	849	9.93876	7	078	9.75647	29	0.24353	520	17
44	596	9.69545	22	834	9.93869	7	116	9.75676	29	0.24324	508	16
45	49622	9.69567	22	86820	9.93862	7	57155	9.75705	30	0.24295	1.7496	15
46	647	9.69589	22	805	9.93855	8	193	9.75735	29	0.24265	485	14
47	672	9.69611	22	791	9.93847	7	232	9.75764	29	0.24236	473	13
48	697	9.69633	22	777	9.93840	7	271	9.75793	29	0.24207	461	12
49	723	9.69655	22	762	9.93833	7	309	9.75822	30	0.24178	449	11
50	49748	9.69677	22	86748	9.93826	7	57348	9.75852	29	0.24148	1.7437	10
51	773	9.69699	22	733	9.93819	8	386	9.75881	29	0.24119	426	9
52	798	9.69721	22	719	9.93811	7	425	9.75910	29	0.24090	414	8
53	824	9.69743	22	704	9.93804	7	464	9.75939	30	0.24061	402	7
54	849	9.69765	22	690	9.93797	8	503	9.75969	29	0.24031	391	6
55	49874	9.69787	22	86675	9.93789	7	57541	9.75998	29	0.24002	1.7379	5
56	899	9.69809	22	661	9.93782	7	580	9.76027	29	0.23973	367	4
57	924	9.69831	22	646	9.93775	7	619	9.76056	30	0.23944	355	3
58	950	9.69853	22	632	9.93768	8	657	9.76086	29	0.23914	344	2
59	975	9.69875	22	617	9.93760	7	696	9.76115	29	0.23885	332	1
60	50000	9.69897		603	9.93753		735	9.76144		0.23856	321	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

60°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	50000	9.69897	22	86603	9.93753	7	57735	9.76144	29	0.23856	1.7321	60
1	025	9.69919	22	588	9.93746	8	774	9.76173	29	0.23827	309	59
2	050	9.69941	22	573	9.93738	7	813	9.76202	29	0.23798	297	58
3	076	9.69963	21	559	9.93731	7	851	9.76231	30	0.23769	286	57
4	101	9.69984	22	544	9.93724	7	890	9.76261	29	0.23739	274	56
5	50126	9.70006	22	86530	9.93717	8	57929	9.76290	29	0.23710	1.7262	55
6	151	9.70028	22	515	9.93709	7	968	9.76319	29	0.23681	251	54
7	176	9.70050	22	501	9.93702	7	58007	9.76348	29	0.23652	239	53
8	201	9.70072	21	486	9.93695	8	046	9.76377	29	0.23623	228	52
9	227	9.70093	22	471	9.93687	7	085	9.76406	29	0.23594	216	51
10	50252	9.70115	22	86457	9.93680	7	58124	9.76435	29	0.23565	1.7205	50
11	277	9.70137	22	442	9.93673	8	162	9.76464	29	0.23536	193	49
12	302	9.70159	21	427	9.93665	7	201	9.76493	29	0.23507	182	48
13	327	9.70180	22	413	9.93658	8	240	9.76522	29	0.23478	170	47
14	352	9.70202	22	398	9.93650	7	279	9.76551	29	0.23449	159	46
15	50377	9.70224	21	86384	9.93643	7	58318	9.76580	29	0.23420	1.7147	45
16	403	9.70245	22	369	9.93636	8	357	9.76609	30	0.23391	136	44
17	428	9.70267	21	354	9.93628	7	396	9.76639	29	0.23361	124	43
18	453	9.70288	22	340	9.93621	7	435	9.76668	29	0.23332	113	42
19	478	9.70310	22	325	9.93614	8	474	9.76697	28	0.23303	102	41
20	50503	9.70332	21	86310	9.93606	7	58513	9.76725	29	0.23275	1.7090	40
21	528	9.70353	22	295	9.93599	8	552	9.76754	29	0.23246	079	39
22	553	9.70375	21	281	9.93591	7	591	9.76783	29	0.23217	067	38
23	578	9.70396	22	266	9.93584	7	631	9.76812	29	0.23188	056	37
24	603	9.70418	21	251	9.93577	8	670	9.76841	29	0.23159	045	36
25	50628	9.70439	22	86237	9.93569	7	58709	9.76870	29	0.23130	1.7033	35
26	654	9.70461	21	222	9.93562	8	748	9.76899	29	0.23101	022	34
27	679	9.70482	22	207	9.93554	7	787	9.76928	29	0.23072	011	33
28	704	9.70504	21	192	9.93547	8	826	9.76957	29	0.23043	1.6999	32
29	729	9.70525	22	178	9.93539	7	865	9.76986	29	0.23014	988	31
30	50754	9.70547	21	86163	9.93532	7	58905	9.77015	29	0.22985	1.6977	30
31	779	9.70568	22	148	9.93525	8	944	9.77044	29	0.22956	965	29
32	804	9.70590	21	133	9.93517	7	983	9.77073	28	0.22927	954	28
33	829	9.70611	22	119	9.93510	8	59022	9.77101	29	0.22899	943	27
34	854	9.70633	21	104	9.93502	7	061	9.77130	29	0.22870	932	26
35	50879	9.70654	21	86089	9.93495	8	59101	9.77159	29	0.22841	1.6920	25
36	904	9.70675	22	074	9.93487	7	140	9.77188	29	0.22812	909	24
37	929	9.70697	21	059	9.93480	8	179	9.77217	29	0.22783	898	23
38	954	9.70718	21	045	9.93472	7	218	9.77246	28	0.22754	887	22
39	979	9.70739	22	030	9.93465	8	258	9.77274	29	0.22726	875	21
40	51004	9.70761	21	86015	9.93457	7	59297	9.77303	29	0.22697	1.6864	20
41	029	9.70782	21	000	9.93450	8	336	9.77332	29	0.22668	853	19
42	054	9.70803	21	85985	9.93442	7	376	9.77361	29	0.22639	842	18
43	079	9.70824	22	970	9.93435	8	415	9.77390	28	0.22610	831	17
44	104	9.70846	21	956	9.93427	7	454	9.77418	29	0.22582	820	16
45	51129	9.70867	21	85941	9.93420	8	59494	9.77447	29	0.22553	1.6808	15
46	154	9.70888	21	926	9.93412	7	533	9.77476	29	0.22524	797	14
47	179	9.70909	22	911	9.93405	8	573	9.77505	28	0.22495	786	13
48	204	9.70931	21	896	9.93397	7	612	9.77533	29	0.22467	775	12
49	229	9.70952	21	881	9.93390	8	651	9.77562	29	0.22438	764	11
50	51254	9.70973	21	85866	9.93382	7	59691	9.77591	28	0.22409	1.6753	10
51	279	9.70994	21	851	9.93375	8	730	9.77619	29	0.22381	742	9
52	304	9.71015	21	836	9.93367	7	770	9.77648	29	0.22352	731	8
53	329	9.71036	22	821	9.93360	8	809	9.77677	29	0.22323	720	7
54	354	9.71058	21	806	9.93352	8	849	9.77706	28	0.22294	709	6
55	51379	9.71079	21	85792	9.93344	7	59888	9.77734	29	0.22266	1.6698	5
56	404	9.71100	21	777	9.93337	8	928	9.77763	28	0.22237	687	4
57	429	9.71121	21	762	9.93329	7	967	9.77791	29	0.22209	676	3
58	454	9.71142	21	747	9.93322	8	60007	9.77820	29	0.22180	665	2
59	479	9.71163	21	732	9.93314	7	046	9.77849	28	0.22151	654	1
60	504	9.71184		717	9.93307		086	9.77877		0.22123	643	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	51504	9.71184	21	85717	9.93307	8	60086	9.77877	29	0.22123	1.6643	60
1	529	9.71205	21	702	9.93299	8	126	9.77906	29	0.22094	632	59
2	554	9.71226	21	687	9.93291	7	165	9.77935	28	0.22065	621	58
3	579	9.71247	21	672	9.93284	8	205	9.77963	29	0.22037	610	57
4	604	9.71268	21	657	9.93276	7	245	9.77992	28	0.22008	599	56
5	51628	9.71289	21	85642	9.93269	8	60284	9.78020	29	0.21980	1.6588	55
6	653	9.71310	21	627	9.93261	8	324	9.78049	28	0.21951	577	54
7	678	9.71331	21	612	9.93253	7	364	9.78077	29	0.21923	566	53
8	703	9.71352	21	597	9.93246	8	403	9.78106	29	0.21894	555	52
9	728	9.71373	20	582	9.93238	8	443	9.78135	28	0.21865	545	51
10	51753	9.71393	21	85567	9.93230	7	60483	9.78163	29	0.21837	1.6534	50
11	778	9.71414	21	551	9.93223	8	522	9.78192	28	0.21808	523	49
12	803	9.71435	21	536	9.93215	8	562	9.78220	29	0.21780	512	48
13	828	9.71456	21	521	9.93207	7	602	9.78249	28	0.21751	501	47
14	852	9.71477	21	506	9.93200	8	642	9.78277	29	0.21723	490	46
15	51877	9.71498	21	85491	9.93192	8	60681	9.78306	28	0.21694	1.6479	45
16	902	9.71519	20	476	9.93184	7	721	9.78334	29	0.21666	469	44
17	927	9.71539	21	461	9.93177	8	761	9.78363	28	0.21637	458	43
18	952	9.71560	21	446	9.93169	8	801	9.78391	28	0.21609	447	42
19	977	9.71581	21	431	9.93161	7	841	9.78419	29	0.21581	436	41
20	52002	9.71602	20	85416	9.93154	8	60881	9.78448	28	0.21552	1.6426	40
21	026	9.71622	21	401	9.93146	8	921	9.78476	29	0.21524	415	39
22	051	9.71643	21	385	9.93138	7	960	9.78505	28	0.21495	404	38
23	076	9.71664	21	370	9.93131	8	61000	9.78533	29	0.21467	393	37
24	101	9.71685	20	355	9.93123	8	040	9.78562	28	0.21438	383	36
25	52126	9.71705	21	85340	9.93115	7	61080	9.78590	28	0.21410	1.6372	35
26	151	9.71726	21	325	9.93108	8	120	9.78618	29	0.21382	361	34
27	175	9.71747	20	310	9.93100	8	160	9.78647	28	0.21353	351	33
28	200	9.71767	21	294	9.93092	8	200	9.78675	29	0.21325	340	32
29	225	9.71788	21	279	9.93084	7	240	9.78704	28	0.21296	329	31
30	52250	9.71809	20	85264	9.93077	8	61280	9.78732	28	0.21268	1.6319	30
31	275	9.71829	21	249	9.93069	8	320	9.78760	29	0.21240	308	29
32	299	9.71850	20	234	9.93061	8	360	9.78789	28	0.21211	297	28
33	324	9.71870	21	218	9.93053	7	400	9.78817	28	0.21183	287	27
34	349	9.71891	20	203	9.93046	8	440	9.78845	29	0.21155	276	26
35	52374	9.71911	21	85188	9.93038	8	61480	9.78874	28	0.21126	1.6265	25
36	399	9.71932	20	173	9.93030	8	520	9.78902	28	0.21098	255	24
37	423	9.71952	21	157	9.93022	8	561	9.78930	29	0.21070	244	23
38	448	9.71973	21	142	9.93014	7	601	9.78959	28	0.21041	234	22
39	473	9.71994	20	127	9.93007	8	641	9.78987	28	0.21013	223	21
40	52498	9.72014	20	85112	9.92999	8	61681	9.79015	28	0.20985	1.6212	20
41	522	9.72034	21	096	9.92991	8	721	9.79043	29	0.20957	202	19
42	547	9.72055	20	081	9.92983	7	761	9.79072	28	0.20928	191	18
43	572	9.72075	21	066	9.92976	8	801	9.79100	28	0.20900	181	17
44	597	9.72096	20	051	9.92968	8	842	9.79128	28	0.20872	170	16
45	52621	9.72116	21	85035	9.92960	8	61882	9.79156	29	0.20844	1.6160	15
46	646	9.72137	20	020	9.92952	8	922	9.79185	28	0.20815	149	14
47	671	9.72157	20	005	9.92944	8	962	9.79213	28	0.20787	139	13
48	696	9.72177	21	84989	9.92936	7	62003	9.79241	28	0.20759	128	12
49	720	9.72198	20	974	9.92929	8	043	9.79269	28	0.20731	118	11
50	52745	9.72218	20	84959	9.92921	8	62083	9.79297	29	0.20703	1.6107	10
51	770	9.72238	21	943	9.92913	8	124	9.79326	28	0.20674	097	9
52	794	9.72259	20	928	9.92905	8	164	9.79354	28	0.20646	087	8
53	819	9.72279	20	913	9.92897	8	204	9.79382	28	0.20618	076	7
54	844	9.72299	21	897	9.92889	8	245	9.79410	28	0.20590	066	6
55	52869	9.72320	20	84882	9.92881	7	62285	9.79438	28	0.20562	1.6055	5
56	893	9.72340	20	866	9.92874	8	325	9.79466	29	0.20534	045	4
57	918	9.72360	21	851	9.92866	8	366	9.79495	28	0.20505	034	3
58	943	9.72381	20	836	9.92858	8	406	9.79523	28	0.20477	024	2
59	967	9.72401	20	820	9.92850	8	446	9.79551	28	0.20449	014	1
60	992	9.72421		805	9.92842		487	9.79579		0.20421	003	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	52992	9.72421	20	84805	9.92842	8	62487	9.79579	28	0.20421	1.6003	60
1	53017	9.72441	20	789	9.92834	8	527	9.79607	28	0.20393	1.5993	59
2	041	9.72461	21	774	9.92826	8	568	9.79635	28	0.20365	983	58
3	066	9.72482	20	759	9.92818	8	608	9.79663	28	0.20337	972	57
4	091	9.72502	20	743	9.92810	7	649	9.79691	28	0.20309	962	56
5	53115	9.72522	20	84728	9.92803	8	62689	9.79719	28	0.20281	1.5952	55
6	140	9.72542	20	712	9.92795	8	730	9.79747	29	0.20253	941	54
7	164	9.72562	20	697	9.92787	8	770	9.79776	28	0.20224	931	53
8	189	9.72582	20	681	9.92779	8	811	9.79804	28	0.20196	921	52
9	214	9.72602	20	666	9.92771	8	852	9.79832	28	0.20168	911	51
10	53238	9.72622	21	84650	9.92763	8	62892	9.79860	28	0.20140	1.5900	50
11	263	9.72643	20	635	9.92755	8	933	9.79888	28	0.20112	890	49
12	288	9.72663	20	619	9.92747	8	973	9.79916	28	0.20084	880	48
13	312	9.72683	20	604	9.92739	8	63014	9.79944	28	0.20056	869	47
14	337	9.72703	20	588	9.92731	8	055	9.79972	28	0.20028	859	46
15	53361	9.72723	20	84573	9.92723	8	63095	9.80000	28	0.20000	1.5849	45
16	386	9.72743	20	557	9.92715	8	136	9.80028	28	0.19972	839	44
17	411	9.72763	20	542	9.92707	8	177	9.80056	28	0.19944	829	43
18	435	9.72783	20	526	9.92699	8	217	9.80084	28	0.19916	818	42
19	460	9.72803	20	511	9.92691	8	258	9.80112	28	0.19888	808	41
20	53484	9.72823	20	84495	9.92683	8	63299	9.80140	28	0.19860	1.5798	40
21	509	9.72843	20	480	9.92675	8	340	9.80168	27	0.19832	788	39
22	534	9.72863	20	464	9.92667	8	380	9.80195	28	0.19805	778	38
23	558	9.72883	19	448	9.92659	8	421	9.80223	28	0.19777	768	37
24	583	9.72902	20	433	9.92651	8	462	9.80251	28	0.19749	757	36
25	53607	9.72922	20	84417	9.92643	8	63503	9.80279	28	0.19721	1.5747	35
26	632	9.72942	20	402	9.92635	8	544	9.80307	28	0.19693	737	34
27	656	9.72962	20	386	9.92627	8	584	9.80335	28	0.19665	727	33
28	681	9.72982	20	370	9.92619	8	625	9.80363	28	0.19637	717	32
29	705	9.73002	20	355	9.92611	8	666	9.80391	28	0.19609	707	31
30	53730	9.73022	19	84339	9.92603	8	63707	9.80419	28	0.19581	1.5697	30
31	754	9.73041	20	324	9.92595	8	748	9.80447	27	0.19553	687	29
32	779	9.73061	20	308	9.92587	8	789	9.80474	28	0.19526	677	28
33	804	9.73081	20	292	9.92579	8	830	9.80502	28	0.19498	667	27
34	828	9.73101	20	277	9.92571	8	871	9.80530	28	0.19470	657	26
35	53853	9.73121	19	84261	9.92563	8	63912	9.80558	28	0.19442	1.5647	25
36	877	9.73140	20	245	9.92555	9	953	9.80586	28	0.19414	637	24
37	902	9.73160	20	230	9.92546	8	994	9.80614	28	0.19386	627	23
38	926	9.73180	20	214	9.92538	8	64035	9.80642	27	0.19358	617	22
39	951	9.73200	19	198	9.92530	8	076	9.80669	28	0.19331	607	21
40	53975	9.73219	20	84182	9.92522	8	64117	9.80697	28	0.19303	1.5597	20
41	54000	9.73239	20	167	9.92514	8	158	9.80725	28	0.19275	587	19
42	024	9.73259	19	151	9.92506	8	199	9.80753	28	0.19247	577	18
43	049	9.73278	20	135	9.92498	8	240	9.80781	27	0.19219	567	17
44	073	9.73298	20	120	9.92490	8	281	9.80808	28	0.19192	557	16
45	54097	9.73318	19	84104	9.92482	9	64322	9.80836	28	0.19164	1.5547	15
46	122	9.73337	20	088	9.92473	8	363	9.80864	28	0.19136	537	14
47	146	9.73357	20	072	9.92465	8	404	9.80892	27	0.19108	527	13
48	171	9.73377	19	057	9.92457	8	446	9.80919	28	0.19081	517	12
49	195	9.73396	20	041	9.92449	8	487	9.80947	28	0.19053	507	11
50	54220	9.73416	19	84025	9.92441	8	64528	9.80975	28	0.19025	1.5497	10
51	244	9.73435	20	009	9.92433	8	569	9.81003	27	0.18997	487	9
52	269	9.73455	19	83994	9.92425	9	610	9.81030	28	0.18970	477	8
53	293	9.73474	20	978	9.92416	8	652	9.81058	28	0.18942	468	7
54	317	9.73494	19	962	9.92408	8	693	9.81086	27	0.18914	458	6
55	54342	9.73513	20	83946	9.92400	8	64734	9.81113	28	0.18887	1.5448	5
56	366	9.73533	19	930	9.92392	8	775	9.81141	28	0.18859	438	4
57	391	9.73552	20	915	9.92384	8	817	9.81169	27	0.18831	428	3
58	415	9.73572	19	899	9.92376	9	858	9.81196	28	0.18804	418	2
59	440	9.73591	20	883	9.92367	8	899	9.81224	28	0.18776	408	1
60	464	9.73611		867	9.92359		941	9.81252		0.18748	399	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	54464	9.73611	19	83867	9.92359	8	64941	9.81252	27	0.18748	1.5399	60
1	488	9.73630	20	851	9.92351	8	982	9.81279	28	0.18721	389	59
2	513	9.73650	19	835	9.92343	8	65024	9.81307	28	0.18693	379	58
3	537	9.73669	20	819	9.92335	9	065	9.81335	27	0.18665	369	57
4	561	9.73689	19	804	9.92326	8	106	9.81362	28	0.18638	359	56
5	54586	9.73708	19	83788	9.92318	8	65148	9.81390	28	0.18610	1.5350	55
6	610	9.73727	20	772	9.92310	8	189	9.81418	27	0.18582	340	54
7	635	9.73747	19	756	9.92302	9	231	9.81445	28	0.18555	330	53
8	659	9.73766	19	740	9.92293	8	272	9.81473	27	0.18527	320	52
9	683	9.73785	20	724	9.92285	8	314	9.81500	28	0.18500	311	51
10	54708	9.73805	19	83708	9.92277	8	65355	9.81528	28	0.18472	1.5301	50
11	732	9.73824	19	692	9.92269	9	397	9.81556	27	0.18444	291	49
12	756	9.73843	20	676	9.92260	8	438	9.81583	28	0.18417	282	48
13	781	9.73863	19	660	9.92252	8	480	9.81611	27	0.18389	272	47
14	805	9.73882	19	645	9.92244	9	521	9.81638	28	0.18362	262	46
15	54829	9.73901	20	83629	9.92235	8	65563	9.81666	27	0.18334	1.5253	45
16	854	9.73921	19	613	9.92227	8	604	9.81693	28	0.18307	243	44
17	878	9.73940	19	597	9.92219	8	646	9.81721	27	0.18279	233	43
18	902	9.73959	19	581	9.92211	9	688	9.81748	28	0.18252	224	42
19	927	9.73978	19	565	9.92202	8	729	9.81776	27	0.18224	214	41
20	54951	9.73997	20	83549	9.92194	8	65771	9.81803	28	0.18197	1.5204	40
21	975	9.74017	19	533	9.92186	9	813	9.81831	27	0.18169	195	39
22	999	9.74036	19	517	9.92177	8	854	9.81858	28	0.18142	185	38
23	55024	9.74055	19	501	9.92169	8	896	9.81886	27	0.18114	175	37
24	048	9.74074	19	485	9.92161	9	938	9.81913	28	0.18087	166	36
25	55072	9.74093	20	83469	9.92152	8	65980	9.81941	27	0.18059	1.5156	35
26	097	9.74113	19	453	9.92144	8	66021	9.81968	28	0.18032	147	34
27	121	9.74132	19	437	9.92136	9	063	9.81996	27	0.18004	137	33
28	145	9.74151	19	421	9.92127	8	105	9.82023	28	0.17977	127	32
29	169	9.74170	19	405	9.92119	8	147	9.82051	27	0.17949	118	31
30	55194	9.74189	19	83389	9.92111	9	66189	9.82078	28	0.17922	1.5108	30
31	218	9.74208	19	373	9.92102	8	230	9.82106	27	0.17894	099	29
32	242	9.74227	19	356	9.92094	8	272	9.82133	28	0.17867	089	28
33	266	9.74246	19	340	9.92086	9	314	9.82161	27	0.17839	080	27
34	291	9.74265	19	324	9.92077	8	356	9.82188	27	0.17812	070	26
35	55315	9.74284	19	83308	9.92069	9	66398	9.82215	28	0.17785	1.5061	25
36	339	9.74303	19	292	9.92060	8	440	9.82243	27	0.17757	051	24
37	363	9.74322	19	276	9.92052	8	482	9.82270	28	0.17730	042	23
38	388	9.74341	19	260	9.92044	9	524	9.82298	27	0.17702	032	22
39	412	9.74360	19	244	9.92035	8	566	9.82325	27	0.17675	023	21
40	55436	9.74379	19	83228	9.92027	9	66608	9.82352	28	0.17648	1.5013	20
41	460	9.74398	19	212	9.92018	8	650	9.82380	27	0.17620	004	19
42	484	9.74417	19	195	9.92010	8	692	9.82407	28	0.17593	1.4994	18
43	509	9.74436	19	179	9.92002	9	734	9.82435	27	0.17565	985	17
44	533	9.74455	19	163	9.91993	8	776	9.82462	27	0.17538	975	16
45	55557	9.74474	19	83147	9.91985	9	66818	9.82489	28	0.17511	1.4966	15
46	581	9.74493	19	131	9.91976	8	860	9.82517	27	0.17483	957	14
47	605	9.74512	19	115	9.91968	9	902	9.82544	27	0.17456	947	13
48	630	9.74531	18	098	9.91959	8	944	9.82571	28	0.17429	938	12
49	654	9.74549	19	082	9.91951	9	986	9.82599	27	0.17401	928	11
50	55678	9.74568	19	83066	9.91942	8	67028	9.82626	27	0.17374	1.4919	10
51	702	9.74587	19	050	9.91934	9	071	9.82653	28	0.17347	910	9
52	726	9.74606	19	034	9.91925	8	113	9.82681	27	0.17319	900	8
53	750	9.74625	19	017	9.91917	9	155	9.82708	27	0.17292	891	7
54	775	9.74644	18	001	9.91908	8	197	9.82735	27	0.17265	882	6
55	55799	9.74662	19	82985	9.91900	9	67239	9.82762	28	0.17238	1.4872	5
56	823	9.74681	19	969	9.91891	8	282	9.82790	27	0.17210	863	4
57	847	9.74700	19	953	9.91883	9	324	9.82817	27	0.17183	854	3
58	871	9.74719	18	936	9.91874	8	366	9.82844	27	0.17156	844	2
59	895	9.74737	19	920	9.91866	9	409	9.82871	28	0.17129	835	1
60	919	9.74756		904	9.91857		451	9.82899		0.17101	826	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	55919	9.74756	19	82904	9.91857	8	67451	9.82899	27	0.17101	1.4826	60
1	943	9.74775	19	887	9.91849	9	493	9.82926	27	0.17074	816	59
2	968	9.74794	18	871	9.91840	8	536	9.82953	27	0.17047	807	58
3	992	9.74812	19	855	9.91832	9	578	9.82980	28	0.17020	798	57
4	56016	9.74831	19	839	9.91823	8	620	9.83008	27	0.16992	788	56
5	56040	9.74850	18	82822	9.91815	9	67663	9.83035	27	0.16965	1.4779	55
6	064	9.74868	19	806	9.91806	8	705	9.83062	27	0.16938	770	54
7	088	9.74887	19	790	9.91798	9	748	9.83089	28	0.16911	761	53
8	112	9.74906	18	773	9.91789	8	790	9.83117	27	0.16883	751	52
9	136	9.74924	19	757	9.91781	9	832	9.83144	27	0.16856	742	51
10	56160	9.74943	18	82741	9.91772	9	67875	9.83171	27	0.16829	1.4733	50
11	184	9.74961	19	724	9.91763	8	917	9.83198	27	0.16802	724	49
12	208	9.74980	19	708	9.91755	9	960	9.83225	27	0.16775	715	48
13	232	9.74999	18	692	9.91746	8	68002	9.83252	28	0.16748	705	47
14	256	9.75017	19	675	9.91738	9	045	9.83280	27	0.16720	696	46
15	56280	9.75036	18	82659	9.91729	9	68088	9.83307	27	0.16693	1.4687	45
16	305	9.75054	19	643	9.91720	8	130	9.83334	27	0.16666	678	44
17	329	9.75073	18	626	9.91712	9	173	9.83361	27	0.16639	669	43
18	353	9.75091	19	610	9.91703	8	215	9.83388	27	0.16612	659	42
19	377	9.75110	18	593	9.91695	9	258	9.83415	27	0.16585	650	41
20	56401	9.75128	19	82577	9.91686	9	68301	9.83442	28	0.16558	1.4641	40
21	425	9.75147	18	561	9.91677	8	343	9.83470	27	0.16530	632	39
22	449	9.75165	19	544	9.91669	9	386	9.83497	27	0.16503	623	38
23	473	9.75184	18	528	9.91660	9	429	9.83524	27	0.16476	614	37
24	497	9.75202	19	511	9.91651	8	471	9.83551	27	0.16449	605	36
25	56521	9.75221	18	82495	9.91643	9	68514	9.83578	27	0.16422	1.4596	35
26	545	9.75239	19	478	9.91634	9	557	9.83605	27	0.16395	586	34
27	569	9.75258	18	462	9.91625	8	600	9.83632	27	0.16368	577	33
28	593	9.75276	18	446	9.91617	9	642	9.83659	27	0.16341	568	32
29	617	9.75294	19	429	9.91608	9	685	9.83686	27	0.16314	559	31
30	56641	9.75313	18	82413	9.91599	8	68728	9.83713	27	0.16287	1.4550	30
31	665	9.75331	19	396	9.91591	9	771	9.83740	28	0.16260	541	29
32	689	9.75350	18	380	9.91582	9	814	9.83768	27	0.16232	532	28
33	713	9.75368	18	363	9.91573	8	857	9.83795	27	0.16205	523	27
34	736	9.75386	19	347	9.91565	9	900	9.83822	27	0.16178	514	26
35	56760	9.75405	18	82330	9.91556	9	68942	9.83849	27	0.16151	1.4505	25
36	784	9.75423	18	314	9.91547	9	985	9.83876	27	0.16124	496	24
37	808	9.75441	18	297	9.91538	8	69028	9.83903	27	0.16097	487	23
38	832	9.75459	19	281	9.91530	9	071	9.83930	27	0.16070	478	22
39	856	9.75478	18	264	9.91521	9	114	9.83957	27	0.16043	469	21
40	56880	9.75496	18	82248	9.91512	8	69157	9.83984	27	0.16016	1.4460	20
41	904	9.75514	19	231	9.91504	9	200	9.84011	27	0.15989	451	19
42	928	9.75533	18	214	9.91495	9	243	9.84038	27	0.15962	442	18
43	952	9.75551	18	198	9.91486	9	286	9.84065	27	0.15935	433	17
44	976	9.75569	18	181	9.91477	8	329	9.84092	27	0.15908	424	16
45	57000	9.75587	18	82165	9.91469	9	69372	9.84119	27	0.15881	1.4415	15
46	024	9.75605	19	148	9.91460	9	416	9.84146	27	0.15854	406	14
47	047	9.75624	18	132	9.91451	9	459	9.84173	27	0.15827	397	13
48	071	9.75642	18	115	9.91442	9	502	9.84200	27	0.15800	388	12
49	095	9.75660	18	098	9.91433	8	545	9.84227	27	0.15773	379	11
50	57119	9.75678	18	82082	9.91425	9	69588	9.84254	26	0.15746	1.4370	10
51	143	9.75696	18	065	9.91416	9	631	9.84280	27	0.15720	361	9
52	167	9.75714	19	048	9.91407	9	675	9.84307	27	0.15693	352	8
53	191	9.75733	18	032	9.91398	9	718	9.84334	27	0.15666	344	7
54	215	9.75751	18	015	9.91389	8	761	9.84361	27	0.15639	335	6
55	57238	9.75769	18	81999	9.91381	9	69804	9.84388	27	0.15612	1.4326	5
56	262	9.75787	18	982	9.91372	9	847	9.84415	27	0.15585	317	4
57	286	9.75805	18	965	9.91363	9	891	9.84442	27	0.15558	308	3
58	310	9.75823	18	949	9.91354	9	934	9.84469	27	0.15531	299	2
59	334	9.75841	18	932	9.91345	9	977	9.84496	27	0.15504	290	1
60	358	9.75859		915	9.91336		70021	9.84523		0.15477	281	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

35°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	57358	9.75859	18	81915	9.91336	8	70021	9.84523	27	0.15477	1.4281	60
1	381	9.75877	18	899	9.91328	9	064	9.84550	26	0.15450	273	59
2	405	9.75895	18	882	9.91319	9	107	9.84576	27	0.15424	264	58
3	429	9.75913	18	865	9.91310	9	151	9.84603	27	0.15397	255	57
4	453	9.75931	18	848	9.91301	9	194	9.84630	27	0.15370	246	56
5	57477	9.75949	18	81832	9.91292	9	70238	9.84657	27	0.15343	1.4237	55
6	501	9.75967	18	815	9.91283	9	281	9.84684	27	0.15316	229	54
7	524	9.75985	18	798	9.91274	8	325	9.84711	27	0.15289	220	53
8	548	9.76003	18	782	9.91266	9	368	9.84738	26	0.15262	211	52
9	572	9.76021	18	765	9.91257	9	412	9.84764	27	0.15236	202	51
10	57596	9.76039	18	81748	9.91248	9	70455	9.84791	27	0.15209	1.4193	50
11	619	9.76057	18	731	9.91239	9	499	9.84818	27	0.15182	185	49
12	643	9.76075	18	714	9.91230	9	542	9.84845	27	0.15155	176	48
13	667	9.76093	18	698	9.91221	9	586	9.84872	27	0.15128	167	47
14	691	9.76111	18	681	9.91212	9	629	9.84899	26	0.15101	158	46
15	57715	9.76129	17	81664	9.91203	9	70673	9.84925	27	0.15075	1.4150	45
16	738	9.76146	18	647	9.91194	9	717	9.84952	27	0.15048	141	44
17	762	9.76164	18	631	9.91185	9	760	9.84979	27	0.15021	132	43
18	786	9.76182	18	614	9.91176	9	804	9.85006	27	0.14994	124	42
19	810	9.76200	18	597	9.91167	9	848	9.85033	26	0.14967	115	41
20	57833	9.76218	18	81580	9.91158	9	70891	9.85059	27	0.14941	1.4106	40
21	857	9.76236	17	563	9.91149	8	935	9.85086	27	0.14914	097	39
22	881	9.76253	18	546	9.91141	9	979	9.85113	27	0.14887	089	38
23	904	9.76271	18	530	9.91132	9	71023	9.85140	26	0.14860	080	37
24	928	9.76289	18	513	9.91123	9	066	9.85166	27	0.14834	071	36
25	57952	9.76307	17	81496	9.91114	9	71110	9.85193	27	0.14807	1.4063	35
26	976	9.76324	18	479	9.91105	9	154	9.85220	27	0.14780	054	34
27	999	9.76342	18	462	9.91096	9	198	9.85247	26	0.14753	045	33
28	58023	9.76360	18	445	9.91087	9	242	9.85273	27	0.14727	037	32
29	047	9.76378	17	428	9.91078	9	285	9.85300	27	0.14700	028	31
30	58070	9.76395	18	81412	9.91069	9	71329	9.85327	27	0.14673	1.4019	30
31	094	9.76413	18	395	9.91060	9	373	9.85354	26	0.14646	011	29
32	118	9.76431	17	378	9.91051	9	417	9.85380	27	0.14620	002	28
33	141	9.76448	18	361	9.91042	9	461	9.85407	27	0.14593	1.3994	27
34	165	9.76466	18	344	9.91033	10	505	9.85434	26	0.14566	985	26
35	58189	9.76484	17	81327	9.91023	9	71549	9.85460	27	0.14540	1.3976	25
36	212	9.76501	18	310	9.91014	9	593	9.85487	27	0.14513	968	24
37	236	9.76519	18	293	9.91005	9	637	9.85514	26	0.14486	959	23
38	260	9.76537	17	276	9.90996	9	681	9.85540	27	0.14460	951	22
39	283	9.76554	18	259	9.90987	9	725	9.85567	27	0.14433	942	21
40	58307	9.76572	18	81242	9.90978	9	71769	9.85594	26	0.14406	1.3934	20
41	330	9.76590	17	225	9.90969	9	813	9.85620	27	0.14380	925	19
42	354	9.76607	18	208	9.90960	9	857	9.85647	27	0.14353	916	18
43	378	9.76625	17	191	9.90951	9	901	9.85674	26	0.14326	908	17
44	401	9.76642	18	174	9.90942	9	946	9.85700	27	0.14300	899	16
45	58425	9.76660	17	81157	9.90933	9	71990	9.85727	27	0.14273	1.3891	15
46	449	9.76677	18	140	9.90924	9	72034	9.85754	26	0.14246	882	14
47	472	9.76695	17	123	9.90915	9	078	9.85780	27	0.14220	874	13
48	496	9.76712	18	106	9.90906	10	122	9.85807	27	0.14193	865	12
49	519	9.76730	17	089	9.90896	9	167	9.85834	26	0.14166	857	11
50	58543	9.76747	18	81072	9.90887	9	72211	9.85860	27	0.14140	1.3848	10
51	567	9.76765	17	055	9.90878	9	255	9.85887	26	0.14113	840	9
52	590	9.76782	18	038	9.90869	9	299	9.85913	27	0.14087	831	8
53	614	9.76800	17	021	9.90860	9	344	9.85940	27	0.14060	823	7
54	637	9.76817	18	004	9.90851	9	388	9.85967	26	0.14033	814	6
55	58661	9.76835	17	80987	9.90842	10	72432	9.85993	27	0.14007	1.3806	5
56	684	9.76852	18	970	9.90832	9	477	9.86020	26	0.13980	798	4
57	708	9.76870	17	953	9.90823	9	521	9.86046	27	0.13954	789	3
58	731	9.76887	17	936	9.90814	9	565	9.86073	27	0.13927	781	2
59	755	9.76904	18	919	9.90805	9	610	9.86100	26	0.13900	772	1
60	779	9.76922		902	9.90796		654	9.86126		0.13874	764	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	58779	9.76922	17	80902	9.90796	9	72654	9.86126	27	0.13874	1.3764	60
1	802	9.76939	18	885	9.90787	10	699	9.86153	26	0.13847	755	59
2	826	9.76957	17	867	9.90777	9	743	9.86179	27	0.13821	747	58
3	849	9.76974	17	850	9.90768	9	788	9.86206	26	0.13794	739	57
4	873	9.76991	18	833	9.90759	9	832	9.86232	27	0.13768	730	56
5	58896	9.77009	17	80816	9.90750	9	72877	9.86259	26	0.13741	1.3722	55
6	920	9.77026	17	799	9.90741	10	921	9.86285	27	0.13715	713	54
7	943	9.77043	18	782	9.90731	9	966	9.86312	26	0.13688	705	53
8	967	9.77061	17	765	9.90722	9	73010	9.86338	27	0.13662	697	52
9	990	9.77078	17	748	9.90713	9	055	9.86365	27	0.13635	688	51
10	59014	9.77095	17	80730	9.90704	10	73100	9.86392	26	0.13608	1.3680	50
11	037	9.77112	18	713	9.90694	9	144	9.86418	27	0.13582	672	49
12	061	9.77130	17	696	9.90685	9	189	9.86445	26	0.13555	663	48
13	084	9.77147	17	679	9.90676	9	234	9.86471	27	0.13529	655	47
14	108	9.77164	17	662	9.90667	10	278	9.86498	26	0.13502	647	46
15	59131	9.77181	18	80644	9.90657	9	73323	9.86524	27	0.13476	1.3638	45
16	154	9.77199	17	627	9.90648	9	368	9.86551	26	0.13449	630	44
17	178	9.77216	17	610	9.90639	9	413	9.86577	26	0.13423	622	43
18	201	9.77233	17	593	9.90630	10	457	9.86603	27	0.13397	613	42
19	225	9.77250	18	576	9.90620	9	502	9.86630	26	0.13370	605	41
20	59248	9.77268	17	80558	9.90611	9	73547	9.86656	27	0.13344	1.3597	40
21	272	9.77285	17	541	9.90602	10	592	9.86683	26	0.13317	588	39
22	295	9.77302	17	524	9.90592	9	637	9.86709	27	0.13291	580	38
23	318	9.77319	17	507	9.90583	9	681	9.86736	26	0.13264	572	37
24	342	9.77336	17	489	9.90574	9	726	9.86762	27	0.13238	564	36
25	59365	9.77353	17	80472	9.90565	10	73771	9.86789	26	0.13211	1.3555	35
26	389	9.77370	17	455	9.90555	9	816	9.86815	27	0.13185	547	34
27	412	9.77387	18	438	9.90546	9	861	9.86842	26	0.13158	539	33
28	436	9.77405	17	420	9.90537	10	906	9.86868	26	0.13132	531	32
29	459	9.77422	17	403	9.90527	9	951	9.86894	27	0.13106	522	31
30	59482	9.77439	17	80386	9.90518	9	73996	9.86921	26	0.13079	1.3514	30
31	506	9.77456	17	368	9.90509	10	74041	9.86947	27	0.13053	506	29
32	529	9.77473	17	351	9.90499	9	086	9.86974	26	0.13026	498	28
33	552	9.77490	17	334	9.90490	10	131	9.87000	27	0.13000	490	27
34	576	9.77507	17	316	9.90480	9	176	9.87027	26	0.12973	481	26
35	59599	9.77524	17	80299	9.90471	9	74221	9.87053	26	0.12947	1.3473	25
36	622	9.77541	17	282	9.90462	10	267	9.87079	27	0.12921	465	24
37	646	9.77558	17	264	9.90452	9	312	9.87106	26	0.12894	457	23
38	669	9.77575	17	247	9.90443	9	357	9.87132	26	0.12868	449	22
39	693	9.77592	17	230	9.90434	10	402	9.87158	27	0.12842	440	21
40	59716	9.77609	17	80212	9.90424	9	74447	9.87185	26	0.12815	1.3432	20
41	739	9.77626	17	195	9.90415	10	492	9.87211	27	0.12789	424	19
42	763	9.77643	17	178	9.90405	9	538	9.87238	26	0.12762	416	18
43	786	9.77660	17	160	9.90396	10	583	9.87264	26	0.12736	408	17
44	809	9.77677	17	143	9.90386	9	628	9.87290	27	0.12710	400	16
45	59832	9.77694	17	80125	9.90377	9	74674	9.87317	26	0.12683	1.3392	15
46	856	9.77711	17	108	9.90368	10	719	9.87343	26	0.12657	384	14
47	879	9.77728	16	091	9.90358	9	764	9.87369	27	0.12631	375	13
48	902	9.77744	17	073	9.90349	10	810	9.87396	26	0.12604	367	12
49	926	9.77761	17	056	9.90339	9	855	9.87422	26	0.12578	359	11
50	59949	9.77778	17	80038	9.90330	10	74900	9.87448	27	0.12552	1.3351	10
51	972	9.77795	17	021	9.90320	9	946	9.87475	26	0.12525	343	9
52	995	9.77812	17	003	9.90311	10	991	9.87501	26	0.12499	335	8
53	60019	9.77829	17	79986	9.90301	9	75037	9.87527	27	0.12473	327	7
54	042	9.77846	16	968	9.90292	10	082	9.87554	26	0.12446	319	6
55	60065	9.77862	17	79951	9.90282	9	75128	9.87580	26	0.12420	1.3311	5
56	089	9.77879	17	934	9.90273	10	173	9.87606	27	0.12394	303	4
57	112	9.77896	17	916	9.90263	9	219	9.87633	26	0.12367	295	3
58	135	9.77913	17	899	9.90254	10	264	9.87659	26	0.12341	287	2
59	158	9.77930	16	881	9.90244	9	310	9.87685	26	0.12315	278	1
60	182	9.77946		864	9.90235		355	9.87711		0.12289	270	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Sin Log.	d.	Nat. Cos	Cos Log.	d.	Nat. Tan	Tan Log.	c.d.	Cot Log.	Cot Nat.	
0	60182	9.77946	17	79864	9.90235	10	75355	9.87711	27	0.12289	1.3270	60
1	205	9.77963	17	846	9.90225	9	401	9.87738	26	0.12262	262	59
2	228	9.77980	17	829	9.90216	10	447	9.87764	26	0.12236	254	58
3	251	9.77997	16	811	9.90206	9	492	9.87790	27	0.12210	246	57
4	274	9.78013	17	793	9.90197	10	538	9.87817	26	0.12183	238	56
5	60298	9.78030	17	79776	9.90187	9	75584	9.87843	26	0.12157	1.3230	55
6	321	9.78047	16	758	9.90178	10	629	9.87869	26	0.12131	222	54
7	344	9.78063	17	741	9.90168	9	675	9.87895	27	0.12105	214	53
8	367	9.78080	17	723	9.90159	10	721	9.87922	26	0.12078	206	52
9	390	9.78097	16	706	9.90149	10	767	9.87948	26	0.12052	198	51
10	60414	9.78113	17	79688	9.90139	9	75812	9.87974	26	0.12026	1.3190	50
11	437	9.78130	17	671	9.90130	10	858	9.88000	27	0.12000	182	49
12	460	9.78147	16	653	9.90120	9	904	9.88027	26	0.11973	175	48
13	483	9.78163	17	635	9.90111	10	950	9.88053	26	0.11947	167	47
14	506	9.78180	17	618	9.90101	10	996	9.88079	26	0.11921	159	46
15	60529	9.78197	16	79600	9.90091	9	76042	9.88105	26	0.11895	1.3151	45
16	553	9.78213	17	583	9.90082	10	088	9.88131	27	0.11869	143	44
17	576	9.78230	16	565	9.90072	9	134	9.88158	26	0.11842	135	43
18	599	9.78246	17	547	9.90063	10	180	9.88184	26	0.11816	127	42
19	622	9.78263	17	530	9.90053	10	226	9.88210	26	0.11790	119	41
20	60645	9.78280	16	79512	9.90043	9	76272	9.88236	26	0.11764	1.3111	40
21	668	9.78296	17	494	9.90034	10	318	9.88262	27	0.11738	103	39
22	691	9.78313	16	477	9.90024	10	364	9.88289	26	0.11711	095	38
23	714	9.78329	17	459	9.90014	9	410	9.88315	26	0.11685	087	37
24	738	9.78346	16	441	9.90005	10	456	9.88341	26	0.11659	079	36
25	60761	9.78362	17	79424	9.89995	10	76502	9.88367	26	0.11633	1.3072	35
26	784	9.78379	16	406	9.89985	9	548	9.88393	27	0.11607	064	34
27	807	9.78395	17	388	9.89976	10	594	9.88420	26	0.11580	056	33
28	830	9.78412	16	371	9.89966	10	640	9.88446	26	0.11554	048	32
29	853	9.78428	17	353	9.89956	9	686	9.88472	26	0.11528	040	31
30	60876	9.78445	16	79335	9.89947	10	76733	9.88498	26	0.11502	1.3032	30
31	899	9.78461	17	318	9.89937	10	779	9.88524	26	0.11476	024	29
32	922	9.78478	16	300	9.89927	9	825	9.88550	27	0.11450	017	28
33	945	9.78494	16	282	9.89918	10	871	9.88577	26	0.11423	009	27
34	968	9.78510	17	264	9.89908	10	918	9.88603	26	0.11397	001	26
35	60991	9.78527	16	79247	9.89898	10	76964	9.88629	26	0.11371	1.2993	25
36	61015	9.78543	17	229	9.89888	9	77010	9.88655	26	0.11345	985	24
37	038	9.78560	16	211	9.89879	10	057	9.88681	26	0.11319	977	23
38	061	9.78576	16	193	9.89869	10	103	9.88707	26	0.11293	970	22
39	084	9.78592	17	176	9.89859	10	149	9.88733	26	0.11267	962	21
40	61107	9.78609	16	79158	9.89849	9	77196	9.88759	27	0.11241	1.2954	20
41	130	9.78625	17	140	9.89840	10	242	9.88786	26	0.11214	946	19
42	153	9.78642	16	122	9.89830	10	289	9.88812	26	0.11188	938	18
43	176	9.78658	16	105	9.89820	10	335	9.88838	26	0.11162	931	17
44	199	9.78674	17	087	9.89810	9	382	9.88864	26	0.11136	923	16
45	61222	9.78691	16	79069	9.89801	10	77428	9.88890	26	0.11110	1.2915	15
46	245	9.78707	16	051	9.89791	10	475	9.88916	26	0.11084	907	14
47	268	9.78723	16	033	9.89781	10	521	9.88942	26	0.11058	900	13
48	291	9.78739	17	016	9.89771	10	568	9.88968	26	0.11032	892	12
49	314	9.78756	16	78998	9.89761	9	615	9.88994	26	0.11006	884	11
50	61337	9.78772	16	78980	9.89752	10	77661	9.89020	26	0.10980	1.2876	10
51	360	9.78788	17	962	9.89742	10	708	9.89046	27	0.10954	869	9
52	383	9.78805	16	944	9.89732	10	754	9.89073	26	0.10927	861	8
53	406	9.78821	16	926	9.89722	10	801	9.89099	26	0.10901	853	7
54	429	9.78837	16	908	9.89712	10	848	9.89125	26	0.10875	846	6
55	61451	9.78853	16	78891	9.89702	9	77895	9.89151	26	0.10849	1.2838	5
56	474	9.78869	17	873	9.89693	10	941	9.89177	26	0.10823	830	4
57	497	9.78886	16	855	9.89683	10	988	9.89203	26	0.10797	822	3
58	520	9.78902	16	837	9.89673	10	78035	9.89229	26	0.10771	815	2
59	543	9.78918	16	819	9.89663	10	082	9.89255	26	0.10745	807	1
60	566	9.78934		801	9.89653		129	9.89281		0.10719	799	0
	Nat. Cos	Cos Log.	d.	Nat. Sin	Sin Log.	d.	Nat. Cot	Cot Log.	c.d.	Tan Log.	Tan Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	61566	9.78934	16	78801	9.89653	10	78129	9.89281	26	0.10719	1.2799	60
1	589	9.78950	17	783	9.89643	10	175	9.89307	26	0.10693	792	59
2	612	9.78967	16	765	9.89633	9	222	9.89333	26	0.10667	784	58
3	635	9.78983	16	747	9.89624	10	269	9.89359	26	0.10641	776	57
4	658	9.78999	16	729	9.89614	10	316	9.89385	26	0.10615	769	56
5	61681	9.79015	16	78711	9.89604	10	78363	9.89411	26	0.10589	1.2761	55
6	704	9.79031	16	694	9.89594	10	410	9.89437	26	0.10563	753	54
7	726	9.79047	16	676	9.89584	10	457	9.89463	26	0.10537	746	53
8	749	9.79063	16	658	9.89574	10	504	9.89489	26	0.10511	738	52
9	772	9.79079	16	640	9.89564	10	551	9.89515	26	0.10485	731	51
10	61795	9.79095	16	78622	9.89554	10	78598	9.89541	26	0.10459	1.2723	50
11	818	9.79111	17	604	9.89544	10	645	9.89567	26	0.10433	715	49
12	841	9.79128	16	586	9.89534	10	692	9.89593	26	0.10407	708	48
13	864	9.79144	16	568	9.89524	10	739	9.89619	26	0.10381	700	47
14	887	9.79160	16	550	9.89514	10	786	9.89645	26	0.10355	693	46
15	61909	9.79176	16	78532	9.89504	9	78834	9.89671	26	0.10329	1.2685	45
16	932	9.79192	16	514	9.89495	10	881	9.89697	26	0.10303	677	44
17	955	9.79208	16	496	9.89485	10	928	9.89723	26	0.10277	670	43
18	978	9.79224	16	478	9.89475	10	975	9.89749	26	0.10251	662	42
19	62001	9.79240	16	460	9.89465	10	79022	9.89775	26	0.10225	655	41
20	62024	9.79256	16	78442	9.89455	10	79070	9.89801	26	0.10199	1.2647	40
21	046	9.79272	16	424	9.89445	10	117	9.89827	26	0.10173	640	39
22	069	9.79288	16	405	9.89435	10	164	9.89853	26	0.10147	632	38
23	092	9.79304	15	387	9.89425	10	212	9.89879	26	0.10121	624	37
24	115	9.79319	16	369	9.89415	10	259	9.89905	26	0.10095	617	36
25	62138	9.79335	16	78351	9.89405	10	79306	9.89931	26	0.10069	1.2609	35
26	160	9.79351	16	333	9.89395	10	354	9.89957	26	0.10043	602	34
27	183	9.79367	16	315	9.89385	10	401	9.89983	26	0.10017	594	33
28	206	9.79383	16	297	9.89375	11	449	9.90009	26	0.09991	587	32
29	229	9.79399	16	279	9.89364	10	496	9.90035	26	0.09965	579	31
30	62251	9.79415	16	78261	9.89354	10	79544	9.90061	25	0.09939	1.2572	30
31	274	9.79431	16	243	9.89344	10	591	9.90086	26	0.09914	564	29
32	297	9.79447	16	225	9.89334	10	639	9.90112	26	0.09888	557	28
33	320	9.79463	15	206	9.89324	10	686	9.90138	26	0.09862	549	27
34	342	9.79478	16	188	9.89314	10	734	9.90164	26	0.09836	542	26
35	62365	9.79494	16	78170	9.89304	10	79781	9.90190	26	0.09810	1.2534	25
36	388	9.79510	16	152	9.89294	10	829	9.90216	26	0.09784	527	24
37	411	9.79526	16	134	9.89284	10	877	9.90242	26	0.09758	519	23
38	433	9.79542	16	116	9.89274	10	924	9.90268	26	0.09732	512	22
39	456	9.79558	15	098	9.89264	10	972	9.90294	26	0.09706	504	21
40	62479	9.79573	16	78079	9.89254	10	80020	9.90320	26	0.09680	1.2497	20
41	502	9.79589	16	061	9.89244	11	067	9.90346	25	0.09654	489	19
42	524	9.79605	16	043	9.89233	10	115	9.90371	26	0.09629	482	18
43	547	9.79621	15	025	9.89223	10	163	9.90397	26	0.09603	475	17
44	570	9.79636	16	007	9.89213	10	211	9.90423	26	0.09577	467	16
45	62592	9.79652	16	77988	9.89203	10	80258	9.90449	26	0.09551	1.2460	15
46	615	9.79668	16	970	9.89193	10	306	9.90475	26	0.09525	452	14
47	638	9.79684	15	952	9.89183	10	354	9.90501	26	0.09499	445	13
48	660	9.79699	16	934	9.89173	11	402	9.90527	26	0.09473	437	12
49	683	9.79715	16	916	9.89162	10	450	9.90553	25	0.09447	430	11
50	62706	9.79731	15	77897	9.89152	10	80498	9.90578	26	0.09422	1.2423	10
51	728	9.79746	16	879	9.89142	10	546	9.90604	26	0.09396	415	9
52	751	9.79762	16	861	9.89132	10	594	9.90630	26	0.09370	408	8
53	774	9.79778	15	843	9.89122	10	642	9.90656	26	0.09344	401	7
54	796	9.79793	16	824	9.89112	11	690	9.90682	26	0.09318	393	6
55	62819	9.79809	16	77806	9.89101	10	80738	9.90708	26	0.09292	1.2386	5
56	842	9.79825	15	788	9.89091	10	786	9.90734	25	0.09266	378	4
57	864	9.79840	16	769	9.89081	10	834	9.90759	26	0.09241	371	3
58	887	9.79856	16	751	9.89071	11	882	9.90785	26	0.09215	364	2
59	909	9.79872	15	733	9.89060	10	930	9.90811	26	0.09189	356	1
60	932	9.79887		715	9.89050		978	9.90837		0.09163	349	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	62932	9.79887	16	77715	9.89050	10	80978	9.90837	26	0.09163	1.2349	60
1	955	9.79903	15	696	9.89040	10	81027	9.90863	26	0.09137	342	59
2	977	9.79918	16	678	9.89030	10	075	9.90889	25	0.09111	334	58
3	63000	9.79934	16	660	9.89020	11	123	9.90914	26	0.09086	327	57
4	022	9.79950	15	641	9.89009	10	171	9.90940	26	0.09060	320	56
5	63045	9.79965	16	77623	9.88999	10	81220	9.90966	26	0.09034	1.2312	55
6	068	9.79981	15	605	9.88989	11	268	9.90992	26	0.09008	305	54
7	090	9.79996	16	586	9.88978	10	316	9.91018	25	0.08982	298	53
8	113	9.80012	15	568	9.88968	10	364	9.91043	26	0.08957	290	52
9	135	9.80027	16	550	9.88958	10	413	9.91069	26	0.08931	283	51
10	63158	9.80043	15	77531	9.88948	11	81461	9.91095	26	0.08905	1.2276	50
11	180	9.80058	16	513	9.88937	10	510	9.91121	26	0.08879	268	49
12	203	9.80074	15	494	9.88927	10	558	9.91147	25	0.08853	261	48
13	225	9.80089	16	476	9.88917	11	606	9.91172	26	0.08828	254	47
14	248	9.80105	15	458	9.88906	10	655	9.91198	26	0.08802	247	46
15	63271	9.80120	16	77439	9.88896	10	81703	9.91224	26	0.08776	1.2239	45
16	293	9.80136	15	421	9.88886	11	752	9.91250	26	0.08750	232	44
17	316	9.80151	15	402	9.88875	10	800	9.91276	25	0.08724	225	43
18	338	9.80166	16	384	9.88865	10	849	9.91301	26	0.08699	218	42
19	361	9.80182	15	366	9.88855	11	898	9.91327	26	0.08673	210	41
20	63383	9.80197	16	77347	9.88844	10	81946	9.91353	26	0.08647	1.2203	40
21	406	9.80213	15	329	9.88834	10	995	9.91379	25	0.08621	196	39
22	428	9.80228	16	310	9.88824	11	82044	9.91404	26	0.08596	189	38
23	451	9.80244	15	292	9.88813	10	092	9.91430	26	0.08570	181	37
24	473	9.80259	15	273	9.88803	10	141	9.91456	26	0.08544	174	36
25	63496	9.80274	16	77255	9.88793	11	82190	9.91482	25	0.08518	1.2167	35
26	518	9.80290	15	236	9.88782	10	238	9.91507	26	0.08493	160	34
27	540	9.80305	15	218	9.88772	11	287	9.91533	26	0.08467	153	33
28	563	9.80320	16	199	9.88761	10	336	9.91559	26	0.08441	145	32
29	585	9.80336	15	181	9.88751	10	385	9.91585	25	0.08415	138	31
30	63608	9.80351	15	77162	9.88741	11	82434	9.91610	26	0.08390	1.2131	30
31	630	9.80366	16	144	9.88730	10	483	9.91636	26	0.08364	124	29
32	653	9.80382	15	125	9.88720	11	531	9.91662	26	0.08338	117	28
33	675	9.80397	15	107	9.88709	10	580	9.91688	25	0.08312	109	27
34	698	9.80412	16	088	9.88699	11	629	9.91713	26	0.08287	102	26
35	63720	9.80428	15	77070	9.88688	10	82678	9.91739	26	0.08261	1.2095	25
36	742	9.80443	15	051	9.88678	10	727	9.91765	26	0.08235	088	24
37	765	9.80458	15	033	9.88668	11	776	9.91791	25	0.08209	081	23
38	787	9.80473	16	014	9.88657	10	825	9.91816	26	0.08184	074	22
39	810	9.80489	15	76996	9.88647	11	874	9.91842	26	0.08158	066	21
40	63832	9.80504	15	76977	9.88636	10	82923	9.91868	25	0.08132	1.2059	20
41	854	9.80519	15	959	9.88626	11	972	9.91893	26	0.08107	052	19
42	877	9.80534	16	940	9.88615	10	83022	9.91919	26	0.08081	045	18
43	899	9.80550	15	921	9.88605	11	071	9.91945	26	0.08055	038	17
44	922	9.80565	15	903	9.88594	10	120	9.91971	25	0.08029	031	16
45	63944	9.80580	15	76884	9.88584	11	83169	9.91996	26	0.08004	1.2024	15
46	966	9.80595	15	866	9.88573	10	218	9.92022	26	0.07978	017	14
47	989	9.80610	15	847	9.88563	11	268	9.92048	25	0.07952	009	13
48	64011	9.80625	16	828	9.88552	10	317	9.92073	26	0.07927	002	12
49	033	9.80641	15	810	9.88542	11	366	9.92099	26	0.07901	1.1995	11
50	64056	9.80656	15	76791	9.88531	10	83415	9.92125	25	0.07875	1.1988	10
51	078	9.80671	15	772	9.88521	11	465	9.92150	26	0.07850	981	9
52	100	9.80686	15	754	9.88510	11	514	9.92176	26	0.07824	974	8
53	123	9.80701	15	735	9.88499	10	564	9.92202	25	0.07798	967	7
54	145	9.80716	15	717	9.88489	11	613	9.92227	26	0.07773	960	6
55	64167	9.80731	15	76698	9.88478	10	83662	9.92253	26	0.07747	1.1953	5
56	190	9.80746	16	679	9.88468	11	712	9.92279	25	0.07721	946	4
57	212	9.80762	15	661	9.88457	10	761	9.92304	26	0.07696	939	3
58	234	9.80777	15	642	9.88447	11	811	9.92330	26	0.07670	932	2
59	256	9.80792	15	623	9.88436	11	860	9.92356	25	0.07644	925	1
60	279	9.80807		604	9.88425		910	9.92381		0.07619	918	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	64279	9.80807	15	76604	9.88425	10	83910	9.92381	26	0.07619	1.1918	60
1	301	9.80822	15	586	9.88415	11	960	9.92407	26	0.07593	910	59
2	323	9.80837	15	567	9.88404	10	84009	9.92433	25	0.07567	903	58
3	346	9.80852	15	548	9.88394	11	059	9.92458	26	0.07542	896	57
4	368	9.80867	15	530	9.88383	11	108	9.92484	26	0.07516	889	56
5	64390	9.80882	15	76511	9.88372	10	84158	9.92510	25	0.07490	1.1882	55
6	412	9.80897	15	492	9.88362	11	208	9.92535	26	0.07465	875	54
7	435	9.80912	15	473	9.88351	11	258	9.92561	26	0.07439	868	53
8	457	9.80927	15	455	9.88340	10	307	9.92587	25	0.07413	861	52
9	479	9.80942	15	436	9.88330	11	357	9.92612	26	0.07388	854	51
10	64501	9.80957	15	76417	9.88319	11	84407	9.92638	25	0.07362	1.1847	50
11	524	9.80972	15	398	9.88308	10	457	9.92663	26	0.07337	840	49
12	546	9.80987	15	380	9.88298	11	507	9.92689	26	0.07311	833	48
13	568	9.81002	15	361	9.88287	11	556	9.92715	25	0.07285	826	47
14	590	9.81017	15	342	9.88276	10	606	9.92740	26	0.07260	819	46
15	64612	9.81032	15	76323	9.88266	11	84656	9.92766	26	0.07234	1.1812	45
16	635	9.81047	14	304	9.88255	11	706	9.92792	25	0.07208	806	44
17	657	9.81061	15	286	9.88244	10	756	9.92817	26	0.07183	799	43
18	679	9.81076	15	267	9.88234	11	806	9.92843	25	0.07157	792	42
19	701	9.81091	15	248	9.88223	11	856	9.92868	26	0.07132	785	41
20	64723	9.81106	15	76229	9.88212	11	84906	9.92894	26	0.07106	1.1778	40
21	746	9.81121	15	210	9.88201	10	956	9.92920	25	0.07080	771	39
22	768	9.81136	15	192	9.88191	11	85006	9.92945	26	0.07055	764	38
23	790	9.81151	15	173	9.88180	11	057	9.92971	25	0.07029	757	37
24	812	9.81166	14	154	9.88169	11	107	9.92996	26	0.07004	750	36
25	64834	9.81180	15	76135	9.88158	10	85157	9.93022	26	0.06978	1.1743	35
26	856	9.81195	15	116	9.88148	11	207	9.93048	25	0.06952	736	34
27	878	9.81210	15	097	9.88137	11	257	9.93073	26	0.06927	729	33
28	901	9.81225	15	078	9.88126	11	308	9.93099	25	0.06901	722	32
29	923	9.81240	14	059	9.88115	10	358	9.93124	26	0.06876	715	31
30	64945	9.81254	15	76041	9.88105	11	85408	9.93150	25	0.06850	1.1708	30
31	967	9.81269	15	022	9.88094	11	458	9.93175	26	0.06825	702	29
32	989	9.81284	15	003	9.88083	11	509	9.93201	26	0.06799	695	28
33	65011	9.81299	15	75984	9.88072	11	559	9.93227	25	0.06773	688	27
34	033	9.81314	14	965	9.88061	10	609	9.93252	26	0.06748	681	26
35	65055	9.81328	15	75946	9.88051	11	85660	9.93278	25	0.06722	1.1674	25
36	077	9.81343	15	927	9.88040	11	710	9.93303	26	0.06697	667	24
37	100	9.81358	14	908	9.88029	11	761	9.93329	25	0.06671	660	23
38	122	9.81372	15	889	9.88018	11	811	9.93354	26	0.06646	653	22
39	144	9.81387	15	870	9.88007	11	862	9.93380	26	0.06620	647	21
40	65166	9.81402	15	75851	9.87996	11	85912	9.93406	25	0.06594	1.1640	20
41	188	9.81417	14	832	9.87985	10	963	9.93431	26	0.06569	633	19
42	210	9.81431	15	813	9.87975	11	86014	9.93457	25	0.06543	626	18
43	232	9.81446	15	794	9.87964	11	064	9.93482	26	0.06518	619	17
44	254	9.81461	14	775	9.87953	11	115	9.93508	25	0.06492	612	16
45	65276	9.81475	15	75756	9.87942	11	86166	9.93533	26	0.06467	1.1606	15
46	298	9.81490	15	738	9.87931	11	216	9.93559	25	0.06441	599	14
47	320	9.81505	14	719	9.87920	11	267	9.93584	26	0.06416	592	13
48	342	9.81519	15	700	9.87909	11	318	9.93610	26	0.06390	585	12
49	364	9.81534	15	680	9.87898	11	368	9.93636	25	0.06364	578	11
50	65386	9.81549	14	75661	9.87887	10	86419	9.93661	26	0.06339	1.1571	10
51	408	9.81563	15	642	9.87877	11	470	9.93687	25	0.06313	565	9
52	430	9.81578	14	623	9.87866	11	521	9.93712	26	0.06288	558	8
53	452	9.81592	15	604	9.87855	11	572	9.93738	25	0.06262	551	7
54	474	9.81607	15	585	9.87844	11	623	9.93763	26	0.06237	544	6
55	65496	9.81622	14	75566	9.87833	11	86674	9.93789	25	0.06211	1.1538	5
56	518	9.81636	15	547	9.87822	11	725	9.93814	26	0.06186	531	4
57	540	9.81651	14	528	9.87811	11	776	9.93840	25	0.06160	524	3
58	562	9.81665	15	509	9.87800	11	827	9.93865	26	0.06135	517	2
59	584	9.81680	14	490	9.87789	11	878	9.93891	25	0.06109	510	1
60	606	9.81694		471	9.87778		929	9.93916		0.06084	504	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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41°

′	Nat. Sin	Log. Sin	d.	Nat. Cos	Log. Cos	d.	Nat. Tan	Log. Tan	c.d.	Log. Cot	Nat. Cot	
0	65606	9.81694	15	75471	9.87778	11	86929	9.93916	26	0.06084	1.1504	60
1	628	9.81709	14	452	9.87767	11	980	9.93942	25	0.06058	497	59
2	650	9.81723	15	433	9.87756	11	87031	9.93967	26	0.06033	490	58
3	672	9.81738	14	414	9.87745	11	082	9.93993	25	0.06007	483	57
4	694	9.81752	15	395	9.87734	11	133	9.94018	26	0.05982	477	56
5	65716	9.81767	14	75375	9.87723	11	87184	9.94044	25	0.05956	1.1470	55
6	738	9.81781	15	356	9.87712	11	236	9.94069	26	0.05931	463	54
7	759	9.81796	14	337	9.87701	11	287	9.94095	25	0.05905	456	53
8	781	9.81810	15	318	9.87690	11	338	9.94120	26	0.05880	450	52
9	803	9.81825	14	299	9.87679	11	389	9.94146	25	0.05854	443	51
10	65825	9.81839	15	75280	9.87668	11	87441	9.94171	26	0.05829	1.1436	50
11	847	9.81854	14	261	9.87657	11	492	9.94197	25	0.05803	430	49
12	869	9.81868	14	241	9.87646	11	543	9.94222	26	0.05778	423	48
13	891	9.81882	15	222	9.87635	11	595	9.94248	25	0.05752	416	47
14	913	9.81897	14	203	9.87624	11	646	9.94273	26	0.05727	410	46
15	65935	9.81911	15	75184	9.87613	12	87698	9.94299	25	0.05701	1.1403	45
16	956	9.81926	14	165	9.87601	11	749	9.94324	26	0.05676	396	44
17	978	9.81940	15	146	9.87590	11	801	9.94350	25	0.05650	389	43
18	66000	9.81955	14	126	9.87579	11	852	9.94375	26	0.05625	383	42
19	022	9.81969	14	107	9.87568	11	904	9.94401	25	0.05599	376	41
20	66044	9.81983	15	75088	9.87557	11	87955	9.94426	26	0.05574	1.1369	40
21	066	9.81998	14	069	9.87546	11	88007	9.94452	25	0.05548	363	39
22	088	9.82012	14	050	9.87535	11	059	9.94477	26	0.05523	356	38
23	109	9.82026	15	030	9.87524	11	110	9.94503	25	0.05497	349	37
24	131	9.82041	14	011	9.87513	12	162	9.94528	26	0.05472	343	36
25	66153	9.82055	14	74992	9.87501	11	88214	9.94554	25	0.05446	1.1336	35
26	175	9.82069	15	973	9.87490	11	265	9.94579	25	0.05421	329	34
27	197	9.82084	14	953	9.87479	11	317	9.94604	26	0.05396	323	33
28	218	9.82098	14	934	9.87468	11	369	9.94630	25	0.05370	316	32
29	240	9.82112	14	915	9.87457	11	421	9.94655	26	0.05345	310	31
30	66262	9.82126	15	74896	9.87446	12	88473	9.94681	25	0.05319	1.1303	30
31	284	9.82141	14	876	9.87434	11	524	9.94706	26	0.05294	296	29
32	306	9.82155	14	857	9.87423	11	576	9.94732	25	0.05268	290	28
33	327	9.82169	15	838	9.87412	11	628	9.94757	26	0.05243	283	27
34	349	9.82184	14	818	9.87401	11	680	9.94783	25	0.05217	276	26
35	66371	9.82198	14	74799	9.87390	12	88732	9.94808	26	0.05192	1.1270	25
36	393	9.82212	14	780	9.87378	11	784	9.94834	25	0.05166	263	24
37	414	9.82226	14	760	9.87367	11	836	9.94859	25	0.05141	257	23
38	436	9.82240	15	741	9.87356	11	888	9.94884	26	0.05116	250	22
39	458	9.82255	14	722	9.87345	11	940	9.94910	25	0.05090	243	21
40	66480	9.82269	14	74703	9.87334	12	88992	9.94935	26	0.05065	1.1237	20
41	501	9.82283	14	683	9.87322	11	89045	9.94961	25	0.05039	230	19
42	523	9.82297	14	664	9.87311	11	097	9.94986	26	0.05014	224	18
43	545	9.82311	15	644	9.87300	12	149	9.95012	25	0.04988	217	17
44	566	9.82326	14	625	9.87288	11	201	9.95037	25	0.04963	211	16
45	66588	9.82340	14	74606	9.87277	11	89253	9.95062	26	0.04938	1.1204	15
46	610	9.82354	14	586	9.87266	11	306	9.95088	25	0.04912	197	14
47	632	9.82368	14	567	9.87255	12	358	9.95113	26	0.04887	191	13
48	653	9.82382	14	548	9.87243	11	410	9.95139	25	0.04861	184	12
49	675	9.82396	14	528	9.87232	11	463	9.95164	26	0.04836	178	11
50	66697	9.82410	14	74509	9.87221	12	89515	9.95190	25	0.04810	1.1171	10
51	718	9.82424	15	489	9.87209	11	567	9.95215	25	0.04785	165	9
52	740	9.82439	14	470	9.87198	11	620	9.95240	26	0.04760	158	8
53	762	9.82453	14	451	9.87187	12	672	9.95266	25	0.04734	152	7
54	783	9.82467	14	431	9.87175	11	725	9.95291	26	0.04709	145	6
55	66805	9.82481	14	74412	9.87164	11	89777	9.95317	25	0.04683	1.1139	5
56	827	9.82495	14	392	9.87153	12	830	9.95342	26	0.04658	132	4
57	848	9.82509	14	373	9.87141	11	883	9.95368	25	0.04632	126	3
58	870	9.82523	14	353	9.87130	11	935	9.95393	25	0.04607	119	2
59	891	9.82537	14	334	9.87119	12	988	9.95418	26	0.04582	113	1
60	913	9.82551		314	9.87107		90040	9.95444		0.04556	106	0
	Nat. Cos	Log. Cos	d.	Nat. Sin	Log. Sin	d.	Nat. Cot	Log. Cot	c.d.	Log. Tan	Nat. Tan	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	66913	9.82551	14	74314	9.87107	11	90040	9.95444	25	0.04556	1.1106	60
1	935	9.82565	14	295	9.87096	11	093	9.95469	26	0.04531	100	59
2	956	9.82579	14	276	9.87085	12	146	9.95495	25	0.04505	093	58
3	978	9.82593	14	256	9.87073	11	199	9.95520	25	0.04480	087	57
4	999	9.82607	14	237	9.87062	12	251	9.95545	26	0.04455	080	56
5	67021	9.82621	14	74217	9.87050	11	90304	9.95571	25	0.04429	1.1074	55
6	043	9.82635	14	198	9.87039	11	357	9.95596	26	0.04404	067	54
7	064	9.82649	14	178	9.87028	12	410	9.95622	25	0.04378	061	53
8	086	9.82663	14	159	9.87016	11	463	9.95647	25	0.04353	054	52
9	107	9.82677	14	139	9.87005	12	516	9.95672	26	0.04328	048	51
10	67129	9.82691	14	74120	9.86993	11	90569	9.95698	25	0.04302	1.1041	50
11	151	9.82705	14	100	9.86982	12	621	9.95723	25	0.04277	035	49
12	172	9.82719	14	080	9.86970	11	674	9.95748	26	0.04252	028	48
13	194	9.82733	14	061	9.86959	12	727	9.95774	25	0.04226	022	47
14	215	9.82747	14	041	9.86947	11	781	9.95799	26	0.04201	016	46
15	67237	9.82761	14	74022	9.86936	12	90834	9.95825	25	0.04175	1.1009	45
16	258	9.82775	13	002	9.86924	11	887	9.95850	25	0.04150	003	44
17	280	9.82788	14	73983	9.86913	11	940	9.95875	26	0.04125	1.0996	43
18	301	9.82802	14	963	9.86902	12	993	9.95901	25	0.04099	990	42
19	323	9.82816	14	944	9.86890	11	91046	9.95926	26	0.04074	983	41
20	67344	9.82830	14	73924	9.86879	12	91099	9.95952	25	0.04048	1.0977	40
21	366	9.82844	14	904	9.86867	12	153	9.95977	25	0.04023	971	39
22	387	9.82858	14	885	9.86855	11	206	9.96002	26	0.03998	964	38
23	409	9.82872	13	865	9.86844	12	259	9.96028	25	0.03972	958	37
24	430	9.82885	14	846	9.86832	11	313	9.96053	25	0.03947	951	36
25	67452	9.82899	14	73826	9.86821	12	91366	9.96078	26	0.03922	1.0945	35
26	473	9.82913	14	806	9.86809	11	419	9.96104	25	0.03896	939	34
27	495	9.82927	14	787	9.86798	12	473	9.96129	26	0.03871	932	33
28	516	9.82941	14	767	9.86786	11	526	9.96155	25	0.03845	926	32
29	538	9.82955	13	747	9.86775	12	580	9.96180	25	0.03820	919	31
30	67559	9.82968	14	73728	9.86763	11	91633	9.96205	26	0.03795	1.0913	30
31	580	9.82982	14	708	9.86752	12	687	9.96231	25	0.03769	907	29
32	602	9.82996	14	688	9.86740	12	740	9.96256	25	0.03744	900	28
33	623	9.83010	13	669	9.86728	11	794	9.96281	26	0.03719	894	27
34	645	9.83023	14	649	9.86717	12	847	9.96307	25	0.03693	888	26
35	67666	9.83037	14	73629	9.86705	11	91901	9.96332	25	0.03668	1.0881	25
36	688	9.83051	14	610	9.86694	12	955	9.96357	26	0.03643	875	24
37	709	9.83065	13	590	9.86682	12	92008	9.96383	25	0.03617	869	23
38	730	9.83078	14	570	9.86670	11	062	9.96408	25	0.03592	862	22
39	752	9.83092	14	551	9.86659	12	116	9.96433	26	0.03567	856	21
40	67773	9.83106	14	73531	9.86647	12	92170	9.96459	25	0.03541	1.0850	20
41	795	9.83120	13	511	9.86635	11	224	9.96484	26	0.03516	843	19
42	816	9.83133	14	491	9.86624	12	277	9.96510	25	0.03490	837	18
43	837	9.83147	14	472	9.86612	12	331	9.96535	25	0.03465	831	17
44	859	9.83161	13	452	9.86600	11	385	9.96560	26	0.03440	824	16
45	67880	9.83174	14	73432	9.86589	12	92439	9.96586	25	0.03414	1.0818	15
46	901	9.83188	14	413	9.86577	12	493	9.96611	25	0.03389	812	14
47	923	9.83202	13	393	9.86565	11	547	9.96636	26	0.03364	805	13
48	944	9.83215	14	373	9.86554	12	601	9.96662	25	0.03338	799	12
49	965	9.83229	13	353	9.86542	12	655	9.96687	25	0.03313	793	11
50	67987	9.83242	14	73333	9.86530	12	92709	9.96712	26	0.03288	1.0786	10
51	68008	9.83256	14	314	9.86518	11	763	9.96738	25	0.03262	780	9
52	029	9.83270	13	294	9.86507	12	817	9.96763	25	0.03237	774	8
53	051	9.83283	14	274	9.86495	12	872	9.96788	26	0.03212	768	7
54	072	9.83297	13	254	9.86483	11	926	9.96814	25	0.03186	761	6
55	68093	9.83310	14	73234	9.86472	12	92980	9.96839	25	0.03161	1.0755	5
56	115	9.83324	14	215	9.86460	12	93034	9.96864	26	0.03136	749	4
57	136	9.83338	13	195	9.86448	12	088	9.96890	25	0.03110	742	3
58	157	9.83351	14	175	9.86436	11	143	9.96915	25	0.03085	736	2
59	179	9.83365	13	155	9.86425	12	197	9.96940	26	0.03060	730	1
60	200	9.83378		135	9.86413		252	9.96966		0.03034	724	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

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′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	68200	9.83378	14	73135	9.86413	12	93252	9.96966	25	0.03034	1.0724	60
1	221	9.83392	13	116	9.86401	12	306	9.96991	25	0.03009	717	59
2	242	9.83405	14	096	9.86389	12	360	9.97016	26	0.02984	711	58
3	264	9.83419	13	076	9.86377	11	415	9.97042	25	0.02958	705	57
4	285	9.83432	14	056	9.86366	12	469	9.97067	25	0.02933	699	56
5	68306	9.83446	13	73036	9.86354	12	93524	9.97092	26	0.02908	1.0692	55
6	327	9.83459	14	016	9.86342	12	578	9.97118	25	0.02882	686	54
7	349	9.83473	13	72996	9.86330	12	633	9.97143	25	0.02857	680	53
8	370	9.83486	14	976	9.86318	12	688	9.97168	25	0.02832	674	52
9	391	9.83500	13	957	9.86306	11	742	9.97193	26	0.02807	668	51
10	68412	9.83513	14	72937	9.86295	12	93797	9.97219	25	0.02781	1.0661	50
11	434	9.83527	13	917	9.86283	12	852	9.97244	25	0.02756	655	49
12	455	9.83540	14	897	9.86271	12	906	9.97269	26	0.02731	649	48
13	476	9.83554	13	877	9.86259	12	961	9.97295	25	0.02705	643	47
14	497	9.83567	14	857	9.86247	12	94016	9.97320	25	0.02680	637	46
15	68518	9.83581	13	72837	9.86235	12	94071	9.97345	26	0.02655	1.0630	45
16	539	9.83594	14	817	9.86223	12	125	9.97371	25	0.02629	624	44
17	561	9.83608	13	797	9.86211	11	180	9.97396	25	0.02604	618	43
18	582	9.83621	13	777	9.86200	12	235	9.97421	26	0.02579	612	42
19	603	9.83634	14	757	9.86188	12	290	9.97447	25	0.02553	606	41
20	68624	9.83648	13	72737	9.86176	12	94345	9.97472	25	0.02528	1.0599	40
21	645	9.83661	13	717	9.86164	12	400	9.97497	26	0.02503	593	39
22	666	9.83674	14	697	9.86152	12	455	9.97523	25	0.02477	587	38
23	688	9.83688	13	677	9.86140	12	510	9.97548	25	0.02452	581	37
24	709	9.83701	14	657	9.86128	12	565	9.97573	25	0.02427	575	36
25	68730	9.83715	13	72637	9.86116	12	94620	9.97598	26	0.02402	1.0569	35
26	751	9.83728	13	617	9.86104	12	676	9.97624	25	0.02376	562	34
27	772	9.83741	14	597	9.86092	12	731	9.97649	25	0.02351	556	33
28	793	9.83755	13	577	9.86080	12	786	9.97674	26	0.02326	550	32
29	814	9.83768	13	557	9.86068	12	841	9.97700	25	0.02300	544	31
30	68835	9.83781	14	72537	9.86056	12	94896	9.97725	25	0.02275	1.0538	30
31	857	9.83795	13	517	9.86044	12	952	9.97750	26	0.02250	532	29
32	878	9.83808	13	497	9.86032	12	95007	9.97776	25	0.02224	526	28
33	899	9.83821	13	477	9.86020	12	062	9.97801	25	0.02199	519	27
34	920	9.83834	14	457	9.86008	12	118	9.97826	25	0.02174	513	26
35	68941	9.83848	13	72437	9.85996	12	95173	9.97851	26	0.02149	1.0507	25
36	962	9.83861	13	417	9.85984	12	229	9.97877	25	0.02123	501	24
37	983	9.83874	13	397	9.85972	12	284	9.97902	25	0.02098	495	23
38	69004	9.83887	14	377	9.85960	12	340	9.97927	26	0.02073	489	22
39	025	9.83901	13	357	9.85948	12	395	9.97953	25	0.02047	483	21
40	69046	9.83914	13	72337	9.85936	12	95451	9.97978	25	0.02022	1.0477	20
41	067	9.83927	13	317	9.85924	12	506	9.98003	26	0.01997	470	19
42	088	9.83940	14	297	9.85912	12	562	9.98029	25	0.01971	464	18
43	109	9.83954	13	277	9.85900	12	618	9.98054	25	0.01946	458	17
44	130	9.83967	13	257	9.85888	12	673	9.98079	25	0.01921	452	16
45	69151	9.83980	13	72236	9.85876	12	95729	9.98104	26	0.01896	1.0446	15
46	172	9.83993	13	216	9.85864	13	785	9.98130	25	0.01870	440	14
47	193	9.84006	14	196	9.85851	12	841	9.98155	25	0.01845	434	13
48	214	9.84020	13	176	9.85839	12	897	9.98180	26	0.01820	428	12
49	235	9.84033	13	156	9.85827	12	952	9.98206	25	0.01794	422	11
50	69256	9.84046	13	72136	9.85815	12	96008	9.98231	25	0.01769	1.0416	10
51	277	9.84059	13	116	9.85803	12	064	9.98256	25	c.01744	410	9
52	298	9.84072	13	095	9.85791	12	120	9.98281	26	0.01719	404	8
53	319	9.84085	13	075	9.85779	13	176	9.98307	25	0.01693	398	7
54	340	9.84098	14	055	9.85766	12	232	9.98332	25	0.01668	392	6
55	69361	9.84112	13	72035	9.85754	12	96288	9.98357	26	0.01643	1.0385	5
56	382	9.84125	13	015	9.85742	12	344	9.98383	25	0.01617	379	4
57	403	9.84138	13	71995	9.85730	12	400	9.98408	25	0.01592	373	3
58	424	9.84151	13	974	9.85718	12	457	9.98433	25	0.01567	367	2
59	445	9.84164	13	954	9.85706	13	513	9.98458	26	0.01542	361	1
60	466	9.84177		934	9.85693		569	9.98484		0.01516	355	0
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′

46°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	d.	Nat. Tan	Log.	c.d.	Log. Cot	Nat.	
0	69466	9.84177	13	71934	9.85693	12	96569	9.98484	25	0.01516	1.0355	**60**
1	487	9.84190	13	914	9.85681	12	625	9.98509	25	0.01491	349	59
2	508	9.84203	13	894	9.85669	12	681	9.98534	26	0.01466	343	58
3	529	9.84216	13	873	9.85657	12	738	9.98560	25	0.01440	337	57
4	549	9.84229	13	853	9.85645	13	794	9.98585	25	0.01415	331	56
5	69570	9.84242	13	71833	9.85632	12	96850	9.98610	25	0.01390	1.0325	**55**
6	591	9.84255	14	813	9.85620	12	907	9.98635	26	0.01365	319	54
7	612	9.84269	13	792	9.85608	12	963	9.98661	25	0.01339	313	53
8	633	9.84282	13	772	9.85596	13	97020	9.98686	25	0.01314	307	52
9	654	9.84295	13	752	9.85583	12	076	9.98711	26	0.01289	301	51
10	69675	9.84308	13	71732	9.85571	12	97133	9.98737	25	0.01263	1.0295	**50**
11	696	9.84321	13	711	9.85559	12	189	9.98762	25	0.01238	289	49
12	717	9.84334	13	691	9.85547	13	246	9.98787	25	0.01213	283	48
13	737	9.84347	13	671	9.85534	12	302	9.98812	26	0.01188	277	47
14	758	9.84360	13	650	9.85522	12	359	9.98832	25	0.01162	271	46
15	69779	9.84373	12	71630	9.85510	13	97416	9.98863	25	0.01137	1.0265	**45**
16	800	9.84385	13	610	9.85497	12	472	9.98888	25	0.01112	259	44
17	821	9.84398	13	590	9.85485	12	529	9.98913	26	0.01087	253	43
18	842	9.84411	13	569	9.85473	13	586	9.98939	25	0.01061	247	42
19	862	9.84424	13	549	9.85460	12	643	9.98964	25	0.01036	241	41
20	69883	9.84437	13	71529	9.85448	12	97700	9.98989	26	0.01011	1.0235	**40**
21	904	9.84450	13	508	9.85436	13	756	9.99015	25	0.00985	230	39
22	925	9.84463	13	488	9.85423	12	813	9.99040	25	0.00960	224	38
23	946	9.84476	13	468	9.85411	12	870	9.99065	25	0.00935	218	37
24	966	9.84489	13	447	9.85399	13	927	9.99090	26	0.00910	212	36
25	69987	9.84502	13	71427	9.85386	12	97984	9.99116	25	0.00884	1.0206	**35**
26	70008	9.84515	13	407	9.85374	13	98041	9.99141	25	0.00859	200	34
27	029	9.84528	12	386	9.85361	12	098	9.99166	25	0.00834	194	33
28	049	9.84540	13	366	9.85349	12	155	9.99191	26	0.00809	188	32
29	070	9.84553	13	345	9.85337	13	213	9.99217	25	0.00783	182	31
30	70091	9.84566	13	71325	9.85324	12	98270	9.99242	25	0.00758	1.0176	**30**
31	112	9.84579	13	305	9.85312	13	327	9.99267	26	0.00733	170	29
32	132	9.84592	13	284	9.85299	12	384	9.99293	25	0.00707	164	28
33	153	9.84605	13	264	9.85287	13	441	9.99318	25	0.00682	158	27
34	174	9.84618	12	243	9.85274	12	499	9.99343	25	0.00657	152	26
35	70195	9.84630	13	71223	9.85262	12	98556	9.99368	26	0.00632	1.0147	**25**
36	215	9.84643	13	203	9.85250	13	613	9.99394	25	0.00606	141	24
37	236	9.84656	13	182	9.85237	12	671	9.99419	25	0.00581	135	23
38	257	9.84669	13	162	9.85225	13	728	9.99444	25	0.00556	129	22
39	277	9.84682	12	141	9.85212	12	786	9.99469	26	0.00531	123	21
40	70298	9.84694	13	71121	9.85200	13	98843	9.99495	25	0.00505	1.0117	**20**
41	319	9.84707	13	100	9.85187	12	901	9.99520	25	0.00480	111	19
42	339	9.84720	13	080	9.85175	13	958	9.99545	25	0.00455	105	18
43	360	9.84733	12	059	9.85162	12	99016	9.99570	26	0.00430	099	17
44	381	9.84745	13	039	9.85150	13	073	9.99596	25	0.00404	094	16
45	70401	9.84758	13	71019	9.85137	12	99131	9.99621	25	0.00379	1.0088	**15**
46	422	9.84771	13	70998	9.85125	13	189	9.99646	26	0.00354	082	14
47	443	9.84784	12	978	9.85112	12	247	9.99672	25	0.00328	076	13
48	463	9.84796	13	957	9.85100	13	304	9.99697	25	0.00303	070	12
49	484	9.84809	13	937	9.85087	13	362	9.99722	25	0.00278	064	11
50	70505	9.84822	13	70916	9.85074	12	99420	9.99747	26	0.00253	1.0058	**10**
51	525	9.84835	12	896	9.85062	13	478	9.99773	25	0.00227	052	9
52	546	9.84847	13	875	9.85049	12	536	9.99798	25	0.00202	047	8
53	567	9.84860	13	855	9.85037	13	594	9.99823	25	0.00177	041	7
54	587	9.84873	12	834	9.85024	12	652	9.99848	26	0.00152	035	6
55	70608	9.84885	13	70813	9.85012	13	99710	9.99874	25	0.00126	1.0029	**5**
56	628	9.84898	13	793	9.84999	13	768	9.99899	25	0.00101	023	4
57	649	9.84911	12	772	9.84986	12	826	9.99924	25	0.00076	017	3
58	670	9.84923	13	752	9.84974	13	884	9.99949	26	0.00051	012	2
59	690	9.84936	13	731	9.84961	12	942	9.99975	25	0.00025	006	1
60	711	9.84949		711	9.84949		10000	0.00000		0.00000	000	**0**
	Nat. Cos	Log.	d.	Nat. Sin	Log.	d.	Nat. Cot	Log.	c.d.	Log. Tan	Nat.	′