PLANE TRIGONOMETRY

AS FAR AS THE

SOLUTION OF TRIANGLES.

BY

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FOURTH EDITION,

WITH NUMEROUS EXAMPLES, AND FOUR-FIGURE TABLES OF LOGARITHMS OF NUMBERS AND OF THE TRIGONOMETRICAL RATIOS.

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LOGARITHMS.

1. The common logarithm of a number is the index of the power to which *ten* must be raised in order to produce that number; so that in the equation A logarithm defined.

$$10^x = N,$$

x is the logarithm of the number N, and this is written

$$x = \log N.$$

In general the logarithm of a number to a given base is the index of the power to which the base must be raised in order to be equal to the given number. So that if $a^x = N$, x is said to be the logarithm of N to the base a. This relation is also thus expressed, $x = \log_a N$.

Thus, since $7^2 = 49$, 2 may be said to be the logarithm of 49 to base 7, or $2 = \log_7 49$.

Any positive number, except unity, might be taken as the base of a system of logarithms; in practice, however, only two bases are used, the common base 10, and the Napierian base, 2·7182818..... In the following pages, unless the contrary is stated, the word logarithm means common logarithm, 10 being the base.

2. The logarithms of numbers which are integral powers of ten are immediately known; for example:

$$\begin{array}{llll} 10^3 = 1000, & \log 1000 = 3, \\ 10^2 = 100, & \log 100 = 2, \\ 10^1 = 10, & \log 10 = 1, \\ 10^0 = 1, & \log 1 = 0, \\ 10^{-1} = 0{\cdot}1, & \log 0{\cdot}1 = -1, \\ 10^{-2} = 0{\cdot}01, & \log 0{\cdot}01 = -2, \\ 10^{-3} = 0{\cdot}001, & \log 0{\cdot}001 = -3, \end{array}$$

For numbers greater than ten, the logarithms will be positive integers or mixed numbers; for numbers between 10 and 1, the logarithms will be positive decimals; for numbers less than 1, the logarithms will be negative quantities; the logarithm of zero is negative infinity, and negative numbers have no logarithms.

Characteristic and Mantissa.

3. When the logarithm of a number is a negative quantity, it is convenient to express it so that the integral part alone is negative, the decimal part remaining always positive, and the negative sign is written *over* the integral part to indicate this:

$$\begin{aligned}\text{Thus, } \log 0{\cdot}05 &= -(1{\cdot}30103)\\ &= -1-0{\cdot}30103\\ &= -2+(1-0.30103)\\ &= -2+0{\cdot}69897\\ \text{and this is written } &= \bar{2}{\cdot}69897.\end{aligned}$$

With this convention, the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa.*

4. Since numbers which have $(n+1)$ figures in their integral part commence with 10^n and run up to 10^{n+1}, their logarithms will commence with n and run up to $(n+1)$, and the characteristic for all such numbers will therefore be n. Again, since pure decimals in which the first significant digit occurs in the n^{th} place from the decimal point commence with 10^{-n} and run up to $10^{-(n-1)}$, their logarithms will commence with $-n$ and run up to $-(n-1)$, that is, will be $-n$ increased by some decimal, and the characteristic for all such will therefore be $\bar{n}$. Hence we have the following rule for finding the characteristic of the logarithm for any number:

Rule for finding the characteristic.

If the number be an integer or a mixed number, the characteristic is positive and is less by unity than the number of figures in the integral part; if the number be a decimal the characteristic is the number of the place of the first significant digit, counting from the decimal point, and is negative.

Thus for the following numbers

12345, 12·345, 1·23, 0·54, 0·000543,

the characteristics are respectively

$4, \ 1, \ 0, \ \bar{1}, \ \bar{4}.$

5. The following are the rules on which are founded the uses of logarithms in performing arithmetical operations:

Investigation of the rules for using logarithms in arithmetical operations.

(1) .... $\log (a\, b) = \log a + \log b.$

Let

$$x = \log a,\ y = \log b.$$

so that

$$10^x = a,\ 10^y = b.$$

Then,

$$a\, b = 10^x \times 10^y = 10^{x+y}$$

so that $x + y$ is the logarithm of $(a\, b)$,

or, $\log (a\, b) = \log a + \log b.$

(2) .... $\log \frac{a}{b} = \log a - \log b.$

Let $x = \log a,\ y = \log b,$

so that $10^x = a,\ 10^y = b.$

Then,

$$\frac{a}{b} = \frac{10^x}{10^y} = 10^{x-y}$$

so that $x - y$ is the logarithm of $\frac{a}{b}$,

or, $\log \left(\frac{a}{b}\right) = \log a - \log b.$

(3) .... $\log \left(a^n\right) = n \log a.$

Let $x = \log a$, so that $10^x = a.$

Then

$$a^n = \left(10^x\right)^n = 10^{nx}$$

so that nx is the logarithm of a^n,

or, $\log \left(a^n\right) = n \log a.$

(4) .... $\log \left(\sqrt[n]{a}\right) = \frac{1}{n} \log a.$

Let $x = \log a$, so that $10^x = a.$

Then $\sqrt[n]{a} = a^{\frac{1}{n}} = \left(10^x\right)^{\frac{1}{n}} = 10^{\frac{x}{n}},$

so that $\frac{x}{n}$ is the logarithm of $\sqrt[n]{a}$,

or, $$\log(\sqrt[n]{a}) = \frac{1}{n}\log a.$$

6. Any of these operations may be combined: thus

$$\log(abcd) = \log a + \log b + \log c + \log d;$$

$$\log\left(\frac{a}{bc}\right) = \log a - \log b - \log c;$$

$$\log\frac{a\sqrt{b}}{c^2\sqrt[3]{d}} = \log a + \tfrac{1}{2}\log b - 2\log c - \tfrac{1}{3}\log d.$$

The mantissa independent of the place of the decimal point in the number.

7. *The mantissa of the logarithm is the same for all numbers which differ only in the position of the decimal point.*

Let a be a number for which the characteristic is c and the mantissa m, so that

$$\log a = c + m.$$

The decimal point in a will be thrown n places to the right upon being multiplied by 10^n; and

$$\log(a \times 10^n) = \log a + \log 10^n = \log a + n$$
$$= (c + n) + m,$$

which has the characteristic $(c + n)$, and the same mantissa as before. Again, the decimal point in a will be thrown n places to the left on being multiplied by 10^{-n}; and

$$\log(a \times 10^{-n}) = \log a + \log 10^{-n} = \log a - n$$
$$= (c - n) + m,$$

which has the characteristic $(c - n)$, and the same mantissa.

It must be observed that the mantissa always retains its positive sign, while that of the characteristic may change.

Arrangement of tables of logarithms.

8. In the tables of logarithms of numbers, the mantissas alone are given (exact to a certain number of decimals), and the characteristics must be supplied by the rule of § 4. The number of figures in the given mantissas determines the number of figures for which the logarithm is given with suf-

ficient accuracy in these tables. Thus when six figures are given in the mantissas, the tables will be available only for numbers consisting of six figures or less, that is (disregarding the decimal point) for numbers ranging from 1 to 1000000. The mantissas, however, are not entered for *all* those numbers, but only for those terminating in the hundreds; for the intermediate numbers, the mantissas must be calculated by aid of the principle that the difference between the logarithms of two numbers is proportional to the difference between the numbers, when the numbers are taken sufficiently close. Thus the difference between two consecutive mantissas in the table corresponds to a difference of 100 between the numbers, and we obtain by a simple proportion the difference of mantissa corresponding to any less difference than 100 in the numbers.

E.g. .. Required the mantissa for the logarithm of 675347.
From the tables,

Number,	675400;	mantissa,	829561
"	675300;	"	829497
Difference,	100;	difference,	64

Then, by the principle,

$$\text{required difference for } 47 = \frac{47}{100} \times 64 = 30{\cdot}08,$$

and therefore the mantissa for 675347 is 829497 + 30, or 829527.

Table of proportional parts.

In many tables, the trouble of performing the multiplication in the above is avoided by the insertion of tables of *proportional parts*, in which are set down the products of the difference for 100 by the respective units, so that these products can at sight be taken out and added to the mantissa.

Thus in the previous example,

From the table, Number	675300;	Mantissa,	829497
From table of *p.p.*, for	40,	difference,	25·2
....	7,	....	4·4
Therefore for Number	675347,	Mantissa,	829527·

According to the usual rule in decimals, in carrying out to a certain number only of places, the last figure must be increased by 1 when the first of the neglected figures is 5 or a higher digit.

To take out the logarithm of a number.

9. The following is then the rule for finding the logarithm of a number of six or less figures.

Disregarding the decimal point, look in the table for the first 3 figures of the number in the left-hand column, and for the fourth figure in the top line; at the intersection of the corresponding line and column will be found the mantissa; for the fifth figure, look in the table of proportional parts and take out the number for that column; and for the sixth figure, also from the table of proportional parts, take out the corresponding number, removing the decimal point one place to the left. Add those two latter numbers to the mantissa previously found, and then, by consideration of the position of the decimal point in the original number, prefix the proper characteristic.*

Example. Required the logarithm of 327·695.

From the table,	3276..,	Mantissa	515344
From *p.p.*,	9 ,	diff.	119·7
....	5,	"	6·6
	327695,	Mant.	515470

and the characteristic is 2; therefore the logarithm of 327·695 is 2·515470.

To take out the number corresponding to a given logarithm.

10. The reverse process of finding the number corresponding to a given logarithm is performed on the same principle. Disregarding the characteristic, look out in the tables for the mantissa next below the given mantissa. In the corresponding line and column will be found the first three and the fourth

* There are numerous tables of logarithms published. Four-figure logarithms of numbers are given on a single card (with anti-logarithms on the reverse face) published by Laytons, London, for three figures given directly, and for the fourth by table of proportional parts. For five-figure logarithms the Nautical and Astronomical Tables by Gregory, Woolhouse and Hann, (Co. of Stationers, London) are admirably arranged; four figures are given and the fifth is interpolated for. For six-figure logarithms, Law's Tables (Copp, Clark & Co., Toronto) require the fifth and sixth figures to be interpolated for. The tables by Bremiker (Berlin) give for five figures direct, and require only the sixth figure to be found from the proportional parts. Of seven-figure tables, those in most common use are Chambers' (Edinburgh) Mathematical Tables; five figures being given and the sixth and seventh interpolated for.

For general purposes the five-figure tables by Gregory mentioned above, are recommended.

figures of the number. Then taking the difference between the mantissa thus found and the given one, and also that between the former and the next higher in the tables (which will be the difference for 100 in the number), by a simple proportion the tens and units in the required number are found. The decimal point must then be inserted by consideration of the characteristic of the given logarithm.

Example. Find the number corresponding to the given logarithm $\bar{2}.767198$. The mantissa next below is 767156, and the corresponding number is 585000. The difference between the two mantissas is 42.

Again in the tables,

Mantissa corresponding to	585100	is	767230	
....	585000	"	767156	
Difference of mantissa for	100	is	74.	

Then, by the proportion, the required difference in the number for a difference of 42 in the mantissa is

$$100 \times \frac{42}{74} = 56{\cdot}7,$$

and the number for this mantissa is 585000 + 57, or 585057. The characteristic in the given logarithm being $\bar{2}$, the number required will be 0.0585057.

Table of proportional parts.

As in the previous case, the trouble of performing the division in the above is avoided by the tables of proportional parts in which the quotients corresponding to the division are set down. Thus, having taken the difference between the given mantissa and the one next below it in the tables, look out in the corresponding table of proportional parts for the number next below this difference, and the column in which this is found gives the fifth figure: again take the difference between the previous difference and the number found in the table of proportional parts, and removing the decimal point in it one place to the right, look out again in the table of proportional parts for the number nearest to it, and the column in which this is found gives the sixth figure.

The previous example would be thus worked:

Given mantissa	767198;			
Mantissa next below, 767156, corresponding number, 5850 . .				
Difference	42,			
In table of p. p., diff. next below is	37·0,	"	"	5
Residual difference	5·0	"	"	7

This gives for the six figures, 585057, and the number required is therefore 0·0585057.

Use of logarithms in multiplication.

We shall now exemplify the rules for performing arithmetical operations by aid of logarithms, demonstrated in § 5, using five-figure logarithms only.

11. To multiply numbers together.

Rule. *Add the logarithms of the numbers, and take from the tables the number corresponding to this sum as a logarithm.*

Ex. (1).	Multiply	379·45 into 2·4672.		
	Number,	379·45;	log,	2·57915
	"	2·4672;	" ,	0·39220
	Product,	936·16;	log,	2·97135

Ex. (2).	Multiply	997 into 0·0325		
	Number,	997;	log,	2·99870
	"	0·0325;	" ,	$\bar{2}$·51188
	Product,	32·403;	log,	1·51058

Observe that the addition is + 2 + (− 2) + 0·9 ... + 0·5 ...

Ex. (3).	Multiply	7240000 into 93201.		
	Number,	7240000;	log,	6·85974
	"	93201;	" ,	4·96942
	Product,	674780000000.;	log,	11·82916

Here the product has 12 figures in its integral part, of which only five are determined; the remaining 7 being unknown are replaced by cyphers.

Ex. (4). Multiply 0·076905 into 0·000094397.

Number,	0·076905;	log, $\bar{2}$·88595
"	0·000094397;	" , $\bar{5}$.97496
Product,	0.0000072596;	log, $\bar{6}$·86091

Here the addition is — 2 — 5 + 0·8 .. + 0·9 ..

12. To divide one number by another. Division.

Rule. *Subtract the logarithm of the divisor from that of the dividend, and take from the tables the number corresponding to this difference as a logarithm.*

Ex. (1). Divide 32·495 by 7·6993.

Dividend,	32·495;	log, 1·51182
Divisor,	7·6993;	" 0·88645
Quotient,	4·2206;	log, 0·62537

Ex. (2). Divide 2·7045 by 312·79.

Dividend,	2·7045;	log, 0·43209
Divisor,	312·79;	log, 2.49525
Quotient,	0·0086465;	log, $\bar{3}$·93684

Here the subtraction is 1·43.... — 0·49.... —2 — 1.

Ex. (3). Divide 465·94 by 0·793.

Dividend,	465·94;	log, 2·66833
Divisor,	0·793;	" $\bar{1}$·89927
Quotient,	587·57;	log, 2·76906

Here the subtraction is 2·6.... —0·8.... —(—1).

Ex. (4). Divide 0·0037095 by 0·00001605.

Dividend,	0·0037095;	log, $\bar{3}$·56932
Divisor,	0·00001605;	log, $\bar{5}$·20548
Quotient,	231·12;	log, 2·36384

Here the subtraction is 0·5.... — 0·2.... + (—3) —(—5).

Use of arithmetical complements.

13. It is convenient to convert the process of subtraction into one of addition by the use of what is called the *arithmetical complement*. Thus if b is to be subtracted from a, instead of subtracting b, add $10 - b$, and subtract 10 from the result ; for

$$a - b = a + (10 - b) - 10.$$

This quantity $(10 - b)$ is called the arithmetical complement of b, and is found by subtracting the first significant digit, beginning from the right hand, from 10, and each following digit from 9, including, in the case of a logarithm, the characteristic with its proper sign.

For example,

Number,	239·31 ;	log, 2·37896 ;	co-log, 7·62104 ;
"	0·0025177 ;	log, $\bar{3}$·40100 ;	co-log, 12.59900.

The working of the previous examples would then stand thus,

Ex. (1).

Dividend,	32·495 ;	log,	1·51182
Divisor,	7·6993 ;	co-log,	9·11355
			0·62537

Ex. (2).

Dividend,	2·7045 ;	log,	0·43209
Divisor,	312·79 ;	co-log,	7·50475
			$\bar{3}$·93684

Ex. (3).

Dividend,	465·94 ;	log,	2·66833
Divisor,	0·793 ;	co-log,	10·10073
			2·76906

Ex. (4).

Dividend,	0·0037095 ;	log,	$\bar{3}$·56932
Divisor,	0·00001605 ;	co-log,	14·79452
			2·36384

14. To raise a number to any power. Involution.

Rule. *Multiply the logarithm of the number by the power, and take from the tables the number corresponding to this product as a logarithm.*

Ex. (1). Find the sixth power of 23·91.

Number, 23·91 ;	log, 1·37858
	6
Required power, 186840000 ;	log, 8·27148

Here the power has 9 figures in its integral part, of which only 5 are determined, the remaining 4, being unknown, are replaced by cyphers.

Ex. (2). Find $(0{\cdot}032507)^{10}$.

Number, 0·032507 ;	log, $\bar{2}{\cdot}51198$
	10
Power = 0·00000000QQ00013177 ;	log, $\overline{15}{\cdot}11980$

Here the multiplication is 10(—2) + 10(·5..).

15. To extract any root of a number. Evolution.

Rule.—*Divide the logarithm of the number by the root, and take from the tables the number corresponding to this quotient as a logarithm.*

Ex. (1). Required the fifth root of 2.

Number, 2 ; log,	0·30103
5	
Required root, 1·1487 ; log,	0·06021

Ex. (2). Required the 8th root of 0·79635.

Number, 0·79635 ; log,	$\bar{1}{\cdot}90110$
8	
Root, 0·97194 ; log,	$\bar{1}{\cdot}98764$

Here the characteristic being negative and not exactly divisible by the root, we add to it a sufficient number (negative) to make it exactly divisible, and therefore the same number (positive) to the mantissa. Thus

$$-8+7\cdot 9.., \text{ which on division gives } -1+0\cdot 9.. \text{ or } \bar{1}\cdot 9..$$

Combined operations.

16. As before remarked, any of these operations may be combined, but when more than one arithmetical complement is used, a ten must be subtracted from the result for each complement.

Ex. (1). Find the value of $\dfrac{(12\cdot 345)^5}{670\cdot 59 \times 50\cdot 323}$.

Number, 12·345;	log, 1·09149			
	5			
	5·45745	....	5·45745
" 670·59;	log, 2·82646	co-log,	7·17354
" 50·323;	log, 1·70177	"	8·29823
Required value,	8.4961	log,	0·92922

Ex. (2). Find $\sqrt[3]{\frac{5}{6}}$.

Number, 5;	log,	0·69897
" 6;	co-log,	9·22185
		3)$\bar{1}$·92082
Required value, 0·94105;	log,	$\bar{1}$·97361

The operation here is this:

$$\log \sqrt[3]{\tfrac{5}{6}} = \tfrac{1}{3} \log \tfrac{5}{6} = \tfrac{1}{3} (\log 5 - \log 6)$$
$$= \tfrac{1}{3} (\log 5 + \text{co-log } 6 - 10).$$

EXERCISE I.

1. What are the characteristics of the logarithms of the following numbers to base 10: 3740, 33·492, 76495·9, ·34781, ·0000053?

2. Shew that $\log \dfrac{144}{35} = 5 \log 2 + 2 \log 3 - \log 7 - 1.$

3. Shew that $\log \frac{14040}{648} = 1 + \log 13 - \log 2 - \log 3.$

4. Shew that $\log 8 + \log 25 = 2 + \log 2.$

5. Prove $\log \sqrt[3]{\frac{351}{560}} = \log 3 - \log 2 + \frac{1}{3}(\log 13 - \log 7 - 1).$

6. Given $\log 6 = a$, $\log 15 = b$, find $\log 8$ and $\log 9$.

7. Find the value of $\log \frac{1}{8}$ in terms of $\log 25$.

8. Multiply $\bar{1}\cdot 3724801$ by 5.

9. Divide $\bar{3}\cdot 0213569$ by 5.

10. Find the value of $\bar{1}\cdot 4873051 - \bar{3}\cdot 4920021 - \frac{1}{3}(\cdot 4721053)$.

11. From $\frac{2}{3}$ of $\bar{1}\cdot 4214036$ take $\frac{1}{3}$ of $3\cdot 4729104$.

12. Given $\log 7\cdot 3335 = \cdot 8653113$, $\log 7\cdot 3336 = \cdot 8653172$, find $\log \cdot 07333572$.

13. Find the logarithm of ·06919583.
$\log 6\cdot 9195 = \cdot 8400747$, $\log 6\cdot 9196 = \cdot 8400810$.

14. Find the logarithm of 56201·25.
$\log \cdot 056201 = \bar{2}\cdot 7497440$, $\log 56\cdot 202 = 1\cdot 7497518$.

15. Find the logarithm of 2965845.
$\log 2\cdot 9658 = \cdot 4721419$, dif. $= 146$.

16. Having given log 2 and log 3, find the logarithms of the following numbers: 18, 25, 216, $1\frac{1}{8}$, 6·480, ·0054, $\frac{4}{9}$, $-43\cdot 2$, 720, $\sqrt{1\frac{1}{2}}$, $\sqrt[3]{1\frac{1}{5}}$, $(5\frac{1}{3})^{-\frac{1}{4}}$.
$\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$.

17. Given log 2, find $\log \cdot 00016$, $\log (\cdot 000016)^{\frac{1}{9}}$

18. Find the logarithm of $\sqrt{\left(\frac{\sqrt[4]{32} \times \sqrt[3]{48}}{2\sqrt{27}}\right)}$, having given log 2 and log 3.

19. Given $\log 2\cdot 6201 = \cdot 4183179$, $\log 262\cdot 02 = 2\cdot 4183344$, find the number whose logarithm is $\bar{3}\cdot 4183253$.

20. Given $\log 56248 = 4.7501071$, $\log 56249 = 4\cdot 7501148$, find the number whose logarithm is $2\cdot 7501113$.

21. Given log 30·413 = 1·4830593, dif. = 142, find the number whose logarithm is 4·4830651.

22. Given log 49553 = 4·6950700, dif. = 87, find the number whose logarithm is $\bar{3}$.6950741.

23. Given log 5·6043 = ·7485214, log 5·6044 = ·7485291, form the table of Proportional Parts, and employ it to find log 560·4356, and also the number corresponding to the logarithm $\bar{3}$·7485282.

24. Find the value of $\sqrt[10]{(7\cdot2489)^3 \times \sqrt[4]{2\cdot3456}}$.
log 72.489 = 1·8602721, log 2·3456 = ·3702540, log 185·07 = 2·2673380.

25. Find the value of $\sqrt[24]{\frac{84}{72}}$

log 2 = ·3010300, log 3 = ·4771213, log 7 = ·8450980, log 10065 = 4·0027894.

26. Find the value of $\frac{\sqrt[3]{3}}{\sqrt[4]{4}\,.\,\sqrt[5]{5}}$.

Given log 2 = 3010300, log 3 = ·4771213, log 7·39148 = ·8687314.

27. Find by logarithms the value of $\frac{600 \times \cdot03 \times \cdot105}{\cdot00000432}$.

log 2 = ·3010300, log 7 = ·8450980, log 4374999 = ·6409780.

28. Find by logarithms the value of $\frac{\sqrt[3]{1\cdot25}}{4^3} \times \frac{12\cdot8}{\sqrt{3}}$.

log 3 = ·4771213, log 1·24385 = ·0947728.

29. Find by logarithms the value of $\frac{\cdot6}{\sqrt{2\cdot7}} \times 54 \times (5\cdot76)^{\frac{1}{5}}$.

log 2 = ·3010300, log 3 = ·4771213, log 2·79865 = ·4469478.

30. Find by logarithms the value of $\frac{\cdot00075}{3\cdot15} \times \frac{\sqrt{1\cdot8}}{\sqrt[3]{\cdot064}}$

log 2 = ·3010300, log 7 = ·8450980, log 7·98595 = ·9023270.

EXERCISE II.

1. Find the logarithms of the following numbers to the assigned bases:

(1). 256 to the base 2. (4). 343 to the base 7.

(2). 32 to the base 4. (5). 64 to the base 16.

(3). 243 to the base 27. (6). $\frac{81}{16}$ to the base 2·25.

2. What is the characteristic of 476 to base 8?

3. What is the characteristic of ·0156 to base 3?

4. If $10^x = 5^y$, find the ratio of x to y.

5. If $20^x = 100$, find x.

6. Find x from the equation $12^x = 180$, having given log 2 = ·3010300 and log 3 = ·4771213.

7. Solve the equations

(1). $a^{mx}\, b^{nx} = c$. (3). $4^{3x}.\, 3^{4-x} = 8^{2x-1}$.

(2) $\frac{1}{8^x} = 1·25$. (4). $\begin{cases} a^x\, b^y = c. \\ x + y = d. \end{cases}$

8. Given $\frac{4^x}{2^{x+y}} = 8$, and $x = 3y$, find x and y.

9. If a series of numbers are in G. P., shew that their logarithms are in A. P.

10. If x, y are the logarithms of two numbers M, N, shew that $\log \sqrt{\text{M N}} = \frac{1}{2}(x + y)$. Hence shew that 1·5 is the logarithm of 31·622...

11. Assuming that log 250 and log 256 differ by ·0103, shew that log 2 = ·30103.

12. How many figures will 2^{40} contain?

13. Find the number of cyphers between the decimal point and the first significant figure of $\frac{1}{5^{40}}$.

14. Given $1 + \log x = 0$, find x.

15. Given $\log x + \log \sqrt{x} = 1$, find x.

16. Given $1 - \log x = \log 4 - \log 2$, find x.

17. Given $\log x\,(\log x^2 - 1) = 6$, find x.

18. Given log 2 = ·30103, log 3 = ·47712, find the logarithm of 12 to the base 40.

19. Given log 3 = ·47712, $\log x$ = ·43429, find the logarithm of 3 to the base x.

20. Shew that $\log_2 \sqrt{20} = 1 + \log_4 5$.

THE TRIGONOMETRICAL RATIOS.

The trigonometrical ratios of an angle defined:

Fig. 1.

17. It is proved by Euclid that in a right-angled triangle, when one of the other angles is given, the ratios of the sides are also given. To these ratios, six in number, distinctive names are attached, and they are called the *trigonometrical ratios* of the given angle. Thus in the triangle ABC (fig. 1), having the angle C right, with reference to the angle A, calling the side opposite to A the perpendicular, the other side the base, that opposite to C being the hypothenuse, the ratio of perpendicular to hypothenuse is called the *sine* of the angle A; the ratio of perpendicular to base, the *tangent*; and the ratio of hypothenuse to base, the *secant*; or, as they are written,

$$\frac{BC}{AB} = \sin A,$$

$$\frac{BC}{AC} = \tan A,$$

$$\frac{AB}{AC} = \sec A.$$

The other three ratios—namely:

$$\frac{AC}{AB}, \qquad \frac{AC}{BC}, \qquad \frac{AB}{BC},$$

are evidently the sine, tangent, and secant with reference to the angle B, and this angle being the complement of A, the term "sine of the complement of A" is abbreviated into the *cosine* of A; and similarly the names, *cotangent, cosecant* are formed for the other two. These are written,

$$\frac{AC}{AB} = \cos A,$$

$$\frac{AC}{BC} = \cot A,$$

$$\frac{AB}{BC} = \operatorname{cosec} A.$$

18. These ratios, when the angle is given, are independent of the magnitude of the triangle, and are in effect determinate positive numbers. Since the perpendicular and base are always less than the hypothenuse, it is plain that the sines and cosines are proper fractions, while the secants and co-secants are whole numbers or improper fractions, but the tangents and cotangents may have any positive values. Their nature

19. As the angle A increases, retaining the same hypothenuse, the perpendicular increases and the base diminishes continually, and therefore the sine, tangent, and secant increase, while the cosine, cotangent, and cosecant diminish, and when A approaches near to $90°$, the perpendicular approaches to coincidence with the hypothenuse, while the base vanishes, and we have therefore for $90°$, Their changes of value.

Sin $90° = 1$, tan $90° = \infty$, sec $90° = \infty$, cos $90° = 0$, cot $90° = 0$, co-sec $90° = 1$. Particular values for 90°, 0°, 45°, 30°, 60°.

Also since $0°$ is the complement of $90°$, these values give cos $0° = 1$, cot $0 = \infty$, co-sec $0 = \infty$, sin $0 = 0$, tan $0 = 0$, sec $0 = 1$.

20. The following intermediate values may be noticed.

Take a right-angled isosceles triangle, (fig. 2), in which the perpendicular and base are each $= 1$, and the hypothenuse therefore $= \sqrt{2}$. Fig. 2.

Then either angle being $45°$, it is seen by inspection that sin $45° =$ cos $45° = \frac{1}{\sqrt{2}}$; tan $45° =$ cot $45° = 1$; sec $45°$ $=$ co-sec $45° = \sqrt{2}$.

Hence also the tangent of an angle less than $45°$ is less than 1, and of an angle greater than $45°$ is greater than 1, while the reverse is the case for the cotangent.

Again, take an equilateral triangle, (fig. 3), each of whose sides $= 2$, and from one of the vertexes drop a perpendicular on the opposite side; this perpendicular bisects both the side and the angle, giving two right-angled triangles with the angles $30°$, $60°$, and the length of this perpendicular is $\sqrt{3}$. Hence by inspection

$$\sin 30^\circ \text{ or } \cos 60^\circ = \frac{1}{2}; \quad \cos 30^\circ \text{ or } \sin 60^\circ = \frac{\sqrt{3}}{2};$$

$$\tan 30^\circ \text{ or } \cot 60^\circ = \frac{1}{\sqrt{3}}; \quad \cot 30^\circ \text{ or } \tan 60^\circ = \sqrt{3};$$

$$\sec 30^\circ \text{ or } \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}; \quad \operatorname{cosec} 30^\circ \text{ or } \sec 60^\circ = 2.$$

Five independent relations connect them.

21. It is also proved by Euclid that when the ratio of two sides in a right-angled triangle is given, the angles are also given. Consequently when any one of the six trigonometrical ratios of an angle is given, the angle itself is determinate, and the other five ratios can be found. Hence there must be five independent relations connecting the six ratios of an angle. By inspection it is seen that the sine and cosecant, the tangent and cotangent, the cosine and secant are reciprocals, so that

$$\sin A = \frac{1}{\operatorname{cosec} A}, \quad \tan A = \frac{1}{\cot A}, \quad \cos A = \frac{1}{\sec A}.$$

Again,

$$\frac{\sin A}{\cos A} = \frac{BC}{AB} \div \frac{AC}{AB} = \frac{BC}{AC} = \tan A.$$

These are four of the relations; a fifth, connecting sine and cosine is given by Euclid, B. I. Prop. 47*; for

$$AB^2 = BC^2 + AC^2,$$

and therefore

$$1 = \left(\frac{BC}{AB}\right)^2 \times \left(\frac{AC}{AB}\right)^2,$$

$$= (\sin A)^2 + (\cos A)^2,$$

or, as it is usually written,

$$\sin^2 A + \cos^2 A = 1.$$

Numerous other relations exist between these ratios, but they are all deducible from the five above given, which enable us by a simple algebraic process to express any one ratio in terms of any other.

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* Another proof, not depending on this proposition, will be subsequently given.

$\left(\frac{BC}{AB}\right)^2 + \left(\frac{AC}{AB}\right)^2 = 1$

EXERCISE III.

Prove the following identities :

1. $\sin^2 A = 1 - \cos^2 A$.
2. $\tan^2 A + 1 = \sec^2 A$.
3. $\cot^2 A + 1 = \operatorname{cosec}^2 A$.
4. $\sec^2 \theta = \dfrac{1}{1 - \sin^2 \theta}$.
5. $\tan^2 A = \dfrac{1 - \cos^2 A}{\cos^2 A}$.
6. $\cos A = \cot A \sin A$.
7. $\cot^2 \theta = \dfrac{\cos^2 \theta}{1 - \cos^2 \theta}$.
8. $\cos^2 \theta = \dfrac{1}{1 + \tan^2 \theta}$.
9. $\sin^2 \theta = \dfrac{\tan^2 \theta}{1 + \tan^2 \theta}$.
10. $\sec A = \tan A \operatorname{cosec} A$.
11. $\sec^2 A = \dfrac{\operatorname{cosec}^2 A}{\operatorname{cosec}^2 A - 1}$.
12. $\sec A + 1 = \dfrac{1 + \cos A}{\cos A}$.
13. $1 + \cos A = \dfrac{\sin^2 A}{1 - \cos A}$.
14. $\cos^2 A = \dfrac{\cot^2 A}{1 + \cot^2 A}$
15. $(1 - \sin A) \sec A = \dfrac{\cos A}{1 + \sin A}$
16. $\sin^4 \theta - \cos^4 \theta = \sin^2 \theta - \cos^2 \theta$.
17. $\cot^2 \varphi + \tan^2 \varphi = \sec^2 \varphi \operatorname{cosec}^2 \varphi - 2$.
18. $1 + \tan A = \sqrt{\sec^2 A + 2 \tan A}$.
19. $\cot^2 A . \sin^2 A + \sin^2 A = 1$.

20. $\sec^2\theta - 1 = \sin^2\theta . \sec^2\theta$.

21. $(\cos^2 A - 1)(\cot^2 A + 1) = -1$.

22. $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A . \tan B$.

23. $(\operatorname{cosec} A - \cot A)^2 = \dfrac{1 - \cos A}{1 + \cos A}$.

24. $\cot A + \dfrac{\sin A}{1 + \cos A} = \operatorname{cosec} A$.

25. $\cos^2\theta \cos^2\varphi - \sin^2\theta \sin^2\varphi = \cos^2\theta - \sin^2\varphi$.

26. $\tan^2\alpha \tan^2\beta - 1 = \dfrac{\sin^2\alpha - \cos^2\beta}{\cos^2\alpha \cos^2\beta}$.

27. $\sec A \left\{ 1 + \operatorname{cosec} A (\cos^2 A - \sin^2 A) \right\} = \cot A$.

28. $\sin^2 A \tan^2 A + \cos^2 A \cot^2 A = \tan^2 A + \cot^2 A - 1$.

29. $\sin^2 A \tan A + \cos^2 A \cot A + 2 \sin A \cos A = \sec A \operatorname{cosec} A$.

30 $\cot^2 x - \tan^2 x = (\cos^2 x - \sin^2 x) \sec^2 x \operatorname{cosec}^2 x$.

EXERCISE IV.

The formulæ of § 21 enable us, having given the numerical value of one of the trigonometrical ratios of an angle, to find the numerical values of the other trigonometrical ratios of the same angle. Thus, given $\sin A = \frac{1}{2}$, to find $\cot A$:

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A} = \frac{\sqrt{1 - \frac{1}{4}}}{\frac{1}{2}} = \sqrt{3}.$$

1. Given $\sin A = \dfrac{3}{5}$, find $\cos A$.

2. Given $\cos A = \dfrac{5}{13}$, find $\sin A$.

3. Given $\tan\theta = 4$, find $\sin\theta$.

4. If $\sin A = \dfrac{a}{b}$, find $\tan A$.

5. Given $\sin x = \frac{3}{5}$, find $\cot x$.

6. Given $\sec x = \frac{a}{b}$, find $\cot x$.

7. Given $\sec x = \sqrt{2}$, find $\operatorname{cosec} x$.

8. Given $\sin \theta + \cos \theta = \frac{7}{5}$, find $\cos \theta$.

9. Given $\operatorname{cosec} x + 6 \sin x = 5$, find $\sin x$.

10. If $(1 + \tan^2 A) \cos A = 2$, find $\cos A$.

22. The values of all these ratios are calculated for all angles between 0 and 90°, and are entered in tables called *natural sines*, &c.; but these values are not so useful as the logarithms of them which form the tables called *logarithmic sines*, &c. Since, however, the sines and cosines are proper fractions, and so also are some of the tangents and cotangents, their logarithms will have negative characteristics, and to avoid the inconvenience of printing these, every logarithm of a trigonometrical ratio is increased by 10 before being entered in the table. To distinguish therefore the real logarithm from that given in the tables, the latter will always be written with an italic capital L, and it must always be borne in mind that 10 is to be taken from each such logarithm when used instead of the real logarithm, the operation being either expressed or understood.

Tables of their values.

The tabular logarithm as distinguished from the real logarithm.

For instance

$$\sin 30^\circ = \tfrac{1}{2} = 0{\cdot}5,$$
$$\log \sin 30^\circ = \log (0{\cdot}5) = \bar{1}{\cdot}69897,$$
$$L \sin 30^\circ = 9{\cdot}69897.$$

Also,

$$\tan 45^\circ = 1,$$
$$\log \tan 45^\circ = 0,$$
$$L \tan 45^\circ = 10{\cdot}00000.$$

23. Again, since

$$\sin A \times \operatorname{cosec} A = 1, \text{ we have}$$
$$\log \sin A + \log \operatorname{cosec} A = 0,$$
$$L \sin A - 10 + L \operatorname{cosec} A - 10 = 0,$$

or,

$$L \sin A + L \operatorname{cosec} A = 20.$$

And similarly,

$$L \tan A + L \cot A = 20,$$
$$L \cos A + L \sec A = 20.$$

Also,

$$\tan A = \frac{\sin A}{\cos A},$$

$$\log \tan A = \log \sin A - \log \cos A,$$
$$L \tan A - 10 = L \sin A - 10 - (L \cos A - 10),$$
$$L \tan A = L \sin A + 10 - L \cos A.$$

By aid of these formulas, if $L \sin A$ and $L \cos A$ be tabulated from 0 to 45°, the values of the other logarithmic functions from 0 to 90° can be formed.

Arrangement of the tables of logarithmic sines, &c.

24. In the ordinary tables, these logarithmic sines, cosines' &c., are given for all angles from 0° to 90° at intervals of one minute, and it will be sufficient for most purposes to take out any required angle to the nearest minute; but if greater accuracy be needed, recourse must be had to the principle of proportional parts already explained in discussing the logarithms of numbers.

The usual arrangement is that the angles from 0 to 45° are placed at intervals of one degree at the head of the page, the minutes running down the left-hand column, while the angles from 45° to 90° are placed at the foot of the page, and the minutes run up the right-hand column. By this arrangement the same column is used for the sine of an angle and for the cosine of its complement; and in the same way for the tangent and cotangent, and for secant and cosecant.

25. Since sines and cosines are proper fractions, the tabular logarithms of them will always be less than 10; and since secants and cosecants are integers or improper fractions, their tabular logarithms will always be greater than 10. The logarithmic tangents will be less than 10 up to 45°, and after this will be greater than 10, and the reverse will be

the case for the cotangents. The following table exhibits the changes as the angle passes from 0 to 90°:

sine increases	from 0 to 1;	*L* sin increases	from $-\infty$ to 10
cosine decreases	" 1 " 0;	*L* cos decreases	" 10 " $-\infty$
tangent increases	" 0 " ∞;	*L* tan increases	" $-\infty$ " $+\infty$
cotangent decreases	" ∞ " 0;	*L* cot decreases	" $+\infty$ " $-\infty$
secant increases	" 1 " ∞;	*L* sec increases	" 10 " $+\infty$
cosecant decreases	" ∞ " 1;	*L* cosec decreases	" $+\infty$ " 10.

L tan and *L* cot are each 10 at 45°.

EXERCISE V.

1. Find the tabular logarithms of the trigonometrical ratios of 30° and 45°, having given log 2, and log 3.

2. Given *L* sin 22° 26′ = 9·5816177, *L* sin 22° 27′ = 9·5819236, find *L* sin 22° 26′ 45″.

3. Given *L* sin 38° 24′ = 9·7931940, *L* sin 38° 25′ = 9·7933543, find *L* sin 38° 24′ 27″.

4. Having given *L* cos 34° 18′ = 9·9170317, *L* cos 34° 19′ = 9·9169455, find *L* cos 34° 18′ 25″.

5. Given *L* cos 57° 12′ = 9·7337654, *L* cos 57° 13′ = 9·7335693, find *L* cos 57° 12′ 24″.

6. Given *L* tan 32° 29′ = 9·8039085, *L* tan 32° 30′ = 9·8041873, find *L* tan 32° 29′ 27″.

7. Given *L* cot 16° 58′ = 10·5155654, *L* cot 16° 59′ = 10·5151130, find *L* cot 16° 58′ 18″.

8. Given *L* sec 73° 24′ = 10·5441074, *L* sec 73° 25′ = 10·5445314, find *L* sec 73° 24′ 36″.

9. Given *L* cosec 69° 34′ = 10·0282238, *L* cosec 69° 35′ = 10·0281767, find *L* cosec 69° 34′ 54″.

10. Given *L* sin 69° 7′ = 9·9704902, *L* sin 69° 8′ = 9·9705383, find the angle whose *L* sin is 9·9705261.

11. Given *L* sin 16° 19′ = 9·4486227, *L* sin 16° 20′ = 9·4490540, find the angle whose *L* sin is 9·4488105.

12. Given L cos 22° 28′ = 9·9657199, L cos 22° 29′ = 9·9656677, find the angle whose L cos is 9·9656913.

13. Given L tan 51° 17′ = 10·0960267, L tan 51° 18′ = 10·0962856, find the angle whose L tan is 10·0962548.

14. Given L tan 30° 21′ = 9·7035329, L tan 30° 22′ = 9·7037486, find A from the equation L tan A = 9·7036421.

15. Given L cot 42° 12′ = 10·0425150, L cot 42° 13′ = 10·0422611, find the angle whose L cot is 10·0423485.

16. Given L sec 47° 30′ = 10·1703167, L sec 47° 31′ = 10·1704546, find the angle whose L sec is 10·1703541.

17. Given L cosec 15° 21′ = 10·5772220, L cosec 15° 22′ = 10·5767620, find the angle whose L cosec is 10·5769821.

18. Having given L cosec 34° 31′ = 10·2466882, L cosec 34° 32′ = 10·2465046, find A from the equation L cosec A = 10·2466153.

19. Given L sin 28° 10′ = 9·6739769, L sin 28° 11′ = 9·6742128, find L cos 61° 49′ 25″.

20. Given L cos 71° 45′ = 9·4957716, L cos 71° 46′ = 9·4953883, find L sin 18° 14′ 10″.

21. Given L tan 52° 35′ = 10·1163279, L tan 52° 36′ = 10·1165897, find L cot 37° 24′ 50″.

22. Given L cot 36° 19′ = 10·1337003, L cot 36° 20′ = 10·1334356, find L tan 53° 40′ 45″.

23. Given L cos 42° 26′ = 9·8680934, Diff. = 1154, find L sin 47° 34′ 47″.

24. Given L sin 20° 15′ = 9·5392230, L sin 20° 16′ = 9·5395653, find the angle whose L cos is 9·5394128.

25. Given L cos 44° 20′ = 9·8544799, Diff. = 1234, find the angle whose L sin is 9·8545671.

26. Given L cot 57° 16′ = 9·8080829, L cot 57° 17′ = 9·8078052, find the angle whose L tan is 9·8080431.

27. In the tables why do the same columns of differences answer for both sine and cosecant, tangent and cotangent, secant and cosine?

28. In the tables for what angles will a column, which when read from the top is that for the sines of certain angles, answer for cosines when read from the bottom?

29. Shew that the sum of the tabular logarithms of the sine and cosecant of any (the same) angle is 20,—also that the same is true of the cosine and secant, and of the tangent and cotangent.

30. In increasing the true logarithm by 10 to form the tabular logarithm, by what are we multiplying the trigonometrical ratio?

SOLUTION OF RIGHT-ANGLED TRIANGLES.

26. Taking the triangle ABC, where C is 90°, and denoting the lengths of the sides opposite to each angle by the small letter corresponding, the definitions of the trigonometrical ratios give the following relations: *Direct relations connecting the sides and the trigonometrical ratios of one of the angles in a right-angled triangle.*

Fig. 4.

$$\sin A = \frac{a}{c}, \quad \text{or,} \quad a = c \sin A;$$

$$\tan A = \frac{a}{b} \quad \ldots\ldots\ldots \quad a = b \tan A;$$

$$\sec A = \frac{c}{b} \quad \ldots\ldots\ldots \quad c = b \sec A;$$

$$\cos A = \frac{b}{c} \quad \ldots\ldots\ldots \quad b = c \cos A;$$

$$\cot A = \frac{b}{a} \quad \ldots\ldots\ldots \quad b = a \cot A;$$

$$\operatorname{cosec} A = \frac{c}{a} \quad \ldots\ldots\ldots \quad c = a \operatorname{cosec} A.$$

27. From these relations, any two of the four quantities a, b, c, A being given, the other two could be found by aid of the tables of *natural* sines, cosines, &c.; and the remaining angle B, which is the complement of A, being thus found *Two parts being given (one at least being a line), the triangle can be solved.*

also, the triangle would be completely determined. Such a mode of solution would however be inconvenient, as involving long processes of multiplication, and we shall proceed to discuss the different cases of the solution of right-angled triangles by means of the logarithmic tables.

Four cases of solution.

28. Four distinct cases will arise, (1), an angle and a side; (2), an angle and the hypothenuse; (3), the two sides; (4), a side and the hypothenuse. In cases (1) and (2), it is indifferent which angle be given, as the other is at once known. The solution will be effected in each case by picking out from among the foregoing relations one which connects the quantity sought for with two quantities which have been given or found, and it will be noticed that in each case there will be *two* of these relations which would serve this purpose. If one involves a process of addition, and the other a process of subtraction, we shall always take the former.

Case 1. A side and an angle given.

Case (I). Given a, A; to find B, b, c.

$B = 90° - A$. .... B found.

$b = a \cot A.$

Taking the logarithms of both sides.

$\log b = \log a + \log \cot A$

$\log b = \log a + L \cot A - 10$......$b$ found.

$c = a \operatorname{cosec} A$

or

$\log c = \log a + L \operatorname{cosec} A - 10$....$c$ found.

Case 2. The hypothenuse and an angle given.

Case (II). Given c, A; to find B, a, b.

$B = 90° - A$. .... B found.

$a = c \sin A,$

$\log a = \log c + L \sin A - 10$ .... a found.

$b = c \cos A,$

$\log b = \log c + L \cos A - 10$ .... b found.

Case 3. The two sides given.

Case (III). Given a, b; to find A, B, c.

$\tan A = \frac{a}{b},$

$\log \tan A = \log a - \log b,$

$L \tan A - 10 = \log a + \operatorname{colog} b - 10$

and therefore

$$L \tan A = \log a + \text{colog}\, b. \quad \ldots\ldots A \text{ found.}$$
$$B = 90° - A. \quad \ldots\ldots\ldots\ldots\ldots\ldots B \text{ found.}$$
$$c = a \text{ cosec } A,$$
$$\log c = \log a + L \text{ cosec } A - 10 \;..\; c \text{ found.}$$

In this case it is indifferent whether we determine A from the formula $\tan A = \frac{a}{b}$, or from $\cot A = \frac{b}{a}$. Also there is not among our relations one connecting c with the given quantities a, b, and although we know from Euclid that $c^2 = a^2 + b^2$, this formula is not convenient for logarithmic computation, and we therefore determine c by means of A, which though not *given* has been already found. We might also have determined c by means of $c = b \sec A$.

Case (IV). Given a, c; to find A, B, b. Case 4. A side and the hypothenuse given.

$$\sin A = \frac{a}{c},$$
$$\log \sin A = \log a - \log c$$
$$L \sin A - 10 = \log a + \text{colog}\, c - 10,$$

and therefore

$$L \sin A = \log a + \text{colog}\, c \;\ldots\ldots\ldots\; A \text{ found.}$$
$$B = 90° - A \;\ldots\ldots\ldots\ldots\ldots\ldots\; \text{B found.}$$
$$b = a \cot A$$
$$\log b = \log a + L \cot A - 10 \;\ldots\; b \text{ found.}$$

In this case it is indifferent whether we determine A from the formula $\sin A = \frac{a}{c}$, or from $\text{cosec } A = \frac{c}{a}$. Also, there being none of the relations which connects b directly with the given quantities a, c, it is determined by means of A which has previously been found; it might also have been found from the formula $b = c \cos A$. It is known from Euclid that $b^2 = c^2 - a^2$, and b might have thus been found directly, but the formula is not convenient for logarithms.

29. The solution of an isosceles triangle can be effected by aid of the preceding; for such a triangle can be divided by a perpendicular dropped from the vertex on the base into two right-angled triangles, equal in all respects, and by solving these, the parts of the isosceles triangle also are determined. Isosceles triangles solved.

Examples.

30. Examples of right-angled triangles.

Case (I). Given $a = 129{\cdot}5$, $A = 37° \ 07'$.

$B = 90° - A$.

$$\begin{array}{r} 90° \ 00' \\ A = 37° \ 07' \\ \hline B = 52° \ 53' \end{array}$$ (*B* found.)

$\log b = \log a + L \cot A - 10$.

$a = 129{\cdot}5$;	$\log a$,	$2{\cdot}11227$
$A = 37° \ 07'$;	$L \cot A$,	$10{\cdot}12105$
$b = 171{\cdot}13$;	$\log b$,	$2{\cdot}23332$ (*b* found.)

$\log c = \log a + L \operatorname{cosec} A - 10$.

	$\log a$,	$2{\cdot}11227$
$A = 37° \ 07'$;	$L \operatorname{cosec} A$,	$10{\cdot}21937$
$c = 214{\cdot}61$;	$\log c$,	$2{\cdot}33164$ (*c* found.)

Case (II). Given $c = 31459$, $A = 46° \ 32'$.

$B = 90° - A$.

$$\begin{array}{r} 90° \ 00' \\ A = 46 \quad 32 \\ \hline B = 43° \ 28' \end{array}$$ (*B* found.)

$\log a = \log c + L \sin A - 10$.

$c = 31459$;	$\log c$,	$4{\cdot}49774$
$A = 46° 32$;	$L \sin A$,	$9{\cdot}86080$
$a = 22832$;	$\log a$,	$4{\cdot}35854$ (*a* found.)

$\log b = \log c + L \cos A - 10$.

	$\log c$,	$4{\cdot}49774$
$A = 46° \ 32'$;	$L \cos A$,	$9{\cdot}83755$
$b = 21642$;	$\log b$,	$4{\cdot}33529$ (*b* found.)

Case (III). Given $a = 2{\cdot}7039$, $b = 3{\cdot}4505$.

$$L \tan A = \log a + \operatorname{colog} b.$$

$a = 2{\cdot}7039$;	$\log a$,	$0{\cdot}43199$
$b = 3{\cdot}4505$;	colog b,	$9{\cdot}46212$
$A = 38^\circ\ 05'$;	$L \tan A$,	$9{\cdot}89411$

(*A* found.)

$$B = 90^\circ - A.$$

	$90^\circ\ 00'$
$A =$	$38\ \ 05'$
$B =$	$51^\circ\ 55'$

(*B* found.)

$$\log c = \log a + L \operatorname{cosec} A - 10.$$

	$\log a$,	$0{\cdot}43199$
$A = 38^\circ\ 05'$;	$L \operatorname{cosec} A$,	$10{\cdot}20985$
$c = 4{\cdot}3837$;	$\log c$,	0.64184

(*c* found.)

Case (IV). Given $a = 21$, $c = 21{\cdot}981$.

$$L \sin A \quad \log a + \operatorname{colog} c.$$

$a = 21$;	$\log a$,	$1{\cdot}32222$
$c = 21{\cdot}981$;	colog c,	$8{\cdot}65795$
$A = 72^\circ\ 49'$;	$L \sin A$,	$9{\cdot}98017$

(*A* found.)

$$B = 90^\circ - A.$$

	$90^\circ\ 00'$
$A =$	$72^\circ\ 49$
$B =$	$17^\circ\ 11'$

(B found.)

$$\log b = \log a + L \cot A - 10.$$

	$\log a$,	$1{\cdot}32222$
$A = 72^\circ\ 49'$;	$L \cot A$,	$9{\cdot}49029$
$b = 6{\cdot}4940$;	$\log b$,	$0{\cdot}81251$

(*b* found.)

The use of "traverse-tables" may be briefly noted in connection with "plane sailing" and surveys. The earth's surface being considered a plane, a straight line drawn upon it is called the *distance*, and the acute angle between the direction in which this distance is drawn and the North or South line is called the *angle of the course*. A right-angled triangle being constructed of which the distance is the hypothenuse, and the base is drawn east or west, while the perpendicular is north or south, the base is called the *departure*, and the perpendicular the *difference of latitude*, these lines being the products of the distance by the sine and cosine of the angle of the course respectively. The values of the departure and difference of latitude are set down in traverse-tables for different values of the distance and angle of course. When different distances are run consecutively at the same angle, the simple addition of the corresponding departures gives the departure for the whole distance; and similarly for the differences of latitude. When consecutive distances are run at different angles, if the departures when eastward are reckoned positive, and when westward negative, the algebraic sum gives the resultant departure with the same convention of signs; and so for the differences of latitude, if those northwards are reckoned positive, and southwards negative. The resultant departure and difference of latitude give the position at the end of the distances. In this way the position of a vessel is ascertained from knowing the distances run and the corresponding angles of the course. The same method applies in running a survey on the earth's surface considered as a plane.

Sometimes the angle of the course is reckoned in *points* instead of degrees, each point being $11\frac{1}{4}°$.

EXERCISE VI.

1. $a = 50$, $b = 50$, $C = 90°$; solve the triangle.
2. $c = 240$, $A = 45°$, $C = 90°$; solve the triangle.
3. $a = 100$, $A = 45°$, $C = 90°$; solve the triangle.
4. $c = 24$, $A = 30°$, $C = 90°$; solve the triangle.
5. $a = 30$, $A = 60°$, $C = 90°$; solve the triangle.
6. $a = 480$, $B = 60°$, $C = 90°$; solve the triangle.
7. $c = 96$, $a = 48$, $C = 90°$; solve the triangle.
8. $a = 198$, $b = 201{\cdot}5$, $C = 90°$; find A.

log 1·98 = ·2966652, log 2·015 = ·3042751, L tan 44° 29′ = 9·9921670,
L tan 44° 30′ = 9·9924197.

9. $a = 742{\cdot}196$, $c = 1025$, $C = 90$; find A.

log 74219 = 4·8705151, log 1·025 = ·0107239, L sin 46° 23′ = 9·8597213.
log 74220 = 4·8705210 L sin 46° 24′ = 9·8598416.

10. $a = 138$, $b = 246 \cdot 5$, $C = 90°$; find B.

log 138 = 2·1398791. log 246·5 = 2·3918169, L cot 60° 45′ = 9·7482089,
L cot 60° 46′ = 9·7479125.

11. $a = 3$, $c = 5$, $C = 90°$; find A and B.

log 3 = ·4771213, log 5 = ·6989700, L sin 36° 52′ = 9·7781186
L sin 36° 53′ = 9·7782870.

12. $a = 600$, $c = 1400$, $C = 90°$; find A and B.

log 3 = ·4771213, log 7 = ·8450980, L sin 25° 22′ = 9·6318591
L sin 25° 23′ = 9·6321255.

13. $a = 12$, $b = 19$, $C = 90°$; find A and B.

log 12 = 1·0791812, log 19 = 1·2787536, L tan 32° 16′ = 9·8002769, L tan 32° 17′ = 9·8005567.

14. $a = 20$, $b = 27$, $C = 90°$; find A and B.

log 2 = ·3010300, log 3 = ·4771213, L cot 36° 31′ = 10·1305269
L cot 36° 32′ = 10·1302628.

15. $a = 200$, $c = 500$, $C = 90°$; find b.

log 3 = ·4771213, log 45825 = 4·6611025
log 7 = ·8450980, log 45826 = 4·6611120.

16. $c = 3000$, $A = 80°$, $C = 90°$, find a and b.

L sin 80° = 9·9933515, L cos 80° = 9·2396702,
log 2·9544 = ·4704693, log 5·2094 = ·7167877
log 2·9545 = ·4704840, log 5·2095 = ·7167960.

17. $c = 4000$, $A = 70°$, $C = 90°$; find a and b.

L cos 70° = 9·5340517, log 3·7587 = ·5750377, log 1·3680 = ·1360861
L cos 20° = 9·9729858 diff. = 115 diff. = 317.

18. $a = 480$, $A = 70°$, $C = 90°$; find b and c.

L sin 70° = 9·9729858, log 480 = 2·6812412, log 5·1080 = ·7082509,
L cos 70° = 9·5340517. diff. = 85
log 1·7470 = ·2422929.
diff. = 249.

19. $b = 3600$, $A = 75°$, $C = 90°$; find a and c.

L cos 75° = 9·4129962, log 1·3909 = ·1432959, log 1·3435 = ·1282377
L tan 75° = 10·5719475 diff. = 312 diff. = 323.

20. $a = 124 \cdot 6$, $A = 64° 20'$, $C = 90°$; find b and c.

L sin 64° 20′ = 9·9548834, log 1·246 = ·0955180, log 5·9876 = ·7772528.
L tan 64° 20′ = 10·3182604, log 1·3824 = ·1406346. log 5·9877 = ·7772600.

21. $c = 294$, $A = 23° 30'$, $C = 90°$; find a and b,
log 2·94 = ·4683473,
L sin 23° 30′ = 9·6006997, log 11·723 = 1·0690388, diff. = 370.
L cos 23° 30′ = 9·9623978, log 26·961 = 1·4307360, dift. = 161.

22. $c = 328$, $a = 192$, $C = 90°$; find A and b.
log 3·28 = ·5158738, L sin 35° 49′ = 9·7672996, diff. = 1750, log 2·6593 = ·4247673, diff. = 163, log 1·92 = ·2833012, L cos 35° 49′ = 9·9089639, diff. = 912.

23. $a = 6·23$, $A = 64° 20'$, $C = 90°$; find b and c.
log 6·23 = ·7944880,
L tan 64° 20′ = 10·3182604, log 299·38 = 2·4762228, diff. = 145
L sin 64° 20′ = 9·9548834, log 691·20 = 2·8396037, diff. = 63.

24. $a = 70·5$, $b = 96·5$, $C = 90°$; find A and c.
log 70·5 = 1·8481891, L tan 36° 9′ = 9·8636500, diff. = 2652.
log 119·50 = 2·0773679, diff. = 363.
log 96·5 = 1·9845273, L sin 36° 9′ = 9·7707793, diff. = 1729

25. $b = 1218$, $c = 1282$, $C = 90°$; find a and B.
log 1218 = 3·0856473, log 1282 = 3·1078880, L sin 71° 49′ = 9·9777523.
L sin 71° 50′ = 9·9777938.

Tables (seven-figure) will be required for the remainder of the Exercise.

26. $A = 36° 21' 20''$, $c = 74·8234$, $C = 90°$; find b.

27. $a = 784·325$, $A = 60° 34'$, $C = 90°$; find b.

28. $b = 29784$, $A = 43° 24' 30''$, $C = 90°$; find c.

29. $b = 200$, $c = 249$ C $= 90°$; find a.

30. $a = 416$, $c = 740$, $C = 90°$; solve the triangle.

31. $A = 37° 10'$, $a = 124$, $C = 90°$; find b and c.

32. $a = 5$, $c = 13$, $C = 90°$; solve the triangle.

33. $a = 1100$, $c = 1109$, $C = 90°$; solve the triangle.

34. The base of an iosceles triangle is 10, and the height 20; find the vertical angle.

35. The side of an isosceles triangle is 30, and the base 10; find the vertical angle.

36. The side of an isosceles triangle is 30, and the height 20; find the vertical angle.

37. The sides of a triangle are 746·232, 746·232, and 400; find the angles.

38. From the extremity of the diameter of a circle whose radius is 20, a chord is drawn, whose length is 12; what is the angle between the chord and the diameter?

39. The sides of a rectangle are 10 and 6; what is the smaller angle contained by the diagonals?

40. The hypothenuse of a right-angled triangle is 20, and one of its angles is 32°; what is the length of the perpendicular to the hypothenuse?

41. Four lines *OA*, *OB*, *OC*, and *OD*, meeting in the point *O*, make each of the angles *AOB*, *BOC*, *COD*, *DOE* equal to 25°; *AB* is at right angles with *OB*, *BC* with *OC*, and *CD* with *OD*; what is the length of *OD*, if *OA* be 24 inches?

Area of a parallelogram = base × perp. height. (Euc. Bk. I., 35).
Area of a triangle $= \frac{1}{2}$ base × perp. height. (Euc. Bk. I., 41).

42. Find the area of the isosceles triangle whose side is 24 inches, and vertical angle 32°.

43. Find the area of the parallelogram whose two adjacent sides are 36 and 28, and the included angle 64°.

44. Find the area of the regular pentagon whose side is 2.

45. The base of a triangle is $62\frac{1}{2}$ inches; a line 32 inches in length, drawn from the vertex to the base, makes an angle of 70° with the base; what is the area of the triangle?

46. Two sides of a triangle are 8 and 25 inches, and the included angle 56°; find its area.

47. The diagonals of a parallelogram are 16 and $12\frac{1}{2}$ inches, and the included angle 68°; find the area of the parallelogram.

48. The diagonals of an irregular quadrilateral are 64 and $31\frac{1}{4}$ respectively, and make with each other an angle of 42°; what is the area of the quadrilateral?

49. The two sides of a triangle are $6\frac{2}{5}$ and $3\frac{1}{8}$; what must be the angle between them that the area may be 6·9466 inches?

50. The area of a parrallelogram is $88\frac{2}{3}$ square inches, and its diagonals 25 and 16 inches; what is the angle between them?

EXERCISE VII.

The angle which a line joining the eye of an observer and a distant object makes with the horizontal plane is called the *angle of elevation* if the object be above the observer, and the *angle of depression* if the object be below the observer.

1. A person wishing to ascertain the height of a tower standing on a declivity, ascends to a point 80 feet below its base, and it then subtends an angle of 30°; find the height of the tower, the inclination of the side of the hill to the horizon being 30°.

2. A person standing at a distance of 82 ft. 4 in. from the base of a tower, observes that the altitude of the tower is exactly 45°; find the height of the tower without referring to the tables, the eye of the observer being 5 ft. 2 in. from the ground.

3. A person standing at the edge of a river observes that the top of a tower on the edge of the opposite side subtends an angle of 60° with a line drawn from his eye parallel to the horizon; receding 30 ft., he finds it to subtend an angle of 45°. Determine the breadth of the river.

4. The angles of depression of the top and bottom of a column observed from the top of a tower 108 ft. high are 30° and 60° respectively; find the height of the column.

5. The angles of depression and elevation of the top of a column observed from the top and bottom of a tower 108 ft. high, are 30° and 60° respectively. Find the height of the column.

6. *A* and *B* are two stations on a hill side; the inclination of the hill to the horizon is 45°; the distance between *A* and *B* is 500 yards. *C* is the summit of another hill in the same vertical plane as *A* and *B*, on a level with *A*, but at *B* its elevation above the horizon is 30°. Find the distance from *A* to *C*.

7. A ship which is known to be sailing due East at 12 miles an hour, was observed to be 30° to the East of South; 1*h*. 30*m*. afterwards it was seen in the South East. Find the distance of the ship when first seen.

8. At the foot of a mountain the elevation of its summit is found to be 45°. After ascending for two miles at a slope of 30° towards its summit, its elevation is found to be 60°. Determine the height of the mountain.

9. A person at a distance af 20 yards from the nearer of two towers in the same straight line with him, and 10 yards apart, observes them to subtend the same angle. Passing the nearer tower a certain distance, he observes them again subtend the same angle, the complement of the former. Find the heights of the towers.

10. A person standing on the bank of a river observes the elevation of the top of a tree on the opposite bank to be 51°; and when he retires 30 feet from the river's edge, he observes the elevation to be 46°. Determine the breadth of the river. Given tan 51° = 1·2381, tan 46° = 1·0355.

11. A person on a tower whose height is 90 feet, observes the angles of depression of two objects on the horizontal plane which are in the same straight line with the tower to be 60° and 34° 10′ 40″. Find the distance of the objects from each other.

log 3 = ·4771213, L tan 55° 49′ = 10·1680189, diff. = 2718; log 1·3254 = ·1223521.

12. Two spectators at two stations distant 200 feet from each other, observe the elevation of a kite to be 75° 21′ at each station, and the angle subtended by the kite and the other station to be 60°; find the height of the kite.

log 2 = ·3010300, L sin 75° 21′ = 9·9856460, log 1·9350 = ·2866760.

13. A person travelling along a straight road observes the elevation (10° 12′) of a church spire, the nearest distance of which from the road is 600 feet. At the same time he observes the angular distance (45°) of the bottom of the tower from an object in the road. Find the height of the tower.

log 6 = ·7781513, log 2 = ·3010300, L tan 10° 12′ = 9·2550997, by 1·5267 = ·1837537, diff. = 284.

14. At noon in Toronto (N. lat. 43° 39′) on the longest day in the year a flag pole was observed to cast a shadow 30 feet long; find the height of the pole.

log 3 = ·4771213, L tan 69° 49′ = 10·4346267, log 8·1611 = ·9117480.

15. On a day on which the north declination of the sun was known from the Almanac to be 10° 30′, the shadow (north) of a stick 10 feet long was observed at noon to be 8 feet long; find the latitude of the place.

log 2 = ·3010300, L tan 51° 20′ = 10·0969100.

Tables (seven-figure) will be required for the remainder of this Exercise.

16. A hill whose slope makes an angle of 12° with the horizon is one mile long; find the height of the hill.

17. A person in a balloon (stationary) whose elevation is 37° drops a stone which falls 560 yards from the observer; find the height of the balloon.

18. To ascertain the distance of an object A from B, I measure a base line BC of 200 feet at right angles to AB, and find the angle ACB to be 24° 12′ 20″; find the distance of A from B.

19. A person standing at a distance of 100 feet from the base of a tower, finds that the altitude of the tower is 50°; what is the height of the tower, the eye of the observer being 5 ft 3 in. from the ground.

20. A person wishing to know the distance of an inaccessible object A on the opposite bank of a river, views it from a station B; he then moves over a distance of 72 ft in a direction at right angles to AB, to a second station C, and observes the angle ACB to be 75°. Find the distance of A from C.

21. A river AC the breadth of which is 200 feet, flows at the foot of a tree CB, which subtends an angle BAC of 25° 10′ at the edge of the bank. Find the height of the tree.

22. Two persons A and B start at the same time from two points distant 400 yards. B starts at right angles to the line joining the two points at the rate of 90 yards a minute. A starts in a direction to catch B as soon as possible at the rate of 150 yards a minute. Find how long he will be before he catches him, and the direction in which he must walk.

23. A staff 1 foot long stands on the top of a tower 200 feet high. Find the angle it subtends at a place 100 feet from the foot of the tower.

24. The shadow cast at noon on the longest day of the year by a tower situated 51° 31′ N. latitude was 124 feet; find the height of the tower.

25. Find the height of a cloud whose elevation is 33° 10′, and depression 45° when seen by reflection in a lake from a station at a height of 150 feet above the surface of the lake.

Extension of the definition of the trigonometrical ratios to the case of an angle greater than 90°.

31. It is necessary now to extend our definitions to the case of an angle greater than one, but less than two, right angles. Let CAB be such an angle, and be denoted by A. Produce CA through A and drop BC' perpendicularly upon it. The angle BAC' is called the *supplement* of A, and $= 180° - A$. We now define the trigonometrical ratios of the angle A to be the corresponding ratios for the angle BAC' in the triangle $BC'A$, with the convention that AC' is to be considered a negative magnitude. Let p, b, h be the numerical values of the lengths of the perpendicular, base, and hypothenuse in the triangle; then

Relations between the ratios of an angle and its supplement.

$$\sin A = \frac{BC'}{AB} = \frac{p}{h} = \sin BAC' = \sin (180° - A);$$

$$\tan A = \frac{BC'}{AC} = \frac{p}{-b} = -\frac{p}{b} = -\tan BAC' = -\tan (180° - A);$$

$$\sec A = \frac{AB}{AC'} = \frac{h}{-b} = -\frac{h}{b} = -\sec BAC' = -\sec (180° - A);$$

$$\cos A = \frac{AC'}{AB} = \frac{-b}{h} = -\frac{b}{h} = -\cos BAC' = -\cos (180° - A);$$

$$\cot A = \frac{AC'}{BC'} = \frac{-b}{p} = -\frac{b}{p} = -\cot BAC' = -\cot (180° - A);$$

$$\operatorname{cosec} A = \frac{AB}{BC'} = \frac{h}{p} = \operatorname{cosec} BAC' = \operatorname{cosec} (180° - A).$$

32. It will be seen on inspection that the ratios according to this extended definition will satisfy the same five fundamental relations as before; and although the complement of an angle (A) which is greater than 90°, being $= 90° - A$, is a negative quantity, and ceases at present to have any signification, we shall still say that the cosine, cotangent, cosecant of such an angle are the sine, tangent, and secant of its complement, and hereafter, if necessary, give a consistent interpretation to the quantity.

The ratios for angles greater than 90° found from those of angles less than 90°.

33. From the above it is seen that the trigonometrical ratio of any angle is the same in numerical value as the corresponding ratio of its supplement, but bears a different sign except in the cases of sine and cosecant which bear the same sign. It is therefore unnecessary to construct additional

tables for angles greater than 90°, as the ratios for such angles can be found from those of their supplements, which are less than 90°. Further, for such angles the tangents, secants, cosines, and cotangents being negative quantities, have no logarithms, and it is only for the sines and cosecants that the logarithms have real values, being the same as those given in the tables for the supplements of these angles.

34. We can now proceed to the discussion of triangles in general, to the angles of which, whether acute or obtuse, our definitions of the ratios will now apply.

Three independent relations connect the six parts of an oblique triangle.

The triangle being $A\ B\ C$, the lengths of the sides opposite to the respective angles will be denoted by the small letters corresponding. The triangle then is said to have six parts:—namely, the three angles, A, B, C, and the three sides a, b, c. It is proved by Euclid that when three of these parts are given (one of them being a side), the other parts can be found There must therefore be three independent relations connecting these six quantities. One such relation is already established by Euclid, namely :

One relation

$$A + B + C = 180° \ldots\ldots\ldots\ldots (1)$$

Two others we proceed to investigate.

From C drop the perpendicular CD on AB (fig 6) or on BA produced (fig. 7).

Then in the right angled-triangle CBD,

$$CD = BC \sin CBD = a \sin B$$

And in the right-angled triangle CAD,

$$CD = AC \sin CAD = b \sin A, \text{ in fig. 6,}$$

$$= b \sin (180° - A) = b \sin A, \text{ in fig. 7}$$

Hence

$$a \sin B = b \sin A$$

Similarly, by dropping a perpendicular from A, we should obtain

$$b \sin C = c \sin B,$$

And hence

other.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \ldots\ldots\ldots\ldots (2)$$

Another relation found independently, but actually deducible from the above.

35. From these three relations (1), (2), all others can be deduced, but for such as we require at present, it will sometimes be easier to give proofs which do not directly depend on these.

Resuming the figures and construction of the previous proposition,

$$\begin{aligned} AB &= DB + AD, \text{ in fig. 6.} \\ &= BC \cos CBD + AC \cos CAD \\ &= a \cos B + b \cos A. \end{aligned}$$

Also,

$$\begin{aligned} AB &= DB - AD, \text{ in fig. 7.} \\ &= BC \cos CBD - AC \cos CAD \\ &= a \cos B - b \cos (180^\circ - A) \\ &= a \cos B + b \cos A. \end{aligned}$$

*Hence, universally,

$$c = a \cos B + b \cos A \ldots\ldots\ldots\ldots\ldots (3)$$

Deduction of certain general formulas.

36. Multiplying the respective terms of this equation by the equal quantities $\frac{\sin C}{c}$, $\frac{\sin A}{a}$, $\frac{\sin B}{b}$,

$$\sin C = \sin A \cos B + \cos A \sin B,$$

but C is the supplement of $(A+B)$; therefore

Sin $(A + B)$.

$$\sin (A+B) = \sin A \cos B + \cos A \sin B \ldots . (4)$$

37. In the preceding, instead of A write $180^\circ - A$; then

$$\sin \{180^\circ - (A - B)\}$$
$$= \sin (180^\circ - A) \cos B + \cos (180^\circ - A) \sin B,$$

---

*In this formula, writing it

$$1 = \frac{a}{c} \cos B + \frac{b}{c} \cos A$$

suppose that C is a right angle. Then $\cos B = \sin A$, $\frac{a}{c} = \sin A$, $\frac{b}{c} = \cos A$, and, making these substitutions, it becomes

$$1 = (\sin A)^2 + (\cos A)^2.$$

This is the proof alluded to on page 17, as not depending on Euclid, Bk. I., Prop. 47, but in fact being also a proof of that proposition.

or

$$\sin (A - B) = \sin A \cos B - \cos A \sin B \ldots . \quad (5)$$ $\sin (A - B)$.

Again, in this for A write $90° - A$; then

$$\sin \{90° - (A + B)\}$$

$$= \sin (90° - A) \cos B - \cos (90° - A) \sin B,$$

or

$$\cos (A + B) = \cos A \cos B - \sin A \sin B. \ldots . \quad (6)$$ $\cos (A + B)$.

The above proof of the last three formulas restricts the angles A and B to have their sum less than 180°. The formulas, however, are universal, but it is not necessary to extend them beyond this case, as it is the only case in which their use is at present required. In the appendix a general proof will be found, applicable to angles of any magnitude.

38. In (4), and (6), putting $B = A$, we obtain

$$\sin 2A = \sin A \cos A + \cos A \sin A$$

$$= 2 \sin A \cos A.$$

$$\cos 2A = \cos A \cos A - \sin A \sin A$$

$$= \cos {}^2A - \sin {}^2A,$$

and therefore, (since $\cos^2 A + \sin_2 A = 1$),

$$= 2 \cos^2 A - 1$$

$$\text{or} = 1 - 2 \sin^2 A.$$

Writing $\frac{1}{2}A$ instead of A, these become

$\sin A$ & $\cos A$ in terms of $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$.

$$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A, \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

$$\cos A = 2 \cos {}^2\tfrac{1}{2}A - 1 = 1 - 2 \sin {}^2\tfrac{1}{2}A \quad \ldots . \quad (8)$$

39. Adding (4) and (5), we obtain

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cos B,$$

And subtracting (5) from (4),

$$\sin (A + B) - \sin (A - B) = 2 \cos A \sin B.$$

Dividing the terms of these two equalities, we obtain

$$\frac{\sin (A+B) + \sin (A - B)}{\sin (A+B) - \sin (A - B)} = \frac{2 \sin A \cos B}{2 \cos A \sin B}$$

$$= \frac{\sin A}{\cos A} \div \frac{\sin B}{\cos B}$$

$$= \frac{\tan A}{\tan B}.$$

In this formula, instead of $(A+B)$ write A, and instead of $(A-B)$ write B, and therefore also instead of A write $\frac{1}{2}(A+B)$, and instead of B write $\frac{1}{2}(A-B)$, and we obtain

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}\ldots\ldots\ldots\ldots\ldots(9)$$

EXERCISE VIII.

(The Examples in this Exercise should be attentively studied.)

Prove the following relations :

1. $\text{Cos}(A-B)=-\cos\left\{180°-(A-B)\right\}=-\cos\left\{(180°-A)+B\right\}=$
$-\cos(180°-A)\cos B+\sin(180°-A)\sin B$
$=\cos A\cos B+\sin A\sin B.$

2. $\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}=\frac{\tan A+\tan B}{1-\tan A\tan B}.$

3. $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$

4. $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$; or $\tan A=\frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}.$

5. $\cot(A+B)=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}=\frac{\cot A\cot B-1}{\cot A+\cot B}.$

6. $\cot 2A=\frac{\cot^2 A-1}{2\cot A}.$

7. $\sin(A+B)=\sin A\cos B+\cos A\sin B,$
$\sin(A-B)=\sin A\cos B-\cos A\sin B;$
$\therefore \sin(A+B)+\sin(A-B)=2\sin A\cos B.$

8. $\sin(A+B)-\sin(A-B)=2\cos A\sin B.$

9. $\cos(A-B)+\cos(A+B)=2\cos A\cos B.$

10. $\cos(A-B)-\cos(A+B)=2\sin A\sin B.$

11. Since $A=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)$, and $B=\frac{1}{2}(A+B)-\frac{1}{2}(A-B)$;
$\therefore \sin A+\sin B=\sin\left\{\frac{1}{2}(A+B)+\frac{1}{2}(A-B)\right\}+\sin\left\{\frac{1}{2}(A+B)-\frac{1}{2}(A-B)\right\}$
$=$, Ex. 7, $2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B).$

12. $\sin A-\sin B=2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B).$

13. $\cos B+\cos A=2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B).$

14. $\cos B-\cos A=2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B).$

15. $\dfrac{1-\cos A}{1+\cos A}=\dfrac{1-(1-2\sin^2\frac{1}{2}A)}{1+(2\cos^2\frac{1}{2}A-1)}=\tan^2\frac{1}{2}A.$

16. $\sin(A+B+C)=\sin(A+B)\cos C+\cos(A+B)\sin C.$
$=\sin A\cos B\cos C+\sin B\cos C\cos A$
$+\sin C\cos A\cos B-\cos A\cos B\cos C.$

17. $\cos(A+B+C)=\cos A\cos B\cos C-\cos A\sin B\sin C$
$-\cos B\sin C\sin A-\cos C\sin A\sin B.$

18. $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$

19. $\sin 3A=\sin(2A+A)=\sin 2A\cos A+\cos 2A\sin A,$
$=2\sin A\cos^2 A+\sin A-2\sin^3 A,$
$=3\sin A-4\sin^3 A.$

20. $\cos 3A=\cos(2A+A)=\cos 2A\cos A-\sin 2A\sin A,$
$=2\cos^3 A-\cos A-2\sin^2 A\cos A,$
$=4\cos^3 A-3\cos A.$

21. $\tan 3A=\tan(2A+A)=\dfrac{\tan 2A+\tan A}{1-\tan 2A\tan A},$

$$=\dfrac{\dfrac{2\tan A}{1-\tan^2 A}+\tan A}{1-\dfrac{2\tan A}{1-\tan^2 A}\cdot\tan A},$$

$$=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$$

(Examples 19, 20, 21 may, of course, be obtained from Examples 16, 17, 18, by putting $A=B=C$, $1-\sin^2 A$ for $\cos^2 A$, and $1-\cos^2 A$ for $\sin^2 A$.)

22. Let $A=18°$. Then, since $\sin 36°=\cos(90°-36°)=\cos 54°$,
$\therefore \sin 2A=\cos 3A,$
$2\sin A\cos A=4\cos^3 A-3\cos A.$

Divide by $\cos A$;
$\therefore 4\sin^2 A+2\sin A=1,$

or $\sin 18°=\sin A=\dfrac{\sqrt{5}-1}{4}$, taking + sign, since $\sin 18°$ is positive.

EXERCISE IX.

Prove the following relations:

1. $\sin 15° = \sin (45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30° = \frac{\sqrt{3} - 1}{2\sqrt{2}}$.

2. $\cos 15° = \frac{\sqrt{3} + 1}{2\sqrt{2}} = \sin 75°$.

3. $\cot 15° = \tan 75° = 2 + \sqrt{3}$.

4. $\tan 15° = \cot 75° = 2 - \sqrt{3}$.

5. If $\sin a = \frac{2}{3}$, $\sin \beta = \frac{3}{5}$, then $\sin (a + \beta) = \frac{8 + 3\sqrt{5}}{15}$, $\cos (a + \beta) = \frac{4\sqrt{5} - 6}{15}$.

6. If $\sin A = \frac{3}{4}$, $\cos B = \frac{1}{3}$, then $\sin (A + B) = \frac{3 + 2\sqrt{14}}{12}$, $\cos (A + B) = \frac{\sqrt{7} - 6\sqrt{2}}{12}$.

7. If $\sin A = \frac{2}{3}$, $\sin B = \frac{3}{5}$, then $\sin (A - B) = \frac{8 - 3\sqrt{5}}{15}$.

8. If $\sin a = \frac{2}{3}$, then $\sin 2a = \frac{4\sqrt{5}}{9}$, $\cos 2a = \frac{1}{9}$.

9. If $\sin A = \frac{5}{13}$, $\sin B = \frac{4}{5}$, then $\sin (45° + A + B) = \frac{79\sqrt{2}}{130}$.

10. Given $\sin 18° = \frac{\sqrt{5} - 1}{4}$, shew that $\cos 36° = \frac{\sqrt{5} + 1}{4}$, $\sin 36° = \sqrt{\frac{5 - \sqrt{5}}{8}}$, $\sin 72° = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$.

EXERCISE X.

Establish the following:

1. $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$.

2. $\sin 4A = 4 \cos 2A \cos A \sin A$.

3. $1 - \frac{1}{2} \sin 2a \tan a = \cos^2 a$.

4. $\cos A \pm \sin A = \sqrt{1 \pm \sin 2A}$.

5. $\cos A - \sin A = \sqrt{2} \sin (45° - A)$. Substitute here for sin (45° - A)

6. $\sin A + \cos A = 2 \sin 45° \cos (45° - A)$.

7. $\cos^4 a - \sin^4 a = \cos 2a$.

8. $\sin 4A - \sin 2A = \sin (3A + A) - \sin (3A - A) =$
$2 \cos 3A \sin A$.

9. $\cos 2A - \cos 4A = 2 \sin A \sin 3A$.

10. $2 \sin 2A \cos A = \sin 3A + \sin A$.

11. $\cos 3A + \cos A = 2 \cos A \cos 2A$.

12. $\cos \theta - \cos 7\theta = 2 \sin 4\theta \sin 3\theta$.

13. $\sin A + \sin 2A = 2 \sin \frac{3A}{2} \cos \frac{A}{2}$. apply Example 11 page 42.

14. $\cos 3A - \cos 6A = 2 \sin \frac{9A}{2} \sin \frac{3A}{2}$.

15. $\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A$.

16. $\frac{\sin 5\theta + \sin 3\theta}{\cos 3\theta - \cos 5\theta} = \cot \theta$.

17. $\frac{\sin 2A + \sin A}{\cos 2A + \cos A} = \tan \frac{3A}{2}$.

18. $\frac{\sin 4A + \sin 3A}{\cos 4A + \cos 3A} = \tan \frac{7A}{2}$.

19. $\frac{\sin 5A - \sin 4A}{\cos 4A - \cos 5A} = \cot \frac{9A}{2}$.

20. $\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}{2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)} =$
$\tan \frac{1}{2}(A + B)$.

21. $\frac{\sin A - \sin B}{\cos B - \cos A} = \cot \frac{A + B}{2}$.

22. $\dfrac{\sin A - \sin B}{\cos B + \cos A} = \tan \dfrac{A - B}{2}$.

23. $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \dfrac{\cot \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$.

24. $\cot \theta + \tan \theta = 2 \operatorname{cosec} 2\theta$.

25. $\cot \theta - \tan \theta = 2 \cot 2\theta$.

26. $\operatorname{cosec} 2\theta + \cot 2\theta = \cot \theta$.

27. $\tan 2\theta + \sec 2\theta = \dfrac{\sin 2\theta + 1}{\cos 2\theta} =$

$$\frac{2 \sin\theta \cos\theta + \sin^2\theta + \cos^2\theta}{\cos^2\theta - \sin^2\theta} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$$

28. $\dfrac{1 - 2\sin^2 A}{1 + \sin 2A} = \dfrac{1 - \tan A}{1 + \tan A}$.

29. $\dfrac{1 + \sin A}{\cos A} = \dfrac{(\cos \frac{1}{2}A + \sin \frac{1}{2}A)^2}{\cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A} = \dfrac{1 + \tan \frac{1}{2}A}{1 - \tan \frac{1}{2}A} =$
$\tan (45^\circ + \frac{1}{2}A)$.

30. $\dfrac{1 - \sin A}{\cos A} = \tan \left(45^\circ - \dfrac{A}{2}\right)$.

31. Given $L \sin 31^\circ 23' = 9{\cdot}7166387$, $L \sin 31^\circ 24' = 9{\cdot}7168458$, find $L \sin 148^\circ 36' 42''$.

32. Given $L \operatorname{cosec} 25^\circ 34' = 10{\cdot}3649578$, $L \operatorname{cosec} 25^\circ 35' = 10{\cdot}3646938$, find $L \operatorname{cosec} 154^\circ 25' 36''$.

EXERCISE XI.

Establish the following:

1. $\cos (30^\circ - \alpha) - \cos (30^\circ + \alpha) = \sin \alpha$.

2. $\sin (\alpha + \beta) \cos \alpha - \cos (\alpha + \beta) \sin \alpha = \sin \beta$.

3. $\sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta = \cos^2 \beta - \cos^2 \alpha$.

4. $\cos^2 (\alpha - \beta) - \sin^2 (\alpha + \beta) = \cos 2\alpha \cos 2\beta$.

5. $\cos (\alpha + \beta) \sin (\alpha - \beta) = \sin \alpha \cos \alpha - \sin \beta \cos \beta$.

6. $\sin(\alpha-\beta)\sin\gamma+\sin(\beta-\gamma)\sin\alpha+\sin(\gamma-\alpha)\sin\beta=o.$

7. $\cos\beta\cos(2\alpha+\beta)=\cos^2(\alpha+\beta)-\sin^2\alpha.$

8. $\dfrac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos^2\alpha\cos^2\beta}=\tan^2\alpha-\tan^2\beta.$

9. $\cos\alpha+\cos(\alpha+2\beta)=\cos\{(\alpha+\beta)-\beta\}+\cos\{(\alpha+\beta)+\beta\}=2\cos(\alpha+\beta)\cos\beta.$

10. $\sin(\alpha+2\beta)=\sin\alpha+2\sin\beta\cos(\alpha+\beta).$

11. $\sin(\alpha-2\beta)=\sin\alpha-2\sin\beta\cos(\alpha-\beta).$

12. $\cos(\alpha+2\beta)=2\cos\beta\cos(\alpha+\beta)-\cos\alpha.$

13. $\sin(\alpha+2\beta)=\sin\alpha-2\sin\alpha\sin^2\beta+2\cos\alpha\sin\beta\cos\beta.$

14. $\sin 8\theta=8\cos 4\theta\cos 2\theta\cos\theta\sin\theta.$

15. $\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1.$

16. $\sin 3A-\sin A=2\sin A\cos 2A.$

17. $\cos 2A+\cos 3A=2\cos\dfrac{5A}{2}\cos\dfrac{A}{2}.$

18. $\cos 3A-\cos 4A=2\sin\dfrac{7A}{2}\sin\dfrac{A}{2}.$

19. $\sin\dfrac{3A}{2}+\sin 2A=2\sin\dfrac{7A}{4}\cos\dfrac{A}{4}.$

20. $\sin 3A-\sin\dfrac{A}{2}=2\cos\dfrac{7A}{4}\sin\dfrac{5A}{4}.$

21. $4\cos A\cos 2A\cos 3A=(\cos A+\cos 3A)\,2\cos 3A.$
$=2\cos A\cos 3A+2\cos^2 3A.$
$=\cos 2A+\cos 4A+1+\cos 6A.$

22. $4\sin A\sin 2A\sin 3A=\sin 2A+\sin 4A-\sin 6A.$

23. $4\cos A\cos 2A\sin 3A=\sin 2A+\sin 4A+\sin 6A.$

24. $8\sin A\sin 2A\sin 3A\sin 4A=1-\cos 6A-\cos 8A+\cos 10A.$

25. $\cos A + \cos 2A + \cos 3A = 4 \cos \frac{A}{2} \cos A \cos \frac{3A}{2} - 1.$

26. $\cos 9A + 3 \cos 7A + 3 \cos 5A + \cos 3A = 8 \cos^3 A \cos 6A.$

27. $\cot^2 A - \tan^2 A = \frac{\cos^4 A - \sin^4 A}{\sin^2 A \cos^2 A} = \frac{8 \cos 2A}{2 \sin^2 2A} = \frac{8 \cos 2A}{1 - \cos 4A}.$

28. $\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A.$

29. $\sin 2a + \cos a = \frac{2 \tan a + \sec a}{1 + \tan^2 a}.$

30. $\frac{\operatorname{cosec} 2\varphi - \cot 2\varphi}{\operatorname{cosec} 2\varphi + \cot 2\varphi} = \tan^2 \varphi.$

31. $\cos^6 A + \sin^6 A = (\cos^2 A + \sin^2 A) \left\{(\cos^2 A + \sin^2 A)^2 - 3 \cos^2 A \sin^2 A\right\} = 1 - \frac{3}{4} \sin^2 2A.$

32. $\cos^6 A - \sin^6 A = \cos 2A \left(\frac{7 + \cos 4A}{8}\right).$

33. $\cos 6A = \cos 3(2A) = 4 \cos^3 2A - 3 \cos 2A = \cos 2A (4 \cos^2 2A - 3) = \cos 2A \left\{2(\cos 4A + 1) - 3\right\} = \cos 2A (2 \cos 4A - 1).$

34. $\frac{\cos 3A + \sin 3A}{\cos A - \sin A} = 1 + \sin 2A.$

35. $\frac{\cos 3A - \sin 3A}{\cos A + \sin A} = 1 - 2 \sin 2A.$

36. $\frac{\tan 3A}{\tan A} = \frac{3 - \tan^2 A}{1 - 3 \tan^2 A}.$

37. $\frac{2 \sin 2A - \sin 4A}{2 \sin 2A + \sin 4A} + 1 = \sec^2 A.$

38. $\frac{\cos na - \cos (n+2) a}{\sin (n+2) a - \sin na} = \tan (n+1) a.$

39. $\cos^2 (\theta + \psi) + \cos^2 (\theta - \psi) - \cos 2\theta \cos 2\psi = 1.$

40. $\dfrac{\cos 3A - 2\cos A}{\sin 3A + 2\sin A}\tan A = \dfrac{2\cos 2A - 3}{2\cos 2A + 3}$.

If $A + B + C = 180°$, prove the following, 41–45 :

41. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
Obtained from $\tan(A + B) = \tan(180° - C)$.

42. $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$.

43. $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$.

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} = 2\cos\frac{C}{2}\cos\frac{A-B}{2}.$$

$$\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2} = 2\cos\frac{C}{2}\cos\frac{A+B}{2}.$$

$$\therefore \sin A + \sin B + \sin C = 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right\}$$

$$= 4\cos\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}.$$

44. $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.
Obtained similarly to preceding.

45. $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$.

By 17 of Ex. VIII, $0 = \cos\left(\dfrac{A}{2} + \dfrac{B}{2} + \dfrac{C}{2}\right) = \cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} - \ldots.$

Solve the following equations :

46. $\cos\theta + \cos 7\theta = \cos 4\theta$.
Here $2\cos 3\theta\cos 4\theta = \cos 4\theta$; $\therefore \cos 4\theta = 0$, or $\cos 3\theta = \frac{1}{2}$.
$\therefore \theta = 22\frac{1}{2}°$ or $20°$.

47. $\cos\theta - \cos 3\theta = \sin 2\theta$.

48. $\cos 4\theta + \cos 2\theta = \cos\theta$.

49. $\cos 2x + \sin x = 1$.

50. $\sin 5x\cos 3x = \sin 9x\cos 7x$.

40. To express the cosine of an angle of a triangle in terms of the sides.

cos A in terms of a, b, c.

Resuming (3),

$$c = a \cos B + b \cos A.$$

From the analogy we see that

$$a = b \cos C + c \cos B.$$
$$b = c \cos A + a \cos C.$$

If from these 3 equations we eliminate cos B and cos C, the required result will be obtained. Multiplying the first by c, and the third by b, and then adding, we have

$$\begin{aligned} c^2 + b^2 &= a\, c \cos B + a\, b \cos C + 2\, b\, c \cos A \\ &= a\,(c \cos B + b \cos C) + 2\, b\, c \cos A \\ &= a^2 + 2\, b\, c \cos A, \text{ (from the second),*} \end{aligned}$$

or,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\, b\, c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Analogous expressions can now be written down for cos B and cos C. These expressions are not adapted to logarithmic calculation, and we thereby proceed to modify them.

The previous expression modified for logarithmic use.

41. From (8),

$$\begin{aligned} 2 \sin^2 \tfrac{1}{2} A &= 1 - \cos A \\ &= 1 - \frac{b^2 + c^2 - a^2}{2\, b\, c} \quad \ldots\ldots \text{ from (10)} \\ &= \frac{a^2 - (b^2 - 2\, b\, c + c^2)}{2\, b\, c} \\ &= \frac{a^2 - (b - c)^2}{2\, b\, c} \\ &= \frac{(a + b - c)\,(a - b + c)}{2\, b\, c}. \end{aligned}$$

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* Written in the form,

$$a^2 = b^2 + c^2 - 2\, b\, c \cos A,$$

this is identical with Euclid, pp. 12, 13, *B. II*; for in fig. (6) $A\,D = b \cos A$, and in fig. (7), $AD = -b \cos A$, and therefore

$$BC^2 = AC^2 + AB^2 \mp 2\, AB.\, AD,$$

− or + according as A is acute or obtuse.

Again from (8),

$$2\cos^2\tfrac{1}{2}A = 1 + \cos A$$
$$= 1 + \frac{b^2 + c^2 - a^2}{2\,b\,c}$$
$$= \frac{(b^2 + 2\,b\,c + c^2) - a^2}{2\,b\,c}$$
$$= \frac{(b+c)^2 - a^2}{2\,b\,c}$$
$$= \frac{(b+c+a)\,(b+c-a)}{2\,b\,c}.$$

Now putting

$$a + b + c = 2\,s$$ s the se perimeter

and therefore

$$a + b - c = 2\,(s - c)$$
$$b + c - a = 2\,(s - a)$$
$$c + a - b = 2\,(s - b),$$

these become

$$\left.\begin{aligned}\sin^2\tfrac{1}{2}A &= \frac{(s-b)\,(s-c)}{b\,c}\\ \cos^2\tfrac{1}{2}A &= \frac{s\,(s-a)}{b\,c}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots (11)$$

And dividing the former by the latter,

$$\tan^2\tfrac{1}{2}A = \frac{(s-b)\,(s-c)}{s\,(s-a)},$$

or

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}\ \ldots\ldots\ldots\ldots (12)$$ tan $\frac{1}{2}A$ in terms of s and the sides.

Analogous expressions may now be written down for tan $\frac{1}{2}B$, tan $\frac{1}{2}C$.

EXERCISE XII.

In any triangle, right-angled at C, prove

1. $a^2 + b^2 = a\,c\cos B + b\,c\cos A$.

2. $a\,(a\sin A + b\sin B + c) = 2\,c^2\sin A$.

3. $a - b = \sqrt{2}\, c \sin \frac{1}{2} (A - B)$.

4. $a + b = \sqrt{2}\, c \cos \frac{1}{2} (A - B)$.

5. $(a + c) \sin \frac{B}{2} = b \cos \left(45^\circ - \frac{A}{2}\right)$.

6. $\frac{1 + \cot \frac{1}{2} B}{\cot \frac{1}{2} A} = \frac{2a}{b + c - a}$.

In any triangle establish the following relations between the sides and angles:

7. $\frac{\sin A + \sin B}{\sin B} = \frac{a + b}{b}$.

8. $\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{a + b}{a - b}$.

9. $\frac{\sin A - \sin B}{a - b} = \frac{\sin C}{c}$.

10. $\frac{\sin A + \sin B - \sin C}{\sin A - \sin B + \sin C} = \frac{a + b - c}{a - b + c}$.

11. $\frac{a + c - b}{4c} = \frac{b \sin^2 \frac{1}{2} A}{a + b - c}$.

12. $a^2 + b^2 + c^2 = 2ab \cos C + 2bc \cos A + 2ca \cos B$

13. $\cos A + \cos B = 2 \frac{a + b}{c} \sin^2 \frac{C}{2}$.

14. $a (b \cos C - c \cos B) = b^2 - c^2$.

15. $\cot A - \cot B = \frac{b^2 - a^2}{ab \sin C}$.

$\left(\text{True if } \frac{c}{a} \cos A - \frac{c}{b} \cos B = \frac{b^2 - a^2}{ab}\right)$.

16. $a + b + c = (a + b) \cos C + (b + c) \cos A + (c + a) \cos B$.

17. $(a + b) \sin^2 \frac{1}{2} C + (b + c) \sin^2 \frac{1}{2} A + (c + a) \sin^2 \frac{1}{2} B = \frac{1}{2} (a + b + c)$.

18. $(a + b)(1 - \cos C) = c(\cos A + \cos B)$.

19. $(a + b) \sin^2 \frac{1}{2} C = c(1 - \sin^2 \frac{1}{2} A - \sin^2 \frac{1}{2} B)$.

20. $\tan B = \dfrac{b \sin C}{a - b \cos C}$.

21. $\dfrac{c^2}{a\,b \sin C} = \cot A + \cot B$.

22. $\dfrac{\sin (B - C)}{\sin A} = \dfrac{b^2 - c^2}{a^2}$.

23. $a^2 \sin 2 B + b^2 \sin 2 A = 2\,a\,b \sin C$.

24. $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$.

25. $b \cos^2 \dfrac{C}{2} + c \cos^2 \dfrac{B}{2} = \dfrac{a + b + c}{2}$.

26. $(b - c) \cos \dfrac{A}{2} = a \sin \dfrac{B - C}{2}$.

27. $\dfrac{\sin^2 \frac{B}{2}}{b} + \dfrac{\sin^2 \frac{C}{2}}{c} = \dfrac{s - a}{b\,c}$.

28. $b\,c \cos^2 \dfrac{A}{2} + c\,a \cos^2 \dfrac{B}{2} + a\,b \cos^2 \dfrac{C}{2} = \frac{1}{4}(a+b+c)^2$.

29. $1 - \tan \dfrac{A}{2} \tan \dfrac{B}{2} = \dfrac{2\,c}{a + b + c}$.

30. $\dfrac{\cot \frac{B}{2} + \cot \frac{C}{2}}{\cot \frac{A}{2}} = \dfrac{2\,a}{b + c - a}$.

31. If $s(s - c) = \frac{1}{2}\,a\,b$, one angle is a right angle.

32. If $c \cos B = b \cos C$, shew that the triangle is isosceles.

33. If $a \sec B = 2\,c$, the triangle is isosceles.

34. If $a = b$, $c = 2\,a \sin \dfrac{C}{2}$.

35. If $c = 2\,a \sin \dfrac{C}{2}$ then either the triangle is isosceles, or $c^2 = a(a - b)$.

SOLUTION OF OBLIQUE TRIANGLES.

Solution of oblique triangles.

42. Four distinct cases occur in the solution of oblique triangles, according to the way in which three parts out of the six which compose the triangle are selected, one at least of the given parts being a side.

Four cases.

These are,

(1), two angles and a side. (Euclid, B. I., Prop. 26.)

(2), the three sides. (.... Prop. 8)

(3), two sides and the included angle. (.... Prop. 4)

(4), two sides and an angle not included. (...... The omitted case.)

Case I.

Two angles and a side given.

43. Case I. Given A, B, a; to find C, b, c.

To find C,

$$C = 180° - A - B. \quad \ldots\ldots\ldots\ldots \quad C \text{ found.}$$

To find b,

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

or

$$b = \frac{a \sin B}{\sin A} = a \sin B \operatorname{cosec} A,$$

and taking logarithms,

$$\begin{aligned}\log b &= \log a + L \sin B - 10 + L \operatorname{cosec} A - 10 \\ &= \log a + L \sin B + L \operatorname{cosec} A - 20\end{aligned}$$

from which there is b found.

To find c,

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

or

$$c = a \sin C \operatorname{cosec} A$$

$$\log c = \log a + L \sin C + L \operatorname{cosec} A - 20,$$

from which there is c found.

In this case it is indifferent which of the sides is given, as all three angles are at once known.

44. Case II. Given a, b, c; to find A, B, C. Case II.

The three sides given.

To find A, we have, (where $s = \frac{1}{2}(a + b + c)$),

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \text{ .... from (12)},$$

and taking logarithms,

$$L \tan \tfrac{1}{2} A - 10 = \tfrac{1}{2} \log \frac{(s-b)(s-c)}{s(s-a)}$$

$$= \tfrac{1}{2} \left\{\log (s-b) + \log (s-c) - \log s - \log (s-a)\right\}$$

$$= \tfrac{1}{2} \left\{\log (s-b) + \log (s-c) + \operatorname{colog} s + \operatorname{colog} (s-a) - 20\right\}$$

and therefore

$$L \tan \tfrac{1}{2} A = \tfrac{1}{2} \left\{\log (s-b) + \log (s-c) + \operatorname{colog} (s-a) + \operatorname{colog} s\right\},$$

from which there is $\frac{1}{2} A$ and therefore A found.

By the analogous formula, B can be found, and then C, which is $180° - A - B$. It is however better in practice to find C also by its analogous formula, and the sum of the three angles amounting to 180° will serve as verification.

We might also have used either of the formulas (11), for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, but that for the tangent is practically preferable. If the sum of two of the quantities, a, b, c, be not greater than the third, one of the quantities $s-a$, $s-b$, $s-c$, will be negative, and its logarithm imaginary.

45. Case III. Given a, b, C; to find A, B, c. $(a > b)$. Case III.

Two sides and the included angle given.

To find A, B.

$$\frac{\sin A}{a} = \frac{\sin B}{b},$$

or

$$\frac{a}{b} = \frac{\sin A}{\sin B},$$

and therefore

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \text{ .... from (9)}$$

or

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A+B).$$

Now

$$\tfrac{1}{2}(A+B)=\tfrac{1}{2}(180^\circ - C)=90^\circ-\tfrac{1}{2}C, \text{ and is known;}$$

also $$\tan\tfrac{1}{2}(A+B)=\cot\tfrac{1}{2}C,$$

and therefore

$$\tan\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\tfrac{1}{2}C,$$

and taking logarithms,

$$L\tan\tfrac{1}{2}(A-B)-10=\log(a-b)+\text{colog}(a+b)-10$$
$$+L\cot\tfrac{1}{2}C-10,$$

or

$$L\tan\tfrac{1}{2}(A-B)=\log(a-b)+\text{colog}(a+b)+L\cot\tfrac{1}{2}C-10,$$

from which $\frac{1}{2}(A-B)$ is found; also $\frac{1}{2}(A+B)$ being known, we have by addition and subtraction...... A and B found.

A having thus been found, we obtain c from the formula

$$\frac{\sin C}{c}=\frac{\sin A}{a},$$

$$c=a\sin C\,\text{cosec}\,A$$

$$\log c=\log a+L\sin C+L\,\text{cosec}\,A-20,$$

(c found,)

in which formula b, B might also be used in place of a, A.

In this case c is known directly in terms of the given parts from

$$c^2=a^2+b^2-2\,a\,b\cos C,$$

but this formula is not adapted to logarithmic calculation, and it is preferable to find c by aid of one of the angles which have been previously found.

Case IV.

46. Case IV. Given A, a, b; to find B, C, c.

Two sides and an angle not included by them given. The ambiguity discussed.

Fig. 8.

Fig. 9.

In this case there are sometimes two triangles which have the given parts. For let A be acute, and (fig. 8) drop the perpendicular CD, which is equal to $b\sin A$; then there can be drawn two lines, each $=a$, one on each side of CD, and if both these fall (as CB_1, CB_2) on the right of b, the two triangles ACB_1, ACB_2 will have the same three given parts. This requires a to be less than b and greater than CD; if however $a=CD$, the two triangles coincide in a right-angled triangle, and if a be less than CD, no triangle exists having the given quantities for parts. Also if $a=b$, the triangle ACB_2 vanishes, and only one is left, and if a be greater than b, the line CB_2 falls to the left of b, and the triangle so formed would not have the angle A, And in this case there is only one triangle.

Again if A be obtuse (fig. 10), in order that a triangle may exist, a must be greater than b, and the other line equal to a will fall to the left of b, so that only one triangle exists. Fig. 10.

Collecting these results, we see that, when A is acute, if $a < b \sin A$, there is no triangle; if $a = b \sin A$, there is one only; if $a > b \sin A$ and $< b$, there are two; if $a =$ or $> b$, there is only one; and when A is obtuse, if $a <$ or $= b$, there is no triangle; and if $a > b$, there is one only. If A be a right angle, then a must be $> b$, and the two triangles on opposite sides of b are equal in every respect, and therefore only give the *same* triangle in different positions.

The analytical solution which follows will of itself shew which of these varieties occurs in any particular case.

To find B; Solution.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

or

$$\sin B = \frac{b \sin A}{a}$$

and taking logarithms

$$L \sin B - 10 = \log b + \text{colog } a - 10 + L \sin A - 10,$$

whence

$$L \sin B = \log b + \text{colog } a + L \sin A - 10.$$

This gives $L \sin B$, but as the $L \sin$ of an angle is the same as the $L \sin$ of the supplement of that angle, there are two angles which have this value of $L \sin$, and both must be taken. Let B_1, B_2 be these two angles, the former being less than 90° and taken direct from the tables, the latter being its supplement. Let C_1, C_2 be the corresponding values for C, so that

$$C_1 = 180° - A - B_1,$$
$$C_2 = 180° - A - B_2.$$

If both these values are positive, two triangles exist.

Let c_1, c_2 be the corresponding values of c. To find them;

$$\frac{\sin C_1}{c_1}=\frac{\sin A}{a}$$

$$c_1 = a \sin C_1 \operatorname{cosec} A.$$

$$\log c_1 = \log a + L \sin C_1 + L \operatorname{cosec} A - 20.$$

Similarly

$$\log c_2 = \log a + L \sin C_2 + L \operatorname{cosec} A - 20.$$

If the second value of C be 0 or negative, the second solution has no existence; and if both values of C are negative, no solution exists. Also if the value of $L \sin B$ be greater than 10, there is no solution.

Examples. 47. Examples.

Case I.

A, B, a given to find [C.] Given $A = 120° 08'$, $B = 24° 40'$, $a = 981{\cdot}23$.

$$C = 180° - A - B.$$

$A =$	120° 08′
$B =$	24 40
	144 48
	180
$C =$	35° 12′ (C found.)

[b.] $\log b = \log a + L \sin B + L \operatorname{cosec} A - 20.$

$a = 981{\cdot}23$;	$\log a$,	2·99177
$B = 24° 40'$;	$L \sin B$,	9·62049
$A = 120° 08'$;	$L \operatorname{cosec} A$,	10·06305
$b = 473{\cdot}49$;	$\log b$,	2·67531 (b found.)

[c.] $\log c = \log a + L \sin C + L \operatorname{cosec} A - 20.$

	$\log a$,	2·99177
$C = 35° 12'$;	$L \sin C$,	9·76075
	$L \operatorname{cosec} A$,	10·06305
$c = 653{\cdot}99$;	$\log c$,	2·81557 (c found.)

Case II.

Given $a = 753{\cdot}09$, $b = 333{\cdot}33$, $c = 666{\cdot}66$. *a, b, c given, to find*

$a = 753{\cdot}09$
$b = 333{\cdot}33$
$c = 666{\cdot}66$

$2s = 1753{\cdot}08$

	log	colog
$s = 876{\cdot}54$	2·94277	7·05723
$s - a = 123{\cdot}45$	2·09149	7·90850
$s - b = 543{\cdot}21$	2·73497	7·26503
$s - c = 209{\cdot}88$	2·32197	7·67803

$L \tan \frac{1}{2} A = \frac{1}{2}\{\log(s-b) + \log(s-c) + \text{colog}(s-a) + \text{colog}\, s\}$. [A.]

log $(s-b)$,	2·73497
log $(s-c)$,	2·32197
colog $(s-a)$,	7·90850
colog s,	7·05723
	2)20·02267
$\frac{1}{2} A = 45° \; 45'$; $L \tan \frac{1}{2} A$,	10·01133

$A = 91° \; 30'$ (*A* found.)

$L \tan \frac{1}{2} B = \frac{1}{2}\{\log(s-c) + \log(s-a) + \text{colog}(s-b) + \text{colog}\, s\}$. [B.]

log $(s-c)$,	2·32197
log $(s-a)$,	2·09149
colog $(s-b)$,	7·26503
colog s,	7·05723
	2)18·73572
$\frac{1}{2} B = 13° \; 08'$; $L \tan \frac{1}{2} B$,	9·36786

$B = 26° \; 16'$ (*B* found.)

$L \tan \frac{1}{2} C = \frac{1}{2}\{\log(s-a) + \log(s-b) + \text{colog}(s-c) + \text{colog}\, s\}$. C.]

log $(s-a)$,	2·09149
log $(s-b)$,	2·73497
colog $(s-c)$,	7·67803
colog s,	7·05723
	2)19·56172
$\frac{1}{2} C = 31° \; 07'$; $L \tan \frac{1}{2} C$,	9·78086

$C = 62° \; 14'$. (*C* found.)

Verification. **Verification.**

$$A = 91^\circ\ 30'$$
$$B = 26\ \ 16$$
$$C = 62\ \ 14$$

$$A + B + C = 180^\circ.$$

Case III.

a, *b*, *C* given, to find

Given $a = 209{\cdot}88$, $b = 333{\cdot}33$, $C = 112^\circ\ 26'$.

Here, b being greater than a, we must interchange a, A with b, B in the formulas of solution.

$$90^\circ\ 00$$
$$C = 122^\circ\ 26'; \quad \tfrac{1}{2} C = 61\ \ 13$$

$$\tfrac{1}{2}(B + A) = 90^\circ - \tfrac{1}{2} C = 28^\circ\ 47'$$

[*A* and *B*.] $L \tan \tfrac{1}{2}(B - A) = \log(b - a) + \text{colog}(b + a) + L \cot \tfrac{1}{2} C - 10.$

$$b = 333{\cdot}33$$
$$a = 209{\cdot}88; \ \log, 2{\cdot}32197$$

$$b - a = 123{\cdot}45; \ \log, 2{\cdot}09149$$
$$b + a = 543{\cdot}21; \ \log, 2{\cdot}73497; \ \text{colog}, 7{\cdot}26503$$

$$\log(b - a), \ 2{\cdot}09149$$
$$\text{colog}(b + a), \ 7{\cdot}26503$$
$$\tfrac{1}{2} C = 61^\circ\ 13'; \ L \cot \tfrac{1}{2} C, \ 9{\cdot}73987$$

$$\tfrac{1}{2}(B - A) = 7^\circ\ \ 07'; \ L \tan \tfrac{1}{2}(B - A), \ 9{\cdot}09639$$
$$\tfrac{1}{2}(B + A) = 28\ \ 47$$

$$B = 35^\circ\ 54'$$
$$A = 21^\circ\ 40' \qquad (B \text{ and } A \text{ found.})$$

[*c*.] $\log c = \log a + L \sin C + L \text{cosec } A - 20.$

$$\log a, \ 2{\cdot}32197$$
$$C = 122^\circ\ 26'; \ L \sin C, \ 9{\cdot}92635$$
$$A = 21^\circ\ 40'; \ L \text{cosec } A, \ 10{\cdot}43273$$

$$c = 479{\cdot}97; \quad \log c, \ 2{\cdot}68105$$

(c found.)

Case IV.

Ex. (1). Given $A = 57° \; 34'$, $a = 47{\cdot}979$, $b = 54{\cdot}324$. A, a, b given to find [B.]

$L \sin B = \log b + \text{colog}\, a + L \sin A - 10.$

$b = 54{\cdot}321$;	$\log b$,	$1{\cdot}73497$
$a = 47{\cdot}979$;	colog a,	$8{\cdot}31895$
$A = 57° \; 34'$;	$L \sin A$,	$9{\cdot}92635$
$B_1 = 72° \; 52'$;	$L \sin B$,	$9{\cdot}98027$
$B_2 = 107° \; 08'$.		

(B_1 and B_2 found.) [C.]

$C_1 = 180 - (A + B_1)$ $\qquad$ $C_2 = 180 - (A + B_2)$

$A = 57° \; 34'$ $\qquad$ $A = 57° \; 34'$

$B_1 = 72 \;\; 52$ $\qquad$ $B_2 = 107 \;\; 08$

$C_1 = 180° - 130° \; 26'$ $\qquad$ $C_2 = 180° - 164 \;\; 42$ Two solutions.

$= 49° \; 34'$ $\qquad$ $= 15° \; 18'$.

Hence there are two solutions.

$\log c_1 = \log a + L \sin C_1 + L \text{ cosec } A - 20.$ [c_1.]

$a = 47{\cdot}979$;	$\log a$	$1{\cdot}68105$
$C_1 = 49° \; 34'$;	$L \sin C_1$,	$9{\cdot}88148$
$A = 57° \; 34'$;	L cosec A,	$10{\cdot}07365$
$c_1 = 43{\cdot}269$.	$\log c_1$,	$1{\cdot}63618$ (c_1 found.)

$\log c_2 = \log a + L \sin C_2 + L \text{ cosec } A - 20.$ [c_2.]

	$\log a$,	$1{\cdot}68105$
$C_2 = 15° \; 18'$;	$L \sin C_2$,	$9{\cdot}42139$
	L cosec A,	$10{\cdot}07365$
$c_2 = 15$.	$\log c_2$,	$1{\cdot}17609$

(c_2 found.)

Ex. 2. Given $A = 49° \; 41'$, $a = 323{\cdot}1$, $b = 21{\cdot}808$. A, a, b given, to find B,

$L \sin B = \log b + \text{colog}\, a + L \sin A - 10.$

$b = 21{\cdot}808$;	$\log b$,	$1{\cdot}33862$
$a = 323{\cdot}1$;	colog a,	$7{\cdot}49066$
$A = 49° \; 41'$;	$L \sin A$,	$9{\cdot}88223$
$B_1 = 2° \; 57'$;	$L \sin B$,	$8{\cdot}71151$
$B_2 = 177° \; 03'$		

(B_1 and B_2 found.)

[C.] $C_1 = 180 - (A + B_1)$ $\qquad$ $C_2 = 180 - (A + B_2)$

$$\begin{array}{rlrl} A & = 49^\circ\ 41' & A & = 49^\circ\ 41' \\ B_1 & = 2^\circ\ 57' & B_2 & = 177^\circ\ 03' \\ \hline \end{array}$$

One solution

$$C_1 = 180^\circ - 52^\circ\ 38' \qquad C_2 = 180^\circ - 226^\circ\ 44'$$
$$= 127^\circ\ 22' \qquad = -$$

[c_1.] The second solution does not exist. The value of c_1 can be found as in the previous example.

A, a, b given, to find B.

Ex. (3). Given $A = 30^\circ$, $a = 18{\cdot}4$, $b = 38{\cdot}9$.

$$L \sin B = \log b + \text{colog } a + L \sin A - 10.$$

$$\begin{array}{rlr} b = 38{\cdot}9\,; & \log b, & 1{\cdot}58995 \\ a = 18{\cdot}4\,; & \text{colog } a, & 8{\cdot}73518 \\ A = 30^\circ\,; & L \sin A, & 9{\cdot}69897 \\ \hline & L \sin B, & 10{\cdot}02410 \end{array}$$

No solution.

No solution exists.

A, a, b given, to find B.

Ex. (4). Given $A = 128^\circ\ 57'$, $a = 21700$, $b = 19342$.

$$L \sin B = \log b + \text{colog } a + L \sin A - 10.$$

$$\begin{array}{rlr} b = 19342\,; & \log b, & 4{\cdot}28650 \\ a = 21700\,; & \text{colog } a, & 5{\cdot}66354 \\ A = 128^\circ\ 57\,; & L \sin A, & 9{\cdot}89081 \\ \hline \left\{\begin{array}{l} B_1 = 43^\circ\ 53'; \\ B_2 = 136^\circ\ 07'. \end{array}\right. & L \sin B, & 9{\cdot}84085 \end{array}$$

C.] $C_1 = 180 - (A + B_1)$ $\qquad$ $C_2 = 180 - (A + B_2)$

$$\begin{array}{rlrl} A & = 128^\circ\ 57' & A & = 128^\circ\ 57' \\ B_1 & = 43\ \ 53 & B_2 & = 136\ \ 07 \\ \hline \end{array}$$

One solution

$$C_1 = 180^\circ - 172^\circ\ 50' \qquad C_2 = 180^\circ - 265^\circ\ 04'$$
$$= 7^\circ\ 10'. \qquad = -$$

The second solution does not exist.

A, a, b given, to find B,

Ex. (5). Given $A = 163^\circ\ 24'$, $a = 42$, $b = 53{\cdot}004$.

$$L \sin B = \log b + \text{colog } a + L \sin A - 10.$$

$$\begin{array}{rlr} b = 53{\cdot}004\,; & \log b, & 1{\cdot}72431 \\ a = 42\,; & \text{colog } a, & 8{\cdot}37675 \\ A = 163^\circ\ 24'; & L \sin A, & 9{\cdot}45589 \\ \hline \left\{\begin{array}{l} B_1 = 21^\circ\ 08'; \\ B_2 = 158^\circ\ 52' \end{array}\right. & L \sin B, & 9{\cdot}55695 \end{array}$$

$C_1 = 180° - (A + B_1)$ $\qquad$ $C_2 = 180° - (A + B_2)$ [C.

$$\begin{array}{rl} A &= 163°\ 24' \\ B_1 &= \ \ 21°\ 08' \\ \hline \end{array} \qquad \begin{array}{rl} A &= 163°\ 24' \\ B_2 &= 158°\ 52' \\ \hline \end{array}$$

$C_1 = 180° - 184°\ 32'$ $\qquad$ $C_2 = 180° - 322°\ 16'$ No solution.

$= -$ $\qquad$ $= -$

No solution exists.

EXERCISE XIII.

1. $\sin B = \cdot 25$, $a = 5$, $b = 2\cdot 5$; find A.

2. $A = 30°$, $a = \sqrt{2}$, $b = 2$; find B.

3. Given $\dfrac{b}{a} = \frac{1}{2}$, $C = 60°$; find the other angles.

4. $A = 60°$, $a = \sqrt{6}$, $b = 2$; find B.

5. $A = 135°$, $a = 2$, $b = \sqrt{2}$; find B.

6. $A = 135°$, $a = 2$, $b = \sqrt{6}$; solve the triangle.

7. $b = 20$, $B = 82°$, $C = 75°$; find a.

   $\log 2 = \cdot 3010300$, $L \sin 23° = 9\cdot 5918780$,
   $\log 7\cdot 8914 = \cdot 8971541$, $L \sin 82° = 9\cdot 9957528$.
   Diff. $= 55$.

8. $a = 36$, $B = 44°$, $C = 104°$; find b.

   $\log 36 = 1\cdot 5563025$, $L \sin 32° = 9\cdot 7242097$,
   $\log 4\cdot 7191 = \cdot 6738592$, $L \sin 44° = 9\cdot 8417713$.
   Diff. $= 92$.

9. $b = 564\cdot 8$, $A = 40°\ 32'\ 16''$, $B = 104°\ 41'\ 32''$; find a.

   $\log 5\cdot 648 = \cdot 7518947$, $L \sin 40°\ 32' = 9\cdot 8128401$,
   $\log 3\cdot 7950 = \cdot 5792118$, Diff. for $1' = 1477$,
   Diff. $= 114$ $\qquad$ $L \sin 75°\ 18' = 9\cdot 9855467$,
   Diff. for $1' = 331$.

10. $a = \sqrt{56},\ b = 1,\ c = 7$; find A.

$\log 2 = \cdot 3010300,\ L\cos 57^\circ 41' = 9\cdot 7280275,$
$\log 7 = \cdot 8450980,\ L\cos 57^\circ 42' = 9\cdot 7278277.$

11. Given $a = 18,\ b = 20,\ c = 22$; find A.

$\log 2 = \cdot 3010300,\ L\tan 25^\circ 14' = 9.6732745,$
$\log 3 = \cdot 4771213,$ Diff. for $1' = 3275.$

12. Given $a = 4,\ b = 5,\ c = 6$; find B.

$\log 2 = \cdot 3010300,\ L\cos 27^\circ 53' = 9\cdot 9464040,$
$\log 5 = \cdot 6989700,\ L\cos 27^\circ 54' = 9\cdot 9463371.$

13. $a = 1900,\ b = 100,\ C = 60^\circ$; find A and B.

$\log 3 = \cdot 4771213,\ L\tan 57^\circ 19' = 10\cdot 1927506,$
$L\tan 57^\circ 20' = 10\cdot 1930286.$

14. $a = 18,\ b = 2,\ C = 55^\circ$; find A and B.

$L\tan 56^\circ 56' = 10\cdot 1863769,\quad \log 2 = \cdot 3010300,$
Diff. for $1' = 2763.\quad L\cot 27^\circ 30' = 10\cdot 2835233.$

15. $a = 9,\ b = 7,\ C = 64^\circ 12'$; find A and B.

$\log 2 = \cdot 3010300,\quad L\tan 11^\circ 16' = 9\cdot 2993216,$
$L\cot 32^\circ 6' = 10\cdot 2025255,$ Diff. for $1' = 6588.$

16. $b = 159\cdot 0643,\ B = 62^\circ 6' 51'',\ C = 53^\circ 27' 20''$; find a.

$\log 159\cdot 06 = 2\cdot 2015610,\quad L\sin 64^\circ 25' = 9\cdot 9551864,$
$\log 159\cdot 07 = 2\cdot 2015883,\quad L\sin 64^\circ 26' = 9\cdot 9552469,$
$\log 162\cdot 33 = 2\cdot 2103988,\quad L\sin 62^\circ 6' = 9\cdot 9463371,$
$\log 162\cdot 34 = 2\cdot 2104255,\quad L\sin 62^\circ 7' = 9\cdot 9464040.$

17. Given $a = 222,\ b = 318,\ c = 406$; find A.

$\log 4\cdot 73 = \cdot 6748611,\quad \log 2\cdot 51 = \cdot 3996737,$
$\log 4\cdot 06 = \cdot 6085260,\quad L\cos 16^\circ 28' = 9\cdot 9818117,$
$\log 3\cdot 18 = \cdot 5024271,\quad L\cos 16^\circ 29' = 9\cdot 9817744.$

18. $a = 85\cdot 63,\ b = 78\cdot 21,\ C = 48^\circ 24'$; solve the triangle.

$\log 16384 = 4\cdot 2144199,\quad L\cot 24^\circ 12' = 10\cdot 3473497,$
$\log 742 = 2\cdot 8704039,\quad L\tan 5^\circ 45' = 9\cdot 0030066,$
Diff. for $1' = 12655.$
$\log 67502 = 4\cdot 8293166,\quad L\sin 24^\circ 12' = 9\cdot 6127023,$
$\log 67501 = 4\cdot 8293102,\quad L\cos 5^\circ 45' 15'' = 9\cdot 9978062.$

19. $a = 212{\cdot}5$, $b = 836{\cdot}4$, $A = 14^\circ\ 24'\ 25''$; find B.

$\log 212{\cdot}5 = 2{\cdot}3273589$, $L \sin 14^\circ\ 24' = 9{\cdot}3956581$,
$\log 836{\cdot}4 = 2{\cdot}9224140$, Diff. for $1' = 4918$,
$L \sin 78^\circ\ 19' = 9{\cdot}9909077$,
Diff. for $1' = 261$.

20. $a = 23$, $B = 18^\circ$, $C = 23^\circ\ 42'\ 43''$; find b.

$\log 23 = 1{\cdot}3617278$, $L \sin 18^\circ = 9{\cdot}4899824$,
$\log 10{\cdot}681 = 1{\cdot}0286119$, $L \sin 41^\circ\ 42' = 9{\cdot}8229721$,
Diff. $= 407$, Diff. for $1' = 1417$.

21. Given $a = 25$, $b = 26$, $c = 27$; find the angles.

$\log 35 = 1{\cdot}5440680$, $L \tan 28^\circ\ 7'\ 30'' = 9{\cdot}7279568$,
$\log\ 3 = {\cdot}4771213$, $L \tan 28^\circ\ 7'\ 40'' = 9{\cdot}7280074$,
$L \tan 31^\circ 56' 50'' = 9{\cdot}7948986$,
$L \tan 31^\circ\ 57' = 9{\cdot}7949455$,

22. $a = 1{\cdot}5$, $b = 13{\cdot}5$, $C = 65^\circ$; find A and B.

$\log 2 = {\cdot}3010300$, $L \tan 51^\circ\ 28' = 10{\cdot}0988763$,
$L \cot 32^\circ\ 30' = 10{\cdot}1958127$, $L \tan 51^\circ\ 29' = 10{\cdot}0991355$,

23. $a = 445$, $b = 565$, $A = 44^\circ\ 29'\ 53''$; find B.

$\log 445 = 2{\cdot}6483600$, $L \sin 44^\circ\ 29' = 9{\cdot}8455332$,
$\log 565 = 2{\cdot}7520484$, Diff. for $1' = 1286$,
$L \sin 62^\circ\ 51' = 9{\cdot}9492997$,
Diff. for $1' = 648$.

24. $A = 78^\circ$, $B = 54^\circ$, $a = 274$; find b and c.

$\log 274 = 2{\cdot}43775$, $L \sin 48^\circ = 9{\cdot}87107$,
$\log 226{\cdot}63 = 2{\cdot}35531$, $L \sin 54^\circ = 9{\cdot}90796$,
$\log 208{\cdot}17 = 2{\cdot}31842$, $L \sin 78^\circ = 9{\cdot}99040$.

25. Given $a = 330$, $b = 310$, $c = 144$; find the angles.

$\log 392 = 2{\cdot}59329$, $\log 82 = 1{\cdot}91381$,
$\log\ 62 = 1{\cdot}79239$, $\log 330 = 2{\cdot}51851$,
$\log 310 = 2{\cdot}49136$, $L \cos 42^\circ\ 27' = 9{\cdot}86798$,
$\log 144 = 2{\cdot}15836$, $L \cos 34^\circ\ 40' = 9{\cdot}91512$.

26. $a = 30$, $b = 20$, $C = 78°$; find c.

$\log 2 = \cdot 30103$, $L \tan 13° 52' = 9 \cdot 39245$,
$L \cot 39° = 10 \cdot 09163$, $L \tan 13° 53' = 9 \cdot 39299$.
$L \sin 39° = 9 \cdot 79887$, $L \cos 13° 53' = 9 \cdot 98712$,
$\log 3 \cdot 2412 = \cdot 51070$, $L \cos 13° 52' = 9 \cdot 98715$.

27. $a = 13$, $b = 37$, $A = 18° 55' 29''$; find B.

$\log 13 = 1 \cdot 1139434$, $L \sin 18° 55' = 9 \cdot 5108031$,
$\log 37 = 1 \cdot 5682017$, Diff. for $1' = 3685$,
$L \sin 67° 22' = 9 \cdot 9651953$,
Diff. for $1' = 527$.

28. $b = 149$, $A = 69° 59' 2''$, $C = 70° 42' 30''$; find a.

$\log 149 = 2 \; 1731863$, $L \sin 39° 18' = 9 \cdot 8016649$,
$\log 22099 = 4 \cdot 3443726$, $L \sin 39° 19' = 9 \cdot 8018192$,
$\log 221 = 2 \cdot 3443923$, $L \sin 69° 59' = 9 \; 9729398$,
$L \sin 70° = 9 \cdot 9729858$.

29. If $a = 22$, $b = 23$, $c = 25$; find B.

$\log 2 = \cdot 3010300$, $L \sin 29° 5' = 9 \cdot 6867088$,
$\log 11 = 1 \cdot 0413927$, Diff. for $1' = 2271$.
$\log 13 = 1 \cdot 1139434$,

30. $a = 75$, $b = 85$, $C = 75°$; find A and B.

$\log 160 = 2 \cdot 20412$, $L \tan 52° 30' = 10 \cdot 11502$,
$L \tan 4° 40' = 8 \cdot 9109$.

31. $a = 2820 \cdot 9385$, $b = 1430 \cdot 8485$, $A = 14° 59' 49''$; find B.

$\log 2 \cdot 8209 = \cdot 4503877$, $L \sin 14° 59' = 9 \cdot 4125245$,
$\log 2 \cdot 8210 = \cdot 4504031$, $L \sin 15° = 9 \cdot 4129962$,
$\log 1 \cdot 4308 = \cdot 1555789$, $L \sin 7° 32' = 9 \cdot 1176125$,
$\log 1 \cdot 4309 = \cdot 1556093$, $L \sin 7° 33' = 9 \cdot 1185667$.

32. $c = 100$, $A = 50°$, $B = 70°$; find a and b.

$\log 2 = \cdot 3010300$, $L \sin 50° = 9 \cdot 8842540$,
$\log 3 = \cdot 4771213$, $L \sin 70° = 9 \cdot 9729858$,
$\log 8 \cdot 8455 = \cdot 9467224$, $\log 10850 = 4 \cdot 0354297$,
Diff. $= 49$, Diff. $= 401$.

33. $a = 230$, $b = 240$, $c = 12$; find B.

$\log 11 = 1{\cdot}0413927$, $L \tan 72° 48' = 10{\cdot}5092668$
$\log 229 = 2{\cdot}3598355$, Diff. for $1' = 4474$.
$\log 241 = 2{\cdot}3820170$,

34. $a : b = 7 : 3$, $C = 6° 37' 24''$; find the other angles.

$\log 2 = {\cdot}3010300$, $L \tan 8° 13' = 9{\cdot}1595646$,
$L \tan 3° 18' 42'' = 8{\cdot}7624080$, $L \tan 8° 14' = 9{\cdot}1604569$.

35. $a = 21{\cdot}217$, $b = 12{\cdot}543$, $A = 29° 51'$; find B and C.

$\log 2{\cdot}1217 = {\cdot}3266840$, $L \sin 17° 6' 40'' = 9{\cdot}4686806$,
$\log 1{\cdot}2543 = {\cdot}0984014$, $L \sin 17° 6' 50'' = 9{\cdot}4687490$,
$L \sin 29° 51' = 9{\cdot}6969947$.

36. The ratio of two sides of a triangle is 9 : 7, and the included angle is 47° 25′; find the other angles.

$\log 2 = {\cdot}3010300$, $L \tan 15° 53' = 9{\cdot}4541479$,
$L \tan 66° 17' 30'' = 10{\cdot}3573942$, Diff. for $1' = 4797$.

37. $a = 462$, $b = 220{\cdot}5$, $A = 124° 34'$; find B and C.

$\log 2{\cdot}205 = {\cdot}3434086$, $L \sin 55° 26' = 9{\cdot}9156460$,
$\log 4{\cdot}62 = {\cdot}6646420$, $L \sin 23° 8' = 9{\cdot}5942513$,
$L \sin 23° 9' = 9{\cdot}5945469$.

38. $a = 95{\cdot}372$, $b = 74{\cdot}896$, $C = 59°$; find A, B and c.

$\log 2{\cdot}0476 = {\cdot}31125$, $L \cot 29° 30' = 10{\cdot}24736$,
$\log 1{\cdot}70268 = {\cdot}23113$, $L \tan 12° = 9{\cdot}32748$,
$\log 9{\cdot}5372 = {\cdot}97942$, $L \sin 59° = 9{\cdot}93307$,
$\log 8{\cdot}5718 = {\cdot}93307$, $L \sin 72° 30' = 9{\cdot}97942$.

39. $A = 41° 10'$, $a = 145{\cdot}3$, $b = 178{\cdot}3$; find B and C.

$L \sin 41° 10' = 9{\cdot}8183919$, $L \sin 53° 52' = 9{\cdot}9072216$,
$\log 1453 = 3{\cdot}1622656$, $L \sin 53° 53' = 9{\cdot}9073138$,
$\log 1783 = 3{\cdot}2511513$.

40. $a - b = 4013{\cdot}166$, $a + b = 7906{\cdot}72$, $C = 36°$; find A and B.

$\log 4{\cdot}0131 = {\cdot}6034800$, $L \cot 18° = 10{\cdot}4882240$,
$\log 4{\cdot}0132 = {\cdot}6034908$, $L \tan 57° 22' = 10{\cdot}1935848$,
$\log 7{\cdot}9067 = {\cdot}8979953$, $L \tan 57° 23' = 10{\cdot}1938630$.
$\log 7{\cdot}9068 = {\cdot}8980008$,

EXERCISE XIV.

Six or seven figure tables will be required for the following Exercise.

1. $a = 31$, $b = 24$, $c = 11$; find A.

2. $A = 23° \ 42' \ 43''$, $B = 18°$, $a = 207$; find b.

3. The sides of a triangle are 32, 40, 66; find the greatest angle.

4. $a = 1{\cdot}125$, $b = {\cdot}875$, $C = 64° \ 12'$; find A and B.

5. $a = 70$, $b = 35$, $C = 36° \ 52' \ 12''$; find A and B.

6. $a = 14000$, $b = 15906{\cdot}43$, $A = 45°$; find the other angles.

7. $c = 3727{\cdot}593$, $A = 50°$, $B = 57° \ 53' \ 9''$; find a.

8. The sides of a triangle are as 4, 5, 6; find the largest angle.

9. $b = 1{\cdot}125$, $c = {\cdot}875$, $A = 54°$; find B and C.

10. $b = 16{\cdot}25$, $c = 13{\cdot}75$, $A = 63°$; find B and C.

11. $b = 1$, $c = 3{\cdot}02943$, $B = 19°$; find C.

12. $c = 100$, $A = 45°$, $B = 10°$; find a.

13. The sides of a triangle are 4, 5, 6; find the smallest angle.

14. $b = 21$, $c = 9$, $A = 6° \ 37' \ 24''$; find B and C.

15. $11\,b = 14\,c$, $A = 60°$; find B and C.

16. $a = 100$, $c = 125$, $C = 45°$; solve the triangle.

17. $a = 500$, $B = 45°$, $C = 10°$; find b.

18. $a : b = 21 : 11$, $C = 34° \ 42' \ 30''$; find A and B.

19. The angles of a triangle are in $A.\ P.$, and the greatest side is to the least as 5 to 4; find the angles.

20. One angle of a triangle is 60°, and the ratio of the side opposite to it to the difference of the sides containing it, is $9\sqrt{3} : 2$; find the other angles.

NOTE.—It will be well also to work the two foregoing Exercises and Exercise VI., using the tables at the end of the book, in which case seconds will be omitted and digits to right in values of the sides, when such values contain more than four digits. The results so obtained will be found sufficiently close to the answers given.

EXERCISE XV.

In this Exercise the tables at the end of the book have been used.

1. $a = 74{\cdot}5$, $B = 69° \; 59'$, $C = 70° \; 43'$; find b and c.

2. $a = 4730$, $b = 4016$, $A = 71° \; 4'$; find B.

3. $a = 420$, $B = 76° \; 42'$, $C = 52° \; 29'$; find c.

4. $a = 759$, $b = 1130$, $A = 40° \; 32'$; find B.

5. $a = {\cdot}1063$, $b = {\cdot}4182$, $A = 14° \; 24'$; find B.

6. $a = 44{\cdot}28$, $b = 14{\cdot}76$, $C = 100° \; 30'$; find A and B.

7. $a = 22\frac{1}{4}$, $b = 28\frac{1}{4}$, $A = 44° \; 30'$; find B.

8. $a = 872{\cdot}5$, $b = 632{\cdot}7$, $C = 80°$; find A and B.

9. $a = 26$, $b = 74$, $A = 18° \; 55'$; find B.

10. Two angles of a triangle are 76° and 54°, and the side opposite the latter 80·9; find the other sides.

11. The sides of a triangle are 112, 88, 76; find the angles.

12. Two sides of a triangle are 1·732 and 1·414, and the included angle 75°; find the remaining angles.

13. Two angles of a triangle are 65° and 85°, and the interjacent side is 12·5; find the other sides.

14. The sides of a triangle are 2376, 1782, 1188; find the angles.

15. Two sides of a triangle are 908 and 640, and the included angle 62°; find the other side.

16. The sides of a triangle are 189·5, 188·5, 123·6; find the angles.

17. Two sides of a triangle are 77·99 and 83·39, and the included angle is 72° 15′; find the other angles.

18. The sides of a triangle are 102, 168, 128; find the angles.

19. Of three towns, the first is 165 miles from the second, the second 155 miles from the third, and the third 72 miles from the first; find the difference in the bearing of the second and third from the first.

20. The angles of a triangle are in *A. P.*, the greatest being twice the smallest, and the greatest side is 984·8; find the other sides.

EXERCISE XVI.

HEIGHTS AND DISTANCES.

1. Describe the observations and calculations necessary to determine the breadth of a river from stations on one of its banks.

2. A tree 51 feet high has a mark at the height of 25 feet from the ground; find at what distance the two parts subtend equal angles to an eye at the height of 5 feet from the ground.

3. A pole is fixed on the top of a hill, and the angles of elevation of the top and bottom of the pole are 60° and 45°; shew that the hill is $\frac{1}{2}(\sqrt{3}+1)$ times as high as the pole.

4. The angular elevation of an object at a place A due south of it is 30°; at a place B due west of A, and at a distance a from it, the elevation is 18°. Shew that the height of the object is $\dfrac{a}{\sqrt{(2\sqrt{5}+2)}}$.

5. An object 6 feet high, placed at the top of a tower, subtends an angle whose tangent is ·015 at a place whose horizontal distance from the base of the tower is 100 feet; shew that the height of the tower is 170·23 feet nearly.

6. A person stationed on a promontory first observes a ship at a point due north of him; in a quarter of an hour it bears due east; and after another quarter of an hour is seen to the south-east of him. Find the course the ship was steering, and shew that it was nearest to the observer 12 minutes after he first saw her.

Ans.—An angle whose tangent is $\frac{1}{2}$, to east of south.

7. A person wishing to know the height of an inaccessible object, measures equal distances AB, BC in a horizontal straight line, and observes the angles of elevation at A, B, C to be 30°, 45° and 60° respectively. Shew that the height of the object is $AB\sqrt{\frac{3}{2}}$, and its distance from ABC is $AB\frac{1}{\sqrt{2}}$.

8. The elevation of two clouds to a person in the same line with them is a. When vertically below the lower one, the elevation of the other is $2a$. Shew that the heights of the clouds are as $2\cos^2 a : 1$.

9. The elevation of a tower on a horizontal plane is observed; on advancing a feet nearer its elevation is found to be the complement of the former; on again advancing its elevation is found to be double of its first elevation; shew that the last station is $\frac{a}{2}$ feet from the foot of the tower.

10. A person on the top of a mountain observes the depression (45°) of an object on the plane below him: he then turns through an angle of 30°, and observes the depression of another object on the same plane to be 30°. On descending the mountain he finds the distance between the objects is d. Shew that the height of the mountain is also d.

11. At noon a column in the E.S.E. cast on the ground a shadow the extremity of which was in the direction N.E.; the angle of elevation of the column being a, and the distance of the extremity of the shadow from the column c, shew that the length of the column is $c \tan a \sqrt{2 - \sqrt{2}}$.

12. The elevation of a tower standing on a horizontal plane is observed; a feet nearer it is found to be 45°; b feet nearer still it is the complement of what it was at the first station. Shew that the height of the tower is $\frac{a\,b}{a - b}$ feet.

13. From the summits of two rocks A, B at sea, the dips, a, β, of the horizon are observed, and it is remarked that the summit of B is in a horizontal line through the summit of A; shew that the rocks subtend at the earth's centre an angle whose cosine is $\sec a \cos \beta$.

(The dip of the horizon is the angle a line drawn to touch the earth makes with the horizontal plane).

14. A person wishing to determine the length of an inaccessible wall places himself due south of one end and due west of the other, at such distances that the angles the wall subtends at the two positions are each equal to a. If a be the distance between the two positions, the length of the wall is $a \tan a$.

15. A ship, the summit of whose top mast is 90 feet from the water, is sailing towards an observer at the rate of 10 miles an hour, and takes 1 hour 12 minutes to reach him from the time of its first appearance. Shew that the earth's radius is 4224 miles, the tangent from the mast head to the earth's surface being considered equal to the arc beneath it.

16. A person walking along a straight road observes that the greatest angle that two objects make with each other is a; from the point where this happens he walks a yards, and the objects there appear in the same straight line making an angle β with the road. The distance between the objects is

$$\frac{2\,a \sin a \sin \beta}{\cos a + \cos \beta}.$$

17. A balloon considered as a vertical object of given height floats at a constant height above the earth, and subtends angles α, β at a place when the elevations of its lowest point are A and B respectively; prove

$$\tan (A + \alpha) \cot A = \tan (B + \beta) \cot B.$$

18. AB is a tower at the foot of a hill of inclination θ; C, D are two stations directly up the hill from B such that $BC = CD$; $ACD = \alpha$, $ADC = \beta$. Shew that

$$\cot \theta = \frac{\sin \alpha \sin \beta}{2 \cos \alpha \sin \beta + \cos \beta \sin \alpha}.$$

19. From the top of a tower the depressions α, β of two objects in the same horizontal plane with the foot of the tower are observed, and also the angle ω which they subtend; the distance a between them is known. The height of the tower is

$$\frac{a \sin \alpha \sin \beta}{\sqrt{(\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \cos \omega)}}.$$

20. At each of three stations in the same horizontal plane, and at given distances a, b, c from each other, the elevation of a tower is observed to be α; shew that the height of the tower is, if $a + b + c = 2s$,

$$\frac{a b c \tan \alpha}{\sqrt{s (s - a) (s - b) (s - c)}}.$$

21. The angles of elevation A, B, C, of a balloon were taken at the same time by three observers placed respectively at the ends and middle point of a base a measured on the earth's surface. Shew that height of balloon is

$$\frac{a}{\sqrt{2 (\cot^2 A + \cot^2 C - 2 \cot^2 B)}}.$$

22. To ascertain the height of a mountain, a base of a feet was measured, and at either extremity of this base were taken the angles α, β, formed by the summit and the other extremity; also at the extremity at which the latter was taken the elevation of the mountain was γ; shew that its height is $\dfrac{a \sin \alpha \sin \gamma}{\sin (\alpha + \beta)}$.

23. A column on a pedestal 20 feet high subtends an angle of 45° to a person on the ground; on approaching 20 feet it again subtends an angle of 45°. The height of the column is 100 feet.

24. In the ambiguous case where a, b and A are given to determine the triangle, if c', c'' be the two values found for the third side of the triangle, prove that

$$c'^2 - 2\,c'\,c''\cos 2\,A + c''^2 = 4\,a^2\cos^2 A.$$

25. In the case where the solution of the triangle is ambiguous, if k, k' be the areas of the two triangles which satisfy the given conditions, prove

$$\frac{k^2 + k'^2 - 2\,k\,k'\cos 2\,A}{(k + k')^2} = \frac{a^2}{b^2},$$

A, a and b being given.

EXPRESSIONS FOR THE AREA OF A TRIANGLE.

The area of a triangle.

48. It is proved by Euclid (B.I., prop. 41) that the area of a triangle is half that of a rectangle having the same base and height. Now the number of square units in the area of a rectangle is equal to the product of the numbers of linear units in the base and height respectively, which is briefly expressed by saying that the area of a rectangle is the product of the base and height. Hence the area of a triangle is half the product of its base and height.

Fig. 6, 7.

In fig. 6, 7, area of triangle $A\,B\,C$

$$= \tfrac{1}{2}\,A\,B.\ \ C\,D,$$
$$= \tfrac{1}{2}\,c\,b\sin A$$
$$= \tfrac{1}{2}\,b\,c\sin A.$$

Again

$$\sin A = 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A \quad \text{.................from (7)}$$

$$= 2\sqrt{\frac{(s-b)\,(s-c)}{b\,c}}\sqrt{\frac{s\,(s-a)}{b\,c}} \quad \text{....from (11)}$$

$$= \frac{2}{b\,c}\sqrt{\{s\,(s-a)\,(s-b)\,(s-c)\}}.$$

Therefore the area

$$= \sqrt{\{s\,(s-a)\,(s-b)\,(s-c)\}}.$$

EXERCISE XVII.

1. The sides of a triangle are 3, 5, 7; find its area.

2. The sides of a triangle are 5, 6, 7; find its area.

3. Find the area of a triangle whose sides are 60, 70 and 110.

4. Find the area of a triangular field whose sides are 471, 406 and 635.

5. If p, q, r be the perpendiculars drawn from each of the angles of a triangle to the opposite sides, shew that the area is equal to

$$\tfrac{1}{2} \sqrt[3]{p\, q\, r.\ a\, b\, c}\ .$$

6. Shew that the area of a triangle is equal to

$$s\,(s - a) \tan \frac{A}{2}\ .$$

7. Shew that the area of a triangle is equal to

$$\frac{2\,a\,b\,c}{a + b + c} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

8. If $2\,s = a + b + c$, prove that the area of the triangle is equal to

$$s^2 \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C.$$

9. The sides a, b, c of a triangle are in $A. P.$, shew that the area

$$= \frac{b}{4} \sqrt{3\,(3\,b - 2\,a)\ (2\,a - b)}.$$

10. The sides of a triangle are in $A. P.$, and the area is four-fifths the area of an equilateral triangle having the same perimeter; shew that the sides are as 7, 10, 13.

11. If the three sides of a triangle be $a + b$, $b + c$, $c + a$, and $2\,s = a + b + c$, shew that the area is equal to

$$\sqrt{2\,s.\ a\,b\,c}\ .$$

12. If the three sides of a triangle be $\sqrt{a+b}$, $\sqrt{b+c}$, $\sqrt{c+a}$, shew that its area is equal to

$$\tfrac{1}{2}\sqrt{a\,b+b\,c+c\,a}\ .$$

13. If p, q, r be the reciprocals of the perpendiculars drawn from each of the angles of a triangle to the opposite sides, shew that the area of the triangle is equal to

$$\frac{1}{\sqrt{(p+q+r)\,(q+r-p)\,(p+r-q)\,(p+q-r)}}.$$

14. Prove that the area of a triangle is equal to

$$\frac{a^2-b^2}{2}\cdot\frac{\sin A\sin B}{\sin(A-B)}.$$

15. Prove that the area of a triangle is equal to

$$\tfrac{1}{4}\sqrt[3]{2\,a^2b^2c^2\,(\sin 2\,A+\sin 2\,B+\sin 2\,C)}.$$

16. Shew that the area of a triangle is equal to

$$\frac{2\,s^2\sin A\sin B\sin C}{(\sin A+\sin B+\sin C)^2}.$$

17. Shew that the area of a triangle is equal to

$$\frac{a^2+b^2+c^2}{4\,(\cot A+\cot B+\cot C)}.$$

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18. Shew that the radius of the circle described about a triangle is equal to

$$\frac{a}{2\sin A}=\frac{a\,b\,c}{4\times\text{area of triangle}}.$$

19. Prove that the area of the circle inscribed in a triangle is equal to

$$\frac{2\times\text{area}}{a+b+c}.$$

20. Shew that the radius of the circle described to touch the side BC, and the sides AB, AC produced (called an escribed circle) is equal to

$$\frac{\text{area}}{s-a}.$$

EXAMINATION PAPERS

OF THE

UNIVERSITY OF TORONTO.

SENIOR MATRICULATION, 1874.

1. Define the logarithm of a number to base 10, and deduce the properties which make logarithms of value in facilitating arithmetical operations.

2. Find the following:

L cosec $85° \, 10' \, 33''$; $L \tan \tan^{-1} \dfrac{362}{201}$;

$$\log \sqrt[3]{\frac{39 \cdot 008 \times 100 \cdot 48}{2010}}.$$

3. Find the numbers and trigonometrical ratios corresponding to the following logarithms: $\bar{1} \cdot 9045$; $4 \cdot 591$; $9 \cdot 51$-7220 (sin); $9 \cdot 998460$ (cos).

4. Perform the following operations by logarithms:

$$\sqrt{\frac{39 \times 201}{362 \times 200}};\ 100 \cdot 1 \times \frac{\cos 85° \, 25'}{\cos 4° \, 50'}.$$

5. Define the trigonometrical ratios and co-ratios of an angle less than 90°.

Express all the trigonometrical ratios in terms of the cosine.

6. Find $\sin (A + B)$ and $\cos (A - B)$.

What are the values of sin 105°, tan 75°, cos 15°?

7. Express $\sin A$ and $\cos A$ (1) in terms of $\dfrac{A}{2}$, (2) of $\cos 2A$.

8. Prove that in any triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}; \quad \cos A = \frac{b^2 + c^2 - a^2}{2\,bc};$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}; \ \sin A = \frac{2}{bc}\sqrt{\left\{ s(s-a)(s-b)(s-c) \right\}}.$$

9. Obtain the logarithmic formulæ of solution in the following cases:

Having given (1) a, A, B; (2) a, b, C; (3) a, b, c.

10. Having given

$a = 240$, $b = 362$, $c = 200$, find A to seconds;
$a = 100{\cdot}1$, $A = 85^\circ\ 10'$, $B = 90^\circ\ 15'$, find b and c to three decimal places.

11. The perimeter of an equilateral triangle being 10, find its height and the radius of the circumscribed circle.

22. C and D are two points lying directly south of A and B respectively, and such that $A\,B$ subtends equal angles at each; having given the distances $A\,C$, $B\,D$, and the area of $A\,B\,C\,D$, determine the distance $A\,B$ and its bearing.

No.	Log.	Diff.	Angle.	Log.	Diff.
362	558709		sin 19° 12′	9·517020	362
3900	591065	111	cos 15′	9·999996	
2010	303196	216	sin 85° 10′	9·998453	11
1001	000434	432	sin 4° 35′	8·902596	
8027	904553	54			
2000	301030	217			
1004	001734	432			

SENIOR MATRICULATION, 1875.

1. Write down the characteristics of the logarithms of 235, 2·368, ·806, ·00025.

2. State the numerical limits between which the numbers lie whose logarithms have characteristics 5 and $\overline{2}$.

3. State the rules for finding the logarithms of products, quotients, powers, and roots.

Find the logarithm of $\sqrt[10]{\cdot 67234} \times (3.8826)^{10}$.

4. Given log 2 = ·30103, find log ·00025.

Calculate the values of

$$\frac{\sqrt[3]{67 \cdot 234}}{38 \cdot 826}, \text{ and } \sqrt{\frac{672 \cdot 34 \times 388 \cdot 26}{412 \cdot 67 \times 462 \cdot 75}}.$$

5. Explain how the size of an angle is expressed in Trigonometry.

Find the complement of 66° 41′ 4″ and the supplement of 100° 5′ 25″.

6. Define the Trigonometrical ratios of an angle less than two right angles.

Find the sin, cos, and tan of 30° and 60°.

7. In a triangle ($A > B$) prove

$$\sin (A + B) = \sin A \cos B + \cos A \sin B,$$

$$\tan \tfrac{1}{2} (A - B) = \frac{a - b}{a + b} \cot \tfrac{1}{2} C.$$

8. Prove the formulas

$$\tan^2 \tfrac{1}{2} x = \frac{1 - \cos x}{1 + \cos x}.$$

$$\tan \tfrac{1}{2} x = \frac{1 + \sin x - \cos x}{1 + \sin x + \cos x}.$$

9. (1) Given $c = 672 \cdot 34$, $A = 35° 16' 25''$, $C = 90°$, solve the triangle.

(2) Given $A = 50° 38' 52''$, $B = 60° 7' 25''$, $a = 412 \cdot 67$, solve the triangle.

10. If $s =$ the semi-perimeter of the triangle ABC, prove that the radii of the inscribed and circumscribed circles, are respectively.

$$s \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C, \ \tfrac{1}{4} s \sec \tfrac{1}{2} A \sec \tfrac{1}{2} B \sec \tfrac{1}{2} C.$$

No.	Log.	Angle	Log.
10473	02006	sin 35° 16′ 25″	9·76154
11691	06788	cos 35° 16′ 25″	9·91190
38826	58913	cosec 50° 38′ 52″	10·11167
41267	61560	sin 60° 7′ 25″	9·93807
46275	66535	sin 69° 13′ 43″	9·97081
49899	69809		
54890	73949		
67234	82759		

FIRST YEAR, 1876.

1. State and prove the rule for finding the characteristic of the logarithms of whole numbers.

Given $\log \cdot 25 = - \cdot 60206$, find how many digits there will be in the integral part of $(2{\cdot}5)^{20}$.

2. Prove $\log a^x = x \log a, \log \dfrac{a}{b} = \log a - \log b$.

Evaluate the following by using logarithms:

$$\sqrt[4]{80} \times \sqrt[3]{2{\cdot}7}, \ \sqrt[5]{-5} \times 18^{-\frac{1}{5}}$$

Find the tabular logarithms of sin 45°, tan 60°, cos 30°.

3. Shew that the logarithms of the trigonometrical ratios need not be entered for angles greater than 45°; take $\sin A$, $\tan A$ as examples, where A has any value from 0° to 180°.

If $\sec 120° = \dfrac{a}{40}$, can a be found by logarithms?

Adapt $\sin A - \tan \frac{1}{2} A$ to logarithmic computation.

4. Prove the following relations :

$$\sin^2 A = 1 - \cos^2 A$$
$$\tan^2 A = \sec^2 A - 1.$$
$$\text{cosec } A - \sin A = \cos A \cot A$$
$$\sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A.$$

5. A person standing on one bank of a river observes that an object on the opposite bank has an angle of elevation of 45°, and going back 150 feet, the corresponding angle is 30°. Find the breadth of the river.

6. A vertical stick whose height is 10 feet throws on a horizontal plane a shadow 7·74 feet long. Find the sun's altitude.

Indicate how the problem would be solved if the shadow fell on a plane through the foot of the stick inclined at an angle θ to the horizon, the line of intersection of the plane and horizon being perpendicular to the plane through the sun and stick.

7. Prove $\cos(A + B) = \cos A \cos B - \sin A \sin B$,

$$2 \sin^2 \tfrac{1}{2} A = 1 - \cos A,$$
$$(\cos A - \sin A)^2 = \cos 2A \tan(45^\circ - A),$$
$$\frac{\cos \theta + \cos 3\theta}{\sin \theta + \sin 3\theta} = \frac{1 - \tan^2 \theta}{2 \tan \theta}.$$

8. In any triangle establish the following :

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$
$$\cos \tfrac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}},$$
$$2 \times \text{area} = b c \sin A = \frac{c^2}{\cot A + \cot B}.$$

9. In a triangle

$B = 123^\circ 40'$, $b = 100$, $c = 60$, find A and C.

$A = 112^\circ 40'$, $b = 213{\cdot}4$, $c = 213{\cdot}4$, solve the triangle.

$a = 200$, $b = 77{\cdot}4$, $C = 41^\circ 50'$, find the area.

10. If $(\sin \theta + \cos \theta) = 3 \sin \theta + \sin 2\theta$, find θ in degrees, &c.

7

If $1 + \sin\theta = 2\cos\frac{1}{2}\theta\ (\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta)$, find θ in degrees, &c.

11. In any triangle shew that

$$2(1 - \sin C) > (\cos A - \sin B)^2.$$

No.	Log.	Angle.	Log.
20000	30103	tan 52° 15′	10·11110
30000	47712	tan 52° 16′	10·11136
41645	61956	sin 56° 20′	9·92027
77400	88874	sin 29° 57′ 30″	9·69842
21340	32919	sin 41° 50′	6·82410
17761	24946	sin 19° 28′	9·52278
51623	71284	sin 19° 29′	9·52314
		tan 26° 33′	9·69868
		tan 26° 34′	9·69900

FIRST YEAR, 1877.

1. Define the trigonometrical ratios of an angle, and write down the five relations connecting the six trigonometrical functions,—sine, cosine, tangent, cotangent, secant, and cosecant.

2. Explain the nature and use of logarithms, and find the common logarithms of $2\frac{1}{2}$, $2\frac{1}{4}$, and $\sqrt[7]{(\cdot 0162)^3}$.

3. Perform the following operation by logarithms :

$$\frac{1\cdot 28}{1\cdot 25} \times \frac{(216)^{\frac{5}{3}}}{\cdot 81} \times \frac{5}{\sqrt[4]{1\cdot 2}}.$$

4. Prove the following :

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)},$$

$$\cot\frac{A}{2} = \cot A + \operatorname{cosec} A,$$

$$\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A,$$

$$\cos A = \frac{\cot\frac{A}{2} - \tan\frac{A}{2}}{\cot\frac{A}{2} + \tan\frac{A}{2}}.$$

5. Find the values of $\sin 2A$ and $\cos 2A$ in terms of the simple angle A.

If $\sin 2A = \cos 3A$, find A; also its sine and cosine.

6. In any triangle, $c^2 = a^2 + b^2 - 2ab \cos C$.

From this single equation prove that any two sides of a triangle are together greater than the third.

7. In any triangle prove the following relations:

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \text{ Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

$$c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}.$$

8. The sides of a triangle are 20, 21, 29, find the area, and the angle opposite the greatest side.

9. If in any triangle $\frac{a+c}{b+c} = \frac{b}{a-c}$, then will $A = 120°$.

10. In determining an angle by its tangent, when the angle is near 90°, how would you proceed?

Shew also how to find the sine of a very small angle accurately from the tables.

11. A person undertakes to measure the distance between two points, A and B, and proceeds 50 yards to C in a straight line towards B; and then meets an impassable barrier, and as he has no instrument for measuring angles, he measures off a line in an unknown direction $CD = 60$ yards, and then measures AD 90 yards, and BD 90 yards; find AB.

In the same question, if the person has neither instruments for measuring angles, nor any trigonometrical tables, find AB.

No.	Log.	Angle.	Log.
20000	30103	tan 54° 44′	10·15048
30000	47712	tan 54° 45′	10 15075
46961	67174	sin 70° 31′	9·97439
		sin 70° 32′	9·97444
		sin 38° 56′	9·79825
		sin 38° 57′	9·79840

JUNIOR MATRICULATION, 1877.

1. Define the logarithm of a number to a given base.

Prove $\log m\,n = \log m + \log n$.

$$\log_a N \log_x a = \log_b N \log_x b.$$

2. What are the advantages of employing 10 as a base?

Shew how to find the characteristic of a number, part of which is integral.

3. The mantissæ of the logarithms of all numbers which differ only in the position of the decimal point, are the same.

What is the object of always making the mantissæ positive?

4. Given $\log 32953 = 4{\cdot}5178950$, $\log 3{\cdot}2954 = {\cdot}5179081$, find $\log {\cdot}003295345$.

Find this also by forming and employing a Table of Proportional Parts. By this table determine the number corresponding to the logarithm $3{\cdot}5179025$.

Find by logarithms the value of

$$\sqrt{{\cdot}0128} \times (12)^{-\frac{1}{4}} \times \frac{{\cdot}0279}{1{\cdot}24} \times 12{\cdot}5.$$

5. Prove

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

$$\cos B - \cos A = 2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B).$$

$$\cos^2 A - \cos^2 3A = \sin 4A \sin 2A.$$

6. In any triangle shew that

$$a = b \cos C + c \cos B.$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{b\,c}}.$$

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

$$\frac{\sin A - \sin B}{\sin(A - B)} = \frac{c}{a+b}.$$

7. $a=300$, $b=400$, $c=500$; solve the triangle.

$a=7$, $b=5$, $c=4$; solve the triangle.

$a=450$, $b=250$, $C=12° \ 36'$; find A and B.

8. If an angle, the opposite side, and the sum of the other two sides of a triangle be given, shew how to solve the triangle.

9. If the sides of a triangle be in *A. P.*, the tangents of half the angles are in *H. P.*

10. Three circles, two of which are equal, touch one another, and a fourth, lying between, touches them. Shew that the radius of this circle is

$$\frac{r\,(r'+r)\sin^2\frac{1}{4}\theta}{r'\cos^2\frac{1}{4}\theta - r\sin^2\frac{1}{4}\theta},$$

where θ is the angle between lines drawn from the centres of the equal circles to the centre of the other, and r' is the radius of each of the equal circles, and r that of the other.

No.	Log.	Angle.	Log.
20000	3010300	cos 53° 7′	9·7782870
30000	4771213	cos 53° 8′	9·7781186
49353	6933116	sin 50° 46′	9·8890644
70000	8450980	sin 50° 47′	9·8891675
		sin 22° 12′	9·5773088
		sin 22° 13′	9·5776183
		tan 6° 18′	9·0429731
		tan 68° 52′	10·4128096
		tan 68° 53′	10·4131853

FIRST YEAR, 1878.

1. Define the trigonometrical ratios of an angle; and shew which of them may have any magnitude whatever, positive or negative, and which of them never can have a value between $+1$ and -1.

Shew that the versed sine of an angle is equal to twice the square of the sine of half the angle.

2. Prove the formula chord $A = 2 \sin \frac{1}{2} A$, and hence shew that the chord of an angle will be positive while the angle increases from 0° to 360°, and negative while it increases from 360° to 720°.

3. Deduce formulas for expressing the sines and cosines of the sum and difference of two angles in terms of the sines and cosines of the angles themselves.

Find $\sin \theta$ in terms of $\sin 2\theta$ and $\cos 2\theta$.

Prove that $2 \operatorname{vers}(\frac{1}{2}A) = (\sin A - \sin \frac{1}{2}A)^2 + (\cos A - \cos \frac{1}{2}A)^2$.

4. Define the logarithm of a number to any given base, and shew how to deduce the common logarithm of a number from the Napierian logarithm.

Prove that $\log_a m = \log_a b . \log_b c . \log_c d \ldots \log_l m$.

5. Shew how to find the area of a triangle when (1) the sides are given, (2) when two sides and an angle opposite to one of them are given.

If a, b, are the perpendiculars from two angles of an equilateral triangle upon a straight line drawn through the other angle, then the area of the triangle will be

$$\frac{a^2 - a\,b + b^2}{\sqrt{3}}$$

6. In any triangle prove the relation

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

and hence deduce $\sin^2 C = \sin^2 A + \sin^2 B$, when C is a right angle.

Also deduce the same equations from formulas referred to in question 3.

7. Solve the following triangles:

(1) $a = 2{\cdot}469$, $b = 6{\cdot}9024$, $c = 9{\cdot}0642$.

(2) $a = 26{\cdot}91$, $b = 69{\cdot}09$, $C = 146° 30'$.

8. Explain the difference between the proper and the tabular logarithms of the trigonometrical functions, and the reason for it.

If A be any angle shew that

$$L \sec A + L \sec \tfrac{1}{2} A + L \operatorname{cosec} \tfrac{1}{2} A = 20{\cdot}60206 + L \operatorname{cosec} 2 A.$$

9. Solve the equation $(\frac{1}{8})^{x}(125)^{1-\frac{x}{2}} = (\frac{1}{4})^{3x+2}(\frac{1}{5})^{x}$.

Find the logarithm of $\dfrac{12 \times {\cdot}92178 \div 3{\cdot}072}{125\ ({\cdot}8436 \times {\cdot}067488)} \times 4\frac{1}{2}$.

10. Prove the formulas

(1) $\cos (A + B) \cos (A - B) = \cos^2 B - \sin^2 A$.

(2) $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2} (A + B)}{\tan \frac{1}{2} (A - B)}$.

(3) $\sin 2 A + \sin 2 B + \sin 2 C = 4 \cos A \cos B \cos C$, when $A + B + C = 90°$.

(4) $\sin A + \sin 2 A = \sin 3 A + 4 \sin \dfrac{A}{2} \sin \dfrac{2 A}{2} \sin \dfrac{3 A}{2}$.

(5) $\dfrac{\cos A + \sin C - \sin B}{\cos B + \sin C - \sin A} = \dfrac{1 + \tan \frac{A}{2}}{1 + \tan \frac{B}{2}}$,

when $A + B + C = 90°$.

11. If a, b, c are the sides of a triangle, and A, B, C the angles opposite them respectively, and $s = \frac{1}{2} (a + b + c)$, and S = the area; prove the following formulas

(a) $S = \frac{1}{2} b c \sin A = \sqrt{s (s - a) (s - b) (s - c)}$

$= \dfrac{a b c}{s} \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$

$= \frac{1}{4} \sqrt{(a^2 + b^2 + c^2)^2 - 2 (a^4 + b^4 + c^4)}$.

(b) $\cos A = \dfrac{b^2 + c^2 - a^2}{2 b c}$.

(c) $c = (a + b) \dfrac{\sin \frac{1}{2} C}{\cos \phi}$, when $\tan \phi = \dfrac{a - b}{a + b} \cot \frac{1}{2} C$.

(d) $\tan \left(\dfrac{A}{2} + B\right) = \dfrac{c + b}{c - b} \tan \dfrac{A}{2}$.

12. Shew how to find the radii of the inscribed and escribed circles of a triangle in terms of its sides and angles.

If R be the radius of the circumscribing circle, prove that the product of the perpendiculars from the angles upon the opposite sides

$$= \frac{a^2b^2c^2}{8\,R^3}.$$

No.	Log.	Angle.			Log.
23154	36463	$4° 19\frac{1}{2}'$	.. tan ..		8·87859
67488	82923	$12° 25\frac{1}{2}'$	.. tan ..		9·34319
1536	18639	$7° 32'$	.. tan ..		9·12141
92178	96461	$73° 15'$	.. tan ..		10·52143
20000	30103				
96000	98227				
42180	62511				

JUNIOR MATRICULATION, 1878.

1. If θ be the circular measure of an angle between $0°$ and $90°$, shew that $\sin\theta > \theta - \frac{1}{4}\theta^3$.

Shew approximately what the dip of the horizon is for every mile of distance.

2. Shew that $\sin 18° = \frac{1}{4}(\sqrt{5} - 1)$, and hence shew how to find the sines and cosines of all angles being multiples of $9°$ from $0°$ to $90°$.

3. If A and B are any angles, prove

(1) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.

(2) $\sec A + \tan A = \tan(45° + \frac{1}{2}A)$.

(3) $\tan\dfrac{A}{2} = \dfrac{\text{vers } A}{\sin A}$.

If $x = \cos A \cot A$, $y = \sin A \tan A$, eliminate A.

4. In every triangle prove the truth of the following formulas :

(1) Area $= \frac{1}{2} a b \sin C = \sqrt{s (s - a) (s - b) (s - c)}$.

(2) $c = a \cos B + b \cos A$.

(3) $\sin \frac{A}{2} = \sqrt{\frac{(s - b) (s - c)}{b c}}$.

(4) $\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$.

5. Define the terms *logarithm, characteristic* and *mantissa*.

How are the logarithms of numbers less than unity to be found from the tables, and how are they represented ?

Given log 4·9353 = ·6933116, find the logs of ·49353 and of $(\cdot 049353)^{\frac{1}{3}}$.

6. Having ascertained the logarithm of four digits of a number from the tables, shew how to proceed to find the logarithm of the whole number.

Given log 2 = ·30103, log 3 = ·4771213, log 7 = ·845098, find the logarithms of $28^{\frac{1}{2}}$, $63^{\frac{1}{3}}$, $98^{\frac{1}{4}}$ and $126^{\frac{1}{5}}$.

7. If a, b, c are the lengths of three straight lines drawn from a point making equal angles with one another, and straight lines be drawn respectively joining the extremities of a, b, c, the area of the whole triangle thus formed will be $\frac{\sqrt{3}}{4} (bc + ca + ab)$.

8. Shew how to solve a triangle when two sides and the included angle are given.

Ex. $a = 765 \cdot 432$, $b = 1006 \cdot 62$, $C = 70^{\circ}$.

9. Find the radius of the circumscribed circle of a triangle in terms of its sides and angles.

If the centres of the escribed circles of a triangle be joined forming another triangle, shew that the circle circumscribing this latter triangle is four times the size of the circle circumscribing the first triangle.

10. A person at the top of a light-house descries a vessel just on the horizon; shew that he can ascertain the distance of the vessel approximately by taking the square root of one and a-half times the height of the light-house in feet, and calling the result miles.

No.	Log.	Angle.	Log.
1772052	2484765	sin 70°	9·9729858
241188	3823555	sin 66°	9·9607302
100662	0028656	tan 35°	9·8452268
103543	0151212	tan 11°	9·2886522
		tan 22°	9·6064096

FIRST YEAR, 1879.

1. Define the logarithm of a number. Shew how the logarithm of a number to base e may be converted to the corresponding logarithm to base 10.

2. Prove the rule for finding the *characteristics* of logarithms. Why are the *mantissas* only inserted in the tables?

3. Prove $\log \frac{ab}{c} = \log a + \log b - \log c$.

$$\log \sqrt[n]{a} = \frac{1}{n} \log a.$$

4. Having given mantissa log 173300 = 238799
" " 173400 = 239049

construct a table of proportional parts for intermediate numbers. Find log 173·344; and write down the number whose log is $\bar{2}$·238854.

5. Given log 2 = 0·301030, log 3 = 0·477121,

find the value of $\frac{3^{-5} \times \sqrt[3]{4}}{21 \cdot 6}$.

Find the tabular logarithms of cos 30°, sec 45° and tan 120°.

6. Prove the formulas

(1) $\tan A = \frac{\sin A}{\cos A}$. (2) $\cos A = \sqrt{1 - \sin^2 A}$.

(3) $\sin (A + B) = \sin A \cos B + \cos A \sin B$.

(4) $\sin n A = 2 \cos A \sin (n - 1) A - \sin (n - 2) A$.

7. In any triangle prove the following:

(1) $\sin \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{b c}}$.

(2) $\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}$.

(3) $\sin \frac{A - B}{2} \sin \frac{C}{2} + \sin \frac{B - C}{2} \sin \frac{A}{2} +$

$$\sin \frac{C - A}{2} \sin \frac{B}{2} = 0.$$

(4) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

8. In a triangle

(1) $A = 53° 7' 48''$, $B = 75°$, $a = 1056$, find b and C.

(2) $a = 48$, $b = 30$, and perpendicular from C upon opposite side $= 24$. Find A, B, C, and c.

9. Obtain expressions for the area of a triangle.

Ex. $a = 70$, $b = 80$, $c = 90$; find area.

If l, m, n be the bisectors of the opposite sides of a triangle, and $2 \sigma = l + m + n$, shew that

$$\text{Area} = \tfrac{4}{3} \sqrt{\sigma (\sigma - l)(\sigma - m)(\sigma - n)}.$$

10. The altitude of a mountain from A is 30°, from B 36°, and from C 45°. A, B, and C are in the same straight line, and $AB = BC = 1000$ yards. Find the height of the mountain.

No.	Log.	Angle.	Log.
30243	480629	sin 75°	9·984944
18944	277478	cot 36°	10·138739
30773	488168	sin 53° 7′ 48″	9·903090
10560	023664		
12750	105518		

JUNIOR MATRICULATION, 1879.

1. Explain the terms *characteristic* and *mantissa*, and state the rule for writing down the characteristic of the logarithm of any number.

Write down the characteristic of ·5, ·0007 and 60050·3.

What would be the characteristics of these numbers to base 100, and also to base $\frac{1}{10}$?

2. Find the logarithms of $\sqrt[5]{\cdot 007}$ and $(\cdot 5)^{-3}$.

Find the index of the power to which 7 must be raised to produce 300.

3. Having given

$$L \cot 57^\circ\ 30' = 9{\cdot}804187$$

$$\text{Difference} = 279,$$

find L cot 57° 30′ 15″, and find the angle, the Log of whose tangent is 9·804251.

4. Find the values of sin 30°, cos 30°, and sec 45°.

Write down the tabular logarithms of these ratios.

5. Prove the formulas,

(1) $\sin A = \sin(180^\circ - A) = \cos(90^\circ - A)$.

(2) $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

(3) $\sin 2A = 2 \sin A \cos A$.

The angle BAC is bisected by AD. BC and BD are perpendicular to AC and AD. Prove that

$$BA \cdot BC = 2\, BD \cdot AD,$$

$$\text{and } BA \cdot AC = AD^2 - BD^2.$$

6. Shew that

(1) $\sin 18^\circ \sin 54^\circ = \frac{1}{4}$.

(2) $16 \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = 1$.

7. In any triangle, prove the formulas

(1) $c = a \cos B + b \cos A$.

(2) $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

In the triangle ABC, BD is drawn at right angles to AB meeting AC in D. Find BD in terms of the sides of the triangle.

8. Solve the equations

(1) $\sin^2 \theta + \sin^2 2\theta = 1$.

(2) $\begin{cases} \sin^2 (\theta + \phi) - \sin^2 (\theta - \phi) = \frac{\sqrt{3}}{2}. \\ \operatorname{cosec} 2\theta + \cos 2\phi = \frac{2}{\sqrt{3}}. \end{cases}$

9. Solve the triangles

(1) $A = 21° \, 10'$, $C = 90°$, $a = 314{\cdot}16$.

(2) $A = 74° \, 53'$, $B = 37° \, 55'$, $c = 300$.

10. Find the area of the circle inscribed in the triangle whose sides are 50, 68 and 78.

($\pi = 3 \,.\, 1416$).

No.	Mantissa.	Angle.	Logarithm.
20000	30103	tan 21° 10′	9·65205
30000	47712	sin 74° 53′	9·98470
70000	84510	sin 37° 55′	9·78858
31416	49715	sin 67° 12′	9·96467
92323	96531		

ANSWERS.

I. (Page 12).

1. 3, 1, 4, −1, −6. 2. $= \log \dfrac{2^5 \times 3^2}{7 \times 10} =$. 4. $= \log 100 \times 2 =$.
6. $\log 8 = \frac{3}{2}(a - b + 1)$, $\log 9 = a + b - 1$. 7. $\frac{3}{2} \log 25 - 3$.
8. $\bar{4}\cdot8624005$. 9. $\bar{1}\cdot4042714$. 10. 1·8379346. 11. $\bar{2}\cdot4566323$.
12. $\bar{2}\cdot8653155$. 13. $\bar{2}\cdot8400799$. 14. 4·7497460. 15. 6·4721485.
16. 1·2552725, 1·3979400, 2·3344539, ·0511526, ·8115752, $\bar{3}\cdot7323939$, $\bar{1}\cdot6478174$, 1·6354839, 2·8573326, ·0880456, ·0263938, 1·6365006. 17. $\bar{4}\cdot20412$, $\bar{1}\cdot4671244$. 18. $\bar{1}\cdot9599947$.
19. ·002620145. 20. 562·4855. 21. 30413·41. 22. ·004955347.
23. 2·7485257, ·005604388. 24. 1·8507. 25. 1·0065.
26. ·739148. 27. 437499·9. 28. ·124385. 29. 27·9865.
30. ·000798595.

II. (Page 14).

1. 8, $\frac{5}{2}$, $\frac{5}{3}$, 3, $\frac{3}{2}$, 2. 2. 2. 3. $\bar{4}$. 4. ·69897. 5. 1·537.
6. 2.089. 7. (1) $\dfrac{\log c}{m \log a + n \log b}$. (2) $\dfrac{3 \log 2 - 1}{3 \log 2}$.
(3) $\dfrac{4 \log 3 + 3 \log 2}{\log 3}$. (4) $\dfrac{\log c - d \log b}{\log a - \log b}$, $\dfrac{\log c - d \log a}{\log b - \log a}$.
8. $\frac{9}{2}$, $\frac{3}{2}$. 10. $1\cdot5 = \frac{1}{2}(2 + 1) = \log (100 \times 10)^{\frac{1}{2}} = \log 31\cdot62\ldots$
11. $\cdot0103 = \log \dfrac{256}{250} = \log \dfrac{2^{10}}{1000}$, &c. 12. 13. 13. 27. 14. ·1.
15. $10^{\frac{2}{3}}$. 16. 5. 17. 100 or $\sqrt{\cdot001}$. 18. ·67362. 19. If y be reqd., $\log x^y = 3$, or $y = \dfrac{\log 3}{\log x} = 1\cdot09862$. 20. True if
$\log_2 (20)^{\frac{1}{2}} = \log_2 2 + \frac{1}{2} \log_2 5 = \log_2 (4 \times 5)^{\frac{1}{2}}$.

III. (Page 19).

Examples such as this exercise contains may easily be worked by expressing the trigonometrical ratios in terms of a single one; the identities being thus reduced to ordinary algebraic identities, may be verified as such. It will frequently be found convenient to express the ratios in terms of sine and cosine, simplify, and substitute the relation between sine and cosine. Thus Ex. 17,

$$\cot^2 \phi + \tan^2 \phi = \sec^2 \phi \operatorname{cosec}^2 \phi - 2,$$

$$\text{if } \frac{\cos^2 \phi}{\sin^2 \phi} + \frac{\sin^2 \phi}{\cos^2 \phi} = \frac{1}{\cos^2 \phi \sin^2 \phi} - 2,$$

$$\text{if } \cos^4 \phi + \sin^4 \phi = 1 - 2 \cos^2 \phi \sin^2 \phi,$$

$$\text{if } (\cos^2 \phi + \sin^2 \phi)^2 = 1.$$

IV. (PAGE 20).

1. $\frac{4}{5}$. 2. $\frac{12}{13}$. 3. $\frac{4}{\sqrt{17}}$. 4. $\frac{a}{\sqrt{b^2 - a^2}}$. 5. $\frac{4}{3}$.

6. $\frac{b}{\sqrt{a^2 - b^2}}$. 7. $\sqrt{2}$. 8. Becomes $\sqrt{1 - \cos^2 \theta} = \frac{7}{5} - \cos \theta$, an ordinary quadratic; and $\cos \theta = \frac{4}{5}$ or $\frac{3}{5}$. 9. $\frac{1}{2}$ or $\frac{1}{3}$. 10. $\frac{1}{2}$.

V. (PAGE 23).

1. Log 2 = ·3010300, log 3 = ·4771213. For 30°. 9·6989700, 9·9375306, 9·7614394, 10·2385606, 10·0624694, 10·3010300; for 45°, 9·8494850, 9·8494850, 10, 10, 10·1505150, 10·1505150.

2. $L \sin 22° \, 27' = 9·5819236$
$L \sin 22° \, 26' = 9·5816177$

Diff. for $60'' = 3059$, $\therefore$ diff. for $45'' = \frac{45}{60}$ of $3059 = 2294$;
$\therefore L \sin 22° \, 26' \, 45'' = 9·5816177 + ·0002294 = 9·5818471$.

Or thus,—Diff. for $15'' = 765$; $\therefore L \sin 22° \, 26' \, 45'' = 9·5819236 - ·0000765 = 9·5818471$.

It will be noted that the sine, tangent and secant increase as the angle (if less than 90°) increases; hence the difference must be added or subtracted according as we have found the difference between the required Log. and the Log. of the less angle, or between the required Log. and the Log. of the greater angle. In the case of cosine, cotangent and cosecant, the ratio decreases as the angle increases; hence the difference must be subtracted or added according as we have found the difference between the required Log. and the Log. of the less angle, or between the required Log. and the Log. of the greater angle.

3. 9·7932666. 4. 9·9169962. 5. 9·7336870. 6. 9·8040340. 7. 10·5154297. 8. 10·5443618. 9. 10·0281814. 10. 69° 7′ 45″ or 110° 52′ 15″ (See § 31). 11. 16° 19′ 26″ or 163° 40′ 34″. 12. 22° 28′ 33″. 13. 51° 17′ 53″. 14. 30° 21′ 30·3″. 15. 42° 12′ 39″. 16. 47° 30′ 16″. 17. 15° 21′ 31″ or 164° 38′ 29″. 18. 34° 31′ 23·8″ or 145° 28′ 36·2″. 19. cos 61° 49′ 25″ = sin 28° 10′ 35″, 9·6741145. 20. 9·4954522. 21. 10·1163715. 22. 10·1336341. 23. 9·8681838. 24. 69° 44′ 27″. 25. 45° 40′ 42″ or 134° 19′ 18″. 26. 32° 43′ 51″. 27. $\sin x = \frac{1}{\operatorname{cosec} x}$; $\therefore L \sin x = - L \operatorname{cosec} x$, $L \sin y = - L \operatorname{cosec} y$; $\therefore L \sin x - L \sin y = -(L \operatorname{cosec} x - L \operatorname{cosec} y)$. 28. When one angle is the complement of the other. 29. $\sin A = \frac{1}{\operatorname{cosec} A}$; $\therefore L \sin A - 10 = -(L \operatorname{cosec} A - 10)$; $\therefore L \sin A + L \operatorname{cosec} A = 20$. 30. By 10^{10}, for $\log 10^{10} \sin A = 10 + \log \sin A = L \sin A$.

VI. (Page 30).

1. $c = 70 \cdot 7$, $A = B = 45°$. 2. $a = b = 169 \cdot 68$, $B = 45°$. 3. $b = 100$, $c = 141 \cdot 4$, $B = 45°$. 4. $a = 12$, $b = 20 \cdot 784$, $B = 60°$. 5. $b = 17 \cdot 32$, $c = 34 \cdot 64$, $B = 30°$. 6. $b = 831 \cdot 36$, $c = 960$, $A = 30°$. 7. $b = 83 \cdot 136$, $A = 30°$, $B = 60°$. 8. $44° 29' 53''$. 9. $46° 23' 37''$. 10. $60° 45' 30''$. 11. $A = 36° 52' 12''$, $B = 53° 7' 48''$. 12. $A = 25° 22' 37''$, $B = 64° 37' 23''$. 13. $A = 32° 16' 32°$, $B = 57° 43' 28''$. 14. $A = 36° 31' 44''$, $B = 53° 28' 16''$. 15. 458·257. 16. $a = 2954 \cdot 42$, $b = 520 \cdot 945$. 17. $a = 3758 \cdot 77$, $b = 1368 \cdot 08$. 18. $b = 174 \cdot 706$, $c = 510 \cdot 805$. 19. $a = 13435 \cdot 4$, $c = 13909 \cdot 3$. 20. $b = 59 \cdot 8767$, $c = 138 \cdot 24$. 21. $a = 117 \cdot 232$, $b = 269 \cdot 616$. 22. $A = 35° 49' 44''$, $b = 265 \cdot 932$. 23. $b = 2 \cdot 99383$, $c = 6 \cdot 91201$. 24. $A = 36° 9' 3''$, $c = 119 \cdot 509$. 25. $a = \sqrt{(1282)^2 - (1218)^2} = \sqrt{64 \times 2500} = 400$, $B = 71° 49' 10''$. 26. 60·2593. 27. 442·546. 28. 40997·9. 29. 148·327. 30. $b = 612$, $A = 34° 12' 20''$, $B = 55° 47' 40''$. 31. $b = 163 \cdot 5614$, $c = 205 \cdot 2519$. 32. $b = 12$, $A = 22° 37' 11''$, $B = 67° 22' 49''$. 33. $b = 141$, $A = 82° 41' 44''$, $B = 7° 18' 16''$. 34. $28° 4' 20''$. Divide triangle into two right-angled triangles. 35. $19° 11' 18''$. 36. $96° 22' 42''$. 37. Triangle is isosceles, and may be divided into right-angled triangles. $31° 5' 32''$, $74° 27' 14''$, $74° 27' 14''$. 38. $72° 32' 33''$. 39. $61° 55' 39''$. 40. 8·988. 41 17·867 ; $OD = 24 (\cos 25°)^3$. 42. 152·62. 43. 905·98 ; $36 \sin 64° =$ altitude. 44. 6·882. 45. 939·7 in. 46. 82·904 in. ; $8 \sin. 56° =$ altitude. 47. By Geometry we may shew that its area is equal to that of a triangle whose sides are the diagonals and included angle same as theirs, 92·72 in. 48. 669·13 in. 49. 44° or 136°. 50. $26° 19'$.

VII. (Page 34).

1. 80 ft. 2. 87 ft. 6 in. 3. 40·98 ft. 4. 72 ft. 5. 81 ft. 6. 258 + yds. 7. 49·2 mls. 8. 2·732 mls. 9. $4\sqrt{5}$, $6\sqrt{5}$. 10. 153·3 ft. 11. 80·58 ft. 12. 193·50 ft. 13. 152·674 ft. 14. 81·611 ft. 15. $49° 10'$. 16. 1097·77 ft. 17. 421·99 yds. 18. 89·9069 ft. 19. 124·4 ft. 20. 278·18 ft. 21. 93·97 ft. 22. 3 min. 20 seconds, $36° 52' 12''$. 23. $7'$ nearly. 24. 233·2 ft. 25. 715·93 ft.

X. (Page 44).

4. If A lie between 0° and 135° $\cos A + \sin A = + \sqrt{1 + \sin 2A}$; between 0° and 45°, $\cos A - \sin A = + \sqrt{1 - \sin 2A}$; between 45° and 135°, $\cos A - \sin A = - \sqrt{1 - \sin 2A}$; between 135° and 180°, $\cos A \pm \sin A = - \sqrt{1 \pm \sin 2A}$. 5. $\sqrt{2} \sin (45° - A)$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos A - \frac{1}{\sqrt{2}}\sin A\right) = \ldots$ 7. $=(\cos^2 a + \sin^2 a)(\cos^2 a - \sin^2 a) = \ldots$ 13. $= \sin(\frac{3}{2}A - \frac{1}{2}A) + \sin(\frac{3}{2}A + \frac{1}{2}A) = \ldots$ 15. $= \frac{\cos(2A - A) - \cos(2A + A)}{\sin(2A + A) - \sin(2A - A)} = \ldots$

24. $= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = \ldots$

26. $= \frac{1 + \cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \ldots$

28. $= \frac{\cos^2 A - \sin^2 A}{(\cos A + \sin A)^2} = \ldots$ 31. 9·7167008. 32. 10·3648522.

XI. (Page 46).

47. 0° or 30°. 48. 20° or 90°. 49. 0° or 30°. 50. Equation equivalent to $\sin 8x + \sin 2x = \sin 16x + \sin 2x$; thence $x = 0$ or $7\frac{1}{2}$°.

XII. (Page 51).

3. $\frac{a}{c} - \frac{b}{c} = \sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B) = \ldots$

5. $\left(\frac{a}{c}+1\right)\sin\frac{B}{2} = (\cos B + 1)\sin\frac{B}{2} = 2\cos^2\frac{B}{2}\sin\frac{B}{2} = \sin B\cos\frac{B}{2} = \frac{b}{c}\cos\left(45° - \frac{A}{2}\right)$. 6. $\frac{2a}{b+c-a} = \frac{2\sin A}{\sin B + 1 - \sin A} = \frac{2\sin A}{\cos A + 1 - \sin A} = \frac{4\sin\frac{1}{2}A\cos\frac{1}{2}A}{2\cos^2\frac{1}{2}A - 2\sin\frac{1}{2}A\cos\frac{1}{2}A} = \frac{2\cot\frac{1}{2}A}{\cot\frac{1}{2}A - 1} =$ &c. 11. $\sin^2\frac{1}{2}A = \frac{1}{2}(1 - \cos A)$. 13. $2(a+b)\sin^2\frac{1}{2}C = a+b-(a+b)\cos C = b\cos C + c\cos B + a\cos C + c\cos A - (a+b)\cos C =$ &c. 20. True if $\frac{\sin B}{\cos B} = \frac{\sin B\sin C}{\sin A - \sin B\cos C}$. 21. True if $c \cdot \frac{c}{\sin C} = b\cos A \cdot \frac{a}{\sin A} + a\cos B \cdot \frac{b}{\sin B}$. 26. $\frac{b-c}{a} = \frac{\sin B - \sin C}{\sin A} = \frac{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B-C)}{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B+C)} =$ &c. 31. Reduces to $a^2 + b^2 = c^2$. 32. For then perp. bisects base. 35. Condtn. reduces to $(c^2 - a^2 + ab)(a - b) = 0$.

XIII. (Page 63).

1. 30° or 150°. 2. 45° or 135°. 3. $A = 90°, B = 30°$. 4. 45°. 5. 30°. 6. No solution. 7. 7·89142. 8. 47·1915. 9. 379·5003. 10. 115° 22′ 36″. 11. 50° 28′ 44″. 12. 55° 46′ 16″.

13. $A = 117° 19' 11''$, $B = 2° 40' 49''$. 14. $A = 119° 26' 51·3''$, $B = 5° 33' 8·7''$. 15. $A = 69° 10' 10''$, $B = 46° 37' 50''$. 16. 162·3358. 17. 32° 57′ 8″. 18. $A = 71° 33' 15''$, $B = 60° 2' 45''$, $c = 67·502$; for latter part use $\dfrac{c}{a+b} = \dfrac{\sin \frac{1}{2} C}{\cos \frac{1}{2}(A-B)}$. 19. 78° 19′ 24″ or 101″ 40′ 36″. 20. 10.6816. 21. $A = 56° 15' 4''$, $B = 59° 51' 10''$, $C = 63° 53' 46''$. 22. $A = 6° 1' 53·9''$, $B = 108° 58' 6·1''$. 23. 62° 51′ 33″ or 117° 8′ 27″. 24. $b = 226·63$, $c = 208·17$. 25. $A = 84° 54'$, $B = 69° 20'$, $C = 25° 46'$. 26. 32·412 ; in latter part use formula in ans. to 18. 27. 62° 22′ 50, or 117° 37′ 10″. 28. 220·999. 29. 58° 10′ 44″. 30. $A = 47° 50'$, $B = 57° 10'$. 31. 7° 32′ 31″. 32. $a = 88·4552$, $b = 108·506$. 33. 145° 37′ 30″. 34. $A = 168° 27' 25·4''$, $B = 4° 55' 10·6''$. 35. $B = 17° 6' 45''$, $C = 133° 2' 15''$. 36. 82° 10′ 49″, 50° 24′ 11″. 37. $B = 23° 8' 33''$ $C = 32° 17' 27''$. 38. $A = 72° 30'$, $B = 48° 30'$, $c = 85·718$. 39. $B = 53° 52' 36''$, or 126° 7′ 24″ ; $C = 84° 57' 24$, or 12° 42′ 36″. 40. $A = 129° 22' 28''$, $B = 14° 37' 32''$.

XIV. (Page 68).

1. 120°. 2. 159·0643. 3. 132° 34′ 34″. 4. 69° 10′ 10″, 46° 37′ 50″. 5. 116° 33′ 54″, 26° 33′ 54″. 6. $B = 53° 27' 20''$ or 126° 32′ 40″, $C = 81° 32' 40''$ or 8′ 27′ 20″. 7. 3652·28. 8. 82° 49′ 10″. 9. 76° 47′ 2″, 49° 12′ 58″. 10. 66° 14′ 38″, 50° 45′ 22″. 11. 80° 30° or 99° 30′. 12. 86·3218. 13. 41° 24′ 34″. 14. 168° 27′ 25″, 4° 55′ 11″. 15. 71° 44′ 30″, 48° 15′ 30″. 16. $A = 34° 27'$, $B = 100° 33'$, $b = 173·7883$. 17. 431·609. 18. 117° 38′ 45″, 27° 38′ 45″. 19. 70° 53′ 36″, 60°, 49° 6′ 24″. 20. 66° 22′ 45″, 53° 37′ 15″.

XV. (Page 69).

1. 110·5, 111. 2. 53° 25′. 3. 429·6. 4. 75° 23′ or 104° 37′ 5. 78° 7′ or 101° 53′. 6. 62° 20′, 17° 10′. 7. 62° 51′ or 117° 9′. 8. 60° 45′, 39° 15′. 9. 67° 19′ or 112° 41′. 10. 97, 76·6. 11. 85° 50′, 51° 36′, 42° 34′. 12. 60°, 45°. 13. 22·66, 24·9. 14. 104° 30′, 46° 34′, 28° 56′. 15. 829·6. 16. 71° 22′, 70° 30′, 38° 8′. 17. 56° 30′, 51° 16. 18. 37° 20′, 93° 10′, 49° 30′. 19. 69° 20′. 20. 642·9, 866.

XVII. (Page 75).

1. 6·496. 2. 14·69. 3. 1897. 4. 95523·5, or 95520 if four-figure tables be used. 6. For $\tan \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

12. $\sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)}$, which use. 13. For p, q, r, substitute $\frac{a}{2\text{ area}}$, $\frac{b}{2\text{ area}}$, $\frac{c}{2\text{ area}}$. 14. Readily obtained by using 22 of Ex. XII. 15. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$, – obtained in same way as 43 of Ex. XI. 16. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$, – 43 of Ex. XI. 17. For $\cot A$, &c., substitute $\frac{b\,c \cos A}{2 \text{ area}}$, &c., and employ 12 of Ex. XII. 18. Let $A\,B\,C$ be the triangle, and O the centre of the circle; then $B\,O\,C$ at centre $= 2A$; $\therefore \sin A = \frac{\frac{1}{2}a}{\text{radius}}$. 19. If O be centre of circle, area $A\,B\,C = O\,A\,B + O\,B\,C + O\,C\,A = \frac{1}{2} r\,c + \frac{1}{2} r\,a + \frac{1}{2} r\,b$
20. If O be centre of escribed circle touching $B\,C$, area $A\,B\,C = O\,A\,B + O\,A\,C - O\,B\,C = \frac{1}{2} r\,c + \frac{1}{2} r\,b - \frac{1}{2} r\,a = r\,(s-a)$.

EXAMINATION PAPERS.

Senior Matriculation: 1874.

1. § § 1, 5. 2. 10·001541; 10·255513; $\tan^{-1}\frac{362}{201}$ means the angle whose tangent is $\frac{362}{201}$, so that $\tan \tan^{-1}\frac{362}{201} = \frac{362}{201}$; ·096679. 3. ·80279; 39006·; sin 19° 12′ 33″; cos 4° 49′ 22″. 4. Sin 19° 12′ 39″; 8·0274. 5. § 17. $\sin A = \sqrt{1-\cos^2 A}$; $\tan A = \frac{\sqrt{1-\cos^2 A}}{\cos A}$; &c. 6. § 36 and Ex. VIII. $\sin 105° = \sin(45° + 60°) = \frac{1+\sqrt{3}}{2\sqrt{2}}$; 3; $\frac{1+\sqrt{3}}{2\sqrt{2}}$. 7. § 38. 8. (1) § 34. (2). § 40. (3). § 41. (4). § 48. 9. (1). § 43. (2). § 45. (3). § 44. 10. (1). 38° 25′ 18″. (2). 100·456, ·802. 11. $\frac{5}{\sqrt{3}}$, $\frac{10}{3\sqrt{3}}$. 12. A circle may be described about $A\,B\,D\,C$. Knowing the area and $A\,C$, $B\,D$, we may find the perp. distance bet. $A\,C$, $B\,D$. We may then proceed in dif. ways, *e.g.*, $AB^2 = (\text{perp.})^2 + \left\{\frac{1}{2}(B\,D \sim A\,C)\right\}^2$; then find $B\,C$, and thence angle $C\,A\,B$.

Senior Matriculation: 1875.

1. 2, 0, $\bar{1}$, $\bar{4}$. 2. 100,000 to 9,999,999·9 . . . ; ·01 to ·0999 . . . 3. § 5. 5·87406. 4. $\bar{4}$·39794. ·10473, 1·1691. 5. 23° 18′ 56″, 79° 54′ 35″. 6. § § 17, 31. § 20. 7. § 36. § 45.
8. $\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{1}{2}x}{2\cos^2\frac{1}{2}x} =$. $\frac{1+\sin x - \cos x}{1+\sin x + \cos x} =$

$\frac{2\sin^2\frac{1}{2}x + 2\sin\frac{1}{2}x\cos\frac{1}{2}x}{2\cos^2\frac{1}{2}x + 2\sin\frac{1}{2}x\cos\frac{1}{2}x} =$ &c. 9. (1). $a = 388{\cdot}26$, $b = 548{\cdot}90$, $B = 54^\circ\ 43'\ 35''$. (2). $b = 462{\cdot}75$, $c = 498{\cdot}99$, $C = 69^\circ\ 13'\ 43''$. 10. (1). $r = \frac{\text{area}}{s}$, and $\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C = \frac{1}{s^2}\sqrt{s\,\overline{s-a}\,\overline{s-b}\,\overline{s-c}} = \frac{\text{area}}{s^2}$. (2). $R = \frac{a\,b\,c}{4 \times \text{area}}$, and $\sec\frac{1}{2}A\sec\frac{1}{2}B\sec\frac{1}{2}C = \frac{a\,b\,c}{s\sqrt{s(s-a)(s-b)(s-c)}} = \frac{a\,b\,c}{s \times \text{area}}$.

First Year: 1876.

1. § 4. 8. 2. § 5. 4·1645, – ·774. 9·84949, 10·23856, 9·93753. 3. Thus $L\sin 120^\circ = L\sin 60^\circ = L\cos 30^\circ$. a must of course be negative; we may however neglect the sign, find numerical value of a, and prefix –. $= 2\sin\frac{1}{2}A\cos\frac{1}{2}A - \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \tan\frac{1}{2}A\,(2\cos^2\frac{1}{2}A - 1) = \tan\frac{1}{2}A\cos A$. 4. (4). $\sqrt{\frac{1+\sin A}{1-\sin A}} = \sqrt{\frac{(1+\sin A)^2}{1-\sin^2 A}} = \frac{1+\sin A}{\cos A} =$ &c. 5. $\tan 30^\circ = \frac{x}{x+150}$; $\therefore x = \frac{150}{\sqrt{3}-1}$. 6. $52^\circ\ 15'\ 37''$. Produce stick (if necessary) to meet horizontal plane through end of shadow; and data are evidently sufficient to determine parts of triangles so found. 7. (3). $\cos 2A\tan(45^\circ - A) = (\cos^2 A - \sin^2 A)\frac{1-\tan A}{1+\tan A} =$ &c. (4). $\frac{\cos(2\theta - \theta) + \cos(2\theta + \theta)}{\sin(\quad) + \sin(\quad)} = \tan 2\theta =$ &c. 8. (4). True if $b\sin A = \frac{c}{\cot A + \cot B} = \frac{c\sin A\sin B}{\sin(A+B)}$, if $b = \frac{c\sin B}{\sin C}$. 9. (1). $A = 26^\circ\ 22'\ 30''$, $C = 29^\circ\ 57'\ 30''$. (2) Triangle isosceles $a = 355{\cdot}22$, $B = C = 33^\circ\ 40'$, (3). 516·23. 10. (1). $1 = 3\sin\theta$; $\theta = 19^\circ\ 28'\ 17''$. (2). Eq. reduces to $1 + \sin\theta = \cos\theta + 1 - \sin\theta$, or $\tan\theta = \frac{1}{2}$; $\therefore\ \theta = 26^\circ\ 33''\ 54'$. 11. True if $2 - 2\sin A\cos B - 2\cos A\sin B > \cos^2 A + \sin^2 B - 2\cos A\sin B$, if $2 - 2\sin A\cos B > 1 - \sin^2 A + 1 - \cos^2 B$, if $(\sin A - \cos B)^2 > 0$.

First Year: 1877.

1. §§ 17, 21. 2. $\log 2\frac{1}{2} = \log\frac{10}{2^2} = {\cdot}39794$; ·35218; $\bar{1}{\cdot}23265$, last no. $= \left(\frac{2 \times 3^4}{10^4}\right)^{\frac{3}{7}}$. 3. 46961·; only logs of 2 and 3 are used until last operation. 4. (1) In left-hand side of identity, for A put its equivalent $\frac{1}{2}\overline{A+B} + \frac{1}{2}\overline{A-B}$, and for

B, $\frac{1}{2}\overline{A+B} - \frac{1}{2}\overline{A-B}$. (2). $\cot A + \operatorname{cosec} A = \frac{\cos A + 1}{\sin A} = \frac{2\cos^2 \frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A} =$. (3). $\sin A = \sin(3A - 2A)$; $\sin 5A = \sin(3A + 2A)$, &c.; $\therefore$ left-hand side of identity $= \frac{\sin 3A + 2\sin 3A\cos 2A}{\cos 3A + 2\cos 3A \cos 2A} = \ldots$ (4). Right-hand side $= \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2}} = \cos A$. 5. § 38. $\cos 3A = 4\cos^3 A - 3\cos A = 2\sin A\cos A$; $\therefore \cos A = 0$, or $A = \frac{\pi}{2}$ or any odd multiple of $\frac{\pi}{2}$; also $4\cos^2 A - 3 = 2\sin A$; $\therefore \sin A = \frac{\pm\sqrt{5} - 1}{4}$, or $\sin\left\{n\pi + (-1)^n A\right\} = \frac{\pm\sqrt{5}-1}{4}$, giving other values of A, n being any integer. $\cos A = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}}$.

6. § 40. (2). $\therefore c^2 < a^2 + b^2 + 2ab$; $c < a + b$. 7. § 41, § 48. (3). $a^2\sin^2 B = b^2 \sin^2 A$; $\therefore a^2\cos^2 B = a^2 - b^2\sin^2 A$; $\therefore c = b\cos A + a\cos B = b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}$. 8. $\tan\frac{1}{2}$ (angle opp. gst. side) $= 1$; $\therefore$ triangle is right-angled, area $= 210$, angle reqd. $= 90°$. 9. $\therefore a^2 = b^2 + c^2 + bc$; but $a^2 = b^2 + c^2 - 2bc\cos A$; $\therefore \cos A = -\frac{1}{2}$, or $A = 120°$. 10. See Chambers' Tables,—"Explanation of the Tables," If angle be near 90°, the method is same as in case of small angle; thus $\tan a = \frac{1}{\tan(90° - a)}$. The rules may readily be proved by one acquainted with Euler's Series for sin and cos. Thus if a be the small angle expressed in circular measure, $\sin a = a - \frac{a^3}{\lfloor 3}$; $\frac{1}{\sec^{\frac{1}{3}} a} = 1 - \frac{a^2}{\lfloor 3}$; $\therefore \sin a = \frac{a}{\sec^{\frac{1}{3}} a}$, or $L\sin a = \log a'' + \log\frac{\pi}{648000} + 10 - \frac{1}{3}(L\sec a - 10)$, if a'' be the number of seconds in a; and $\therefore a = \frac{\pi}{180 \times 60 \times 60} \times a''$. 11. $ACD = 109° 28' 18''$, $A = 38° 56' 52'' = B$; $\therefore BDC = 70° 31' 26'' = BCD$; $\therefore BC = 90$, and $AB = 140$. (2). Let DN be the perp. from D on AB. Then from Euclid, B. II., 12, $CN = 20$, $\therefore DN = 40\sqrt{2}$; $\therefore BN = 70$, and $AB = 140$.

Junior Matriculation: 1877.

1. § 1. § 5. From definition $a^{\log_a N} = N$, $x^{\log_x a} = a$; $\therefore$ $x^{\log_a N \log_x a} = N$; similarly $x^{\log_b N \log_x a} = N$; $\therefore$ &c. 2. Mantissas same for same sequence of figures and characteristic determined by inspection. § 4. 3. § 7. By making mantissas always positive, same sequence of figures always has same mantissa. 4. $\bar{3}\cdot5179009$. 3295.357. ·0049353. 5. Ex. VIII. Ex. VIII. $= (\cos A + \cos 3A)(\cos A - \cos 3A) = 2 \cos 2A \cos A \times 2 \sin 2A \sin A = \sin 4A \sin 2A$. 6. § 35. § 41. § 45. $\frac{c}{a+b} = \frac{\sin C}{\sin A + \sin B} = \frac{\sin (A+B)}{\sin A + \sin B}$; substitute and reduce. 7. (1.) Triangle is right-angled, $- B = 53° 7' 48''$, $A = 36° 52' 12''$. (2). $A = 101° 32' 14''$, $B = 44° 24' 56''$, $C = 34° 2' 50''$. (3). $A = 152° 34' 24''$, $B = 14° 49' 36''$. 8. $\frac{c}{a+b} = \frac{\sin C}{\sin A + \sin B} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \frac{\sin \frac{1}{2} C}{\cos \frac{1}{2}(A-B)}$, from which if $a+b$, c, C be given, $A - B$ may be found, and thence A and B. 9 $a - b = b - c$; $\therefore \sin A - \sin B = \sin B - \sin C$; $\therefore \sin \frac{1}{2} C \sin \frac{1}{2}(A - B) = \sin \frac{1}{2} A \sin \frac{1}{2}(B - C)$; $\therefore \tan \frac{1}{2} C (\tan \frac{1}{2} A - \tan \frac{1}{2} B) = \tan \frac{1}{2} A (\tan \frac{1}{2} B - \tan \frac{1}{2} C)$; &c. 10. If x be this radius, $(r' + x)^2 = (r + r')^2 + (r + x)^2 - 2(r + r')(r + x) \cos \frac{1}{2}\theta$; $\therefore x = \frac{r(r+r')(1 - \cos \frac{1}{2}\theta)}{r'(1 + \cos \frac{1}{2}\theta) - r(1 - \cos \frac{1}{2}\theta)} =$ &c.

First Year: 1878.

1. tan and cot may have any value; sec and cosec cannot lie bet. 1 and -1. versin $A = 1 - \cos A = 2 \sin^2 \frac{1}{2} A$. 2. Radius is unity. 3. § 36. (2). $2 \sin \theta = \pm \sqrt{1 + \sin 2\theta} \mp \sqrt{1 - \sin 2\theta}$, $\sin \theta = \pm \sqrt{\frac{1}{2}(1 - \cos 2\theta)}$. (3). True if $2(1 - \cos \frac{1}{2} A) = 2 - 2(\cos A \cos \frac{1}{2} A + \sin A \sin \frac{1}{2} A)$, if &c. 4. From $\log_{10} N = \log_e N \div \log_e 10$, where e is the base of the Napierian system. Let $b = a^x$, $c = b^y, \ldots m = l^z$; $\therefore m = a^{xy \ldots z}$, $\log_a m = xy \ldots z =$ &c. 5. (1). § 48. (2). Area $= \frac{1}{2} c \sin B (c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B})$. If x be side and θ its inclination to the line, $\frac{a}{\sin \theta} = x = \frac{b}{\sin (120 - \theta)}$; $\therefore \frac{a\sqrt{3}}{2b - a} = \tan \theta = \frac{a}{\sqrt{x^2 - a^2}}$; whence x is found and thence area. 6. (2). $a^2 +$

$b^2 = c^2$. (3). $\sin C = 1$; $\therefore \sin^2 C = \sin C = \sin(A+B) = \sin A \cos B + \cos A \sin B = \sin^2 A + \sin^2 B$. 7. (1). $A = 8° 39'$, $B = 24° 51'$, $C = 146° 30'$. (2). $A = 9° 13'$, $B = 24° 17'$. 8. § 22. Equivalent to shewing that $\sec A \sec \frac{1}{2} A \operatorname{cosec} \frac{1}{2} A = 4 \operatorname{cosec} 2A$. 9. $x = \frac{6 + 2 \log 2}{1 - 7 \log 2}$. (2). ·35730. 10. (1). $= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B =$ &c. (2). § 39. (3). See Ex. XI., 43. (4). $\sin A + \sin 2A - \sin 3A = 2 \sin \frac{3}{2} A \cos \frac{1}{2} A - \sin 3A = 2 \sin \frac{3}{2} A (\cos \frac{1}{2} A - \cos \frac{3}{2} A) =$ &c. (5). $=$

$$\frac{\cos A(1+\cos B) - \sin B(1+\sin A)}{\cos B(1+\cos A) - \sin A(1+\sin B)}$$

$$= \frac{\dfrac{1+\cos B}{\cos B} - \tan B \, . \, \dfrac{1+\sin A}{\cos A}}{\dfrac{1+\cos A}{\cos A} - \tan A \, . \, \dfrac{1+\sin B}{\cos B}},$$

which is at once convertible into an expression involving only $\tan \frac{1}{2} A$, $\tan \frac{1}{2} B$. 11. (*a*). (3). For $\cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C = \frac{s}{a\,b\,c}\sqrt{s(s-a)(s-b)(s-c)}$. (4). In (2) for s put $\frac{1}{2}(a+b+c)$. (*b*). § 40. (*c*). By § 45, $\frac{1}{2}(A-B) = \varphi$; and $\frac{c}{a+b} = \frac{\sin C}{\sin A + \sin B} = \frac{2 \sin \frac{1}{2} C \cos \frac{1}{2} C}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} =$ &c. (*d*). By § 45, $\frac{c+b}{c-b} \tan \frac{A}{2} = \cot \frac{1}{2}(C-B)$, and $\tan\left(\frac{A}{2} + B\right) = \cot \frac{1}{2}(C-B)$, if $\cos(\frac{1}{2} A + B + \frac{1}{2} C - \frac{1}{2} B) = 0$. 12. Radius of inscribed circle $= \frac{S}{s}$; radii of escribed circles $= \frac{S}{s-a}$, &c.; radius of circumscribed circle $= \frac{a}{2 \sin A} = \frac{a\,b\,c}{4S}$. Product of perpendiculars $= \frac{8S^3}{a\,b\,c} = \frac{8}{a\,b\,c} \times \frac{a^3 b^3 c^3}{64 R^3} =$.

Junior Matriculation: 1878.

1. (1). See Todhunter's Trigonometry. (2). $\frac{1}{8000\pi}$ of 360°. The dip of the Horizon is the angle a horizontal line makes with the line drawn from the observer's eye tangent to the earth's surface. (2). Find sin 9°, cos 9°, and then such formulas may be used as $\sin(A + 9°) + \sin(A - 9°) = 2 \sin A \cos 9°$, giving $\sin(A + 9°)$, since ratios of other angles in formula would be already known. 3. (1). See Ex. VIII. (2). $\sec A + \tan A = \frac{1 + \sin A}{\cos A} =$

$\frac{\cos\frac{1}{2}A + \sin\frac{1}{2}A}{\cos\frac{1}{2}A - \sin\frac{1}{2}A} =$ &c. (3). $\frac{\text{vers } A}{\sin A} = \frac{1 - \cos A}{\sin A} = \frac{2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A} =$ &c. (4). $x\sin A + y\cos A = 1, \frac{y}{x} = \tan \tan^3 A$; $\therefore x\sqrt[3]{\frac{y}{x}} + y = \sqrt{1 + \left(\frac{y}{x}\right)^{\frac{2}{3}}}$. 4. (1). § 48. (2). § 35. (3). § 41. (4). Obtained from formula of § 40. 5. (1). §§ 1, 3, (2). § 3. (3). $\bar{1}$·6933116, $\bar{1}$·5644372. 6. (1). § 8. (2). ·7235790. ·5997802, ·4978065, ·4200741. 7. Area $= \frac{1}{2}ab\sin 120° + \cdots =$ &c. 8. § 45. $A = 44°$, $B = 66°$, $c = 1035·43$. 9. $R = \frac{a}{2\sin A}$. The radius of the circle may be shewn to be $\left(b\frac{\cos\frac{1}{2}C}{\cos\frac{1}{2}B} + c\frac{\cos\frac{1}{2}B}{\cos\frac{1}{2}C}\right) \div 2\cos\frac{1}{2}A$, and this may be shewn equal to $\frac{a}{\sin A}$. 10. Let A be top, B bottom of lighthouse; O centre of earth; AC a tangent to earth's surface. Then $AE^2 = AB\,(2\,BO + AB) = \frac{h}{5280}\left(8000 + \frac{h}{5280}\right) = \frac{8000}{5280}h$, nearly, $= \frac{3}{2}h$ nearly; or $AE = \sqrt{\frac{3}{2}h}$ miles.

First Year: 1879.

1. § 1. $\log_{10}N = \log_e N \div \log_e 10$. 2. § 4. 3. § 5. 4. Diff. for 1 is 25, for 2 is 50, &c. 2·239159; ·0173322. 5. ·00030243, 9·937531, 10·150515; tan 120° is negative, and ∴ has no log. 6. (1), (2). § 21. (3). § 36. (4). $\sin nA + \sin(n-2)A = \sin(\overline{n-1}\,A + A) + \sin(\overline{n-1}\,A - A) = 2\sin(n-1)A\cos A$. 7. (1), § 41. (2). § 45. (3). Expand and cancel. (4). Ex. XI., 41. 8. (1). $b = 1275$, $C = 51° 52' 12''$. (2) $A = 52° 7' 48''$, $B = 30$, $C = 96° 52' 12''$, $c = 18 + 24\sqrt{3}$. 9. § 48. Area $= 1200\sqrt{5}$. $b^2 + c^2 = 2l^2 + \frac{1}{2}a^2$ &c.; $\therefore a^2 + b^2 + c^2 = \frac{4}{3}(l^2 + m^2 + n^2)$. Also $2l^2 + \frac{3}{2}a^2 = a^2 + b^2 + c^2 = 2m^2 + \frac{3}{2}b^2 = 2n^2 + \frac{3}{2}c^2$. Thence we may prove $a^4 + b^4 + c^4 = \frac{16}{9}(l^4 + m^4 + n^4)$. Area $= \frac{1}{4}\sqrt{(a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4)} = \frac{4}{3}\cdot\frac{1}{4}\sqrt{(l^2 + m^2 + n^2)^2 - 2(l^4 + m^4 + n^4)} =$ &c. 10. If D be foot of mountain, and x represent height $x^2 + 3x^2 = 2BD^2 + 2(1000)^2$; $\therefore \cot^2 36° = 2 - \left(\frac{1000}{x}\right)^2$. Now numerical value of $\cot^2 36°$ may be found from given table, and is 1·8944. Thence $x = 3077·3$ yards.

Junior Matriculation: 1879.

1. (1). § § 3, 4. (2). $\bar{1}$, $\bar{4}$, 4. (3). $\bar{1}$, $\bar{2}$, 2; 0, 3, $\bar{5}$. 2. $\bar{1}$·56902, ·90309. 2·9311 +. 3. 9·804117, 32° 30′ 14″. 4. $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\sqrt{2}$. 9·69897, 9·93753, 10·15052. 5. (1). § § 31, 17. (2). See Ex. VIII. (3). § 38. (4). True if $\sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$. True if $BA \cdot AC = AB^2 - 2\,BD^2$, if $\cos A = 1 - 2 \sin^2 \frac{1}{2} A$. 6. (1). $\sin 72° = 2 \sin 36° \cos 36° = 4 \sin 18° \cos 18° \sin 54°$, and $\sin 72° = \cos 18°$, ∴ &c. (2). $\sin 20° = \sin 160° = 2 \sin 80° \cos 80° = \ldots = 8 \sin 20° \cos 20° \cos 40° \cos 80°$, and $\cos 60° = \frac{1}{2}$. 7. (1). § 35. (2). § 41. (3). $BD = c \tan A = 2c \frac{\tan \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} A}$ = &c. 8. (1). $\sin^2 \theta = \cos^2 2\theta$, $1 - \cos 2\theta = 2 \cos^2 2\theta$, &c., $\theta = 90°$ or $30°$. (2). First equation reduces to $\sin 2\theta \sin 2\psi = \frac{\sqrt{3}}{2}$; ∴ from second $\frac{2}{\sqrt{3}} \sin 2\psi + \sqrt{1 - \sin^2 2\psi} = \frac{2}{\sqrt{3}}$. $\theta = 30°$, $\psi = 45°$; also $\sin 2\psi = {}^1$, $\sin 2\theta = \frac{7}{2}\sqrt{3}$, impos. 9. (1). $b = 700$, $B = 68° \; 50'$ (2). $C = 67° \; 12'$, $a = 314·16$, $b = 200$. 10. Radius $= \frac{\text{area of triangle}}{s}$; ∴ area of circle $= \frac{(s-a)(s-b)(s-c)}{s} \times 3·1416 = 923·23$.

APPENDIX.

SIN $(A+B)$, &c.

1. To find the value of $\sin(A+B)$ and $\cos(A+B)$.

Let BAC (Fig. 11) represent the angle A, and CAD the angle B; then BAD represents $A+B$. Fig. 11.

From any point P in AD draw PM, PQ perpendicular to AB, AC respectively. From Q draw QR perpendicular to PM, and QN perpendicular to AB.

Then $\angle QPR = 90° - PQR = RQA = QAN = A$.

$$\text{Now } \sin(A+B) = \sin PAM = \frac{PM}{AP}$$

$$= \frac{QN+PR}{AP}$$

$$= \frac{QN}{\cdot} \cdot \frac{\cdot}{AP} + \frac{PR}{\cdot} \cdot \frac{\cdot}{AP}$$

$$= \frac{QN}{AQ} \cdot \frac{AQ}{AP} + \frac{PR}{PQ} \cdot \frac{PQ}{AP}$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\text{Also } \cos(A+B) = \cos PAM = \frac{AM}{AP}$$

$$= \frac{AN - QR}{AP}$$

$$= \frac{AN}{\cdot} \cdot \frac{\cdot}{AP} - \frac{QR}{\cdot} \cdot \frac{\cdot}{AP}$$

$$= \frac{AN}{AQ} \cdot \frac{AQ}{AP} - \frac{QR}{PQ} \cdot \frac{PQ}{AP}$$

$$= \cos A \cos B - \sin A \sin B.$$

2. To find the value of $\sin(A-B)$ and $\cos(A-B)$.

Let BAC (Fig. 12) represent the angle A, and CAD the angle B; then BAD represents $A-B$. Fig. 12.

From any point P in AD draw PM, PQ, perpendicular to AB, AC respectively. From Q draw QR perpendicular to MP produced, and QN perpendicular to AB.

Then $\angle QPR = 90° - PQR = CQR = CAB = A$.

$$\text{Now } \sin(A-B) = \sin PAM = \frac{PM}{AP}$$
$$= \frac{QN - PR}{AP}$$
$$= \frac{QN}{\cdot} \cdot \frac{\cdot}{AP} - \frac{PR}{\cdot} \cdot \frac{\cdot}{AP}$$
$$= \frac{QN}{AQ} \cdot \frac{AQ}{AP} - \frac{PR}{PQ} \cdot \frac{PQ}{AP}$$
$$= \sin A \cos B - \cos A \sin B.$$

$$\text{Also } \cos(A-B) = \cos PAM = \frac{AM}{AP}$$
$$= \frac{AN + QR}{AP}$$
$$= \frac{AN}{\cdot} \cdot \frac{\cdot}{AP} + \frac{QR}{\cdot} \cdot \frac{\cdot}{AP}$$
$$= \frac{AN}{AQ} \cdot \frac{AQ}{AP} + \frac{QR}{PQ} \cdot \frac{PQ}{AP}$$
$$= \cos A \cos B + \sin A \sin B.$$

3. In Article 1 we have taken A and B each less than a right angle, and in Article 2 their sum is less than a right angle. The same results, however, are obtained whatever be the magnitudes of A and B.

Fig. 13. Thus let A and B (Fig. 13) have the magnitudes indicated by the figure, the lettering and construction being the same as in the preceding Article.

Then $\angle QPR = 90° - PQR = RQA = QAN = 180° - A$.

$$\text{And } \cos(A-B) = \cos PAB = -\frac{AM}{AP}$$
$$= -\frac{AN - QR}{AP}$$
$$= -\frac{AN}{\cdot} \cdot \frac{\cdot}{AP} + \frac{QR}{\cdot} \cdot \frac{\cdot}{AP}$$
$$= -\frac{AN}{AQ} \cdot \frac{AQ}{AP} + \frac{QR}{PQ} \cdot \frac{PQ}{AP}$$
$$= -(\cos 180° - A) \cos B$$
$$+ \sin(180° - A) \sin B$$
$$= \cos A \cos B + \sin A \sin B.$$

And similarly in any other case.

We may accordingly assume these formulas to hold whatever be the magnitudes of A and B.

LINE-DEFINITIONS OF THE TRIGONOMETRICAL RATIOS.

4. The following definitions of the Trigonometrical Ratios, formerly given by most English writers, but now falling into disuse, are still sometimes referred to.

Take any arc AB (Fig. 14), subtending at the centre the angle ACB, and draw BP, AT at right angles to AC. Let AT meet CB produced in T. Draw OT', BP' at right angles to OC. Fig. 14.

Then BP is called the sine of the angle ACB to radius CB, AT is called the tangent, and CT the secant. Also BP' (or CP), OT' and CT', being corresponding lines for the angle BCO, which is the complement of ACB, are called respectively the cosine, cotangent and cosecant of ACB. AP is called the versed sine of A.

If we take the arc Ab, greater than one quadrant and less than two, then bp is called the sine, Cp the cosine, At the tangent, Ot' the cotangent, Ct the secant, Ct' the cosecant, and Ap the versed sine.

The radius is the whole sine, or sine of 90°.

5. If the radius of the circle be the unit of length, or, as it is expressed, to radius *unity*, it will be seen that the above definitions exactly coincide with those already given in Article 17. For then each line in the figure will have for its numerical value the number of times it contains the radius, that is, the ratio it bears to the radius. Hence to radius *unity*, the values of the ratios $\frac{BP}{CB}$, $\frac{CP}{CB}$, &c., *i.e.*, of the sine, cosine, &c., will be the same as the numerical values of the *lines* BP, CP, &c.; in other words, the numerical values of the trigonometrical ratios from the definitions given of them in Article 17, will be precisely the same as the numerical values obtained from the *line*-definitions given above.

6. The *line*-definitions explain the origin of the names sine, tangent, &c.

The name sine, from *sinus*, bosom, is given to BP as being (half) the string of the *arcus*, or bow of which BA is half, which is brought up to the *breast* of the archer in discharging it. The tangent AT is the *touching* line. The secant TC is the *cutting* line. The cosine, cotangent and cosecant are so called as being the sine, tangent and secant of the complement.

FORMULAS, &c.

7. $\log 10 = 1$, $\log 1 = 0$, $\log 0 = -\infty$.

$\log (ab) = \log a + \log b$.

$\log \frac{a}{b} = \log a - \log b$.

$\log a^n = n \log a$.

$\log \sqrt[n]{a} = \frac{1}{n} \log a$.

Any trigonometrical ratio of an angle is the co-ratio of the complement.

$$\sin A = \frac{1}{\text{cosec } A}, \tan A = \frac{1}{\cot A}, \cos A = \frac{1}{\sec A}, \tan A = \frac{\sin A}{\cos A},$$

$$\sin^2 A + \cos^2 A = 1.$$

$$\sin 30° = \tfrac{1}{2},\ \cos 30° = \frac{\sqrt{3}}{2},\ \tan 45° = 1.$$

As the angle changes from 0 to 90°.

sin increases from ..	0 to 1 ;	*L* sin increases from	$-\infty$ to 10
tan................	0 .. ∞ ;	*L* tan	$-\infty$ .. $+\infty$
sec................	1 .. ∞ ;	*L* sec	10 .. $+\infty$
cos decreases	1 .. 0 ;	*L* cos decreases....	10 .. $-\infty$
cot	∞ .. 0 ;	*L* cot	$+\infty$ .. $-\infty$
cosec	∞ .. 1 ;	*L* cosec	$+\infty$ .. 10

$$L \tan 45° = 10 = L \cot 45°.$$

In a right-angled triangle, *C* the right angle,

$a = c \sin A$; $a = b \tan A$; $b = c \cos A$; $b = a \cot A$; $c = b \sec A$;

$$c = a \text{ cosec } A.$$

$\sin A = \sin (180° - A)$, $\quad \text{cosec } A = \text{cosec } (180° - A)$;

$L \sin A = L \sin (180° - A)$, $\quad L \text{ cosec } A = L \text{cosec} (180° - A)$;

$\cos A = -\cos (180° - A)$, $\quad \sec A = -\sec (180° - A)$;

$\tan A = -\tan (180° - A)$, $\quad \cot A = -\cot (180° - A)$;

General formulas,

$\sin (A+B) = \sin A \cos B + \cos A \sin B$ .... (4)

$\sin (A-B) = \sin A \cos B - \cos A \sin B$ .... (5)

$\cos (A+B) = \cos A \cos B - \sin A \sin B$ .... (6)

$\cos (A-B) = \cos A \cos B + \sin A \sin B$.

$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A$ .... (7)

$\cos A = 2 \cos^2 \tfrac{1}{2} A - 1 = 1 - 2 \sin^2 \tfrac{1}{2} A$ ...... (8)

$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$ .... (9)

In any triangle ABC,

$$A+B+C = 180^\circ \quad\ldots\ldots (1)$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad\ldots\ldots (2)$$

$$c = a \cos B + b \cos A \quad\ldots\ldots (3)$$

$$\left.\begin{aligned} a^2 &= b^2+c^2-2\,b\,c \cos A \\ \cos A &= \frac{b^2+c^2-a^2}{2\,bc} \end{aligned}\right\} \quad\ldots\ldots (10)$$

$$\left.\begin{aligned} \sin \tfrac{1}{2} A &= \sqrt{\frac{(s-b)\ s-c)}{bc}} \\ \cos \tfrac{1}{2} A &= \sqrt{\frac{s\ (s-a)}{bc}} \end{aligned}\right\} \quad\ldots\ldots (11)$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)\ s-c)}{s\ (s-a)}} \quad\ldots\ldots (12)$$

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \tfrac{1}{2} C.$$

Area of triangle $= \frac{1}{2}$ (*base* $\times$ *height*)

$= \frac{1}{2}\, b\, c \sin A$

$= \sqrt{\{ s\,(s-a)\,(s-b)\ s-c) \}}$.

8. Circumference of a circle of radius $r = 2\pi r$.

Area .... $= \pi r^2$.

π is an incommensurable quantity of which an approximate value is $\frac{22}{7}$; and a still more approximate, 3·14159.

Length of arc in a circle (radius r) which subtends an angle of A degrees at the centre $= \frac{A}{180}\ \pi r$.

Angle subtended at the centre of a circle (radius r) by an arc of length $l = \frac{180^\circ}{\pi} \times \frac{l}{r}$.

Length of arc in a circle (radius r) subtending at the centre an angle of $1^\circ = \frac{\pi}{180}\, r = (\cdot 01745) \times r$.

Angle subtended at the centre of a circle by an arc whose length is equal to the radius $= \frac{180^\circ}{\pi} = 57^\circ\ \cdot 29578$.

Sin $1' = 0\cdot 000291 =$ tan $1'$.

Sin $1'' = 0\cdot 000004848 =$ tan $1''$.

Surface of a sphere (radius r) $= 4\pi r^2$.

Volume .... $= \frac{4}{3}\pi r^3$.

Volume of a pyramid or cone $= \frac{1}{3}$ (*base* $\times$ *height*).

TABLES OF LOGARITHMS.

9. In the following four-figure tables of logarithms of numbers, the first two figures of the number whose logarithm is sought will be found in the column marked N, and the third in the column at the top ; and opposite the first two figures and under the third will be found the mantissa corresponding to the first three figures. The proportional parts are given in the columns to the right ; the part corresponding to the fourth figure will be found here beneath the fourth figure and opposite the first two figures.

Thus to find logarithm of 83·47.

$$\begin{aligned} \text{Mantissa corresponding to } 834 &= 9212 \\ \text{Proportional Part for } 7 \ldots\ldots &= 4 \\ \hline \therefore \log 83{\cdot}47 &= 1{\cdot}9216 \end{aligned}$$

To find the number corresponding to the logarithm 2·7648.

7648
7642 is mantissa corresponding to 581

6 is proportional part for 8 ;

∴ 2·7648 is logarithm of 581·8.

The tables will obtain numbers correct to four figures only. If, however, in any number a fifth figure be given, we may obtain approximate results by neglecting the fifth figure or increasing the fourth by unity, according as such fifth figure be less or not less than 5.

10. In the tables of the logarithms of the trigonometrical ratios, the angles are given at intervals of 10′ between 0° and 20° and between 70° and 90°, and at intervals of 1° between 20° and 70°. The reason for this arrangement is that when the tables give angles between 20° and 70° at intervals of 1°, we can interpolate as accurately for the minutes of any angle as when the tables give angles between 0° and 20°, and 70° and 90° at intervals of 10 minutes. In interpolating we must be careful to notice whether the number given in the column of differences be a difference for 10 minutes or for 60 minutes.

Thus to find $L \sin 33° 25'$.

$L \sin 33° = 9{\cdot}7361$ Diff. for $60' = 115$;

48 $\therefore$ ...... $25' = \frac{25}{60}$ of $115 = 48$.

$\therefore L \sin 33° 25' = 9{\cdot}7409$.

To find $L \cot 79° 35'$.

$L \cot 79° 30' = 9{\cdot}2680$ Diff. for $10' = 71$;

36 $\therefore$ ...... $5' = \frac{5}{10}$ of $71 = 36$.

$\therefore L \cot 79° 35' = 9{\cdot}2644$.

The Tables will obtain angles correct to minutes only. If, however, seconds be given in the case of any angles we may obtain approximate results by neglecting the seconds, or considering them equal to 1 minute, according as their number is less or not less than 30.

LOGARITHMS OF NUMBERS.

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	20	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7

LOGARITHMS OF NUMBERS.—(*Continued.*)

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

LOGARITHMS OF TRIGONOMETRICAL RATIOS.

ANGLES AT INTERVALS OF 10′.

	Sine.	Diff. for 10′.	Tang.	Com. Diff. for 10′.	Cotang.	Cosine.	Diff. for 10′.	
0° 0′	In. Neg.		In. Neg.		Infinite.	10·0000	0	90° 0′
10′	7·4637	3011	7·4637	3011	12·5363	10·0000	0	50′
20′	7·7648	1760	7·7648	1761	12·2352	10·0000	0	40′
30′	7·9408	1250	7·9409	1249	12·0591	10·0000	0	30′
40′	8·0658	969	8·0658	969	11·9342	10·0000	0	20′
50′	8·1627	792	8·1627	792	11·8373	10·0000	0	89° 10′
1° 0′	8·2419	669	8·2419	670	11·7581	9·9999	0	89° 0′
10′	8·3088	580	8·3089	580	11·6911	9·9999	0	50′
20′	8·3668	511	8·3669	512	11·6331	9·9999	0	40′
30′	8·4179	458	8·4181	457	11·5819	9·9999	0	30′
40′	8·4637	413	8·4638	415	11·5362	9·9998	1	20′
50′	8·5050	378	8·5053	378	11·4947	9·9998	0	88° 10′
2° 0′	8·5428	348	8·5431	348	11·4569	9·9997	1	88° 0′
10′	8·5776	321	8·5779	322	11·4221	9·9997	0	50′
20′	8·6097	300	8·6101	300	11·3899	9·9996	1	40′
30′	8·6397	280	8·6401	281	11·3599	9·9996	0	30′
40′	8·6677	263	8·6682	263	11·3318	9·9995	1	20′
50′	8·6940	248	8·6945	249	11·3055	9·9995	0	87° 10′
3° 0′	8·7188	235	8·7194	235	11·2806	9·9994	1	87° 0′
10′	8·7423	222	8·7429	223	11·2571	9·9993	1	50′
20′	8·7645	212	8·7652	213	11·2348	9·9993	0	40′
30′	8·7857	202	8·7865	202	11·2135	9·9992	1	30′
40′	8·8059	192	8·8067	194	11·1933	9·9991	1	20′
50′	8·8251	185	8·8261	185	11·1739	9·9990	1	86° 10′
4° 0′	8·8436	177	8·8446	178	11·1554	9·9989	1	86° 0′
10′	8·8613	170	8·8624	171	11·1376	9·9989	0	50′
20′	8·8783	163	8·8795	165	11·1205	9·9988	1	40′
30′	8·8946	158	8·8960	158	11·1040	9·9987	1	30′
40′	8·9104	152	8·9118	154	11·0882	9·9986	1	20′
50′	8·9256	147	8·9272	148	11·0728	9·9985	1	85° 10′
5° 0′	8·9403	142	8·9420	143	11·0580	9·9983	2	85° 0′
10′	8·9545	137	8·9563	138	11·0437	9·9982	1	50′
20′	8·9682	134	8·9701	135	11·0299	9·9981	1	40′
30′	8·9816	129	8·9836	130	11·0164	9·9980	1	30′
40′	8·9945	125	8·9966	127	11·0034	9·9979	1	20′
50′	9·0070	122	9·0093	123	10·9907	9·9977	2	84° 10′
	Cosine.	Diff. for 10′.	Cotang.	Com. Diff. for 10′.	Tang.	Sine.	Diff. for 10′.	

ANGLES AT INTERVALS OF 10′—*(Continued.)*

	Sine.	Diff. for 10′.	Tang.	Com. Diff. for 10′.	Cotang.	Cosine.	Diff. for 10′.	
6° 0′	9·0192	119	9 0216	120	10·9784	9·9976	1	84° 0′
10′	9·0311	115	9·0336	117	10·9664	9·9975	1	50′
20′	9·0426	113	9·0453	114	10·9547	9·9973	2	40′
30′	9·0539	109	9·0567	111	10·9433	9·9972	1	30′
40′	9·0648	107	9·0678	108	10·9322	9·9971	1	20′
50′	9·0755	104	9·0786	105	10·9214	9·9969	2	83° 10′
7° 0′	9·0859	102	9·0891	104	10·9109	9·9968	1	83° 0′
10′	9·0961	99	9·0995	101	10·9005	9·9966	2	50′
20′	9·1060	97	9·1096	98	10·8904	9·9964	2	40′
30′	9·1157	95	9·1194	97	10·8806	9·9963	1	30′
40′	9·1252	93	9·1291	94	10·8709	9·9961	2	20′
50′	9·1345	91	9·1385	93	10·8615	9·9959	2	82° 10′
8° 0′	9·1436	89	9·1478	91	10·8522	9·9958	1	82° 0′
10′	9·1525	87	9·1569	89	10·8431	9 9956	2	50′
20′	9·1612	85	9·1658	87	10·8342	9·9954	2	40′
30′	9·1697	84	9·1745	86	10·8255	9·9952	2	30′
40′	9·1781	82	9·1831	84	10·8169	9·9950	2	20′
50′	9·1863	80	9·1915	82	10·8085	9·9948	2	81° 10′
9° 0′	9·1943	79	9·1997	81	10·8003	9·9946	2	81° 0′
10′	9·2022	78	9·2078	80	10·7922	9·9944	2	50′
20′	9·2100	76	6·2158	78	10·7842	9·9942	2	40′
30′	9·2176	75	9·2236	77	10·7764	9·9940	2	30′
40′	9·2251	73	9·2313	76	10·7687	9·9938	2	20′
50′	9·2324	73	9·2389	74	10·7611	9·9936	2	80° 10′
10° 0′	9·2397	71	9·2463	73	10·7537	9·9934	2	80° 0′
10′	9·2468	70	9·2536	73	10·7464	9·9931	3	50′
20′	9·2538	68	9·2609	71	10·7391	9·9929	2	40′
30′	9·2606	68	9·2680	70	10·7320	9·9927	2	30′
40′	9·2674	66	9·2750	69	10·7250	9·9924	3	20′
50′	9·2740	66	9·2819	68	10·7181	9·9922	2	79° 10′
11° 0′	9·2806	64	9·2887	66	10·7113	9·9919	3	79° 0′
10′	9·2870	64	9·2953	67	10·7047	9·9917	2	50′
20′	9·2934	63	9·3020	65	10·6980	9·9914	3	40′
30′	9·2997	61	9·3085	64	10·6915	9·9912	2	30′
40′	9·3058	61	9·3149	63	10·6851	9·9909	3	20′
50′	9·3119	60	9·3212	63	10·6788	9·9907	2	78° 10′
12° 0′	9·3179	59	9·3275	61	10·6725	9·9904	3	78° 0′
10′	9·3238	58	9·3336	61	10·6664	9·9901	3	50′
20′	9·3296	57	9·3397	61	10·6603	9·9899	2	40′
30′	9·3353	57	9·3458	59	10·6542	9·9896	3	30′
40′	9·3410	56	9·3517	59	10·6483	9·9893	3	20′
50′	9·3466	55	9·3576	58	10·6424	9·9890	3	77° 10′
	Cosine.	Diff. for 10′.	Cotang.	Com. Diff. for 10′.	Tang.	Sine.	Diff. for 10′.	

ANGLES AT INTERVALS OF 10′—*(Continued.)*

	Sine.	Diff. for 10′.	Tang.	Com. Diff. for 10′.	Cotang.	Cosine.	Diff. for 10′.	
13° 0′	9·3521	54	9·3634	57	10·6366	9·9887	3	77° 0′
10′	9·3575	54	9·3691	57	10·6309	9·9884	3	50′
20′	9·3629	53	9·3748	56	10·6252	9·9881	3	40′
30′	9·3682	52	9·3804	55	10·6196	9·9878	3	30′
40′	9·3734	52	9·3859	55	10·6141	8·9875	3	20′
50′	9·3786	51	9·3914	54	10·6086	9·9872	3	76° 10′
14° 0′	9·3837	50	9·3968	53	10·6032	9·9869	3	76° 0′
10′	9·3887	50	9·4021	53	10·5979	9·9866	3	50′
20′	9·3937	49	9·4074	53	10·5926	9·9863	3	40′
30′	9·3986	49	9·4127	51	10·5873	9·9859	4	30′
40′	9·4035	48	9·4178	52	10·5822	9·9856	3	20′
50′	9·4083	47	9·4230	51	10·5770	9·9853	3	75° 10′
15° 0′	9·4130	47	9·4281	50	10·5719	9·9849	4	75° 0′
10′	9·4177	46	9·4331	50	10·5669	9·9846	3	50′
20′	9·4223	46	9·4381	49	10·5619	9·9843	3	40′
30′	9·4269	45	9·4430	49	10·5570	9·9839	4	30′
40′	9·4314	45	9·4479	48	10·5521	9·9836	3	20′
50′	9·4359	44	9·4527	48	10·5473	9·9832	4	74° 10′
16° 0′	9·4403	44	9·4575	47	10·5425	9·9828	4	74° 0′
10′	9·4447	44	9·4622	47	10·5378	9·9825	3	50′
20′	9·4491	42	9·4669	47	10·5331	9·9821	4	40′
30′	9·4533	43	9·4716	46	10·5284	9·9817	4	30′
40′	9·4576	42	9·4762	46	10·5238	9·9814	3	20′
50′	9·4618	41	9·4808	45	10·5192	9·9810	4	73° 10′
17° 0′	9·4659	41	9·4853	45	10·5147	9·9806	4	73° 0′
10′	9·4700	41	9·4898	45	10·5102	9·9802	4	50′
20′	9·4741	40	9·4943	44	10·5057	9·9798	4	40′
30′	9·4781	40	9·4987	44	10·5013	9·9794	4	30′
40′	9·4821	40	9·5031	44	10·4969	9·9790	4	20′
50′	9·4861	39	9·5075	43	10·4925	9·9786	4	72° 10′
18° 0′	9·4900	39	9·5118	43	10·4882	9·9782	4	72° 0′
10′	9·4939	38	9·5161	42	10·4839	9·9778	4	50′
20′	9·4977	38	9·5203	42	10·4797	9·9774	4	40′
30′	9·5015	37	9·5245	42	10·4755	9·9770	4	30′
40′	9·5052	38	9·5287	42	10·4713	9·9765	5	20′
50′	9·5090	36	9·5329	41	10·4671	9·9761	4	71° 10′
19° 0′	9·5126	37	9·5370	41	10·4630	9·9757	4	71° 0′
10′	9·5163	36	9·5411	40	10·4589	9·9752	5	50′
20′	9·5199	36	9·5451	40	10·4549	9·9748	4	40′
30′	9·5235	35	9·5491	40	10·4509	9·9743	5	30′
40′	9·5270	36	9·5531	40	10·4469	9·9739	4	20′
50′	9·5306	35	9·5571	40	10·4429	9·9734	5	70° 10′
20° 0′	9·5341	34	9·5611	39	10·4389	9·9730	4	70° 0′
	Cosine.	Diff. for 10′.	Cotang.	Com. Diff. for 10′.	Tang.	Sine.	Diff. for 10′.	

ANGLES AT INTERVALS OF 1°.

	Sine.	Diff. for 1°.	Tang.	Com. Diff. for 1°.	Cotang.	Cosine.	Diff. for 1°.	
20°	9·5341	202	9·5611	231	10·4389	9·9730	27	70°
21°	9·5543	193	9·5842	222	10·4158	9·9702	28	69°
22°	9·5736	183	9·6064	215	10·3936	9·9672	30	68°
23°	9·5919	174	9·6279	207	10·3721	9·9640	32	67°
24°	9·6093	166	9·6486	201	10·3514	9·9607	33	66°
25°	9·6259	159	9·6687	195	10·3313	9·9573	34	65°
26°	9·6418	152	9·6882	190	10·3118	9·9537	36	64°
27°	9·6570	146	9·7072	185	10·2928	9·9499	38	63°
28°	9·6716	140	9·7257	181	10·2743	9·9459	40	62°
29°	9·6856	134	9·7438	176	10·2562	9·9418	41	61°
30°	9·6990	128	9·7614	174	10·2386	9·9375*	43	60°
31°	9·7118	124	9·7788	170	10·2212	9·9331	44	59°
32°	9·7242	119	9·7958	167	10·2042	9·9284	47	58°
33°	9·7361	115	9·8125	165	10·1875	9·9236	48	57°
34°	9·7476	110	9·8290	162	10·1710	9·9186	50	56°
35°	9·7586	106	9·8452	161	10·1548	9·9134	52	55°
36°	9·7692	103	9·8613	158	10·1387	9·9080	54	54°
37°	9·7795	98	9·8771	157	10·1229	9·9023	57	53°
38°	9·7893	96	9·8928	156	10·1072	9·8965	58	52°
39°	9·7989	92	9·9084	154	10·0916	9·8905	60	51°
40°	9·8081	88	9·9238	154	10·0762	9·8843	62	50°
41°	9·8169	86	9·9392	152	10·0608	9·8778	65	49°
42°	9·8255	83	9·9544	153	10·0456	9·8711	67	48°
43°	9·8338	80	9·9697	151	10·0303	9·8641	70	47°
44°	9·8418	77	9·9848	152	10·0152	9·8569	72	46°
45°	9·8495	74	10·0000	152	10·0000	9·8495	74	45°
	Cosine.	Diff. for 1°.	Cotang.	Com. Diff. for 1°.	Tang.	Sine.	Diff. for 1°.	

* L. Sin 60 = 9.9375306

Circular Measure = 57°.29578 ——