PLANE TRIGONOMETRY

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PREFACE.

SOME of the features of this book are:

(1) Abandonment of the "Academic Triangle."

The "academic triangle" is one, certain of whose parts are known exactly and whose remaining parts are calculable free from all errors except such as are due to the "place" of the tables used. In the academic triangle one side may show one figure and another seven figures, and the angles may show any readings the caprice of the maker may suggest.

(2) Replacement of the "Academic Triangle" by triangles having variety of form, but always such form as they should have if they had been actually measured in the field:

(i) Showing in all measured lines of a diagram the same number of significant figures, thus indicating about the same care in each measurement.

(ii) Giving in all calculated lines of a diagram not more significant figures than in its measured lines, since such extra figures are misleading and of no value.

(iii) Letting the angles of a diagram show such readings as exhibit care in angle-reading in keeping with that indicated in line-measurement.

While the student is learning to calculate, he might as well learn at the same time what form intelligent measurement should take and what reliability calculated results have as related to the data.

(3) What place tables to use to suit the data and how to cut a large table to suit short data.

(4) How checks should check to suit the data.

(5) The fundamental principles of calculation with approximate data (Chapter II.).

(6) The suggestion of simple laboratory exercises in connection with fundamental things.

(7) Exercises from related topics, as Physics, Analytical Geometry, Surveying, etc.

(8) Graphs, using rectangular and polar coördinate paper.

(9) Geometric treatment of DeMoivre's Theorem.

(10) Order of treatment of the trigonometric functions.

When many different ideas are presented simultaneously to the mind the result is confusion. When the same idea is presented again and again, with a suitable interval between presentations, its difficulties disappear. Most of the difficulties connected with any trigonometric function are those of every other such function. I have ventured, therefore, to present the subjects in the following order:

(i) The sine, — its inverse and reciprocal.

(ii) The cosine, — its inverse and reciprocal.

(iii) The sine family and cosine family in union.

(iv) The tangent, — its inverse and reciprocal.

(v) All the functions in union.

(vi) The quantity $\sqrt{-1}$ in trigonometry.

I am indebted to Dr. Coddington and Dr. Kuhn of the Ohio State University, and to Dr. J. W. Young of the Northwestern University, for valuable suggestions. Dr. Coddington has tested all the exercises.

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CONTENTS.

CHAPTER I.

ELEMENTARY DISCUSSION OF LOGARITHMS. — HOW TO USE TABLES AND WHAT PLACE TABLES TO USE.

§ 1. Definition of a Logarithm.

The expressions

$$a^x = y, \qquad (1)$$

$$\log_a y = x, \qquad (2)$$

$$y = \log_a{}^{-1} x, \qquad (3)$$

indicate the same relation between the three quantities, a, x, y; namely, that x is the logarithm of y to the base a. *The logarithm of any number, y, to the base, a, is the exponent, x, of the power to which a must be raised to produce y.*

Expression (2) is read, "the logarithm of y to the base a is x," or, as is usual among engineers, "log y, base a, is x." Expression (3) is read, "y is the number whose logarithm to the base a is x," or, more briefly, "y is anti-log x, base a." Among engineers it is customary, both in writing and in speech, to cut the word logarithm to "log."

Expression (1) is called the exponential form for a logarithm; (2), (3) are called the logarithmic forms.

$2^3 = 8$, (1), is equivalent to $\log_2 8 = 3$, (2), and to $8 = \log_2{}^{-1} 3$, (3); and $10^2 = 100$, (1), is equivalent to $\log_{10} 100 = 2$, (2), and to $100 = \log_{10}{}^{-1} 2$, (3). Read these expressions.

EXERCISES.

1. What is the value of $a^{\log_a m}$? Of $e^{\log_e m}$? Of $10^{\log_{10} x}$? Of $\log_a(\log_a{}^{-1} x)$? Of $\log_a{}^{-1}(\log_a y)$?

2. What is the base when the logarithm of 81 is 4? Express this in the three forms, (1), (2), (3).

3. What number has 3 for its logarithm when the base is 6? Express this in the three forms, (1), (2), (3).

4. What is the logarithm of 1000 to the base 10? Express this in the three forms, (1), (2), (3).

5. Make up ten examples like Exs. 2, 3, 4, and solve.

6. If 10 is the base, what are the logarithms of 10, 100, 1000, 10000, 10^n? If $0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$, what is the logarithm of 0.01? Of 0.1, 0.001, 0.0001, 0.00001? Of $\frac{1}{10^n}$?

7. Give the usual illustration from algebra from which is drawn the interpretation that $a^0 = 1$. In accordance with this interpretation, what would you say is the log of 1 for all bases?

8. Express in the form of an identity the relation of a^{-x} to a^x. How, then, is the logarithm of a number related to that of its reciprocal?

9. If in the expression $a^x = y$, a is positive, what sign has y, both for positive real values of x and for negative real values of x? Give some numerical illustrations. What would you say, then, as to the possibility of negative numbers having real logs, if the base is positive?

10. If the base is 10, between what limits will lie the logs of all numbers between 1 and 10? Between 10 and 100? Between 100 and 1000? How many digits to the left of the decimal point have numbers which lie between 10 and 100? Between 1 and 10? Between 100 and 1000? Write three four-figured numbers which lie between 1 and 10. Between 10 and 100. Between 100 and 1000. Calling the integral part of a log its *characteristic*, can you make up, from your answers to the preceding questions, the rule as to the number which is the characteristic as compared with the number of digits to the left of the decimal point in the number whose log is desired?

11. Express 0.1, 0.01, 0.001, 0.0001, 0.00001, in exponential form similar to $(10)^{-2} = 0.01$, where the base is 10 and the exponents negative. From these expressions draw conclusions as to the limits between which lie the logs of all numbers between 1 and 0.1, and between 0.1 and 0.01. Between 0.01 and 0.001. Between 0.001 and 0.0001. Write four numbers which lie between 1 and 0.1. Between 0.1 and 0.01. Between 0.01 and 0.001. Between 0.001 and 0.0001. What sign has the log of a decimal fraction?

It is customary to make the integral part of such logs negative and the decimal part positive, taking for the integral part the log of the power of 10 next below the given decimal whose log is sought. Can you make out, from your answers to the preceding questions, the rule as to the negative number which is the characteristic as related to the number of zeros to the right of the decimal point before a figure other than zero is reached in the number whose log is sought?

§ 2. Working Rules for Logarithms.

(*a*) *The logarithm of the product of two or more numbers is the sum of the logarithms of the numbers,*

or, $$\log(mn) = \log m + \log n;$$
$$\log(mnp) = \log m + \log n + \log p.$$

Proof.

If $a^x = m$, (1), then $\log_a m = x$, (2),

and if $a^y = n$, (3), then $\log_a n = y$, (4).

Multiplying (1) by (3), $a^{x+y} = mn$,

or, $$\log_a(mn) = x + y$$
$$= \log_a m + \log_a n, \text{ by (2), (4).}$$

Similarly for a product of more than two factors.

EXERCISES.

1. If n is a positive integer, show that $\log(a^n) = n \cdot \log a$.

2. Given $\log_{10} 2 = 0.30103$ and $\log_{10} 3 = 0.47712$, find, to the base 10, the logs of the following numbers: 6; 4; 9; 27; 8; 12; 36; 2^{10}; 3^8; 2^n; 3^m; $2^n \cdot 3^m$ (n, m, integers, positive).

3. Write down all the prime numbers from 1 to 100, and show that if the logs of these numbers are known, the logs of the other numbers below 100 can be calculated. For what other numbers could the logs be calculated?

4. If e is the base, show that $1 + \log_e n = \log_e(en)$. If 10 is the base, show that $1 + \log_{10} n = \log_{10}(10 \cdot n)$. Show also

$2 + \log_e n = \log_e(e^2 \cdot n)$; $3 + \log_e n = \log_e(e^3 \cdot n)$;
$n + \log_e n = \log_e(e^n . n)$; $4 + \log_{10} n = \log_{10}(10000 \cdot n)$;
$n + \log_{10} n = \log_{10}(10^n \cdot n)$.

(*b*) *The logarithm of an indicated quotient* $\left(\frac{m}{n}\right)$ *is the logarithm of the numerator minus the logarithm of the denominator,*

or $$\log\left(\frac{m}{n}\right) = \log m - \log n.$$

PROOF.

If $a^x = m$, (1), then $\log_a m = x$, (2),
and if $a^y = n$, (3), then $\log_a n = y$, (4).

Dividing (1) by (3),

$$a^{x-y} = \frac{m}{n}, \quad (5),$$

or,
$$\log_a\left(\frac{m}{n}\right) = x - y$$
$$= \log_a m - \log_a n, \text{ by (2), (4).}$$

EXERCISES.

1. If $\log_{10} 2 = 0.30103$ and $\log_{10} 3 = 0.47712$, find to the base 10, the logs of the following numbers: $\frac{2}{3}$; 1.5; 4.5; 6.75; $1\frac{1}{3}$; $2\frac{2}{3}$; $\frac{4}{9}$; $\frac{16}{27}$; $\frac{2^n}{3^m}$; $\left(\frac{2}{3}\right)^n$ (n, m, integers, positive or negative).

2. Express the logs of the proper fractions above with negative characteristic and positive decimal part (mantissa) to the log. For example, $\log \frac{2}{3} = \bar{1}.82331$, where 1 alone is negative and .82391 is positive.

3. Show that $\log\left(\frac{1}{a}\right) = -\log a$.

4. Show that

$$1 - \log_e x = \log_e \frac{e}{x}; \qquad 1 - \log_{10} n = \log_{10} \frac{10}{n};$$

$$\log_e x - 3 = \log_e\left(\frac{x}{e^3}\right); \qquad \log_{10} n - 5 = \log_{10}\left(\frac{n}{10^5}\right).$$

5. Show that
$$\log_{10} 27.34 = \log 2734 - 2;$$
$$\log_{10} 27.34 = \log 273.4 - 1;$$
$$\log_{10} 3.415 = \log 3415 - 3.$$

6. In general, show that if a and b are two numbers with the same sequence of digits, but with the decimal point differently placed, that $a = 10^n \cdot b$, and that, consequently, the logs of a and b to the base 10 differ only by the integer n. What is the relation between the decimal parts of the logs of numbers with the same sequence of figures?

7. From $\log_{10} 2$, find $\log_{10} 5$; then from $\log_{10} 3$, find $\log_{10} 15$.

(c) *The logarithm of any power of a given number is the logarithm of the given number multiplied by the exponent of the power,*

or,
$$\log(m^p) = p \log m.$$

PROOF.

If $a^x = m$ (1), $\log_a m = x$, (2).

Then $a^{px} = m^p$, (3).

or $\log_a(m^p) = px$

$= p \log_a m.$

NOTE. — We assume that the student's knowledge of algebra is such that he will grant that (3) is true for all values of p, positive, negative, integral, fractional, rational, irrational. Then,

$$\log (m^{\frac{q}{s}}) = \frac{q}{s} \log m, \quad (1)$$

$$\log (m^{-\frac{q}{s}}) = -\frac{q}{s} \cdot \log m, \quad (2)$$

$$\log m^{\frac{1}{n}} = \log \sqrt[n]{m} = \frac{1}{n} \log m, \quad (3)$$

(1), (2), (3) are true no matter what base is taken. (3) is so important, it may be set in words as a rule:

(*d*) *The logarithm of the nth root of a number is one nth of the logarithm of the number.*

§ 3. Special Case of Logarithms to the Base 10.

(*a*) *Rule for Characteristic.*

$$\begin{aligned}
10^5 &= 100000\\
10^4 &= 10000\\
10^3 &= 1000\\
10^2 &= 100\\
10^1 &= 10\\
10^0 &= 1\\
10^{-1} &= 0.1\\
10^{-2} &= 0.01\\
10^{-3} &= 0.001\\
10^{-4} &= 0.0001\\
10^{-5} &= 0.00001
\end{aligned}$$

The logarithms of integral powers of 10 are integers. Numbers which are not integral powers of 10 lie between consecutive integral powers of 10. Thus the logarithms of such numbers lie between consecutive integers.

There are two cases:

(1) *Numbers with digits to the left of the decimal point.* Numbers with one digit to the left of the decimal point lie between 1 and 10. Their logarithms, as the table on page 5 shows, lie between 0 and 1. Numbers with two digits to the left of the decimal point lie between 10 and 100. Thus their logarithms lie between 1 and 2; that is, the logarithm is 1 plus a decimal fraction. Similarly, the logarithm of a number with three digits to the left of the decimal point is 2 plus a decimal fraction. And if the number has n digits to the left of the decimal point, its logarithm is $n - 1$ plus a decimal fraction.

The integral part of a logarithm is called its *Characteristic;* the decimal part, its *Mantissa.*

For numbers with digits to the left of the decimal, the characteristic is the positive integer one less than the number of such digits.

The characteristic for log 8 is 0
The characteristic for log 8.3 is 0
The characteristic for log 8.349 is 0
The characteristic for log 83 is 1
The characteristic for log 83.4 is 1
The characteristic for log 83.459 is 1

The characteristic is determined wholly by the number of digits to the left of the decimal point. Those to the right of the decimal point have nothing to do with the characteristic in numbers which have digits to the left of the decimal point.

EXERCISES.

The teacher may assign some numbers at random, and question as to the characteristic.

(2) *Numbers with no digits to the left of the decimal point.* All such numbers which have no zero immediately to the

right of the decimal point, lie between 1 and 0.1, as 0.348, 0.8925, 0.726952, etc. Thus the logarithms of all such numbers lie between -1 and 0. Similarly, all such numbers with only one zero immediately to the right of the decimal point, as 0.043, 0.0987654, etc., all lie between 0.1 and 0.01, and have their logarithms between -2 and -1. And so, in general, if there are n zeros immediately to the right of the decimal point, and no digits to the left of the decimal point, the logarithm of the number lies between $-(n+1)$ and $-n$.

The characteristic in such cases is usually taken as the negative number which is the larger numerically, *so that the mantissa for such logarithms is positive.* An important exception is mentioned later.

Thus, to get the characteristic of the logarithm of a number with no digits to the left of the decimal point, take the negative number which is one more than the number of zeros immediately to the right of the decimal point.

The sign of the characteristic is set immediately over it, when negative.

The characteristic of log 0.23 is $\bar{1}$
The characteristic of log 0.2348 is $\bar{1}$
The characteristic of log 0.023 is $\bar{2}$
The characteristic of log 0.02345 is $\bar{2}$
The characteristic of log 0.00004 is $\bar{5}$

EXERCISES.

The teacher may set some numbers at random and see if the student can state the characteristic immediately.

NOTE. — Instead of using negative characteristics, many calculators add to the characteristic such a multiple of 10 as will make it positive, and allow for such tens in the calculation.

For $\bar{1}.7632$ they take $9.7632 - 10$.
For $\bar{2}.7632$ they take $8.7632 - 10$; and so on.

This always seemed to me an unnecessary procedure, but it is quite commonly followed by astronomers, who are the great calculators. It is necessary for the student to be acquainted with both processes, as some tables and some books follow one plan and some the other. The fact that a special process is followed by the astronomers is almost a proof that it is best as a calculation process.

(b) *Rule for the Mantissa.* All numbers with the same sequence of digits have logarithms with the same mantissa. For if a, b, are any two numbers with the decimal point differently placed, but with the same sequence of digits,

$$a = b.10^n, \text{ where } n \text{ is an integer}$$
$$\therefore \log_{10} a = \log_{10} b + n.$$

Thus the logarithms of a and b differ only by the integer n $\therefore$ *the mantissas are the same.*

For this reason the tables give only the mantissa of the logarithm of a number. The calculator supplies the proper characteristic.

The mantissa is, as a rule, an unending, non-repeating decimal. There are tables which give the mantissa to four places of decimals; to five; to six; to seven; to ten. A table which gives the mantissa to four places is called a four-place table; and so on.*

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* Tables of the first rank are:

F. G. Gauss's Five Place Tables (German).
Hussey's Five Place Tables.
Bremiker's Six Place Tables (German).
Lodge's Bremiker's Six Place Tables (English).
Vega's Seven Place Tables (German).
Vega's Thesaurus Logarithmorum, or Ten Place Tables (published 1794 and still considered the best ten-place table).
Zech's Addition-Subtraction Tables (German).

In case the student cannot read the explanations of the tables which are given in German in Gauss's Tables, Hussey's Mathematical Tables will be found a satisfactory substitute. In my judgment no five-place table surpasses that of F. G. Gauss. This table has been made the basis of frequent piracy, with its best features omitted. His Tables cannot be changed without hurting them. This is the general opinion of all calculators who have used them. For this reason it is recommended that Gauss's or Hussey's tables be used in connection with this book, when it is necessary to calculate seconds in angles.

(*c*) *Logarithms of roots, when the characteristic is negative and not a multiple of the root index.*

Given $\log 0.002 = \bar{3}.30103$, what is $\log \sqrt{0.002}$?

(1) First Method (characteristic negative).

$$\log \sqrt{0.002} = \tfrac{1}{2} \log 0.002 = \frac{\bar{3}.30103}{2} = \frac{\bar{4} + 1.30103}{2} = \bar{2}.65052$$

The general process is to bring the characteristic to the next integer above itself into which the root index will go exactly, balancing this subtraction by the addition of the corresponding number to the positive mantissa, as in

$$\log \sqrt[3]{0.0002} = \frac{\bar{4}.30103}{3} = \frac{\bar{6} + 2.30103}{3} = \bar{2}.76701$$

$$\log \sqrt[7]{0.0002} = \frac{\bar{4}.30103}{7} = \bar{1}.47158$$

(When the sixth figure of decimals in the quotient is 5 or more, increase the terminal figure (fifth) by unity.)

EXERCISES.

The teacher may assign them at will.

(2) Second Method (characteristic positive by aid of tens).

$$\log \sqrt{0.002} = \frac{7.30103 - 10}{2} = \frac{17.30103 - 20}{2} = 8.65052 - 10$$

$$\log \sqrt[3]{0.0002} = \frac{6.30103 - 10}{3} = \frac{26.30103 - 30}{3} = 8.76701 - 10$$

$$\log \sqrt[7]{0.0002} = \frac{6.30103 - 10}{7} = \frac{66.30103 - 70}{7} = 9.47158 - 10$$

In general, a multiple of ten one less than the root-index is added and subtracted.

EXERCISES.

The teacher may assign some at will, selecting also some fractional indices which modify the general rules above.

(*d*) *Logarithms when negative factors occur.*

Since
$$a^{-x} = \frac{1}{a^x},$$
in the expression
$$a^{\pm x} = y,$$
if a is positive, y is positive for negative real values of x as well as for positive real values of x.

Thus, with a positive base negative numbers have no real logarithms.

Logarithms can be used, however, in getting products, quotients, powers, roots, of negative numbers, by using the logarithms as if the numbers were positive, *attaching the proper sign to the result.*

It is customary to write (n) after the log of the factor which is negative. An even number of n's will indicate a positive result and an odd number a negative result.

EXERCISES.

1. Find the value of $\frac{a \cdot b \cdot c}{d \cdot e}$, where $a = 17$, $b = 13$, $c = -.14$, $d = -23$, $e = -9$.

The result is negative. The process is:

log 17 =			
log 13 =		log 23 =	(n)
log 14 =	(n)	log 9 =	(n)
Sum =	(1)	Sum =	(2)

Subtract (2) from (1) and look up in the tables the number corresponding to the resulting logarithm. (Directions for doing this will be found in the accompanying tables.)

2. Given $\log_{10} 2 = 0.30103$, and $\log_{10} 3 = 0.47712$, find logs of following numbers to base 10: 2^3, $2^{\frac{3}{2}}$, $2^{\frac{2}{3}}$, $2^{\frac{5}{3}}$, $3^{\frac{5}{9}}$, 3^5, $3^{\frac{9}{5}}$, $\sqrt[7]{2}$, $\sqrt[5]{0.2}$, $\sqrt[5]{0.002}$, $\sqrt[3]{0.003}$, $\sqrt[3]{0.0006}$, $(-12)^{\frac{3}{7}}$, $(-6)^{\frac{2}{3}}$, $(-2)^{\frac{5}{3}}$, $(-2)^{-\frac{5}{9}}$, $(-0.02)^{\frac{4}{3}}$, $(-2)^8(-3)^8$, $(2)^5(-3)^7$.

(*e*) *Use of the Cologarithm in Calculations.* — The logarithm of the reciprocal of a number taken so that the *mantissa is positive* is called the *cologarithm* of the number. The characteristic may be positive or negative. When negative it can be made positive by the addition and subtraction of an appropriate multiple of 10, if one is following the calculation process pointed out in the Note on page 7. In such procedure one must allow, in the final result of a summation of cologarithms, for the tens thus introduced.

$$\text{colog}\,(a) = \log\left(\frac{1}{a}\right) = -\log\,(a).$$

$$\log 20 = 1.30103;\ \text{colog}\ 20 = \bar{2}.69897, \text{ or } 8.69897 - 10.$$

$$\log 0.02 = \bar{2}.30103;\ \text{colog}\ 0.02 = 1.69897.$$

Rule for writing the cologarithm of a number: *Subtract each digit of the mantissa from* 9, *from left to right, except the terminal digit to the right. Subtract this from* 10. *Add* 1, *algebraically, to the characteristic and change the sign.*

In calculating the value of $\frac{a \cdot b \cdot c}{d \cdot e}$, one may add the logs of a, b, and c. Also add in a separate column the logs of d, e. Then subtract the second sum from the first, and look up the number corresponding to this log. Or by the aid of cologarithms, the calculation may all be carried out in one column, as follows:

$$\begin{aligned} \log a &= \\ \log b &= \\ \log c &= \\ \text{colog}\ d &= \\ \text{colog}\ e &= \\ \hline \text{Sum} &= \end{aligned}$$

EXERCISES.

1. Given log 2 = 0.30103, find cologs of 20; 800; 4,000,000.

2. Given log 3 = 0.47712, find cologs of 0.3; 0.003; $\sqrt[15]{0.9}$.

Whether to use the colog or not depends on circumstances. Frequently it is best not to use it; frequently it lends itself to a handy calculation-scheme. For example, later we shall use the expression,

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c},$$

where x, y, z are the *sines* of the angles of a triangle. If b, c have to be calculated,

$$b=y\cdot\frac{a}{x};\qquad c=z\cdot\frac{a}{x}.$$

And the calculation-scheme is,

$\log b =$	(6)
$\log y =$	(1)
$\log a =$	(2)
$\text{colog}\, x =$	(3)
$\log z =$	(4)
$\log c =$	(5)

where (6) is (1) + (2) + (3)
and (5) is (2) + (3) + (4),
one and the same scheme giving b, c. In adding, cover the log not to be used.

(*f*) *Using tables of logarithms.*

It is recommended that at this point the student be taught how to look up the logarithms of numbers in a four-place table and in some five-place table, both in finding the logarithms corresponding to numbers and in finding the numbers corresponding to logarithms, taking, for the present, only the cases where the logarithms and numbers can be found exactly, or to a close approximation, in the tables, without interpolation. The principles on which interpolation are based can be understood only after further study, as set forth in the next several articles.

The student should carry out the operations with the tables and without the tables and compare results. How to use the four-place tables of this book is explained in the preface to the tables.

EXERCISES.

(For a four-place table.)

Calculate with logarithms and without logarithms the values of the following expressions and compare results. Give results to not more than three figures and to only the nearest approximation which the tables will give without interpolation. In the case of fractions, calculate their values with cologs and without cologs, using negative characteristics and also characteristics positive by the aid of 10's. Find b, c, in Exs. 4, 5, 6.

1. 2×3; 3×4; 4×5; 5×6; 6×7; 7×8; 8×9; 9×10; 10×11; 11×12; 12×13.

2. $2 \times 3 \times 4$; $3 \times 4 \times 5$; $4 \times 5 \times 6$; $5 \times 6 \times 7$; $6 \times 7 \times 8$; $7 \times 8 \times 9$; $8 \times 9 \times 10$; $2 \times 3 \times 9$; $3 \times 4 \times 9$; $4 \times 5 \times 9$; $5 \times 6 \times 8$.

3. $\frac{12 \times 13}{7}$; $\frac{13 \times 14}{9}$; $\frac{16 \times 17}{11}$; $\frac{18 \times 19}{13}$; $\frac{25 \times 27}{32}$; $\frac{35 \times 37}{43}$; $\frac{45 \times 47}{53}$.

4. $\frac{45 \times 49 \times 85}{71 \times 73}$; $\frac{56 \times 57 \times 73}{81 \times 91}$; $\frac{96 \times 83 \times 85}{61 \times 77}$; $\frac{0.21}{0.31} = \frac{0.41}{b} = \frac{0.51}{c}$.

5. $\frac{243 \times 244}{231}$; $\frac{256 \times 295}{331}$; $\frac{259 \times 831}{923}$; $\frac{0.031}{0.016} = \frac{0.023}{b} = \frac{0.033}{c}$.

6. $\frac{341 \times 441 \times 551}{536 \times 437}$; $\frac{349 \times 837 \times 624}{555 \times 989}$; $\frac{.317}{.218} = \frac{.231}{b} = \frac{.314}{c}$.

7. $\frac{\sqrt{341} \cdot \sqrt[3]{225}}{\sqrt[4]{316} \cdot \sqrt[5]{719}}$; $\frac{\sqrt[3]{0.034} \cdot \sqrt{0.0014}}{\sqrt[4]{0.0015} \cdot \sqrt[5]{0.00042}}$.

8. $\frac{(-95) \cdot (-34) \cdot (-65)}{27 \cdot (-86)}$; $\frac{(-32) \cdot (-41) \cdot (-51) \cdot (64)}{(-17) \cdot (-29) \cdot (-43)}$.

9. Find to three figures:

$$\sqrt{2};\ \sqrt{3};\ \sqrt{5};\ \sqrt{6};\ \sqrt[3]{2};\ \sqrt[3]{3};\ \sqrt[3]{4};\ \sqrt[3]{5};\ \sqrt[4]{2};\ \sqrt[4]{3};\ \sqrt[4]{5};\ \sqrt[4]{6}.$$

10. $\frac{(6.6)^3 \cdot (1.02)^4}{(2.5)^5 \cdot (3.15)^2}$; $\frac{(2.51)^5 \cdot (3.03)^3}{(2.47)^3 \cdot (1.78)^4}$.

11. $\sqrt{\frac{(3.11)^3 \cdot (2.71)^5}{(2.14)^5 \cdot (3.19)^3}}$; $\sqrt[3]{\frac{(47)^2 \cdot (-85)^5}{(23)^2 \cdot (71)^4}}$.

§ 4. Systems of Logarithms.

While any finite positive number, other than 1, may be made the base, there are only two bases of importance. These are 10 and the number indicated by the letter e, where

$$e = 2.7182818284 \cdots.$$

The decimal part of e is unending and non-repeating. The number e is the limit toward which the expression

$$\left(1 + \frac{1}{x}\right)^x$$

approaches as x becomes infinite; that is, as will be shown presently, e is the limit toward which the infinite series

$$1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 5} + \cdots$$

approaches, as the number of terms approaches infinity. The logarithms with the base 10 are called the *Briggs System*, or *Common Logarithms*. The logarithms with the base e are called, in honor of Baron Napier, the inventor of logarithms, the *Napierian System*. They are also called the *Natural Logarithms*.*

§ 5. Changing from One System to Another.

Logarithms are first calculated, in the manner shown in § 10, to the base e. From the table to the base e, the table to the base 10 is then calculated, by dividing each log in the e-table by the log of 10 in the e-table. That is, the number $\frac{1}{\log_e 10}$ used as a multiplier for the e-table will give the 10-table.

The number $\frac{1}{\log_e 10} = 0.4342944819 \cdots$ is called *the Modulus of the Briggs System, or Common Logarithms, with reference to the Napierian Logarithms.*

In general, any table of logs can be converted into a table to another base by dividing every log in the table by the log of the new base in the given table.

PROOF.

Let $$a^x = m = b^y, \tag{1}$$
so that $$x = \log_a m, \tag{2}$$
and $$y = \log_b m. \tag{3}$$

---

* As indices had not found a place in Algebra at the time of Napier, his logarithms were very different from those now called, in his honor merely, Napierian Logarithms. The student interested in the subject may consult Cajori's "History of Mathematics"; the section "Logarithms" in the Encyclopædia Britannica; also article by J. W. Young in *Am. Math. Monthly*, 1903.

Take the logarithm of (1) to the base a,

$$\therefore\ x = y \log_a b, \qquad (4)$$

or

$$y = \frac{x}{\log_a b}. \qquad (5)$$

By (5), logarithms of the b-system are those of the a-system divided by the logarithm of b in the a-system; whence the general rule for changing logarithms of a given system to a new system:

Divide the logarithms of the given system by the logarithm of the base of the new system taken from the old system.

The number, $\frac{1}{\log_a b}$, is called the modulus of the b-system with reference to the a-system.

Thus the modulus of the Common Logarithms with reference to Napierian Logarithms is

$$\frac{1}{\log_e 10} = 0.43429 \cdots.$$

And the modulus of the Napierian Logarithms with reference to Common Logarithms is

$$\frac{1}{\log_{10} e} = 2.303 \cdots.$$

Thus, to convert a Napierian table to a Common table, multiply each logarithm by $0.43429 \cdots$.

And to convert a Common table to a Napierian table, multiply each logarithm by $2.303 \cdots$.

§ 6. $\log_a b \cdot \log_b a = 1$.

PROOF.

Let

$$a^x = b, \qquad (1)$$

so that

$$\log_a b = x. \qquad (2)$$

Take the logarithm of (1) to the base b,

$$\therefore\ x \cdot \log_b a = 1, \qquad (3)$$

$\therefore$ by (2),

$$\log_a b \cdot \log_b a = 1, \qquad (4)$$

$$\therefore\ \frac{1}{\log_a b} = \log_b a. \qquad (5)$$

Thus the modulus of the b-system with reference to the a-system is either $\dfrac{1}{\log_a b}$, or $\log_b a$.

In particular, $\dfrac{1}{\log_e 10} = \log_{10} e$.

Thus the modulus of the Common system is either the reciprocal of the logarithm of 10 *in the e-system, or it is the logarithm of e in the* 10-*system.*

EXERCISES.

1. Look up $\log e$ in tables and compare with the modulus.

2. How would you convert an ordinary table of logs to the base 10 into a Napierian table?

Ans. Divide each log by 0.4342 · · ·, or multiply each by 2.303 · · ·.

3. Of what number in the 10-system is the modulus, 0.4342 · · ·, the log? Of what number in the e-system is 2.303 · · · the log?

4. From the relation
$$a^x = m = b^y,$$
show that
$$\log_a m = \frac{\log_b m \cdot \log_c b}{\log_c a}.$$

5. If
$$\log_{10} 37.31 = 1.57183$$
and
$$\log_{10} 2 = 0.30103,$$
find
$$\log_2 37.31 \text{ and } \log_2 373.1$$
and
$$\log_2 3731 \text{ and } \log_2 37310.$$

6. Given
$$\log_{10} e = 0.43429,$$
$$\log_{10} 2 = 0.30103,$$
$$\log_{10} 3 = 0.47712,$$
$$\log_{10} 7 = 0.84510,$$
find to the base e, the logs of the following numbers: 2; 3; 4; 5; 8; 9; $\frac{2}{3}$; $\frac{3}{2}$; $\frac{4}{3}$; $\frac{3}{4}$; 27; $\frac{7}{120}$.

7. By the aid of the preceding logarithms, find x in the expressions: $2^x = 3$; $3^x = 2$; $5^x = 12$; $16^x = 10$; $27^x = 4$.

(*a*) Using e-logs. (*b*) Using 10-logs.

8. Given
$$\log_e 2 = 0.69315,$$
$$\log_e 3 = 1.09861,$$
and
$$\log_e 10 = 2.30259,$$
find to base e the logs of following numbers: 6; 12; 4; 9; 8; 27; 5; 15; 25; 125; $\frac{3}{2}$; $\frac{2}{3}$; $\frac{5}{4}$; $\frac{4}{5}$.

9. From the values of $\log_e 2$, $\log_e 3$, given in Ex. 8, calculate $\log_{10} 2$, $\log_{10} 3$, and compare with values previously given.

10. If $e^x = \sqrt{y^2+1} + y$, then $e^x - e^{-x} = 2y$; $e^x + e^{-x} = 2\sqrt{y^2+1}$.

11. If $e^x = y - \sqrt{y^2-1}$, then $e^x + e^{-x} = 2y$; $e^x - e^{-x} = 2\sqrt{y^2-1}$.

§ 7. The Base of the Napierian System of Logarithms.

$$e = \lim_{x=\infty}\left(1+\frac{1}{x}\right)^x.$$

By the binomial theorem, when $y<1$,

$$(1+y)^n = 1 + ny + \frac{n\cdot n-1}{1\cdot 2}y^2 + \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3}y^3 + \text{ etc.},$$

$$\therefore\ \left(1+\frac{1}{x}\right)^x = 1 + x\left(\frac{1}{x}\right) + \frac{x\cdot x-1}{1\cdot 2}\left(\frac{1}{x}\right)^2 + \frac{x\cdot x-1\cdot x-2}{1\cdot 2\cdot 3}\left(\frac{1}{x}\right)^3 + \text{ etc.},$$

$$= 1 + 1 + \frac{1\left(1-\frac{1}{x}\right)}{1\cdot 2} + \frac{1\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)}{1\cdot 2\cdot 3} + \text{ etc.}$$

Let x approach the value infinity,

$$\lim_{x=\infty}\left(1+\frac{1}{x}\right)^x = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \text{ etc.}$$

The value of e may be calculated to five places as follows:

	1.000000
2	1.000000
3	0.500000
4	0.166667
5	0.041667
6	0.008333
7	0.001388
8	0.000198
9	0.000025
	0.000003
e = Sum =	2.71828 . . . .

To ten places, $e = 2.7182818284\ldots.$

e is an *irrational* number, or an unending, non-repeating decimal.

COROLLARIES.

1. $\left(1+\frac{1}{x}\right)_{x=\infty}^{-x}=\frac{1}{e}\cdot$

2. $\left(1-\frac{1}{x}\right)_{x=\infty}^{-x}=\left(1+\frac{1}{y}\right)^{y}=e.$

3. $\left(1-\frac{1}{x}\right)_{x=\infty}^{x}=\frac{1}{e}\cdot$

4. $\left(1+\frac{1}{x}\right)_{x=\infty}^{nx}=e^{n}.$

5. $\left(1+\frac{1}{x}\right)_{x=\infty}^{-nx}=\frac{1}{e^{n}}\cdot$

6. $\left(1+\frac{n}{x}\right)_{x=\infty}^{x}=\left(1+\frac{n}{x}\right)^{\frac{x}{n}\cdot n}=e^{n}.$

7. $\left(1-\frac{n}{x}\right)_{x=\infty}^{x}=\frac{1}{e^{n}}\cdot$

8. $\left(1+\frac{1}{nx}\right)_{x=\infty}^{x}=\sqrt[n]{e}.$

9. $\left(1-\frac{1}{nx}\right)_{x=\infty}^{x}=\frac{1}{\sqrt[n]{e}}\cdot$

10. $(1+x)_{x=0}^{\frac{1}{x}}=e.$

11. $(1+x)_{x=0}^{-\frac{1}{x}}=\frac{1}{e}\cdot$

12. $(1+nx)_{x=0}^{\frac{1}{x}}=e^{n}\cdot$

13. $(1+x)_{x=0}^{\frac{n}{x}}=e^{n}.$

14. $(1-nx)_{x=0}^{\frac{1}{x}}=\frac{1}{e^{n}}\cdot$

15. $\left(1+\frac{x}{a}\right)_{x=0}^{\frac{1}{x}}=\sqrt[a]{e}.$

16. $\left(1-\frac{x}{a}\right)_{x=0}^{\frac{1}{x}}=\frac{1}{\sqrt[a]{e}}\cdot$

§ 8. The Exponential Series, or Exponential *x*.

$$e^{y}=1+y+\frac{y^{2}}{\underline{|2}}+\frac{y^{3}}{\underline{|3}}+\frac{y^{4}}{\underline{|4}}+\text{etc.},$$

$$e=\lim_{x=\infty}\left(1+\frac{1}{x}\right)^{x},$$

$$\therefore\quad e^{y}=\lim_{x=\infty}\left(1+\frac{1}{x}\right)^{xy},$$

or, by the binomial theorem,

$$e^{y}=\lim_{x=\infty}\text{ of }\left(1+xy\left(\frac{1}{x}\right)+\frac{xy\,(xy-1)}{1\cdot 2}\left(\frac{1}{x}\right)^{2}+\text{etc.}\right)$$

$$=\lim_{x=\infty}\text{ of }\left[1+y+\frac{y\left(y-\frac{1}{x}\right)}{1\cdot 2}+\frac{y\cdot\left(y-\frac{1}{x}\right)\left(y-\frac{2}{x}\right)}{1\cdot 2\cdot 3}+\text{etc.}\right]$$

or, $$e^y = 1 + y + \frac{y^2}{\lfloor 2} + \frac{y^3}{\lfloor 3} + \frac{y^4}{\lfloor 4} + \text{ etc.}$$

Similarly, $$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \text{ etc.}$$

This series is called the *Exponential Series*, or *Exponential x*. It is true for all finite values of x, that is, it converges to a definite limit for all values of x, except $x = \infty$. No matter how large x is, if finite, the numbers in the factorial denominators finally overtake it in value. From that point on, the terms decrease in value. Thus, the ratio of the general term, $\frac{x^n}{\lfloor n}$ to the preceding term, $\frac{x^{n-1}}{\lfloor n-1}$, is $\frac{x^n}{\lfloor n} \div \frac{x^{n-1}}{\lfloor n-1} = \frac{x}{n}$. This ratio has the limit zero, when $n = \infty$, for any finite x. Thus, the series is convergent. (See any College Algebra.)

NOTE. — The teacher may find it advisable here to give the proof of the convergency of an infinite series, as covered by the statements:

(*a*) If the ratio of the nth term to the preceding is < 1, when $n = \infty$, the series is convergent.

(*b*) If this ratio is > 1, the series is divergent.

(*c*) If this ratio is $= 1$, the matter is in doubt.

EXERCISES.

1. $e^{-1} = \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \frac{1}{\lfloor 6} \text{ etc.}$

2. $\frac{1}{2}(e + e^{-1}) = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 6} + \frac{1}{\lfloor 8} + \text{etc.}$

3. $\left(1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \text{etc.}\right)\left(1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \text{etc.}\right) = 1.$

4. $\left(1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 6} + \text{etc.}\right)^2 = 1 + \left(1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \text{etc.}\right)^2.$

5. $1 + \frac{2}{\lfloor 3} + \frac{3}{\lfloor 5} + \frac{4}{\lfloor 7} = \frac{e}{2}.$

6. $\frac{2}{\lfloor 3} + \frac{4}{\lfloor 5} + \frac{6}{\lfloor 7} + \text{etc.} = \frac{1}{e}.$

7. $\dfrac{\frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 6} + \text{etc.}}{1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \text{etc.}} = \frac{e-1}{e+1}.$

8. $1+\frac{2^3}{\underline{|2}}+\frac{3^3}{\underline{|3}}+\frac{4^3}{\underline{|4}}+\text{etc.}=5e.$

9. $1+\frac{1+2}{\underline{|2}}+\frac{1+2+3}{\underline{|3}}+\frac{1+2+3+4}{\underline{|4}}+\text{etc.}=\frac{3}{2}e.$

§ 9. The Logarithmic Series.

$$\log_e(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}\cdots.$$

$$-\log_e(1-z)=z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}\cdots.$$

PROOF.

If $$e^y=1+z, \tag{1}$$

then $$\log_e(1+z)=y.$$

But $$e=\left(1+\frac{1}{x}\right)^x_{x=\infty}.$$

And $$\left(1+\frac{y}{x}\right)^x_{x=\infty}=\left(1+\frac{y}{x}\right)^{\frac{x}{y}\cdot y}_{x=\infty}=e^y.$$

$\therefore$ by (1), $$\left(1+\frac{y}{x}\right)^x_{x=\infty}=1+z.$$

$$\therefore\ 1+\frac{y}{x}=(1+z)^{\frac{1}{x}},\ \text{when}\ x=\infty.$$

If $z<1$, we may expand $(1+z)^{\frac{1}{x}}$ by the binomial theorem.

$$\therefore\ 1+\frac{y}{x}=1+\frac{1}{x}z+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{1\cdot 2}z^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{1\cdot 2\cdot 3}z^3+\text{ etc.}$$

Cancel the 1's and multiply by x.

$$\therefore\ y=z+\frac{\left(\frac{1}{x}-1\right)z^2}{1\cdot 2}+\frac{\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{1\cdot 2\cdot 3}z^3+\text{ etc.,}$$

when $x=\infty$.

$$\therefore\ y=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4},\ \text{etc.,}$$

or $$\log_e(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4},\ \text{etc.} \tag{1}$$

This series has been derived under the supposition that $z < 1$, numerically, and is therefore subject to that limitation.

Changing z to $-z$,

$$\log_e (1-z) = -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \text{etc.}\right). \qquad (2)$$

§ 10. Calculation of Napierian Logarithms.

The series (1), (2), above, are not adapted to the calculation of logarithms. They hold only when z is numerically less than 1; (1) holding also when z is 1. And even within the field of their convergency, they are very slowly convergent, that is, a large number of terms have to be taken to get an approximate value of the logarithm of a proper fraction.

A rapidly convergent series, suitable for computation purposes, is obtained thus:

Subtract (2) from (1).

$$\therefore\ \log_e \frac{1+z}{1-z} = 2\left[z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \text{etc.}\right]. \qquad (3)$$

Since z is to be less than 1, let

$$z = \frac{1}{2x+1},$$

where x is a positive integer.

Then $$\frac{1+z}{1-z} = \frac{1 + \frac{1}{2x+1}}{1 - \frac{1}{2x+1}} = \frac{2x+2}{2x} = \frac{x+1}{x}.$$

Now x and $x+1$ are consecutive numbers, when x is an integer.

And $\log \frac{x+1}{x} = \log (x+1) - \log x = \log \frac{1+z}{1-z}.$

$\therefore\ \log_e (x+1)$

$$= \log_e x + 2\left[\frac{1}{2x+1} + \frac{1}{3}\,\frac{1}{(2x+1)^3} + \frac{1}{5}\,\frac{1}{(2x+1)^5} + \text{etc.}\right]. \qquad (4)$$

As soon as x is large, a very few terms of this series will give $\log_e(1+x)$ to a close approximation, when $\log_e x$ is known.

Since $\log_e 1 = 0$, the Napierian table is constructed as follows:

$$\log_e 2 = 2\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\text{etc.}\right),$$

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}+\text{etc.}\right),$$

$$\log_e 4 = 2\log_e 2,$$

$$\log_e 5 = \log_e 4 + 2\left(\frac{1}{9}+\frac{1}{3}\cdot\frac{1}{9^3}+\frac{1}{5}\cdot\frac{1}{9^5}+\text{etc.}\right).$$

And so on.

The number of terms taken in the series is dependent upon the place of the table to be constructed.

EXERCISES.

1. Calculate to four decimals $\log_e 2$; $\log_e 3$; $\log_e 4$; $\log_e 5$; $\log_e 6$; $\log_e 7$; $\log_e 8$; $\log_e 9$; $\log_e 10$. Reduce each such logarithm to the base 10, and compare with the logarithms in the table accompanying this book.

Ans.
$$\begin{cases} \log_e 2 = 0.6951; & \log_e 5 = 1.6094; & \log_e 8 = 2.0794; \\ \log_e 3 = 1.0986; & \log_e 6 = 1.7918; & \log_e 9 = 2.1972; \\ \log_e 4 = 1.3863; & \log_e 7 = 1.9459; & \log_e 10 = 2.3026. \end{cases}$$

2. What relation has $\log_e 10$ to the modulus of the e-system as compared with the 10-system? What relation has it to the modulus, 0.4343, of the Common system as related to the e-system?

§ 11. Series for the Logarithm of Any Number whatever, without Reference to the Adjacent Number.

If in $\frac{1+z}{1-z}$ we let $z = \frac{x-1}{x+1}$, so that $z < 1$, if x is any positive number,
$$\frac{1+z}{1-z} = \frac{1+\frac{x-1}{x+1}}{1-\frac{x-1}{x+1}} = x.$$

∴ Series (3) becomes

$$\log_e x = 2\left(\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \text{etc.}\right), \quad (5)$$

$$\therefore\ \log_{10} x = 0.8686\left(\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \text{etc.}\right). \quad (6)$$

Series (5), (6) define what is meant by $\log_e x$ and $\log_{10} x$ as related to x, in calculation form, 0.8686 being twice the modulus, approximately.

EXERCISES.

1. Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots = \log_e 2$. Is this the best series from which to calculate $\log_e 2$? Is it rapidly or slowly convergent?

2. Show $\log_e 3 - \log_e 2 = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} - \frac{1}{4}\cdot\frac{1}{2^4}$, etc.

3. Show that if $x > 1$, series (3) becomes

$$\log_e \frac{x+1}{x-1} = 2\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \text{etc.}\right).$$

4. Show that $\log_e a - \log_e b = \frac{a-b}{a} + \frac{1}{2}\left(\frac{a-b}{a}\right)^2 + \frac{1}{3}\left(\frac{a-b}{a}\right)^3 + \text{etc.}$

5. Show that $\log_e (1 + 3x + 2x^2) = 3x - \frac{5x^2}{2} + \frac{9x^3}{3} - \frac{17x^4}{4} + \cdots$

if $2x$ be not > 1.

6. Show that $2\log_e x - \log_e (x+1) - \log_e (x-1)$

$$= \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} + \cdots, \text{ if } x > 1.$$

7. Show $\log_e 2 = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$.

8. Show $\log_e 2 - \frac{1}{2} = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \cdots$.

§ 12. The Rule of Proportional Parts in Logarithms.

Let x and $x + \epsilon$ be two numbers, where ϵ is small compared with x.

Then $\log_e (x+\epsilon) - \log_e x = \log_e \frac{x+\epsilon}{x} = \log_e\left(1 + \frac{\epsilon}{x}\right)$.

But $\log_e\left(1 + \frac{\epsilon}{x}\right) = \frac{\epsilon}{x} - \frac{1}{2}\left(\frac{\epsilon}{x}\right)^2 + \frac{1}{3}\left(\frac{\epsilon}{x}\right)^3$, etc. (§ 9).

If $\frac{\epsilon}{x}$ is small, $\left(\frac{\epsilon}{x}\right)^2$, $\left(\frac{\epsilon}{x}\right)^3$, are still smaller. For instance, if $\frac{\epsilon}{x} = 0.00001$, $\left(\frac{\epsilon}{x}\right)^2 = 0.0000000001$.

$$\therefore \log_e\left(1 + \frac{\epsilon}{x}\right) = \frac{\epsilon}{x}, \text{ approximately,}$$

and
$$\log_{10}\left(1 + \frac{\epsilon}{x}\right) = 0.4343\frac{\epsilon}{x} = \frac{M}{x}\epsilon, \text{ nearly,}$$

where M is the modulus.

$$\therefore \log_{10}(x + \epsilon) - \log_{10} x = \frac{0.4343}{x}\epsilon, \text{ nearly.}$$

Thus the difference of the logs of any two numbers which differ by a quantity relatively small, is proportional, approximately, to the difference, ε, of the numbers.

This is called the *Rule of Proportional Parts.*

In the tables of logarithms the numbers differ by unity. Thus $\epsilon = 1$, and

$$\frac{M}{x} = \frac{0.43429\cdots}{x} \text{ is the } \textit{Tabular Difference,}$$

in tables to the base 10.

Thus the tabular difference is large at the beginning of the table, and decreases as the numbers increase. Also, a given difference in numbers at the beginning of the tables will make a larger difference in the logarithms of the corresponding numbers than will the same difference in numbers occurring at any other part of the table.

Four-place tables are generally arranged to give directly the logarithms of three-figured numbers, and five-place tables (arranged like Gauss's) give directly the logarithms of four-figured numbers. Since the differences in logarithms at the beginning of the table are large and vary rapidly, it is advisable that a four-place table should give directly the logarithms of four-figured numbers at the beginning of the table. The table in this book follows this plan in numbers up to 1709.

§ 13. Applications of Tabular Difference.

By § 12, $$\log_e(x+\epsilon) - \log_e x = \frac{M}{x}\epsilon,$$

or **log-difference = tabular difference times number-difference.**

(a) *To find the logarithm of a number not in the table.*

Given $\log 2054 = 3.31260$

and $\log 2055 = 3.31281.$ What is log 2054.2?

The tabular difference is 21.

The number-difference is 0.2.

∴ the log-difference is 0.2 of 21, or 4.2, or 4.

∴ log 2054.2 = log 2054 plus 0.00004 = 3.31264.

NOTE. — The usual rule for "throwing away" is followed in the log-differences. The above difference, 4.2, is counted as 4. If it had been 4.8 or 4.5, it would have been taken as 5. When the number to the right of the point is less than 5, it is thrown away; when it is 5 or more than 5, the number to the left of the point is increased by 1.

Since the mantissa with the base 10 is independent of the position of the decimal point in the number, when the logarithm of a number is to be interpolated from the table-logarithms a decimal point is set in the given number after the number of digits for which the table gives the logarithm. For instance, if the table gives logarithms of numbers with four digits, and the logarithm of 2.3459 is desired, consider this number, so far as mantissa is concerned, as 2345.9, and interpolate for 0.9, as above for 0.2. Similarly, for log 456789 interpolate for 0.89 from log 4567 and 4568.

Generally an interpolation for more than two figures cannot be made, since if numbers differ by 0.001 or by 0.009 beyond the table figures, their logarithms agree to the place of the table. Often the second figure is a matter of indifference.

For, $$\log_{10}(x+\epsilon) - \log_{10} x = \frac{.4343}{x}\cdot\epsilon;$$

or, log-difference $= \frac{.4343}{x}$ times number-difference, and number-difference $= 2.3\,x$ times log-difference.

Thus the smaller x is, the larger is the log-difference for a given number-difference. If we are using a five-place table, like Gauss's, giving logarithms of four-figured numbers, the smallest value of x, so far as mantissa is concerned, is 1000, and in a five-place table the smallest appreciable log-difference is 0.000005. Thus if the number-difference were 0.009 in the neighborhood of 1000, the log-difference would be 0.0004342 times 0.009, or about 0.0000039, not enough to affect the logarithm in the fifth place.

Thus in no table (which gives directly logarithms for numbers of one figure less than the *place* of the table) is it possible to interpolate for more than two figures beyond the reading of the table; often not for a second figure.

(*b*) *To find the number corresponding to a logarithm lying between two logarithms in the table.*

Given $3.74687 = \log_{10} 5583$

and $3.74695 = \log_{10} 5584,$

$$1.74689 = \log_{10} ?$$

The tabular difference for a unit number-difference is 8. The log-difference between the given log and that next under it in the table is 2.

$$\textbf{Number-difference} = \frac{\textbf{log-difference}}{\textbf{tabular difference}}$$

$$= \tfrac{2}{8} = \tfrac{1}{4} = .25.$$

$$\therefore\ 1.74689 = \log_{10} 55.8325.$$

The characteristic fixes the position of decimal point.

To how many places should such divisions be carried? This depends upon the number x.

For, by the equations above,

$$\frac{\epsilon}{x} = \frac{\text{log-difference}}{0.43429\cdots}.$$

For a five-place table the smallest appreciable log-difference is 0.000005; that is, 5 in the sixth place.

$$\therefore \text{ here } \frac{\epsilon}{x} = \frac{0.000005}{0.43429} = \frac{1}{86859}, \textit{ to be appreciable.}$$

Thus in using a five-place table, the division should not be carried to a place where 1 in that place bears to the number being found a ratio smaller than 1 to 86859 (ϵ to x).

For instance, above we carried the division to two places, getting 25, where the number found was 55.8325. The ratio of 1 in the final place to the number is 1 to 558325. This is less than 1 to 86859. Thus the division was carried too far. In fact, since

$$\frac{5}{558325} < \frac{1}{86859},$$

the final 5 would not affect the logarithms to five places. That is, the logarithm of 558325 is the same as that of 55832 to five places.

The general rule for any table giving logarithms directly for numbers of one figure less than the table is:

Do not carry the division to the place where 1 *in that place has to the number being found a ratio less than* 1 *to the double modulus, considered as an integer to as many figures as the place of the table.*

Thus, in four-place tables not beyond 1 to 8686,
in five-place tables not beyond 1 to 86859,
in six-place tables not beyond 1 to 868599,
and so on.

It is thus apparent that the division should never be carried in such tables beyond two places; generally not beyond one place.

It is apparent from what precedes that in a five-place table, as soon as a number rises above 86859 (the double modulus) one in the fifth place of the number does not affect the logarithm in the first five figures. Thus in interpolating for numbers corresponding to logarithms in the part of the table above the double modulus, it is not possible to get the fifth figure within a unit of accuracy. In that part of the table no interpolation for the fifth figure should be made.

So in general for any place table, we should not interpolate for figures in the place of the table, when the logarithms indicate a number beyond twice the modulus.

EXERCISES.

1. Look up the logarithms of 234 and 235 in the four-place table, and from them find the logarithms of 23.48, 2.341, 2.346, 23.49, 234.5, 0.2341.

2. Look up in a five-place table the logarithms of 4357 and 4358, and from them determine the logarithms of 43.572, 4.3578, 432.56, 43579.

3. Write five logarithms at random, and find in the tables the numbers corresponding to them. Take four-place logarithms and five-place logarithms.

4. Examine your table, and see at what part of the table the tabular differences are largest.

5. Take any two consecutive numbers greater than twice the modulus and having as many figures as the place of the table, and examine as to whether in the table they have the same logarithms to the place of the table.

6. The teacher may exercise the class at will on examples like Exs. 1, 2, 3, so selecting them that the first part of the table, the middle of the table, and the part of the table above the double modulus are all considered, until the class is thoroughly familiar with how to interpolate, when to interpolate, how far to interpolate, and when not to interpolate.

§ 14. What Place Table to Use.

When we have before us a five-place table, we may consider that it has been made from a six-place table by dropping the sixth figure, following the usual rule as to the final figure as compared with 5 (see § 13, *a*, Note). In any particular logarithm we do not know, then, when we have a five-place table, what the sixth figure of the logarithm is. The logarithm may be in error by almost 5 in the sixth place, in either direction. We must thus allow in any given logarithm of five places a possible error of almost ± 0.000005. In § 13 we found

$$\frac{\epsilon}{x} = \frac{\text{log-difference}}{\text{modulus}}.$$

If, then, there is an error of ± 0.000005 in a logarithm, the corresponding value of $\frac{\epsilon}{x}$, or *the relative error in the corresponding number*, is

$$\frac{\pm 0.000005}{0.43429\ldots} = \frac{\pm 1}{86859}, \text{ nearly.}$$

Thus, if we have a single logarithm *and we are certain that it is correct to five places*, the corresponding number can be determined to within *one 86859th part of its value.* This is far beyond the degree of accuracy with which measurements in engineering work are usually carried out.

Similarly, if a logarithm is known accurately to four places, the corresponding number can be determined to within one 8686th of its true value. Hence, for a great many engineering operations *a four-place table is sufficient.*

It must be stressed here that we say, *If the logarithm is known to be correct to five figures, or four figures.*

If the given logarithm has itself resulted from a manipulation of values themselves uncertain in the next place beyond their given place, the resulting logarithm may itself be uncertain in one or more of its terminal places.

The effect of such uncertainty is considered in the next chapter, so far as it relates to ordinary trigonometrical calculations.

Later it will appear that if the sides of a diagram show only one significant figure, or only two, or only three, or only four, the angles should not show seconds, and a four-place table is sufficient. When the angles show seconds, a five-place table is called for. When the angles show tenths of seconds, a six-place table is called for; hundredths of seconds, a seven-place table.

An extension of the table by one place implies, as is clear from what precedes, a diminution of allowable error in readings and in the results of calculation by one-tenth, as also in the data.

A diagram with its sides reading to one, two, or three significant figures and showing seconds in angles would indicate that angles had been measured more carefully than lines, and would thus be absurd.

When we use a table of more places than the significant figures of the data call for, we must not take calculated results to the degree of accuracy of the table, but cut them back to the pattern of the data. This will be made clear in the next chapter.

EXERCISES.

Calculate to four significant figures the value of the following:

1. 27.34×13.56.

2. 2.374×0.0732.

3. 314.3×0.3164.

4. $27.31 \times 273.1 \times 0.003416$.

5. $\dfrac{316.1}{21.32}$.

6. $\dfrac{\sqrt{36.73}}{\sqrt[3]{318.9}}$.

7. $\dfrac{71.36 \times 21.27}{3763 \times 0.003721}$.

8. $\dfrac{\sqrt[5]{61.31} \times \sqrt[3]{7183}}{\sqrt{0.003456} \times \sqrt[4]{56.73}}$.

9. $(26.31)^{2.141} \times 0.3465$.

10. $(2.341)^{(1.361)}$.

Calculate to five significant figures the following:

11. 73.214×3.2154.

12. 384.62×2.7184.

13. $\dfrac{56.732 \times 87.563}{7.2134}$.

14. $\dfrac{7.8321 \times 0.032564}{2.3146 \times 0.056378}$.

15. $\sqrt{73.856} \times \sqrt[3]{7.3214}$.

16. $\dfrac{\sqrt{0.0035678} \times \sqrt[3]{0.043785}}{\sqrt[4]{0.0073214} \times \sqrt[5]{0.00032156}}$.

If there be more than one calculator, what is the best plan for checking such calculations as the above? What, if there be only one calculator?

§ 15. Negative Mantissa.

It occasionally happens that a calculation can be carried out more readily by having the logarithms that are usually part positive and part negative all negative, as, for example, in getting the value of

$$x = (0.00347)^{0.0567},$$

and similar expressions.

The usual logarithms would be

$$\log x = 0.0567 \log (0.00347)$$
$$= 0.0567 \times \bar{3}.54033.$$
$$\therefore \log (\log x) = \log (0.0567) + \log (\bar{3}.54033).$$

The last number is part positive (mantissa) and part negative (characteristic). Getting its logarithm in this form is not possible. Instead, we use

$$\log (\log x) = \log (0.0567) + \log (2.45967)\ (n)$$
$$= \log (0.0567) + \log (2.460)\ (n)$$
$$= \bar{2}.75358 + 0.39094\ (n)$$
$$= \bar{1}.14452\ (n)$$
$$\therefore \log x = -0.13948$$

or

$$\log x = \bar{1}.86052$$
$$\therefore x = 0.7253.$$

In similar expressions of the form

$$y = a^{b^c},$$

it is best to calculate the value of $\log (b^c) = \log x$, as in the preceding example, for

$$\log y = x \log a,$$

and

$$\log \log y = \log x + \log \log a.$$

EXERCISES.

Find the values of

1. $(0.003468)^{0.004378}$ to four significant figures.
2. $(0.0056143)^{0.00036213}$ to five significant figures.
3. $(0.3468)^{(0.003214)^{0.0003614}}$ to four significant figures.
4. $(\sqrt[3]{0.0346})^{(\sqrt{0.347})^{\sqrt[5]{0.361}}}$ to three significant figures.

CHAPTER II.

CALCULATION VICES AND DEVICES.

[NOTE. — When to stop "figuring" is what the student must know in calculations. In Trigonometry is the first place where the mature student faces this question. This chapter is intended as a *preparation* for judicious calculation.]

§ 16. **When a calculation involving only multiplications and divisions is to be carried out,** one may

(1) Do the work as in arithmetic.

(2) Shorten the arithmetic process by dropping the figures which fall, in the final result, beyond the place (decimal or integer) of possible accuracy. (See § 26 and § 40, for *the shortened process of multiplication and division.*)

(3) Use logarithms.

(4) Use a sliding-rule (which mechanically adds and substracts logarithms and indicates the corresponding number).*

(5) Use a calculating machine designed for office work.

Which process to use depends upon the extent of work to be done. A single multiplication, like 83 times 72, or even like 217.3 times 31.46, can be carried out more quickly, perhaps, directly than by logarithms, since, in the latter case, one must find the table, then look up the logarithm of each number, and then look up the corresponding number. However, if more than three operations are necessary, and the numbers

* Every engineer who has much calculating to do, will find a pocket sliding-rule of great convenience. The makers furnish an explanatory pamphlet. A description of the theory on which they are based will also be found in Raymond's "Surveying."

are not small, there is a great saving of time in using logarithms. Speed, however, in using logarithms, as in any other labor-saving device, depends on a thorough acquaintance, from long practice, with the process.

§ 17. **The numbers occurring in calculations** may be

(1) Known exactly.
(2) Known only approximately.
(3) Known exactly, but used only approximately.

Since most of the numbers used in calculations in engineering work come from measurements which are of necessity made only approximately, or else come from other calculations carried out only approximately, we may make the broad statement that an engineer is concerned chiefly, in calculations, with *approximate numbers*. The chief vice, then, of the engineering student (we might also include many engineers), as a calculator, is a more or less total disregard of the fact that the numbers used are only approximate, with the consequent superabundant care in the extent of "figuring" done. His vice is "ciphering" gone mad.

§ 18. Significant Figures.

All figures, other than zero, are significant in a number. A zero is significant unless used merely to locate the decimal point. In 207, 2.07, or 0.207, the central zero is significant. A zero used merely to locate the decimal point is not significant. Thus in 0.0003, the zeros are not significant, for an error in the next place to the right of 3 would be the same relative error in 0.0003 as in tenths' place for 3. Thus 0.0003 is not called a number of four significant figures, but of one significant figure. A final zero may be quite significant. In such cases it should always be retained. Thus if a line measured to the nearest hundredth of an inch is found nearer 29.30 than to 29.31 or to 29.29, the result should be written 29.30 and not 29.3, for the latter would mean that the measurement had been made only to the nearest tenth. Frequently, however, ending zeros are not significant. For example, if it is said that a certain building cost about $35,000, evidently the three zeros are not significant. When the context does not indicate the degree of inaccuracy of an approximate number, there should be some accompanying statement to make the matter clear. Frequently a number is written in the form 35.246, ± 0.012, with the understanding that the error may be as great in either direction as 0.012, and that, consequently, the true value lies somewhere between 35.258 and 35.234. The same notation is used in **Least Squares** in an entirely different sense, namely, that it is an even chance that the error in the result is larger or smaller than the indicated error. The inaccuracy to which a result is liable may also be expressed as a per cent. Thus 217, ± 2.17, means the same as 217 with possible inaccuracy of 1%. A statement like 31.145693, ± 0.032, would be absurd, as the last three figures, 693, would then be meaningless. This should be written, 31.146, ± 0.032. With the exception already mentioned in the case of zero, all figures in a number are significant, that is, any change in the figures would indicate a different number.

§ 19. Rejection Error.

Approximate numbers of very frequent occurrence are: logarithms; square roots of numbers like 2, 7, etc.; numbers like $\pi = 3.14159\ldots$ and $e = 2.71\ldots$; numbers like those treated later in this book and called sines, cosines, etc., which, with a few exceptions, belong to the large class of non-terminating, non-ending decimals. Approximate values of such numbers are obtained by stopping at any place, increasing the figure in that place by 1 when it is followed by a 5 or by a figure larger than 5, — otherwise, leaving the final figure unchanged. Thus, approximate values of π are 3.1, 3.14, 3.142, 3.1416. Such approximations are always nearer the true value than 5 in the next place to the right. Thus 3.14 is too small for π, but not by 0.005; 3.142 is too large, but not by 0.0005. The errors introduced into calculations by thus cutting the number of figures in a number are called *rejection errors*. The limit for the extreme value of the rejection error in any approximate number entering directly into a calculation is 5 in the next place to the right of the ending place. This error in the elements of a calculation can effect very materially the final result of the calculation.

§ 20. Effect of Errors in the Terms of an Addition-subtraction Result.

If a is liable to an error x_1,
and b is liable to an error x_2,

evidently $a \pm b$ is liable to the error $\pm(x_1 + x_2)$, for the error in a and b may both chance to lie the same way.

Similarly, for any number of such operations, the largest possible error is $\pm(x_1 + x_2 + x_3 + x_4 + \cdots + x_n)$.

It is extremely unlikely, of course, that in a very large number of such terms, taken at random, where the error in each term is as likely to lie one way as the other, the errors in all the terms will lie the same way. Experience has shown that in such cases there is a balancing of errors. When, however, the number of terms is small, it is not at all uncommon to find the errors all running one way. The student can give himself some acquaintance with this matter by selecting logarithms, in groups of two, of three, of four — as will be the case in most of the calculations in trigonometry — at random, from a four-place table, and afterwards looking up the fifth figure in a five-place table, or higher place table. The errors will be far from balancing. It is quite a common error to apply the deductions of the Method of Least Squares, based as they are on "the long run," to a very short run.

A not uncommon disregard of the preceding deductions is shown in adding (subtracting) approximate numbers.

For example, in the addition,

$$\begin{array}{r} 27.31 \\ 346.2159 \\ \hline 373.5259 \end{array},$$

if these numbers are liable to rejection error, the final result is quite misleading, for it makes it appear that the sum is liable only to the error 0.00005, whereas it is 0.00505. In such cases the number with many decimal places should be cut back to the pattern of the other.

$$\text{Thus,}\quad \begin{array}{r} 27.31 \\ \underline{346.22} \\ 373.53 \end{array}$$
where the final 3 may be "off" by 1.

In additions and subtractions of approximate numbers, subject to rejection error, let each term show the same number of decimal places.

§ 21. Applications of the Suggestions of § 20.

If the number of terms of an addition-subtraction expression is n, and each term is liable to rejection error beyond the rth decimal place, the extreme limit for the accumulation-effect of these rejection errors in the algebraic sum, is $5n$ in the $(r+1)$th place. Suppose 20 terms lead to the result, 1000, the smallest sequence of four significant figures, with 20 times 5 in tenths' place as extreme accumulation rejection error, or 10. This is 1% of the sum. If the result had been 9999, the largest four-figured number, the accumulation rejection error would still be 10, but as a per cent, only a little more than one-tenth of 1%. Similarly, for 10000 and 99999, the smallest and largest five-figured numbers, the per cents are $\frac{1}{10}$ of 1% and $\frac{1}{100}$ of 1%.

Since per cents are independent of the position of the decimal point, the following computation rule is sometimes advised:

If an addition-subtraction expression is desired within the range of $\frac{1}{10}$ of 1% to 1% possible inaccuracy, let the final result show four significant figures. If the permissible error is to lie between $\frac{1}{10}$ of 1% and $\frac{1}{100}$ of 1%, let the final result show five significant figures.

This rule, deduced as just shown, takes the worst cases and the best cases, presuming in each case that the effect of rejection error lies in each term all the time the same way, and with the rather unusual number of 20 terms. The "factor of safety" is thus so great that in the case of 2, 3, 4, 5, 6 terms, which are far more common than numbers of terms very much larger, it is very much better to disregard the rule and take the number of figures to suit the individual example, as illustrated in the following examples.

EXERCISES.

1. Find to within 0.4 of 1% the value of

$$55.6342 + 71.8371 - 4.32713.$$

By inspection, the result is about 120, of which 0.4 of 1% is about 0.5. Since there are only three terms, the maximum accumulation rejection error is 0.15, if hundredths are dropped. This is within the limit required. Therefore as follows:

$$55.6 + 71.8 - 4.3 = 123.1, \ \pm 0.15.$$

2. Find to within 0.6% the value of

$$47.3489 + 174.32825 - 5.62147.$$

By inspection, the result is about 200, of which 0.6% is 1.2. Thus, hundredths may be dropped, and we have

$$43.3 + 174.3 - 5.6 = 216.0, \ \pm 0.15.$$

3. Make up and solve some examples like the preceding, some with three terms, some with four, some with five, six, seven terms.

4. Add 273.1415 and 564.1. To what error is the sum liable?

5. Add 32.14156, 32.718, 32.6. To what error is the sum liable?

6. Find the value of $314.2156783 + 315.61 - 2.3124 - 87.4$.

7. Find the sum of 36.732, $\pm$ 0.021, and 71.2468, $\pm$ 0.0004.

The uncertainty in the first number being the greater, that decides the number of decimal places to retain in the second. Since 0.0004 added to the final 8 of the second number makes the 6 uncertain by 1, we may use

$$\begin{array}{r} 36.732 \pm 0.021 \\ 71.246 + 0.001 \\ \hline 107.978 \pm 0.022 \end{array}$$

the negative limit being not quite this.

8. Add 3.607($\pm$ 0.022) and 5.84($\pm$ 0.28).

9. Reduce to a single term

$$7.0845(\pm 0.0012) + 7.364(\pm 0.001) - 3.2748(\pm 0.012).$$

10. Find the value of

$$53.3456(\pm 0.05\,\%) + 167.3578(\pm 0.01\,\%) - 6.2315(\pm 0.03\,\%).$$

§ 22. Application to Logarithmic Work.

The most important practical application of the results reached in the preceding sections concerning the effect of accumulation rejection error is in the use of logarithms. The logarithm of a product is the sum of the logarithms of its factors, etc.

Take the case of two logarithms added (subtracted). Consider two four-place logarithms as a special case. The result may possibly be uncertain, in the extreme case, by almost 1 in the fourth place. If ten such logarithms are added (subtracted), the result, in the extreme case, may be uncertain by 5 in the fourth place, so that the resulting logarithm has only the accuracy of a logarithm taken from a three-place table. These simple facts are constantly disregarded in the use of four-place tables, as, for example, when one interpolates to get a fifth figure in a number corresponding to a logarithm obtained by additions (subtractions) of logarithms.

It has been pointed out in § 12, page 24, that if two logarithms differ by the small quantity ϵ, the ratio of the difference of the two corresponding numbers to the smaller number (the "relative error" in the two numbers) is $\dfrac{\epsilon}{\text{modulus}}$ or $\dfrac{\epsilon}{0.4342\ldots}$. In constructing a four-place table, 0.00014 counts as 1 in the fourth place, while 0.00015 counts as 2, as would also 0.00024. Thus the "relative error" corresponding to the accumulation rejection error 1 in the fourth place, in the case of an addition (subtraction) of two four-place logs may be $\dfrac{0.00014}{0.4342\ldots}$ or about $\dfrac{1}{3100}$ or $\dfrac{2}{6200}$ or $\dfrac{3}{9300}$, etc.

Thus if two four-place logs are added, giving a log corresponding to a number lying between 3100 and 6200 (the decimal point falling where it may), the rejection error of 1 in the fourth place will leave the corresponding number in doubt as to three consecutive numbers (and of course all intermediate numbers). If the corresponding number lies between 6200 and 9300, it will be in doubt among five consecutive numbers and the intermediates. It will be in doubt among seven consecutive numbers and their intermediates when it lies above 9300.

From this one will readily draw the conclusion that in using a four-place table one ought not, as a rule, to interpolate for a fifth figure, and one sees that the fourth figure is in doubt as soon as the number corresponding to a given log which has arisen by adding two logs, or subtracting them, rises above 3100. This can be observed in a four-place table.

EXERCISES.

1. Two logs added (subtracted) give 2.6042; what is the corresponding number?

Ans. 401.9 or 402.0 or 402.1 or any intermediate number.

2. Two logs added (subtracted) give 2.8482; what is the corresponding number?

Ans. 704.8 or 704.9 or 705.0 or 705.1 or 705.2 or any number intermediate to these.

3. Two logs added (subtracted) give 3.9845; what is the corresponding number?

Ans. 9647; 9648; 9649; 9650; 9651; 9652; 9653; or any number intermediate to these.

§ 23. Conclusion as to What Place Table to Use.

From the preceding section it will be clear that when the number of additions (subtractions) of logs is few, and where, consequently, a balancing of errors is not to be counted on, one ought to use a table giving at least one figure beyond the number of figures desired in the calculated results. In 20 additions with a seven-place table, it may happen that the fifth figure is in error by 1. If the characteristic calls for more than four figures in using a four-place table, the figures following the fourth should be filled with non-significant zeros. Such a characteristic indicates that a four-place table ought not to be used.

§ 24. Application to Some of the Problems of Trigonometry.

1. It will be found later in solving right-angled triangles that two logs are added (subtracted). The effect is, therefore, as pointed out in § 22.

2. In solving oblique-angled triangles, except when three sides are given, it will be found later that three logs are united. In using a four-place table, the fourth place becomes doubtful by 1.5. This counts as 2 in the table construction. So does 2.4.

$$\text{“Relative error”} = \frac{0.00024}{0.4342\ldots} = \frac{1}{1770} = \frac{2}{3500} = \frac{3}{5300} = \frac{4}{7000} = \frac{5}{8900},$$ as approximate results.

Consequently, in such work, if the resulting logarithm corresponds to a number (irrespective of decimal point) which lies

between 1770 and 3500, there is doubt among three consecutives,
between 3500 and 5300, there is doubt among five consecutives,
between 5300 and 7000, there is doubt among seven consecutives,
between 7000 and 8900, there is doubt among nine consecutives,

and all intermediate numbers.

EXERCISES.

If three logs added (subtracted) give the following logs, state the corresponding numbers:

1. 2.2989. **2.** 1.5955. **3.** 2.7404. **4.** 1.8633. **5.** 3.9294.

3. In solving a triangle when the three sides are given, it will be found later that four logs are added (subtracted), and this result is divided by 2 on account of a square root. The resulting doubt in the fourth place is therefore 1, and this case is the same as for solving right-angled triangles.

§ 25. Errors in Products of Two Factors.

(*a*) If a is liable to an error $\pm x_1$, and b is liable to an error $\pm x_2$, to find the error to which the product ab is liable.

Let a_1, b_1, be the true values of a, b,

$$a_1 = a \pm x_1,$$
$$b_1 = b \pm x_2,$$
$$a_1 b_1 = ab \pm ax_2 \pm bx_1 + x_1 x_2.$$

If x_1, x_2 are small, $x_1 x_2$ may be neglected.

$$\therefore \text{ error in } ab \text{ is } \pm (ax_2 + bx_1),$$

or, *a times b's error plus b times a's error.*

(*b*) If a is liable to an error of p_1 per cent, and b is liable to an error of p_2 per cent, to what per cent error is the product ab liable?

Here, $$x_1 = \frac{ap_1}{100},$$

and $$x_2 = \frac{bp_2}{100}.$$

$$ax_2 + bx_1 = \frac{ab}{100}(p_1 + p_2).$$

∴ *per cent error in the product ab is the sum of the per cents for the factors.*

This fact is frequently stated in a manner quite misleading, namely, "A product cannot be more accurate than its least accurate factor." Since when the errors are unknown the worst case must be allowed for, it is not how small the error may be, but how large, that is the important matter. If a is liable to 2% error and b to 3% error, ab is liable to 5% error. It is not sufficient to say that ab cannot be more accurate than b.

(*c*) If x_1, x_2 are rejection errors, to find the accumulation rejection error in ab, let

$$a = \frac{A}{(10)^{d_1}}, \text{ and } b = \frac{B}{(10)^{d_2}},$$

which signifies that A, B are the numbers a, b, when the decimal point is disregarded, and d_1, d_2 are the numbers of decimal places in a, b respectively.

x_1 is, at most, 5 in the (d_1+1)th decimal place,

that is, x_1 is, at most, $\frac{1}{2}$ in the d_1th decimal place,

and x_2 is, at most, $\frac{1}{2}$ in the d_2th decimal place.

$$ax_2 + bx_1 = \frac{1}{2} \cdot \frac{A+B}{(10)^{d_1+d_2}}.$$

$\therefore$ *to get the extreme accumulation rejection error in the product ab, add a and b without regard to decimal point, and then point off a number of decimal places equal to the sum of the decimal places in a and b; then divide by* 2.

"Relative error" in ab is $\frac{1}{2}\left(\frac{1}{A}+\frac{1}{B}\right)$.

EXERCISE.

What is the extreme accumulation rejection error in 3.142 times 36.52?

SOLUTION.

$$\begin{array}{r} 3142 \\ \underline{3652} \\ 2\overline{).06794} \\ .03397 \end{array}$$

Consequently the hundredths' place in the product is uncertain by more than 3.

When such a product is formed by the usual arithmetic process, a large part of the "ciphering" is a waste of time, since figures beyond the place of doubt are written down. In forming a product like that just given, the numbers being subject to rejection error, writing figures in places to the right of hundredths is an absurdity. The actual multiplication, when logs are not used, should be carried out by the method of § 26. When logs are used, the result should be given only as far as to the doubtful place.

§ 26. The Shortened Process of Multiplication.

First, determine the uncertain place in the product, as above. Set the numbers so that decimal points fall one under the other. In the product, set down no figures to the right of the doubtful place. Perform no multiplications two or more places to the right of the doubtful place. Do this only one place to the right of the doubtful place, but set down no results; use such products only to "carry" to the doubtful place. From a product like 27 "carry" 3 (an extra 1, because 7 is more than 5); the same for 25; for 24, carry 2, and so on. Carry out the multiplication from left to right on the multiplier, instead of from right to left, as in arithmetic. Start the multiplication with that figure of the multiplicand which gives a figure one place to the right of the doubtful place. As one moves to the right one figure each time on the multiplier, start the multiplication one figure to the left on the multiplicand with each such move along the multiplier.

ILLUSTRATIVE EXAMPLES.

Find 3.142 × 36.53. As already shown, hundredths are doubtful by 0.03.

By old-fashioned process.	By shortened process (hundredths doubtful).
36.53	36.53
3.142	3.142
7306	109.59
14612	3.65
3653	1.46
10959	7
114.77726	114.77 ± 0.03

Other examples by the shortened process:

Units doubtful.	Tenths doubtful.	Hundredths doubtful.
43.29	27.314	96.789
3.8	71.65	21.579
130.	1912.0	1935.78
34.	27.3	96.79
164. ± 2	16.4	48.39
	1.4	6.77
	1957.1 ± 0.2	.86
		2088.59 ± 0.06

In the first example, since hundredths are doubtful, the multiplication is started by using 3 of the multiplier with the 3 on the right of the multiplicand. In taking next the 1 of the multiplier, the multiplication is started on the multiplicand with the 5, — motion to the right on the multiplier with motion to the left on the multiplicand. There is in this line no carrying from the 1 times 3, since this is less than 5. In the next line, where 4 of the multiplier is used, the multiplication is started with the 6 of the multiplicand, there being a "carrying" of 2 from the 4 times 5, one place to the right of the doubtful place. In the next line, where the 2 of the multiplier is used, the start is on the left-hand 3 of the multiplicand, with 1 carried from the 2 times 6, this being one place to the right of the doubtful place.

In the second example, where units are doubtful, the start is made by 3 times the unit 3 of the multiplicand, with 1 carried from 3 times the 2 to the left of this figure.

Where to start the multiplication on the multiplicand is easily seen. Consider the third example, where tenths are doubtful; the initial 7 of the multiplier is 70. If 70 is multiplied by the 4 on the right of the multiplicand, this is in fact multiplying by 0.004, with the result 0.28, and we should thus have the figure 8 in hundredths' place, whereas tenths' place is the doubtful place. Consequently, the multiplication is started by taking 7 times the 1 of the multiplicand, carrying 3 from the 7 times the 4, which is one place to the right of the doubtful place. A careful study of the examples worked out will make the process clear.

The place in the multiplicand at which to make the initial start is given by the following rule:

Calling units' place the zeroth place; tenths' place, place 1; hundredths' place, place 2, and so on; tens' place as place -1; hundreds' place as place -2, and so on; the starting place in the multiplicand from which *to set down figures* is given by subtracting from the doubtful place of the product the place of the extreme left-hand figure of the multiplier. *Carry from one place to the right of this.*

Thus, in the first example, the doubtful place of the product is 2, and the place of the extreme left-hand figure of the multiplier is zero. The start is on the place 2 of the multiplicand. In the last example, the doubtful place of the product is 2, and the place of the left-hand figure of the multiplier is -1. The start is on place 3 of the multiplicand.

EXERCISES.

Determine the extreme accumulation rejection error in the following products. Find the first place which is uncertain. Name the extent of uncertainty in that place. Carry out the multiplications by the shortened process.

1. 6.043 times 6.043.
2. 12.65 times 111.7.
3. 78.21 times 1450.
4. 23.57 times 612.3.
5. 23.57 times 61.23.
6. 0.65 times 0.32.
7. 78.21 times 1.45.

Carry out some of the preceding multiplications by the ordinary arithmetic process, and see if there is any saving of labor. Carry out the same multiplications by logs, and compare results. Make sure that a log table which will give correctly the number of figures in the final products is used, and that the products are not taken beyond the doubtful place. A great deal can be learned about a log table from these exercises by making the comparisons here recommended.

§ 27. To find the Accumulation Rejection Error in the Product *ab* as a Per Cent Error.

Since the product is
$$\frac{AB}{(10)^{d_1+d_2}}$$
and the error is
$$\frac{1}{2}\cdot\frac{A+B}{(10)^{d_1+d_2}},$$
the per cent error is
$$\frac{100}{2}\left(\frac{1}{A}+\frac{1}{B}\right).$$

$\therefore$ *the per cent accumulation rejection error is one-half of one hundred times the sum of the reciprocals of a and b when taken without the decimal point.*

EXERCISES.

1. To compare the rejection error in

$$1000 \times 1000 \text{ and } 10.00 \times 1.000.$$

The per cent error in each product is

$$\tfrac{100}{2}\left(\tfrac{1}{1000} + \tfrac{1}{1000}\right), \text{ or } \tfrac{1}{10} \text{ of } 1\,\%.$$

2. To compare the rejection error in

$$9999 \times 9999 \text{ and } 99.99 \times 9.999.$$

The per cent rejection error is in each case

$$\frac{100}{2}\left(\frac{1}{9999} + \frac{1}{9999}\right), \text{ or } \frac{1}{99.99} \text{ of } 1\,\%;$$

that is, a little more than $\frac{1}{100}$ of 1 %.

3. A comparison of Exs. 1 and 2 will show that the accumulation rejection error in a single product of two factors of four significant figures may vary from $\frac{1}{10}$ of 1 % to $\frac{1}{100}$ of 1 % of the product.

4. Deduce in the same way the corresponding relations for the product of two factors of five significant figures; six significant figures; seven significant figures.

Summary.

(1) For three significant figures, $< 1 > \frac{1}{10}$ of 1%.
(2) For four significant figures, $< \frac{1}{10} > \frac{1}{100}$ of 1%.
(3) For five significant figures, $< \frac{1}{100} > \frac{1}{1000}$ of 1%.
(4) For six significant figures, $< \frac{1}{1000} > \frac{1}{10000}$ of 1%.
etc., etc.

§ 28. Approximate Values for Per Cent Accumulation Rejection Error in a Product of Two Factors.

Let the 100 in $\frac{100}{2}\left(\frac{1}{A} + \frac{1}{B}\right)$ cancel the two terminal figures of A, B. Whence the rule:

(1) *The per cent rejection error in the product ab is given approximately by dropping the decimal points and the two terminal figures of A, B, observing the usual rule with reference to the next figure being 5, more or less; invert; add; take half.*

An approximation somewhat rougher, but in many cases close enough, is given by the following:

(2) *Drop the decimal points and two terminal figures; change each remaining figure, except the initial figure, to zero; retain the initial figure, observing the usual rule with reference to the next figure being* 5, *or more or less than* 5; *invert; add; take half.*

EXERCISES.

1. A square has its side measured to the nearest hundredth inch, giving 25.32. What is the maximum per cent accumulation rejection error in its area? *Ans.* About $\frac{1}{25}$ of 1 %.

2. A rectangle has its sides measured, giving 24.32 and 378.2. What is the maximum per cent accumulation rejection error in the area? *Ans.* About $\frac{1}{28}$ of 1 %.

3. Apply the approximation rules to find the maximum per cent rejection error in each product of § 26.

§ 29. Relation of the Two Factors of a Given Product when the Effect of Accumulation Rejection Error is Least.

Per cent error is $\frac{100}{2}\left(\frac{1}{A}+\frac{1}{B}\right)$, or $\frac{100}{2}\left(\frac{A+B}{AB}\right)$.

Since AB is fixed by hypothesis, the per cent error is least when $A + B$ is least.

Let $$A = \sqrt{AB} + X, \quad (1)$$

$$B = \sqrt{AB} - Y. \quad (2)$$

Then $$A + B = 2\sqrt{AB} + X - Y. \quad (3)$$

Also $(\sqrt{A} - \sqrt{B})^2$ is positive, when A is not B.

$$A + B - 2\sqrt{AB} > 0, \text{ or } A + B > 2\sqrt{AB}.$$

∴ Y cannot be more than X in (3). ∴ the least value which $A + B$ can have in (3) is $2\sqrt{AB}$, which occurs when $A = B$.

EXERCISES.

1. Compare the effect of accumulation rejection error in the area of a square whose side is 12.24, and the rectangle of about the same area, its sides being 3.06 and 48.96.

2. Compare the accumulation rejection error in $(31.62)^2$, and 9999×0.1000, the products being each about 999.9000.

§ 30. To find the Per Cent Error in the Product of Three or More Factors, Each Factor being Liable to a Given Per Cent Error.

If a is liable to p_1 per cent error, and b to p_2 per cent error, it has been shown that ab is liable to the error (p_1+p_2) per cent. Evidently, then, $(ab)c$ is liable to the per cent error $(p_1+p_2+p_3)$, c being liable to the error p_3 per cent.

So, in general, the per cent error to which a product of any number of factors is liable, is the sum of the per cents of the factors.

It is assumed here, of course, that the absolute error in each factor is small, for only in this case is the error in ab a times b's error plus b times a's error. In the above deductions all errors arising from product of errors have been neglected.

The tendency of multiplication of factors, each subject to error, is to increase the maximum error to which the product may be liable.

This is a consequence of the preceding section, it being assumed all the while that the worst possible case may occur by the errors chancing to fall all the same way. When the number of factors is comparatively few, so that a balancing of errors is not to be relied on, the tendency of multiplication is to subject the result to an increasing possible error.

From this it appears that the measured quantities which form the basis of a calculation should be measured with a greater degree of accuracy than that expected in the calculated results based on them.

All measurements are subject to rejection error.

Later on, therefore, when the measured sides of a diagram are given to one significant figure, it will be absurd to let the calculated sides show two or more significant figures.

The calculated parts of a diagram should never show more significant figures than the measured parts.

It is hopeless to give calculated results seven-place accuracy when they are based on field measurements with three-place accuracy.

§ 31. To find Numerical Expressions for the Per Cent Accumulation Rejection Errors in Products of Several Factors.

For two factors it is $\frac{100}{2}\left(\frac{1}{A}+\frac{1}{B}\right)$.

Similarly, for three factors $\frac{100}{2}\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)$

Similarly, for four factors $\frac{100}{2}\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}\right)$,

where A, B, C, D are a, b, c, d with the decimal points dropped. (See § 27).

Approximate results can be obtained by an extension of the two approximation rules given in § 28.

§ 32. On the Effect of the Number of Significant Figures in the Factors of a Product on the Accumulation Rejection Error of the Product.

From the approximation rules of the preceding section, it is apparent that if four significant figures in each factor give a certain per cent rejection error in the product, five significant figures in each factor will, approximately, cut this error to $\frac{1}{10}$ of its value, and three significant figures will multiply it by 10, approximately, and so on.

From this it is easy to determine, in any special case, how many significant figures to take in each factor in order to secure a result true to within a specified per cent of error.

ILLUSTRATIVE EXAMPLES.

1. Given $\pi = 3.14159$, $r = 8.43276$, $h = 9.76438$, find the volume of the corresponding cylinder from the formula $\pi \cdot r \cdot r \cdot h \cdot$, to within 1%.

Taking only integers, the per cent rejection will be approximately $\frac{100}{2}\left(\frac{1}{3} + \frac{1}{8} + \frac{1}{8} + \frac{1}{10}\right)$, or about $\frac{205}{6}$. The use of any additional significant figure practically divides this result by 10.

2. Show that if the volume of a cylinder is gotten from $\pi = 3.142$, $r = 6.043$, $h = 12.65$, the accumulation rejection error is less than $\frac{3}{40}$ of 1%.

3. Use the shortened process of multiplication to find the volumes of the cylinders in Exs. 1 and 2, to within 1% and to within $\frac{1}{10}$ of 1% respectively.

Solution of Example 2 to within 1%.

Hundredths doubtful.	Tenths doubtful.	Doubtful by less than 12.
6.04	36.5	12.7
6.04	3.14	115.
36.48	109.5	1270.
	3.7	127.
	1.4	64.
	114.6	1451.

The student will find it valuable exercise to calculate the error $ax_2 + bx_1$ for each product as formed, and notice its growth.

4. What would be the maximum effect of accumulation rejection error in Exs. 1 and 2, if we take

in Ex. 1, $\pi = 3.1$, $r = 8.4$, $h = 9.8$;
in Ex. 2, $\pi = 3.1$, $r = 6.0$, $h = 13$?

5. The area of a circle $(\pi \cdot r \cdot r)$ is obtained from $\pi = 3.142$, $r = 6.043$; in what place is the area uncertain from accumulation rejection error?

The easiest way to answer a question like this is to determine the maximum per cent rejection error, and from this get the doubtful place in the area from a rough estimate of the area.

6. Determine the area of the circle in Ex. 5, using four significant figures and the shortened process of multiplication, dropping in each product, as in the solution of Ex. 2 above, the doubtful place as new products are formed.

To within what per cent will $\pi = 3.1$, $r = 6.0$ give the area of the circle? What if we take $\pi = 3$ and $r = 6$?

7. If in the cylinder of Ex. 2, $r = 6.0428$, $\pm \frac{1}{4}$ of 1% and $h = 12{,}653$, $\pm \frac{1}{10}$ of 1% and $\pi = 3.1416$, to what per cent error is the volume liable, neglecting the error in π? *Ans.* About $\frac{7}{10}$ of 1%.

How many significant figures should be taken in each factor of $\pi \cdot r \cdot r \cdot h$ to get within this error? *Ans.* Four.

8. Consider the volume of the cone $r = 6.18$, $h = 127$, $\pi = 3.14$, in the same manner as the preceding examples.

§ 33. On the Reasonableness of Retaining the Same Number of Significant Figures in the Factors of a Product.

Two measurements showing the same number of significant figures indicate about the same degree of relative accuracy of measurement. Thus, in 36.27 and 4134, the former means something between 36.265 and 36.275, and the latter something between 4133.5 and 4134.5, so that the largest relative error in the first is $\frac{1}{7254}$ and in the second $\frac{1}{8268}$, which are reasonably near together. The relative rejection error in the first is $\frac{5}{36270}$ and in the second $\frac{5}{41340}$, or $\frac{1}{7254}$ and $\frac{1}{8268}$. The widest possible discrepancy occurs, of course, between the largest and smallest sequence of the given number of places, as, for example, in 1000 and 9999, for four figures. Thus, unless some unusual reason can be given why one factor of a product should be measured with far greater accuracy than others, all factors should show the same number of significant figures, as a rule.

All the measured parts of a diagram should thus show the same number of significant figures, and the calculated parts should not show more significant figures than the measured parts. They may show, generally, the same number of significant figures.

EXERCISES.

1. Find 21746893 × 1.53, these representing measurements.

Since the rejection error in 1.53 is 0.005, ax_2 is large and bx_1 is small. Thus the larger quantity may be cut back to the number of significant figures of the smaller.

Therefore find 21700000 × 1.53, as close enough.

2. What is the product of 372 and 0.0001, these numbers representing measurements?

Here the first number shows three significant figures and the second only one.

Therefore take 400 × 0.0001.

The rejection error, as a per cent, is in the first case $\frac{100}{2}(\frac{1}{372} + \frac{1}{1})$ and in the second case $\frac{100}{2}(\frac{1}{400} + \frac{1}{1})$. The difference is slight.

3. What is the relative accuracy of measurement in each factor of the products

$$1.000 \times 0.0001;\ 10.00 \times 0.01;\ 1.000 \times 0.0001?$$

Find the per cent rejection error in each product.

§ 34. Some Extreme Cases and Computation Rules based on them.

(1) Suppose there are 20 factors in a product, each factor with four significant figures, and each factor about 1000.

The accumulation rejection error as a per cent is about

$$\tfrac{100}{2}(\tfrac{1}{1000} + \tfrac{1}{1000} + \text{etc., to twenty terms})$$

or 1%.

(2) Taking now the other extreme case of four significant figures, where each factor is about 9999 (without regard to decimal place).

The per cent rejection error is now

$$\tfrac{100}{2}(\tfrac{1}{9999} + \tfrac{1}{9999} + \text{etc., to twenty terms})$$

or about $\frac{1}{10}$ of 1%.

Similarly, for numbers of five significant figures the results are from $\frac{1}{10}$ of 1% to $\frac{1}{100}$ of 1%, and so on.

From these results arise the following computation rules occasionally advised:

For an inaccuracy ranging from $\frac{1}{10}$ of 1% to 1% in the final result, use four significant figures in each factor of the product and in each partial product as formed and in the final product.

When the range of inaccuracy is to be from $\frac{1}{10}$ of 1% to $\frac{1}{100}$ of 1%, use, similarly, five significant figures, and so on.

Since a product with 20 terms is uncommon, and since in the majority of cases the number of factors in the product to be found is quite few, the rough approximation rules given in § 28, for the determination of the per cent error in the final product, will, in any individual case, generally shorten the work sufficiently to justify finding the maximum error likely to arise. Then the number of significant figures to take to keep within the prescribed error can be determined. Illustrative examples have already been given in § 32.

The foregoing computation rules are sometimes stated incorrectly, thus: "For an inaccuracy of 1% or *worse*, use four significant figures, etc." The error cannot be worse than 1% for less than 20 factors. That is the extreme case, as already shown.

§ 35. On what Forms of Products Accumulation Rejection Error has Least Effect.

We have already shown that in a product of two factors the accumulation rejection error is least, for a given product, when the two factors are the same, decimal point being disregarded. (See § 29.)

Similarly for three factors in a given product, the effect of such error is least when $A = B = C$, and for four factors, making a given product, when $A = B = C = D$, and so on.

EXERCISES.

1. A square and rectangle of about the same area have their sides measured and areas calculated. For the square, the side is 24.82. For the rectangle, the sides are 12.41, 49.64. Determine by the shortened process of multiplication the areas of each and the per cents of rejection error in each area.

2. Consider in the same way the cube whose edge is 12.36 and the parallelopipedon whose edges are 6.06, 4.09, 73.2.

3. Consider in the same way two cylinders of about 1450 cubic inches volume, where in one $r = 6.043$, $h = 12.65$, and in the other $r = 3.022$, $h = 50.60$.

4. What shaped cylinder can be measured most accurately to five places for volume? *Ans.* $r = h = \pi = 3.1416$.

5. What circle? *Ans.* $r = \pi$.

6. Which can be measured with greater accuracy for area, a square or rectangle, both being of about the same area?

7. For volume, cube or parallelopipedon, both being of about the same volume?

8. What shaped triangle for area, the base and altitude being measured?

9. What shaped right-angled triangle for area, if the two legs are measured?

10. What shaped cone?

§ 36. Approximate Values of $\frac{1}{1+x}$ and $\frac{1}{1-x}$ when x is Small.

By actual division, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + R_1$.

By actual division, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + R_2$, where R denotes "the rest" after the terms written down.

If x is less than unity, x^2, x^3, etc., are smaller than x. If x is 0.0001, x^2 is 0.00000001, and so on. When x is on the verge of the smallness taken into account in any measurement, x^2, x^3, etc., may be neglected.

Thus, approximately, $$\frac{1}{1+x} = 1 - x,$$

and $$\frac{1}{1-x} = 1 + x.$$

EXERCISES.

1. $\frac{1}{0.9} = \frac{1}{1 - 0.1} = 1.1$, nearly.
2. $\frac{1}{0.99} = \frac{1}{1 - 0.01} = 1.01$, nearly.
3. $\frac{1}{0.999} = \frac{1}{1 - 0.001} = 1.001$, nearly.
4. $\frac{1}{1.0001} = 1 - 0.0001 = 0.9999$.

§ 37. Approximate Values of $\frac{1}{a \pm x}$ when x is Small.

$$\frac{1}{a \pm x} = \frac{1}{a\left(1 \pm \frac{x}{a}\right)} = \frac{1}{a}\left(1 \mp \frac{x}{a}\right) = \frac{1}{a} \mp \frac{x}{a^2}.$$

EXERCISES.

1. $\frac{1}{9.1} = \frac{1}{9} - \frac{1}{810}$.
2. $\frac{1}{10.03} = \frac{1}{10} - \frac{3}{10000} = 0.0997$, nearly.
3. $\frac{1}{9.999} = \frac{1}{10 - 0.001} = \frac{1}{10} + \frac{1}{100000} = 0.10001$.

§ 38. If a is liable to an error x, where x is small, to what error is $\frac{1}{a}$ liable?

By § 37 the error is $\frac{x}{a^2}$.

§ 39. If a is liable to an error x_1 and b to an error x_2, to what error is the quotient $\frac{a}{b}$ liable?

$$\frac{a}{b} = a\left(\frac{1}{b}\right).$$

This is a product, and the error in such a product is a times the error in $\frac{1}{b}$ plus $\frac{1}{b}$ times the error in a.

$$\therefore \text{ error} = a \cdot \frac{x_2}{b^2} + \frac{1}{b} \cdot x_1 = \frac{ax_2 + bx_1}{b^2}.$$

Thus the error in the quotient $\frac{a}{b}$ *is that in the product* ab *divided by* b^2.

As a per cent this error is $\frac{100(ax_2 + bx_1)}{ab}$, which is the same as it would be in the product ab.

Thus all that has been said concerning products heretofore holds also for quotients, so far as per cent errors is concerned.

§ 40. The Shortened Process of Division.

Special Example: find the value of $\frac{1578.53}{718.2}$.

The first step is to determine to how many places the division may be carried accurately, assuming the terms as representing approximate numbers and thus subject to rejection error. This can be determined by calculating the error in the quotient by the formula of § 39. This is too much trouble -- as much as performing the division itself. It is better to get the rejection error approximately as a per cent, and calculate roughly the quotient, and thus calculate the inaccurate place in the quotient.

In the quotient above, the per cent error is (by § 27)

$$\frac{100}{2}\left(\frac{1}{157853}+\frac{1}{7182}\right),$$

or, approximately, $\frac{1}{2}\cdot\frac{1}{72}=\frac{1}{144}$ of 1%. The quotient is roughly 2, of which $\frac{1}{144}$ of 1% will be a figure in the fourth decimal place. The division may thus be carried accurately not beyond the third decimal place.

```
           2.198
7182)15785.3
     14364
      14213
       7182
       7031
       6464
        567
        574
```

Explanation.

(1) Make the denominator an integer by multiplying both numerator and denominator by an appropriate power of 10. The quotient is set in the top line, and so that the decimal points fall the one under the other.

(2) In carrying out the division, one proceeds as in arithmetic until the final significant figure of the dividend is used. This, above, is until the remainder 7031 is reached.

(3) When such figure is reached, then, instead of drawing down zeros in the dividend, as in arithmetic, the divisor is cut one figure, making it 718, and 718 goes into 7031, 9 times, giving the third figure of the quotient. One carries from the rejected 2 of the divisor in multiplying 718 by 9; 9 times 2 is 18, and 2 is carried to the 9 times 8, making

74. At the next step the divisor is cut to 71, which goes into 568, 8 times. In multiplying by 8, 6 is carried from 8 times the rejected 8 of the divisor.

A second example: $\frac{15678}{673.2}$

```
           23.29
6732)156780
       13464
        2214
        2020
         194
         135
          59
          60
```

EXERCISES.

Determine to how many places the divisions following should be carried, and then do the work as in the preceding examples:

$$\frac{15374}{624.2};\ \frac{327.3}{31.8384};\ \frac{427}{61.7};\ \frac{4.3}{7.3}.$$

The teacher may assign other examples at pleasure.

In the divisions which occur in practical work, arising from measurements intelligently taken, both the divisor and dividend will, as already pointed out, show the same number of significant figures, and the limit of accuracy takes care of itself in the actual division by the disappearance of the divisor itself under the pruning process to which the method subjects it.

A number of examples should be selected, in which numerator and denominator show the same number of significant figures, the number of places to which the division may be carried accurately determined by the methods given, and then the division by the shortened process should be carried out, noting that the divisor vanishes opportunely.

Test results by using logarithms.

§ **41.** If two numbers subject to rejection error are multiplied together by the use of logarithms, what effect have such errors on the logarithms and on the final result as calculated by logarithms?

It has been pointed out in § 12 that if two numbers differ by ϵ, a small number, their logarithms will differ by

$$(0.4342\ldots)\frac{\epsilon}{x}.$$

Also $\log(abc\ldots l) = \log a + \log b + \cdots + \log l.$

Thus, if $a, b, c\ldots l$, have the errors $x_1, x_2\ldots x_n$, the error in the $\log(abc\ldots l)$ is

$$(0.4342\ldots)\left(\frac{x_1}{a}+\frac{x_2}{b}\cdots\frac{x_n}{l}\right).$$

In the case where the x's are rejection errors, this becomes, as in the cases already considered,

$$\frac{(0.4342\ldots)}{2}\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\cdots+\frac{1}{L}\right).$$

Corresponding to this error in the log of the product will be the relative error in the product itself,

$$\frac{1}{2}\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\cdots+\frac{1}{B}\right).$$

This is exactly the same as would be the relative error due to rejection error if the multiplications were carried out directly. (See § 25 and § 31.)

Consequently the use of logarithms has no effect on the final result other than that arising from rejection error in the logs themselves, and which has been considered already in § 22.

§ 42. Approximate Values of Powers and Roots.

When x is small compared with a, then

$$(a+x)^2 = a + 2\,ax; \quad (a+x)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2}\cdot\frac{x}{a^{\frac{1}{2}}}.$$

$$(a+x)^2 = a + 3\,ax; \quad (a+x)^{\frac{1}{3}} = a^{\frac{1}{3}} + \frac{1}{3}\cdot\frac{x}{a^{\frac{2}{3}}}.$$

$$(a+x)^n = a + nx; \quad (a+x)^{\frac{1}{n}} = a^{\frac{1}{n}} + \frac{1}{n}\cdot\frac{x}{a^{\frac{n-1}{n}}}.$$

If x is a percentage $= \dfrac{az}{100}$ (z per cent), the above results become

$$(a+x)^2 = a^2 + 2\,a^2\cdot\frac{z}{100}; \quad (a+x)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2}\cdot a^{\frac{1}{2}}\cdot\frac{z}{100}.$$

$$(a+x)^3 = a^3 + 3\,a^3\cdot\frac{z}{100}; \quad (a+x)^{\frac{1}{3}} = a^{\frac{1}{3}} + \frac{1}{3}\cdot a^{\frac{1}{3}}\cdot\frac{z}{100}.$$

$$(a+x)^n = a^n + na^n\cdot\frac{z}{100}; \quad (a+x)^{\frac{1}{n}} = a^{\frac{1}{n}} + \frac{1}{n}\cdot a^{\frac{1}{n}}\cdot\frac{z}{100}.$$

Thus the per cent of error in the square of a number is, approximately, twice that in the number; in a cube, three times that in the number, and so on.

The per cent of error in a square root is, approximately, one-half that in the number; in a cube root, one-third, and so on.

EXERCISES.

1. If $x = 0.000008468$, what is the value of $\sqrt{1-x}$?

$$\begin{aligned}\sqrt{1-x} &= 1 - \tfrac{1}{2}x, \text{ nearly,}\\ &= 1 - 0.000004234,\\ &= 0.9999958.\end{aligned}$$

2. If the side of a square is 1.0002, what is the area approximately?

$$(1.0002)^2 = 1 + 2 \times 0.0002 = 1.0004, \text{ nearly}.$$

3. Find the approximate value of $(1.0001)^3$.

4. Find the approximate value of $\sqrt[3]{1.0003}$.

5. Find the approximate value of $\sqrt{0.9999996}$.

6. If the mantissa is zero, what modification is made in the rule on page 11 for getting the cologarithm? What is the colog 0.01? Colog $(10)^{1000000}$?

CHAPTER III.

ANGLES AND ANGLE-UNITS.

§ 43. Angles in Formation and in Sign.

The student beginning the study of trigonometry is already familiar with the word angle. Some extension of his ideas, as to size and sign of angles, is essential. Imagine *OA*, *OP*, to be two straight lines hinged together at *O*; and that *OA* is held fixed in position while *OP* turns in a plane about *O*, starting from coincidence with *OA*. Then the amount that *OP* has turned, as compared with some *unit-turn*, is a measure of the *angle* between *OA* and *OP*. *OP* may turn as do the hands of a clock (clockwise), or in the opposite way (counter-clockwise). To distinguish between the two, when this is desired, angles described by a counter-clockwise turn of *OP*, as *M*, are called *positive* angles; the reverse, like *N*, *negative* angles. The line *OA* is called the *initial line;* *OP*, the *terminal line*, or simply the *terminal*, of the angle. When, therefore, significance is attached to the sign of an angle, a distinction is made between its border lines. One is the *initial line;* the other, the *terminal line.* The latter must, for positive angles, lie in the order of a counter-clockwise turn from the former; the reverse, for negative angles.

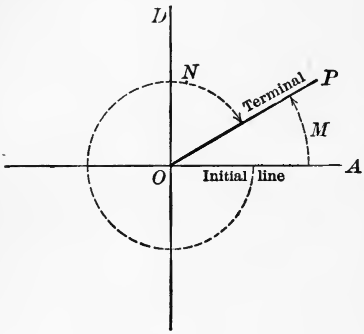


FIG. 1.

§ 44. Adding and Subtracting Angles.

Angles are added as are numbers. To add *A* to *B*, pictorially, first lay out *A*, paying attention to both magnitude and sign. Then from the terminal of A as a new initial line, lay out an angle equal to B and of the same sign. The old initial line of A and the new terminal of B form the initial line and terminal of $A+B$. This applies to $\pm A+(+B)$ as well as to $\pm A+(-B)$. To *subtract* the angle B from the angle A, lay from the terminal of A an angle equal in magnitude to B, but of opposite sign. The initial line of A and the new terminal of B form the border lines in order for $A-B$. This applies to $\pm A-(+B)$ and to $\pm A-(-B)$. In Fig. 1, OA, OD border many different angles, so that the expression, angle AOD, is indefinite. However, any angle AOD is the *sum* of two represented by AOP, POD. It is also the *difference* of two represented by AOP and DOP. So is AOP equal to $AOD+DOP$, or, it is $AOD-POD$.

EXERCISES.

1. What is the method used in geometry for constructing an angle equal to a given angle?

2. Draw with a straight-edge some angles at random. Construct an angle equal to the sum of two selected positive angles; then to three. Add a positive and a negative angle, selected at will. Subtract a positive angle from a given larger positive angle; from a given smaller positive angle; a positive from a negative angle. Make the number and variety of exercises sufficient unto proficiency, but not unto weariness.

NOTE.—The teacher using this book is advised against assigning to tiresomeness exercises that are alike. Nothing deadens intellect more than having to do too much of a tedious thing, which one can see readily how to do, but does not wish to do.

§ 45. The Initial Line Par Excellence and the Quadrants.

When angles are considered singly, with reference to size and sign, and with reference to certain related magnitudes, called sines, cosines, etc., which vary with the size and sign of angles, it is customary to take as initial line a right-hand horizontal line, as OA in Fig. 1 or Fig. 2. The point O is called the *origin;* also the *pole*. Continuing OA

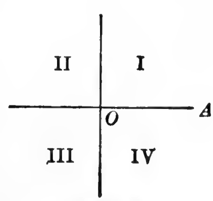
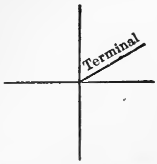
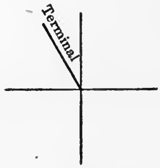


FIG. 2.

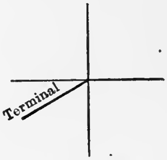
backward through the origin, O, and drawing through O, at right angles to OA, another line, and extending both lines in imagination infinitely in both directions, we divide the plane of angles into four quadrants. These quadrants are numbered and named, as indicated in Fig. 2, in counter-clockwise order, the *first quadrant* (I), the *second quadrant* (II), the *third quadrant* (III), the *fourth quadrant* (IV).



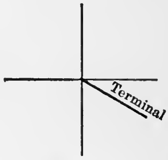
Angle of 1st quadrant
FIG. 3.



Angle of 2nd
FIG. 4.



Angle of 3rd
FIG. 5.



Angle of 4th
FIG. 6.

An angle whose terminal is in the first quadrant (I), no matter what its size or sign, is said to be an angle of the first quadrant. When the terminal is in the second quadrant, the angle is said to be an angle of the second quadrant; similarly, with reference to the third and fourth quadrants. In all these cases the right-hand horizontal line is reckoned the initial line.

§ 46. Angles unlimited in Size.

In geometry the angles considered are, for the most part, less than two right angles. In trigonometry no limit is set upon the size of angles. That is, no limit is set upon the extent of turn of OP in Fig. 7. It may pass any position any number of times, finite or infinite, and in either direction. In Fig. 7 the arrow M indicates the smallest positive angle, M, corresponding to that

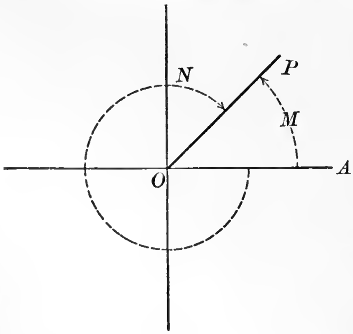


FIG. 7.

position of the terminal. If T denotes one complete turn of OP, from the position OA, back to that position again, then $M \pm T$, $M \pm 2\,T$, $M \pm 3\,T$, $M \pm nT$ (where n is any integer), are angles whose terminals are coincident with that of M. Denoting by the arrow N, in Fig. 7, the smallest negative angle, numerically, corresponding to the given position of the terminal of M, then also, evidently, $N \pm T$, $N \pm 2\,T$, $N \pm nT$, are angles whose terminals are coincident with that of M.

Thus $M + nT$ is a general formula for all angles whose terminals are coincident with that of M, n being any positive or negative integer.

The smallest angle, numerically, locating any given terminal, is called the *principal angle* of the terminal.

§ 47. Measuring Angles.

To measure an angle is to determine its relation in magnitude to some *unit-angle*, that is, its *number* (or its ratio to the unit-angle). This is done in practice, in outdoor work, by means of graduated circles, with some means of "pointing," more or less accurate, ranging in accuracy from the "sights" on two uprights, as in the surveyor's compass, to the telescope of a transit instrument. Detailed descriptions of such apparatus will be found in books on surveying. Angles on paper are measured by means of a protractor. This is taken up in the following chapter. We are concerned here, for the present, with the *unit-angle*.

(*a*) *The Sexagesimal Angle Measure*, or *Degree Measure*. In America, England, and Germany (and some other countries) the *unit-angle* is *one degree* ($1°$), or $\frac{1}{360}$ of a complete turn. Fractional parts of this unit are expressed, ordinarily, in minutes and seconds; sometimes in decimal parts of a degree. The student is assumed to be familiar, from his work in arithmetic, with the notation for degrees, minutes, and seconds, as in $24°\,13'\,43''$, and with the table:

60 seconds make a minute,
60 minutes make a degree,

as also with the conversions which may arise in connection with the table.

From the prominence of 60 in the table, this system of angle measurement is frequently called the *Sexagesimal System;* more frequently, however, merely the *Degree Measure.*

To the Babylonians is ascribed the credit of originating this system. It is undoubtedly of very great antiquity. The "360 degrees in a circumference" dates back to that remote period when astronomy was in its infancy and men thought the year contained 360 days. Thus the unit-angle, one degree, is but what was thought then the daily step (gradus) of the sun in his apparent annual walk around the ecliptic, his step being each day about twice his angular diameter.

Why "60 minutes make a degree" and "60 seconds make a minute," is lost in obscurity. A fairly good guess, but only a guess, is that the Babylonians had noticed that the radius of a circle, used as a chord, would go just six times around the circle, subtending 60 degrees, thus making 60 a sort of "charmed number."

(*b*) *The Centesimal Angle Measure*, or *the Grade System*, or *French System.* In this system of units, the right angle is divided into 100 equal parts, instead of 90, as in degree measure, and one hundredth of a right angle is the primary unit-angle. This unit is called a *grade.* A minute and a second in this system are, respectively, the hundredth and ten-thousandth of the grade, or,

100 seconds make a minute,
100 minutes make a grade.

In this system the angle is, therefore, always expressed decimally. The expression $19^g.3552$ is 19 grades, 35 minutes, 52 seconds, while $3^g.0407$ is 3 grades, 4 minutes, 7 seconds. Also $12^g.3456$ is $1234'.56$ and $123456''$, the unit being changed by merely moving the decimal point. The decimal system has here, as in all other cases, great advantages in adaptability to calculations.

Can you mention any serious practical difficulty which stands in the way of a change from degree measure to grade measure in America?

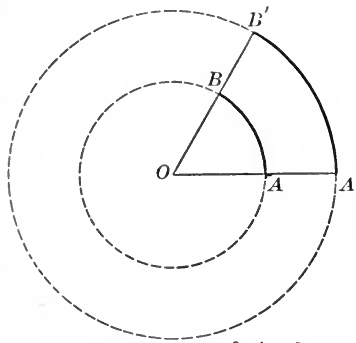
(c) *Radian Measure*, or *Circular Measure*, and *π-Measure*. (1) While the degree and grade are the units for practical mathematics, as in surveying and astronomy, in theoretical mathematics, for a reason which the student is not yet in a position to appreciate (it will appear later), another unit, called the *radian*, is used. The *radian* is the angle subtended at the centre of any circle by an arc of its circumference equal in length to its radius. Since in unequal circles the arcs subtending equal central angles are to each other as the radii of the circles, *the radian is the same for all circles.* Later (§ 49) it will be shown that 3.14 ⋯, or π-radians, make 180°. Frequently π-radians are taken as the unit. This measure might be called the π-measure.

(2) Adjoining is a picture of the radian as an angle, the angle AOB, where

$$\text{arc } AB = \text{radius } OA$$

and where

$$\text{arc } A'B' = \text{radius } OA'.$$



Angle $AOB = 57° 17' 44.8''$
= Radian

FIG. 8.

LABORATORY EXERCISES.

Construct five circles of pasteboard, and of different sizes, with smooth edges. Wrap on each circumference a string equal in length to its radius, and cut out from each circle the corresponding sector. See if these angles fit each other. Get the size of a radian thus fixed in the mind. Measure this angle with the protractor.

Devise a plan for dividing a radian into tenths. Use the radian you have constructed to get in radian measure the values of 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, to the nearest tenth. What relation have your answers to $\pi = 3.14$? Determine, mechanically, how many radians make 360°. How are rims and rods graduated?

(3) *The Radian Protractor (Polar Coördinate Paper).* In Fig. 9 we give a central clipping from the radian polar coördinate paper of Professor B. F. Groat of the University of Minnesota. The divisions are in radians and tenths of a radian.*

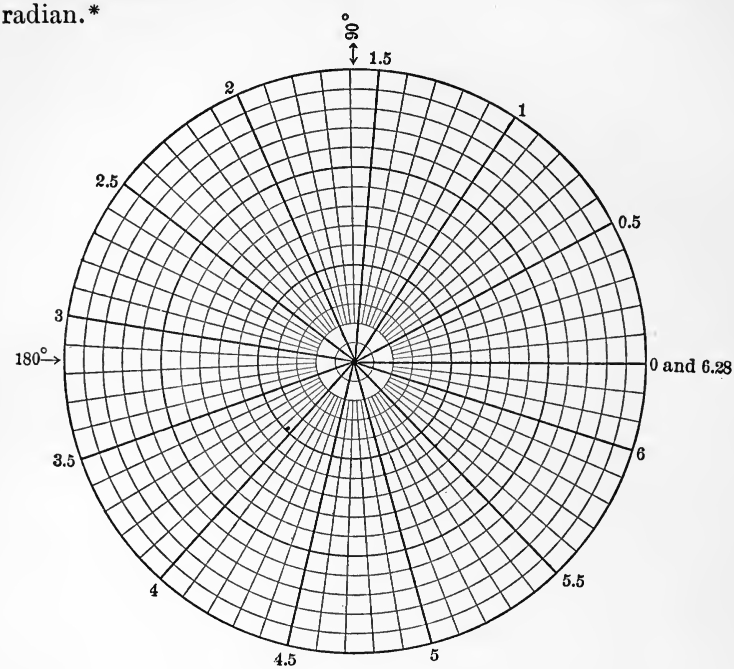


FIG. 9.

EXERCISE.

Set arrows on the diagram at 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°, 300°, 315°, 330°, 360°, and estimate with the eye the corresponding values in radians, tenths, and hundredths. Compare these values with the calculated values obtained by § 48.

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* This paper is recommended for use, with this book, in drawing diagrams to scale (Chap. IV.), locating points in polar coördinates (§ 55 *b*), curve-tracing (Graphs, §§ 102, 132), etc. Here it should be used to estimate with the eye the size in radians of degree-angles, that the student may really know what a radian is. Groat's Polar Coördinate Paper can be supplied by the H. W. Wilson Co., Minneapolis, Minn., in both radian measure and degree measure.

(4) *The radian measure of an angle can also be pictured as a line (arc).* Let AOP be the given angle. Draw about O the unit circle: the portion of this circle-arc included between the initial line and terminal line of the angle is a picture of the radian measure of the angle; that is, the *number* representing the length of the arc BD (between the arrows) is the same as that representing the angle BOD. For

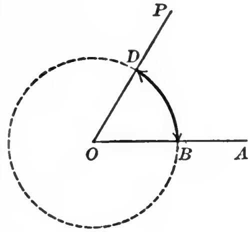

FIG. 10.

$$\frac{\text{arc } BD}{\text{radian's arc}} = \frac{\text{angle } BOD}{\text{radian}} = BOD \text{ in radian measure.}$$

But the radian's arc is the radius, which is here unity.

$$\therefore \text{ arc } BD = \text{radian measure of the angle } BOD.$$

Since the scale-unit is quite arbitrary, so is the size of the adjoining picture.

LABORATORY EXERCISE.

Construct a pasteboard circle of one inch radius (or of one foot radius). Lay out five central angles on the circle. Cut strings equal in length to arcs of these angles. Get the lengths of the strings on the inch rule (or in decimals of a foot, if a circle of one foot radius is used). These lengths are the circular measures of the angles. Measure the angles with the protractor. Reduce the protractor measure to circular measure (§ 48). Compare results with string lengths.

(5) Similar to the preceding is the *surface picture* of the *spherical measure* of a solid angle. Draw about the vertex of the solid angle a sphere of unit radius: the portion of the surface of this sphere included within the solid angle, and bordered by arcs of great circles, is the spherical measure of the solid angle.

(6) It is becoming customary, as is done in this book, to denote angles expressed in radian (circular) measure by the small letters of the Greek alphabet, while the capital letters of the English alphabet are used to denote angles expressed in degree measure. Since many students

taking up trigonometry are not familiar with the Greek letters, we give here the Greek alphabet for reference. No attempt to memorize it is recommended. When the student meets a new Greek letter acquaintance (there are but few of them not met with in mathematical journals and astronomical publications), he can look it up here.

GREEK ALPHABET.

Α	α	alpha	Ν	ν	nu
Β	β	beta	Ξ	ξ	xi
Γ	γ	gamma	Ο	ο	omikron
Δ	δ	delta	Π	π	pi
Ε	ε	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σ	sigma
Η	η	eta	Τ	τ	tau
Θ	θ or ϑ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	φ	phi
Κ	κ	kappa	Χ	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
Μ	μ	mu	Ω	ω	omega

(7) *Circular Arcs in Radian Measure.* Take AB as any given arc on a circle of radius $OA = r$. Draw the unit circle, CDC, concentric with the given circle. Then

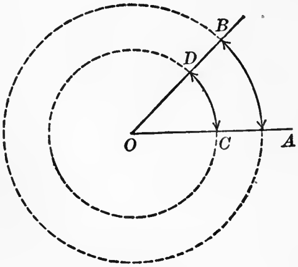

FIG. 11.

$$\frac{\text{arc } AB}{\text{arc } CD} = \frac{\text{radius } OA}{\text{radius } OC} = \frac{\text{radius } OA}{1}.$$

$$\therefore \text{ arc } AB = \text{radius} \times \text{arc } CD.$$

But arc CD is the circular (radian) measure of the angle AOB (as shown in (4), page 67) or θ.

$$\therefore \text{ arc } AB = r \cdot \theta.$$

That is, *the arc of any circle is its radius, r, multiplied by the circular measure, θ, of the central angle which it subtends.*

Also
$$\theta = \frac{\text{arc } AB}{r},$$

that is, *the circular measure of an angle can be expressed as the ratio of a line to a line (the radius), making it of the same char-*

acter as the ratios which form later the chief study of this book. Herein lies the importance of this system of measurement in theoretic mathematics.

LABORATORY EXERCISE.

Construct from pasteboard a circle of one inch radius. Construct five other circles of pasteboard, some larger, some smaller, than the inch circle. Lay out equal central angles on all the circles. Measure with strings the lengths of the corresponding arcs. Get the lengths of the strings (in inches) and the lengths of the radii. Show that for each circle the length of the arc is its radius times the length of the arc on the inch circle.

(8) Similarly, the portion of the surface of any sphere, when the boundary lines are great circles, is r^2 times the spherical measure of the corresponding solid angle.

(9) *Areas of Circular Sectors in Terms of Radian Measure of their Angles.* It is shown in geometry that the area of such a sector is one-half its radius times its arc.

$$\therefore \text{ area} = \tfrac{1}{2} r \cdot r\theta = \tfrac{1}{2} r^2\theta,$$

where θ is the circular measure of the angle.

(10) Similarly, the volume of a spherical sector is,

$$\tfrac{1}{3} r \cdot r^2\phi = \tfrac{1}{3} r^3\phi,$$

where ϕ is the spherical measure of its solid angle.

§ 48. Conversion of Angles from One Measure to Another.

(*a*) *Degrees to Grades and Vice Versa.* One right angle is 90 degrees and is 100 grades. Let x denote the number of degrees in an angle, and y the number of grades in the same angle.

$$\therefore\ y = \tfrac{10}{9} x = x + \tfrac{1}{9} x, \tag{1}$$

$$\text{and } x = \tfrac{9}{10} y = y - \tfrac{1}{10} y. \tag{2}$$

Thus, by (1), *to convert degrees and decimal parts of a degree to grades, add one-ninth of the given number to itself.*

When minutes and seconds are given, they should first be changed to the decimal part of a degree.

And by (2), *to convert grades to degrees and the decimal part of a degree, subtract one-tenth of the given number of grades from itself.*

This is not a matter of very great importance. The teacher may assign a few practice examples, selected at random.

(*b*) *Degrees to Radians and Vice Versa.* This is a matter of considerable importance. Denoting by π the number 3.14159 ⋯, as in geometry, the circumference of a circle of radius r, is $2\pi r$, and any two arcs of the same circle are to each other as the central angles which they subtend.

$$\therefore \frac{\text{radian}}{\text{4 right angles}} = \frac{\text{radius}}{\text{circumference}} = \frac{r}{2\pi r} = \frac{1}{2\pi}.$$

$$\therefore \text{radian} = \frac{\text{4 right angles}}{2\pi} = \frac{180°}{\pi} = \frac{180°}{3.14159 \cdots}.$$

$\therefore$ radian $= 57°.2957 \cdots$ degrees, or, $57°.3$, approximately,
$= 3437'.747 \cdots$
$= 206264''.806 \cdots = 57°\ 17'\ 44''.8$.

$\therefore$ The rule for converting degrees to radians:

(i) *When the angle is in degrees and decimals*, divide by 57.2957 ⋯, cutting according to the accuracy desired.

(ii) *When the angle is expressed in minutes and decimals*, divide by 3437.747 ⋯, cutting according to the accuracy desired.

(iii) *When it is given in seconds*, divide by 206265.

Division by the preceding numbers is the same as multiplication by the following, respectively:

(iv) For degrees, 0.017453 ⋯.
(v) For minutes, 0.000291 ⋯.
(vi) For seconds, 0.00000485 ⋯.

When the relation of degree measure to radians is merely to be *indicated*, the work being left unperformed, we write:

$$x^\circ = \frac{\pi}{180}x;\ x' = \frac{\pi}{10800}x;\ x'' = \frac{\pi}{648000}x.$$

To change radian measure to degree measure, *use the preceding divisors for multipliers and the multipliers for divisors.*

When numbers indicate radians, it is customary, in case special attention is to be called to the fact, to use r or c as an index. Thus 2^r or 2^c would mean two radians, or $114^\circ.59$.

EXERCISES.

1. Express in degrees (decimally) the angles: 3^r; $2^r.5$; $4^r.7$; $4^r.23$. Test the results by working them backwards.

2. Express in radians the angles: 180°; 360°; 90°; 45°; 30°; 60°; 235°; 270°; 191°.37; 424°.76; 5°.39; 17° 46′ 35″; 5° 43′ 26″; 10′; 10″; 1°; 1′; 1″. Test by working backwards.

3. If the radius of a circle is 57.3 feet, how long is the arc whose central angle is 1°? If 3437.7 feet, how long for 1′? If 206265 feet, how long for 1″?

4. If a man 6 feet tall, leaning on the rim of a circle, covers the arc of a radian, what are the circumference and radius of the circle?

5. If an object subtending an angle at the eye of $2\frac{1}{2}'$ is the smallest object that is visible, how far away is a brick building when the horizontal lines of plaster are just discernible?

6. Find the ratio of 13° 24′ 56″ to 4.57 radians.

7. Assuming the radius of the earth as 4000 miles, find the length of a radian on the equator; of a degree; of 1′; of 1″.

8. The circular measure of two of the angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$; what is the third angle in degrees and in radians?

9. The angles of a quadrilateral are in arithmetic progression, and the greatest is double the least; express each angle in radians and the least angle in degrees.

10. The angles of a triangle are in arithmetic progression, and the number of radians in the least angle is to the number of degrees in the mean as 1 : 120; what are the angles in radians?

11. The diameter of the whispering gallery in St. Paul's is 108 feet; how long is the arc of a radian? Of π radians? What is the area of the circular sector whose arc is a radian in this circle?

12. The large hand of the Westminster clock is 11 feet long; how many yards does its moving extremity travel in passing over a radian? What area is described?

13. What is the circumference of the circle on which the radian arc is 3 inches? What is the area of the sector of this circle whose arc is the radius?

14. The apparent diameter of the moon is $30'$ and that of the sun is $32'$; how many moons, placed tangentially to each other on the circumference of a circle, would it take to cover a radian? How many suns? Look in the almanac for the time of sunrise and sunset to-morrow, and determine how many suns, placed tangentially, would cover the day-arc of the sun's motion that day. The night-arc.

15. A railway train is going at the rate of 60 miles an hour on a circular curve whose radius is $\frac{2}{3}$ of a mile; how long does it take to pass over a radian?

16. How long does it take the minute hand of a clock to pass over a radian?

17. If π is taken as $\frac{22}{7}$, what common fraction expresses the number of degrees in a radian? What if π is taken as $\frac{355}{113}$?

18. What is the area of a circular sector whose angle is $\frac{2}{3}^r$, the radius being 10 feet?

19. A radian being $206265''$, at what distance does 1 foot (assumed as a circular arc) subtend an angle of $1''$? Of $1'$? Of $1°$?

20. The circumference of a circle is divided into 5 parts which are in arithmetic progression, and the greatest part is 6 times the least; find in radians the angles subtended at the centre by the parts.

§ 49. Special Angles in Radian Measure (π-Measure).

By (*b*) of the preceding section, $180°$ in radian measure is $\frac{\pi}{180}$ times 180 radians, or π radians.

Thus the ratio of $180°$ to the unit of radian measure is the same as the ratio of the circumference of a circle to its diameter, or $3.14159 \cdots$. Consequently π is the Greek letter representative (see (6), page 67) of the angle which is the equivalent of a half-turn, or $180°$. It is customary to write $180°$, in circular measure, simply as π, instead of π^c, or π^r. Thus:

$$360° = 2\pi \qquad 90° = \frac{\pi}{2} \qquad 120° = \tfrac{2}{3}\pi$$

$$540° = 3\pi \qquad 45° = \frac{\pi}{4} \qquad 270° = \tfrac{3}{2}\pi$$

$$720° = 4\pi \qquad 30° = \frac{\pi}{6} \qquad 240° = \tfrac{4}{3}\pi$$

$$\pm 180° = \pm\pi \qquad \pm 60° = \pm\frac{\pi}{3} \qquad 135° = \tfrac{3}{4}\pi, \text{ etc.}$$

EXERCISES.

1. Express in degrees, in radians, and in terms of π radians, the central angles subtended by the sides of each of the first twelve regular polygons. Express in terms of π radians the angles whose terminals are bisectors and trisectors of the quadrants.

2. Do the same for external angles of the same polygons.

3. What is the central angle of a regular polygon of n sides in terms of π? In radians? In degrees?

4. Show that $A°$ expressed in π-measure is $\frac{A}{180}\pi$.

5. Reduce some numerical angles in degree measure to π-measure.

LABORATORY EXERCISE.

Construct of pasteboard a circle of one foot radius. Lay out the angles noted in § 49. Use strings to measure the arcs, get the lengths of the strings (in feet), and test the above results numerically, in comparison with $\pi = 3.1$.

§ 50. Special Terminals located in Radian Measure (π-Measure).

If n is any positive or negative integer (including zero), the angles $2n\pi$ radians (or, as it is usually written, $2n\pi$) will have their terminals coincident with the initial line OA. Thus if α is any angle whose terminal is given, all the angles $2n\pi + \alpha$ will have that same terminal. The lines of the adjoining diagram indicate the quadrant and semi-quadrant lines. Thus all angles

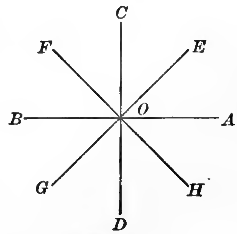

FIG. 12.

with terminal OA have the form $2n\pi$, as 0, 2π, 4π, etc.;
with terminal OB, the form $(2n+1)\pi$, as π, 3π, 5π, etc.;
with terminal OC, the form $(2n+\frac{1}{2})\pi$, or $(4n+1)\frac{\pi}{2}$,

as $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, etc.;

with terminal OD, the form $(2n+\frac{3}{2})\pi$, or $(4n+3)\frac{\pi}{2}$,

as $\frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$, etc.;

with terminal OE, the form $(2n+\frac{1}{4})\pi$, or $(8n+1)\frac{\pi}{4}$,

as $\frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$, etc.;

with terminal OF, the form $(2n+\frac{3}{4})\pi$, or $(8n+3)\frac{\pi}{4}$,

as $\frac{3\pi}{4}, \frac{11\pi}{4}, \frac{19\pi}{4}$, etc.;

with terminal OG, the form $(2n+\frac{5}{4})\pi$, or $(8n+5)\frac{\pi}{4}$,

as $\frac{5\pi}{4}, \frac{13\pi}{4}, \frac{21\pi}{4}$, etc.;

with terminal OH, the form $(2n+\frac{7}{4})\pi$, or $(8n+7)\frac{\pi}{4}$,

as $\frac{7\pi}{4}, \frac{15\pi}{4}, \frac{23\pi}{4}$, etc.

EXERCISES.

1. Show that if n is an integer running from $-\infty$ to $+\infty$, $2n+1$ and $2n-1$ represent the same set of numbers. Find expressions for every alternate even number; every alternate odd number; every fourth even number; every fourth odd number.

2. Consider the first twelve regular polygons as central at O, with one vertex on the initial line, and, denoting by the symbol (VI, 4) the terminal line for the fourth vertex of the hexagon, counting that on the initial line as first and the order being counter-clockwise, write the general expression for all angles whose terminal is that of any specified vertex of any one of these regular figures.

For example, for (VI, 4) it is $(2n+4\times\frac{2}{6})\pi$ or $(12n+4)\frac{2}{6}\pi$.

3. Taking four different angles (two positive and two negative) whose terminal is OE in Fig. 12, write out from the general formula $2n\pi+\alpha$, where α is each of the four angles in turn, four sets of ten angles each, placing them in parallel vertical columns, with the angles obtained by the same value of n in the same horizontal line. Compare the sets and see, if n were given all integer values from positive infinity to negative infinity, whether or not the four sets of angles would be identical. Can one set be brought to coincidence with any other set by sliding? Is the proposition true that if α is *any* angle of a given terminal, the formula $2n\pi+\alpha$ will give all the angles corresponding to this terminal and only these?

4. State for each of the following angles the quadrant to which it belongs:

$\frac{2}{3}\pi$; $\frac{3\pi}{4}$; $\frac{7}{2}\pi$; $\frac{6}{5}\pi$; 9π; $-\frac{3}{4}\pi$; $-\frac{\pi}{4}$; $-\frac{3}{4}\pi$; $-\frac{7}{2}\pi$; $n\pi+\frac{\pi}{6}$; $(2n+1)\pi+\frac{2}{3}\pi$; $(2n-1)\pi-\frac{2}{3}\pi$; $2n\pi+\frac{\pi}{4}$; $2n\pi-\frac{\pi}{4}$; $-\frac{8}{3}\pi$; $-\frac{5}{3}\pi$; $-\frac{5}{2}\pi$.

5. State for each of the following angles the quadrant to which it belongs: $50°$; $120°$; $240°$; $396°$; $-50°$; $-60°$; $-101°$; $-380°$; $-1000°$; $-1080°$; $-90°$; $90°$; $180°$; $-180°$; $270°$; $360°$; $425°$; $370°$; $425°$; $590°$; $-750°$; $-39°$.

6. Give for each of the angles of Exs. 4 and 5, two other angles which have the same terminal, expressing those for Ex. 4 in radian measure.

7. Give the general expression in degrees for the difference of all angles whose terminals are coincident; the general expression in circular measure.

8. Give the general expression in degree measure for all angles whose terminals are symmetric to the vertical with the terminal of $30°$; the general expression in circular measure.

9. Give in degree measure the general expression for all angles whose terminals coincide with that of $45°$; $-45°$.

10. Give in degree measure the general expressions for all angles which have for terminals the border-lines of the quadrants; the general expressions in circular measure in terms of π.

11. Give in degree measure all angles whose terminals are coincident with that of $A°$; symmetric to that of $A°$, with the vertical; with the horizontal.

12. Give in all three systems of angle measure two angles whose terminals are symmetric to the initial line; to the $90°$ line; to the $180°$ line; to the $270°$ line; to the lines bisecting the quadrants.

13. Give the general expressions for some selected three of the angles of Ex. 2.

14. What are the expressions for the length of a circular arc and for the area of the corresponding sector? Selecting two angles involving degrees, minutes, and seconds, calculate the lengths of the corresponding arc and areas of the corresponding sectors, on a circle of 10 inch radius.

15. Find the length of the arc subtending an angle of 2^r on a circle of 8 inch radius. Express in terms of radians the corresponding lengths and areas for 30°, 45°, 60°, 120°, 135°, 180°, 210°.

16. If the radius of a circle is 100 feet, what angle in radians does a 10 foot arc subtend? Express the same result in terms of π radians. Make up and solve four other examples like this.

17. Find in degree measure, grade measure, and radian measure the angle between the hour and minute hands of a clock at 20 minutes of 6; at 2:30.

18. The angle of a circular sector is $22\frac{1}{2}°$ and the diameter of the circle is 10 feet; what is the area of the sector and the length of its arc?

19. The area of a sector of a circle of unit radius is 10 square feet; what is its angle in all three measures?

20. A strip of paper a mile long is rolled tightly into a circular cylinder. The paper is 0.001 inch thick; what is the radius of the cylinder and the volume of the cylindrical sector whose angle is a radian?

21. Three circles, each of 10 inch radius, are tangent to each other; find the length of the arcs between the points of tangency, the area of the sectors and the area bounded by the circular arcs, assuming $\pi = \frac{22}{7}$.

22. If a geographical mile on the earth's surface subtends $1'$ at the centre of the earth (radius 3960 miles), how far off is the sun if the earth's radius subtends at the centre of the sun an angle of $8.76''$?

23. The moon subtends an angle of about $30'$ from the centre of the earth, and is about 60 earth's radii distant; what is its diameter, approximately, and what angle does the earth subtend at the moon?

24. The radius of a circle is 8 feet; how many radians does an arc of 13.2 feet subtend?

25. If the number of degrees in one angle of a triangle equals the number of grades in another and the number of radians in the third, what is that number?

26. Show that $n\pi + \frac{\pi}{2}$ will give angles whose terminals are either on the upright vertical or downright vertical, n being any positive or negative integer.

CHAPTER IV.

CONSTRUCTION OF ANGLES AND OF STRAIGHT-LINE DIAGRAMS TO SCALE, AND THE MEASUREMENT OF ANGLES.

[NOTE TO THE TEACHER. — *It is advised that in solutions of examples diagrams to scale be made and the ungiven parts be measured and compared with the calculated parts as a test. The diagram will serve as a check on numerical solutions. Even a rough free-hand sketch will serve often as a check on results, giving a "common sense" check, saving one from writing* 139 *for* 1.39, *and the like. Drawing to scale is valuable exercise in itself, in preparation for "graphic solutions," so useful in all engineering work.*]

§ 51. The Protractor (in Degree Measure and in Radian Measure).

The protractor is an instrument for laying out angles of a given size on a diagram and for determining the size of the angles of a given diagram.*

(*a*) *To lay out at a Given Point on a Straight Line a Given Angle.* Place the straight edge of the protractor on the given line and the middle point of the straight edge at the angle-vertex. Then with a sharp pencil, or with a pin, make a dot opposite the given angle on the circular rim of the protractor. Connect the dot and the angle-centre by a straight line.

(*b*) *How to measure a given angle with the protractor* will occur to the student without direction.

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* A protractor (in degree measure) and a scale of equal parts in inches and fractions of an inch should be used in connection with this book. Groat's Coördinate Paper in degree measure and in radian measure furnishes a cheap protractor, and can be used to fix in the mind the ability to *estimate* angles as well as measure them. Protractors can be bought made of paper, of horn, of brass, etc. See Johnson's "Surveying," or the catalogues of the instrument makers, as that of Dietzgen, New York.

LABORATORY EXERCISES.

1. Make radial clippings, from Groat's Coördinate Paper, of angles of various sizes, and study them until you feel able to estimate fairly well, in degree measure, radian measure, and π-measure, the size of any angle selected at random.

2. Construct some angles at random. Guess at their size. Write down your guess. Measure the angles and test yourself as a guesser on angles. Repeat the process until you feel like congratulating yourself.

3. Construct angles of special sizes until you can readily estimate by eye the number of degrees and the number of radians in a given angle.

4. Construct the angles of the first ten regular polygons, beginning with a triangle. Construct a radian and a degree.

5. What angle is subtended at the end of a pencil line of average width by the width at different points along the line?

6. What angle is subtended by the opposite edges of a chalk-spot just visible on a blackboard? What angle is subtended by the distance between the pairs of a double star just apparently double? *

§ 52. Drawing to Scale.

A straight-line diagram to scale is one similar to the object represented, that is, having all its angles the corresponding angles of the object, and any pair of its sides bearing to each other the same ratio as do the corresponding lines of the object. Architects' plans of buildings, surveyors' plots of fields, are familiar examples. The ratio of any line of the drawing to the corresponding line of the object is the *scale* of the diagram. This scale should be indicated on the drawing. This may be done by noting on the map, "scale 1 to 10," or "scale 1 inch to 1 mile,"

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* This angle is, for most very good eyes, about $2\frac{1}{2}'$. You can test your eye by seeing at which corner of the parallelogram under the bright star Vega there is a double star. The stars are about $2\frac{1}{2}'$ apart, and but few students can correctly report the corner. This should convince the student that when he calculates seconds, tenths of seconds, and even hundredths of seconds, in angles, as many text-books do, he is splitting hairs very fine. Visibility varies with the eye and with contrast in color of object and background, running from about $30''$ to $2\frac{1}{2}'$. Mr. F. L. O. Wadsworth has shown that much of the "fine measurement" with instruments is merely an optical delusion.

as the case may be. Look this matter up on some railway map, or map in a geography, and report as to how the scale was indicated. The scale will, of course, depend on the size of the object as compared with the size of the drawing desired. It may be an inch to many miles, or an inch to a few feet, according to the amount of detail desired. The drawing may be on reduced scale, as railway maps, or on enlarged scale, as in drawings of microscopic objects.

EXERCISES.

(Using protractor and straight edge.)

1. Draw to scale a triangle, given two sides and the included angle, the sides being 217, 250, angle 63° 15′, the longest side of the diagram not to be over 5 inches. Measure the other two angles. Test.

2. Given two sides and the angle opposite one of them. Sides, 240, 224; angle, 47° 30′, opposite 224. How many triangles are possible? What change would be necessary in one of the given sides to make just one triangle possible? Can the sides be so given as to make the triangle impossible? How many triangles are possible when the angle given is to be opposite the larger of the two sides?

3. Given two angles and the included side. Angles, 30°, 85° 30′; side, 10 feet. Make the diagram to scale.

4. Given three sides, 5.2, 8.2, 9.3. Make the diagram to scale.

5. Given the three angles. How many triangles?

6. In all the preceding cases, use the scale and protractor to determine the ungiven parts. Do you know of any way by which you can test your results?

7. Plot to scale a right-angled triangle of which one side is 20 miles, and one angle 57° 30′.

8. Plot to scale a right-angled triangle, one of whose angles is 76° 30′, and whose hypothenuse is 500 feet.

9. Plot to scale a right-angled triangle whose two legs are 15 and 12.

10. Measure the ungiven parts in the preceding right-angled triangles, and from these measurements and the scale calculate their values. Test the same by other calculations.

11. Look up the map of your state, in some geography or wall map, and calculate from the given scale of the map the air-line distances between its five largest cities.

12. Calculate to three decimal places the ratio of the sides of each of the right-angled triangles in Exs. 7, 8, 9, and the ratio of each side to the hypothenuse.

13. Plot a four-sided field whose sides are, in feet, 500, 240, 120, 180, and whose angles are, following the order of sides, 80°, 70°, 110°, A. How large is A?

14. Draw the diagonals in Ex. 13, and calculate from the scale their lengths. Measure in the diagram all angles not given, and test them by the three angle tests which hold, together with the corner tests.

§ 53. The Two Half Terminals, Three Third Terminals, etc.

If the angle A is the principal angle of a given terminal, this terminal is also located by the angles,

$$A, \quad A \pm 360°, \quad A \pm 720°, \quad \text{etc.}$$

Thus the half angles corresponding to a given terminal are

$$\frac{A}{2}, \quad \frac{A}{2} \pm 180°, \quad \frac{A}{2} \pm 360°, \quad \text{etc.}$$

These locate two, and only two, terminals. These terminals are 180° apart.

Similarly, one-third of the angle locating a given terminal will fall under the forms,

$$\frac{A}{3}, \quad \frac{A}{3} \pm 120°, \quad \frac{A}{3} \pm 240°, \quad \text{etc.}$$

These locate three, and only three, terminals, 120° apart.

Similarly, there are four terminals for one-fourth of the angle corresponding to a given terminal, and so on.

EXERCISES.

1. Use the protractor to draw the terminals for the half angles corresponding to the initial line.

2. Do the same for the terminals corresponding to 90°, 180°, 270°.

3. Show that if one vertex of an equilateral triangle is on the initial line, the other vertices are on the terminals corresponding to one-third of the angle of the initial line.

4. Show the corresponding proposition for the square, regular pentagon, regular hexagon, etc.

5. If one vertex of an equilateral triangle is on the terminal of 180°, show that the other vertices are on the terminals of the third of the general angle corresponding to the terminal of 180°.

6. Locate the terminals for one-quarter of the general angle corresponding to the terminal of 180°; for one-fifth; for one-sixth.

NOTE. — Later it will be shown that the geometric operations above correspond to the algebraic solution of the equations,

$$x^2 = \pm 1, \quad x^3 = \pm 1, \quad x^4 = \pm 1, \quad x^5 = \pm 1, \text{ etc.,}$$

by De Moivre's theorem.

§ 54. Instruments for Measuring Angles.

It is advisable that along with the study of trigonometry the student be given field-work in measuring angles with the surveyor's compass, with a sextant, and with a transit. In no other way can he learn the limits of accuracy of measurement, and the folly of calculating hundredths of seconds when he ought not to do it.

EXERCISES.

1. Investigate and report on the following topics: (*a*) The accuracy of the outfit and processes of a county surveyor in your state. (*b*) The same for a city surveyor. (*c*) The same for a steam railroad surveyor. (*d*) The same for the surveyor of an electric railway. (*e*) The means of measuring small angles in engineering and astronomy and the smallest readings in different sorts of surveying. (*f*) Measuring base-lines in geodetic surveys. (*g*) Limits of accuracy in government land-surveys.

2. A city lot, about 29 feet by 290 feet, was sold at $1000 a square foot, on the measurements 29.3 feet by 293.7 feet. Show that, possibly, the owner lost about $14,600, by not having the short side measured with the same relative accuracy as the long side. Which would entail the larger loss, dropping .3 on the short side or .7 on the long side?

3. Show that $\dfrac{\log_e a}{\log_e b} = \log_b a$. Show that $a^x = e^{x \log_e a}$, and from the series for e^x (§ 8) deduce a series for a^x.

4. Show that if a set of numbers form a G. P., their logarithms form an A. P. Show that the arithmetic mean of two logarithms is the logarithm of the geometric mean of the two numbers corresponding to the given logarithms. Show that the logarithm of any number can be calculated to any desired degree of accuracy by the continued insertion of arithmetic and geometric means. Calculate $\log_{10} 2$ to four figures in this way.

CHAPTER V.

THE SINE, ANTI-SINE, RECIPROCAL SINE (COSECANT), AND COVERSED SINE OF AN ANGLE.

§ 55. Coördinates.

(*a*) *Rectangular Coördinates and Rectangular Coördinate Paper.* Draw through O a pair of mutually perpendicular lines, giving the quadrants as explained in § 45. O is called the *origin of coördinates*, also the *origin*, also the *pole*. The border lines of the quadrants are called the *axes of coördinates*. And for distinction, the old initial line (right and left) is called the *axis of abscissas*, and the vertical line the *axis of ordinates*. From any point, P, in the terminal OP, of a given angle AOP, drop a perpendicular PM, on the initial line OA (produced if necessary). Then OM is called the *abscissa* of the point P, and is indicated by the letter x; MP, its *ordinate* or y; and OP its *modulus*, and also its *radius vector* (we shall use the word modulus and the letter r). Notice particularly *the order in which these lengths are written:* It is OM, not MO; it is MP, not PM; it is OP, not PO. This is of primary importance, for reversal of the order in which a length is written, when sign is considered, is a reversal of direction and equivalent to a change of sign. As to sign, it is now universally agreed that all lines taken upward, no matter where starting, are +;

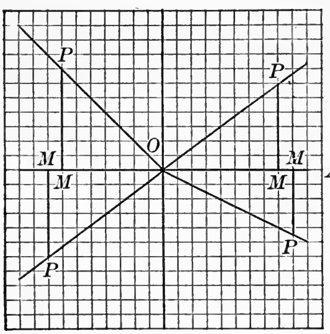


FIG. 13.

downward, –; to the right, +; to the left, –. Where no sign is stated, + is understood. A point in the plane of the axes is located by giving its coördinates in a parenthesis, as (2, 3), the abscissa being in the lead. The point (2, 3) is two scale units to the right of the vertical axis and three scale units above the horizontal axis. To reach it, go two units from *O* along the axis of abscissas to the right; then three units vertically up from the point so reached on the abscissa axis. Similarly, to locate (– 2, 3), go two units along the abscissa axis to the left from *O*, and then three units vertically up from the point so reached. How are (2, – 3), and (– 2, – 3) located?

The teacher may select at random sufficient practice examples to make sure that the matter is understood by the class.

In quadrants I and II, ordinates are +; in III, IV, –. In quadrants IV, I, abscissas are +; in II, III, –.

The modulus *OP*, of the point *P*, is always counted plus *when it lies along the terminal of the given angle.* For all points in the *opposite terminal* (the terminal continued backwards through the origin), *it is minus.* Thus, while *OP* might be counted plus for a certain angle, this same line *OP* would be minus for the angle which is 180° more than the given angle, or any odd integral multiple of 180° more than the given angle. Even when the terminal falls on the left-hand horizontal line or on the downward vertical, the modulus for any point on it is counted plus, while in the first case the ordinate is zero and the abscissa negative; and in the second, the reverse. Any measurement *along the terminal, away from the origin, is, by agreement, plus always, no matter whether the angle is plus or minus, and no matter what its size.*

Paper like that on which Fig. 13 is constructed is called *rectangular coördinate paper.* It can be bought in small sheets or by the yard, and of various divisions to the inch (or centimeter). That 10 × 10 to the inch is convenient for locating points in a drawing.

(*b*) *Polar Coördinates and Polar Coördinate Paper.* A point on a plane is also located by giving the length of its modulus and the angle which the modulus makes with the initial line. These are given in a parenthesis as (r, θ), the modulus being set first. By the previous agreement as to sign of the modulus, a point indicated by (2, 30°) would be two units from the origin along the terminal of 30°, while (– 2, 30°) would be two units along the opposite terminal from the origin. Where will (2, – 30°) and (– 2, – 30°) fall ?

EXERCISES.

1. Calling the small divisions on radii units, locate on Fig. 14 the points (4, 5°), (5, 5°), (6, 5°), (4, 20°), (4, 25°), (4, 50°), (4, 90°), (– 5, 5°), (– 5, 50°), (– 5, 180°), (– 5, 275°), (– 8, – 20°), (– 8, – 80°), (– 8, – 180°), (– 8, – 270°), (– 8, 3000°).

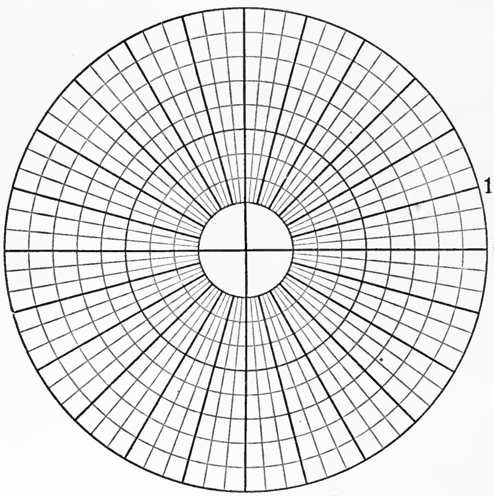


FIG. 14.*

2. Similarly locate on Fig. 15 the following points, the second figure in each parenthesis being radian measure: (6, 3), (7, 3), (8, 3), (8, 4.3), (9, 5.5), (6, 6.3), (7, – 0.5), (4, – 0.6), (4, 3.14159), $\left(4, \frac{\pi}{2}\right)$, $(4, -\pi)$, $(-4, -\pi)$, $\left(-4, -\frac{\pi}{3}\right)$, $\left(-4, -\frac{\pi}{4}\right)$, (– 4, – 3), (– 4, – 6).

3. Construct to scale the following points in rectangular coördinates, using, preferably, rectangular coördinate paper: (2, 3), (2, – 3), (– 2, 3),

* Figure 14 is in 5° angle-spaces; Fig. 15 is in tenths of radians. The divisions on the radius are in each case arbitrary.

(– 2, – 3), (7, 8), (15, – 3), (– 5, 4), (– 9, – 3), (3, – 3), (3, – 1), (150, 160), (900, –800), (3.3, 4.1), (–3.3, – 4.1), (5.6, – 7.3).

4. Where are (0, 0), (0, 1), (– 1, 0), (1, 0), (1, 0), (0, – 1)?

5. Use scale and protractor, or Groat's coördinate paper, to construct the following points: (2, 30°), (2, – 30°), (– 2, 30°), (– 2, – 30°), (2, π), (3, 60°), (3, – 60°), (– 3, – 45°), (3, 45°), (5, – π), $\left(-5, \frac{-\pi}{6}\right)$, $\left(5, \frac{-\pi}{3}\right)$, (150, 2 π), $\left(150, \frac{-3\pi}{2}\right)$, (2, 1^r), (– 2, 3^r), (– 2, – 1^r), (2, –3^r).

6. In what quadrant is a point whose abscissa and ordinate are both plus? Both minus? Abscissa plus, ordinate minus? Ordinate plus, abscissa minus?

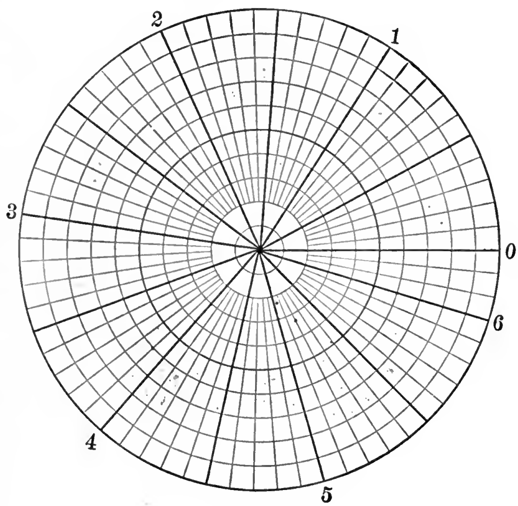


FIG. 15.

7. Use Groat's coördinate paper, taking the small divisions on the radii as units, to locate to the extent of the sheet the points (radian measure for angles): (0, 0), (1, 0.1), (2, 0.2), (3, 0.3), (4, 0.4), etc. Then join all these points by a smooth curve. This will represent a particular form of the Spiral of Archimedes, namely,

$$r = 10\,\theta,$$

the general curve being $r = a\theta$. Note that in each point above r is 10 times θ, unit for unit.

8. Use rectangular coördinate paper, taking the small divisions as units, to locate the points: (0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), etc., (– 1, –1), (– 2, – 2), (– 3, – 3), (– 4, – 4), etc. Join these points. What then is $y = x$?

9. Use rectangular coördinate paper to locate (0, 2), (1, 5), (2, 8), (3, 11), (4, 14), where y is always $3x + 2$. Join these points. What do you get? What are some negative points on the same figure? What, then, does $y = 3x + 2$ represent? What does $y = mx + b$ represent, if m and b are fixed numbers?

10. If a point is located by $(r, A°)$, what variety of form may r and $A°$ take to locate one and the same point?

§ 56. The Sine of an Angle.

The sine of an angle is the ratio of the ordinate of any point on the terminal of the angle to the modulus of the same point, or, in Fig. 16,

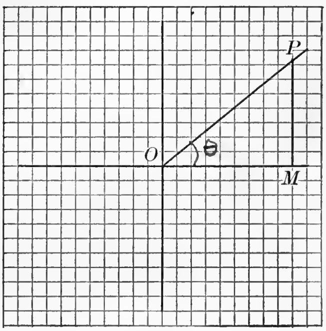


FIG. 16.

$$\text{Sine of } \theta = \frac{MP}{OP}.$$

This is known as the *ratio-definition of the sine*, or as *Hassler's definition*, to distinguish it from the so-called *line-definition*. See § 67.

It must be considered more as a working definition than as one of theoretic exactness. Sines are not calculated by laying out the angle with a protractor, measuring ordinate and modulus and then dividing the former by the latter. Later it will be found that connected with every angle θ, *expressed in circular measure*, is a certain *number*, the sine of the angle, which can be calculated for any given value of θ by the following series, to any desired degree of exactness:

$$\text{Sin } \theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} \ldots, \text{ etc., to infinity.}$$

This series gives the correct theoretic definition of the sine of the angle θ. The sine of an angle is thus a number connected with the angle in a specific manner, as indicated by the above series, and calculable when the angle is given.

These *numbers* have been calculated, approximately, and tabulated for certain angles, forming what is called a Table of Natural Sines.

All that Hassler's definition means is that if the ordinate were measured with exactness and the modulus with exactness, and then the division of the corresponding numbers carried out as indicated, there would be obtained a number agreeing with that obtained from the series for the same angle.

Calculations of the sine from the series give the sine in the form of a decimal fraction. Since the series has an infinite number of terms, the sine of an angle can be calculated, as a rule, only approximately. The tables give the sines to 3, 4, 5, 6, 7, 10 places. A table of sines giving them to three places of decimals is called a three-place table; one giving them to four places of decimals, a four-place table, and so on.

Ordinates and moduli and angles can be measured only approximately. Consequently, if the sine of a measured angle were obtained from Hassler's definition and the result compared with the tables, the closeness of agreement would depend upon the degree of accuracy with which the measurements of lines and angle had been carried out.

The practical value of Hassler's definition is, therefore, this: It indicates that if the ordinate and modulus of an angle in a diagram, or in the field, are measured, and their ratio obtained and compared with the tables, the tables will show an angle (one of many) which is, approximately, the angle of the diagram, or one from which that angle can be determined, approximately. The closeness of the approximation will depend upon the accuracy with which the measurements are made.

The sine of an angle is thus a number connected with the angle and having under Hassler's definition a threefold practical value:

(1) *When ordinate and modulus are given, it indicates the angle.*

(2) *When the angle and modulus are given, it is the number which multiplied by the modulus will give the ordinate.*

(3) *When the angle and ordinate are given, it is the number by which the ordinate is divided to give the modulus.*

These three uses to which the sine may be put constitute its claim to importance as a calculation device and give to Hassler's definition its value.

LABORATORY EXERCISES.

That the student may get firmly fixed in his mind the definition of the sine as a ratio, the teacher may here show him how to look up in the tables the sines of angles in degrees (omitting minutes and seconds). The student may then lay out to scale some five such selected angles, measure the corresponding ordinate and modulus, divide (to one or to two figures), and compare with the tables.

He may also take the modulus one inch long and show that the ordinate (in decimals of an inch) is the table-sine. Do the same with a modulus ten inches long. Make ten such constructions and measure.

EXERCISES.

(Make diagrams to scale.)

1. If the modulus is 5 and the ordinate 3, what is the sine of the corresponding angle? Calculate the sine to two decimal places and get the angle when the modulus is 34 and the ordinate 25; to three places with ordinate 25.3 and modulus 34.7; to four places with ordinate 25.37 and modulus 34.72.

2. The side of a square is $\sqrt{2}$ inches; calculate to four decimal places the sine of the angle which the diagonal makes with the side, and compare with the tables. Show that the sine of the angle which the diagonal makes with the side is independent of the length of the side.

3. Assuming the side of an equilateral triangle as 2, calculate the sine of 60° to four decimal places. Do the same when the side is a.

4. By drawing an equilateral triangle whose side is 2, so that its median is horizontal and the initial line, calculate to four decimal places the sine of 30°. Compare the result with the table-value.

5. Give the sines of 30°, 45°, 60°, in the form of radicals.

6. The modulus for an angle whose sine is 0.35 is 27, what is the ordinate? How high up on the side of a house will a ladder 50 feet long reach when tilted at an angle of 30° with the ground? 45°? 60°? How high when the sine of the angle of tilt is 0.42? Give all these results to only two figures.

7. The ordinate for an angle whose sine is 0.24 is 53, what is the modulus? If the roof of a house is inclined to the horizontal at an angle of 30° and the ridge-pole is 15 feet, how long are the rafters? How long when the angle is 45°? 60°? How long when the sine of the angle is 0.32? Give results to only two figures.

8. The student will devise seven other examples like the above, solve them and hand them in. Let five of them be practical examples (roofs, etc.) within the student's experience.

§ 57. The sine has many interesting and important properties, aside from its use in calculations. All these can be deduced directly from the series definition, without reference to Hassler's definition. In fact, a text-book could be based on the series definition. However, for that the times seem not yet ripe.

§ 58. Hassler's definition holds for all positions of the terminal, no matter in which quadrant it falls.

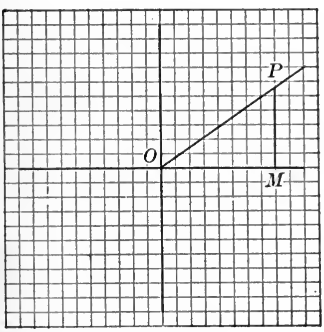


FIG. 17.

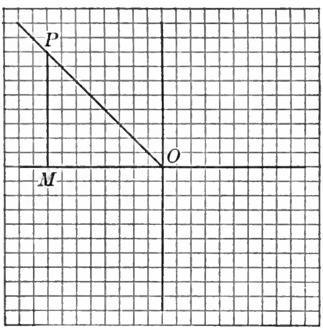


FIG. 18.

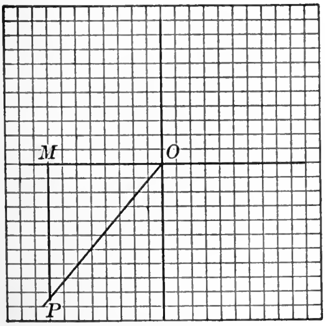


FIG. 19.

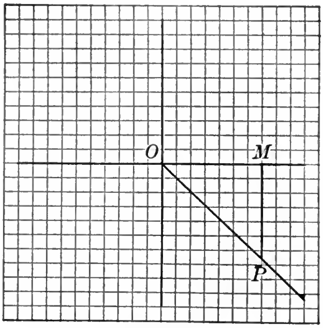


FIG. 20.

In each of the preceding diagrams (Figs. 17–20), corresponding to the four quadrants, if θ is any one of the angles corresponding to any one of the indicated terminals,

$$\text{Sine of } \theta = \frac{\text{ordinate } MP}{\text{modulus } OP} = \frac{y}{r}.$$

It is customary to contract sine of θ to $\sin\theta$. It is read, "sine θ," omitting "of."

§ 59. That the value of the sine is independent of the length of the modulus, is apparent at once from the accompanying diagram (Fig. 21).

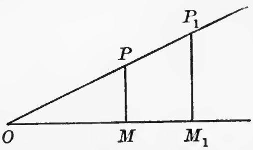


FIG. 21.

The triangles OMP, $O_1M_1P_1$ are similar.

$$\therefore \frac{MP}{OP} = \frac{M_1P_1}{OP_1}.$$

The sine of an angle is thus a ratio, or number, whose value depends only on the position of the terminal line as related to the initial line. It refers not so much to a specific angle as to a specific position of the terminal.

LABORATORY EXERCISE.

Lay out an angle of 18° with the protractor. Select five different moduli (one of them an inch). Measure the moduli and ordinates. Divide, compare results with each other and with the table. Carry out the divisions only so far as the means of measurement used justify.

All ratios are numbers. They are sometimes called pure numbers to distinguish them from the so-called denominate numbers. Some special remarks on ratios may be appropriate here.

§ 60. General Remarks on Ratios. The Sine a Continuum.

Ratios among real numbers are of three kinds:

(*a*) Positive or negative integers.
(*b*) Positive or negative common fractions.
(*c*) Unending, non-repeating decimals, positive or negative.

Ratios of the second kind, (*b*), are of two classes:

(i) Ending decimals, like $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, etc.

(ii) Repeating decimals, like $\frac{1}{3} = 0.333 \cdots$, $\frac{1}{6} = 0.1666 \cdots$.

All common fractions whose denominators are 2 or 5, or powers of 2, or of 5, or of 2 and 5, will form ending decimals. For example:

$$\tfrac{1}{2} = 0.5;\ \tfrac{1}{5} = 0.2;\ \tfrac{1}{4} = 0.25;\ \tfrac{1}{25} = 0.04, \text{ etc.}$$

All other common fractions, when expressed decimally, give repeating decimals. This will become evident by considering some special example, as $\frac{251}{7}$, and carrying out the division as follows, setting the remainders in the upper horizontal line:

```
   46.451326
7)251.000000
   35.857142
```

The remainder, when the zeros first come into use, is 6. Since, now, in dividing by 7, there can be only six different remainders, and since the division is to be non-terminating, this special remainder, 6, appearing in connection with the use of the added zeros, must, sooner or later, reappear, as shown in the actual division. As soon as this remainder reappears, there will be a repetition of the quotient figures obtained since the 6 first appeared as remainder. What has happened here with 7 is readily seen to be general. For with any other denominator, as, say, 213, there can be only 212 different remainders for non-terminating division, and that one present when the zeros are first used must again reappear, giving a repetition of quotient figures.

EXERCISES.

Show that the following fractions form repeating decimals:

$$\tfrac{2}{3};\ \tfrac{17}{7};\ \tfrac{2}{9};\ \tfrac{151}{11};\ \tfrac{234}{13}.$$

Examples of numbers of the third class, (*c*), the unending non-repeating decimals, are:

$\pi = 3.14159 \cdots$, the ratio of the circumference of a circle to its diameter.

$e = 2.7182818 \cdots$, the base of the Napierian logarithms.

$\sqrt{2} = 1.41 \cdots$,

and all other real numbers which do not belong to classes (*a*) and (*b*). Originally, only the roots (of rational numbers) which could not be extracted exactly were called the *irrational* numbers. Since, however, there is no essential difference between such roots and numbers like π, e, in so far as being neither ending nor repeating is concerned, all real numbers not of classes (*a*), (*b*), may be called *irrational.*

If the numbers of (*a*), the positive and negative integers, are plotted on a straight line, as in algebra, they are represented by separate points, a unit distance apart. The numbers of (*b*) when plotted on the same straight line help to fill in the points between those occupied by the numbers of (*a*); but (*a*) and (*b*) do not occupy completely all the points on the straight line. Class (*c*) comes in to take up the remaining points. The complete set of numbers of classes (*a*), (*b*), (*c*), occupy the line completely, so that each number occupies a point, and each point represents a number, passage from point to point along the line representing the growth of one number into another. Such a set of numbers is called a *continuum*, in contradistinction to the *discreet* set of numbers, the units, or the units and common fractions, or any set, or pairs of sets, of the numbers (*a*), (*b*), (*c*).

The sines of angles take up, as will be seen later, but a very limited portion of this continuum, namely, that from $+1$ to -1, these limits included; but *they occupy this limited region completely, forming themselves a continuum* (§ 69, *m*).

§ 61. The Sign of the Sine.

Since the modulus is always plus along the terminal of an angle (§ 55), the *sign of the sine is that of the ordinate.*

∴ All angles of whatever size or sign, with terminals in quadrants I, II, have positive sines; with terminals in quadrants III, IV, negative sines.

Quadrant	I	II	III	IV
Sine	+	+	−	−

FIG. 22.

EXERCISES.

(Make diagrams to scale.)

1. What signs have the sines of the following angles (degrees): 30, − 30, 135, − 135, 185, − 185, 275, − 275, 361, − 361, 455, − 455 570, − 570, 650, − 650, 1700, − 1700.

2. Determine the signs of the sines of the following angles;

$\frac{\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}, \frac{4\pi}{3}, -\frac{4\pi}{3}, \frac{7\pi}{4}, -\frac{7\pi}{4}, \frac{7\pi}{3}, -\frac{7\pi}{3}, \frac{\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{5}, -\frac{7\pi}{5}, \frac{17\pi}{6}, -\frac{17\pi}{6}.$

§ 62. Angles with the Same Sine.

The sine for the terminal position OP (Fig. 23) is $\frac{MP}{OP}$.

∴ (i) All angles with the same terminal have the same sine.

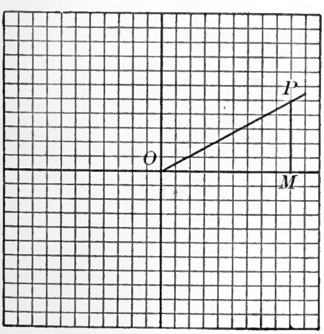


FIG. 23.

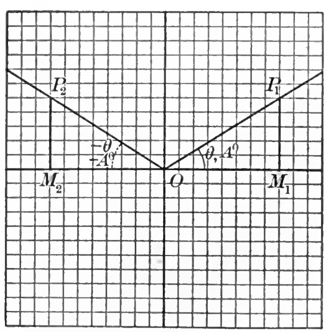


FIG. 24.

Taking θ (circular measure), $A°$ (Fig. 24) as the principal angle of OP_1 (see § 46), the angles of this terminal are

$$2n\pi + \theta, \text{ or } 2n \cdot 180° + A°.$$

$$\therefore \sin(2n\pi + \theta) = \sin\theta,$$

or,
$$\sin(2n \cdot 180° + A°) = \sin A°,$$

where n is any positive or negative integer.

(ii) Terminals symmetrically inclined to the vertical, as OP_1, OP_2 (Fig. 24), or as OP_3, OP_4 (Fig. 25), belong to angles with the same sine, since for such angles equal terminals give equal ordinates.

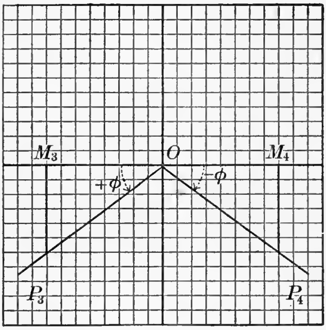


FIG. 25.

The angles of the terminal OM_1 (Fig. 24) are $2n\pi$ or $2n \cdot 180°$, and those of OM_2 are $(2n+1)\pi$, or $(2n+1) \cdot 180°$. The position OP_1 (Fig. 24) is best reached by going forward the principal angle θ, or $A°$, from the terminal OM_1; the position OP_2, by going backward the same angle from the position OM_2.

$$\therefore \sin\{(2n+1)\pi - \theta\} = \sin\theta,$$

or,
$$\sin\{(2n+1)180° - A°\} = \sin A°.$$

In particular,
$$\sin(\pi - \theta) = \sin\theta,$$

and
$$\sin(180° - A°) = \sin A°.$$

(iii) Any angle of terminal OP_2 (Fig. 24) has the same sine as any angle of terminal OP_1.

$$\therefore \sin\{(2n+1)\pi - \theta\} = \sin(2m\pi + \theta),$$

or,
$$\sin\{(2n+1)180° - A°\} = \sin(2m \cdot 180° + A°),$$

where m, n are any positive or negative integers.

(iv) Thus, in general, all angles having the same sine as θ or $A°$, fall under the forms

$$\left\{\begin{matrix} 2n\pi + \theta \\ (2n+1)\pi - \theta \end{matrix}\right\} \text{ or } \left\{\begin{matrix} 2n \cdot 180° + A° \\ (2n+1)180° - A° \end{matrix}\right\}.$$

Since $(-1)^m$ is $+1$ when m is even and -1 when m is odd, the two preceding sets of formulas are both included in these :

$$m\pi + (-1)^m\theta,$$
$$m \cdot 180° + (-1)^m \cdot A°,$$

where m is any positive or negative integer.

(v) The general result of (iv) is so important that a verbal statement may also be given:

If any given angle in degree measure is added to an even multiple of 180°, *or subtracted from an odd multiple of* 180°, *an angle is obtained having the same sine as the given angle.*

The student may give the corresponding statement for angles in radian measure.

(vi) While in the preceding formulas any angle of the given terminal may be taken, it is customary to take the *principal* angle of the terminal. For example, while 187° and $-173°$ locate the same terminal, one would use as the general expression for all angles whose sine is that of this terminal,

$$m \cdot 180 - (-1)^m 173°,$$

rather than $$m \cdot 180 + (-1)^m 187°,$$

though they both give the same set of values, when m is taken from $-\infty$ to $+\infty$.

EXERCISES.

1. Find twelve angles, six being plus and six minus, having the same sine as 30°. In which quadrants will these angles lie? Give the diagram.

2. Do the same for $-60°$, and for $\frac{\pi}{4}$ and $\frac{-\pi}{4}$.

3. Select sufficient examples of the same character to fix the formula in mind. Stop when you know it thoroughly.

4. From Fig. 25, show $\sin(m\pi + (-1)^m\phi) = \sin\phi$.

5. Show that $\sin(a^x \cdot b^x) = \sin(m\pi + (-1)^m e^{x \log_e ab})$

and that $$\sin\frac{a^x}{b^x} = \sin(m\pi + (-1)^m e^{x \log_e \frac{a}{b}}).$$

6. Find the angles which satisfy the equation $\sin 9\theta = \sin 8\theta$.

All angles having the same sine as 8θ are of the form $2n\pi + 8\theta$, or of the form $(2n+1)\pi - 8\theta$. To satisfy the given equation, 9θ must fall under the one or the other of these forms:

$$\therefore\ 9\theta = 2n\pi + 8\theta, \tag{1}$$

or,

$$9\theta = (2n+1)\pi - 8\theta. \tag{2}$$

By (3), $\theta = 2n\pi$, and by (4), $\theta = (2n+1)\frac{\pi}{17}$.

The student may make an illustrative diagram and note the positions of the terminals.

7. Solve similarly and give diagrams for

(*a*) $\sin 7\theta = \sin \theta$,

(*b*) $\sin 2\theta = \sin 3\theta$,

(*c*) $\sin m\theta = \sin r\theta$.

8. Make up three examples like those of Ex. 7, and solve; illustrate by diagrams.

§ 63. Angles with Opposite Sines.

Numbers which are equal numerically but opposite in sign are said to be *opposite*, or *oppositely equal.*

(i) Angles with opposite terminals have opposite sines.

Here for equal moduli the ordinates are oppositely equal.

$$\therefore\ \sin(\pm 180° + A°) = -\sin A°,$$
$$\sin(\pm \pi + \theta) = -\sin\theta.$$

And, in general,

$$\sin\{(2n+1)180° + A°\} = -\sin A°,$$
$$\sin\{(2n+1)\pi + \theta\} = -\sin\theta,$$

where n is any positive or negative integer.

(ii) Terminals symmetric to the horizontal give for equal moduli oppositely equal ordinates. Thus all angles whose terminals are symmetric to the horizontal have opposite sines. Angles numerically equal but opposite in sign form a special case.

$$\therefore\ \sin(-\theta) = -\sin\theta,$$
$$\sin(-A°) = -\sin A°.$$

(iii) Any angle of a terminal has the opposite sine of any angle of the opposite terminal.

$$\sin\{(2n+1)\pi+\theta\}=-\sin(2m\pi+\theta),$$

or, $$\sin\{(2n+1)180^\circ+A^\circ\}=-\sin A^\circ,$$

where m, n are any positive or negative integers.

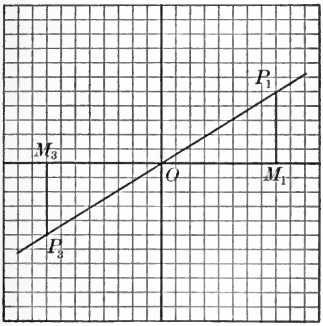


FIG. 26.

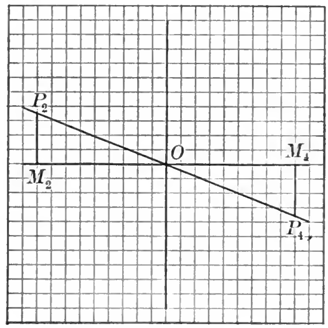


FIG. 27.

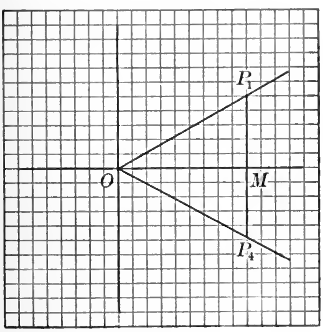


FIG. 28.

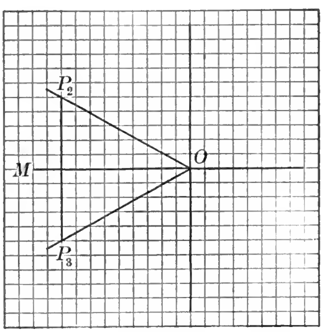


FIG. 29.

(iv) Any angle of any terminal has the opposite sine of any angle of the terminal symmetric to the horizontal.

$$\sin(2n\pi+\theta)=-\sin(2m\pi-\theta),$$

$$\sin(2n\cdot 180+A^\circ)=-\sin(2m\,180^\circ-A^\circ),$$

where m, n are any positive or negative integers.

(v) Thus, while adding an angle in degree measure to an even multiple of 180° or subtracting it from any odd multiple of 180° leaves the sine unchanged, the reverse changes the sign of the sine.

(vi) The student is advised against attempting to memorize the preceding formulas as mere facts. It is better to hold in mind the effect on the terminal of any given angle, of its addition to, or subtraction from, given multiples of 180°, or of π radians, noting that:

(α) *For terminals symmetric to the vertical, sines are equal.*

(β) *For terminals symmetric to the horizontal, sines are opposite.*

(γ) *For terminals which are opposite, the sines are opposite.*

EXERCISES.

1. Determine from a diagram five positive angles having the opposite sine of 30°, also five negative angles. Let one-half the number of angles be on the opposite terminal, the remainder on the terminal symmetric to the horizonal. Compare the results obtained from the diagram with the formulas.

2. Do the same for 45°; 60°; 120°.

3. Do the same for – 135°; – 210°; – 300°.

4. Do the same for 1780°; 2117°; 3185°.

5. Do the same for $\frac{\pi}{3}$; $\frac{\pi}{6}$; $-\frac{\pi}{4}$; $-\frac{3}{4}\pi$.

6. What effect on the terminal of an angle has the addition or subtraction of a double-even multiple of 90°? What a double-odd multiple of 90°? Find, among the first fifty numbers, those multiples of $\frac{\pi}{2}$ radians which may be added to or subtracted from a given angle without changing its sine. Also among the second fifty numbers, those multiples of 90° which change the sign of the sine, when added or subtracted.

§ 64. The Angles in a Table of Sines.

From the preceding sections, it is seen that the sine of any angle can be connected, in equality or opposite equality, with the sine of an angle of the first quadrant, between 0° and 90°, inclusive.

To find such angle for any given angle, note:

(*a*) The sine of a negative angle is the negative of that of the same angle taken positive:

$$\sin(-A^\circ) = -\sin A^\circ.$$

(*b*) All multiples of 360° may be dropped. Call the remainder R.

(*c*) If R is less than 90°, it is the required angle.

(*d*) If R is $> 90^\circ$ and $< 180^\circ$, take $180^\circ - R$.

(*e*) If R is $> 180^\circ$ and $< 270^\circ$, take $R - 180^\circ$.

(*f*) If R is $> 270^\circ$ and $< 360^\circ$, take $360^\circ - R$.

(*g*) And then if A° is the given angle,

(i)	$\sin(180 - R) = \sin A^\circ$,	case (*d*).
(ii)	$\sin(R - 180^\circ) = -\sin A^\circ$,	case (*e*).
(iii)	$\sin(360 - R) = -\sin A^\circ$,	case (*f*).

In practice it is best to sketch the terminal of the given angle.

(*h*) If the terminal is in the first quadrant, its principal angle is the required angle.

(*i*) If the terminal is in the second quadrant, draw the terminal symmetric to the vertical and take its principal angle.

(*j*) If the terminal is in the third quadrant, draw the opposite terminal and take its principal angle (opposite equality).

(*k*) If the terminal is in the fourth quadrant, draw the terminal symmetric to the horizontal and take its principal angle (opposite equality).*

EXERCISE.

Give the angles less than 90° and greater than 0°, which have sines equal, in numerical magnitude, to those of the following angles, stating in each case the proper sign, and using a diagram:

136°, 172°, 185°, 200°, 275°, 246° 34′ 53″, 301° 23′, − 30° 10′, 100° 45′, − 185° 54′, − 13° 13′ 13″, 400°, − 400°, 500°, − 500°, 1200°, − 1290°, 1800°, − 1800°, 2500°, − 2500°.

---

* Some tables are so arranged that the sine of any angle less than 360° can be taken from the table directly. See Hussey's Tables.

§ 65. Supplementary Angles.

Any two angles whose sum in degree measure is 180°, or, in radian measure, π radians, are said to be *supplementary.* Each is called the *supplement* of the other.

They have the forms $180° - A°, A°$; $\pi - \theta, \theta$. To construct $180° - A°$, lay out from the left-hand horizontal line an angle equal to the angle $A°$, reversed in sign. Thus the terminal of an angle and that of its supplement are symmetrically inclined to the vertical.

$$\therefore \sin(180° - A°) = \sin A°; \ \sin(\pi - \theta) = \sin\theta,$$

or, *the sine of an angle is the sine of its supplement.*

EXERCISES.

1. Show from a diagram that $\sin(180° - A°) = \sin A°$.

2. If the sines of two angles are equal, are the angles necessarily supplementary?

3. Does it follow that because the terminals of supplementary angles are symmetric to the vertical, all angles whose terminals are symmetrical to the vertical are supplementary?

4. Name five pairs of supplementary angles which have the same pair of terminals. How many pairs of supplementary angles have the same pair of terminals? If A is an angle of a given terminal, give the general formulas for all supplementary pairs having this terminal.

5. Select five angles of the first quadrant and determine their supplements, illustrating the positions of the terminals by diagrams. Do the same for five angles in each of the other quadrants.

6. Do the same for five negative angles of each quadrant.

7. If A, B, C, are the angles of a triangle, give three formulas connecting these angles by sines.

8. Connect by sines pairs of opposite angles of a parallelogram.

9. Connect by sines the adjacent angles, when one straight line meets another. Also the exterior angle of a triangle with the sum of the interior opposite angles.

10. Select five angles in circular measure and determine their supplements.

11. What is the supplement of $45° - A°$? Of $\frac{\pi}{4} + \theta$?

REVIEW EXERCISES.

1. If 45° is a special solution of the equation $\sin A = \frac{1}{+\sqrt{2}}$, what is the general solution?

2. How are the terminals of $180° - 18°$, $180° + 18°$, $360° - 18°$, related to that of 18°? How are the sines of these angles related to that of 18°?

3. How are the terminals of all angles having the same sine as 30°, related to the terminal of 30°?

4. How are the terminals of all angles having the opposite sine to that of 60°, related to the terminal of 60°?

5. How is the terminal of the supplement of an angle related to that of the angle?

6. What is the supplement of 120°? Of $-120°$? Of 360°? Of $-800°$?

7. What is the supplement of $\frac{\pi}{3}$? Of $-\frac{\pi}{3}$? Of 4π? Of -8π?

8. If the sine of $A = \frac{3}{5}$, what is the sine of the supplement of A?

9. State how the terminals of the following angles are related to the border lines of the quadrants, and give the sign of the sine:

$\pm 45°$; $\pm 60°$; $\pm\frac{\pi}{6}$; $\pm\frac{\pi}{12}$; $\pm\frac{5\pi}{6}$; $\pm\frac{7\pi}{6}$; $\pm 180°$; $\pm 135°$; $\pm 120°$; $\pm 240°$; $\pm 225°$; $\pm 330°$; $\pm 315°$; $\pm\frac{\pi}{2}$; $\pm\frac{3\pi}{2}$; $\pm 2n\pi + \frac{\pi}{4}$; $\pm(2n+1)\pi - \frac{\pi}{6}$; $\pm(2n-1)\pi + \frac{2\pi}{3}$.

10. State with reference to each of the following angles the quadrant to which it belongs, the sign of its sine, and the angle of the first quadrant whose sine is numerically the same:

150°; 120°; 240°; 225°; 330°; 105°; 165°; 195°; 255°; 480°; 498°; 510°; 585°; 555°; 975°; 1305°; 1590°; 1665°; $\frac{2\pi}{3}$; $\frac{7\pi}{4}$; $\frac{5\pi}{3}$; $\frac{11\pi}{6}$; $\frac{11\pi}{2}$; $2n\pi + \frac{3\pi}{4}$; $(2n+1)\pi - \frac{\pi}{3}$; $(2n-1)\pi + \frac{\pi}{13}$; $2n\pi - \frac{\pi}{6}$; $(2n-1)\pi + \frac{5\pi}{12}$; $(2n+1)\pi + \frac{\pi}{2}$; $-405°$; $-390°$; $-420°$; $-840°$; $-1200°$; $-1305°$; $-1020°$; $-1665°$; $-7\frac{2}{3}\pi$; $-5\frac{3}{4}\pi$; $-3\frac{2}{3}\pi$; $-8\frac{1}{5}\pi$; $-6\frac{1}{4}\pi$; $-13\frac{11}{12}\pi$.

11. Express in radians 46° 30′; $\frac{1}{24}$ of a right angle; 49° 43′ 45″; 1′.

12. Express the angles in Ex. 11 in terms of π radians.

13. Express in degrees, minutes, and seconds: $\frac{3\pi^r}{5}$; $2\pi^r$; $.8^r$; 3.1416^r; 5^r; $2\frac{3}{4}^r$.

14. Taking π as $\frac{22}{7}$, how far off is a 1-foot line when it subtends an angle of 1″?

15. What angle at the earth's centre does the moon subtend, supposing her diameter 4000 miles and her distance 240,000 miles, taking $\pi = 3.1$?

16. How far off is a bright object 350 feet tall when just visible? ($2\frac{1}{2}'$ is the limit of average eye-vision for a bright object on a black background.)

17. Find two angles whose difference is 1° and whose sum is 1^r.

18. Divide 77° into two parts such that the number of English seconds in one is the number of French seconds in the other.

19. Name the quadrants to which the following points belong: (2, 4); (– 3, 4); (3, – 4); (– 3, – 4); (2, 30°); (– 2, 30°); (2, – 30°); (– 2, – 30°).

20. Define the sine of an angle.

21. If modulus is 27.8 and sine = 0.345, what is the ordinate?

22. If the ordinate is 3.18 and the sine is 0.432, what is the modulus?

23. If the modulus is 31.4 and the ordinate is ± 16.3, what are the sines, to three places?

24. A tree is 50 feet high and subtends an angle whose sine is 0.37 from a point in the horizontal plane of its ground-line; how far is it from that point to the top of the tree?

§ 66. Construction of the Terminals for Given Values of the Sine.

Required the terminals when $\sin\theta = \frac{5}{9}$. Consider 9 as modulus and 5 as ordinate. Draw a circle of radius 9 about the origin. Lay out on the axis of ordinates a distance OD equal to + 5. Then draw NDM parallel to the initial line, cutting the circle in N, M. The required positions of the terminals are OM, ON. If θ is some one of the angles locating the terminal OM, as, for example, the principal angle as in the diagram, then,

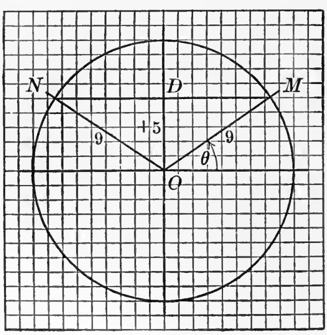


FIG. 30.

$$\left.\begin{matrix} 2n\pi + \theta \\ (2n+1)\pi - \theta \end{matrix}\right\}, \text{ or, } m\pi + (-1)^m\theta,$$

are the complete set of angles whose sine is $\frac{5}{9}$ (§ 62).

If the sine had been given as $-\frac{5}{9}$, the only change in the preceding construction would have been to lay out OD 5 units downward from O, instead of upward.

In constructing terminals for table-sines, cut the table-value down to the nearest tenth or nearest hundredth.

EXERCISES.

1. Construct the double terminals for each of the following values taken as sines (using coördinate paper):

$$-\tfrac{2}{3}, \tfrac{3}{4}, -\tfrac{3}{4}, \tfrac{4}{5}, -\tfrac{4}{5}, \tfrac{5}{4}, -\tfrac{5}{4}, \tfrac{7}{8}, -\tfrac{7}{8}.$$

Are any of the preceding values impossible?

2. Selecting a line to represent unity, construct a line to represent $+\sqrt{2}$; a line for $\sqrt{3}$; for $\sqrt{5}$. Can you see from the construction for $\sqrt{2}$, what angles have $\frac{+1}{\sqrt{2}}$ for sine? What $\frac{-1}{\sqrt{2}}$? What $\frac{\sqrt{3}}{2}$? What $\frac{-\sqrt{3}}{2}$?

3. Where is the terminal when the sine is 1? -1? 0? What are the general expressions for the corresponding angles?

4. Construct terminals for five sines in the table of sines, and measure the angles. Compare with table-values.

§ 67. Line Picture of the Sine.

Since $\text{sine} = \frac{\text{ordinate}}{\text{modulus}} = \frac{MP}{OP} = \frac{M_1P_1}{OP_1}$, if the modulus is taken as unity, the sine becomes, *on the same scale*, the ordinate. If, therefore, OP_1 (Fig. 31) is taken as a unit, M_1P_1 is the sine, or

$\sin\theta = \text{ordinate } M_1P_1$, (A).

(A) is frequently spoken of as *the line definition of the sine.*

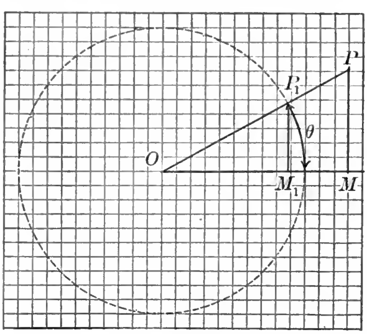


FIG. 31.

Not many years ago all text-books on trigonometry used this definition exclusively. Two objections have led to its abandonment. (1) Its use gives a false impression at the outset, namely, that the sine is a line, and a line of a certain length, that of the diagram in the text-book studied. The fact is that the sine is not a line, but a ratio, as already seen. The diagram means merely that the line MP represents the sine on the same scale (any scale) on which OP represents unity. Since the unit-representative is arbitrary in length, so is the line representing the sine. If the line representing a unit is taken half as long as that above, the sine-line is reduced one-half. (2) The second objection to the use of the line definition is that those who learn trigonometry with unity as the modulus, find trouble in introducing any other modulus, whereas, with the general modulus, r, used from the outset, no difficulty is met in assigning to r the special value, 1.

The ratio definition was first used in America at the University of Virginia, by Professor Bonnycastle.

The line definition continued to be used in American text-books to a very recent day, and is still in very general use in engineering field books. Instead of abandoning completely the line definition, as is now the drift in text-books, it appears to the author advisable to hold it for such use as is made in § 69. It presents *to the eye* certain properties, and thus teaches these properties with greater lucidity than does the ratio definition.

§ 68. The Expression "Sine of the Arc."

It has been shown (§ 47, *c*, 4) that when the radius is taken as unity, the *number* representing the arc is the same as that representing the corresponding central angle in radian measure. Consequently, when the line definition was in use, the sine of the angle, as now used, was called *the sine of the arc.* The notation for an angle whose sine is any special value, as $\frac{2}{3}$, is $\sin^{-1}\frac{2}{3}$. This is generally read, "the angle whose sine is $\frac{2}{3}$," frequently, "the arc whose sine is $\frac{2}{3}$." This is a lingering influence of the line definitions, as is the designation used in German text-books, "arcus sinus." The relative magnitude of arc θ and ordinate M_1P_1 (Fig. 32) expresses graphically the size of an angle compared with its sine.

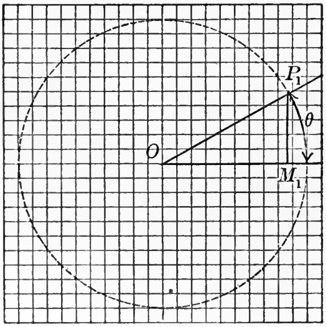


FIG. 32.

§ 69. Line Picture of the Sines of all Angles, and its Story.

Draw the unit-circle about O. It cuts the terminal OP at P_1 (Fig. 33). Then M_1P_1 is the sine of θ. The ordinate at any point represents the sine of all angles whose terminals pass through that point. Thus the ordinates of the unit-circle make up the sum total of all possible values of the sines of all angles.

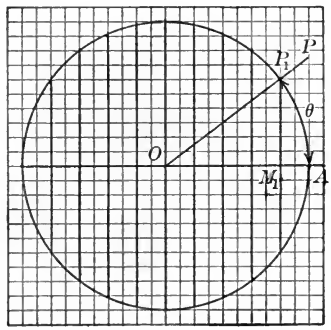


FIG. 33.

From the diagram of the line values of sines (Fig. 33) there can be learned *more readily than in any other way* — because seen by the eye — certain very important properties of the sine.

(*a*) When the terminal is close to the initial line, the sine is small.

(*b*) When the terminal is on or opposite the initial line, the sine is *zero*.

(*c*) Starting the terminal from coincidence with the initial line, and letting it move counter-clockwise around the origin, the sine sets out with the value zero, and increases as do the numbers of a continuum, until the terminal reaches the upright vertical, when the sine becomes the radius, or 1. As the terminal passes the position of the upright vertical, the sine begins to decrease, and, as the terminal moves on, the sine continues decreasing as do the numbers of a continuum, until it reaches the value 0, when the terminal coincides with the left-hand horizontal. The sine continues to decrease as do the numbers of a continuum, until the terminal is vertical, downward, when it is -1. It then begins to increase, and increases as do the numbers of a continuum until the terminal is again in coincidence with the initial line, when it returns to the value with which it started out, zero.

As the terminal moves again over its circuit the sines go again through the same order of values; and so on, over and over again, as the terminal continues to turn.

(*d*) With a clockwise turn of the terminal, the same thing takes place in the reverse order. (The student may state verbally what happens.) Thus:

(*e*) The sine is never greater, numerically, than 1.

(*f*) It can take all numerical values lying between $+1$ and -1, inclusive.

(*g*) It is zero whenever the terminal is on or opposite the initial line.

(*h*) It is $+1$ whenever the terminal is the upright vertical, and -1 whenever the terminal is the downright vertical.

(*i*) When the sine is zero, the terminal is on or opposite the initial line, and, therefore, the angle is an even or odd number of half turns, or

$$2n\pi, \text{ or } (2n+1)\pi,$$

or,

$$2n\cdot 180°, \text{ or } (2n+1)180°.$$

Important special cases are:

$$\sin 0° = 0,\ \sin(\pm 180°) = 0,\ \sin(\pm 360°) = 0,$$

$$\sin(\pm\pi) = 0,\ \sin(\pm 2\pi) = 0,$$

where the general formula is

$$\mathbf{\sin m\pi = 0, \text{ or } \sin m\cdot 180° = 0,}$$

where m is any positive or negative integer.

(*j*) When the sine is $+1$, the terminal is the upward vertical and the angle is of the form

$$2n\pi + \frac{\pi}{2}, \text{ or } 2n\cdot 180° + 90°,$$

or,

$$(4n+1)\frac{\pi}{2}, \text{ or } (4n+1)\,90°.$$

Therefore, in general,

$$\mathbf{\sin(4n+1)\frac{\pi}{2} = 1, \text{ or } \sin(4n+1)\,90° = 1.}$$

And in particular,

$$\sin\frac{\pi}{2}=1,\ \sin 5\cdot\frac{\pi}{2}=1,\ \text{etc.}$$

$$\sin 90^\circ=1,\ \sin 5\times 90^\circ=1,\ \text{etc.}$$

$$\sin\left(-3\frac{\pi}{2}\right)=1,\ \sin\left(-7\frac{\pi}{2}\right)=1,\ \text{etc.}$$

$$\sin(-3\times 90^\circ)=1,\ \sin(-7\times 90^\circ)=1,\ \text{etc.}$$

(*k*) When the sine is -1, the terminal is the downward vertical, therefore,

$$\sin(-90^\circ)=-1,\ \sin(-5\times 90^\circ)=-1,\ \text{etc.}$$

$$\sin\left(-\frac{\pi}{2}\right)=-1,\ \sin\left(-5\cdot\frac{\pi}{2}\right)=-1,\ \text{etc.,}$$

with the general statement,

$$\mathbf{\sin(4\,n-1)\,90^\circ=-1,}$$

or,

$$\mathbf{\sin(4\,n-1)\frac{\pi}{2}=-1,}$$

where, as in all formulas of this kind, here as elsewhere, n is any positive or negative integer.

(*l*) The sine of an angle is less than the corresponding arc on the unit-circle, that is, the arc of the principal angle. This is evident at once on the unit-circle, for the arc PAP_1 (Fig. 34) is longer than the straight line PP_1; therefore, the half of arc PAP_1 is longer than the half of the straight line PP_1,

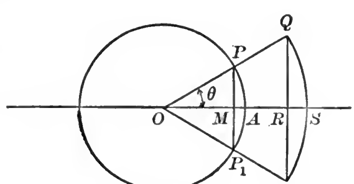


Fig. 34.

or, $MP < \text{arc } AP$,

or, $$\sin\theta < \text{arc } AP.$$

It has been mentioned in § 68 that when the radius is taken as unity, the number representing the arc on the unit-circle is the same as the number representing the corresponding central angle. Therefore an equivalent statement to that above is,

The sine of an angle is less than its radian measure.

The same proposition is also readily proven for any other radius than unity. For in Fig. 34,

$$\text{straight line } GRQ < \text{arc } GSQ,$$

$$\therefore \text{ ordinate } RQ < \text{arc } SQ,$$

$$\therefore \frac{\text{ordinate } RQ}{\text{modulus } OQ} < \frac{\text{arc } SQ}{\text{radius } OQ},$$

or, $\sin\theta <$ radian measure,

or, $\sin\theta < \theta.$

(*m*) A slight change in the angle makes a slight change in the sine.

For the angle AOP_1 (Fig. 35), the sine is M_1P_1. For the larger angle AOP_2, the sine is M_2P_2.

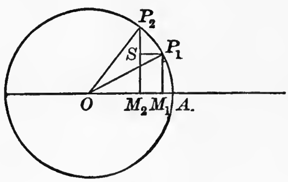


FIG. 35.

$$\therefore \sin AOP_2 - \sin AOP_1$$
$$= M_2P_2 - M_1P_1 = SP_2.$$

Evidently, then, with diminishing difference in the angles, SP_2 diminishes, becoming zero when the angle difference is zero.

When the angle difference becomes less than any assignable quantity, so also does the sine difference. An equivalent statement to this, and the one most commonly used is:

The sine varies continuously as the angle varies continuously.

This is also differently expressed by saying that if angle values form a continuum from A to B, the sine values form a continuum from the sine of A to the sine of B. For example, as the angle passes from zero to 90°, the sine passes through all those values indicated as ordinates on the unit-circle from 0 to 1. Similarly, as the angle changes continuously from 45° to 135°, the sine passes through all values from $\frac{\sqrt{2}}{2}$ to 1, and then from 1 to $\frac{\sqrt{2}}{2}$ again. It waxes in length as does a stretching rubber string, and wanes in length as does a contracting rubber string. The illustration expresses fairly well what is meant by a continuous change, a change without sudden jumps in value, without omitting intermediate values of a continuum. One can easily, however, get a false impression from the illustration, if any idea of the rate of change along the continuum is allowed to enter the mind. Suppose, for instance, one considers the ordinates of the straight line AB (Fig. 36) from A to B. They form a continuum. The ordinates of the circle APB form exactly the same continuum,

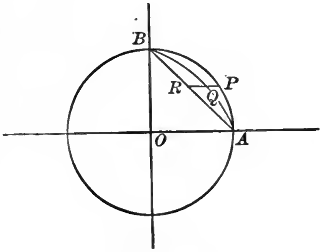


FIG. 36.

as do also the ordinates of any other curve, like AQB, which has no ordinate between A and B higher than OB. This is seen by drawing a line RQP parallel to the initial line. This gives three equal ordinates, so that for any ordinate of any one of the curves (line), there is an equal ordinate on the others. Thus the sets of ordinates form the same continuum, whether from the straight line or from the circle. Evidently the rate of rise of the ordinates of the circle is not uniform. The rise is rapid at the outset at A, and is a diminishing rate of rise as the terminal turns uniformly from the position OA to OB.

As much of the notion of continuity as we desire at present to give, is covered by the statement:

If two angles in circular measure, or, what is the same thing, two arcs on the unit-circle, differ by less than any assignable magnitude, so will their sines.

This follows from the diagram (Fig. 35), for the straight line SP_2 is shorter that the curve P_1P_2, that is, the sine difference is always less than the arc difference.

(n) For a given small change in the angle, the change in the sine is smaller the nearer the terminal of the angle is to the vertical; and the larger, the nearer the terminal is to the horizontal. In particular, if two angles are very near 90°, the difference in their sines is very small compared with the difference in the angles; whereas, if the two angles are very near zero, the difference in the sines is very nearly equivalent to the difference in the angles in radian measure. The student is expected to see this only in so far as it is clear from the diagram, that is, see it with the eye. Later the same thing will be investigated by the aid of formulas. Often we see on the diagram what a formula never makes any clearer.

(*o*) When an angle is small, doubling the angle very nearly doubles the sine.

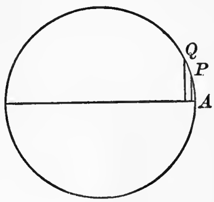


FIG. 37.

Motion along the arc at A, on the circle, is very nearly vertical, and the increase in the ordinate is very nearly the same as in the arc.

(*p*) However, $\sin 2\alpha$ is never $2\sin\alpha$, except when α is zero or some multiple of π.

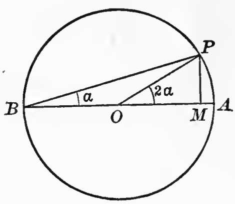


FIG. 38.

For, let APB (Fig. 38) be a semicircle, with $\alpha > 0$ and $< \frac{\pi}{2}$. Then,

$$\sin 2\alpha = \frac{MP}{OP}, \quad \text{and} \quad \sin\alpha = \frac{MP}{BP}.$$

In the triangle BOP,

$$BP < BO + OP, \text{ i.e. } BP < 2\,OP.$$

$$\therefore \sin\alpha > \frac{MP}{2\,OP}, \text{ or } 2\sin\alpha > \frac{MP}{OP},$$

or, $2\sin\alpha > \sin 2\alpha$, numerically.

This is true for all values of α. See § 139 (1). We introduce the word "numerically" above, to correspond with the result which would be obtained if the diagram were turned over about BA as an axis.

We give the statement, $\sin 2\alpha$ is not $2\sin\alpha$ here, because the desire to treat 2 as a factor with the word "sin" manifests itself quite promptly among average students.

EXERCISE.

Deduce all the propositions from (*a*) to (*p*), using the ratio definition for the sine, and without making use of the unit-circle.

LABORATORY EXERCISES.

1. Draw a circle of one foot radius. Divide it into five-degree spaces; measure the ordinates in decimals of a foot, and compare with a table of sines, and show that the sines for the second, third, and fourth quadrants can be expressed in terms of those of the first quadrant.

2. On the same circle measure the sine of 10° and the sine of 5°, and see that $\sin 10° < 2 \sin 5°$. Test several other double angles.

3. Show from the diagram of Ex. 1 that $\sin 15° < 3 \sin 5°$.

4. Show from the diagram of Ex. 1 that $\sin 20° < 4 \sin 5°$.

§ 70. The Sine as a Function of the Angle.

Whenever a quantity y is so related to another quantity x, that y can be calculated when x is given, then y is said to be a function of x. Calculations involve four operations, — addition, subtraction, multiplication, division. In fact, *these are the only operations involved in calculations.* When y is reached by a finite number of calculation operations on x, y is said to be an *algebraic function* of x. Examples are:

$$y = 2x,$$
$$y = 2x \pm 3,$$
$$y = \left\{\frac{3x^3 + 6x^2 - 8x}{9x^5 - 6x^2 + 3}\right\}^3, \text{ etc.}$$

Here y is in each case derived from x by a finite number of calculation operations. Roots of algebraic functions, where the root-indices are integers or common fractions, are also reckoned as algebraic functions, even though it may not be possible to calculate the function exactly.

Functions that are not algebraic are said to be *transcendental*, since their calculation implies an *infinite number* of one or more of the fundamental mathematical operations. Such functions can evidently be calculated only approximately, this being, however, as indicated above, no test of the transcendental character of a function.

That the sine of an angle is a function of the angle, in the sense of the foregoing definition of a function, is evident at once from the definition of a sine $\left(\frac{\text{ordinate}}{\text{modulus}}\right)$. For one can draw the ordinate, measure it and the corresponding modulus, divide, measure the angle. It is evident, however, that while such a procedure establishes the fact that the sine is a function of the angle, the value of the sines of all angles could not be obtained with any great degree of accuracy from this procedure without great care in the measurements. The ratio definition of the sine is not given with a view to its use in calculating the tabulated sines of angles, but rather with the object of calculating the ordinate when the angle and modulus are given, or calculating the modulus when the angle and ordinate are given. That is, the definition is not used to calculate a table of sines of angles. Frequently, however, from measured ordinates and moduli, sines are calculated and the corresponding angle determined by comparison of the resulting calculated sine with a table of sines. It will be shown in § 156 that if θ is the radian measure of an angle,

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \frac{\theta^9}{\underline{|9}} \ldots \text{ etc., without end.}$$

From this relation the sine of any angle can be calculated to any desired degree of accuracy, but never when θ is other than zero can it be determined completely from this series, since an infinite number of calculation operations is indicated. The sine is thus a transcendental function of the angle, and in most cases can be known only approximately.

While the formula above for the sine would be used to calculate the sine of any given angle, if that were unknown, it is not the formula from which tables of sines of angles have been calculated. It will be shown later that when the sine of any angle is known, that of an angle differing from it slightly can be determined more readily than by using this formula. The chief point we wish to make here is, that the sine is a function of the angle; that is, that when the angle is given, the sine can be calculated. These calculations have been made with great care for all angles between 0° and 90°, at intervals of every "second of arc" for small angles; for every 10″ for larger, but still small, angles; then for every 1′, or for every 10′, the angle-interval depending upon the place of the table.

Such tables of sines are frequently called *tables of natural sines*, to distinguish them from the tables of *logarithms* of the same numbers, called *tables of logarithmic sines*.

§ 71. Many-valued Functions.

When to each value of the quantity x corresponds a single value of the calculated quantity y, y is said to be a single-valued function, or unique function, of the quantity x. Examples are the algebraic functions, free from radicals, previously given. If $y = \sqrt{x^2 + 3x - 7}$, to each value of x will correspond two values of y. In this case, and all similar ones, y is called a two-valued function of x. While, as already seen, the sine is a single-valued function of the angle, the angle is a many-valued function of the sine, since all angles with the same terminal, or with terminals symmetric to the vertical, have the same sine (§ 62).

§ 72. The Sine as a Periodic Function.

When it is desired to indicate that y is a function of x without stating the character of the relation specifically, we write

$$y = F(x).$$

This is read, "y equals a function of x." If two or more functions are to be used at the same time, some other letter may take the place of F. Thus we might have

$$y = F(x),\ \ y = \theta(x),\ \ y = \phi(x),\ \ y = G(x), \text{ etc.}$$

In such cases, if the functions are read in succession, distinction is made among them by reading the function letter, as "y equals the F-function of x," "y equals the theta-function of x," etc.

If $y = F(x)$, then $y_1 = F(x_1)$ means that y_1 is the special value which y takes when x is given the special value x_1.

For example, if

$$y = x^2 + 3x - 7 = F(x),$$

$$F(x_1) = x_1^2 + 3x_1 - 7,$$

and $$F(2) = 2^2 + 3 \cdot 2 - 7 = 3, \text{ etc.}$$

Similarly, if $y = F(x)$, then $y_1 = F(x_1 + h)$ means that y_1 is the special value which y takes when x takes the special value $x_1 + h$.

With these agreements, if $y = F(x)$, then

$$F(x_1) = F(x_1 + h)$$

signifies that y takes the same value for $x = x_1$ as for $x = x_1 + h$, and

$$F(x_1) = F(x_1 + h) = F(x_1 + 2h) = F(x_1 + 3h) = F(x_1 + 4h), \text{ etc.}$$
$$= F(x_1 - h) = F(x_1 - 2h) = F(x_1 - 3h), \text{ etc.},$$

signifies that all values of x separated from x_1 by multiples of the value h, give the same value to the function y.

When we write

$$F(x) = F(x + nh),$$

where n is any integer, positive or negative, and x is free from subscripts, it signifies that for any value whatever of x, values of x separated from this value by multiples of h give all the same value to the function as did the initial value. Such a function is called a *periodic* function, and h is called the *period*.

We have seen that for the sine,

$$\sin(\theta) = \sin(\theta + n \cdot 2\pi),$$

for all integral values of n, and for all values of θ (§ 62), and that this is the only angle relation which leaves the sine unchanged both in magnitude and sign.

Thus the sine is a periodic function whose period is 2π.

EXERCISE.

Point out the periodicity of the sine on the unit-circle, taking the arc, its number being the same as that of the corresponding angle, to represent x, while the sine, or ordinate, is y.

§ 73. Inverse Functions. Notation for Angles having a Given Sine.

When y is expressed directly in terms of x, y is said to be an explicit, or direct, function of x, as in $y = x^2 + 2x - 1$ or $y = \sin x$. When x is given here, y can be calculated directly. Generally, if y is a function of x, x is also a function of y; that is, if y is given, x can be calculated. In the first of the relations above, if x is 2, y is 7. The *inverse* process of finding what x is when y is given as 7, is quite different. For when

$$x^2 + 2x - 1 = 7,$$

by the usual process for solving quadratics,

$$x = \frac{-2 \pm \sqrt{4 + 32}}{2} = 2 \text{ or } -4.$$

Here, while y was single-valued, x is two-valued.

Similarly, if y is a cubic in x, as

$$y = 2x^3 - 5x^2 + 4x - 3,$$

y will be single-valued, and x, as shown in algebra, three-valued, with corresponding relations for quartics, quintics, etc.

From the foregoing it may readily appear that if one quantity, y, is expressed directly in terms of another, x, it may be quite difficult, or impossible, to express x in terms of y, or to calculate x when y is given. For example, if y is a quintic in x, as

$$y = x^5 + ax^4 + bx^3 + cx^2 + dx + e,$$

x cannot be expressed in terms of y, and the quantities a, b, c, d, e, in terms of radicals.

A like relation holds for algebraic functions of higher degree than the fifth, when the coefficients are given the general values, a, b, c, etc.

However, if the quantities a, b, c, d, e, y, etc., are *numbers*, x can be calculated to any desired degree of accuracy, as explained in algebra (Horner's Method).

We may assume, in general, that if y is a function of x, x is also a function of y. The one function is said to be the *inverse* function of the other.

The inverse function of any function F is indicated by F^{-1}. Thus, if

$$y = F(x), \qquad (a)$$

$$x = F^{-1}(y). \quad \text{(Read: } x \text{ equals anti-}F \text{ of } y) \qquad (b)$$

These two notations imply the same equation-relation, or an equivalent equation-relation, between x and y. So do

$$y = \sin x, \qquad (1)$$

$$x = \sin^{-1} y. \qquad (2)$$

As already pointed out, y is here a one-valued function of x, while x is a many-valued function of y. In fact (1) is nothing but

$$y = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \text{etc.} \qquad (3)$$

The infinite degree of the second member would carry with it, to one familiar with algebra, an infinite number of values of x for any value of y. We have already seen that for any given value of the sine there are an infinite number of angles, differing by multiples of 2π, or 180°.

The direct expression of x in terms of y from (3) we do not take up here. Read "Reversion of Series" in any college algebra.

Relation (2) is read in several different ways:

(i) x is an angle whose sine is y;
(ii) x is an arc whose sine is y;
(iii) x is anti-sine y;
(iv) x is the inverse sine of y;

(iii) is preferred, on account of its brevity. It is, perhaps, best to use (i) until the significance of the symbol is well in mind.

The statements

$$\begin{Bmatrix} \sin\cdot\sin^{-1}y \ \ (3) \\ \sin^{-1}\cdot\sin y \ \ (4) \end{Bmatrix} \text{ are read } \begin{Bmatrix} \text{sine anti-sine } y \\ \text{anti-sine sine } y \end{Bmatrix}.$$

(3) is also read: "The sine of the angle whose sine is y," and thus makes immediately the mental impression that its value is none other than y itself. After some practice, and after the notation is familiar, "sine anti-sine" will make the same impression. Expression (3) is thus single-valued. On the contrary, (4) may be read "any angle whose sine is that of y." It is thus many-valued, being any one of the angles

$$m\pi + (-1)^m\theta,$$

where θ is any particular angle of the set (§ 62).

§ 74. Origin of the Notation, $\sin^{-1}$.

$a\cdot\frac{1}{a}\cdot x$ is x. That is, $a\cdot a^{-1}\cdot x$ is x.

Therefore a and a^{-1}, considered as *operators* on x, annihilate the effect of each other.

Corresponding to this is the effect of the expressions "sine of" and "whose sine is," as in the expression "the sine of an angle whose sine is x, is x."

Thus, while

$$\begin{Bmatrix} \sin\cdot\sin^{-1}x = x \\ a\cdot a^{-1}\cdot x = x \end{Bmatrix} \ (1) \quad \text{correspond to each other,}$$

$$\begin{Bmatrix} \sin^{-1}\cdot\sin x \\ a^{-1}\cdot a\cdot x \end{Bmatrix} \ (2) \quad \text{do not;}$$

for while the last is x, the next to the last is, as already pointed out, *any* angle whose sine is that of x.

However, the origin of the notation, $\sin^{-1}x$, is in the similarity here noted in (1); so also $\log_e^{-1}x$.

EXERCISES.

1. Read in all possible ways each of the following expressions:

$$\sin^{-1}\frac{2}{3};\quad \sin^{-1}\frac{3}{4};\quad \sin^{-1}\left(-\frac{2}{3}\right);\quad \sin^{-1}\left(-\frac{3}{4}\right);\quad \sin^{-1}\frac{1}{\sqrt{2}};$$

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right);\quad \sin^{-1}\cdot\frac{\sqrt{2}}{2};\quad \sin^{-1}\frac{1}{2};\quad \sin^{-1}\left(-\frac{1}{2}\right).$$

2. Read the following:

$$\sin^{-1}\frac{1}{2}+\sin^{-1}\left(-\frac{2}{3}\right);\quad \sin^{-1}\frac{\sqrt{3}}{2}-\sin^{-1}\cdot\frac{1}{\sqrt{2}};$$

$$\sin^{-1}\frac{2}{3}+\sin^{-1}\left(-\frac{3}{4}\right);\quad \sin^{-1}\frac{\sqrt{3}}{2}+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right);\quad \sin^{-1}\frac{1}{2}\pm\sin^{-1}\left(-\frac{1}{2}\right).$$

3. If θ, A are the radian and degree measure, respectively, for the principal angle of any expression of Ex. 1, what is the general expression? Write the general expressions for the angles of Ex. 2, in both radian and degree measure.

4. Read the following expressions in all possible ways:

$$\sin^{-1}\frac{2}{3}+3\sin^{-1}\frac{3}{4};\quad 2\sin^{-1}\frac{\sqrt{2}}{2}-5\sin^{-1}1;\quad 2\sin^{-1}\frac{1}{2}-3\sin\frac{\sqrt{2}}{2};$$

$$\sin^{-1}\frac{1}{3}+2\sin^{-1}\frac{1}{2};\quad 4\sin\frac{-\sqrt{3}}{2}-2\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right);\quad \sin^{-1}\left(-\frac{2}{3}\right)-\sin^{-1}\frac{2}{3};$$

and write the general expressions in circular measure and in degree measure for the corresponding angles, using the tables when necessary.

5. Construct the terminals corresponding to each expression in Exs. 1, 2, and 4.

GENERAL EXERCISES.

1. Construct the terminals when $\sin A$ has the following values:

$$\tfrac{2}{3};\ -\tfrac{4}{5};\ 1;\ -1;\ 0;\ \frac{\sqrt{2}}{2};\ \frac{\sqrt{3}}{2};\ \frac{-\sqrt{2}}{2};\ \frac{-\sqrt{3}}{2};\ \frac{1}{2};\ -\frac{1}{2}.$$

2. Give, with diagrams, general solutions of the equations:

$$\sin\theta=1;\qquad \sin\theta=-1;\qquad \sin\theta=0;$$
$$\sin 2\theta=1;\qquad \sin 3\theta=-1;\qquad \sin 4\theta=0;$$
$$\sin m\theta=1;\qquad \sin(-m\theta)=-1;\qquad \sin 2m\theta=0.$$

3. Name two angles not in the same quadrant whose sines are the same as that of 17°, with diagram for terminals.

4. Name two angles not in the same quadrant whose sines are the opposite of that of 17°, with diagram for terminals.

5. What is the best plan for determining for any given angle what angle of the first quadrant has numerically the same sine?

6. What are the general values of θ when $\theta=\sin^{-1}1$, $\theta=\sin^{-1}(-1)$, $\theta=\sin^{-1}(0)$? Of A when $3A=\sin^{-1}(+1)$, $mA=\sin^{-1}(-1)$, $7A=-\sin^{-1}(0)$?

7. Write the general form of a periodic function of x whose period is K.

8. Write the sine equation which indicates that the sine is a periodic function.

9. If $\sin 3\,\theta = 15$, may we divide by 3 and write $\sin \theta = 5$?

10. Give the general formula for all angles whose terminals are either on the upright or downright vertical.

11. Find the number of seconds in the angle 0.3^r; the number of minutes in 0.4^r; the number of degrees, minutes, and seconds in 0.5^r and in $\frac{3}{13}\pi^r$.

12. Express 11″ in radian measure.

13. The three angles of a triangle have the same numerical measure, one being in degree measure, another in grade measure, the third in radians; find the third angle in terms of π.

14. Construct the terminals of θ, when $2\,\theta = \sin^{-1}\frac{2}{3}$; when $3\,\theta = \sin^{-1}(-\frac{4}{5})$; when $4\,\theta = \sin^{-1}(1)$; giving, in each case, all possible solutions.

15. What angle of the first quadrant has the same numerical sine as 2317°?

16. How are the sines of $\pi + \theta$, $\pi - \theta$, $2\,\pi - \theta$ related to $\sin \theta$?

17. Express $3^g.12$ in radian measure in terms of π.

18. When the modulus is 25.4 and the sine -0.312, what is the ordinate?

19. When the ordinate is ± 37.2 and the sine is ± 0.341, what is the modulus?

20. When the ordinate is ± 21.3 and the modulus is 32.4, what are the sines to three figures?

§ 75. Angles whose Sines can be determined readily from a Diagram.

We have already stated (§ 70) that to calculate the sine of any angle given in radian measure and selected at random, one must use the relation:

$$\sin \theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \text{etc.}$$

There are, however, a few angles whose sines can be determined quite readily without the use of this formula.

Prominent among these are the angles whose terminals pass through the vertices of the regular polygons of 3, 4, 6, 8, 12, sides, the polygons being central at the origin, with one vertex on the initial line. Such terminals will form the bisectors and trisectors of the quadrantal angles, together

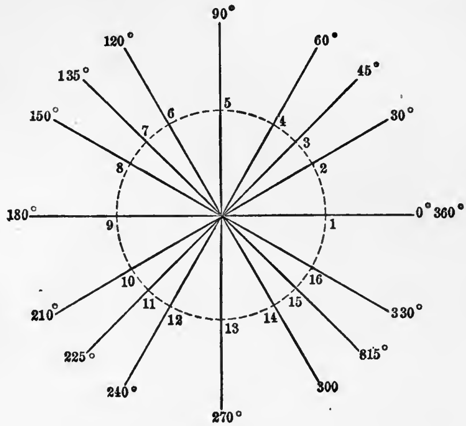


FIG. 39.

with the border lines of the quadrants, in all sixteen terminals, as in the above diagram, in which the terminals are indicated by numerals and also in degree measure:

(*a*) For terminals 1, 5, 9, 13, the sines have already been given (§ 69). The student may restate the results, giving the general formulas. Try it without looking at the reference, and then compare results with the reference.

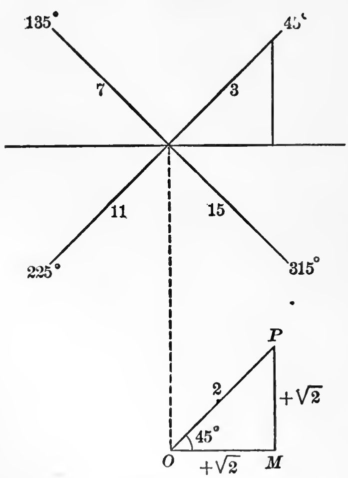


FIG. 40.

(*b*) For terminals 3, 7, 11, 15, use the diagram adjoining, in which a right-angled triangle whose hypothenuse, coincident with terminal 3, is 2 units of length, and taken with one vertex at the origin and one leg (length $\sqrt{2}$) on the initial line.

(i) Then, $\sin 45° = \frac{MP}{OP} = \frac{+\sqrt{2}}{2} = \sin\frac{\pi}{4}$.

(ii) Terminals 3, 7, are symmetric to the vertical, and therefore belong to angles of the same sine (§ 62),

$$\therefore \sin 135° = \sin 45° = \frac{+\sqrt{2}}{2} = \sin\frac{\pi}{4} = \sin\frac{3\pi}{4}.$$

(iii) Terminals 3, 11, are opposite and therefore belong to angles of opposite sine (§ 63),

$$\therefore \sin 225° = -\sin 45° = \frac{-\sqrt{2}}{2} = -\sin\frac{\pi}{4} = \sin\frac{5\pi}{4}.$$

(iv) Terminals 3, 15, are symmetric to the horizontal, and therefore belong to angles of opposite sine (§ 63),

$$\therefore \sin 315° = -\sin 45° = \frac{-\sqrt{2}}{2} = -\sin\frac{\pi}{4} = \sin\frac{7\pi}{4}.$$

EXERCISES.

1. Find the sines of the negative angles numerically equal to those above.

2. Give in degree measure and in radian measure the general formulas for all angles having the same sine as 45°. Also the general formulas for all angles having the opposite sine of 45°.

3. Find from the diagram (Fig. 40) six positive angles having the same sine as 45°. Which lines may be their terminals? What values of m in the general formula give the angles which you have found?

4. Determine similarly six negative angles.

(*c*) For terminals 2, 8, 10, 16, use the diagram adjoining, in which an equilateral triangle whose side is of length 2, is set with one vertex at the origin and a bisector on the initial line. The side OP (Fig. 41) will fall on terminal 2, MP will be 1, and OM will be $\sqrt{3}$.

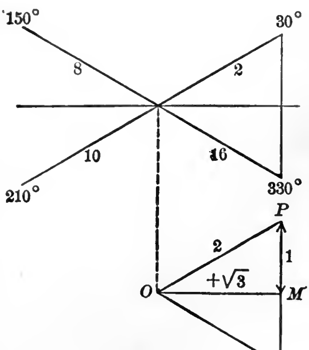


FIG. 41.

Then, exactly as in the preceding case,

(i) $\sin 30° = \frac{1}{2} = \sin\frac{\pi}{6}$;

(ii) $\sin 150° = \sin 30° = \frac{1}{2} = \sin\frac{\pi}{6} = \sin\frac{5\pi}{6}$;

(iii) $\sin 210° = -\sin 30° = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\frac{7\pi}{6}$;

(iv) $\sin 330° = -\sin 30° = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\frac{11\pi}{6}$.

EXERCISES.

Take those on page 121, replacing 45° by 30°.

(*d*) For terminals 4, 6, 12, 14, use Fig. 42, an equilateral triangle, whose side is of length 2, one side coinciding with the initial line and another with terminal 4.

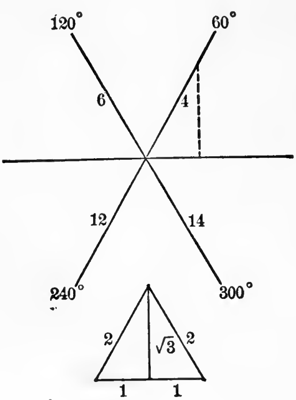


FIG. 42.

(i) $\sin 60° = \frac{+\sqrt{3}}{2} = \sin\frac{\pi}{3}$;

(ii) $\sin 120° = \sin 60° = \frac{+\sqrt{3}}{2}$

$= \sin\frac{\pi}{3} = \sin\frac{2\pi}{3}$;

(iii) $\sin 240° = -\sin 60° = \frac{-\sqrt{3}}{2}$

$= -\sin\frac{\pi}{3} = \sin\frac{4\pi}{3}$;

(iv) $\sin 300° = -\sin 60° = \frac{-\sqrt{3}}{2}$

$= \sin\frac{\pi}{3} = \sin\frac{5\pi}{3}$.

EXERCISES.

Those on page 121, replacing 45° by 60°.

NOTE. — The student is advised against attempting to memorize the foregoing results as mere facts. Let the diagrams be called up when needed, and read from the mental diagram the value sought:

(*a*) For 45° and the related angles, use the diagram shown in Fig. 43.
(*b*) For 30° and the related angles, use the diagram shown in Fig. 44.
(*c*) For 60° and the related angles, use the diagram shown in Fig. 45.

These diagrams determine the sine in magnitude. The sign is then determined by the location of the terminal.

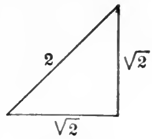


FIG. 43.

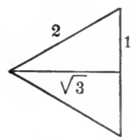


FIG. 44.

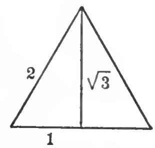


FIG. 45.

These results may be tabulated as follows:

Angles										
	degrees	0,	30,	45,	60,	90,	120,	135,	150,	180,
	radians	0,	$\frac{\pi}{6}$,	$\frac{\pi}{4}$,	$\frac{\pi}{3}$,	$\frac{\pi}{2}$,	$\frac{2\pi}{3}$,	$\frac{3\pi}{4}$,	$\frac{5\pi}{6}$,	π,

whose sines are $0,\ \frac{1}{2},\ \frac{+\sqrt{2}}{2},\ \frac{+\sqrt{3}}{2},\ 1,\ \frac{+\sqrt{3}}{2},\frac{+\sqrt{2}}{2},\frac{1}{2},\ 0.$

EXERCISES.

1. What are the sines of the corresponding negative angles?

2. Read from mental diagrams the sines of 0°, 45°, 30°, 60°, 90°.

3. Considering the terminals of the angles in Ex. 2 as the fundamental terminals, state the relative position of any other one of the sixteen terminals of Fig. 39 with reference to some one of these, and thereby determine at once the sine of the corresponding angle.

The class may be drilled several days in succession at this, until the sine of any one of the sixteen terminals can be read off at once from the proper mental diagram.

4. Using the principal angle corresponding to each of the following sines, determine the general angles for the following expressions:

$\sin^{-1}\left(+\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(+\frac{\sqrt{3}}{2}\right)$; $\qquad \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)+4\sin^{-1}\frac{\sqrt{2}}{2}$;

$\sin^{-1}\frac{\sqrt{2}}{2}+\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$; $\qquad 3\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$;

$\sin^{-1}\frac{1}{2}+\sin^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$; $\qquad 5\sin^{-1}\frac{1}{2}$;

$\sin^{-1}\frac{\sqrt{3}}{2}+2\sin^{-1}\left(-\frac{1}{2}\right)$; $\qquad 2\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)+5\sin\left(-\frac{\sqrt{3}}{2}\right)$;

$\sin^{-1}\left(-\frac{1}{2}\right)+\sin\left(-\frac{\sqrt{2}}{2}\right)$; $\sin^{-1}(-1)+3\sin^{-1}\left(-\frac{1}{2}\right)-2\sin^{-1}\frac{\sqrt{3}}{2}$;

$\sin^{-1}0+\sin^{-1}1$; $2\sin^{-1}0$;

$3\sin^{-1}1+2\sin^{-1}\left(\frac{1}{2}\right)-4\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$; $6\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)-5\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$;

and so on, until the sines of these fundamental angles are known.

5. Verify the following statements (sign not considered):

$\sin 2.45^\circ < 2\sin 45^\circ$; $\sin 2.30^\circ < 2\sin 30^\circ$;

$\sin 2.90^\circ < 2\sin 90^\circ$; $\sin 2.60^\circ < 2\sin 60^\circ$;

$\sin 2.120^\circ < 2\sin 120^\circ$; $\sin 2.135^\circ < 2\sin 135^\circ$;

$\sin 2.150^\circ < 2\sin 150^\circ$; $\sin 2.180^\circ = 2\sin 180^\circ$.

6. Consider similarly the remaining positive angles less than 360° corresponding to the sixteen terminals of Fig. 39.

7. Compare results when the angles of Ex. 5 are taken as negative.

8. Give three special solutions and the general solution of:

$\sin^2 x = 0$; $\sin^2 x = 1$; $\sin^2 x = \frac{1}{2}$; $\sin^2 x = \frac{3}{4}$;

$\sin^2 x = \frac{1}{4}$; $\sin 3x = \frac{1}{2}$; $\sin^2\frac{3x}{2} = \frac{3}{4}$; $\sqrt{2}\sin 5x = 1$.

9. Using the general formula for angles of a given sine, determine the general values of the angles represented by the expressions in Ex. 4.

10. Taking the general form of the quadratic equation as $ax^2 + bx + c = 0$, show that the general values of x are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Sufficient special numerical examples may here be assigned to make sure that the class can write at once the solution of any quadratic equation without going through the usual process of completing the square.

11. Find general expressions for the angles satisfying the following equations:

$$4\sin^2 x - 2(1+\sqrt{2})\sin x + \sqrt{2} = 0.$$
$$4\sin^2 x + 2(1+\sqrt{2})\sin x + \sqrt{2} = 0.$$
$$4\sin^2 x + 2(1-\sqrt{2})\sin x - \sqrt{2} = 0.$$
$$4\sin^2 x + 2(\sqrt{2}-1)\sin x - \sqrt{2} = 0.$$
$$4\sin^2 x - 2(\sqrt{3}+1)\sin x + \sqrt{3} = 0.$$
$$4\sin^2 x + 2(\sqrt{3}+1)\sin x + \sqrt{3} = 0.$$
$$4\sin^2 x + 2(\sqrt{3}-1)\sin x - \sqrt{3} = 0.$$

$$4 \sin^2 x - 2(\sqrt{3} - 1) \sin x - \sqrt{3} = 0.$$
$$4 \sin^2 x - 2(\sqrt{2} + \sqrt{3}) \sin x + \sqrt{6} = 0.$$
$$4 \sin^2 x + 2(\sqrt{2} + \sqrt{3}) \sin x + \sqrt{6} = 0.$$
$$2\sqrt{2} \sin^2 x - \sqrt{2}(\sqrt{2} - \sqrt{3}) \sin x - \sqrt{3} = 0.$$
$$2\sqrt{2} \sin^2 x - \sqrt{2}(\sqrt{3} - \sqrt{2}) \sin x - \sqrt{3} = 0.$$
$$2 \sin^2 x - (\sqrt{2} + 2) \sin x + \sqrt{2} = 0.$$
$$2 \sin^2 x + (\sqrt{2} + 2) \sin x + \sqrt{2} = 0.$$
$$2 \sin^2 x + (\sqrt{2} - 2) \sin x - \sqrt{2} = 0.$$
$$2 \sin^2 x - (\sqrt{2} - 2) \sin x - \sqrt{2} = 0.$$

12. Find what values of x will satisfy the following equations, giving x in degrees and in terms of π in each case:

$\sin^{-1}(x^2 - x) = 30°$;
$\sin^{-1}(x^2 - 5x) = -30°$;
$\sin^{-1}(-4x^2 + 7x) = -45°$;
$\sin^{-1}(x^2 + 6x) = 60°$;
$\sin^{-1}(3x^2 - 8x) = -60°$;
$\sin^{-1}(4x^2 - 4x) = 120°$;
$\sin^{-1}(3x^2 + 7x) = -120°$;
$\sin^{-1}(2x^2 - 4x) = \pm 135°$;
$\sin^{-1}(x^2 + x) = \pm 150°$;
$\sin^{-1}(x^2 - x) = \pm 180°$;
$\sin^{-1}(x^2 - x) = \pm 210°$;
$\sin^{-1}(x^2 - 2x) = \pm 90°$;
$\sin^{-1}(4x^2 - x) = \pm 270°$;
$\sin^{-1}(7x^2 - x) = \pm 225°$;
$\sin^{-1}(2x^2 - 3x) = \pm 240°$;
$\sin^{-1}(3x^3 + 5x) = \pm 360°$;
$\sin^{-1}(x^2 - x) = \pm 300°$.

13. Name two angles for each of the examples of Ex. 12, which might replace the second member of the equation without altering the final result.

14. Show that, knowing the sines of 30°, 45°, 60°, one can, by constructing a diagram, find the sines of the following angles:

± 405°; ± 390°; ± 420°; ± 840°; ± 660°; ± 690°; ± 1050°; ± 960°; ± 1305°; ± 1200°; ± 1020°; ± 945°; ± 855°; ± 870°; ± 810°; ± 900°; ± 780°; ± 570°.

§ 76. Calculations using Sines, without use of Tables except as Check.

Since the three quantities, sine, ordinate, modulus, are connected by the relation of definition,

$$\text{sine} = \frac{\text{ordinate}}{\text{modulus}},$$

if any two are given, the third can be calculated as follows:

(i) When ordinate and modulus are given,

$$\text{sine} = \frac{\text{ordinate}}{\text{modulus}}.$$

(ii) When angle and modulus are given,

$$\text{ordinate} = \text{modulus times sine}.$$

(iii) When angle and ordinate are given,

$$\text{modulus} = \frac{\text{ordinate}}{\text{sine}}.$$

These three formulas are to be memorized each as a primary relation, so there will be no hesitation in writing any one of them, due to halting to deduce it from some other. The author's experience is that where students learn trigonometry from the ratio definitions, the tendency is to hold in mind as a primary relation that of (i), there being a halt always in seeing (ii), (iii), due to their deduction from (i). The class should be drilled on exercises like the following until the three relations are known equally well.

EXERCISES.

N.B. *Never carry the number of figures in calculated results beyond the number in the given quantities. All measured sides should show the same number of significant figures. All lengths and angles given must be assumed to represent measurements and are thus approximate.*

1. Draw diagrams to scale, illustrating the following data, and calculate the corresponding ordinates. Test each result by calculating backward the given modulus from the calculated ordinate. Also draw the ordinate in the diagram, measure it, and compare the result with the calculated value. Measure the angles in the diagram and check by comparing the given (or calculated) sine with the table-sine.

Modulus.	Sin A.	Ordinate?	Modulus.	$A°$.	Ordinate?
250	$\pm\frac{2}{3}$		316	± 30	
350	$\pm\frac{4}{5}$		250	± 45	
32	$\pm\frac{1}{4}$		216	± 60	
16	$\pm\frac{1}{2}$		215	± 90	
85	$\pm\frac{1}{3}$		16	± 120	
90	$\pm\frac{1}{3}$		80	± 135	
100	$\pm\frac{2}{5}$		100	± 150	
25	$\pm\frac{1}{3}$		120	± 210	
17	$\pm\frac{5}{6}$		150	± 225	

Modulus.	Sin A.	Ordinate?	Modulus.	A°.	Ordinate?
50	$\pm \frac{1}{4}$		30	± 240	
8	$\pm \frac{1}{5}$		70	± 300	
10	$\pm \frac{1}{5}$		80	± 315	
25	$\pm \frac{1}{6}$		90	± 330	
80	$\pm \frac{3}{4}$		100	± 360	
20	$\pm \frac{3}{5}$		200	± 390	
60	$\pm \frac{7}{10}$		8	± 405	
75	$\pm \frac{4}{5}$		10	± 420	
80	$\pm \frac{1}{62}$		12	± 450	
100	$\pm \frac{3}{20}$		14	± 270	
200	$\pm \frac{7}{8}$		18	± 0	

2. Interchange the words "ordinate" and "modulus" in the above table, construct to scale the corresponding diagram, calculate the corresponding moduli, test as in Ex. 1; measure; compare.

Note. — Construction to scale will be found quite helpful in fixing in mind the sine as a ratio. The actual lines of the diagram will be seen to bear to each other the ratio indicated by the sine.

3. Construct to scale diagrams illustrating the following data, and calculate the corresponding sines to as many decimal places as the data.

Modulus.	Ordinate.	Sin A.	Modulus.	Ordinate.	Sin A.
5	$\pm$ 3		61.2	$\pm$ 60.1	
8	$\pm$ 4		61	$\pm$ 11	
13	$\pm$ 12		6.53	$\pm$ 3.32	
13	$\pm$ 8.2		65	$\pm$ 56	
17	$\pm$ 15		65.24	$\pm$ 16.32	
17	$\pm$ 17		6.5	$\pm$ 6.3	
17	$\pm$ 7.1		73	$\pm$ 48	
25	$\pm$ 24		7.31	$\pm$ 48.2	
29	$\pm$ 20		0.85	$\pm$ 0.36	
2.9	$\pm$ 2.1		85	$\pm$ 77	
37	$\pm$ 12		36	$\pm$ 9.2	
37	$\pm$ 35		85.3	$\pm$ 84.2	
41	$\pm$ 9.2		89	$\pm$ 80	
41	$\pm$ 19		9.721	$\pm$ 6.532	
53	$\pm$ 28		61	$\pm$ 16	
5.3	$\pm$ 4.5		43	$\pm$ 34	

4. On the diagrams representing the data above, measure any other ordinate and its modulus; divide, and compare the resulting sine with that gotten from the given numbers. Also measure the angle and compare calculated sine with the table sine.

§ 77. Selecting and Cutting a Table of Sines.

As already indicated, the number of angles for which the sine can be determined directly from a diagram is quite limited, and, in fact, such angles here, as in other text-books, are given far more prominence than they deserve. It is customary to use the tables for all angles except multiples of 90°. What place table to use depends upon the accuracy with which the measurements have been made and upon the accuracy to which the calculated result is desired. As pointed out in §§ 20, 22, 24, the tendency of calculation manipulations is to make the calculated results less accurate than the measurements on which they are based. The resulting inaccuracy depends on the number of additions, multiplications, etc., made.

It must be assumed in calculations with triangles that the lines whose lengths are given have been actually measured, as also the angles given. All the given parts of a diagram should indicate about the same degree of accuracy in measurement.

In calculating triangles, the number of *operations being always few*, the following rules may be laid down:

(1) All measured lines must show the same number of significant figures. (See § 33.)

(2) Calculated lines, sines, and logarithmic sines must not show more significant figures than the measured lines. They may generally show the same number. (See § 30.)

(3) When the measured lines of a diagram show only one significant figure, measured angles and calculated angles may read to the nearest five degrees; with only two significant figures in lines, angles may read to the nearest half-degree, as with a surveyor's compass.

(4) When the measured lines of a diagram show only three significant figures, angles may read to the nearest five minutes; while with only four significant figures, the measured angles and the calculated angles may read to the near-

est minute when a four-place table is used, and to the nearest ten seconds when a five-place table is used.

(5) In cases (3), (4), a four-place table is sufficiently accurate, generally. At times four significant figures in measured results call for the use of five-place tables. (See § 23.) Whatever table is used should be cut back to one place more than the data.

(6) When the measured lines show five significant figures, a five-place table or a six-place table should be used. The measured angles and calculated angles may then read to the nearest second with a six-place table, and to the nearest five seconds with a five-place table.

(7) In astronomical calculations, when the number of terms makes a balancing of table-errors likely, a five-place table is used when the given angles and calculated angles are given to the nearest second; a six-place table, when tenths of seconds appear; a seven-place table, when hundredths of seconds appear.

(8) Do not use a seven-place table when a four-place table is sufficient, nor a four-place table when a seven-place table is necessary. One place beyond the data is all that is ever needed. The smaller the suitable table, the greater is the saving of time in manipulation.

Proof of the Rules.

Sin $x = 0.1$ means that $\sin x$ lies between 0.05 and 0.15. Taking from the tables the angles opposite the sines 0.05 and 0.15, which stand closest to the sine 0.1, we get $3° 26'$ and $8° 38'$, whose mean may be taken as x when $\sin x = 0.1$, and half whose difference, or $2° 36'$, will be the possible error in x. Similarly, $\sin x = 0.2$ has as extremity angles, $9° 15'$ and $14° 29'$, whose mean may be taken as x, with the possible error of $2° 37'$. Treating 0.3, 0.4, ⋯, 0.9, the same way, the *average* error for the whole table is about $3° 30'$ in determining an angle from one figure in the sine. Treating 0.10, 0.20, etc., in the same way, the average is about $20'$.

Safe working limits then are 5° and 30′, respectively, as in rule (3). An increase of one figure in the line-data, when the table has at least one place more than the data, divides the error by 10. This and § 23 give the remaining rules.

§ 78. A Four-place Table.

In the tables accompanying this book is a four-place table of sines and of logarithmic sines. It is not to be used:

(1) When the measured lines of the diagram show more than four significant figures.

(2) When the measured angles of a diagram show seconds.

(3) For small angles (less than about 5°) when the sines (angles) cannot be obtained directly from the table without interpolation. Here the sines change too rapidly to use their differences. (See § 186.)

How to use these tables is explained in the preface of the tables.

In practical work in electrical and mechanical engineering the data are largely three-figured. A four-place table meets most of the requirements.

§ 79. Some Calculations using Sines of the Tables and Logarithmic Sines. (Four-place Tables.)

From the relations

(i) $\text{sine} = \dfrac{\text{ordinate}}{\text{modulus}}$,

(ii) $\text{ordinate} = \text{modulus times sine}$,

(iii) $\text{modulus} = \dfrac{\text{ordinate}}{\text{sine}}$,

follow the relations

(iv) $\log \sin = \log \text{ord} - \log \text{mod}$,

(v) $\log \text{ord} = \log \text{mod} + \log \sin$,

(vi) $\log \text{mod} = \log \text{ord} - \log \sin$.

A. Examples on Calculating the Ordinate (case ii).

(1)

Given $\begin{cases} \text{modulus} = 8 \\ \text{angle} = 25° \end{cases}$

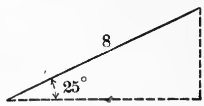


Fig. 46.

Find ordinate = 3

Ordinate = modulus times sine.
log ord = log mod + log sin.

Modulus being given to only one significant figure, the ordinate should be calculated to only one significant figure. By natural sines, to as many figures as the data, is the proper method. We give logs here and in (2) as a mere check, or rather as a mere sample process in cutting logs to one figure beyond the data.

SOLUTION BY NATURAL SINES.

$$\begin{aligned} \sin 25° &= 0.4 \\ \text{mod} &= \underline{8} \\ \therefore \text{ord} &= 3 \end{aligned}$$

Possible error = ± 0.6

SOLUTION BY LOGARITHMS.

$$\begin{aligned} \log 8 &= 0.90 \\ \log \sin 25° &= \underline{\bar{1}.63} \\ \log \text{ord} &= 0.53 \\ \therefore \text{ord} &= 3 \end{aligned}$$

(2) Given, modulus = 27, angle = 22°.

SOLUTION BY NATURAL SINES.

$$\begin{aligned} \sin 22° &= 0.38 \\ \text{mod} &= \underline{27} \\ & 7.6 \\ & \underline{2.6} \\ & 10 \quad \pm 0.3 \end{aligned}$$

SOLUTION BY LOGARITHMS.

$$\begin{aligned} \log 27 &= 1.431 \\ \log \sin 22° &= \underline{\bar{1}.574} \\ \log \text{ord} &= 1.005 \\ \therefore \text{ord} &= 10 \end{aligned}$$

The solution by natural sines is to be carried out by the shortened process of multiplication. (See § 26.) The numbers 0.6 and 0.3 after the ordinates show the possible error in these results if tenths were written down, and are obtained by § 25.

The student is advised to follow some scheme like that above. Place the data on one side of a diagram. Place the things to be found on the other side. Draw the given parts of the diagram in heavy lines; the parts to be found in dots. Make the diagram to scale on coördinate paper; measure as a check. Place the formulas to be used under the diagram. Check by combining the given and calculated quantities in some new relation, directly or by logarithms.

NOTE. — The student may solve the following examples both by natural sines and by logarithmic sines, and compare results. He may also work the examples backward, as a test of accuracy. Since every calculation ought to be tested in some way by the person making it, no answers are given. With one-figured and two-figured data, practical engineers use the natural functions.

EXERCISES.

Make all diagrams to scale on coördinate paper. Measure the ungiven part as a check on calculation.

Keep in mind, in making the diagrams, that with data like Ex. 1, a *measured* modulus 9 means not 9.0, but something between 8.5 and 9.5, and that the angle 35°, given as going with such a modulus, means that 35° is the angle to the nearest 5°, and that the angle lies between 30° and 40°, but nearer 35° than to 30° or to 40°. If diagrams are made representing the extreme cases of measurement signified by the data, it will become very apparent why calculated lines should not be carried to more places than the data. Make such diagrams for the following examples. In examples like 5, 6, 7, 8, modify the angle readings to suit the table used, according to the rules on page 129.

1. Modulus 9, angle $\pm 35°$; modulus 7, angle $\pm 55°$; modulus 6, angle 85°.

2. Modulus 9, angle $\pm 105°$; modulus 8, angle $\pm 200°$; modulus 7, angle $\pm 305°$.

3. Modulus 23, angle $\pm 34°$; modulus 5.6, angle $\pm 123°$; modulus 6.8, angle $\pm 196° 30'$; modulus 84, angle $\pm 302° 30'$; modulus 23, angle $\pm 4325°$.

4. Modulus 45.3, angle $\pm 34° 5'$; modulus 4.57, angle $\pm 174° 10'$; modulus 34.8, angle $\pm 267° 20'$; modulus 41.3, angle $\pm 27942° 15'$.

5. Modulus 45.41, angle $\pm 23° 14'$; modulus 4.587, angle $\pm 174° 23'$; modulus 0.4752, angle 234° 43′; modulus 678.4, angle $\pm 4545° 2'$.

6. Modulus 345.61, angle 34° 25′ 43″; modulus 23.598, angle 172° 34′ 56″; modulus 4567.8, angle 256° 31′ 36″; modulus 0.58923, angle 295° 17′ 13″; modulus 7.8519, angle 862° 44′ 44″ (at least five-place table).

7. Modulus 234.567, angle 45° 47′ 53″.2 (at least six-place table).

8. Modulus 2564.321, angle 82° 13′ 27″.34 (at least seven-place table).

9. What data would suit an uncut ten-place table?

10. What data would suit a three-place table?

B. Examples on Calculating the Modulus (case iii).

(1)

Given $\begin{cases} \text{ordinate} = 8 \\ \text{angle} \quad = 17° \end{cases}$ Find modulus = 30

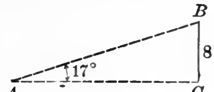


Fig. 47.

$$\text{Modulus} = \frac{\text{ordinate}}{\text{sine}}.$$

$$\therefore \log \text{mod} = \log \text{ord} - \log \text{sin}.$$

SOLUTION BY SINES.

sin

$.3\overline{)8}$

26

which must be taken as 30, since ordinate has only one significant figure. Are the data consistent?

SOLUTION BY LOGARITHMS.

$$\log 8 = 0.90$$
$$\log \sin 17° = \bar{1}.47$$
$$\log \text{mod} = 1.43$$
$$\therefore \text{mod} = 27$$
$$\therefore \text{mod} = 30$$

(2)

Given $\begin{cases} \text{ordinate} = 13.27 \\ \text{angle} \quad = 42° 17' \end{cases}$ Find modulus = 19.73

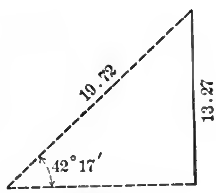


Fig. 48.

$$\text{Modulus} = \frac{\text{ordinate}}{\text{sine}}.$$

$$\therefore \log \text{mod} = \log \text{ord} - \log \text{sin}.$$

SOLUTION BY NATURAL SINES.

(Shortened division process.)

sin 19.73 mod

$6728\overline{)132700}$

673

654

605

49

47

2

SOLUTION BY LOGARITHMS.

$$\log \text{ord} = 1.1229$$
$$\log \text{sin} = \bar{1}.8279$$
$$\log \text{mod} = 1.2950$$
$$\text{mod} = 19.73$$

The solution by natural sines is made by the shortened process of division. (See § 40.) To determine to how many places the division may be carried, use may be made of § 27 and § 28, from which it follows that the per cent rejection error in the modulus is $\frac{100}{2}(\frac{1}{6728} + \frac{1}{1372})$ per cent, or about $\frac{1}{2}(\frac{1}{67} + \frac{1}{13})$, or about $\frac{1}{2}(\frac{1}{65} + \frac{1}{13})$, or about $\frac{3}{65}$ of 1 per cent, or about $\frac{1}{20}$ of 1 per cent. The modulus being about 20, the error is about 0.01. The division should not be carried, therefore, beyond hundredths. By the shortened process of division, the operation is self-limited, when the data have the same number of significant figures. It ceases at the limit of possible inaccuracy.

The results by natural sines and by logs might have differed by 1 in the last place.

The work may also be tested by the backward calculation,

ordinate = sine times modulus.

```
 0.6728  sin
19.73    mod
-------
 6.728
 6.055
  .470
  .020
-------
13.27    ord
```

EXERCISES.

Interchange the words "ordinate" and "modulus" in the exercises of (A), page 132, and solve. Determine in each case the maximum error in the result due to rejection error in the ordinate and sine. Make diagrams to scale on coördinate paper, and measure the ungiven parts as a check on calculation.

C. Examples on Calculating the Sine and Angle from the Ordinate and Modulus (case i).

Given $\begin{cases} \text{ordinate} = 36.16 \\ \text{modulus} = 67.32 \end{cases}$

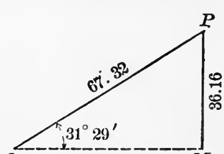


Fig. 49.

Find angle = 31° 29′

$$\text{sine} = \frac{\text{ordinate}}{\text{modulus}}$$

$$\log \sin = \log \text{ord} - \log \text{mod}.$$

SOLUTION BY NATURAL SINES.

(Shortened division.)

```
          0.5371 sin
6732)3616.
     3366
      250
      202
       48
       47
        1
```

Table angle, 32° 29′.

General angle,

$m \cdot 180° + (-1)^m\, 32°\ 29'$.

SOLUTION BY LOGARITHMS.

$$\log \text{ord} = 1.5582$$
$$\log \text{mod} = 1.8281$$
$$\log \sin = \bar{1}.7301$$

Table angle, 32° 29′.

General angle,

$m \cdot 180 + (-1)^m\, 32°\, 29'$.

ADDITIONAL TEST OF ACCURACY.

ordinate = modulus times sine.

```
67.32   mod
 .5371  sin
33.660
 2.020
  .471
  .007
36.16   = ord
```

To determine the number of places to which the sine may be determined from the data, assuming rejection error, use may be made of § 27 and § 28.

Per cent rejection error in sine is

$$\tfrac{100}{2}\left(\tfrac{1}{6732} + \tfrac{1}{3616}\right), \text{ or about } \tfrac{1}{2}\left(\tfrac{1}{67} + \tfrac{1}{36}\right),$$
$$\text{or about } \tfrac{1}{2}\left(\tfrac{1}{70} + \tfrac{1}{35}\right), \text{ or about } \tfrac{3}{140} \text{ of } 1\%.$$

Now, $\frac{3}{140}$ of 1% of the sine, 0.5371, is about 0.0001. So the division may not be carried beyond four figures.

The results here and in the previous cases agree with the general statement made in rule (2) of § 77, that calculated results must not be carried beyond the number of significant figures in the data.

It will be observed in any large table of sines, where seconds are taken into account, that over a large part of the table, when the angles differ by only a second, the sines agree in the first four figures. Thus, in four-figured data, calculated angles cannot, as a rule, be determined to the nearest second. It is sufficient to give only minutes.

EXERCISES.

Determine the sine and angle in the following cases:

Ordinate.	Modulus.	Ordinate.	Modulus.	Ordinate.	Modulus.
72.16	95.73	71.6	94.3	2.31	8.56
1.831	3.924	54.3	83.9	0.5673	1.906
21	67	5	7	5.43	9.56

Take the ordinates as both plus and minus.

§ 80. Using a Five-place Table.

Engineering students should have some good five-place tables, as Gauss's or Hussey's. The teacher may here explain the use of such a table, if it is to be used. He may also make five-figured data, corresponding to the cases above, and have the class make the corresponding calculations. Given angles will read now to seconds; also calculated angles will be given to seconds, or five seconds. See § 77, (6).

Similarly, some practice may be given on examples where a six-place or seven-place table is called for.

§ 81. Solution of Right-angled Triangles by Sines.

The selection of the position of the initial line is altogether arbitrary, as is also the direction for positive or negative turn. If in Fig. 50 AH is taken as initial line and AB as modulus, with the counter-clockwise turn as positive, then HB is ordinate,

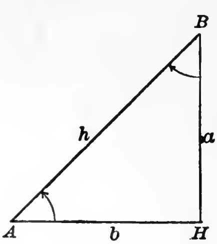


Fig. 50.

$$\therefore \sin A = \frac{HB}{AB}.$$

Similarly, counting BH as initial line, BA as modulus, and the clockwise turn as positive, then HA is ordinate,

$$\therefore \sin B = \frac{HA}{BA}.$$

In considering right-angled triangles, it is not customary to pay any attention to the question of direction, either of turns or of lines, but to use the general statements

$$\text{sine} = \frac{\text{side opposite the angle}}{\text{hypothenuse}}.$$

The angles being acute, their sines are always positive. Using A, a; B, b, to denote angles and their opposite sides, or pairs of opposites, h being the hypothenuse, we have:

(1) $$\sin A = \frac{a}{h}; \ \sin B = \frac{b}{h}, \text{ or,}$$

the sine of either angle is the side opposite divided by the hypothenuse.

(2) $$a = \sin A \cdot h; \ b = \sin B \cdot h, \text{ or,}$$

either side is the sine of the opposite angle times the hypothenuse.

(3) $$h = \frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or,}$$

the hypothenuse is either side divided by the sine of its opposite angle.

(4) $$h = \sqrt{a^2 + b^2}.$$

The relation (4) can be used advantageously to determine h only when a, b are small. After h is found, either angle can be calculated from its sine. When a, b are large, the angles are best found by a method which will be given later. (See § 169.)

These relations, (1), (2), (3), are easily memorized, and should be memorized thoroughly, each as a primary fact. Note that in (1), (2), opposites, as A, a, etc., are on opposite sides of the relations. Thus, as in (1), when $\sin A$ is written, write at once on the other side of the equation its opposite, a, and then under a write h, since a sine is a ratio. Likewise, when a is placed on one side of the equation, write the sine of its opposite, sin A, on the other, and then h, to make the relation homogeneous, — a line equal to a line.

Examples in right-angled triangles can be of only four types, corresponding to the four sets of relations above:

(*a*) *Given the hypothenuse and a side, to find the two angles and the remaining side.*

For example, given a, h,

then $$\sin A = \frac{a}{h};\quad B = 90 - A;\quad b = \sin B \cdot h.$$

(*b*) *Given either angle and the hypothenuse, to find the other angle and the two sides.*

For example, given A, h,

$$B = 90 - A;\quad a = \sin A \cdot h;\quad b = \sin B \cdot h.$$

(*c*) *Given a side and an angle, to find the remaining parts.*

For example, given a, A,

$$h = \frac{a}{\sin A};\quad B = 90 - A;\quad b = \sin B \cdot h.$$

(*d*) *Given the two sides, to find the remaining parts.*

$$h = \sqrt{a^2 + b^2};\quad \sin A = \frac{a}{h};\quad B = 90 - A.$$

See note above in connection with (4), page 137.

§ 82. Testing the Solution.

With $A + B = 90°$, $h = \dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

$$\therefore a \cdot \sin B = b \cdot \sin A.$$
$$h^2 - a^2 = b^2.$$

Thus the log-checks:

$$\log a + \log \sin B = \log b + \log \sin A.$$
$$\log (h - a) + \log (h + a) = 2 \log b.$$

In applying log-checks to "answers," the logarithms of such answers must be looked up in the tables, instead of checking with the logarithms of the calculation scheme. The latter may be correct and yet the "answers" be incorrect, an error having been made in looking up the numbers corresponding to logarithms.

Logarithmic checks are not to be held to a closer agreement than the data call for. It is always sufficient, as pointed out in Chapter II., if they agree within 2 in the last place, *such last place not being necessarily the last place of the table used, but the place to which the results are called upon to be correct, in accordance with the data.* For example, if the data are three-figured, the check is met if the logs differ by not more than 2 in the third place. (See the example, p. 140.)

In extended astronomical calculations, where the number of terms is sufficient to allow for balancing of errors, it is also reckoned sufficient if the log-checks agree to within 2 in the last place.

EXERCISES.

Test the following triangles for consistency, using log-checks:

1. $a = 42,\ b = 56,\ h = 70;\ a = 231,\ b = 95.7,\ h = 250.$

2. $a = 231,\ A = 67° 30',\ b = 95.7,\ B = 22° 30'.$

3. $a = 229,\ A = 37° 20',\ b = 300,\ B = 52° 40'.$

§ 83. Solution Scheme for solving Right-angled Triangles when Logarithms are used.

Construct to scale a diagram on which the parts given are indicated by heavy lines, and the parts to be found, by dotted lines. Write on the given parts their numerical values. Do the same with the dotted parts, after they have been found.

Set the parts given in one column, as in the following example. Set off a similar column for the parts to be found, in the order in which they will be found. Fill in with the parts found after the calculations have been carried out. Give under the diagram the formulas to be used.

The following calculation scheme is suitable for all cases except when the two sides are given. Merely the order of filling in the scheme will change with change of data, the scheme itself remaining unchanged.

Model Example.

Given $\begin{cases} h = 27.3 \\ B = 35^\circ\, 15' \end{cases}$

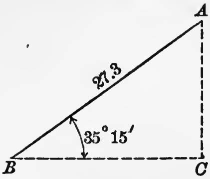


Fig. 51.

Find $\begin{cases} A = 54^\circ\, 45' \\ b = 15.8 \\ a = 22.3 \end{cases}$

Solution formulas $\begin{cases} A = 90 - B \\ b = \sin B \cdot h. \ \therefore \log b = \log \sin B + \log h \\ a = \sin A \cdot h. \ \therefore \log a = \log \sin A + \log h \end{cases}$

Solution Scheme.

LOGARITHMS.

(4)	b	$= 1.1975$
(2)	$\sin B$	$= \bar{1}.7613$
(1)	h	$= 1.4362$
(3)	$\sin A$	$= \bar{1}.9120$
(5)	a	$= 1.3482$

TEST.

$a \sin B = b \sin A$

LOGARITHMS.

1.1987	1.3483
$\bar{1}.9120$	$\bar{1}.7613$
1.1107	1.1096

Here, in applying the checks, we have taken the logs of 15.8 and 22.3, the nearest three-figured approximations corresponding to the logs of b, a, and not the logs of b, a in the scheme. *We are checking our answers. The data being three-figured, the answers are three-figured.* The log-check checks to within 2 in the third place, which is all that is required in the case of three-figured data. (See § 82.)

The order of filling in the solution scheme is indicated by the numerals, (1), (2), (3), (4), (5).

First look up logarithms of (1), (2), (3). Then add (1), (2), and set the result after (4). Next add (1), (3), and set the result after (5). Then look up the numbers corresponding to logarithms (4), (5), to as many places as the data, and enter the results after "Find" in the calculation scheme.

It is generally best not to enter numbers on the solution scheme, but letters, taking their logarithms to correspond to their values in the data.

If the given parts are a, A, the same solution scheme is used, but the order of filling in would be as indicated by the numerals following:

		LOGARITHMS.
(5)	$b =$	
(3)	$\sin B =$	
(4)	$h =$	
(2)	$\sin A =$	
(1)	$a =$	

Enter the logs (1), (2), (3). Next subtract (2) from (1) for (4). Then add (3), (4), for (5).

EXERCISES.

1. Indicate the order of filling in for any other case which may arise.

2. Make schemes into which cologs enter where subtractions occur above. Which do you prefer?

NUMERICAL EXAMPLES.

(Make diagrams to scale, for check.)

Determine both by natural sines and by logarithms, using the scheme given above, the ungiven parts of the following triangles, testing the final results as indicated above, finding also to how many significant figures the results should be taken if the data are assumed subject to rejection error:

1. Given $h=234.5$, $a=136.8$; $h=234$, $a=156$; $h=23$, $a=15$.
2. Given $h=46.7$, $b=23.8$; $h=46$, $a=25$; $h=4$, $a=2$.
3. Given $h=9.68$, $a=5.76$; $h=95$, $a=57$; $h=9$, $a=5$.
4. Given $h=468.2$, $A=46°\,34'$; $h=465$, $A=45°\,35'$; $h=46$, $A=45°$.
5. Given $h=34.43$, $B=38°\,23'$; $h=54.4$, $A=56°\,15'$; $h=5$, $A=40°$.
6. Given $h=543.2$, $A=68°\,43'$; $h=543$, $A=66°\,45'$; $h=54$, $A=80°$.
7. Given $a=45.6$, $A=76°\,35'$; $a=46$, $A=45°\,30'$; $h=4$, $A=45°$.
8. Given $b=213.4$, $B=64°\,46'$; $b=21.3$, $A=54°\,5'$; $h=21$, $A=54°$.
9. Given $a=96.2$, $B=49°\,45'$; $a=96$, $B=49°\,30'$; $h=9$, $B=40°$.

§ 84. Solution with Five-place Table.

The teacher may construct and assign corresponding data calling for the use of a five-place table.

§ 85. Calculations with Sines, with Practice Examples in the Use of Logarithms and the Shortened Processes of Multiplication and Division on Exact Data.

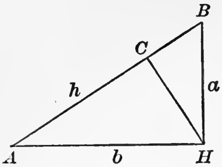


FIG. 52.

Let AHB (Fig. 52) be a right-angled triangle. Drop the perpendicular HC on AB. Then BC is called the projection of a on AB, and will be denoted by a_p. Similarly AC is b's projection, or b_p. The angle AHC is the same as B, while BHC is the same as A.

$$a_p = a \cdot \sin CHB = a \cdot \sin A = a \cdot \frac{a}{h} = \frac{a^2}{h}. \qquad (1)$$

$$b_p = b \cdot \sin CHA = b \cdot \sin B = b \cdot \frac{b}{h} = \frac{b^2}{h}. \qquad (2)$$

$$CH = h_h = a \cdot \sin CBH, \text{ or } b \cdot \sin CAH = \frac{ab}{h}. \qquad (3)$$

(*a*) *Oral exercises* in connection with the preceding triangle:

Taking a, b, h from the table of triangles, page 145, as in the adjoining diagram for Ex. 1 of that table, excellent practice on the sine may be had, as follows:

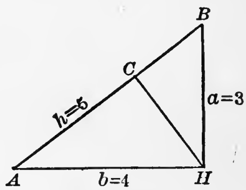


FIG. 53.

TEACHER (QUESTION).	STUDENT (ANSWER).
$\sin A$?	$\frac{3}{5}$
$\sin B$?	$\frac{4}{5}$
$\sin CHB$?	$= \sin A = \frac{3}{5}$
$\sin CHA$?	$= \sin B = \frac{4}{5}$
CH?	$= \sin B$ times $BH = \frac{4}{5}$ of $3 = \frac{12}{5}$ or, $\sin A$ times $AH = \frac{3}{5}$ of $4 = \frac{12}{5}$
AC?	$= \sin CHA$ times $AH = \frac{4}{5}$ of $4 = \frac{16}{5}$
BC?	$= \sin CHB$ times $HB = \frac{3}{5}$ of $3 = \frac{9}{5}$

The exercise may be varied by tilting the triangle in various attitudes, until the student is thoroughly familiar with:

(i) The sine is the side opposite divided by the hypothenuse.

(ii) Any side is the sine of the opposite angle times the hypothenuse.

(iii) The hypothenuse is any side divided by the sine of opposite angle.

(*b*) *Numerical exercises* in connection with the triangle, Fig. 52, and the table, page 145 :

In addition to (1), (2), (3), for practice in the use of logarithms and in the shortened processes of division and multiplication, one may calculate the area, Δ, of the triangle *AHB*, and the radii of the circles tangential to the sides of this triangle.

The area of *AHB* is $\frac{AH \cdot BH}{2}$

or, $$\frac{ab}{2} = \Delta. \qquad (4)$$

The radius of any tangential circle is found by joining its centre to the vertices *A*, *H*, *B* (Fig. 54). In the case of the internal circle the sum of the three new triangles formed is the original triangle *AHB*. In the case of an external circle (an *escribed circle*, it is generally called), the original triangle is the sum of the two new triangles which have for sides the two sides of the original triangle touched on the border of the original triangle, minus that new triangle whose side is that side of the original triangle touched on the border of the domain outside the original triangle.

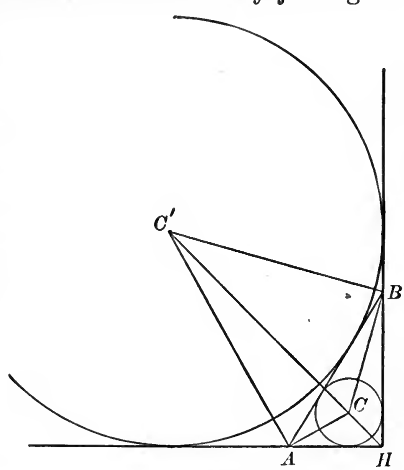


FIG. 54.

For the internal circle, whose centre is C,

$$\triangle CAH + \triangle CHB + \triangle CBA = \triangle AHB,$$

or,
$$r_i(a + b + h) = a \cdot b,$$

or,
$$r_i = \frac{a \cdot b}{a + b + h}. \qquad (5)$$

For the escribed circle touching AB and whose centre is C',

$$\triangle C'AH + \triangle C'HB - \triangle C'BA = \triangle AHB,$$

or,
$$r_h(a + b - h) = a \cdot b,$$

or,
$$r_h = \frac{a \cdot b}{a + b - h}. \qquad (6)$$

EXERCISES.

1. Deduce expressions corresponding to (6) for r_a, r_b.

2. Show that if Δ is the area of any triangle, right-angled or oblique, and s its semiperimeter,

$$r_a = \frac{\Delta}{s - a}, \quad r_b = \frac{\Delta}{s - b}, \quad r_c = \frac{\Delta}{s - c}, \quad r_i = \frac{\Delta}{s},$$

where r_i is the radius of the inscribed circle.

3. Show that $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r_i}$, for all triangles.

4. If r_0 is the radius of the circle circumscribing any triangle,

$$r_0 = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4 \Delta}.$$

5. Show that $r_0 r_i = \frac{abc}{2(a + b + c)}$.

6. Show that the lengths of tangents from the vertices A, B, C to inscribed circle are $s - a$, $s - b$, $s - c$, respectively; and for the escribed circles s, $s - b$, $s - c$; s, $s - c$, $s - a$; s, $s - a$, $s - b$.

(c) Table of triangles (taken from Dr. Conradt's "Trigonometry") for use in connection with exercises (a), (b) preceding.

The table on page 145 gives a large number of integers for which $a^2 + b^2 = h^2$, and from them a great variety of exercises on sines (and later cosines and tangents) will suggest themselves to the teacher. The numbers here being *exact*, and not representing measurements, calculated parts can be carried to any number of decimals.

	1	2	3	4	5	6	7	8	9	10	11	12
a	4	12	15	24	21	35	40	45	60	56	63	55
b	3	5	8	7	20	12	9	28	11	33	16	48
h	5	13	17	25	29	37	41	53	61	65	65	73
h_h	2.4	4.62	7.06	6.72	14.48	11.35	8.78	23.77	10.82	28.43	15.51	36.16
a_p	3.2	11.08	13.23	23.04	15.21	33.11	39.02	38.21	59.02	48.25	61.06	41.44
b_p	1.8	1.92	3.76	1.96	13.79	3.89	1.98	14.79	1.98	16.75	3.94	31.56
Δ	6	30	60	84	210	210	180	630	330	924	504	1320
r	1	2	3	3	6	5	4	10	5	12	7	15
r_h	6	15	20	28	35	42	45	63	66	77	72	88
A	53° 8′	67° 23′	61° 56′	73° 44′	46° 24′	71° 5′	77° 19′	58° 7′	79° 37′	59° 29′	75° 45′	48° 53′

	13	14	15	16	17	18	19	20	21	22	23	24
a	77	84	80	72	99	91	112	117	105	143	144	140
b	36	13	39	65	20	60	15	44	88	24	17	51
h	85	85	89	97	101	109	113	125	137	145	145	149
h_h	32.61	12.85	35.06	48.25	19.60	50.09	14.87	41.18	67.45	23.67	16.88	47.92
a_p	69.75	83.01	71.91	53.44	97.04	75.97	111.01	109.51	80.47	141.03	143.01	131.54
b_p	15.25	1.99	17.09	43.56	3.96	33.03	1.99	15.49	56.53	3.97	1.99	17.46
Δ	1386	546	1560	2340	990	2730	840	2574	4620	1716	1224	3570
r	14	6	15	20	9	21	7	18	28	11	8	21
r_h	99	91	104	117	110	130	120	143	165	156	153	170
A	64° 57′	81° 12′	64° 1′	47° 55′	78° 35′	56° 36′	82° 22′	69° 23′	50° 2′	80° 28′	83° 16′	69° 59′

§ 86. Vertical Projections.

If perpendiculars AA_1, BB_1 (Fig. 55), are drawn from the points A, B, of the straight line AB, to the vertical straight line MN, A_1B_1 is called the vertical projection of AB on MN.

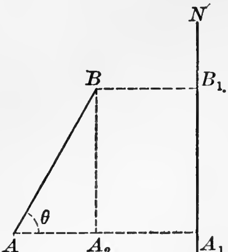


FIG. 55.

$$A_1B_1 = A_2B = AB \cdot \sin\theta.$$

∴ the length of the vertical projection of a given length, l, is l times the sine of the projecting angle, this angle being measured from the right-hand horizontal line as initial line, the starting-point for the length being taken as the pole, A for AB, B for BA.

Similarly, the projection of ABC (Fig. 56) is $l_1 \cdot \sin\theta_1 + l_2 \cdot \sin\theta_2$, or, *the projection of a sum is the sum of the projections.* Counting projections upward as plus and downward as minus, *the projection of any closed outline is evidently zero.* That of Fig. 57 is

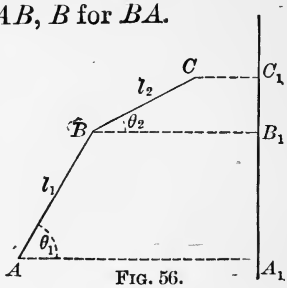


FIG. 56.

$$A_1B_1 + B_1C_1 + C_1D_1 + D_1E_1 + E_1F_1 + F_1A_1,$$

or zero, since there is a return to the starting-point. In surveying this is expressed by saying that *the sum of the northings is the same as the sum of the southings*, in a closed diagram representing a plotted survey. The lengths of the sides being l_1, l_2, l_3, . . . l_n, the sum of the projections is

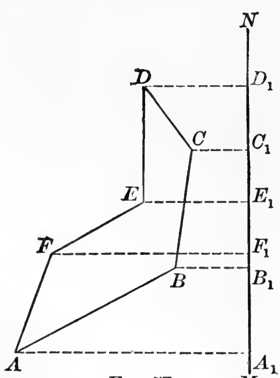


FIG. 57.

$$l_1 \cdot \sin\theta_1 + l_2 \cdot \sin\theta_2 + l_3 \cdot \sin\theta_3 + \ldots + l_n \cdot \sin\theta_n = 0.$$

What is here termed vertical projection is identical with difference of latitude in surveying, or, as commonly called, *latitude*.

∴ Difference of latitude of two points is their distance apart multiplied by the sine of the bearing of their connecting line (or "course") from the east and west line.

LABORATORY EXERCISE.

Draw a line and its projection on a vertical line. Measure the lines and the angle. Compare the measured length of the projection with its calculated length. Taking the projection as given, calculate the length of the projected line and compare with the measurement.

Carry out the preceding exercise in the field instead of on paper.

Make the projection of a closed broken line carefully in a drawing, measure each projection, and add algebraically. See how near the result is to zero.

Do the same in the field.

EXERCISES.

Find the latitude of each of the following courses, both by natural sines and by logarithms, and test the results. The lengths of the courses are given in chains, and the bearing, or direction from the north and south line, is indicated by the corresponding letters. Thus, N 20° E, 10 chains, signifies that the course is 10 chains long, and is directed 20° toward the east from the north line. So is S 20° E, 20° east from the line running south from the beginning of the course.

1. N 20° 13′ W, 18.34 chains.

2. N 34° E, 17.35 chains.

3. S 54° 34′ E, 85.45 chains.

4. S 47° 47′ W, 24.17 chains.

5. N 56° 56′ E, 34.34 chains.

6. Calling north latitude plus and south latitude minus, what is the bearing of a course whose latitude is + 17.21 and length 56.13 chains? What the bearing when the latitude is − 19.23 and length 75.34? How many different courses satisfy these conditions?

7. The notes of a survey give the following courses; determine whether the sum of the northings is numerically the same as the sum of the southings. Solve by natural sines and by logarithms and compare results:

N 69° E, 437.0 feet. S 19° E, 236.0 feet.
S 27° W, 244.0 feet. N 71° W, 324.0 feet.
N 19° W, 183.0 feet.

8. In an example like 7, with the data on sides reading to four significant figures, are the angles given sufficiently exact? Should they not read to minutes? Would dropping the final zeros leave the indicated accuracy unchanged?

9. Make up an example calling for the use of a five-place table.

10. One calling for a six-place table.

11. One calling for a seven-place table.

12. If a velocity along a line inclined at an angle $A°$ to the horizontal is represented by a line of given length, what line will represent the vertical component of such velocity?

13. What is the initial vertical velocity of a projectile which comes from a gun at an angle θ to the vertical with a velocity of V ft. per second? What is such velocity when $V = 26.3$ and $\theta = 21°$?

14. If a weight of w pounds rests (tied) on a smooth plane inclined at an angle θ to the vertical, what is the pressure on the plane? What is the pressure when $\theta = 23°$, and the weight 10 pounds? What is the pull on a string held parallel to the plane and holding such a weight?

15. A lever a feet long and inclined at an angle $A°$ to the vertical has a weight of p pounds at one end, what is the moment of this weight about the other end of the lever? What is it when $p=10$ pounds, $a=10$ feet, $A°=10°$?

§ 87. Horizontal Projections.

To use the sine in horizontal projections, the upright vertical is taken as the initial line and the clockwise turn as positive. Then projections running toward the right are plus, and those toward the left, minus.

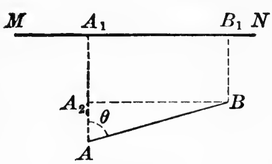


FIG. 58.

What has been said with reference to sums of vertical projections holds also for sums of horizontal projections. In surveying, the horizontal projection of a course is called its *departure*, or the difference in longitude of its extremities. The algebraic sum of the departures of a closed survey is zero.

EXERCISES.

1. Determine the departures of the courses given in Exs. 1–5 on page 147.

2. In Ex. 6, page 147, change the word "latitude" to departure, the word "north" to east, the word "south" to west, and solve.

3. Determine whether the sum of the eastings is the same as the sum of the westings in the survey of Ex. 7, page 147.

EXERCISES TO BE SOLVED WITHOUT THE USE OF TABLES.

Express the results in terms of the sines given in § 75, and in general terms.

1. A regular polygon of n sides is inscribed in a circle of radius r; find the general expression for the length of its sides in terms of the radius and half the central angle subtended by a side. To what inscribed polygons may the expression be applied without using the tables? Determine the sides of such polygons when the radius is 250 feet.

2. Determine the general expression for the perpendicular from the centre upon the sides of the polygons of Ex. 1, in terms of the radius and the half of the vertex angle. To what polygons may this be applied without using the tables? Make the calculations when the radius is 100 feet.

3. If a building is 50 ft. wide, calculate the length of rafters when the pitch of the roof is 30°, 45°, 60°, respectively, the roof being an isosceles triangle. Also the rise of the roof in each case. If A is the pitch and l the length of rafter, what is the rise? What the width? If b is the breadth and A the pitch, what is the length of rafter? What the rise? If p is the rise and A the pitch, what is the length of rafter? What the width?

4. A ladder 50 feet long is tilted at angle of 30° to the horizontal and leans against a building; find the distance of the top of the ladder from the ground, and of the foot from the building? Answer the same questions when the tilt is 45°; 60°.

5. A kite is flying 50 feet high; what is the length of string when the angle between the string and ground is 30°, assuming the string straight? When the angle is 45°? 60°?

6. The upper part of a tree broken over by the wind makes an angle of 30° with the ground, and the top lies 50 feet from the root; how tall was the tree?

7. The angle of elevation of a hill (its rise from a horizontal line) is 30°; what distance would one travel in walking up the hill to an elevation of 100 feet? What if the angle were 45°? 60°? What if the angle were A and the height h?

8. Find x, y, in the adjoining diagram. If 30°, 60° are changed to A, B, and 10 yards to a yards, find expressions for x, y, in terms of sines. (Drop perpendicular from B on AC.)

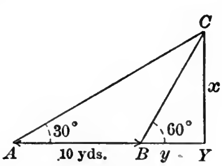


FIG. 59.

9. A regular polygon of n sides is circumscribed about a circle of radius r; find an expression for the length of the diameter of the polygon in terms of the sine of half the central angle; find also an expression for the length of the side of the polygon. Taking the radius as 50 feet, find the lines just mentioned for all regular polygons for which they may be found without using the tables.

10. A person on the top of a tower 80 feet high observes two objects on a straight road in line with the tower. The angle of depression of one object below a horizontal line passing through the top of the tower is 30°, and that of the other is 45°. Find the distance apart of the two objects. Solve the same problem when the angles are A, B, and the height h.

11. A man wishing to know the width of a river, selects a tree on the opposite bank, and finds its angle of elevation to be 45°. Going back 150 feet in a straight line with the tree and first point of observation, the elevation is 30°; how wide is the river? Solve the same problem when the angles are 60° and 30°; 60° and 45°; also when they are A, B, with the distance 150 changed to d.

12. How far from the polar axis of the earth is a point whose latitude is 30° (the radius of the earth being taken as 4000 miles)? 45°? 60°? Solve the same problem when the latitude is L and radius r.

13. Find the area of an equilateral triangle whose side is 100 feet. Solve the same problem when the side is s. Find also the distance of the centre of gravity of such a triangle from the vertex and from a side.

14. The hypothenuse of a right-angled triangle is 15 yards; what is its area when one of its angles is 30°? What when one of its angles has its sine equal to $\frac{2}{3}$?

15. The base of a right-angled triangle is 24 feet; what is its area when one of its angles is 30°? What when one of its angles has its sine equal to $\frac{3}{4}$?

EXERCISES USING THE TABLES.

1. Make up from each of the preceding fifteen examples an example for whose solution it is necessary to use the tables, by taking the angles other than 30°, 45°, 60°, etc. Solve the same, and apply in each case a suitable test. Solve each example by Natural Sines and by Logarithmic Sines, and compare results. Make in each case an example calling for the use of a four-place table, and one calling for the use of a five-place table. Suggest data suitable for a six-place table; for a seven-place table.

2. A smooth lever, turning in a vertical plane, has a 1-pound weight attached to its end; what connection with a table of sines (of the angles which the lever makes with the vertical) have the pressures at right angles to the lever? What if the weight is 10 pounds?

§ 88. Solving by Sines Triangles Other than Right-angled Triangles.

To solve an oblique-angled triangle there must be given not less than three parts, including a side. Only three different cases, so far as method of solution is concerned, can arise :

(α) A pair of opposites (an angle and its opposite side) is among the parts given.

(β) The three sides are given.

(γ) Two sides and the included angle are given.

Case (α) is always solved by sines. Case (β) may be solved by sines, though this is not the best solution when all three angles are desired (see § 185). Case (γ) cannot be solved by sines conveniently when the sides are long. It can be solved by dividing the triangle into right-angled triangles.

Case (α). *A pair of opposites are given.*

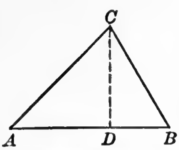


FIG. 60.

Drop a perpendicular, p, from C on AB (Fig. 60), or AB produced (Fig. 61). Then from the two triangles formed we have

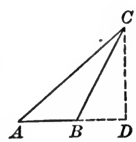


FIG. 61.

$$p = a \cdot \sin B,$$

$$p = b \cdot \sin A,$$

$$\therefore\ a \cdot \sin B = b \cdot \sin A.$$

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B}.$$

By symmetry, each one of these expressions is the same as $\frac{c}{\sin C}$. This may be proven, if thought necessary, by dropping the perpendicular from some other vertex than C. However, it is advisable to note early the principle of symmetry, that where certain parts of a diagram are involved symmetrically in an expression, the remaining parts are similarly involved.

It is evidently a matter of indifference whether the perpendicular falls within the triangle or without, since the sine of the exterior angle of a triangle is the same as the sine of the interior adjacent supplementary angle. Thus, in all cases,

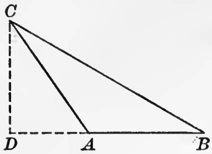


FIG. 62.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Each of these expressions is, in fact, the diameter of the circle circumscribing the triangle.

The central angle BOC (Fig. 63) is twice the angle A on the circumference. Taking OD perpendicular to BC, angle BOD is, therefore, A.

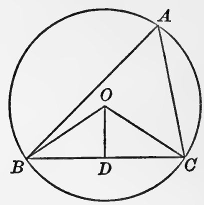


FIG. 63.

$$\therefore\ DB = OB \cdot \sin DOB;$$

or, $$\frac{a}{2} = r \cdot \sin A.$$

$$\therefore\ \frac{a}{\sin A} = 2\,r, \text{ or the diameter of the circle.}$$

Show how to construct the diagram to bring in the angle B or C. The same relation holds also, of course, for right-angled triangles. Then O is the middle point of AB, and

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{1} = \frac{c}{\sin 90^\circ} = \frac{c}{\sin C}.$$

In using the relations above in solving an oblique triangle, one should proceed as in right-angled triangles, writing on the first side of an equality the part desired, then immediately on the other side the part opposite, followed by the remaining given parts, in the order, sine/side or side/sine, according as the first member is sine or side, as in

$$a = \sin A \cdot \frac{b}{\sin B}, \text{ when } A,\ B,\ b \text{ are given.}$$

$$a = \sin A \cdot \frac{c}{\sin C}, \text{ when } A,\ C,\ c \text{ are given.}$$

$$\sin A = a \cdot \frac{\sin B}{b}, \text{ when } a,\ b,\ B \text{ are given.}$$

$$\sin A = a \cdot \frac{\sin C}{c}, \text{ when } a,\ c,\ C \text{ are given.}$$

The student may write the corresponding formulas for the eight other possible cases.

In solving triangles by these formulas, only two different cases can arise:

(1) *Given a pair of opposites and another angle.*
(2) *Given a pair of opposites and another side.*

In Case (2) an angle is to be found from its sine, and since it is an angle of a triangle, and therefore less than 180°, there are two angles which have the calculated sine; for if x be one, $180° - x$ is another. It is necessary in all cases to determine which angle to take, or whether both are to be taken.

In Case (2) the given angle is $\begin{cases} > 90°, & \text{(i)} \\ < 90°. & \text{(ii)} \end{cases}$

In Case (i) each of the remaining angles must be less than 90°, and there is no ambiguity.

In Case (ii) there may arise four different cases:

(δ) The side of the given pair of opposites is longer than the other given side. For example, given a, A, b, with $a > b$. Since, by hypothesis, A is acute, so is B, for if $a > b$, so is $A > B$.

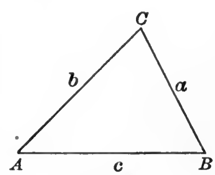


Fig. 64.

(ϵ) The side of the given pair of opposites is less than the other given side.

Under this there may fall three subcases, as in the following diagrams:

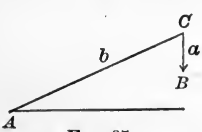


Fig. 65.

(ϵ_1) In Fig. 65, the side opposite the given angle is not long enough to reach to the border of the angle. Here

$$a < b \text{ and also } < b \cdot \sin A,$$

and there is no triangle.

(ϵ_2) In Fig. 66, a is just long enough to reach AB. Here

$$a < b, \text{ but } a = b \cdot \sin A,$$

and there is only one triangle.

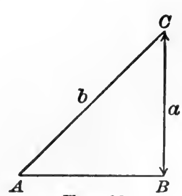


Fig. 66.

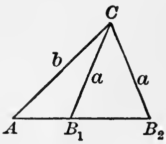


FIG. 67.

(ϵ_3) In Fig. 67 the side a is longer than just necessary to reach the base, so that it may rest in two positions, CB_1, CB_2. Here

$$a < b, \text{ but } > b \cdot \sin A.$$

This is the only case, when a pair of opposites are given, in which both possible angles determined by the sine are to be taken. It is known as the *ambiguous case.*

In solving problems where a pair of opposites and another side are given, one should note, therefore:

(ζ) If the given angle is greater than 90°, the calculated angle is to be taken less than 90°.

(η) If the given angle is less than 90°, with the opposite side greater than the other given side, the calculated angle is to be taken less than 90°.

(κ) If the given angle is less than 90°, with its opposite side less than the other given side, there may be no triangle, one triangle, two triangles.

Which of the three cases of Case (κ) is present, may be settled by making a preliminary sketch to scale of the data, with the results already indicated in (ϵ_1), (ϵ_2), (ϵ_3). Lay out the given angle with the protractor. From its vertex, lay out along one of its terminals the side bordering the angle. From the other extremity of this side as a centre, describe a circle having for its radius the side opposite the given angle. There will be either no triangle, or one triangle, or two triangles (Fig. 68). A preliminary sketch is, however, not essential. The calculation of the unknown parts will itself show which case is present.

FIG. 68.

For example, if a, A, b, are given, with $A < 90°$,

$$\sin B = b \frac{\sin A}{a}.$$

(λ) If $a < b \cdot \sin A$, $\sin B$ will be greater than 1, showing there is no triangle. Using logarithms, $\log \sin B$ would appear greater than zero.

(μ) If $a = b \cdot \sin A$, $\sin B = 1$, and there is but one triangle, with $B = 90°$. Using logarithms, $\log \sin B$ will be zero.

(ν) If $a > b \cdot \sin A$, $\sin B < 1$, and there are two solutions. Using logarithms, $\log \sin B$ will appear less than zero.

Model Examples.

1. Given $\begin{cases} a = 65 \\ A = 73° \\ b = 97 \end{cases}$ Find $\begin{cases} B \\ C \\ c \end{cases}$ No triangle.

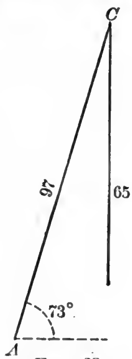


Fig. 69.

$$\left.\begin{aligned} \sin B &= b\frac{\sin A}{a} \\ C &= 180 - (A + B) \\ c &= \sin C \cdot \frac{a}{\sin A} \end{aligned}\right\}$$ The order here is that in which the quantities will be found, as that also after "Find" above.

$$\log \sin B = \log b + \log \sin A - \log a$$
$$\log b = \log a + \log \sin A$$
$$\log c = \log \sin C + \log a - \log \sin A.$$

SOLUTION BY SINES.

$\sin A = 0.9563$
$b = 97.$
86.0
6.7
$65 \,|\, 93$
$\therefore \sin > 1$
$\therefore$ no triangle.

SOLUTION BY LOGARITHMS.

$\log b = 1.9868$
$\log a = 1.8129$
$\log b - \log a = 0.1739$
$\log \sin A = \bar{1}.9806$
$\log \sin B = 0.1545$
This is > 0
$\therefore$ no triangle.

2. Given $\begin{cases} a = 1072 \\ A = 12°46' \\ b = 4849 \end{cases}$

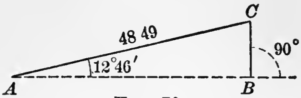


FIG. 70.

Find $\begin{cases} B = 90° \\ C = 77°14' \\ c = 4731 \end{cases}$

$$\sin B = b\frac{\sin A}{a}$$

$$C = 180 - (A + B)$$

$$c = \sin C\frac{a}{\sin A}$$

$$\log \sin B = \log b - \log a + \log \sin A$$

$$\log c = \log \sin C + \log a - \log \sin A.$$

SOLUTION BY NATURAL SINES.

$$b = 4849.$$

$$\sin A = \underline{\quad 0.2210}$$

969.8
97.0
4.8

1072, about

$\therefore \sin B = 1$, about.

SOLUTION BY LOGARITHMS.

$$\log b = 3.6857$$

$$\log a = 3.0302$$

$$\log b - \log \cdot a = 0.6555$$

$$\log \sin A = \bar{1}.3444$$

$$\log \sin B = \bar{1}.9999$$

$\log \sin B = 0$, about.

Thus the triangle is seemingly a right-angled triangle.

In determining c, one should not set down again any logs already set down. Both B and c should be determined in the same scheme, as follows :

$$\log b = 3.6857 \quad (1)$$

$$\log a = 3.0302 \quad (2)$$

$$\log b - \log a = 0.6555 \quad (4)$$

$$\log \sin A = \bar{1}.3444 \quad (3)$$

$$\log \sin B = \bar{1}.9999 \quad (5)$$

$$\log a - \log \sin A = 3.6858 \quad (6)$$

$$\log \sin C = \bar{1}.9891 \quad (7)$$

$$\log c = 3.6749 \quad (8)$$

The corresponding values of B, c are entered after "Find" above.

A triangle like the preceding can occur only by design. Whether the triangle is exactly a right-angled triangle, cannot be settled by the tables. It might appear to be right-angled by a four-place table, where a seven-place table would show otherwise. No table can settle the matter except within its own degree of approximation.

Testing the preceding results: A combination of the parts in some new form is a test. Here we may use

$$c \cdot \sin A = a \cdot \sin C, \text{ or } \log \cdot c + \log . \sin A = \log \cdot a + \log \cdot \sin C.$$

The logs are to be looked up in the tables, and not taken from the preceding scheme, in order to test the quantities found.

$$\begin{aligned} \log a &= 3.0302 \\ \log \sin C &= \overline{1}.9891 \\ & \overline{3.0193} \end{aligned} \qquad \begin{aligned} \log c &= 3.6749 \\ \log \sin A &= \overline{1}.3444 \\ & \overline{3.0193} \end{aligned}$$

The sums agree. They might have differed by 2 in the final figure and the solution be considered correct (§ 82).

3. Given $\begin{cases} a = 52.1 \\ A = 31° 25' \\ b = 61.2 \end{cases}$ Find $\begin{cases} B = 37° 45' \\ \text{or } 142° 15' \\ C = 110° 50' \\ \text{or } 6° 20' \\ c = 93.4 \\ \text{or } 11.0. \end{cases}$

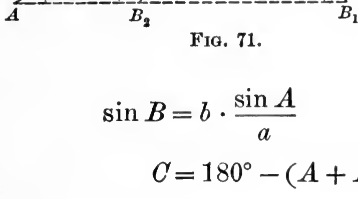


FIG. 71.

$$\sin B = b \cdot \frac{\sin A}{a}$$

$$C = 180° - (A + B)$$

$$c = \sin C \frac{a}{\sin A}$$

$$\log \sin B = \log b + \log \frac{\sin A}{a}$$

$$\log c = \log \sin C + \log \frac{a}{\sin A}.$$

SOLUTION BY NATURAL SINES.

$$
\begin{array}{r}
\sin A = 0.521 \\
b = \underline{61.2} \\
31.26 \\
.52 \\
\underline{.10} \\
31.9 = b \sin A \\
0.612 = \sin B
\end{array}
$$

$$
\begin{array}{r|l}
521 & 319 \\
 & \underline{313} \\
 & 6 \\
 & \underline{5} \\
 & 1
\end{array}
$$

$\sin B < 1$; $\therefore$ two B's.

$$
\begin{array}{r}
\sin C_1 = 0.935 \\
a = \underline{52.1} \\
46.75 \\
1.87 \\
\sin A \quad \underline{.09} \quad c_1 \\
0.5215 \,|\, 48.71 \,|\, 93.4 \\
\underline{46.93} \\
\underline{1.78} \\
\underline{1.56} \\
.22 \\
\underline{.20}
\end{array}
$$

$$
\begin{array}{r}
\sin C_2 = 0.110 \\
a = \underline{52.1} \\
5.50 \\
.22 \\
\sin A \quad \underline{.01} \quad c_2 \\
0.522 \,|\, 5.73 \,|\, 11.0 \\
\underline{5.22} \\
.51 \\
\underline{.52}
\end{array}
$$

SOLUTION BY LOGARITHMS.

LOGARITHMS.

$\sin B = \bar{1}.7870$	(6) two B's.	
$b = 1.7868$		(3)
$\dfrac{\sin A}{a} = \bar{2}.0002$		(4)
$\sin A = \bar{1}.7170$		(1)
$a = 1.7168$		(2)
$\dfrac{a}{\sin A} = 1.9998$		(5)
$\sin C_2 = \bar{1}.0426$		(7)
$\sin C_1 = \bar{1}.9706$		(8)
$c_2 = 1.0424$		(9)
$c_1 = 1.9704$		(10)

Check: $b \sin C = c \sin B$.

$$
\begin{array}{r}
\log b = 1.7868 \\
\log \sin C_1 = \underline{\bar{1}.9706} \\
1.7574 \\
\log c_1 = 1.9703 \\
\log \sin B = \underline{\bar{1}.7869} \\
1.7572
\end{array}
$$

This log check is required to check to within 2 in the third place, since the lines of data are three-figured.*

---

* Example 3 is taken, as to data, from a book on Trigonometry, except that I have made the angle read 25′ instead of 23′, since three-figured data in lines call for angles reading not closer than to the nearest five minutes, as previously explained. In the book referred to,

4. Given $\begin{cases} a = 7.42 \\ A = 105° \ 15' \\ b = 3.39 \end{cases}$ Find $\begin{cases} B = \\ C = \\ c = \end{cases}$

Since $A > 90°$, with $a > b$, there is one and but one solution. The student may make the calculations, using both natural sines and logs, determining also how far the significant figures of the calculated results should be taken.

5. The data as in Ex. 4, with the values of a, b, interchanged.

The solution is impossible, since if $a < b$, then is $A < B$. As A is obtuse, this would require B to be obtuse. A triangle cannot have two obtuse angles.

6. Given

$a = 20.24$

$A = 103° \ 36'$

$B = 19° \ 21'$

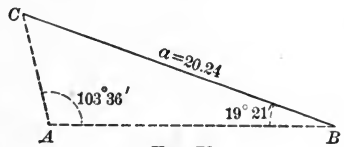


FIG. 72.

Find

$C = 57° \ 3'$

$b = 6.901$

$c = 17.48$

$$C = 180° - (A + B),\ b = \frac{\sin B \cdot a}{\sin A},\ c = \frac{\sin C \cdot a}{\sin A}.$$

Order of logs $\begin{cases} \log b = \log a - \log \sin A + \log \sin B \\ \log c = \log a - \log \sin A + \log \sin C \end{cases}$

Since $A > 90°$, with $a > b$, there is only one solution.

---

this example is worked out completely as a "model example." The calculated sides are carried to *seven significant figures*, and the calculated angles to *hundredths of a second.* A procedure like that is justified only in the case of data known *exactly*, which is never the case with any triangle met "on land or sea." When one is called upon to measure a line and finds himself satisfied on knowing the length to within one five-hundredth of its value, the spirit of calculation in him seems somewhat insatiable if nothing short of pushing the apparent error in results to within one ten-millionth of the calculated line meets his wishes. This refinement of calculation is all a sham. The figures beyond the place of the data are altogether worthless.

SOLUTION BY NATURAL SINES.

```
sin B =  0.3314
    a = 20.24
        6.628
         .066
sin A    .013      b
0.9720 | 6.707 | 6.900
         5.832
          .875
          .875

sin C =  0.8392
    a = 20.24
        16.784
          .168
sin A     .033      c
0.9720 | 16.985 | 17.48
          9.720
          7.265
          6.804
           .461
           .389
           .072
```

SOLUTION BY LOGARITHMS.

$\log a = 1.3062$
$\log \sin A = \bar{1}.9876$
diff. $= 1.3186$
$\log \sin B = \bar{1}.5203$
$\log \sin C = \bar{1}.9238$
$\log b = 0.8389$
$\log c = 1.2424$

(The student may supply a log-check.)

§ 89. Schemes for Solution.

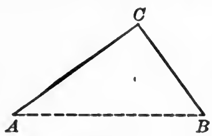
The preceding examples have been worked out to serve as models for subsequent solutions of the same sort. The student will note the forms followed in the calculations. He will state in tabular form the parts given, followed by a tabular statement of the parts to be found, *in the order in which they will be found*, accompanied by a diagram to scale on which the given parts are noted in full lines, with the lines to be calculated indicated by dotted lines. As the parts calculated are determined, they may be entered in the table for such parts. The formulas to be used in calculating the parts are to follow the data as in the examples above, and when logs are to be used, the order in which they are to follow each other in the scheme of solution may be indicated as above, while the student is still in the novitiate stage.

The order in which the logs follow each other in the solution scheme should be such as to make the manipulations easy. Some regular scheme ought always to be followed. That adopted in the preceding examples seems as economical as any.

Having settled on a scheme, *enter on the scheme all logs which are called for before manipulating these logs at all.*

(A) The complete scheme, using logs (and not cologs) when a pair of opposites and another side are given (assuming only one solution), is for a, A, b as follows:

$$\text{Given}\begin{cases} a = \\ A = \\ b = \end{cases}$$



$$\text{Find}\begin{cases} B = \\ C = \\ c = \end{cases}$$

FIG. 73.

$$\text{Formulas}\begin{cases} \sin B = b \cdot \dfrac{\sin A}{a} & (1) \\ C = 180° - (A + B) & (2) \\ c = \sin C \cdot \dfrac{a}{\sin A} & (3) \end{cases}$$

$$\text{Log order}\begin{cases} \log \sin B = \log b + \log\left(\dfrac{\sin A}{a}\right) \\ \log c = \log \sin C + \log\left(\dfrac{a}{\sin A}\right) \end{cases}$$

LOG SOLUTION SCHEME.

LOGARITHMS.

$\sin B =$	(6)
$b =$	(3)
$\dfrac{\sin A}{a} =$	(4)
$\sin A =$	(1)
$a =$	(2)
$\dfrac{a}{\sin A} =$	(5)
$\sin C =$	(7)
$c =$	(8)

In working examples, *make the scheme first.* Then fill it in. First with (1); then (2), (3); subtract (1) from (2) for (4); then (2) from (1) for (5); add (3), (4), for (6). Then find B. Then C. Next fill in (7); add (5), (7), for (8). Then look up c.

TESTING RESULTS.

The student should test his own results. Combining the logs in some new relation is an appropriate test here, as in

$$b \cdot \sin C = c \cdot \sin B.$$

$\log b =$	$\log c =$
$\log \sin C =$	$\log \sin B =$

Make sure to look up all logs (of "Ans.") *in the tables.* The log-check should check by 2 or less in the final figure, the *final* figure being the figure in *the place* to which the data go. For two-figured data, the second figure of mantissa is the final place, etc. Test also by making diagrams to scale.

(B) Solution scheme for case (A) when cologarithms are used:

$\log \sin B =$	(6)
$\log b =$	(3)
$\text{colog}\ a =$	(4)
$\log \sin A =$	(1)
$\text{colog} \sin A =$	(5)
$\log a =$	(2)
$\log \sin C =$	(7)
$\log c =$	(8)

The numerals indicate the ordering of performing the work: (1), (4), (3) added give (6). Then find B. Then C. Then (7). Then add (5), (2), (7), for (8). Then find c.

EXERCISES.

Arrange calculation schemes for all similar cases, and the logarithmic tests. Arrange similar schemes when cologarithms are used.

Solve the following examples both by logarithms and by natural sines. Compare results. Give also the logarithmic tests. Make diagrams to scale. Measure the ungiven parts, and compare with the results of calculation. Make sure the results are given appropriately as to significant figures and angle readings.

1. a 8, A 35°, b 3.

2. a 65, A 47°, b 35.

3. a 5.4, A 67° 30′, b 6.7.

4. a 71.5, A 65°, b 63.2.

5. a 7.45, A 47° 25′, b 9.36.

6. a 67.34, A 45° 43′, b 87.34.

7. a 6715, A 63° 13′, b 7123.

8. a 23.579, A 34° 52′ 23″, b 25.431.

9. a 234.671, A 34° 45′ 67″.3, b 234.678.

10. a 2345.76, A 56° 57′ 58″.45, b 2531.89.

11. Construct similar examples for a, A, c, taking data to one significant figure in lines; two significant figures; three; four; five; six; seven. See that the angles read appropriately. Solve and test them.

12. Do the same for b, B, c, as given.

13. Do the same for C, c, b, as given.

14. Construct some examples which have no solution. Test.

15. Construct some examples with two solutions. Solve.

16. Construct some examples with only one solution. Solve.

(*C*) The complete scheme when a pair of opposites and another angle are given, is similar to the following, assuming a, A, B given:

$$\text{Given}\begin{cases} a = \\ A = \\ B = \end{cases}$$

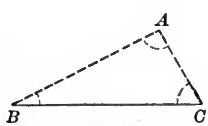


Fig. 74.

$$\text{Find}\begin{cases} C = \\ b = \\ c = \end{cases}$$

$$\left\{\begin{aligned} C &= 180^\circ - (A + B) \\ b &= \sin B \cdot \frac{a}{\sin A} \\ c &= \sin C \cdot \frac{a}{\sin A} \end{aligned}\right\}.$$

$$\text{Log order}\begin{cases} \log b = \log a - \log\sin A + \log\sin B \\ \log c = \log a - \log\sin A + \log\sin C \end{cases}$$

LOG SOLUTION SCHEME.

$$\log b = \qquad (7)$$
$$\log \sin B = \qquad (3)$$
$$\log\left(\frac{a}{\sin A}\right) = \qquad (5)$$
$$\log \sin A = \qquad (2)$$
$$\log a = \qquad (1)$$
$$\log\left(\frac{a}{\sin A}\right) = \qquad (6)$$
$$\log \sin C = \qquad (4)$$
$$\log c = \qquad (8)$$

SOLUTION TEST.

$$b \cdot \sin C = c \cdot \sin B, \text{ or}$$
$$\log b + \log \sin C = \log c + \log \sin B$$

$\log b =$	$\log c =$
$\log \sin C =$	$\log \sin B =$
sum =	sum =

The solution scheme is arranged from its centre, (6) being (5) repeated, to bring it adjacent to log sin C and log sin B. In extended calculations, where the same log is to be added (subtracted) several times, it may be written on the edge of a card and held adjacent to the log with which it is to be combined. On which edge it should be written, or whether on both, will suggest itself. The card while being used is not to cover the part of the scheme on which you wish to write.

EXERCISES.

Construct the corresponding schemes for all similar cases. Also the schemes when cologs are used.

Solve the following triangles:

1. a 8, A 55°, B 50°.

2. a 3.1, A 50°, B 54°.

3. a 43.1, A 47°, B 76°.

4. a 451, A 59° 15′, B 58° 25′.

5. a 46.76, A 67° 54′, B 68° 32′.

6. a 341.67, A 56° 32′ 31″, B 54° 54′ 54″.

7. a 3214.51, A 51° 51′ 51″.2, B 65° 56′ 56″.3.

8. a 2345.56, A 68° 56′ 32″.12, B 36° 36′ 36″.36.

9. Construct similar examples for other possible similar cases, seeing that the angles read correctly, to suit the line data.

GENERAL EXERCISES ON THE SINE.

Construct diagrams illustrating the following problems and solve both by natural sines and by logs. Devise also a suitable independent test for each numerical example.

1. To determine the height of an inaccessible vertical tower, its angle of elevation at two points, A, B, in the same horizontal line as the foot of the tower is measured. Calling these angles α, β, show that if a is the distance between A and B, the height of the tower is

$$a\frac{\sin\alpha\sin\beta}{\sin(\beta\sim\alpha)}.$$

2. A tree 85 feet tall subtends an angle of 58° 10′ at a point A on the river's edge. At a point B on the other side of the river, in a horizontal straight line with A and the foot of the tree, the angle of elevation is 13° 50′. Calculate the width of the river along the line.

3. At a point on a horizontal plane the angle of elevation of a mountain is 34° 10′, and at another point on the plane, a mile distant from the first and in the same plane of elevation, the mountain's angle of elevation is 13° 30′. Find the height of the mountain.

4. From the top of a tree, 77 feet tall, the angle of depression of an object on one bank of a river is 31° 20′, and the angle of depression of an object on the opposite bank is 20° 10′. Assuming the foot of the tree and the two objects as in the same horizontal straight line, directly across the river, how broad is the river?

5. A person in a balloon observes the angle of depression of the camp of an army to be about 40°. After rising about 500 feet higher, vertically over the first place of observation, the angle of depression is about 45°; about how far away is the camp?

6. AB is a vertical object, and A, D, C, are three points in the same horizontal plane, but not in a straight line. These are measured angles, BDA as ϕ, BDC as θ, BCD as β; prove,

$$AB = CD\frac{\sin\phi\sin\beta}{\sin(\beta+\phi)}.$$

7. At each end of a horizontal base of length $2a$ it is found that the angular height of a certain peak is θ and that at the middle point of the base it is ϕ. Prove that the vertical height of the peak is

$$\frac{a \sin\theta \sin\phi}{\sqrt{(\sin\phi - \sin\theta)(\sin\phi + \sin\theta)}}.$$

8. A flagstaff, NP, stands on level ground. A base, AB, is measured at right angles to AN, the points A, B, N being in the same horizontal plane. The angles PAN, PBN are θ, ϕ respectively. Prove that the height of the flagstaff is

$$AB \frac{\sin\theta \sin\phi}{(\sin\theta - \sin\phi)(\sin\theta + \sin\phi)}.$$

9. Solve Ex. 7 for θ, $24° 23'$; ϕ, $34° 56'$; a, 317.4.

10. Solve Ex. 8 for θ, $67° 45'$; ϕ, $34° 24'$; AB, 213.3.

11. A man standing due south of a tower on a horizontal plane observes the elevation of the top of the tower to be $50° 45'$. Going 100 yards due east, he finds the elevation to be $46° 25'$. Find the height of the tower.

12. A man in a balloon observes the angle of depression of a ship at anchor due north of him to be 40°. After the balloon has drifted 3 miles due west at the same elevation, the angle of depression of the ship is 30°. Find the height of the balloon.

13. From the extremities of a horizontal base, AB, whose length is b, the angles BAF, ABF are measured, F being the foot of a tower and in the same horizontal plane as A, B. At A the angle of elevation of the tower is also measured. Call this α; BAF, θ; ABF, ϕ. Show that the height of the tower is

$$b \frac{\sin\phi \cdot \sin\alpha}{\sin(90° - \alpha)\sin(\theta + \phi)}.$$

14. A vertical object stands on a hill whose angle of slope is $A°$. At a distance a from the foot of the object, this distance being measured along the slope of the hill and down the hill, the object subtends an angle $B°$. Show that the height of the object is

$$\sin B \frac{a}{\sin(90° - A - B)}.$$

15. A house stands on a hill whose slope is 15°, and at a point 80 feet from the house down the hill the house subtends an angle of 33°. Find the height of the house, and the distance from the point of observation to the top of the house.

16. A vertical object stands on a hill whose angle of slope to the horizon is $A°$. At two points, whose distance apart measured along the slope of the hill is a, the object subtends angles $B°$, $C°$. Show that the height of the object is

$$a\frac{\sin B \sin C}{\sin (C - B) \sin (90° + A)}.$$

17. Solve Ex. 16 when $A = 20°$, $B = 13°$, $C = 19°$, $a = 553$ feet.

18. A person standing on the slope of a hill whose angle of slope to the horizon is A, finds that the line running up hill to the top of a building from the point where he stands makes an angle B with the face of the hill. The observer is a feet from the building, this distance being measured along the slope of the hill; show that the height of the building is

$$a\frac{\sin B}{\sin (90° - A - B)}.$$

19. Solve Ex. 18 when a is 75, A is 18°, and B is 35°.

20. By means of the relation that the sines of the angles of triangle are to each other as the opposite sides, prove that if a line is drawn bisecting one angle of a triangle, it divides the opposite side into segments proportional to the sides about the bisected angle.

21. A regular pyramid on a square base has an edge 150 feet long, and the length of the side of the base is 200 feet. Find the sine of the inclination of the face of the pyramid to the base, and the corresponding angle.

22. A pyramid has a square base, the length of whose side is a. The vertex of the pyramid is over the centre of the base, and at a distance h from this centre. Show that the angle between the two lateral faces of the pyramid is given by the equation

$$\sin A = \frac{2h\sqrt{2a^2 + 4h^2}}{a^2 + 4h^2}.$$

Find also the sine of the angle between a lateral face and the base.

23. A flagstaff 100 feet high stands in the centre of an equilateral triangle. From the top of the flagstaff each side of the triangle subtends an angle of 60°. The triangle is horizontal and the flagstaff vertical. Show that the length of the side of the triangle is $50\sqrt{6}$ feet.

24. A rectangular target faces south, being vertical and standing on a horizontal plane. If the sun's angular altitude is $A°$ while the sun is $B°$ from the south, show that the area of the shadow of the target on the horizontal plane is found by multiplying the area of the target by

$$\frac{\sin (90° - A) \sin (90° - B)}{\sin A}.$$

25. The angles of elevation of the top of a tower, standing on a horizontal plane, from two points distant a, b, from the foot of the tower and in a straight line with the foot of the tower, are together equal to 90°. Prove that the height of the tower is a mean proportional between a and b. Prove also that the sine of the angle subtended at the top of the tower by the line joining the two points is $\frac{a-b}{a+b}$.

26. A cloud observed simultaneously from two points on a north and south line and distant a miles apart, appears in the south from one point and at an elevation of A, and in the north from the other point and at an elevation of B; find the distance of the cloud from each point, and its vertical height.

27. A man ascends a mountain by a direct course, the inclination of his path to the horizon being at first α and afterward changing suddenly to β, which continues to the top. If the height of the mountain is a feet, and the angle of depression of the starting-point as observed from the top is γ, show that the length of the ascent is

$$\frac{a}{\sin\gamma}\left\{\sin(\beta-\gamma)+\sin(\gamma-\alpha)\right\}\frac{1}{\sin(\beta-\alpha)}.$$

28. At noon a person on a cliff h feet above sea level observes the altitude of a cloud in the plain of the meridian to be α, and the angle of depression of its shadow to be β. If γ is the angular elevation of the sun at the time of observation, show that the height of the cloud above sea level is

$$h\frac{\sin\gamma\sin(\alpha+\beta)}{\sin\beta\sin(\gamma\pm\alpha)},$$

the plus sign preceding α when, if the cloud is in the south, the sun is in the north; the minus holding when both sun and cloud are on the same side of the observer.

29. Two circles of radii a, b, touch each other externally; find expressions for the sines of the half angles between all possible pairs of common tangents.

30. A ship's observer notices that two objects are in a straight line whose bearing from the north is $A°$ toward the east. The ship sails a miles in a course $B°$ west from north, and then the bearing of the more distant object is $D°$ east by north from the direction of the ship's course, and that of the other object is $C°$ east by north from the direction of the ship's course; show that the distance between the objects is

$$a\frac{\sin(A+B)\sin(C-D)}{\sin(C-B-A)\sin(D-A-B)}.$$

31. Show that if, in the preceding problem, the bearing of the line of objects is 15° to the east of north, and the ship's course northwest, and the bearings of the objects east and northeast, the distance between the objects is

$$a\frac{\sin 60° \sin 45°}{\sin 75° \sin 30°}.$$

32. A ship observed another ship $A°$ from the north, sailing in a direction parallel to its own. In p hours its bearing was $B°$ from the north, and in q hours afterward its bearing was $C°$ from the north. Show that if $D°$ is the bearing from the north of the ships' courses, the following equation holds:

$$\frac{\sin(D-C)}{\sin(D-B)}=\frac{p\cdot\sin(B-C)}{q\cdot\sin(A-B)}.$$

§ 90. Use of the Sine in Calculating the Areas of Triangles.

Drop a perpendicular from any vertex, as C, on the opposite side. This perpendicular is $a\cdot\sin B$, or $b\cdot\sin A$, in all cases. The double area of the triangle is base times altitude.

$$\therefore\ 2\Delta = ac\cdot\sin B, \qquad (1)$$
$$= bc\cdot\sin A, \qquad (2)$$
$$= ab\cdot\sin C,\text{ by symmetry.} \qquad (3)$$

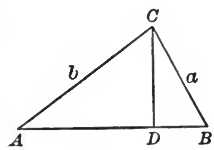


FIG. 75.

The area of a triangle is one-half the product of any pair of its sides and the sine of their included angle.

We have previously shown (§ 88) that

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}. \qquad (4)$$

$$\therefore\ \frac{abc}{bc\sin A}=\frac{abc}{ac\sin B}=\frac{abc}{ab\sin C}=\frac{abc}{2\Delta}.$$

Thus each expression in (4) is $\frac{abc}{2\Delta}$.

$$\therefore \text{ diameter of circumscribing circle} = \frac{abc}{2\Delta}\ (\S\ 88).$$

$$\therefore\ 2\Delta=\frac{abc}{\text{diameter}}.$$

By means of (4), the area expressions (1), (2), (3), may be changed so as to involve only one side and the three angles.

For example, by (4), $c = \sin C \cdot \frac{a}{\sin A}$. Setting this in (1),

$$\text{Double area} = \frac{a^2}{\sin A} \cdot \sin B \cdot \sin C$$

$$= \frac{b^2}{\sin B} \cdot \sin C \cdot \sin A, \text{ by symmetry}$$

$$= \frac{c^2}{\sin C} \cdot \sin A \cdot \sin B, \text{ by symmetry.}$$

The student may express these relations in a single verbal statement.

EXERCISES.

1. Find the areas, the three altitudes, and diameters of the circumscribing circles for the following triangles, without using the tables:

$a = 14,\ b = 10,\ C = 30°;\quad a = 3,\ b = 6,\ C = 60°;$
$b = 15,\ c = 12,\ A = 120°;\quad a = 6,\ c = 5,\ B = 150°;$
$a = 15,\ c = 17,\ B = 135°;\quad a = 3,\ b = 5,\ C = 180°;$
$a = 21,\ c = 12,\ B = 45°;\quad a = 3,\ A = 30°,\ B = 30°;$
$b = 21,\ A = B = C.$

2. Calculate by natural sines and by logs the areas of the following triangles and the diameters of circles circumscribing them. Test.

$a = 23.4,\ b = 34.5,\ C = 34° 25';\quad b = 3423,\ c = 2145,\ A = 34° 23';$
$a = 3.45,\ c = 5.43,\ B = 45° 45';\quad a = 231,\ b = 432,\ C = 23° 45';$
$a = 23.4,\ A = 45° 25',\ B = 67° 35';\quad a = 23.1,\ A = 32° 25',\ C = 43° 35';$
$c = 6.45,\ A = 57° 13',\ B = 67° 53'.$

3. Show that the double area of any quadrilateral is the product of its two diagonals into the sine of the angle between them.

4. Show that the perimeter of any triangle is the diameter of the circumscribing circle into the sum of the sines of its angles, and the area of any triangle is the product of the radii of the inscribed and circumscribed circles into the sum of the sines of its angles.

5. Show that if triangles of equal area are circumscribed about the same circle, their perimeters are the same. What is the sum of the sines of their angles?

6. The diagonals of a quadrilateral are 150, 131, and their angle 45° 15'; what is its area?

7. The diagonals of a parallelogram are 123, 321; find its sides, angles, and area, when the inclination of the diagonals is 45° 25′.

8. The sides of a parallelogram are 150, 130, and one of its angles is 37° 15′; find its area.

9. Construct examples similar to Ex. 2 for one-figured data; two-figured; four-figured; five-figured. Solve. Test.

§ 91. To Determine the Sine of Any Angle of a Triangle, and the Area of the Triangle, when the Three Sides are Given.

Let ABC be the given triangle, with CD perpendicular to AB, and lengths as in the diagram.

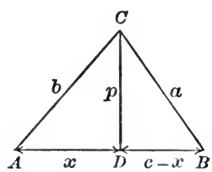


FIG. 76.

$$p^2 = b^2 - x^2, \quad (1)$$

$$p^2 = a^2 - (c - x)^2. \quad (2)$$

$$\therefore\ a^2 - (c - x)^2 = b^2 - x^2. \quad (3)$$

$$\therefore\ a^2 - c^2 + 2\,cx = b^2. \quad (4)$$

$$\therefore\ x = \frac{b^2 + c^2 - a^2}{2\,c}. \quad (5)$$

$\therefore$ by (1),

$$p^2 = (b + x)(b - x)$$

$$= \left(b + \frac{b^2 + c^2 - a^2}{2\,c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2\,c}\right)$$

$$= \frac{2\,bc + b^2 + c^2 - a^2}{2\,c} \cdot \frac{2\,bc - b^2 - c^2 + a^2}{2\,c}$$

$$= \frac{(b + c)^2 - a^2}{2\,c} \cdot \frac{a^2 - (b - c)^2}{2\,c}$$

$$= \frac{(b + c + a)(b + c - a)(a - b + c)(a + b - c)}{4\,c^2}. \quad (6)$$

Let $$a + b + c = 2\,s. \quad (7)$$

Then $$-a + b + c = 2(s - a), \quad (8)$$

$$a - b + c = 2(s - b), \quad (9)$$

$$a + b - c = 2(s - c). \quad (10)$$

Now, $\sin A = \frac{p}{b}.$

$$\therefore \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}. \qquad (11)$$

By symmetry, $$\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}, \qquad (12)$$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}. \qquad (13)$$

By § 90, $\Delta = \text{area of triangle} = \frac{ab \cdot \sin C}{2}$, etc.

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

(10) may be used to find the area of a triangle when the three sides are given. (7), (8), (9) may be used to find the *sines* of the angles. The angles themselves are not uniquely determined, since two angles less than 180° have the same sine. For this reason any angle is found from the sine of its half rather than from the foregoing expressions, since the half angle is acute.

§ 92. The Sine of the Half Angle of a Triangle in Terms of the Sides.

Inscribe in the given triangle a circle (Fig. 77).

$AD = AF$, $BD = BE$, $CF = CE$.

$\therefore 2AD + 2BC$

$= \text{perimeter of triangle}.$

$\therefore AD = s - a. \qquad (1)$

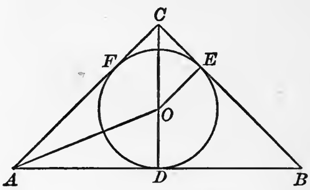


FIG. 77.

The three triangles AOB, BOC, COA make up the triangle ABC.

$$\therefore r(a+b+c) = 2\Delta = 2\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad (2)$$

Now, $$\sin\frac{A}{2} = \frac{DO}{AO} = \frac{r}{AO}. \qquad (3)$$

But $\overline{AO}^2 = r^2 + \overline{AD}^2$

$$= \frac{(s-a)(s-b)(s-c)}{s} + (s-a)^2 \quad \text{by (2), (1)}$$

$$= \frac{s-a}{s}\{(s-b)(s-c) + s(s-a)\}$$

$$= \frac{(s-a)}{4s}\{(a-(b-c))(a+(b-c)) + (b+c+a)(b+c-a)\}$$

$$= \frac{s-a}{s} \cdot bc, \text{ when multiplied out.}$$

$$\therefore \sin\frac{A}{2} = \frac{r}{AO} = \frac{\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}}{\sqrt{\dfrac{s-a}{s} \cdot bc}} = \sqrt{\frac{(s-b)(s-c)}{bc}}. \quad (4)$$

$$\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad (5), \text{ by symmetry.}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}, \quad (6), \text{ by symmetry.}$$

§ 93. Calculation of the Half Angles of a Triangle when the Three Sides are Given.

It is best, when using logs, to set the sines in the forms

$$\sin\frac{A}{2} = \sqrt{\frac{(s-a)(s-b)(s-c)}{abc} \cdot \frac{a}{s-a}},$$

$$\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-b)(s-c)}{abc} \cdot \frac{b}{s-b}},$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)(s-c)}{abc} \cdot \frac{c}{s-c}}.$$

The scheme of solution will then be, the numerals giving the order of filling in, all logs being looked up before manipulation begins:

$$a = \qquad (1)$$
$$b = \qquad (2)$$
$$c = \qquad (3)$$
$$2s = \qquad (4)$$
$$s = \qquad (5)$$
$$s - a = \qquad (6)$$
$$s - b = \qquad (7)$$
$$s - c = \qquad (8) \quad \textit{Check: } (6)+(7)+(8)=(5)$$
$$\log(s-a) = \qquad (9)$$
$$\log(s-b) = \qquad (10)$$
$$\log(s-c) = \qquad (11)$$
$$\log(s-a)(s-b)(s-c)$$
$$= \qquad (15), \text{ by adding (9), (10), (11)}$$
$$\log a = \qquad (12)$$
$$\log b = \qquad (13)$$
$$\log c = \qquad (14)$$
$$\log abc = \qquad (16), \text{ by adding (12), (13), (14)}$$
$$\log \frac{(s-a)(s-b)(s-c)}{abc}$$
$$= \qquad (17), \text{ by subtracting (16) from (15)}$$
$$\log \frac{a}{s-a} = \qquad (18), \text{ by subtracting (9) from (12)}$$
$$\log \frac{b}{s-b} = \qquad (19), \text{ by subtracting (10) from (13)}$$
$$\log \frac{c}{s-c} = \qquad (20), \text{ by subtracting (11) from (14)}$$
$$2 \log \sin \frac{A}{2} = \qquad (21), \text{ by adding (17), (18)}$$
$$2 \log \sin \frac{B}{2} = \qquad (22), \text{ by adding (17), (19)}$$
$$2 \log \sin \frac{C}{2} = \qquad (23), \text{ by adding (17), (20)}$$
$$\log \sin \frac{A}{2} = \qquad (24), \text{ by taking half of (21)}$$

$$\log \sin \frac{B}{2} = \qquad (25), \text{ by taking half of } (22)$$

$$\log \sin \frac{C}{2} = \qquad (26), \text{ by taking half of } (23)$$

$$\therefore \frac{A}{2} = \qquad (27)$$

$$\frac{B}{2} = \qquad (28)$$

$$\frac{C}{2} = \qquad (29)$$

$$\therefore A = \qquad (30)$$

$$B = \qquad (31)$$

$$C = \qquad (32)$$

Test. Sum = 180°, (33), *approximately.*

EXERCISES.

Determine, to the appropriate reading, the angles of the following triangles, by means of the sines of the half angles. Test by adding the angles. Their sum should be approximately 180°, for data of three or more figures. When four-place tables are used on four-figured data, the logarithm of the sine for the half angle may be in error by almost 1 in the fourth place. (See § 24, 3.) By looking at a four-place table, it will be observed that for angles between 6° and 10°, the difference for 1′ is about 9, and that when the angle is about 45°, the difference for 1′ is about 1. The effect on the angle, then, of an error of 1 in the fourth place of logs varies with the size of the angle. By a glance at the table for angles in the neighborhood of the angles calculated, noting what the angle difference is there for an error of 1 in the fourth place of logs, the student can at once see in any special case whether the sum of the calculated angles is sufficiently near 180°. For example, if the triangle is approximately equilateral, each half angle is about 30°. In that part of the table, 1 in the fourth place of logs corresponds to about half a minute in the angle. So that each angle of the triangle may be in error by about 1′ on account of the tables. All the calculations may thus be correct, and the sum of the angles differ from 180° by as much as 3′. Similar considerations hold for any place table. The angles may sum to 180° and each angle be wrong by balancing errors. A log test should be used.

1. The triangle has the sides 23, 25, 30; 4, 5, 7.

2. The triangle has the sides 234, 240, 251.

3. The triangle has the sides 34.5, 35.6, 289.

4. The triangle has the sides 34.51, 22.43, 16.85.

5. The triangle has the sides 345.3, 185.3, 298.7.

6. The teacher may assign data suitable for a five-place table.

7. For one-figured data on lines, how close should the test $A + B + C = 180°$ hold?

8. How close for two-figured data?

9. How close for three-figured data?

10. How close for four-figured data?

11. How close for five-figured data?

12. How close for six-figured data?

13. How close for seven-figured data?

§ 94. Use of the Sine in constructing Given Angles.

It is frequently necessary, in solving problems in mechanics by graphical methods, to lay out angles of given size more accurately than can be done by means of a protractor (subject to errors of warping, centring, alligning, etc.). For angles not very small the sine may be used advantageously.

In Fig. 78 the chord of the arc AMB is twice the radius times the sine of the half of the angle AOB.

$$AB = 2\,AN = 2\,OA \cdot \sin AON.$$

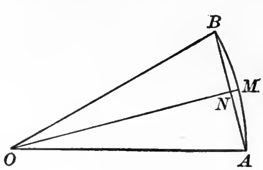


FIG. 78.

Thus, if OA is the unit of the scale of the diagram, twice the sine of the half angle taken from the tables will be the chord AB, to the same scale.

Thus, to lay out a given angle at a given point on a given line, draw about the given point as centre, a circle whose radius is the scale unit. From the point where this arc cuts the given line as a centre and with twice the sine of half the given angle, on the same scale as radius, draw an arc cutting the given arc. Join this point of intersection of arcs and the given vertex point.

EXERCISES.

The teacher may make selections.

§ 95. The Reciprocal Sine, or Cosecant.

The reciprocal of the sine is $\frac{1}{\text{sine}}$. This is called the *cosecant.* The cosecant of x is written cosec x, or csc x.

The practical importance of the cosecant is small, for multiplying by it is the same as dividing by the sine, and *vice versa.* Also $\log \operatorname{cosec} x = -\log \sin x$. For these reasons it is neither customary to give the cosecant in tables, nor to use it in calculations. "Cambria Steel" gives cosecants.

One must not confuse the reciprocal sine with the inverse sine (§ 73); that is, $(\sin x)^{-1}$ with $\sin^{-1} x$.

The sign of the cosecant is evidently the same as that of the sine.

The range of values of the cosecant is very different from that of the sine. As the angle runs from 0° to 360°, the sine, it will be remembered, runs through the continuum from 0 to 1 to 0 to -1 to 0. At the same time the cosecant runs through the reciprocal set of values: from plus infinity to unity, then back to plus infinity; then suddenly changes from plus infinity to minus infinity; then goes on continuously to minus unity; then back to minus infinity, as the angle approaches 360°. As the angle passes 360°, there is a sudden change in value of the cosecant from minus infinity to plus infinity. Then there is a repetition of what has just happened from 0° to 360°, etc.

The cosecant is periodic with the same period as the sine; but while the sine is a continuous function of the angle, the cosecant is *discontinuous*, there being jumps from plus infinity to minus infinity when the angle passes through odd multiples of 180°, and from minus infinity to plus infinity when the angle passes through any even multiple of 180°, counter-clockwise motion being assumed.

§ 96. Line Pictures of the Cosecant in Each Quadrant.

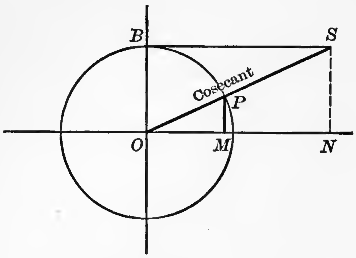


FIG. 79.

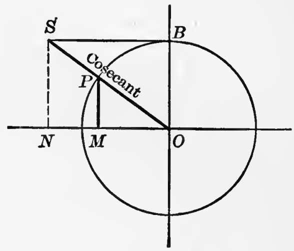


FIG. 80.

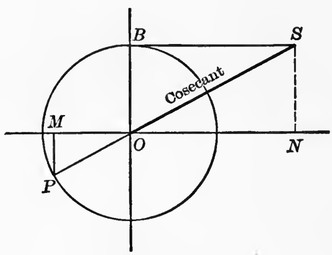


FIG. 81.

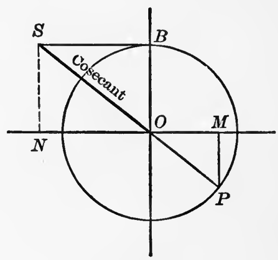


FIG. 82.

In the unit-circle (Figs. 79, 80, 81, 82) MP is the sine, and, consequently, OS the cosecant.

$$\text{For} \quad \frac{MP}{OP} = \frac{NS}{OS}, \text{ or } \frac{MP}{1} = \frac{1}{OS}. \quad \therefore OS = \frac{1}{MP}.$$

Since MP is the sine, OS is the cosecant.

The cosecant is obtained by drawing the tangent at the 90° point of the unit circle, and prolonging it to meet the terminal of the angle. In the first and second quadrants the direct terminal is met, and the cosecant is positive. In the third and fourth quadrants it is the opposite terminal which is met, and the cosecant is negative.

§ 97. Line Picture of the Cosecants of All Angles.

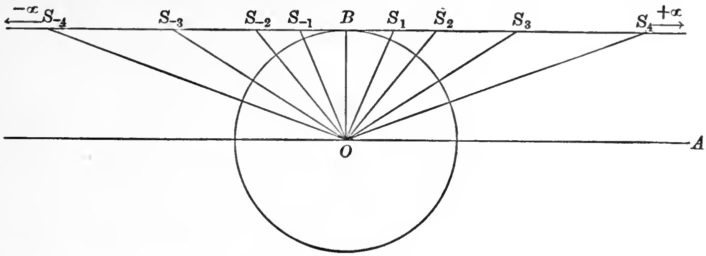


Fig. 83.

For the position, OA, of the terminal the cosecant is positive infinity ($+\infty$). It diminishes as the terminal turns, counter-clockwise, taking the values (where $OB = 1$)

$$+\infty \cdots OS_4 \cdots OS_3 \cdots OS_2 \cdots OS_1 \cdots OB.$$

Then the values

$$OB \cdots OS_{-1} \cdots OS_{-2} \cdots OS_{-3} \cdots +\infty,$$

as the terminal goes from 90° to 180°.

As the terminal passes through 180°, there is the discontinuity from $+\infty$ to $-\infty$, since the cosecant passes to the *opposite* terminal, and the values indicated above, with signs changed, are passed over, as the terminal moves from 180° to 360°.

As the terminal passes 360° there is again a discontinuity, from $-\infty$ to $+\infty$, since the cosecant passes from the opposite terminal to the direct terminal.

As the terminal continues to turn there is a repetition of what has just taken place.

EXAMPLES.

1. Solve the equation $\sin x + \operatorname{cosec} x = \frac{5}{2}$.

Hint: Let $\operatorname{cosec} x = \dfrac{1}{\sin x}$. Solve the resulting quadratic for $\sin x$. Recall the corresponding angles.

2. Can there be an angle whose cosecant is $\frac{1}{2}$? What is the least value of the cosecant? When the sine increases, does the cosecant increase or decrease?

3. Construct an angle whose cosecant is $\frac{5}{4}$; $-\frac{5}{4}$.

4. From the corresponding matter in connection with sines, read the following expressions:

$$\operatorname{cosec}^{-1}\tfrac{6}{5};\quad \operatorname{cosec}^{-1}(-\tfrac{9}{2});\quad 2\operatorname{cosec}^{-1}\tfrac{7}{3};\quad 3\operatorname{cosec}^{-1}(-\tfrac{8}{5}).$$

Construct the terminals for these expressions. How many terminals for any given value of the cosecant?

5. Give the general value of all angles which have the same cosecant as any given angle A, α.

6. What is the relation of the terminals of angles which have the same cosecant? Opposite cosecants?

7. Determine and tabulate the cosecants corresponding to the sixteen terminals of § 75.

8. Solve the equation $\sin x + \operatorname{cosec} x = -\frac{5}{2}$ (general value).

9. Solve the equation $\sin x + \operatorname{cosec} x = \pm\dfrac{3\sqrt{2}}{2}$ (general value).

10. Solve with tables the equation $\sin x + 3\operatorname{cosec} x = 3.1432$.

11. Solve the equation $4\sin\theta = \operatorname{cosec}\theta$ (general value).

12. Solve the equation $4\sin\theta = 3\operatorname{cosec}\theta$ (general value).

13. Solve the equation $\sin\theta + 3\operatorname{cosec}\theta = \frac{7}{2}$ (general value).

14. Solve the equation $3\sin\theta - 8\operatorname{cosec}\theta = -\dfrac{23\sqrt{3}}{6}$ (general value).

15. Solve the equation $\sin x + \operatorname{cosec} x = 2.5618$.

§ 98. The Graph of a Function in Rectangular Coördinates.

Suppose y is a single-valued function of x, and that for any arbitrary value, a, of x, y has been calculated, giving b. This pair of corresponding values may be taken as the coördinates of a point in a plane, the x as abscissa, and the y as ordinate. Let this point be constructed. Suppose it is A in Fig. 84, with the abscissa $OM = a$ and the ordinate $MA = b$. Suppose a second point, B, obtained in the same way, with $x = ON = c$, $y = NB = d$. If we imagine now that x takes all values of the continuum from $x = a$ to $x = c$, and that the corresponding values of y are calculated and the corresponding points plotted, as were A, B, we shall have on the paper a set of points which viewed as an assemblage will constitute a line from A to B, straight or zigzag or crinkly or curved, or, as commonly called, a curve (this term including also straight lines as a special case). Such a curve is called the *graph* of the function, y, from A to B. In practice we cannot construct an infinite number of points. A sufficiently large number of points to give us a more or less close approximation to the true curve, depending on the use that is to be made of the result, are plotted and joined by a smooth curve. (In practice a so-called "French Curve" is used.) The graph shows to the eye the relation between the function, y, and the variable, x, both as to corresponding magnitudes and as to rate of change of the function as the variable changes. The graph of a function is frequently called the *locus of the equation* connecting x, y; that is, it is the *places* of all the points whose coördinates give pairs of values of x, y, which satisfy the equation. A graph is also called a *locus* of a point, moving in a specified manner.

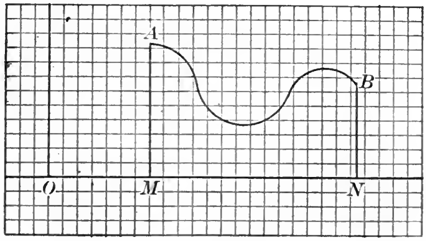


Fig. 84.

In the case of a two-valued function, each value of x will give two values of y. So two points will have the same abscissa. The corresponding curve, or graph, will have two *branches*. Similarly, an n-valued function will have a graph of n branches.

A graph can also be used to show pictorially the connection between quantities whose relation to each other is not expressible in the form of an equation. For example, we may take, from the census reports, the population of a city for every ten years for a number of years. Laying out the years, to any scale, as abscissas, and the population as corresponding ordinates, to any convenient scale (500 inhabitants to an inch, or any other convenient number), then connecting the plotted points by a smooth curve, we have the population graph. The ordinate for any selected year will give the population, approximately, for that year. The graph will thus enable one to calculate the population fairly well for years other than the census years.

Such graphs are used extensively in engineering work, also in economic reports for the exhibition of statistics. They tell the whole story pictorially, and enlighten where the same report given in figures would confuse or make no impression at all.

ILLUSTRATIVE EXAMPLES.

1. $y = x$.

This is evidently satisfied by any point on a straight line passing through the origin and bisecting the first and third quadrants. This line is the graph.

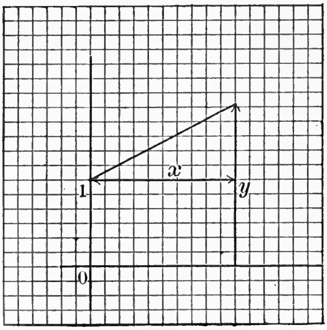


FIG. 85.

2. $y = -x$.

This is satisfied by any point on the bisector of the second and fourth quadrants.

3. $y = x + 1$.

This is evidently satisfied by any point on a line passing through the point (0, 1) on the ordinate axis and making an angle of 45° with abscissa axis.

4. $y = x - 1$.

This is a straight line passing through (0, − 1) on the ordinate axis and making 45° with the abscissa axis.

5. $x = \pm 2$.

This is a pair of straight lines parallel to the ordinate axis and at distances 2 to the right and left of the origin.

6. $y = \pm 2$.

This is a pair of lines horizontal and at distances 2 above and below the abscissa axis.

7. $x^2 + y^2 = 25.$

This is satisfied by any point on a circle of radius 5, with its centre at the origin. This is an example of a two-valued function, since for any x there are two y's,

$$y = \pm\sqrt{25 - x^2}.$$

Thus, there are two points equidistant from the abscissa axis which satisfy the given equation, both points having the same abscissa, or x. Similarly for x in terms of y.

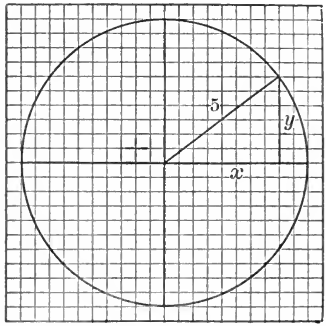


FIG. 86.

8. The preceding examples are those of functions whose graphs can be seen readily without calculation. To illustrate the calculation process, we may take

$$y^2 = 4x.$$

Giving to x the values following, one can calculate the corresponding y's, or get them from a table of squares:

x	$\pm y$	x	$\pm y$	x	$\pm y$	x	± 1
0	0	1.0	2.0	2.0	2.8	4.3	4.1
.1	.6	1.1	2.1	2.2	3.0	4.6	4.3
.2	.9	1.2	2.2	2.4	3.1	4.9	4.4
.3	1.1	1.3	2.3	2.6	3.2	5.2	4.6
.4	1.3	1.4	2.4	2.8	3.3	5.5	4.7
.5	1.4	1.5	2.5	3.0	3.5	5.8	4.8
.6	1.5	1.6	2.5	3.2	3.6	6.1	4.9
.7	1.7	1.7	2.6	3.4	3.7		
.8	1.8	1.8	2.7	3.6	3.8		
.9	1.9	1.9	2.8	3.8	3.9		

These values plotted to scale (1 inch = 1, say), taking y to the nearest 10th, give the curve. The student may plot this curve. Use coördinate paper.

9. Construct the graphs of the following:

$y^2 = -4x$, $x^2 = 4y$, $x^2 = -4y$, $y = 2x + 3$, $y = 3x - 4$, $y = -2x + 3$, $y = -2x - 1$, $3x + 2y = 6$, $3x - 2y = 6$, $-3x + 2y = 6$, $-3x - 2y = 6$.

10. Look up in the census reports the population of the three most important cities in your state, and construct corresponding graphs to the same scale and compare. Calculate from the graphs the population for some year not a census year.

11. On rectangular coördinate paper make graphs of $y = \log_{10} x$ and $y = \log_e x$, to the same scale. What is the relation between corresponding y's for the same x? (See § 5.)

§ 99. Graph of the Sine, or Locus of the Equation $y = \sin x$.

(*a*) Imagine the unit circle made of wire, with each ordinate a wire hinged at its extremity to the unit circle. Clip the circle wire at *a*, and stretch it out straight, with the positive ordinates pointing up and the negative ordinates down. The ends of the wire ordinates give the graph of the sine for all values of x from zero to 2π radians.

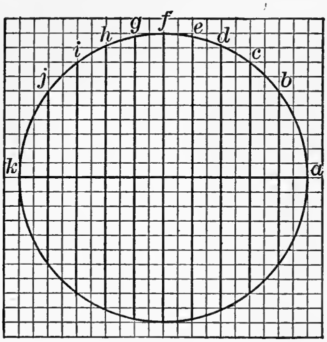


FIG. 87.

(*b*) *An easy practical process for making the graph of the sine.* Draw a circle of any radius on a piece of pasteboard. Draw on it a large number of ordinates. Cut the circle out with a sharp knife, so as to give a smooth edge. Roll the circle in a vertical position on a straight line of a horizontal sheet of paper. Dot the line with a pencil at the places where the ends of ordinates touch it, laying out at such dots the corresponding ordinates, in length and sign. Join the ends by a smooth curve. This is the sine graph corresponding to the radius of the selected circle as unit.

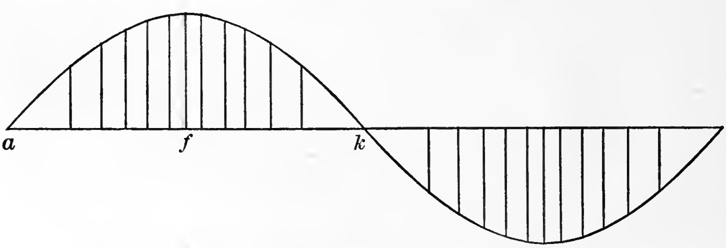


FIG. 88.—Sine graph, $x = 0$ to $x = 2\pi$.

(*c*) *Continuous graph of the sine.* Imagine the preceding curve (Fig. 88) repeated indefinitely to the right and left; we have the sine graph for all values of x from minus infinity to plus infinity. (See another method in Ex. 8, page 191.)

After the curve from a to the end of the central ordinate at f has been plotted, a pasteboard cutting of this may be made and used to continue the curve indefinitely forwards and backwards.

FIG. 89.

The curve of Fig. 89 shows the periodicity of the sine, by the equality of ordinates separated by distances which are multiples of 2π. (See § 72.)

(*d*) *The units in the graph of the sine.* The angle must be expressed in circular measure, for the sines (the ordinates) are expressed as ratios in terms of the modulus, or radius. Consequently the horizontal distances (the angles) of the graph must be expressed in terms of the same unit (the radius), that the graph may give a correct impression of the relation of the two connected quantities, the sine and angle.

Thus $57°.3$ is the unit for horizontal distances when using degrees. The same length which represents $57°.3$ will represent the sine of $90°$, while the horizontal distance for $90°$ will be $\frac{90}{57.3}$. Thus the height of the curve will be less than half the distance between the points where it crosses the horizontal line.

(*e*) *Using the table of natural sines to construct the sine curve.* Take $57°\ 18'$ as the unit (an inch, say, on the scale). Tabulate the sines of all eighths, or tenths, of $57°\ 18'$, to run from $0°$ to $90°$. Plot corresponding values. Join plotted points by a smooth curve. Make pasteboard cutting of this and use it to continue the curve.

(*f*) *The sine curve as a wave.* The sine curve is of great importance in mathematical physics. Many sorts of physical phenomena, periodic in their character, can be represented by a sine curve, or by two or more sine curves, in the same plane, or in different planes, combined with each other.

If we make graphs of $y = \sin x$ (1) and $y = \sin(x + a)$ (2)

and compare them, the graph of (2) will be that of (1) slid along the horizontal axis a distance a, forward or backward, according to the sign of a. The ordinate of (2) at any point is that of (1) at a distance a to the right or left, according as a is minus or plus. Two such sine curves are said to be in *different phase*, and a is called the *phase-difference*. The effect of superposing two curves is obtained by plotting them in connection with the same pair of coordinate axes, and then making a new curve whose ordinate at any point is the algebraic sum of those of the two curves. Similarly for any number of curves.

Two equal sine curves are said to be in opposite phase when their ordinates at the same point of the abscissa axis are equal and opposite. If two such curves are superposed, the resulting curve is the horizontal axis. This has its counterpart in two waves of sound producing silence. If two sine curves of slightly different phase are superposed, the hills and valleys of the resulting curve will be more pronounced than those of the given curves. This has its physical representative in the familiar phenomenon of "beats," when two musical notes are "slightly out of tune."

Many different waves may be superposed, — waves different in phase and in size and period and plane. Such are ocean waves. The topic is an extensive one, and is discussed at length in mathematical treatises on sound, light, electricity. Here we have given merely a hint. It plays an important practical rôle in the discussion of Alternating Currents, as in Steinmetz's book on this subject.

§ 100. Harmonic Motion.

Imagine a terminal line turning with uniform angular velocity of α units per unit of time (a second, generally). At the end of t time units, counting time from the moment when the terminal coincides with the initial line, the angle AOP in Fig. 90 is αt. Let MP be the ordinate from P.

$$\therefore MP = r \cdot \sin(\alpha t). \qquad (1)$$

If instead of counting time from the moment when OP coincides with OA, it is counted from the position ON (β),

$$MP = r \cdot \sin(\alpha t + \beta). \tag{2}$$

Draw now PP_1 perpendicular to the vertical axis. P_1 is called the projection of P on this axis. As P moves uniformly around the circle, P_1 moves back and forth along the vertical axis. The motion of P_1 is said to be *harmonic*, being similar to that of a point in a vibrating string, or to that of the wood particles in a piano sounding board.

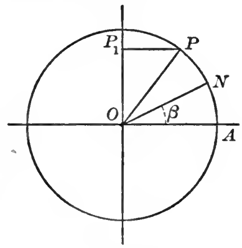


FIG. 90.

The radius of the circle, or the length r in (1), (2), is called the *amplitude of motion*, being the half swing of the point P_1.

If graphs were made for (1), (2), r would be the wave height.

If one has before him a graph of (1) or (2), and imagines, as he gazes at the graph on its page, that the graph travels across the page with the flow of time and at the rate of α units per second, he has a picture of a simple ocean wave. Here α would be called the *wave velocity* instead of the angular velocity. If instead of watching the whole graph move, the observer fixes his eye upon a single point of the horizontal axis and watches the changing height of the ordinates at that point, as the moving graphs pass it, he has a picture of harmonic motion at a point. It is approximately the motion of the straight part of the piston rod of a stationary engine as the driving wheel turns uniformly.

β of (2), without reference to (1), is called the *phase* at zero-time. It is the position of the terminal at zero-time, in considering harmonic motion. In the sine graph, it is the angle whose sine is the ratio of the wave height on the vertical axis at the origin to the total wave height.

In (1), (2), $\frac{2\pi}{\alpha}$ is called the periodic time; for if in (1), (2), t is changed by $\frac{2\pi}{\alpha}$, then calling the time at the first instant t_1 and that at the second t_2,

$$t_2 = t_1 + \frac{2\pi}{\alpha}.$$

The ordinate for t_1 is $r \cdot \sin(\alpha t_1 + \beta)$.
The ordinate for t_2 is $r \cdot \sin(\alpha t_2 + \beta)$;

or, $$r \cdot \sin\left(\alpha\left(t_1 + \frac{2\pi}{\alpha}\right) + \beta\right);$$

or, $$r \cdot \sin(\alpha t_1 + \beta + 2\pi);$$

or, $$r \cdot \sin(\alpha t_1 + \beta)$$

$\therefore$ the ordinates are equal.

This is also evident by considering the uniform turn of OP. The time of a complete turn, if the rate is α units per second, is $\frac{2\pi}{\alpha}$, α being in radian measure. After a complete turn of OP from any position, P_1 is again in the same state of motion as when OP formerly occupied this position.

This is but a touch on a topic of great practical importance in the discussion of alternating electrical currents, light, etc.

§ 101. Graph of the Cosecant.

The graph of the cosecant is readily constructed from that of the sine, since sin times cosec = 1. In Fig. 91, representing a right-angled triangle in a circle, with CM perpendicular to AB, $AC^2 = AB \cdot AM$, from the similarity of the triangles AMC, ABC.

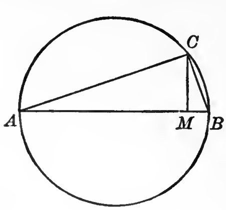


Fig. 91.

If then at any point M, Fig. 92, of the sine graph, we draw the line MC perpendicular to the ordinate AM, and with A as a centre and the height of the sine curve as radius, draw an arc, cutting MC in C, CB drawn

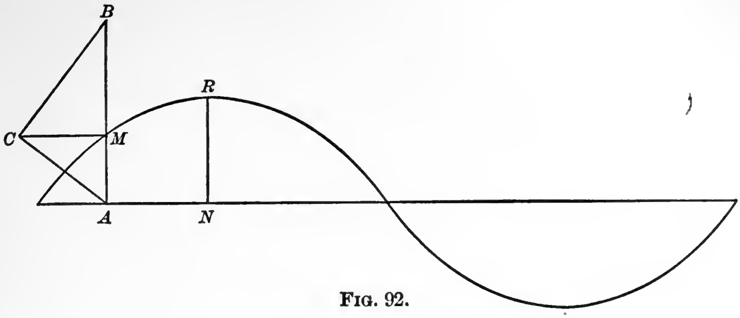


Fig. 92.

at right angles to AC will cut AM prolonged in B, giving AB as the length of the cosecant at the point A, and B as a point on the cosecant graph.

In the same way any number of points on the graph can be found.

The appearance of the cosecant curve as related to the sine curve is that of Fig. 93.

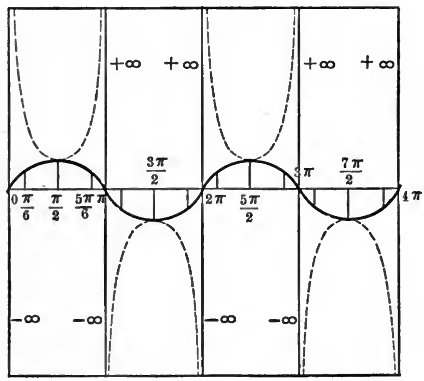


Fig. 93.

LABORATORY EXERCISE.

Construct together some sine curve and its corresponding cosecant curve. Tilt a sine curve, to give an anti-sine curve. Tilt a cosecant curve, to give an anti-cosecant curve.

§ 102. Graphs in Polar Coördinates.

When an equation is given connecting r, θ, we can, by giving one of them special values, calculate the other. Plotting a number of corresponding points, not too far apart, and joining them by a smooth curve, we have the graph. For plotting such curves Groat's polar coördinate paper will be found very useful.

What is $r=a$, where a is a constant? What is $r^2-5r+6=0$? What is $F(r)=0$, where the function is many valued? What is $\theta=0$? What is $\theta=\alpha$, where α is a constant? What is $\theta^2-5\theta+6=0$? What is $F(\theta)=0$, where F is many valued?

EXERCISES.

1. Use Groat's radian polar coördinate paper to make the graph of $r=10\,\theta$, $r=20\,\theta$, $r=a\theta$. (Plan: give values to θ, as 0.1 radian, 0.2 radian, etc., and calculate corresponding values of r. Then plot the corresponding points and join them by a smooth curve. The points will be for the first curve: (0, 0), (1, 0.1), (2, 0.2), etc. These curves are spirals of Archimedes.)

2. Use Groat's radian polar coördinate paper to make a graph of $r=\frac{1}{\theta}$, $r=\frac{2}{\theta}$, $r=\frac{a}{\theta}$. (Hyperbolic spirals.) (Such a curve with r about an inch, or inch and a half, when $\theta=\frac{\pi}{2}$, will be found almost as useful, when made in hard wood, for drawing purposes as a "French Curve." Such a curve was found most useful in plotting Professor Orton's Reports on Ohio Clays.) Plot $r^2\theta=a$. (Lituus.)

3. Show that a circle is the graph for $r=a\sin\theta$, a being the diameter (vertical), θ being measured from a horizontal tangent.

4. Use Groat's degree-measure polar coördinate paper to plot $r=10\sin\theta$ by points, taking as radial unit one of the small divisions on the paper. See if your curve looks like a circle.

5. Show that $r\sin\theta=a$ is a straight line parallel to the x-axis at a distance a.

6. Use Groat's degree-measure polar coördinate paper to make a graph for $r\sin\theta=10$.

7. Use Groat's degree-measure polar coördinate paper to make graphs of $r\sin 2\,\theta=a$; $r=a\sin 2\,\theta$; $r\sin 3\,\theta=a$; $r=a\sin 3\,\theta$. Give a some convenient numerical value. See if the number of loops in the graph of $r=a\sin n\theta$ depends upon n being even or odd.

8. Draw a circle. Take a horizontal tangent as initial line and its point of tangency as pole. Prolong each radial line (vector) from this point a distance equal to a diameter beyond the circle. Connect their ends by a smooth curve. This is the graph then of $r=a(\sin\theta+1)$. It has the shape of a heart, and is called a cardioid.

9. Use Groat's polar coördinate paper (degree measure) to plot $r = a(1 + \sin\theta)$; $r = a(1 - \sin\theta)$; $r(1 + \sin\theta) = a$; $r(1 - \sin\theta) = \theta$, for selected values of a.

10. Draw a horizontal line. Select a point above it at a distance a as pole. Sketch points that are as far from the pole as from the line. The curve is a parabola. Show that any point on it satisfies the equation, $r(1 - \sin\theta) = a$. What is $r(1 + \sin\theta) = a$?

11. Make the graphs for $\theta = \log_e r$; $\theta = \log_{10} r$; $r = e^\theta$; $r = 10^\theta$.

§ 103. The Coversed Sine.

When the line definition was in use, with the corresponding diagram for the sine to the radius 1, the balance of the radius beyond the sine was given the name, the *Coversed Sine.* And still,

$$\text{coversin } A = 1 - \sin A.$$

The coversed sine is used in engineering field books.

EXERCISES.

1. Can the coversed sine be negative?

2. What are its limits of value?

3. Construct terminals when $\text{coversin } A = \frac{2}{3}$; $\text{coversin } A = \frac{4}{3}$.

4. What are the coversed sines for angles whose terminals border the quadrants?

5. What are the coversed sines for angles whose terminals are the bisectors and trisectors of the quadrants?

6. How could a table of coversed sines be made for angles from 0° to 90°?

7. Make the graph for the coversed sine, $y = \text{coversin } x$.

8. Use the following method to construct a sine-curve: draw a circle of one inch radius, and a horizontal line through its centre, A, AO being a horizontal radius to the right. Divide the circumference into 9° spaces from O. Lay out from O, right and left, on the horizontal line, to the extent of the paper used, equal distances, each 0.16 inch (9° in radians, inches). Prolong the abscissas of the extremities of the 9° divisions to meet the ordinates at the ends of the horizontal divisions. Intersections of corresponding lines will be points on the sine curve. Use a similar process to make a cosecant curve.

CHAPTER VI.

THE COSINE, INVERSE COSINE, RECIPROCAL COSINE (SECANT), AND VERSED SINE (1 – COSINE) OF AN ANGLE.

§ 104. **The Cosine of an Angle** *is the ratio of the abscissa of any point on the terminal of the angle to the modulus of the same point*, or

$$\cos\theta = \frac{OM}{OP}.$$

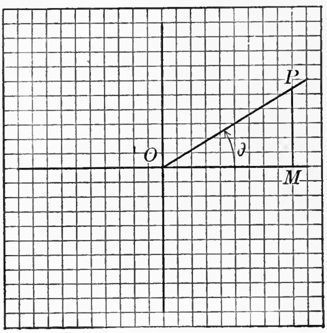


FIG. 94.

This is the ratio-definition, or Hassler's definition. The numerical relation of an angle in circular measure to its cosine is given by the following series, which may be considered also as a definition of the cosine:

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \cdots.$$

Similar remarks to those made with reference to the definitions of the sine hold with reference to those of the cosine.

As with the sine, the cosine has a threefold use in calculations:

(1) *When the abscissa and modulus are given, their ratio, expressed as a decimal fraction, and compared with a table of cosines, will indicate the corresponding angle, or angles.*

(2) *When the modulus and angle are given, the cosine of the angle is the number which, used as a multiplier with the modulus, will give the abscissa.*

(3) *When the abscissa and angle are given, the cosine of the angle is the number by which to divide the abscissa to obtain the modulus.*

LABORATORY EXERCISES.

Construct with the protractor an angle of 26°. Lay out on it five different moduli and corresponding abscissas, including the modulus of unity (one inch or one foot). Measure moduli and abscissas.

Divide each abscissa by its modulus to one or two figures. Compare results with one another and with the table.

Do the same for five different angles, each with a single modulus and its abscissa. Compare results with the tables.

Lay out an angle with the protractor. Measure the modulus. Multiply it by the table-cosine (to one or two figures). See if the result agrees with the measured abscissa.

Test for another angle the measured modulus with the value obtained by dividing the measured abscissa by the table-cosine.

EXERCISES.

1. Take the exercises under § 76, changing the word "sine" to "cosine," and "ordinate" to "abscissa."

2. The student may make and solve five simple practical examples which can be solved by the cosine; by the sine.

3. If the initial velocity of a projectile at an angle $A°$ to the horizontal is v, what are the vertical and horizontal components? Solve a numerical example.

4. Assuming that the velocity due to gravity is $32t$ (t in seconds), how long will the projectile in Ex. 3 rise before gravity destroys the initial vertical velocity? How far will the projectile drift horizontally in that time (range)?

5. A smooth plane, inclined $A°$ to the vertical, has a weight of $\frac{1}{2}$ pound tied on by a string running from the upper end. What is the pressure normal to the plane? What is the pull on the string? Solve a numerical example.

6. How far from the centre of the earth is the centre of the 40° parallel of latitude? What is the radius of this parallel? Give general formulas for sphere of radius r.

7. A particle is moving with uniform angular velocity α in a circle. What are the component velocities along horizontal and vertical diameters at the time t (seconds), counting time from the moment when the particle is at the right-hand end of the horizontal diameter? Solve a numerical example.

8. A particle is acted on by a force, F, in the direction $A°$ to the horizontal line. What are the horizontal and vertical components of the force? Solve a numerical example.

9. A particle has a velocity v in the direction $A°$ to the horizontal line. What are the horizontal and vertical components? Solve a numerical example.

10. Compare numerical solutions of the foregoing problems with their graphical solutions.

Hassler's definition holds for all positions of the terminal, no matter in which quadrant it may fall.

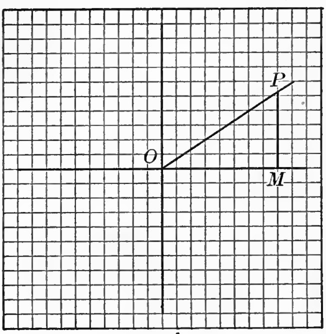


FIG. 95.

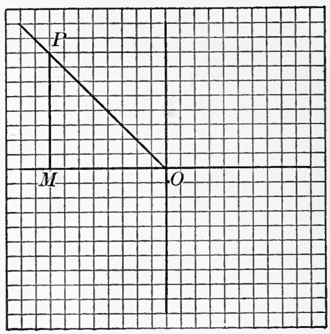


FIG. 96.

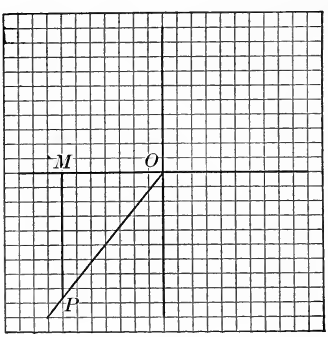


FIG. 97.

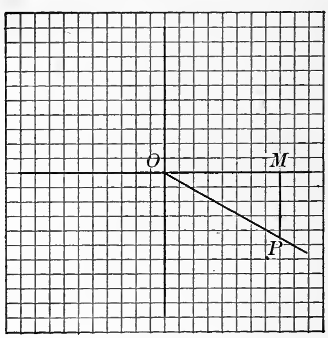


FIG. 98.

Representing by θ any one of the angles corresponding to the terminal in any one of the above diagrams,

$$\text{cosine of } \theta = \frac{\text{abscissa } OM}{\text{modulus } OP} = \frac{x}{r} = \cos\theta.$$

§ 105. The Sign of the Cosine.

The sign of the cosine is determined by that of the abscissa, as that of the sine by the ordinate.

Quadrant	I	II	III	IV
Cosine	+	−	−	+
Sine	+	+	−	−

FIG. 99.

EXERCISES.

1. By means of the protractor, locate the terminals of the following angles (in degree measure). Determine the signs of their sines and cosines: 30; −30; 60; −60; 110; −110; 130; −130; 271; −271; 375; −375; 456; −456; 548; −548; 638; −638.

Do you notice any connection in size and sign between the cosine of positive and negative angles numerically equal? Can the sign of the sine and cosine be determined without knowing accurately the position of the terminal?

2. Give the signs of the sines and cosines of the following angles (radian measure), taking each angle both positively and negatively:

1; 2; 3; 4; 5; 7; 9; 10; 12; $\frac{\pi}{6}$; $\frac{\pi}{3}$; $\frac{\pi}{4}$; $\frac{2\pi}{3}$; $\frac{3\pi}{4}$; $\frac{5\pi}{6}$; $\frac{5\pi}{4}$; $\frac{7\pi}{4}$; $\frac{9\pi}{4}$; $\frac{13\pi}{4}$; $\frac{\pi}{2}$; π; $\frac{3\pi}{2}$.

§ 106. Angles with the Same Cosine.

The cosine, like the sine, depends only on the position of the terminal.

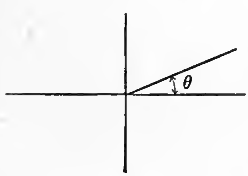


FIG. 100.

(i) All angles with the same terminal have the same cosine.

$$\therefore \cos(2n\pi + \theta) = \cos\theta,$$

$$\cos(2n \cdot 180° + A°) = \cos A°,$$

where n is any positive or negative integer.

(ii) All angles with their terminals symmetric to the horizontal have the same cosine, since points with equal moduli on such terminals have the same abscissa.

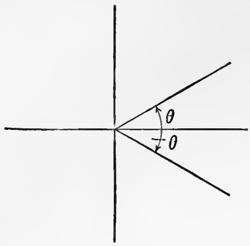


FIG. 101.

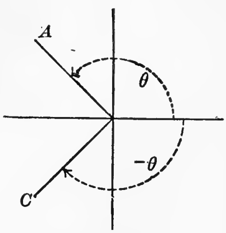


FIG. 102.

∴ as a special case,

$$\cos\theta = \cos(-\theta), \text{ in radian measure,}$$
$$\cos A^\circ = \cos(-A^\circ), \text{ in degree measure,}$$

or, *positive and negative angles numerically equal have the same cosine.*

And generally,

$$\cos(2n\pi + \theta) = \cos(2m \cdot \pi - \theta),$$
$$\cos(2n \cdot 180^\circ + A^\circ) = \cos(2m \cdot 180^\circ - A^\circ),$$

where m, n are any positive or negative integers.

(iii) Thus, in general, any angle having the same cosine as θ, A° are of the forms

$$2n\pi \pm \theta, \text{ in radian measure,}$$
$$2n \cdot 180^\circ \pm A^\circ, \text{ in degree measure,}$$

where n is any positive or negative integer.

The student may give a verbal statement for each of these formulas.

(iv) In the preceding formulas any angle of the given terminal may be taken as θ, A°. It is customary, however, to take the principal angle. For example, 185° and − 175° locate the same terminal. One would use

$$2n \cdot 180^\circ \pm 175^\circ,$$

rather than

$$2n \cdot 180^\circ \pm 185^\circ,$$

for the set of angles having the cosine of this terminal.

EXERCISES.

1. Find five positive angles and five negative angles having the same cosine as 30°, and also give in degree measure and in radian measure the general formulas for all such angles.

2. Do the same, replacing the word "cosine" by the word "sine." Give a diagram for Exs. 1 and 2.

3. Consider in the same way as in Exs. 1 and 2, the angles 100°, 193°, 275°, using the principal angle of the terminals.

4. For terminals in the first quadrant, where are the terminals for all angles having the same cosine? For terminals in the second quadrant? In the third? In the fourth?

5. Give the general formulas, both in radian measure and in degree measure, for all angles having the same cosine as $\frac{\pi}{4}$; $\frac{\pi}{3}$; $\frac{2\pi}{3}$; $\frac{3\pi}{4}$; $\frac{7\pi}{4}$; $\frac{\pi}{6}$; $\frac{5\pi}{6}$. For the same angles give the general formulas, in both measures, for all angles having the same sine. Give illustrating diagrams.

6. Find the two angles numerically less than 180° which have the same cosine as ± 1085°. Do the same for ± 365°; ± 800°; ± 1100°; ± 3000°; ± 185°; ± 275°. Give in radian measure and in degree measure, for each of these angles, the general formulas for all angles having the same cosines.

7. Consider Ex. 6, replacing the word "cosine" by the word "sine."

8. Using π radians as the unit angle, give the general formulas for Exs. 5 and 6.

9. Solve the equations:

$\cos 9\theta = \cos 8\theta$; $\cos 7\theta = \cos\theta$; $-\cos 3\theta = \cos 5\theta$;
$\cos 2\theta = \cos\theta$; $-\cos 3\theta = \cos\theta$; $\cos n\theta = \cos\theta$;
$\cos m\theta = \cos n\theta$; $\sin 2\theta = \sin 6\theta$; $-\sin 8\theta = \sin 7\theta$;
$\sin m\theta = \sin n\theta$.

Illustrate each solution by a diagram.

Solution of the first equation in Ex. 9.

$$\cos 9\theta = \cos 8\theta.$$

Here 9θ is not necessarily 8θ, but 9θ may be any one of the angles whose cosine is that of 8θ.

But $$\cos 8\theta = \cos(2n\pi \pm 8\theta).$$
$$\therefore \cos 9\theta = \cos(2n\pi \pm 8\theta).$$
$$\therefore 9\theta = 2n\pi + 8\theta, \text{ or}$$
$$9\theta = 2n\pi - 8\theta.$$

$$\therefore \theta = 2n\pi; \quad (1)$$
or, $$17\theta = 2n\pi; \quad (2)$$
or, $$\theta = n\cdot\frac{2\pi}{17}. \quad (3)$$

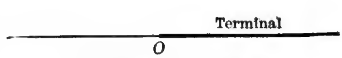


FIG. 103. $(\theta = 2n\pi.)$

For (1) the terminal line is the initial line. Turning that line nine times or eight times around leaves it unchanged. The diagram is Fig. 103.

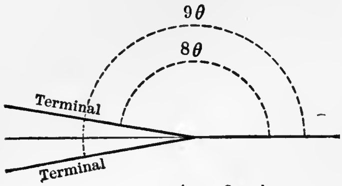


FIG. 104. $\left(\theta = \frac{2\pi}{17}.\right)$

For (3),

$9\,\theta$ is $9\,n \cdot \frac{2}{17}\pi$, or $n(1-\frac{8}{17})2\,\pi$;
$8\,\theta$ is $8\,n \cdot \frac{2}{17}\pi$, or $n(1-\frac{9}{17})2\,\pi$.

Thus the terminal for $9\,\theta$ is symmetric to the horizontal with that of $8\,\theta$. The diagram for $n = 1$ is Fig. 104.

The student may give similar solutions for the remaining examples, and state what the general process is.

§ 107. Angles with Opposite Cosines.

(*a*) Any pair of terminals symmetric to the vertical give angles with opposite cosines, since for equal moduli the abscissas are oppositely equal.

∴ (i) Supplementary angles have opposite cosines,

or, $\cos(180° - A°) = -\cos A°$,
$\cos(\pi - \theta) = -\cos\theta$.

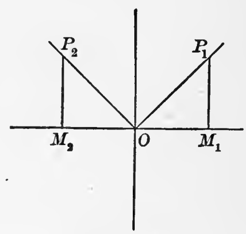


FIG. 105.

(ii) Any angle of a terminal has the opposite cosine of any angle of the terminal symmetric to the vertical.

$$\cos\{(2\,n+1)\,\pi - \theta\} = -\cos(2\,m \cdot \pi + \theta),$$
$$\cos\{(2\,n+1)\,180° - A°\} = -\cos(2\,m \cdot 180° + A°).$$

(iii) Thus all angles of the form

$$(2\,n+1)\,\pi - \theta,$$
$$(2\,n+1)\,180° - A°,$$

have the opposite cosine of θ, $A°$.

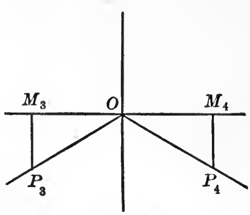


FIG. 106.

In words, *subtracting an angle from an odd multiple of* 180° (*or* π *radians*) *gives an angle of opposite cosine.*

(*b*) Angles of opposite terminals have opposite cosines, since for equal moduli the abscissas are oppositely equal.

$$\therefore \cos(180° + A°) = -\cos A°,$$

$$\cos(\pi + \theta) = -\cos\theta,$$

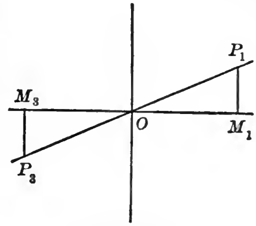


FIG. 107.

or, in general,

$$\cos((2n+1)180° + A°) = -\cos(2m \cdot 180° + A°),$$

$$\cos((2n+1)\pi + \theta) = -\cos(2m \cdot \pi + \theta).$$

In general, angles having the opposite cosine of θ, $A°$ are of the form

$$(2n+1)\pi \pm \theta,$$

$$(2n+1)180° \pm A°.$$

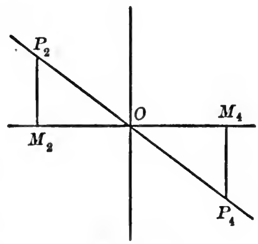


FIG. 108.

Thus, while adding an angle to an even multiple of 180° *or subtracting it from such an angle leaves the cosine unchanged* (§ 106), *an odd multiple of* 180°, *similarly treated, gives the opposite cosine.*

EXERCISES.

In solving examples, it is advised that the formulas be not used, but rather the location of the terminal, as with the sine.

1. Write five positive angles and five negative angles having the opposite cosine of 30°; of $\pm 60°$; of $\pm 45°$. Let one-half the angles found in each case have the opposite terminal, the remaining half the terminal symmetric to the vertical. Give the general formulas in each case.

2. Do the same for $\frac{\pi}{3}$; $\frac{\pi}{6}$; $\frac{2\pi}{3}$; $\frac{-2\pi}{3}$; $\frac{3\pi}{4}$; 2^r; 3^r; 4^r.

3. If an angle has its terminal in the first quadrant, in which quadrants will the angles of opposite cosines have their terminals? Consider each quadrant similarly.

4. Change the word "cosines" in Ex. 3 to "sines," and solve.

§ 108. On the Effect on a Diagram of a Quarter-turn, Half-turn, Three-quarter-turn, Counter-clockwise.

By a "quarter-turn counter-clockwise" is meant a tilt through 90° counter-clockwise. Evidently after such a tilt:

A line now vertically up was originally right-flat.
A line now right-flat was originally vertical-down.
Or, a line now a positive ordinate was a positive abscissa.
A line now a positive abscissa was a negative ordinate.

To the word "ordinate" we may consider the word "sine" attached; to positive ordinate, positive sine; to negative ordinate, negative sine. The word "cosine" we attach similarly to the word "abscissa."

Thus, "a positive ordinate was a positive abscissa," is the same as "a positive sine was a positive cosine," or, "a sine was a cosine"; while "a positive abscissa was a negative ordinate" means "a cosine was a negative sine."

$$\therefore \sin(90^\circ + A^\circ) = \cos A^\circ; \ \cos(90^\circ + A^\circ) = -\sin A^\circ.$$

Similarly, in considering any question of this kind, *it is necessary only to determine where the ordinate, upright after the tilt, was before the tilt, and where the abscissa, flat to the right after the tilt, was before the tilt.*

What effect has a half-turn counter-clockwise?

A line *up*, was *down; right-flat* was *left-flat*.

Thus, a positive ordinate was a negative ordinate, and a positive abscissa was a negative abscissa.

$$\therefore \sin(180^\circ + A^\circ) = -\sin A^\circ; \ \cos(180^\circ + A^\circ) = -\cos A^\circ.$$

What effect has a three-quarter counter-clockwise turn?

A positive ordinate was a negative abscissa, and a positive abscissa was a positive ordinate.

$$\therefore \sin(270^\circ + A^\circ) = -\cos A^\circ; \ \cos(270^\circ + A^\circ) = \sin A^\circ.$$

What effect has a complete turn?

The diagram is unaffected.

$$\therefore\ \sin(360° + A°) = \sin A°;\ \ \cos(360° + A°) = \cos A°.$$

What effect on a diagram has any number of quarter-turns, counter-clockwise?

From all such tilts all complete turns may be dropped, as being without effect. There will remain a quarter-turn, a half-turn, or a three-quarter-turn. The corresponding angles are:

$$4n \cdot 90°,\ (4n+1)\,90°,\ (4n+2)\,90°,\ (4n+3)\,90°.$$

$$\therefore \left\{ \begin{array}{l} \sin(4n \cdot 90° + A°) = \sin A°. \\ \cos(4n \cdot 90° + A°) = \cos A°. \end{array} \right\}$$

$$\left\{ \begin{array}{l} \sin((4n+1)\,90° + A°) = \cos A°. \\ \cos((4n+1)\,90° + A°) = -\sin A°. \end{array} \right\}$$

$$\left\{ \begin{array}{l} \sin((4n+2)\,90° + A°) = -\sin A°. \\ \cos((4n+2)\,90° + A°) = -\cos A°. \end{array} \right\}$$

$$\left\{ \begin{array}{l} \sin((4n+3)\,90° + A°) = -\cos A°. \\ \cos((4n+3)\,90° + A°) = \sin A°. \end{array} \right\}$$

If B, C, D, are three angles in degree measure free from multiples of 360°, and of the second, third, fourth quadrant, respectively, then $B - 90°$, $C - 180°$, $D - 270°$, are three angles, each less than 90°, whose terminals, by a quarter-turn, half-turn, three-quarter-turn, respectively, come into coincidence with those of B, C, D, respectively. Thus,

$$\left\{ \begin{array}{l} \sin B = \cos(B - 90°). \\ \cos B = -\sin(B - 90°). \end{array} \right\} (i)$$

$$\left\{ \begin{array}{l} \sin C = -\sin(C - 180°). \\ \cos C = -\cos(C - 180°). \end{array} \right\} (ii)$$

$$\left\{ \begin{array}{l} \sin D = -\cos(D - 270°). \\ \cos D = \sin(D - 270°). \end{array} \right\} (iii)$$

Formulas (*i*), (*ii*), (*iii*), are of great practical importance in using the tables, obviating subtractions on minutes and seconds. The teacher may assign practice exercises.

EXERCISES.

1. Cut out a right-triangle from paper, carry out the tilts indicated in § 108, and thus prove the formulas.

2. Determine what the formulas above become when n is a negative integer.

3. Change A to $-A$ in the formulas of this section, and from the facts $\sin A = -\sin(-A)$, $\cos A = \cos(-A)$, deduce the values of the following expressions in terms of A: $\sin(90° - A)$; $\cos(90° - A)$; $\sin(180° - A)$; $\cos(180° - A)$; $\sin(270° - A)$; $\cos(270° - A)$; $\sin(360° - A)$; $\cos(360° - A)$; $\sin(4n \cdot 90° - A)$; $\cos(4n \cdot 90° - A)$; $\sin((4n+1)90° - A)$; $\cos((4n+1)90° - A)$; $\sin((4n+2)90° - A)$; $\cos((4n+2)90° - A)$; $\sin((4n+3)90° - A)$; $\cos((4n+3)90° - A)$.

4. Solve Ex. 3 by diagrams.

5. Determine the values of the expressions in Ex. 3 when $-90°$ is written for 90°.

6. Construct diagrams for the expressions $90° \pm A$, etc., of this section and the preceding examples, and show the results directly from the diagrams.

§ 109. The following generalization may be made from the preceding results :

If the subtracted (added) 90°'s are even, the word "sine" remains the word "sine," and the word "cosine" remains the word "cosine." When, however, their number is odd, the word "sine" changes to "cosine" and the word "cosine" to "sine." The proper sign is then to be attached, according to the relative position of the original terminal and the new terminal.

EXERCISE.

Determine by the method of tilts and by the preceding suggestion, the effect on the sine and cosine of adding (subtracting) the first fifty multiples of 90° to an angle A, taking A in each quadrant.

For example, what is the effect of adding twenty-seven 90°'s to the angle 123° ?

Dividing 27 by 4, the remainder is 3. The effect, then, is the same as that of a three-quarter tilt, counter-clockwise.

The positive ordinate was the negative abscissa.
The positive abscissa was the positive ordinate.
∴ sine was a negative cosine, and cosine was a sine.

$$\therefore \sin(27 \times 90° + 123°) = -\cos 123°,$$

and $$\cos(27 \times 90° + 123°) = \sin 123°.$$

Using the second method :

Since 27 is an odd number, the words interchange, "sine" becoming "cosine" and "cosine," "sine." Omitting from 27 all multiples of 4, the remainder is 3. The angle 123° is of the second quadrant. Adding three 90°'s will bring the new terminal to the first quadrant. A sine in the first quadrant is opposite in sign to a cosine of the second.

$$\therefore \sin(27 \times 90° + 123°) = -\cos 123°.$$

But a cosine of the first quadrant is of the same sign as a sine of the second.

$$\cos(27 \times 90° + 123°) = \sin 123°.$$

I prefer the method of tilts. The student may take his choice, or devise a better method.

§ 110. Cosines of all Angles are First Quadrant Cosines.

It is evident that if the cosines of all angles of the first quadrant, between 0° and 90°, are known, the cosines of all other angles are known in magnitude.

For a terminal of quadrant II, use the symmetric terminal of quadrant I, with change of sign of the cosine.

For a terminal of quadrant III, use the opposite terminal of quadrant I, with change of sign of the cosine.

For terminal of quadrant IV, use the symmetric terminal of quadrant I, without change of sign of the cosine.

Take in each case the principal angle of the corresponding terminal in quadrant I.

Without constructing the terminal one may proceed thus:

(*a*) Disregard the sign of the given angle, since $\cos(-A) = \cos A$.

(*b*) Drop all multiples of 360°, as being without effect.

(*c*) For remainder, R, if $R > 90°$ and $< 180°$, take $-\cos(180° - R)$, or $-\sin(R - 90°)$ (§ 108).

(*d*) For remainder, R, if $R > 180°$ and $< 270°$, take $-\cos(R - 180°)$.

(*e*) For remainder, R, if $R > 270°$ and $< 360°$, take $\cos(360° - R)$, or $\cos(R - 270°)$ (§ 108).

The student will observe, on comparing these statements with terminal diagrams, that they are nothing but:

Terminals symmetric to the vertical, give opposite cosines.

Terminals opposite, give opposite cosines.

Terminals symmetric to the horizontal, give the same cosines.

EXERCISES.

1. Give the angles between 0° and 90° which have the same cosine in numerical magnitude as the following degree-measure angles, and give the sign in each case, this being obtained from the quadrant of the given angle:

140°; 175°; 187°; 200°; 280°; 305° 23′; − 30° 23′; − 100° 32′; − 185° 43′; − 5° 43′; − 150° 25′; ± 432°; ± 403°; ± 506°; ± 603°; ± 1850°; ± 2317°.

2. Solve the preceding when for the word "cosine" there is read the word "sine."

3. Express two of the preceding angles in radian measure, (i) in terms of a radian, (ii) in terms of π radians.

4. Solve Ex. 1 by constructing terminals.

§ 111. Construction of the Terminals of Angles having a Given Cosine.

(i) Let $\cos A = \frac{2}{3}$.

Draw a circle with the radius 3. Lay out on the abscissa axis the distance $OM = 2$; then draw through M the vertical line, cutting the circle in P_1 P_4. OP_1, OP_4 are the required terminals.

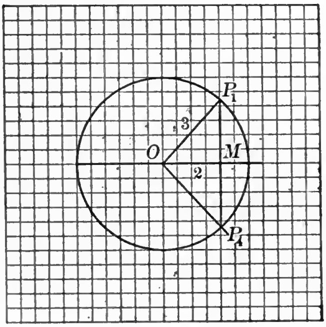


FIG. 109.

For every given value of the cosine, other than ± 1, there are two and only two positions of the terminal. These two positions are always symmetric to the horizontal, as were terminals for a given sine symmetric to the vertical (§ 66).

The angles of the two preceding terminals are

$$2n \cdot 180° \pm A° (\S\ 106, \text{iii}).$$

(ii) Let $\cos A = -\frac{2}{3}$.

The construction is the same as that above, except that OM is laid out to the left, or $OM = -2$.

EXERCISES.

1. Construct terminals for the cosines $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{1}{2}$, and give the general value of the angles, using tables. Test with protractor.

2. Show that taking some line for unity, one can construct lengths representing the square roots of 2, 3, 5, 6, 7, 8, or any other integer. Use the process to construct the terminals for the angles whose cosines are $\pm\frac{\sqrt{2}}{2}$, $\pm\frac{\sqrt{3}}{2}$, $\pm\frac{\sqrt{5}}{2}$.

3. Can you determine from the constructions, without measurement or tables, what angles have the cosines $\pm\frac{\sqrt{2}}{2}$, $\pm\frac{\sqrt{3}}{2}$, $\pm\frac{1}{2}$, ± 1, 0?

4. Change the word "cosines" to "sines" in Exs. 1, 2, and 3, and solve.

5. Construct terminals for some cosines in the tables. Measure the angle with the protractor and compare with the table.

§ 112. Line Picture of the Cosine.

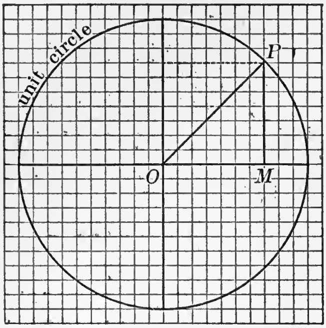


FIG. 110.

As on the unit-circle the ordinates represent the sines for the corresponding terminals, so do the abscissas represent cosines.

For the terminal OP, the cosine is

$$\frac{OM}{OP} = \frac{OM}{1} = OM.$$

This is the line definition of the cosine. Compare § 67.

LABORATORY EXERCISE.

Divide a circumference whose radius is one foot into 5° spaces; measure the abscissas; compare with the table-cosines.

§ 113. Line Pictures of the Cosines of All Angles.

As all the ordinates of the unit-circle represent, as a continuum, the sines of all angles (see § 69), so do the abscissas give, pictorially, a correct impression of the relative magnitude of the cosines of all angles, with their manner and extent of variation as the terminal passes from the initial position OA, counter-clockwise, through a perigon, and continues its motion, with a repetition of the cosine values.

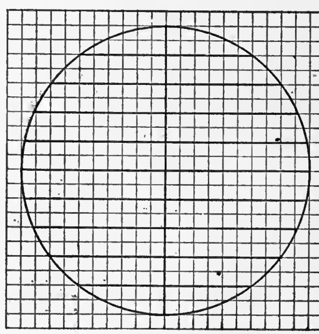


FIG. 111.

From the diagram (Fig. 111) one can readily draw the following conclusions:

(*a*) *When the terminal is horizontal to the right, the cosine is* $+1$.

$$\therefore \cos 0° = 1 = \cos(\pm 360°) = \cos(\pm 720°), \text{ etc.},$$

or, in general,

$$\cos(2n \cdot 180°) = 1 = \cos(2n \cdot \pi),$$

where n is any $\pm$ integer.

That is, *the cosine of all even multiples of* 180°, *or of* π *radians, is* $+1$.

(*b*) *When the terminal is horizontal to the left, the cosine is* -1.

$$\therefore \cos(\pm 180°) = -1 = \cos(\pm 540°), \text{ etc.},$$

or, in general,

$$\cos(2n+1)\,180° = -1 = \cos(2n+1)\,\pi.$$

What is the $\cos(\pm\pi)$?

The cosine of all odd multiples of 180°, *or of* π *radians, is* -1.

(*c*) *When the terminal is vertical, up or down, the cosine is zero.*

$$\therefore \cos(\pm 90°) = 0 = \cos(\pm 270°) = \cos(\pm 540°), \text{ etc.}$$

$$\cos\left(\pm\frac{\pi}{2}\right) = 0 = \cos\left(\pm\frac{3\pi}{2}\right) = \cos\left(\pm\frac{5\pi}{2}\right), \text{ etc.}$$

In general,

$$\cos(2n+1)\,90° = 0 = \cos(2n+1)\frac{\pi}{2}.$$

The cosine of all odd multiples of 90°, *or of* $\frac{\pi}{2}$ *radians, is zero.*

(*d*) *Thus, for terminals bordering the quadrants:*

Terminal	Right	Up	Left	Down	Right
Cosine	+1	0	−1	0	+1
Sine	0	+1	0	−1	0

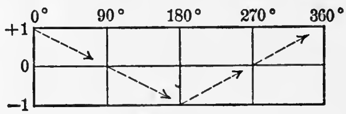
Fig. 112.

EXERCISE.

Name five positive angles and five negative angles in degree measure and in radian measure for each terminal corresponding to Fig. 112.

(*e*) As the terminal passes from 0° to 360°, the cosine, starting with the value + 1, diminishes continuously from 1 to 0, which is reached at 90°; then diminishes continuously to −1, which is reached at 180°; then increases continuously to 0, which is reached at 270°; then increases continuously to + 1, which is reached at 360°. As the terminal passes again around the circuit, the same set of values is repeated, and so on. Thus, as the angles form a continuum from 0° to 360°, the cosines form a double continuum from + 1 to − 1, and − 1 to + 1.

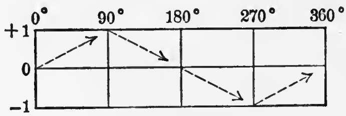
The Cosine.



Repeating, forward and backward, without end.

FIG. 113.

The Sine.



Repeating, forward and backward, without end.

FIG. 114.

EXERCISE.

What effect has a reversal of the order of angle description on the order of description of the double cosine-continuum? On the sine-continuum?

(*f*) *For any position on the terminal, a slight change in the terminal position makes a slight change in the corresponding cosine.*

In Fig. 115, OM_1, OM_2 are the cosines of two angles nearly equal. The cosine-difference is M_1M_2. As P_2 is brought nearer and nearer to P_1, the smaller becomes M_1M_2. When the angle-difference is less than any assignable quantity, so is the cosine-difference, for M_1M_2 is less than P_1P_2, this arc-difference on the unit-circle being the same, numerically, as the angle-difference in radian measure. See (4), page 67.

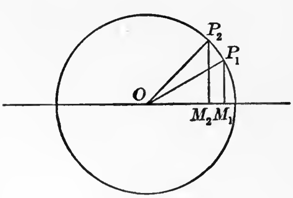


FIG. 115.

Thus the cosine is a continuous function of the angle.

(*g*) For all angles whose terminals are in the first or second quadrant, increasing the angle diminishes the cosine.

Therefore all angles with positive sines have for an increase in the angle a decrease in the cosine.

Similarly, considering the third and fourth quadrants, all angles with negative sines have, for an increase in the angle, an increase in the cosine.

(*h*) The points at which a function changes from an increasing function to a decreasing function, or *vice versa*, are called the *turning-points of the function.*

EXERCISES.

1. In what position is the terminal when the cosine is at a turning-point?

2. When the sine is at a turning-point?

3. Give in circular measure and in degree measure general expressions for all angles whose cosines are at a turning-point. Do the same for the sine.

(*i*) *When the terminal is nearly flat, a slight change in its position does not affect the cosine very materially. When, however, the terminal is very nearly vertical, slight changes in its position are accompanied by marked changes in the cosine.*

From this it follows that small angles cannot be determined with great accuracy from the cosine, just as angles near 90° cannot be determined accurately from the sine. In practical work, it is well to avoid, when possible, observations which lead to the determination of an angle from its sine, when the angle is within 5 or 6 degrees of 90°, or from its cosine, when the angle is within 5 or 6 degrees of zero. Information in detail will be given on this point later. See §§ 185, 186.

EXERCISE.

Examine four-place tables and five-place tables for the changes made in the sine and cosine for a change of 1′ in the angle at different parts of the tables. Examine also the corresponding changes in log sin, log cos.

§ 114. The Cosine as a Function of the Angle.

The relation between the cosine and the corresponding angle in radian measure is, as will be shown later,

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \cdots, \text{ etc.}$$

Compare this with the sine series, or sine function, as given in § 70.

The cosine, like the sine, is thus a transcendental function of the angle.

While the sine involves only odd powers of the angle, the cosine is expressed wholly in terms of the even powers. This is essential, since

$$\sin(\theta) = -\sin(-\theta),$$

and

$$\cos(\theta) = \cos(-\theta).$$

The cosine is thus an *even function* of the angle, while the sine it an *odd function* of the angle.

EXERCISES.

1. Show from the function expressions for sine and cosine that $\sin(\theta) = -\sin(-\theta)$ and $\cos(\theta) = \cos(-\theta)$.

2. Show in general that if any function is expressed in powers of x and if $F(x) = F(-x)$, there can be only even powers of x, while if $F(x) = -F(-x)$, there can be only odd powers of x.

3. Use the sine and cosine function expressions to calculate the sines and cosines, to two significant figures, of the following angles:

$$0,\ 1'',\ 1',\ 1^\circ,\ \pm 2^\circ,\ \pm 3^\circ,\ \pm 5^\circ,\ 1^r,\ 2^r,\ \pi^r.$$

§ 115. The Cosine as a Periodic Function.

Since

$$\cos\theta = \cos(\theta + n \cdot 2\pi),$$

the cosine is, like the sine, periodic, with the period, 2π.

Reread § 72.

§ 116. The Anti-cosine, or Inverse-cosine.

When $y = \cos x$, then $x = \cos^{-1} y$.
Reread § 73.

EXERCISES.

1. Read in all possible ways, as in the corresponding exercise on the sine, the expressions:

$\cos^{-1}\frac{3}{4}$; $\cos^{-1}\frac{2}{3}$; $\cos^{-1}\frac{\sqrt{2}}{2}$; $\cos^{-1}\frac{\sqrt{3}}{2}$; $\sin^{-1}\frac{1}{2}$; $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$; $\sin(\cos^{-1}\frac{1}{2})$; $\cos(\sin^{-1} 0)$; $\cos(\cos^{-1} x)$; $\sin(\cos^{-1} 0)$; $3\sin^{-1}\frac{1}{2} + 2\cos^{-1}\frac{\sqrt{2}}{2}$; $3\sin(\cos^{-1}\frac{1}{2}) - 2\cos(\sin^{-1}\frac{2}{3})$.

2. Construct the terminals for such of the expressions in Ex. 1 as represent angles.

3. How many values have the following expressions?

$$\sin(\sin^{-1} x);\ \sin(\cos^{-1} x);\ \cos(\cos^{-1} x);\ \cos(\sin^{-1} x).$$

Read these expressions in all possible ways.

4. How many values have the following expressions?

$$\cos^{-1}(\cos x);\ \sin^{-1}(\sin x);\ \sin^{-1}(\cos x);\ \cos^{-1}(\sin x).$$

Read these expressions in all possible ways.

5. What are the values of the following expressions?

$\sin(\cos^{-1} 0)$; $\sin(\cos^{-1} 1)$; $\sin(\cos^{-1}(-1))$; $\sin(-\cos^{-1} 1)$; $\sin(-\cos^{-1}(-1))$; $\sin(-\cos^{-1} 0)$.

6. Give the values of the expressions in Ex. 5 when the words "sine" and "cosine" are interchanged. When both words are "sine." When both words are "cosine."

7. Give two special values and the general values of each of the following expressions:

$\sin^{-1}(\cos 0°)$; $\sin^{-1}(\cos(\pm 90°))$; $\sin^{-1}(\cos(\pm 180°))$; $\sin^{-1}(\cos(\pm 270°))$; $\sin^{-1}\cos(\pm 360°))$.

8. Interchange the words "sine" and "cosine" in Ex. 7 and solve.

9. Also answer Ex. 7 when the word "cosine" is preceded by the negative sign. Then interchange the word "sine" and "cosine," and answer.

§ 117. Some Angles whose Cosines can be determined readily from a Diagram.

These are the same angles for which the sines were determined in § 75. The diagrams used for sines give also the cosines. These diagrams are:

For 45° and all angles of the form	For 60° and all angles of the form	For 30° and all angles of the form
$2n\pi \pm \frac{\pi}{4}$	$2n\pi \pm \frac{\pi}{3}$	$2n\pi \pm \frac{\pi}{6}$
$2n \cdot 180° \pm 45°$	$2n \cdot 180° \pm 60°$	$2n \cdot 180° \pm 30°$

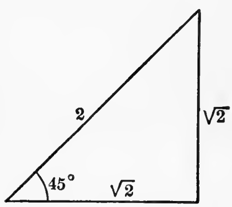


FIG. 116.

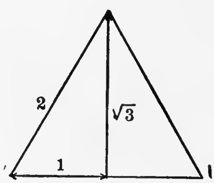


FIG. 117.

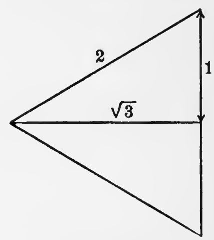


FIG. 118.

The cosines of all angles noted in § 75 are readily obtained from these diagrams by observing whether, as related to the diagram terminal, the new terminal is

(i) opposite,

(ii) symmetric to the horizontal,

(iii) symmetric to the vertical, remembering:

Angles with opposite terminals have $\begin{cases}\text{opposite sines,}\\ \text{opposite cosines.}\end{cases}$

Angles with terminals symmetric to the horizontal have $\begin{cases}\text{opposite sines,}\\ \text{same cosine.}\end{cases}$

Angles with terminals symmetric to the vertical have $\begin{cases}\text{same sine,}\\ \text{opposite cosines.}\end{cases}$

EXERCISES.

1. Name five angles having their terminals opposite to that of 45°, and give their sines and cosines.

2. Name five angles having their terminals symmetric to the horizontal with reference to the terminal of 45°, and give their sines and cosines.

3. In Ex. 2, replace the word "horizontal" by "vertical," and answer.

4. In Exs. 1, 2, and 3, replace 45° by 30°, and answer.

5. In Exs. 1, 2, and 3, replace 45° by 60°, and answer.

6. Verify the following tables:

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1

Angle	0°	−30°	−45°	−60°	−90°	−120°	−135	−150°	−180°
Sine	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{2}}{2}$	$-\frac{1}{2}$	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1

And fill in the following tables:

Angle	0°	210°	225°	240°	270°	300°	315°	330°	360°
Sine									
Cosine									

Angle	0°	−210°	−225°	−240°	−270°	−300°	−315°	−330°	−360°
Sine									
Cosine									

NOTE. — The student is advised against attempting to memorize the results of the preceding tables as independent facts. It is better to hold in mind the diagrams, and read from these mental diagrams the numerical values of the sines and cosines, attaching the proper sign according to the location of the terminal.

7. Determine from the diagrams the smallest angles (numerically) satisfying the following expressions, together with some two other values; also the general values:

$\sin^{-1}\frac{\sqrt{2}}{2} \pm \sin^{-1}\frac{\sqrt{3}}{2}$; $\pm \sin^{-1}0 \pm 2\sin\frac{\sqrt{3}}{2}$; $3\sin^{-1}0 \pm 4\cos^{-1}0$;

$\pm 2\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \pm 3\cos^{-1}\frac{\sqrt{3}}{2}$; $\pm \sin^{-1}(1) \pm 5\cos^{-1}(1)$;

$\pm \sin^{-1}(\frac{1}{2}) \pm 2\cos^{-1}(\frac{1}{2})$; $3\sin^{-1}(-\frac{1}{2}) - 2\cos^{-1}(-\frac{1}{2})$;

$2\sin\left(-\frac{\sqrt{2}}{2}\right) \pm 3\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$; $\sin^{-1}(0) + 2\sin^{-1}\frac{\sqrt{2}}{2} - 3\cos\frac{\sqrt{2}}{2}$;

$\sin^{-1}0 + \sin^{-1}\frac{\sqrt{2}}{2} + \sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{2} + \cos^{-1}0 + \cos^{-1}\frac{\sqrt{2}}{2} + \cos^{-1}\frac{\sqrt{3}}{2}$,
and this last when the sines and cosines are negative.

8. Solve the exercises at the end of § 75, changing the word "sine" to "cosine" where possible.

9. Verify the following results (using diagrams and not the tables):

(*a*) $\sin 30° \cos 60° + \cos 30° \sin 60° = \sin 90° = \cos 0° = -\sin 270°$.

(*b*) $\sin 45° \cos 60° \pm \cos 45° \sin 60° = \frac{\sqrt{2}}{4}(1 \pm \sqrt{3})$.

(*c*) $\sin 60° \cos 30° + \cos 60° \sin 30° = \sin 90°$.

(*d*) $\cos 60° \cos 30° + \sin 60° \sin 30° = \cos 30°$.

(*e*) $\cos 60° \cos 30° - \sin 60° \sin 30° = \cos 90°$.

(*f*) $\sin 45° \cos 0° \pm \cos 45° \sin 0° = \cos 45°$.

(*g*) $\cos^2 30° + \sin^2 30° = 1$.

(*h*) $\cos^2 45° + \sin^2 45° = 1$.

(*i*) $\cos^2 60° + \sin^2 60° = 1$.

(*j*) $\cos^2 120° + \sin^2 120° = 1$.

(*k*) $\cos^2 30° - \sin^2 30° = \cos 60°$.

(*l*) $\cos^2 45° - \sin^2 45° = \cos 90°$.

(*m*) $\cos^2 60° - \sin^2 60° = \cos 120°$.

(*n*) If $A = 60°$, $\sin\frac{A}{2} = +\sqrt{\frac{1-\cos A}{2}}$, and $\cos\frac{A}{2} = +\sqrt{\frac{1+\cos A}{2}}$.

(*o*) Can you show that $\sin^2 A + \cos^2 A = 1$, where A is any angle?

10. Determine from the diagrams the value of

$$\sin A + \cos B,$$

and

$$\sin A \cos B \pm \cos A \sin B,$$

when A, B are any pair of angles appearing in the tables on page 213. Do not use the table for values, but get the values direct from a diagram.

11. Verify from diagrams the following results, when $A = 30°, 45°, 60°$, or any angle of tables, page 213:

(*a*) $\sin 2A = 2 \sin A \cos A$.

(*b*) $\sin 3A = 3 \sin A - 4 \sin^3 A$.

(*c*) $\cos 2A = \cos^2 A - \sin^2 A$.

(*d*) $\cos 2A = 1 - 2 \sin^2 A$.

(*e*) $\cos 2A = 2 \cos^2 A - 1$.

(*f*) $\cos 3A = 4 \cos^3 A - 3 \cos A$.

§ 118. Complementary Angles.

Any pair of angles whose sum in degree measure is 90°, or radian measure, $\frac{\pi}{2}$, are called a *complementary pair.* Each is said to be the *complement* of the other.

Their form is $A°, 90° - A°; \ \theta, \frac{\pi}{2} - \theta$.

To construct the terminal of $90° - A°$, lay out from the upright vertical an angle, in the reversed direction, equal to A. The terminals of an angle and its complement are, therefore, inclined to the right-hand horizontal line and the upright vertical in opposite equality.

Thus, if the upright vertical were made the initial line and abscissa axis, and counter-clockwise motion turned into clockwise motion, and *vice versa*, angles would become their complementary angles by equal turns.

This is equivalent to changing ordinates into abscissas, and abscissas into ordinates, or

sines into cosines

and *cosines into sines.*

Thus, *the sine of an angle is the cosine of its complement,*

and *the cosine of an angle is the sine of its complement.*

EXERCISES.

1. Show on a diagram that if equal moduli are taken on the terminals of an angle and its complement, the ordinate of one modulus is the abscissa of the other, and *vice versa*. Use each quadrant.

2. Show that the sine (cosine) of any angle can always be expressed in terms of the sine or cosine of an angle less than 45°. Illustrate by diagrams, with the terminal in each quadrant. Make up and solve five numerical examples.

This is made use of in trigonometric tables; columns headed "sine" at the top are marked "cosine" at the bottom. The teacher may point this out in whatever five-place table is used, showing that such tables need go only to 45° direct and that the sine (cosine) of any angle less than 360° can be taken directly from the table, by the aid of formulas (*i*), (*ii*), (*iii*), page 201. See special arrangement in Hussey's Tables.

§ 119. The Complementary Arc.

In the early treatment of trigonometry, with the unit circle, and with arcs used instead of angles, their numbers, or numerical measures, being the same, arcs like *AP* and *BP* (Fig. 119) were taken as complementary arcs. *NP*, the sine of the arc *BP*, is equal to the cosine, *OM*, of the arc *AP*. So also the sine, *MP*, of the arc *AP* is the same in length as *ON*, the cosine of the arc *BP*. The complementary arc, *BP*, behaved toward the upright vertical just as the arc itself to the horizontal. The *co-* of *cosine* thus indicated the sine of the complementary arc.

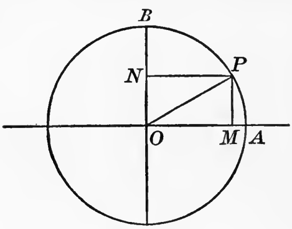


Fig. 119.

EXERCISES.

1. Select a terminal in each quadrant, construct its complementary terminal, and show from the diagram that

$$\sin(90° - A) = \cos A; \ \cos(90° - A) = \sin A,$$

using both the ratio definitions and the line definitions.

2. Select at random two angles (involving minutes and seconds) in each quadrant, and find their complements.

3. Test by diagrams the relations $\sin(90° - A) = \cos A$, $\cos(90° - A) = \sin A$, for each of the following angles in degree measure, taking each both positive and negative: 0; 30; 45; 60; 90; 120; 135; 150; 180; 210; 225; 240; 270; 300; 315; 330; 360.

4. Show that if $\sin A = \cos B$, the terminals of A, B are equally inclined to the vertical and horizontal respectively, but that A, B are not necessarily complementary. Find the general formula connecting A, B, when $\sin A = \cos B$, and when $\cos A = \sin B$, as in Ex. 5.

5. Solve the equation $\sin 2\theta = \cos 3\theta$ by sines.

SOLUTION.

$$\sin 2\theta = \cos 3\theta = \sin\left(\frac{\pi}{2} - 3\theta\right).$$

Thus $\frac{\pi}{2} - 3\theta$ must be some one of the angles which have the same sine as 2θ.

$$\therefore \frac{\pi}{2} - 3\theta = 2n\pi + 2\theta; \tag{1}$$

or

$$\frac{\pi}{2} - 3\theta = (2n+1)\pi - 2\theta. \tag{2}$$

Remembering n is any $\pm$ integer, its sign, in transposing, is not considered.

$$\therefore \text{by (1)}, \quad 5\theta = (2n + \tfrac{1}{2})\pi; \tag{3}$$

$$\text{by (2)}, \quad \theta = (2n - \tfrac{1}{2})\pi; \tag{4}$$

$$\text{by (3)}, \quad \theta = (4n+1)\frac{\pi}{10}; \tag{5}$$

or

$$\text{by (4)}, \quad \theta = (4n-1)\frac{\pi}{2}. \tag{6}$$

6. Locate the terminals for a few of the angles determined by (5), (6), of the preceding solution and show that the terminals of 2θ, 3θ are equally tilted toward the upright vertical and right-hand horizontal, so that $\sin 2\theta = \cos 3\theta$.

7. Show from (5) and (6), of Ex. 5, that 2θ, 3θ as determined by (5) are complementary only when $n = 0$, giving $\theta = 18°$, and that as determined from (6) they are never complementary.

8. Solve Ex. 5 by cosines instead of by sines.

9. Give the general solution of $\sin A = \cos B$ by sines, and show that $A + B = (4n+1)90°$ or $A - B = (4n+1)90°$. For what value of n do we get complementary angles? For what value would we get $A = 90° + B$?

10. Show from the results of Ex. 9 that $\sin((4n+1)90° + A) = \cos A$; $\sin(90° + A) = \cos A$; $\cos((4n+3)90° + A) = \sin A$; $\cos(270° + A) = \sin A$. Compare with the formulas on page 201.

11. Give the general solution of $\sin A = \cos B$ by cosines, and deduce the results in Ex. 10.

12. Solve by sines or by cosines the equations:

$$-\sin A = \cos B;$$
$$\sin A = \sin B;$$
$$\sin A = -\sin B;$$
$$\cos A = \cos B;$$
$$\cos A = -\cos B;$$

and deduce the results of § 108, as in Ex. 10.

13. Solve by sines or by cosines the following equations:

$$\sin 5A = \cos 6A;$$
$$\sin 7A = -\cos 9A;$$
$$\sin (sA) = \cos (tA);$$
$$\sin (sA) = -\cos (tA).$$

14. State the general method of solving equations like those in Exs. 12 and 13. It is given in Ex. 5.

§ 120. Calculations using the Cosine, without the Use of Tables.

From the relation of definition connecting the three quantities, cosine, abscissa, modulus, when any two of the quantities are given, the third can be calculated:

(i) *When modulus and cosine are given*, abscissa = modulus times cosine.

(ii) *When modulus and abscissa are given*, cosine $= \dfrac{\text{abscissa}}{\text{modulus}}$.

(iii) *When abscissa and cosine are given*, modulus $= \dfrac{\text{abscissa}}{\text{cosine}}$.

EXERCISES.

Change the words "ordinate" and "sine" in the examples of § 76, to "abscissa" and "cosine," respectively, and solve.

§ 121. Calculations with Cosines, using the Tables.

These calculations may be made either with the tables of natural cosines or with logarithms. As the student is now calculating for calculation's sake, merely to learn, he may use both processes with each example, the one as a check on the results of the other.

For natural cosines $\begin{cases} \text{abscissa} = \text{modulus times cosine}, \\ \text{cosine} = \dfrac{\text{abscissa}}{\text{modulus}}, \\ \text{modulus} = \dfrac{\text{abscissa}}{\text{cosine}}. \end{cases}$

For log cosines:

$$\begin{aligned} \log \text{abscissa} &= \log \text{modulus} + \log \text{cosine}, \\ \log \text{cosine} &= \log \text{abscissa} - \log \text{modulus}, \\ \log \text{modulus} &= \log \text{abscissa} - \log \text{cosine}. \end{aligned}$$

EXERCISES.

After the manner of the corresponding exercises on the sine, § 79, carry out the following calculations, with tests of accuracy.

Let calculated parts show the same number of significant figures as the data. In one-figured data on lines, read angles to the nearest five degrees; in two-figured data on lines, read angles to the nearest half-degree; in three-figured data on lines, read angles to the nearest five minutes; in four-figured data, to the nearest minute; in five-figured data, to the nearest second, etc. (See § 77.)

A. Find the general value of the angle in degrees and in circular measure in terms of π:

1. Modulus 9, abscissa ± 7; modulus 8, abscissa ± 3.

2. Modulus 34, abscissa ± 23; modulus 6.7, abscissa ± 3.4.

3. Modulus 23.4, abscissa ± 15.4; modulus 4.56, abscissa ± 2.34.

4. Modulus 23.78, abscissa ± 17.34; modulus 2.674, abscissa ± 1.789.

5. Modulus 234.98, abscissa ± 189.90 (five-place table).

6. Modulus 453.764, abscissa ± 389.031 (six-place table).

7. Construct a set of examples similar to the six preceding and consistent as representing measurements.

B. Calculate the abscissa, given:

8. Modulus 5, angle $\pm 25°$; modulus 9, angle $\pm 40°$.

9. Modulus 34, angle $\pm 55°$; modulus 7.8, angle $\pm 67°$.

10. Modulus 35.7, angle $\pm 43°$; modulus 3.42, angle $\pm 67° 25'$.

11. Modulus 23.45, angle $\pm 67° 23'$; modulus 271.8, angle $\pm 45° 32'$.

12. Modulus 234.67, angle $\pm 34° 45' 56''$ (five-place table).

13. Modulus 4578.67, angle $\pm 23° 45' 56.7''$ (six-place table).

14. Construct a set of examples similar to Exs. 8–13 preceding and consistent as representing measurements.

15. Express the angles of Exs. 8–13 in terms of π; also in radian measure.

C. Calculate the modulus, given:

16. Abscissa 7, angle 15°; abscissa 9, angle 45°.

17. Abscissa 34, angle 23°; abscissa 7.9, angle 70° 30′.

18. Abscissa 45.7, angle 34° 50′; abscissa 3.89, angle 59° 5′.

19. Abscissa 34.91, angle 73° 23′; abscissa 6.789, angle 45° 21′.

20. Abscissa 45.831, angle 56° 56′ 56″ (five-place table).

21. Abscissa 2345.67, angle 34° 45′ 56″.7 (six-place table).

22. Abscissa 45.69879, angle 67° 23′ 34″.51 (seven-place table).

23. Construct a similar set of consistent examples representing measurements.

§ 122. Calculations with Right-angled Triangles, using the Sine and Cosine.

Right-angled triangles may be solved by using the sine alone, as already shown in § 81, or by using the cosine alone, since the acute angles are complementary.

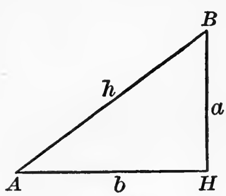


Fig. 120.

$$\text{Sine} = \frac{\text{side opposite the angle}}{\text{hypothenuse}},$$

$$\text{Cosine} = \frac{\text{side adjacent to the angle}}{\text{hypothenuse}}.$$

We have thus for the solution of right-angled triangles:

USING SINES.	USING COSINES.
$\sin A = \frac{a}{h}$	$\cos A = \frac{b}{h}$
$\sin B = \frac{b}{h}$	$\cos B = \frac{a}{h}$
$a = \sin A \cdot h$	$a = \cos B \cdot h$
$b = \sin B \cdot h$	$b = \cos A \cdot h$
$h = \frac{a}{\sin A} = \frac{b}{\sin B}$	$h = \frac{a}{\cos B} = \frac{b}{\cos A}$

It is customary to use a combination of sines and cosines, the sine in connecting the hypothenuse and the side opposite an angle and the cosine in connecting the hypothenuse and the side adjacent to the angle. We may think of opposite the angle as *in* the angle, and adjacent to the angle as *on* the angle. As *in* is in *sine* and the *o* of *on* is in *cosine*, you will remember that the side *in the angle* involves the sine and hypothenuse, while the side *on the angle* brings in the cosine and hypothenuse. This may be helpful when the triangles are tilted away from a horizontal (vertical) position.

ORAL EXERCISE.

Practise with the cosine and sine in connection with the Conradt collection of right-angled triangles on page 145.

§ 123. Calculation Scheme for Right-angled Triangles, using Log Sines and Log Cosines.

MODEL EXAMPLES.

Given $\begin{cases} B = 35^\circ\, 12' \\ h = 27.34 \end{cases}$

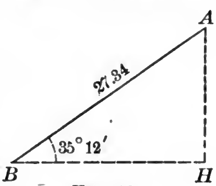


FIG. 121.

Find $\begin{cases} A = 54^\circ\, 48' \\ b = 15.76 \\ a = 22.34 \end{cases}$

$$A = 90^\circ - B.$$

$$b = \sin B \cdot h. \quad \therefore \log b = \log \sin B + \log h,$$

$$a = \cos B \cdot h. \quad \therefore \log a = \log \cos B + \log h.$$

SOLUTION SCHEME.

LOGARITHMS.

(4)	b	1.1975
(2)	$\sin B$	$\bar{1}.7607$
(1)	h	1.4368
(3)	$\cos B$	$\bar{1}.9123$
(5)	a	1.3491

LOG-CHECK.

$a \cdot \sin B = b \cdot \sin A.$

LOGARITHMS.

1.3491	1.1975
$\bar{1}.7607$	$\bar{1}.9123$
1.1098	1.1098

Order of work is indicated by the numerals. First enter the logs (1), (2), (3). Then add (1), (2), for (4), and (1), (3), for (5). Then look up the numbers corresponding to logs (4), (5), and enter the results in the column marked "Find."

If A takes the place of B in the scheme, a, b are interchanged.

The order of filling in the scheme will change with the data, but not the scheme itself.

EXERCISES.

Arrange the data and results to be found, diagram, formulas, log-formulas, and calculation scheme for the following examples. Give scheme for other possible cases, omitting the case when two sides are given.

1. b 8, B 65°; b 6, B 35°; a 5, B 55°; a 7, A 25°.

2. b 12, B 46°; b 3.4, B 76° 30′; a 2.7, B 61°; a 26, A 26° 30′.

3. b 12.4, B 45°; b 34.5, B 56° 50′; a 678, A 45° 15′.

4. b 23.45, A 34° 45′; b 345.7, A 45° 53′.

5. b 234.56, B 56° 34′ 31″.

6. h 8, B 45°; h 9, A 15°.

7. h 23, B 34°; h 4.5, A 67° 30′.

8. h 23.8, B 23°; h 765, A 34° 25′.

9. h 236.9, B 56° 21′; h 4.798, A 34° 51′.

10. h 234.59, B 45° 51′ 34″; h 34.761, A 48° 48′ 48″.

11. h 234.786, B 45° 43′ 21″.3.

SINE AND COSINE PROBLEMS IN PHYSICS.

(Such of these as do not lie outside the students' training in physics may be used.)

1. A force F acting at an angle $A°$ to a horizontal plane has what vertical and horizontal components? Solve a numerical example, getting the sums of the horizontal and vertical components when three forces act at a point in a plane in different directions in the plane. Select the directions so that some of the resolved forces are negative.

2. A right-angled triangle in a vertical plane, with its hypothenuse horizontal and right angle above, has a uniform chain covering the other two sides. Find the pressure of the chain on each side, and also the force acting down each side, and see if a continuous chain on such inclines would be in "perpetual motion."

3. A string inclined at an angle $A°$ to a smooth plane inclined $B°$ to the horizontal sustains a weight, W, on the plane. Find the pull on the string and the pressure on the plane, and also in what position the string is most effective in preventing the weight sliding down the plane.

4. A mule hitched to a canal-boat is effective only 75%. What is the angle of the pulling rope with the tow-path?

5. Three horses of equal strength, and two mules, the one as strong as the other and the two as strong as the three horses, pull on five ropes tied to another rope which passes over a pulley and is attached to a weight, W. The angles of the five ropes with the single rope are in order for the horses 17°, 19°, 21° on the same side, and for the mules, on the other side, 18°, 20°, — all in the same plane. How many times the pull of a horse is the lifted weight, and will the single rope maintain its direction as the animals pull?

6. The resultant of two forces is 8. One of them is 5, and the direction of the other 35° from the resultant. What is the angle between the two forces?

7. Two velocities represented by 100, 125, give a resultant velocity 153 directly north. What equations connect the angles that the components make with the north and south line?

8. The force 816 makes angles 25°, 35° with its components. Calculate the components.

9. A movable pulley rests on a rope with one end fixed, the other end of the rope passing over a smooth peg in a horizontal line with the fixed end of the rope. A weight is attached to the movable pulley. Find the equation connecting the pull on the rope and the weight. Does the pull increase or decrease as the weight rises?

10. A sphere of weight W and radius a rests in the angle of two smooth planes inclined at angles $A°$, $B°$ to the horizon. Find the pressure on each plane.

11. A man weighing 200 lbs. stands at the apex of an ordinary triangular roof (equal rafters). What part of his weight tends to spread the walls?

12. When a roof like that in Ex. 11 is covered uniformly with snow, what part of the weight of the snow tends to spread the walls?

§ 124. Application of Sines and Cosines to a Problem in Field-surveying (a Projection Problem).

What is meant by the bearing of a course of a survey has already been explained (§ 86). While the latitude of a course is its length multiplied by the cosine of its bearing, its longitude is its length multiplied by the sine of its bearing, — bearings being taken from the north and south line. In the diagram, the latitude AL is $AP \cos B$, and the longitude, or departure, AM is $AP \sin B$, B being the bearing.

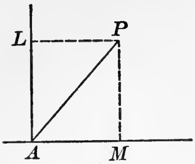


FIG. 122.

North latitudes are counted plus, and south latitudes minus; east departures, plus; west, minus.

In a closed survey, carried out with absolute accuracy, the algebraic sum of the latitudes is zero, as is the algebraic sum of the departures. In practice this is never reached. The difference between the sum of the northings and the sum of the southings is called the *error in latitude;* the difference between the sum of the eastings and sum of the westings, the *error in departure.* The square root of the sum of the squares of these two errors is called the *error in closing.* If the survey is plotted on a diagram, the error in closing is the distance between the beginning and end of the plot when expressed in the unit of length in which the lengths of the courses are given.

The error of closing is a measure of the accuracy of the field work. When it lies within the limit allowable for the kind of land surveyed, it is customary to "balance the survey." This consists in distributing the errors of latitude and departure among the courses in proportion to their length as compared with the length of the border of the survey.

The following calculation scheme applies to each course. The scheme for the entire survey is made by placing such schemes one under another in the order of the survey. At the bottom of such a scheme will appear the errors in latitude and departure, with the error in closing.

	Logarithms	Latitude		Departure		Balance Errors		Balanced	
		$N+$	$S-$	$E+$	$W-$	Lat.	Dep.	Lat.	Dep.
(4) D									
(2) $\sin B$									
(1) s									
(3) $\cos B$									
(5) L									

FIG. 123.

Under Balance Errors and Balanced Latitude and Departure, the proper sign is given, along with the magnitudes.

The numerals indicate, as heretofore, the order of entry. Logs (1) and (2), added, give (4); (1) and (3), added, give (5). Then the numbers corresponding to (4) and (5) are looked up and entered under Departures and Latitudes, in the proper columns.

LABORATORY EXERCISE.

If apparatus is obtainable, make a survey of a few fields, and carry out the calculations as indicated above. Get also the area of each survey. The best laboratory for calculation-trigonometry is the field. A real stump and a real river are much more difficult to manage than paper stumps and paper rivers.

EXERCISES.

Fill out the scheme for the following survey, the numbers 1, 2, 3, etc., indicating the courses in order. Calculate the latitudes and departures of the courses, and the errors of latitude, departure, and closing. Balance the survey. Make the plot.

1. N. 69° E., 437 ft.
2. S. 19 E., 236 ft.
3. S. 27° W., 244 ft.
4. N. 71° W., 324 ft.
5. N. 19° W., 184 ft.

Do the same for the following survey:

1. S. 89° 55′ E., 25.42 ch.
2. N. 27° 40′ E., 34.68 ch.
3. N. 19° 10′ W., 7.40 ch.
4. N. 86° 50′ W., 25.58 ch.
5. S. 47° 30′ W., 1.50 ch.
6. S. 77° 45′ W., 13.60 ch.
7. S. 89° W., 3.53 ch.

§ 125. Use of the Cosine as a Check on Solutions of Triangles.

In the triangle *ABC*, if *CD* is perpendicular to *AB*, *AD* is called the projection of *AC* on *AB*, or the projection of *b* on *c*. *DB* is the projection of *a* on *c*.

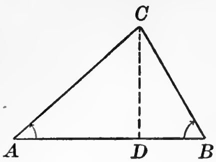


FIG. 124.

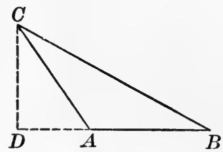


FIG. 125.

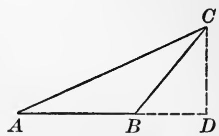


FIG. 126.

$$AD = b \cdot \cos A.$$
$$DB = a \cdot \cos B.$$

But $AD + DB = AB$, for all positions of *A*, *B*, *D*, when signs are taken into account.

$$\therefore\ \boldsymbol{c = a \cdot \cos B + b \cdot \cos A} \quad (1),$$

and $$\boldsymbol{b = a \cdot \cos C + c \cdot \cos A} \quad (2), \text{ by symmetry;}$$
$$\boldsymbol{a = b \cdot \cos C + c \cdot \cos B} \quad (3), \text{ by symmetry.}$$

Since the cosine of an angle between 90° and 180° is negative, the preceding formulas hold for all triangles, when signs are taken into account. In Figs. 124, 125, and 126,

$$AD + DB = AB,$$

in all cases.

EXERCISES.

Solve the following examples, and use (1), (2), (3) above to test the correctness of the solutions:

1. Given *B*, 41° 41′; *b*, 43.21; *c*, 49.32.

2. Given *A*, 37° 37′; *a*, 37.37; *b*, 41.24.

3. Given *A*, 135° 24′; *c*, 45.45; *a*, 54.54.

4. Given *a*, 37.38; *b*, 38.39; *c*, 40.41.

5. Given *B*, 137° 37′; *b*, 50.51; *a*, 30.31.

6. Construct some appropriate examples where the sides have one significant figure; two; three; five; six; seven. Solve. Determine in each case how close the test should hold.

§ 126. Use of the Cosine in Connection with a Projection Proposition of Plane Geometry.

In all books on ordinary geometry will be found a proposition reading somewhat like this: The square on the side of a triangle opposite an $\left\{\begin{matrix}\text{acute}\\ \text{obtuse}\end{matrix}\right\}$ angle is equal to the sum of the squares on the other two sides $\left\{\begin{matrix}\text{diminished}\\ \text{increased}\end{matrix}\right\}$ by twice the rectangle contained by one of these sides and the projection of the other on it.

The trigonometric statement for this proposition is the same, whether the angle is obtuse or acute, and is this: *The square on the side of a triangle is the sum of the squares of the other two sides diminished by twice the product of these two sides and the cosine of their included angle.*

When the angle is obtuse, its cosine is negative, so that the trigonometric and geometric statements agree.

PROOF: In Fig. 127, CD is perpendicular to AB, and the lengths are as indicated.

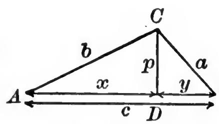


FIG. 127.

$$\begin{aligned} a^2 &= p^2 + y^2 \\ &= p^2 + (c - x)^2 \\ &= p^2 + x^2 + c^2 - 2\,cx \\ &= b^2 + c^2 - 2\,c \cdot b \cdot \cos A. \qquad (1) \end{aligned}$$

For $\qquad p^2 + x^2 = b^2,$

and $\qquad x = b \cdot \cos A.$

In Fig. 128, taking signs into account, $AD + DB = AB$,

or, $\qquad x + y = c.$

$$\therefore\ y = c - x;$$

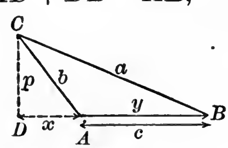


FIG. 128.

and no change is made in the proof.

We have thus:

$$a^2 = b^2 + c^2 - 2\,bc \cdot \cos A, \quad (1)$$

$$b^2 = c^2 + a^2 - 2\,ca \cdot \cos B, \quad (2)$$

$$c^2 = a^2 + b^2 - 2\,ab \cdot \cos C. \quad (3)$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \quad (4)$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca}, \quad (5)$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}. \quad (6)$$

Are these formulas adapted to logarithmic calculation?

EXERCISES.

1. Given a 12, b 27, C 30°, find c to two significant figures without using the tables. Do the same when C has the value 45°; 60°; 120°; 135°; 150°.

2. Given a 9, b 7, C 35°, find c to one significant figure.

3. Given a 12, c 13, B 48° 30′, find b to two significant figures.

4. Given b 23.4, c 54.1, A 56°, find a to three significant figures.

5. Determine to one significant figure the cosine of the angles of a triangle whose sides are 7, 8, 9. Can the angles be determined uniquely (§ 71) from these cosines? Why is the case different when the sine of the angle of a triangle is given?

6. Show that the diagonals of a parallelogram are

$a^2 + b^2 \pm 2\,ab \cos \theta$, where θ is the angle between the sides a, b.

7. Determine the diagonals to the appropriate number of significant figures, given a 3, b 5, θ 35°.

8. Show that twice the length of the median of a triangle (from angle A) is

$$\sqrt{b^2 + c^2 + 2\,bc \cos A}.$$

9. If a, b, c, d are the sides, in order, of a quadrilateral inscribed in a circle, show that if B lies between a, b,

(1) $$a^2 + b^2 - 2\,ab \cos B = c^2 + d^2 + 2\,cd \cos B.$$

(2) $$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

(3) $\sin^2 B = \frac{(a+b+c-d)(b+c+d-a)(c+d+a-b)(d+a+b-c)}{4(ab+cd)^2}.$

(4) $$(ab + cd)\sin B = 2\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $$2s = a + b + c + d.$$

(5) Area of quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

(6) Find the area when the sides are 3, 5, 7, 9.

(7) Show that the sine of the angle between the diagonals is

$$\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}.$$

10. Show that the sines of the angles which the median of a triangle (from the angle A) makes with sides c, b are

$$\frac{a \sin C}{\sqrt{b^2+c^2+2bc\cos A}}, \quad \frac{a \sin B}{\sqrt{b^2+c^2+2bc\cos A}}.$$

11. Two circles of radii a, b cut each other at an angle θ. Prove that the length of their common chord is

$$\frac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}.$$

12. The revolving crank-arm R of a stationary engine is connected by a link-bar l to a piston rod, whose stroke S is in a straight line not passing through the centre of revolution of the crank-arm. Draw the positions of the crank at the dead points and show that a triangle will be formed whose sides are S, $l-R$, $l+R$. Show that if θ is the angle between the positions of the crank-arm at the dead points,

$$S^2 = 2(l^2 + R^2) - 2(l^2 - r^2)\cos\theta.$$

Show that $S = 2R$ when the stroke line passes through the centre of revolution of the crank-arm.

§ 127. Determination of the Angles of a Triangle from the Cosines of the Half Angles.

Since there is only one angle less than 180° and positive which has a given cosine, the formulas of the preceding section determine the angles of a triangle uniquely from the cosine. However, they are not suitable for use when the sides are expressed by numbers of more than one or two significant figures. The cosine of the half angle is suitable for logarithmic computation.

In § 92 it was shown that in Fig. 77

$$AD = s - a,$$

and
$$\overline{AO}^2 = \frac{s-a}{s} \cdot bc.$$

$$\therefore \cos\frac{A}{2} = \frac{AD}{AO} = \sqrt{\frac{s(s-a)}{bc}}.$$

Similarly,
$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}},$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

What sign must be given to these radicals, A, B, C being the angles of a triangle?

When all three angles are to be calculated, these formulas may be used to best advantage in the forms:

$$\cos\frac{A}{2} = \sqrt{\frac{s}{abc} \cdot a(s-a)},$$

$$\cos\frac{B}{2} = \sqrt{\frac{s}{abc} \cdot b(s-b)},$$

$$\cos\frac{C}{2} = \sqrt{\frac{s}{abc} \cdot c(s-c)},$$

corresponding to which we have the following calculation-scheme, the numerals indicating the order of filling in:

CALCULATION-SCHEME.

$a=$	(1)	$s-c=$	(8)
$b=$	(2)	$\log(s-a)=$	(9)
$c=$	(3)	$\log(s-b)=$	(10)
$2s=$	(4)	$\log(s-c)=$	(11)
$s=$	(5)	$\log a=$	(12)
$s-a=$	(6)	$\log b=$	(13)
$s-b=$	(7)	$\log c=$	(14)

$\log abc =$ (16), by adding (12), (13), (14)

$\log s =$ (15)

$\log \frac{s}{abc} =$ (17), by subtracting (16) from (15)

$\log a(s-a) =$ (18), by adding (9), (12)

$\log b(s-b) =$ (19), by adding (10), (13)

$\log c(s-c) =$ (20), by adding (11), (14)

$2 \log \cos \frac{A}{2} =$ (21), by adding (17), (18)

$2 \log \cos \frac{B}{2} =$ (22), by adding (17), (19)

$2 \log \cos \frac{C}{2} =$ (23), by adding (17), (20)

$\therefore \log \cos \frac{A}{2} =$ (24), by taking half of (21)

$\log \cos \frac{B}{2} =$ (25), by taking half of (22)

$\log \cos \frac{C}{2} =$ (26), by taking half of (23)

$\therefore \frac{A}{2} =$ (27)

$\frac{B}{2} =$ (28)

$\frac{C}{2} =$ (29)

$\therefore A =$ (30)

$B =$ (31)

$C =$ (32)

TEST. $A+B+C=180°$, approximately. (33)

NOTE. — Here, as in § 93, all logs, (9) to (15) inclusive, are to be looked up before their manipulation begins with (16) and the following numerals.

The test, $A + B + C = 180°$, is subject to the same limitations here as in the sine-calculation. (See page 175.)

EXERCISES.

1. Solve the following by cosines and by sines and compare results:

(i) Given a, 54.2; b, 59.5; c, 67.3.

(ii) Given a, 9.847; b, 8.352; c, 7.283.

2. Solve by cosines the following and test by the formulas of § 125:

a, 543; b, 586; c, 600. a, 437.2; b, 564.3; c, 498.3.

Test the same by $a \sin B = b \sin A$.

3. Is there any convenient way of determining whether the test of § 125 is met with sufficient accuracy?

4. What is an appropriate logarithmic test for the accuracy of the solution by cosines? If one uses the test $a \sin B = b \sin A$, how close ought it to check?

5. Make up and solve a triangle requiring the use of a five-place table. Check the same.

§ 128. The Reciprocal Cosine, or Secant, and Graphs.

Related to the cosine as the cosecant to the sine is another function of the angle called the secant.

$$\text{Secant} = \frac{1}{\text{cosine}}; \quad \text{cosecant} = \frac{1}{\text{sine}}.$$

The origin of the words "secant" and "cosecant," as that of the "cosine," is clear on the unit-circle of the old-time trigonometry.

Draw a tangent at the initial point A of the unit-circle. Prolong the terminal of the angle to cut the tangent. The secant of the angle, or arc, is the amount *cut off* from the terminal by the initial tangent.

In each of the following diagrams (Figs. 129–132) *OS* is the secant of the angles of the corresponding terminals.

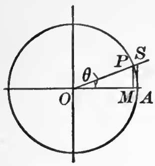


FIG. 129.

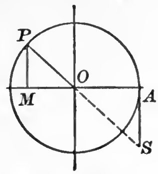


FIG. 130.

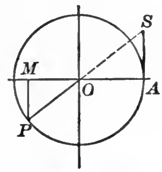


FIG. 131.

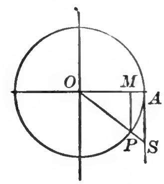


FIG. 132.

For $\frac{OS}{OA} = \frac{OP}{OM}$; but both *OA* and *OP* are unity, and *OM* is the cosine.

$$\therefore\ OS = \frac{1}{\text{cosine}}, \text{ or the secant.}$$

By looking up the line picture of the cosecant (§ 96) it will be observed that the cosecant is the secant for the complementary angle, or arc, just as the cosine is the sine of the complementary angle, or arc. Thus the *co-* of *co-secant* signifies the secant of the complement.

The sign of the secant is the same as that of the cosine. The range of values of the secant is determined by considering that of the cosine and taking the reciprocal range.

As the angle runs from 0° to 360°, the cosine runs over the continuum from 1 to 0 to − 1 to 0 to 1.

At the same time the secant runs over the reciprocal set of values. It goes from 1 to positive infinity, while the angle passes from 0 to 90°. As the angle passes over 90°, the secant makes a jump from positive infinity to negative infinity. As the angle passes on from 90° to 180°, the secant goes from negative infinity to − 1. As the angle changes from 180° to 270°, the secant changes from − 1 back to negative infinity. As the terminal passes over 270°, there is another break in the continuity of the secant, by a spring from negative infinity to positive infinity. As the terminal moves on from 270° to 360°, the secant passes over the continuum from positive infinity to 1.

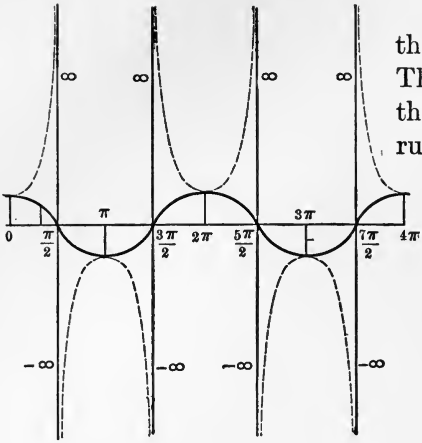


Fig. 133.

If $\sin(90°+A)=\cos A$, then $\operatorname{cosec}(90°+A)=\sec A$. Therefore, if we look on the sine and cosecant as running through a set of values, the cosine and secant will be running through the same set of values 90° behind the sine and cosecant, respectively.

Thus the graphs of the cosine and secant are the same as those of the sine and cosecant, respectively, if we pull these back along the horizontal axis a distance representing 90°. (Compare Figs. 93, 133.)

The diagram shown in Fig. 134, the circle being the unit-circle, illustrates the secants of all angles. The distances OS_1, OS_2, OS_3, etc., represent the secants for the terminal positions OP_1, OP_2, OP_3, etc., and their opposites. When the terminal is almost the upright vertical, the secant is almost infinite, positive. Just after the terminal has passed the position of the upright vertical, the secant is on the opposite terminal, and is thus negative and almost infinite. Just what the secant is when the terminal is the upright vertical or downright vertical, it is impossible to say. It is customary to take it as $+\infty$ for 90 and as $-\infty$ for 270°.

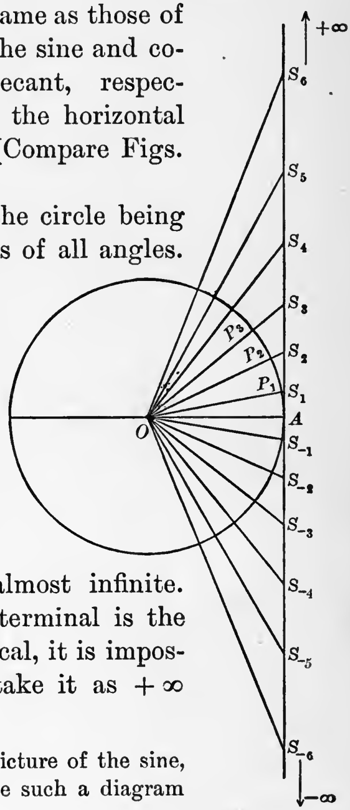


Fig. 134.

Illustrate on a single diagram the line-picture of the sine, cosine, secant, cosecant of an angle. Give such a diagram for each quadrant.

§ 129. Line Picture of the Secants of All Angles.

The secants are (Fig. 134) $OA_1 \cdots OS_1 \cdots OS_2 \cdots OS_3 \cdots + \infty$, then $-\infty \cdots OS_{-4} \cdots OS_{-3} \cdots OS_{-2} \cdots OS_{-1} \cdots OA$, with repetitions.

EXERCISES.

1. Change the word "sine" to "cosine," and the word "cosecant" to "secant" in the Exercises of § 97, and solve.

2. By what tilt can the graphs of $y = \cos^{-1} x$ and $y = \sec^{-1} x$ be obtained at once from those for $y = \cos x$ and $y = \sec x$?

§ 130. Cosine Waves and Cosine Harmonic Motion.

Since the graph of the cosine is the same as that of the sine pulled back a distance representing 90°, the cosine may be used in connection with waves, just as the sine. (See §§ 99, 100.)

EXERCISES.

1. If $y = r \cdot \sin(\alpha t + a)$, and $y = r \cdot \cos(\alpha t + \beta)$, are two waves, how far behind one wave is the other at the same time-instant?

2. Draw a diagram illustrating the superposition of the waves of Ex. 1.

For harmonic motion on the horizontal axis the cosine plays the same rôle as does the sine on the vertical axis. (Reread § 100.)

§ 131. The Versed Sine, Exsecant and Coexsecant.

When the line definition was in use, the balance of the radius (unity) beyond the cosine was called the *Versed Sine*. And still

$$\text{versin } A = 1 - \cos A.$$

The versed sine is used in engineering field books, as also the

$$\text{exsecant} = \text{secant} - 1.$$

Also

$$\text{coexsecant} = \text{cosecant} - 1.$$

EXERCISES.

1. Take the exercise on the coversed sine (§ 103), reading *versed sine* for *coversed sine.*

2. Show how to plot a circular railroad curve, using sines and versed sines as offsets (coördinates).

§ 132. Cosine Graphs.

EXERCISES.

1. Make graphs (Groat's degree-measure polar coördinate paper) for: $r = a\cos\theta$ (circle); $r\cos\theta = a$ (straight line); $r = a\cos 2\theta$; $r\cos 2\theta = a$; $r = a\cos 3\theta$; $r\cos 3\theta = a$. See if the number of loops is affected by n being even or odd in $r = a\sin n\theta$ and in $r = a\cos n\theta$.

2. Make graphs for:

$$r = a(1 + \cos\theta);\ r(1 + \cos\theta) = a,$$
$$r = a(1 - \cos\theta);\ r(1 - \cos\theta) = a.$$

(Cardioids and parabolas.)

Show that the cardioid can be drawn readily by means of a circle.

3. Make a graph for

$$\begin{Bmatrix} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{Bmatrix}.$$

Show that the resulting curve is a cycloid (the curve described by a point in the rim of a circular wheel rolled on a horizontal roadway).

4. Make a graph for

$$r = a\sec\theta \pm b. \qquad (a > b;\ a = b;\ a < b.)$$

(Conchoid of Nicomedes.)

Show that the conchoid can be drawn readily by means of a straight line.

5. Make a graph for

$$\begin{Bmatrix} x = a\cos\theta \\ y = a\sin\theta \end{Bmatrix}. \quad \text{Show it is a circle.}$$

(Use rectangular coördinate paper.)

6. Make a graph for

$$\begin{Bmatrix} x = a\cos\theta \\ y = b\sin\theta \end{Bmatrix}. \quad \text{(An ellipse.)}$$

Show that points on this curve are readily located mechanically by means of two concentric circles of radii a, b, prolonging ordinates of the one to meet abscissas of the other.

CHAPTER VII.

THE SINE AND THE COSINE IN UNION.

§ 133. Relation of the Sine of Any Angle to its Cosine.

In Fig. 135, $\overline{OM}^2 + \overline{MP}^2 = \overline{OP}^2$. This is true for all positions of the terminal, OP.

Thus, for all positions of OP, and for each quadrant:

$$\left(\frac{OM}{OP}\right)^2 + \left(\frac{MP}{OP}\right)^2 = 1,$$

or, $(\cos x)^2 + (\sin x)^2 = 1$, (1)
for all values of x.

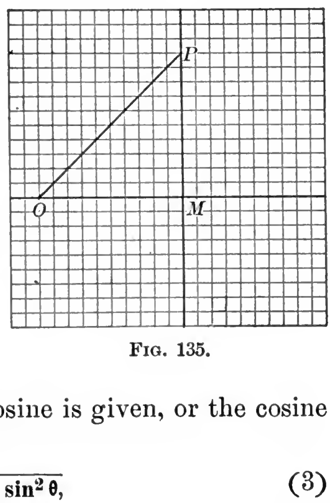


FIG. 135.

For any special angle, θ, this is written in the form:

$$\cos^2\theta + \sin^2\theta = 1, \qquad (2)$$

The relation (2) serves to determine the sine of θ when the cosine is given, or the cosine when the sine is given.

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}, \qquad (3)$$

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}. \qquad (4)$$

Explain on a diagram the significance of the double sign.

When the sine (or cosine) is given in the form of a *common fraction or a simple decimal* (*tenths*), it is better to calculate the cosine (or sine) from a diagram than from the formulas (3), (4).

For example, given the sine $= \frac{4}{5}$, what is the cosine?

Construct the terminals (§§ 66 and 111), assuming the modulus as 5 and the ordinate as 4 (Fig. 136); calculate the remaining side. Here it is ± 3. The cosine is thus $\pm \frac{3}{5}$. Similarly, for sine $-\frac{4}{5}$, the adjoining diagram (Fig. 137), where the cosine is $\pm \frac{3}{5}$.

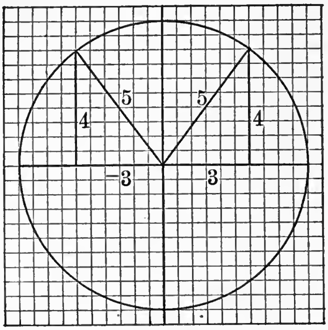


Fig. 136.

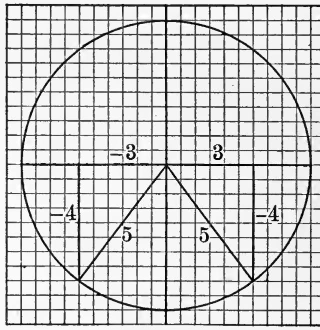


Fig. 137.

EXERCISES.

1. Determine by constructions the cosines and secants when the sines are the following, taking each fraction both positively and negatively:

$$\tfrac{5}{13},\ \tfrac{8}{17},\ \tfrac{7}{25},\ \tfrac{20}{29},\ \tfrac{12}{37},\ 0.2,\ 0.4,\ 0.75.$$

2. Determine by constructions the sines and cosecants when the cosine has the following values, taken with both signs:

$$\tfrac{9}{41},\ \tfrac{28}{53},\ \tfrac{11}{61},\ \tfrac{33}{65},\ \tfrac{16}{65},\ 0.3,\ 0.1,\ 0.8.$$

3. Determine by constructions the secants and cosines when the cosecants have the following values, taken with both signs:

$$\tfrac{89}{39},\ \tfrac{97}{65},\ \tfrac{101}{20},\ \tfrac{109}{60},\ \tfrac{113}{112},\ \tfrac{145}{143}.$$

See the table of triangles, page 145, for a large number of integers which may be selected as the sides of right-angled triangles.

When the sine (cosine) is given in the form of a decimal of several figures, it is better to get the cosine (sine) from the formulas (3), (4), than from a diagram.

EXERCISES.

1. Find the cosines to two decimal places when the sine has the values (+ or −): 0.23, 0.36, 0.27, 0.21.

2. Find the sines to two decimal places when the cosine has the values (+ or −): 0.67, 0.32, 0.45, 0.01.

It will be recalled that when z is small

$$\sqrt{1-z} = 1 - \frac{z}{2}, \text{ approximately (§ 42).}$$

$$\therefore\ \sqrt{1-z^2} = 1 - \frac{z^2}{2}.$$

$$\therefore\ \cos\theta = 1 - \frac{\sin^2\theta}{2}, \text{ approximately, when } \theta \text{ is small.}$$

EXERCISES.

1. Given $\sin 10'' = 0.000{,}048{,}481{,}368 \cdots$. Show that
$\sin^2 10'' = 0.000{,}000{,}002{,}350{,}4 \cdots$ and that
$\cos 10'' = 0.999{,}999{,}998{,}825$.

2. Assuming (as will be found true later) that for the number of decimal places given above in $\sin 10''$, $\sin 1''$ is $\frac{1}{10}$ of the sine of $10''$, and $\sin n''$ is $\frac{n}{10}$ times the sine $10''$, when $n < 10$ and positive, calculate the cosines of $1''$, $2''$, $3''$, ... $9''$.

3. Eliminate θ from $x = a\cos\theta$, $y = a\sin\theta$. What is the corresponding graph?

4. Eliminate θ from $x = a\cos\theta$, $y = b\sin\theta$. What is the graph?

5. Eliminate θ from $\left\{\begin{matrix} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{matrix}\right\}$. What graph is this?

6. Eliminate t from $\left\{\begin{matrix} y = \frac{1}{2}gt^2 - v\sin\theta\cdot t \\ x = vt\cos\theta \end{matrix}\right\}$. This is the path of a projectile. If you know the laws of falling bodies under gravity, prove that the above equations are true for a projectile started with initial velocity v at an angle θ with the horizontal line.

§ 134. Use of the Relation, $(\text{cosine } x)^2 + (\text{sine } x)^2 = 1$, in transforming a Trigonometric Expression.

Model Examples.

(1) To show that $\sqrt{\dfrac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A\,(1-\cos A)$.

Multiplying the quantity under the radical, numerator, and denominator by $1-\cos A$, and then writing in place of the new denominator, $1-\cos^2 A$, its value, $\sin^2 A$, the radical becomes $\dfrac{1-\cos A}{\sin A}$, or, $\operatorname{cosec} A\,(1-\cos A)$.

(2) To show that $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A$.

$$\begin{aligned}\cos^4 A - \sin^4 A &= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)\\ &= \cos^2 A - \sin^2 A = \cos^2 A - (1-\cos^2 A)\\ &= 2\cos^2 A - 1.\end{aligned}$$

From this the transformation follows at once.

In working such examples, we may

(i) Transform the first member into the second; or

(ii) Transform the second member into the first; or

(iii) Transform the first and second members into some third expression.

EXERCISES.

(Following from $\cos^2 A + \sin^2 A = 1$.)

1. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1$.
2. $\cos^2 A - \sin^2 A = 1 - 2\sin^2 A$.
3. $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A$.
4. $\sec^2 A + \operatorname{cosec}^2 A = (\sin A \sec A + \cos A \operatorname{cosec} A)^2$.
5. $\sin A \sec A + \cos A \operatorname{cosec} A = \sec A \operatorname{cosec} A$.
6. $2(1-\cos A) - (1-\cos A)^2 = \sin^2 A$.
7. $\dfrac{1-\sin A}{1+\sin A} = \dfrac{1+\sin^2 A - 2\sin A}{\cos^2 A}$.
8. $\operatorname{cosec}^2 A - 1 = \cos^2 A \operatorname{cosec}^2 A$.
9. $(\sin A - \cos A)^2 = 1 - 2\sin A \cos A$.

10. $\sin^3 A + \cos^3 A = (\sin A + \cos A)(1 - \sin A \cos A).$

11. $\sin^3 A - \cos^3 A = (\sin A - \cos A)(1 + \sin A \cos A).$

12. $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cos^2 A.$

13. $\sin^4 A - \cos^4 A = (\sin A + \cos A)(\sin A - \cos A).$

14. $\sin^6 A + \cos^6 A = 1 - 3\sin^2 A \cos^2 A.$

15. $\sin^6 A - \cos^6 A = (\sin^2 A - \cos^2 A)(1 - \sin^2 A \cos^2 A).$

16. $\sin^6 A - \cos^6 A = (2\sin^2 A - 1)(1 - \sin^2 A + \sin^4 A).$

17. $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A \cos^2 A).$

18. $\dfrac{1}{\cos A \operatorname{cosec} A + \sin A \sec A} = \sin A \cos A.$

19. $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = 2 \operatorname{cosec} A,$

and $\dfrac{\cos A}{1 - \sin A} + \dfrac{\cos A}{1 + \sin A} = 2 \sec A.$

20. $\dfrac{1 + \sin^2 A \sec^2 A}{1 + \cos^2 A \operatorname{cosec}^2 A} = \sin^2 A \sec^2 A.$

21. $\dfrac{1 - \sin A}{1 + \sin A} = 1 + 2\sin A \sec^2 A (\sin A - 1).$

22. $\dfrac{\sin A \sec A + \sin B \sec B}{\cos A \operatorname{cosec} A + \cos B \operatorname{cosec} B} = \sin A \sec A \sin B \sec B.$

23. $\sin A (1 + \sin A \sec A) + \cos A (1 + \cos A \operatorname{cosec} A) = \sec A + \operatorname{cosec} A.$

24. $(\sin^2 A - \cos^2 A)^2 = 1 - 4\cos^2 A + 4\cos^4 A.$

25. $\sin A \cos A (\sin A \sec A + \cos A \operatorname{cosec} A) = 1.$

26. $\dfrac{\sin A \sec A + \sec A - 1}{\sin A \sec A - \sec A + 1} = (1 + \sin A) \sec A.$

27. $\sqrt{\dfrac{1 - \sin A}{1 + \sin A}} = \sec A (1 - \sin A).$

28. $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2 \sec^2 A.$

29. $\cos A = \dfrac{\operatorname{cosec} A}{\cos A \operatorname{cosec} A + \sin A \sec A}.$

30. If $\dfrac{\sin A}{\cos A} = \dfrac{m}{n}$, $\sin A = \dfrac{\pm m}{\sqrt{m^2 + n^2}}$, and $\cos A = \dfrac{\pm n}{\sqrt{m^2 + n^2}}.$

31. If a, b, are any real quantities, angles A can be found such that $\sin A = \dfrac{a}{\sqrt{a^2+b^2}}$, $\cos A = \dfrac{b}{\sqrt{a^2+b^2}}$.

32. $$\frac{(1+\cos^2 A)^2+(1-\cos^2 A)^2}{(1+\cos^2 A)^2-(1-\cos^2 A)^2} = \tfrac{1}{2}(\cos^2 A + \sec^2 A).$$

33. $(1+\sin A - \cos A)^2 + (1-\sin A + \cos A)^2 = 4(1 - \sin A \cos A)$.

34. $$\frac{1-\sin A}{\cos A} = \frac{\cos A}{1+\sin A}.$$

35. $$\frac{1-\sin A\cos A}{\cos A(\sec A - \operatorname{cosec} A)}\cdot\frac{\sin^2 A - \cos^2 A}{\sin^3 A + \cos^3 A} = \sin A.$$

36. $\sec^2 A \operatorname{cosec}^2 A - \sec^2 A - 2\cos^2 A = (\sin^4 A + \cos^4 A)\operatorname{cosec}^2 A$.

37. $3(\sin A + \cos A) - 2(\sin^3 A + \cos^3 A) = (\sin A + \cos A)^3$.

38. $\sin^6 A + \sin^4 A\cos^2 A - \sin^2 A\cos^4 A - \cos^6 A = \sin^2 A - \cos^2 A$.

39. $\sin^2 A - \cos^2 B = \sin^2 B - \cos^2 A$.

40. $$\frac{\cos A + \cos B}{\sin A + \sin B} + \frac{\sin A - \sin B}{\cos A - \cos B} = 0.$$

41. $\cos^2 A + \cos^4 A \operatorname{cosec}^2 A = \cos^2 A \operatorname{cosec}^2 A$.

42. $\sec A \cos^3 A(\sec^2 A - 1) = \operatorname{cosec} A \sin^3 A$.

43. $\sec^2 A + \operatorname{cosec}^2 A = \sin^2 A \sec^2 A + \cos^2 A \operatorname{cosec}^2 A + 2$.

44. $\sin^2 B \sec^2 B = \sec^2 B - 1$.

45. If $\cos D = \dfrac{\cos A}{\sin B}$ and $\cos(90° - D) = \dfrac{\cos C}{\sin B}$,

$$\cos^2 A + \cos^2 B + \cos^2 C = 1.$$

46. If $\sin^2 A \operatorname{cosec}^2 B + \cos^2 A \cos^2 C = 1$,

then $$\sin C \sin B \cos A = \sin A \cos B.$$

47. $(1 - \operatorname{cosec} A + \cos A \operatorname{cosec} A)(1 + \sec A + \sin A \sec A) = 2$.

48. $$\frac{\cos A \operatorname{cosec} A - \sin A \sec A}{\cos A + \sin A} = \operatorname{cosec} A - \sec A.$$

49. $2(1-\cos A) + \cos^2 A = 1 + (1-\cos A)^2$.

50. $$\frac{1-\sin A\sec A}{1+\sin A \sec A} = \frac{\cos A \operatorname{cosec} A - 1}{\cos A \operatorname{cosec} A + 1}.$$

51. $$\left(\frac{1}{\sec^2 A - \cos^2 A} + \frac{1}{\operatorname{cosec}^2 A - \sin^2 A}\right)\cos^2 A \sin^2 A = \frac{1-\cos^2 A \sin^2 A}{2+\cos^2 A\sin^2 A}.$$

§ 135. Solution of Equations in Sine and Cosine.

Process: Reduce the equation to a single function by

$$\sin x = \sqrt{1-\cos^2 x} \quad \text{or} \quad \cos x = \sqrt{1-\sin^2 x}.$$

EXAMPLE:

$$\sin x + \cos x = 1. \qquad (1)$$

$$\therefore \sin x - 1 = \sqrt{1-\sin^2 x}. \qquad (2)$$

$$\therefore (\sin x - 1)^2 = 1 - \sin^2 x. \qquad (3)$$

$$\therefore 2\sin^2 x - 2\sin x = 0. \qquad (4)$$

$$\therefore \sin x(\sin x - 1) = 0. \qquad (5)$$

$$\therefore \sin x = 0; \qquad (6)$$

or,

$$\sin x = 1. \qquad (7)$$

By (6) $x = n\pi$, or $n \cdot 180°$.

By (7) $x = (4n+1)\frac{\pi}{2}$, or $(4n+1)\,90°$.

EXERCISES.

For Exs. 1–16 give in every case the general solution in radian measure (in terms of π) and in degree measure, unless the tables are necessary to get the solution. Then give the solution only in degree measure.

1. $\sqrt{3}\sin x - \cos x = 1.$

2. $\cos^2 x = \sin^2 x.$

3. $2\sin x + \cos x = 1.378.$

4. $8\sin x = 4 + \cos x.$

5. $\cos^2 x = 0.037\sin x.$

6. $\sin^2 x = 0.051\cos x.$

7. $3\sin^2 x - 2\cos x - \frac{1}{2} = 0.$

8. $5\cos^2 x - 3\sin x + \frac{1}{4} = 0.$

9. $a\sin^2 x + b\cos x + c = 0.$

10. $a\cos^2 x + b\sin x + c = 0.$

11. $\cos^2 x - \sin x - \frac{1}{4} = 0.$

12. $2\sqrt{3}\cos^2 x = \sin x.$

13. $2\sin^2 x + 3\cos x = 0.$

14. $\cos^2\theta + \cos\theta = 0.673.$

15. $\sin^2\theta - 2\cos\theta + \frac{}{4} = 0.$

16. $\sec\phi + 0.34\,\text{cosec}\,\phi = 2.7.$

17. A line was measured by three different men. One man reported the length as 1 mile; another, as 5280 feet; the third, as 5280.0 feet. Were the reports indentical, so far as care in measurement is concerned? Does it mean the same to say a race-track is a mile long as it does to say that the distance between two towns is a mile? Does it mean the same to say a line is a foot long as to say it is 12 inches, or 12.0 inches, or 12.00 inches? What effect on the sun's distance has a change of one-hundredth of a second in the parallax-angle (angle subtended at the sun by the earth's radius)?

§ 136. The Addition-Subtraction Formulas for Sines, Cosines.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B; \qquad (1)$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B; \qquad (2)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B; \qquad (3)$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B. \qquad (4)$$

It is necessary to prove only (1), (3), since they include (2), (4), when A, B are allowed the double sign.

Now (1) is nothing but the trigonometric expression of the geometric fact that if the sides of a certain right-angled triangle, drawn as presently to be indicated, are projected on any vertical line, the projection of the hypothenuse is the algebraic sum of the projections of the other two sides. Similarly (3) represents a projection of the same triangle on a horizontal line. In each case the line on which the triangle is projected is, of course, in the plane of the triangle.

When a line OP (Fig. 138), counted *positive* along the terminal of an angle x, and directed by that angle, is projected on a vertical line, its projection is

$$OP \cdot \sin x.$$

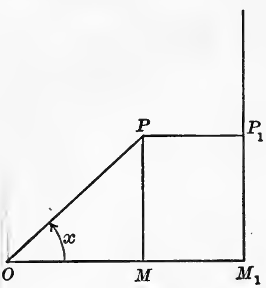


Fig. 138.

If OP, as directed by the angle x, is for any reason *negative*, its projection is the opposite of the preceding case, or $-OP \sin x$,

or, $$OP \cdot \sin(x+180°),$$

where *the negative sign is taken up by the sine* and OP is now merely a *length*.

∴ (α) Thus, when a length l, whose direction is that of the terminal of an angle x, is projected on the vertical,

$$\text{projection} = l \cdot \sin x.$$

(β) When a length l, whose direction is that of the opposite terminal of the angle x, is projected on the vertical,

$$\text{projection} = l \cdot \sin(x+180°),$$

or l times the sine of the directing angle of the opposite terminal.

Suppose that a, Fig. 139, is the terminal of the angle A, laid out from the right-hand horizontal line, and that b is the terminal of B, *laid out from a as a new initial line.* Then b is also the terminal of $A+B$, as laid out from the right-hand horizontal line. When a is the new initial, along it is positive; opposite, negative; while also $A+90°$ is a positive direction, and $A+270°$ is a negative direction.

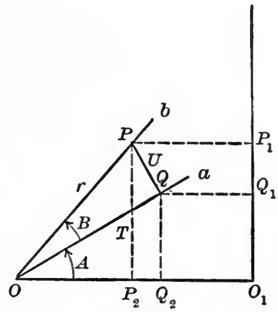


FIG. 139.

The triangle OQP, whose projections give (1), (3), is in all cases drawn as follows:

From any point P, in the terminal b, of $A+B$, drop a perpendicular PQ on a.

No matter what the size or sign of A, B, the sides of the triangle OPQ can only take four different sorts of locations as follows:

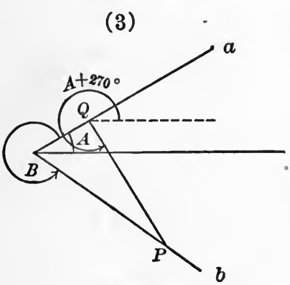


FIG. 140.

OQ may be on the terminal of A, or on the opposite terminal.

QP may be directed by the angle $A+90°$, as in Fig. 139, or by $A+270°$, as in a diagram like Fig. 140.

Let $OQ=T$, or t, according as it lies on A's terminal or its opposite.

Let $QP=U$, or u, according as it is directed by $A+90°$, or by $A+270°$.

Let $OP=r$, which is all the time directed by $A+B$.

Thus, no matter what the size or sign of A, B, there can be only four cases for the triangle OQP:

(i) Sides r, T, U, (iii) Sides r, t, U,
(ii) Sides r, T, u, (iv) Sides r, t, u.

Now, assuming that the projection of the hypothenuse of a right-angled triangle is the algebraic sum of the projections of the other two sides (§ 86), and denoting by T_v the vertical projection of T, we have for the four possible cases:

$$r_v = T_v + U_v, \text{ case (i);}$$
$$r_v = T_v + u_v, \text{ case (ii);}$$
$$r_v = t_v + U_v, \text{ case (iii);}$$
$$r_v = t_v + u_v, \text{ case (iv).}$$

By (α), p. 244, $T_v = T \cdot \sin A$,

and $U_v = U \cdot \sin(A + 90°) = U \cdot \cos A$.

By (β), p, 244, $t_v = t \cdot \sin A$,

and $u_v = u \cdot \sin(A + 90°) = u \cos A$,

for, in the last two cases, the directing angle of the *opposite of an opposite* is the angle itself.

We thus have the four possible cases:

$$\left.\begin{aligned} r \cdot \sin(A+B) &= T \cdot \sin A + U \cdot \cos A, \text{ case (i);} \\ r \cdot \sin(A+B) &= T \cdot \sin A + u \cdot \cos A, \text{ case (ii);} \\ r \cdot \sin(A+B) &= t \cdot \sin A + U \cdot \cos A, \text{ case (iii);} \\ r \cdot \sin(A+B) &= t \cdot \sin A + u \cdot \cos A, \text{ case (iv).} \end{aligned}\right\} \quad (E)$$

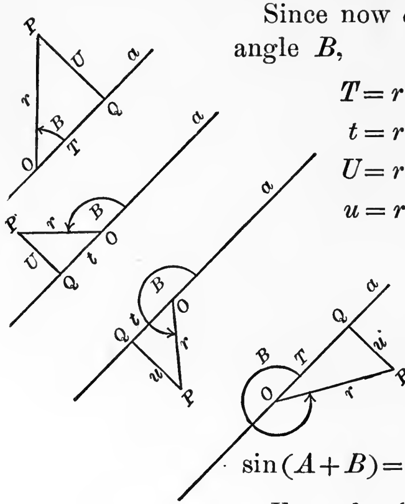


FIG. 141.

Since now a is the initial line for the angle B,

$$\left.\begin{aligned} T &= r \cdot \cos B \\ t &= r \cdot \cos B \\ U &= r \cdot \sin B \\ u &= r \cdot \sin B \end{aligned}\right\}$$, in the appropriate diagram adjoining.

Substituting these values in the equations (E), we have, after dividing by r, the single equivalent of them all:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B. \quad (1)$$

Formula (3) is now readily proven, for when the projection is made on a horizontal line, sin A above becomes cos A, and sin $(A + 90°)$

becomes $\cos(A+90°)$, or $-\sin A$, and $\sin(A+B)$ becomes $\cos(A+B)$. No other change is made.

$$\therefore\ \cos(A+B)=\cos A\cos B-\sin A\sin B. \qquad (3)$$

No special proofs of (2), (4) are needed, for they are, as already mentioned, covered by (1), (3). When B is negative, it is added to A by laying it out clockwise from A's terminal, and no change whatever is made in the preceding proofs.

§ 137. Special Cases of the Addition Theorem.

The addition-subtraction theorems are so important that the student should follow through the proofs in some special cases. For instance, for case (i), Fig. 142:

$$r_v = T_v + U_v;$$

or,
$$O_1P_1 = O_1Q_1 + Q_1P_1;$$

or,
$$r\cdot\sin(A+B) = T\cdot\sin A + U\cdot\sin(A+90°);$$

or,
$$r\cdot\sin(A+B) = r\cdot\cos B\sin A + r\cdot\sin B\cos A.$$

$$\therefore\ \sin(A+B)=\sin A\cos B+\cos A\sin B. \qquad (1)$$

Similarly, for cosines:

$$r_h = T_h + U_h;$$

or,
$$OP_2 = OQ_2 + Q_2P_2.$$

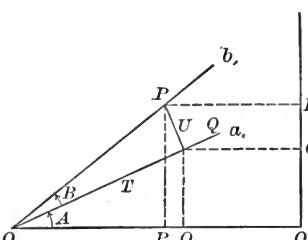


FIG. 142.

The student must note carefully that here, as in all cases, the addition of the projections is *algebraic*. OP_2 is shorter numerically than OQ_2, but the added Q_2P_2 is *negative*. If A, B, C are any three points on a straight line, then, always, no matter what the order of points,

$$AB = AC + CB.$$

The student, up to this point in his career, has probably had this presented to his mind but seldom. The teacher will find it advisable here to draw a variety of diagrams, showing that the above statement is always true in algebraic addition. When the student has once seen that $AB=AC+CB$, for all possible arrangements, the significance of the generality of the $\sin(A+B)$ and $\cos(A+B)$ formulas is clear.

In Fig. 142, $OP_2 = OQ_2 + Q_2P_2$,

or, $r \cdot \cos(A+B) = T \cdot \cos A + U \cdot \cos(A+90°)$,

or, $r \cdot \cos(A+B) = r \cdot \cos B \cdot \cos A - r \cdot \sin B \cdot \sin A$.

$$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B. \qquad (3)$$

Case (ii), Fig. 143.

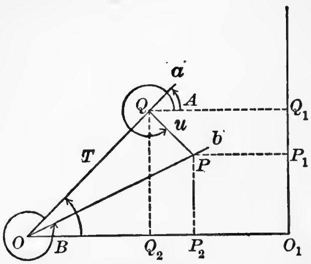


FIG. 143.

$$r_v = T_v + u_v,$$

or, $O_1P_1 = O_1Q_1 + Q_1P_1$

(algebraic addition).

Here, since a is the initial line for B, QP *is a negative line.* It is directed by the angle $A + 270°$. Its *opposite* is directed by the angle $A + 90°$. Thus, by (β) of the preceding section, the projection of QP on the vertical is $QP \sin(A+90°)$, or $u \cdot \cos A$.

Thus,

$O_1P_1 = r \cdot \sin(A+B)$; $O_1Q_1 = T \cdot \sin A$; $Q_1P_1 = u \cdot \cos A$.

$\therefore r \cdot \sin(A+B) = T \cdot \sin A + u \cdot \cos A$, as before.

But $T = r \cdot \cos B$ and $u = r \cdot \sin B$.

$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B$ (1), as before.

Similarly, in Fig. 143,

$$r_h = T_h + u_h,$$

or, $OP_2 = OQ_2 + Q_2P_2$ (algebraic, same as arithmetic),

or,

$r \cdot \cos(A+B) = T \cdot \cos A + u \cdot \cos(A+90°)$,

or, $\cos(A+B) = \cos A \cos B - \sin A \sin B$. (3)

The student may now make diagrams illustrating cases (iii) and (iv) (bottom of page 245), and carry through the proofs, noting how many different orders the points O_1, P_1, Q_1, and O_1, Q_2, P_2, may take, and noting that the addition of segments is always algebraic, the algebraic addition occasionally agreeing with the arithmetic. See Ex. 4, p. 344.

EXERCISE IN PROJECTIONS.

Show that if a straight line is drawn anywhere in the coördinate plane and a perpendicular from the origin is let fall on it, making an angle θ with the x-axis, then $x\cos\theta + y\sin\theta = p$, when p is the length of the perpendicular, (x, y) being any point on the first line.

(If P is (x, y), T the foot of the perpendicular, and MP the ordinate of P, project the broken line $OMPTO$ on OT.)

EXERCISES.

1. Given $\sin A = \frac{4}{5}$, $\cos B = \frac{5}{13}$. Find, in the form of a common fraction, the values of $\sin(A+B)$, $\sin(A-B)$, $\cos(A+B)$, $\cos(A-B)$.

2. Solve Ex. 1 when $\cos A = \frac{11}{60}$, $\cos B = \frac{48}{73}$.

3. Find the sines and cosines, cosecants and secants, of the following angles in the form of radicals and also to two decimal places, without using the tables: $\pm 75°$, $\pm 15°$, $\pm 105°$, $\pm 165°$. What other angles have the same sine as any one of these angles? The same cosine? The same sine and cosine?

Answer in radicals: $\sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$; $\operatorname{cosec} 15° = \sqrt{6}+\sqrt{2}$.

$\cos 15° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$; $\sec 15° = \sqrt{6}-\sqrt{2}$.

$\sin 75° = \cos 15°$; $\cos 75° = \sin 15°$.

4. Take, from the tables, the sines and cosines of two angles, and from these calculate the sines and cosines of their sum and difference, and compare the results with those of the table.

5. Apply the addition-subtraction formulas to the angles $90° \pm A$, $180° \pm A$, $270° \pm A$, $360° \pm A$. Compare the results with those of § 108.

Prove the following:

6. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$.

7. $\cos(A-B)\cos(A+B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

8. $\cos(45°-A)\cos(45°-B) - \sin(45°-A)\sin(45°-B) = \sin(A+B)$.

(Do not expand the expressions in the first member here nor in many of the examples following.)

9. $\sin(45°+A)\cos(45°-B) + \cos(45°+A)\sin(45°-B) = \cos(A-B)$.

10. $\sin(n+1)A\sin(n-1)A + \cos(n+1)A\cos(n-1)A = \cos 2A$.

11. $\sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A = \cos A$.

12. $\cos\alpha\cos(\gamma-\alpha) - \sin\alpha\sin(\gamma-\alpha) = \cos\gamma$.

13. $\cos(\alpha+\beta)\cos\gamma - \cos(\beta+\gamma)\cos\alpha = \sin\beta\sin(\gamma-\alpha)$.

14. $\sin(\alpha+\beta)\cos\alpha - \cos(\alpha+\beta)\sin\alpha = \sin\beta$.

15. $\cos(\alpha+\beta)\cos(\alpha-\beta) + \sin(\alpha+\beta)\sin(\alpha-\beta) = \cos 2\beta$.

16. $\sin(n-1)\alpha\cos(n+1)\alpha + \cos(n-1)\alpha\sin(n+1)\alpha = \sin 2n\alpha$.

17. $\sin(135° - \theta) + \cos(135° + \theta) = 0$.

18. $\sin 105° + \cos 105° = \cos 45°$.

19. $\sin 75° - \sin 15° = \cos 105° + \cos 15°$.

20. $\cos A + \cos(120° - A) + \cos(120° + A) = 0$.

21. By setting $\sin(A+B+C) = \sin(\overline{A+B}+C)$, and applying the addition formulas in succession, show that $\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C$.

22. From Ex. 21, show that if $A+B+C = 180°$ or $(2n+1)180°$, $\sin A\sin B\sin C = \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B$.

23. Find, similarly, $\cos(A+B+C)$, and give the results, when here, or in Ex. 22, one or more of the quantities A, B, C, are negative.

24. $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

25. $\sin(A+B) - \sin(A-B) = 2\sin B\cos A$.

26. $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$.

27. $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$.

28. $\sqrt{2}\sin(A - 45°) = \sin A - \cos A$.

29. $\sqrt{2}\sin(A + 45°) = \sin A + \cos A$.

30. $\sin(45° \pm A) = \cos(45° \mp A)$.

31. $\sqrt{2}\cos(A + 45°) = \cos A - \sin A$.

32. $\sqrt{2}\cos(A - 45°) = \cos A + \sin A$.

33. $\sin(\theta-\phi)\cos\phi + \cos(\theta-\phi)\sin\phi = \sin\theta$.

34. $\cos(\theta+\phi)\cos\theta + \sin(\theta+\phi)\sin\theta = \cos\phi$.

35. $2\sin\left(\alpha+\frac{\pi}{4}\right)\cos\left(\beta-\frac{\pi}{4}\right) = \cos(\alpha-\beta) + \sin(\alpha+\beta)$.

36. $2\sin\left(\frac{\pi}{4}-\alpha\right)\cos\left(\frac{\pi}{4}+\beta\right) = \cos(\alpha-\beta) - \sin(\alpha+\beta)$.

37. $\cos(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}+\beta\right)$.

38. $\cos(\alpha+\beta)-\sin(\alpha-\beta)=2\sin\left(\frac{\pi}{4}-\alpha\right)\cos\left(\frac{\pi}{4}-\beta\right)$.

39. $\cos(n-1)A\cdot\cos A-\sin(n-1)A\sin A=\cos nA$.

40. If $A+B+C=90°$, show that

$$\sin^2A+\sin^2B+\sin^2C=1-2\sin A\sin B\sin C.$$

41. $\sin\theta\sec\theta+\sin\phi\sec\phi=\sin(\theta+\phi)\sec\theta\sec\phi$.

42. $\sin\theta\sec\theta-\sin\phi\sec\phi=\sin(\theta-\phi)\sec\theta\sec\phi$.

43. $\cos\theta\operatorname{cosec}\theta+\sin\phi\sec\phi=\cos(\theta-\phi)\operatorname{cosec}\theta\sec\phi$.

44. $\cos\theta\operatorname{cosec}\theta-\sin\phi\sec\phi=\cos(\theta+\phi)\operatorname{cosec}\theta\sec\phi$.

45. $\dfrac{\sin\theta\sec\theta+\sin\phi\sec\phi}{\sin\theta\sec\theta-\sin\phi\sec\phi}=\sin(\theta+\phi)\operatorname{cosec}(\theta-\phi)$.

46. $\dfrac{\sin\theta\sec\theta\sin\phi\sec\phi+1}{1-\sin\theta\sec\theta\sin\phi\sec\phi}=\cos(\theta-\phi)\sec(\theta+\phi)$.

47. $\dfrac{\sin\theta\sec\theta+\cos\phi\operatorname{cosec}\phi}{\cos\phi\operatorname{cosec}\phi-\sin\theta\sec\theta}=\cos(\theta-\phi)\sec(\theta+\phi)$.

48. $\dfrac{\cos\theta\operatorname{cosec}\theta+\cos\phi\operatorname{cosec}\phi}{\cos\theta\operatorname{cosec}\theta-\cos\phi\operatorname{cosec}\phi}=-\sin(\theta+\phi)\operatorname{cosec}(\theta-\phi)$.

49. $\dfrac{1+\cos\theta\operatorname{cosec}\theta\sin\phi\sec\phi}{\cos\theta\operatorname{cosec}\theta-\sin\phi\sec\phi}=\sin(\theta+\phi)\sec(\theta+\phi)$.

50. $\dfrac{1-\cos\theta\operatorname{cosec}\theta\sin\phi\sec\phi}{\cos\theta\operatorname{cosec}\theta+\sin\phi\sec\phi}=\sin(\theta-\phi)\sec(\theta-\phi)$.

§ 138. The Addition-Subtraction Formulas in Inverse Notation.

If $\quad \sin(A+B)=\sin A\cos B+\cos A\sin B,$ (1)

then $\quad A+B=\sin^{-1}(\sin A\cos B+\cos A\sin B).$ (2)

If $\quad \sin A=x$ and $\sin B=y$,

then, $\quad \cos A=\pm\sqrt{1-x^2}$ and $\cos B=\pm\sqrt{1-y^2}$.

$$\therefore\ A=\sin^{-1}x=\cos^{-1}(\pm\sqrt{1-x^2}),$$

$$B=\sin^{-1}y=\cos^{-1}(\pm\sqrt{1-y^2}).$$

Thus (2) is equivalent to

$$\mathbf{\sin^{-1}x+\sin^{-1}y=\sin^{-1}(\pm x\sqrt{1-y^2}\pm y\sqrt{1-x^2}).} \quad (3)$$

Similarly, when A and B are determined by sines,

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

is equivalent, in inverse notation, to

$$\sin^{-1}x - \sin^{-1}y = \sin^{-1}(\pm x\sqrt{1-y^2} \mp y\sqrt{1-x^2}),$$

while $$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

is equivalent to

$$\sin^{-1}x + \sin^{-1}y = \cos^{-1}(\pm\sqrt{1-x^2}\sqrt{1-y^2} - xy),$$

and $$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

is equivalent to

$$\sin^{-1}x - \sin^{-1}y = \cos^{-1}(\pm\sqrt{1-x^2}\sqrt{1-y^2} + xy).$$

N.B. What has been said in § 73 concerning the multiplicity of values of A when $A = \sin^{-1}x$ (similar statements holding when $A = \cos^{-1}x$) must be borne in mind in connection with the results of this section and the examples. What is meant by the statement in Ex. 3, page 253,

$$\sin^{-1}\tfrac{3}{5} + \sin^{-1}\tfrac{8}{17} = \sin^{-1}\tfrac{77}{85},$$

is, that among the angles whose sine is $\frac{77}{85}$ is *an* angle which can be made up by adding *an* angle whose sine is $\frac{3}{5}$ to *an* angle whose sine is $\frac{8}{17}$. Similarly, with reference to the statement $A = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$, the proper sign is to be given the radical if a special value is to be given to A. For example, $\sin^{-1}\frac{1}{2}$ is not $\cos^{-1}\left(\frac{+\sqrt{3}}{2}\right)$, if for $\sin^{-1}\frac{1}{2}$ the special value 150° is taken. In the exercises following, the radicals are to be taken as having both signs.

EXERCISES.

1. Show that if $\cos A = x$ and $\cos B = y$, the formulas for $\sin(A+B)$, $\sin(A-B)$, $\cos(A+B)$, $\cos(A-B)$ become, respectively,

$$\cos^{-1}x + \cos^{-1}y = \sin^{-1}(y\sqrt{1-x^2} + x\sqrt{1-y^2}),$$

$$\cos^{-1}x - \cos^{-1}y = \sin^{-1}(y\sqrt{1-x^2} - x\sqrt{1-y^2}),$$

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-x^2}\cdot\sqrt{1-y^2}),$$

$$\cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}).$$

2. Give the corresponding formulas when $\sin A = x$, $\cos B = y$, and when $\cos A = x$ and $\sin B = y$.

Prove the following, giving also the general solution, as in the model solution (below) of Ex. 3.

Model Solution of Ex. 3.

Find the general value of $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}$. Let θ be the principal angle of the first quadrant whose sine is $\frac{3}{5}$. Then $\cos\theta=\frac{4}{5}$. Similarly, if B is the principal angle of the first quadrant for which $\sin\phi=\frac{8}{17}$, $\cos\phi=\frac{15}{17}$.

Then $$\sin^{-1}\tfrac{3}{5}=2\,n\pi+\theta,\quad\text{or}\quad(2\,n+1)\,\pi-\theta,$$

and $$\sin^{-1}\tfrac{8}{17}=2\,m\pi+\phi,\quad\text{or}\quad(2\,m+1)\,\pi-\phi.$$

Then, $$\sin^{-1}\tfrac{3}{5}+\sin^{-1}\tfrac{8}{17}=2\,s\pi+\theta+\phi,\text{ or }2\,s\pi-(\theta+\phi),$$
$$\text{or}\quad(2\,s+1)\,\pi+\theta-\phi,\quad\text{or}\quad(2\,s+1)\,\pi+\phi-\theta,$$

where s is any positive or negative integer.

Now $$\sin(2\,s\pi+\theta+\phi)=\sin(\theta+\phi),$$
$$\sin(2\,s\pi-(\theta+\phi))=-\sin(\theta+\phi),$$
$$\sin((2\,s+1)\,\pi+(\theta-\phi))=-\sin(\theta-\phi)=\sin(\phi-\theta),$$
$$\sin((2\,s+1)\,\pi+\phi-\theta)=-\sin(\phi-\theta)=\sin(\theta-\phi).$$

And $$\sin(\theta+\phi)=\tfrac{3}{5}\cdot\tfrac{15}{17}+\tfrac{4}{5}\cdot\tfrac{8}{17}=\tfrac{77}{85},$$
$$\sin(\phi-\theta)=\tfrac{4}{5}\cdot\tfrac{8}{17}-\tfrac{3}{5}\cdot\tfrac{15}{17}=\tfrac{13}{85}.$$

$$\therefore\ \sin^{-1}\tfrac{3}{5}+\sin^{-1}\tfrac{8}{17}=2\,r\pi+\alpha,\quad\text{or}\quad(2\,r+1)\,\pi-\alpha,$$

where α is some angle whose sine is $\frac{77}{85}$; also some angle whose sine is $-\frac{77}{85}$; also some angle whose sine is $\frac{13}{85}$; also some angle whose sine is $-\frac{13}{85}$.

3. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$.

4. $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\frac{253}{325}$.

5. $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$.

6. $\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{3}{5}=90°$.

7. $\sin^{-1}\dfrac{1}{\sqrt{5}}+\cos^{-1}\dfrac{3}{\sqrt{10}}=45°$.

8. $\cos^{-1}\frac{4}{5}-\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{63}{65}$.

9. $\sin^{-1}\sqrt{\dfrac{1+x}{2}}+\cos^{-1}\sqrt{\dfrac{1-x}{2}}=\cos^{-1}x$.

10. $\cos^{-1}\sqrt{\dfrac{2}{3}}-\cos^{-1}\dfrac{\sqrt{6}+1}{2\sqrt{3}}=\dfrac{\pi}{6}$.

11. $\sin^{-1}\dfrac{2}{2.5}+\sin^{-1}\dfrac{1.5}{2.5}=90°$.

12. $\sin^{-1}\frac{4}{5}-\sin^{-1}\frac{12}{13}=\sin^{-1}(?)$.

13. $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{5}{13} = \cos^{-1}(?)$.

14. $\cos^{-1}\frac{7}{25} - \sin^{-1}\frac{8}{17} = \sin^{-1}(?)$.

15. $\cos^{-1}\frac{20}{29} - \sin^{-1}\frac{12}{37} = \cos^{-1}(?)$.

16. $\sin^{-1}\frac{35}{37} - \sin^{-1}\frac{21}{29} = \cos^{-1}(?)$.

17. $2\sin^{-1}x = \sin^{-1}2x\sqrt{1-x^2}$.

18. $2\cos^{-1}x = \sin^{-1}2x\sqrt{1-x^2}$.

19. $2\sin^{-1}x = \cos^{-1}(1-2x^2)$.

20. $2\cos^{-1}x = \cos^{-1}(2x^2-1)$.

21. $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}}$.

22. $\cos^{-1}x = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

23. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{13}{85}=\frac{\pi}{2}$.

24. If $\sin^{-1}m + \sin^{-1}n = \frac{\pi}{2}$, prove that $m\sqrt{1-n^2}+n\sqrt{1-m^2}=1$.

§ 139. To find the Sine and Cosine of twice Any Angle, when the Sine and Cosine of the Angle are known.

From $\quad \sin(A+B) = \sin A\cos B + \cos A \sin B$

and $\quad \cos(A+B) = \cos A\cos B - \sin A\sin B$

follow, by letting $B = A$,

$$\left.\begin{aligned} \sin 2A &= 2\sin A\cos A && (1)\\ \cos 2A &= \cos^2 A - \sin^2 A && (2)\\ &= 2\cos^2 A - 1 && (3)\\ &= 1 - 2\sin^2 A && (4)\end{aligned}\right\} A \text{ being any angle.}$$

That is, by (1), *the sine of any angle is twice the sine of half of it into the cosine of half of it*, and by (2), (3), (4), *the cosine of any angle is the cosine squared of the half angle minus the sine squared of the half angle*, or also *twice the cosine squared of the half angle minus one*, or also *one minus twice the square of the sine of the half angle.*

$$\therefore \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2},$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2},$$

$$\cos A = 2\cos^2\frac{A}{2} - 1,$$

$$\cos A = 1 - 2\sin^2\frac{A}{2}.$$

From which follow the useful formulas:

$$1 + \cos A = 2\cos^2\frac{A}{2},$$

$$1 - \cos A = 2\sin^2\frac{A}{2},$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}},$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}},$$

$$\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}},$$

$$\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}.$$

EXERCISES.

1. Express the sine and cosine of the following angles in terms of their half angles:

$$3A;\ 4A;\ 5A;\ 6A;\ 7A;\ 8A;\ nA;\ 2nA.$$

2. If $\sin\theta = \frac{1}{2}$, find $\sin 2\theta$ and $\cos 2\theta$, $\sin 4\theta$, $\cos 4\theta$, $\sin 8\theta$, $\cos 8\theta$, each to two decimals.

3. If $\cos\theta = \frac{1}{3}$, find the values called for in Ex. 2.

4. $\left(\sin\frac{\theta}{2} \pm \cos\frac{\theta}{2}\right)^2 = 1 \pm \sin\theta$; $(\sin\theta \pm \cos\theta)^2 = ?$

5. If $\sin\theta = \frac{1}{2}$ and $\cos\phi = \frac{1}{3}$, find $\sin(\theta + \phi)$ and $\sin(2\theta + 2\phi)$.

6. If α, β are positive acute angles and $\cos\alpha = \frac{11}{16}$ and $\sin\beta = \frac{4}{5}$, find the values of $\sin^2\frac{\alpha-\beta}{2}$ and $\cos^2\frac{\alpha-\beta}{2}$.

7. If α, β are positive acute, and $\cos\alpha = \frac{3}{5}$ and $\cos\beta = \frac{4}{5}$, find $\cos\frac{\alpha-\beta}{2}$. Solve by Ex. 8; also without using Ex. 8.

8. Prove that

$$(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\frac{\alpha+\beta}{2},$$

$$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{\alpha-\beta}{2},$$

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2},$$

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\sin^2\frac{\alpha+\beta}{2}.$$

9. Look up the sine and cosine of 36° in the tables and calculate sin 72°, and compare the result with the table value. Calculate cos 72° from the three formulas for cos 2 A, and compare results with each other and with that calculated from sin 72° and that of the table.

10. $\text{Cosec}\,2\,\theta = \dfrac{\text{cosec}\,\theta \cdot \sec\theta}{2}$; $\sec 2\,\theta = \dfrac{\text{cosec}^2\,\theta \cdot \sec^2\theta}{\text{cosec}^2\,\theta - \sec^2\theta}$; $\text{cosec}\,8\,\theta = \dfrac{\sec 4\,\theta \cdot \sec 2\,\theta \cdot \sec\theta \cdot \text{cosec}\,\theta}{8}$.

11. Prove $\dfrac{\sin 2\,\theta}{1 + \cos 2\,\theta} = \sin\theta\sec\theta$; $\dfrac{\sin\theta}{1 + \cos\theta} = ?$

12. Prove $\dfrac{\cos 2\,\theta}{1 + \sin 2\,\theta} = \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$; $\dfrac{\cos\theta}{1 + \sin\theta} = ?$

13. Prove $\dfrac{\cos 2\,\theta}{1 - \sin 2\,\theta} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$; $\dfrac{\cos\theta}{1 - \sin\theta} = ?$

14. Prove $\dfrac{1 + \sec 2\,x}{\sec 2\,x} = 2\cos^2 x$; $\dfrac{1 + \sec x}{\sec x} = ?$

15. Prove $\sin A \sec A + \cos A\,\text{cosec}\,A = 2\,\text{cosec}\,2\,A$.

16. Prove $\sin A \sec A - \cos A\,\text{cosec}\,A = -2\cos 2\,A\,\text{cosec}\,2\,A$.

17. Prove $\dfrac{\cos A + \sin A}{\cos A - \sin A} - \dfrac{\cos A - \sin A}{\cos A + \sin A} = 2\sin 2\,A\sec 2\,A$.

18. Prove $\text{cosec}\,2\,A\,(1 + \cos 2\,A) = \cos A\,\text{cosec}\,A$.

19. Prove $\dfrac{\cos 2\,\theta}{1 \mp \sin 2\,\theta} = \sin\left(\dfrac{\pi}{4} \pm \theta\right) \cdot \sec\left(\dfrac{\pi}{4} \pm \theta\right)$; $\dfrac{\cos\theta}{1 \pm \sin\theta} = ?$

20. Prove $\dfrac{\sin\theta + \sin 2\,\theta}{1 + \cos\theta + \cos 2\,\theta} = \sin\theta\sec\theta$; $\dfrac{\sin\dfrac{\theta}{2} + \sin\theta}{1 + \cos\dfrac{\theta}{2} + \cos\theta} = ?$

21. Prove $\cos(A + 15°)\,\text{cosec}\,(A + 15°) - \sin(A - 15°)\sec(A - 15°) = \dfrac{4\cos 2\,A}{1 + 2\sin 2\,A}$.

22. Prove $\text{cosec}\,2\,x\,(1 - \cos 2\,x) = \sin x \sec x$; $\text{cosec}\,x\,(1 - \cos x) = ?$

23. Prove $\sec^2 A\,(1 + \sec 2\,A) = 2\sec 2\,A$.

24. Prove $\text{cosec}\,A\,(1 - \cos 2\,A) = 2\sin A$.

25. Prove $1 + \cos^2 2\,\theta = 2\,(\cos^4\theta + \sin^4\theta)$.

26. Prove $\dfrac{\text{cosec}^2 A}{\text{cosec}^2 A - 2} = \sec 2\,A$.

27. Prove $\dfrac{2 - \sec^2 A}{\sec^2 A} = \cos 2\,A$; $\dfrac{2 - \sec^2 2\,A}{\sec^2 2\,A} = ?$

28. Prove $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \sin\dfrac{\theta}{2}\sec\dfrac{\theta}{2}$; $\dfrac{1 + \sin 4\,\theta - \cos 4\,\theta}{1 + \sin 4\,\theta + \cos 4\,\theta} = ?$

29. Prove $\cos A \operatorname{cosec} A = \frac{1}{2}\left(\cos\frac{A}{2}\operatorname{cosec}\frac{A}{2} - \sin\frac{A}{2}\sec\frac{A}{2}\right)$;

$\cos 2A \operatorname{cosec} 2A = ?$

30. Prove $\dfrac{1-\cos 2A}{1+\cos 2A} = \sin^2 A \sec^2 A$; $\dfrac{1-\cos A}{1+\cos A} = ?$

31. Prove $\dfrac{2\sin A \sec A}{1+\sin^2 A \sec^2 A} = \sin 2A$.

32. Prove $\dfrac{1-\sin^2 A \sec^2 A}{1+\sin^2 A \sec^2 A} = \cos 2A$.

33. $\cos^2 A\,(1+\sin A \sec A)^2 = 1+\sin 2A$.

34. $\sin^2 A\,(\cos A \operatorname{cosec} A - 1)^2 = 1-\sin 2A$.

35. $\left(\dfrac{\sin A \sec A + 1}{\sin A \sec A - 1}\right)^2 = \dfrac{1+\sin 2A}{1-\sin 2A}$.

36. $\dfrac{\cos^3 A + \sin^3 A}{\cos A + \sin A} = \frac{1}{2}(2-\sin 2A)$.

37. $\dfrac{\cos^3 A - \sin^3 A}{\cos A - \sin A} = \frac{1}{2}(2+\sin 2A)$.

38. $\cos^4 A - \sin^4 A = \cos 2A$; $\cos^4 2A - \sin^4 2A = ?$

39. $\cos^6 A + \sin^6 A = \dfrac{1+3\cos^2 2A}{4}$.

40. $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = 2$.

41. $\cos 3A \operatorname{cosec} A + \sin 3A \sec A = 2\cos 2A \operatorname{cosec} 2A$.

42. $\sin 4A \operatorname{cosec} 2A = 2\cos 2A$.

43. $\sin 5A \operatorname{cosec} A - \cos 5A \sec A = 4\cos 2A$.

44. $\sin\frac{5\pi}{12}\operatorname{cosec}\frac{\pi}{12} - \cos\frac{5\pi}{12}\sec\frac{\pi}{12} = 2\sqrt{3}$.

45. $\sin\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}+\theta\right) - \sin\left(\frac{\pi}{4}-\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) = 2\sin 2\theta \sec 2\theta$.

46. $\sin\left(\frac{\pi}{4}-\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) + \cos\left(\frac{\pi}{4}-\theta\right)\operatorname{cosec}\left(\frac{\pi}{4}-\theta\right) = 2\sec 2\theta$.

47. $\cos(A+45°)\sec(A-45°) = \sec 2A\,(1-\sin 2A)$.

48. $\dfrac{(1+\sin A)}{(1-\sin A)} = \sin\left(45°+\frac{A}{2}\right)\sec\left(45°+\frac{A}{2}\right)\cos\left(45°-\frac{A}{2}\right)\operatorname{cosec}\left(45°-\frac{A}{2}\right)$.

49. $\dfrac{\sin^2(45°+A)\sec^2(45°+A)-1}{\sin^2(45°+A)\sec^2(45°+A)+1} = \sin 2A$.

50. $\dfrac{1}{1+\sin^2 x} + \dfrac{1}{1+\cos^2 x} + \dfrac{1}{1+\sec^2 x} + \dfrac{1}{1+\operatorname{cosec}^2 x} = ?$

51. $\dfrac{\sin 2A - \sin A}{1-\cos A+\cos 2A} = \sin A \sec A$.

§ 140. To find the Sine and Cosine of the Half Angle when the Sine and Cosine of the Whole Angle are known.

Case (*a*). *When the cosine of the whole angle is given.*

By (3), (4) of the preceding section

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}, \tag{1}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}. \tag{2}$$

That is, *the sine of the half of an angle is the square root (taken with both sines) of one-half of the expression, one minus the cosine of the angle, and the cosine of the half of an angle is plus or minus the square root of one-half of the sum of one plus the cosine of the angle.*

EXERCISE.

From the value of the cosine of the angle of a triangle in

$$a^2 = b^2 + c^2 - 2\,bc\cos A \quad (\S\,126)$$

deduce by the preceding formulas the values of the $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ given in §§ 92 and 127.

Case (*b*). *When the sine of the whole angle is given.*

By (1) of § 139.

$$2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A. \tag{3}$$

Also $$\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1. \tag{4}$$

$$\therefore \sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A}. \tag{5}$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}. \tag{6}$$

$$\therefore \sin\frac{A}{2} = \frac{\pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}}{2}. \tag{7}$$

$$\cos\frac{A}{2} = \frac{\pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A}}{2}. \tag{8}$$

§ 141. The Significance of the Double Signs in (1), (2), (5), (6), (7), (8) of § 140.

When any *special angle* is taken as A, the signs are not plus **and** minus, but plus **or** minus, the proper sign being determined by the quadrant of the given angle. For example, if $A = 45°$, the signs of (1), (2), (5) are all plus, and (6) is minus. Consequently the signs of (7), (8) are plus and minus, plus and plus. Similarly are the signs settled for (7), (8) in any other special case, by first determining what they are in (5), (6).

When no special angle is taken for A, but when A is assumed as given only by its sine or by its cosine, then the reading of signs above is plus **and** minus and not plus **or** minus.

When the cosine of A is given, there are two positions of the terminal, located symmetric to the horizontal (§ 106), and the angles are $2n \cdot 180° \pm A$ (§ 106). Thus the half angles are $n \cdot 180° \pm \frac{A}{2}$. But the terminal for $n \cdot 180° + \frac{A}{2}$ is that of $\frac{A}{2}$ when n is even and the opposite of that of $\frac{A}{2}$ when n is odd, and the terminal of $n \cdot 180° - \frac{A}{2}$ is similarly that of $-\frac{A}{2}$ when n is even and the opposite when n is odd. Thus the terminals for the half angles are four lines, namely, the terminals of $\frac{A}{2}$ and $-\frac{A}{2}$ and their opposites. The terminals of $\frac{A}{2}$ and $-\frac{A}{2}$ are symmetric to the horizontal, and thus have opposite sines and the same cosine. Opposite terminals give opposite sines and opposite cosines. Thus the four terminals give only two sines (opposites) and two cosines (opposites), as shown in (1), (2).

EXERCISE.

Illustrate the preceding by a diagram.

When the sine of A is given, there are likewise two terminals. They are symmetric to the vertical. The corresponding general angles are $2n \cdot 180° + A$ and $(2n+1)180° - A$. The half angles are $n \cdot 180° + \frac{A}{2}$ and $n \cdot 180° + \frac{180° - A}{2}$, whose terminals are that of $\frac{A}{2}$ and its opposite, and that of half the supplement of A and its opposite. This gives four terminals for the half angle, opposite in pairs, and thus four values of the sine of the half angle, opposite by pairs, this being also the case with the cosines, as shown in (7), (8).

EXERCISE.

Illustrate the preceding by a diagram.

Both the preceding cases are covered by § 53, where it was shown that for a single position of the terminal there are two positions of the terminal of the half angle.

§ 142. What Signs to Use in the formulas

$$\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1 + \sin A}, \qquad (5)$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1 - \sin A}, \qquad (6)$$

for a special angle, is readily settled by a glance at the sine and cosine on the unit-circle. The quadrantal bisectrices mark the angles for which there is numerical equality in the sines and cosines. For terminals between $-45°$ and $+45°$, the cosine of the angle is positive and numerically greater than the sine.

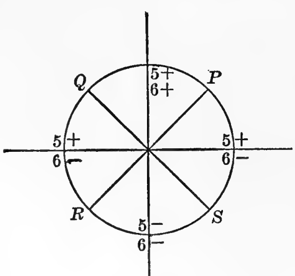


FIG. 144.

Then (5) is + and (6) is −.

The opposite is the case for terminals between the opposites of the preceding terminals.

Thus, for terminals between those of 135° and −135°, (5) is − and (6) is +.

For terminals between those of 45° and 135°, the sine is positive and numerically greater than the cosine.

Then (5) is + and so is (6).

The opposite is the case between the opposite terminals.

The diagram shown in Fig. 144 covers all these cases. In any special case we need only to locate the terminal. After the signs of (5), (6) are settled, those of (7), (8) of § 140 follow.

ILLUSTRATIVE EXAMPLES.

1. If $A = 580°$, what signs have (1), (2), etc., of § 140?

$$\therefore \frac{A}{2} = 290°, \text{ or } 270° + 20°.$$

The terminal of $\frac{A}{2}$ lies between OR and OS in Fig. 144, with (1) –, (2) +, (5) –, (6) –, and consequently with the order of sines in (7) as – –, and that in (8) as – +.

2. Given that the numerical value of the sine and cosine at the quadrantal bisectrices is $\frac{\sqrt{2}}{2}$, or 0.7071, what signs have (1), (2), etc., when $\sin A = 0.3572$ (with $\cos A$ +), when $\sin A = 0.8624$, – 0.4517, or – 0.9653, with $\cos A$ – ?

In the first case the terminal lies between OS, OP; in the second, between OP, OQ; in the third, between OQ, OR; in the fourth, between OR, OS. The corresponding signs are indicated on the diagram. (Fig. 144.)

3. Where lies $\frac{A}{2}$ when the signs of (7) are – – ?

In this case (5) was – and (6) was –. Thus, by Fig. 144, the terminal lies between OR, OS. Therefore the angle lies between

$$2n\pi - \frac{\pi}{4} \text{ and } 2n\pi - \frac{3\pi}{4}.$$

EXERCISES.

1. State the signs for the formulas considered above when A has the following values in degree measure:

44, 45, 382, 560, – 44, – 840, 2040, – 650, 690, 200.

2. Determine the signs for (5), (6), (7), (8), when A lies between 270° and 315°.

3. The same, if possible, when A lies between 270° and 360°.

4. What are the signs in (7) when A lies between 450° and 495°?

5. Where must $\frac{A}{2}$ lie when the signs in (7) are $+ -$?

6. Show that if A lies between 585° and 630° the signs in (8) are $- +$.

7. Determine where $\frac{A}{2}$ must lie when the signs in (7) are $+ +$, $- +$, $- -$.

8. Determine where $\frac{A}{2}$ must lie when the signs in (8) are $+ +$, $+ -$, $- +$, $- -$.

§ 143. The Double-angle in Inverse Notation.

From $\sin 2\theta = 2 \sin\theta \cos\theta$ it follows, if $\sin\theta = x$ and $\cos\theta = \pm\sqrt{1-x^2}$, that

$$\sin 2\theta = \pm 2x\sqrt{1-x^2}.$$

$$\therefore\ 2\theta = \sin^{-1}(\pm 2x\sqrt{1-x^2}).$$

$$\therefore\ \mathbf{2\sin^{-1}x = \sin^{-1}(\pm 2x\sqrt{1-x^2})}.$$

That is, *among the angles whose sine is* $\pm 2x\sqrt{1-x^2}$ *will fall the angle which is twice any special angle whose sine is* x.

For example,

$$\text{Let } \sin\theta = \frac{1}{+\sqrt{2}}.$$

$$\therefore\ \cos\theta = \frac{1}{\pm\sqrt{2}}(+ \text{ or } -).$$

If θ is selected as $\frac{\pi}{4}$, $\cos\theta = \frac{1}{+\sqrt{2}}$.

Then $$2\sin^{-1}x = \sin^{-1}(\pm 2x\sqrt{1-x^2})$$

becomes $$2\sin^{-1}\left(\frac{1}{+\sqrt{2}}\right) = \sin^{-1}(+1).$$

Among the angles, $\sin^{-1}(+1)$, is $\frac{\pi}{2}$, which is twice the angle $\frac{\pi}{4}$, selected as a special angle whose sine is $\frac{1}{+\sqrt{2}}$.

Similarly, from the relations

$$\cos 2\,\theta = \cos^2\theta - \sin^2\theta,$$
$$\cos 2\,\theta = 1 - 2\sin^2\theta,$$
$$\cos 2\,\theta = 2\cos^2\theta - 1,$$

follow

$$\mathbf{2\sin^{-1}x = \cos^{-1}(1 - 2\,x^2),}$$
$$\mathbf{2\cos^{-1}x = \cos^{-1}(2\,x^2 - 1).}$$

These relations are subject to the limitations to which attention has already been invited in connection with the multiplicity of values in the inverse notation.

EXERCISES.

Show that for special values of the angles the following relations are true:

1. $2\sin^{-1}\frac{1}{2} = \cos^{-1}\frac{1}{2}$.
2. $2\sin^{-1}\frac{4}{5} = \cos^{-1}(-\frac{7}{25})$.
3. $2\sin^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$.
4. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2} = 120°$.
5. $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}}$.
6. $\cos^{-1}x = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$.
7. $2\sin^{-1}\frac{5}{13} = \sin^{-1}\frac{120}{169}$.
8. $2\sin^{-1}\frac{8}{17} = \sin^{-1}\frac{240}{289}$.
9. $2\sin^{-1}\frac{7}{25} = \cos^{-1}\frac{527}{625}$.
10. $2\cos^{-1}\frac{5}{12} = \cos^{-1}(-\frac{47}{72})$.
11. $\sin^{-1}\frac{3}{5} + 2\sin^{-1}\frac{12}{13} = \sin^{-1}(?)$.
12. $\cos^{-1}\frac{3}{5} + 2\cos^{-1}\frac{5}{12} = \cos^{-1}(?)$.
13. $\sin^{-1}\frac{7}{25} + 2\cos^{-1}\frac{3}{5} = \sin^{-1}(?)$.
14. $\sin^{-1}\frac{7}{25} - 2\sin^{-1}\frac{8}{17} = \sin^{-1}(?)$.
15. $\cos^{-1}\frac{5}{13} - 2\cos^{-1}\frac{5}{12} = \cos^{-1}(?)$.
16. $\cos^{-1}\frac{3}{5} - 2\cos^{-1}\frac{4}{5} = \sin^{-1}(?)$.
17. $\sin^{-1}(.35) + 2\cos^{-1}(.37) = ?$

§ 144. To find the Sine and Cosine of 3 *A* in Terms of the Sine and Cosine of *A*, respectively, and the Use of the Results in solving Cubics.

$$\sin(3\,A) = \sin(2\,A + A) = \sin 2\,A\cos A + \cos 2\,A\sin A$$
$$= 2\sin A\cos A\cos A + (1 - 2\sin^2 A)\sin A$$
$$= 2\sin A\,(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A.$$

$$\therefore\ \mathbf{\sin 3\,A = 3\sin A - 4\sin^3 A.} \qquad (1)$$

A similar process, expressing the terms as rapidly as possible in terms of cos A instead of sin A as above when sin 3 A is determined, gives:

$$\cos 3A = 4\cos^3 A - 3\cos A. \tag{2}$$

Inasmuch as (1), (2) have the 4, 3 terms interchanged, it may be helpful, in remembering them, to note that each term contains a 3, and that they must both remain true for any special angle. Take for A the angle whose sine is 1 in the sin 3 A result, and for A, in the cos 3 A result, the angle whose cosine is 1; namely, 90° and 0°, respectively.

Then $\sin 3A = \sin 270° = -1,$

so the 4 must follow the 3;

and $\cos 3A = \cos 0 = 1,$

so the 3 must follow the 4.*

The equations (1), (2) show that if sin 3 A, cos 3 A are given, the sine and cosine of A are to be obtained by solving cubics. Thus the ancient problem of "trisecting an angle by means of the ruler and compass," is the same as the algebraic problem of solving a cubic when the second power of the unknown quantity is absent, as is the case in (1), (2).

Any cubic equation, as

$$ax^3 + bx^2 + cx + d = 0,$$

can be freed of the x^2 term by setting $x = y - \frac{b}{3a}$.

For $a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0,$

becomes, on expansion, an expression like

$$ay^3 + py + q = 0. \tag{3}$$

Comparing (3) with

$$4\cos^3 A - 3\cos A - \cos 3A = 0, \tag{2}$$

---

* Or this:

Next "sin" comes "trin," 3
Next "co" comes "fo," 4!

cos A might be y, if y were less than 1, numerically. We may set (3) in the form

$$a(ny)^3 + pn^2(ny) + qn^3 = 0, \tag{4}$$

where n may be so selected that ny may be less than 1, and thus the cos A.

When an expression like (3) is zero, it remains zero when multiplied by any number. Therefore, when two similar expressions are identical and zero, there is proportionality among the coefficients of corresponding terms. So if (2), (4) are identical, and $\cos A = ny$,

$$\frac{a}{4} = \frac{pn^2}{-3} = \frac{qn^3}{-\cos 3A}. \tag{5}$$

$$\therefore n^2 = -\frac{3a}{4p}; \tag{6}$$

$$\cos 3A = \frac{3qn}{p}. \tag{7}$$

(6) shows that n is real only when a, p are opposite in sign; (6) also shows that n may have two values, equal and opposite in sign. However, it is necessary to take only one value of n, since $-n$ in (4) merely changes the sign throughout and gives only the same solutions as $+n$; (7) shows that no real angle, $3A$, exists unless $3qn \overline{\overline{<}} p$, numerically, or $9q^2n^2 \overline{\overline{<}} p^2$, or, by (6), $9q^2\left(\frac{3a}{4p}\right) \overline{\overline{<}} p^2$, numerically,

or, $$27q^2a \overline{\overline{<}} 4p^3, \text{ numerically.} \tag{8}$$

(8) is the test as to whether a cubic of the form of (3) can be solved by comparison with (2), *when a is positive and p negative*, and $3A$ is an ordinary angle,—one whose cosine is real.

MODEL EXAMPLE.

$$x^3 + 6x^2 + 9x + 3 = 0. \tag{1}$$

Here $a = 1$, $b = 6$. $\therefore$ to get rid of the x^2 term, set $x = y - \frac{b}{3a}$, or $x = y - 2$. Then (1) becomes

$$y^3 - 3y + 1 = 0. \tag{2}$$

Here $a = 1$, $p = -3$, $q = 1$; a, p are opposite in sign, with (8) satisfied.

(6) gives $n^2 = \frac{1}{4}$. $\therefore n = \frac{1}{2}$.

(7) gives $\cos 3A = -\frac{1}{2}$.

$$\therefore 3A = 2m \cdot 180° \pm 120°.$$

$$\therefore A = m \cdot 120° \pm 40°.$$

This gives only three different cosines for A, — those of 40°, 160°, 280°.

And since $ny = \cos A$ and $n = \frac{1}{2}$, and $x = y - 2$, the three values of x are $x = \frac{1}{2}\cos 40° - 2$; $x = -\frac{1}{2}\cos 20° - 2$; $x = \frac{1}{2}\cos 80° - 2$. Looking up these cosines in the table, we have an approximate solution of (1), all three values of x being real.

It is this case — *three real roots* — to which the comparison method pointed out above, for solving a cubic, is suited. The student is familiar with the fact that a quadratic has two roots. So the cubic has three, of which only one may be real, with two imaginary, or all three may be real. At least one root must be real.

To make the solution of the cubic complete, we give below the so-called *Cardan's Rule* or process, suited to the case when a, p, are both positive, and to the case where $27q^2a < 4p^3$ (8) is not satisfied with a positive and p negative.

Let $$ay^3 + py + q = 0. \qquad (3)$$

Assume $y = u^{\frac{1}{3}} + v^{\frac{1}{3}}$.

$$\therefore y^3 = u + 3u^{\frac{2}{3}}v^{\frac{1}{3}} + 3u^{\frac{1}{3}}v^{\frac{2}{3}} + v,$$

or, $$y^3 = u + v + 3u^{\frac{1}{3}}v^{\frac{1}{3}}(u^{\frac{1}{3}} + v^{\frac{1}{3}}),$$

or, $$y^3 = (u + v) + 3u^{\frac{1}{3}}v^{\frac{1}{3}} \cdot y,$$

or, $$y^3 - 3u^{\frac{1}{3}}v^{\frac{1}{3}}y - (u + v) = 0. \qquad (U)$$

Dividing (3) by a and considering (3) and (U) as identical, then

$$-u^{\frac{1}{3}}v^{\frac{1}{3}} = \frac{p}{3a}, \qquad (\alpha)$$

and $$-(u + v) = \frac{q}{a}. \qquad (\beta)$$

Cubing (α) and adding four times the result to the square of (β), we have

$$u^2 - 2\,uv + v^2 = \frac{q^2}{a^2} + \frac{4\,p^3}{27\,a^3}. \qquad (\gamma)$$

Now, $u^2 - 2\,uv + v^2 = (u-v)^2$, and, when real, must be positive or zero. The second member of (γ) is always positive if a, p are the same in sign; but if a, p are opposite in sign, we must have

$$\frac{q^2}{a^2} \overline{\gtrless} \frac{4\,p^3}{27\,a^3}, \text{ numerically,}$$

or,

$$27\,aq^2 \overline{\gtrless} 4\,p^3, \text{ numerically.} \qquad (9)$$

Now, compare this with (8). It is the opposite case.

When (9) holds, (γ) gives $u - v$ and (β) gives $u + v$.

Adding and subtracting, etc., we have u and v. Whence, y from $y = u^{\frac{1}{3}} + v^{\frac{1}{3}}$, single values, when only the ordinary, or real, cube roots of u, v are used.

The student can readily see that any real number, like u, v above, has three cube roots, one real and two imaginary.

For suppose $\qquad z^3 = u = u \cdot 1.$

Then $\qquad z = \sqrt[3]{u} \cdot \sqrt[3]{1},$

where we suppose $\sqrt[3]{u}$ is the ordinary cube root of u. We will show that $\sqrt[3]{1}$ has three values.

Let

$$s^3 = 1.$$

$$\therefore\ s^3 - 1 = 0.$$

$$\therefore\ (s-1)(s^2+s+1) = 0.$$

$$\therefore\ s - 1 = 0, \text{ or } s^2 + s + 1 = 0.$$

The first supposition, $s - 1 = 0$, gives $s = 1$.

Solving the quadratic supposition in the usual way, we get

$$s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\sqrt{-1}.$$

The three values of s taken with $\sqrt[3]{u}$, $\sqrt[3]{v}$ give y three values.

Summary for solution of $ay^3 + py + q = 0$, taking a positive:

(i) If p is also positive, use Cardan's process as just given.

(ii) If p is negative and $27\,aq^2 > 4\,p^3$, numerically, use Cardan's process.

(iii) If p is negative and $27\,aq^2 < 4\,p^3$, numerically, use the trigonometric method as in the example on page 265.

(iv) If p is negative and $27\,aq^3 = 4\,p^3$, use either method, or solve by the method of equal roots.

In cases (i), (ii) there will be one real root and two imaginary; in cases (iii), (iv), three real roots, there being equal roots in case (iv).

EXERCISES.

Determine by which method each of the following examples, from Ex. 2 to Ex. 10, is best solved, and solve:

1. $2\,x^3 - 3\,x - 1 = 0.$

2. $x^3 + 3\,x^2 - 1 = 0.$

3. $x^3 - 24\,x - 32 = 0.$

4. $x^3 - 7\,x + 5 = 0.$

5. $x^3 + 4\,x^2 + 2\,x - 1 = 0.$

6. $x^3 - 3\,x - 2 = 0.$

7. $x^3 - x + 6 = 0.$

8. $x^3 - 9\,x - 28 = 0.$

9. $3\,x^3 - 6\,x^2 - 2 = 0.$

10. $x^3 - 15\,x^2 - 33\,x + 847 = 0.$

11. Show $\sin 4\,\theta = 4 \sin\theta \cos^3\theta - 4 \cos\theta \sin^3\theta.$

12. Show $\cos 4\,\theta = 1 - 8\cos^2\theta + 8\cos^4\theta.$

13. Show $\cos 5\,\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

14. Show $\cos 6\,\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$

15. Show $\sin 3\,\theta = (2\cos 2\,\theta + 1)\sin\theta.$

16. Show $\cos 3\,\theta = (2\cos 2\,\theta - 1)\cos\theta.$

17. Show $\dfrac{\cos 3\,\theta + 3\cos\theta}{3\sin\theta - \sin 3\,\theta} = \cos^3\theta \operatorname{cosec}^3\theta.$

18. Show $\sin 3\,\theta - \sin\theta = \sin\theta\,(\cos 3\,\theta + \cos\theta)\sec\theta.$

19. Show $\sin 3\,\theta - \cos 3\,\theta = (\sin\theta + \cos\theta)(2\sin 2\,\theta - 1).$

20. Show $(1 + 2\cos\theta)^2(1 - \cos\theta) = 1 - \cos 3\,\theta.$

21. Show $(\sec 2\,\theta + 1)^3(3\sin\theta - \sin 3\,\theta)^2$
$= (\sec 2\,\theta - 1)^3(3\cos\theta + \cos 3\,\theta)^2.$

22. Find expressions for $\sin 5\,\theta$ and $\sin 6\,\theta$ in terms of functions of θ.

23. Show that $A = 18°$ is a particular solution of the equation

$$\sin 2\,A = \cos 3\,A.$$

Express this equation in terms of $\sin A$ and $\cos A$. Then in terms of $\sin A$, and solve, showing

$$\sin A = \sin 18° = \frac{+\sqrt{5} - 1}{4},$$

$$\cos A = \cos 18° = \frac{+\sqrt{10 + 2\sqrt{5}}}{4}.$$

24. Give the general solution of the equation

$$\sin 2A = \cos 3A.$$

25. Show from the values of $\sin 18°$, $\cos 18°$, in Ex. 13, that

$$\sin 36° = \frac{+\sqrt{10 - 2\sqrt{5}}}{4}.$$

$$\cos 36° = \frac{\sqrt{5}+1}{4},$$

using $$\sin 2x = 2 \sin x \cos x.$$

26. Show that $$\sin 9° + \cos 9° = \sqrt{1 + \sin 18°},$$

$$\sin 9° - \cos 9° = -\sqrt{1 - \sin 18°}.$$

From these equations, show

$$\sin 9° = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4}.$$

$$\cos 9° = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4}.$$

What are the values of $\sin 81°$ and $\cos 81°$?

§ 145. The Addition-multiplication and Subtraction-multiplication Formulas.

It is frequently advisable, either for purposes of simplification or to set an expression in a form suitable for logarithmic computation, to express the sum or difference of two sines (cosines) in the form of a product. From

$$\sin(A+B) = \sin A \cos B + \cos A \sin B, \qquad (1)$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B, \qquad (2)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B, \qquad (3)$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B, \qquad (4)$$

follow by addition (subtraction):

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B, \qquad (5)$$

$$\sin(A+B) - \sin(A-B) = 2 \sin B \cos A, \qquad (6)$$

$$\cos(A-B) + \cos(A+B) = 2 \cos A \cos B, \qquad (7)$$

$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B. \qquad (8)$$

Here $A + B$ and $A - B$ may represent any two angles.

Let $$A + B = x,$$
$$A - B = y.$$

Then $$A = \frac{x + y}{2},$$
$$B = \frac{x - y}{2}.$$

Thus, (5), (6), (7), (8) may be written:

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}, \qquad (5')$$

$$\sin x - \sin y = 2 \sin \frac{x - y}{2} \cos \frac{x + y}{2}, \qquad (6')$$

$$\cos y + \cos x = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}, \qquad (7')$$

$$\cos y - \cos x = 2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}. \qquad (8')$$

These formulas are general, being true for all values of x, y. The subtractions in the second members must be carried out, so far as sines are concerned, in the order indicated, since $\sin(-\theta) = -\sin\theta$; for the cosines, this is a matter of indifference, since $\cos(-\theta) = \cos\theta$.

The order in (7′), (8′) is set different from that in (5′), (6′), as relates to x, y, in deference to angles of the first quadrant, since here the larger the angle, the larger the sine, but the larger the angle, the smaller the cosine.

EXERCISE.

1. Give verbal statements for (5′), (6′), (7′), (8′), paying attention to the order of subtractions.

2. $\sin 60° + \sin 30° = 2 \sin 45° \cos 15°$.

3. $\sin 60° - \sin 30° = 2 \sin 15° \cos 45°$.

4. $\sin 25 - \sin 45° = -2 \sin 10° \cos 35°$.

5. $\cos 60° + \cos 30° = 2 \cos 45° \cos 15°$.

6. $\cos 60° - \cos 30° = -2 \sin 45° \sin 15°$.

7. $\cos 25^\circ - \cos 55^\circ = 2 \sin 40^\circ \sin 15^\circ.$

8. $\sin 115^\circ + \sin 281^\circ = -2 \sin 18^\circ \sin 7^\circ.$

9. $\sin 317^\circ - \sin 151^\circ = -2 \sin 36^\circ \cos 7^\circ.$

10. $\cos 295^\circ + \cos 817^\circ = 2 \cos 16^\circ \sin 9^\circ.$

11. $\cos 370^\circ - \cos 280^\circ = 2 \sin 35^\circ \sin 45^\circ.$

12. Verify some of the preceding results by using the tables.

13. Take a pair of angles in each quadrant and apply (5′), (6′), (7′), (8′), letting (i) $x > y$ in (6′), (8′), and then (ii) $x < y$, expressing the second member in each case in terms of angles less than 45° (if not 45°). Test by the tables.

14. Apply (5′), (6′), (7′), (8′) to pairs of positive angles selected one from one quadrant and the other from another, in the six possible ways for (5′), (7′) and in the twelve possible ways for (6′), (8′), expressing the second members in terms of angles less than (or equal to) 45°. Test some of the results by the tables.

15. Apply (5′), (6′), (7′), (8′) to pairs of negative angles from the same quadrant and to pairs from different quadrants, and test some of the results.

16. Apply (5′), (6′), (7′), (8′) to a pair of angles, one positive and one negative, making the selections show variety of sign in the second members, after the angles there are reduced to angles less than 45°.

17. $\sin 7\theta - \sin 5\theta = 2 \sin \theta \cos 6\theta.$

18. $\sin \theta + \sin 2\theta = 2 \sin \frac{3\theta}{2} \cos \frac{\theta}{2}.$

19. $\sin 2\theta - \sin 4\theta = -2 \sin \theta \cos 3\theta.$

20. $\cos 3\theta + \cos 7\theta = 2 \cos 5\theta \cos 2\theta.$

21. $\cos 3\theta - \cos 5\theta = 2 \sin \theta \sin 4\theta.$

22. $\cos 7\theta - \cos 5\theta = -2 \sin \theta \sin 6\theta.$

23. $\cos 2\theta - \cos \theta = -2 \sin \frac{\theta}{2} \sin \frac{3\theta}{2}.$

24. $\sin (45^\circ + A) + \sin (45^\circ - A) = \sqrt{2} \cos A.$

25. $\cos (45^\circ - A) + \cos (45^\circ + A) = \sqrt{2} \cos A.$

26. $\sin (45^\circ - A) - \sin (45^\circ + A) = -\sqrt{2} \sin A.$

27. $\cos (45^\circ + A) - \cos (45^\circ - A) = -\sqrt{2} \sin A.$

28. $\cos (60^\circ + A) + \cos (60^\circ - A) = \cos A.$

29. In Exs. 24, 25, 26, replace 45° by 60° and find second member.

30. Find second member in Exs. 24, 25, 26, 27, with 53° replacing 45°.

31. $\sin\left(\frac{\pi}{4}+\theta\right)+\sin\left(\frac{3\pi}{4}+\theta\right)=\sqrt{2}\cos\theta.$

32. $\frac{\sin 75^\circ+\sin 15^\circ}{\cos 75^\circ-\cos 15^\circ}=-\frac{\cos 30^\circ}{\sin 30^\circ}=-\sqrt{3}.$

33. $\frac{\sin 75^\circ-\sin 15^\circ}{\cos 75^\circ+\cos 15^\circ}=0.577\cdots.$

34. $\frac{\sin x+\sin y}{\sin x-\sin y}=?$

35. $\frac{\cos x+\cos y}{\cos x-\cos y}=?$

36. $\frac{\sin x+\sin y}{\cos x-\cos y}=?$

37. $\frac{\sin x-\sin y}{\cos x-\cos y}=?$

38. $\sin 50^\circ-\sin 70^\circ+\sin 10^\circ=0.$

39. $\sin 10^\circ+\sin 20^\circ+\sin 40^\circ+\sin 50^\circ=\sin 70^\circ+\sin 80^\circ.$

40. $\frac{\sin 7A-\sin 5A}{\cos 7A+\cos 5A}=\sin A\sec A.$

41. $\frac{\sin A+\sin 3A}{\cos A+\cos 3A}=\sin 2A\sec 2A.$

42. $\frac{\sin 7A-\sin A}{\sin 8A-\sin 2A}=\cos 4A\sec 5A.$

43. $\frac{\cos 2B+\cos 2A}{\cos 2B-\cos 2A}=\cos(A+B)\cos(A-B)\operatorname{cosec}(A+B)\operatorname{cosec}(A-B).$

44. $\frac{\sin 2B+\sin 2A}{\sin 2B-\sin 2A}=-\sin(A+B)\cos(A-B)\sec(A+B)\operatorname{cosec}(A-B).$

45. $\frac{\sin A+\sin 2A}{\cos 2A-\cos A}=-\cos\frac{A}{2}\operatorname{cosec}\frac{A}{2}.$

46. $\frac{\sin 5A-\sin 3A}{\cos 3A+\cos 5A}=\sin A\sec A.$

47. $\frac{\cos 2A-\cos 2B}{\sin 2A+\sin 2B}=-\sin(A-B)\sec(A-B).$

48. $\cos(A+B)+\sin(A-B)=2\sin(45^\circ+A)\cos(45^\circ+B).$

49. $\frac{\cos 3A-\cos A}{\sin 3A-\sin A}+\frac{\cos 2A-\cos 4A}{\sin 4A-\sin 2A}=\frac{\sin A}{\cos 2A\cos 3A}.$

50. $\frac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\sin(A+B)\sec(A+B).$

51. $\frac{\cos 3A+2\cos 5A+\cos 7A}{\cos A+2\cos 3A+\cos 5A}=\cos 2A-\frac{\sin 2A\sin 3A}{\cos 3A}.$

52. $\frac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\sin 4A\sec 4A.$

53. $\dfrac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)}=\sin\theta\sec\theta.$

54. $\dfrac{\sin\theta+2\sin 3\theta+\sin 5\theta}{\sin 3\theta+2\sin 5\theta+\sin 7\theta}=\sin 3\theta\operatorname{cosec} 5\theta.$

55. $\dfrac{\sin(\theta-\phi)+2\sin\theta+\sin(\theta+\phi)}{\sin(\beta-\phi)+2\sin\beta+\sin(\beta+\phi)}=\sin\theta\operatorname{cosec}\beta.$

56. $\dfrac{\sin\theta-\sin 5\theta+\sin 9\theta-\sin 13\theta}{\cos\theta-\cos 5\theta-\cos 9\theta+\cos 13\theta}=\cos 4\theta\operatorname{cosec} 4\theta.$

57. $\cos 3x+\cos 5x+\cos 7x+\cos 15x=4\cos 4x\cos 5x\cos 6x.$

58. $\sin\theta+\sin 2\theta+\sin 4\theta+\sin 5\theta=4\cos\dfrac{\theta}{2}\cos\dfrac{3\theta}{2}\sin 3\theta.$

59. Show, without using (5′), (6′), (7′), (8′), that

$$\frac{\sin x+\sin y}{\cos y+\cos x}=\frac{\cos y-\cos x}{\sin y-\sin x}.$$

60. $\dfrac{\cos\theta-\cos 3\theta}{\sin\theta-\sin 5\theta}=\dfrac{\cos 6\theta-\cos 4\theta}{\sin 8\theta+\sin 2\theta}.$

61. $\dfrac{\cos(x+y+z)+\cos(-x+y+z)+\cos(x-y+z)+\cos(x+y-z)}{\sin(x+y+z)+\sin(-x+y+z)-\sin(x-y+z)+\sin(x+y-z)}=\dfrac{\cos y}{\sin y}.$

62. $\dfrac{\cos\theta+\cos 3\theta}{\cos\theta+\cos 5\theta}=\dfrac{\sec 2\theta}{2-\sec 2\theta}.$

§ 146. Application of the Addition-multiplication, Subtraction-multiplication Formulas to the Solution of Trigonometric Equations of a Special Type.

Model Example.

Solve $\sin\theta+\sin 7\theta=\sin 4\theta.$ (1)

By (5), $2\sin 4\theta\cos 3\theta=\sin 4\theta,$ (2)

or, $\sin 4\theta(2\cos 3\theta-1)=0.$ (3)

$\therefore\ \sin 4\theta=0,$ (4)

or, $2\cos 3\theta-1=0.$ (5)

By (4), $4\theta=n\pi.\quad\therefore\ \theta=n\dfrac{\pi}{4}.$

By (5), $\cos 3\theta=\dfrac{1}{2}.$ (6)

A particular solution is $3\theta=\dfrac{\pi}{3}.$

$\therefore$ the general solution is . $3\,\theta = 2\,n\pi \pm \frac{\pi}{3}$,

or, $\theta = 2\,n\frac{\pi}{3} \pm \frac{\pi}{9}$.

Thus the general solutions of (1) are

$$\theta = n\frac{\pi}{4}, \text{ or } 2\,n\frac{\pi}{3} \pm \frac{\pi}{9}.$$

The applicability of the process indicated in this example is limited to the case where a factor is apparent after the simplification made in (2). An example like

$$\sin\theta + \sin 7\,\theta = \sin 13\,\theta,$$

or like $5\sin\theta + 3\cos\theta = 4$,

or $2\sin\theta + 3\sin 7\,\theta = 2$,

could not be solved by the process given above. Later, a solution of such expressions by the aid of auxiliary angles will be given. (See § 148; also § 180.)

EXERCISES.

Solve the following, giving general values for the angle, θ:

1. $\cos\theta + \cos 7\,\theta = 2\cos 4\,\theta$.
2. $\cos\theta + \cos 7\,\theta = \cos 3\,\theta$.
3. $\sin\theta - \sin 7\,\theta = \sin 3\,\theta$.
4. $\sin 7\,\theta - \sin\theta = \cos 4\,\theta$.
5. $\cos\theta - \cos 7\,\theta = 2\sin 4\,\theta$.
6. $\cos 7\,\theta - \cos\theta = \sin 3\,\theta$.
7. $\cos\theta + \cos 2\,\theta + \cos 3\,\theta = 0$.
8. $\sin\theta + \sin 2\,\theta + \sin 3\,\theta = 0$.
9. $\cos\theta - \cos 2\,\theta + \cos 3\,\theta = 0$.
10. $\sin\theta - \sin 2\,\theta + \sin 3\,\theta = 0$.
11. $\cos\theta + \sin 2\,\theta - \cos 3\,\theta = 0$.
12. $\sin\theta - \cos 2\,\theta - \sin 3\,\theta = 0$.
13. $\sin 2\,\theta - \cos 2\,\theta - \sin\theta + \cos\theta = 0$.
14. $\sin\theta + \sin 2\,\theta = 0$.
15. $\sin\theta - \sin 5\,\theta = 0$.
16. $\cos 2\,\theta + \cos 3\,\theta = 0$.
17. $\cos 7\,\theta - \cos 5\,\theta = 0$.
18. $\sin m\theta \pm \sin n\theta = 0$. $m \lesseqgtr n$.
19. $\cos m\theta \pm \sin n\theta = 0$. $m \gtreqless n$.
20. $\sin\theta \pm \cos 3\,\theta = 0$.
21. $\sin m\theta \pm \cos n\theta = 0$. $m \lesseqgtr n$.
22. $\sin(3\,\theta + \alpha) + \sin(3\,\theta - \alpha) + \sin(\alpha - \theta) - \sin(\alpha + \theta) = \cos\alpha$.
23. $\cos n\theta = \cos(n-2)\theta + \sin\theta$.
24. $\sin\frac{n+1}{2}\theta = \sin\frac{n-1}{2}\theta + \sin\theta$.

25. $\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2\theta$.

26. $\sin 3\theta - 4\sin\theta\sin(\theta+\beta)\sin(\theta-\beta) = 0$.

27. $\cos 3\theta + 2\cos 2\theta = 0$.

28. $\cos 2\theta + 3\cos\theta = 0$.

29. $\dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = 2$.

30. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin 2\theta}{\cos 2\theta} + \dfrac{\sin 3\theta}{\cos 3\theta} = 0$.

31. $\dfrac{\cos^2\theta}{\sin^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta} = 4\dfrac{\cos 2\theta}{\sin 2\theta}$.

32. $\cos 3\theta - \cos 5\theta = \sin\theta$.

33. $\cos 5\theta + \cos 3\theta = \sqrt{2}\cos 4\theta$.

34. $\sin 6\theta + \sin 4\theta = 2\cos\theta$.

35. $\sin(m+1)\theta + \sin(m-1)\theta = \cos\theta$.

36. $\cos m\theta - \cos(m-2)\theta = \sin\theta$.

37. $\sin\theta + \sin 2\theta + \sin 3\theta = \cos\theta + \cos 2\theta + \cos 3\theta$.

38. $\sin 2\theta - \cos 2\theta = \cos\theta - \sin\theta$.

39. $\cos(\alpha - \theta) = \sin(\alpha + \theta)$.

40. $\sin(\theta+\alpha) + \cos(\theta+\alpha) = \sin(\theta-\alpha) + \cos(\theta-\alpha)$.

41. $\cos 3\theta\sin^3\theta + \sin 3\theta\cos^3\theta = 0$.

42. $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$.

43. $\cos 3\theta\sin^3\theta + \sin 3\theta\cos^3\theta = \dfrac{3\sqrt{3}}{8}$.

44. $\sin\theta(\cos 2\theta + \cos 4\theta + \cos 6\theta + \cos 8\theta) = \sin 4\theta$.

45. $2\cos 2\theta\cos 3\theta - \cos\theta = 0$.

46. $4\sin^2\theta + \sin^2 2\theta = 3$.

47. $2\sin^3\theta + 3\cos^2\theta + \sin\theta = 3$.

48. $\cos^3\theta - \cos\theta\sin\theta - \sin^3\theta = 1$.

49. $\sin^3\theta = 0.2341\sin^2\theta$.

§ 147. The Multiplication-addition, Multiplication-subtraction Formulas.

The formulas (5), (6), (7), (8) of § 145, when read backward, may be called the Multiplication-addition, Multiplication-subtraction formulas.

$$\sin A\cos B = \tfrac{1}{2}\{\sin(A+B) + \sin(A-B)\}, \quad (1)$$

$$\sin B\cos A = \tfrac{1}{2}\{\sin(A+B) - \sin(A-B)\}, \quad (2)$$

$$\cos A\cos B = \tfrac{1}{2}\{\cos(A-B) + \cos(A+B)\}, \quad (3)$$

$$\sin A\sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}. \quad (4)$$

From (3), it appears that if both *words* of the product are cosines, the product may be changed to a sum of cosines; from (4), that if both *words* of the product are sines, the product may be changed to a difference of cosines.

For example, $\begin{cases} \cos 40° \cos 60° = \frac{1}{2}(\cos 20° + \cos 100°), \\ \sin 40° \sin 60° = \frac{1}{2}(\cos 20° - \cos 100°). \end{cases}$

In (1), (2), the form of subtraction in the second member assumes A numerically greater than B, so we may say that when the *words* of the product are one sine and the other cosine, then, if sine is connected with the numerically larger angle, a sum of sines is indicated, and if sine is connected with the smaller angle, a difference of sines is indicated.

For example,

$$\sin 60° \cos 40° = \tfrac{1}{2}(\sin 100° + \sin 20°),$$
$$\sin 40° \cos 60° = \tfrac{1}{2}(\sin 100° - \sin 20°).$$

However,

$$\sin x \cos y = \tfrac{1}{2}\{\sin(x + y) + \sin(x - y)\},$$

no matter what the relative size of x and y, when the difference $x - y$ is taken in the order indicated.

For example,

$$\begin{aligned} \sin 40° \cos 60° &= \tfrac{1}{2}\{\sin(40° + 60°) + \sin(40° - 60°)\} \\ &= \tfrac{1}{2}\{\sin 100° + \sin(-20°)\} \\ &= \tfrac{1}{2}(\sin 100° - \sin 20°). \end{aligned}$$

EXERCISES.

1. $\sin 80° \cos 30° = ?$

2. $\sin 30° \cos 80° = ?$

3. $\cos 80° \cos 30° = ?$

4. $\sin 80° \sin 30° = ?$

5. $\sin 7\theta \cos 5\theta = ?$

6. $\sin 5\theta \cos 7\theta = ?$

7. $\cos 5\theta \cos 7\theta = ?$

8. $\sin 5\theta \sin 7\theta = ?$

9. $\sin\theta \cos\phi = ?$

10. $\cos\theta \cos\phi = ?$

11. $\sin\frac{3\theta}{2} \sin\frac{5\theta}{2} = ?$

12. $\sin\frac{3\theta}{2} \cos\frac{5\theta}{2} = ?$

13. $\sin\frac{3\theta}{2} \cos\frac{\theta}{2} = ?$

14. Simplify $2\cos 2\theta\cos\theta - 2\sin 4\theta\sin\theta$.

15. Simplify $\sin\frac{5\theta}{2}\cos\frac{\theta}{2} - \sin\frac{9\theta}{2}\cos\frac{3\theta}{2}$.

16. Prove $2\sin(45° + A)\sin(45° - A) = \cos 2A$.

17. Prove $2\cos(45° + A)\cos(45° - A) = \cos 2A$.

18. Prove that $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta\sin 5\theta$.

19. Prove $\cos 2\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2} = \sin 5\theta\sin\frac{5\theta}{2}$.

20. Prove $\sin\frac{11\theta}{4}\sin\frac{\theta}{4} + \sin\frac{7\theta}{4}\sin\frac{3\theta}{4} = \sin 2\theta\sin\theta$.

21. Prove $2\sin 2\theta\cos\theta + 2\cos 4\theta\sin\theta = \sin 5\theta + \sin\theta$.

22. Prove $\cos 2\theta\cos\theta - \sin 4\theta\sin\theta = \cos 3\theta\cos 2\theta$.

23. Prove $\sin\alpha\sin(\alpha+2\beta) - \sin\beta\sin(\beta+2\alpha) = \sin(\alpha-\beta)\sin(\alpha+\beta)$.

24. Prove $(\sin 3\phi + \sin\phi)\sin\phi + (\cos 3\phi - \cos\phi)\cos\phi = 0$.

25. $\dfrac{2\sin(\theta - \phi)\cos\phi - \sin(\theta - 2\phi)}{2\sin(\beta - \phi)\cos\phi - \sin(\beta - 2\phi)} = \dfrac{\sin\theta}{\sin\beta}$.

26. $\dfrac{\sin\theta\sin 2\theta + \sin 3\theta\sin 6\theta + \sin 4\theta\sin 13\theta}{\sin\theta\cos 2\theta + \sin 3\theta\cos 6\theta + \sin 4\theta\cos 13\theta} = \dfrac{\sin 9\theta}{\cos 9\theta}$.

27. $\dfrac{\cos 2\theta\cos 3\theta - \cos 2\theta\cos 7\theta + \cos\theta\cos 10\theta}{\sin 4\theta\sin 3\theta - \sin 2\theta\sin 5\theta + \sin 4\theta\sin 7\theta} = \dfrac{\cos 6\theta}{\sin 6\theta}\cdot\dfrac{\cos 5\theta}{\sin 5\theta}$.

28. $\cos\theta\sin(\beta - \phi) + \cos\beta\sin(\phi - \theta) + \cos\phi\sin(\theta - \beta) = 0$.

29. $\sin(\beta - \gamma)\cos(\alpha - \delta) + \sin(\gamma - \alpha)\cos(\beta - \delta) + \sin(\alpha - \beta)\cos(\gamma - \delta) = 0$.

30. $\cos(\theta - \phi)\cos(\theta + \phi) = \cos^2\theta - \sin^2\phi = \cos^2\phi - \sin^2\theta$.

31. $\text{versin}(\theta + \phi)\,\text{versin}(\theta - \phi) = (\cos\theta - \cos\phi)^2$.

32. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$.

33. $\cos(36° - A)\cos(36° + A) + \cos(54° + A)\cos(54° - A) = \cos 2A$.

34. $\dfrac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta+\phi}{2}} - \dfrac{\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}} = \dfrac{2\sin\phi}{\cos\theta + \cos\phi}$.

35. $\dfrac{\sin\theta\cdot\sin 2\theta + \sin 2\theta\sin 5\theta + \sin 3\theta\sin 10\theta}{\cos\theta\cdot\sin 2\theta + \sin 2\theta\cos 5\theta - \cos 3\theta\sin 10\theta} = -\sin 7\theta\sec 7\theta$.

36. $\dfrac{\sin 5\theta}{\cos 5\theta} - \dfrac{\sin 3\theta}{\cos 3\theta} - \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{\sin 2\theta}{\cos 2\theta}\cdot\dfrac{\sin 3\theta}{\cos 3\theta}\cdot\dfrac{\sin 5\theta}{\cos 5\theta}$.

§ 148. Use of an Auxiliary, or Subsidiary, Angle in Solving Trigonometric Equations of a Certain Type.

MODEL EXAMPLE.

$$2 \sin x + 3 \cos x = 3. \tag{1}$$

Divide by the square root of the sum of the squares of the coefficients of $\sin x$ and $\cos x$.

$$\therefore \frac{2}{\sqrt{13}} \sin x + \frac{3}{\sqrt{13}} \cos x = \frac{3}{\sqrt{13}}. \tag{2}$$

Since $$\sin^2 y + \cos^2 y = 1$$

and $$\left(\frac{2}{\sqrt{13}}\right)^2 + \left(\frac{3}{\sqrt{13}}\right)^2 = 1,$$

we may set

$$\sin y = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}. \tag{3}$$

$$\cos y = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}. \tag{4}$$

Either (3) or (4), when expressed as a decimal, directly or by logarithms, determines y in the tables. Thus (2) becomes

$$\sin y \sin x + \cos y \cos x = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}. \tag{5}$$

If we determine z from the tables by means of

$$\cos z = \frac{3\sqrt{13}}{13}, \tag{6}$$

(5) becomes $$\cos (x - y) = \cos z; \tag{7}$$

or, $$x - y = 2n\,180° \pm z. \tag{8}$$

$$\therefore x = 2n\,180° + y \pm z. \tag{9}$$

By (3), $$\sin y = \frac{2\sqrt{13}}{13} =$$

$$\therefore y =$$

By (6), $$\cos z = \frac{3\sqrt{13}}{13} =$$

$$\therefore z =$$

(The student may determine y, z from the tables.)

Then, by (9), $x = 2n \cdot 180° + y \pm z$ (as found), where n is any integer.

z and y are called *auxiliary* angles; sometimes *subsidiary* angles.

Example (1) above belongs to the general type.

$$a \sin\theta + b\cos\theta = c, \tag{A}$$

whose solution is, similarly,

$$\frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}.$$

Let $$\sin\phi = \frac{a}{\sqrt{a^2+b^2}},\ \cos\phi = \frac{b}{\sqrt{a^2+b^2}},$$

and $$\cos\alpha = \frac{c}{\sqrt{a^2+b^2}}.$$

$$\therefore\ \cos(\theta-\phi) = \cos\alpha.$$

$$\therefore\ \theta = 2n\pi \pm \alpha + \phi.$$

Since no cosines are greater than unity, evidently α cannot be determined if $c^2 > a^2+b^2$. In this case no solution exists. When a, b are large, use § 180.

EXERCISES.

A. Show that (A) above may be solved by the form: $\sin(\theta\pm\phi)=\sin\alpha$.

B. Determine the general value of θ in the following cases:

1. $\sin\theta \pm \cos\theta = \sqrt{2}$.

2. $\sin 3\theta \pm \cos 3\theta = \dfrac{\sqrt{3}}{\sqrt{2}}$.

3. $\sin 5\theta \pm \cos 5\theta = -\sqrt{2}$.

4. $\sin 2\theta \pm \cos 2\theta = 0$.

5. $\sin m\theta \pm \cos n\theta = 0$.

6. $\sin m\theta \pm \cos n\theta = \pm\sqrt{2}$.

7. $\sin m\theta \pm \cos n\theta = \pm\dfrac{\sqrt{3}}{\sqrt{2}}$.

8. $\sqrt{3}\cos\theta \pm \sin\theta = \pm\sqrt{2}$.

9. $\sqrt{3}\sin\theta \pm \cos\theta = \pm\sqrt{2}$.

10. $1 + \sin\theta = \sqrt{3}\cos\theta$.

11. $1 - \cos\theta = \sin\theta$.

12. $5\cos\theta - 2\sin\theta = \pm\sqrt{29}$.

13. $2\cos\theta \pm 5\sin\theta = \pm\sqrt{29}$.

14. $5.32\cos\theta - 2.32\sin\theta = 2.41$.

15. $2\cos\theta - 5\sin\theta = 2$.

16. $5\cos\theta - 2\sin\theta = 6$.

17. $2\cos 3\theta + 3\cos 3\theta = 3$.

18. $3\sin 4\theta \pm 4\cos 4\theta = 5$.

19. $3\sin 4\theta \pm 4\cos 4\theta = 4$.

20. $3\sin 4\theta \pm 4\cos 4\theta = 6$.

21. $5\sin 5\theta \pm 7\cos 5\theta = 3$.

22. $5\sin\theta + 2\cos\theta = 5$.

23. $6\cos\theta + 8\sin\theta = 9$.

24. By the aid of an auxiliary angle, show the changes in sign and magnitude of $\sin\theta + \cos\theta$, as θ goes from 0° to 360°.

Ans. From 1 to $+\sqrt{2}$ to 0 to $-\sqrt{2}$ to 0 to 1, as θ passes from 0° to 45° to 135° to 225° to 315° to 360°.

25. Show, similarly, the changes in sign and magnitude of:

(*a*) $\sin\theta - \cos\theta$. (*c*) $\sin\theta - \sqrt{3}\cos\theta$. (*e*) $5\cos\theta - 2\sin\theta$.
(*b*) $\sin\theta + \sqrt{3}\cos\theta$. (*d*) $\cos^2\theta - \sin^2\theta$. (*f*) $3\cos\theta + 4\sin\theta$.

26. Is there any limitation in the possibility of solving by the method of auxiliary angles?

§ 149. Use of the Addition-subtraction Formulas in calculating a Table of Sines and Cosines.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B. \tag{1}$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B. \tag{2}$$
$$\therefore \sin(A+B) = 2\sin A\cos B - \sin(A-B). \tag{3}$$
$$\therefore \sin(60° + A) = \sin A + \sin(60° - A). \tag{4}$$

Formulas (3), (4) may be used in calculating a table of sines; (3) being used for all angles, A, below 60°, and (4) to get from these the sines for all angles from 60° to 90°. When the sines are known, so are the cosines, since $\cos A = \sin(90° - A)$.

For example, suppose the table is to read to every 10″, let $B = 10''$ and $A = 10'', 20'', 30'', 40''$, etc., in turn. Then (3) gives:

$$\sin 20'' = 2\sin 10''\cos 10'',$$
$$\sin 30'' = 2\sin 20''\cos 10'' - \sin 10'',$$
$$\sin 40'' = 2\sin 30''\cos 10'' - \sin 20'',$$
$$\sin 50'' = 2\sin 40''\cos 10'' - \sin 30'',$$
etc., etc., etc.

The table may be started with any angle whose sine and cosine are known, when the sine and cosine of the angle-difference of the table are known. For instance, let $A = 30°$, $\sin A = 0.5000$, $\cos A = \frac{\sqrt{3}}{2} = 0.8660$, for a table reading to minutes. Let $B = 1'$. Then

$$\sin 30°\,1' = \sin 30°\cos 1' + \cos 30°\sin 1'.$$

Thus knowing sin 30° 1′, then by (3),

$$\sin 30° 2' = 2 \sin 30° 1' \cos 1' - \sin 30°,$$
$$\sin 30° 3' = 2 \sin 30° 2' \cos 1' - \sin 30° 1',$$
$$\sin 30° 4' = 2 \sin 30° 3' \cos 1' - \sin 30° 2',$$
etc., etc., etc.

Evidently from the foregoing it is necessary to calculate the sine and cosine of the angle-difference of the table, whether this be 10′, 1′, 10″, or 1″, the usual differences.

§ 150. Calculating the Sine and Cosine of the Angle-difference of Tables.

Let θ be any small angle in circular measure.

Then $\begin{cases} \sin\theta < \theta, \\ \cos\theta < 1. \end{cases}$ (*l*), p. 107

We now show $\begin{cases} \sin\theta > \theta - \dfrac{\theta^3}{4}, \\ \cos\theta > 1 - \dfrac{\theta^2}{2}, \end{cases}$

from which it will follow that when for the sine of a small angle its circular measure is taken, the *error is less than one-fourth the cube of the circular measure*, and when for the cosine of a small angle unity is taken, *the error is less than one-half the square of the circular measure.*

DAC (Fig. 145) is a portion of the unit circle. *CB*, *DB* are tangents to this circle.

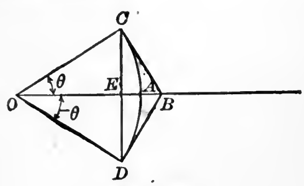


FIG. 145.

The broken line $DBC >$ arc DAC.

$\therefore$ line $BC >$ arc AC. (1)

The *number* for the arc AC is the same as that for the angle AOC.

$$\therefore \text{arc } AC = \theta.$$

Triangles OEC, OCB are similar.

$$\therefore \frac{CB}{OC} = \frac{EC}{OE}.$$

But $$OC = 1,\ EC = \sin\theta,\ OE = \cos\theta.$$

$$\therefore\ CB = \frac{\sin\theta}{\cos\theta}.$$

$\therefore$ by (1), $$\frac{\sin\theta}{\cos\theta} > \theta.$$

$$\therefore\ \sin\theta > \theta\cdot\cos\theta. \qquad (2)$$

That is, *the sine of any small positive angle is greater than the product of its circular measure and cosine.*

$$\therefore\ \sin\frac{\theta}{2} > \frac{\theta}{2}\cos\frac{\theta}{2}. \qquad (3)$$

But $$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

$$\therefore\ \sin\theta > 2\left(\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2},$$

or, $$\sin\theta > \theta\cos^2\frac{\theta}{2}. \qquad (4)$$

$$\therefore\ \sin\theta > \theta\left(1 - \sin^2\frac{\theta}{2}\right). \qquad (5)$$

But $$\frac{\theta}{2} > \sin\frac{\theta}{2}.$$

$$\therefore\ \frac{\theta^2}{4} > \sin^2\frac{\theta}{2}. \qquad (6)$$

By (5), (6), $$\sin\theta > \theta\left(1 - \frac{\theta^2}{4}\right),$$

or, $$\sin\theta > \theta - \frac{\theta^3}{4}.$$

Also $$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}.$$

$$\therefore\ \cos\theta > 1 - 2\left(\frac{\theta^2}{4}\right),$$

or, $$\cos\theta > 1 - \frac{\theta^2}{2}.$$

The circular measure of $10'$

$$= \frac{\pi}{180 \times 6} \text{ radians}$$

$$= 0.002{,}908{,}9, \text{ about.}$$

$$\therefore \sin 10' = 0.002{,}908{,}9,$$

with an error less than one-fourth the cube of this number,

$$\text{or} \quad < \frac{(0.003)^3}{4},$$

$$\text{or} \quad < 0.000{,}000{,}007, \text{ about,}$$

or less than 1 in the eighth decimal place.

Thus, in constructing any place table, up to and including a seven-place table of sines and cosines, we may take sin $10'$ as equal to the circular measure of $10'$ for seven places without appreciable error.

Similarly, $\sin 1' = 0.000{,}290{,}89,$

or, approximately, $\sin 1' = 0.0003$, or, 3 *zeros and* 3.

Similarly, $\sin 1'' = 0.000{,}004{,}848{,}136{,}811,$

or, approximately, $\sin 1'' = 0.000{,}005$, or, 5 *zeros and* 5.

The error in the above value of $\sin 1'$ is less than 0.000,000,000,007, and the error in the value of $\sin 1''$ is less than 0.000,000,000,000,000,03.

From the formulas $\begin{cases} \sin\theta < \theta, \\ \sin\theta > \theta - \dfrac{\theta^3}{4}, \end{cases}$

$$\sin 1'' < 0.000{,}004{,}848{,}136{,}811;$$

$$\sin 1'' > 0.000{,}004{,}848{,}136{,}807.$$

$\therefore$ these agree to 13 figures.

$$\therefore \sin 1'' = 0.000{,}004{,}848{,}136{,}8,$$

which is correct to 13 figures.

$$\cos\theta = \sqrt{1 - \sin^2\theta} = (1 - \sin^2\theta)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\sin^2\theta,$$

approximately (§ 42).

$$\therefore \cos 1'' = 0.999{,}999{,}999{,}988.$$

Since $\cos\theta < 1$ and $\cos\theta > 1 - \frac{\theta^2}{2}$, we may set $\cos 1'' = 1$, with an error less than $\frac{1}{2}(0.000{,}004{,}848)^2$,

$$\text{or} \quad < \tfrac{1}{2}(0.000{,}005)^2,$$

$$\text{or} \quad < 0.000{,}000{,}000{,}013.$$

Thus the first ten figures of $\cos 1''$ are 9's.

Similarly, $\cos 1' = 1$, with an error less than $\frac{1}{2}(0.0003)^2$,

$$\text{or} \quad < 0.000{,}000{,}05,$$

and when we set $\cos 10' = 1$, the error is less than $\frac{1}{2}(0.003)^2$,

$$\text{or} \quad < 0.000{,}005.$$

Thus in constructing a four-place or five-place table we may assume $\cos 10'$, or the cosine of any angles less than $10'$, as 1.

Similarly, for a four-place, five-place, six-place, seven-place table, we may set $\cos 1' = 1$, as also for any smaller angle. And for any table of less than eleven places, we may set $\cos 1'' = 1$.

Thus in making a four-place table reading to every $10'$, we may take

$$\sin 20' = 2 \sin 10';$$
$$\sin 30' = 2 \sin 20' - \sin 10';$$
$$\sin 40' = 2 \sin 30' - \sin 20';$$
$$\sin 50' = 2 \sin 40' - \sin 30'.$$
$$\text{etc., etc.}$$

Thus also any table for which we may assume $\cos 1'' = 1$ (*i.e.* for any place table below an eleven-place table, as above).

$$\sin 2'' = 2 \sin 1'';$$
$$\sin 3'' = 2 \sin 2'' - \sin 1'' = 3 \sin 1'';$$
$$\sin 4'' = 2 \sin 3'' - \sin 2'' = 4 \sin 1'';$$
$$\sin 5'' = 2 \sin 4'' - \sin 3'' = 5 \sin 1''.$$

In general, $\sin n'' = n \sin 1''$, when n is small.

This formula is but another expression of the fact that when an angle is small its sine may be taken as equal to its circular measure.

For $\sin n'' =$ circular measure of n''

$= n$ times circular measure of $1''$

$= n$ times sin of $1''$.

The error is less than $\frac{1}{4}$ (circ. meas.)^3. Thus, for a four-place table, if this error is not to affect the table, it must be less than 0.00005.

Thus, to get the limit for n, so that

$$\sin n'' = n \sin 1''$$

for a four-place table,

$$\tfrac{1}{4}(n \text{ circ. meas. } 1'')^3 < 0.00005;$$

$$n < \frac{\sqrt[3]{0.0002}}{0.000005}.$$

$$\therefore\ n < \text{about } 12{,}000.$$

Therefore the formula

$$\sin n'' = n \sin 1''$$

will hold up to about $n'' = 3°$ in a four-place table.

EXERCISE.

Observe this in some four-place table, making some tests like

$$\sin 20' = 2 \sin 10';$$
$$\sin 40' = 2 \sin 20';$$
$$\sin 60' = 2 \sin 30';$$
$$\sin 3° = 2 \sin 1° 30', \text{ etc.}$$

(Compare (o), p. 110).

§ 151. Useful Practical Method for Determining the Number of Seconds in a Small Angle of Given Circular Measure.

Let $\theta =$ circular measure of n''.

$$\therefore\ \theta = \sin n'' = n \sin 1''.$$

$$\therefore\ n = \frac{\theta}{\sin 1''} = \frac{\theta}{0.000{,}005},$$

which, as already pointed out, will hold for four places up to 3°, and determine n from θ, or θ from n.

EXERCISE.

Determine such limits for five-place, six-place, seven-place tables, and find the seconds in selected small angles in circular measure, and *vice versa*.

§ 152. Basis for the Rule of Proportional Parts in a Table of Sines and Cosines.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B;$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

If $\beta =$ the circular measure of B, then, for as many places as we may assume

$$\cos B = 1 \text{ and } \sin B = \beta, \qquad (§ 150)$$

is $$\sin(A+B) = \sin A + \beta \cdot \cos A;$$

and $$\cos(A+B) = \cos A - \beta \cdot \sin A.$$

$$\therefore \sin(A+B) - \sin A = \beta \cdot \cos A. \qquad (1)$$

$$\cos(A+B) - \cos A = -\beta \cdot \sin A. \qquad (2)$$

Formula (1) shows that, within the limitations noted above, the difference in the sines of two angles is proportional to the difference in the angles.

Formula (2) shows the same thing for cosines.

Let $$\beta = n\alpha.$$

Then $$\sin(A+B) - \sin A = n\alpha \cos A.$$

Also $$\sin\left(A + \frac{B}{n}\right) - \sin A = \alpha \cos A.$$

$$\therefore \sin(A+B) - \sin A = n\left(\sin\left(A + \frac{B}{n}\right) - \sin A\right).$$

That is, *the sine-difference for the angle-difference $n\alpha$ is n times that for α*, which is the rule for proportional parts.

EXERCISES.

1. Show that, in any table of sines, the number under "D" opposite any angle is the cosine of that angle times the circular measure of the angle-difference of the table. Explain columns "P. P." of tables.

2. Show that in any table of cosines the number under "D" opposite any angle is the sine of that angle times the circular measure of the angle-difference of the table.

§ 153. Testing the Accuracy of Calculated Tables of Sines and Cosines.

The method of § 149 for calculating a table of sines is based upon additions and subtractions of results already found. If an error occurs at any part of the table, it affects all the following part of the table.

The following tests of accuracy may be applied:

(*a*) $$\sin A = \tfrac{1}{2}\{\sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}\},$$

$$\cos A = \tfrac{1}{2}\{\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}\},$$

A being any angle less than 45°. Test (*a*) is tedious.

(*b*) $\sin(36° + A) - \sin(36° - A) + \sin(72° - A) - \sin(72° + A) = \sin A.$

For $$\sin(36° + A) - \sin(36° - A) = 2 \sin A \cos 36°,$$

$$\sin(72° - A) - \sin(72° + A) = -2 \sin A \cos 72°.$$

$$\therefore \text{ sum} = 2 \sin A(\cos 36° - \cos 72°),$$

$$= 2 \sin A\left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right) = \sin A.$$

Test (*b*) is Euler's test.

(*c*) $\sin(54° + A) + \sin(54° - A) - \sin(18° + A) - \sin(18° - A) = \cos A.$

This is known as Legendre's test. It is identical with Euler's test, from which it is obtained by writing $90° - A$ for A.

The student interested in practical processes which have been followed in constructing Logarithmic Tables may consult Glaisher's article, "Logarithms," in the *Encyclopædia Britannica.*

EXERCISE.

Test a few sines (cosines) of some table by these formulas.

§ 154. The Limiting Value of $\frac{\sin\theta}{\theta}$ and of $\cos\theta$ as θ approaches the Value Zero.

In § 150 it has been shown that

$$\sin\theta < \theta,$$

and

$$\sin\theta > \theta - \frac{\theta^3}{4}.$$

$$\therefore \frac{\sin\theta}{\theta} < 1 \text{ and } \frac{\sin\theta}{\theta} > 1 - \frac{\theta^2}{4}.$$

Thus $\frac{\sin\theta}{\theta}$ can be made to approach as near as we please to the limiting value 1, as θ is made smaller and smaller. This is expressed in the form

$$\frac{\sin\theta}{\theta} \doteq 1, \text{ when } \theta \doteq 0.$$

Similarly, since

$$\cos\theta < 1,$$

and

$$\cos\theta > 1 - \frac{\theta^2}{2}, \qquad (§ 150)$$

$$\cos\theta \doteq 1, \text{ when } \theta \doteq 0.$$

§ 155. Differentiating the Sine and Cosine.

If $F(x)$ denotes any function of x, and if x is given the value $x + \Delta x$, and the difference $F(x+\Delta x) - F(x)$ is formed and divided by Δx, then the limit toward which the quotient $\frac{F(x+\Delta x) - F(x)}{\Delta x}$ approaches, *when it approaches a definite limit as Δx approaches zero*, is called the differential coefficient of $F(x)$ with respect to x, also, the first derived function of $F(x)$ with respect to x, and is indicated by $F'(x)$. The process here pointed out to which $F(x)$ is subjected is called *differentiating $F(x)$ with respect to x.*

Treating the first derived function as was the original function is called getting *the second differential coefficient*, also getting *the second derived function;* and so on. The second derived function is indicated by $F''(x)$, etc.

For example,

(1) If $F(x) = x^2,$

$$F(x+\Delta x) = (x+\Delta x)^2 = x^2 + 2x \cdot \Delta x + \overline{\Delta x}^2.$$

$$\therefore \frac{F(x+\Delta x) - F(x)}{\Delta x} = 2x + \Delta x.$$

$$\therefore F'(x) = 2x.$$

Similarly, $F''(x) = 2$ and $F'''(x) = 0.$

(2) If $F(x) = \sin x,$

$$F(x+\Delta x) = \sin(x+\Delta x).$$

$$\therefore \frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$= \frac{2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x}. \qquad (\S\ 145)$$

This can be set in the form

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cos\left(x + \frac{\Delta x}{2}\right).$$

By the preceding section, $\dfrac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \doteq 1$, when $\Delta x \doteq 0.$

$\therefore F'(x) = \cos x.$

That is, *the differential coefficient of* $\sin x$, *with respect to* x, *is* $\cos x$, x *being in circular measure.*

(3) If $F(x) = \cos x,$

$$F(x+\Delta x) = \cos(x+\Delta x).$$

$$\therefore \frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

$$= \frac{-2 \sin \Delta x \sin\left(x + \frac{\Delta x}{2}\right)}{\Delta x} = -\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \sin\left(x + \frac{\Delta x}{2}\right).$$

$$\therefore F'(x) = -\sin(x).$$

That is, *the differential coefficient of* $\cos x$ *with respect to* x *is* $-\sin x$, x *being in circular measure.*

(4) We thus have:

When $F(x) = \sin x$,

$$F'(x) = \cos x,$$
$$F''(x) = -\sin x,$$
$$F'''(x) = -\cos x,$$
$$F^{iv}(x) = \sin x,$$

and then a repeat without end.

When $F(x) = \cos x$,

$$F'(x) = -\sin x,$$
$$F''(x) = -\cos x,$$
$$F'''(x) = \sin x,$$
$$F^{iv}(x) = \cos x,$$

and then a repeat without end.

(5) If, now, we denote by $F(0), F'(0), F''(0)$, etc., the special values which these functions assume when $x = 0$, we have

When $F(x) = \sin x$,

$$F(0) = 0,$$
$$F'(0) = 1,$$
$$F''(0) = 0,$$
$$F'''(0) = -1,$$
$$F^{iv}(0) = 0,$$

and a repeat without end.

When $F(x) = \cos x$,

$$F(0) = 1,$$
$$F'(0) = 0,$$
$$F''(0) = -1,$$
$$F'''(0) = 0,$$
$$F^{iv}(0) = 1,$$

and a repeat without end.

(6) If $\quad F(x) = x^n,$

where n is a positive integer,

$$F(x + \Delta x) = (x + \Delta x)^n$$
$$= x^n + nx^{n-1}\Delta x + \frac{n \cdot n - 1}{2} x^{n-2} \Delta x^{-2} + \text{etc.},$$

assuming the binomial theorem for positive integral indices.

$$\therefore \frac{F(x + \Delta x) - F(x)}{\Delta x} = nx^{n-1} + \text{powers of } \Delta x.$$

$$\therefore F'(x) = nx^{n-1}.$$

$$\therefore \text{if } F(x) = x^3, \ F'(x) = 3x^2,$$

and if $\quad F(x) = ax^4, \ F'(x) = 4ax^3$, etc.

§ 156. The Sine-series and Cosine-series.

Assuming that $\sin\theta$ can be expressed in powers of θ, let

$$F(\theta) = \sin\theta = a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5 + \text{etc.}$$

$$\therefore\ F'(\theta) = \cos\theta = b + 2\,c\theta + 3\,d\theta^2 + 4\,e\theta^3 + 5f\theta^4 + \text{etc.},$$

$$F''(\theta) = -\sin\theta = 2\,c + 2\cdot 3\cdot d\theta + 4\cdot 3\,e\theta^2 + 5\cdot 4f\theta^3 + \text{etc.},$$

$$F'''(\theta) = -\cos\theta = 2\cdot 3\cdot d + 4\cdot 3\cdot 2\,e\theta + 5\cdot 4\cdot 3f\theta^2 + \text{etc.},$$

$$F^{iv}(\theta) = \sin\theta = 4\cdot 3\cdot 2\cdot e + 5\cdot 4\cdot 3\cdot 2f\theta + \text{etc.},$$

$$F^{v}(\theta) = \cos\theta = 5\cdot 4\cdot 3\cdot 2f + 6\cdot 5\cdot 4\cdot 3\cdot 2\,g\theta + \text{etc.}$$

etc.

Assuming that these relations hold for all values of θ,

let $\theta = 0.$

$$\therefore\ a = \sin(0) = 0,$$

$$b = \cos(0) = 1,$$

$$2\,c = -\sin(0) = 0. \qquad \therefore\ c = 0.$$

$$2\cdot 3\,d = -\cos(0) = -1. \qquad \therefore\ d = -\frac{1}{\underline{|3}}.$$

$$4\cdot 3\cdot 2\,e = \sin(0) = 0. \qquad \therefore\ e = 0.$$

$$5\cdot 4\cdot 3\cdot 2f = \cos(0) = 1. \qquad \therefore\ f = \frac{1}{\underline{|5}}.$$

etc.

$\therefore$ Setting those values of a, b, c, etc., in the selected series,

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \cdots, \text{ to infinity.}$$

Similarly,

$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \cdots, \text{ to infinity.}$$

These are the series heretofore given as expressing what is really meant by the sine and cosine as related numerically to a given angle θ, θ being in circular measure.

§ 157. The Sine-series and Cosine-series are Convergent.

A series is said to be convergent when the larger and larger the number of terms summed, the nearer and nearer some definite finite quantity is approached.

The sum of the series,

$$1 - 1 + 1 - 1 + 1 - 1 + 1, \text{ etc. } \cdots$$

is not infinite. It is either $+1$, or 0, according to the number of terms taken. Such a series is not convergent under the definition. Many algebras give a wrong impression as to convergency of series, restricting divergent series to those whose sum is infinite.

If the terms of a series are real, and if from some term within a finite number of terms from the beginning-term they diminish numerically, with alternating signs, the series is convergent, *if the order of its terms is not disturbed.*

Let the rth term be a positive term beyond which the terms alternate in sign and diminish. Call this term A_r, and the sum of the preceding terms, S_r. The series is

$$S = S_r + (A_r - A_{r+1}) + (A_{r+2} - A_{r+3}) + (\quad)\cdots$$

or $$S = S_r + A_r - (A_{r+1} - A_{r+2}) - (A_{r+3} - A_{r+4}) - (\quad)\cdots.$$

Since the bracketed terms are all plus in the first arrangement and all minus in the second, S is more than S_r and less than $S_r + A_r$. Thus, the true sum of the series differs from S_r by less than A_r. Evidently, then, by increasing r, the nearer is a definite sum reached. The quantities S_r, S_{r+1}, S_{r+2}, etc., have thus the limit S, which is the sum of the series.

That the order of the terms of such a series must not be disturbed *when the convergency depends on the signs*, was first pointed out by Riemann. For we may make such a series approach any sum whatever if we change the order of terms appropriately. We may add up positive terms until a sum larger than the selected sum is reached. Then take negative terms until a sum less than the selected sum is reached. Then again add positive terms until the selected sum is passed. Then attach negative terms, to bring the sum again below the selected sum, and so on. Evidently, the more and more terms we take *in this manner*, the nearer we approach the selected sum.

In the sine-series the ratio of the $(r+1)$th term to the rth is $-\theta^2/2r(2r+1)$, which is numerically less than 1 so soon as $r>\theta$, or $r>\frac{1}{2}\sqrt{\theta^2+\frac{1}{4}}-\frac{1}{4}$. Thus the sine-series is convergent. So, likewise, the cosine-series. Both of these series remain convergent if all their terms are taken positive, for each would then be a part of the series for e^θ (§ 8), a convergent series. A part of a convergent series, all of whose terms are positive, must be convergent. Thus the convergency of the sine-series and cosine-series is not dependent on the *signs of the terms*, and is consequently independent of the order in which the terms are taken. These series are what is termed *absolutely convergent*. The case is very different with the series for $\log_e(1+z)$, § 9, where the convergency is dependent on the signs, and that series converges to the logarithm only when the order of terms is held as given in § 9. For information more in detail on series, consult Osgood's "Infinite Series."

§ 158. Sine and Cosine Relations for the Angles of a Triangle.

$$A+B+C=180°. \tag{1}$$

$$\therefore\ A+B=180°-C. \qquad \therefore\ \frac{A+B}{2}=90°-\frac{C}{2}.$$

$$\therefore\ \sin(A+B)=\sin C; \qquad \cos(A+B)=-\cos C;$$

$$\sin\frac{A+B}{2}=\cos C; \qquad \cos\frac{A+B}{2}=\sin C;$$

with similar relations for any pair of angles.

From these relations, together with the addition-multiplication, etc., relations, can be deduced many others.

(1) To prove

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

$$\sin 2A + \sin 2B + \sin 2C = \overline{\sin 2A + \sin 2B} + \sin 2C$$

$$= 2 \sin (A + B) \cos (A - B) + 2 \sin C \cos C$$

$$= 2 \sin C \cos (A - B) + 2 \sin C \{- \cos (A + B)\}$$

$$= 2 \sin C \{\cos (A - B) - \cos (A + B)\}$$

$$= 2 \sin C (2 \sin A \sin B)$$

$$= 4 \sin A \sin B \sin C.$$

(2) To prove

$$\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

$$\cos A + \cos B - \cos C = \cos A + \overline{\cos B - \cos C}$$

$$= 2 \cos^2 \frac{A}{2} - 1 + 2 \sin \frac{C - B}{2} \sin \frac{C + B}{2}$$

$$= 2 \cos \frac{A}{2} \sin \left(\frac{C + B}{2}\right) - 1 + 2 \sin \frac{C - B}{2} \cos \frac{A}{2}$$

$$= 2 \cos \frac{A}{2} \left(\sin \frac{C + B}{2} + \sin \frac{C - B}{2}\right) - 1$$

$$= 2 \cos \frac{A}{2} \left(2 \sin \frac{C}{2} \cos \frac{B}{2}\right) - 1$$

$$= -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

EXERCISES.

$(A + B + C = 180°.)$

1. $\sin^2 A + \sin^2 B + \sin^2 C = 2 (1 + \cos A \cos B \cos C)$.

2. $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$.

3. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$.

4. $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$.

5. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

6. $\sin A + \sin B - \sin C = 4 \sin\frac{A}{2} \sin\frac{B}{2} \cos\frac{C}{2}$.

7. $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

8. $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$.

9. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$.

10. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$.

11. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

12. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2 \cos\frac{A}{2} \cos\frac{B}{2} \sin\frac{C}{2}$.

13. $\sin(A + 2B) + \sin(B + 2C) + \sin(C + 2A)$

$$= 4 \sin\frac{A - B}{2} \sin\frac{B - C}{2} \sin\frac{C - A}{2}.$$

14. $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$

$$= 1 + 4 \sin\frac{180^\circ - A}{4} \sin\frac{180^\circ - B}{4} \sin\frac{180^\circ - C}{4}.$$

15. $\sin(A + B - C) + \sin(B + C - A) + \sin(C + A - B)$

$$= 4 \sin A \sin B \sin C.$$

§ 159. Relations connecting the Sides of a Triangle with the Sines and Cosines of its Angles.

We have shown that in any triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}; \qquad \text{(i)}$$

$a = b \cos C + c \cos B$, (ii) with similar equations for b, c;

$a^2 = b^2 + c^2 - 2bc \cos A$, (iii) with similar equations for b, c;

$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, (iv) with similar equations for $\frac{B}{2}, \frac{C}{2}$;

$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, (v) with similar equations for $\frac{B}{2}, \frac{C}{2}$;

$\sin A = 2 \sin\frac{A}{2} \cos\frac{A}{2}$, (vi) with similar equations for B, C.

From these relations, and the treatment of the angles as in the preceding section, follow many other relations. For example, from (i),

$$\frac{a}{b}=\frac{\sin A}{\sin B}.$$

$$\therefore\ \frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}.$$

$$\therefore\ (a+b)\sin\frac{A-B}{2}\cos\frac{A+B}{2}=(a-b)\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

EXERCISES.

1. $(b+c)\sin\frac{A}{2}=a\cos\frac{B-C}{2}.$

2. $\frac{a+b+c}{c}=\frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}.$

3. $(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}.$

4. $a(\cos B+\cos C)=2(b+c)\sin^2\frac{A}{2}.$

5. $a(\cos B-\cos C)=2(c-b)\cos^2\frac{A}{2}.$

6. $a^2+b^2+c^2=2(ab\cos C+bc\cos A+ca\cos B).$

7. $c^2=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}.$

8. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$

9. $\frac{a}{b^2-c^2}\cdot\sin(B-C)=\frac{b}{c^2-a^2}\cdot\sin(C-A)=\frac{c}{a^2-b^2}\cdot\sin(A-B).$

10. $a\sin\frac{A}{2}\sin\frac{B-C}{2}+b\sin\frac{B}{2}\sin\frac{C-A}{2}+c\sin\frac{C}{2}\sin\frac{A-B}{2}=0.$

11. $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)+c^2(\cos^2 A-\cos^2 B)=0.$

12. $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0.$

13. $a+b+c=(a+b)\cos C+(b+c)\cos A+(c+a)\cos B.$

14. $\sin(B-C)=\frac{b^2-c^2}{a^2}\sin A.$

CHAPTER VIII.

THE TANGENT AND RECIPROCAL TANGENT (COTANGENT) OF ANGLES.

§ 160. The Tangent of an Angle *is the ratio of the ordinate of any point on its terminal to the abscissa of the same point.*

In Fig. 146 the tangent of any angle whose terminal is OP is $\frac{MP}{OM}$.

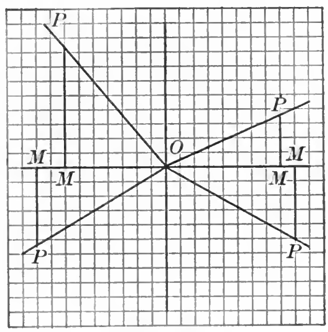


FIG. 146.

LABORATORY EXERCISES.

1. Lay out with the protractor a given angle. Measure five pairs of ordinates and abscissas for this angle, one abscissa being a unit (inch or foot). Divide to as many figures as the plan of measurement will justify and compare results with each other and with the table-tangent for the same angle.

2. Lay out five different angles. Measure for each an ordinate and abscissa. Divide. Compare with the table-tangents.

3. Lay out the following points on coördinate paper. Calculate the tangents to one or two decimal places. Measure the angles with the protractor. Compare with the table-tangents for the same angles: $(2, 3)$; $(-2, -3)$; $(2, -3)$; $(-2, 3)$; $(4, 3)$; $(.5, -4)$; $(-6, 7)$; $(3, -4)$; $(12, -5.2)$; $(-15, -8.3)$; $(24, 7.3)$; $(-21, 20)$; $(35, -12)$.

§ 161. The Sign of the Tangent.

When the ordinate and abscissa have the same sign, the tangent is positive; otherwise it is negative. Thus the tangents of all angles whose terminals are in the first or third quadrant are positive; those for all angles of the second or fourth quadrant are negative.

EXERCISE.

Determine the signs of the tangents of the angles in Exs. 1, 2 under the sine, § 61.

§ 162. Angles with the Same Tangent (Periodicity).

(1) All angles with the same terminal have the same tangent.

$$\therefore \tan(2n \cdot 180° + A°) = \tan A°,$$
$$\tan(2n \cdot \pi + \theta) = \tan \theta.$$

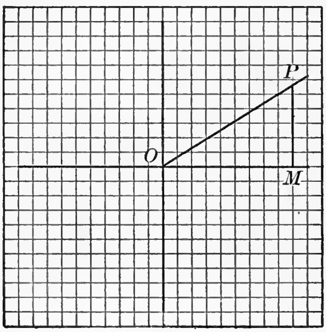


Fig. 147.

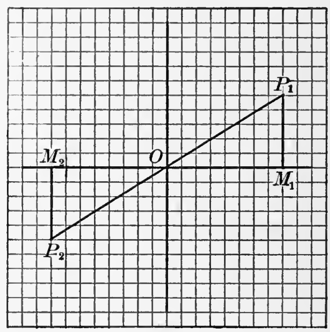


Fig. 148.

(2) All angles with opposite terminals have the same tangent.

For then when the moduli are equal, oppositely equal are the abscissas, and so also the ordinates. If $OP_1 = -OP_2$,

$$\frac{M_1P_1}{OM_1} = \frac{-M_1P_1}{-OM_1} = \frac{M_2P_2}{OM_2}.$$

If any terminal is located by the angle A, it is also located by the angle $2n \cdot 180° + A$, while the opposite terminal is located by $(2n+1)180° + A$. Thus either an even or an odd number of multiples of 180° may be added, positively or negatively, to an angle without altering its tangent;

or, $\tan(n \cdot 180° + A°) = \tan A°$, $\tan(n\pi + \theta) = \tan\theta$,

where n is any positive or negative integer.

Thus the tangent is a periodic function with the period π. What is the period for the sine, cosine, secant, cosecant?

EXERCISES.

1. Determine ten angles on diagrams which have the same tangent as 30°, five of them having the same terminal and five the opposite terminal.

2. Do the same for $-60°$. Solve $\sin 6\theta = \tan 3\theta$.

3. Do the same for $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. Solve $\cos 3\theta = \tan 3\theta$.

4. Find the angles which satisfy the relation, $\tan 9\theta = \tan 8\theta$.

§ 163. Angles with Opposite Tangents.

(1) Terminals symmetric to the horizontal give, for equal moduli, oppositely equal ordinates, while terminals symmetric to the vertical give, for equal moduli, oppositely equal abscissas. In the first case the abscissa is unchanged, and in the second case the ordinate is unchanged.

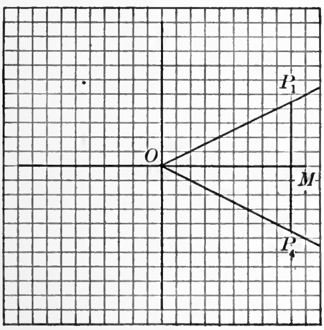


FIG. 149.

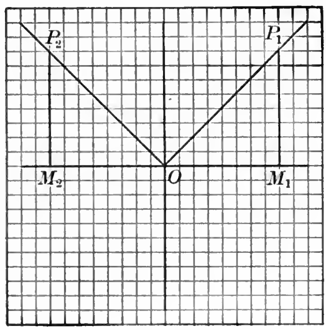


FIG. 150.

Or,
$$\frac{MP_1}{OM} = -\frac{-MP_1}{OM} = -\frac{MP_4}{OM},$$

and
$$\frac{M_1P_1}{OM_1} = -\frac{M_1P_1}{-OM_1} = -\frac{M_2P_2}{OM_2}.$$

Therefore,

(*a*) The tangent of a negative angle is the negative tangent of the corresponding positive angle.

(*b*) The tangent of the supplement of an angle is the negative of the tangent of the angle.

Or,

$$\tan A^\circ = -\tan(-A^\circ); \tag{1}$$

$$\tan\theta = -\tan(-\theta); \tag{2}$$

$$\tan A^\circ = -\tan(180^\circ - A^\circ); \tag{3}$$

$$\tan\theta = -\tan(\pi - \theta); \tag{4}$$

$$\tan A^\circ = -\tan(n\cdot 180^\circ - A^\circ); \tag{5}$$

$$\tan\theta = -\tan(n\cdot\pi - \theta); \tag{6}$$

$$\tan(n\cdot 180^\circ + A^\circ) = -\tan(m\cdot 180^\circ - A^\circ);$$

$$\tan(n\cdot\pi + \theta) = -\tan(m\cdot\pi - \theta).$$

EXERCISES.

1. Diagram five angles whose terminals are symmetric to the horizontal with the terminal for 30° and having tangent oppositely equal to that of 30°. Solve $\sin 5\theta = -\tan 10\theta$.

2. Diagram five angles having their terminals symmetric to the vertical with the terminal of 135° and with their tangent oppositely equal to that of 135°. Solve $\tan 6\theta = -\tan 8\theta$.

§ 164. Tangents of all Angles are First Quadrant Tangents.

It is left for the student to draw conclusions similar to those made for the sine in § 64, and report them. Give also rules for the terminal positions, similar to those in the same article.

EXERCISES.

Take those in § 64, replacing the word "sine" by "tangent."

§ 165. Construction of the Terminals for a Given Value of the Tangent.

Example. Draw the terminals when $\tan A$ is $\frac{2}{3}$.

Construct on coördinate paper the point (3, 2) and join it with the origin.

Draw also the terminal opposite to this line.

EXERCISES.

Construct the terminals corresponding to the following values of the tangent: $\frac{3}{4}$, $-\frac{3}{4}$, $\frac{5}{8}$, $-\frac{5}{8}$, .3, 3.7, 33.7, 0, ∞. Construct terminals for table tangents. Measure the angles and compare with table.

§ 166. Some Angles for which the Tangent is readily calculated without the Use of Tables.

These are the same as those for which the sines were calculated in § 75. It is left as an exercise for the student to establish the following results, using diagrams:

$$\tan 45° = 1; \qquad \tan 135° = -1;$$
$$\tan 30° = \frac{\sqrt{3}}{3}; \qquad \tan 330° = -\frac{\sqrt{3}}{3};$$
$$\tan 60° = \sqrt{3}; \qquad \tan 300° = -\sqrt{3};$$
$$\tan 120° = -\sqrt{3}; \qquad \tan 240° = \sqrt{3};$$
$$\tan 150° = -\frac{\sqrt{3}}{3}; \qquad \tan 210° = \frac{\sqrt{3}}{3};$$
$$\tan 225° = 1; \qquad \tan 315° = -1.$$

EXERCISES.

1. Give the general solution of the equations, $\tan^2 x = 1$; $\tan^2 x = \frac{1}{3}$; $\tan^2 x = 3$; $\tan^2 x = 0$; $\tan^2 x = \sin^2 x$; $\tan^2 x = 3\cos^2 x$; $3\tan^2 x = \sec^2 x$.

2. $\tan^2 x - \frac{3+\sqrt{3}}{3}\tan x + \frac{\sqrt{3}}{3} = 0.$

3. $\tan^2 x - \frac{3-\sqrt{3}}{3}\tan x - \frac{\sqrt{3}}{3} = 0.$

4. $\tan x - \frac{1}{\tan x} = 1 - \frac{\sqrt{3}}{3}.$

5. $\tan^2 x + 8\tan x + 7 = 0.$

§ 167. Value of the Tangent when the Terminal is on a Border Line of the Quadrants.

If the point P of the terminal OP is supposed to describe a circle of radius r about O, then as OP approaches the position of the initial line to the right, OM approaches the value r, while MP approaches the value zero. The value of $\frac{MP}{OM}$ thus approaches the value zero. This is expressed by saying $\tan 0° = 0$. Similarly, when OP approaches the initial line to the left of the origin, OM approaches the value $-r$ and MP approaches the value zero. The tangent thus approaches the value zero when the angle approaches 180°. Thus $\tan 180° = 0$.

When OP approaches the upright vertical from the right, MP approaches the value r and OM approaches zero from the positive side. Thus $\frac{MP}{OM}$ becomes *infinitely great, positively*. If, however, OP approaches the upright position from the left, MP approaches the value r while OM approaches the value zero *from the negative side*. Thus the tangent becomes *infinitely great, negatively*. It is thus impossible to determine the value of the tangent for upright positions of the terminal. It is customary to write

$$\tan 90° = + \infty.$$

And, similarly, $$\tan 270° = - \infty.$$

This is a short way of saying that if the terminal is upright, *almost*, with a slight tilt to the right, the tangent is a very large positive number, while if the terminal is downright, *almost*, with a slight passage beyond the position denoted by 270°, the tangent is a very large negative number. Thus, if we look on the terminal as moving, counterclockwise, about the origin, there is, when it passes the positions denoted by 90° and 270°, a sudden spring in the value of the tangent from plus infinity to minus infinity. Thus in the function $\tan x$ there is what is called a break in continuity of value when x is 90° or 270°, or, in general, when x is $(2n+1)90°$, or $(2n+1)\frac{\pi}{2}$, n being any positive or negative integer.

§ 168. Line Picture of the Tangent.

If in $\tan\theta = \frac{MP}{OM}$, OM is taken as unity, $\tan\theta$ is MP. Thus, on any scale on which OM is taken to represent 1, MP will represent the value of the tangent of all angles corresponding to the position OP of the terminal.

If we imagine the point P' to describe the unit-circle, and if through each position of P' on the unit-circle we imagine a line OP' prolonged to meet the tangent at M in the point P, we shall have for each such position of P' a line picture of

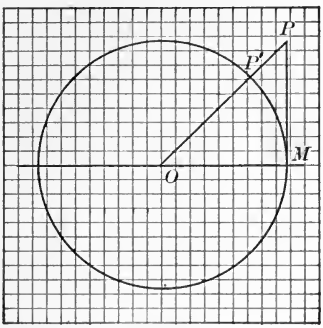


FIG. 151.

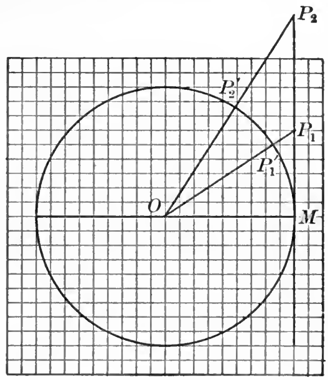


FIG. 152.

the corresponding tangent denoted by MP. Thus, tangent of angle MOP_1' is MP_1; tangent of angle MOP_2' is MP_2, and so on.

When OP' is coincident with OM, the angle is zero, and so is MP. As the angle increases from zero to 90°, the point P' describing the circle counter-clockwise, the point P ascends the line MP, and the tangent increases. When the terminal is almost upright, the point P is *high up* on the line, corresponding to

$$\tan 90° = +\infty,$$

in the preceding section. When OP' passes 90°, P passes from *high up* on the line to *low down* on the line, corresponding to the spring from plus infinity to minus infinity noted in the preceding section, when the angle passes the value 90°. When OP' moves from 90° to 180°, the motion of the corresponding point, P, on the line is from negative infinity on the line, *up to the point M.* As OP' goes from 180° to 270°, P goes on up the line from M to plus infinity. There is again the spring from plus infinity to minus infinity as

OP' passes through the position 270°. That is, as OP' passes through 270° the corresponding point P leaps from high up on the line to low down on the line. As OP' goes from 270° back to the starting point, P moves from minus infinity to zero on the line, that is, from low down on the line up to the point M.

The line-diagram teaches clearly the following propositions:

(*a*) $\tan\theta = -\tan(-\theta)$.

(*b*) $\tan(180° + A°) = \tan A°$.

(*c*) $\tan(180° - A°) = -\tan A°$.

(*d*) $\tan 0° = 0 = \tan 360°$.

(*e*) $\tan 180° = 0 = \tan(-180°)$.

(*f*) $\tan n \cdot 180° = 0$.

(*g*) $\tan 90° = +\infty$ (as already agreed on).

(*h*) $\tan 270° = -\infty$ (as already agreed on).

(*i*) $\tan(4n + 1)90° = \infty$ (as already agreed on).

(*j*) $\tan(4n + 3)90° = -\infty$ (as already agreed on).

(*k*) As the angle passes from 0° to 90°, or from 180° to 270°, the tangent is positive and an increasing function of the angle. As the angle passes from 90° to 180°, or from 270° to 360°, the tangent is negative, but an increasing function of the angle. The tangent is always an increasing function, except on the instantaneous leap from $+\infty$ to $-\infty$.

(*l*) The tangent, like the sine and cosine, is a periodic function of the angle, the period being 360°, or, in circular measure, 2π. It has also the period π, since $\tan\theta = \tan(\theta + n\pi)$, for all integral values of n.

(*m*) For all angles whose terminals are in the first or third quadrant, the tangent is positive; for all angles whose terminals are in the second or fourth quadrant, the tangent is negative.

LABORATORY EXERCISE.

Draw a diagram like the preceding, with the radius as one foot. Measure ten angles and tangents. Compare results with the table-tangents for the same angles. Make a graph for the tangent.

§ 169. Some Simple Calculations with the Tangent.

Since the tangent is the ratio of the ordinate to the abscissa, when any two of the three quantities, ordinate, abscissa, tangent, are given, the third can be calculated.

(*a*) Given ordinate and abscissa.
Tangent equals ordinate divided by abscissa.

(*b*) Given abscissa and tangent.
Ordinate equals abscissa multiplied by tangent.

(*c*) Given ordinate and tangent.
Abscissa equals ordinate divided by tangent.

These relations follow at once from the definition of the tangent:

$$\text{tangent} = \frac{\text{ordinate}}{\text{abscissa}}.$$

Make diagrams and test by measurement the following

EXERCISES.

(Tables not to be used where 30°, 60°, 45° occur.)

1. When the ordinate is 23 and the abscissa is 37, calculate the tangent to two decimal places.

2. When the tangent is 1.32 and the abscissa is 24.5, what is the ordinate to three significant figures?

3. When the tangent is 23.41 and the ordinate is 34.67, what is the abscissa to four significant figures?

(Use the shortened process of division and multiplication in Exs. 1, 2, and 3.)

4. A vertical stick a feet long casts a shadow a feet long, what is the sun's angular elevation? What for shadow lengths, $a\sqrt{3}$, $\frac{a\sqrt{3}}{3}$?

5. If a vertical flagstaff subtends an angle of 30° at a horizontal distance of 150 feet from its foot, show that its height is $50\sqrt{3}$. Calculate this to three significant figures.

6. Calculate in radicals and also to three significant figures, if the angle in Ex. 5 is 60°; 45°.

7. Show that tangents can be used to solve the following problem: A man wishing to determine the height of a church spire, found its angles of elevation from two points, distant apart 100 ft., in the same horizontal line with the base of the spire, to be 45° and 60°, from which he found the height of the spire to be $50(3+\sqrt{3})$. Calculate this to three significant figures. How far are the two points from the base of the tower?

8. From the top of a cliff, 200 ft. high, the angles of depression of the top and bottom of a tower are 30° and 60°; use tangents to calculate the height of the tower.

9. The angle of elevation of a tower due north from a certain point is 30°. At a distance 300 ft. due west from the first point the elevation is 60°. What is the height of the tower in radicals and to three significant figures?

10. What is the angular elevation of the sun when the shadow of a vertical stick is $\sqrt{5}$ times its height?

11. At a certain point the tangent of the angle of elevation of a tower is $\frac{5}{3}$, and at a point 32 ft. nearer the tower in a line directly toward the tower from the first point, the tangent of the angle of elevation of the tower is $\frac{5}{2}$. Calculate the height of the tower.

12. Calculate the height of a chimney, if it is found that on walking 100 ft. in a direct line toward the chimney the angle of its elevation changes from 30° to 45°.

13. Use tangents to calculate the distance apart of two objects which lie in a horizontal line, when it is given that the angles of depression of the two objects from a point 200 ft. directly above the line joining them are 45° and 30°.

14. If the shadow of a tower is 60 ft. longer when the elevation of the sun is 30° than when the elevation is 45°, show that the height of the tower is $30(1+\sqrt{3})$.

15. If a stick 27 ft. long casts a shadow 31 ft. long, what is the tangent of the sun's elevation?

16. To determine the distance between two objects, A and B, on opposite sides of a river, a line 100 ft. long from B to C is measured, BC being at right angles to BA. Calculate the lengths of AB, when the angle BCA is 30°, 45°, 60°. Also when this angle is one whose tangent is 1.34.

17. If the breadth of a house is 47 ft., what is the height of the ridgepole above the top of the wall for a plain roof when the angle of the roof is 30°? 45°? 60°? What when it is an angle whose tangent is 0.42?

18. Given that the ridgepole of a roof is 23 ft. above the line joining opposite walls of a house, what is the width of the house when the slope of the roof is 30°? 45°? 60°? What when the tangent of the angle of slope is 2.3?

19. Express the length of the side of a regular polygon of n sides in terms of the radius of the inscribed circle. Also the radius of the inscribed circle in terms of the side of the polygon. What are the corresponding expressions if the circumscribed circle is used instead of the inscribed circle?

20. From a certain point the angle of elevation of a kite is A. From another point due west of the first and a feet from it, the kite's elevation is B. If the first point is due south of the kite, express the height of the kite in terms of A, B, a.

21. Two lines, AB, AC, in the same horizontal plane and at right angles to each other, are observed from a point vertically above A. The angle subtended by AC is one whose tangent is 3; that subtended by AB has 2 for its tangent. If AC is 150, how long is AB?

22. Express the area of a regular polygon in terms of its side and half the angle subtended at its centre by a side. Also in terms of the radius of the inscribed circle. Also in terms of the radius of the circumscribed circle.

23. If a vertical tower subtends an angle A at a distance b feet from its foot, on the horizontal line through the foot, how high is the tower?

24. The summit of a spire is vertically over the middle point of a horizontal square enclosure whose side is of length a feet. The height of the spire is h feet above the level of the square. If the shadow of the spire just reaches a corner of the square when the sun has an altitude A, prove that

$$h\sqrt{2} = a \tan A.$$

25. Two stations due south of a tower, which leans toward the north, are at distances a, b from its foot. If A, B are the angles of elevation of the top of the tower from these stations, show that the inclination of the tower to the horizontal is one whose tangent is

$$\frac{(b-a)\tan A \tan B}{b \tan B - a \tan A}.$$

26. A pyramid of altitude h stands on an equilateral triangle as base; express the side of the base in terms of h and the angle which the faces of the pyramid make with the base. If the altitude is 100 ft. and the apex-angle is 60°, show that the side is $50\sqrt{6}$.

27. A flagstaff 6 ft. high stands on the top of a pyramid of square base. When the sun's angular altitude is A, the end of the shadow of the flagstaff just reaches the edge of the base of the pyramid, and at a point distant y, x from the ends of a side of the base. Show that the height of the pyramid is

$$\sqrt{\frac{x^2+y^2}{2}} \cdot \tan A - 6.$$

LABORATORY EXERCISE.

Pick out from among the preceding exercises a few that might possibly be of use in practical life, and make in the field the measurements called for.

EXERCISES.—CALCULATIONS WITH THE TANGENT, USING THE TABLES.

(Make diagrams to scale and test by measurement.)

NOTE.—The tangent of an angle is looked up in the tables in the same way as the sine, or else by § 170, to avoid subtraction of minutes and seconds.

Calculated results should show the same number of figures as data. With data to one, two, three, four figures, tangents are to be taken to one, two, three, four figures respectively.

1. If the abscissa and ordinate are 7, 9, calculate the tangent to one decimal place and determine the nearest angle.

2. If the abscissa and ordinate are 13, 17, calculate the tangent to two decimal places, and determine the nearest angle.

3. If the abscissa and ordinate are 31.4, 29.8, calculate the tangent to three figures, and determine the nearest angle.

4. If the abscissa and ordinate are 13.45, 23.61, calculate to four decimal places the tangent, and determine the nearest angle.

5. Solve Exs. 1, 2, 3, and 4 by using logarithms, and compare results with those just obtained by natural tangents.

6. If the abscissa is 3 and the angle 15°, calculate to one significant figure the ordinate. Is it best to use logarithms?

7. If the abscissa is 4.7 and the angle is 43°, calculate by tangent to two figures the ordinate to two figures. Is it best to use logarithms?

8. If the abscissa is 37.2 and the angle is 17° 15′, use the tangent to three figures to calculate the ordinate to three figures. Work the same also by logarithms.

9. If the abscissa is 43.27 and the angle is 34° 17′, calculate the ordinate to four figures, by logarithms and by natural tangents, and compare results.

10. If the ordinate is 7 and the angle 55°, calculate the abscissa to a single figure by the best plan.

11. If the ordinate is 73 and the angle is 34° 30′, calculate to two places, by the best method, the corresponding abscissa.

12. If the ordinate is 65.3 and the angle is 54° 15′, calculate by logarithms and by natural tangents the abscissa to three figures.

13. If the ordinate is 43.76 and the angle is 34° 23′, calculate to four figures, by logarithms and by natural tangents, the abscissa.

14. If the abscissa is − 23 and the angle 137°, calculate the ordinate to two figures.

15. If the ordinate is − 23.4 and the angle 200° 15′, calculate the abscissa to three figures.

16. If the abscissa is 34.53 and the angle is 1715° 13′, calculate to four figures the ordinate.

17. If the ordinate is 5.23 and the angle 760° 15′, calculate the abscissa to three figures.

18. If the abscissa is − 4321 and the angle is − 205° 23′, calculate the ordinate to four figures.

19. If the abscissa and ordinate are − 23.5, 54.2, calculate the principal angle to the nearest five minutes, and give the general formula for the angles.

20. If the abscissa and ordinate are − 54.32, − 67.54, calculate the principal angle to the nearest minute, and the general angle.

21. If the abscissa and ordinate are 23, − 34, calculate the principal angle to the nearest half-degree, and the general angle.

22. If the abscissa and ordinate are 7, − 9, calculate the principal angle to the nearest five degrees, and the general angle.

23. Calculate the area of the regular polygon of eighteen sides, if the length of a side is 34.51.

24. A rectangular field has its sides 42.14, 67.53. What angle does its diagonal make with the shorter side? What with the longer side?

25. A vertical pole subtends an angle of 43° 15′ from a point 34.5 ft. distant in a horizontal line from its foot. Calculate to three figures the height of the pole.

§ 170. The Cotangent.

The cotangent of an angle is the reciprocal of its tangent.

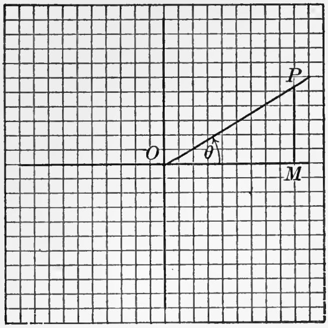


FIG. 153.

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{abscissa}}{\text{ordinate}}.$$

$$\therefore \cot\theta = \frac{OM}{MP}.$$

By § 118, or directly from a diagram, it is clear that the tangent of angle is the cotangent of its complement, or,

$$\tan A = \cot(90^\circ - A);$$
$$\cot A = \tan(90^\circ - A);$$

By § 162,

$$\cot(180^\circ + A) = \cot A; \quad \cot(n\cdot 180^\circ + A) = \cot A,$$

By § 163,

$$\cot(180^\circ - A) = -\cot A; \quad \cot(n\cdot 180^\circ - A) = -\cot A,$$

By § 108, $\tan B = -\cot(B - 90^\circ)$; $\cot B = -\tan(B - 90^\circ)$,
$\tan C = \tan(C - 180^\circ)$; $\cot C = \cot(C - 180^\circ)$,
$\tan D = -\cot(D - 270^\circ)$; $\cot D = -\tan(D - 270^\circ)$.

EXERCISE.

1. Show that the tangent of any angle of any size can be expressed as the tangent or else the cotangent of an angle less than 45°. Use the last six formulas for practice in taking tans and cotans and their logs from the tables, taking *B*, *C*, *D* in the second, third, fourth quadrant, respectively.

§ 171. Calculations with the Cotangent.

By the definition, division by the tangent is multiplication by the cotangent, and *vice versa*.

(1) Cotangent equals abscissa divided by ordinate.

(2) Abscissa equals ordinate multiplied by cotangent.

(3) Ordinate equals abscissa divided by cotangent.

EXERCISES.

Use the set on pages 308, 309, replacing the word "tangent" by "cotangent." Cotangents are looked up in the tables by the last six formulas of § 170.

THEORETIC EXAMPLES ON THE COTANGENT.

1. ABM is a straight line at right angles to the straight line MP. If AB is of length a, express MP, BM in terms of a and the cotangents of angles MBP, MAP.

2. Make up a problem (numerical) corresponding to the preceding, involving a church steeple on one side of a river, and two points, at a known distance apart, on the other side of the river, and calculate the height of the steeple and width of the river. Make a diagram to scale, and test by measurement.

3. ABC is a horizontal triangle, right angled at C. AP is a vertical line. CB is of length a. Angles ACP, ABP are ϕ, θ. Express AP in terms of a, ϕ, θ.

4. Make up a kite problem (numerical) corresponding to Ex. 3, and solve it, with a diagram to scale. Test by measurement.

5. Express the radius of the circle inscribed in a regular polygon of n sides in terms of the side of the polygon and half the central angle in circular measure.

6. If the side of the polygon in Ex. 5 is 34.52 ft., calculate to four figures the radius of the inscribed circle when the number of sides is twenty. Test with a diagram to scale.

7. Express the area of a regular polygon of n sides in terms of its side and half the central angle in circular measure.

8. Calculate the area of the regular polygon of eighteen sides, if each side is 4.32 ft.

9. The shadows of two vertical walls, which are at right angles to each other, and are a_1, a_2 feet in height, are observed, when the sun is due south, to be b_1, b_2 feet broad. Show that if α be the sun's altitude above the horizon, and β the inclination of the first wall to the meridian,

$$\cot^2\alpha = \frac{b_1^2}{a_1^2} + \frac{b_2^2}{a_2^2}, \text{ and } \cot\beta = \frac{a_1 b_2}{a_2 b_1}.$$

10. Apply Ex. 9 to a numerical example. Can logarithms be used?

11. Two stations due south of a leaning tower (leaning toward the north) are at distances a, b from its foot. If α, β are the elevations at the top of the tower from these stations, show that the inclination of the tower to the vertical is an angle whose cotangent is

$$\frac{b\cot\alpha - a\cot\beta}{b - a}.$$

12. Apply Ex. 11 to a numerical problem. Can logarithms be used? Make diagram to scale; test by measurement.

EXAMPLES FROM RELATED STUDIES.

1. Show that if a straight line passes through the origin and is not upright, the coördinates of any point on it are connected by the equation $y = \tan\theta \cdot x$, where θ is the angle of the line with the right-hand x-axis.

2. Show that if the preceding line, instead of passing through the origin, cuts off the distance b from the y-axis,

$$y = \tan\theta \cdot x + b.$$

3. Show that if (x, y) is *any* point on a straight line and (x', y'), (x'', y'') are two fixed points, the tangent of the angle which the line makes with the x-axis is any one of the expressions

$$\frac{y-y'}{x-x'},\ \frac{y-y''}{x-x''},\ \frac{y'-y''}{x'-x''};$$

or

$$\frac{y'-y}{x'-x},\ \frac{y''-y}{x''-x},\ \frac{y''-y'}{x''-x'}.$$

4. A particle has impressed on it instantaneously a vertical velocity of 50 ft. per second, and at the same instant a horizontal velocity of 75 ft. per second. What is the direction and what the magnitude of the resultant?

5. A cylindrical stone tower 20 ft. high and 4 ft. in diameter is tilted, by settling of the earth, until just ready to tumble over. What is the tilt?

6. A cube is held on an inclined plane by friction. To what angle can the plane be tilted before the cube rolls over, when the cube is held in the position giving the easiest solution?

7. Solve Ex. 6, replacing the cube by an upright cylinder and by a parallelopipedon of given dimensions.

8. A body weighing 100 lb. is suspended by a chain. What horizontal force will pull it 10° from the vertical, and what will then be the pull on the chain?

9. PB, PC are two equal rods, held in a vertical plane, hinged at P, with B, C in a horizontal line, B being tied to C by a string, and P being above the line BC. A weight is set at P. What pull will this give the string? Solve some numerical examples.

10. If (x, y) is a point at a distance b from the lower end of a straight beam which slides in a vertical plane, one end being on a horizontal plane and the other on a vertical plane, show that $x = a\cos\phi$, $y = b\sin\phi$, where a is the distance of (x, y) from the upper end of the beam, and ϕ the angle the beam makes with a horizontal line. When is $x^2 + y^2 = a^2$?

CHAPTER IX.

SINES AND COSINES, TANGENTS AND COTANGENTS.

§ 172. Relation of the Tangent and Cotangent to Sine and Cosine.

$$\text{tangent} = \frac{\text{ordinate}}{\text{abscissa}} = \frac{\dfrac{\text{ordinate}}{\text{modulus}}}{\dfrac{\text{abcisssa}}{\text{modulus}}} = \frac{\text{sine}}{\text{cosine}}.$$

$$\therefore \tan A = \frac{\sin A}{\cos A}. \qquad (1)$$

$$\therefore \cot A = \frac{\cos A}{\sin A}. \qquad (2)$$

And since $\sin^2 A + \cos^2 A = 1,$ (3)

$$\therefore \frac{\sin^2 A}{\cos^2 A} + 1 = \frac{1}{\cos^2 A}, \qquad (4)$$

and $$1 + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A}. \qquad (5)$$

(4) is $$1 + \tan^2 A = \sec^2 A. \qquad (6)$$

(5) is $$1 + \cot^2 A = \operatorname{cosec}^2 A. \qquad (7)$$

$$\therefore \tan^2 A = \sec^2 A - 1, \qquad (8)$$

and $$\cot^2 A = \operatorname{cosec}^2 A - 1. \qquad (9)$$

From (3) the sine can be calculated when the cosine is given, or the cosine can be calculated when the sine is given.

From (6) the secant (or cosine) when the tangent is given.

From (7) the cosecant (or sine) when the cotangent is given.

From (8) the tangent when the secant (or cosine) is given.

From (9) the cotangent when the cosecant (or sine) is given.

Note that since squares appear in (3), (4), (6), (7), (8), (9), the calculated function has two opposite values for a given value of the given function. Explain this on a diagram.

EXERCISES.

1. From the sine of 17° 23′ in the tables, calculate the cosine and compare with the table value.

2. From the cosine of 43° 21′ in the tables, calculate the sine and compare with the table.

3. From the sine of 15° 32′ in the table, calculate the tangent through the cosecant.

4. From the cos 67° 41′, calculate the tangent through the secant.

5. From the tan 75° 12′, calculate the cosine through the secant.

6. From the cot 345° 54′, calculate the sine through the cosecant.

7. From sin 23° in the table, calculate all its other functions and test such as the tables give. From tan 25° calculate the other functions.

§ 173. Use of a Diagram in Calculating One Function from Another.

The formulas of § 172 are used in calculating one function from another when the given function is in the form of a decimal of two or more figures. When the given function is in the form of a common fraction, it is best to construct a diagram corresponding to the given function, and then read the other functions from the diagram. This is particularly true if the parts of the given function are integers which with some other integer form a right-angled triangle. (See the set of such figures in the table on page 145.)

EXAMPLE. Given $\sin A = \frac{3}{5}$, calculate all other functions.

Figure 154 shows the two positions of the terminal for $\sin A = \frac{3}{5}$, and gives ± 4 as the third side.

And by the diagram:

$\text{cosec } A = \frac{5}{3}$;
$\tan A = \pm \frac{3}{4}$;
$\cot A = \pm \frac{4}{3}$;
$\cos A = \pm \frac{4}{5}$;
$\sec A = \pm \frac{5}{4}$;
$\text{versin } A = 1 - \cos A = \frac{1}{5} \text{ or } \frac{9}{5}$;
$\text{coversin } A = 1 - \sin A = \frac{2}{5}$.

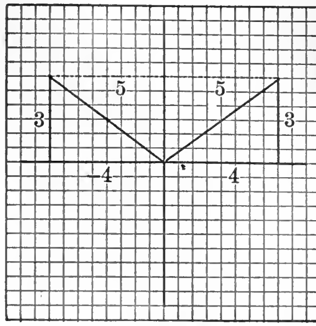


FIG. 154.

EXERCISES.

Plot the dual position for the terminal corresponding to each of the following, and calculate the ungiven functions:

1. $\cos A = \frac{5}{13}$.
2. $\cos A = -\frac{7}{25}$.
3. $\sin A = \frac{20}{29}$.
4. $\sin A = -\frac{12}{37}$.
5. $\tan A = \frac{20}{21}$.
6. $\tan A = -\frac{13}{84}$.
7. $\cot A = \frac{77}{36}$.
8. $\cot A = -\frac{65}{72}$.
9. $\text{cosec } A = \frac{60}{109}$.
10. $\text{cosec } A = -\frac{97}{72}$.
11. $\sec A = \frac{85}{13}$.
12. $\sec A = -\frac{101}{20}$.
13. $\text{versin } A = \frac{98}{113}$.
14. $\text{versin } A = \frac{225}{137}$.
15. $\text{coversin } A = \frac{9}{17}$.
16. $\text{coversin } A = \frac{49}{29}$.

17. Use the diagram for some similar examples when the numbers are selected at random, without reference to making a right-angled triangle whose sides are integers.

§ 174. Use of a Diagram in Expressing All the Functions in Terms of Some One Function.

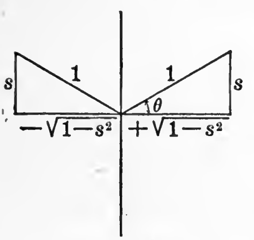


FIG. 155.

(*a*) All in terms of the sine.

Let s = sine.

Lay out the double terminals where s represents the sine to the modulus 1. Then the abscissa is $\pm\sqrt{1-s^2}$.

From the diagram (Fig. 155) one has directly from the definitions of the functions:

$\cos\theta = \pm\sqrt{1-s^2}$; $\tan\theta = \dfrac{s}{\pm\sqrt{1-s^2}}$; $\csc\theta = \dfrac{1}{s}$; $\sec\theta = \dfrac{1}{\pm\sqrt{1-s^2}}$; $\cot\theta = \dfrac{\pm\sqrt{1-s^2}}{s}$; $\text{versin }\theta = 1 \pm\sqrt{1-s^2}$; $\text{coversin }\theta = 1 - s$.

EXERCISES.

1. The diagrams shown in Figs. 156–160 indicate, in order, given, (1) cosine, (2) secant, (3) tangent, (4) cotangent, (5) cosecant. The

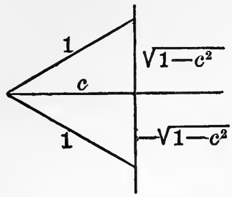


FIG. 156.—Cosine given.

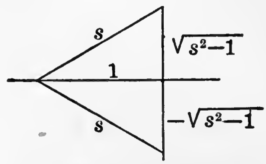


FIG. 157.—Secant given.

student may express, as above with the sine, all functions in terms of each as given, using the diagrams.

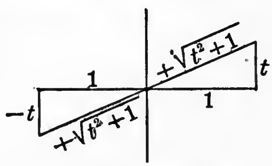


FIG. 158.—Tangent given.

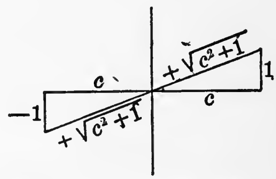


FIG. 159.—Cotangent given.

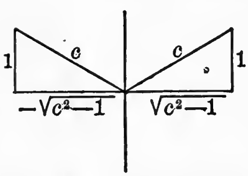


FIG. 160.—Cosecant given.

2. Deduce the expressions found in the preceding exercise and those in terms of the sine, all directly from the formulas of § 172, without the use of a diagram.

3. Show by a diagram that $\sin^{-1}\dfrac{x}{+\sqrt{1+x^2}}=\tan^{-1}x.$

4. Show by a diagram that $\sin^{-1}\dfrac{1}{+\sqrt{1+x^2}}=\cot^{-1}x.$

5. Show by a diagram that $\cos^{-1}\dfrac{x}{+\sqrt{1+x^2}}=\cot^{-1}(x)=\tan^{-1}\left(\dfrac{1}{x}\right).$

6. Show by a diagram that $\tan^{-1}\dfrac{x}{+\sqrt{1-x^2}}=\sin^{-1}x.$

7. What changes are made in Exs. 3, 4, 5, 6, if the radical is allowed the negative sign?

8. If $\theta=\sec^{-1}\dfrac{x}{\pm\sqrt{x^2-y^2}}$, find from a diagram the values of the other functions.

9. If $\theta = \sin^{-1}\dfrac{x}{\pm\sqrt{y^2+x^2}}$, find from a diagram the values of the other functions.

10. If $\theta = \cos^{-1}\dfrac{x}{\sqrt{y^2+x^2}}$, find from a diagram the values of the other functions.

11. If $\theta = \tan^{-1}\dfrac{x}{\sqrt{y^2-x^2}}$, find from a diagram the values of the other functions.

12. If $A = \sin^{-1}\dfrac{x^2-y^2}{x^2+y^2}$, find, in the best way, the other functions.

13. If $\sin A = \dfrac{m^2+2\,mn}{m^2+2\,mn+2\,n^2}$, show, in the best way, that

$$\tan A = \pm\frac{m^2+2\,mn}{2\,mn+2\,n^2}.$$

14. Prove $\sec(\tan^{-1}x) = \pm\sqrt{1+x^2}$.

15. Prove $\sin(\cos^{-1}x) = \pm\sqrt{1-x^2}$.

16. Prove $\cos(\cot^{-1}x) = \dfrac{x}{\pm\sqrt{1+x^2}}$.

17. Prove $\cot(\sin^{-1}x) = \dfrac{\pm\sqrt{1-x^2}}{x}$.

18. Select the cosine of some angle in the table of cosines and calculate the other functions and compare results with the table.

19. Do the same as indicated in Ex. 18 for each function given in the table, selecting for each a different angle.

EXERCISES CONNECTING TANGENT AND COTANGENT WITH THE OTHER FUNCTIONS.

1. $\tan A + \cot A = \sec A \operatorname{cosec} A$.

2. $(\tan A + \cot A)^2 = \sec^2 A + \operatorname{cosec}^2 A$.

3. $\dfrac{1+\tan A}{\sec A} = \sin A + \cos A$.

4. $(1+\tan A)(1+\cot A) = 2 + \sec A \operatorname{cosec} A$.

5. $\dfrac{\operatorname{cosec} A}{\cot A + \tan A} = \cos A$.

6. $\dfrac{1-\tan A}{1+\tan A} = \dfrac{\cot A - 1}{\cot A + 1}$.

7. $\dfrac{1+\tan^2 A}{1+\cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}$.

8. $\dfrac{1}{\cot A + \tan A} = \sin A \cos A.$

9. $\cos^3 A \tan A + \sin^2 A \cos A \tan A = \sin A.$

10. $(\tan A - \sin A) \operatorname{cosec}^3 A = \dfrac{\sec A}{1 + \cos A}.$

11. $\dfrac{\cot A - \tan A}{\cos A - \sin A} = \dfrac{\cos A + \sin A}{\cos A \sin A}.$

12. $\sqrt{\dfrac{1 - \sin A}{1 + \sin A}} = \sec A - \tan A.$

13. $\dfrac{1}{\sec A - \tan A} = \sec A + \tan A.$

14. $\dfrac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2 \sec A \tan A + 2 \tan^2 A.$

15. $\dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = \sec A \operatorname{cosec} A + 1.$

16. $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \sin A + \cos A.$

17. $(\sin A + \cos A)(\cot A + \tan A) = \sec A + \operatorname{cosec} A.$

18. $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A.$

19. $\sec^2 A \operatorname{cosec}^2 A = \tan^2 A + \cot^2 A + 2.$

20. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A.$

21. $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}.$

22. $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B.$

23. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}.$

24. $\dfrac{1 - \sin A}{1 + \sin A} = 1 + 2 \tan A (\tan A - \sec A).$

25. $(\tan A + \operatorname{cosec} B)^2 - (\cot B - \sec A)^2 = 2 \tan A \cot B (\operatorname{cosec} A + \sec B).$

§ 175. The Addition-subtraction Formula for Tangents.

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} \tag{1}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}. \tag{2}$$

Dividing numerator and denominator by $\cos A \cos B$, this becomes

$$\tan(A+B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}, \tag{3}$$

or,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \tag{4}$$

Similarly,

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}, \tag{5}$$

$$\cot(A+B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}, \tag{6}$$

$$\cot(A-B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}. \tag{7}$$

EXERCISES.

1. From the values of $\tan 45°$ and $\tan 30°$ show that $\tan 75° = 2 + \sqrt{3} = 3.73205 \cdots$.

2. Similarly, $\tan 15° = 2 - \sqrt{3} = 0.26795 \cdots$.

3. Tan $R = \frac{1}{2}$, $\tan B = \frac{3}{4}$. Find $\tan(R+B)$, $\tan(R-B)$, $\cot(R+B)$, $\cot(R-B)$, $\tan(2R+B)$, $\tan(2R-B)$.

4. Take D, B as some angles in the tables, and calculate $\tan(D+B)$, $\tan(D-B)$, $\cot(D+B)$, $\cot(D-B)$, and compare results with the tables.

5. $\dfrac{\tan x - \tan y}{\cot x + \tan y} = \tan(x-y) \tan x$.

6. Prove $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$, and find $\tan(A+B+C)$.

7. Prove $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$, and find value of $\cot 3x$.

8. If $\tan L = \frac{1}{2}$ and $\tan B = \frac{1}{2}$, find

$$\tan(L+B),\ \tan(L-B);\ \cot(L+B);\ \cot(L-B).$$

9. Show by formula (5) that $\tan 0° = 0$, and by (4) that $\tan 90° = \infty$.

§ 176. The Addition-subtraction Formula in Inverse Tangents.

If $\tan A = x$ and $\tan B = y$,

$$A + B = \tan^{-1} x + \tan^{-1} y.$$

Thus (4), (5), of the preceding article may be written in the forms

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}. \quad (3')$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}. \quad (4')$$

Similarly, if $\cot A = x$ and $\cot B = y$ (6), (7), take the forms:

$$\cot^{-1} x + \cot^{-1} y = \cot^{-1} \frac{xy-1}{x+y}. \quad (6')$$

$$\cot^{-1} x - \cot^{-1} y = \cot^{-1} \frac{xy+1}{y-x}. \quad (7')$$

Re-read what has been said about multiplicity of values of $\sin^{-1} x$, $\cos^{-1} x$.

EXERCISES.

1. $\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{13} = \tan^{-1} \frac{2}{9}$.

2. $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = n\pi + \frac{\pi}{4}$.

3. $\tan^{-1} \frac{m}{n} - \tan^{-1} \frac{m-n}{m+n} = n\pi + \frac{\pi}{4}$.

4. $\tan^{-1} t + \tan^{-1} \frac{2t}{1-t^2} = \tan^{-1} \frac{3t-t^3}{1-3t^2}$.

5. $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \frac{1}{2} \cos^{-1} \frac{3}{5}$.

6. $\cos^{-1} \frac{4}{5} + \tan^{-1} \frac{3}{5} = \tan^{-1} \frac{27}{11}$.

7. $2 \cos^{-1} \frac{3}{\sqrt{13}} + \cot^{-1} \frac{16}{63} + \frac{1}{2} \cos^{-1} \frac{7}{25}$ has π for one of its values. Give also the general value. Test formulas above by tables.

8. $\sin^{-1} \frac{1}{\sqrt{5}} + \cot^{-1} 3$ has $\frac{\pi}{4}$ for one value. Give also the general value.

9. $\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} - \tan^{-1} \frac{8}{19}$ has $\frac{\pi}{4}$ for one value.

10. If $\tan^{-1} \frac{3}{4} + \tan^{-1} x = \tan^{-1} \frac{5}{3}$, find the general value of x.

11. $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$ (one value).

12. $\tan^{-1}\dfrac{p-qx}{q+px}=\tan^{-1}\dfrac{p}{q}-\tan^{-1}x.$

13. $\tan^{-1}\dfrac{p+qx}{q-px}=\tan^{-1}\dfrac{p}{q}+\tan^{-1}x.$

14. $\tan^{-1}\dfrac{x^{\frac{1}{2}}-x}{1+x^{\frac{3}{2}}}=\tan^{-1}x^{\frac{1}{2}}-\tan^{-1}x.$

15. $\cos^{-1}\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}=2\cot^{-1}x.$ (Set $x=\cot\theta$.)

16. $\sin^{-1}\dfrac{2x}{1+x^2}=2\tan^{-1}x.$ (Set $x=\tan\theta$.)

17. $\sec^{-1}\dfrac{1}{2x^2-1}=2\cos^{-1}x.$ (Set $x=\cos\theta$.)

§ 177. Tangent of the Double Angle.

From $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B},$$

follows, if $$A=B,$$

$$\mathbf{\tan 2A=\frac{2\tan A}{1-\tan^2 A}}.$$

Thus the tangent of any angle is twice the tangent of half of it divided by one minus the square of the tangent of half of it.

$$\therefore \tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}};$$

$$\tan 8x=\frac{2\tan 4x}{1-\tan^2 4x};$$

$$\tan 4x=\frac{2\tan 2x}{1-\tan^2 2x};\ \text{etc.}$$

LABORATORY EXERCISE.

On a ten-inch circle, measure tan 5° and tan 10°, and test the preceding formula.

EXERCISES.

1. Test tan 42° 18′ in the tables as compared with tan 21° 9′.

2. If $\tan A = \frac{3}{4}$, find $\tan 2A$, $\tan 3A$, and $\tan 4A$.

3. Show $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

4. Show $\tan 4A = \dfrac{4(\tan A - \tan^3 A)}{1 - 6\tan^2 A + \tan^4 A}$.

5. Deduce $\tan 2A$ from $\tan 2A = \dfrac{\sin 2A}{\cos 2A}$ by using the formulas for $\sin 2A$ and $\cos 2A$ in terms of functions of A.

6. Prove $\cos A = \dfrac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$.

7. Prove $\sin A = \dfrac{2\tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$.

8. Prove $\tan \dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$.

9. Prove $\tan \dfrac{A}{2} = \dfrac{1 - \cos A}{\sin A}$.

10. Prove $\tan \dfrac{A}{2} = \dfrac{\sin A}{2\cos^2 \frac{A}{2}}$.

11. Prove that $\tan \dfrac{A}{2}$ always has the same sign as $\sin A$. Can this be shown on a diagram?

12. Prove $\dfrac{\sin A}{\operatorname{versin} A} = \cot \dfrac{A}{2}$.

13. Prove $\sin \dfrac{A}{2} = \sqrt{\dfrac{\operatorname{versin} A}{2}}$.

14. Prove $\dfrac{\cot A + 1}{\cot A - 1} = \sec 2A + \tan 2A$.

15. Prove $\tan \dfrac{A}{2} + \cot \dfrac{A}{2} = 2\operatorname{cosec} A$.

16. Prove $\cot A - \tan A = 2\cot 2A$.

17. Prove $2\operatorname{cosec} 2A = \tan A + \cot A$.

18. Prove $\dfrac{2\cot A}{\cot^2 A - 1} = \tan 2A$.

19. Prove $\operatorname{cosec} 2A + \cot 2A = \cot A$.

20. Prove $\dfrac{\cos A + \sin A}{\cos A - \sin A} - \dfrac{\cos A - \sin A}{\cos A + \sin A} = 2\tan 2A$.

21. Prove $(1 + \tan A)\cot A = \dfrac{2\cot 2A}{1 - \tan A}$.

22. Prove $\dfrac{1+\tan^2(45^\circ - A)}{1-\tan^2(45^\circ - A)} = \dfrac{1}{\sin 2A} = \operatorname{cosec} 2A.$

23. $\dfrac{1 \pm \sin 2A}{\cos 2A} = \tan(45^\circ \pm A).$

24. $\tan(45^\circ + 2A) + \tan(45^\circ - 2A) = 2\sec 4A.$

25. If $\tan^2 x = 1$, what is the general value of x?

§ 178. Tangent of the Double Angle in Inverse Notation.

From
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A},$$

follows, if $\tan A = x$ and $A = \tan^{-1} x$,
$$2A = \tan^{-1}\frac{2x}{1-x^2},$$

or,
$$\mathbf{2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}}.\quad \text{Test by tables.}$$

EXERCISES.

1. $2\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{12}{5}$ (one value).

2. $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8} = \dfrac{\pi}{4}$ (one value).

3. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \dfrac{\pi}{4}$ (one value).

4. $3\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{20} = \dfrac{\pi}{4} - \tan^{-1}\frac{1}{1985}$ (one value).

5. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x} = \frac{1}{2}\tan^{-1} x.$

6. $\sec^{-1}\dfrac{x+\dfrac{1}{x}}{x-\dfrac{1}{x}} = 2\cot^{-1} x.$

7. $\cos^{-1}(2x^2-1) = 2\cos^{-1} x.$

8. $\sin^{-1}\dfrac{2x}{1+x^2} = 2\tan^{-1} x.$

9. $\sec^{-1}\dfrac{1}{2x^2-1} = 2\cos^{-1} x.$

10. $\cos^{-1}\dfrac{1-x^2}{1+x^2} = 2\tan^{-1} x.$

11. $\tan^{-1}\dfrac{3x - x^3}{1 - 3x^2} = 3\tan^{-1}x.$

12. $\sin^{-1}(3x - 4x^3) = 3\tan^{-1}\dfrac{x}{\sqrt{1 - x^2}}.$

13. $\cos^{-1}(4x^3 - 3x) = 3\tan^{-1}\dfrac{\sqrt{1 - x^2}}{x}.$

14. $\sin^{-1}2x\sqrt{1 - x^2} = 2\tan^{-1}\dfrac{x}{\sqrt{1 - x^2}}.$

15. $\cos^{-1}(2x^2 - 1) = 2\cot^{-1}\dfrac{x}{\sqrt{1 - x^2}}.$

16. $\cos^{-1}(1 - 2x^2) = 2\tan^{-1}\dfrac{x}{\sqrt{1 - x^2}}.$

§ 179. Tangent of the Half-angle.

Since
$$\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}},$$
if we let $\tan A = a$ and $\tan\frac{A}{2} = x$, x is gotten by solving the quadratic
$$ax^2 + 2x - a = 0.$$
$$\therefore\ x = \frac{-1 \pm \sqrt{1 + a^2}}{a}.$$
$$\therefore\ \tan\frac{A}{2} = \frac{-1 \pm \sqrt{1 + \tan^2 A}}{\tan A}.$$

If $\tan A$ is given, there are two positions of the corresponding terminal, 180° apart. There are thus two positions of the terminal of the half-angle, 90° apart. This explains the dual sign. When A is itself given and not by its tangent, the proper sign is to be selected. Thus
$$\tan 22° = \frac{-1 + \sqrt{1 + \tan^2 45}}{\tan 45°}$$
and
$$\tan 100° = \frac{-1 - \sqrt{1 + \tan^2 200°}}{\tan 200°}.$$

The tangent of the half-angle can also be expressed in terms of other functions.

It has been proven (§ 139) that

$$1 - \cos A = 2 \sin^2 \frac{A}{2}, \tag{1}$$

$$1 + \cos A = 2 \cos^2 \frac{A}{2}, \tag{2}$$

and

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}. \tag{3}$$

$$\therefore \tan \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}, \tag{4}$$

$$\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}, \tag{5}$$

$$\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}. \tag{6}$$

To get (4), divide (1) by (2); to get (5), divide (1) by (3); to get (6), divide (3) by (2). The dual sign in (4) means that if an angle is given by its cosine, there are two positions of the terminal (as in § 106). Thus there are two positions for the half-terminal, and two values of the tangent of the half-angle, oppositely equal. When the angle itself is given, there is only one terminal and the proper sign is to be selected in (4). Thus

$$\tan 11° = +\sqrt{\frac{1 - \cos 22°}{1 + \cos 22°}}, \text{ and } \tan 100° = -\sqrt{\frac{1 - \cos 200°}{1 + \cos 200°}}.$$

EXERCISES.

1. Express tan $22\frac{1}{2}°$ in radicals. Express the radicals in decimals and calculate tan $22\frac{1}{2}°$, and compare with tables. Do the same for tan $(-22\frac{1}{2}°)$.

2. Express tan 15° in radicals and in decimals, and compare with tables.

3. Express tan $7\frac{1}{2}°$ in radicals and in decimals, and compare with tables. Test the four formulas for $\tan \frac{A}{2}$ by tables.

4. If $\cos A = .23$, find $\tan \frac{A}{2}$, and explain the dual sign.

5. If $\sec A = 1\frac{1}{4}$, find $\tan \frac{A}{2}$ and $\tan A$. Express $\frac{A}{2}$ in inverse tangents.

6. $\tan\left(45^\circ + \frac{A}{2}\right) = \sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A.$

7. $\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) = 2\sec 2\theta.$

8. If $\sin\theta + \sin\phi = a$, and $\cos\theta + \cos\phi = b$, find the value of $\tan\frac{\theta-\phi}{2}$.

9. If $\sin A = \frac{3}{5}$, find $\sin 2A$, $\cos 2A$, $\tan 2A$, and $\tan\frac{A}{2}$.

10. If $\cos A = \frac{12}{13}$, find $\sin 2A$, $\cos 2A$, $\tan 2A$, and $\tan\frac{A}{2}$.

11. If $\tan A = \frac{16}{63}$, find $\tan 2A$, $\sin 2A$, and $\cos 2A$.

12. If $\sec A = \frac{5}{4}$, find all the functions of $2A$.

13. If $\operatorname{cosec} A = \frac{13}{5}$, find all the functions of $2A$.

14. $\dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}} = \cos A.$

15. $1 + \cot A\cot\frac{A}{2} = \operatorname{cosec} A\cot\frac{A}{2}.$

16. $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\tan\left(\frac{\pi}{4}-\frac{x}{2}\right) = 1.$

17. $\tan\frac{x}{2} + 2\sin^2\frac{x}{2}\cot x = \sin x.$

18. $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \sec x + \tan x.$

19. $\tan^2\frac{A}{2}\left(1+\cot^2\frac{A}{2}\right)^2 = 4\operatorname{cosec}^2 A = 4(1+\cot^2 A).$

§ 180. The Tangent in Auxiliary Formulas.

When forms like $a\cos x + b\sin x$ (§ 148) are to be calculated, and logarithms are to be used, we write

$$a\cos x + b\sin x = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right).$$

We may set $\dfrac{a}{\sqrt{a^2+b^2}} = \sin\phi$; $\dfrac{b}{\sqrt{a^2+b^2}} = \cos\phi$,

where $\tan\phi = \frac{a}{b}$ is more readily calculated than $\sin\phi$ or $\cos\phi$.

$$\therefore\ a\cos x + b\sin x = \sqrt{a^2+b^2}\sin(x+\phi)$$
$$= \sqrt{a^2+b^2}\sin\left(x + \tan^{-1}\frac{a}{b}\right).$$

EXERCISES.

1. Find x when $317 \sin x + 212 \cos x = 321$.

2. Find x when $8.314 \sin x - 7.215 \cos x = 1.314$.

3. Show $\sqrt{3} \cdot \sin x + \cos x = 2 \sin\left(x + \frac{\pi}{6}\right)$.

4. Show $3 \sin x + 4 \cos x = 5 \sin\left(x + \tan^{-1}\frac{4}{3}\right)$.

5. Express the preceding results in terms of the cosine and cotangent.

6. Express $a \cos x - b \sin x$ in terms of the sine and in terms of the cosine.

7. Solve the examples under § 148 when the tangent expresses the auxiliary angle.

EXERCISES CONNECTING THE TANGENT AND COTANGENT WITH OTHER FUNCTIONS.

1. $$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan\frac{A+B}{2} \cot\frac{A-B}{2}.$$

2. $$\frac{\cos A + \cos B}{\cos B - \cos A} = \cot\frac{A+B}{2} \cot\frac{A-B}{2}.$$

3. $$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan\frac{A+B}{2}.$$

4. $$\frac{\sin A - \sin B}{\cos B - \cos A} = \cot\frac{A+B}{2}.$$

5. $$\frac{\sin 7A - \sin 5A}{\cos 7A + \cos 5A} = \tan A.$$

6. $$\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A.$$

7. $$\frac{\cos 2B + \cos 2A}{\sin 2B - \cos 2A} = \cot(A+B)\cot(A-B).$$

8. $$\frac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot\frac{A}{2}.$$

9. $$\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A.$$

10. $$\frac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \tan(A+B)\cot(A-B).$$

11. $$\frac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = \tan(A-B).$$

12. $\dfrac{\sin(4A - 2B) + \sin(4B - 2A)}{\cos(4A - 2B) + \cos(4B - 2A)} = \tan(A + B)$.

13. $\dfrac{\tan 5A + \tan 3A}{\tan 5A - \tan 3A} = 4\cos 2A \cos 4A$.

14. $\dfrac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \cos 2A - \sin 2A \tan 3A$.

15. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$.

16. $\dfrac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A$.

17. $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$.

18. $\dfrac{\sin 2A}{1 - \cos 2A} = \cot A$.

19. $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A$.

20. $\tan A + \cot A = 2 \operatorname{cosec} 2A$.

21. $\tan A - \cot A = -2\cot 2A$.

22. $\operatorname{cosec} 2A + \cot 2A = \cot A$.

23. $\dfrac{\cos A}{1 \mp \sin A} = \tan\left(45^\circ \pm \dfrac{A}{2}\right)$

24. $\dfrac{\sec 8A - 1}{\sec 4A - 1} = \tan 8A \cdot \cot 2A$.

25. $\dfrac{1 + \tan^2(45^\circ - A)}{1 - \tan^2(45^\circ - A)} = \operatorname{cosec} 2A$.

26. If $\sin\theta + \sin\phi = a$, and $\cos\theta + \cos\phi = b$, find θ and ϕ.

27. $\tan(45^\circ + A) - \tan(45^\circ - A) = 2\tan 2A$.

28. $\dfrac{\cos A + \sin A}{\cos A - \sin A} - \dfrac{\cos A - \sin A}{\cos A + \sin A} = 2\tan 2A$.

29. $\cot(A + 15^\circ) - \tan(A - 15^\circ) = \dfrac{4\cos 2A}{1 + 2\sin 2A}$.

30. $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$.

31. $\dfrac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \tan\dfrac{A}{2}$.

32. $\dfrac{\sin(n+1)A - \sin(n-1)A}{\cos(n+1)A + 2\cos nA + \cos(n-1)A} = \tan\dfrac{A}{2}.$

33. $\dfrac{\sin(n+1)A + 2\sin nA + \sin(n-1)A}{\cos(n-1)A - \cos(n+1)A} = \cot\dfrac{A}{2}.$

34. $\cot\dfrac{A}{2} - \tan\dfrac{A}{2} = 2\cot A.$

35. $\cot A + \cot(60^\circ + A) - \cot(60^\circ - A) = 3\cot 3A.$

36. $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A.$

37. $\sin A = \dfrac{2\tan\dfrac{A}{2}}{1+\tan^2\dfrac{A}{2}}.$ **38.** $\cos A = \dfrac{1-\tan^2\dfrac{A}{2}}{1+\tan^2\dfrac{A}{2}}.$

39. $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1.$

40. Given $\sin(A - x) = \cos(A + x)$, find $\tan x$.

41. Given $\sin(x+a) + \cos(x+a) = \sin(x-a) + \cos(x-a)$, find $\tan x$.

42. Given $\tan A \tan x = \tan^2(A + x) - \tan^2(A - x)$, find $\cos x$.

43. Given $$\frac{m\tan(A-x)}{\cos^2 x} = \frac{n\tan x}{\cos^2(A-x)}.$$

Prove $$\tan(A - 2x) = \frac{n-m}{n+m}\tan A.$$

44. Given $$\tan(A+x)\tan(A-x) = \frac{1-2\cos 2A}{1+2\cos 2A}.$$

Show that $x = 30^\circ$ is a solution.

45. Given $$n\sec^2 x\tan(A-x) = m\sec^2(A-x)\tan x.$$

Show $$\tan 2x = \frac{n\sin 2A}{m + n\cos 2A}.$$

46. Prove $\dfrac{1+\sin A}{1-\sin A} = \tan^2\left(45^\circ + \dfrac{A}{2}\right).$

47. $\tan\left(45^\circ + \dfrac{A}{2}\right) + \cot\left(45^\circ + \dfrac{A}{2}\right) = 2\sec A.$

48. $\tan(30^\circ + A)\tan(30^\circ - A) = \dfrac{2\cos 2A - 1}{2\cos 2A + 1}.$

49. If $$\sin x = \sin A\sin(B + x),$$

show $$\tan x = \frac{\sin A\sin B}{1 - \sin A\cos B}.$$

50. If $m\sin B\cos(A - x) = n\sin A\cos(B + x)$, find $\tan x$.

§ 181. Trigonometric Equations involving All the Functions.

Process for Solution: Reduce the equation to a single function and solve in the most general manner, using tables if necessary.

SAMPLE EXAMPLE:

$$\sin x = \tan^2 x. \quad (1)$$

$$\therefore \sin x = \frac{\sin^2 x}{\cos^2 x}. \quad (2)$$

$$\therefore (1 - \sin^2 x) \sin x = \sin^2 x. \quad (3)$$

$$\therefore \sin x (1 - \sin^2 x - \sin x) = 0. \quad (4)$$

$$\therefore \sin x = 0; \quad (5)$$

or,

$$1 - \sin^2 x - \sin x = 0. \quad (6)$$

By (5), $x = n\pi$, or $n \cdot 180°$.

By (6), $\sin x = \frac{-1 \pm \sqrt{5}}{2} = 0.6180$, or $-(1.6180)$.

The latter is rejected, since $\sin x$ lies between $+1, -1$.

If $x = \sin^{-1} 0.6180$;

$x = 2n\, 180° + 38° 10'$ (tables);

or, $x = (2n + 1)\, 180° - 38° 10'$.

EXERCISES.

Solve in the most general manner:

1. $\sec^2 x = 4 \tan x$.
2. $4 \tan x - \cot x = 3$.
3. $2 \cot^2 \theta = \text{cosec}^2 \theta$.
4. $4 \cos \theta - 3 \sec \theta = 2 \tan \theta$.
5. $\sin^2 \theta - 2 \cos \theta + \frac{1}{4} = 0$.
6. $\cot^2 \theta + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) \cot \theta + 1 = 0$.
7. $\tan^2 \theta + \cot^2 \theta = 2$.
8. $\sec \theta - 1 = (\sqrt{2} - 1) \tan \theta$.
9. $\tan^2 \theta - (1 + \sqrt{3}) \tan \theta + \sqrt{3} = 0$.
10. $2 \cos \theta = \sqrt{3} \cot \theta$.
11. $1.4 \tan \theta + 1.7 \cot \theta = 2.3$.
12. $2134 \sin \theta = 1372 \tan \theta$.
13. $43 \tan \theta = 27 \sin^2 \theta$.
14. $\tan^2 \theta + \cot^2 \theta = \frac{10}{3}$.
15. $\tan^2 x + \text{cosec}^2 x = 3$.
16. $2 \sin^2 \theta + 3 \cos \theta = 0$.

CHAPTER X.

THE TANGENT AND COTANGENT IN THE SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

§ 182. Given Two Sides and the Included Angle.

Since for a solution by sines a pair of opposites must be given, if two sides and the included angle are given, the triangle cannot be solved by sines without splitting the triangle into two right-angled triangles. A solution by the aid of the tangent is possible.

Given $b, c, A.$

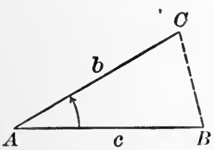


FIG. 161.

$$\frac{\sin B}{\sin C} = \frac{b}{c} \quad (\S\ 88).$$

Assume $b > c.$

$$\therefore \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b - c}{b + c}.$$

$\therefore$ by § 145,

$$\frac{\sin \dfrac{B - C}{2} \cdot \cos \dfrac{B + C}{2}}{\cos \dfrac{B - C}{2} \cdot \sin \dfrac{B + C}{2}} = \frac{b - c}{b + c};$$

or,

$$\tan \frac{B - C}{2} \cdot \cot \frac{B + C}{2} = \frac{b - c}{b + c}.$$

$$\therefore \tan \frac{B - C}{2} = \frac{b - c}{b + c} \cdot \tan \frac{B + C}{2}. \qquad (X)$$

Since $A + B + C = 180°,$

$$\frac{B + C}{2} = 90° - \frac{A}{2}.$$

$$\therefore \tan \frac{B + C}{2} = \tan\left(90° - \frac{A}{2}\right) = \cot \frac{A}{2}.$$

(The tangent of an angle is the cotangent of the complement. § 170.)

$$\therefore \tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}. \qquad (Y)$$

From formula (X), or from (Y), by logarithms, or by natural functions, $\frac{B-C}{2}$ can be found. And since $\frac{B+C}{2}$ is known,

$$B = \frac{B+C}{2} + \frac{B-C}{2},$$

and

$$C = \frac{B+C}{2} - \frac{B-C}{2}.$$

B and C being thus known, there remains to find a.

$$a = \sin A \cdot \frac{b}{\sin B};$$

or,

$$a = \sin A \cdot \frac{c}{\sin C}.$$

In a calculation, the values of B, C should be checked. For this purpose there may be used $b \cdot \sin C = c \cdot \sin B$.

LOGARITHMS.

$b =$	$= c$
$\sin C =$	$= \sin B$
sum	sum

Check to within 2 in "final" figure.

Since a can be found by two sine formulas, that one not used for calculating a may be used as a check on a.

MODEL EXAMPLE.

Given $\begin{cases} b = 48.3 \\ c = 36.0 \\ A = 62° \end{cases}$

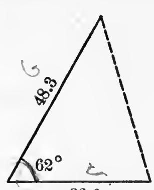


FIG. 162.

Find $\begin{cases} \frac{B+C}{2} = 59° \\ \frac{B-C}{2} = 13° \, 40' \\ B = 72° \, 40' \\ C = 45° \, 20' \\ a = 44.7 \end{cases}$

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \tan \frac{B+C}{2};$$

$$b - c = 12.3; \qquad b + c = 84.3;$$

$$a = \sin A \cdot \frac{c}{\sin C};$$

$$\log\tan\frac{B-C}{2} = \log(b-c) + \log\tan\frac{B+C}{2} - \log(b+c);$$

$$\log a = \log\sin A + \log c - \log\sin C.$$

LOGARITHMS.

$b - c = 1.0899$

$\tan\frac{B+C}{2} = 0.2212$

$\text{sum} = 1.3111$

$b + c = 1.9258$

$\text{diff.} = \bar{1}.3853$

$= \log\tan\frac{B-C}{2}$

This gives $\frac{B-C}{2}$ to the nearest 5 minutes as $13° 40'$.

LOGARITHMIC CHECK FOR B, C.

LOGARITHMS.

$b = 1.6839$	$1.5563 = C$
$\sin C = \bar{1}.8520$	$\bar{1}.9798 = \sin B$
1.5359	1.5361

The data in lines being three-figured and the check holding to within 2 in third place, B, C are assumed correct.

LOGARITHMS.

$\sin A = \bar{1}.9459$

$c = 1.5563$

$\text{sum} = 1.5022$

$\sin C = \bar{1}.8520$

$a = 1.6502$

LOGARITHMIC CHECK FOR a.

$a = 1.6503$	$1.6839 = b$
$\sin B = \bar{1}.9798$	$\bar{1}.9459 = \sin A$
1.6301	1.6298

Agree to three figures within 2. Satisfactory.

EXERCISES.

Prove directly the formulas corresponding to that proven when b, c, A were given, when there are given:

1. a, b, C, with $a > b$.

2. a, b, C, with $a < b$.

3. b, c, A, with $c > b$.

4. b, c, A, with $c < b$.

5. a, c, B, with $a > c$.

6. a, c, B, with $a < c$.

7. What if the two given sides are equal?

8. The student may construct some appropriate examples, solve and test them (§ 77):

(*a*) With lines to one significant figure and angles reading to the nearest 5°.

(*b*) With lines to two significant figures and angles reading to the nearest half degree.

(*c*) With lines to three significant figures and angles reading to the nearest five minutes.

(*d*) With lines showing four significant figures and angles reading to minutes.

(*e*) With lines showing five significant figures and angles reading to seconds.

(*f*) Six-figured data; angles to tenths of seconds.

(*g*) Seven-figured data; angles to hundredths of a second.

9. Show that a can also be calculated from

$$a = (c + b)\frac{\cos\frac{C+B}{2}}{\cos\frac{C-B}{2}} = (c - b)\frac{\sin\frac{C+B}{2}}{\sin\frac{C-B}{2}}.$$

§ 183. To find the Angles of a Triangle by Tangents when the Three Sides are Given.

Since
$$\tan x = \frac{\sin x}{\cos x},$$
we have, from the values of the sines of the half angles of a triangle in § 92 and the values of the cosines in § 127, the following values of the tangents of the half angles of a triangle:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \qquad (1)$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \qquad (2)$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}, \qquad (3)$$

with their reciprocals for the cotangents of the half angles.

For calculation purposes it is best to multiply (1) by $\frac{s-a}{s-a}$ under the radical; (2) by $\frac{s-b}{s-b}$; (3) by $\frac{s-c}{s-c}$.

Then, $$\tan\frac{A}{2} = \frac{1}{s-a}\cdot\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{r_i}{s-a};$$

$$\tan\frac{B}{2} = \frac{1}{s-b}\cdot\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{r_i}{s-b};$$

$$\tan\frac{C}{2} = \frac{1}{s-c}\cdot\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{r_i}{s-c}.$$

Calculation Scheme.

	NUMBERS.		
	$a =$	(1)	
	$b =$	(2)	
	$c =$	(3)	
	$2s =$	(4)	
	$s =$	(5)	
	$s - a =$	(6)	
	$s - b =$	(7)	Check
	$s - c =$	(8)	$(6)+(7)+(8)=(5)$
	LOGARITHMS.		
	$s - a =$	(1)	
	$s - b =$	(2)	
	$s - c =$	(3)	
	sum $=$	(5)	
	$s =$	(4)	
$(5)-(4)$	$r_i^2 =$	(6)	
$(6)\div 2$	$r_i =$	(7)	
$(7)-(1)$	$\tan\frac{A}{2} =$	(8)	
$(7)-(2)$	$\tan\frac{B}{2} =$	(9)	
$(7)-(3)$	$\tan\frac{C}{2} =$	(10)	

The scheme is made before beginning the work, then numbers are entered on the scheme. The logarithms (1), (2), (3), (4) are all looked up before their manipulation begins in (5), etc. The numbers are close enough on the scheme to be added, subtracted, without rewriting. If not, the log (7) may be written on the end of a card and held in turn over (1), (2), (3), and the subtractions be made and written in their appropriate place.

Since the tangent solution requires but four logarithms, those of s, $s-a$, $s-b$, $s-c$, it is more convenient for use than the corresponding formulas for sines and cosines, which required six, seven, logarithms. (See §§ 92, 127.)

As the tangent, over much of the tables, varies more rapidly than the sine and cosine for equal changes in the angle (notice this in the tables), the tangent formulas are, as a rule, more suitable for calculation than the cosine and sine formulas. This is again taken up in Chapter XI. Besides, the log tan will give the angle when it is near 90° as well as when small, even though the tangent is infinite.

Checks.

$$A + B + C = 180° \qquad (1)$$

or $$a \cdot \sin B = b \cdot \sin A \text{ and } a \cdot \sin C = c \cdot \sin A \qquad (2)$$

How close the second check should check we have often had occasion to point out. How close the first check should check depends likewise on the number of significant figures in the data, if they are assumed as measurements and not as exact.

When the sides are given to one, or two, or even three, significant figures, and represent measurements, the summation of angles is not an appropriate test. Better, then, test by logarithms of the sine formulas (second test above).

It is often said that if a five-place table is used the test should hold to within "a few seconds." And so it should, for *exact* data. That is absurd, however, on measurements, for no one of the angles may be knowable to minutes. In one-figured data (measurement) the angles cannot be known even to the nearest degree.

It cannot be impressed upon the student too often that there is no such measurable triangle as one having its sides as 9, 8, 7. The more accurately one tries in a physical laboratory to measure the length of a given line the more will the results differ. One can measure a line roughly and get the same result each time. But if the same line is measured with extreme care each time, the results rarely agree. It is quite absurd, therefore, to say, "if five-placed tables are used, the angles should sum to 180° to within a few seconds." It is childish to use a five-place table on one-figured data to the full extent of the table. When a table of more places than the data is used, calculated results should be cut back to the pattern of the data. The severity of a check must in all cases correspond to the character of the data.

If one is using a five-place table, *with data appropriate to such a table*, it is easy to test the closeness with which the summation of the angles should reach 180°. In §§ 22, 24, it is shown that a logarithmic formula like that of the tangent of the half-angle of a triangle may be in error, on account of the approximate values of the tangents in the tables, by about 1 in the last place of logarithms. How much this means for the angle will depend upon the size of the angle. The possible error for each angle may be determined by examining the table in its neighborhood. Take the sum for the three angles and double it and we have the allowable error in the check.

Consider, for example, an equilateral triangle. The half-angle is 30°. In the neighborhood of 30° in a four-place table a difference of 0.0001 in logarithms of tangents means a difference of about 20″ in angles. So in such a triangle, calculated from its sides by tangents of half-angles, the sum of the three angles may differ from 180° by 2′. The use of a five-place table on data appropriate for the use of such a table to five places, will cut this by the divisor 10 on each angle, making the allowable variation in the check about 12 seconds.

How close the check should check has to be determined for each triangle.

As a general rule holds for the logarithmic check, it is best to use that.

To say, however, that the place of the table used determines how close the check should check, without reference to the character of the data, is to assume the triangle being dealt with a *theoretic triangle.* Such triangles are of no value to the engineer.

MODEL EXAMPLE.

NUMBERS.

$a = 15.47$

$b = 17.39$

$c = 22.88$

$2s = 55.74$

$s = 27.87$

$s - a = 12.40$

$s - b = 10.48$

$s - c = 4.99$

LOGARITHMS.

$s - a = 1.0934$

$s - b = 1.0204$

$s - c = 0.6981$

$Sum = 2.8119$

$s = 1.4451$

$r_i^2 = 1.3668$

$r_i = 0.6834$

$\tan\frac{A}{2} = \bar{1}.5900 \quad \therefore \frac{A}{2} = 21^\circ\ 15'\ (+) \quad \therefore A = 42^\circ\ 31'$

$\tan\frac{B}{2} = \bar{1}.6630 \quad \therefore \frac{B}{2} = 24^\circ\ 43' \quad \therefore B = 49^\circ\ 26'$

$\tan\frac{C}{2} = \bar{1}.9853 \quad \therefore \frac{C}{2} = 44^\circ\ 2' \quad \therefore C = 88^\circ\ 4'$

$\overline{180^\circ\ 1'}$

At 21° 15′, the log difference 0.0001 means about 15″.

At 24° 43′, the log difference 0.0001 means about 20″.

At 44° 2′, the log difference 0.0001 means about $\frac{20''}{55''}$.

So the solution may be considered correct.

It is clear that the angles may sum to 180° and still the individual angles be incorrect. This is rather unlikely.

The use of a five-place table with the data of the foregoing example (representing measurements) could in no wise affect the closeness of the check, since with data to only four significant figures, a fifth figure in the tangents of angles would be without significance. However, if we assume the data as *exact*, representing a *theoretic* triangle, a five-place table will demand a closer check than a four-place table. See § 77.

Logarithmic Test.

LOGARITHMS.

$$
\begin{array}{r|l}
a = 1.1895 & 1.2403 = b \\
\sin B = \bar{1}.8806 & \bar{1}.8298 = \sin A \\
\hline
1.0701 & 1.0701
\end{array}
$$

They agree too well ! C should also be checked.

EXERCISES.

1. The student may select some numerical exercises, solve and test:

(*a*) With sides to one significant figure, calculating angles to the nearest five degrees.

(*b*) With two-figured data; angles to the nearest half degree.

(*c*) With three-figured data; angles to the nearest five minutes.

(*d*) With four-figured data; angles to the nearest minute.

(*e*) With five-figured data; angles to the nearest second.

(*f*) With six-figured data; angles to tenths of a second.

(*g*) With seven-figured data; angles to hundredths of a second.

2. In Fig. 145, line $DEC < DAC < DBC$. Show from this that $\sin\theta < \theta < \tan\theta$, and $\dfrac{\theta}{\sin\theta} \doteq 1$, when $\theta \doteq 0$; also $\dfrac{\theta}{\tan\theta} \doteq 1$, when $\theta \doteq 0$.

3. In Fig. 145, triangle $OEC <$ sector $OAC <$ triangle OBC. Show from this that $\sin\theta < \theta < \tan\theta$.

4. Show that $(1 + \sin\theta)^{\operatorname{cosec}\theta} \doteq e$, when $\theta \doteq 0$.

5. Show that $(1 + 3\tan^2\theta)^{2\operatorname{cosec}^2 4\theta} \doteq e^{\frac{3}{8}}$, when $\theta \doteq 0$.

6. From the series for $\sin\theta$ and $\cos\theta$, find a series for $\tan\theta$.

7. Given $r\sin\phi = 21.71$, $r\cos\phi = 31.41$, find r, ϕ.

8. Given $r\cos\phi\cos\theta = 3.172$, $r\sin\phi\cos\theta = 2.113$, $r\sin\theta = 3.121$, find r, θ, ϕ.

9. Given $3121\tan\phi + 2171\cot\phi = 3141$, find ϕ.

10. Given $\sin(\phi + 16°17') = 0.3142\sin\phi$, find ϕ.

11. Given $\tan(\phi + 31°) = 23\tan\phi$, find ϕ.

12. Change $\sqrt{a^2 - x^2}$ and $\sqrt{a^2 + x^2}$ to trigonometric forms.

CHAPTER XI.

GENERAL REVIEW ON THE SOLUTION OF TRIANGLES. LIST OF FORMULAS TO MEMORIZE.

§ 184. Solution of Triangles.

It is possible, as we have seen, to solve any sort of triangle with the sine alone. Similarly, we might use the cosine alone, or the tangent alone. In fact, since all the trigonometric functions are expressible in terms of any one of them, any one of them might be used alone for the solution of triangles.

EXERCISES.

1. Show that it is possible to solve all possible cases in right-angled triangles by using:

(1) The sine alone.

(2) The cosine alone.

(3) The tangent alone.

2. Show that for triangles not right-angled it is always possible, either directly, or by breaking the triangle into two right-angled triangles, to solve every case:

(1) By the sine alone.

(2) By the cosine alone.

(3) By the tangent alone.

§ 185. The Best Method of Solving Triangles.

What is the best method in any particular case involves two considerations: (α) reaching an answer with the least labor, (β) reaching an answer the most accurate possible from the data. Evidently, if the observer is the computer, he should know beforehand what method of computation is contemplated and avoid observations leading to

(*a*) Angles near 90° to be computed from the sine.

(*b*) Small angles to be computed from the cosine.

EXERCISES.

1. Consider all possible data in right-angled triangles, and determine what formulas will give answers with the least labor.

2. Do the same with triangles not right-angled.

3. In case the three sides of a triangle are given and all three angles are desired, how many logarithms are required if (i) sines are used, (ii) if cosines are used, (iii) if tangents are used?

4. When is it better to calculate by the natural functions rather than by logarithms? Investigate as to whether practical engineers of your acquaintance use logarithms or natural functions in ordinary calculations.

It is frequently stated that an angle can be calculated better from the log tan than from the log sin or log cos.

The table on page 342 will show that, *with a five-place logarithmic table*,

(i) Angles less than about 25° can be calculated as accurately from log sin as from log tan.

(ii) Angles between about 65° and 90° can be calculated as accurately from log cos as from log tan.

(iii) Angles between 25° and 65° can be calculated more accurately from the log tan than from log sin or log cos.

(iv) An angle of any size can be calculated as accurately from log tan as from log sin or log cos.

(v) Small angles cannot be calculated at all accurately from the log cos.

(vi) Angles near 90° cannot be calculated at all accurately from log sin.

(vii) Log tan (log cot) varies the same way at both ends of the table, so, therefore, when an angle is near 90° its value is given as readily by log tan as if it were near 0°, even though $\tan 90° = \infty$.

$A°$	Sin		Tan and Cot		Cos		$A°$
	d	$\frac{60''}{d}$	d	$\frac{60''}{d}$	d	$\frac{60}{d''}$	
1°	724	0″.08	724	0″.08	0.22	273″	89°
2°	362	0.17	362	0.17	0.44	136	88°
3°	241	0.25	242	0.25	0.66	91	87°
4°	181	0.33	182	0.33	0.89	67	86°
5°	144	0.42	145	0.41	1.1	55	85°
10°	72	0.8	74	0.8	2	30	80°
15°	47	1.3	50	1.2	3	20	75°
20°	35	1.7	40	1.5	5	12	70°
25°	27	2.2	33	1.8	6	10	65°
30°	22	2.7	29	2.1	7	9	60°
35°	18	3.3	27	2.2	9	7	55°
40°	15	4.0	26	2.3	11	5	50°
45°	13	4.6	25	2.4	13	5	45°

The columns headed d are the differences in the logarithms for 1′ in the neighborhood of the angles given at the side, and since $\sin A = \cos(90 - A)$, and $\tan A = \cot(90 - A)$, a difference like 724 for sine, opposite 1°, is the difference for cos 89°, etc. Since d is for 1′, a difference of 1 in logarithms corresponds to $\frac{60''}{d}$. The table gives, at various parts of a five-place logarithm table, the difference in seconds corresponding to a difference of 1 in logarithms.

In the neighborhood of 1° a difference of 1 in logarithms corresponds to a difference of about 0.08″ for angles obtained from log sin and for log tan, and to about 273″ for angles obtained from log cos.

Such an angle can, therefore, be determined with equal accuracy from log sin and log tan, but cannot be determined accurately from log cos, where any interpolation is necessary.

Note that up to about 25° the values of $\frac{60''}{d}$ are nearly the same for sine as for tangent and cotangent.

Therefore angles less than about 25° can be determined almost as accurately from the log sin as from log tan, and, consequently, those between 65° and 90° can be determined almost as accurately from the log cos as from log tan.

It is thus clear that the angle can be determined at any place as accurately from the log tan as from the log sin or log cos, while between 25° and 65° it can be determined more accurately from the log tan than from either log sin or log cos, since there a difference of 1 in logarithms corresponds to a smaller difference for tangents than for sines or cosines.

In a table of six, seven, ten places the advantage in calculating from the tangent would appear stronger than it does here with a five-place table.

§ 186. Angles determined when they are Small from Log Sin and Log Tan.

Large tables are arranged to give such angles from the formulas

$$\log \sin x = \log x'' + S$$

$$\log \tan x = \log x'' + T.$$

See Gauss's or Hussey's Tables. The student may be assigned some exercises.

EXERCISES.

1. Explain from the formula $\tan x = \frac{\sin x}{\cos x}$ why at the beginning of the table the difference for log sin is about the same as for log tan, and near 90° the difference for log tan is about the same as for log cos.

2. Explain why the difference for log tan is everywhere the same as for log cot.

3. Explain why the differences for log sin decrease from 0° to 90°.

4. Explain why the differences for log tan and log cot decrease from 0° to 45°, where they are least, and then increase to 90°, being the same at equal distances from 0° and 90°, and reconcile this with $\tan 90° = \infty$ and $\log \tan 90° = \infty$, and explain why it is, in a table running to 45° with function and co-function, like Gauss's or Hussey's, an angle near 89° can be determined as accurately from log tan as an angle near 1°.

WRITTEN REVIEW EXERCISE.

1. Make out a statement for the possible data in right-angled triangles, with the formulas best adapted for calculation in each special case, with suggestions as to what should be done in exceptional cases, when certain angles or lines are small.

2. Do the same for triangles not right-angled.

The following general conclusions should appear:

(*a*) In right-angled triangles:

(i) The cosine (secant) connects the hypothenuse and the side bordering the angle.

(ii) The sine (cosecant) connects the hypothenuse and the side opposite the angle.

(iii) The tangent (cotangent) connects the two sides.

(*b*) In triangles not right-angled:

(i) When a pair of opposites are given, use the sine.

(ii) When three sides are given, use tangents of half-angles.

(iii) When two sides and the included angle are given, get the two remaining angles by tangents (§ 172), and the third side by sines.

3. Make a tabulated statement of the formulas and processes of trigonometry which should be committed to memory. (The extent of this will depend upon the career in view by the class, and the outline is left to the teacher.)

4. Show that $c = a \cos B + b \cos A$, (1), § 125,

becomes, on making

$$a = \sin A \cdot \frac{c}{\sin C}, \quad b = \sin B \cdot \frac{c}{\sin C},$$

since

$$\sin C = \sin (A + B),$$

$$\sin (A + B) = \sin A \cos B + \cos A \sin B \text{ (§ 136)}.$$

Show that this is a general proof.

Show also, from Fig. 127, that

$$-\cos C = \frac{(x+y)^2 - a^2 - b^2}{2\,ab} = \frac{x}{b} \cdot \frac{y}{a} - \frac{p}{b} \cdot \frac{p}{a}.$$

$$\therefore \cos (A + B) = \cos A \cos B - \sin A \sin B.$$

Show that this proof is general.

CHAPTER XII.

THE QUANTITY $\sqrt{-1}$ IN TRIGONOMETRY.

§ 187. The Argand Diagram.

If OA, OC (Fig. 163) are equal in length but opposite in direction,

$$OC = -OA = (-1) \cdot OA.$$

Thus multiplying a line by (-1) is equivalent to a turn through 180°.

Let i (the initial letter of imaginary) stand for $\sqrt{-1}$.

Then $$i \cdot i = -1.$$

$$\therefore OC = (-1) \cdot OA = i \cdot i \cdot OA.$$

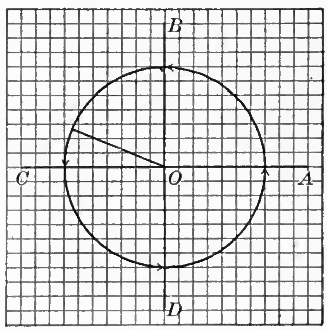


FIG. 163.

Thus, multiplying *twice* by i is equivalent to a turn through 180°. Consequently, a *reasonable interpretation* of multiplying *once* by i is a turn through 90°.

$$\therefore i \cdot OA = OB,$$

$$i \cdot i \cdot OA = (-1) \cdot OA = OC,$$

$$i \cdot i \cdot i \cdot OA = (i \cdot i) \cdot (i \cdot OA) = (-1) \cdot OB = OD,$$

$$i \cdot i \cdot i \cdot i \cdot OA = (i \cdot i) \cdot (i \cdot i) \cdot OA = (-1) \cdot (-1) \cdot OA = OA$$

Thus, if numbers expressed in terms of the unit $(+1)$ are laid out to the right on a horizontal line, those in terms of the unit (-1) are to the left on the same line, while those in terms of the unit $(+i)$ are laid out on the upright vertical and those in terms of the unit $(-i)$ are on the downright vertical.

Numbers of the form $a + bi$, where a, b are in terms of the units (1), (−1), one or both, as

$$5 + 5i, \quad (\alpha)$$
$$-5 + 5i, \quad (\beta)$$
$$-5 - 5i, \quad (\gamma)$$
$$5 - 5i, \quad (\delta)$$

are called *complex* numbers.

They are plotted by laying out a, according to sign, from the vertical axis along the horizontal axis, and then laying out b, according to sign, from the terminal of a, parallel to the vertical axis. The points P_1, P_2, P_3, P_4 (in Fig. 164) represent the numbers (α), (β), (γ), (δ) above, a side of one of the small squares being taken as a unit.

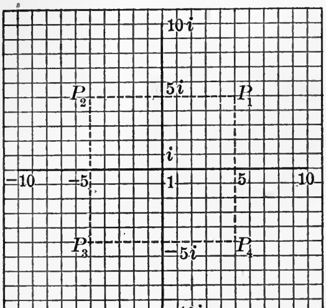


FIG. 164.

When a, b are given all real values, positive, negative, rational, irrational, from zero to infinity, the expression $a + bi$ is a varying quantity, and is represented, in a one to one correspondence, by the points of the plane, each point representing a number and each number represented by a point. Such a diagram-representation of $a + bi$ is called the *Argand Diagram*, from the French mathematician who first made the use of it prominent, though priority of suggestion is accredited to Gauss.

EXERCISES.

On coördinate paper plot the numbers:

$$\pm 1 \pm i;\ \pm 2 \pm 4i;\ \pm 3 \pm 2i;\ \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i;\ \pm\tfrac{1}{2} \pm \frac{\sqrt{3}}{2}i;\ a + bi;$$

$$x + yi;\ \pm\cos 25° \pm i\sin 25°;\ \pm\cos 30° \pm i\sin 30°;\ \pm\cos 100° \pm i\sin 100°;$$

$$\pm\cos 190° \pm i\sin 190°;\ \pm\cos 280° \pm i\sin 280°;\ 2(\cos 30° + i\sin 30°);$$

$$3(\cos 120° + i\sin 120°);\ 5(\cos 103° + i\sin 103°)$$

$$\sqrt{2}(\cos 315° + i\sin 315°);\ \cos 797° + i\sin 797°.$$

§ 188. The Addition (Subtraction) of Complex Numbers.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

P_1 is $a + bi$, if $OM_1 = a$, and $M_1P_1 = b$, (Fig. 165)

and P_2 is $c + di$, if $OM_2 = c$, and $M_2P_2 = d$.

If the parallelogram on OP_1, OP_2 is completed as in Fig. 165, evidently,

P_3 is $(a + c) + (b + d)i$, with $OM_3 = a + c$, and $M_3P_3 = b + d$.

Thus, to add P_2 to P_1, start with P_1, and draw a line, P_1P_3, parallel and equal to OP_2 (and in the same direction). The point reached, P_3, represents the sum, $P_1 + P_2$.

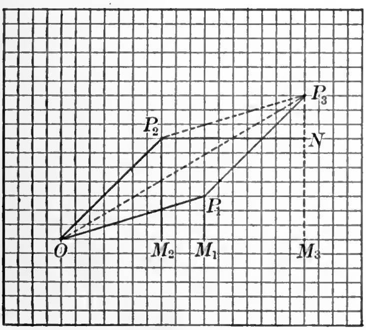


FIG. 165.

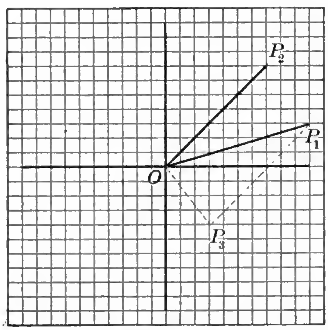


FIG. 166.

Similarly, since subtraction is the reverse operation to addition, to subtract P_2 from P_1 (Fig. 166), start with P_1, and lay out P_1P_3 parallel and equal to P_2O (not OP_2).

Thus, points are added (subtracted) as forces in mechanics.

EXERCISES.

The teacher may select some examples for plotting additions, subtractions, making the selection so that pairs of points fall in the same quadrant, and in different quadrants (all possible combinations). Use coördinate paper (ten by ten to the inch will be found convenient).

§ 189. Points on the Unit-circle, and on Any Circle.

The point P_1 in the unit-circle evidently represents

$$\cos\theta + i\sin\theta.$$

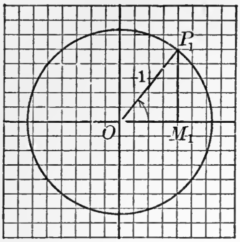


FIG. 167.

Thus, if θ is given all values from 0 to 2π, the expression $\cos\theta + i\sin\theta$ will represent all points on the unit-circle.

Similarly, if a, b vary subject to the condition $\sqrt{a^2+b^2}$ remaining constant, the expression $a + bi$ will represent all points on the circle of radius $\sqrt{a^2+b^2}$.

If a, b are constants, $a + bi$ represents a point at a distance $\sqrt{a^2+b^2}$ from the origin and in the direction $\tan^{-1}\frac{b}{a}$.

Thus a complex number depends upon two things, the distance and the direction of its representing point from the origin. The former is called the *modulus* of the number, also its *absolute value.* The latter is called the *amplitude*, or *angle.* For the amplitude, the smallest *positive* angle less than 360° locating the modulus is taken.

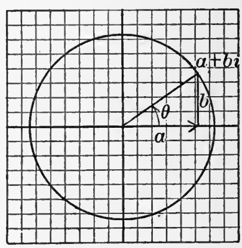


FIG. 168.

$$a + bi = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + i\cdot\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$= \sqrt{a^2+b^2}\,(\cos\theta + i\sin\theta).$$

Thus, $a + bi$ has two *factors*, a length factor and a directing factor.

The radical, $+\sqrt{a^2+b^2} = r$, is the *modulus*, or length factor.

The factor, $\cos\theta + i\sin\theta$, is the *directing factor.*

The angle θ is the *amplitude.*

The radical, $+\sqrt{a^2+b^2}$, is an absolute number, or a number in terms of the unit $(+1)$.

Since $a + bi = (\cos\theta + i\sin\theta)\cdot\sqrt{a^2+b^2}$, and since the first factor represents a point on the unit-circle, we may look upon the point P, representing $a + bi$, as derived from the corresponding point P_1 on the unit-circle by multiplication by the absolute number $+\sqrt{a^2+b^2}$. Thus with

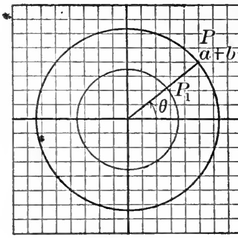


FIG. 169.

$$a + bi = (\cos\theta + i\sin\theta)\cdot\sqrt{a^2+b^2}$$

and
$$c + di = (\cos\phi + i\sin\phi)\cdot\sqrt{c^2+d^2},$$

$$(a+bi)(c+di)$$
$$=\{(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\}\cdot\{\sqrt{a^2+b^2}\cdot\sqrt{c^2+d^2}\}.$$

Hence the multiplication of any two complex numbers can be reduced to the multiplication of two numbers (points) on the unit circle, together with the multiplication of two absolute numbers as in arithmetic.

§ 190. Multiplication (Division) of Points on the Unit-circle.

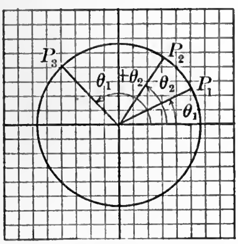


FIG. 170.

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$
$$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).$$
$$\text{or, } P_1\cdot P_2 = P_3.$$

Thus any two points on the unit-circle are multiplied by adding (algebraically) their angles.

Since division is the reverse operation of multiplication, P_1 on the unit-circle is divided by P_2 on the same circle by subtracting (algebraically) the angle of P_2 from that of P_1. Similarly, the point representing the product of three such points has its angle equal to the algebraic sum of the angles of the factors.

COR.: In general, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, when n is a *positive integer.*

EXERCISES.

1. Prove by radicals and direct multiplication that $(\cos 30° + i \sin 30°)^2 = \cos 60° + i \sin 60°$, and that $(\cos 60° + i \sin 60°)^2 = \cos 120° + i \sin 120°$.

2. Prove by direct multiplication, using the tables, that $(\cos 10° + i \sin 10°)(\cos 20° + i \sin 20°) = \cos 30° + i \sin 30°$.

3. Prove by using radicals, with direct multiplication, that $(\cos 30° + i \sin 30°)^3 = i$; $(\cos 120° + i \sin 120°)^3 = 1$; $(\cos 60° + i \sin 60°)^3 = -1$; $(\cos 300° + i \sin 300°)^3 = -1$.

4. Prove by using the tables that $(\cos 36° + i \sin 36°)^2 = \cos 72° + i \sin 72°$.

5. Prove by using radicals, with direct multiplication, that $(\cos 45° + i \sin 45°)^4 = -1$; $(\cos 135° + i \sin 135°)^4 = -1$; $(\cos 225° + i \sin 225°)^4 = -1$; $(\cos 315° + i \sin 315°)^4 = -1$.

6. Use a diagram (Groat's coördinate paper is convenient) to solve the preceding examples.

7. Prove directly, and by diagram, $\dfrac{1}{\cos\theta - i\sin\theta} = \cos\theta + i\sin\theta$, and show that if $P_1 \cdot P_2 = 1$, P_1, P_2 are symmetric to the horizontal axis.

§ 191. Multiplication of Points not on the Unit-circle.

This may be carried out in three ways.

(*a*) *Direct multiplication.*

EXAMPLE :

$$\begin{array}{l} 2 + 3i \\ \underline{4 - 5i} \\ 8 + 2i + 15 = 23 + 2i. \end{array}$$

This method is suited to the case when a, b in $a + bi$ are small.

(*b*) *Put each number in the form of its factors* (*modulus and directing factor*).

Then the product is evidently the product of the moduli into the directing factor of the product, whose angle, as for points on the unit-circle, is the algebraic sum of the angles of the factors. It is necessary to use the tables, except in those few special cases when the angles are such that their sines and cosines are known without use of the tables, as 30°, 45°, etc.

(i) *Example with tables not necessary.*

$$(1+i)(\sqrt{3}+i).$$

$$1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\sqrt{2}(\cos 45^\circ + i\sin 45^\circ).$$

$$\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2(\cos 30^\circ + i\sin 30^\circ).$$

$$\therefore\ (1+i)(\sqrt{3}+i)=2\sqrt{2}(\cos 75^\circ + i\sin 75^\circ).$$

The sine, cosine of 75° may now be taken from the tables, or expressed in radicals.

(ii) *Example in which tables are advisable.*

$$(375+274\,i)(432+548\,i).$$

$$375+274\,i=\sqrt{(375)^2+(274)^2}(\cos A + i\sin A),$$

$$\text{where } A=\tan^{-1}\tfrac{274}{375}.$$

A is best found by the use of logarithms, and the value of the modulus from a table of squares.

The second factor is treated in the same way.

The product is then $r_1r_2\{\cos(A+B)+i\sin(A+B)\}$.

The student may complete calculations.

(*c*) *Pictorial multiplication.*

Since the modulus of the product of two factors is the product of the moduli of the factors, and the angle is the sum of the angles, constructive multiplication is readily carried out by the use of similar triangles. Join one of the points, as P_1 (Fig. 171), representing one of the factors of the product, to the unit point, A.

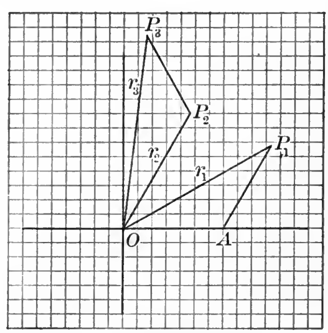


Fig. 171.

Construct the triangle P_2OP_3 similar to the triangle AOP_1, by making the angle P_2OP_3 equal to the angle AOP_1 and the angle OP_2P_3 equal to the angle OAP_1. P_3 represents the product $P_1\cdot P_2$, for $r_3 : r_2 :: r_1 : 1$.

$\therefore\ r_3 = r_1 r_2$, and the angle of P_3 is the sum of those of P_1, P_2. A scale of equal parts to measure OP_3, a protractor to measure the angle AOP_3, and the tables, will give an approximate numerical value for P_3, if it is desired. Groat's paper can be used very advantageously when the points are located by polar coördinates or by forms like $r(\cos\theta + i\sin\theta)$.

EXERCISES.

1. Find the product of $(1+i)(-1-i)$; $(3+4i)(5+2i)$; $(1+i)(-\sqrt{3}+i)$; $(1+\sqrt{2}+i)(1-\sqrt{2}-i)$; $(21+37i)(64-18i)$; $(273+564i)(613+515i)$.

2. Simplify $\dfrac{(\cos\theta - i\sin\theta)^{10}}{(\cos\theta + i\sin\theta)^{12}}$; $\dfrac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$; $\dfrac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 4\theta + i\sin 4\theta)^{12}(\cos 5\theta - i\sin 5\theta)^{-6}}$.

HINT: Make use of Ex. 7, § 190.

§ 192. Division of Points not on the Unit-circle.

(*a*) Since $(a+bi)(a-bi) = a^2 + b^2$, *division by* $a + bi$ *is multiplication* by $\dfrac{a-bi}{a^2+b^2}$ (as in case (*a*) of the preceding section).

EXAMPLE: $\dfrac{2+3i}{3+4i} = \dfrac{(2+3i)(3-4i)}{25} = \dfrac{18+i}{25}$.

This is the best method when a, b are small.

(*b*) *By using the directing angles of the numbers.*

$$\frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{r}\cdot\frac{\cos^2\theta + \sin^2\theta}{\cos\theta + i\sin\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)).$$

$$\therefore\ \frac{a+bi}{c+di} = \frac{r_1}{r_2}\cdot(\cos\theta_1 + i\sin\theta_1)(\cos(-\theta_2) + i\sin(-\theta_2))$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\},$$

where r_1, r_2, are the moduli of $a+bi$, $c+di$; θ_1, θ_2, their directing angles.

Thus the modulus of the quotient is the quotient of the moduli, and the directing angle of the quotient is that of the dividend minus that of the divisor.

Two cases may arise :

(1) *The directing angles may be apparent without the use of tables*, as in

$$\frac{\sqrt{3}+i}{1+i}=\frac{2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)}{\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}=\frac{2}{\sqrt{2}}\cdot\frac{\cos 30^\circ+i\sin 30^\circ}{\cos 45^\circ+i\sin 45^\circ}$$

$$=\sqrt{2}\cdot(\cos 15^\circ - i\sin 15^\circ),$$

and the final numerical value can here be obtained either with or without the use of tables.

Such examples are of infrequent occurrence, when not made to order.

(2) *Tables advisable.*

$$\frac{213+315\,i}{314+426\,i}=\frac{\sqrt{(213)^2+(315)^2}}{\sqrt{(314)^2+(426)^2}}\cdot\frac{\cos x+i\sin x}{\cos y+i\sin y},$$

where $\tan x=\frac{315}{213}$, and $\tan y=\frac{426}{314}$.

x, y may be found from tables and the work carried out as above.

Unless a table of squares is at hand to obtain the values of the radicals, there is no advantage in this method over the direct process of (a).

(c) *Pictorial division.*

This is carried out similarly to pictorial multiplication. Instead of advancing the terminal of the dividend let it retrograde (algebraically) by the angle of the divisor. The triangle P_3OP_2 (Fig. 172) is made similar to IOP_1, by turning OP_2 back (algebraically) an angle equal to IOP_1, and making at P_2 the angle OP_2P_3 equal to IP_1O (if rectangular paper is used). Then P_3 represents $\frac{P_2}{P_1}$, for the angle of P_3 is that of P_2 minus that of P_1, and, from similar triangles, $\frac{r_3}{r_2}=\frac{1}{r_1}$, or

$r_3 = \frac{r_2}{r_1}$. In the diagram P_2 is (14, 50°), P_1 is (7, 15°), I (the unit-point) is (5, 0°); so P_3 is (10, 35°).

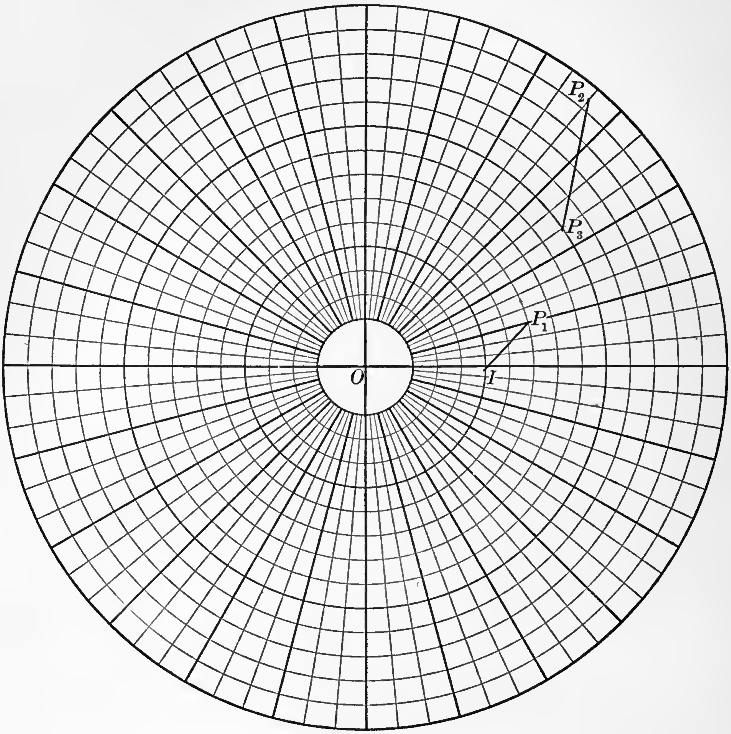


FIG. 172.

EXERCISES.

1. Divide $1 + i$ by $-1 + \sqrt{-3}$ by the first and second methods, without using the tables. Carry out the division also pictorially.

2. Divide $1 + \sqrt{3}\, i$ by $-1 - \sqrt{3}\, i$, as in Ex. 1.

3. Divide $213 + 321\, i$ by $324 - 245\, i$.

4. Find the quotient of any point by its opposite point (the point symmetric to the given point with reference to the origin). What is the result when the points are on the unit-circle? Find also the product of such pairs of points.

5. Carry out pictorial division on pairs of points selected with variety of quadrantal position, using Groat's paper. Pictorial multiplication.

§ 193. Integral Roots of Numbers (Points) on the Unit-circle.

If P_1, P_2, P_3, three points on the unit-circle whose angles are $\frac{A}{3}$, $\frac{A}{3} + 120°$, $\frac{A}{3} + 240°$, be cubed, there will result the single point whose angle is A. Thus, there are three cube roots for any point on the unit-circle, and they are 120° apart at the vertices of an equilateral triangle.

It is also easily apparent that there are only three cube roots, for if the angle A locates a terminal, the same terminal is located by $A + n \cdot 360°$, and since any number divided by three can leave only the remainders 0, 1, 2, the integer n above can have only the three forms $3m$, $3m + 1$, $3m + 2$, where m is an integer. Thus $\frac{A + n \cdot 360°}{3}$ will be

$$\frac{A}{3} + m \cdot 360, \text{ or } \frac{A}{3} + m \cdot 360° + 120°, \text{ or } \frac{A}{3} + m \cdot 360° + 240°.$$

These angles locate *only* the three terminals located by $\frac{A}{3}$, $\frac{A}{3} + 120°$, $\frac{A}{3} + 240°$, as above. Thus there are three, and only three, cube roots for a point on the unit-circle.

Similarly, if the four points on the unit-circle with the angles $\frac{A}{4}$, $\frac{A}{4} + 90°$, $\frac{A}{4} + 180°$, $\frac{A}{4} + 270°$ are raised to the fourth power, there will result the single point whose angle is A. We have thus four, and only four, fourth roots for the point whose angle is A.

And, in general, the nth power of points whose angles are

$$\frac{A}{n}, \quad \frac{A}{n} + \frac{360°}{n}, \quad \frac{A}{n} + 2 \cdot \frac{360°}{n}, \quad \cdots \quad \frac{A}{n} + (n - 1)\frac{360°}{n}, \qquad (\alpha)$$

will be the single point whose angle is A. And these points are the n nth roots of the point whose angle is A. The argument for only three cube roots holds for only n nth roots, for an nth of the angle locating a given terminal, or, $\frac{A + r \cdot 360°}{n}$, can have only the n values (α) above plus multiples of 360°, locating only n different terminals, since r divided by n can leave only n different remainders.

§ 194. Integral Roots of Positive Unity (or Solution of $x^n - 1 = 0$).

This is the special case of the preceding paragraph when $A = 0°$.

Thus, the three cube roots of unity are located by 0°, 120°, 240°, and are 1, $\cos 120° + i \sin 120°$, $\cos 240° + i \sin 240°$, or 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$. (Fig. 173.)

The four fourth roots of unity are located by 0°, 90°, 180°, 270°, and are 1, i, -1, $-i$. (Fig. 174.)

The five fifth roots of unity are located by 0°, 72°, 144°, 216°, 288°, 432°, and are 1, $\cos 72° + i \sin 72°$, etc., and may be calculated by the tables. (Fig. 175.)

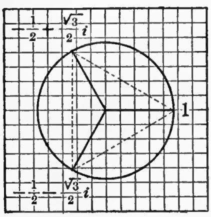


FIG. 173.

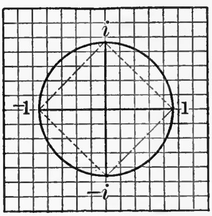


FIG. 174.

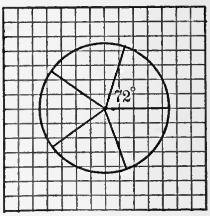


FIG. 175.

The corresponding roots are indicated on Figs. 173, 174, and 175, at the corners of an equilateral triangle; at the corners of a square; at the corners of a regular pentagon, etc.

Similarly, if n is a positive integer, the n nth roots of unity are at the corners of a regular n-gon, one of whose vertices is at unity.

EXERCISES.

1. Prove by direct multiplication that the cube roots of 1 given above produce 1 when cubed. Treat the fourth roots in the same way.

2. Show by pictorial multiplication that each cube root above is a cube root of 1. Treat the fourth and fifth roots the same way, pictorially.

3. Find pictorially the six sixth roots of unity and their values from the diagram, and show by pictorial multiplication that each is a sixth root of unity.

4. Use the tables to calculate the 7th, 8th, 9th, 10th roots of 1.

5. Solve by algebraic processes the equations, $x^3 - 1 = 0$, $x^4 - 1 = 0$, $x^6 - 1 = 0$, and compare the results and labor with the pictorial method of solution.

6. Show pictorially and algebraically that if α is any complex cube root of unity, $\alpha^2 + \alpha + 1 = 0$.

7. Show pictorially and algebraically that the sum of the cube roots of 1 is zero. Also that the sum of the fourth roots is zero; sum of the fifth roots is zero; sum of the nth roots is zero.

8. Show that if α is a complex fourth root of unity, $\alpha^3+\alpha^2+\alpha+1=0$.

9. State and prove the general proposition corresponding to Exs. 6 and 8.

10. Show that the sum of the products of the cube roots of unity, taken two and two, is zero. Show this pictorially and algebraically. Show that the product of the cube roots of unity is $+1$.

11. Show for the fourth roots of unity that the sums of their products, two at a time, three at a time, are zero. Prove that their product is -1. Prove these propositions pictorially and algebraically.

12. If you have some acquaintance with the theory of equations, state and prove the general propositions of which Exs. 10 and 11 are special cases.

13. Use the tables to solve the equations:
$x^3 = \cos 15° + i \sin 15°$, $x^4 = \cos 24° + i \sin 24°$, $x^5 = \cos 25° + i \sin 25°$, $x^6 = \cos 144° + i \sin 144°$, $x^8 = 213 + 185° i$. Give diagrams.

§ 195. Integral Roots of Negative Unity (or Solution of $x^n + 1 = 0$).

This is a special case of § 193 when $A = 180°$. Thus, the three cube roots of -1 are located by $60°$, $60° + 120°$, $60° + 240°$, and are $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, -1, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$, as shown by Fig. 176.

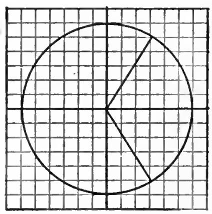


Fig. 176.

The four fourth roots of -1 are located by $45°$, $45°+90°$, $45°+180°$, $45°+270°$, and have the four values $\pm \cos 45° \pm i \sin 45°$, or $\pm \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$, as shown by Fig. 177.

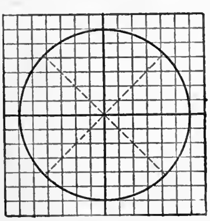


Fig. 177.

Similarly, the n nth roots of -1 are the n values which $\cos\frac{2r+1}{n}180° + i\sin\frac{2r+1}{n}180°$ can take when r is 0, 1, 2, 3, ⋯ $n-1$, these being the only values such an expression can take when r takes all integral values from $-\infty$ to $+\infty$.

EXERCISES.

1. Solve by algebraic process the equations $x^3+1=0$, $x^4+1=0$, and compare the results with the pictorial solution.

2. Solve pictorially the equations $x^2+1=0$, $x^5+1=0$, $x^6+1=0$, $x^7+1=0$, $x^8+1=0$, using the tables when necessary to get the numerical solution.

3. Show pictorially that each root found in the preceding cases is the appropriate root.

4. State and prove for roots of negative unity propositions corresponding to Exs. 4–8 inclusive, of the preceding section.

§ 196. Integral Roots of Any Number, $a + bi$.

Set the given number in the form of *modulus times directing factor*, the latter being taken in its general form. Then an nth root is the ordinary nth root of the modulus times any one of the n nth roots of the directing factor, found as already given for points on the unit-circle.

$$a+bi=\sqrt{a^2+b^2}\,\{\cos(2r\cdot180°+A)+i\sin(2r\cdot180°+A)\},$$

where A is the *smallest positive angle locating* $a+bi$,

$$\text{and}\left\{(a^2+b^2)^{\frac{1}{2n}}\cos\frac{2r\cdot180°+A}{n}+i\sin\frac{2r\cdot180°+A}{n}\right\}^n = a+bi.$$

The n nth roots of $a+bi$ are the n different values which the bracketed expression can take when r is given the values 0, 1, 2, 3, ⋯ $n-1$.

Thus the roots for points on any circle with radius R are located on the same terminal lines as the corresponding roots for corresponding points on the unit-circle, and all lie on the circle whose radius is the ordinary required root of R.

The cube roots of 8 are located on the circle of radius 2, as are the cube roots of unity on the unit-circle.

EXERCISES.

1. Locate the three cube roots of -8; the four fourth roots of 16 and -16; the five fifth roots of 32 and -32. Give the numerical values of these roots.

2. Use the tables to calculate the six sixth roots of $64(\cos 234° + i \sin 234°)$; the three cube roots of $213 + 432\,i$.

§ 197. De Moivre's Theorem in General.

The multiplications, divisions, powers, and roots, given in the preceding paragraphs are all special cases of what is known as De Moivre's theorem, which has two forms:

(*a*) $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)(\;)$
$(\;)\cdots = \cos(\alpha+\beta+\gamma+\cdots) + i\sin(\alpha+\beta+\gamma+\cdots).$

(*b*) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

Form (*a*) is only the multiplication of complex numbers on the unit-circle (§ 190).

Form (*b*) can be proven readily for all real values of n,— integers, fractions, irrational numbers,—positive, negative. We shall not consider the case when n is itself complex.

(1) *n a positive integer.* The proof is the generalization of the multiplication in § 190, for n factors all equal.

(2) *n a negative integer.*

Let $n = -m$, so that m is a positive integer.

Then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$
$$= \frac{1}{\cos m\theta + i\sin m\theta} = \frac{\cos^2 m\theta + \sin^2 m\theta}{\cos m\theta + i\sin m\theta}$$
$$= \cos m\theta - i\sin m\theta = \cos(-m)\theta + i\sin(-m)\theta$$
$$= \cos n\theta + i\sin n\theta.$$

(3) *n a positive common fraction with numerator* 1, *as*

$$n = \frac{1}{q}.$$

In this case, instead of writing

$$(\cos\theta + i\sin\theta)^{\frac{1}{q}} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q},$$

the general angle $2r\pi + \theta$ locating the same terminal that θ does should be taken, since in this case $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$ is q-valued, while $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is single-valued.

By (1),

$$\left(\cos\frac{2r\pi+\theta}{q}+i\sin\frac{2r\pi+\theta}{q}\right)^q = \cos(2r\pi+\theta)+i\sin(2r\pi+\theta)$$
$$= \cos\theta + i\sin\theta.$$

$$\therefore \cos\frac{2r\pi+\theta}{q}+i\sin\frac{2r\pi+\theta}{q} = (\cos\theta + i\sin\theta)^{\frac{1}{q}}.$$

The left-hand member can take q, and only q, values, namely, those it takes when $r = 0, 1, 2 \cdots n-1$.

(4) *n a negative common fraction with numerator* 1.

The proof follows from (3) as (2) from (1).

The result may be set in the form:

$$\cos\frac{2\pi r+\theta}{q} - i\sin\frac{2\pi r+\theta}{q} = (\cos\theta + i\sin\theta)^{-\frac{1}{q}}.$$

(5) *n any positive common fraction, as* $\frac{p}{q}$.

Here also the qth root is q-valued and the general angle is to be taken.

By (1),

$$\left(\cos\frac{2\pi r+p\theta}{q}+i\sin\frac{2\pi r+p\theta}{q}\right)^q$$
$$= \cos(2r\pi+p\theta)+i\sin(2r\pi+p\theta)$$
$$= \cos p\theta + i\sin p\theta$$
$$= (\cos\theta + i\sin\theta)^p$$

$$\therefore \cos\frac{2\pi r+p\theta}{q}+i\sin\frac{2\pi r+p\theta}{q} = (\cos\theta + i\sin\theta)^{\frac{p}{q}}.$$

The q-values which the left-hand member can take when $r = 0, 1, 2, 3 \cdots q-1$, are the q-values of the expression $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$.

(6) *n a negative common fraction.*

The proof follows from (5) as (2) from (1).

(7) *n an irrational number.*

An irrational number like 3.14159 ⋯, can be looked upon as the limit toward which the numbers 3, 3.1, 3.14, 3.141, 3.1415, etc., are approaching. Thus, by taking more and more figures of the irrational number a common fraction can be found from which the irrational number differs by an amount as small as we please. Between two such fractions, one above the irrational number and one below it, lies any irrational number. The theorem holds for the common fractions, and thus in the limit may be proved to hold for the irrational number.

EXERCISES.

(A "cyclic group.")

Prove by direct multiplication, and also on Groat's polar coördinate paper, the following:

1. Given the six roots of $x^6 = 1$: 1; $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$; $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$; $\cos\pi + i\sin\pi$; $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$; $\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$.

(*a*) Show that the product of any two is a third one of the set.

(*b*) Show that $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^x$ where $x = 0, 1, 2, 3, 4, 5$ are the six given numbers. Show that the same is true of $\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$. Is it true of any other one of the set?

(*c*) To what power must each be raised to produce 1?

(*d*) Show that $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{-1} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$, and that $\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)^{-1} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. Is this true of any other two? What ones produce themselves when raised to the power $^{-1}$?

(*e*) Show that all the six numbers can be obtained from $\cos\pi + i\sin\pi$ and $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ by forming their powers and successive products. Is this true of any other two of the set?

(*f*) The above numbers are an example of a *cyclic group.*

2. Write down the roots of $x^{10} = 1$, and of $x^{12} = 1$ and discuss them as the above numbers are discussed.

3. Assuming De Moivre's Theorem, prove that $\sin(\alpha + \beta + \gamma) = \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\gamma\cos\alpha + \sin\gamma\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma$.

§ 198. Use of De Moivre's Theorem to Express Cosines and Sines of Multiples of an Angle in Terms of Powers of Cosines and Sines of the Angle.

If $$a + bi = c + di,$$
then $$a = c, \text{ and } b = d,$$
for each complex number locates but a single point on the Argand diagram. By (*b*), § 197,

$$\cos 2\theta + i \sin 2\theta = (\cos\theta + i\sin\theta)^2$$
$$= \cos^2\theta - \sin^2\theta + i\cdot 2\sin\theta\cos\theta.$$

Equating reals on opposite sides, also imaginaries on opposite sides:
$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$
$$\sin 2\theta = 2\sin\theta\cos\theta, \text{ as in § 139.}$$

The general process is illustrated in this special example. Similarly:

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta \sin\theta - \sin^3\theta).$$
$$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta;$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta. \quad (§ 144)$$

EXERCISES.

1. Express similarly: $\cos 4\theta$, $\sin 4\theta$, $\cos 5\theta$, $\sin 5\theta$, $\cos 6\theta$, $\sin 6\theta$, $\cos n\theta$, $\sin n\theta$.

2. Express in terms of $\tan\theta$, the values of $\tan 2\theta$, $\tan 3\theta$, $\tan 4\theta$, $\tan 5\theta$, $\tan 6\theta$, $\tan 7\theta$.

§ 199. Use of De Moivre's Theorem to Express Powers of Cosines, Sines in Terms of Cosines, Sines of Multiple Angles.

$$\left.\begin{aligned} &\text{Let } \cos\theta + i\sin\theta = z. \\ &\text{Then } \cos\theta - i\sin\theta = \frac{1}{z}, \end{aligned}\right\} \text{ and } \left\{\begin{aligned} &\cos n\theta + i\sin n\theta = z^n. \\ &\cos n\theta - i\sin n\theta = \frac{1}{z^n}. \end{aligned}\right.$$

$$\therefore \left\{\begin{aligned} 2\cos\theta &= z + \frac{1}{z}, \quad (1) \\ 2i\sin\theta &= z - \frac{1}{z}, \quad (2) \end{aligned}\right\} \text{ and } \left\{\begin{aligned} 2\cos n\theta &= z^n + \frac{1}{z^n}, \quad (3) \\ 2i\sin n\theta &= z^n - \frac{1}{z^n}. \quad (4) \end{aligned}\right\}$$

(1), with the aid of (3), will express any power of a cosine in terms of cosines of multiple angles.

(2), with the aid of (3) or (4), according as an even or odd power is given, will express any power of a sine in terms of cosines or sines of multiple angles.

Thus, for $\cos^3\theta$,

by (1), $$2^3\cos^3\theta=\left(z+\frac{1}{z}\right)^3=z^3+3z^2\cdot\frac{1}{z}+3z\left(\frac{1}{z}\right)^2+\frac{1}{z^3}$$
$$=\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)$$
$$=2\cos 3\theta+3\cdot 2\cos\theta.$$
$$\therefore\ \cos^3\theta=\tfrac{1}{4}\{\cos 3\theta+3\cos\theta\}.$$

Similarly, for $\sin^3\theta$,

by (2), $$2^3 i^3\sin^3\theta=\left(z-\frac{1}{z}\right)^3=z^3-\frac{1}{z^3}-3\left(z-\frac{1}{z}\right)$$
$$=2i\sin 3\theta-3\cdot 2i\sin\theta.$$
$$\therefore\ -\sin^3\theta=\tfrac{1}{4}(\sin 3\theta-3\sin\theta).$$

For $\sin^4\theta$,

by (2), $$2^4 i^4\cdot\sin^4\theta=\left(z-\frac{1}{z}\right)^4=\left(z^4+\frac{1}{z^4}\right)-4\left(z^2+\frac{1}{z^2}\right)+6$$
$$=2\cos 4\theta-4\cdot 2\cos 2\theta+6$$
$$\therefore\ \sin^4\theta=\tfrac{1}{8}(\cos 4\theta-4\cos 2\theta+3).$$

A little study of the preceding examples will make it possible to write the value for any power of sine or cosine without making the actual expansion by the binomial theorem:

(*a*) The division factor for cubes is 4, or 2^2; for fourth powers it is 2^3; for nth powers it is 2^{n-1}.

(*b*) For powers of cosines there are only cosines in the second member, starting with the same multiple angle as the power given, dropping two each term, the coefficients being those of the first half of the binomial expansion.

(*c*) Even powers of sines turn into cosines, odd powers into sines, with alternation of sign, the coefficients being those of the first half of the binomial expansion. The sign of the first member is determined by that of the corresponding power of i. Since $i^4=1$, the value of any power of i is i, i^2,

i^3, or 1. Divide the given power by 4; sign is + for remainders 0, 1; −, for remainders 2, 3.

(*d*) The binomial coefficients (the first half) are formed by Pascal's rule, as indicated in the following set:

1,	2				
1,	3,	3			
1,	4,	6			
1,	5,	10,	10		
1,	6,	15,	20		
1,	7,	21,	35,	35	
1,	8,	28,	56,	70,	etc.

A number under a number in any line is formed from that immediately above it by the addition to the latter of the number immediately preceding it.

From such a table we can write immediately the value of any power covered by the table. For example:

$$\cos^8\theta = \frac{1}{2^7}(\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35),$$

$$\sin^8\theta = \frac{1}{2^7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35),$$

$$-\sin^7\theta = \frac{1}{2^6}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta).$$

EXERCISES.

1. Fill out the preceding table of binomial coefficients to the twelfth power, and use it to write down the values of the first twelve powers of sine and cosine.

2. Carry out the expansions direct for $\cos^6\theta$, $\sin^6\theta$.

3. Show that an expression for $\cos^5\theta\sin^7\theta$ can be obtained by multiplying together the expansions for $\left(x^2 - \frac{1}{x^2}\right)^5$ and $\left(x - \frac{1}{x}\right)^2$, giving

$$-2^{11}\cos^5\theta\sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta.$$

4. Show that $\sin^m\theta\cos^n\theta$, for any positive integral values of m, n, can be changed to multiple angles, by changing each factor, multiplying results together, and then changing products of functions of multiple angles to sums (differences) by § 147. Change $\sin^8\theta\cos^{10}\theta$.

5. Expand $x \cos v + y \sin v + iz$ to the second, third, fourth, fifth powers, and show that in each case the coefficients of the sines (cosines) of the multiple angles are homogeneous functions of x, y, z, and that in no case are the coefficients linearly connected, that is, connected in a first degree additive (subtractive) relation.

§ 200. Relation of Sines, Cosines to the Exponential Imaginary.

We have shown in §§ 8, 156 that

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \cdots. \tag{1}$$

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \cdots. \tag{2}$$

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \frac{x^6}{\lfloor 6} + \frac{x^7}{\lfloor 7} + \cdots. \tag{3}$$

Assuming that (3) holds for $x = i\theta$,

$$e^{i\theta} = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} \cdots + i\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} \cdots\right).$$

$$\therefore\ e^{i\theta} = \cos\theta + i\sin\theta. \tag{4}$$

and

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{5}$$

$$\therefore\ \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta. \tag{6}$$

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta. \tag{7}$$

$$\therefore\ \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = \tan\theta. \tag{8}$$

The preceding equations, (4), (5), (6), (7), (8), give the connection between the trigonometric and exponential functions.

EXERCISES.

1. Use (6), (7) to prove the addition-subtraction formulas.

2. Use (6), (7) to prove $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$, and the other corresponding formulas.

§ 201. Periodicity of the Exponential Function.

By equation (4) of § 200,

$$e^{i\theta} = \cos\theta + i\sin\theta = \cos(2n\pi + \theta) + i\sin(2n\pi + \theta) = e^{i\theta + 2n\pi i}.$$

$$\therefore\ e^{i\theta} = e^{i\theta+2n\pi i} = e^{i\theta}\cdot e^{2n\pi i}.\quad \therefore\ e^{2n\pi i} = 1.\quad \therefore\ e^{x+2n\pi i} = e^{x}.$$

Thus, exponential $x,(e^x)$, is periodic, the period being $2\pi i$.

§ 202. Every Number has an Infinite Number of Logarithms.

(*a*) Base *e*. $y = e^x = e^{x+2n\pi i}$, by § 201.

$\therefore\ x + 2n\pi i$, n taking all integer values from minus infinity to plus infinity, is the infinite set of logarithms of y, x being the ordinary table-logarithm.

(*b*) Any base.

If $$a^x = y \text{ and } e^{\log_e a} = a,$$

then $$y = e^{x\log_e a} = e^{x\log_e a + 2n\pi i},$$

and $x\log_e a + 2n\pi i$ makes up, as before, the infinite set of logarithms.

§ 203. Logarithmic Picture of the Points in a Plane.

The numbers $a + bi$, when a, b take all real values, cover a plane completely (§ 187). We may now take the logarithms of all numbers in one plane and plot these logarithms on another plane.

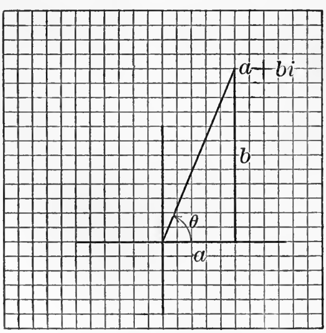


FIG. 178.—*a*, *b* plane.

Since

$$a + bi = \sqrt{a^2+b^2}(\cos\theta + i\sin\theta),$$

where $\tan\theta = \frac{b}{a}$, any number can be set in the form $R\cdot e^{i\theta}$, where R is its modulus.

$$\therefore\ N = R\cdot e^{i\theta} = R\cdot e^{i\theta+2n\pi i}.$$

$$\therefore\ \log_e N = \log_e R + i(\theta + 2n\pi)$$
$$= x + iy.$$

Here y is infinite-valued, as n goes from $-\infty$ to $+\infty$.

If N is a point on the unit-circle, $\log_e R = 0$, and thus points on the unit-circle of one plane plot into points on the y-axis of the other plane. Any point of one plane plots into an infinite number of points, 2π distant from each other, on a line parallel to the y-axis, at a distance x, or $\log_e R$ from it.

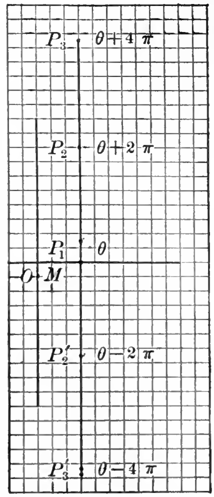


Fig. 179. — x, y plane.

The point P thus plots into

$$P_1, P_2, P_3, \cdots, P'_2, P'_3, \cdots,$$

where

$$OM = \log_e R,$$
$$MP_1 = \theta,$$
$$P_1P_2 = 2\pi, \text{ etc.}$$

§ 204. Mercator's Projection.

If A, B are the extremities of a diameter of a sphere, and if a plane touches the sphere at A, and if a straight line is then drawn from B through each point of the sphere back to the plane, we have what is called a *polar projection* of the sphere on the plane. If A, B represent the north and south poles of the earth, the earth's meridian curves become rays from A on the plane and the latitude curves become concentric circles about A.

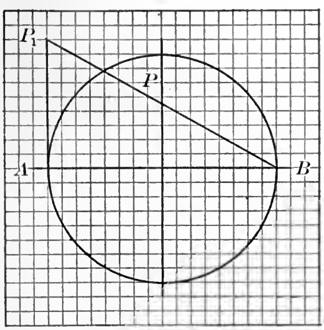


Fig. 180.

If now, we take the logarithm of this polar projection, we have, when these logarithms are plotted, *Mercator's Projection.* Which circle on the polar projection is taken for the unit-circle is, of course, quite arbitrary. This unit-circle plots (§ 203) into the y-axis. By § 203, circles concentric with and outside the unit-circle plot into lines parallel to the y-axis, and to the right of it, while those within the unit-circle plot into lines to the left of the y-axis and parallel to it. Radial lines on the polar projection plot into lines parallel to the x-axis on the Mercator map. The radius whose directing angle is a radian goes a unit's distance up, etc.

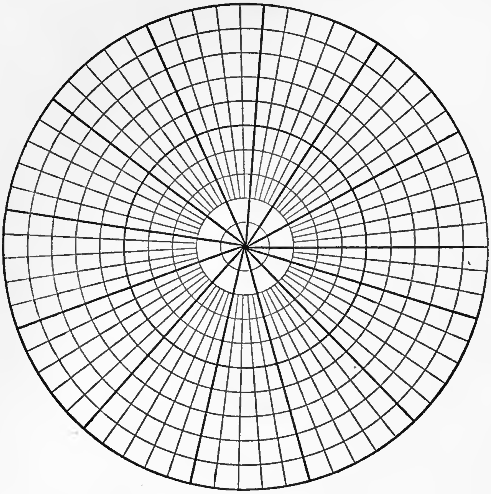


Fig. 181. — Polar projection.

Thus, if we take the logarithm of the numbers represented by a sheet of Groat's polar coördinate paper in radian measure (a picture of a polar projection of the earth) we have the numbers represented by a sheet of rectangular coördinate paper (a picture of Mercator's projection).

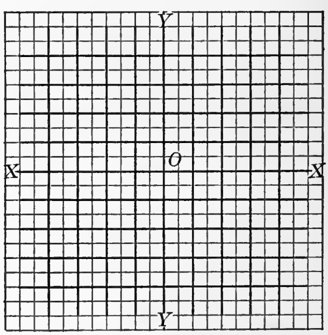


Fig. 182. — Mercator's projection.

On a polar projection the curve representing a direct sail from Liverpool to New York would be represented by a curve cutting all meridian circles (the radii) at the same angle. This curve is called an equiangular spiral. The logarithms of points on this spiral pass into a straight line on the Mercator map, namely, the line joining the point which is the logarithm of the point representing Liverpool to that which is the logarithm of the point representing New York.

If $\log(a + bi) = \log_e R + i(\theta + 2n\pi)$

any point on the unit-circle will become, in the logarithmic-picture, an infinite number of points on the y-axis, separated from each other by a distance 2π. If we imagine a point to start at the initial line and move around the unit-circle, how would the points representing the logarithms move? Circles outside the unit-circle and concentric with it will turn into what? If R is less than 1, $\log_e R$ is negative; thus a point within the unit-circle will lie to the left of the y-axis in the logarithmic-picture. A circle within the unit-circle, and concentric with it, will become a line parallel to the y-axis and to the left of it.

Since for points on a ray through the origin in the a, b picture, θ is a constant, the corresponding points in the logarithmic-picture will lie on lines parallel to the x-axis, at distances $\theta + 2n\pi$ from it. Thus any such ray will turn into an infinite number of parallel lines. The part of a ray beyond the unit-circle goes into what? What represents the part of the ray within the unit-circle?

§ 205. The *i*th Power and *i*th Roots of Numbers.

Since $a + bi$ can be put in the form $R \cdot e^{i\theta}$, every number can be set in the form of two factors, the real factor giving the distance of its representing point from the origin and the exponential imaginary factor indicating its angle.

$$\therefore\ (a+bi)^i = (R \cdot e^{i\theta})^i = e^{-\theta} \cdot R^i = \frac{1}{e^\theta} \cdot e^{i \log_e R}.$$

Thus, if the ith power of a number on one plane is plotted on another plane, the new modulus is the reciprocal of the exponential of the old angle, while the new angle is the logarithm of the old modulus.

Thus the ith power of all points on the unit-circle will lie on the initial line.

Similarly, $(a+bi)^{\frac{1}{i}} = (R \cdot e^{i\theta})^{\frac{1}{i}} = e^\theta \cdot R^{\frac{1}{i}}$

$$= e^\theta \cdot R^{-i} = e^\theta \cdot e^{-i \log R}.$$

EXAMPLE: Find the ith power of i.

The angle of i is $\frac{\pi}{2}$ and its modulus is 1.

$$\therefore\ i^i = \frac{1}{e^{\frac{\pi}{2}}} e^{i \log 1} = \frac{1}{e^{\frac{\pi}{2}}}.$$

EXERCISE.

Make maps of the ith power and ith roots of all points on a selected curve on a plane.

§ 206. The Hyperbolic Functions.

If x, y are the coördinates of any point on a circle of radius a, as in Fig. 183, $x^2 + y^2 = a^2$.

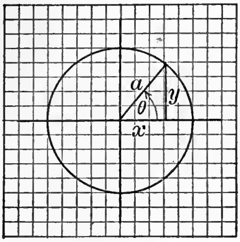


FIG. 183.

Thus, if we call $\frac{x}{a}$ the cosine of θ and $\frac{y}{a}$ the sine of θ, then characteristic of the trigonometry of the circle is

$$\cos^2\theta + \sin^2\theta = 1.$$

Similarly, if x, y are connected by the relation

$$x^2 - y^2 = a^2, \text{ or } \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2 = 1,$$

the point x, y (Fig. 184) will lie on the curve which has the name equilateral hyperbola, or rectangular hyperbola, a curve obtained by making a sketch of points (in a plane) such that the distance of each from a fixed point in that plane bears to its distance from a fixed line (vertical) of that plane the ratio of the hypothenuse of a 45° right-angled triangle to a side of the same triangle.

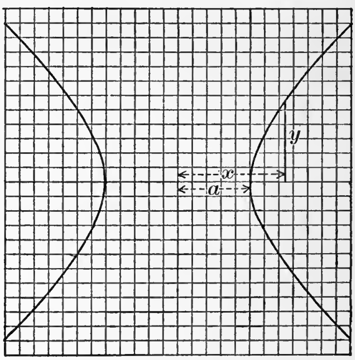


FIG. 184.

In correspondence to the trigonometry of the circle, we may call $\frac{x}{a}$, as related to the hyperbola, the *hyperbolic cosine*, and $\frac{y}{a}$ the *hyperbolic sine*, so that

$$(\text{hyperbolic cosine})^2 - (\text{hyperbolic sine})^2 = 1.$$

There is thus a trigonometry related to such an hyperbola as is the ordinary trigonometry to the circle.

Abbreviated names of the functions in this trigonometry are the same as in those of the circle, with an h interpolated: cosh for cos; sinh for sin; sech for sec, etc. These names may be read "cosine hyperbolic," "sine hyperbolic," etc. It is becoming customary, for brevity's sake, to throw the h in the pronunciation where it will make the abbreviated form pronounceable. Thus cosh x is pronounced as here written, "cosh" x. Sinh x is "shin" x. Sech x is "shec" x. Tanh x is "than" x (th having its sound in *thing*), etc.

We found $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$. (§ 200)

As defining relations for cosh x and sinh x, the i is omitted, and

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2}.$$

Whence

$$\cosh^2 x - \sinh^2 x = 1,$$

corresponding to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$.

§ 207. Relation of Hyperbolic Trigonometry to Circular Trigonometry.

From the exponential relations of §§ 200, 206 follow at once:

$$\cosh x = \cos (ix) \quad (1)$$

$$i \cdot \sinh x = \sin (ix). \quad (2)$$

Thus, assuming that circular trigonometry holds for quantity of the form (ix), the corresponding hyperbolic trigonometry comes at once by changing the word *cosine* to *cosh* and the word *sine* to *i · sinh.*

For example:

$$\sin (x+y) = \sin x \cos y + \cos x \sin y.$$

$$\therefore i \cdot \sinh (x+y) = (i \cdot \sinh x)(\cosh y) + (\cosh x)(i \cdot \sinh y),$$

or,

$$\sinh (x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

A second example is $\cos^2 x + \sin^2 x = 1$.

$$\therefore\ (\cosh x)^2 + (i \cdot \sinh x)^2 = 1.$$

$$\therefore\ \cosh^2 x - \sinh^2 x = 1.$$

From (1), (2) above it follows that

tan changes to $i \cdot$ tanh,
cot changes to $-\,i \cdot$ coth,
sec changes to sech,
cosec changes to $-\,i \cdot$ cosech.

EXERCISES.

Prove from the defining relations for the hyperbolic functions, or from the changes in circular functions noted above, as may seem best, the following:

1. $\sinh 2x = 2 \sinh x \cosh x$.

2. $\cosh 2x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1$.

3. $\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y$.

4. $\cosh (x + y) - \cosh (x - y) = 2 \sinh x \sinh y$.

5. $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$.

6. $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$.

7. $\tanh (x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.

8. $\sinh (x + y) \cosh (x - y) = \frac{1}{2} (\sinh 2x + \sinh 2y)$.

9. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$.

10. $\sinh 0 = 0$; $\cosh 0 = 1$.

11. $\sinh \dfrac{\pi i}{2} = i$; $\cosh \dfrac{\pi i}{2} = 0$; $\sinh \pi i = 0$; $\cosh \pi i = -1$.

12. $\text{sech}^2 x + \tanh^2 x = 1$; $\coth^2 x - \text{cosech}^2 x = 1$.

13. $\sinh x - \sinh y = 2 \sinh \dfrac{x - y}{2} \cosh \dfrac{x + y}{2}$.

14. $\sinh x + \sinh y = 2 \sinh \dfrac{x + y}{2} \cosh \dfrac{x - y}{2}$.

15. $\cosh x - \cosh y = 2 \sinh \dfrac{x - y}{2} \sinh \dfrac{x + y}{2}$.

16. $\cosh x + \cosh y = 2 \cosh \dfrac{x - y}{2} \cosh \dfrac{x + y}{2}$.

17. Plot the curves $y = e^x$ and $y = e^{-x}$ and thus obtain graphs for $y = \sinh x$ and $y = \cosh x$. Show that the latter has the form taken by a string hung on two pegs in the same horizontal line (catenary).

§ 208. Relation of the Anti-Hyperbolic Functions to Logarithms.

$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}. \qquad (\S\ 206)$$

Let $\sinh x = y$, so that $x = \sinh^{-1} y$.

$$\therefore e^x - e^{-x} = 2y. \quad \therefore e^x - \frac{1}{e^x} = 2y.$$

$$\therefore e^{2x} - 2y \cdot e^x - 1 = 0.$$

$$\therefore e^x = y \pm \sqrt{y^2 + 1}.$$

$$\therefore \boldsymbol{x = \sinh^{-1} y = \log_e (y \pm \sqrt{y^2 + 1})}. \qquad (1)$$

Similarly, $$\boldsymbol{\cosh^{-1} y = \log_e (y \pm \sqrt{y^2 - 1})}. \qquad (2)$$

Since the quantity in the bracket after log in (1) is negative when the minus sign is given the radical, and since negative numbers have no real logarithms, it is customary, when the generality of the logarithm has not been considered, to take the sign of the radicals in (1), (2) as plus. Here, as the student is familiar with $\log (a + bi)$, for all values of a, b, we need not restrict the sign of the radical.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = y.$$

$$\therefore \frac{e^{2x} - 1}{e^{2x} + 1} = y. \quad \therefore e^{2x} = \frac{1 + y}{1 - y}.$$

$$\therefore x = \tanh^{-1} y = \frac{1}{2} \log \frac{1 + y}{1 - y}.$$

Thus, in general, making $y = \frac{z}{a}$,

$$\boldsymbol{\sinh^{-1} \frac{z}{a} = \log \frac{z \pm \sqrt{z^2 + a^2}}{a}},$$

$$\boldsymbol{\cosh^{-1} \frac{z}{a} = \log \frac{z \pm \sqrt{z^2 - a^2}}{a}},$$

$$\boldsymbol{\tanh^{-1} \frac{z}{a} = \frac{1}{2} \log \frac{z + a}{z - a}}.$$

EXERCISES.

1. Show that the hyperbolic functions are periodic.

2. Show that the anti-hyperbolic functions are many-valued.

§ 209. Relation of the Hyperbolic Functions to the Circular Functions of the Eccentric Angle of the Hyperbola.

For the hyperbola, $x^2 - y^2 = a^2$.

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{a^2} = 1.$$

In the diagram, MT being a tangent to the circle,

$$\frac{x}{a} = \sec \phi,$$

ϕ being called the *eccentric angle.*

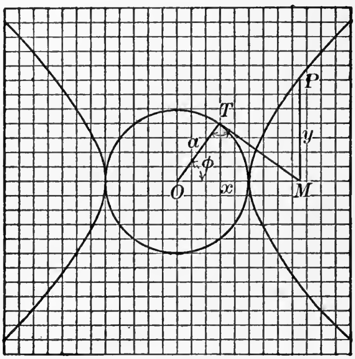


FIG. 185.

And since $\sec^2\phi - \tan^2\phi = 1$, $\therefore \frac{y}{a} = \tan \phi$.

Also $\cosh^2\theta - \sinh^2\theta = 1$.

$$\therefore \cosh \theta = \sec \phi, \quad (1)$$

and $$\sinh \theta = \tan \phi. \quad (2)$$

$$\therefore \phi = \sec^{-1}(\cosh \theta) = \tan^{-1}(\sinh \theta).$$

The angle ϕ as defined by these relations is called the *Gudermannian* of θ. (1), (2) may be put in the forms

$$\cos \phi = \text{sech } \theta,$$

$$\sin \phi = \cos \phi \cdot \sinh \theta = \tanh \theta.$$

$$\therefore \text{gd } \theta = \cos^{-1} \text{sech } \theta = \sin^{-1} \tanh \theta = \tan^{-1} \sinh \theta.$$

EXERCISE.

Show that if $u = \log \tan \left(\frac{\pi}{4} + \frac{x}{2}\right)$,

$u = \text{gd}^{-1} x$.