Florian Cajori

PLANE AND SPHERICAL TRIGONOMETRY

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PLANE AND SPHERICAL TRIGONOMETRY

BY

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PREFACE

Of late years, in the writing of textbooks of trigonometry, a tendency to amplification has shown itself, doubtless with the idea that amplification means simplification. Unfortunately the amplification has spent itself upon details rather than upon principles, which latter have too often been inadequately treated. The result has been textbooks which overlook the comparative maturity of the boys and girls who study trigonometry and which cling almost with affection to the practices of the most elementary mathematics.

The present text aims to present the trigonometry in such a way as to make it interesting to students approaching some maturity, and so as to connect the subject, not only with the mathematics which the student has already had, but also with the mathematics which, in many cases at least, is to follow. A subject may be so burdened with detailed explanations as to become monotonous and lifeless, or, on the other hand, presented in so concise and difficult a manner as to be repellent. The present work endeavors to avoid both extremes. Full explanations are given of important principles, but many simple details are left to the work of the student.

The following points in the text may be noted:

1. Positive and negative angles of any magnitude and the trigonometric functions of such angles, defined by means of a system of rectangular coördinates, are taken up in the beginning of the book; acute angles, with their functions, being mentioned as a special case.

2. Thus the basic trigonometric identities are got at once for all angles.

3. The functions of 0°, 90°, etc., are carefully explained by the theory of limits.

4. The solution of right triangles and related problems are taken up early without the use of logarithms.

5. Logarithms are then very carefully explained and fully discussed, not so much as to their use in computation, but rather so as to clarify their meaning.

6. Right triangles are then solved by the use of logarithms, and the essentially approximate nature of all numerical results is emphasized.

7. The text next returns to trigonometric identities, giving a detailed and accurate proof of the addition formulæ for sines and cosines, with less detailed but sufficient explanation of other fundamental identities. The number of identities to be memorized is reduced to a minimum.

8. The circular measure of an angle and the inverse functions are then taken up, emphasis being laid upon the fact that the latter are angles.

9. There follows the solution of triangles in general. As each case is mentioned the theorems or formulæ needed for its solution are derived.

10. The last subject treated in the plane trigonometry is the solution of trigonometric equations, and the fact is emphasized that the operations are simply the solution of algebraic equations applied to a new class of quantities.

11. The lists of examples and problems are numerous and carefully chosen, many of them being taken from work in analytic geometry and calculus, though, of course, no knowledge of either of these subjects is assumed. Some of the problems are entirely new, being invented for this text, and all problems are chosen with a purpose to indicate the practical interest and value of trigonometry.

12. In the spherical trigonometry, as in the plane, the three chief aims are brevity, clarity, and simplicity; a chapter on the Earth treated as a sphere being given to enliven an otherwise somewhat formal and lifeless subject.

13. The author has *not* tried to revolutionize the teaching

of trigonometry, believing that much that has been done in the past is good though none the less open to improvement. Such improvement has been the aim of this work.

The author wishes to acknowledge the kindness of his colleagues Professor H. W. Tyler, Professor F. L. Hitchcock, and Professor J. Lipka in reading and criticizing the manuscript of his book, and to express his thanks to Professor E. R. Hedrick, editor of the tables appended, for permission to make use of them.

L. M. Passano.

CONTENTS

INTRODUCTION

TRIGONOMETRY is primarily the science concerned with the measurement of plane and spherical triangles, that is, with the determination of three of the parts of such triangles when the numerical values of the other three parts are given. This is done by means of the six trigonometric functions, defined in article 4 following. But these functions enter so intimately into many branches of mathematical and physical science not directly concerned with the measurement of angles, that their analytical properties are of fundamental importance. Analytical trigonometry, that is, the proof and use of various algebraic relations among the trigonometric functions of the same or related angles, is therefore, in modern times, of equal importance with the trigonometry which deals with triangular solutions.

The same functions which enable one to solve triangles constructed in a plane suffice also for the solution of spherical triangles. But the solution of triangles of which the sides are geodetic lines, that is, lines which are the shortest distances between pairs of points on the surface, on a spheroidal surface such as the Earth, requires the use of other functions than those needed for the solution of plane or spherical triangles. This spheroidal trigonometry is very complex, and becomes necessary only in the accurate survey of very large tracts of the Earth's surface. For ordinary purposes of surveying and for the solution of triangles on the Earth's surface over small areas, plane and spherical trigonometry are sufficient.

The study of trigonometry, as ancillary to astronomy, dates from very early times. Among the Greeks, who, however, were more famous as geometers than as investigators in other branches of mathematics, the names of Hipparchus (about 150 B.C.) and of Ptolemy (who lived in the second century of the Christian era), both astronomers, are prominent. Hipparchus left no mathematical writings, but we are told by an ancient writer that he created the science of trigonometry. Ptolemy, making use of the investigations and discoveries of Hipparchus, perfected the form of the science. The theorems of these two astronomers are still the basis of trigonometry.

Ptolemy calculated a table of chords, which were used in those earliest days of the science, as we now use the sines of angles. The radius of a circle he divided into sixty equal parts. Each of these he divided again into sixty equal parts, called, in the Latin translation of his work the Almagest, "**partes minutae primae**"; and each of these in turn into sixty, called "**partes minutae secundae**"; whence have come the names "minutes" and "seconds" for the subdivisions of the angular degree. Ptolemy, however, was not the first to calculate a table of chords, Hipparchus, among others, having done so previously, but he invented theorems by means of which the calculations could be more readily made.

The Hindus, more skillful calculators than the Greeks, acquired the knowledge of the latter and improved upon it, notably in that they calculated tables of the half-chord, or sine, instead of the whole chord of the angle. The Arabs also were acquainted with the Almagest, and with the investigations of the Hindus. It was an Arab, Al Battani or Albategnius, who first calculated a table of what may be called cotangents, by computing the lengths of shadows of a vertical object cast by the sun at different altitudes. Another Arab invented, as a separate function, the tangent, which had previously been used only as an abbreviation of the ratio sine to cosine. Curiously enough this invention was afterwards forgotten until the tangent was re-invented in England in the fourteenth century by Bradwardine, and in the fifteenth century by the German, Johannes Müller, called Regiomontanus, who wrote the first complete European treatise on trigonometry.

When Napier* invented logarithms, in 1614, they were at once adopted in trigonometric calculations, and the first tables of logarithmic sines and tangents were made by Edmund Gunter, an English astronomer (1581–1626). He it was who first used the names cosine, cotangent, and cosecant. During the following century the science of trigonometry progressed slowly, becoming more analytical in form, until, in the hands of Euler (1707–1783), it became essentially what it is at the present day.

With this brief introduction to the history of trigonometry let us now proceed to become acquainted with that homely, perhaps, but most serviceable handmaid to so many of the arts and sciences,

> " . . . being just as great, no doubt,
> Useful to men, and dear to God, as they!"

* John Napier, 1550–1617.

PLANE TRIGONOMETRY

CHAPTER I

THE TRIGONOMETRIC FUNCTIONS OF ANY ANGLE, AND IDENTICAL RELATIONS AMONG THEM

1. Rectangular Coördinates. Two lines, $x'x$ and $y'y$, drawn in a plane at right angles to each other, as in Fig. 1, form a system of rectangular, Cartesian coördinates. The point O in which the lines intersect is called the origin; the two

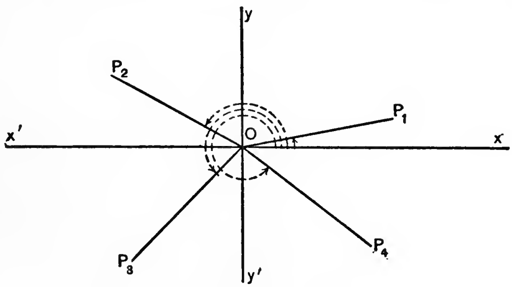

FIG. 1.

lines are called the axes of coördinates. One of these, usually the horizontal line, is called the axis of abscissæ, or the axis of x. The other is called the axis of ordinates, or the axis of y. We shall speak of XOY, YOX', $X'OY'$, and $Y'OX$ as the first, second, third, and fourth quadrants respectively.

2. Angles of any Magnitude. There are many ways in which a system of coördinates is used in mathematics. In trigonometry such a system is used primarily in defining the trigonometric functions, but before we proceed to do so we shall extend our ideas of angles beyond the knowledge we obtained of them in the elementary geometry. There an angle is defined by some such definition as the following: the plane figure formed by two straight lines drawn from the same point. The unit of angles is either the right angle, or the degree, and the largest angle usually dealt with is equivalent to two right angles and is often called a straight angle. In trigonometry, on the other hand, we deal with angles of any magnitude whatever. To do so we introduce the idea of motion, of revolution. Starting from the initial position OX, Fig. 1, we may revolve the line about O in the direction indicated by the arrows, stopping in any desired terminal position OP_1, OP_2, OP_3, OP_4, etc. In this way angles of any number of degrees whatever may be generated. Thus, if we stop in the position OY, we have an angle of 90°; in the position OX', 180°; in the position OP_3, 225°; in the position OY', 270°, and so on. By making one whole revolution we should arrive at an angle of 360°; two and one half revolutions, 900°; etc.

Not only so, but we might revolve from the initial position OX in the opposite direction. Now oppositeness is indicated algebraically by the use of the signs *plus* (+) and *minus* (−). So that if we agree to take the positive direction of revolution counterclockwise, then clockwise will be the negative direction and we can thus generate negative angles of any magnitude whatever. Thus, Fig. 1, the angle XOP_3 is 225° if we have revolved in the positive direction, but is −135° if we have revolved in the negative direction. When an angle lies in value between 0° and 90° it is said to be an angle in the first quadrant since its terminal side lies in the first quadrant. An angle lying in value between 90° and 180° is said to be in the second quadrant; between 180° and 270°, in the third quadrant; between 270° and 360°, in the fourth quadrant.

EXAMPLES

Construct the angles

1. 300°.
2. – 210°.
3. 750°.
4. – 495°.
5. – 1215°.
6. 420°.

Add the following angles graphically:

7. 720° and 30°.
8. – 180° and 60°.
9. – 90° and – 45°.
10. 990° and – 60°.
11. – 45° and 120°.
12. 135° and – 450°.

If A is a positive angle in the first, second, third, or fourth quadrant respectively, add graphically

13. 450° and A.
14. – 270° and A.
15. 180° and – A.
16. – 540° and – A.

3. Abscissa, Ordinate, and Distance. Consider an angle, positive or negative, of any magnitude whatever*, XOP, of

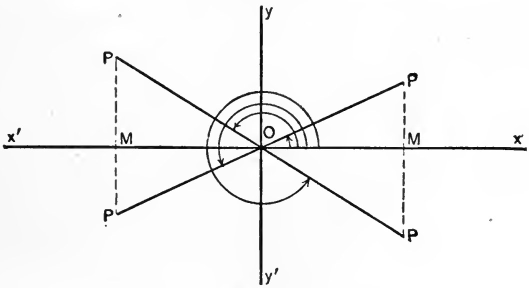

Fig. 2.

Fig. 2. From P, any point in the terminal side of this angle, drop a perpendicular upon the axis of x. The lines of the figure are named as follows: OM is called the abscissa of the point P, MP the ordinate, and OP the distance. The abscissa OM and the ordinate MP are together called the coördinates of the point P. Note very carefully that the abscissa is always read from O to M, the ordinate from M to P; that is, in each case *from* the axis *to* the

* As a matter of convenience we do not consider angles numerically greater than 360°. It is obvious that the discussion applies equally well to such angles.

point. The distance is read from O to P. Thus, for an angle in the second or third quadrant the direction of the abscissa is opposite to that of an angle in the first or fourth quadrant. For an angle in the third or fourth quadrant the direction of the ordinate is opposite to that of the ordinate of an angle in the first or second quadrant. Oppositeness in direction being distinguished as usual by difference in algebraic sign we have the following conventions:

The abscissa measured to the right of the axis of y is positive; to the left, negative. The ordinate measured upward from the axis of x is positive; downward, negative. The distance is measured from the origin outward and is taken positive.

EXAMPLES

1. The abscissa of a point is 3, its ordinate 4; find the distance.
2. The distance of a point is 5, its ordinate 4; find the abscissa.
3. The ordinate of a point is -2, its distance 3; find the abscissa.
4. The ordinate of a point is -5, its abscissa -4; find the distance.
5. Prove that the square of the distance of any point is equal to the sum of the squares of the abscissa and ordinate.
6. Prove that for all points on a straight line through the origin the ratio of the ordinate to the abscissa is constant.

4. The Trigonometric Functions Defined. Let us now proceed to define the six trigonometric functions of an angle; six quantities which depend upon the angle for their values. They are the possible ratios between the various pairs of the three lines named in Art. 3. Thus, Fig. 2, the

$$\text{sine } XOP = \frac{\text{ordinate of } P}{\text{distance of } P} = \frac{MP}{OP},$$

$$\text{cosine } XOP = \frac{\text{abscissa of } P}{\text{distance of } P} = \frac{OM}{OP},$$

$$\text{tangent } XOP = \frac{\text{ordinate of } P}{\text{abscissa of } P} = \frac{MP}{OM},$$

$$\text{cotangent } XOP = \frac{\text{abscissa of } P}{\text{ordinate of } P} = \frac{OM}{MP},$$

$$\text{secant } XOP = \frac{\text{distance of } P}{\text{abscissa of } P} = \frac{OP}{OM},$$

$$\text{cosecant } XOP = \frac{\text{distance of } P}{\text{ordinate of } P} = \frac{OP}{MP}.$$

Three other functions are sometimes used: The versed sine, which is unity minus the cosine; the coversed sine, which is unity minus the sine; the suversed sine, which is unity plus the cosine. They are relatively unimportant.

5. Trigonometric Functions are Ratios. The first thing to be noted about these functions is that, being ratios, they are independent of the actual lengths of the abscissa,

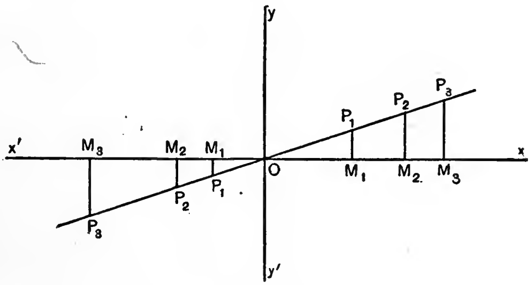

FIG. 3.

ordinate, and distance. Thus, Fig. 3, the triangles OM_1P_1, OM_2P_2, and OM_3P_3 being similar, their homologous sides are proportional, so that

$$\sin XOP = \frac{M_1P_1}{OP_1} = \frac{M_2P_2}{OP_2} = \frac{M_3P_3}{OP_3},$$

$$\tan XOP = \frac{M_1P_1}{OM_1} = \frac{M_2P_2}{OM_2} = \frac{M_3P_3}{OM_3}.$$

Similarly the truth of the statement may be shown for the remaining functions.

6. Signs of the Functions. The second point to be noted is that the signs of the functions vary according to the quadrant in which the angle lies. Thus, Fig. 2, for the angle XOP in the first quadrant the abscissa, ordinate and distance are all positive so that all the functions are positive. For the angle XOP in the second quadrant the ordinate and distance are positive, the abscissa negative. Thus we have for the angle in the second quadrant

$$\sin XOP = \frac{MP}{OP} = \frac{+}{+} = +,$$

$$\cos XOP = \frac{OM}{OP} = \frac{-}{+} = -,$$

$$\tan XOP = \frac{MP}{OM} = \frac{+}{-} = -,$$

$$\cot XOP = \frac{OM}{MP} = \frac{-}{+} = -,$$

$$\sec XOP = \frac{OP}{OM} = \frac{+}{-} = -,$$

$$\csc XOP = \frac{OP}{MP} = \frac{+}{+} = +.$$

The following table gives the signs of the functions in the four quadrants.

Quad.	I	II	III	IV	Quad.
sine	+	+	−	−	cosecant
cosine	+	−	−	+	secant
tangent	+	−	+	−	cotangent

EXAMPLES

Determine the algebraic signs of

1. cos 218°. **3.** sin 1100°. **5.** sec 315°.

2. tan (− 460°). **4.** cot (− 99°). **6.** csc (− 210°).

7. Let the student determine, as above, the signs of the trigonometric functions of angles in the third and fourth quadrants.

7. Functions of Acute Angles. A special set of definitions for the functions of acute angles, which are sometimes useful and should be known, follows directly as a special

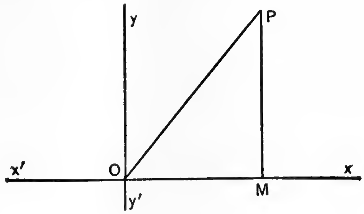

FIG. 4.

case of the general definitions given above. Thus, Fig. 4, in which the angle XOP lies in a right triangle,

$$\sin XOP = \frac{\text{ordinate}}{\text{distance}} = \frac{\text{opposite side}}{\text{hypotenuse}},$$

$$\cos XOP = \frac{\text{abscissa}}{\text{distance}} = \frac{\text{adjacent side}}{\text{hypotenuse}},$$

$$\tan XOP = \frac{\text{ord.}}{\text{abs.}} = \frac{\text{opp. side}}{\text{adj. side}},$$

$$\cot XOP = \frac{\text{abs.}}{\text{ord.}} = \frac{\text{adj. side}}{\text{opp. side}},$$

$$\sec XOP = \frac{\text{dist.}}{\text{abs.}} = \frac{\text{hypot.}}{\text{adj. side}},$$

$$\csc XOP = \frac{\text{dist.}}{\text{ord.}} = \frac{\text{hypot.}}{\text{opp. side}}.$$

These definitions, it must be noted, completely agree with the more general definitions, but are applicable only to angles less than ninety degrees, since angles greater than ninety degrees cannot occur in right triangles.

8. Reciprocal Functions. Two questions would naturally suggest themselves at this point: Are the trigonometric functions of an angle related to each other in any particular way? and, second, if there be a definite relation between two given angles will the functions of those angles bear some special relation to each other? We shall proceed to answer the first of these questions affirmatively, but shall leave the discussion of the second question to a later chapter (Chap. II). Thus, if α be any angle, it follows by the definitions of the trigonometric functions that

$$\sin\alpha = \frac{\text{ordinate}}{\text{distance}} = \frac{1}{\dfrac{\text{dist.}}{\text{ord.}}} = \frac{1}{\csc\alpha},$$

$$\cos\alpha = \frac{\text{abscissa}}{\text{distance}} = \frac{1}{\dfrac{\text{dist.}}{\text{abs.}}} = \frac{1}{\sec\alpha},$$

$$\tan\alpha = \frac{\text{ordinate}}{\text{abscissa}} = \frac{1}{\dfrac{\text{abs.}}{\text{ord.}}} = \frac{1}{\cot\alpha},$$

or, the *sine* and *cosecant*, the *cosine* and *secant*, the *tangent* and *cotangent* respectively of the same angle *are reciprocals of each other.*

9. Tangent, Sine and Cosine. Again, by definition, and by Art. 8,

$$\tan\alpha = \frac{\text{ordinate}}{\text{abscissa}} = \frac{\dfrac{\text{ordinate}}{\text{distance}}}{\dfrac{\text{abscissa}}{\text{distance}}} = \frac{\sin\alpha}{\cos\alpha},$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{\dfrac{\sin\alpha}{\cos\alpha}} = \frac{\cos\alpha}{\sin\alpha}.$$

These relations may be proved otherwise, thus, Fig. 5:

$$\tan XOP = \frac{MP}{OM} = \frac{\dfrac{MP}{OP}}{\dfrac{OM}{OP}} = \frac{\sin XOP}{\cos XOP},$$

$$\cot XOP = \frac{OM}{MP} = \frac{\frac{OM}{OP}}{\frac{MP}{OP}} = \frac{\cos XOP}{\sin XOP}.$$

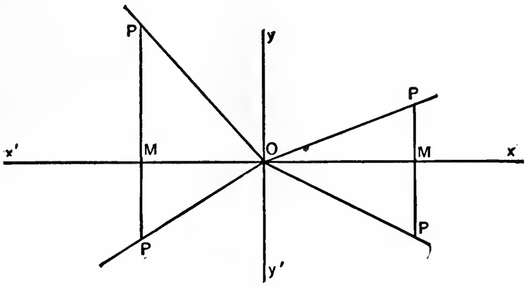

FIG. 5.

10. Sine and Cosine. Also, Fig. 5, it is obvious that

$$\overline{MP}^2 + \overline{OM}^2 = \overline{OP}^2.$$

Dividing each term by $\overline{OP}^2$ gives

$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1.$$

Whence, by definition,

$$(\sin XOP)^2 + (\cos XOP)^2 = 1$$

or, as it is usually written, letting $\alpha = \angle XOP$,

$$\sin^2 \alpha + \cos^2 \alpha = 1.$$

11. Tangent and Secant. Similarly, writing the first equation of Art. 10 in the form

$$\overline{OP}^2 = \overline{MP}^2 + \overline{OM}^2$$

and dividing each term by $\overline{OM}^2$, we have

$$\left(\frac{OP}{OM}\right)^2 = \left(\frac{MP}{OM}\right)^2 + 1.$$

That is $(\sec XOP)^2 = (\tan XOP)^2 + 1$

or,
$$\sec^2 \alpha = \tan^2 \alpha + 1.$$

In the same way we obtain the relation

$$\csc^2 \alpha = \cot^2 \alpha + 1.$$

12. Fundamental Relations. These relations, summarized below, are of great importance and must be memorized.

$$\sin \alpha = \frac{1}{\csc \alpha}, \quad \cos \alpha = \frac{1}{\sec \alpha}, \quad \tan \alpha = \frac{1}{\cot \alpha}, \qquad (1)$$

$$\csc \alpha = \frac{1}{\sin \alpha}, \quad \sec \alpha = \frac{1}{\cos \alpha}, \quad \cot \alpha = \frac{1}{\tan \alpha}.$$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}, \quad \cot \alpha = \frac{\cos \alpha}{\sin \alpha}. \qquad (2)$$

$$\sin^2 \alpha + \cos^2 \alpha = 1. \qquad (3)$$

$$\sec^2 \alpha = 1 + \tan^2 \alpha, \qquad \csc^2 \alpha = 1 + \cot^2 \alpha. \qquad (4)$$

13. By means of the identities of Art. 12 the value of any one of the trigonometric functions may be expressed in terms of each of the other five. Thus, by (3)

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha}.$$

By (2), (1), and (4),

$$\sin \alpha = \tan \alpha \cdot \cos \alpha = \frac{\tan \alpha}{\sec \alpha} = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}},$$

where the radical may be either plus or minus.

14. To Compute the Values of the Other Functions when One Function of an Angle is Given. By means of the relations of the preceding article if the value of any one function of an angle be given, the values of the remaining functions may be found, but a simpler method of obtaining them is illustrated by the following examples.

Example 1. Given $\sin A = -\frac{2}{3}$, find the values of the remaining functions.

$$\sin A = \frac{\text{ord.}}{\text{dist.}} = -\frac{m \cdot 2}{m \cdot 3} = -\frac{2}{3} = \frac{-2}{3}.$$

The distance being always positive, the minus sign necessarily is taken with the ordinate. Therefore, Fig. 6, construct an angle whose ordinate is -2 and whose distance is 3, or any multiple (m) of -2 and 3. The third side of the right triangle is $\pm\sqrt{9-4} = \pm\sqrt{5}$. This is the value of the abscissa and we may write the values of the six functions from the definitions.

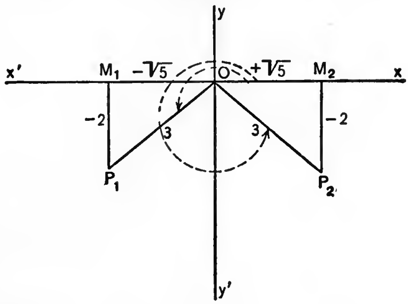

Fig. 6.

$\sin XOP_1 = -\frac{2}{3}$,	$\sin XOP_2 = -\frac{2}{3}$,
$\cos XOP_1 = -\frac{\sqrt{5}}{3}$,	$\cos XOP_2 = \frac{\sqrt{5}}{3}$,
$\tan XOP_1 = \frac{2}{\sqrt{5}}$,	$\tan XOP_2 = -\frac{2}{\sqrt{5}}$,
$\cot XOP_1 = \frac{\sqrt{5}}{2}$,	$\cot XOP_2 = -\frac{\sqrt{5}}{2}$,
$\sec XOP_1 = -\frac{3}{\sqrt{5}}$,	$\sec XOP_2 = \frac{3}{\sqrt{5}}$,
$\csc XOP_1 = -\frac{3}{2}$.	$\csc XOP_2 = -\frac{3}{2}$.

Example 2. Given $\cot A = -\frac{5}{4}$, find the remaining functions.

$$\cot A = \frac{\text{abs.}}{\text{ord.}} = -\frac{m \cdot 5}{m \cdot 4} = -\frac{5}{4} = \frac{-5}{4} \text{ or } \frac{5}{-4},$$

since either the abscissa or the ordinate may be negative.

Construct an angle XOP_1, Fig. 7, having an abscissa -5 and an ordinate $+4$, and an angle XOP_2 having an abscissa

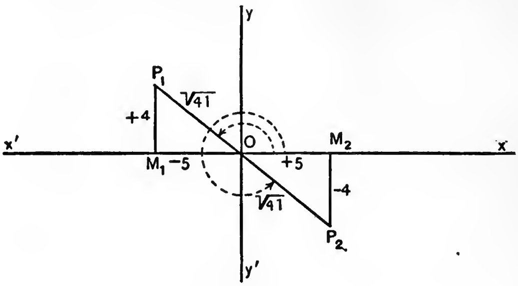

FIG. 7.

$+5$ and an ordinate -4. In each case the distance is found to be $\sqrt{25+16} = \sqrt{41}$, and we may write

$$\sin XOP_1 = \frac{4}{\sqrt{41}}, \qquad \sin XOP_2 = -\frac{4}{\sqrt{41}},$$

$$\cos XOP_1 = -\frac{5}{\sqrt{41}}, \qquad \cos XOP_2 = \frac{5}{\sqrt{41}},$$

$$\tan XOP_1 = -\frac{4}{5}, \qquad \tan XOP_2 = -\frac{4}{5},$$

$$\cot XOP_1 = -\frac{5}{4}, \qquad \cot XOP_2 = -\frac{5}{4},$$

$$\sec XOP_1 = -\frac{\sqrt{41}}{5}, \qquad \sec XOP_2 = \frac{\sqrt{41}}{5},$$

$$\csc XOP_1 = \frac{\sqrt{41}}{4}. \qquad \csc XOP_2 = -\frac{\sqrt{41}}{4}.$$

It will be seen that the ambiguity of the two sets of values will occur in every case, no matter what function be given and no matter whether the sign of the given function be plus or minus.

EXAMPLES

Find the values of the remaining functions, given that

1. $\sin\alpha = \frac{1}{2}$.
2. $\cos\alpha = -\frac{3}{8}$.
3. $\cot\alpha = -3$.
4. $\tan\alpha = \frac{9}{4}$.
5. $\sec\alpha = 4$.
6. $\csc\alpha = -\frac{25}{7}$.

7. If $\sin x = 5$, can the values of the remaining functions be found? Why?

8. If $\sec x = \frac{1}{2}$, can the values of the remaining functions be found? Why?

9. If $\tan x = -4$, can the values of the remaining functions be found? Why?

10. Given $\sec\alpha = \frac{5}{4}$, find the functions of $90° - \alpha$.

11. Given $\cot\alpha = x$, find the functions of $90° - \alpha$.

Prove the following relations:

12. $\cos\alpha = \dfrac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}$.

13. $\cot\alpha = \dfrac{1}{\sqrt{\sec^2\alpha - 1}}$.

14. $\sec\alpha = \dfrac{\csc\alpha}{\sqrt{\csc^2\alpha - 1}}$.

15. $\sin\alpha = \dfrac{1}{\sqrt{1+\cot^2\alpha}}$.

16. $\tan\alpha = \dfrac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$.

17. $\tan\alpha = \dfrac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$.

18. $\cot\alpha = \dfrac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}$.

19. $\csc\alpha = \dfrac{\sqrt{1+\tan^2\alpha}}{\tan\alpha}$.

20. $\dfrac{\cos\alpha}{1-\sin\alpha} = \dfrac{1+\sin\alpha}{\cos\alpha}$.

21. $\dfrac{\tan\alpha}{\sec\alpha} = \sin\alpha$.

22. $\dfrac{\sec^2\alpha}{\sin^2\alpha} = \csc^2\alpha + \sec^2\alpha$.

23. $\dfrac{\cot\alpha}{\csc\alpha} = \cos\alpha$.

24. $\tan\alpha\cdot\csc\alpha = \sec\alpha$.

25. $\cos\alpha\cdot\sec\alpha = \dfrac{1}{\cos\alpha\cdot\sec\alpha}$.

26. $\sec^2\alpha - \csc^2\alpha = \tan^2\alpha - \cot^2\alpha$.

CHAPTER II

IDENTICAL RELATIONS AMONG THE FUNCTIONS OF RELATED ANGLES. THE VALUES OF THE FUNCTIONS OF CERTAIN ANGLES

15. Functions of Negative Angles. We shall now proceed to determine the relations which exist among the functions of two angles when those angles are related in some par-

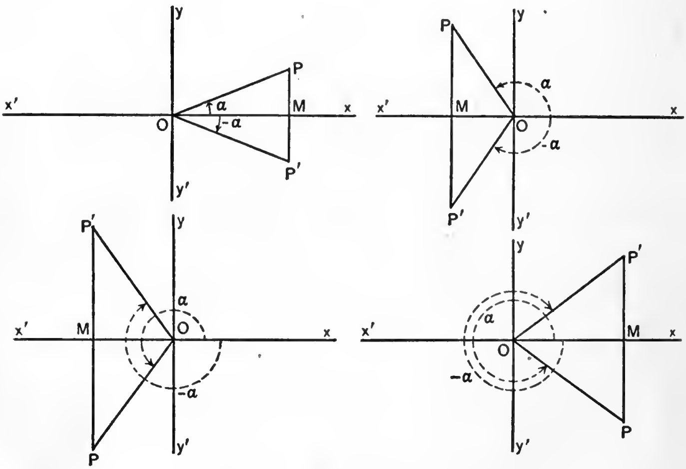

FIG. 8.

ticular way. Let us consider first two angles one of which is the negative of the other, Fig. 8. Let the value of the positive angle XOP be α, and of the numerically equal negative angle XOP' be $-\alpha$. On the terminal sides of these angles lay off the equal distances OP and OP', and drop perpendiculars from P and P' upon the axis of x. These perpendiculars will obviously cut the axis of x in the same point, M, and the two right triangles MOP and MOP' will be congruent. Therefore, $OF' = OP$, $OM = OM$, and $MP' = -MP$. We then have

$$
\begin{aligned}
\sin(-\alpha) &= \frac{MP'}{OP'} = \frac{-MP}{OP} = -\sin\alpha, \\
\cos(-\alpha) &= \frac{OM}{OP'} = \frac{OM}{OP} = \cos\alpha, \\
\tan(-\alpha) &= \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\alpha, \\
\cot(-\alpha) &= \frac{OM}{MP'} = \frac{OM}{-MP} = -\cot\alpha, \\
\sec(-\alpha) &= \frac{OP'}{OM} = \frac{OP}{OM} = \sec\alpha, \\
\csc(-\alpha) &= \frac{OP'}{MP'} = \frac{OP}{-MP} = -\csc\alpha.^*
\end{aligned}
\qquad (5)
$$

Thus any function of a negative angle is equal, numerically, to the same function of an equal positive angle. The algebraic sign is determined by the quadrant which $-\alpha$ lies in when α is acute.

16. Functions of $90° - \alpha$. Consider next two angles, α and $90° - \alpha$, Fig. 9. Let XOP be the angle α and XOP' be $90° - \alpha$. Lay off on the terminal sides of these angles the equal distances OP and OP', and from P and P' drop perpendiculars PM and $P'M'$ upon the axis of x. Then obviously the right triangles MOP and $M'OP'$ are congruent, and $OP' = OP$, $M'P' = OM$, and $OM' = MP$. Therefore,

$$
\begin{aligned}
\sin(90° - \alpha) &= \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\alpha, \\
\cos(90° - \alpha) &= \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\alpha,
\end{aligned}
$$

* By virtue of Art. 12, it is necessary to memorize only $\sin(-\alpha)$ and $\cos(-\alpha)$.

$$\tan(90° - \alpha) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\alpha,$$

$$\cot(90° - \alpha) = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\alpha,$$

$$\sec(90° - \alpha) = \frac{OP'}{OM'} = \frac{OP}{MP} = \csc\alpha,$$

$$\csc(90° - \alpha) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\alpha.$$

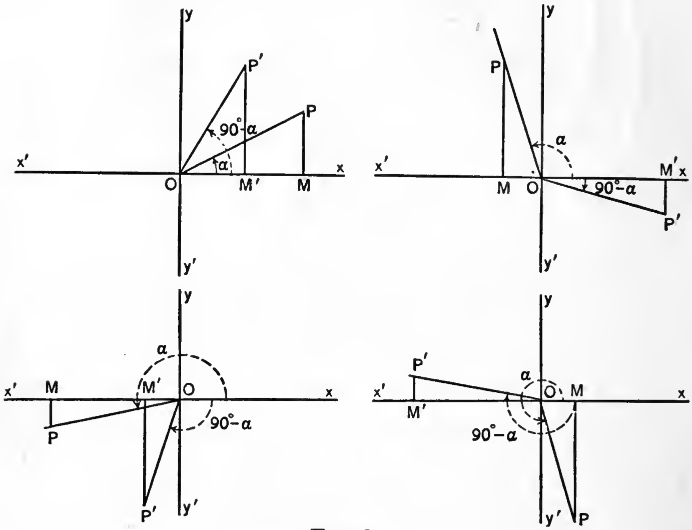

Fig. 9.

Thus we see that each of the functions of $90° - \alpha$ is equal, numerically, to the co-function of the angle α. The important case is when α is acute, and α and $90° - \alpha$ are complementary angles. Indeed it was because of this relation that the cosine, cotangent, and cosecant received their names. They are the sine, tangent, and secant of the complementary angle.

17. Functions of $90° + \alpha$. We shall consider two more cases, limiting the discussion to acute values of α, although the results will be equally true for any value of α whatever. In Fig. 10 let the angle XOP be α and XOP' be $90° + \alpha$. On the terminal sides of these angles lay off the equal distances OP and OP', and from P and P' drop perpendiculars PM and $P'M'$ upon the axis of x. It follows that the two

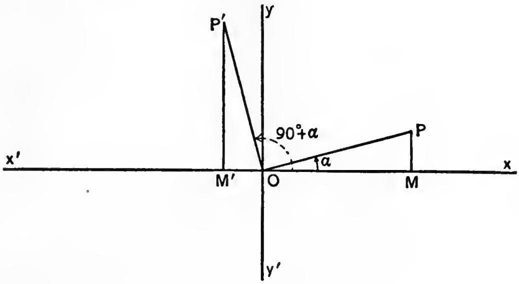

FIG. 10.

triangles MOP and $M'OP'$ are congruent and that $OP' = OP$, $M'P' = OM$, $OM' = -MP$. Therefore,

$$\sin(90° + \alpha) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\alpha,$$

$$\cos(90° + \alpha) = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\sin\alpha,$$

$$\tan(90° + \alpha) = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot\alpha,$$

$$\cot(90° + \alpha) = \frac{OM'}{M'P'} = \frac{-MP}{OM} = -\tan\alpha,$$

$$\sec(90° + \alpha) = \frac{OP'}{OM'} = \frac{OP}{-MP} = -\csc\alpha,$$

$$\csc(90° + \alpha) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\alpha.$$

18. Functions of 180° + α. In Fig. 11 let the angle XOP be α and XOP' be $180° + \alpha$. On the terminal sides of these angles lay off the equal distances OP and OP', and from P and P' drop perpendiculars PM and $P'M'$ upon the axis

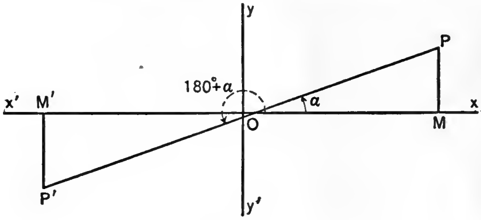

FIG. 11.

of X. Then the triangles MOP and $M'OP'$ are congruent and $OP' = OP$, $M'P' = -MP$, $OM' = -OM$. Therefore,

$$\sin(180° + \alpha) = \frac{M'P'}{OP'} = \frac{-MP}{OP} = -\sin\alpha,$$

$$\cos(180° + \alpha) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\alpha,$$

$$\tan(180° + \alpha) = \frac{M'P'}{OM'} = \frac{-MP}{-OM} = \tan\alpha,$$

$$\cot(180° + \alpha) = \frac{OM'}{M'P'} = \frac{-OM}{-MP} = \cot\alpha,$$

$$\sec(180° + \alpha) = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\alpha,$$

$$\csc(180° + \alpha) = \frac{OP'}{M'P'} = \frac{OP}{-MP} = -\csc\alpha.$$

19. Generalization. In a similar manner may be found analogous relations connecting the functions of an angle α with the functions of any integral multiple of 90° plus or minus α. Upon examining these relations we are led, by induction, to express them in the following general rule.

Any function of an **even** * *multiple of 90° plus or minus* α *is the* **same** *function of the angle* α.

* Zero is taken as an even number, so that the rule includes the case of Art. 15.

Any function of an **odd** *multiple of 90° plus or minus* α *is the* **co-function** *of the angle* α.

The algebraic sign of the value is determined by the quadrant (counting in the positive direction) in which the terminal side of the angle lies when α *is acute.*

Examples.

1. $\sin(720° - \alpha) = -\sin\alpha$, since 720° is an even multiple of 90° and the terminal side of $720° - \alpha$, when α is acute, lies in the fourth quadrant.

2. $\cot(-90° - \alpha) = \tan\alpha$, since $-90°$ is an odd multiple of 90° and the terminal side of $-90° - \alpha$, when α is acute, lies in the third quadrant.

3. $\sec(-180° + \alpha) = -\sec\alpha$, since $-180°$ is an even multiple of 90° and the terminal side of $-180° + \alpha$, α acute, lies in the third quadrant.

4. $\tan 281° = \tan(270° + 11°) = -\cot 11°$, or
$\tan 281° = \tan(360° - 79°) = -\tan 79°$,

since $\begin{cases} 270° \\ 360° \end{cases}$ is an $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ multiple of 90° and the terminal side of 281° lies in the fourth quadrant.

EXAMPLES

By means of a geometrical construction express each of the following as a function of α, where α is an acute angle. Check your results by the rule given above.

1. $\cos(270° + \alpha)$.
2. $\sin(180° - \alpha)$.
3. $\csc(-90° + \alpha)$.
4. $\tan(540° + \alpha)$.
5. $\cot(270° - \alpha)$.
6. $\sec(270° - \alpha)$.
7. $\sin(-180° - \alpha)$.
8. $\cos(-270° + \alpha)$.

Express as a function of an acute angle

9. $\sin 324°$.
10. $\cos(-375°)$.
11. $\tan 457°$.
12. $\csc(-801° 32')$.
13. $\sec(-537°)$.
14. $\cot 1140°$.
15. $\tan 495°$.
16. $\cos(-480°)$.

20. Functions of Certain Angles. We see by the preceding article that the functions of angles greater than 90°, and of negative angles, can be expressed in terms of the functions of angles lying between 0° and 90°. It follows that if we wish to use the trigonometric functions for computation or for other purposes we need find their values only for all positive acute angles. We shall not discuss the methods by means of which these values are computed in general, but shall proceed to find the values of the functions of certain angles which frequently occur. We shall then, in the following chapter, show how we may find the values of the functions of any angle from tables with which we are provided. We shall see, also, how the values thus found may be used in the solution of triangles; that is, in finding the unknown parts, angles or sides, of a triangle from parts which are given.

21. Functions of 30° and 60°. Let the angle XOP, Fig. 12, be an angle of 30°, and from P drop a perpendicular, PM,

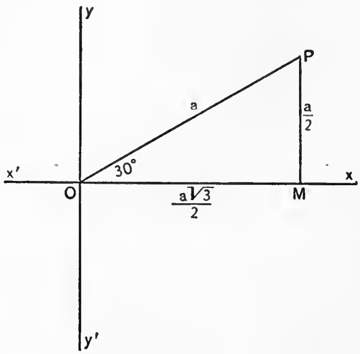

FIG. 12.

upon the axis of x. Then, as we know, the angle OPM is 60°, and if OP have the value a, MP must be equal to $\frac{a}{2}$ and OM equal to $\frac{a\sqrt{3}}{2}$. Therefore, by definition,

$$\sin 30° = \frac{MP}{OP} = \frac{\frac{a}{2}}{a} = \frac{1}{2}, \qquad \cos 30° = \frac{OM}{OP} = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2},$$

$$\tan 30° = \frac{MP}{OM} = \frac{\frac{a}{2}}{\frac{a\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}, \qquad \cot 30° = \frac{OM}{MP} = \frac{\frac{a\sqrt{3}}{2}}{\frac{a}{2}} = \sqrt{3},$$

$$\sec 30° = \frac{OP}{OM} = \frac{a}{\frac{a\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}, \qquad \csc 30° = \frac{OP}{MP} = \frac{a}{\frac{a}{2}} = 2.$$

By a similar construction, or by the relations of Art. 16, the following values may be derived:

$$\sin 60° = \frac{\sqrt{3}}{2}, \qquad \tan 60° = \sqrt{3}, \qquad \sec 60° = 2,$$

$$\cos 60° = \frac{1}{2}, \qquad \cot 60° = \frac{1}{\sqrt{3}}, \qquad \csc 60° = \frac{2}{\sqrt{3}}.$$

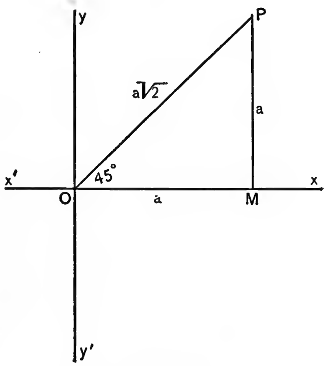

Fig. 13.

22. Functions of 45°. Let the angle XOP, Fig. 13, be an angle of 45°, and from P drop a perpendicular, PM, upon the axis of x. Then the angle OPM is an angle of 45°, and if OM have the value a, MP also will be equal to a and OP will be $a\sqrt{2}$. Therefore, by definition,

$$\sin 45° = \frac{MP}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}, \qquad \cos 45° = \frac{OM}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\tan 45° = \frac{MP}{OM} = \frac{a}{a} = 1, \qquad \cot 45° = \frac{OM}{MP} = \frac{a}{a} = 1,$$

$$\sec 45° = \frac{OP}{OM} = \frac{a\sqrt{2}}{a} = \sqrt{2}, \qquad \csc 45° = \frac{OP}{MP} = \frac{a\sqrt{2}}{a} = \sqrt{2}.$$

23. Functions of Other Angles Readily Found. By similar constructions the functions of 120°, 150°, 135°, etc., or, in general, any integral multiple of 90° plus or minus 30°, 60°, or 45°, may be found. They may be found more readily, however, by using the rule given in Art. 19. Thus,

$$\sin 120° = \sin (90° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$$

or

$$\sin 120° = \sin (180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}.$$

EXAMPLES

Find the values of the functions of

1. 120°.
2. 135°.
3. 150°.
4. 210°.
5. 225°.
6. 240°.
7. 300°.
8. 315°.
9. 330°.

Prove that

10. $\sin 210° \tan 300° = \sin 120°$.
11. $\sec 315° \sec 300° = \sec 240° \sec 225°$.
12. $\tan 210° : \cos 150° = \tan 150° : \cos 330°$.
13. $\csc 330° \sec 315° \sin 225° = - \sec 120°$.

24. Functions of Zero. Let the value of the angle *XOP*, Fig. 14, be represented by α, and from *P*, any point in the terminal side of the angle, drop a perpendicular, *PM*, upon the axis of x. By definition,

$$\sin \alpha = \frac{MP}{OP} \text{ and } \cos \alpha = \frac{OM}{OP}.$$

Now, for the sake of convenience keeping the distance OP constant in length, let the line OP approach nearer and nearer to the position OX. Then the angle α can be made,* smaller than any angle that may be assigned, however

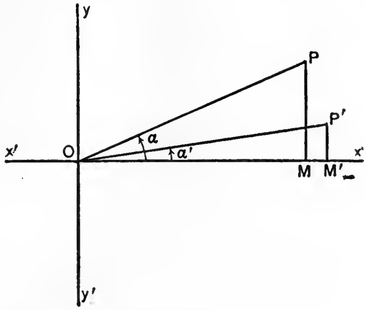

FIG. 14.

small, or, as it is otherwise expressed, α will approach the limit zero. At the same time MP will approach zero as a limit, and OM will approach OP as a limit. Then $\frac{MP}{OP}$ will approach the limit zero and $\frac{OM}{OP}$ will approach the limit unity. Thus, as the angle approaches the limit zero (or, becomes smaller than any value that may be assigned, however small) its sine approaches the limit zero (or, becomes smaller than any value that may be assigned, however small) and its cosine approaches the limit unity (or, differs from unity by a number smaller than any number that may be assigned, however small). This may be written

$$\lim_{\alpha \doteq 0} \sin \alpha = 0, \qquad \lim_{\alpha \doteq 0} \cos \alpha = 1.$$

Again, by definition, $\csc \alpha = \frac{OP}{MP}$, and as α grows smaller OP remains constant and MP grows smaller, so that $\frac{OP}{MP}$ becomes continually greater. Finally, when α approaches zero as a limit, MP becomes smaller than any number that

* And will remain.

may be assigned, however small, and $\frac{OP}{MP}$ becomes greater than any number that may be assigned, however great. This we express by saying that $\frac{OP}{MP}$ approaches the limit infinity, or increases without limit. We may then write

$$\lim_{\alpha \doteq 0} \csc \alpha = \infty.$$

Similarly it may be shown that

$$\lim_{\alpha \doteq 0} \tan \alpha = 0, \qquad \lim_{\alpha \doteq 0} \cot \alpha = \infty, \qquad \lim_{\alpha \doteq 0} \sec \alpha = 1.$$

These relations are often briefly expressed,

$$\begin{array}{lll} \sin 0^\circ = 0, & \tan 0^\circ = 0, & \sec 0^\circ = 1, \\ \cos 0^\circ = 1, & \cot 0^\circ = \infty, & \csc 0^\circ = \infty. \end{array} \qquad (6)$$

to which there is no objection if we remember that these are merely abbreviations of the preceding statements, and that

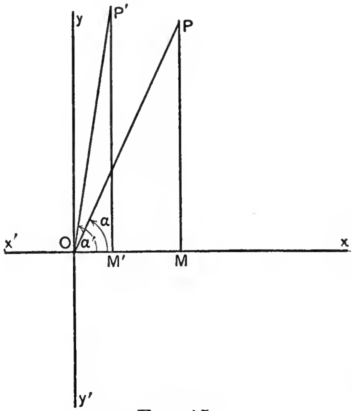

FIG. 15.

0° means, not that we have no angle, but that we are dealing with an angle which becomes smaller than any value that may be assigned, however small; and that when this happens the sine of the angle also becomes smaller than any value that may be assigned, however small, the cotangent becomes greater than any value that may be assigned, however great, the cosine approaches the limit unity, etc.

25. Functions of 90°. Let the angle XOP, Fig. 15, be represented by α, and let OM, MP, and OP be respectively the abscissa, ordinate, and distance of P. Also, keeping the distance OP constant, let the line OP approach OY as its limiting position. Then,

α approaches the limit 90°,
OM approaches the limit zero,
MP approaches the limit OP.

Therefore,

$$\operatorname*{limit}_{\alpha \doteq 90^\circ} \sin\alpha = \text{limit}\,\frac{MP}{OP} = 1,$$

$$\operatorname*{limit}_{\alpha \doteq 90^\circ} \cos\alpha = \text{limit}\,\frac{OM}{OP} = 0,$$

$$\operatorname*{limit}_{\alpha \doteq 90^\circ} \tan\alpha = \text{limit}\,\frac{MP}{OM} = \infty,$$

$$\operatorname*{limit}_{\alpha \doteq 90^\circ} \cot\alpha = \text{limit}\,\frac{OM}{MP} = 0,$$

$$\operatorname*{limit}_{\alpha \doteq 90^\circ} \sec\alpha = \text{limit}\,\frac{OP}{OM} = \infty,$$

$$\operatorname*{limit}_{\alpha \doteq 90^\circ} \csc\alpha = \text{limit}\,\frac{OP}{MP} = 1.$$

With the same understanding as in the preceding article these may be written

$$\begin{aligned} &\sin 90^\circ = 1, && \tan 90^\circ = \infty, && \sec 90^\circ = \infty, \\ &\cos 90^\circ = 0, && \cot 90^\circ = 0, && \csc 90^\circ = 1. \end{aligned} \tag{7}$$

26. The student should find, as in Arts. 24 and 25, the following:

$$\begin{aligned} &\sin 180^\circ = 0, && \tan 180^\circ = 0, && \sec 180^\circ = -1, \\ &\cos 180^\circ = -1, && \cot 180^\circ = \infty, && \csc 180^\circ = \infty. \end{aligned} \tag{8}$$

$$\begin{aligned} &\sin 270^\circ = -1, && \tan 270^\circ = \infty, && \sec 270^\circ = \infty, \\ &\cos 270^\circ = 0, && \cot 270^\circ = 0, && \csc 270^\circ = -1. \end{aligned} \tag{9}$$

27. Limiting Values of the Functions. We have seen (Art. 20) that all possible numerical values of the trigonometric functions are given by angles lying between 0° and 90°. Let us now see between what limits the values of the functions lie. From the discussion and figures of articles 24 to 26 we see that

the *sine* and *cosine* of an angle *lie between* -1 and $+1$,
the *tangent* and *cotangent* lie *between* $-\infty$ and $+\infty$,
the *secant* and *cosecant* lie *between* 1 and ∞ or *between* -1 and $-\infty$.

It is well to note also, for angles in the first quadrant, that as the angle increases the direct functions increase, the co-functions decrease.

A very convenient and simple way to remember the range of values and the signs of the trigonometric functions is by

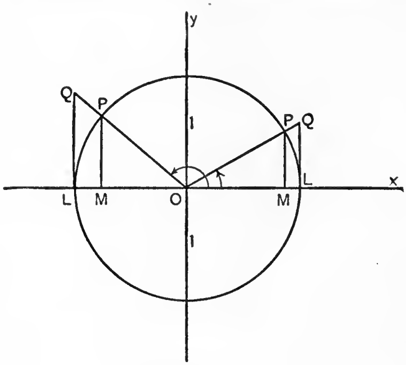

Fig. 16.

means of the unit circle, a circle with unit radius, which need not be actually drawn but merely visualized. Draw such a circle, Fig. 16, with its center at the origin of coordinates, and let XOP be any angle. Drop the perpendicular PM upon the axis of X, and draw LQ tangent to the circle at L and meeting OP produced in Q. Then, by definition,

$$\sin XOP = \frac{MP}{OP} = \frac{MP}{1} = MP,$$

$$\cos XOP = \frac{OM}{OP} = \frac{OM}{1} = OM,$$

$$\tan XOP = \frac{LQ}{OL} = \frac{LQ}{\pm 1} = \pm LQ, \text{ etc.}$$

If now the line *OP* be pictured as revolving from the position *OL*, the sine of the angle *XOP*, namely *MP*, will be seen to increase from zero and approach unity as the angle approaches 90°. The cosine, namely *OM*, decreases from unity to zero, and the tangent (*LQ*) increases without limit. Also, as the angle increases beyond 90°, the directions of the lines *MP* and *OM* indicate the signs of the sine and cosine. The other functions follow directly from these two by virtue of the relations of Art. 12.

CHAPTER III

THE SOLUTION OF RIGHT TRIANGLES. LOGARITHMS AND COMPUTATION BY MEANS OF LOGARITHMS

28. Solution of Right Triangles. With the definitions of the trigonometric functions and tables giving their numerical values we are now prepared to solve right triangles; that is, to find the values of the unknown parts from those that are known. Two parts in addition to the right angle must be known, and one at least of these parts must be a side. We have then the general rule of procedure: *Select that trigonometric function which involves the two known parts and one unknown part.* The value of the unknown part can then be computed by elementary algebraic processes.

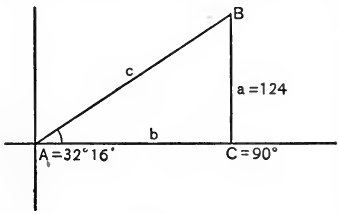

FIG. 17.

Example 1. Given $A = 32° 16'$, $a = 124$, $C = 90°$, find B, b, and c. See Fig. 17.

Obviously $B = 90° - A = 90° - 32° 16' = 57° 44'$. Then

$$\cot A = \frac{b}{a}, \qquad \sin A = \frac{a}{c},$$

or

$$b = a \cot A. \qquad c = \frac{a}{\sin A}.$$

From the tables we find

$$\cot A = 1.5839. \qquad \sin A = .5338.$$

Therefore,

$$b = 124 \times 1.5839 \qquad\qquad c = \frac{124}{.5338}$$

$$= 196.4. \qquad\qquad = 232.3.$$

Example 2. Given $a = 50$, $b = 60$, $C = 90°$, find A, B, and c.

In this case,

$$\tan A = \frac{a}{b} = \frac{50}{60} = .8333.$$

$$A = 39° \ 48'.$$

$$B = 90° - A = 50° \ 12'.$$

To find c we may use either

$$\sin A = \frac{a}{c} \qquad \text{or} \qquad c^2 = a^2 + b^2$$

$$c = \frac{a}{\sin A} \qquad\qquad c = \sqrt{a^2 + b^2}$$

$$= \frac{50}{.6402} \qquad\qquad = \sqrt{2500 + 3600}$$

$$= 78.1. \qquad\qquad = 78.1.$$

EXAMPLES

Solve the following right triangles:

1. $a = 250$, $A = 36° \ 22'$.
2. $a = 37.5$, $b = 40.1$.
3. $a = .55$, $c = .70$.
4. $B = 72° \ 6'$, $c = 502$.
5. $A = 59° \ 58'$, $b = 412$.
6. $B = 24° \ 33'$, $a = 211$.

PROBLEMS

7. What is the height of a flagpole if at a horizontal distance of 200 feet from the foot of the pole the angle of elevation of its top is 19° 28′ ?

8. A rope is stretched taut from the top of a building to the ground, and is found to make an angle of 58° 56′ with the horizontal. If the building is 61 feet high how long is the rope?

9. If a tree 74.3 feet high casts a shadow 42.6 feet long, how many degrees above the horizon is the sun?

10. A man walking on level ground finds, at a certain point, that the angle of elevation of the top of a tower is 30°. He walks directly toward the tower for a distance of 300 feet and then finds the angle of elevation of the top to be 60°. What is the height of the tower?

11. At a point, A, south of a tower the angle of elevation of the top of the tower is 60°. At another point 300 feet east of A the angle of elevation is 30°. What is the height of the tower?

12. The angles of a right triangle are 42° and 48°; the hypotenuse is 200 feet. What is the length of the perpendicular from the right angle to the hypotenuse?

13. The height of a gable roof is 20 feet, its width 42 feet. What is the pitch of the roof; that is, the angle it makes with the horizontal?

14. From where I stand a tree 50 feet away has an angle of elevation of 43° 31′. From the same point another tree, 75 feet distant, has an angle of elevation of 32° 20′. Which tree is the taller and by how much?

29. Logarithms. The solution of right triangles as thus explained is simple in theory but may become laborious in practice because of the arithmetic computation involved. Fortunately we have in logarithms a device for simplifying such computation. The base of a system of logarithms is, in general, any arbitrarily chosen number.* In practice two systems are used: the Briggsian or common system of which the base is 10; and the Napierian system of which the base is $e = 2.718 \cdots$. The logarithm of a number to a given base (a) is the exponent of the power to which the base (a) must be raised to produce the number. Thus, if $a^x = m$, then x is the logarithm of m to the base a; written $x = \log_a m$.

The word power is used here in its broader sense to include fractional and negative exponents. Defining fractional and negative exponents in such a way that the laws of exponents — $a^m a^n = a^{m+n}$; $(a^m)^n = a^{mn}$ — hold for negative numbers and fractions as well as for positive integers,

* Some numbers, unity, for example, cannot be so used.

values of x may be found to satisfy, approximately at least, such an equation as $a^x = b$, no matter what values a and b may have. Thus, given any number, a, by raising it to a suitable power, p, and extracting a suitable root, q, of the result, we can obtain any other number, b; that is, $\sqrt[q]{a^p} = b$. But this may be written $a^{\frac{p}{q}} = b$ or $a^x = b$, where $x = \frac{p}{q}$, the division of p by q being carried out to any desired number of decimal places. We then call x the logarithm of b to the base a.

30. The Common System. For purposes of computation the common system, base 10, is used. Let us form a table of powers of 10 and express the relations in terms of logarithms.

$10^{-3} = .001,$	or	$\log_{10} .001 = -3.$
$10^{-2} = .01,$		$\log_{10} .01 = -2.$
$10^{-1} = .1,$		$\log_{10} .1 = -1.$
$10^0 = 1,$		$\log_{10} 1 = 0.$
$10^1 = 10,$		$\log_{10} 10 = 1.$
$10^2 = 100,$		$\log_{10} 100 = 2.$
$10^3 = 1000,$		$\log_{10} 1000 = 3,$*
etc.		etc.

This table could be extended indefinitely in either direction. If we examine the table we notice that to produce a number between 1 and 10 we must raise the base 10 to a positive power between 0 and 1; to produce a number between 10 and 100, the exponent of the base must lie between 1 and 2; for a number between 100 and 1000, the exponent must lie between 2 and 3, and so on. In other words, the logarithm of a number between 1 and 10 lies between 0 and 1, and is, therefore, a fraction, always expressed as a decimal. The logarithm of a number between 10 and 100 lies between 1 and 2, or is 1 plus a decimal. The logarithm of

* Hereafter in this work we shall not write the base 10. Thus log 7 means $\log_{10} 7$. In general, however, except in works on trigonometry, if no base is written, $e = 2.718 \cdots$ is understood.

a number between 100 and 1000 is 2 plus a decimal. The logarithm of a number is thus seen to consist, in general, of two parts, an integral part and a decimal part. The *integral part* is called the *characteristic* of the logarithm; the *decimal part* is called the *mantissa.* The results of our observations may be summarized thus:

Number Between	Contains	Characteristic of Logarithm
1 and 10	1 integral digit	0
10 and 100	2 integral digits	1
100 and 1000	3 integral digits	2
. . . . . .	. . . .	. . .

Whence we formulate the law: *The characteristic of the logarithm of a number is one less than the number of digits in the integral part of the number.*

On the other hand, we observe from the table of this article that if a number contain no integral digits, that is, if it be purely decimal, its logarithm is negative. The characteristic in this case can be got by counting the number of zeros before the first significant figure, prefixing the minus sign. It is usual, and better, however, except for special purposes, not to write the characteristic of the logarithm of a decimal number in the form just stated, for reasons which will now be pointed out.

31. The Mantissa. In the common system the *mantissa* of the logarithm of a number can be made to depend only upon the sequence of digits in the number, and be independent of the position of the decimal point. Let us assume that we know the logarithm of 1.285 to be 0.1089. It follows, multiplying successively by ten, that

$$
\begin{aligned}
10^{0.1089} &= 1.285, &\quad \text{or} \quad \log 1.285 &= 0.1089.\\
10^{1.1089} &= 12.85, & \log 12.85 &= 1.1089.\\
10^{2.1089} &= 128.5, & \log 128.5 &= 2.1089.\\
10^{3.1089} &= 1285, & \log 1285 &= 3.1089.\\
10^{4.1089} &= 12850, & \log 12850 &= 4.1089.\\
&\cdots\cdots & &\cdots\cdots
\end{aligned}
$$

which verifies the law we have stated. If, however, we divide $10^{0.1089}$ successively by 10 we find

$$10^{0.1089-1} = 10^{-0.8911} = .1285, \quad \text{or} \quad \log .1285 = -0.8911.$$
$$10^{-1.8911} = .01285, \quad \log .01285 = -1.8911.$$
$$10^{-2.8911} = .001285, \quad \log .001285 = -2.8911.$$
$$\cdots \qquad \cdots$$

This is the true form of the logarithm of a purely decimal number, and for certain purposes this is the form which must be used.*

It is obvious from the preceding discussion that the mantissa corresponding to a given sequence of digits remains the same as long as the sequence contains one or more integral digits, but that as soon as the sequence is a purely decimal number the mantissa changes. To obviate this difficulty and to keep the mantissa the same for a given sequence of digits regardless of the position of the decimal point, we note that the number -0.8911 may be written, without change of value, in the form $9.1089 - 10$. We have added 10 and subtracted 10, and have therefore left the value unchanged. We may then say

$$\log .1285 = -0.8911 = 9.1089 - 10,$$

and if we agree to use the latter form † we see that the mantissa of the logarithm of .1285 (that is, 1089) is the same as the mantissa of the logarithm of the sequence 1285 when it contains integral digits. We may now write

$$\log 1.285 = 0.1089 \qquad \log .1285 = 9.1089 - 10$$
$$\log 12.85 = 1.1089 \qquad \log .01285 = 8.1089 - 10$$
$$\cdots \qquad \cdots$$

and make the statement: *In the common system the mantissa of a logarithm is unique for a given sequence of digits. The*

* For example, in dividing one logarithm by another.

† This form, $9.1089 - 10$, is perfectly convenient as long as the operations to be performed are addition and subtraction, which are the usual operations in dealing with logarithms.

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characteristic is one less than the number of integral digits. If a number be purely decimal, count the decimal point and the zeros before the first significant figure. The result subtracted from 10 minus 10 will be the characteristic.

32. Four Computation Theorems. The use of logarithms in computation depends upon the four following theorems:

I. *In any system the logarithm of a product is equal to the sum of the logarithms of its factors.*

To prove, $\log_a mn \cdots s = \log_a m + \log_a n + \cdots + \log_a s.$

Let $\log_a m = x$ then $a^x = m$

$\log_a n = y \qquad a^y = n$

$\cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot$

$\log_a s = z \qquad a^z = s.$

Whence $a^x \cdot a^y \cdots a^z = a^{x+y+\cdots+z} = mn \cdots s,$

or, by the definition of a logarithm,

$$\log_a mn \cdots s = x + y + \cdots + z.$$

That is, $\log_a mn \cdots s = \log_a m + \log_a n + \cdots + \log_a s.$

This theorem replaces the operation of multiplication by the simpler operation of addition.

II. *In any system the logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

To prove, $\log_a \frac{m}{n} = \log_a m - \log_a n.$

Let $\log_a m = x$ then $a^x = m$

$\log_a n = y \qquad a^y = n.$

Whence

$$\frac{a^x}{a^y} = a^{x-y} = \frac{m}{n},$$

or, by the definition of a logarithm,

$$\log_a \frac{m}{n} = x - y.$$

That is,

$$\log_a \frac{m}{n} = \log_a m - \log_a n.$$

This theorem replaces the operation of division by the simpler operation of subtraction.

III. *In any system the logarithm of a power of a number is equal to the exponent of the power times the logarithm of the number.*

To prove, $\log_a m^n = n \log_a m.$

Let $\log_a m = x$ or $a^x = m.$

Whence $(a^x)^n = a^{nx} = m^n,$

or, by the definition of a logarithm,

$$\log_a m^n = nx.$$

That is $\log_a m^n = n \log_a m.$

This theorem replaces the operation of involution, or successive multiplications, by the simpler operation of a single multiplication.

IV. *In any system the logarithm of a root of a number is equal to the quotient of the logarithm of the number by the index of the root.*

To prove $\log_a \sqrt[n]{m} = \frac{\log_a m}{n}.$

Let $\log_a m = x$ or $a^x = m.$

Whence $\sqrt[n]{a^x} = a^{\frac{x}{n}} = \sqrt[n]{m},$

or, by the definition of a logarithm,

$$\log_a \sqrt[n]{m} = \frac{x}{n}.$$

That is $$\log_a \sqrt[n]{m} = \frac{\log_a m}{n}.$$

This theorem replaces the operation of evolution, or extraction of roots, by the simpler operation of division.

Another theorem, important in the theory of logarithms, but of which no application is made in the study of trigonometry is the following:

$$\log_a m = \frac{\log_b m}{\log_b a}.$$

Proof: Let $$\log_b m = x \quad \text{then} \quad b^x = m$$

$$\log_b a = y \qquad\qquad b^y = a.$$

Whence $$m = b^x = (b^y)^{\frac{x}{y}} = a^{\frac{x}{y}}$$

or, $$\log_a m = \frac{x}{y} = \frac{\log_b m}{\log_b a}.$$

By means of this theorem the logarithm of a number to any base can be found if the logarithms of numbers to some one base are known. Thus, assuming that logarithms to the base 10 are known,

$$\log_e 71.24 = \frac{\log_{10} 71.24}{\log_{10} e} = \frac{\log_{10} 71.24}{\log_{10} 2.718} = \frac{1.8527}{0.4343} = 4.2659.$$

As a corollary of the above theorem we have, putting $b = m$,

$$\log_a m = \frac{1}{\log_m a}.$$

33. Special Properties of Logarithms. In addition to the preceding theorems we may note the following properties of logarithms:

1. In any system the logarithm of 1 is 0. For, by the definition of zero exponent, $a^0 = 1$. Therefore, $\log_a 1 = 0$.

2. In any system the logarithm of the base is 1. For $a^1 = a$. Therefore, $\log_a a = 1$.

3. In any system whose base is greater than 1 the logarithm of 0 is $-\infty$. For, $a > 1$, $a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0$. Therefore, $\log_a 0 = -\infty$. That is, the base of the system being greater than unity, the logarithm of a number which becomes smaller than any assigned number however small, is negative and numerically greater than any assigned number however great.

4. The cologarithm of a number is the logarithm of the reciprocal of the number.

Thus, the base being 10,

$$\operatorname{colog} n = \log \frac{1}{n} = \log 1 - \log n,$$

or,

$$\operatorname{colog} n = 0 - \log n,$$

which may be written,

$$\operatorname{colog} n = (10 - 10) - \log n.$$

Therefore, to find the *cologarithm* of a number to the base 10 *subtract the logarithm* of the number *from 10 − 10.*

It may be noted that

$$\log \frac{m}{n} = \log m \cdot \frac{1}{n} = \log m + \log \frac{1}{n} = \log m + \operatorname{colog} n.$$

Therefore we may, instead of subtracting the logarithm of a number, add its cologarithm. It is found convenient to do so in most cases.

34. The following example will illustrate the use of logarithms in making numerical computations.

Example.

Find the value of

$$\sqrt[3]{\frac{.0005616 \times \sqrt[7]{-424.65}}{(6.73)^4 \times (.03194)^{\frac{5}{6}}}}$$

We note first that, with the definition of logarithms we have adopted, negative numbers have no logarithms. But the numerical result of operations of multiplication and division is the same no matter what the combination of algebraic signs. We therefore find the numerical value of any expression, treating all numbers as positive, and determine the algebraic sign of the result by considering the operations indicated. Thus, in the above example the factors are all positive except $\sqrt[7]{-424.65}$. Therefore, the number of which we are to extract the cube root is negative and the final result will be negative.

$$\begin{aligned} \log .0005616 &= 6.7494 - 10 \\ \tfrac{1}{7} \log 424.65 &= 0.3754 \\ 4 \operatorname{colog} 6.73 &= 6.6880 - 10 \\ \tfrac{5}{6} \operatorname{colog} .03194 &= 1.2464 \\ &\; 3\overline{)5.0592 - 10} \\ \log N &= 8.3531 - 10 \\ N &= .02255. \end{aligned}$$

Therefore

$$\sqrt[3]{\frac{.0005616 \times \sqrt[7]{-424.65}}{(6.73)^4 \times (.03194)^{\frac{5}{6}}}} = -.02255.$$

NOTE. The colog $6.73 = 9.1720 - 10$, which being multiplied by four gives $36.6880 - 40$; subtracting and adding 30 this becomes $6.6880 - 10$, the desired form of "a number minus 10." Similarly to divide $5.0592 - 10$ by three we first add and subtract 20. Also, in finding five sixths of the cologarithm of .03194, we first multiply by 5 and then divide by 6, in order that any error arising from inexact division by 6 may not be increased 5-fold.

35. We may now return to the problems of Art. 28 and solve them by the use of logarithms.

Example 1. Given $A = 32° 16'$, $a = 124$, $C = 90°$, find B, b, and c.

As before,

$$b = a \cot A, \qquad c = \frac{a}{\sin A}.$$

Therefore

$$\log b = \log a + \log \cot A, \qquad \log c = \log a - \log \sin A.$$

$$\begin{array}{rl} \log a & = 2.0934 \\ \log \cot A & = 0.1997 \\ \hline \log b & = 2.2931 \\ b & = 196.4, \end{array} \qquad \begin{array}{rl} \log a & = 12.0934 - 10 \\ \log \sin A & = 9.7274 - 10 \\ \hline \log c & = 2.3660 \\ c & = 232.3. \end{array}$$

Example 2. Given $a = 50$, $b = 60$, $C = 90°$.
As before,

$$\tan A = \frac{a}{b}, \text{ therefore } \log \tan A = \log a - \log b.$$

$$\begin{array}{rl} \log a & = 11.6990 - 10 \\ \log b & = 1.7782 \\ \hline \log \tan A & = 9.9208 - 10 \\ A & = 39° \, 48'. \end{array}$$

Also

$$c = \frac{a}{\sin A}, \text{ or } \log c = \log a - \log \sin A.$$

$$\begin{array}{rl} \log a & = 11.6990 - 10 \\ \log \sin A & = 9.8063 - 10 \\ \hline \log c & = 1.8927 \\ c & = 78.1. \end{array}$$

It must be emphasized that results obtained by logarithmic computation are approximate. The value of the logarithm of a number cannot, in general, be found exactly, but only approximately to four, five, or any desired number of decimal places. *The results of numerical computation by means of logarithms are not*, in any case, *correct beyond the number of decimal places in the logarithms* used to make the computation. In the same way, the values of the trigonometric functions being, in general, not exact but approximate to four, five, or more decimal places, the solutions of triangles got by their use, with or without logarithms, are *approximate solutions only, to the degree of accuracy of the tables used.*

Indeed, in all but the simplest problems in applied mathematics the results are necessarily approximate, the data of a problem being themselves approximate. It is useless to try to make results "more accurate" by using tables of logarithms or other functions carried to seven places when the data are correct only to, say, three figures. In general if data are given to three figures, three-place tables should be used; if to seven figures, seven-place tables, etc. On the other hand, no matter to how many figures the data may be given, if we are using, say, four-place tables, the data should be used and results found to four figures only. To illustrate these points the following simple example will be worked in four ways: 1. by actual multiplication; 2. by using four-place tables; 3. by using five-place tables; 4. by using seven-place tables.

Example. Find the value of

$$123045 \times 200368.$$

1. By actual multiplication the result is 24,654,280,560.

2.
$$\begin{array}{rl} \log 123045 & \\ 123|0 = & 5.0899 \\ \log 200368 & \\ 200|4 = & 5.3019 \\ \hline \log \text{product} = & 10.3918 \\ \text{product} = & 24{,}650{,}000{,}000 \end{array}$$

which agrees with the first result to four figures.

3.
$$\begin{array}{rl} \log 1230|45 = & 5.09007 \\ \log 2003|68 = & 5.30183 \\ \hline \log \text{product} = & 10.39190 \\ \text{product} = & 24{,}655{,}000{,}000 \end{array}$$

which does not agree with the first result to the fifth figure. It will be noted that there was an accumulation of errors all in one direction. Result 3 is nearer to result 1, however, than is result 2.

4.
$$\begin{array}{r} \log 12304\,|\,5 = 5.0900640 \\ \log 20036\,|\,8 = 5.3018284 \\ \hline \log \text{product} = 10.3918924 \\ \text{product} = 24{,}654{,}280{,}000 \end{array}$$

which agrees with the first result to seven figures.

EXAMPLES

What is the value of

1. $10^{\log 7.218}$.
2. $\log 10^{2.6994}$.
3. Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, find $\log 12$.
4. Prove $10^{\log a+1} = 10\,a$.
5. Is $\log 14 = \log 2 \cdot \log 7$? Why?
6. Is $\dfrac{\log 10}{\log 3} = \dfrac{10}{3}$? Why?
7. Is $\dfrac{\log 10}{\log 3} = \log\left(\dfrac{10}{3}\right)$? Why?

Find the value of

8. $\dfrac{\log .00365}{\log .05312}$.
9. $\dfrac{\log 77.95}{\log .00684}$.

Find the value of x in the following equations:

10. $\log_{10} x = 3$.
11. $\log_{10} x = \frac{3}{2}$.
12. $\log_e x = 2$.
13. $\log_e x = \frac{5}{4}$.
14. $\log_{10} x \cdot \log_e x = \log_{10} x^2$.
15. $a \log_{10} x - b \log_{10} x = a^2 - b^2$.
16. $\log_e x^3 - \log_e x^2 = 5$.
17. $\frac{1}{2} \log_{10} x^{10} - \log_{10} x^3 = 4$.

Express as the logarithm of a fraction:

18. $\log (x^2 - a^2)^2 - \log (x^2 - a^2) - \log (x + a)$.
19. $\log \sqrt{x^2 + a^2} - \log \sqrt[4]{x^2 + a^2} + \log (x^2 + a^2)^{\frac{3}{4}}$.

Solve the equations:

20. $e^x + e^{-x} = 2$.
21. $e^x - e^{-x} = 0$.
22. $e^{2(x-1)} - 2\,e^{x-1} + 1 = 0$.
23. $e^{2(x-1)} + 2\,e^{x-1} + 1 = 0$.
24. Given $10^x = 400$, prove that $x = 2 + \log_{10} 4$.
25. Solve the equation

$$a^2e^{-ax} - b^2e^{-bx} = 0. \qquad \text{(Assume } a > b\text{)}$$

Find the value of $a^2e^{-ax} - b^2e^{-bx}$:

26. When $x = \dfrac{\log a - \log b}{a - b}$. $(a > b)$

27. When $x = \dfrac{2(\log a - \log b)}{a - b}$. $(a > b)$

Compute the values of the following:

28. $\log_e 1$. **29.** $\log_e 2$. **30.** $\log_e 3$. **31.** $\log_e 4$.

By means of logarithms compute the values of:

32. $\sqrt[3]{.01236}$. **33.** $\sqrt{1.1193}$. **34.** $\sqrt[5]{-.002807}$.

35. $\dfrac{(56.333)^{\frac{2}{3}}}{\sqrt{11.11}}$. **36.** $\dfrac{99.02}{\sqrt[4]{.02983}}$. **37.** $\dfrac{(-16.65)^4}{\sqrt[3]{-.00986}}$.

38. $\dfrac{\log 771.2}{40.04}$. **39.** $[\log(.00915)]^{\frac{1}{2}}$. **40.** $\dfrac{693.08}{\log .00598}$.

41. $\dfrac{62.85 \times \sqrt{3111.59}}{-999.9 \times .002008}$. **42.** $\dfrac{(56.3)^3 \times \sqrt[3]{56.3}}{\sqrt{.08888 \times 40.19}}$.

43. $\sqrt{(.001)^3}$. **44.** $(.01)^{\frac{5}{7}}$. **45.** $(.0001)^{-\frac{1}{2}}$.

46. $\sqrt[3]{\dfrac{485.7 \times 22.01 \times 11.79}{-55.5 \times -66.66}}$. **48.** $\dfrac{(.0002635)^{-\frac{1}{3}}}{(5362)^{-\frac{1}{2}}}$.

47. $\dfrac{\sqrt{.00298 \times .00384}}{\sqrt[4]{632 \times .06302}}$. **49.** $\dfrac{(88)^2 \times (999)^{\frac{1}{2}}}{\sqrt[3]{1000000}}$.

EXAMPLES: SOLUTION OF RIGHT TRIANGLES

Solve the following right triangles, and find their areas:

1. $B = 24° 23'$, $b = .02126$.

2. $B = 55° 45'$, $c = 4116$.

3. $B = 43° 30'$, $a = 26185$.

4. $a = 77.38$, $c = 91.08$.

5. $B = 76° 34'$, $b = 2423$.

6. $A = 67° 47'$, $c = .00954$.

7. $b = .2072$, $a = .4212$.

8. $A = 82° 6'$, $b = .08937$.

9. $a = .8478$, $c = 1.234$.

10. $B = 60° 14'$, $c = .007745$.

11. $A = 14° 53'$, $a = 1353$.

12. $B = 39° 22'$, $a = 121.2$.

13. $b = 156.6$, $c = 856.4$.

14. $B = 43° 46'$, $a = 66650$.

15. $B = 74° 17'$, $b = .00002039$.

16. $A = 29° 56'$, $c = .0007814$.

17. $b = 8.243$, $c = 9.275$.

18. $B = 58° 39'$, $c = 35.73$.

19. $A = 35° 8'$, $a = 17270$.

20. $b = 3814$, $a = 3651$.

PROBLEMS

21. A road rises 348.9 feet in a horizontal distance of one half mile. Another road rises the same height in a distance of 3019 feet along the road. Which road is the steeper and by how much ?

22. From a ship sailing due east at the rate of 7.6 miles per hour a headland bears due north at 10.35 A.M. At 12.46 P.M. the headland bears 33° west of north. How far was the headland from the ship in each position ?

23. At a distance of 502.3 feet, horizontally, from the center of a bridge the sidewalk rises at an angle of elevation of 5°. The roadway, beginning 203.5 feet farther away from the center, has an angle of elevation of 4° 25′. If a pedestrian and a team enter the bridge at the same moment, which will reach the center first, the man, walking 3.4 miles per hour, or the team, going 5.6 miles per hour ?

24. A flagpole 20 feet long stands on the corner of a building 143.6 feet high. Find the angle subtended by the flagpole from a point 100 feet distant from the foot of the building in a horizontal line.

25. If the radius of a circle is 835.4 feet, what is the length of the chord which subtends an arc of 45° 37′ ?

26. In a circle whose radius is 35.37 inches is inscribed a regular polygon of fifteen sides. Find the length of a side.

27. A tree 214.8 feet high casts a shadow 167.4 feet long. How many degrees is the sun above the horizon ? What is the time of day if the sun rose at six o'clock and will set at six o'clock ?

[Assume that the sun passes through the zenith.]

28. A gable roof is 23.4 feet high and 90.6 feet broad. By how much must the height be reduced to reduce the pitch of the roof 40 per cent ?

NOTE. The pitch of a roof is the angle between the slope of the roof and the horizontal line.

29. From the top of a cliff 378.6 feet above the sea, the angles of depression of a boat and a buoy, in line with the observer, are found to be 29° 20′ and 11° 50′ respectively. Is the boat or the buoy farther from the base of the cliff ? How much farther ?

30. The point B is 1249 feet due east of the point A, and the point C is 376 feet due east of B. The angle of elevation of B above A is 9° 13′ ; of C above B, 7° 23′. A railroad runs from A to C via B. What is the increase in altitude from A to C ?

31. If, in Example 30, the railway could, by grading, be made to run in a straight line from A to C, what would be the angle of elevation of the new route ?

32. How much shorter would the railway of Example 31 be than the railway of Example 30 ?

33. Taking the earth as a sphere of radius 3956 miles, what is the length of the radius of the Arctic Circle, latitude 66° 32′ N. ?

34. Taking the earth as a sphere of radius 3956 miles, what is the latitude of a place which is 2113 miles from the earth's axis ?

35. A vessel sailing due south at a uniform rate observes at 7.15 A.M. that a lighthouse bears 70° east of south. At 8.05 A.M. the lighthouse is 12.75 miles due east from the ship. How far from the ship, and in what direction, will the lighthouse be at 9.30 A.M. ?

36. A ship sailing due south at a uniform rate observes, at 6 A.M., a lighthouse 11.25 miles away, due east. At 6.30 A.M. the lighthouse bears 17° 57′ north of east. What will be the bearings of the lighthouse from the ship at 9 A.M.? How fast does the vessel sail ?

37. Taking the Earth as a sphere with diameter 7912 miles, what is the distance of the farthest point on the Earth's surface visible from the top of a mountain 8200 feet in height ?

38. The towns B and C lie due east from the town A, B being halfway from A to C, which are 5 miles apart. The towns B, C, and D are equally distant from each other. How far is D from A and in what direction ?

39. A ray of light from a source, A, strikes a mirror, 102 mm. broad, at a point two thirds of the way from the edge. The ray is then reflected to E at a perpendicular distance 25.7 mm. from the mirror. Find the length of the path traveled by the ray.

40. From a window of a house, on a level with the bottom of a spire, the angle of elevation of the top of the spire was 41°. From another window, 20.5 feet directly above the former, the like angle was 37° 31′. What was the height of the spire ?

41. Having at a certain (unknown) distance measured the angle of elevation of a cliff, a surveyor walked 60 yards on a level toward the cliff. The angle of elevation from this second station was the complement of the former angle. The surveyor then walked 20 yards nearer the cliff, in the same line, and found the angle of elevation from the third station to be double the first angle. How high was the cliff ?

CHAPTER IV

FUNDAMENTAL IDENTITIES

36. In this chapter we shall discuss some of the important relations of analytical trigonometry. The number of such relations is, of course, unlimited, but there are a few, of frequent occurrence and of fundamental importance, upon which the others depend; it is this fundamental group with which we shall now deal. Let us first observe how the need for some of the relations may arise. We have seen (Art. 27) that as the angle increases from 0° the sine of the angle also increases. But does the sine increase at the same rate as the angle, so that, for instance, if the angle be made twice as large the sine also becomes twice as large? This is obviously not so, for, as we have seen, the sine of 60° is not twice the sine of 30°. What then are the relations, if there be any such, by which we may find the functions of twice an angle when the functions of the angle are given? or again, is there any relation connecting the functions of the sum of two angles with the functions of the angles separately? Such questions as these we shall now proceed to answer.

37. The Addition Formulæ. Let x and y be two acute angles, whose sum may be an angle either in the first quadrant or in the second. Construct, Fig. 18, the angle XOP equal to x and add to it the angle POQ equal to y. Then the angle XOQ is equal to $x + y$. From any point, A, in the terminal side of the combined angle $x + y$ draw AB perpendicular to the axis of x which is the initial side of the angle x. Then OB, BA, and OA are respectively the abscissa, ordinate, and distance of the point A and we may write any function of the angle $x + y$. But as we wish to express the functions of $x + y$ in terms of the functions of x and y, we proceed to draw lines which will give us those functions. Thus, from A draw AC perpendicular to the terminal side of the angle x, and from C draw CD perpendicular to the axis of x and CE perpendicular to AB.

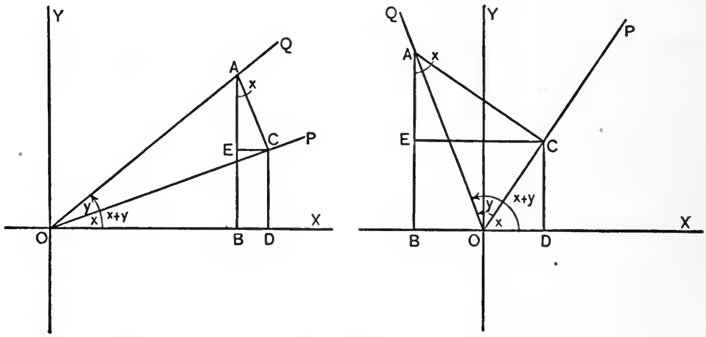


FIG. 18.

Then since AE is perpendicular to OX and AC to OP, the angle EAC is equal to the angle x, each of the angles being acute. We now have

$$\sin(x+y) = \frac{BA}{OA} = \frac{BE + EA}{OA} = \frac{DC}{OA} + \frac{EA}{OA}.$$

But these last two ratios are not functions of any of the angles in the figure. To obtain a function of x or y we must use with DC either OD or OC, and with OA either OC or CA. Therefore we shall multiply and divide $\frac{DC}{OA}$ by the common line OC. Similarly with EA and OA we use CA. Thus we may write

$$\sin(x+y) = \frac{DC}{OC} \cdot \frac{OC}{OA} + \frac{EA}{CA} \cdot \frac{CA}{OA}$$

or

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

In the same way

$$\cos(x+y) = \frac{OB}{OA} = \frac{OD - BD}{OA} = \frac{OD}{OA} - \frac{EC}{OA}$$

$$= \frac{OD}{OC} \cdot \frac{OC}{OA} - \frac{EC}{AC} \cdot \frac{AC}{OA},^{*}$$

or

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

38. The Addition Formulæ (*continued*). Again let x and y be two acute angles where x may be either greater or less

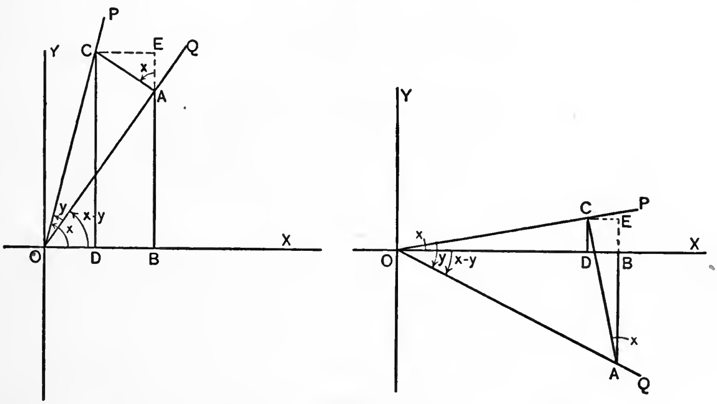


FIG. 19.

than y. Construct, Fig. 19, the angle XOP equal to x and from it subtract the angle QOP equal to y. Then the angle XOQ is equal to $x - y$. From A, any point in the terminal side of the combined angle, draw AB perpendicular to the axis of x which is the initial side of the angle x. Then OB, BA, and OA are, respectively, the abscissa, ordinate and distance of A. From A draw AC perpendicular to the terminal side of the angle x, and from C draw CD perpendicular to the axis of x and CE perpendicular to BA pro-

* Note that in $\frac{EC}{AC}$ we use AC as the positive direction of the line, therefore AC must be positive in the ratio $\frac{AC}{OA}$ also.

duced. Then since AE is perpendicular to OX and AC to OP, the angle EAC is equal to the angle x, each being acute. We now have as in Art. 37,

$$\sin(x-y)=\frac{BA}{OA}=\frac{BE-AE}{OA}=\frac{DC}{OA}-\frac{AE}{OA}$$

$$=\frac{DC}{OC}\cdot\frac{OC}{OA}-\frac{AE}{AC}\cdot\frac{AC}{OA},$$

or,

$$\sin(x-y)=\sin x\cos y-\cos x\sin y.$$

Also

$$\cos(x-y)=\frac{OB}{OA}=\frac{OD+DB}{OA}=\frac{OD}{OA}+\frac{CE}{OA}$$

$$=\frac{OD}{OC}\cdot\frac{OC}{OA}+\frac{CE}{CA}\cdot\frac{CA}{OA},$$

or,

$$\cos(x-y)=\cos x\cos y+\sin x\sin y.$$

39. We have thus proved the formulæ

$$\begin{aligned}\sin(x\pm y)&=\sin x\cos y\pm\cos x\sin y,\\ \cos(x\pm y)&=\cos x\cos y\mp\sin x\sin y,\end{aligned}\qquad(10)$$

for values of x and y less than 90°. It now remains to be proved that these relations are true for all values of x and y. This may be done by a geometric construction as in the cases given, but the following method is preferable.

40. Let x be an angle in the second quadrant and y an angle in the third quadrant. Then we may put $x=90°+a$ and $y=180°+b$, where a and b are acute. We may now write

$$\cos(x+y)=\cos(\overline{90°+a}+\overline{180°+b})$$

$$=\cos(270°+\overline{a+b})$$

$$=\sin(a+b).\qquad\text{(Art. 19)}$$

$$=\sin a\cos b+\cos a\sin b.\qquad\text{(Art. 39)}$$

But $a=-90°+x$ and $b=-180°+y$.

Therefore

$$\cos(x+y) = \sin(-90^\circ + x)\cos(-180^\circ + y) + \cos(-90^\circ + x)\sin(-180^\circ + y)$$

$$= (-\cos x)(-\cos y) + (\sin x)(-\sin y) \qquad \text{(Art. 19)}$$

or

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

which is the same as the relation of Art. 39.

Again, let x be an angle in the first quadrant and y an angle in the third. We may put $y = 180^\circ + b$, where b is acute, and write

$$\sin(x-y) = \sin(x - \overline{180^\circ + b})$$

$$= \sin(-180^\circ + \overline{x - b})$$

$$= -\sin(x - b) \qquad \text{(Art. 19)}$$

$$= -\sin x \cos b + \cos x \sin b. \qquad \text{(Art. 39)}$$

But $b = -180^\circ + y$, and therefore,

$$\sin(x-y) = -\sin x \cos(-180^\circ + y) + \cos x \sin(-180^\circ + y)$$

$$= -\sin x(-\cos y) + \cos x(-\sin y) \qquad \text{(Art. 19)}$$

$$= \sin x \cos y - \cos x \sin y.$$

Thus it may be proved that the equations of Art. 39 are true for all values of x and y.

The importance of these four relations, (10) of Art. 39, can hardly be over-emphasized. From them, together with those given in Art. 12, may be derived all other trigonometric identities. The method of so doing is shown in the following articles, and is illustrated by the following examples:

Example 1. Prove the relation

$$\sin(45^\circ + a)\cos(45^\circ - b) + \cos(45^\circ + a)\sin(45^\circ - b) = \cos(a - b).$$

This is simply a case of the first formula of (10) where

$$x = 45° + a,\ y = 45° - b.$$

We may write

$$\sin(45° + a)\cos(45° - b) + \cos(45° + a)\sin(45° - b)$$
$$= \sin(\overline{45° + a} + \overline{45° - b}) = \sin(90° + a - b) = \cos(a - b).$$

EXAMPLES

Prove that

1. $\sin 105° + \cos 105° = \cos 45°.$
2. $\cos(45° - x)\cos(45° + x) - \sin(45° - x)\sin(45° + x) = 0.$
3. $\sin x \cos(90° - x) - \cos x \sin(90° - x) = -\cos 2x.$
4. $\cos(30° - 45°) - \cos(30° + 45°) = \sin 45°.$
5. Given $\sin x = \frac{3}{5}$, $\cos y = \frac{3}{4}$, find $\sin(x + y)$.
6. Given $\cos x = \frac{1}{3}$, $\cos y = \frac{1}{4}$, find $\cos(x - y)$.

Given $\tan x = 2$, $\tan y = 3$, find

7. $\sin(x + y)$. 8. $\cos(x + y)$. 9. $\sin(x - y)$. 10. $\cos(x - y)$.

41. Tangent of a Sum. To derive an expression for the tangent of the sum or difference of two angles we proceed as follows:

$$\tan(x \pm y) = \frac{\sin(x \pm y)}{\cos(x \pm y)}$$

$$= \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y \mp \sin x \sin y}$$

$$= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} \pm \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} \mp \dfrac{\sin x \sin y}{\cos x \cos y}}.$$

or

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}. \qquad (11)$$

In a similar manner may be proved

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}. \qquad (12)$$

EXAMPLES

Prove that

1. $\tan(45° + x) = \dfrac{1 + \tan x}{1 - \tan x}$.

2. $\tan(45° + x)\tan(135° + x) + 1 = 0$.

3. $\tan(45° + x)\tan(45° - x) = 1$.

4. Given $\tan a = 2$, $\tan b = 4$, find $\tan(a + b)$.

5. Given $\sin a = \frac{1}{2}$, $\cos b = \frac{1}{3}$, find $\tan(a - b)$.

6. Given $\sec a = 3$, $\csc b = 4$, find $\tan(a + b)$.

7. Given $\tan a = \frac{5}{6}$, $\tan b = \frac{1}{11}$, find $a + b$.

8. Given $\sin a = \frac{3}{5}$, $\cos b = \frac{4}{5}$, find $\tan(a + b)$.

9. Given $\sin a = \frac{3}{5}$, $\sin b = \frac{4}{5}$, find $a + b$.

Prove the following identities.

10. $\dfrac{\sin(x + y)}{\cos(x - y)} = \dfrac{\cot x + \cot y}{1 + \cot x \cot y}$.

11. $\dfrac{\tan x - \tan(x - y)}{1 + \tan x \tan(x - y)} = \tan y$.

42. Functions of the Double Angle. The equations of Arts. 39 and 41 being true for all values of x and y, let us assume that $y = x$. Substituting x for y in the functions of the sum of two angles we obtain

$$\sin 2x = 2 \sin x \cos x \tag{13}$$

$$\cos 2x = \cos^2 x - \sin^2 x. \tag{14}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \tag{15}$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}. \tag{16}$$

43. The student should clearly understand that the equations of Art. 42 give the values of functions of twice an angle in terms of functions of the angle, no matter what the value or form of the angle may be. For example, the following relations are all true, being merely the equations of Art. 42 changed slightly in form.

$$2 \sin \frac{a}{2} \cos \frac{a}{2} = \sin a.$$

$$\cos \frac{3x}{2} = \cos^2 \frac{3x}{4} - \sin^2 \frac{3x}{4}.$$

$$\tan (2\alpha + \beta) = \frac{2 \tan \left(\alpha + \frac{\beta}{2}\right)}{1 - \tan^2 \left(\alpha + \frac{\beta}{2}\right)}.$$

44. Functions of the Half-angle. We may write the two proved relations

$$\sin^2 x + \cos^2 x = 1 \text{ and } \cos 2x = \cos^2 x - \sin^2 x$$

in the form

$$\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} = 1.$$

$$\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos x.$$

Subtracting and adding these, we have

$$\left.\begin{aligned} 2 \sin^2 \frac{x}{2} &= 1 - \cos x. \\ 2 \cos^2 \frac{x}{2} &= 1 + \cos x. \end{aligned}\right\} \quad (17)$$

Dividing the last two equations one by the other we obtain

$$\left.\begin{aligned} \tan^2 \frac{x}{2} &= \frac{1 - \cos x}{1 + \cos x}, \\ \cot^2 \frac{x}{2} &= \frac{1 + \cos x}{1 - \cos x}. \end{aligned}\right\} \quad (18)$$

Thus we have equations which give the sine, cosine, tangent, and cotangent of one half an angle in terms of the cosine of that angle.

EXAMPLES

Given $\sin\theta = \frac{1}{3}$, find

1. $\sin 2\theta$.
2. $\cos 2\theta$.
3. $\tan 2\theta$.
4. $\sin\frac{\theta}{2}$.
5. $\cos\frac{\theta}{2}$.
6. $\tan\frac{\theta}{2}$.

Prove the following identities.

7. $\cos^4 x - \sin^4 x = \cos 2x$.
8. $(\sin x + \cos x)^2 = 1 + \sin 2x$.
9. $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}$.
10. $\sin^2\frac{x}{2} = \dfrac{\sec x - 1}{2\sec x}$.
11. $\tan\frac{x}{2} = \dfrac{1 - \cos x}{\sin x}$.
12. $\tan\frac{x}{2} = \dfrac{\sin x}{1 + \cos x}$.
13. $\dfrac{\cos 2x}{1 + \sin 2x} = \tan(45° - x)$.
14. $2\cos^2\frac{x}{2} = \dfrac{1 + \sec x}{\sec x}$.
15. $\tan(45° + x) + \tan(45° - x) = 2\sec 2x$.

45. Sum of Sines or Cosines. By addition and subtraction of the two equations

$$\sin(x + y) = \sin x\cos y + \cos x\sin y,$$
$$\sin(x - y) = \sin x\cos y - \cos x\sin y,$$

we obtain

$$\sin(x + y) + \sin(x - y) = 2\sin x\cos y,$$
$$\sin(x + y) - \sin(x - y) = 2\cos x\sin y.$$

If now we let $x + y = \alpha$, $x - y = \beta$, so that $x = \frac{1}{2}(\alpha + \beta)$ and $y = \frac{1}{2}(\alpha - \beta)$, we obtain from the last two identities

$$\begin{aligned}\sin\alpha + \sin\beta &= 2\sin\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha - \beta),\\ \sin\alpha - \sin\beta &= 2\cos\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\alpha - \beta).\end{aligned} \tag{19}$$

Proceeding in the same way with the equations

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y,$$

we obtain two more equations of importance

$$\begin{aligned}\cos\alpha + \cos\beta &= 2\cos\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha - \beta),\\ \cos\alpha - \cos\beta &= -2\sin\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\alpha - \beta).\end{aligned} \tag{20}$$

EXAMPLES

Express each of the following as the algebraic sum of sines or cosines.

1. $\sin 6x \cos 2x$.
2. $\cos 4x \sin 2x$
3. $\cos \frac{x}{2} \sin \frac{3x}{2}$.
4. $\sin (x + 2y) \cos (x - y)$.
5. $\sin (30° + x) \sin (30° - x)$.
6. $\cos 3x \cos (x - y)$.

Prove the following identities.

7. $\cos (30° - x) \cos (60° - x) = \frac{1}{4}(2 \sin 2x + \sqrt{3})$.
8. $\cos 3x \sin 2x - \cos 4x \sin x = \cos 2x \sin x$.
9. $\sin x \cos (x + y) - \cos x \sin (x - y) = \cos 2x \sin y$.

46. Identities and Equations. It should be borne in mind that all of the equations of this chapter are *identities*, that is, they are true no matter what values the angles may have. We shall deal later on, in Chapter VII, with trigonometric equations of condition, where we shall find that not every value but only particular values of the angles involved will satisfy the equations. Also, in connection with this chapter attention should be again called to the group of fundamental identities in Art. 12.

ILLUSTRATIVE EXAMPLES

Example 1. Prove that $\sec 2x = 1 + \tan x \tan 2x$.

$$\sec 2x = \frac{1}{\cos 2x} = \frac{1}{\cos^2 x - \sin^2 x} = \frac{\frac{1}{\cos^2 x}}{1 - \tan^2 x} = \frac{\sec^2 x}{1 - \tan^2 x}$$

$$= \frac{1 + \tan^2 x}{1 - \tan^2 x} = 1 + \frac{2 \tan^2 x}{1 - \tan^2 x}$$

$$= 1 + \tan x \cdot \frac{2 \tan x}{1 - \tan^2 x} = 1 + \tan x \tan 2x.$$

By the above method we begin with $\sec 2x$ and deduce or derive the required result. Another method of procedure is as follows:

Assume that

$$\sec 2x = 1 + \tan x \tan 2x.$$

Then

$$\sec 2x = 1 + \frac{2\tan^2 x}{1 - \tan^2 x}$$

$$= \frac{1 + \tan^2 x}{1 - \tan^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x}$$

$$= \frac{1}{\cos 2x}.$$

$$= \sec 2x.$$

Therefore, the original assumption is correct.

Example 2. Prove that $\csc 2x = \frac{1}{2}\sec x \csc x$.
First method.

$$\csc 2x = \frac{1}{\sin 2x} = \frac{1}{2\sin x \cos x} = \frac{\sec x \csc x}{2}.$$

Second method. Take the reciprocals of both members.

$$\sin 2x = 2\sin x \cos x.$$

EXAMPLES

Without the use of tables find the following:

1. Sine and cosine of 15°.
2. Sine and cosine of 22° 30′.
3. Tangent of 15°.
4. Tangent of 22° 30′.
5. Find the value of $\sin 3x$ in terms of $\sin x$.
6. Find the value of $\cos 3x$ in terms of $\cos x$.
7. Find the value of $\tan 3x$ in terms of $\tan x$.
8. Find the value of $\tan 4x$ in terms of $\tan x$.
9. Find the value of $\sin 4x$ in terms of functions of x.
10. Find the value of $\cos 4x$ in terms of functions of x.
11. Given $\sin 4x = a$, $\cos 4x = b$, find $\sin 8x$ and $\cos 8x$.
12. Given $\tan 3x = a$, find $\tan 6x$.

Prove the following identities.

13. $\sin(90° + x + y) = \cos x \cos y - \sin x \sin y$.

14. $\sin\frac{a}{2}\cos\frac{b}{2} + \cos\frac{a}{2}\sin\frac{b}{2} = \sin\frac{1}{2}(a+b)$.

15. $\cos 6a = 1 - 2\sin^2 3a$.

16. $\frac{\sin 4a + \sin 2a}{\sin 4a - \sin 2a} = \frac{3 - \tan^2 a}{1 - 3\tan^2 a}$.

17. $\frac{1}{1 + \cos^2\theta} = \frac{\sec^2\theta}{\tan^2\theta + 2}$.

18. $\sin^2 x \cos^2 x = \frac{1 - \cos 4x}{8}$.

19. $\cos 5x \cos 2x = \frac{1}{2}(\cos 7x + \cos 3x)$.

20. $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$.

21. $\frac{\sin 3x + \sin 5x}{\sin 4x} = 2\cos x$.

22. $\cos 4x \sin x = \frac{1}{2}(\sin 5x - \sin 3x)$.

23. $\frac{\sin 4x + 2\sin 3x + \sin 2x}{\sin 3x} = 4\sin^2\frac{x}{2}$.

24. $\sin x = \frac{2\tan\frac{1}{2}x}{1 + \tan^2\frac{1}{2}x}$.

25. $\sin x = \frac{2}{\cot\frac{1}{2}x + \tan\frac{1}{2}x}$.

26. $\tan 2x = \frac{2}{\cot x - \tan x}$.

27. $\frac{\sin(x + 2y) - \sin(x - 2y)}{\sin y} = 4\cos x \cos y$.

28. $\frac{\cos(2x - y) - \cos(2x + y)}{\cos y} = 4\sin x \sin y$.

29. $\tan^2\frac{x}{2} + 2\cot x \tan\frac{x}{2} = 1$.

30. $\tan^2\frac{x}{2} - 2\csc x \tan\frac{x}{2} + 1 = 0$.

31. $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$.

32. $\tan\frac{1}{2}x = \csc x - \cot x$.

33. $\cot\frac{x}{2} = \csc x + \cot x$.

34. $\tan \frac{1}{2} x = \frac{1 + \sin x - \cos x}{1 + \sin x + \cos x}$.

35. $\frac{\sec^2 (x + 45°)}{2 \tan (x + 45°)} = \sec x$.

36. $\frac{\cos x - \cos 3x}{\sin 3x - \sin x} = \tan 2x$.

37. $\sin x (1 + \tan x) + \cos x (1 + \cot x) = \sec x + \csc x$.

38. $\cot x - \cot 2x = \csc 2x$.

39. $\frac{1 + \sin 2x}{\cos 2x} = \frac{\cos x + \sin x}{\cos x - \sin x}$.

40. $\sin x \cos^3 x - \cos x \sin^3 x = \frac{\sin 4x}{4}$.

41. Given $\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$, find the value of $\cos x$.

42. Given $r^2 = a^2 \sin 2\theta$,

Prove $$\frac{r^2 \cos \theta + a^2 \cos 2\theta \sin \theta}{a^2 \cos 2\theta \cos \theta - r^2 \sin \theta} = \tan 3\theta.$$

43. Given $r = a \sec^2 \frac{\theta}{2}$,

Prove $$\frac{r \cos \theta + a \sec^2 \frac{\theta}{2} \tan \frac{\theta}{2} \sin \theta}{a \sec^2 \frac{\theta}{2} \tan \frac{\theta}{2} \cos \theta - r \sin \theta} = -\cot \frac{\theta}{2}.$$

44. Given $r = a (1 - \cos \theta)$,

Prove $$\frac{r \cos \theta + a \sin^2 \theta}{a \sin \theta \cos \theta - r \sin \theta} = \tan \frac{3\theta}{2}.$$

Prove the following identities.

45. $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x} = 2 \cot 2x$.

46. $1 + \cos 2x \cos 2y = 2 (\sin^2 x \sin^2 y + \cos^2 x \cos^2 y)$.

47. $\frac{\sin^2 x \cos^2 y - \cos^2 x \sin^2 y}{\cos^2 x \cos^2 y - \sin^2 x \sin^2 y} = \tan (x + y) \tan (x - y)$.

48. $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \frac{1}{2} \sin 2x$.

49. $\frac{\cos 2\theta + \cos \theta + 1}{\sin 2\theta + \sin \theta} = \cot \theta$.

50. $(\sec 2\theta + 1)\sqrt{\sec^2\theta - 1} = \tan 2\theta.$

51. $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} - \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\tan 2\theta.$

52. $\tan 2\theta - \sec\theta \sin\theta = \tan\theta \sec 2\theta.$

53. $\dfrac{1 - \cos x + \cos y - \cos(x + y)}{1 + \cos x - \cos y - \cos(x + y)} = \dfrac{\tan \frac{1}{2} x}{\tan \frac{1}{2} y}.$

54. Given $\left.\begin{aligned} x &= -3\cos^2\theta \sin\theta \\ y &= 3\sin^2\theta\cos\theta \end{aligned}\right\}$ Prove that $2\sqrt{x^2 + y^2} = 3\sin 2\theta.$

55. Given $x = a\cos\theta - r\sin\theta,$
$y = a\sin\theta + r\cos\theta,$
$r^2 = 2\cos 2\theta,$
$a = -\dfrac{2\sin 2\theta}{\sqrt{2\cos 2\theta}},$
Prove, $x^2 + y^2 = 2\sec 2\theta.$

CHAPTER V

THE CIRCULAR OR RADIAN MEASURE OF AN ANGLE. INVERSE TRIGONOMETRIC FUNCTIONS

47. Circular or Radian Measure of an Angle. Any convenient unit may be chosen for the measurement of angles. We have hitherto used the degree, subdivided into minutes and seconds, as the unit,* but we shall now introduce another unit called the *radian*, the unit angle in the circular measure of angles.

The *radian* is an *angle at the centre of a circle whose subtending arc is equal to the radius* of the circle.

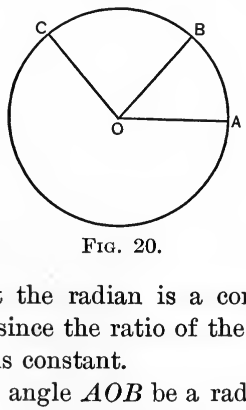


Fig. 20.

It is obvious that the radian is a constant angle, is the same in all circles, since the ratio of the circumference of a circle to its radius is constant.

In Fig. 20 let the angle AOB be a radian, that is, let the arc AB be equal to OA, the radius of the circle. Also let the angle AOC be an angle to be measured in radians. To measure a quantity is to find its ratio to another quantity of the same kind chosen as the unit. Therefore,

* It may be noted that the right angle is used as a unit in the study of geometry, partly because it is an angle easily constructed.

$$\text{Circular measure } AOC = \frac{AOC}{AOB} \text{ radians.}$$

But
$$\frac{AOC}{AOB} = \frac{\text{arc } AC}{\text{arc } AB} = \frac{\text{arc } AC}{\text{radius } OA}.$$

Therefore, *to measure an angle in circular measure*, or in other words to express the angle in radians, *find the ratio of the arc subtending the angle* in any circle *to the radius* of the circle.

If we represent the angle, measured in radians, by x, the length of the arc subtending the angle by s, and the radius of the circle by r, we have the relation $x = \frac{s}{r}$. This is an algebraic equation involving three quantities. If any two of the quantities are known the third can be found. Thus

$$x = \frac{s}{r}, \quad s = rx, \quad r = \frac{s}{x}.$$

Example 1. What is the radius of a circle in which an arc of 12 inches subtends an angle of $1\frac{1}{2}$ radians?

$$r = \frac{s}{x} = \frac{12}{1\frac{1}{2}} = 8 \text{ inches.}$$

Example 2. If the radius of a circle is 15 feet what length of arc subtends an angle of two-thirds of a radian?

$$s = 15 \times \tfrac{2}{3} = 10 \text{ feet.}$$

We know that the ratio of a semicircumference to its radius is $\pi = 3.1416$. It follows, therefore, that the angle which is sometimes called a straight angle, and which is expressed as 180°, may also be expressed as π radians. Thus,

$$\pi \text{ radians} = 180°$$

$$1 \text{ radian} = \frac{180°}{\pi} = 57.296°, \text{ approximately.}$$

Also
$$180° = \pi \text{ radians}$$

$$1° = \frac{\pi}{180} \text{ radians.}$$

By means of these two relations we can readily reduce a given angle from either system of measurement to the other.

Example 1. Express the angle 7 radians * in degrees.

$\pi = 180°$. Therefore,

$$1 = \frac{180°}{\pi}. \qquad 7 = 7 \times \frac{180°}{\pi} = \frac{1260°}{\pi} = 401.03°.$$

Example 2. Express the angle $\frac{2\pi}{3}$ in degrees.

Since $\pi = 180°$, $\frac{2\pi}{3} = \frac{2}{3} \times 180° = 120°$.

Example 3. Express the angle 110° 32′ 30″ in circular measure.

$$110°\ 32'\ 30'' = 110\tfrac{13}{24}° = \tfrac{2653}{24}°.$$

Since $180° = \pi$, therefore,

$$1° = \frac{\pi}{180}, \qquad \frac{2653°}{24} = \frac{\pi}{180} \times \frac{2653}{24} = \frac{2653\pi}{4320}.$$

An angle in circular measure is usually expressed as a multiple of π.

EXAMPLES

Express the following angles in degrees.

1. 6.5 radians.
2. $\frac{2\pi}{5}$.
3. $\frac{\pi}{7}$.
4. 3.8 radians.
5. $\frac{7\pi}{6}$.
6. $\frac{5\pi}{8}$.

Express the following angles in circular measure.

7. 270°.
8. 13° 24′.
9. 25° 16′.
10. − 450°.
11. 208° 30′.
12. − 98°.
13. What is the ratio of a radian to a right angle ?

* The name radian is often omitted. An angle written $\frac{\pi}{3}$ and read "pi over three," means "pi over three" radians, or about 1.05 radians.

How many right angles are there in each of the following angles?

14. $\frac{\pi}{5}$. **15.** $\frac{3\pi}{7}$. **16.** $\frac{2\pi}{3}$. **17.** 5 radians.

18. Through how many radians do the minute and hour hands of a clock turn in 30 minutes?

19. Through how many radians does the minute hand of a clock turn in 35 minutes?

20. Through how many radians does the hour hand of a clock turn in 18 minutes?

21. The front wheel of a cart is 2 feet in diameter, the hind wheel 3 feet. Through how many radians will the hind wheel turn while the front wheel is turning through 600°?

22. Through how many radians does the earth revolve about its axis in a week? Is the result the same in 45° north latitude as at the equator?

23. A wheel turns 50 revolutions per minute. Express its angular velocity in radians per second.

24. A wheel has an angular velocity of 20 radians per second. How many revolutions does it make per minute?

25. Through how many miles will a point on the equator of the earth travel as the earth turns through $1\frac{1}{2}$ radians?

26. Through how many miles will a point at 45° north latitude travel as the earth turns through one radian?

27. The radius of a graduated quadrant is 2 feet, and the graduations are 5′ apart. What is the distance between successive graduations?

28. What must be the radius of a graduated quadrant if the distance between graduations 5′ apart is to be $\frac{1}{10}$ inch?

48. Inverse Trigonometric Functions. Let us suppose that y is the sine of the angle x. We express this briefly in mathematical symbols as $y = \sin x$. Suppose now that we wish to make the inverse statement that x is the angle whose sine is y. To express this in mathematical symbols we write $x = \sin^{-1} y$, where, it must be noted, the *minus unity is not an exponent.* Having expressed our idea in symbols we next note that x depends upon y for its value, is a function of y, and we name the function the anti-sine or inverse sine. Similarly $a = \tan^{-1} b$ means that a is the angle whose tangent is b, and we say that a is the anti-tangent of b. In this way we have a group of six inverse trigonometric functions,

$$\sin^{-1} x, \qquad \tan^{-1} x, \qquad \sec^{-1} x,$$
$$\cos^{-1} x, \qquad \cot^{-1} x, \qquad \csc^{-1} x.$$

These six quantities, it must be remembered, are angles.

49. General Value of an Angle. Identities connecting the various inverse trigonometric functions exist and may be derived or proved by methods analogous to those of Chapter IV. Before taking them up, however, one important dif-

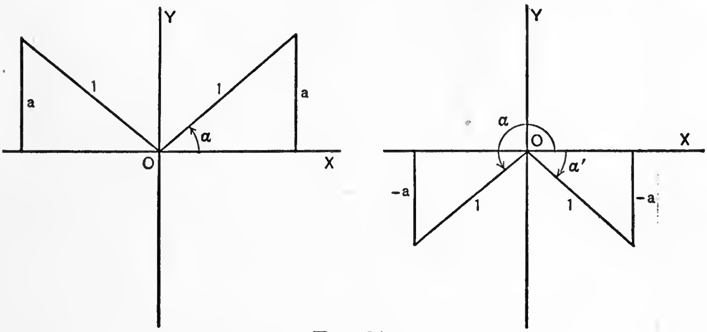


FIG. 21.

ference between the direct and the inverse trigonometric functions must be noted.

If $y = \sin x$, and if we give x a particular value, say 30°, then y will have one and only one value, one-half. On the other hand, if $x = \sin^{-1} y$ and if we give y a particular value, say $\frac{1}{2}$, then x does not have one value only but an infinite number of values, 30°, 150°, 390°, $-210°$ etc. This being so it is well to get an expression that will represent all the angles which have a given value of the sine, cosine, etc.

Let $\sin x = \pm a$, a being a positive number, or $x = \sin^{-1}(\pm a)$ and let α be the smallest angle* which has for

* That is, α if we use positive angles only; α' if negative angles also are used. Either method may be adopted.

its sine the value $\pm a$. By Fig. 21 we see that the possible values of x are

$$\alpha, \quad \pi-\alpha, \quad 2\pi+\alpha, \quad 3\pi-\alpha, \quad 4\pi+\alpha, \cdots,$$
$$-\pi-\alpha, \quad -2\pi+\alpha, \quad -3\pi-\alpha, \quad -4\pi+\alpha, \cdots,$$

which may be written in the general form

$$x=\sin^{-1} a = n\pi + (-1)^n \alpha \qquad (21)$$

where n is any positive or negative integer, including zero, and α is the least angle whose sine is a. This is called the *general value* of the angle and α is called the *principal value.*

Since the cosecant is the reciprocal of the sine we may write

$$x=\csc^{-1} a = n\pi + (-1)^n \alpha. \qquad (22)$$

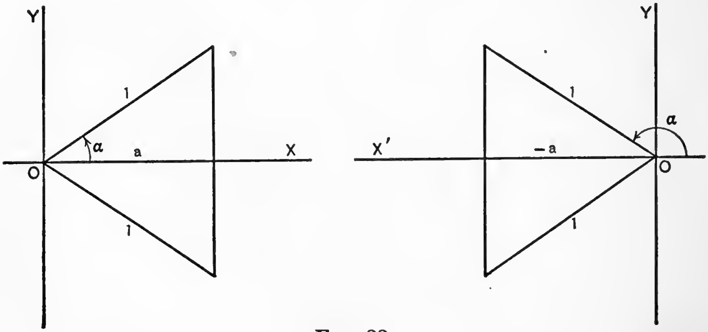


Fig. 22.

Let $\cos x = \pm a$ or $x = \cos^{-1}(\pm a)$ and let α be the least angle whose cosine is $\pm a$. By Fig. 22 we see that possible values of x are

$$\alpha, \; 2\pi\pm\alpha, \; 4\pi\pm\alpha, \cdots,$$

or, in the general form

$$x=\cos^{-1} a = 2n\pi \pm \alpha \qquad (23)$$

where n is any positive or negative integer, including zero, and α is the least angle whose cosine is a.

Since $\sec x = \frac{1}{\cos x}$ we may write for the general value, α

being the principal value,

$$x = \sec^{-1} a = 2\,n\pi \pm \alpha. \tag{24}$$

Let $\tan x = \pm a$ or $x = \tan^{-1}(\pm a)$, and let α be the least angle whose tangent is $\pm a$. By Fig. 23 we see that pos-

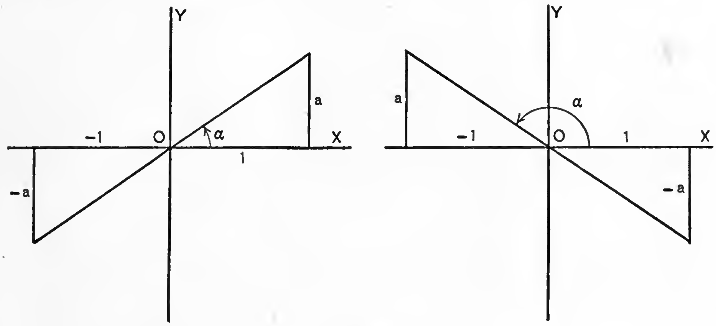


FIG. 23.

sible values of x are

$$\alpha, \quad \pi + \alpha, \quad 2\pi + \alpha, \quad 3\pi + \alpha, \cdots,$$
$$-\pi + \alpha, \quad -2\pi + \alpha, \quad -3\pi + \alpha, \cdots,$$

or, for the general value,

$$x = \tan^{-1} a = n\pi + \alpha \tag{25}$$

where n is any positive or negative integer, including zero, and α is the principal value.

Giving n and α the same meaning we may write, since $\cot x = \dfrac{1}{\tan x}$,

$$x = \cot^{-1} a = n\pi + \alpha. \tag{26}$$

One need not make use of the formulæ, 21–26, but may proceed as follows: Find the *two smallest angles*, positive or negative, which correspond to the given value of the function. If we call these angles α and β then the complete series of angles will be given by

$$2\,n\pi + \alpha \text{ and } 2\,n\pi + \beta.$$

Example 1. Write the general value of $\cos^{-1} .9205$.

From the tables of trigonometric functions we find $\alpha = 23°$. Therefore

$$x = \cos^{-1} .9205 = 2\,n\pi \pm 23° = 2\,n\pi \pm \frac{23\,\pi}{180}.$$

Example 2. Write the general value of $\sin^{-1} 1$. We know $\alpha = 90° = \frac{\pi}{2}$. Therefore

$$x = \sin^{-1} 1 = n\pi + (-1)^n \frac{\pi}{2}.$$

Example 3. Prove the identity

$$2 \sin^{-1} a = \sin^{-1} (2\,a\sqrt{1 - a^2}).$$

Let $\sin^{-1} a = x$, then $\sin x = a$. Substituting these values in the formula to be proved we have

$$2\,x = \sin^{-1} (2 \sin x\sqrt{1 - \sin^2 x}),$$

or

$$\sin 2\,x = 2 \sin x\sqrt{1 - \sin^2 x} = 2 \sin x \cos x.$$

Q. E. D.

Or, we may proceed as follows:

We know $\sin 2\,x = 2 \sin x \cos x$, which may be written

$$2\,x = \sin^{-1}(2 \sin x \cos x).$$

Let $\sin^{-1} a = x$, $\sin x = a$ and substitute:

$$2 \sin^{-1} a = \sin^{-1} (2\,a\sqrt{1 - a^2}).$$ Q. E. D.

Example 4. Find the principal value of $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$. We note that this is the sum of two angles each given by the value of its tangent. We therefore write, formula (11),

$$\tan (\tan^{-1} \tfrac{1}{2} + \tan^{-1} \tfrac{1}{3}) = \frac{\tan (\tan^{-1} \frac{1}{2}) + \tan(\tan^{-1} \frac{1}{3})}{1 - \tan(\tan^{-1} \frac{1}{2}) \cdot \tan (\tan^{-1} \frac{1}{3})}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1.$$

Therefore, $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} 1 = \frac{\pi}{4}$.

EXAMPLES

Write the general values of the following angles:

1. $\sin^{-1}\frac{1}{2}$. 3. $\tan^{-1}(-1)$. 5. $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$. 7. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

2. $\cos^{-1}0$. 4. $\cot^{-1}\frac{1}{\sqrt{3}}$. 6. $\csc^{-1}\sqrt{2}$. 8. $\cot^{-1}(-\sqrt{3})$.

Find the value of

9. $\sin(\sin^{-1}a)$. 11. $\tan(\tan^{-1}y)$. 13. $2\cos(\cos^{-1}.523)$.

10. $\cos^{-1}(\cos x)$. 12. $\sec^{-1}(\sec 30°)$. 14. $\cot(\cot^{-1}2.718)$.

15. $\sin^{-1}(\cos 35°)$.

16. $\tan^{-1}(\cot 40°)$.

17. $\sin\left(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{\sqrt{2}}\right)$.

18. $\cos\left(\tan^{-1}1+\sec^{-1}\frac{2}{\sqrt{3}}\right)$.

19. $\tan(\sin^{-1}1+\cos^{-1}\frac{1}{2})$.

20. $\cot\left(\tan^{-1}\sqrt{3}+\tan^{-1}\frac{1}{\sqrt{3}}\right)$.

21. Prove that $x=\sec^{-1}\sqrt{1+\tan^2 x}$.

22. Prove that $\tan^{-1}y=\sec^{-1}\sqrt{1+y^2}$.

Prove the following:

23. $\tan^{-1}(\sqrt{2}+1)-\tan^{-1}(-\sqrt{2}-1)=135°$.

24. $\tan^{-1}\sqrt{3}-\tan^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\tan^{-1}(-3\sqrt{3})$.

25. $\tan^{-1}\frac{1}{3}-\tan^{-1}(-1)=\tan^{-1}2$.

26. $\tan^{-1}\frac{a+b}{a-b}-\tan^{-1}\frac{b+a}{b-a}=\tan^{-1}\frac{b^2-a^2}{2ab}$.

27. $\tan^{-1}\frac{b}{\sqrt{a^2-b^2}}-\tan^{-1}\frac{-b}{\sqrt{a^2-b^2}}=\tan^{-1}\frac{2b\sqrt{a^2-b^2}}{a^2-2b^2}$.

28. $\tan^{-1}\left(-\frac{a^2}{b^2}\right)-\tan^{-1}\left(\frac{b^2}{a^2}\right)=\frac{\pi}{2}$.

29. $\tan^{-1}\frac{2b\sqrt{a^2-b^2}}{a^2-2b^2}=2\sin^{-1}\frac{b}{a}$.

30. $\sin^{-1}x+\cos^{-1}y=\tan^{-1}\frac{xy+\sqrt{1-x^2}\sqrt{1-y^2}}{y\sqrt{1-x^2}-x\sqrt{1-y^2}}$.

31. $\sec^{-1}x-\csc^{-1}y=\cos^{-1}\frac{\sqrt{x^2-1}+\sqrt{y^2-1}}{xy}$.

CHAPTER VI

THE SOLUTION OF GENERAL TRIANGLES

50. Four Cases. — As in the case of right triangles the solution of any triangle means the finding of the values of unknown parts from the parts that are known. Of the six parts (three angles and three sides) there must be given three, one of which at least is a side, in order that the triangle may be solved. Consider any triangle, Fig. 24.

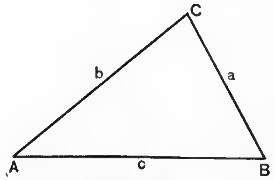


FIG. 24.

The following cases may be enumerated:

I. *Given a side and two angles; say,* a, A, B.

II. *Given two sides and the angle opposite one of them; say,* a, b, A.

III. *Given two sides and the included angle; say,* a, b, C.

IV. *Given the three sides;* a, b, c.

51. The Law of Sines. — Cases I and II may be solved by means of the following theorem.

In any triangle the sides are proportional to the sines of the opposite angles. That is, Fig. 25,

$$a : b : c = \sin A : \sin B : \sin C. \tag{27}$$

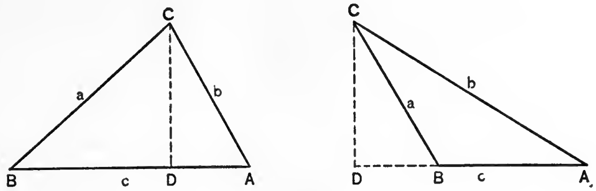


FIG. 25.

Proof: In the triangle BAC draw CD perpendicular to BA. Then

$$\sin A = \frac{DC}{AC} = \frac{DC}{b} \text{ and } \sin B = \frac{DC}{BC} = \frac{DC}{a}.$$

Therefore

$$DC = b \sin A = a \sin B.$$

Whence

$$a : b = \sin A : \sin B.$$

Similarly the theorem may be proved for the other pairs of sides and angles.

52. Case I. *Given a side and two angles ; a, A, C.*

To find the third angle we have

$$B = 180° - (A + C).$$

To find B and C we have

$$\frac{b}{a} = \frac{\sin B}{\sin A}, \qquad \frac{c}{a} = \frac{\sin C}{\sin A},$$

selecting in each case that proportion, from (27), which involves an unknown side, b or c, and three known parts.

From these two proportions we have

$$\log b = \log a + \log \sin B + \text{colog} \sin A$$

$$\log c = \log a + \log \sin C + \text{colog} \sin A$$

Example. Given $a = 412.7$, $A = 50° 38'$, $C = 69° 13'$, find B, b, c.

$$B = 180° - 119° 51' = 60° 9'.$$

$$\begin{array}{rl} \log a &= 2.6157 \\ \log \sin B &= 9.9382 \\ \text{colog} \sin A &= 0.1118 \\ \hline \log b &= 2.6657 \\ b &= 463.1 \end{array} \qquad \begin{array}{rl} \log a &= 2.6157 \\ \log \sin C &= 9.9708 \\ \text{colog} \sin A &= 0.1118 \\ \hline \log c &= 2.6983 \\ c &= 499.2 \end{array}$$

53. Case II. *Given two sides and the angle opposite one of them; a, b, A.*

We have, to find B,

$$\frac{\sin B}{\sin A} = \frac{b}{a}.$$

Whence, $\log \sin B = \log \sin A + \log b + \text{colog}\, a.$

Also, $$C = 180° - (A + B).$$

Then $$\frac{c}{a} = \frac{\sin C}{\sin A}.$$

Whence $$\log c = \log a + \log \sin C + \text{colog} \sin A.$$

Example. Given $a = 31.24$, $b = 49$, $A = 32° 18'$, find B, C, c.

$$\begin{array}{rl} \log \sin A &= 9.7278 \\ \log b &= 1.6902 \\ \text{colog}\, a &= 8.5053 \\ \hline \log \sin B &= 9.9233 \\ B &= 56° 56' \end{array}$$

But since B is found from the log sine it may have two values; namely, $56° 56'$ and $180° - 56° 56' = 123° 4'$. To determine which value is correct or whether both are possible we recall the theorem of geometry which states that if the given angle is acute and the side opposite is less than the other given side, then it may be possible to construct two triangles from the given parts, two sides and an opposite angle. In the above example the given angle A is acute and its opposite side a is less than b; there are two solutions and both values of B must be used. Figure 26 explains the case graphically.

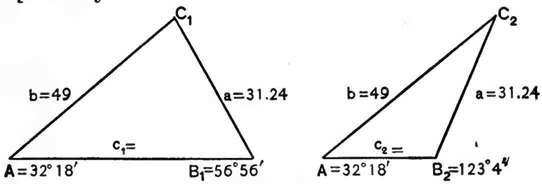


FIG. 26.

Consequently there are two values of C, namely,

$$C_1 = 180° - (A + B_1) \qquad C_2 = 180° - (A + B_2)$$
$$= 90°\ 46' \qquad\qquad = 24°\ 38'$$

and two values of c, got as follows:

$$\begin{array}{rl} \log a = & 1.4947 \\ \log \sin C_1 = & 0.0000 \\ \text{colog} \sin A = & 0.2722 \\ \hline \log c_1 = & 1.7669 \\ c_1 = & 58.46 \end{array} \qquad \begin{array}{rl} \log a = & 1.4947 \\ \log \sin C_2 = & 9.6199 \\ \text{colog} \sin A = & 0.2722 \\ \hline \log c_2 = & 1.3868 \\ c_2 = & 24.37 \end{array}$$

If the given angle be obtuse there will be only one solution. If the given angle, A, be acute and the side a be greater than the side b, there will be one solution only. If A be acute and a be equal to the perpendicular from C to AB, there will be only one solution, a right triangle. In this case $B = 90°$ and $\log \sin B = 0.0000$. If, A being acute, a be less than the perpendicular from C to AB, there is no solution. In this case $\log \sin B$ will be greater than zero, which is impossible since $\sin B$ cannot be greater than unity.

54. Case III may be solved by means of the theorem following:

In any triangle the sum of two sides is to their difference as the tangent of half the sum of the angles opposite the two sides is to the tangent of half their difference.

Proof: By Art. 51

$$a : b = \sin A : \sin B.$$

Whence, by composition and division,

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}$$

or,

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}. \qquad (28)$$

55. Case III.* *Given two sides and the included angle; b, c, A.*

By Art. 54 we have

$$\frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{b-c}{b+c}.$$

The sides b and c are known, and also

$$\tfrac{1}{2}(B+C) = \tfrac{1}{2}(180^\circ - A) = 90^\circ - \tfrac{1}{2}A,$$

since $\quad A + B + C = 180^\circ.$

Therefore we may write

$$\tan \tfrac{1}{2}(B-C) = \frac{b-c}{b+c} \cdot \tan \tfrac{1}{2}(B+C),$$

or,

$$\log \tan \tfrac{1}{2}(B-C) = \log(b-c) + \text{colog}(b+c) + \log \tan \tfrac{1}{2}(B+C).$$

Thus $\frac{1}{2}(B-C)$ is found, and by finding the sum and difference of $\frac{1}{2}(B+C)$ and $\frac{1}{2}(B-C)$ the values of B and C are known. Finally, to determine a we have, as in Case I:

$$a : b = \sin A : \sin B.$$

Example. Given $b = .06239$, $c = .02348$, $A = 110^\circ\ 32'$; find B, C, a.

$$\tfrac{1}{2}(B+C) = 90^\circ - \tfrac{1}{2}A = 90^\circ - 55^\circ\ 16' = 34^\circ\ 44'.$$

$$b + c = .08587 \qquad b - c = .03891.$$

* See also Art. 61 following.

Whence we have

$$\log (b - c) = 8.5900$$
$$\text{colog } (b + c) = 1.0661$$
$$\log \tan \tfrac{1}{2}(B + C) = 9.8409$$
$$\log \tan \tfrac{1}{2}(B - C) = 9.4970$$
$$\tfrac{1}{2}(B - C) = 17° \ 26'.$$

And as $\tfrac{1}{2}(B + C) = 34° \ 44'$

We have $B = 52° \ 10'$ $C = 17° \ 18'$.

Then

$$\log b = 8.7951$$
$$\log \sin A = 9.9715$$
$$\text{colog } \sin B = 0.1025$$
$$\log a = 8.8691$$
$$a = .07398$$

56. The Law of Cosines. Case IV may be solved by means of the following theorem:

In a triangle the square of any side is equal to the sum of the squares of the other two sides minus twice the product of those sides by the cosine of their included angle.

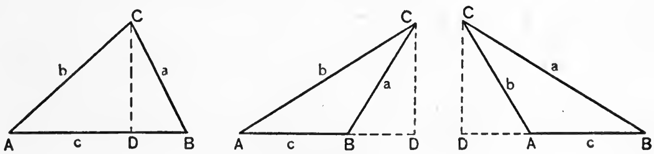


FIG. 27.

That is, Fig. 27, where CD is perpendicular to AB,

$$a^2 = b^2 + c^2 - 2\,bc \cos A.$$

We have proved the geometrical theorem

$$a^2 = b^2 + c^2 - 2\,c \cdot AD.^*$$

But $\cos A = \dfrac{AD}{b}$, or $AD = b \cos A$.

* Note that in the first two triangles of Fig. 27, AD, the projection of b, is read left to right and is positive; in the third triangle from right to left and is negative.

Therefore, $a^2 = b^2 + c^2 - 2\,bc \cos A.$ Q.E.D.

Obviously
$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} \tag{29}$$

and in the same way

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}$$

so that the three angles may be found.

57. The objection to the formulæ of Art. 56 is that they are not adapted to logarithmic computation. To remove this objection we proceed as follows: From (29) we have

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{a^2 - b^2 + 2\,bc - c^2}{2\,bc} = \frac{a^2 - (b-c)^2}{2\,bc}$$
$$= \frac{(a - b + c)(a + b - c)}{2\,bc},$$

or, by (17), Art. 44,

$$2 \sin^2 \tfrac{1}{2} A = \frac{(a - b + c)(a + b - c)}{2\,bc}.$$

Let $a + b + c = 2\,s.$

Then $a - b + c = 2(s - b), \ a + b - c = 2(s - c),$

Whence $2 \sin^2 \tfrac{1}{2} A = \dfrac{2(s - b) \cdot 2(s - c)}{2\,bc},$

or
$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}. \tag{30}$$

Similarly

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s - c)(s - a)}{ca}}, \quad \sin \tfrac{1}{2} C = \sqrt{\frac{(s - a)(s - b)}{ab}}.$$

These formulæ may be used for logarithmic computation.

58. Again, from (29), Art. 56, we have

$$1+\cos A = 1+\frac{b^2+c^2-a^2}{2\,bc}=\frac{(b+c)^2-a^2}{2\,bc}$$

$$=\frac{(a+b+c)(b+c-a)}{2\,bc},$$

or, by (17), Art. 44,

$$2\cos^2 \tfrac{1}{2}A=\frac{(a+b+c)(b+c-a)}{2\,bc}.$$

As before, letting $a+b+c=2\,s$, this becomes

$$2\cos^2 \tfrac{1}{2}A=\frac{2\,s\cdot 2(s-a)}{2\,bc},$$

or

$$\cos \tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}. \qquad (31)$$

Similarly

$$\cos \tfrac{1}{2}B=\sqrt{\frac{s(s-b)}{ca}}, \qquad \cos \tfrac{1}{2}C=\sqrt{\frac{s(s-c)}{ab}}.$$

59. These formulæ also may be used for logarithmic computation, but a more convenient set is obtained by dividing the formulæ of Art. 57 by those of Art. 58. We thus obtain

$$\tan \tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad (32)$$

$$\tan \tfrac{1}{2}B=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \qquad \tan \tfrac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

A comparison of these three sets of formulæ (30), (31), (32), will show that for the complete solution of a triangle when the three sides are given, the first set (30), requires six different logarithms, the second set seven, the third set four. In addition to this slight advantage the tangent set, (32), gives more accurate results than the other two when the angles involved happen to be very small or very near ninety degrees.

60. Case IV. We will now solve a triangle when the *three sides* are known.

Example. Given $a = 10$, $b = 12$, $c = 14$; find A, B, C.

Here $2s = a + b + c = 36$, so that we have

$$s = 18, \quad \log s = 1.2553, \quad \text{colog } s = 8.7447 - 10.$$
$$s - a = 8, \quad \log (s - a) = 0.9031, \quad \text{colog } (s - a) = 9.0969 - 10.$$
$$s - b = 6, \quad \log (s - b) = 0.7782, \quad \text{colog } (s - b) = 9.2218 - 10.$$
$$s - c = 4, \quad \log (s - c) = 0.6021, \quad \text{colog } (s - c) = 9.3979 - 10.$$

$$\begin{aligned} \log (s - b) &= 0.7782 \\ \log (s - c) &= 0.6021 \\ \text{colog } s &= 8.7447 \\ \text{colog } (s - a) &= 9.0969 \\ \hline & 2)19.2219 \end{aligned} \qquad \begin{aligned} \log (s - c) &= 0.6021 \\ \log (s - a) &= 0.9031 \\ \text{colog } s &= 8.7447 \\ \text{colog } (s - b) &= 9.2218 \\ \hline & 2)19.4717 \end{aligned}$$

$$\begin{aligned} \log \tan \tfrac{1}{2} A &= 9.6110, \\ \tfrac{1}{2} A &= 22° \, 13', \\ A &= 44° \, 26', \end{aligned} \qquad \begin{aligned} \log \tan \tfrac{1}{2} B &= 9.7359, \\ \tfrac{1}{2} B &= 28° \, 34', \\ B &= 57° \, 8', \end{aligned}$$

$$\begin{aligned} \log (s - a) &= 0.9031 \\ \log (s - b) &= 0.7782 \\ \text{colog } s &= 8.7447 \\ \text{colog } (s - c) &= 9.3979 \\ \hline & 2)19.8239 \end{aligned}$$

$$\begin{aligned} \log \tan \tfrac{1}{2} C &= 9.9120, \\ \tfrac{1}{2} C &= 39° \, 14', \\ C &= 78° \, 28'. \end{aligned}$$

CHECK: $A + B + C = 180° \, 2'$.

A common method of solving Case IV is by means of an auxiliary quantity,

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

We may write

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}, \quad \tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

In using this method $\log r$ is first found, whence the log-tangents of the three half-angles are readily obtained.

61. Case III; Other Methods of Solution. The formulæ of Art. 56 may sometimes be used to advantage in solving Case III.

Example (see Art. 55). Given $b = .06239$, $c = .02348$, $A = 110° 32'$, to solve the triangle.

We have $$a^2 = b^2 + c^2 - 2\,bc\cos A.$$

Then

$$a^2 = (.06239)^2 + (.02348)^2 - 2(.06239)(.02348)\cos 110° 32'.$$

$\log b = 8.7951$	$\log c = 8.3708$
2	2
$\log b^2 = 7.5902$	$\log c^2 = 6.7416$
$b^2 = .003893$	$c^2 = .0005516$
$+ c^2 = .0005516$	$\log 2 = 0.3010$
$-2\,bc\cos A = .001028$	$\log b = 8.7951$
$a^2 = .005473$	$\log c = 8.3708$
	$\log \cos A = 9.5450$
	$\log 2\,bc\cos A = 7.0119$
	$2\,bc\cos A = -.001028$

Then $\log a = \frac{1}{2}\log a^2 = 8.8691$ and $a = .07398$.

To find B and C we have the formulæ of Art. 51.

The above computation can in some cases be done best and quickest without the use of logarithms.

Another Method of Solution for Case III, preferred by many, is as follows:

From Fig. 27 we see that

$$DC = b\sin A,\; AD = b\cos A,\; DB = c - AD.$$

Then $\tan B = \dfrac{DC}{DB}$, whence B is known, and $C = 180° - (A + B)$. To find a we use $a = \dfrac{DC}{\sin B}$.

Applying this method to the example above we have

$$\begin{aligned} \log b &= 8.7951 \\ \log \sin A &= 9.9715 \\ \hline \log DC &= 8.7666 \\ \log DB &= 8.6567 \\ \hline \log \tan B &= 0.1099 \\ \\ B &= 52°\ 10' \\ C &= 17°\ 18' \end{aligned}$$

$$\begin{aligned} \log b &= 8.7951 \\ \log^{-} \cos A &= 9.5450 \\ \hline \log^{-} AD &= 8.3401 \\ AD &= -\ .02188 \\ DB &= .02348 + .02188 = .04536. \\ \\ \log DC &= 8.7666 \\ \log \sin B &= 9.8975 \\ \hline \log a &= 8.8691 \\ a &= \ .07398 \end{aligned}$$

NOTE. The fundamental importance of the *law of sines* and the *law of cosines* should be noted. By their use, direct or indirect, any triangle whatever may be solved.

AREAS OF TRIANGLES

62. Right Triangles.

Case I. *Given the two legs a and b*, Fig. 28.

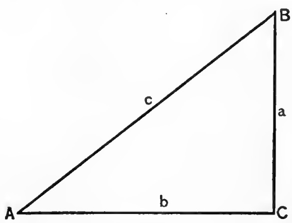


FIG. 28.

Representing the area of the triangle by K, it is obvious that

$$2K = ab. \tag{33}$$

Case II. *Given the hypotenuse and an acute angle, c and A.*

Then $$a = c \sin A, \quad b = c \cos A.$$

Whence, by (33),

$$2K = c^2 \sin A \cos A = \tfrac{1}{2} c^2 \sin 2A,$$

or $$\mathbf{4K = c^2 \sin 2A.}$$

Case III. *Given an angle and the adjacent leg, A and b.*

Then $$a = b \tan A.$$

Whence, by (33),

$$\mathbf{2K = b^2 \tan A.}$$

Case IV. *Given the hypotenuse and a leg, c and a.*

Then $$b^2 = c^2 - a^2 \text{ or } b = \sqrt{(c+a)(c-a)}.$$

Whence, by (33),

$$\mathbf{2K = a\sqrt{(c+a)(c-a)}.}$$

63. Oblique Triangles.

Case I. *Given two sides and the included angle, a, b, C.*

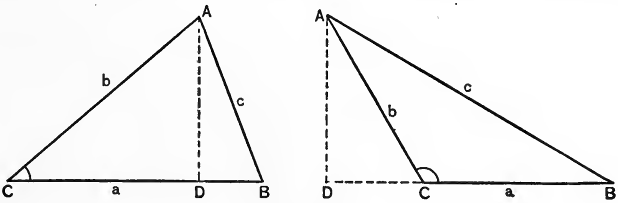


FIG. 29.

In Fig. 29 the line AD is perpendicular to BC.
It is obvious that $2K = a \times DA$.

But $$\sin C = \frac{DA}{b}, \text{ or } DA = b \sin C.$$

Therefore, $$\mathbf{2K = ab \sin C.} \qquad (34)$$

Case II. *Given a side and the angles, a, A, B, C.*

By Art. 51,

$$\frac{b}{a}=\frac{\sin B}{\sin A} \text{ or } b=\frac{a \sin B}{\sin A}.$$

Whence, by (34),

$$2K=\frac{a^2 \sin B \cdot \sin C}{\sin A}. \quad (35)$$

Case III. *Given the three sides, a, b, c.*

The formula (34), of Case I may be written, by Art. 42, (13),

$$2K=2ab \sin \tfrac{1}{2} C \cdot \cos \tfrac{1}{2} C.$$

Substituting in this the values of $\sin \frac{1}{2} C$ and $\cos \frac{1}{2} C$ given in Arts. 57 and 58, we have

$$2K=2ab\sqrt{\frac{(s-a)(s-b)}{ab}} \cdot \sqrt{\frac{s(s-c)}{ab}},$$

or

$$K=\sqrt{s(s-a)(s-b)(s-c)}. \quad (36)$$

EXAMPLES

Solve the following triangles, in each case obtaining also the area of the triangle:

1. $a=1419$, $B=29° 59'$, $C=16° 1'$.
2. $a=3.384$, $b=9.828$, $B=109°$.
3. $a=302$, $b=427$, $C=134° 29'$.
4. $a=56.22$, $b=63.91$, $c=70.54$.
5. $b=38.65$, $c=48.12$, $B=34° 32'$.
6. $c=5141$, $A=96° 3'$, $C=55° 46'$.
7. $b=56.2$, $c=63.9$, $A=71° 33'$.
8. $b=268.5$, $c=282.9$, $C=75° 20'$.
9. $b=6.362$, $A=76° 13'$, $C=35° 17'$.
10. $a=5499$, $c=2959$, $A=133° 34'$.
11. $b=.5042$, $a=.3618$, $B=74° 43'$.
12. $a=.03574$, $b=.02921$, $c=.01853$.
13. $b=.2792$, $a=.2271$, $A=65° 45'$.
14. $a=.01044$, $A=26° 32'$, $B=146° 26'$.
15. $a=31.49$, $b=49.88$, $B=44° 35'$.

16. $c = .0357$, $a = .0292$, $B = 31° 7'$.

17. $a = 32.15$, $b = 67.54$, $A = 28° 26'$.

18. $b = 4621$, $a = 6473$, $B = 31° 7'$.

19. $b = .4312$, $c = .8901$, $A = 29° 55'$.

20. $a = 6.743$, $b = 3.025$, $c = 4.271$.

21. $c = .01825$, $b = .02893$, $B = 83° 30'$.

PROBLEMS IN THE SOLUTION OF TRIANGLES

22. A man owns a triangular lot on the corner of two streets which do not intersect at right angles. The frontage on one street is 300 feet, on the other 250 feet. The back line of the lot is 350 feet long. If he buys land to add 275 feet to the 300-foot frontage, by how much is his lot increased in size?

23. A man owns a triangular lot on the corner of two streets which intersect at an angle of 62°. The frontage on one street is 200 feet, on the other 150 feet. If the land is worth one dollar a square foot and the man has $1200 with which to increase the size of his lot, by how much can he lengthen the 150-foot frontage?

24. The perimeter of a triangle is 100 feet, and the perpendicular from the vertex C to the base AB is 30 feet. The angle A is 50°. Find the length of the base AB.

25. What is the perpendicular height of a hill which is known to rise 72 feet for every 100 feet of length of its slope, if the angle of elevation of the hilltop from a point 100 yards from the base of the hill is 31°?

26. From where I stand, 50 feet from the bank of a stream, the angles of depression of the near and far banks of the stream are respectively 15° 37′ and 6° 24′. How wide is the stream? How far am I above the level of the stream?

27. A man 5 feet 6 inches tall, standing on a bluff 40 feet high, measures the angles of depression of the near and far shores of a bay. The angles are 46° 52′ and 5° 3′ respectively. How wide is the bay?

28. A man 5 feet tall, standing on the edge of a pond, finds the angle of elevation of the top of a tree on the other bank to be 44° 26′. The angle of depression of the reflection of the treetop is 60° 47′. Find the height of the tree.

The reflection of an object appears as far below the surface as the object is above the surface.

29. The frontage on the beach (AB) of a quadrangular lot $ABCD$ cannot be measured. The sides BC, CD, and DA are found to be 236, 155 and 105 feet respectively. The angles DAC and DBC are 32° 20′ and 29° 50′ respectively. Find the length of AB.

30. The bases of a trapezoid are 48.25 and 94.75 feet. The angles at the ends of the longer base are 63° 52′ and 70° 55′. Find the lengths of the other two sides.

31. Two sides of a triangle are 8.53 and 7.41. The difference between the angles opposite these sides is 18° 23′. Solve the triangle.

32. The area of a triangle is 979 square feet. The angle A is 56° 22′ and the side b is 44.80 feet. Solve the triangle.

33. Two sides of a parallelogram are 8005 and 5008. The included angle is 60° 53′. Find the lengths of the diagonals.

34. The diagonals of a quadrangular field $ABCD$ intersect at O at an angle of 78° 3′. The lines AO, BO, CO, and DO are 27.5, 31.8, 58.5 and 63.2 feet respectively. What is the area of the field?

35. Two sides of a triangle are $b = 302$ and $c = 40.8$. Find the angle A so that the triangle may have the same area as the triangle whose sides are 62, 51 and 30. If b were 30.2 and c were 40.8 could A be found? Why?

36. Two vessels start from the same point and sail, one northeast at the rate of 6 miles per hour, and the other east 30° south at the rate of 8 miles per hour. How far apart will the ships be after $2\frac{1}{2}$ hours?

37. A submarine in submerging drifts back 5 feet for every 20 feet it sinks. After the submarine has sunk vertically 300 yards, at what angle must a torpedo be shot from a cruiser one mile away to hit the submarine, if the latter drifts away from the cruiser?

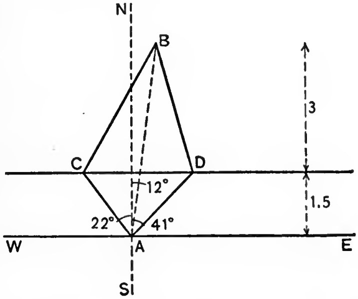
38. A post 6 feet high casts a shadow 10 feet long. What is the length of a flagpole that casts a shadow 60 feet long if the pole makes an angle of 82° with the horizontal on the side away from the sun?

39. In problem 38 find the length of the flagpole if the angle made with the horizontal is 82° on the side towards the sun.

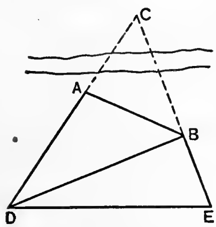
40. Two yachts begin a race by sailing from a point A, along the windward leg of the course in the direction northeast until they reach a buoy B. They then sail before the wind, east 32° south, until they reach a point C, 5 miles east along a straight coast from A. The first yacht sails to windward 5 miles per hour, and before the wind 6.5 miles per hour; the second 5.8 miles per hour to windward and 6 miles before the wind. Which yacht wins the race and by how much?

41. A triangular beach lot has a frontage on the sea of 100 yards. The boundary lines running from the beach make, on the inner side of the lot, angles of 60° and 50° respectively with the shore line. How must a line be drawn from the middle point of the shore line to form two equal lots?

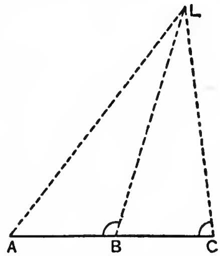
42. A point A, on the south bank of a river 1.5 miles broad and flowing due east is to be connected by bridge and road with a town B, 3 miles back in a straight line from the north bank of the river. It is found that the bridge can be built to a point, C, on the farther bank lying north 22° west from A, or to D lying north 41° east from A. The town B lies north 12° east from A. If the bridge costs $2000 per mile to build and the road $500 per mile, which route is the more economical and by how much? (See Fig.)



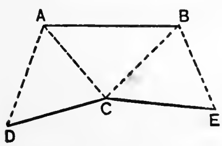
43. The distances of a point C, on the far side of a river from two points A and B on the near side, are to be found but can not be directly measured. In the direction CA a distance $AD = 150$ feet is measured, and in the direction CB a distance $BE = 250$ feet. The distance from A to B is 279.5 feet, and by measurement it is found that $BD = 315.8$ feet, $DE = 498.7$ feet. How far is C from A and B? (See Fig.)



44. The distance from a point, A, on the coast to a lighthouse, L, is to be found. A straight line is run from A along the coast, and on the line two points, B and C, are taken from which the lighthouse is visible. By measurement it is found that $AB = 236.7$ feet, $BC = 215.9$ feet, the angle $ABL = 142°\ 37'$, the angle $ACL = 76°\ 14'$. How far is the lighthouse from A? (See Fig.)



45. On the north side of a river lie two points A and B both of which can be seen from C, and from no other point, on the south side of the stream. From a point D, whose distance from C is 425.3 feet, A and C are sighted. It is found that the angle $ADC = 37°\ 15'$, and the angle $ACD = 42°\ 35'$. From another point E, whose distance from C is 405.4 feet, and from which B and C are visible, the angles $CEB = 53°\ 15'$, and $ECB = 58°\ 5'$ are measured. The angle ACB is also measured and found to be 65° 11'. What is the distance from A to B? (See Fig.)



CHAPTER VII

THE SOLUTION OF TRIGONOMETRIC EQUATIONS

64. The trigonometric equations hitherto dealt with have been identical equations; equations, that is, true for any values of the variables involved. We shall now deal with trigonometric equations which are not identities, and shall examine the methods by which such equations are solved. No methods applicable to all such equations can be given, but methods applicable to several important classes will be discussed. In general it may be said that all such equations are algebraic in form, the one difference being that now the unknown quantities are the trigonometric functions, sine, tangent, etc., or, occasionally, the inverse functions. Therefore, all methods applicable to the solution of algebraic equations are applicable to the solution of trigonometric equations. Moreover, in the case of trigonometric equations we have the various fundamental identities, treated in former chapters, which being true for all values of the variables involved can be used in connection with any equation whose solution is desired.

65. For example, given the equation

$$2\sin^2 x - \cos^2 x + \tfrac{1}{4} = 0$$

to find the value of x.

In form this is an algebraic, quadratic equation in two unknowns, $\sin x$ and $\cos x$. To find the values of two unknowns we must have two consistent and independent equations. But we also know that $\cos^2 x = 1 - \sin^2 x$. Therefore, our equation may be written

$$2\sin^2 x - (1 - \sin^2 x) = -\tfrac{1}{4},$$

$$3\sin^2 x = \tfrac{3}{4},$$

whence $$\sin x = \pm\tfrac{1}{2},$$

and $$x = \sin^{-1}(\pm\tfrac{1}{2}).$$

The principal values of x are, therefore, $\pm\frac{\pi}{6}$ and the general values are

$$x = n\pi + (-1)^n\left(\pm\frac{\pi}{6}\right) = n\pi \pm \frac{\pi}{6};$$

or we may proceed thus:

Given $$2\sin^2 x - \cos^2 x = -\tfrac{1}{4}.$$

We know $$\sin^2 x + \cos^2 x = 1.$$

Adding the two equations,

$$3\sin^2 x = \tfrac{3}{4}, \text{ etc.}$$

Example 2. Solve the equation

$$\cos x - \sqrt{3}\sin x + 1 = 0.$$

For $\sin x$ substitute $\sqrt{1 - \cos^2 x}$.

Then

$$\cos x - \sqrt{3}\cdot\sqrt{1 - \cos^2 x} + 1 = 0,$$

$$\cos x + 1 = \sqrt{3}\cdot\sqrt{1 - \cos^2 x},$$

$$\cos^2 x + 2\cos x + 1 = 3 - 3\cos^2 x,$$

$$2\cos^2 x + \cos x - 1 = 0,$$

a quadratic equation in $\cos x$ whose roots are

$$\cos x = -1 \text{ or } \tfrac{1}{2}.$$

Therefore,

$$x = \cos^{-1}(-1) = 2n\pi + \pi = (2n + 1)\pi,$$

and $$x = \cos^{-1}\left(\frac{1}{2}\right) = 2n\pi \pm \frac{\pi}{3}.$$

These roots, as in every case, must be tested by substitution in the original equation. It is found that $2n\pi - \frac{\pi}{3}$ does not satisfy the equation, while the other two values do. The roots are, therefore,

$$x = (2n+1)\pi \text{ and } 2n\pi + \frac{\pi}{3}.$$

Another method of solving the last equation is as follows:

Given $$\cos x - \sqrt{3}\sin x = -1.$$

Divide by 2,

$$\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = -\frac{1}{2}.$$

But $$\frac{1}{2} = \cos\frac{\pi}{3} \text{ and } \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}.$$

Therefore we may write,

$$\cos\frac{\pi}{3}\cos x - \sin\frac{\pi}{3}\sin x = -\frac{1}{2}$$

or $$\cos\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}$$

Whence $$x + \frac{\pi}{3} = \cos^{-1}\left(-\frac{1}{2}\right) = 2n\pi \pm \frac{2\pi}{3},$$

and $$x = (2n-1)\pi \text{ or } 2n\pi + \frac{\pi}{3}.$$

Note that the two general solutions $(2n+1)\pi$ and $(2n-1)\pi$ are identical since each represents any odd multiple of π.

66. Special Types of Equations. This last solution is an example of the *type* of equation

1. $$a\cos x + b\sin x = c,\ c \leqq \sqrt{a^2+b^2}.$$

To solve equations of this type divide by $\sqrt{a^2+b^2}$.

$$\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x = \frac{c}{\sqrt{a^2+b^2}}.$$

Now $$\frac{a}{\sqrt{a^2+b^2}} = \cos\alpha \text{ and } \frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$$

since

$$\cos^2\alpha + \sin^2\alpha = \left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1.$$

Therefore we may write

$$\cos\alpha\cos x + \sin\alpha\sin x = \frac{c}{\sqrt{a^2+b^2}},$$

$$\cos(x-\alpha) = \frac{c}{\sqrt{a^2+b^2}},$$

$$x - \alpha = \cos^{-1}\frac{c}{\sqrt{a^2+b^2}} = 2n\pi \pm \cos^{-1}\frac{c}{\sqrt{a^2+b^2}},$$

and $$x = 2n\pi \pm \cos^{-1}\frac{c}{\sqrt{a^2+b^2}} + \alpha.$$

Another *type* of equation is

2. $\tan a\theta = \cot b\theta$ or $\sin a\theta = \cos b\theta$.

We may put

$$\cot b\theta = \tan\left(\frac{\pi}{2} - b\theta\right); \qquad \sin a\theta = \cos\left(\frac{\pi}{2} - a\theta\right).$$

Therefore

$$\tan a\theta = \tan\left(\frac{\pi}{2} - b\theta\right); \qquad \cos b\theta = \cos\left(\frac{\pi}{2} - a\theta\right).$$

$$a\theta = n\pi + \left(\frac{\pi}{2} - b\theta\right); \qquad b\theta = 2n\pi \pm \left(\frac{\pi}{2} - a\theta\right).$$

or

$$\theta = \frac{n\pi + \frac{\pi}{2}}{a+b}. \qquad \theta = \frac{2n\pi \pm \frac{\pi}{2}}{b \pm a}.$$

Example. Given $\tan 3\theta = \cot 2\theta$, find θ.

$$\cot 2\theta = \tan\left(\frac{\pi}{2} - 2\theta\right).$$

Therefore,

$$\tan 3\theta = \tan\left(\frac{\pi}{2} - 2\theta\right),$$

$$3\theta = n\pi + \left(\frac{\pi}{2} - 2\theta\right),$$

$$\theta = \frac{n\pi + \frac{\pi}{2}}{5} = \frac{(2n+1)\pi}{10}.$$

A third *type* of equation is

3.
$$\begin{aligned} \sin ax + \sin bx + \sin cx &= 0, \\ \cos ax + \cos bx + \cos cx &= 0, \\ \cos ax + \cos bx + \sin cx &= 0, \\ \sin ax + \sin bx + \cos cx &= 0. \end{aligned}$$

To solve equations of this type, formulæ (19) and (20) are used.

Example. Solve the equation $\sin 5x - \sin 3x + \sin x = 0$.

We may write

$$\begin{aligned} \sin 5x - \sin 3x &= 2\cos\tfrac{1}{2}(5x + 3x)\sin\tfrac{1}{2}(5x - 3x) \\ &= 2\cos 4x \sin x. \end{aligned}$$

Therefore

$$2\cos 4x \sin x + \sin x = 0,$$

$$\sin x(2\cos 4x + 1) = 0.$$

Whence,

$\sin x = 0$ or $2\cos 4x + 1 = 0$

$x = \sin^{-1} 0$ $\qquad$ $4x = \cos^{-1}(-\tfrac{1}{2})$

$x = \sin^{-1} 0 = n\pi$ $\qquad$ $4x = \cos^{-1}(-\tfrac{1}{2}) = 2n\pi \pm \frac{2\pi}{3}$

$$x = \frac{n\pi}{2} \pm \frac{\pi}{6}.$$

67. Simultaneous equations involving trigonometric functions can in many cases be solved.

Example 1. Given $y = 1 - \cos x$, $y = 1 + \sin x$, find x and y.

We have $$1 + \sin x = 1 - \cos x,$$

$$\sin x = -\cos x,$$

$$\tan x = -1.$$

$$x = \tan^{-1}(-1) = n\pi + \frac{3\pi}{4},$$

and $$y = 1 + \sin x = 1 - \cos x = 1 \pm \frac{1}{\sqrt{2}}.$$

Example 2. Given $r \cos\left(\theta - \frac{\pi}{3}\right) = a$, $r \cos\left(\theta - \frac{\pi}{6}\right) = a$, find r and θ.

We have

$$r \cos\left(\theta - \frac{\pi}{3}\right) = r \cos\left(\theta - \frac{\pi}{6}\right),$$

$$\cos\left(\theta - \frac{\pi}{3}\right) = \cos\left(\theta - \frac{\pi}{6}\right),$$

$$\theta - \frac{\pi}{3} = 2n\pi \pm \left(\theta - \frac{\pi}{6}\right),$$

$$\theta = n\pi + \frac{\pi}{4},$$

$$r = a \sec\left(\theta - \frac{\pi}{3}\right) = a \sec\left(\theta - \frac{\pi}{6}\right)$$

$$= a \sec\left(n\pi - \frac{\pi}{12}\right) = a \sec\left(n\pi + \frac{\pi}{12}\right).$$

When n is even, $\sec\left(n\pi - \frac{\pi}{12}\right) = -\sec\frac{\pi}{12}$,

$$\sec\left(n\pi + \frac{\pi}{12}\right) = +\sec\frac{\pi}{12}.$$

When n is odd:

$$\sec\left(n\pi - \frac{\pi}{12}\right) = \sec\left(n\pi + \frac{\pi}{12}\right) = -\sec\frac{\pi}{12}.$$

Therefore, $$r = -a\sec\frac{\pi}{12}.$$

68. Equations Involving Inverse Trigonometric Functions may, in general, be solved by transforming to other, equivalent equations involving the direct functions. The method of solution is illustrated by the following examples.

Example 1. Solve the equation $2\tan^{-1}x = \cot^{-1}x$.

We have $$\cot(2\tan^{-1}x) = \cot(\cot^{-1}x)$$

or $$\frac{[\cot(\tan^{-1}x)]^2 - 1}{2\cot(\tan^{-1}x)} = x.$$

That is, $$\frac{\frac{1}{x^2} - 1}{\frac{2}{x}} = x \text{ or } 3x^2 = 1;\text{ whence } x = \pm\frac{1}{\sqrt{3}}.$$

Example 2. Solve the equation $\cos^{-1}x + \sin^{-1}2x = 0$.

We have $$\sin(\cos^{-1}x + \sin^{-1}2x) = 0,$$

or

$$\sin(\cos^{-1}x)\cdot\cos(\sin^{-1}2x) + \cos(\cos^{-1}x)\cdot\sin(\sin^{-1}2x) = 0.$$

That is, $\pm\sqrt{1-x^2}\cdot\sqrt{1-4x^2} + x\cdot 2x = 0$.

Whence, $(1-x^2)(1-4x^2) = 4x^4$, $5x^2 = 1$, $x = \pm\dfrac{1}{\sqrt{5}}$.

A second method of solving example 2 is as follows:

$$\cos^{-1}x = -\sin^{-1}2x,$$

$$\sin(\cos^{-1}x) = \sin(-\sin^{-1}2x),$$

$$\sqrt{1-x^2} = -2x,$$

$$5x^2 = 1,$$

$$x = \pm \frac{1}{\sqrt{5}}.$$

In every case the values must be checked by substitution in the original equation.

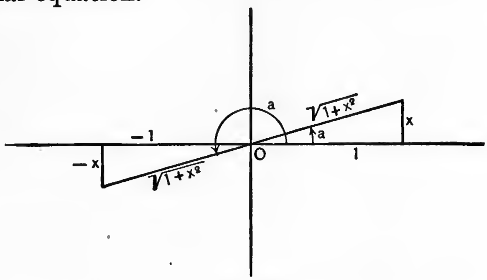


FIG. 30.

It is often convenient, in dealing with inverse functions, to assume the angle whose function is given and to construct a figure to show the values of the remaining functions. Thus, in example 1, we wish to find $\cot(\tan^{-1} x)$. Let $\tan^{-1} x = a$ and construct the angle a, Fig. 30, with ordinate equal to $\pm x$ and abscissa equal to ± 1. The distance is

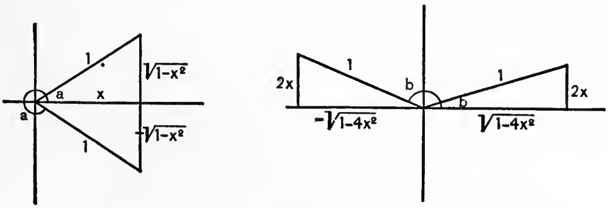


FIG. 31.

then $\sqrt{1+x^2}$, and all the functions are readily found. Thus, $\cot(\tan^{-1} x) = \frac{1}{x}$. Similarly, in example 2, we wish to find $\sin(\cos^{-1} x)$ and $\cos(\sin^{-1} 2x)$. By figure 31 we see assuming $\cos^{-1} x = a$ and $\sin^{-1} 2x = b$, that $\sin(\cos^{-1} x) = \pm\sqrt{1-x^2}$ and $\cos(\sin^{-1} 2x) = \pm\sqrt{1-4x^2}$.

EXAMPLES

Solve the following equations:

1. $\sin 5x - \sin 3x + \sin x = 0$.
2. $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.
3. $\sin 4x - \sin 2x - \cos 3x = 0$.
4. $6\sin\theta + \cos\theta = 2$.
5. $2\sin\theta + \cos\theta = 2$.
6. $\sin 2x = \sin x$.
7. $\cos 2x = \sin x$.
8. $\sin 3\theta = \cos\theta$.
9. $\tan x = \cos x$.
10. $\cos 2x = \frac{1}{2}\cos x$.
11. $\sin 2\theta \cos 2\theta + 2\sin\theta = 0$.
12. $\sin 4x - 2\sin x \cos 2x = 0$.
13. $\sin 4\theta = \cos 2\theta$.
14. $\cos 2\theta = \sin 2\theta - 1$.
15. $\cos(x - a)\cos x - \sin(x - a)\sin x = 0$.
16. $\sec^2 x = 3\csc^2 x$.
17. $\sin\theta\cos\theta - \sin\left(\frac{\pi}{3} - \theta\right)\cos\left(\frac{\pi}{3} - \theta\right) = 0$.
18. $27\csc\theta\cot\theta = 8\sec\theta\tan\theta$.
19. $25\sin\theta - 12.8\csc^2\theta = 0$.
20. $\cos^3\theta - 2\sin^2\theta\cos\theta = 0$.

Solve the following simultaneous equations for x and y, or r and θ.

21. $y = 1 - \cos 2x$,
    $y = 1 + \sin 2x$.
22. $r = \sec^2\frac{\theta}{2}$,
    $r = \csc^2\frac{\theta}{2}$.
23. $r = a\sin\theta$,
    $r = a\sin 2\theta$.
24. $y = 2a\cos x$,
    $y = 2a\sin x\tan x$.
25. $y\cos x = 2a$,
    $y = 5a\sin x$.
26. $r^2\sin 2\theta = 1$,
    $r^2\cos 2\theta = 1$.
27. $r = 3\sin\theta + 2\cos\theta$,
    $r = 3\cos\theta + 2\sin\theta$.
28. $y^2 = 4\sin 2x$,
    $y^2\sin 2x = 1$.
29. $y = \sin x$,
    $y = \sin(x + a)$.
30. $r\cos\left(\theta - \frac{\pi}{2}\right) = \frac{3a}{4}$,
    $r = a\sin\theta$.
31. $r^2 = \sin\theta$,
    $r^2 = \sin 3\theta$.
32. $r^2\sin 2\theta = 8$,
    $r = 2\sec\theta$.
33. $r^2\sin 2\theta = 4$,
    $r^2 = 16\sin 2\theta$.

Solve the following equations:

34. $\sin^{-1} x = \sin^{-1} a + \sin^{-1} b.$

35. $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}.$

36. $\sin^{-1}\frac{4}{x} + \sin^{-1}\frac{8}{x} = \frac{\pi}{2}.$

37. $\sin^{-1} x = \cos^{-1}(-x).$

38. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{3\pi}{4}.$

39. $\tan^{-1} x = 2\cos^{-1}\frac{x}{2}.$

40. $\tan^{-1}(x+1) - \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1} 2.$

41. $\cos^{-1}\frac{x^2-1}{x^2+1} + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}.$

42. $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{x}{x-1} = \tan^{-1}(-2).$

Solve the following equations, finding only the principal values of angles:

43. $\cos 5x - \cos 3x + \sin x = 0.$

44. $\sin 4x + \sin 2x + 2\cos x = 0.$

45. $\cos 3x = \cos x.$

46. $\cot x = \cos x.$

47. $y = 1 + \cos 2x,$
$y = 1 - \sin 2x.$

48. $r = a\cos\theta,$
$r = a\sin 2\theta.$

49. $y = \tan 2x,$
$y = \cos 2x.$

50. $r = \sin\theta,$
$r = \sin\left(\theta + \frac{\pi}{4}\right).$

51. $\sin 4\theta = \cos\theta.$

52. $3\sin\theta + \cos\theta = 2.$

SPHERICAL TRIGONOMETRY

CHAPTER VIII

FUNDAMENTAL RELATIONS

69. Spherical trigonometry deals with the relations among the sides and angles of a spherical triangle; that is, of a portion of the surface of a sphere bounded by the intersecting arcs of three great circles. It deals also with the computation of unknown parts of such a triangle from parts which are known, the process being called, as in plane trigonometry, the solution of the triangle. The sides of a spherical triangle, being arcs of circles, are expressed in degrees, minutes, and seconds, and, as is customary, we shall consider only those triangles in which each part (angle or side) is less than one hundred and eighty degrees.

70. Law of Cosines. There is one theorem, the law of cosines, which may be called the fundamental theorem of

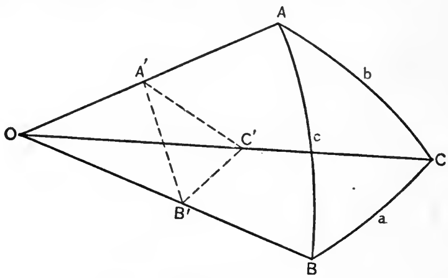


FIG. 32.

spherical trigonometry, because by means of the theorem any spherical triangle may be solved when three of its parts are known. We shall proceed to prove the *Law of Cosines.*

Let ABC, Fig. 32, be a spherical triangle on a sphere whose center is O, and let the sides b and c be less than 90°.

Through any point, A', on OA pass a plane perpendicular to OA cutting the planes OAC, OAB, and OBC, in $A'C'$, $A'B'$, and $B'C'$, respectively. Then the angle $B'A'C'$ is the measure of the diedral angle $B\text{–}OA\text{–}C$ and, therefore, of the spherical angle A. Also, by the construction, the angles $OA'B'$ and $OA'C'$ are right angles. In the triangle $A'B'C'$

$$\overline{B'C'}^2 = \overline{B'A'}^2 + \overline{C'A'}^2 - 2\,B'A' \cdot C'A' \cos A,$$

and in the triangle $B'OC'$

$$\overline{B'C'}^2 = \overline{B'O}^2 + \overline{C'O}^2 - 2\,B'O \cdot C'O \cos a.$$

Whence

$$\overline{B'O}^2 + \overline{C'O}^2 - 2\,B'O \cdot C'O \cos a$$
$$= \overline{B'A'}^2 + \overline{C'A'}^2 - 2\,B'A' \cdot C'A' \cos A,$$

or

$$2\,B'O \cdot C'O \cos a$$
$$= \overline{B'O}^2 - \overline{B'A'}^2 + \overline{C'O}^2 - \overline{C'A'}^2 + 2\,B'A' \cdot C'A' \cos A.$$

But $B'OA'$ and $C'OA'$ are right triangles, and therefore,

$$\overline{B'O}^2 - \overline{B'A'}^2 = \overline{OA'}^2;\quad \overline{C'O}^2 - \overline{C'A'}^2 = \overline{OA'}^2.$$

We then have

$$B'O \cdot C'O \cos a = \overline{OA'}^2 + B'A' \cdot C'A' \cos A,$$

or

$$\cos a = \frac{OA'}{B'O} \cdot \frac{OA'}{C'O} + \frac{B'A'}{B'O} \cdot \frac{C'A'}{C'O} \cdot \cos A.$$

But

$$\frac{OA'}{B'O} = \cos BOA = \cos c, \quad \frac{OA'}{C'O} = \cos AOC = \cos b,$$

$$\frac{B'A'}{B'O} = \sin BOA = \sin c, \quad \frac{C'A'}{C'O} = \sin AOC = \sin b.$$

Hence $$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

which is the law of cosines.

In the above demonstration the sides b and c were taken less than 90° in order that the construction of the right triangles $B'OA'$ and $C'OA'$ might be possible. The resulting theorem, however, is true in all cases. Let us assume $90° < b < 180°$ and $90° < c < 180°$. Then, Fig. 33, produce the arcs AB and AC to meet in A', thus forming a lune. In the triangle $A'BC$, b' and c' are less than 90°.

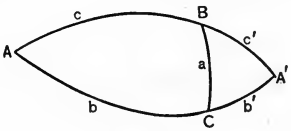


FIG. 33.

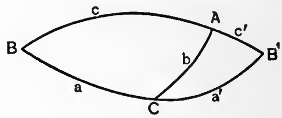


FIG. 34.

The law of cosines is, therefore, true for the triangle $A'BC$, so that, since $A' = A$,

$$\cos a = \cos b' \cos c' + \sin b' \sin c' \cos A.$$

But $$b' = 180° - b \text{ and } c' = 180° - c.$$

Whence

$$\cos a = \cos (180° - b) \cos (180° - c) + \sin (180° - b) \sin (180° - c) \cos A,$$

or, $$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad \text{Q.E.D.}$$

Again, let $b < 90°$ and $90° < c < 180°$. Produce the arcs BA and BC, Fig. 34, to meet in B', thus forming a lune. Then, in the triangle $AB'C$, $b < 90°$ and $c' < 90°$, and, therefore,

$$\cos a' = \cos b \cos c' + \sin b \sin c' \cos CAB'.$$

But

$$a' = 180° - a,\ c' = 180° - c, \text{ and } CAB' = 180° - A.$$

Hence

$$\cos (180° - a) = \cos b \cos (180° - c) + \sin b \sin (180° - c) \cos (180° - A),$$

or, $$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

which proves the law of cosines for all cases.

We thus have the three fundamental equations, the *law of cosines:*

$$\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A, \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B, \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C, \end{aligned} \tag{37}$$

by means of which any spherical triangle may be solved.

For example, given $a = 60°$, $b = 70°$, $A = 65°$.

We have

$$\cos 60° = \cos 70° \cos c + \sin 70° \sin c \cos 65,°$$

or,

$$.500 = .342 \cos c + .940 \sin c \times .423,$$

$$.342 \cos c + .398 \sin c = .500,$$

$$\frac{.342 \cos c + .398 \sin c}{\sqrt{(.342)^2 + (.398)^2}} = \frac{.500}{\sqrt{(.342)^2 + (.398)^2}},$$

$$\frac{.342}{.525} \cos c + \frac{.398}{.525} \sin c = \frac{.500}{.525},$$

$$.651 \cos c + .758 \sin c = .952,$$

$$.651 = \cos 49.4°, \; .758 = \sin 49.4°.$$

Therefore,

$$\cos (c - 49.4°) = .952,$$

$$c - 49.4° = \cos^{-1} .952 = 18.2° \text{ and } c = 67.6°.$$

Similarly the other parts may be found. The equations are not, however, adapted to logarithmic computation, so that for practical use, as will presently be shown, they must be transformed in various ways.

71. Law of Cosines Applied to the Polar Triangle. The law of cosines, being true for any triangle, is true for the polar triangle of ABC. Therefore, denoting the six parts of the polar triangle by the same letters accented, we have

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

But

$$a' = 180° - A,\ b' = 180° - B,\ c' = 180° - C,\ A' = 180° - a.$$

Whence,

$$\cos (180° - A) = \cos (180° - B) \cos (180° - C) + \sin (180° - B) \sin (180° - C) \cdot \cos (180° - a),$$

or

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a,$$

so that the truth of the three following equations is obvious:

$$\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b, \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c. \end{aligned} \qquad (38)$$

72. Law of Sines. Another theorem of importance in the solution of spherical triangles, known as the law of sines, is as follows: *In any spherical triangle the sines of the sides are proportional to the sines of the opposite angles.* That is,

$$\sin a : \sin b : \sin c = \sin A : \sin B : \sin C. \qquad (39)$$

From equations (37) we have,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Whence

$$1 - \cos^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c},$$

or,

$$\begin{aligned} \sin^2 A &= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \\ &= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \\ &= \frac{1 - \cos^2 b - \cos^2 c - \cos^2 a + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}. \end{aligned}$$

Whence,

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c},$$

or,

$$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

where the positive sign is taken because A and a are each less than 180°. The right-hand member of this expression is symmetric in a, b, and c, so that if we started with $\cos B$ or $\cos C$ instead of with $\cos A$, the final result for the right-hand member would be identical with that written above. Therefore, obviously, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

the law of sines which was to be proved.

CHAPTER IX

THE SOLUTION OF RIGHT SPHERICAL TRIANGLES

73. Special Formulæ for Right Triangles. If we let C be the right angle in a right spherical triangle, and put $C = 90°$ in the third equation of (37), we have

$$\cos c = \cos a \cos b. \tag{40}$$

The third equation of (38) gives

$$0 = -\cos A \cos B + \sin A \sin B \cos c,$$

or,

$$\cos c = \cot A \cot B. \tag{41}$$

The first two equations of (38) give

$$\left.\begin{aligned} \cos A &= \sin B \cos a, \\ \cos B &= \sin A \cos b. \end{aligned}\right\} \tag{42}$$

Using the proportions of (39) when $C = 90°$, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{1}{\sin c}.$$

Whence,

$$\left.\begin{aligned} \sin a &= \sin A \sin c, \\ \sin b &= \sin B \sin c. \end{aligned}\right\} \tag{43}$$

From (42) by (43) we have

$$\cos A = \sin B \cos a = \frac{\sin b}{\sin c} \cdot \cos a,$$

or, by (40)

$$\cos A = \frac{\sin b}{\sin c} \cdot \frac{\cos c}{\cos b} = \frac{\tan b}{\tan c}.$$

Similarly,

$$\cos B = \frac{\tan a}{\tan c}. \tag{44}$$

By (43), (44), and (40),

$$\tan A = \frac{\sin A}{\cos A} = \frac{\dfrac{\sin a}{\sin c}}{\dfrac{\tan b}{\tan c}} = \frac{\sin a}{\sin c} \cdot \frac{\tan c}{\tan b}$$

$$= \frac{\sin a}{\tan b \cos c} = \frac{\sin a}{\tan b \cdot \cos a \cos b} = \frac{\tan a}{\sin b}.$$

In the same way,

$$\tan B = \frac{\tan b}{\sin a}. \tag{45}$$

74. The formulæ (40) to (45) may be assembled, in a slightly different form, as follows:

$$\begin{aligned} \sin A &= \frac{\sin a}{\sin c}. & \sin B &= \frac{\sin b}{\sin c}. \\ \cos A &= \frac{\tan b}{\tan c}. & \cos B &= \frac{\tan a}{\tan c}. \\ \tan A &= \frac{\tan a}{\sin b}. & \tan B &= \frac{\tan b}{\sin a}. \\ \cos A &= \cos a \sin B. & \cos B &= \cos b \sin A. \\ \cos c &= \cos a \cos b. & \cos c &= \cot A \cot B. \end{aligned} \tag{46}$$

A device, known as Napier's Rules, was formulated by Napier to facilitate the remembering of the above formulæ. Let us take for the *five* parts of a right triangle the sides a and b, and the complements of A, B, and c. These five parts, Fig. 35, arrange themselves so that each is a *middle* to *two adjacent parts* and a *middle* to *two opposite parts.*

NAPIER'S RULES state

I. *The sine of the middle part equals the product of the tangents of the adjacent parts.*

II. *The sine of the middle part equals the product of the cosines of the opposite parts.*

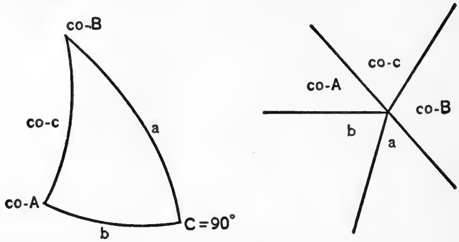


FIG. 35.

By applying these rules to the various parts all the formulæ of (46) may be obtained. Thus, for example,

$$\sin(\text{co–}A) = \tan b \cdot \tan(\text{co–}c).$$

That is,

$$\sin(90^\circ - A) = \tan b \cdot \tan(90^\circ - c),$$

or,

$$\cos A = \tan b \cot c = \frac{\tan b}{\tan c}.$$

75. Rules for Solution. In a right triangle, the right angle being always known, only two other parts need be known to solve the triangle. To solve a right triangle by means of the formulæ (46) we have, therefore, the general rule: *Select that equation which involves the two known parts and one unknown part.*

The algebraic signs of the functions must be carefully noted in order to determine the sign of the resulting function and thereby the angle. If the part to be found is got from a cosine, tangent, or cotangent there is no ambiguity, for if these functions are *plus* the part will have a value less than 90°. If they are *minus* the part will have for its value the supplement of the angle found from the tables of trigonometric functions.

On the other hand, if the unknown part is determined by a sine, the sine being positive for all angles between 0° and 180°, the value may be either that got from the tables or its supplement. In general both solutions must be used unless the ambiguity can be removed by the following laws:

1. *If the sides adjacent to the right angle are in the same quadrant, the hypotenuse is less than* 90°; *if they are in different quadrants, the hypotenuse is greater than* 90°.

2. *An angle and its opposite side are in the same quadrant.*

PROOF OF LAW 1. By (40) $\cos c = \cos a \cos b$.

Let $a \lessgtr 90°$ and $b \lessgtr 90°$. Then $\cos a = \pm$, $\cos b = \pm$, and $\cos c = (\pm)(\pm) = +$. Therefore, $c < 90°$. Again, let $a \lessgtr 90°$, and $b \gtrless 90°$. Then $\cos a = \pm$, $\cos b = \mp$, $\cos c = (\pm)(\mp) = -$, and $c > 90°$.

PROOF OF LAW 2. By (45) $\sin b = \dfrac{\tan a}{\tan A}$.

Since $\sin b$ is necessarily positive, it follows that $\tan a$ and $\tan A$ are both *plus* or both *minus*. Therefore a and A are each less than 90° or each greater than 90°.

76. The solution of right triangles is illustrated by the following examples:

Example 1. Given $A = 33° 50'$, $b = 108°$, find B, a, and c.

From the formulæ (46) we select

$$\tan A = \frac{\tan a}{\sin b}, \qquad \cos A = \frac{\tan b}{\tan c}, \qquad \cos B = \cos b \sin A,$$

or,

$$\tan a = \tan A \sin b, \quad \cot c = \cos A \cot b, \quad \cos B = \sin A \cos b.$$

$^{+}\log \tan A = 9.8263$	$^{+}\log \cos A = 9.9194$
$^{+}\log \sin b = 9.9782$	$^{-}\log \cot b = 9.5118$
$^{+}\log \tan a = 9.8045$	$^{-}\log \cot c' = 9.4312$
	$c' = 74° 54'$
$a = 32° 31'$	$c = 105° 6'$

$$\begin{aligned} {}^{+}\log \sin A &= 9.7457 \\ {}^{-}\log \cos b &= 9.4900 \\ \hline {}^{-}\log \cos B' &= 9.2357 \\ B' &= 80^\circ\ 6' \\ B &= 99^\circ\ 54' \end{aligned}$$

To check the results we select a formula involving the three parts to be found; a, c, and B. Thus $\cos B = \frac{\tan a}{\tan c}$, or,

$$\log \cos B = \log \tan a + \log \cot c$$

$$9.2357 = 9.8045 + 9.4312 \qquad \text{CHECK.}$$

Example 2. Given $a = 47^\circ\ 30'$, $c = 120^\circ\ 20'$, find A, B, and b.

We have

$$\sin A = \frac{\sin a}{\sin c}, \qquad \cos B = \frac{\tan a}{\tan c}, \qquad \cos b = \frac{\cos c}{\cos a}.$$

$\log \sin a = 9.8676$	${}^{+}\log \tan a = 0.0379$	${}^{-}\log \cos c = 9.7033$
$\log \sin c = 9.9361$	${}^{-}\log \tan c = 0.2327$	${}^{+}\log \cos a = 9.8297$
$\log \sin A = 9.9315$	${}^{-}\log \cos B' = 9.8052$	${}^{-}\log \cos b' = 9.8736$
	$B' = 50^\circ\ 19'$	$b' = 41^\circ\ 38'$
$A = 58^\circ 40'$	$B = 129^\circ\ 41'$	$b = 138^\circ\ 22$

CHECK. $\cos B = \cos b \sin A$

$$\log \cos B = \log \cos b + \log \sin A$$

$$9.8052 = 9.8736 + 9.9315 = 9.8051.$$

NOTE. The value of A less than 90° is taken by virtue of law 2, Art. 75.

Example 3. Given $B = 105^\circ\ 59'$, $b = 128^\circ\ 33'$, find A, a, and c.

We have,

$$\cos B = \cos b \sin A, \qquad \tan B = \frac{\tan b}{\sin a}, \qquad \sin B = \frac{\sin b}{\sin c},$$

or,

$$\sin A = \frac{\cos B}{\cos b}, \qquad \sin a = \tan b \cot B, \qquad \sin c = \frac{\sin b}{\sin B}.$$

$^-\log \cos B = 9.4399$	$^-\log \cot B = 9.4570$	$\log \sin b = 9.8932$
$^-\log \cos b = 9.7946$	$^-\log \tan b = 0.0986$	$\log \sin B = 9.9828$
$\log \sin A = 9.6453$	$\log \sin a = 9.5556$	$\log \sin c = 9.9104$
$A_1 = 26° \, 14'$	$a_1 = 21° \, 4'$	$c_1 = 54° \, 27'$
$A_2 = 153° \, 46'$	$a_2 = 158° \, 56'$	$c_2 = 125° \, 33'$

CHECK. $\sin a = \sin A \sin c.$

$$\log \sin a = \log \sin A + \log \sin c$$

$$9.5556 = 9.6453 + 9.9104 = 9.5557.$$

By law 2 both sets of values must be used; but by law 1 the acute value c_1 belongs with the obtuse values of A and a, the obtuse value c_2 with the acute values of A and a. Thus the two solutions are:

1. $A = 26° \, 14'$, $a = 21° \, 4'$, $c = 125° \, 33'$.
2. $A = 153° \, 46'$, $a = 158° \, 56'$, $c = 54° \, 27'$.

NOTE. *A quadrantal spherical triangle* is one which has a side equal to a quadrant. The polar triangle of a quadrantal triangle is right. Therefore, to solve a quadrantal triangle solve its polar triangle and take the supplements of the parts thus found.

EXAMPLES

Solve the following triangles in which $C = 90°$.

1. $A = 40° \, 13'$, $a = 26° \, 25'$.
2. $B = 83° \, 15'$, $b = 76° \, 46'$.
3. $B = 110° \, 50'$, $b = 118° \, 30'$.
4. $b = 127° \, 36'$, $c = 94° \, 52'$.
5. $a = 165° \, 19'$, $c = 46° \, 50'$.
6. $b = 40° \, 49'$, $c = 135° \, 40'$.
7. $a = 21° \, 18'$, $b = 49° \, 55'$.
8. $a = 78° \, 32'$, $b = 132° \, 25'$.
9. $a = 144° \, 1'$, $b = 123° \, 6'$.
10. $A = 59° \, 17'$, $B = 51° \, 46'$.
11. $A = 137° \, 18'$, $B = 119° \, 30'$.
12. $A = 71° \, 46'$, $B = 148° \, 3'$.

13. $A = 20° 34'$, $c = 23° 18'$.

14. $B = 97° 36'$, $c = 96° 31'$.

15. $A = 100° 38'$, $c = 51° 44'$.

16. $B = 59° 54'$, $a = 6° 50'$.

17. $B = 47° 34'$, $a = 144° 24'$.

18. $A = 102° 49'$, $b = 10° 19'$.

19. $A = 98° 17'$, $a = 143° 8'$.

20. $a = 172° 28'$, $c = 124° 39'$.

21. $a = 4° 54'$, $b = 169° 27'$.

22. $A = 76° 17'$, $B = 144° 1'$.

23. $B = 82° 43'$, $c = 99° 26'$.

24. $B = 99° 47'$, $a = 26° 43'$.

25. $A = 97° 24'$, $a = 103° 12'$.

26. $b = 164° 10'$, $c = 133° 50'$.

27. $b = 34° 3'$, $a = 54° 26'$.

28. $A = 156° 30'$, $B = 104° 50'$.

29. $A = 165° 1'$, $c = 50° 30'$.

30. $B = 37° 56'$, $a = 157° 12'$.

Solve the following quadrantal spherical triangles ($c = 90°$):

31. $A = 30° 12'$, $a = 72° 29'$.

32. $B = 118° 16'$, $a = 137° 57'$.

33. $b = 51° 33'$, $C = 25° 48'$.

34. $A = 141° 13'$, $C = 49° 35'$.

35. $A = 159° 20'$, $a = 136° 30'$.

36. $b = 18° 41'$, $A = 39° 24'$.

CHAPTER X

THE SOLUTION OF OBLIQUE SPHERICAL TRIANGLES

77. Six Cases may be enumerated in the solution of oblique spherical triangles.

1. *Given the three sides,* a, b, c.

2. *Given the three angles,* A, B, C.

3. *Given two sides and the included angle,* a, b, C.

4. *Given two angles and the included side,* A, B, c.

5. *Given two sides and the angle opposite one of them,* a, b, A.

6. *Given two angles and the side opposite one of them,* A, B, a.

We shall proceed to consider these cases in the order named.

78. Case 1. *Given the three sides.* The law of cosines is sufficient to solve this case, but the equations are not adapted to logarithmic computation. We therefore develop them as follows:

We have proved the formula

$$\tan \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{1 + \cos A}}.$$

By the law of cosines

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Whence

$$\frac{1-\cos A}{1+\cos A}=\frac{\sin b\sin c+\cos b\cos c-\cos a}{\sin b\sin c-\cos b\cos c+\cos a}$$

$$=\frac{\cos(b-c)-\cos a}{\cos a-\cos(b+c)}=\frac{\cos a-\cos(b-c)}{\cos(b+c)-\cos a}.$$

But

$$\cos a-\cos(b-c)=-2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a-b+c),$$

$$\cos(b+c)-\cos a=-2\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(b+c-a).$$

Hence

$$\frac{1-\cos A}{1+\cos A}=\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}.$$

Let $a+b+c=2s$; then $a+b-c=2(s-c)$,

$a-b+c=2(s-b)$, and $b+c-a=2(s-a)$.

Therefore, $$\frac{1-\cos A}{1+\cos A}=\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)},$$

or $$\tan\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}. \qquad (47)$$

Similarly,

$$\tan\tfrac{1}{2}B=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}},$$

$$\tan\tfrac{1}{2}C=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}.$$

We may write

$$\tan\tfrac{1}{2}A=\frac{1}{\sin(s-a)}\cdot\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

or, putting

$$k=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan\tfrac{1}{2}A=\frac{k}{\sin(s-a)}, \qquad (48)$$

$$\tan\tfrac{1}{2}B=\frac{k}{\sin(s-b)},\quad \tan\tfrac{1}{2}C=\frac{k}{\sin(s-c)}.$$

Either set of formulæ (47) or (48) may be used in the solution of Case 1. If a *check* is desired in the solution the law of sines may be so used. Thus, since

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c},$$

it follows that

$$\begin{aligned}\log\sin A-\log\sin a&=\log\sin B-\log\sin b\\&=\log\sin C-\log\sin c.\end{aligned}$$

It must be remembered, however, that results may check and still be incorrect. If they do not check they are wrong; if they check they *may* be right, or may be wrong, since errors may compensate each other. It is important to check one's work, but far more important to learn, by careful attention, to work accurately.

Example. Given $a=103°$, $b=53°$, $c=61°$, find A, B, and C.

Using the formulæ (48) we find

$$s=\tfrac{1}{2}(103°+53°+61°)=108°\,30',$$

$$s-a=5°\,30',\quad s-b=55°\,30',\quad s-c=47°\,30'.$$

$$\begin{aligned}\log\sin(s-a)&=\ \ 8.9816\\\log\sin(s-b)&=\ \ 9.9160\\\log\sin(s-c)&=\ \ 9.8676\\\log\csc s&=\ \ \underline{0.0230}\\&\ \ 2\overline{)18.7882}\\\log k&=\ \ 9.3941\end{aligned}$$

$$\begin{aligned}\log k&=9.3941\\\log\sin(s-a)&=\underline{8.9816}\\\log\tan\tfrac{1}{2}A&=0.4125\\\tfrac{1}{2}A&=68°\,51',\\A&=137°\,42',\end{aligned}\qquad\qquad\begin{aligned}\log k&=9.3941\\\log\sin(s-b)&=\underline{9.9160}\\\log\tan\tfrac{1}{2}B&=9.4781\\\tfrac{1}{2}B&=16°\,44',\\B&=33°\,28',\end{aligned}$$

$$\begin{aligned} \log k &= 9.3941 \\ \log \sin (s-c) &= 9.8676 \\ \hline \log \tan \tfrac{1}{2} C &= 9.5265 \\ \tfrac{1}{2} C &= 18° \, 35', \\ C &= 37° \, 10'. \end{aligned}$$

CHECK.

$$\begin{array}{lll} \log \sin A = 9.8280 & \log \sin B = 9.7415 & \log \sin C = 9.7811 \\ \log \sin a = 9.9887 & \log \sin b = 9.9023 & \log \sin c = 9.9418 \\ \hline 9.8393 & 9.8392 & 9.8393 \end{array}$$

79. Case 2. *Given the three angles.* This case may be solved by the same formulæ that are used in Case 1, by making use of the *principle of polar triangles.* Thus, using accented letters to represent the corresponding parts of the polar triangle, we have $a' = 180° - A$, $b' = 180° - B$, $c' = 180° - C$. Knowing the sides a', b', c', we can find the angles A', B', C', as in Art. 78. Then the sides of the original triangle will be

$$a = 180° - A', \; b = 180° - B', \; c = 180° - C'.$$

Example. Given $A = 123°$, $B = 43°$, $C = 64°$, find a, b, c.

Here $a' = 180° - A = 57°$, $b' = 137°$, $c' = 116°$,

$$s = \tfrac{1}{2}(57° + 137° + 116°) = 155°,$$

$$s - a' = 98°, \; s - b' = 18°, \; s - c' = 39°.$$

$$\begin{aligned} \log \sin (s - a') &= 9.9958 \\ \log \sin (s - b') &= 9.4900 \\ \log \sin (s - c') &= 9.7989 \\ \log \csc s &= 0.3741 \\ \hline & 2)19.6588 \\ \log k &= 9.8294 \end{aligned}$$

$$\begin{aligned} \log k &= 9.8294 \\ \log \sin (s - a') &= 9.9958 \\ \hline \log \tan \tfrac{1}{2} A' &= 9.8336 \\ \tfrac{1}{2} A' &= 34° \, 17', \\ A' &= 68° \, 34', \end{aligned}$$

$$\begin{aligned} \log k &= 9.8294 \\ \log \sin (s - b') &= 9.4900 \\ \hline \log \tan \tfrac{1}{2} B' &= 0.3394 \\ \tfrac{1}{2} B' &= 65° \, 24', \\ B' &= 130° \, 48', \end{aligned}$$

$$
\begin{aligned}
\log k &= 9.8294 \\
\log \sin (s - c') &= 9.7989 \\
\hline
\log \tan \tfrac{1}{2} C' &= 0.0305 \\
\tfrac{1}{2} C' &= 47^\circ\ 1', \\
C' &= 94^\circ\ 2'.
\end{aligned}
$$

Therefore

$$a = 111^\circ\ 26', \qquad b = 49^\circ\ 12', \qquad c = 85^\circ\ 58'.$$

CHECK.

$\log \sin A = 9.9236$	$\log \sin B = 9.8338$	$\log \sin C = 9.9537$
$\log \sin a = 9.9689$	$\log \sin b = 9.8791$	$\log \sin c = 9.9989$
9.9547	9.9547	9.9548

NOTE. Using the law of cosines as stated in (38), Art. 71, whence

$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C},$$

and proceeding as in Art. 78, the following formulæ may be got:

$$\cot \tfrac{1}{2} a = \sqrt{-\frac{\cos (S - B) \cos (S - C)}{\cos S \cos (S - A)}},$$

or

$$\cot \tfrac{1}{2} a = \frac{K}{\cos (S - A)},$$

where

$$K = \sqrt{-\frac{\cos (S - A) \cos (S - B) \cos (S - C)}{\cos S}},$$

and similar formulæ for $\cot \frac{1}{2} b$ and $\cot \frac{1}{2} c$. These formulæ are simple and convenient, but it is unnecessary to burden the memory with them.

80. Cases 3 and 4, *two sides and the included angle, two angles and the included side,* are solved by means of **Napier's Analogies,** which we shall proceed to derive.

From (48), Art. 78, we may write

$$\tan \frac{A}{2} \tan \frac{B}{2} = \frac{k^2}{\sin (s - a) \sin (s - b)},$$

or, since
$$k^2 = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s},$$

$$\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} \cdot \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} = \frac{\sin(s-c)}{\sin s}. \qquad (\alpha)$$

Whence,
$$1 - \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = 1 - \frac{\sin(s-c)}{\sin s},$$

or,
$$\frac{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin s - \sin(s-c)}{\sin s}.$$

That is,
$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{2\cos\frac{1}{2}(2s-c)\sin\frac{1}{2}c}{\sin s}.$$

Whence, since $2s - c = a + b + c - c = a + b$,
$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{2\cos\frac{1}{2}(a+b)\sin\frac{1}{2}c}{\sin s}. \qquad (\beta)$$

Also, from (α) above,
$$1 + \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = 1 + \frac{\sin(s-c)}{\sin s},$$

which being transformed in the same manner gives
$$\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{2\sin\frac{1}{2}(a+b)\cos\frac{1}{2}c}{\sin s}. \qquad (\gamma)$$

Dividing (β) by (γ) we have

$$\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a+b)}. \tag{49}$$

Again, from (48) Art. 78 we may write

$$\frac{\tan \frac{A}{2}}{\tan \frac{B}{2}} = \frac{\sin (s-b)}{\sin (s-a)},$$

or,

$$\frac{\sin \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \sin \frac{B}{2}} = \frac{\sin (s-b)}{\sin (s-a)}.$$

Whence,

$$\frac{\sin \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \sin \frac{B}{2}} \pm 1 = \frac{\sin (s-b)}{\sin (s-a)} \pm 1$$

$$\frac{\sin \frac{A}{2} \cos \frac{B}{2} \pm \cos \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{A}{2} \sin \frac{B}{2}} = \frac{\sin (s-b) \pm \sin (s-a)}{\sin (s-a)}.$$

Using the upper signs,

$$\frac{\sin \frac{1}{2}(A+B)}{\cos \frac{A}{2} \sin \frac{B}{2}} = \frac{2 \sin \frac{1}{2}(2s-a-b) \cos \frac{1}{2}(a-b)}{\sin (s-a)}$$

$$= \frac{2 \sin \frac{1}{2} c \cos \frac{1}{2}(a-b)}{\sin (s-a)}. \tag{δ}$$

Using the lower signs,

$$\frac{\sin \frac{1}{2}(A-B)}{\cos \frac{A}{2} \sin \frac{B}{2}} = \frac{2 \cos \frac{1}{2}(2s-a-b) \sin \frac{1}{2}(a-b)}{\sin (s-a)}$$

$$= \frac{2 \cos \frac{1}{2} c \sin \frac{1}{2}(a-b)}{\sin (s-a)}. \tag{ϵ}$$

Dividing (δ) by (ϵ),

$$\frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a-b)}. \qquad (50)$$

Applying (49) and (50) to the polar triangles we obtain

$$\frac{\cos \frac{1}{2}(A'+B')}{\cos \frac{1}{2}(A'-B')} = \frac{\tan \frac{1}{2} c'}{\tan \frac{1}{2}(a'+b')},$$

$$\frac{\sin \frac{1}{2}(A'+B')}{\sin \frac{1}{2}(A'-B')} = \frac{\tan \frac{1}{2} c'}{\tan \frac{1}{2}(a'-b')}.$$

Remembering that $A' = 180° - a$, $a' = 180° - A$, etc. these become

$$\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A+B)}, \qquad (51)$$

$$\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A-B)}. \qquad (52)$$

The formulæ (49), (50), (51), and (52), called **Napier's Analogies** because of their similarity to formula (28) of the plane trigonometry, can obviously be written in other forms by the cyclical interchange of the letters.

81. Case 3. Example. Given $a = 100° 30'$, $b = 40° 20'$, $C = 46° 40'$, find A, B, c. Napier's analogies (51) and (52) may be written

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b) \cot \frac{1}{2} C}{\cos \frac{1}{2}(a+b)},$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b) \cot \frac{1}{2} C}{\sin \frac{1}{2}(a+b)},$$

which will determine A and B. Then to find c we may use either (49) or (50). The latter may be written

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B) \tan \frac{1}{2}(a-b)}{\sin \frac{1}{2}(A-B)}.$$

We have

$$\tfrac{1}{2}(a+b)=70^\circ\,25',\ \tfrac{1}{2}(a-b)=30^\circ\,5',\ \tfrac{1}{2}C=23^\circ\,20'.$$

$$\begin{aligned}\log\cos\tfrac{1}{2}(a-b)&=9.9371\\ \log\cot\tfrac{1}{2}C&=0.3652\\ \log\sec\tfrac{1}{2}(a+b)&=0.4748\\ \hline \log\tan\tfrac{1}{2}(A+B)&=0.7771\\ \tfrac{1}{2}(A+B)&=80^\circ\,31',\end{aligned}\qquad \begin{aligned}\log\sin\tfrac{1}{2}(a-b)&=9.7001\\ \log\cot\tfrac{1}{2}C&=0.3652\\ \log\csc\tfrac{1}{2}(a+b)&=0.0259\\ \hline \log\tan\tfrac{1}{2}(A-B)&=0.0912\\ \tfrac{1}{2}(A-B)&=50^\circ 59'.\end{aligned}$$

Whence $A=131^\circ\,30'$, $B=29^\circ\,32'$.

$$\begin{aligned}\log\sin\tfrac{1}{2}(A+B)&=9.9940\\ \log\tan\tfrac{1}{2}(a-b)&=9.7629\\ \log\csc\tfrac{1}{2}(A-B)&=0.1096\\ \hline \log\tan\tfrac{1}{2}c&=9.8665\\ \tfrac{1}{2}c&=36^\circ\,20',\\ c&=72^\circ\,40'.\end{aligned}$$

The signs are all *plus* in the above computation.

Case 4. Example. Given $B=110^\circ\,40'$, $C=100^\circ\,36'$, $a=76^\circ\,38'$, find b, c, A.

Napier's analogies (49) and (50) may be written

$$\tan\tfrac{1}{2}(b+c)=\frac{\cos\tfrac{1}{2}(B-C)\tan\tfrac{1}{2}a}{\cos\tfrac{1}{2}(B+C)},$$

$$\tan\tfrac{1}{2}(b-c)=\frac{\sin\tfrac{1}{2}(B-C)\tan\tfrac{1}{2}a}{\sin\tfrac{1}{2}(B+C)},$$

which will determine b and c. To find A either (51) or (52) may be used. The latter is

$$\cot\tfrac{1}{2}A=\frac{\sin\tfrac{1}{2}(b+c)\tan\tfrac{1}{2}(B-C)}{\sin\tfrac{1}{2}(b-c)}.$$

Here $\tfrac{1}{2}(B+C)=105^\circ\,38'$, $\tfrac{1}{2}(B-C)=5^\circ\,2'$, $\tfrac{1}{2}a=38^\circ\,19'$.

$$\begin{aligned}{}^{+}\log\cos\tfrac{1}{2}(B-C)&=9.9983\\ {}^{+}\log\tan\tfrac{1}{2}a&=9.8977\\ {}^{-}\log\sec\tfrac{1}{2}(B+C)&=0.5695\\ \hline {}^{-}\log\tan\tfrac{1}{2}(b+c)&=0.4655\\ 180^\circ-\tfrac{1}{2}(b+c)&=71^\circ\,6',\\ \tfrac{1}{2}(b+c)&=108^\circ\,54'.\end{aligned}\qquad \begin{aligned}{}^{+}\log\sin\tfrac{1}{2}(B-C)&=8.9432\\ {}^{+}\log\tan\tfrac{1}{2}a&=9.8977\\ {}^{+}\log\csc\tfrac{1}{2}(B+C)&=0.0164\\ \hline {}^{+}\log\tan\tfrac{1}{2}(b-c)&=8.8573\\ \tfrac{1}{2}(b-c)&=4^\circ\,7'.\end{aligned}$$

Whence $b = 113° 1'$, $c = 104° 47'$.

$$\begin{array}{r} \log^+ \sin \tfrac{1}{2}(b+c) = 9.9759 \\ \log^+ \tan (B-C) = 8.9449 \\ \log^+ \csc \tfrac{1}{2}(b-c) = 1.1440 \\ \hline \log^+ \cot \tfrac{1}{2} A = 0.0648 \\ \tfrac{1}{2} A = 40° 44', \\ A = 81° 28'. \end{array}$$

Note that the algebraic signs are not all *plus*, and that the quadrant in which the angle lies is determined by the sign in the case of the tangent, cotangent, or cosine.

82. Cases 5 and 6, *two sides and an opposite angle* or *two angles and an opposite side*, may be solved by the law of sines together with Napier's analogies. Thus, if a, b and A are given, we may write

$$\sin B = \frac{\sin b \sin A}{\sin a}$$

which, however, does not determine B unambiguously, since B may be either acute or obtuse. In this case, indeed, there may be two solutions, one solution, or none. We know, however, that if two sides (or angles) of a spherical triangle are unequal the angles (or sides) opposite are unequal, and the greater angle (or side) lies opposite the greater side (or angle). These theorems enable us to determine which values of the angle (or side) are possible. Thus *if* $b \lesseqgtr a$, *then only values of* B *which are* $\left\{\begin{matrix} \textit{less} \\ \textit{greater} \end{matrix}\right\}$ *than* A *are possible;* both values of B may be so, or only one value. *If the sine of* B *is greater than unity;* that is, *if* $\log \sin B$ *is positive, no solution is possible.* These same considerations obviously apply to case 6 also.

Another method of removing the ambiguity of Cases 5 and 6 is as follows: Two angles are of the *same species* when they are both acute or both obtuse. Also, since each side and angle of a spherical triangle is less than 180°, we see that $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are each less than 180°; while $\frac{1}{2}(A-B)$ and $\frac{1}{2}c$ are each less than 90°. It follows, in Napier's first analogy,

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)},$$

that $\tan\frac{1}{2}c$ and $\cos\frac{1}{2}(A-B)$ are both positive. Then $\cos\frac{1}{2}(A+B)$ and $\tan\frac{1}{2}(a+b)$ must have the same algebraic sign, and, therefore, $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ *are of the same species.* Thus, when a and b are given and A or B is to be found, *if* $\frac{1}{2}(a+b) \gtrless 90°$ *must also* $\frac{1}{2}(A+B) \gtrless 90°$; and the values of A or B must be so chosen as to satisfy this condition.

Having thus found B, say, (whether there be two values or only one) we may complete the solution of the triangle by the use of Napier's analogies.

Example 1. Given $a = 46° 30'$, $b = 30° 20'$, $B = 36° 40'$, solve the triangle.

We have
$$\sin A = \frac{\sin a \sin B}{\sin b}$$

$$\begin{aligned}
\log \sin a &= 9.8606\\
\log \sin B &= 9.7761\\
\log \csc b &= 0.2967\\
\hline
\log \sin A &= 9.9334\\
A = 59° 4' &\text{ or } 120° 56' = A'.
\end{aligned}$$

Here $a > b$, and, therefore, must $A > B$. This is *true of both values of* A found, so that there are *two possible solutions* of the triangle. To find C and c we may use (52) and (50).

$$\cot\frac{C}{2}=\frac{\sin\frac{1}{2}(a+b)\tan\frac{1}{2}(A-B)}{\sin\frac{1}{2}(a-b)}$$

$$\tan\frac{c}{2}=\frac{\sin\frac{1}{2}(A+B)\tan\frac{1}{2}(a-b)}{\sin\frac{1}{2}(A-B)}.$$

We have

First solution	*Second solution*
$\frac{1}{2}(a+b) = 38° 25'$,	$\frac{1}{2}(a+b) = 38° 25'$,
$\frac{1}{2}(a-b) = 8° 5'$,	$\frac{1}{2}(a-b) = 8° 5'$,
$\frac{1}{2}(A+B) = 47° 52'$,	$\frac{1}{2}(A'+B) = 78° 48'$,
$\frac{1}{2}(A-B) = 11° 12'$.	$\frac{1}{2}(A'-B) = 42° 8'$.

$$\begin{array}{rll} \log\sin\frac{1}{2}(a+b) = 9.7934 & \text{or} & 9.7934 \\ \log\tan\frac{1}{2}(A-B) = 9.2966 & & 9.9565 \\ \underline{\log\csc\frac{1}{2}(a-b) = 0.8519} & & \underline{0.8519} \\ \log\cot\frac{C}{2} = 9.9419 & \text{or} & 0.6018 \end{array}$$

$$\frac{C}{2} = 48° 49', \quad \frac{C'}{2} = 14° 3',$$

$$C = 97° 38'. \quad C' = 28° 6'.$$

$$\begin{array}{rll} \log\sin\frac{1}{2}(A+B) = 9.8701 & \text{or} & 9.9916 \\ \log\tan\frac{1}{2}(a-b) = 9.1524 & & 9.1524 \\ \underline{\log\csc\frac{1}{2}(A-B) = 0.7117} & & \underline{0.1734} \\ \log\tan\frac{c}{2} = 9.7342 & \text{or} & 9.3174 \end{array}$$

$$\frac{c}{2} = 28° 28', \qquad \frac{c'}{2} = 11° 44',$$

$$c = 56° 56'. \qquad c' = 23° 38'.$$

The two complete solutions are, therefore,

$$\begin{array}{rll} A = 59° 4', & \text{or} & 120° 56', \\ C = 97° 38', & & 28° 6', \\ c = 56° 56'. & & 23° 28'. \end{array}$$

Example 2. Given $a = 126°$, $c = 70°$, $A = 56°$, solve the triangle.

Using the formula

$$\sin C = \frac{\sin A \sin c}{\sin a}$$

we have

$$\begin{array}{r}\log\sin A = 9.9186\\ \log\sin c = 9.9730\\ \log\csc a = 0.0920\\ \hline \log\sin C = 9.9836\\ C = 74^\circ\ 20' \text{ or } 105^\circ\ 40'.\end{array}$$

But since $\boldsymbol{a} > \boldsymbol{c}$, must $\boldsymbol{A} > \boldsymbol{C}$. Therefore, there is *no solution.* Otherwise thus:

$\frac{1}{2}(a+c) = 98^\circ$, $\frac{1}{2}(A+C) = 65^\circ\ 10'$ or $80^\circ\ 50'$, which are *not* of the same species.

Example 3. Given $A = 84^\circ$, $C = 19^\circ$, $a = 28^\circ$, solve the triangle.

Using the law of sines, $\sin c = \dfrac{\sin C \sin a}{\sin A}$

$$\begin{array}{r}\log\sin C = 9.5126\\ \log\sin a = 9.6716\\ \log\csc A = 0.0024\\ \hline \log\sin c = 9.1866\\ c = 8^\circ\ 50' \text{ or } 171^\circ\ 10'.\end{array}$$

But since $\boldsymbol{C} < \boldsymbol{A}$, must $\boldsymbol{c} < \boldsymbol{a}$, and the *second value* is *impossible.*

To find b use (50).

$$\begin{array}{r}\log\sin\frac{1}{2}(A+C) = 9.8935\\ \log\tan\frac{1}{2}(a-c) = 9.2275\\ \log\csc\frac{1}{2}(A-C) = 0.2698\\ \hline \log\tan\frac{1}{2}b = 9.3908\\ \frac{1}{2}b = 13^\circ\ 49',\ b = 27^\circ\ 38'.\end{array}$$

To find B we may use (52), which has the advantage of giving an unambiguous result, or the law of sines. Selecting the latter method we have

$$\begin{array}{r}\log\sin C = 9.5126\\ \log\sin b = 9.6663\\ \log\csc c = 0.8137\\ \hline \log\sin B = 9.9926\\ B = 79^\circ\ 27' \text{ or } 100^\circ\ 33'.\end{array}$$

But since $b < a$, must $B < A$, and the *second value* is *impossible*. The complete solution is, therefore,

$$c = 8° \, 50', \quad b = 27° \, 38', \quad B = 79° \, 27'.$$

83. Delambre's Analogies or Gauss's Equations.

Using the law of cosines we may write

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Whence

$$1 - \cos A = 2 \sin^2 \tfrac{1}{2} A = \frac{(\cos b \cos c + \sin b \sin c) - \cos a}{\sin b \sin c}$$

or,

$$2 \sin^2 \frac{A}{2} = \frac{\cos(b-c) - \cos a}{\sin b \sin c} = \frac{2 \sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(a-b+c)}{\sin b \sin c}.$$

That is,

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}$$

with similar formulæ for $\sin \frac{B}{2}$ and $\sin \frac{C}{2}$.

In the same manner, by adding unity to each side of the first equation of this article, may be obtained formulæ of which the type is

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}.$$

From these obviously follows

$$\sin \frac{A}{2} \cos \frac{B}{2} = \frac{\sin (s-b)}{\sin c} \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}$$

$$= \frac{\sin (s-b)}{\sin c} \cos \frac{C}{2}. \qquad (\alpha)$$

In the same way we obtain

$$\cos \frac{A}{2} \sin \frac{B}{2} = \frac{\sin (s-a)}{\sin c} \cos \frac{C}{2}. \qquad (\beta)$$

Adding (α) and (β),

$$\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}=\frac{\sin(s-a)+\sin(s-b)}{\sin c}\cos\frac{C}{2}.$$

Whence

$$\sin\tfrac{1}{2}(A+B)=\frac{\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)}{\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\frac{C}{2},$$

or

$$\sin\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\frac{C}{2}. \qquad \text{I}$$

Similarly may be obtained

$$\sin\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}\cos\frac{C}{2}, \qquad \text{II}$$

$$\cos\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\sin\frac{C}{2}, \qquad \text{III}$$

$$\cos\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\frac{C}{2}, \qquad \text{IV}$$

which are the analogies or equations sought. These important equations may be conveniently used in the solution of Cases 3 and 4 of oblique triangles.

Example. Given $a=132°\,47'$, $b=59°\,50'$, $C=56°\,28'$, solve the triangle.

We have

$$\tfrac{1}{2}(a+b)=96°\,19', \quad \tfrac{1}{2}(a-b)=36°\,29', \quad \tfrac{1}{2}C=28°\,14'.$$

$\log\sin\frac{1}{2}(a+b)=9.9973,$ $\log\sin\frac{1}{2}(a-b)=9.7742,$

$\log\cos\frac{1}{2}(a+b)=9.0414,$ $\log\cos\frac{1}{2}(a-b)=9.9053,$

$$\log\sin\frac{C}{2}=9.6749, \qquad \log\cos\frac{C}{2}=9.9450.$$

From equations II and IV,

$$\log\{\sin\tfrac{1}{2}c\sin\tfrac{1}{2}(A-B)\}=9.7742+9.9450=9.7192,$$
$$\log\{\sin\tfrac{1}{2}c\cos\tfrac{1}{2}(A-B)\}=9.9973+9.6749=9.6722.$$

Whence

$$\log \tan \tfrac{1}{2}(A - B) = 0.0470,$$

$$\tfrac{1}{2}(A - B) = 48° 6'.$$

From I and III,

$$\log^{+} \{\cos \tfrac{1}{2} c \sin \tfrac{1}{2}(A + B)\} = 9.9053 + 9.9450 = 9.8503,$$

$$\log^{-} \{\cos \tfrac{1}{2} c \cos \tfrac{1}{2}(A + B)\} = 9.0414 + 9.6749 = 8.7163.$$

Whence $\log^{-} \tan \tfrac{1}{2}(A + B) = 1.1340.$

$$180° - \tfrac{1}{2}(A + B) = 85° 48', \quad \tfrac{1}{2}(A + B) = 94° 12'.$$

Therefore, $A = 142° 18', \quad B = 46° 6'.$

Also $\log \sin \tfrac{1}{2}(A - B) = 9.8718.$

$$\begin{array}{r} \text{Therefore, } \log \{\sin \tfrac{1}{2} c \sin \tfrac{1}{2}(A - B)\} = 9.7192 \\ \log \sin \tfrac{1}{2}(A - B) = 9.8718 \\ \hline \log \sin \tfrac{1}{2} c = 9.8474 \end{array}$$

$$\tfrac{1}{2} c = 44° 44', \quad c = 89° 28'.$$

Possibility of Solution by Inspection of Data. Before attempting the solution of a spherical triangle it may be desirable to determine whether the triangle is possible with the given data.

Case 1. Given the three sides. The triangle is always possible if the sum of the sides is less than 360°, and if no one side is greater than the sum of the other two. This follows at once from well-known geometrical theorems.

Case 2. Given the three angles. This case can be readily tested by the criteria of Case 1 applied to the polar triangle. For example, the triangle $A = 78°$, $B = 100°$, $C = 160°$ is impossible because the sides of the polar triangle, $a' = 102°$, $b' = 80°$, $c' = 20°$, are such that $a' > b' + c'$.

Case 3, given two sides and the included angle, and **Case 4**, two angles and the included side, are always possible.

Cases 5 and 6 have been discussed in Art. 82.

EXAMPLES

1. $a = 68^\circ 25'$, $b = 71^\circ 11'$, $c = 56^\circ 57'$.
2. $a = 100^\circ 8'$, $b = 50^\circ 2'$, $c = 60^\circ 6'$.
3. $A = 51^\circ 59'$, $B = 83^\circ 55'$, $C = 58^\circ 54'$.
4. $A = 142^\circ 33'$, $B = 27^\circ 53'$, $C = 32^\circ 27'$.
5. $b = 42^\circ 10'$, $c = 96^\circ 11'$, $A = 110^\circ 5'$.
6. $a = 146^\circ$, $c = 69^\circ 20'$, $B = 125^\circ 10'$.
7. $a = 90^\circ 50'$, $c = 117^\circ 50'$, $B = 120^\circ 6'$.
8. $B = 41^\circ 6'$, $C = 122^\circ 10'$, $a = 37^\circ$.
9. $A = 135^\circ$, $C = 50^\circ 50'$, $b = 68^\circ 50'$.
10. $A = 147^\circ 30'$, $C = 163^\circ 10'$, $b = 76^\circ 25'$.
11. $a = 29^\circ 2'$, $b = 14^\circ 3'$, $A = 49^\circ 5'$.
12. $b = 98^\circ$, $c = 36^\circ$, $C = 163^\circ$.
13. $a = 132^\circ$, $b = 56^\circ$, $A = 116^\circ 18'$.
14. $a = 111^\circ 20'$, $c = 41^\circ 30'$, $C = 25^\circ 10'$.
15. $A = 159^\circ 1'$, $C = 36^\circ$, $a = 9^\circ 5'$.
16. $A = 25^\circ 20'$, $C = 153^\circ 30'$, $a = 73^\circ 33'$.
17. $B = 142^\circ 30'$, $C = 71^\circ 20'$, $c = 39^\circ 35'$.
18. $A = 110^\circ 5'$, $B = 123^\circ 20'$, $b = 126^\circ 55'$.
19. $a = 59^\circ 34'$, $b = 136^\circ 11'$, $c = 150^\circ 2'$.
20. $a = 109^\circ 24'$, $c = 81^\circ 50'$, $A = 107^\circ 40'$.
21. $a = 99^\circ 50'$, $c = 64^\circ 10'$, $A = 96^\circ 13'$.
22. $A = 35^\circ 31'$, $B = 24^\circ 43'$, $C = 138^\circ 25'$.
23. $A = 31^\circ 20'$, $C = 122^\circ 40'$, $b = 40^\circ 40'$.
24. $a = 120^\circ 45'$, $c = 70^\circ 25'$, $B = 50^\circ 16'$.
25. $A = 120^\circ 21'$, $B = 130^\circ 22'$, $C = 140^\circ 7'$.
26. $c = 109^\circ 20'$, $b = 80^\circ 20'$, $C = 106^\circ 50'$.
27. $A = 132^\circ$, $B = 139^\circ 50'$, $b = 127^\circ 10'$.
28. $A = 79^\circ$, $B = 40^\circ$, $c = 108^\circ$.
29. $a = 40^\circ$, $b = 118^\circ 21'$, $A = 29^\circ 25'$.
30. $C = 148^\circ$, $B = 22^\circ 20'$, $c = 136^\circ$.
31. $a = 114^\circ$, $c = 148^\circ$, $C = 135^\circ 7'$.
32. $A = 73^\circ$, $B = 81^\circ 50'$, $a = 122^\circ 47'$.
33. $B = 61^\circ 40'$, $C = 140^\circ 15'$, $c = 150^\circ 25'$.
34. $a = 125^\circ 16'$, $b = 151^\circ 37'$, $c = 75^\circ 55'$.
35. $A = 60^\circ 40'$, $C = 105^\circ$, $a = 64^\circ 30'$.
36. $a = 55^\circ 5'$, $c = 138^\circ 5'$, $A = 42^\circ 28'$.
37. $B = 116^\circ 6'$, $C = 73^\circ 50'$, $c = 80^\circ$.
38. $B = 134^\circ$, $C = 51^\circ$, $a = 70^\circ 20'$.
39. $b = 108^\circ$, $c = 40^\circ$, $C = 39^\circ$.

40. $a = 58° 20'$, $b = 138° 5'$, $c = 116° 3'$.

41. $a = 61°$, $c = 97°$, $B = 110°$.

42. $A = 70° 5'$, $B = 122°$, $C = 95° 4'$.

43. $a = 60°$, $b = 120°$, $c = 50°$.

44. $A = 115°$, $B = 60°$, $C = 135°$.

45. $a = 150°$, $b = 160°$, $B = 10°$.

46. $a = 112° 30'$, $b = 108° 40'$, $c = 140° 10'$.

47. $A = 20° 30'$, $B = 32° 30'$, $C = 124° 30'$.

CHAPTER XI

THE EARTH AS A SPHERE

84. Distances on the Earth. As we remarked in the introductory chapter, plane trigonometry is sufficient for the survey of small areas. For larger areas and in navigation, except in the most refined work, the Earth is treated as a sphere and we make use of the principles of spherical trigonometry already enunciated.

The shortest distance between two points on the Earth is the arc of a great circle joining them. If we know the number of degrees in that arc we can compute its length by the formula $s = xr$ (Art. 47), where s is the length of arc, x the angle in circular measure, and r the radius of the sphere; in this case 3960 miles, the radius of the Earth.

Example. Find the length of an arc of 26° on the Earth's surface.

$$26^\circ = \frac{26\,\pi}{180} = \frac{13\,\pi}{90} \text{ radians.}$$

Therefore,

$$s = \frac{13\,\pi}{90} \times 3960 = \frac{13}{90} \cdot \frac{22}{7} \cdot 3960 = 1798 \text{ miles.}$$

It is convenient to compute and remember the number of miles in one degree of arc for the purpose of finding lengths of arcs.

$$s = \frac{\pi}{180} \times 3960 = 69.1 \text{ miles, approximately.}$$

85. Position and direction. The *position* of a point on the Earth is determined by its *latitude* and *longitude;* that is, by the number of degrees the point lies north or south of the equator, and the number of degrees east or west of a great circle, through the Earth's axis, chosen as a reference line. We shall use the great circle, or meridian, through Greenwich.

A point moving along a great circle of the Earth, unless that circle be a meridian or the equator, is constantly changing its direction, or *course*. For example, Fig. 36, at A the compass points north along AN, and a ship at A is

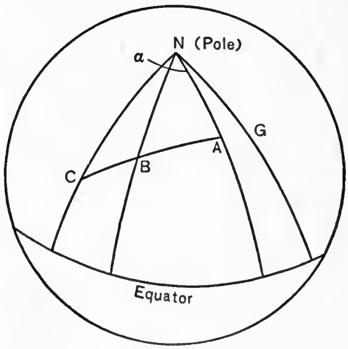


Fig. 36.

sailing, say, due west. When the ship has reached B the compass points north along BN and the ship is sailing west 30° south. On the other hand if a ship sailed constantly on a course, say, west 30° south it would move around the Earth in a spiral approaching continually nearer to the South Pole.

86. Bearings. To illustrate the use of spherical trigonometry in determining positions, directions, and distances on the Earth's surface, consider, Fig. 36, a ship sailing from C to A along the great circle CBA. The lines NC, NB, NA, and NG are meridians, the last being the meridian of Greenwich. Suppose the latitude and longitude of C are 44° 40′ N., 63° 35′ W.; of A, 53° 24′ N., 3° 4′ W. The longitude of G, obviously, is 0°. The positions of the points C and A being thus known, let us find the directions (called *bearings*) of A from C and of C from A, and the distance from C to A. From the meaning of longitude we have

$GNA = 3° 4'$, $GNC = 63° 35'$, whence $\alpha = ANC = 60° 31'$.

Also, by the meaning of latitude,

$AN = 90° - 53° 24' = 36° 36'$; $CN = 90° - 44° 40' = 45° 20'$.

We therefore have, in the spherical triangle CNA, $CN = a = 45° 20'$, $AN = c = 36° 36'$ and the included angle $\alpha = 60° 31'$, which is case 3 in the solution of spherical triangles. The data:

$\frac{1}{2}(a + c) = 40° 58'$, $\frac{1}{2}(a - c) = 4° 22'$, $\frac{1}{2}\alpha = 30° 15.5'$.

$$\begin{array}{rl}
\log\cos\frac{1}{2}(a-c) &= 9.9987 \\
\log\cot\frac{\alpha}{2} &= 0.2340 \\
\log\sec\frac{1}{2}(a+c) &= 0.1220 \\
\hline
\log\tan\frac{1}{2}(A+C) &= 0.3547 \\
\frac{1}{2}(A+C) &= 66° 10' \\
\text{Whence}\quad A &= 77° 26'.
\end{array}
\qquad
\begin{array}{rl}
\log\sin\frac{1}{2}(a-c) &= 8.8816 \\
\log\cot\frac{\alpha}{2} &= 0.2340 \\
\log\csc\frac{1}{2}(a+c) &= 0.1834 \\
\hline
\log\tan\frac{1}{2}(A-C) &= 9.2990 \\
\frac{1}{2}(A-C) &= 11° 16'. \\
C &= 54° 54'.
\end{array}$$

Therefore the bearings of C from A are N. 77° 26′ W.; of A from C, N. 54° 54′ E.

To find the side $CA = x$ we have

$$\begin{array}{rl}
\log\sin\frac{1}{2}(A+C) &= 9.9613 \\
\log\tan\frac{1}{2}(a-c) &= 8.8829 \\
\log\csc\frac{1}{2}(A-C) &= 0.7092 \\
\hline
\log\tan\frac{x}{2} &= 9.5534
\end{array}$$

$$\tfrac{1}{2}x = 19° 47.5', \qquad x = 39° 35'.$$

Therefore length $CA = 39.6° \times 69.1$ miles $= 2736$ miles.

If only the distance sailed is required it is simpler to use the law of cosines. Thus, $\cos x = \cos c \cos a + \sin c \sin a \cos \alpha$.

$$\begin{array}{rl} \log\cos c = & 9.9046 \\ \log\cos a = & 9.8469 \\ \hline & 9.7515 \\ \text{number} = & .5643 \\ & .2087 \\ \hline \cos x = & 0.7730, \\ x = & 39^\circ\, 23' \end{array} \qquad \begin{array}{rl} \log\sin c = & 9.7754 \\ \log\sin a = & 9.8520 \\ \log\cos \alpha = & 9.6921 \\ \hline & 9.3195 \\ \text{number} = & .2087 \end{array}$$

and the distance $= 39.4^\circ \times 69.1 = 2723$ miles.

87. The course of the ship at C would be N. 54° 54′ E. To show how the ship's course changes as it sails along CA let us find the course as the ship crosses the meridian 38° W. at the point B, Fig. 35. In the triangle NCB we have $b = CN = 45^\circ\ 20'$, $C = NCB = 54^\circ\ 54'$, $N = CNB = 63^\circ\ 35' - 38^\circ = 25^\circ\ 35'$; that is, two angles and the included side.

$\frac{1}{2}(C + N) = 40^\circ\ 14.5'$, $\frac{1}{2}(C - N) = 14^\circ\ 39.5'$, $\frac{1}{2}b = 22^\circ\, 40'$.

$$\begin{array}{rl} \log\cos\frac{1}{2}(C-N) = & 9.9856 \\ \log\tan\frac{1}{2}b = & 9.6208 \\ \log\sec\frac{1}{2}(C+N) = & 0.1173 \\ \hline \log\tan\frac{1}{2}(c+n) = & 9.7237 \\ \frac{1}{2}(c+n) = & 27^\circ\ 53.5' \end{array} \qquad \begin{array}{rl} \log\sin\frac{1}{2}(C-N) = & 9.4033 \\ \log\tan\frac{1}{2}b = & 9.6208 \\ \log\csc\frac{1}{2}(C+N) = & 0.1897 \\ \hline \log\tan\frac{1}{2}(c-n) = & 9.2138 \\ \frac{1}{2}(c-n) = & 9^\circ\ 17.5'. \end{array}$$

Whence $c = BN = 37^\circ\ 11'$, $n = CB = 18^\circ\ 36'$.

The latitude of B is $90^\circ - BN = 52^\circ\ 49'$ N., and the distance sailed is

$$CB = 18.6^\circ \times 69.1 \text{ miles} = 1285 \text{ miles}.$$

To find the angle $B = CBN$ we have

$$\begin{array}{rl} \log\sin\frac{1}{2}(c+n) = & 9.6700 \\ \log\tan\frac{1}{2}(C-N) = & 9.4176 \\ \log\csc\frac{1}{2}(c-n) = & 0.7919 \\ \hline \log\cot\frac{1}{2}B = & 9.8795 \end{array}$$

$\frac{1}{2}B = 52^\circ\, 51'$, $B = 105^\circ\ 42'$, and $NBA = 180^\circ - B = 74^\circ\ 18'$.

Therefore the ship's course at B (the bearing of A from B) is N. 74° 18′ E.

88. The Area of a Spherical Triangle may be found as follows: The theorem has been proved that the area of a spherical triangle is equal to its spherical excess (the excess of the sum of its angles over two right angles) times the area of the tri-rectangular triangle; it being understood that the right angle is the unit of angles. Thus, using Δ to represent the area of a triangle whose angles (in degrees) are A, B, and C; and noting that the tri-rectangular triangle is one eighth of the surface of the sphere; we have

$$\Delta = \frac{A + B + C - 180^\circ}{90^\circ} \cdot \frac{4\pi r^2}{8} = \frac{(A + B + C - 180)\pi r^2}{180^\circ}.$$

Example. Given $A = 105^\circ$, $B = 80^\circ$, $C = 95^\circ$, and taking $r = 3960$ miles, the radius of the earth,

$$\Delta = \frac{(105^\circ + 80^\circ + 95^\circ - 180^\circ)}{180^\circ} \cdot \pi(3960)^2 = \frac{5\pi(3960)^2}{9}.$$

$$\begin{aligned} \log 5 &= 0.6990 \\ \log \pi &= 0.4971 \\ 2 \log r &= 7.1954 \\ \text{colog } 9 &= 9.0458 - 10 \\ \hline \log \Delta &= 7.4373 \end{aligned}$$

and $\Delta = 27{,}370{,}000$ square miles.

TABLE OF LATITUDE AND LONGITUDE

Baltimore	39° 17′ N.,	76° 37′ W.
Boston	42° 21′ N.,	71° 4′ W.
Chicago	41° 53′ N.,	87° 38′ W.
Greenwich	51° 29′ N.,	0° W.
Honolulu	21° 18′ N.,	157° 55′ W.
Liverpool	53° 24′ N.,	3° 4′ W.
New York	40° 43′ N.,	74° W.
Pernambuco	8° S.,	34° W.
Rio de Janeiro	22° 54′ S.,	43° 10′ W.
San Francisco	37° 48′ N.,	122° 24′ W.
Washington	38° 54′ N.,	77° 3′ W.

EXAMPLES

In the following problems assume that one can travel directly along the arc of a great circle between the points named.

1. A ship sails from Baltimore to Boston. Find the course of the ship as she leaves Baltimore, her course as she crosses the meridian of New York, and the entire distance she sails. What are the bearings of Baltimore from Boston, and of Boston from Baltimore?

2. Find the course at Liverpool, the course at 55° W., and the total distance sailed by a ship going from Liverpool to New York. What are the bearings of these cities from each other?

3. A ship sails from Baltimore to Rio de Janeiro. She sails first to a point off Pernambuco in latitude 8° S., longitude 34° W., and from there to Rio. How far does she sail, and what is her course off Pernambuco?

NOTE. In the Southern Hemisphere latitudes are taken as algebraically negative. Use the north-polar distances of places as sides in solving triangles.

4. In problem 3 what course will the ship be sailing after she has gone 1000 miles? What will be her latitude and longitude at that point?

5. How far is the Washington Observatory from the Greenwich Observatory? What are the bearings of the two observatories from each other?

6. A ship sails from Boston on a course East 12° South. At what distance would she be sailing due East? What are her latitude and longitude at that instant?

7. A ship sails northwest from San Francisco. What would be the highest latitude she would reach? What would be the ship's longitude at that instant?

8. Find the number of square miles in the triangle whose vertices are Baltimore, Boston, and Chicago.

9. A ship sails from Honolulu to San Francisco. Find the entire distance sailed, and the course of the ship when she has gone halfway.

10. An aëroplane sails from Washington to Chicago along a great circle arc one mile above the surface of the Earth. In what time is the flight made at a rate of 75 miles per hour?

11. Find the number of square miles in the triangle whose vertices are Baltimore, New York, and Chicago.

12. Find the face and edge angles of a regular triangular pyramid.

13. What is the latitude of three points on the Earth equally distant from each other and from the North pole ?

14. Each face of a triangular pyramid is a triangle whose sides are 3, 4, and 5 respectively. Find the face and edge angles of the pyramid.

ANSWERS

CHAPTER III

Art. 28

1. $b = 340$ $c = 422$	2. $A = 43° 5$ $c = 54.9$	3. $A = 51° 47'$ $b = .433$
4. $b = 478$ $a = 154$	5. $a = 713$ $c = 823$	6. $b = 96.4$ $c = 232$

7. 70.7 ft. 8. 71.2 ft. 9. $60° 10'$ 10. 260 ft. 11. 212 ft.

12. 99.5 ft. 13. $43° 36'$ 14. Heights equal.

Art. 35

8. 1.912 9. $-.874$ 14. c^2 15. 10^{a+b} 16. e^5

17. 100 20. 0 21. 0 22. 1 23. Impossible.

25. $\frac{2(\log_e a - \log_e b)}{a-b}$ 26. $(a-b)\frac{b^{\frac{a}{a-b}}}{a^{\frac{b}{a-b}}}$ 27. 0 29. .6931

30. 1.099 31. 1.386 32. .2312 33. 1.029 34. $-.3088$

35. 4.408 36. 238.2 37. -358300 38. .07212 39. Impossible.

40. -312.1 41. -1747 42. 57090 43. .00003162

44. .03728 45. 100 46. 3.241 47. .001347 48. 1142

49. 2448

SOLUTIONS OF RIGHT TRIANGLES

1. $a = .04691$ $c = .05151$ $K = .0004988$	2. $a = 2316$ $b = 3402$ $K = 3,941,000$	3. $b = 24850$ $c = 36100$ $K = 325,400,000$
4. $A = 58° 10'$ $b = 48.04$ $K = 1859$	5. $a = 578.8$ $c = 2491$ $K = 701,000$	6. $a = .00883$ $b = .003607$ $K = .00001593$
7. $A = 63° 48'$ $c = .4694$ $K = .04364$	8. $a = .6441$ $c = .6503$ $K = .02879$	9. $A = 43° 24'$ $b = .8966$ $K = .3801$

10. $a = .003845$, $b = .006723$, $K = .00001293$ 11. $b = 5091$, $c = 5268$, $K = 3,444,000$ 12. $b = 99.43$, $c = 156.8$, $K = 6030$

13. $A = 79° 28'$, $a = 842$, $K = 65,900$ 14. $b = 63,840$, $c = 92,280$, $K = 2,128,000,000$ 15. $a = .000005737$, $c = .00002118$, $K = .0000000000585$

16. $a = .0003899$, $b = .0006772$, $K = .000,000,1321$ 17. $A = 27° 17'$, $a = 4.252$, $K = 17.53$ 18. $a = 18.59$, $b = 30.51$, $K = 283.7$

19. $b = 24,540$, $c = 30,010$, $K = 211,900,000$ 20. $A = 43° 45'$, $c = 5280$, $K = 6,970,000$

21. First steeper by 54′. 22. 24.7 mi. and 29.5 mi.
23. Team by 15 seconds. 24. 3° 25′ 25. 648 ft. 26. 14.7 in.
27. 9 hr. 28 min. A.M. or 2 hr. 32 min. P.M.
28. Reduced by 10.1 ft. 29. Buoy farther by 1133 ft.
30. Increase in altitude 251.4 ft. 31. 8° 45′. 32. 1 ft. shorter.
33. 1575 mi. 34. 57° 43′ N. or S. 35. N. 58° 15′ E. 15 mi.
36. E. 62° 46′ N. 7.29 mi. per hour.
37. 112.5 mi. 38. E. 30° N. or S. 4.33 mi. 39. 127.9 mm.
40. 155.1 ft. 41. 74.17 yd.

CHAPTER IV

Art. 40

5. $\frac{9 \pm 4\sqrt{7}}{20}$ 6. $\frac{1 \pm 2\sqrt{30}}{12}$ 7. $\frac{\pm\sqrt{2}}{2}$ 8. $\frac{\pm\sqrt{2}}{2}$

9. $\frac{\pm\sqrt{2}}{10}$ 10. $\frac{\pm 7\sqrt{2}}{10}$

Art. 41

4. $-\frac{6}{7}$ 5. $\frac{\pm 9\sqrt{3} \pm 8\sqrt{2}}{5}$ 6. $\frac{\pm 32\sqrt{2} \pm 9\sqrt{15}}{7}$

7. One value, 45° 8. $\pm \frac{24}{7}$, 0 9. Two values, ± 90°

Art. 44

1. $\pm\frac{4\sqrt{2}}{9}$ 2. $\frac{7}{9}$ 3. $\pm\frac{4\sqrt{2}}{7}$ 4. $\pm\frac{\sqrt{2} \pm 1}{\sqrt{6}}$

5. $\pm\frac{\sqrt{2} \pm 1}{\sqrt{6}}$ 6. $\pm(3 \pm 2\sqrt{2})$

CHAPTER V

Art. 47

18. Minute hand, π 19. $\frac{7\pi}{6}$ 20. $\frac{\pi}{20}$ 21. $\frac{20\pi}{9}$ 22. 14π
Hour hand, $\frac{\pi}{12}$

23. $\frac{5\pi}{3}$ radians per second. 24. 191 revolutions per minute.

25. 5934 mi. 26. $1978\sqrt{2}$ mi. 27. .035 in. 28. 5.7 ft.

Art. 49

17. $\frac{\sqrt{2}+\sqrt{6}}{4}$ 18. $\frac{\sqrt{6}-\sqrt{2}}{4}$ 19. $-\frac{\sqrt{3}}{3}$ 20. 0

CHAPTER VI

1. $b = 986$, $c = 544.3$, $K = 193{,}000$
2. $A = 19°$, $C = 52°$, $c = 8.19$, $K = 13.1$
3. $A = 18° 39'$, $B = 26° 52'$, $c = 673.9$, $K = 45{,}990$
4. $A = 49° 8'$, $B = 59° 18'$, $C = 71° 36'$, $K = 1705$
5. $A = 100° 35'$ or $10° 21'$; $C = 44° 53'$ or $135° 7'$; $a = 67.02$ or 12.25; $K = 914$ or 167.1
6. $a = 6184$, $b = 2937$, $K = 7{,}510{,}000$
7. $B = 49° 8'$, $C = 59° 19'$, $a = 70.48$, $K = 1703$
8. $A = 37° 58'$, $B = 66' 42'$, $a = 179.9$, $K = 23{,}370$
9. $a = 6.64$, $c = 3.95$, $K = 12.21$
10. $B = 23° 29'$, $C = 22° 57'$, $b = 3024$, $K = 3{,}243{,}000$
11. $A = 43° 49'$, $C = 61° 28'$, $c = .4592$, $K = .0802$
12. $A = 94° 16'$, $B = 54° 36'$, $C = 31° 8'$, $K = .0002699$
13. No solution.
14. $b = .01292$, $c = .002861$, $K = .00000826$
15. $A = 26° 19'$, $C = 109° 6'$, $c = 67.14$, $K = 742$
16. $b = .0185$, $A = 54° 40'$, $C = 94° 13'$, $K = .000002694$
17. $B = 90°$, $c = 59.39$, $K = 955$

18. $A = 46° 23'$ or $133° 37'$
$C = 102° 30'$ $15° 16'$
$c = 8730$ 2354
$K = 14{,}600{,}000$ $3{,}938{,}000$

19. $B = 22° 37'$
$C = 127° 28'$
$a = .5593$
$K = .0958$

20. $A = 134° 22'$
$B = 18° 42'$
$C = 26° 56'$
$K = 4.622$

21. $A = 57° 41'$
$C = 38° 49'$
$a = .02461$
$K = .0002232$

22. 33,695 sq. ft. **23.** 15 ft. **24.** 35.6 ft. **25.** 428 ft.

26. Width 74.6 ft.; height above stream, 14 ft. **27.** 472 ft.

28. 17.1 ft. **29.** 109 ft. **30.** 61.93 and 58.81 ft.

31. $A = 61° 43'$ $B = 80° 7'$ $C = 38° 10'$ $c = 5.20$

32. $B = 53° 26'$ $a = 46.45$ ft. $c = 52.48$ ft.

33. 11,320 and 7082. **34.** 3997 sq. ft. **35.** $A = 7° 5'$; no.

36. 21.7 mi. **37.** $9° 17'$ with the surface of the water.

38. 33.5 ft. **39.** 40 ft. **40.** Second yacht by 1 min. 12 sec.

41. At $82° 33'$ with shore on the side towards the $60°$ angle.

42. A–C–B by $560. **43.** $AC = 152.1$ ft. $BC = 319.4$ ft.

44. 441 ft. **45.** 336.9 ft.

CHAPTER VII

1. $n\pi$
$\frac{n\pi}{2} \pm \frac{\pi}{6}$

2. $n\pi \pm \frac{\pi}{4}$
$2n\pi \pm \frac{2\pi}{3}$

3. $2n\pi \pm \frac{\pi}{2}$
$n\pi + (-)^n \frac{\pi}{6}$

4. $2n\pi + \cos^{-1}\frac{2 \mp 6\sqrt{33}}{37}$

5. $2n\pi \pm \frac{\pi}{2}$
$2n\pi + \cos^{-1}\frac{4}{5}$

6. $n\pi$
$2n\pi \pm \frac{\pi}{3}$

7. $2n\pi - \frac{\pi}{2}$
$\frac{2n\pi}{3} + \frac{\pi}{6}$

8. $n\pi + \frac{\pi}{4}$
$\frac{n\pi}{2} + \frac{\pi}{8}$

9. $n\pi + (-)^n \sin^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$

10. $2n\pi \pm \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$.

11. $n\pi$.

12. $n\pi \pm \frac{\pi}{4}$
$2n\pi$
$(2n+1)\frac{\pi}{3}$

13. $n\pi + \frac{\pi}{4}$

$\frac{n\pi}{3} + \frac{\pi}{12}$.

14. $\frac{n\pi}{4}$

15. $n\pi + \frac{a}{2} \pm \frac{\pi}{4}$

16. $2n\pi \pm \frac{\pi}{3}$

$2n\pi \pm \frac{2\pi}{3}$

17. $\frac{n\pi}{2} + \frac{\pi}{6}$

18. $n\pi + \tan^{-1}\frac{3}{2}$

19. $n\pi + (-)^n \sin^{-1}\frac{4}{5}$

20. $2n\pi \pm \frac{\pi}{2}$, $n\pi \pm \tan^{-1}\frac{\sqrt{2}}{2}$

21. $x = \frac{n\pi}{2} - \frac{\pi}{8}$

$y = 1 \pm \frac{\sqrt{2}}{2}$

22. $\theta = 2n\pi \pm \frac{\pi}{2}$

$r = 2$

23. $\theta = n\pi$, $2n\pi \pm \frac{\pi}{3}$

$r = 0$, $\frac{\pm a\sqrt{3}}{2}$

24. $x = \frac{n\pi}{2} + \frac{\pi}{4}$

$y = \pm a\sqrt{2}$

25. $x = \frac{1}{2}\{n\pi + (-)^n \sin^{-1}\frac{4}{5}\}$

$y = \frac{\pm a\sqrt{5}}{5}$, n even

$y = \pm\frac{2a\sqrt{5}}{5}$, n odd

26. $\theta = \frac{n\pi}{2} + \frac{\pi}{8}$

$r = \pm\sqrt[4]{2}$

27. $\theta = n\pi + \frac{\pi}{4}$

$r = \pm\frac{5\sqrt{2}}{2}$

28. $x = \frac{n\pi}{2} + (-)^n\frac{\pi}{12}$

$y = \pm\sqrt{2}$

29. $x = \frac{(2n+1)\pi - a}{2}$

$y = \pm\cos\frac{a}{2}$

30. $\theta = n\pi \pm \frac{\pi}{3}$

$r = \frac{\pm a\sqrt{3}}{2}$

31. $\theta = \frac{n\pi}{2}$, $r = 0$, n even

$\theta = \frac{n\pi}{4}$, $r = \pm\frac{1}{\sqrt[4]{2}}$, n odd

32. $\theta = n\pi + \frac{\pi}{4}$

$r = \pm 2\sqrt{2}$

33. $\theta = \frac{n\pi}{2} \pm \frac{\pi}{12}$

$r = \pm 2\sqrt{2}$

34. $a\sqrt{1 - b^2} + b\sqrt{1 - a^2}$

35. $\frac{\pm\sqrt{21}}{14}$

36. $\pm 4\sqrt{5}$

37. $\pm\frac{\sqrt{2}}{2}$

38. 1 or $-\frac{1}{6}$

39. 0, $\pm\sqrt{3}$

40. ± 1

41. $\pm\sqrt{3}$, $\frac{\pm\sqrt{3}}{3}$

42. -1 or $\frac{1}{2}$

43. 0, $\frac{\pi}{24}$

44. $\frac{\pi}{2}$, $-\frac{\pi}{6}$

45. 0.

46. $\frac{\pi}{2}$

47. $x = -\frac{\pi}{8}$

$y = \frac{2 + \sqrt{2}}{2}$

48. $\theta = \frac{\pi}{2}$, $\frac{\pi}{6}$

$r = 0$, $\frac{a\sqrt{3}}{2}$

49. $x = \frac{1}{2}\sin^{-1}\left(\frac{\sqrt{5} - 1}{2}\right)$

$y = \frac{\sqrt{2\sqrt{5} - 2}}{2}$

50. $\theta = \frac{3\pi}{8}$, $r = \sin\frac{3\pi}{8}$

51. $\frac{\pi}{10}$

52. $\cos^{-1}\frac{2 - 3\sqrt{6}}{10}$

CHAPTER IX

1. $B = 58° 30'$, $121° 30'$; $b = 35° 59'$, $144° 1'$; $c = 43° 33'$, $136° 27'$

2. $A = 30° 53'$, $149° 7'$; $a = 30° 13'$, $149° 47'$; $c = 78° 35'$, $101° 25'$

3. $A = 48° 11'$, $131° 49'$; $a = 44° 29'$, $135° 31'$; $c = 109° 52'$, $70° 8'$

4. $A = 83° 39'$, $B = 127° 20'$, $a = 82° 1'$

5. $A = 159° 39'$, $B = 104° 14'$, $b = 135°$

6. $A = 147° 34'$, $B = 66° 3'$, $a = 157° 26'$

7. $A = 27°$, $B = 73°$, $c = 53° 8'$

8. $A = 81° 29'$, $B = 131° 50'$, $c = 97° 42'$

9. $A = 139° 5'$, $B = 110° 57'$, $c = 63° 47'$

10. $a = 49° 26'$, $b = 43° 58'$, $c = 62° 5'$

11. $a = 147° 37'$, $b = 136° 32'$, $c = 52° 11'$

12. $a = 53° 45'$, $b = 153° 17'$, $c = 121° 53'$

13. $a = 7° 59'$, $b = 21° 58'$, $B = 70° 59'$

14. $a = 49° 11'$, $b = 100°$, $A = 49° 37'$

15. $a = 129° 30'$, $b = 166° 50'$, $B = 163° 8'$

16. $A = 30° 47'$, $b = 11° 36'$, $c = 13° 26'$

17. $A = 126° 53'$, $b = 32° 29'$, $c = 133° 18'$

18. $a = 141° 47'$, $c = 140° 37'$, $B = 16° 25'$

19. $B = 10° 23'$, $169° 37'$; $b = 6° 16'$, $173° 44'$; $c = 142° 41'$, $37° 19'$

20. $A = 170° 50'$, $B = 84° 45'$, $b = 55° 1'$

21. $A = 25° 5'$, $B = 114° 38'$, $c = 168° 23'$

22. $a = 66° 12'$, $b = 146° 25'$, $c = 109° 39'$

23. $a = 142° 40'$, $b = 78° 7'$, $A = 142° 3'$

24. $A = 28° 19'$, $b = 110° 59'$, $c = 108° 39'$

25. $b = 33° 37'$, $146° 23'$; $c = 101°$, $79°$; $B = 34° 20'$, $145° 40'$

26. $A = 74° 12'$, $B = 157° 47'$, $a = 43° 57'$

27. $A = 68° 11'$, $B = 39° 43'$, $c = 61° 11'$

28. $a = 161° 32'$, $b = 129° 57'$, $c = 52° 28'$

29. $a = 168° 30'$, $b = 130° 29'$, $B = 99° 40'$

30. $A = 124° 32'$, $b = 16° 48'$, $c = 151° 57'$

31. $b = 20° 23'$, $159° 37'$; $B = 10° 35'$, $169° 25'$; $C = 148° 10'$, $31° 50'$

32. $A = 141° 32'$, $C = 111° 46'$, $b = 108° 29'$

33. $A = 163° 16'$, $B = 19° 55'$, $a = 138° 36'$

34. $a = 124° 38'$, $b = 46° 49'$, $B = 33° 43'$

35. $B = 23° 26'$, $156° 34'$; $C = 30° 51'$, $149° 9'$; $b = 50° 50'$, $129° 10'$

36. $B = 12° 7'$, $C = 139° 5'$, $a = 75° 40'$

CHAPTER X

1. $A = 76°$, $B = 81°$, $C = 61°$

2. $A = 138° 18'$, $B = 31° 12'$, $C = 35° 52'$

3. $a = 38° 2'$, $b = 51° 2'$, $c = 42° 2'$

4. $a = 101° 2'$, $b = 49°$, $c = 60°$

5. $B = 41° 32'$, $C = 79° 2'$, $a = 108° 10'$

6. $A = 145° 23'$, $C = 108° 3'$, $b = 126° 24'$

7. $A = 105° 57'$, $C = 121° 45'$, $b = 115° 56'$

8. $b = 47° 49'$, $c = 72° 37'$, $A = 32° 16'$

9. $a = 120° 25'$, $c = 71° 1'$, $B = 49° 52'$

10. $a = 124° 57'$, $c = 153° 47'$, $B = 140° 26'$

11. $B = 22° 13'$, $C = 112° 8'$, $c = 36° 30'$

12. No solution

13. $B = 90°$, $C = 138° 32'$, $c = 146° 42'$

14. $A = 36° 42'$, $143° 18'$; $B = 160° 32'$, $38° 52'$; $b = 148° 44'$, $78°$

15. No solution

16. $c = 90°$, $b = 18° 15'$, $B = 8° 2'$

17. $a = 138° 34'$, $b = 155° 50'$, $A = 100° 16'$

18. $a = 63° 59'$, $116° 1'$; $c = 156° 10'$, $72° 54'$; $C = 155° 2'$, $87° 36'$

19. $A = 110° 4'$, $B = 131° 2'$, $C = 147° 2'$

20. $b = 115° 19'$, $B = 114° 2'$, $C = 90°$

21. $B = 96° 16'$, $C = 65° 14'$, $b = 99° 52'$

22. $a = 61^\circ\ 2'$
$b = 39^\circ\ 2'$
$c = 92^\circ\ 2'$

23. $B = 37^\circ\ 30'$
$a = 33^\circ\ 49'$
$c = 64^\circ\ 15'$

24. $b = 69^\circ\ 46'$
$A = 67^\circ\ 37'$
$C = 25^\circ\ 17'$

25. $a = 90^\circ\ 58'$
$b = 118^\circ$
$c = 132^\circ\ 2'$

26. $a = 120^\circ\ 29'$
$A = 119^\circ\ 3'$
$B = 90^\circ$

27. $a = 66^\circ\ 40'$, $113^\circ\ 20'$
$c = 160^\circ\ 54'$, $102^\circ\ 2'$
$C = 164^\circ\ 38'$, $127^\circ\ 42'$

28. $C = 109^\circ\ 58'$
$a = 96^\circ\ 42'$
$b = 40^\circ\ 34'$

29. $B = 42^\circ\ 15'$, $137^\circ\ 45'$
$C = 160^\circ\ 10'$, $49^\circ\ 50'$
$c = 153^\circ\ 38'$, $89^\circ\ 56'$

30. $A = 44^\circ\ 54'$
$a = 112^\circ\ 14'$
$b = 29^\circ\ 53'$

31. No solution.

32. No solution

33. $A = 89^\circ\ 36'$, $26^\circ\ 48'$
$a = 138^\circ\ 54'$, $25^\circ\ 28'$
$b = 42^\circ\ 48'$, $137^\circ\ 12'$

34. $A = 142^\circ$
$B = 159^\circ$
$C = 133^\circ$

35. $B = 58^\circ\ 6'$
$b = 61^\circ\ 31'$
$c = 90^\circ$

36. $b = 96^\circ\ 24'$
$B = 54^\circ\ 54'$
$C = 146^\circ\ 38'$

37. $A = 72^\circ\ 54'$
$a = 78^\circ\ 30'$
$b = 112^\circ\ 57'$

38. $A = 51^\circ\ 16'$
$b = 119^\circ\ 47'$
$c = 69^\circ\ 41'$

39. $a = 129^\circ\ 44'$, $95^\circ\ 40'$
$A = 131^\circ\ 8'$, $76^\circ\ 58'$
$B = 68^\circ\ 36'$, $111^\circ\ 24'$

40. $A = 70^\circ\ 44'$
$B = 132^\circ\ 12'$
$C = 94^\circ\ 54'$

41. $b = 110^\circ\ 50'$
$A = 61^\circ\ 35'$
$C = 86^\circ\ 27'$

42. $a = 62^\circ\ 42'$.
$b = 126^\circ\ 44'$
$c = 109^\circ\ 42'$

CHAPTER XI

1. Course at Baltimore : E. 37° 55′ N.
Course at meridian of New York : E. 36° 14′ N.
Bearings Boston from Baltimore : E. 37° 55′ N.
Bearings Baltimore from Boston : W. 34° 17′ S.
Distance sailed : 359 mi.

2. Course at Liverpool : W. 14° 51′ N.
Course at 55° W. : W. 26° 56′ S.
Bearings New York from Liverpool : W. 14° 51′ N.
Bearings Liverpool from New York : E. 40° 31′ N.
Distance sailed : 3303 mi.

3. Course at Pernambuco : S. 36° 35′ E. arrives.
Course at Pernambuco : S. 29° 33′ W. departs.
Distance sailed : 5141 mi.

4. Course : S. 42° 32′ E.
Position : 29° 12′ N., 64° 1′ W.

5. Bearings Greenwich from Washington : E. 40° 41′ N.
Bearings Washington from Greenwich : W. 18° 33′ N.
Distance : 3669 mi.

6. Distance : 11,550 mi.
Position : 43° 42′ S., 91° 25′ E.

7. Position : 56° 2′ N., 153° 54′ W.

8. 117,700 sq. mi.

9. Distance : 2398 mi.
Course : E. 29° 6′ N.

10. 8 hours, nearly.

11. 35,580 sq. mi.

12. Face angle, 60° ; edge angle, 70° 32′.

13. 19° 28′ S.

14. Face : 36° 52′, 53° 8′, 90°.
Edge : 180°, 0°, 0°.

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AND

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Edited by EARLE RAYMOND HEDRICK
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