PLANE TRIGONOMETRY

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SECOND PRINTING, CORRECTED

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED

Printed in U. S. A.

TECHNICAL COMPOSITION CO.
CAMBRIDGE, MASS., U. S. A. 4-41

PREFACE

While the importance of the function concept for elementary mathematics has become recognized by many writers of college algebra texts and of "unified freshman mathematics" books, it has received little recognition from writers on elementary trigonometry. To emphasize this importance has been the leading motive in writing the present book. A somewhat detailed study of the graphs of the trigonometric functions (Chapter V) and of the inverse functions (Chapter VIII) has been introduced for this purpose. Much more could and should be done in this direction; perhaps the present effort may suffice as a first step.

The opportunity afforded by the writing of a new text has been used to make some changes in the presentation of the traditional material. Circular measurement of angles is introduced in the first chapter so as to be available for use throughout the course. The fundamental theorems on projections are presented early and are used subsequently so that the student may be familiar with them when they are applied in a general proof of the addition theorems, based on a method quite generally followed by continental writers. Recognizing the value of the "solution of triangles," a good deal of space has been devoted to this subject, and an attempt has been made to develop it in such a manner that the student can appreciate the reasons for the different methods that are discussed.

On the question of "applied problems," I have taken a definite position. I do not think it feasible to introduce into an elementary text technical material from the applied sciences, important though such material may be. Without such material, however, applications cannot well be anything but problems which use the language of the applied sciences without really belonging to them. An elementary text can render useful service, even to applied science, by stressing the fundamental concepts of trigonometry and by setting problems which connect with the student's actual experience and which suggest ways in which these concepts may be applied, leaving actual applications to the fields to which they belong.

It has not seemed desirable to add to the number of tables of logarithms already available. The elementary treatment of logarithms in Chapter III and the problems scattered throughout the book call for the use of a set of five-place tables, of which there are many excellent ones in existence.

No attempt at logical completeness has been made, but rather has it been my aim to adapt the treatment to the stage of logical development which may be expected of students who begin the study of trigonometry. I am aware of the fact that a fuller discussion might be made in several instances and I shall be happy if the treatment as given should arouse the critical powers of some students and develop in them a desire for more penetrating analysis.

The material as here presented was used originally in mimeographed form by a few classes in the University of Wisconsin. I am under a debt of gratitude to the Department of Mathematics for allowing the material to be thus tried out. And it is with special pleasure that I recognize my indebtedness to colleagues in that department and to Professor T. M. Simpson, now of the University of Florida, to some for suggestions and criticisms, to others for assistance in the reading of the proofs. If I add to this my appreciation of the courtesy shown by the publishers of the book, I am ready to rest my case with the jury consisting of the teachers and students of trigonometry.

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University of Wisconsin
March, 1921

CONTENTS

CHAPTER I

POSITIVE AND NEGATIVE LINES AND ANGLES. COÖRDINATES. RADIAN MEASUREMENT

CHAPTER II

THE TRIGONOMETRIC RATIOS. SIMPLE IDENTITIES

CHAPTER III

LOGARITHMS

CHAPTER IV

SOLUTION OF RIGHT TRIANGLES. APPLICATIONS

CHAPTER V

THE GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

CHAPTER VI

THE ADDITION FORMULAE

CHAPTER VII

THE SOLUTION OF TRIANGLES

CHAPTER VIII

INVERSE TRIGONOMETRIC FUNCTIONS. TRIGONOMETRIC EQUATIONS

PLANE TRIGONOMETRY

CHAPTER I

POSITIVE AND NEGATIVE LINES AND ANGLES. CO-ORDINATES. RADIAN MEASUREMENT

1. Directed magnitudes. The use of positive and negative numbers is a familiar method of indicating temperatures either above or below a certain fixed temperature, which is taken as the zero of the scale. In the same way, it is convenient to use positive and negative numbers to designate the measures of other magnitudes, the values of which may fall on either side of a certain fixed value, taken as a point of departure. For example, northern latitude may be designated as positive, southern latitude as negative; eastern longitude as positive, western longitude as negative; a credit balance as positive, a debit balance as negative; altitude above sea level as positive, altitude below sea level as negative, etc.

2. Points on a line. Directed lines. A simple graphical representation of such magnitudes is obtained by means of a straight line, called the **axis,** upon which are indicated a fixed point, called the **origin,** a **unit distance** and a **positive direction.**

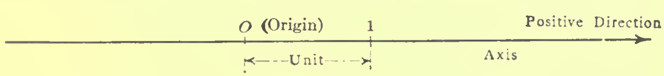


Fig. 1

For every number, positive or negative, there is a single corresponding point, P, on this line. P is the point whose distance from the origin, O, measured in terms of the unit distance, is, in magnitude and direction, equal to the given number. Conversely, to every point, P, on the line there corresponds a real number, viz., the measurement of the distance OP in terms of the unit distance, prefixed by a plus or minus sign. With the conventions of Figure 1, positive numbers will correspond to points to the right of O, negative numbers to points to the left of O. The number which is in this way associated with a point P is called the **coördinate of P.**

3. Points in a plane. A simple extension of this method enables us to designate the position of *a point in a plane* by means of a pair of numbers. We take two mutually perpendicular lines, called the **axes of coördinates,** one horizontal and one vertical. Their point of intersection, called the **origin of coördinates,** is used as origin on each axis, as explained in the preceding section. Moreover, a unit distance and a positive direction are specified on each axis. The horizontal axis is called the **axis of abscissae** or the **X-axis**; the vertical line the **axis of ordinates** or the **Y-axis.** The position of an arbitrary point P in the plane is now determined by the horizontal distance from the Y-axis to P, measured according to the unit and positive direction on the X-axis, and by the vertical distance from the X-axis to P, measured in accordance with the specifications on the Y-axis. In this way two real numbers are obtained, called respectively, the **abscissa** or **x-coördinate** (or simply the "x") and the **ordinate** or **y-coördinate** (or simply the "y") of the point P. The position of the point P is indicated by means of these two numbers in parentheses, the x-coördinate being placed first, and the y-coördinate second. Thus the points A, B and C of Figure 2 are represented by the symbols (3,5) $(-3,-2)$ and $(2,-3)$ respectively. Conversely, for every pair of real numbers, such as $(-2,4)$, a point is uniquely determined of which these numbers are the x- and y-coördinates.

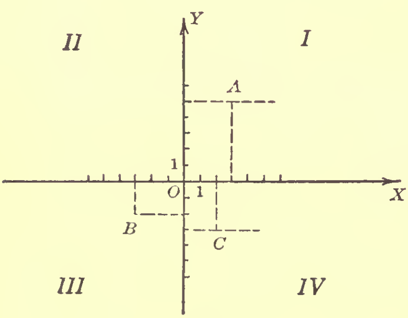


FIG. 2

The four parts into which the two axes of coördinates divide the plane are called the four **quadrants**; they are numbered as indicated in Figure 2. We say that A lies in the first quadrant (in I), B in the third quadrant (in III) and C in the fourth quadrant (in IV).

4. Exercises.

1. Locate the points A (0,0), X (3,5), Y (−4,1), Z (0,3), V (−2,−2), W (5,−2), U (−2,0), T (4,−3).

2. Determine the coördinates of the third vertex of an equilateral triangle of which the origin and the point (6,0) are the other two vertices.

3. Determine the coördinates of the center of the circle which passes through the points (0,0), (4,0), and (0,−4).

4. Show, by construction, that the points (0,2), (1,5), (3,11) and (−2,−4) lie on a straight line.

5. Show, by construction, that the points (−1,5), (6,4), (−1,−3) and (6,−2) lie on the circumference of a circle of which the point (2,1) is the center.

6. Determine the coördinates of the center of the circle which passes through the points A (5,0), B (0,−5) and C (−5,0).

7. Determine the coördinates of the points midway between the points A (2,7) and B (4,3); P (3,−4) and Q (7,4); X (−1,3) and Y (3,−1).

8. Find the distance from the origin of each of the following points: A (3,−4), B (−5,4), C (−4,−6), D (5,5).

9. If r denote the distance of a point from the origin, determine both coördinates for each of the following points:

A: $r = 5$, $y = 3$; B: $r = 13$, $x = -12$; C: $r = 3$, $y = -4$; D: $r = 2$, $x = 1$.

10. The point P lies on a circle about the origin as a center and with a radius equal to 10; its ordinate is twice its abscissa, the two coördinates having like signs. Determine the coördinates of P.

5. A fundamental theorem.

THEOREM I. **If three points A, B and C be taken arbitrarily on a directed straight line, the sum of the three segments AB, BC and CA is equal to zero.**

Proof. Reading from left to right, the three points must lie in one of the following six orders:

A, B, C; A, C, B;
B, A, C; B, C, A;
C, A, B; C, B, A.

FIG. 3

In the third case, illustrated in Figure 3, the segments BA, BC and AC are positive segments, satisfying the relation $BA + AC = BC$, or $-BA + BC - AC = 0$. But, $-BA = AB$ and $-AC = CA$. We conclude therefore

$$AB + BC + CA = 0.$$

In all the other cases the proof proceeds in entirely analogous manner; the details are left to the reader.

COROLLARY. **If any number of points be taken arbitrarily on a directed straight line, the sum of the segments from the first point to the second point, from the second point to the third point, etc., and from the last point to the first point, is equal to zero.**

Proof. Let the points be A, B, C, D, E and F. We know then

$$AB + BC + CA = 0, \qquad CD + DE + EC = 0,$$

$$EF + FA + AE = 0, \quad \text{and} \quad CA + AE + EC = 0.$$

If we now add the first three of these equalities and subtract the last one from their sum, we obtain

$$AB + BC + CD + DE + EF + FA = 0,$$

which proves the corollary.

6. Projections of line segments. In elementary geometry, the **projection** of the segment AB of a line l upon a line m is defined as the segment A_1B_1 of the line m determined by the feet A_1 and B_1 of the perpendiculars dropped from A and B respectively upon m. We prove now the following important theorem.

THEOREM II. **The sum of the projections upon a directed line m of the directed segments of a closed broken line is equal to zero.**

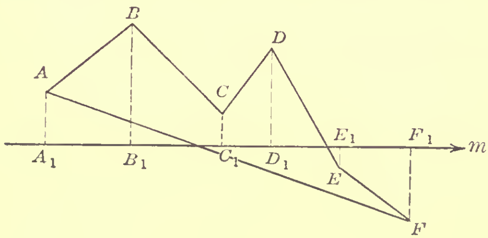


FIG. 4

Proof. Let the broken line be $ABCDEFA$ (see Fig. 4). The projections of the directed segments AB, BC, CD, DE, EF and FA upon the directed line m are the directed segments A_1B_1, B_1C_1, C_1D_1, D_1E_1, E_1F_1, and F_1A_1. In virtue of the Corollary to Theorem I,

$$A_1B_1 + B_1C_1 + C_1D_1 + D_1E_1 + E_1F_1 + F_1A_1 = 0;$$

hence the theorem is proved.

COROLLARY. **The sum of the projections upon a directed line m of the directed segments of an open broken line $ABC \ldots XP$ is equal to the projection upon m of the directed line AP.**

Proof. If the projections of $A, B, C, \ldots X, P$ upon m be denoted by $A_1, B_1, C_1, \ldots X_1, P_1$, we have to show that

$$A_1B_1 + B_1C_1 + \cdots + X_1P_1 = A_1P_1,$$

or that
$$A_1B_1 + B_1C_1 + \cdots + X_1P_1 + P_1A_1 = 0.$$

But this follows, in virtue of Theorem II, from the fact that $ABC \ldots XPA$ is a closed broken line.

7. Exercises.

1. Plot the points A (3,4), B (5,7) and C (7,2). Project the segments AB, BC and CA upon the X-axis and prove that the sum of the projections is equal to zero.

2. Proceed in a similar manner with the points P (1,−3), Q (3,2), R (5,4) and S (7,−4).

3. Determine the length of the projections upon the X- and Y-axes of the broken line O (0,0), A (4,0), P (4,4).

4. Prove that the sum of the projections upon the Y-axis of the segments AB, BC and CA of Exercise 1 is equal to zero.

5. Show that the projections upon each of the coördinate axes of the straight line PS of Exercise 2 is equal to the sum of the projections of the segments of the broken line $PQRS$.

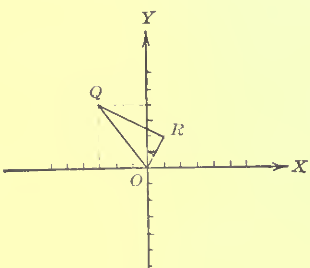


Fig. 5

6. The points O (0,0), R (1,2) and Q (−3,4) (see Fig. 5) are the vertices of a right triangle. Show that the projection of the hypotenuse OQ upon each of the coördinate axes is equal to the sum of the projections of the legs OR and RQ.

7. Plot the points A (−4,3) and B (−2, −7). Show that the projections upon each of the axes of the line AB is equal to the sum of the projections of AO and OB.

8. Plot the points Q (−3,−7) and R (1,−3). Show that the projections upon the coördinate axes of OQ is equal to the sum of the projections of OR and RQ.

9. Proceed as in 8 for the points Q (4,−4) and R (−2,−3).

8. Directed angles. In the preceding paragraphs we have enlarged the concept of line segment by attributing to it not only magnitude, but also direction. We proceed now to extend the concept of *angle* in a similar way, viz., by giving an angle *sense* as well as magnitude. We distinguish between the two sides of an angle by calling one the **initial side**, the other the **terminal**

side. The angle is now to be thought of as having been generated by rotating a line from the position of the initial side to the position of the terminal side; and the angle is measured by the amount of this rotation. If the rotation is **clockwise,** the resulting angle is called **negative;** if the rotation is **counterclockwise,** the angle is called **positive.** The initial and terminal sides of an angle, as well as the sense of rotation, are indicated by means of an arrow, as in Figure 6.

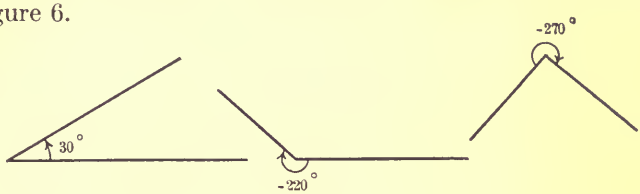


FIG. 6

The most common unit of measurement for angles is the **degree,** which is the 360th part of the angle obtained by one complete revolution. An angle of 180° (180 degrees) is called a **straight angle,** an angle of 90° (90 degrees) is called a **right angle.** Angles smaller than a right angle are called **acute;** angles greater than a right angle are called **obtuse.** Two angles whose sum is equal to a straight angle are called **supplementary angles;** two angles whose sum is equal to a right angle are called **complementary angles.**

9. Exercises.

1. Draw angles of $-45°$, $457°$, $-312°$, $583°$, $1080°$, $-630°$.

2. Determine the complements of the following angles: $25°$, $78°$, $-23°$, $154°$, $217°$, $-112°$, $325°$, $427°$, $-508°$. Construct these angles and their complements.

3. Determine the supplements of the following angles: $89°$, $127°$, $-212°$, $195°$, $-315°$, $287°$, $513°$, $-459°$.

4. Draw the following angles: $120° + 70°$, $65° + 145°$, $180° - 25°$, $270° - 105°$, $180° + 55°$, $270° + 132°$.

10. Radian measure. The *degree,* in which the angles presented so far have been measured, is the unit commonly used in practical work.* For theoretical purposes, another unit, called the **radian** is to be preferred. A **radian** is an angle, which sub-

* In France, the "grade," one four-hundredth part of a complete revolution, is frequently used as a unit of measurement. It has the advantage of fitting into the system of decimal notation.

tends on the circumference of any circle described with the vertex of the angle as a center, an arc equal to the radius of that circle.

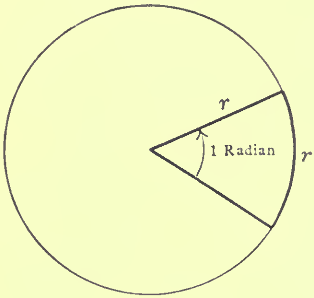


FIG. 7

Since the circumference of a circle of radius r is equal to $2\pi r$, the radius of any circle will be contained 2π times in its circumference. Hence the following relations hold:

$$360° = 2\pi \text{ radians};$$

$$1° = \frac{\pi}{180} \text{ radians} = 0.01745 \ldots \text{ radians};$$

$$1 \text{ radian} = \frac{180°}{\pi} = 57°.29578 \cdots = 57° 17' 44''.81 \ldots .$$

By means of these relations the measurement of an angle can be converted from radian measurement into degree measurement and *vice versa*. The radian measurement of an angle is frequently called the **circular** measure of the angle. When no unit is indicated, it is understood that circular measure is meant. Whenever possible, the circular measure of an angle is expressed by means of multiples and submultiples of π.*

The radian measurement of an angle at the center of a circle bears a simple relation to the arc which this angle intercepts on the circumference of the circle. For every radian in the angle, we will have on the circumference an arc equal to the radius. Hence, if the radian measurement of the angle be denoted by θ and the radius of the circle by r, we shall have

$$\text{length of arc} = \theta r.$$

* A submultiple of a number is the quotient of that number by an integer; e.g., $a/5$ is a submultiple of a; $\pi/3$ is a submultiple of π.

11. Exercises.

1. Determine the circular measure of the following angles: 45°, 30°, −225°, 330°, 540°. −150°, 60°, −270°, −810°, 210°, 480°, −650°.

2. Determine the number of degrees in each of the following angles:

$$\frac{3\pi}{2}, \frac{-4\pi}{3}, \frac{2\pi}{5}, \frac{5\pi}{4}, \frac{-\pi}{2}, 1.5, 2.3.$$

3. Determine the circular measures of the complements and of the supplements of the following angles: $2\pi/3$, $-3\pi/4$, 75°, $7\pi/6$, −112°, $-5\pi/3$, 135°, $\pi/8$, $2\pi/9$, −325°, $7\pi/6$, $-3\pi/2$.

4. Construct the following angles: $-\pi/2$, $5\pi/6$, $7\pi/3$, $-11\pi/4$, $\pi/6$, $3\pi/4$, $-2\pi/3$, 2π, $-11\pi/6$, -3π, 2, −5.2.

5. A wheel makes 10 revolutions per second. Determine the circular measure of the angle described by one of the spokes in 15 seconds.

6. How large an arc will a central angle of 2.5 radians subtend on a circle whose radius is 4 feet?

7. How large an angle, at the center of a circle whose diameter is 10 feet, will subtend an arc whose length is equal to 4 feet?

8. An angle of 2 radians placed at the center of a circle intercepts an arc of 4 feet. What is the radius of the circle?

9. If a wheel makes 50 revolutions per second, how large an angle does its radius describe in 1 minute?

10. A carriage wheel covers a distance of 1 mile in 1000 revolutions. How large is its radius?

CHAPTER II

THE TRIGONOMETRIC RATIOS. SIMPLE IDENTITIES

12. Standard position. For the purpose of comparing different angles we place them in such a way that their vertices coincide with the origin of a system of coördinate axes, and that their initial sides fall along the positive half of the X-axis of this system. Angles so placed are said to be in **standard position.** According as the terminal side of the angle in standard position falls in I, II, III or IV, the angle is said to lie in the first, second, third or fourth quadrant respectively.

13. Exercises.

1. Draw the following angles in standard position: $75°$, $-\pi/3$, $120°$, $\pi/2$, $-165°$, $5\,\pi/6$, $-213°$, $-9\,\pi/5$, $325°$, $195°$, $-3\,\pi$, $155°$.

2. Draw, in standard position, the complements of the angles given in Ex. 1.

3. Draw, in standard position, the supplements of the angles given in Ex. 1.

14. Trigonometric ratios. On the terminal side of the angle θ, placed in standard position, an arbitrary point P is taken. The coördinates of P and its distance OP from O, called the **radius vector** of P, are denoted by x, y and r respectively; it is agreed that the direction from O to P shall always be the positive direction along the terminal side, so that r is always a positive number. While now the numbers x, y and r will be different for different positions of the point P on the same terminal side, it follows from the properties of similar triangles that any one of the six ratios which subsist among these three numbers will be the same for one position of the point P as for any other position and will depend upon the position of the terminal side only.

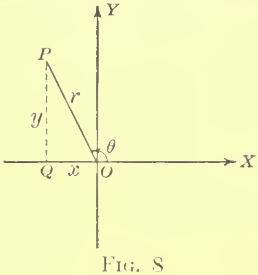


FIG. 8

These six ratios are called the **trigonometric ratios of the angle θ**, and are fundamental in the whole field of mathematics. They are known by the following names:

$\frac{y}{r}$ **is called the sine of the angle θ, written $\sin\theta$,**

$\frac{x}{r}$ **is called the cosine of the angle θ, written $\cos\theta$,**

$\frac{y}{x}$ **is called the tangent of the angle θ, written $\tan\theta$,**

$\frac{x}{y}$ **is called the cotangent of the angle θ, written $\cot\theta$,**

$\frac{r}{x}$ **is called the secant of the angle θ, written $\sec\theta$,**

$\frac{r}{y}$ **is called the cosecant of the angle θ, written $\operatorname{cosec}\theta$.**

Notice that the two ratios occurring in each of the three groups are obtained, one from the other, by replacing x by y and y by x, and that interchanging x and y replaces each ratio by its **co-ratio.***

15. Exercises.

1. Draw each of the following angles in standard position, measure the coördinates and the radius vector of some point on the terminal side of each and determine the trigonometric ratios: $25°$, $\pi/6$, $2\pi/3$, $145°$, $7\pi/5$, $220°$, $7\pi/4$, $315°$, $11\pi/5$, $425°$, $21\pi/8$, $700°$.

2. Proceed as in Exercise 1 with the following angles: $-30°$, $\pi/4$, $-130°$, $-5\pi/6$, $-210°$, $-4\pi/3$, $-320°$, $-5\pi/4$, $-112°$, $-19\pi/6$, $-670°$, $-17\pi/4$.

16. Signs of the ratios. Because the radius vector is always positive, and because the signs of the coördinates of a point depend only upon the quadrant in which the point lies, the *algebraic signs of the trigonometric ratios of an angle are determined by the quadrant in which the angle lies.* The agreements as to the signs of the coördinates of a point (see **3**) lead to the following table of signs for the trigonometric ratios at the top of p. 11.

We observe, furthermore, that the numerical values of the ratios of an angle in any quadrant are equal to the ratios of some angle in the first quadrant. A careful inspection of the diagram will enable the student to determine, for any given angle, an angle in the first quadrant whose ratios are numerically equal to those

* For the origin of the names of the trigonometric functions see, for instance, *Cajori*, "A History of Mathematics," p. 109.

	I	II	III	IV
sine..................	+	+	−	−
cosine................	+	−	−	+
tangent...............	+	−	+	−
cotangent............	+	−	+	−
secant................	+	−	−	+
cosecant..............	+	+	−	−

of the given angle. By the use of a table of the trigonometric ratios of *acute* angles the ratios of an arbitrary angle can therefore be found.

17. Exercises.

Draw each of the following angles in standard position; determine the trigonometric ratios by measurement, as in **14.** Determine an angle in the first quadrant whose trigonometric ratios are numerically equal to those of the given angle, and verify the results of the graphical determination by means of a table of the trigonometric ratios of acute angles:

1. 130°. **2.** 215°. **3.** $11\pi/6$. **4.** −493°. **5.** 1300°. **6.** $-9\pi/5$. **7.** $4\pi/5$. **8.** −1105°. **9.** 600°. **10.** $11\pi/8$. **11.** 310°. **12.** $17\pi/6$.

18. Reduction to first quadrant. We turn now to the problem of determining for every arbitrary angle θ an angle θ' in I, such that the ratios of θ' shall be numerically equal to the ratios of θ.

Consider, as an example, the angle θ in III, in Figure 9a. If

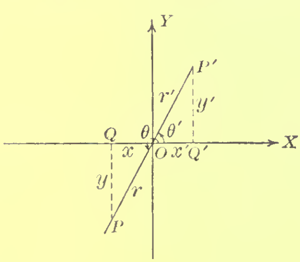


FIG. 9a

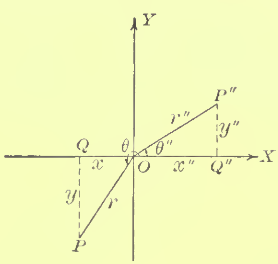


FIG. 9b

we construct in I an angle θ' equal in magnitude to the acute angle between the terminal side of θ and the X-axis, and take a point P' on the terminal side of θ', such that $OP' = OP$, we find by considering the two triangles $OP'Q'$ and OPQ, that

$$r' = r, \ -y' = y, \text{ and } -x' = x. \qquad \text{Why?}$$

It follows from this that the ratios of θ are numerically equal to those of θ', the acute angle made by the terminal side of θ with the X-axis.

If we construct in I an angle θ'' equal to the acute angle between the terminal side of θ and the Y-axis, and take on the terminal side of θ'' a point P'' such that $OP'' = OP$ (see Fig. 9b), we find upon considering the triangles $OP''Q''$ and OPQ, that

$$r = r'', \quad x = -y'', \quad \text{and } y = -x''. \qquad \text{Why?}$$

It follows from this that the ratios of θ are numerically equal to the co-ratios of the angle θ'', the acute angle made by the terminal side of θ with the Y-axis.

These facts are summarized in the following theorem:

THEOREM I. **The numerical values of the trigonometric ratios of any angle θ are equal to the ratios of the positive acute angle θ' which the terminal side of θ makes with the X-axis; they are also equal to the co-ratios of the positive acute angle θ'' which the terminal side of θ makes with the Y-axis.**

The algebraic signs of the ratios are determined by the quadrant in which the angle θ lies (see **16**).

19. Exercises.

1. Apply the theorem of **18** to angles in I and show that the trigonometric ratios of a positive acute angle are equal to the co-ratios of its complement.

2. Apply the theorem of **18** to angles in II and show that the trigonometric ratios of an angle that is less than 180°, are numerically equal to those of its supplement.

3. Apply the theorem of **18** to angles in IV and show that the trigonometric ratios of a negative acute angle are numerically equal to those of the corresponding positive angle.

4. Determine all the trigonometric ratios for the following angles:

(*a*) 237°, (*b*) $3\pi/5$, (*c*) $11\pi/8$,
(*d*) 338°, (*e*) 163°, (*f*) $17\pi/9$.

5. Also for the following angles:

(*a*) $-248°$, (*b*) $-512°$, (*c*) $\frac{-7\pi}{3}$, (*d*) $\frac{-11\pi}{6}$, (*e*) $\frac{5\pi}{4}$ (*f*) $\frac{-5\pi}{6}$.

20. Ratios of acute angles. For acute angles, the definitions of **14** may be put in a special form equivalent to that of **14**, but frequently better adapted to the use that is to be made of them. The abscissa, ordinate and radius vector of any point P on the terminal side of the angle θ are, in this case, the sides and hypotenuse of a right triangle OPQ, *in which the given angle θ actually occurs as an acute angle.* The scaffolding, constituted by the coördinate axes, may now be removed, and the ratios of the angle θ may be defined with reference to this triangle OPQ instead of with reference to the coördinate axes, by putting

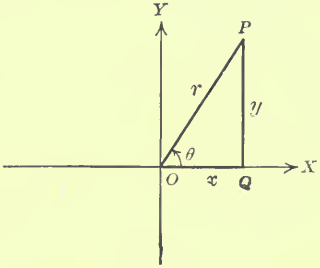


Fig. 10

"side opposite the angle" in place of y,
"side adjacent to the angle" in place of x,
"hypotenuse" in place of r.

In this way we obtain the following form for the definition of the **trigonometric ratios of acute angles:**

$$\textbf{sine}\ \theta = \frac{\textbf{side opposite}\ \theta}{\textbf{hypotenuse}}, \qquad \textbf{cosine}\ \theta = \frac{\textbf{side adjacent to}\ \theta}{\textbf{hypotenuse}},$$

$$\textbf{tangent}\ \theta = \frac{\textbf{side opposite}\ \theta}{\textbf{side adjacent to}\ \theta}, \qquad \textbf{cotangent}\ \theta = \frac{\textbf{side adjacent to}\ \theta}{\textbf{side opposite}\ \theta},$$

$$\textbf{secant}\ \theta = \frac{\textbf{hypotenuse}}{\textbf{side adjacent to}\ \theta}, \qquad \textbf{cosecant}\ \theta = \frac{\textbf{hypotenuse}}{\textbf{side opposite}\ \theta}.$$

21. Ratios of 30°, 60°, 45°. This special form of the definitions enables us to find simple numerical values for the ratios of certain angles:

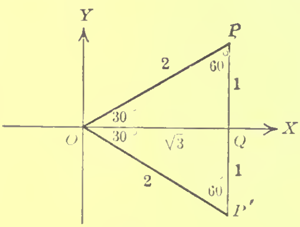


Fig. 11

(*a*) To determine the ratios of an angle of 30°, or of an angle of 60°, a diagram like the one in Figure 11 is constructed, in which $QP' = PQ$, and $\angle OQP = \angle OQP' = 90°$. It follows that $\angle POP' = \angle OPP' = \angle OP'P = 60°$, so that $\triangle\ OPP'$ is equilateral. Choosing PQ as the unit of measurement, we find $OP = 2$, $PQ = 1$, $OQ = \sqrt{3}$. The definitions of **20** lead then to the following results:

THEOREM II. **The trigonometric ratios of angles of 30° and 60° are given by the following formulae:**

$$\sin 30° = 1/2; \quad \cos 30° = \sqrt{3}/2; \quad \tan 30° = 1/\sqrt{3} = \sqrt{3}/3 \quad \cot 30° = \sqrt{3};$$
$$\sec 30° = 2/\sqrt{3} = 2\sqrt{3}/3; \quad \operatorname{cosec} 30° = 2;$$

$$\sin 60° = \sqrt{3}/2; \quad \cos 60° = 1/2; \quad \tan 60° = \sqrt{3}; \quad \cot 60° = 1/\sqrt{3} = \sqrt{3}/3;$$
$$\sec 60° = 2; \ \operatorname{cosec} 60° = 2/\sqrt{3} = 2\sqrt{3}/3.$$

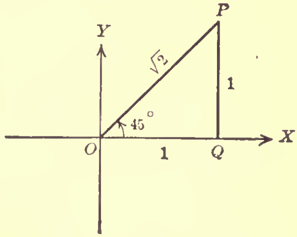


FIG. 12

(*b*) For the ratios of 45°, a diagram like the one in Fig. 12 is used, in which *OQ* is taken as the unit of measurement. The results are summarized in

THEOREM III. **The trigonometric ratios of an angle of 45° are given by the following formulae:**

$$\sin 45° = 1/\sqrt{2} = \sqrt{2}/2; \quad \cos 45° = 1/\sqrt{2} = \sqrt{2}/2; \quad \tan 45° = \cot 45° = 1;$$
$$\sec 45° = \operatorname{cosec} 45° = \sqrt{2}.$$

22. Exercises.

1. Determine the trigonometric ratios of the following angles, without the use of the tables:

(*a*) 120°, 135° and 150°. (*b*) 210°, 225° and 240°. (*c*) 300°, 315° and 330°.

2. Determine the value of each of the following expressions, without the use of tables:

(*a*) $\sin 30° \cos 60° + \cos 30° \sin 60°$;

(*b*) $\cos 120° \cos 30° - \sin 120° \sin 30°$;

(*c*) $\dfrac{\tan 120° + \tan 60°}{1 - \tan 120° \tan 60°}$;

(*d*) $\cos 150° \sin 120° - \cos 210° \sin 150°$;

3. Similarly for:

(*a*) $\sin 315° \cos^2 45° + \tan^2 30° \sec 135°$;

(*b*) $\sin^2 240° \cot 225° - \cos^2 330° \tan 315°$;*

(*c*) $\operatorname{cosec}^2 300° \sin 60° \tan 150° + \sec^2 210° \cot 240° \cos^2 30°$;

(*d*) $\cot^3 150° \operatorname{cosec} 240° - \tan 330° \sec^3 150°$.

* The notation $\sin^2 \theta$ means $(\sin \theta)^2$. Similar notations are used for other powers of the trigonometric ratios.

23. Ratios of 90°, 180°, 270°, 360°. For the ratios of an angle whose terminal side coincides with one of the coördinate axes, a special consideration is necessary, because application of the definitions of **14** would require division by 0, an operation which is impossible. To divide a number a by 0 would require the determination of another number b such that b multiplied by 0 would yield a, which is obviously impossible, unless a were 0.

When $\theta = 0°$, we have $x = r$, $y = 0$. Hence:

$$\sin 0° = 0/r = 0, \quad \cos 0° = r/r = 1, \quad \tan 0° = 0/r = 0,$$
$$\cot 0° = r/0 = ?, \quad \sec 0° = r/r = 1, \quad \operatorname{cosec} 0° = r/0 = ?.$$

For an angle of 90°, $x = 0$, and $y = r$, so that we now find:

$$\sin 90° = 1, \quad \cos 90° = 0, \quad \tan 90° = r/0 = ?,$$
$$\cot 90° = 0, \quad \sec 90° = r/0 = ?, \quad \operatorname{cosec} 90° = 1.$$

Similar results are obtained for angles of 180° and 270°; but it should be noticed, that for an angle of 180°, the x and the r are equal numerically but opposite in sign, in accordance with the agreements as to the signs of these quantities, so that we have in this case $x = -r$. For a similar reason, we find that for an angle of 270°, $y = -r$. Consequently the ratios of these angles have the following values:

$$\sin 180° = 0, \quad \cos 180° = -1, \quad \tan 180° = 0,$$
$$\cot 180° = -r/0 = ?, \quad \sec 180° = -1, \quad \operatorname{cosec} 180° = r/0 = ?.$$
$$\sin 270° = -1, \quad \cos 270° = 0, \quad \tan 270° = -r/0 = ?,$$
$$\cot 270° = 0, \quad \sec 270° = r/0 = ?, \quad \operatorname{cosec} 270° = -1.$$

The question marks put after cot 0°, cosec 0°; tan 90°, sec 90°, cot 180°, cosec 180°; and tan 270°, sec 270° are intended to indicate that these ratios cannot be determined because a division by 0 would be involved therein.

These facts are expressed in the following theorem:

THEOREM IV. **The cotangent and the cosecant of angles of 0° and 180° do not exist; the secant and the tangent of angles of 90° and 270° do not exist. The remaining ratios of these angles are given by the following formulae:**

$$\sin 0° = \tan 0° = 0, \qquad \cos 0° = \sec 0° = 1;$$
$$\sin 90° = \operatorname{cosec} 90° = 1, \qquad \cos 90° = \cot 90° = 0;$$
$$\sin 180° = \tan 180° = 0, \qquad \cos 180° = \sec 180° = -1;$$
$$\sin 270° = \operatorname{cosec} 270° = -1, \qquad \cos 270° = \cot 270° = 0.$$

24. Trigonometric functions. When an angle is changing, as, e.g., the angle described by the spoke of a revolving wheel or the angle described by a swinging pendulum, there is in general a value of each of the trigonometric ratios corresponding to every value of the variable angle. Whenever such a correspondence exists between two varying magnitudes, whereby to each value of one there corresponds a value of the other, it is said that one of the variables is a function of the other. In the present case, the trigonometric ratios are functions of the angle; they are called **trigonometric functions**.

If we consider a variable angle θ, which while steadily remaining in I, increases towards $\pi/2$, the variation of the tangent of θ may best be studied by keeping either x or y fixed as the angle varies, as in Figures 13a and 13b respectively. In either case, the

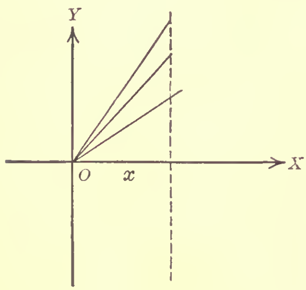


FIG. 13a

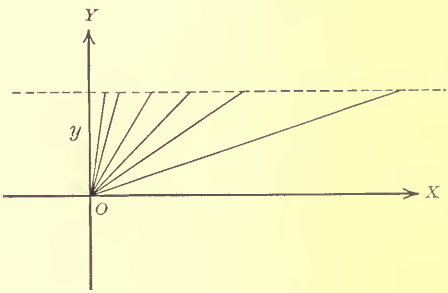


FIG. 13b

ratio y/x remains positive; its numerical value increases indefinitely. These facts are expressed by the following statement:

"*As θ tends towards 90°, being always less than 90°, tan θ increases indefinitely,* and by the formula:

$$\lim_{\theta \to 90^\circ -} \tan\theta = +\infty."$$

In a similar manner it is seen that if an angle tends towards 90°, while steadily remaining in II, the ratio y/x is negative throughout, while the numerical value increases indefinitely; i.e.,

"*As θ tends towards 90°, being always greater than 90°, tan θ decreases indefinitely,* expressed by the formula:

$$\lim_{\theta \to 90^\circ +} \tan\theta = -\infty."$$

The fact that $\lim_{\theta \to 90^\circ -} \tan\theta = +\infty$, and that $\lim_{\theta \to 90^\circ +} \tan\theta = -\infty$ corroborates our former conclusion that for an angle of 90° the tangent does not exist.

25. Exercises.

1. Interpret and prove the following statements:

(*a*) $\lim_{\theta \to 90^\circ -} \sec \theta = +\infty.$

(*b*) $\lim_{\theta \to 90^\circ +} \sec \theta = -\infty.$

(*c*) $\lim_{\theta \to 180^\circ +} \cot \theta = +\infty.$

(*d*) $\lim_{\theta \to 180^\circ -} \cot \theta = -\infty.$

(*e*) $\lim_{\theta \to 180^\circ -} \operatorname{cosec} \theta = +\infty.$

(*f*) $\lim_{\theta \to 180^\circ +} \operatorname{cosec} \theta = -\infty.$

2. Also:

(*a*) $\lim_{\theta \to 270^\circ -} \tan \theta = +\infty.$

(*b*) $\lim_{\theta \to 270^\circ +} \tan \theta = -\infty.$

(*c*) $\lim_{\theta \to 270^\circ +} \sec \theta = +\infty.$

(*d*) $\lim_{\theta \to 270^\circ -} \sec \theta = -\infty.$

(*e*) $\lim_{\theta \to 360^\circ +} \cot \theta = +\infty.$

(*f*) $\lim_{\theta \to 360^\circ -} \cot \theta = -\infty.$

(*g*) $\lim_{\theta \to 360^\circ +} \operatorname{cosec} \theta = +\infty.$

(*h*) $\lim_{\theta \to 360^\circ -} \operatorname{cosec} \theta = -\infty.$

3. Determine the value of the following expressions:

(*a*) $\sin^2 \pi/2 \cos^2 \pi/6 - \tan^2 3\,\pi/4 \sec^2 11\,\pi/3.$

(*b*) $\cos^2 \pi/2 \sin^2 3\,\pi/2 + \cot^2 11\,\pi/4 \operatorname{cosec}^2 11\,\pi/6.$

(*c*) $\sin 3\,\pi \cot 3\,\pi/2 - \tan 5\,\pi/6 \cos 3\,\pi.$

(*d*) $\sec 3\,\pi \sec^2 9\,\pi/4 + \operatorname{cosec} 3\,\pi/2 \operatorname{cosec}^2 17\,\pi/6.$

26. Periodicity. In the preceding paragraph we have discussed the properties of the trigonometric functions of the variable angle θ for values of θ which increase or decrease towards 0°, 90°, 180° and 270°. We consider next some other properties of the trigonometric functions, the first to be considered being the property of **periodicity.** A function is said to be periodic if there exists a constant number a, such that the function assumes the same value for $\theta + a$ as for θ, no matter what value θ may have. The number a is called the **period** of the function. Examples of periodic functions are furnished by many natural phenomena, such as the swinging pendulum, the motion of the tides, the movement of a vibrating string, the wave motion of sound, light and electricity, etc. The fact that the trigonometric functions possess this property of periodicity is one of the reasons for their great importance in the study of natural phenomena.

If two angles which differ by an integral multiple of 360° ($2\,\pi$) are placed in standard position, they have the same terminal side; hence their trigonometric ratios are equal; e.g.,

$$\sin(\theta + m \cdot 360^\circ) = \sin(\theta + m \cdot 2\pi) = \sin\theta,$$

for every value of θ, and for every positive or negative integral value of m. We have therefore the following theorem:

THEOREM V. **The trigonometric functions are periodic; they have a period of 2π.**

In Chapter V we shall see how the trigonometric functions, in particular the sine and the cosine, may be used for the representation of more general periodic functions.

27. Relations between the trigonometric functions. From the definitions of the trigonometric ratios in **14**, there follow immediately the following theorems:

THEOREM VI. **The sine and the cosecant, the cosine and the secant, the tangent and the cotangent of any angle are reciprocals of each other; i.e.:**

$$\sin\theta \cdot \operatorname{cosec}\theta = 1; \quad \cos\theta \cdot \sec\theta = 1; \quad \tan\theta \cdot \cot\theta = 1.$$

THEOREM VII. **The tangent of any angle is equal to the quotient of its sine by its cosine; the cotangent of any angle is equal to the quotient of its cosine by its sine; i.e.:**

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Since (see Fig. 8), the abscissa, ordinate and radius vector of any point P are respectively the legs and the hypotenuse of a right triangle, we have for any angle θ:

$$x^2 + y^2 = r^2.$$

Dividing both sides of this equality in succession by r^2, by x^2 and by y^2, and making use of the definitions given in **14**, we obtain:

THEOREM VIII. **The sum of the squares of the sine and the cosine of any angle is equal to unity; the square of the secant of any angle diminished by the square of the tangent, and the square of the cosecant of any angle diminished by the square of the cotangent are each equal to unity; i.e.:**

$$\sin^2\theta + \cos^2\theta = 1; \quad \sec^2\theta - \tan^2\theta = 1; \quad \operatorname{cosec}^2\theta - \cot^2\theta = 1.$$

Theorems VI, VII and VIII may be used to determine all the trigonometric ratios of an angle as soon as one of them is known. By means of these theorems a great many other relations may be proved to hold between the trigonometric functions.

28. Exercises.

1. Given $\sin\theta = -\frac{2}{3}$. Determine the remaining ratios of θ.

Since $\sin\theta$ is negative, θ may lie in III or in IV. In either case, we can draw the terminal side of θ, by determining a point for which $y = -2$ and $r = 3$. We find then $x = -\sqrt{5}$ or $x = \sqrt{5}$ according as θ is in III or IV. Knowing x, y, and r, we can obtain the trigonometric ratios of θ.

Proceeding without direct application of the definitions, we can use the formula $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$; after that the other ratios can all be determined by means of Theorems VI and VII of **27.**

2. Determine all the ratios of θ, when it is known that $\tan\theta = 4$.

3. Similarly, when $\cos\theta = \frac{1}{2}$ and θ lies in IV.

4. Also when $\cot\theta = -7$ and θ lies in II.

Prove that the following identities result from the formulae of **27:**

5. $\sin\theta = \tan\theta\cos\theta$.

6. $\cot\theta = \cos\theta\operatorname{cosec}\theta$.

7. $\tan^2\theta + \sin^2\theta + \cos^2\theta = \sec^2\theta$.

8. $\operatorname{cosec}\theta - \sin\theta = \cot\theta\cos\theta$.

9. $\sin^2\theta - \sin^4\theta = \sin^2\theta\cos^2\theta$.

10. $\sec^2\theta + \cot^2\theta = \operatorname{cosec}^2\theta + \tan^2\theta$.

11. $\sin^2\theta - \cos^2\theta = \sin^4\theta - \cos^4\theta$.

12. $\sin^2\theta = \dfrac{\tan^2\theta}{1+\tan^2\theta} = \dfrac{\sec^2\theta - 1}{\sec^2\theta}$.

13. $\cos^2\theta = \dfrac{\cot^2\theta}{1+\cot^2\theta} = \dfrac{\operatorname{cosec}^2\theta - 1}{\operatorname{cosec}^2\theta}$.

14. $\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta$.

15. $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$.

16. $\operatorname{cosec}^2\theta\sec^2\theta = \cot^2\theta\sec^2\theta + \operatorname{cosec}^2\theta\tan^2\theta$.

17. $\dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta} = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$.

18. $\sec\theta - \tan\theta = \dfrac{\cos\theta}{1+\sin\theta}$.

19. $\operatorname{cosec}\theta - \cot\theta = \dfrac{\sin\theta}{1+\cos\theta}$.

20. $2(\cos^6\theta + \sin^6\theta) - 3(\cos^4\theta + \sin^4\theta) = -1$.

21. $\dfrac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha\cdot\tan\beta$.

22. $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$.

23. $\dfrac{\cot\theta}{\operatorname{cosec}\theta} + \sin\theta\tan\theta = \sec\theta$.

24. $\sin\theta + \cot\theta\cos\theta = \operatorname{cosec}\theta$.

25. $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$.

26. $\cos\theta(\cot\theta + \tan\theta) = \operatorname{cosec}\theta$.

27. $\sec\theta - \cos\theta = \tan\theta\sin\theta$.

CHAPTER III

LOGARITHMS

29. Theory of Exponents. In order to facilitate the calculations involved in the application of the trigonometric ratios to the solution of problems, we take up the study of logarithms. We recall first the following parts of the theory of exponents.

(*a*) A continued product, all of whose factors are equal to the same number, a, is called a **power** of that number. The number a is called the **base of** the power; the number of factors in the product is called the **exponent** of the power. Powers are classified according to their exponents; they are represented in abbreviated form by the base and the exponent.

Thus $a \cdot a \cdot a \cdot a \cdot a$, the 5th power of a, is represented by a^5; a being the base of the power and 5 the exponent.

(*b*) Besides powers whose exponents are positive integral numbers, defined in (*a*), we consider powers whose exponents are zero, negative or fractional. Such powers are defined as follows:

The **zero-th** power of any number is equal to 1; e.g., $a^0 = 1$.

A power of a whose exponent is **negative** is the *reciprocal* of the power of a, whose exponent is the corresponding positive number: $a^{-n} = \frac{1}{a^n}$.

A power of a positive number, a, whose exponent is a **unit fraction** is the *real positive root* of a whose index is equal to the denominator of the fraction: $a^{1/q} = \sqrt[q]{a}$.

Thus, e.g., $2^{-5} = 1/32$; $36^{\frac{1}{2}} = 6$; $7^0 = 1$.

(*c*) From these definitions are derived the following fundamental theorems, usually referred to as the "Laws of Exponents":

I. The product of two powers of a number a is equal to that power of a, whose exponent is equal to the sum of the exponents of the factors: $a^p \times a^q = a^{p+q}$.

II. The quotient of two powers of a number a is equal to that power of a, whose exponent is equal to the exponent of the dividend diminished by the exponent of the divisor: $a^p \div a^q = a^{p-q}$.

III. A power of a power of a number a is equal to that power of a whose exponent is equal to the product of the exponents of the given powers: $(a^p)^q = a^{pq}$.

Thus, e.g.: $\sqrt[3]{2} \times \frac{1}{4} = 2^{\frac{1}{3}} \times 2^{-2} = 2^{-\frac{5}{3}} = \sqrt[3]{\frac{1}{32}}$;

$$\sqrt[5]{3^4} = (3^4)^{\frac{1}{5}} = 3^{\frac{4}{5}};$$

$$25 \div \tfrac{1}{125} = 5^2 \div 5^{-3} = 5^5 = 3{,}125.$$

30. The use of exponents in calculation. The fundamental idea underlying the use of logarithms in calculations is that the laws of exponents may be utilized for the purpose of multiplication and division of numbers, for raising numbers to a power, or for extracting roots. If, e.g., we had a list of the successive positive integral powers of 2 from 2^1 to 2^{150}, the product and quotient of two of them, their powers and roots, could, within certain limits, be found by addition, subtraction, multiplication and division. For example, we would have:

$$2^8 \times 2^{17} = 2^{8+17} = 2^{25}; \quad \frac{2^{49}}{2^{63}} = 2^{49-63} = 2^{-14} = \frac{1}{2^{14}};$$

$$(2^{17})^6 = 2^{17\times 6} = 2^{102}; \quad \sqrt[12]{2^{144}} = (2^{144})^{\frac{1}{12}} = 2^{12}.$$

If, therefore, we could write *every* number as a power of some fixed number, all multiplications and divisions could be reduced essentially to additions and subtractions, while powers and roots of numbers could be found by simple multiplications and divisions.

31. Definition of the logarithm. Accordingly, we lay down the following definition:

Definition I. **The logarithm of any number p with respect to the base a is the exponent of that power of a which is equal to p.**

This logarithm is designated by the symbol $\log_a p$. We shall consider only logarithms of positive numbers with respect to bases which are positive.*

As a consequence of this definition, we can say:

$\log_2 16 = 4$, because $2^4 = 16$; $\log_3 \frac{1}{9} = -2$, because $3^{-2} = \frac{1}{9}$;
$\log_5 \sqrt{5} = \frac{1}{2}$, because $5^{\frac{1}{2}} = \sqrt{5}$; $\log_7 1 = 0$, because $7^0 = 1$.

* It does not fall within the scope of this book to consider whether for any number p and any base a, there always exists an unique logarithm of p with respect to the base a. We take for granted that if a and p are positive, there always exists uniquely a real number $\log_a p$. The further study of this question belongs to the Theory of Functions.

32. Exercises.

1. Determine $\log_3 27$, $\log_3 \frac{1}{27}$, $\log_3 \sqrt[3]{3}$, $\log_3 1$, $\log_3 \sqrt[5]{27}$, $\log_3 3$.

2. Determine $\log_{10} 10$, $\log_{10} 1000$, $\log_{10} .01$, $\log_{10} .0001$, $\log_{10} 1$.

3. Determine $\log_2 8$, $\log_2 \sqrt[4]{2}$, $\log_2 \frac{1}{16}$, $\log_2 1/\sqrt{2}$, $\log_2 4/\sqrt{8}$.

4. Determine $\log_8 4$, $\log_4 8$, $\log_9 27$, $\log_{27} \frac{1}{9}$, $\log_{16} \frac{1}{2}$.

5. It is known that $\log_{10} 17 = 1.23045$, $\log_{10} 29.5 = 1.46982$, $\log_{10} 83 = 1.91908$. Determine $10^{1.46982}$, $10^{1.23045}$, $10^{1.91908}$, $10^{3.23045}$, $10^{.46982}$

6. Determine $10^{\log_{10} 7}$, $5^{\log_5 7}$, $2^{\log_2 7}$, $a^{\log_a 7}$.

33. The fundamental theorems on logarithms. From the "Laws of Exponents," quoted in **29** (c), we derive now, by the aid of the definition of **31,** the following fundamental theorems on logarithms, in which we *prove* what was merely suggested in the last paragraph of **30.**

THEOREM I. **The logarithm of the product of two numbers with respect to the base a is equal to the sum of the logarithms of the factors:**

$$\log_a (pq) = \log_a p + \log_a q.$$

Proof. The theorem evidently says nothing else than that the exponent of that power of a which equals pq is equal to the sum of the exponents of the powers of a which are equal to p and q. For, let

$$\log_a p = x, \text{ and } \log_a q = y;$$

i.e., let
$$a^x = p, \text{ and } a^y = q.$$

Then $a^{x+y} = pq$; i.e., $\log_a pq = x + y = \log_a p + \log_a q$.

THEOREM II. **The logarithm of the quotient of two numbers with respect to the base a is equal to the logarithm of the dividend diminished by the logarithm of the divisor:**

$$\log_a \frac{p}{q} = \log_a p - \log_a q.$$

THEOREM III. **The logarithm of a power of a number with respect to the base a is equal to the exponent of the power multiplied by the logarithm of the number:**

$$\log_a p^n = n \cdot \log_a p.$$

The proofs of Theorems II and III are left to the student.

We proceed to illustrate the use of these theorems:

(a) To determine $\log_{10} 14$ when it is known that $\log_{10} 2 = .30103$ and $\log_{10} 7 = .84510$.

From Theorem I, it follows that $\log_{10} 14 = \log_{10} 2 + \log_{10} 7 = 1.14613$.

(*b*) To determine $\log_{10} \frac{14 \times 25}{27}$, when $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$, $\log_{10} 5 = .69897$ and $\log_{10} 7 = .84510$.

By the use of Theorems I and II, we find:

$$\log_{10} \frac{14 \times 25}{27} = \log_{10} 14 + \log_{10} 25 - \log_{10} 27.$$

But, from Theorem III it follows that $\log_{10} 25 = \log_{10} 5^2 = 2 \log_{10} 5 = 1.39794$, and that $\log_{10} 27 = \log_{10} 3^3 = 3 \log_{10} 3 = 1.43136$.

$$\therefore \quad \log_{10} \frac{14 \times 25}{27} = \log_{10} 14 + 2 \log_{10} 5 - 3 \log_{10} 3 = 1.11271.$$

(*c*) To determine $\log \sqrt[3]{\frac{25 \times 32}{81}}$, when $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$ and $\log_{10} 5 = .69897$.

By the use of Theorems I, II and III, we find:

$$\log_{10} \sqrt[3]{\frac{25 \times 32}{81}} = \tfrac{1}{3} [2 \log_{10} 5 + 5 \log_{10} 2 - 4 \log_{10} 3]$$
$$= \tfrac{1}{3} [1.39794 + 1.50515 - 1.90848] = \tfrac{1}{3} .99461$$
$$= .33154.$$

34. Exercises.

Given, that $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$, and $\log_{10} 7 = .84510$, determine

1. $\log_{10} 5$.

2. $\log_{10} 21$.

3. $\log_{10} 35$.

4. $\log_{10} 70$.

5. $\log_{10} \frac{49}{64}$

6. $\log_{10} \sqrt[5]{32}$.

7. $\log_{10} \frac{125 \times 32}{27}$.

8. $\log_{10} \sqrt{\frac{28 \times 25}{81}}$.

9. $\log_{10} \sqrt[5]{\frac{27 \times 49}{50}}$.

10. $\log_{10} \frac{81 \times 64}{25 \times \sqrt[3]{2}}$.

11. $\log_{10} \frac{\sqrt{8} \times \sqrt[3]{9}}{\sqrt[6]{7}}$.

12. $\log_{10} \frac{\sqrt[3]{32 \times 75}}{\sqrt{245 \times 54}}$.

35. Common logarithms. It will now be recognized that the plan suggested in 30 can actually be carried out, provided we can "write every number as a power of some fixed number," i.e., provided we can find the logarithm of every number with respect to some fixed base. The Differential Calculus furnishes methods for solving this problem and for constructing a "table of logarithms." For the purpose of computation, logarithms with respect to the base 10, called **common logarithms** are most useful. Without considering here the problem of constructing a table of common logarithms, we shall see how a comparatively small table of logarithms can be made to serve our purpose.*

From now on, the symbol "log" will be used to designate $\log_{10}$, i.e., common logarithm.

We know then that $\log 1000 = 3$, $\log 100 = 2$, $\log 10 = 1$, $\log 1 = 0$, $\log .1 = -1$, $\log .01 = -2$, $\log .001 = -3$, etc.; i.e., if we arrange numbers in the geometrical progression of scale A, their common logarithms will form the arithmetical progression of scale B.

Scale A......	N	.001	.01	.1	1	10	100	1000
Scale B......	log N	−3	−2	−1	0	1	2	3

We assume now, without proof, that as the number N increases, its logarithm will also increase. It will then follow that if a number N lies between two successive terms of scale A, $\log N$ will lie between the two corresponding integers of scale B. Hence if we have determined the place of a number N in scale A, we shall know the integral part of $\log N$, and conversely, if we know the integral part of $\log N$, we shall know the position of N in scale A.

The integral part of the logarithm of a number is called the **characteristic**; the fractional part, taken positively, is called the **mantissa**. Here it is to be understood that if the logarithm of a number is negative, it will be written as the sum of a negative integer (the characteristic) and a positive fraction (the mantissa). For example, $\log \frac{1}{\sqrt{10}} = \log 10^{-\frac{1}{2}} = -\frac{1}{2}$ will be written in the

* Any set of five-place tables of logarithms of numbers and of trigonometric functions may be used in connection with this text.

form $-1 + .50000$, or $9.50000 - 10$; the characteristic is -1 (or 9–10), the mantissa is .50000.*

We conclude that the position of a number in scale A and the position of its logarithm in scale B are related to each other in the way expressed by the following theorem:

THEOREM IV. **The characteristic of the common logarithm of a number N is the lesser of the two integers in scale B, corresponding to the two numbers in scale A between which N lies; conversely, a number N will lie between those two numbers in scale A, which correspond to the characteristic of log N and the next greater integer in scale B.**

This theorem may be reduced at once to the following working rule:

The characteristic of log N is equal to the integer in scale B, corresponding to that number in scale A which is pointed off like N; conversely, N is to be pointed off like that number in scale A, which corresponds to the characteristic of log N in scale B.

It remains to determine the mantissa of a logarithm; it is found from a table. It is important to observe however that the mantissa of the logarithm is the same for all numbers which differ only in the position of the decimal point. Let us compare, for instance, log 475.3 and log .04753. Since $475.3 = 10^4 \times .04753$, it follows that $\log 475.3 = 4 + \log .04753$. But the addition of 4 to log .04753 will simply increase its characteristic by 4, without affecting the mantissa. In general, if A and B differ only in the position of the decimal point, the greater one, say A, can be obtained by multiplying the smaller one, B, by a power of 10, i.e. $A = 10^k \times B$, where k is a positive integer. Therefore $\log A = k + \log B$, so that log A is obtained from log B by adding to it the integer k. This however will merely increase the characteristic by k and will not alter the mantissa. This fact finds expression in the following theorem:

THEOREM V. **The mantissa of the common logarithm of a number is determined by its sequence of digits only; conversely, the sequence of digits of a number is determined by the mantissa of its logarithm.**

* When a five-place table is used, all mantissas are written out to five places of decimals.

36. Use of a table of logarithms. Theorems IV and V enable us to find from a five-place table of logarithms the logarithm to five places of any number of five significant figures; * and to determine five significant figures of a number whose mantissa is given to five places. It only remains to become familiar with the arrangement of the tables. This is best explained by means of examples.

Example 1. To find log 47.316.

Since 47.316 lies between 10 and 100, (or, since 47.316 is pointed off like 10) the characteristic of log 47.316 is 1. To determine the mantissa we look up in the tables the sequence of digits 47316. The first three digits are found in the left-hand column of the table of logarithms of numbers, usually headed N, the fourth digit is found in the line at the top or bottom of the page. In this way we find that

for 47320 the mantissa is 67504; while
for 47310 the mantissa is 67495.

For an increase of 10 in the number, there is therefore an increase of 9 in the mantissa. Now we approximate to the value of the mantissa for 47316 by assuming that as the number increases by ten equal steps from 47310 to 47320, the mantissa will also increase by ten equal steps from 67495 to 67504. We must then add to 67495 six tenths of 9. The table of proportional parts (usually headed P.P.) shows that $\frac{6}{10}$ of 9 is equal to 5.4, which we round off to 5. We conclude that the mantissa for 47316 is 67500 and that log 47.316 = 1.67500.

Example 2. To find log .089327.

Since .089327 lies between .01 and .1, (i.e. since .089327 is pointed off like .01) the characteristic of the logarithm is -2 or $8 - 10$.

The mantissa for 89330 is 95100; the mantissa for 89320 is 95095. A difference of 10 in the number makes a difference of 5 in the mantissa.

From the table of proportional parts, we find that $\frac{7}{10}$ of 5 is 3.5, which we round off to 4, so that the mantissa for 89327 is 95099 and therefore log .089327 = 8.95099 -10.

* The figures of a number significant for its logarithm are the digits left after the ciphers at the beginning and end of the number have been removed.

Example 3. Given $\log N = 2.23130$. To determine N.

We now reverse the process followed in the preceding examples and begin by searching among the mantissas given in the table for 23130. We find that

23147 is the mantissa for 17040, and that
23121 is the mantissa for 17030, so that a difference of
26 in the mantissa makes a difference of 10 in the number.

The given mantissa 23130 exceeds 23121 by 9; from the table of proportional parts, we find that $\frac{3}{10}$ of 26 equals 7.8, while $\frac{4}{10}$ of 26 equals 10.4. Of these, the former is closer to 9 than the latter; hence 23130 is the mantissa for 17033, which must therefore be the sequence of digits for the required number N. Since the characteristic is 2, N must be pointed off like 100 so that the decimal point must be placed between the 0 and the 3. We conclude that the required number N is 170.33.

Example 4. Given $\log N = 9.07025 - 10$. To determine N.

We find that 07041 is the mantissa for 11760, while 07004 is the mantissa for 11750, so that a difference of 37 in the mantissa makes a difference of 10 in the number. Hence a difference of 21 in the mantissa corresponds to a difference of 6 in the number; the sequence of digits of N must therefore be 11756. Since the characteristic is -1, the number must be pointed off like .1, so that we conclude that $N = .11756$.

The table of the logarithms of the trigonometric functions gives directly the logarithms of the trigonometric ratios, both characteristic and mantissa, except that from the logarithms of sines and cosines, and from the logarithms of tangents of angles less than 45° and of cotangents of angles greater than 45°, the term -10 is usually omitted. The arrangement of the table will be readily understood from the following examples.

Example 5. To determine $\log \tan 37° 23' 45''$.

We find $\log \tan 37° 23' = 9.88315 - 10$ and $\log \tan 37° 24' = 9.88341 - 10$, so that a difference of 1 minute in the angle causes a difference of 26 in the logarithm of its tangent. Accordingly a difference of 45″ in the angle will cause a difference of 20 in the logarithm of the tangent. Hence $\log \tan 37° 23' 45'' = 9.88335 - 10$.

Example 6. To determine log cos 54° 29′ 13″.

We find log cos 54° 29′ = 9.76413 − 10, while log cos 54° **30′** = 9.76395 − 10, so that as the angle *increases* by 1 minute, the logarithm of its cosine *decreases* by 18. Consequently for an increase of 13″ in the angle there will be a decrease of $\frac{13}{60}$ of 18, i.e., of 4 in the logarithm of the cosine; hence log cos 54° 29′ 13″ = 9.76409 − 10.

Example 7. Given log sin θ = 9.47468 − 10. To determine θ

We find that log sin 17° 21′ = 9.47452 − 10, while log sin 17° **22′** = 9.47492 − 10, so that a difference of 40 in the logarithm of the sine corresponds to a difference of 1 minute in the angle. Consequently, a difference of 16 in the logarithm of the sine corresponds to a difference of $\frac{16}{40}$ of a minute, i.e., of 24″ in the angle. We conclude that θ = 17° 21′ 24″.

37. Exercises.

Determine:

1. log 56.387.
2. log .084923.
3. log 1.0576.
4. log .20458.
5. log 978.94.
6. log .00073299.
7. log sin 27° 15′ 20″.
8. log tan 68° 37′ 35″.
9. log cos 47° 58′ 15″.
10. log cot 15° 47′ 50″.
11. log sin 78° 29′ 40″.
12. log cos 36° 35′ 45″.

Determine the number N, when:

13. log N = 1.65783.
14. log N = 9.04987 − 10.
15. log N = .27586.
16. log N = .09675.
17. log N = 7.80880 − 10.
18. log N = 3.97538.

Determine the angle θ, when:

19. log tan θ = .27725.
20. log cos θ = 9.88247 − 10.
21. log sin θ = 9.48030 − 10.
22. log cot θ = 8.83225 − 10.
23. log cos θ = 9.68552 − 10.
24. log tan θ = 9.96795 − 10.

38. Calculation by means of logarithms. We are now prepared to apply the results of the preceding articles. In all calculations, it is important to arrange the work in such a way as to secure the greatest possible accuracy with least effort. This is accomplished best by making a plan of the entire calculation before looking up the logarithms, as illustrated in the following examples:

Example 1. To determine $N = \sqrt[3]{\dfrac{47.321 \times .015732}{.9763}}$.

First we apply Theorems I, II and III; in this way we find that $\log N = \frac{1}{3}$ [log 47.321 + log .015732 − log .9763].

Accordingly we make the following plan for the calculation:

$$
\begin{array}{rl}
\log 47.321 = & 1\ldots\ldots \\
\log .015732 = & 8\ldots\ldots -10 \\
& \overline{}\ A \\
\log .9763 = & 9\ldots\ldots -10 \\
& \overline{}\ S \\
& \ldots\ldots\ldots\ldots \\
3 & \overline{} \\
\log N = & \ldots\ldots\ldots\ldots \\
N = & \ldots\ldots\ldots\ldots
\end{array}
$$

After having completed the plan, we turn to the tables to determine the mantissas and to complete the calculation; this gives the following results:

$$
\begin{array}{rl}
\log 47.321 = & 1.67505 \\
\log .015732 = & 8.19678 - 10 \\
& \overline{}\ A \\
& 19.87183 - 20 \\
\log .9763 = & 9.98958 - 10 \\
& \overline{}\ S \\
& 29.88225 - 30 \\
3 & \overline{} \\
\log N = & 9.96075 - 10 \\
N = & .91358.
\end{array}
$$

In order to make the subtraction possible without introducing a negative mantissa, we wrote the characteristic of the minuend in the form 19 − 20 instead of 9 − 10; to make the division possible without introducing a negative fraction, we wrote the mantissa of the dividend in the form 29 − 30, instead of 9 − 10.

Example 2. To determine $N = \left[\dfrac{356.12 \times (.56836)^2}{51.834 \times \sqrt{.0843}}\right]^5$.

We find that $\log N = 5\,[\log 356.12 + 2 \log .56836 - (\log 51.834 + \frac{1}{3} \log .0843)]$, so that the calculation is carried out in the following form:

$$\begin{array}{rr}
 & \log 356.12 = 2.55159 \\
\log .56836 = 9.75462 - 10; & 2 \log .56836 = 19.50924-20 \\
 & \overline{}\ A \\
 & 2.06083 \\
\log 51.834 = 1.71461 & \\
\log .0843 = 28.92583-30; \quad \frac{1}{3} \log .0843 = 9.64194 - 10 & \\
\overline{}\ A & \\
1.35655 \ldots\ldots & 1.35655 \\
 & \overline{}\ S \\
 & .70428 \\
 & \overline{}\ 5 \\
 & \log N = 3.52140 \\
 & N = 3322.0
\end{array}$$

Example 3. To determine $N = \dfrac{.075869 \times \sin 47^\circ\, 15'\, 36''}{\tan 68^\circ\, 23'}$.

$$\log N = \log .075869 + \log \sin 47^\circ\, 15'\, 36'' - \log \tan 68^\circ\, 23'.$$

$$\begin{array}{rl}
\log .075869 = & 8.88006 - 10 \\
\log \sin 47^\circ\, 15'\, 36'' = & 9.86596 - 10 \\
 & \overline{}\ A \\
 & 8.74602 - 10 \\
\log \tan 68^\circ\, 23' = & .40201 \\
 & \overline{}\ S \\
\log N = & 8.34401 - 10 \\
N = & .022081
\end{array}$$

39. Exercises.

Calculate:

1. $\sqrt{\dfrac{9^3 \sqrt{2}}{105}}$.

2. $\sqrt[3]{\dfrac{4^5 \sqrt{15}}{507}}$.

3. $\dfrac{4.357 \times (.08356)^3}{\sqrt[4]{.057}}$.

4. $\dfrac{(.4316)^{\frac{1}{3}} \times 52.07}{(.083)^2}$.

5. $\sqrt[5]{\dfrac{54.7 \times .0287}{15.8}}$.

6. $\sqrt[4]{\dfrac{14.03 \times 1.028}{(.005734)^{\frac{3}{5}}}}$.

7. $\dfrac{4.3857 \times .1296}{.00097435}$.

8. $\sqrt{\dfrac{6.7248 \times .098376}{57.849 \times .0001574}}$.

9. $\dfrac{51.86 \times \sin 35^\circ\, 18'}{\sin 49^\circ\, 23'}$.

10. $\tan 28^\circ\, 37'\, 15'' \times \cos 56^\circ\, 13' \times \operatorname{cosec} 75^\circ\, 20'\, 20''$.

11. $9.63 \times \sin 42^\circ\ 17' \times \sin\ 18^\circ\ 29'\ 30''$.

12. $\dfrac{.1875 \times \tan 65^\circ\ 34'\ 10''}{\sin 17^\circ\ 54'\ 20''}$.

13. $\sqrt{\dfrac{4.268 \times \sin 81^\circ 20'}{\tan 14^\circ\ 38'}}$.

14. $\left[\dfrac{78.643 \times \cos 21^\circ\ 17'}{9.5064 \times \cot 68^\circ\ 23'}\right]^3$.

15. $\dfrac{1.0567 \times \cos 28^\circ\ 43'\ 50''}{2.3981}$.

16. $\sqrt{\dfrac{57.8 \times 67.845 \times 23.593}{98.627}}$.

17. $\sqrt{\dfrac{.0024856 \times .57321 \times .009847}{\sin 9^\circ\ 15'\ 15''}}$.

18. $\sqrt{\dfrac{.08734 \times .09586 \times .06792}{.15647}}$.

19. $\dfrac{.5487 \times \sin 38^\circ\ 27'\ 15''}{.98346}$.

20. $\dfrac{3.75 \times \sin 53^\circ\ 27'\ 30''}{\sin 145^\circ\ 35'\ 40''}$.

CHAPTER IV

THE SOLUTION OF RIGHT TRIANGLES. APPLICATIONS

40. The right triangle. If enough elements of a right triangle are given to determine it completely, every other magnitude connected with the triangle can be determined by means of the trigonometric ratios of the angles. A right triangle is completely determined by any one of the following sets of data:

(*a*) two legs, (*b*) one leg and the hypotenuse, (*c*) one acute angle and a leg, (*d*) one acute angle and the hypotenuse.

In the case of the right triangle the remaining elements can be determined in each case by the use of the special form of the definitions for acute angles, given in **20.** It is customary to denote the lengths of the sides of a triangle by small Roman letters, the vertices of the opposite angles by the corresponding Roman capitals, the vertex of the right angle and the hypotenuse in a right triangle usually being denoted by C and c, respectively.

In cases (*a*) and (*b*) the knowledge of the sides enables us to find one of the ratios of either of the acute angles, and hence these angles themselves. In cases (*c*) and (*d*) we proceed in the opposite manner: from the given side and appropriate trigonometric ratios of the given angle the other sides are found. The second acute angle is the complement of the given angle.

From the definitions given in **20** we obtain immediately the following theorem:

THEOREM I. **In a right triangle, the following relations hold between the sides and angles:**

the side opposite an acute angle = hypotenuse × the sine of the angle;

the side adjacent to an acute angle = hypotenuse × the cosine of the angle;

the side opposite an acute angle = the adjacent side × the tangent of the angle.

41. Example (see Fig 14).

Given. $\angle C = 90°$, $\angle A = 47° 13'$, $b = 14.35$.

Required. B, a, c.

Solution. We find $\angle B = 90° - 47° 13' = 42° 47'$.

Moreover, we have by Theorem I,

$$a = b \tan A;$$

also, $$b = c \cos A,$$

whence $$c = b/\cos A.$$

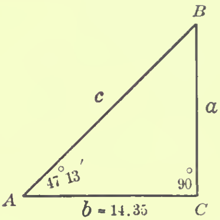


Fig. 14

The unknown elements of the triangle have now been expressed in terms of the known elements. It remains to calculate a and c. This may be done by the use of logarithms (see **38**) as follows:

$$\log b = \log 14.35 = 1.15685; \quad \log b = \log 14.35 = 1.15685$$

$$\log \tan A = \log \tan 47° 13' = .03364;$$

$$\log \cos A = \log \cos 47° 13' = 9.83202 - 10$$

$$\text{———— } A \qquad \text{———— } S$$

$$\log a = 1.19049 \qquad \log c = 1.32483$$

$$a = 15.506 \qquad c = 21.127$$

Using tables giving the values of the ratios themselves (the so-called natural values) instead of those of their logarithms, we obtain:

$$a = 14.35 \times \tan 47° 13' = 14.35 \times 1.0805 = 15.505.$$

$$b = 14.35 \div \cos 47° 13' = 14.35 \div .6792 = 21.128.$$

42. Accuracy of the calculation. Checking the results. The values of the logarithms and of the trigonometric ratios found in a table are correct only to within the limit of accuracy of the table, i.e., to within .000005 if 5-place tables are used, or .00005 in the case of 4-place tables.

Therefore, since the sum and difference of two approximate numbers are more readily determined and are frequently more nearly accurate than their product or quotient, it follows that in most cases the calculation by means of logarithms is to be preferred, particularly if the data of the problem are themselves approximations.*

* A comparison of the advantages of calculations with and without the use of logarithms requires a much more detailed discussion than can here be devoted to it. Most of the calculations in this book have been made by means of logarithms, on account of the greater convenience of this method.

To secure a higher degree of certainty as to the correctness of the final numerical results of a calculation, these results should be checked. A rough check can be obtained by drawing the figure to scale by use of ruler and compass and then measuring the required elements. A more accurate, numerical check involves the testing of the results, together with the data, in other relations between the sides and angles of the triangle than those used in the solution of the problem.

For the right triangle, the Pythagorean relation $c^2 = a^2 + b^2$ serves the purpose especially well, since it involves all three sides of the triangle. If written in the form

$$a^2 = c^2 - b^2 = (c - b)(c + b) \text{ or } b^2 = c^2 - a^2 = (c - a)(c + a),$$

it is well adapted to logarithmic calculation. For the example of the preceding section, it furnishes the following check:

$$\begin{aligned} c &= 21.127 \\ a &= 15.506 \\ \hline c + a &= 36.633 \qquad \log (c + a) = 1.56388 \\ c - a &= 5.621 \qquad \log (c - a) = .74981 \end{aligned}$$

$$\begin{aligned} \log (c + a)(c - a) &= 2.31369 \quad A \\ 2 \,|\!\!\underline{} \\ \log b &= 1.15685 \end{aligned}$$

This value should agree to within four units of the last decimal place with the value of log b derived from the value of b as given.

43. Isosceles triangles. An isosceles triangle is divided into two congruent right triangles by a perpendicular from the vertex to the base. The methods explained in **40, 41** and **42** suffice therefore for the treatment of such triangles.

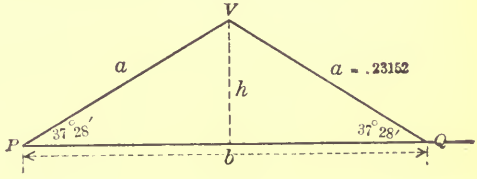


Fig. 15

Example (see Fig. 15).

Given. $\angle P = \angle Q = 37° 28'$, $a = .23152$.

Required. V, h, b.

Solution. We find $\angle V/2 = 90° - 37° 28' = 52° 32'$; $\therefore \angle V = 105° 4'$.

Furthermore

$$\sin P = h/a, \quad \text{i.e.,} \quad h = a \sin P,$$

and $$\cos P = \tfrac{1}{2} b/a, \quad \text{i.e.,} \quad \tfrac{1}{2} b = a \cos P,$$

whence $$b = 2a \cos P.$$

$$\begin{aligned} \log a &= 9.36459 - 10 \\ \log \sin P &= 9.78412 - 10 \quad A \\ \hline \log h &= 9.14871 - 10 \\ h &= .14087 \end{aligned}
\qquad
\begin{aligned} \log a &= 9.36459 - 10 \\ \log \cos P &= 9.89966 - 10 \\ \log 2 &= .30103 \quad A \\ \hline \log b &= 9.56528 - 10 \\ b &= .36752 \end{aligned}$$

Check. $h^2 = a^2 - (b/2)^2 = (a + b/2)(a - b/2)$

$$\begin{aligned} a &= .23152 \\ b/2 &= .18376 \end{aligned}$$

$$\begin{aligned} a + b/2 &= .41528; \quad \log (a + b/2) = 9.61834 - 10 \\ a - b/2 &= .04776; \quad \log (a - b/2) = 8.67906 - 10 \quad A \\ &\log [a^2 - (b/2)^2] = 8.29740 - 10 \\ &\quad 2 \underline{\qquad\qquad} \\ &\log h = 9.14870 - 10 \end{aligned}$$

44. Exercises.

Calculate the unknown elements of the triangles ABC in which $\angle C = 90°$ and in which the following elements are given; check the results:

1. $a = 373$, $b = 526$.

2. $a = .1432$, $b = .0756$.

3. $a = 2.146$, $c = 4.292$.

4. $b = 13.071$, $c = 19.603$.

5. $A = 68° 25'$, $b = 8732$.

6. $B = 27° 13'$, $b = .06315$.

7. $A = 84° 35'$, $c = 378$.

8. $A = 44° 35'$, $c = 378$.

9. $A = 45° 3'$, $a = .08512$.

10. $a = .06891$, $c = .09004$.

11. $b = 13.683$, $a = 3.9857$.

12. $B = 5° 2'$, $c = 1.0059$.

Determine the remaining elements of the following isosceles triangles (the notation being in accordance with Fig. 15) and check the results:

13. $b = 26.804$, $P = 57° 13'$.

14. $b = 35.96$, $V = 128° 46'$.

15. $a = 12.05$, $h = 8.041$.

16. $h = 1.0203$, $V = 44° 52'$.

17. $b = 24.192$, $h = 16.387$.

18. $h = .05831$, $P = 10° 19'$.

19. $b = 9.0834$, $a = 9.9457$.

20. $a = 6.8032$, $P = 25° 27'$.

21. $a = 16584$, $V = 90°$.

45. Projection. The projection of a directed segment AB of a directed line l upon another directed line m, which makes with l an angle θ can now be readily determined. Let the length of AB (see Fig. 16) be equal to a; then, since the direction of AB

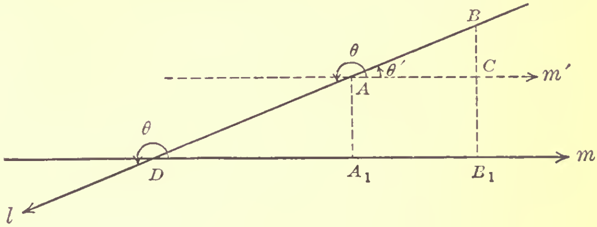


FIG. 16

is opposite to the positive direction upon l, $AB = -a$. The projection of AB upon m is A_1B_1. Through A draw a line m' parallel to m, cutting BB_1 in C. Then $AC = A_1B_1$ (Why?), and $\angle CAD = \theta$ (Why?). Moreover, $\cos\theta = -\cos\theta'$ (see **18**). Since now A_1B_1 is in the positive direction upon m, we have

$$A_1B_1 = AC = a\cos\theta' = -a\cos\theta = AB\cos\theta.$$

This result finds expression in the following theorem:

THEOREM II. **The projection of a directed segment AB of a directed line l, upon a directed line m, is equal, in magnitude and direction, to $AB\cos\theta$, where θ is the angle which l makes with m.**

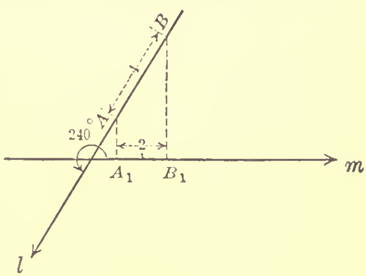


FIG. 17

This theorem is illustrated by the following examples:

(*a*) In Figure 17, l makes with m an angle of 240°; since AB is a negative segment of the directed line l, we have $AB = -4$. Hence, $A_1B_1 = \text{Proj}_m AB = -4 \cdot \cos 240° = +2$.

(b) In Figure 18, l makes with m an angle of 135° and $AB = 3$. Hence $A_1B_1 = \text{Proj}_m AB = 3 \cdot \cos 135° = \dfrac{-3\sqrt{2}}{2} = -2.1$.

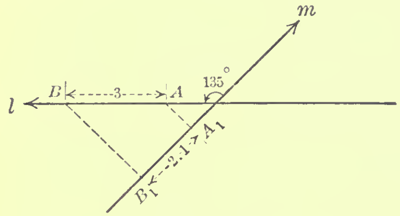


FIG. 18

46. Application of Theorem II. We return now to Theorem II of **6** and calculate the projections of the segments by means of Theorem II of the present chapter. Consider, for example, the equilateral triangle ABC of Figure 19, of which the side AC

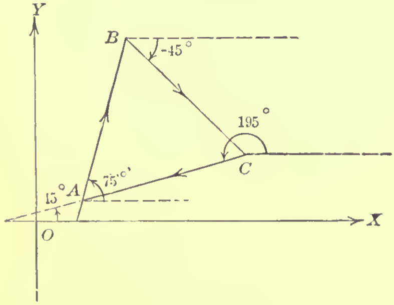


FIG. 19

makes an angle of 15° with the positive X-axis.

The side AB then makes with OX an angle of 75°,
the side BC makes with OX an angle of −45°,
the side CA makes with OX an angle of 195°.

If the length of the side of the triangle be denoted by a and the projections of the vertices upon the X-axis by A_1, B_1, and C_1, then we find:

$$\begin{aligned} A_1B_1 + B_1C_1 + C_1A_1 &= \text{Proj}_X AB + \text{Proj}_X BC + \text{Proj}_X CA \\ &= a\,(\cos 75° + \cos 45° - \cos 15°) \\ &= a\,(.2588 + .7071 - .9659) = 0. \end{aligned}$$

47. Exercises.

1. Determine the projection upon the X-axis of a segment, 5 feet long, of a line which makes with the X-axis an angle of 30°.

2. Determine the projection of the same line upon the Y-axis.

3. Determine the projections upon the X- and Y-axes of a segment, 7 feet long, of a line parallel to the Y-axis.

4. Show that the sum of the projections upon the X-axis of the sides of the square of Figure 20 is equal to zero. Show that the same result holds for the projections upon the Y-axis.

5. (See Fig. 21.) Show that the projection of AC upon the X-axis (Y-axis) is equal to the sum of the projections of AB and BC upon the X-axis (Y-axis).

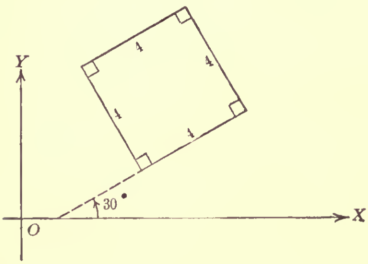


Fig. 20

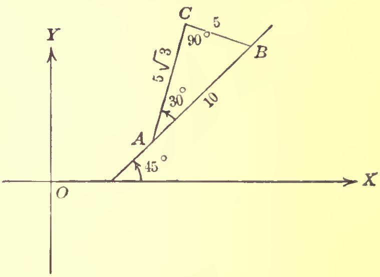


Fig. 21

48. Applications. Numerous problems in different fields of science can be solved by means of the methods developed in the preceding paragraphs. To solve such problems there is required, however, besides a knowledge of trigonometry, an understanding of the technical terms used in those different fields. We explain below a few technical terms which will be used in the exercises of the following paragraph and in Chapter VII. For further applications the student is referred to books on surveying, navigation, astronomy, artillery, etc.

The angle made by the line along which an object is sighted with a horizontal line through the point of observation and in the same vertical plane as the line of sight, is called the **angle of elevation** or the **angle of depression** of the object, according as the object is higher or lower than the point of observation.

The surveyor frequently designates the direction of a line by means of its **bearing**, i.e., the acute angle which the line makes with a N. S. line through the point of observation, indicating at the same time whether the line runs to the east or the west of the

N. S. line. The bearings of the lines OA, OB and OC in Figure 22 are denoted as N 23° E (read 23 degrees East of North),

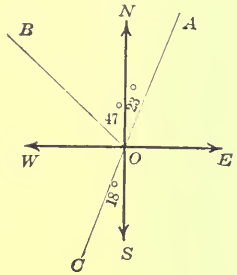


FIG. 22

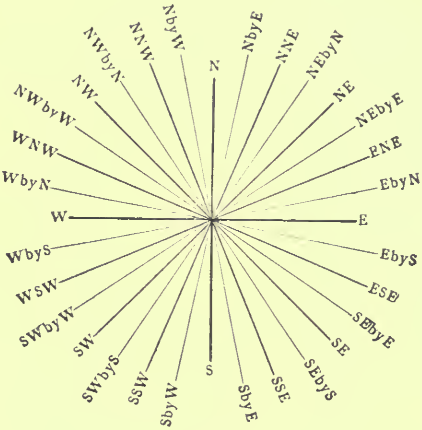


FIG. 23

N 47° W (read 47 degrees West of North) and S 18° W (read 18 degrees West of South) respectively.

The points of the compass, as used by the navigator, are indicated on Figure 23.

49. Exercises.

1. The angle of elevation of the top of a mountain from a point A, situated in a plane 1500 feet below the top, is 19° 27′. Determine the distance from A to the top in an air line; also the horizontal distance from A to the foot of the mountain.

2. From a lightship L, at a distance of 500 feet from a point A on shore, the angle of elevation of a water tower vertically above A is 28° 33′. Determine the height of the tower above the level of the ship.

3. The angle of depression of a point P as seen from an airship 1800 feet above the ground is 63° 27′. What is the straight line distance of the airship from P?

4. An observation tower T is 40 feet high and stands 20 feet from the edge of a vertical cliff, whose top is 400 feet above sea level. A ship S is in the vertical plane through T and on a line at right angles to the shore line; its angle of depression from T is 9° 23′ 16″. How far is S from the shore?

5. From a point A on the edge of a stream and 5 feet above the ground, the angle of elevation of the top of a tree straight across the stream is 20° 13′. The height of the tree above the ground is estimated to be 15 feet. How wide is the stream at that point?

6. A vessel is observed directly south from a lighthouse L and S 43° 27′ W from a lighthouse M known to be 50 miles due East from L. What are the distances of the vessel from each of the lighthouses?

7. From points P and Q, 150 feet apart and both lying in the same vertical plane through a spire S, the angles of elevation of the spire are observed to be 12° 13′ and 28° 36′ respectively. How high is the spire if P and Q are on the same side of the spire; if P and Q are on opposite sides of the spire?

8. Two points, P and Q, known to be a mile apart on a level road which lies in a vertical plane through the top of a mountain A are observed from A. The angles of depression of P and Q are 23° 17′ and 32° 27′ respectively. How high is the mountain top above the road?

9. The bearing of the line CA is N 12° E; the bearing of the line CB is S 78° E, while A bears N 15° W from B. The distance CB is known to be 257 feet. Determine the distances from A to B and to C.

10. From a point B on one side of a stream and 5 feet above the ground, the angle of elevation of a point P directly across on the opposite shore is found to be 34° 13′. A level line BC, 100 feet long is laid off at right angles to the stream, and from C the angle of elevation of P is found to be 20° 43′. Determine the height of P and the width of the stream.

11. A railroad track consists of a horizontal piece followed by a downward grade 4 miles long, making an angle of 12° with the horizontal. It is proposed to replace the 12° grade by a 4° grade. How much of the horizontal track must be removed to accomplish this change?

12. How much track mileage would be saved by the change proposed in Problem 11?

13. The Washington monument is 555 feet high. From a point P on a level with the base, the angle of elevation of the top is 60°. How far is P from the bottom of the monument? How far is P from the top?

14. Raising the ridgepole of a roof 3 feet, changes the angle under which the rafters slope from 32° to 40°. How high was the roof originally, and how long are the tie beams?

15. A lightship is observed due West at 10 A.M. and N40°W at 2 P.M. by a vessel that is traveling due South at 20 miles per hour. How far is the vessel from the lightship at 10 A.M., at noon, at 2 P.M.?

CHAPTER V

THE GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

50. Graphs of sin θ and cos θ. Many properties of the trigonometric functions appear in a new light if we consider their graphical representation. In this chapter we shall be concerned principally with the graphical representations (usually called the **graphs**) of the functions $\sin\theta$ and $\cos\theta$ and of the sine and cosine functions of expressions of the form $a\theta + b$, i.e., of $\sin(a\theta + b)$ and $\cos(a\theta + b)$. For their construction we make use of the graphical representation of pairs of numbers used in Chapter I.

Angles θ measured by an arbitrary unit will be represented by distances along the X-axis (abscissae); with each abscissa will be associated as ordinate a line representing the value of that function of the angle θ whose graph we wish to obtain. The determination of these ordinates can be carried out graphically by the following simple device.

After having established the unit of measurement on the X- and Y-axis, we describe a circle whose radius is equal to this

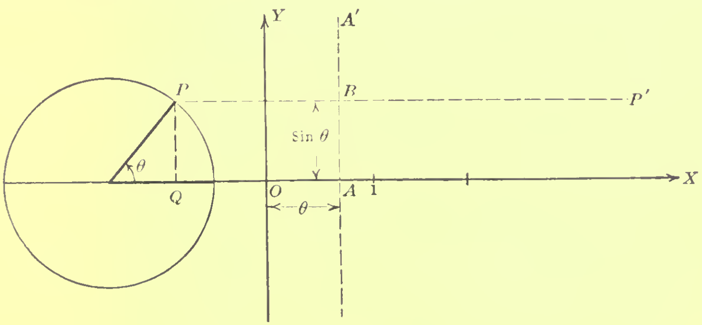


FIG. 24

unit and whose center is on the X-axis (see Fig. 24). At the center of this circle we construct an angle θ whose initial side lies in the positive direction along the X-axis. From the point P, where the terminal side of θ meets the circle, we drop a perpendicular PQ upon the X-axis. Since for P the radius vector is equal to unity, the *sine of angle θ is measured in magnitude and direction by the ordinate of P, i.e., by the line QP.* Through P we draw a line PP', parallel to the X-axis. We then lay off on the X-axis a distance OA, representing the angle θ in magnitude and in sense, through A we draw a line AA' parallel to the Y-axis, meeting the line PP' in B; this point B is then a point on the graph of sin θ.

If the angle θ be laid off with its initial side in the positive direction along the Y-axis and on the vertical diameter of the

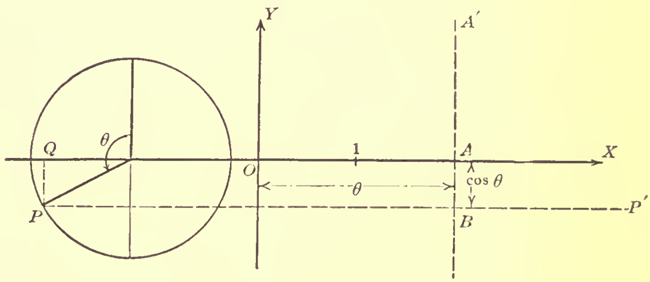


Fig. 25

circle (see Fig. 25), we obtain an *ordinate QP which measures in magnitude and direction the cosine of θ.*

51. Examples of graphs.

(*a*) *Construct the graph of sin θ for θ varying from* 0° *to* 360° (see Fig. 26).

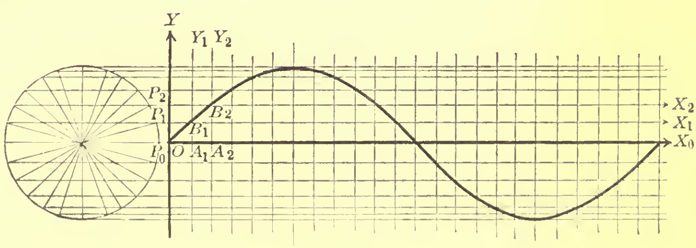


Fig. 26

At the center of the auxiliary circle we lay off angles of 0°, 15°, 30°, . . . 330°, 345°, 360°, with initial sides in the positive direction along the X-axis and with terminal sides cutting the circle in points P_0, P_1, P_2, etc. On the X-axis we lay off distances $OA_0 = O$, $OA_1 = \pi/12$, $OA_2 = \pi/6$, etc., representing the radian measurements of these angles. Through P_0, P_1, P_2, etc. we draw lines P_0X_0, P_1X_1, P_2X_2, etc. parallel to the X-axis; through A_0, A_1, A_2, etc. we draw lines A_0Y_0, A_1Y_1, A_2Y_2, etc. parallel to the Y-axis. The intersections B_0, B_1, B_2, etc. of the lines A_0Y_0 and P_0X_0, A_1Y_1 and P_1X_1, A_2Y_2 and P_2X_2, etc. respectively, are points whose x-coördinates measure angles and whose y-coördinates measure the sines of these angles. A smooth curve drawn through the points B_0, B_1, B_2, etc. is the graph of the function $\sin\theta$ for θ varying from 0° to 360°.

(*b*) *Construct the graph of* $\cos\theta$ *for* θ *varying from* 0° *to* 360° (see Fig. 27).

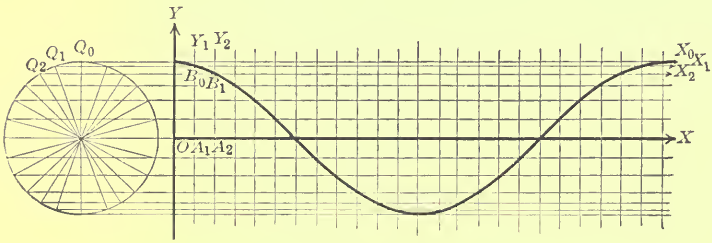


FIG. 27

We construct at the center of the auxiliary circle angles of 0°, 15°, 30°, . . . 345°, 360° with initial sides in the positive direction along the *Y-axis* and with terminal sides cutting the circle in points Q_0, Q_1, Q_2, etc. On the X-axis we determine the points A_0 A_1, A_2, etc. as in example (*a*). Lines Q_0X_0, Q_1X_1, Q_2X_2, etc. parallel to the X-axis will meet lines A_0Y_0, A_1Y_1, A_2Y_2, etc. parallel to the Y-axis in points B_0, B_1, B_2, etc. whose x-coördinates measure angles and whose y-coördinates measure the cosines of these angles. A smooth curve drawn through these points B_0, B_1, B_2, etc. is the graph of the function $\cos\theta$ for θ varying from 0° to 360°.

(*c*) *Construct the graph of* $\sin(2\theta - 30°)$ *for* θ *varying from* 0° *to* 360°. We construct as in example (*a*) the lines P_0X_0, P_1X_1, P_2X_2, etc. and A_0Y_0 A_1Y_1, A_2Y_2, etc. By means of these lines we determine a point B_0, whose x-coördinate measures an angle of 0° and whose y-coördinate measures the sine of $2 \cdot 0° - 30°$, i.e., the sine of $-30°$; a point B_1, whose x-coördinate measures an angle of 15°, and whose y-coördinate measures the sine of $2 \cdot 15° - 30°$, i.e., the sine of 0°; a point B_2, whose x-coördinate measures an angle of 30° and whose y-coördinate measures the sine of $2 \cdot 30° - 30°$, i.e., the sine of 30°, etc. A smooth curve through the points B_0, B_1, B_2, etc. is the required graph.

(*d*) *Construct the graph of* $\cos(\theta/3 + 45°)$ *for* θ *varying from* 0° *to* 360° (see Fig. 28).

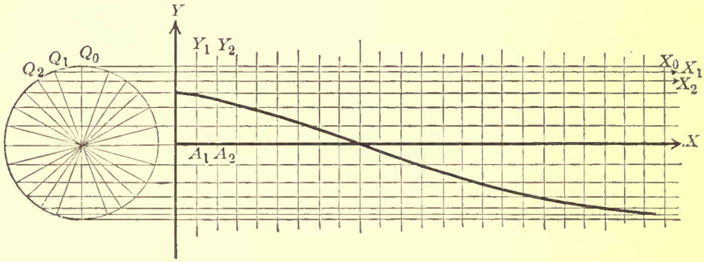


FIG. 28

We construct, as in example (*b*) the lines Q_0X_0, Q_1X_1, Q_2X_2, etc. and A_0Y_0, A_1Y_1, A_2Y_2, etc. By means of these lines we determine a point B_0, whose x-coördinate measures an angle of 0° and whose y-coördinate measures the cosine of $0°/3 + 45°$, i.e., cos 45°; a point B_1 whose x-coördinate measures an angle of 15° and whose y-coördinate measures the cosine of $15°/3 + 45°$, i.e., cos 50°; a point B_2, whose x-coördinate measures an angle of 30°, and whose y-coördinate measures the cosine of $30°/3 + 45°$, i.e., cos 55°, etc. A smooth curve drawn through the points B_0, B_1, B_2, etc. is the required graph.

52. Exercises.

Construct the graphs of the following functions, for θ varying from 0° to 360°:

1. $\cos 2\theta$.
2. $\cos \theta/2$.
3. $\sin 2\theta$.
4. $\sin \theta/2$.
5. $2\cos\theta$.
6. $2\sin\theta$.
7. $\frac{1}{2}\sin\theta$.
8. $\frac{1}{2}\cos\theta$.
9. $\sin 3\theta$.
10. $3\sin\theta$.
11. $3\cos\theta/3$.
12. $\cos 3\theta$.
13. $\frac{1}{3}\cos 3\theta$.
14. $3\sin\theta/3$.
15. $\frac{1}{3}\sin 3\theta$.
16. $\frac{1}{3}\cos 3\theta$.
17. $\cos(-\theta)$.
18. $\sin(\theta + 90°)$.
19. $\cos(\theta - 180°)$.
20. $\cos(\theta + 180°)$.
21. $\cos(\theta - 90°)$.
22. $\sin(\theta - 90°)$.
23. $\cos(\theta + 270°)$.
24. $\sin(\theta - 270°)$.

53. Operations on graphs. The graph of the function $\sin 3\theta$ associates with any particular value of θ the same ordinate that the graph of $\sin \theta$ associates with 3θ, a point three times as far from the origin as θ. Hence to every point A on the graph of the function $\sin \theta$ there corresponds a point B on the graph of the function $\sin 3\theta$, three times as near to the Y-axis as the point A. It follows from this that the graph of the function $\sin 3\theta$ may be obtained by contracting the graph of the function $\sin \theta$ in the direction of the X-axis in the ratio $3:1$. If we apply these same considerations to the general case, we obtain the following theorem:

THEOREM I. **The graphs of the functions $\sin a\theta$, $\cos a\theta$ may be obtained by contracting the graphs of the functions $\sin \theta$ and $\cos \theta$ respectively towards the Y-axis in the ratio $a:1$, if a is a positive rational number.**

Here it is to be understood that if $a < 1$, the contraction becomes an enlargement in the ratio $1 : 1/a$.

The graph of $\sin(-\theta)$ associates with each value of θ the same ordinate that the graph of $\sin \theta$ associates with $-\theta$. Hence to every point A on the graph of $\sin \theta$ there corresponds a point B on the graph of $\sin(-\theta)$ obtained by reflecting A in the Y-axis as a mirror.* This leads to the following theorem:

THEOREM II. **The graphs of the functions $\sin(-\theta)$, $\cos(-\theta)$ may be obtained by reflecting the g aphs of the functions $\sin \theta$ and $\cos \theta$ respectively in the Y-axis as a mirror.**

The graph of the function $\sin(2\theta + 180°)$ associates with any particular value of θ the same ordinate that the graph of $\sin 2\theta$ associates with $\theta + 90°$, a point on the X-axis 90° farther to the right than θ. Hence, to every point A on the graph of $\sin 2\theta$ there will correspond a point B on the graph of $\sin(2\theta + 180°)$, a distance of 90° to the left of A. It follows from this that the graph of $\sin(2\theta + 180°)$ can be obtained by translating the graph of $\sin 2\theta$ in the direction of the negative X-axis a distance of 90°. In the same way, we see that the graph of $\sin(2\theta - 180°)$ may be obtained by translating the graph of $\sin 2\theta$ in the direction of the positive X-axis a distance of 90°.

* B is said to be the reflection of A in the Y-axis as a mirror if the Y-axis is the perpendicular bisector of the line AB.

If we carry through the same considerations for the general case, we obtain the following theorem:

THEOREM III. **The graphs of the functions sin $(a\theta + b)$, cos $(a\theta + b)$ may be obtained by translating the graphs of sin $a\theta$ and cos $a\theta$ respectively a distance of $|b/a|$,* in the direction of the negative X-axis, if b/a is positive; in the direction of the positive X-axis, if b/a is negative.**

Theorems I, II and III may be used to obtain the graphs of the functions sin $(a\theta + b)$ and cos $(a\theta + b)$, where a and b are arbitrary rational numbers by translating, reflecting and contracting (enlarging) the graphs of the functions sin θ and cos θ respectively. This will now be illustrated by examples.

(1) *To obtain the graph of the function sin* $(3\,\theta + 30°)$, we first contract the graph of the function sin θ horizontally in the ratio 3 : 1, obtaining in this way the graph of the function sin $3\,\theta$ (see Fig. 29), in virtue of Theorem I.

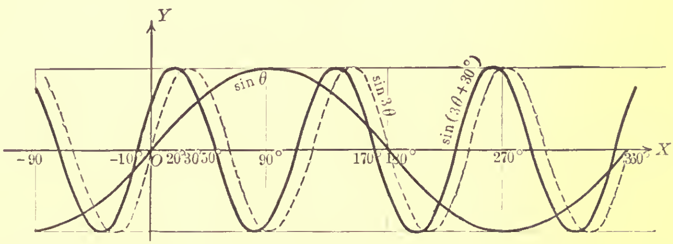


FIG. 29

We now translate the graph of sin $3\,\theta$ a distance of 10° to the left and obtain in this way, in virtue of Theorem III, the graph of sin $(3\,[\theta + 10°]) = \sin(3\,\theta + 30°)$.

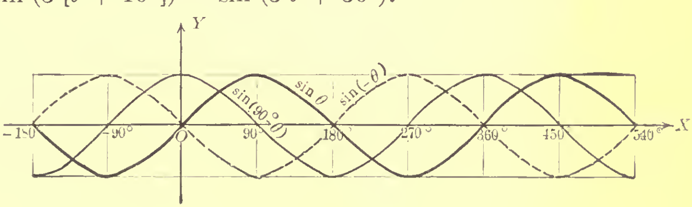


FIG. 30

(2) *To construct the graph of the function sin* $(90° - \theta)$, we reflect the graph of sin θ in the Y-axis as a mirror, which gives us the graph of sin $(-\theta)$, in virtue of Theorem II (see Fig. 30).

* The symbol $|b/a|$ designates the numerical value of b/a. Thus, if $a = -1$, $b = 90°$, $|b/a| = |-90°| = 90°$; similarly, $|-4| = 4$, $|-\frac{7}{8}| = \frac{7}{8}$.

Translating this latter graph a distance of 90° to the right, we shall obtain, in virtue of Theorem III, the graph of $\sin(-[\theta - 90°])$ $= \sin(90° - \theta)$.

54. Exercises.

Construct the graphs of the following functions:

1. $\cos(90° + \theta)$.	**7.** $\cos(\theta + 180°)$.	**13.** $\cos(3\theta - 60°)$.
2. $\sin(90° - \theta)$.	**8.** $\cos(180° - \theta)$.	**14.** $\sin(\theta - 180°)$.
3. $\cos(90° - \theta)$.	**9.** $\sin(270° + \theta)$.	**15.** $\sin(3\theta + 45°)$.
4. $\sin(\theta - 90°)$.	**10.** $\sin(270° - \theta)$.	**16.** $\cos(90° - 2\theta)$.
5. $\sin(90° + \theta)$.	**11.** $\sin(2\theta + 60°)$.	**17.** $\sin(60° + 3\theta)$.
6. $\cos(\theta - 90°)$.	**12.** $\cos(2\theta - 90°)$.	**18.** $\cos(180° - 4\theta)$.

55. Applications of graphs. The graphs of the trigonometric functions $\sin(a\theta + b)$ and $\cos(a\theta + b)$, which we have learned to construct in the foregoing paragraphs, will now be used to illustrate and verify some of the important properties of the functions $\sin\theta$ and $\cos\theta$.

(1) The functions $\sin ax$ and $\cos ax$ are periodic functions whose period is equal to $2\pi/|a|$, where a represents any rational number. (See footnote on p. 46 and **26**.)

(2) The graphs of the functions $\sin(-\theta)$ and $\sin\theta$ are symmetric with respect to the X-axis, i.e., for any value of θ, the corresponding values of these functions are equal but opposite in sign. We conclude therefore:

(1) $\quad$ **$\sin(-\theta) = -\sin\theta$,** for every value of θ.

The graphs of the functions $\cos(-\theta)$ and $\cos\theta$ coincide, so that we have:

(2) $\quad$ **$\cos(-\theta) = \cos\theta$,** for every value of θ.

Formula (1) shows an analogy between the sine function and an *odd* power of a variable; for we know that, e.g., $(-\theta)^{15} = -\theta^{15}$. On the other hand formula (2) shows an analogy between the cosine function and an *even* power, for, e.g., $(-\theta)^{16} = \theta^{16}$. On account of this analogy, formulae (1) and (2) are frequently expressed in the form "**the sine is an odd function,**" and "**the cosine is an even function.**"

(3) The graph of the function $\sin(90° - \theta)$ coincides with the graph of $\cos\theta$; from this we conclude:

(3) $\sin(90° - \theta) = \cos\theta$, for every value of θ. (See **19**, Ex. 1.)

Formula (3) as well as most of those which appear in the exercises below can also be obtained by the use of Theorem I in Chapter II. The advantage gained by deriving them by the present method lies in the fact that it emphasizes more sharply the fact that these relations hold for *every* value of the variable, i.e., that they are properties of the trigonometric *functions*.

56. Exercises.

Prove graphically:

1. $\sin(180° - \theta) = \sin\theta$.
2. $\sin(\theta - 180°) = -\sin\theta$.
3. $\sin(\theta + 180°) = -\sin\theta$.
4. $\cos(90° + \theta) = -\sin\theta$.
5. $\cos(\theta - 90°) = \sin\theta$.
6. $\cos(180° + \theta) = -\cos\theta$.
7. $\sin(90° + \theta) = \cos\theta$.
8. $\sin(\theta - 90°) = -\cos\theta$.
9. $\cos(90° - \theta) = \sin\theta$.
10. $\cos(180° - \theta) = -\cos\theta$.
11. $\cos(\theta - 180°) = -\cos\theta$.
12. $\sin(\theta + 270°) = -\cos\theta$.
13. $\cos(\theta + 270°) = \sin\theta$.
14. $\sin(270° - \theta) = -\cos\theta$.
15. $\cos(270° - \theta) = -\sin\theta$.
16. $\cos(\theta - 270°) = -\sin\theta$.
17. $\sin(\theta - 270°) = \cos\theta$.
18. $\sin(\theta - 45°) + \sin(\theta + 45°) = \sqrt{2}\cdot\sin\theta$.
19. $\sin(\theta + 45°) - \sin(\theta - 45°) = \sqrt{2}\cdot\cos\theta$.
20. $\sin\theta + \cos\theta = \sqrt{2}\cdot\sin(\theta + 45°)$.

57. Graphs of $\tan\theta$ and $\cot\theta$. We proceed now to a brief consideration of the graphs of he functions $\tan\theta$ and $\cot\theta$. For this purpose we begin by developing a graphical method for determining an ordinate which measures the tangent of a given angle θ, analogous to the method developed for the sine in **50**.

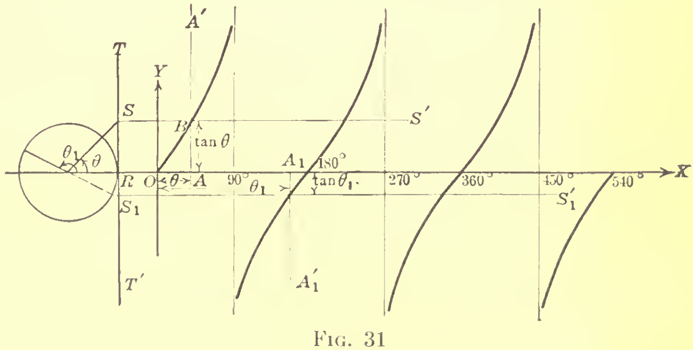


FIG. 31

We return to the auxiliary circle used in **50** and **51** and draw a line TT' tangent to this circle at the right hand extremity of its horizontal diameter (see Fig. 31). At the center of the circle we draw an angle θ whose initial side lies in the positive direction along the X-axis. From the point S, where the terminal side of θ or, for angles in II and III, the terminal side of θ produced through the origin, cuts the tangent line TT', we draw a perpendicular SR upon the X-axis. Since for S the abscissa is equal to unity the tangent of θ is measured by the ordinate of S, i.e., by the line RS. Through S we draw a line SS' parallel to the X-axis. Furthermore, we lay off on the X-axis, as in **50**, a distance OA, representing the angle θ; through A we draw a line AA' parallel to the Y-axis, meeting the line SS' in B; this point B is then a point on the graph of tan θ.

Now, following the general method explained in **51**, we readily construct the graph of the function tan $(a\theta + b)$.

To obtain an ordinate measuring the cotangent of θ, we draw a line UU' tangent to the circle at the left hand extremity of its horizontal diameter. We lay off the angle θ at the center of the circle, with its initial side in the positive direction along the Y-axis, and determine the point S where the terminal side of θ, or the terminal side produced through the origin, meets this tangent line; from S we drop a perpendicular SR upon the X-axis. It is clear (see Fig. 32) that if we consider this position of θ as the

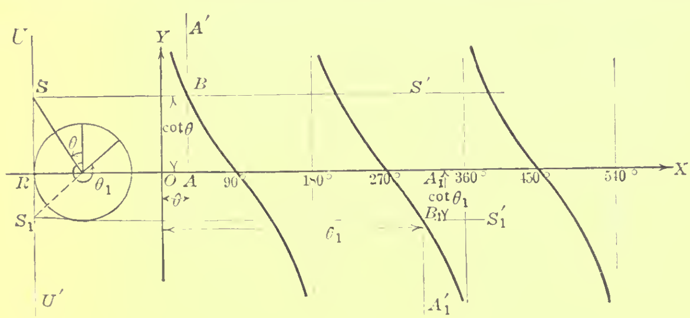


FIG. 32

standard position, the ordinate for the point S is equal to unity, so that the cotangent of θ is measured by the abscissa, i.e., by the line RS, no matter how large the angle θ may be. Through S we draw a line SS' parallel to the X-axis; we lay off, as before, a distance OA on the X-axis, measuring the angle θ, draw the line AA' parallel to the Y-axis and determine the point B, in which the lines SS' and AA' meet. This point B is a point on the graph of $\cot\theta$. The graph of the general function $\cot(a\theta + b)$ is now constructed by the methods used in **51.**

58. Exercises.

Construct the graphs of the following functions, for θ varying from 0° to 360°:

1. $\tan 2\theta$.

2. $\cot \theta/2$.

3. $\tan(\theta + 90°)$.

4. $\cot(\theta - 90°)$.

5. $\tan(\theta + 180°)$.

6. $\cot(\theta + 180°)$.

7. $\tan \theta/3$.

8. $\cot 3\theta$.

9. $\cot(\theta + 270°)$.

10. $\tan(\theta - 270°)$.

11. $\tan(180° - \theta)$.

12. $\cot(180° - \theta)$.

13. $2 \tan\theta$.

14. $3 \tan \theta/3$.

15. $\tan(2\theta + 90°)$.

16. $\cot(2\theta - 180°)$.

17. $\cot(180° + 2\theta)$.

18. $\tan(180° - 2\theta)$.

59. Graphs of $\tan(a\theta + b)$ and $\cot(a\theta + b)$. Mere repetition of the arguments of **53** will enable us at once to establish the following theorems, analogous to Theorems I, II, III of **53.**

THEOREM Ia. **The graphs of the functions $\tan a\theta$, $\cot a\theta$ may be obtained by contracting the graphs of the functions $\tan\theta$ and $\cot\theta$ respectively towards the Y-axis in the ratio $a:1$, if a is a positive rational number; if $a < 1$, the contraction becomes an enlargement in the ratio $1:1/a$.**

THEOREM IIa. **The graphs of the functions $\tan(-\theta)$, $\cot(-\theta)$ may be obtained by reflecting the graphs of the functions $\tan\theta$ and $\cot\theta$ respectively in the Y-axis as a mirror.**

THEOREM IIIa. **The graphs of the functions $\tan(a\theta + b)$, $\cot(a\theta + b)$ may be obtained by translating the graphs of the functions $\tan a\theta$ and $\cot a\theta$ respectively a distance of $|b/a|$, in the direction of the negative X-axis, if b/a is positive; in the direction of the positive X-axis, if b/a is negative.**

These theorems enable us to obtain the graphs of functions of the form $\tan(a\theta + b)$ and $\cot(a\theta + b)$, where a and b are arbitrary rational numbers, by translating, reflecting and contracting (enlarging) the graphs of $\tan\theta$ and $\cot\theta$ respectively.

(*a*) *To construct the graph of tan* $(2\theta + 45°)$, we first contract the graph of $\tan\theta$ horizontally in the ratio $2:1$, so as to obtain the graph of $\tan 2\theta$. Then we translate the latter graph a distance of $22\frac{1}{2}°$ in the direction of the negative X-axis. (See Fig. 33.)

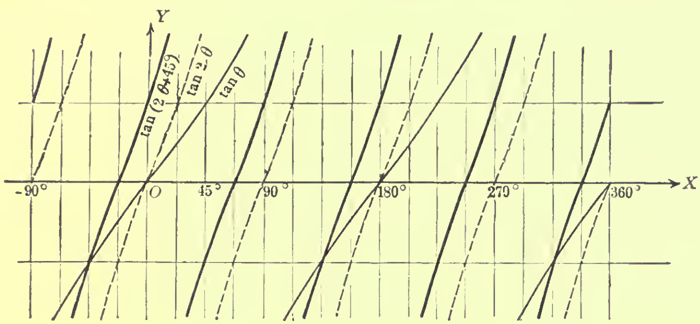


Fig. 33

(b) *To construct the graph of cot* $(180° - 2\theta)$, we reflect the graph of $\cot\theta$ in the Y-axis as a mirror, and contract the latter in the ratio 2 : 1 in the direction of the X-axis, obtaining in this way the graph of $\cot(-2\theta)$. If we now translate this graph a distance of 90° in the direction of the positive X-axis, we obtain the graph of the function $\cot(-2[\theta - 90°]) = \cot(180° - 2\theta)$, which proves to be identical with the graph of $\cot(-2\theta)$. (See Fig. 34.)

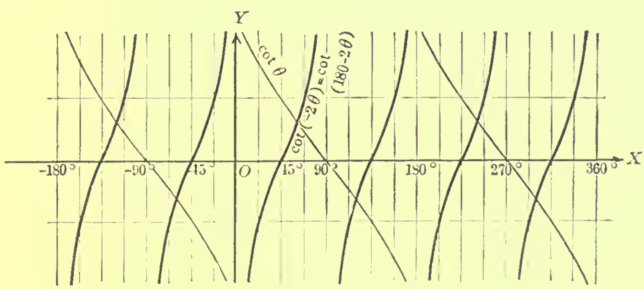


Fig. 34

60. Applications. The graphs of the functions $\tan(a\theta + b)$ and $\cot(a\theta + b)$ may now be used to derive some of the properties of the functions $\tan\theta$ and $\cot\theta$, by the method explained in **55**. We obtain in this way the following results:

(1) the functions $\tan a\theta$ and $\cot a\theta$ are periodic functions whose period is $\pi / |a|$.

It is to be observed that the period of the tangent and cotangent functions is half as large as that of the corresponding sine and cosine functions.

(2) the tangent of $\pi/2$ and $3\,\pi/2$, and the cotangent of 0 and π do not exist (see Theorem IV, Chapter II). Moreover

$$\lim_{\theta \to 90^\circ -} \tan\theta = +\infty. \qquad \lim_{\theta \to 90^\circ +} \tan\theta = -\infty.$$

$$\lim_{\theta \to 180^\circ -} \cot\theta = -\infty. \qquad \lim_{\theta \to 180^\circ +} \cot\theta = +\infty. \quad \text{(See 24, 25.)}$$

(3) the tangent and cotangent are odd functions, i.e.,

$$\mathbf{tan}\,(-\theta) = -\mathbf{tan}\,\theta, \qquad \mathbf{cot}\,(-\theta) = -\mathbf{cot}\,\theta.$$

(4) $\tan\,(180^\circ - \theta) = -\tan\theta$, etc.
$\tan\,(90^\circ - \theta) = \cot\theta$, etc. (See exercises below.)

61. Exercises.

Construct the graphs of the following functions:

1. $\tan\,(90^\circ + \theta)$.
2. $\cot\,(-\theta)$.
3. $\tan\,(90^\circ - \theta)$.
4. $\cot\,(90^\circ + \theta)$.
5. $\tan 2\,\theta$.
6. $\tan\,(180^\circ + \theta)$.
7. $\cot\,(\theta - 180^\circ)$.
8. $\cot\theta/2$.
9. $\tan\,(3\,\theta - 90^\circ)$.
10. $\tan\,(\theta/2 + 90^\circ)$.
11. $\cot\,(2\,\theta - 90^\circ)$.
12. $\tan\,(90^\circ - 2\,\theta)$.

Prove graphically:

13. $\tan\,(\theta + 180^\circ) = \tan\theta$.
14. $\cot\,(\theta + 180^\circ) = \cot\theta$.
15. $\tan\,(90^\circ + \theta) = -\cot\theta$.
16. $\cot\,(90^\circ - \theta) = \tan\theta$.
17. $\tan\,(\theta + 270^\circ) = -\cot\theta$.
18. $\cot\,(90^\circ + \theta) = -\tan\theta$.
19. $\tan\,(\theta - 180^\circ) = \tan\theta$.
20. $\cot\,(270^\circ + \theta) = -\tan\theta$.
21. $\tan\,(270^\circ - \theta) = \cot\theta$.
22. $\cot\,(\theta - 270^\circ) = -\tan\theta$.
23. $\tan\,(45^\circ + \theta) = \cot\,(45^\circ - \theta)$.
24. $\tan\,(45^\circ - \theta) = \cot\,(45^\circ + \theta)$.

62. Graphs of sec θ and cosec θ. For the sake of completeness, we add a method for the construction of the graphs of the functions $\sec\theta$ and $\operatorname{cosec}\theta$. For the former, we use the diagram of Figure 31. Since for the point S, the abscissa is equal to unity, the secant of θ is measured by the radius vector MS. If we describe an arc with M as center and MS as radius we may determine on the vertical diameter of the circle a point S_1, whose perpendicular distance from the X-axis is equal in magnitude and direction to sec θ.* Through S_1 we draw a line S_1S_1' parallel to the X-axis meeting the line AA' in a point B, which is a point on the graph of sec θ. The graph is completed by means of the method of **51** (see Fig. 35).

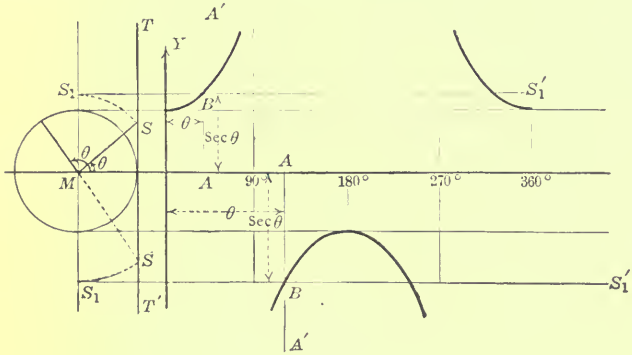


FIG. 35

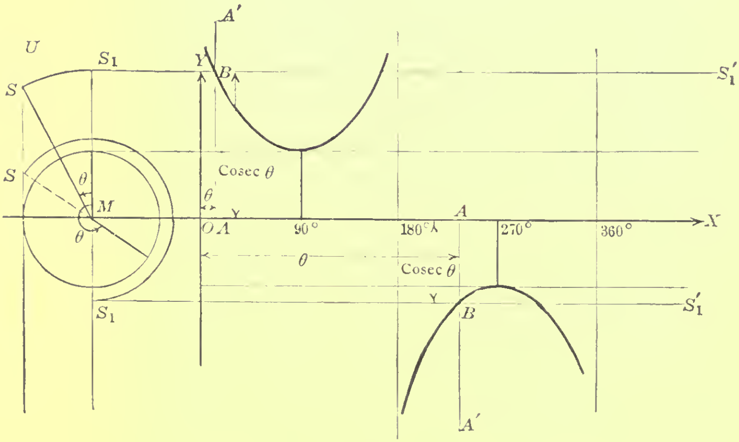


FIG. 36

For the graph of the cosecant function we use a diagram similar to the one used for the cotangent (see Fig. 32). The ordinate of S

* Here it is to be understood that S_1 is to be determined on the positive half of the diameter, if S lies on the terminal side of θ; on the negative half of the diameter if S lies on the terminal side of θ produced through the origin.

being equal to unity, the cosecant of θ is measured by the radius vector MS. Describing an arc with center at M and radius equal to MS, we find on the vertical diameter of the circle a point S_1 whose distance from the X-axis is equal in magnitude and direction to the cosecant of θ (see footnote on p. 53). Drawing a line S_1S_1' parallel to the X-axis, we determine upon the line AA', drawn parallel to the Y-axis, a point B which is a point of the graph of cosec θ. Repeating this construction for various values of θ, as explained in **51**, we obtain the graph of the function cosec θ (see Fig. 36).

For the functions sec θ and cosec θ we can now develop theorems analogous to theorems I, II, III of **53**, by means of which the graphs of the general functions sec $(a\theta + b)$ and cosec $(a\theta + b)$ can be obtained by translating, reflecting and contracting (enlarging) the graphs of sec θ and cosec θ. This in turn enables us to prove graphically various properties of these functions.

63. Exercises.

Construct the graphs of the following functions:

1. sec $2\,\theta$.

2. sec $(90° + \theta)$.

3. cosec $(-\theta)$.

4. sec $(-\theta)$.

5. cosec $\theta/2$.

6. cosec $(180° + \theta)$.

7. cosec $(\theta - 90°)$.

8. sec $(3\,\theta - 270°)$.

9. cosec $3\,\theta$.

10. sec $(\theta + 270°)$.

11. cosec $(\theta/2 + 90°)$.

12. sec $(180° - \theta)$.

Prove graphically:

13. The secant function is an even function.

14. The cosecant function is an odd function.

15. sec $(90° - \theta)$ = cosec θ.

16. cosec $(90° - \theta)$ = sec θ.

17. sec $(180° + \theta)$ = − sec θ.

18. cosec $(180° + \theta)$ = − cosec θ.

19. sec $(90° + \theta)$ = − cosec θ.

20. sec $(\theta - 90°)$ = cosec θ.

21. cosec $(90° + \theta)$ = sec θ.

22. cosec $(\theta - 90°)$ = − sec θ.

23. sec $(180° - \theta)$ = − sec θ.

24. sec $(\theta - 180°)$ = − sec θ.

25. sec $(270° + \theta)$ = cosec θ.

26. cosec $(270° + \theta)$ = − sec θ.

27. cosec $(180° - \theta)$ = cosec θ.

28. cosec $(\theta - 180°)$ = − cosec θ.

29. sec $(\theta - 270°)$ = − cosec θ.

30. cosec $(\theta - 270°)$ = sec θ.

CHAPTER VI

THE ADDITION FORMULAE

64. A special case. In the preceding chapter we have proved formulae like $\sin(\theta + 90°) = \cos\theta$; $\cos(90° - \theta) = \sin\theta$, etc. We next inquire how the trigonometric functions of the sum and difference of any two angles may be expressed in terms of the functions of these angles; i.e., we ask in the first place for formulae for $\sin(\alpha + \beta)$, $\sin(\alpha - \beta)$, $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$ in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$.

We begin by deriving in a slightly different manner some of the formulae mentioned above. From the graphs of the functions $\sin(\alpha + 90°)$ and $\cos\alpha$; $\cos(\alpha + 90°)$ and $\sin\alpha$, we have already concluded, that

(1) $\mathbf{\sin(\alpha + 90°) = \cos\alpha}$; and that (2) $\mathbf{\cos(\alpha + 90°) = -\sin\alpha}$.

If in these formulae we replace α by $\alpha - 90°$, we obtain

(3) $\mathbf{\sin\alpha = \cos(\alpha - 90°)}$, and (4) $\mathbf{\cos\alpha = -\sin(\alpha - 90°)}$,

which may also be derived directly from the graphs.

65. Addition formulae for the sine and the cosine. We proceed now to the general case. We place the angle α in standard position and we bring the initial side of β into coincidence with the terminal side of α; let OA then be the terminal side of β.

The angle $\alpha + \beta$ is then in standard position with respect to the axes OX and OY (see Fig. 37), but not the angle β. In order to make possible the discussion of the trigonometric ratios of β, we introduce as auxiliary axes the lines OP and OQ. The positive direction on OP is that of the initial side of β; the positive direction on OQ is so determined that when XOY rotates about O until OX coincides with OP, then OY will coincide with OQ. From this it follows that

$$\angle XOP = \angle YOQ = \alpha.$$

We take now a point on the terminal side of $\alpha + \beta$ (which is at the same time terminal side of β) and construct its coördinates

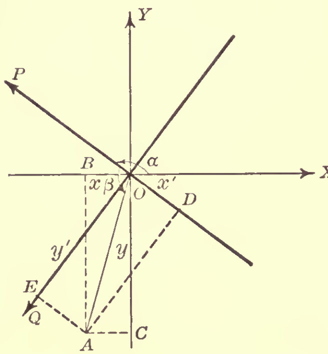
x and y with respect to the axes OX and OY, and also its coördinates x' and y' with respect to the auxiliary axes OP and OQ. Moreover, we select the line OA as our unit of measurement. In this way we find:

(1) $\sin\beta = OE/OA = OE$, and $\cos\beta = OD/OA = OD$.

Furthermore, we see that

$$\sin(\alpha + \beta) = OC/OA = OC = \text{Proj}_Y OA.$$

In order to express $\sin(\alpha + \beta)$ in terms of the trigonometric ratio of α and β separately, we proceed as in **46** and **47**, making use of the Corollary of **6**, with triangle ODA. Thus we find:



(2) $$\sin(\alpha + \beta) = \text{Proj}_Y OA$$
$$= \text{Proj}_Y OD + \text{Proj}_Y DA$$
$$= \text{Proj}_Y OD + \text{Proj}_Y OE.$$

But, OD is a segment of the directed line OP, which makes with the Y axis the angle $\alpha - 90°$; hence, by Theorem II of **45**,

$$\text{Proj}_Y OD = OD \cdot \cos(\alpha - 90°)$$
$$= \cos\beta\cos(\alpha - 90°). \quad \text{Why?}$$

And, OE is a segment of the directed line OQ, which makes with OY the angle α; hence $\text{Proj}_Y OE = OE \cdot \cos\alpha = \sin\beta\cos\alpha$. Why?

Substituting the last two results in (2), we find that

$$\sin(\alpha + \beta) = \cos\beta\cos(\alpha - 90°) + \sin\beta\cos\alpha.$$

Finally, we make use of **64** to obtain the result

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

To obtain a formula for $\cos(\alpha + \beta)$ we proceed in the same way, projecting on the X-axis instead of on the Y-axis. This leads to the following development:

$$\cos(\alpha + \beta) = OB/OA = OB = \text{Proj}_X OA$$
$$= \text{Proj}_X OD + \text{Proj}_X DA = \text{Proj}_X OD + \text{Proj}_X OE$$
$$= OD\cos\alpha + OE\cos(\alpha + 90°)$$
$$= \cos\beta\cos\alpha + \sin\beta\cos(\alpha + 90°) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Thus we have obtained the following important result:

(3) $$\boldsymbol{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,}$$
$$\boldsymbol{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta;}$$

i.e., **the cosine of the sum of two angles is equal to the product of the cosines of these angles diminished by the product of their sines; the sine of the sum of two angles is equal to the product of the sine of one by the cosine of the other plus the product of the sine of the other by the cosine of the first.**

Formulae (3) are known as the **addition formulae** for the sine and cosine functions. The proof given here is entirely independent of the quadrant in which the angles lie. The student should however, carry the proof through for various positions of the terminal sides of the angles.

These addition formulae may now be used in the first place to derive some of the other results of Chapter V. We have, for instance:

$$\sin(\alpha + 180°) = \sin\alpha\cos 180° + \cos\alpha\sin 180° = -\sin\alpha,$$
$$\cos(\alpha + 180°) = \cos\alpha\cos 180° - \sin\alpha\sin 180° = -\cos\alpha.$$

66. Subtraction formulae. In the second place we use the addition formulae to express the sine and cosine of $\alpha - \beta$ in terms of the sine and cosine of α and β. We know from **55**, (2) that the sine is an odd function and that the cosine is an even function, i.e., that

$$\sin(-\beta) = -\sin\beta, \qquad \cos(-\beta) = \cos\beta.$$

We replace β by $-\beta$ in the addition formulae, and make use of the above formulae. In this way we find:

$$\begin{aligned}\mathbf{cos}\,(\alpha - \beta) &= \cos[\alpha + (-\beta)] = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)\\ &= \mathbf{cos}\,\alpha\,\mathbf{cos}\,\beta + \mathbf{sin}\,\alpha\,\mathbf{sin}\,\beta,\end{aligned}$$

$$\begin{aligned}\mathbf{sin}\,(\alpha - \beta) &= \sin[\alpha + (-\beta)] = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)\\ &= \mathbf{sin}\,\alpha\,\mathbf{cos}\,\beta - \mathbf{cos}\,\alpha\,\mathbf{sin}\,\beta.\end{aligned}$$

67. Exercises.

1. Express sin 75° and cos 75° in terms of the ratios of 45° and 30°, and use the results for calculating the ratios of 75°.

2. Calculate the ratios for 105°.

3. Calculate the ratios for 15°. Compare the results with those obtained from the tables.

4. Determine the angles in the third quadrant whose ratios may be calculated without the use of tables, by means of the addition and subtraction formulae.

Verify the following formulae by means of the addition and subtraction formulae:

5. $\cos(180° - \theta) = -\cos\theta$.

6. $\sin(180° - \theta) = \sin\theta$.

7. $\cos(270° \pm \theta) = \pm\sin\theta$.

8. $\sin(270° - \theta) = -\cos\theta$.

9. $\sin(\theta + 45°) + \sin(\theta - 45°) = \sqrt{2}\sin\theta$.

10. $\cos(\theta + 45°) + \cos(\theta - 45°) = \sqrt{2}\cos\theta$.

11. $\sin(\theta + 30°) + \cos(\theta + 60°) = \cos\theta$.

12. $\sin(\theta - 60°) = -\cos(\theta + 30°)$.

13. $\cos(45° + \theta) = \sin(45° - \theta) = \frac{1}{2}\sqrt{2}(\cos\theta - \sin\theta)$.

14. $\sin(45° + \theta) = \cos(45° - \theta) = \frac{1}{2}\sqrt{2}(\cos\theta + \sin\theta)$.

15. $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha + \cos^2\beta - 1$.

16. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$

17. $\cos(n\pi + \theta) = (-1)^n\cos\theta$.

18. $\sin(n\pi + \theta) = (-1)^n\sin\theta$.

19. $\cos(n\pi - \theta) = (-1)^n\cos\theta$.

20. $\sin(n\pi - \theta) = (-1)^{n+1}\sin\theta$.

21. $\cos[(2n + 1)\pi/2 - \theta] = (-1)^n\sin\theta$.

22. $\sin[(2n + 1)\pi/2 - \theta] = (-1)^n\cos\theta$.

23. $\cos[(2n + 1)\pi/2 + \theta] = (-1)^{n+1}\sin\theta$.

24. $\sin[(2n + 1)\pi/2 + \theta] = (-1)^n\cos\theta$.

68. Addition and subtraction formulae for the tangent and cotangent. Since $\tan\theta = \sin\theta/\cos\theta$ and $\cot\theta = \cos\theta/\sin\theta$ (Why?), the addition and subtraction formulae for the sine and cosine enable us to obtain corresponding formulae for the tangent and cotangent, viz.,

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta},$$

and

$$\cot(\alpha + \beta) = \frac{\cos(\alpha + \beta)}{\sin(\alpha + \beta)} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}.$$

The results may be expressed in terms of tangents alone by dividing the numerators and the denominators of each of these fractions by $\cos\alpha\cos\beta$, or in terms of cotangents alone by dividing them by $\sin\alpha\sin\beta$. In this manner we find:

$$\mathbf{\tan(\alpha + \beta)} = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \mathbf{\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}},$$

and

$$\cot(\alpha+\beta) = \frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} - \dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta} + \dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}} = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}.$$

Starting with the formulae

$$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} \quad \text{and} \quad \cot(\alpha-\beta) = \frac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)},$$

and proceeding in exactly the same way as above, we find:

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} \quad \text{and} \quad \cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}.$$

69. Exercises.

1. Determine tan 75° and cot 75° by means of the addition formulae for the tangent and cotangent.

2. Determine tan 15° and cot 15° by means of the subtraction formulae.

3. Determine tan 105° and cot 105°. (It would not be advisable to write 90° + 15° in place of 105° in this problem. Why not?)

4. Prove that $\tan(45° + \theta) = \cot(45° - \theta) = \dfrac{1+\tan\theta}{1-\tan\theta} = \dfrac{\cot\theta+1}{\cot\theta-1}$.

5. Also that $\tan(45° - \theta) = \cot(45° + \theta) = \dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\cot\theta-1}{\cot\theta+1}$.

6. Express $\tan(\alpha+\beta)$ in terms of the cotangents of α and β.

7. Express $\cot(\alpha+\beta)$ in terms of $\tan\alpha$ and $\tan\beta$.

8. Prove that $\tan(30° + \theta) = \cot(60° - \theta) = \dfrac{1+\sqrt{3}\tan\theta}{\sqrt{3}-\tan\theta}$.

9. Prove that $\tan(60° + \theta) = \cot(30° - \theta) = \dfrac{1+\sqrt{3}\cot\theta}{\cot\theta-\sqrt{3}}$.

70. Double angle formulae and half-angle formulae. From the addition formulae we derive, by putting $\alpha = \beta = \theta$, the following formulae, by means of which our knowledge of the ratios of any angle enables us to find the ratios of an angle twice as large; in other words, formulae which express the ratios of any angle in terms of the ratios of an angle half as large; we find:

$$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \mathbf{\cos^2\theta - \sin^2\theta}.$$

$$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = \mathbf{2\sin\theta\cos\theta},$$

$$\tan 2\theta = \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} = \frac{2\tan\theta}{1-\tan^2\theta}$$

If we put $\theta = \phi/2$, and therefore $2\theta = \phi$, we obtain an equivalent form of these formulae, bringing out more vividly the fact that they express the ratios of an arbitrary angle in terms of the ratios of one half of that angle, viz.,

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},\quad \cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2},\quad \tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},$$

where we have again written θ in place of ϕ.

The second of these formulae leads to another important set of results, in the following manner:

To the two members of the identity

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2},$$

we add the corresponding members of the identity

$$1 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2},$$

yielding the result
$$1 + \cos\theta = 2\cos^2\frac{\theta}{2} \tag{1}$$

If we subtract the first of these identities from the second we obtain:

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}. \tag{2}$$

Upon solving the resulting identities (1) and (2) for $\cos\theta/2$ and $\sin\theta/2$ respectively, we obtain:

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}},\qquad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}},$$

formulae which express the ratios of one half of an angle in terms of the ratios of that angle. The plus or minus sign is to be used according to the quadrant in which $\theta/2$ falls.

Dividing the second of the latter formulae by the first, we find:

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta},$$

the last two forms being derived from the first by multiplying the numerator and the denominator of the fraction under the radical sign by $1 + \cos\theta$ and $1 - \cos\theta$ respectively. In the last two expressions the double sign is not necessary, because $1 + \cos\theta$

and $1 - \cos\theta$ are always positive, while $\tan\theta/2$ and $\sin\theta$ always have the same sign.

71. Exercises.

1. Determine the functions of 30° by means of those of 60°.

2. Determine the ratios of 30° by means of those of 15°, found in **67,** 3.

3. From the ratios of 15°, derive those of 7° 30′.

4. Obtain a formula which expresses $\cot\theta/2$ rationally in terms of $\sin\theta$ and $\cos\theta$.

5. Prove that $\operatorname{cosec} 2\,\theta = \frac{1}{2}\sec\theta\operatorname{cosec}\theta$.

6. Prove that $\sec\frac{\theta}{2} = \pm\sqrt{\frac{2\sec\theta}{1+\sec\theta}}$.

7. Derive the double angle formula for the tangent from the double angle formulae for sine and cosine.

8. Derive the half angle formula for the tangent from the double angle formula for the tangent.

72. Factorization formulae. We return once more to the addition formulae for the sine and cosine in order to derive from them a set of formulae which will enable us to convert the sum (or the difference) of the sines or cosines of two angles into a product. Such a conversion is of great importance, as is evident from Theorem I of Chapter III, whenever we wish to carry out logarithmic calculations with expressions which involve sums or differences of sines or cosines, and also for many theoretical purposes.

We know:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

and

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Adding the corresponding members of these two identities, we find:

$$(1)\qquad \sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta.$$

Subtracting them, we find:

$$(2)\qquad \sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta.$$

These formulae can be put in a slightly different form, more useful for the purpose for which we are deriving them, by putting

$\alpha + \beta = \theta$ and $\alpha - \beta = \phi$, whence we obtain by addition and subtraction

$$\alpha = \frac{\theta + \phi}{2} \quad \text{and} \quad \beta = \frac{\theta - \phi}{2}.$$

Substitution of these values for $\alpha + \beta$, $\alpha - \beta$, α and β in formulae (1) and (2) gives us the following results:

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

and

$$\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}.$$

We now proceed in exactly the same manner with the formulae

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

and

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

and find the formulae:

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

and

$$\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}.$$

73. Exercises.

Convert the following sums and differences into products:

1. $\sin 55° + \sin 65°$.
2. $\cos 75° - \cos 15°$.
3. $\sin 27° - \sin 18°$.
4. $\cos 87° + \cos 42°$.
5. $\cos 312° - \cos 252°$.
6. $\sin 213° + \sin 237°$.

Calculate by means of logarithms:

7. $(\sin 49° + \sin 35°)(\cos 37° - \cos 51°)$.
8. $\dfrac{\sin 57° + \sin 24°}{\sin 57° - \sin 24°}$.
9. $(\cos 137° + \cos 84°)^3$.
10. $\sqrt{\cos 306° - \cos 246°}$.

Prove the following identities:

11. $\dfrac{\sin\theta + \sin\phi}{\sin\theta - \sin\phi} = \tan\dfrac{\theta+\phi}{2}\cot\dfrac{\theta-\phi}{2}$.
12. $\dfrac{\cos\theta - \cos\phi}{\cos\theta + \cos\phi} = -\tan\dfrac{\theta+\phi}{2}\tan\dfrac{\theta-\phi}{2}$.
13. $\dfrac{\sin\theta + \sin\phi}{\cos\theta + \cos\phi} = \tan\dfrac{\theta+\phi}{2}$.
14. $\dfrac{\cos\theta - \cos\phi}{\sin\theta - \sin\phi} = -\tan\dfrac{\theta+\phi}{2}$.

74. Summary of formulae proved in Chapter VI.

1. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, see **65.**
2. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, see **65.**
3. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, see **66.**
4. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$, see **66.**
5. $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$, see **68.**
6. $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$, see **68.**
7. $\cos 2\theta = \cos^2\theta - \sin^2\theta$, see **70.**
8. $\sin 2\theta = 2\sin\theta\cos\theta$, see **70.**
9. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, see **70.**
10. $\cos\theta/2 = \pm\sqrt{(1 + \cos\theta)/2}$, see **70.**
11. $\sin\theta/2 = \pm\sqrt{(1 - \cos\theta)/2}$, see **70.**
12. $\tan\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$, see **70.**
13. $\sin\theta + \sin\phi = 2\sin\dfrac{\theta + \phi}{2}\cos\dfrac{\theta - \phi}{2}$, see **72.**
14. $\sin\theta - \sin\phi = 2\cos\dfrac{\theta + \phi}{2}\sin\dfrac{\theta - \phi}{2}$, see **72.**
15. $\cos\theta + \cos\phi = 2\cos\dfrac{\theta + \phi}{2}\cos\dfrac{\theta - \phi}{2}$, see **72.**
16. $\cos\theta - \cos\phi = -2\sin\dfrac{\theta + \phi}{2}\sin\dfrac{\theta - \phi}{2}$, see **72.**

75. Miscellaneous Exercises on Chapter VI. (The more difficult examples are marked with a *.)

1. Determine the ratios of 165° (*a*) by means of the addition formulae: (*b*) by means of the half-angle formulae.

2. Prove that $\tan\pi/8 = \sqrt{2} - 1$.

3. Prove the identity: $\sin(60° + \theta) - \cos(30° + \theta) = \sin\theta$.

4. Calculate: $\dfrac{(\sin 67° - \sin 34°)\sin 23°}{\cos 17°}$.

Prove the following identities:

5. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

6. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

***7.** $\tan(\alpha + \beta) = \dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cos\alpha - \sin\beta\cos\beta}$.

8. $\sin^2 3\theta - \sin^2 2\theta = \sin 5\theta\sin\theta$.

9. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

***10.** $\dfrac{1+\sin\theta}{1-\sin\theta} = \tan^2\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$.

11. $\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\alpha\cos\gamma + \sin\gamma\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma$.

12. $\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\cos\gamma - \sin\beta\sin\gamma\cos\alpha - \sin\alpha\sin\gamma\cos\beta$.

***13.** $\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1$, provided $\alpha+\beta+\gamma = 90°$.

14. $\dfrac{\sin 2\alpha}{1+\cos 2\alpha} \cdot \dfrac{\cos\alpha}{1+\cos\alpha} = \tan\dfrac{\alpha}{2}$.

15. $\cos(\theta + 30°) + \sin(\theta + 240°) = -\sin\theta$.

16. $\sin(60° + \theta) + \cos(\theta + 30°) = \sqrt{3}\cos\theta$.

17. $\dfrac{2}{\cot\alpha/2 + \tan\alpha/2} = \sin\alpha$.

18. $\sin\theta + \sin(\theta + 2\pi/3) + \sin(\theta + 4\pi/3) = 0$.

19. $\cos\theta + \cos(\theta + 2\pi/3) + \cos(\theta + 4\pi/3) = 0$.

20. $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$.

21. $\cos(\alpha+\beta) + \cos(\alpha-\beta) + \cos(-\alpha+\beta) + \cos(-\alpha-\beta) = 4\cos\alpha\cos\beta$.

22. $\dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2} = \cos\alpha$.

23. $\tan(\theta + 45°) + \tan(\theta - 45°) = 2\tan 2\theta$.

***24.** $\tan 2\alpha - \tan\alpha = \dfrac{2\sin\alpha}{\cos\alpha + \cos 3\alpha}$.

25. $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

***26.** $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$, if $\alpha+\beta+\gamma = 180°$.

27. $\sin 3\theta = 4\sin\theta\sin(\pi/3 + \theta)\sin(\pi/3 - \theta)$.

28. Calculate the functions of $3\pi/8$.

***29.** $\tan(\pi/4 + \theta/2) + \cot(\pi/4 + \theta/2) = 2\sec\theta$.

***30.** $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1 - 4\cos\alpha\cos\beta\cos\gamma$, if $\alpha+\beta+\gamma = 180°$.

CHAPTER VII

THE SOLUTION OF TRIANGLES

76. The Law of Sines; the area of a triangle. We consider in this chapter the problem of "solving an arbitrary triangle ABC," i.e., the determination of the unknown elements of a triangle ABC of which sufficient elements are given. The problem is solved by the use of various relations subsisting between the sides and angles of a triangle.

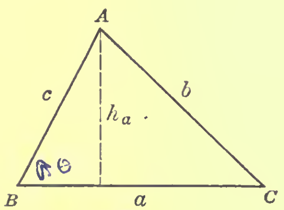


FIG. 38

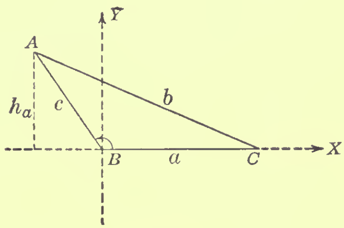


FIG. 38a

Denoting the length of the perpendicular dropped from the vertex A upon the opposite side BC by h_a, and using analogous notations for the other perpendiculars, we have,

(1) $$h_a = c \sin B = b \sin C, \qquad h_b = a \sin C = c \sin A,$$
$$h_c = b \sin A = a \sin B.$$

For h_a in Fig. 38a, the above result is obtained most readily, if we remember that $\angle B$ is placed in standard position with reference to the axes indicated in the diagram.

From these formulae we derive:

(a) THEOREM I. **The sines of the angles of a triangle are proportional to the sides opposite the angles.*** (LAW OF SINES.)

* It is a familiar theorem of plane geometry that of two unequal sides of a triangle the greater side lies opposite the greater angle. The law of sines may be looked upon as completing this theorem by stating how the unequal sides are related to the angles opposite them.

To prove this law, we divide the two expressions for h_a in equations (1) by bc, those for h_b by ac. In this way we obtain:

$$\sin B/b = \sin C/c = \sin A/a,$$

which was to be proved.

(b) THEOREM II. **The area of a triangle is equal to one half of the product of any two sides multiplied by the sine of the angle included by them.**

Proof. Denoting the area of triangle ABC by Δ, we have:

$$\Delta = a/2 \cdot h_a = b/2 \cdot h_b = c/2 \cdot h_c.$$

If in these expressions we substitute for the altitudes h_a, h_b and h_c the values which they have in equations (1), we find

$$\Delta = \tfrac{1}{2}\, ab \sin C = \tfrac{1}{2}\, bc \sin A = \tfrac{1}{2}\, ca \sin B,$$

which was to be proved.

77. Two angles and one side. A triangle is determined when two angles and a side are given. The law of sines suffices to determine the remaining elements in this case. For, suppose that A, B, and a are given; we have then $C = 180° - (A + B)$.

Moreover $\sin B/b = \sin A/a$ and $\sin C/c = \sin A/a$; hence

$$b = \frac{a \sin B}{\sin A}, \quad \text{and} \quad c = \frac{a \sin C}{\sin A},$$

which completes the solution.

If the given elements are measured by short numbers the calculation may be carried out directly by means of a table of natural values; the expressions for b and c, however, are well adapted to calculation by means of logarithms.

Example.

Given. $a = 43.257$, $A = 57° 23'$, $C = 49° 47'$.

Required. B, b and c.

Solution. $B = 180° - (A + C) = 180° - 107° 10' = 72° 50'$.

$$\frac{b}{\sin B} = \frac{a}{\sin A}, \quad \text{i.e.,} \quad b = \frac{a \sin B}{\sin A} = \frac{43.257 \quad \sin 72° 50'}{\sin 57° 23'}.$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}, \quad \text{i.e.,} \quad c = \frac{a \sin C}{\sin A} = \frac{43.257 \quad \sin 49° 47'}{\sin 57° 23'}.$$

$\log 43.257 = 1.63606$	$\log 43.257 = 1.63606$
$\log \sin 72° 50' = 9.98021-10$ A	$\log \sin 49° 47' = 9.88287-10$ A
$11.61627-10$	$11.51893-10$
$\log \sin 57° 23' = 9.92546-10$ S	$\log \sin 57° 23' = 9.92546-10$ S
$\log b = 1.69081$	$\log c = 1.59347$
$b = 49.069$	$c = 39.216$

78. Exercises.

1. $A = 39° 27'$, $B = 108° 51'$, $b = .43215$. Determine C, a, c.

2. $C = 30°$, $A = 45°$, $c = 123$. Determine B, a, b.

3. $B = 27° 45' 15''$, $C = 89° 19' 20''$, $c = 14.302$. Determine A, a, b.

4. $A = 37° 12' 30''$, $C = 58° 26' 40''$, $a = 103.47$. Determine B, b, c.

5. We wish to determine the distance from a point A to a point B, situated in a marsh, visible but not accessible from A (see Fig. 39). For this

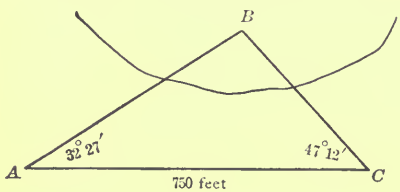


FIG. 39

purpose, we measure the distance from A to a point C, from which both A and B are visible and we measure the angles BAC and ACB. Calculate AB, if $AC = 750$ feet, $\angle BAC = 32° 27'$ and $\angle ACB = 47° 12'$.

6. Devise a method for finding the distance from a point A on one bank of a river to a point B on the opposite bank.

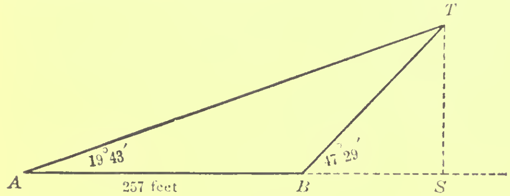


FIG. 40

7. To determine the height of a tower TS (see Fig. 40) we measure the angle of elevation of its top T from two points A and B lying on the same side of T, in the same vertical plane with T and a known distance apart.

Calculate the height of the tower, if we find that $AB = 257$ feet, $\angle TAB = 19° 43'$ and $\angle TBS = 47° 29'$. (Compare the present method of solving this problem with the method used for solving problem 10 in **49**.)

8. Solve problem 10 in **49** by the method used in the preceding problem.

79. Two sides and an angle opposite one of them. If two sides of a triangle and an angle opposite one of them are given, the triangle is not completely determined. Let there be given the sides a and b and the acute angle A (see Fig. 41). To con-

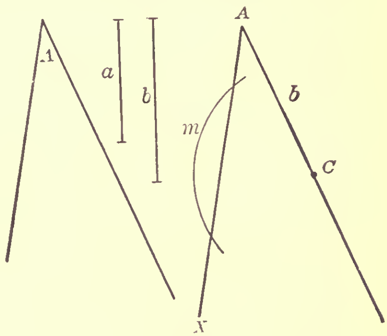


Fig. 41

struct the triangle ABC we lay off a line equal to b on one of the legs of the angle A. The vertex C is then located. We then strike an arc with C as a center and a as a radius. The intersection of this arc m with the second leg AX of angle A determines the third vertex of the triangle, B.

It is now clear that if a is shorter than the perpendicular distance p from C to AX, as in Figure 42a, then the arc m will not

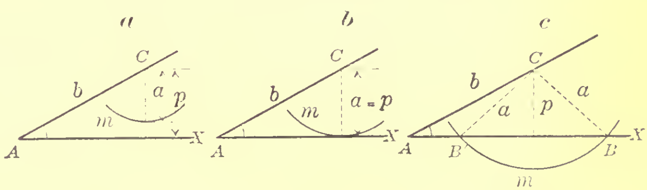


Fig. 42a Fig. 42b Fig. 42c

intersect AX at all, so that no triangle can be constructed. If a is equal to the perpendicular p, as in Figure 42b, we obtain a single right-angled triangle. If a is greater than p, but less than b (see Fig. 42c), then the arc m will cut AX in two points, B and B', and we will obtain two triangles, ACB and ACB', both of which will satisfy the requirements of the problem. Since $CB = CB'$, $\angle CB'B = \angle CBB'$, and therefore the angles B and B' occurring in triangles ABC and $AB'C$ respectively are supplementary angles. If $a > b$ (see Fig. 42d) the arc m will meet AX again at two points

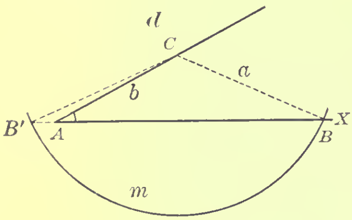


FIG. 42d

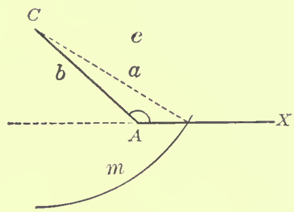


FIG. 42e

B and B', but one of these points will fall on XA produced, so that the triangle $AB'C$, to which it gives rise, does not contain the given angle A. Hence in this case there is only one triangle which satisfies the requirements of the problem, viz., triangle ABC. If the given angle A is obtuse the construction of triangle ABC proceeds in the same way as before (see Fig. 42e). It is evident that in this case there is no solution possible, unless $a > b$. We observe moreover that in all the cases which we have considered the perpendicular p is equal to $b \sin A$. Hence we can put the results of this discussion in the following form:

THEOREM III. **If two lines and an angle, such as a, b, A are given, then there may be no triangle, one triangle or two triangles of which the given lines are sides and of which the given angle is an angle opposite one of these sides, viz.,**

if $a < b \sin A$, there is no triangle;

if $a = b \sin A$ and A is acute, there is one triangle;

if $a > b \sin A$, $a < b$ and A is acute, there are two triangles;

if $a = b$ and A is acute, there is one triangle;

if $a > b$, there is one triangle.

The different cases mentioned in this theorem are illustrated in the examples which follow. The solution proceeds in each case according to the following plan:

From the law of sines, we find: $\sin B = \dfrac{b \sin A}{a}$, which enables us to determine B. Next we can calculate C, since $C = 180° - (A + B)$. Finally we use the law of sines to find c, viz., $c = \dfrac{a \sin C}{\sin A}$.

If $a < b \sin A$, we find that $\sin B > 1$; hence no angle B can be found, and therefore no triangle exists. If $a = b \sin A$, then $\sin B = 1$, $B = 90°$ and we have a single right-angled triangle.

If $a > b \sin A$, then $\sin B < 1$ and we can determine angle B from the tables. But, besides the acute angle B found from the tables, we must also consider the supplement of B, whose sine is equal to the sine of B; we have therefore $B' = 180° - B$. Moreover if $a < b$, then we must have $B > A$ (see footnote on p. 65); hence both angles, B and B', can be used, if A is acute, while neither can be used if A is right or obtuse. If, on the other hand, $a > b$, then we must have $B < A$, so that only the acute angle B, found from the tables, can be used.

Example 1.

Given. $a = 4.73$, $b = 18.65$, $A = 43° 27'$.

Required. c, B, C.

Solution.

$$\frac{\sin B}{b} = \frac{\sin A}{a}, \quad \text{therefore} \quad \sin B = \frac{b \sin A}{a} = \frac{18.65 \sin 43° 27'}{4.73},$$

$$\log 18.65 = 1.27068.$$
$$\log \sin 43° 27' = 9.83741 - 10,$$
$$\overline{\hspace{8em}}\, A$$
$$\log b \sin A = 11.10809 - 10,$$
$$\log a = \log 4.73 = .67486.$$

We notice that $\log b \sin A > \log a$; therefore $a < b \sin A$. Hence no triangle can be found with the given elements.

Example 2.

Given. $a = 14.73$, $b = 18.65$, $A = 43° 27'$.

Required. c, B, C.

Solution. We have now

$$\sin B = \frac{b \sin A}{a} = \frac{18.65 \ \sin \ 43° \ 27'}{14.73},$$

$$\begin{aligned} \log 18.65 &= 1.27068, \\ \log \sin 43° \, 27' &= 9.83741 - 10, \\ &\overline{}\; A \\ \log b \sin A &= 11.10809 - 10, \\ \log 14.73 &= 1.16820, \\ &\overline{}\; S \\ \log \sin B &= 9.93989 - 10. \end{aligned}$$

Here $a > b \sin A$, $a < b$ and A is acute. Hence there are two triangles in this case. We find:

First solution

$B = 60° \, 32' \, 43''$.

We know $A = 43° \, 27'$.

Hence $C = 180° - (A + B)$
$= 76° \, 0' \, 17''$.

Furthermore:

$$\frac{\sin C}{c} = \frac{\sin A}{a}, \text{ and therefore}$$

$$c = \frac{a \sin C}{\sin A} = \frac{14.73 \times \sin 76° \, 17''}{\sin 43° \, 27'},$$

$$\begin{aligned} \log 14.73 &= 1.16820 \\ \log \sin 76° \, 17'' &= 9.98691 - 10 \\ &\overline{}\; A \\ &\, 11.15511 - 10 \\ \log \sin 43° \, 27' &= 9.83741 - 10 \\ &\overline{}\; S \\ \log c &= 1.31770 \\ c &= 20.783 \end{aligned}$$

Second solution

$B' = 119° \, 27' \, 17''$.

We know $A = 43° \, 27'$.

Hence $C' = 180° - (A + B)$
$= 17° \, 5' \, 43''$.

Furthermore:

$$\frac{\sin C'}{c'} = \frac{\sin A}{a}, \text{ and therefore}$$

$$c' = \frac{a \sin C'}{\sin A} = \frac{14.73 \times \sin 17° \, 5' \, 43''}{\sin 43° \, 27'}$$

$$\begin{aligned} \log 14.73 &= 1.16820 \\ \log \sin 17° \, 5' \, 43'' &= 9.46829 - 10 \\ &\overline{}\; A \\ &\, 10.63649 - 10 \\ \log \sin 43° \, 27' &= 9.83741 - 10 \\ &\overline{}\; S \\ \log c' &= .79908 \\ c' &= 6.2963 \end{aligned}$$

Figure 43 is a drawing to scale of the triangles ABC and $AB'C$.

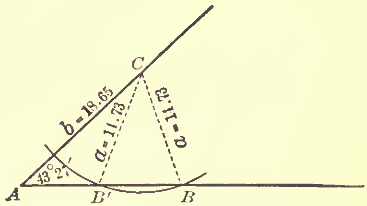


FIG. 43

Example 3.

Given. $a = 24.73$, $b = 18.65$, $A = 143° 27'$.

Required. c, B, C.

Solution. $\sin B = \dfrac{b \sin A}{a} = \dfrac{18.65 \sin 143° 27'}{24.73}$

$$\begin{aligned}
\log 18.65 &= 1.27068 \\
\log \sin 143° 27' = \log \sin 36° 33' &= 9.77490 - 10 \quad A \\
\log b \sin A &= 11.04558 - 10 \\
\log 24.73 &= 1.39322 \quad S \\
\log \sin B &= 9.65236 - 10 \\
B &= 26° 41' 14''
\end{aligned}$$

Since $a > b$ and A is obtuse, there is only one solution and B must be acute. Hence the value found from the table is the only one that can be used in this case.

$$C = 180° - (143° 27' + 26° 41' 14'') = 9° 51' 46''$$

$$c = \frac{a \sin C}{\sin A} = \frac{24.73 \sin 9° 51' 46''}{\sin 143° 27'}$$

$$\begin{aligned}
\log 24.73 &= 1.39322 \\
\log \sin 9° 51' 46'' &= 9.23373 - 10 \quad A \\
\log a \sin C &= 10.62695 - 10 \\
\log \sin 143° 27' &= 9.77490 - 10 \quad S \\
\log c &= .85205 \\
c &= 7.1130
\end{aligned}$$

80. Exercises.

1. $a = 42.3$, $b = 57.03$, $A = 35° 35'$. Determine c, B, C.

2. $c = 507.8$, $b = 751.3$, $B = 23° 47'$. Determine a, A, C.

3. $b = 5.5$, $c = 4.3$, $C = 75° 29'$. Determine a, A, B.

4. $a = 3.207$, $c = 7.831$, $C = 137° 18'$. Determine b, A, B.

5. $a = 37.052$, $b = 49.312$, $A = 19° 25'$. Determine c, B, C.

6. $c = .047031$, $b = .047031$, $C = 28° 31'$. Determine a, A, B.

7. An island, known to be 75 miles wide, subtends an angle of 40° 17′ from a point P, 40 miles distant from one extremity of the island. How far is P from the other extremity of the island?

8. A flagpole, 10 feet high, subtends an angle of 2° 37′ from a point A. If A is 200 feet from the foot of the pole, how far is it from the top?

9. Two lighthouses, M and L are 40 miles apart. At 8.30 A.M. a ship S leaves M and travels at the rate of 12 miles per hour. At 11 A.M. the distance between M and L subtends an angle of 33° from the ship. How far is S from L at that moment?

81. The Law of Cosines. Two sides and the included angle. Three sides. If three sides of a triangle are given, and also if two sides and the included angle are given, the triangle is determined. The law of sines does not suffice, however, to calculate the unknown elements in these cases. We therefore proceed to derive a new relation between the sides and angles of the triangle.

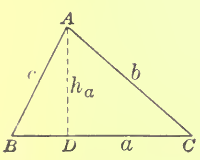


FIG. 44

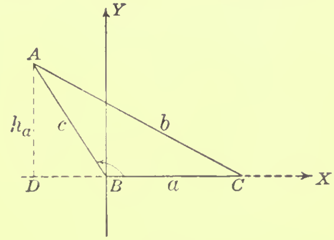


FIG. 44a

In Fig. 44a, the obtuse angle B is in standard position with reference to the axes indicated in the diagram. We have, therefore, in both figures:

$$BD = c \cos B, \qquad h_a = c \sin B, \qquad BC = a.$$

Using now Theorem I of **5**, we find that $BD + DC + CB = 0$.

Therefore, $DC = -BD - CB = BC - BD = a - c \cos B$. (1)

Moreover $\qquad b^2 = h_a^2 + \overline{DC}^2 = c^2 \sin^2 B + \overline{DC}^2$. (2)

Substituting (1) in (2), we find:

$$b^2 = c^2 \sin^2 B + (c \cos B - a)^2$$
$$= c^2 \sin^2 B + c^2 \cos^2 B + a^2 - 2\,ac \cos B;$$

i.e., $$\mathbf{b^2 = c^2 + a^2 - 2\,ac \cos B,}$$

We have therefore proved the following theorem:

THEOREM IV. **The square of one side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the continued product of these two sides and the cosine of the angle included by them.** (LAW OF COSINES.)

By dropping perpendiculars from each of the other two vertices of triangle ABC and proceeding in an exactly similar manner, we obtain two formulae analogous to the one derived above, viz.,

$$\mathbf{c^2 = a^2 + b^2 - 2\,ab \cos C} \quad \text{and} \quad \mathbf{a^2 = b^2 + c^2 - 2\,bc \cos A.}$$

In the form in which they are here given, these formulae enable us to calculate the third side of a triangle of which two sides and the included angle are known. If we solve them for the cosines of the angles, we obtain:

$$\cos A = \frac{b^2+c^2-a^2}{2\,bc}, \quad \cos B = \frac{c^2+a^2-b^2}{2\,ca}, \quad \cos C = \frac{a^2+b^2-c^2}{2\,ab}, \qquad (3)$$

in which form they are well adapted to determining the angles of a triangle whose sides are given.

We observe that none of the formulae derived in the present section are suited to logarithmic calculation. For this reason they are useful only for computations which involve short numbers.

Example 1.

Given. $a = 14$, $b = 27$, $C = 35°$.

Required. c, A, B.

Solution. By means of Theorem IV we find:

$c^2 = a^2 + b^2 - 2\,ab \cos C = 14^2 + 27^2 - 2 \cdot 14 \cdot 27 \cdot \cos 35°;$

$c^2 = 196 + 729 - 756 \times .81915 = 925 - 619.27740 = 305.72260.$

$c = \sqrt{305.72260} = 17.484.$

To complete the calculation, we use the law of sines, and find:

$$\sin A = \frac{a \sin C}{c}, \qquad \text{and} \qquad \sin B = \frac{b \sin C}{c}.$$

log 14 =	1.14613	log 27 =	1.43136
log sin 35° =	9.75859 − 10 (A)	log sin 35° =	9.75859 − 10 (A)
	10.90472 − 10		11.18995 − 10
log 17.484 =	1.24264 (S)	log 17.484 =	1.24264 (S)
log sin A =	9.66208 − 10	log sin B =	9.94731 − 10

$A = 27° 20' 28''$ or $152° 39' 32''$ and $B = 62° 20' 34''$ or $117° 39' 26''$.

Since $a < c$, we must have $A < C$; hence $A = 27° 20' 28''$.

Since $b > a$ and $b > c$, we must have $B > A$ and $B > C$; both values of B satisfy this condition; it is clear, however, that with the smaller value of B, the condition that $A + B + C = 180°$ would not be satisfied; therefore $B = 117° 39' 26''$.

Example 2.

Given. $a = 10$, $b = 15$, $c = 19$.

Required. A, B, C.

Solution. Using formula (3), we find:

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab} = \frac{100 + 225 - 361}{300} = \frac{-36}{300} = -.12000.$$

Since $\cos C$ is negative, $\angle C$ is obtuse and $\cos (180° - C) = -\cos C = .12000$.

We find from the tables, that $180° - C = 83° 6' 29''$; and hence

$$C = 96° 53' 31''.$$

Furthermore,

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca} = \frac{361 + 100 - 225}{380} = \frac{236}{380} = .62105;$$

therefore $B = 51° 36' 27''$.

And finally,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{225 + 361 - 100}{570} = \frac{486}{570} = .85263;$$

whence we obtain $A = 31° 30' 4''$.

Check. $A + B + C = 180° 0' 2''$.

82. Exercises.

1. $a = 120$, $b = 150$, $C = 60°$. Determine c, A, B.

2. $p = 1.3$, $q = 1.4$, $r = 1.5$. Determine P, Q, R.

3. $a = 17$, $b = 15$, $c = 29$. Determine A, B, C.

4. $r = .45$, $s = .78$, $T = 45°$. Determine t, R, S.

5. To determine the width of a lake, the distances of its extreme points A and B from a point P and the angle subtended by AB at P are measured. It is found that $AP = 750$ feet, $BP = 600$ feet, and $\angle P = 32°$.

6. It is desired to make a triangle out of sticks that are 5, 8, and 9 inches long. What angle should the first two of these sticks make, in order that the third one may be just long enough to join their free ends?

7. Prove that $a^2 = b^2 + c^2 - 2\,bc \cos A$.

8. Prove that $a^2 + b^2 + c^2 = 2\,ab \cos C + 2\,bc \cos A + 2\,ca \cos B$.

9. Prove that $\cos C = \dfrac{a^2 + b^2 - c^2}{2\,ab}$.

83. Summary and critique. In the preceding sections we have learned to calculate the unknown parts of triangles of which are given (*a*) one side and two angles (**77**); (*b*) two sides and an angle opposite one of the sides (**79**); (*c*) two sides and the included angle (**81**); or (*d*) three sides (**81**). We know, moreover, from the study of plane geometry, that if a triangle is to be determined by sides and angles, then the given elements must form one of the four sets (*a*), (*b*), (*c*) or (*d*) enumerated above. Hence the general problem of "solving a triangle" has been solved in sections **77–81**, in so far as it relates to triangles determined by means of sides and angles. There are however two criticisms to be made of the theory developed so far, viz.:

(1) The methods developed in **81** for cases (*c*) and (*d*) are not suited to the use of logarithms and are not very useful, therefore, in problems involving long numbers.

(2) There are no convenient methods for accurately checking the calculations in cases (*a*), (*b*) and (*c*).

In order to meet these criticisms some further relations between the sides and angles of a triangle will now be derived.

84. The law of tangents. From Theorem I, we conclude that

$$\frac{\sin A}{\sin B} = \frac{a}{b}. \qquad (1)$$

Adding 1 to both sides of this equation, we obtain:

$$\frac{\sin A + \sin B}{\sin B} = \frac{a + b}{b}; \tag{2}$$

subtracting 1 from both sides of equation (1) gives us:

$$\frac{\sin A - \sin B}{\sin B} = \frac{a - b}{b}. \tag{3}$$

If we divide the sides of (2) by the corresponding sides of (3), we find:

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{a + b}{a - b}. \tag{4}$$

We now make use of formulae (13) and (14) of **74** to factor respectively the numerator and denominator on the left hand side of (4); we also divide the numerator and denominator by 2 and obtain:

$$\frac{a+b}{a-b} = \frac{\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A + B)}$$

$$= \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A - B)}.$$

To justify the last step we observe that $A + B + C = 180°$.

Hence

$$\tfrac{1}{2}(A + B) = 90° - \tfrac{1}{2} C. \tag{5}$$

$$\therefore \quad \tan \tfrac{1}{2}(A + B) = \tan (90° - \tfrac{1}{2} C) = \cot \tfrac{1}{2} C \quad \text{(see 60, (4))}.$$

The result is most conveniently written in the following form,

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \cot \tfrac{1}{2} C,$$

which formula is expressed in the following theorem:

THEOREM V. **The tangent of one half the difference of two angles of a triangle is equal to the quotient of the difference of the sides opposite these angles by their sum, multiplied by the cotangent of one half the angle included by them.** (LAW OF TANGENTS.)

85. Mollweide's Equations. If we multiply both sides of equations (2) and (3) of **84** by the corresponding sides of the equation $\frac{\sin B}{\sin C} = \frac{b}{c}$, we find that

$$\frac{\sin A + \sin B}{\sin C} = \frac{a+b}{c}, \tag{1}$$

and

$$\frac{\sin A - \sin B}{\sin C} = \frac{a-b}{c}. \tag{2}$$

We factor the numerators on the left hand sides of these equations by means of Formulae (13) and (14) of **74.** The denominators we change by writing $\sin C = 2 \sin \frac{1}{2} C \cos \frac{1}{2} C$, which is a consequence of formula (8) of **74.** In this way equations (1) and (2) will assume the following form:

$$\frac{\sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C \cos \frac{1}{2} C} = \frac{a+b}{c}, \tag{3}$$

and

$$\frac{\sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A+B)}{\sin \frac{1}{2} C \cos \frac{1}{2} C} = \frac{a-b}{c}. \tag{4}$$

Moreover, from equation (5) of **84,** it follows that

$$\sin \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2} C$$

and $\cos \frac{1}{2}(A+B) = \sin \frac{1}{2} C$, (see **19** and **55**).

Consequently, equations (3) and (4) may be simplified to the final form:

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C} \quad \text{and} \quad \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C}.$$

These equations are known as ***Mollweide's equations.***

86. Exercises.

1. Prove: $\tan \frac{1}{2}(C-A) = \frac{c-a}{c+a} \cot \frac{1}{2} B$.

2. Prove: $\frac{\sin \frac{1}{2}(B-C)}{\cos \frac{1}{2} A} = \frac{b-c}{a}$.

3. Derive a proof of the law of tangents from Mollweide's equations.

87. Two sides and the included angle. The law of tangents enables us to meet the criticism brought forward in **83** with reference to case (*c*). Moreover, Mollweide's equations are well adapted to serve for checking the calculations in cases (*a*), (*b*) and (*c*), because they involve all the sides and all the angles of the triangle.

Example.

Given. $a = .4503$, $b = .7831$, $C = 43° \; 48'$.

Required. c, A, B.

Solution. We use the law of tangents to determine angles A and B:

$$\tan \tfrac{1}{2}(B - A) = \frac{b - a}{b + a} \cot \tfrac{1}{2} C = \frac{.3328}{1.2334} \cot 21° \; 54'.$$

$$\begin{aligned}
\log .3328 &= 9.52218 - 10 \\
\log 1.2334 &= .09110 \\
&\overline{}\; S \\
& 9.43108 - 10 \\
\log \cot 21° \; 54' &= .39578 \\
&\overline{}\; A \\
\log \tan \tfrac{1}{2}(B - A) &= 9.82686 - 10 \\
\tfrac{1}{2}(B - A) &= 33° \; 52' \; 11'',
\end{aligned}$$

but,

$$\tfrac{1}{2}(B + A) = 90° - \tfrac{1}{2} C = 90° - 21° \; 54' = 68° \; 6' \qquad \overline{}\; A \text{ and } S.$$

Therefore,

$$\angle B = \tfrac{1}{2}(B + A) + \tfrac{1}{2}(B - A) = 101° \; 58' \; 11''$$

and

$$\angle A = \tfrac{1}{2}(B + A) - \tfrac{1}{2}(B - A) = 34° \; 13' \; 49''.$$

Furthermore:

$$c = \frac{a \sin C}{\sin A} = \frac{.4503 \times \sin 43° \; 48'}{\sin 34° \; 13' \; 49''}$$

$$\begin{aligned}
\log .4503 &= 9.65350 - 10 \\
\log \sin 43° \; 48' &= 9.84020 - 10 \\
&\overline{}\; A \\
& 19.49370 - 20 \\
\log \sin 34° \; 13' \; 49'' &= 9.75014 - 10 \\
&\overline{}\; S \\
\log c &= 9.74356 - 10 \\
c &= .55406
\end{aligned}$$

Check.
$$\frac{\sin \frac{1}{2}(B - C)}{\cos \frac{1}{2} A} = \frac{b - c}{a}.$$

$\frac{1}{2}(B - C) = 29° \, 5' \, 6''$, $\frac{1}{2} A = 17° \, 6' \, 55''$, $b - c = .22904$

$\log \sin \frac{1}{2}(B - C) = 9.68673 - 10$ $\log (b - c) = 9.35992 - 10$

$\log \cos \frac{1}{2} A = 9.98032 - 10$ $\log a = 9.65350 - 10$

$\overline{}\, S$ $\overline{}\, S$

$9.70641 - 10$ $9.70642 - 10$

88. Exercises.

1. $c = 27.04$, $b = 84.31$, $A = 112° \, 44'$. Determine a, B, C.

2. $a = 3152$, $c = 4281$, $B = 88° \, 27'$. Determine b, A, C.

3. $p = .0432$, $q = .0586$, $R = 47° \, 36'$. Determine r, P, Q.

4. $x = .8132$, $z = .5817$, $Y = 120°$. Determine y, X, Z.

5. A hill slopes at an angle of 17°. A point P is 39.3 feet up the hillside; a point Q is in the plane, 173.5 feet from the foot of the hill and in the same vertical plane as P. How far is P from Q?

6. The distance from the earth to the sun is approximately 92.9 million miles, the distance from the earth to the moon approximately 239,000 miles. What is the distance from the sun to the moon at a moment when the line sun-moon subtends at the earth an angle of 24° 36′?

89. The half-angle formulae for the angles of a triangle. We start with the first formula for $\tan \frac{1}{2} A$ given in **70** and multiply numerator and denominator of the fraction under the radical sign by $2\,bc$. In this way we find:

$$\tan \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \sqrt{\frac{2\,bc - 2\,bc \cos A}{2\,bc + 2\,bc \cos A}}. \tag{1}$$

If we multiply both sides of formula (3) of **81** by $2\,bc$, we obtain:

$$2\,bc \cos A = b^2 + c^2 - a^2. \tag{2}$$

We substitute (2) in (1); this gives us:

$$\tan \tfrac{1}{2} A = \sqrt{\frac{2\,bc + a^2 - b^2 - c^2}{2\,bc + b^2 + c^2 - a^2}} = \sqrt{\frac{a^2 - (b^2 - 2\,bc + c^2)}{(b^2 + 2\,bc + c^2) - a^2}}$$

$$= \sqrt{\frac{(a + b - c)(a - b + c)}{(b + c + a)(b + c - a)}}. \tag{3}$$

Now, we introduce the abbreviation $2s$ for the perimeter of the triangle; i.e., $a + b + c = 2s$. Subtracting in turn $2a$, $2b$ and $2c$ from both sides of this equality, we obtain:

$$-a + b + c = 2(s - a),$$
$$a - b + c = 2(s - b),$$
and
$$a + b - c = 2(s - c).$$

These expressions are now substituted in equation (3), which then becomes:

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{p}{s-a},$$

where $p = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$.

In a similar way, we obtain the analogous formulae:

$$\tan \tfrac{1}{2} B = \frac{p}{s-b} \quad \text{and} \quad \tan \tfrac{1}{2} C = \frac{p}{s-c}.$$

These formulae will be referred to as the half-angle formulae for the triangle.

90. Three sides. The results of the preceding section make it possible to remove the one remaining point contained in the criticism of **83,** viz., to devise a treatment for case (*d*) adapted to calculation by logarithms.

Example.

Given. $a = 14.931$, $b = 16.902$, $c = 24.315$.

Required. A, B, C.

Solution.

$a = 14.931$
$b = 16.902$
$c = 24.315$
———— *A*
$2s = 56.148$ and therefore $s = 28.074$;

$s - a = 13.143$	and	$\log(s - a) = 1.11870$
$s - b = 11.172$	and	$\log(s - b) = 1.04813$
$s - c = 3.759$	and	$\log(s - c) = .57507$
———— *A* *		———— *A*
$s = 28.074$		2.74190
		$\log s = 1.44830$
		———— *S*
		1.29360
		2 ————

* It is well to use this check upon the calculation of the quantities s, $s - a$, $s - b$, and $s - c$ before proceeding with the rest of the work.

Since $p = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$, we have $\log p = \quad .64680$

Subtracting $\log (s-a)$, $\log (s-b)$ and $\log (s-c)$ in turn from $\log p$, we find by means of the half-angle formulae:

$$\log \tan A/2 = 9.52810 - 10, \quad \log \tan B/2 = 9.59867 - 10,$$
$$\log \tan C/2 = .07173;$$

$$\therefore \quad A/2 = 18^\circ\, 38'\, 33'', \; B/2 = 21^\circ\, 38'\, 52'', \quad C/2 = 49^\circ\, 42'\, 37'',$$

and $A = 37^\circ\, 17'\, 6''$, $B = 43^\circ\, 17'\, 44''$, $C = 99^\circ\, 25'\, 14''$.

Check. $A + B + C = 180^\circ\, 0'\, 4''$.

91. Exercises.

1. $a = .96834$, $b = .94572$, $c = .95902$. Determine A, B, C.

2. $a = 453.67$, $b = 112.34$, $c = 369.85$. Determine A, B, C.

3. $x = 1.004$, $y = 1.705$, $z = 1.526$. Determine X, Y, Z.

4. $a = 4500$, $b = 5400$, $c = 6300$. Determine A, B, C.

5. A triangular piece of land is to be staked off, so that its sides measure, respectively, 73.84 rods, 68.701 rods, and 32.503 rods. How may this be done?

6. Two vessels leave the same harbor at the same moment, both going at the rate of 12 miles per hour. After $2\frac{1}{2}$ hours, the vessels are 24 miles apart. If one of the vessels was sailing in a due easterly course, in what direction was the other vessel going?

92. Inscribed and circumscribed circles. Area. The results of the preceding articles enable us to derive other important relations, viz.:

(1) *A formula for the radius of the inscribed circle* (see Fig. 45).

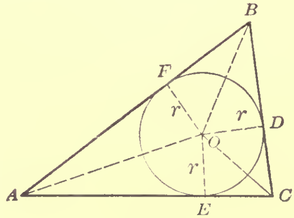


FIG. 45

The points of tangency of the inscribed circle of radius r divide the total perimeter of the triangle into six parts, which are equal in pairs. Hence the sum of the segments of any set of three, of which no two are equal, must be equal to s, one half of the perimeter; e.g.,

$$AF + BD + DC = s \quad \text{or} \quad AF + a = s;$$

hence

$$AF = EA = s - a.$$

In similar manner we show that $FB = BD = s - b$ and $DC = CE = s - c$.

Moreover $\triangle AOF$ is a right-angled triangle (Why?) and $\angle OAF = A/2$. (Why?)

Therefore

$$\tan\frac{A}{2} = \frac{OF}{AF} = \frac{r}{s-a}.$$

On the other hand, we have from the half-angle formulae for the angles of a triangle: $\tan\frac{A}{2} = \frac{p}{s-a}$. From this we conclude that $r = p$, i.e.:

THEOREM VI. **The radius of the inscribed circle of a triangle ABC is given by the formula:**

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

where s designates one half of the perimeter of the triangle.

(2) *A second formula for the area of a triangle.*

We see from Fig. 45, that

$$\begin{aligned}\triangle ABC &= \triangle AOB + \triangle BOC + \triangle COA\\ &= r \cdot c/2 + r \cdot a/2 + r \cdot b/2 = r \cdot \frac{a+b+c}{2} = rs.\end{aligned}$$

Using the formula for the radius of the inscribed circle found above we get:

THEOREM VII. **The area of a triangle ABC is given by the formula:**

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

This formula, known as **Hero's formula**, expresses the area of the triangle in terms of the sides only; it is well known from the study of plane geometry. Together with the formula derived in **76,** it suffices to determine the area of the triangle in all cases which we have considered.

(3) *A formula for the radius R of the circumscribed circle.*

We draw (see Fig. 46) the diameter AOD and connect C with D.

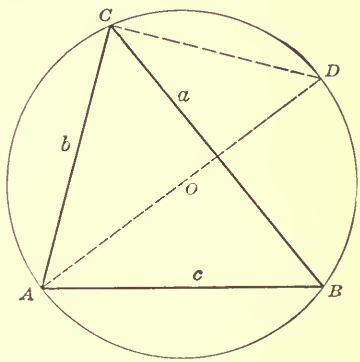


Fig. 46

Then $\angle ACD$ is a right angle (Why?) and $\angle ADC = \angle ABC = B$ (Why?). Hence $\sin B = \sin ADC = \dfrac{AC}{AD} = \dfrac{b}{2R}$. From this it follows hat $R = \dfrac{b}{2 \sin B}$. By means of the law of sines, we derive from this wo other expressions for R, so that we can now state the following theorem:

Theorem VIII. **The radius of the circumscribed circle of a triangle ABC is given by the formulae:**

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}.$$

93. Summary of the results of Chapter VII.

1. Law of sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, see **76.**

2. Law of cosines: $a^2 = b^2 + c^2 - 2\,bc \cos A$, see **81.**

$$b^2 = c^2 + a^2 - 2\,ca \cos B.$$
$$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

3. Law of tangents: $\tan \frac{1}{2}(A - B) = \frac{a-b}{a+b} \cot \frac{1}{2} C$, see **84.**

$$\tan \frac{1}{2}(B - C) = \frac{b-c}{b+c} \cot \frac{1}{2} A.$$

$$\tan \frac{1}{2}(C - A) = \frac{c-a}{c+a} \cot \frac{1}{2} B.$$

4. Half-angle formulae for the triangle:

$$\tan \frac{1}{2} A = \frac{p}{s-a}, \qquad \text{see } \mathbf{89.}$$

$$\tan \frac{1}{2} B = \frac{p}{s-b},$$

$$\tan \frac{1}{2} C = \frac{p}{s-c},$$

where $$p = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

5. Mollweide's equations:

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}, \quad \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C}, \text{ see } \mathbf{85.}$$

$$\frac{b+c}{a} = \frac{\cos \frac{1}{2}(B-C)}{\sin \frac{1}{2} A}, \quad \frac{b-c}{a} = \frac{\sin \frac{1}{2}(B-C)}{\cos \frac{1}{2} A},$$

$$\frac{c+a}{b} = \frac{\cos \frac{1}{2}(C-A)}{\sin \frac{1}{2} B}, \quad \frac{c-a}{b} = \frac{\sin \frac{1}{2}(C-A)}{\cos \frac{1}{2} B}.$$

6. The area of triangle:

a. $\triangle = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$, see **76.**

b. $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$, see **92.**

7. The radius of the inscribed circle:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \qquad \text{see } \mathbf{92.}$$

8. The radius of the circumscribed circle:

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}, \qquad \text{see } \mathbf{92.}$$

9. Methods for solving triangles:

Case	Solved by use of:		Checked by use of:
	Without logs	With logs	
Two angles and one side.	Formulae 1, 6a	Formulae 1, 6a	Formulae 5
Two sides and an angle opposite one of them.	Formulae 1, 6a	Formulae 1, 6a	Formulae 5
Two sides and the included angle.	Formulae 2, 6a	Formulae 3, 6a	Formulae 5
Three sides.	Formulae 2, 6b	Formulae 4, 6b	$\angle A + \angle B + \angle C = 180°$

94. Exercises.

1. Prove the law of sines by means of the method indicated in **92** (3).

2. Show, in Figure 45, that $BD + CE + EA = s$.

3. Prove, in Figure 45, that $CE = DC = s - c$.

4. Use **92,** (3) to show that the area of a triangle is equal to $\frac{abc}{4R}$.

Determine the area, and the radius of the circumscribed circle for each of the following triangles:

5. $a = .473$, $b = .586$, $C = 23° 47' 12''$.

6. $x = 3.045$, $Y = 47° 28'$, $Z = 65° 34'$.

7. $a = 41.35$, $b = 36.78$, $A = 35° 27'$.

8. $c = 632$, $a = 571$, $B = 30°$.

Determine the area, and the radius of the inscribed circle for each of the following triangles:

9. $a = 40.37$, $b = 31.56$, $c = 27.08$.

10. $a = .971$, $b = .506$, $c = .683$.

11. $a = 437$, $c = 856$, $C = 38° 41'$.

12. $a = .0456$, $b = .0731$, $C = 74° 26'$.

95. Miscellaneous exercises on Chapter VII and applications.

1. The area of a triangular piece of land is 43 acres. One side measures 440 yards, and the angle at one extremity of this side is 43°. What must the remaining sides and angles be? (1 acre = 4840 sq. yds.)

2. A lighthouse is observed N 15° W from a vessel which is sailing 15 miles an hour in a due northerly course. Half an hour later the bearing of the same lighthouse is N 37° W. How far is the lighthouse from the second position of the vessel and how long will it be before the lighthouse is sighted due West?

3. From the top of a mountain the angles of depression of two consecutive milestones in the same vertical plane with the top of the mountain are 10° and 15°. How high is the mountain?

4. A forester observes that the angle of elevation of an observation tower from a point P is 5°; after walking towards the tower along a horizontal road for a distance of 500 feet, he observes that the angle of elevation has changed to 35°. How much farther will he have to go to reach the foot of the tower?

Calculate the unknown parts and the area of each of the triangles indicated in Exs. 5–7:

5. $a = 47.032$, $b = 35.614$, $A = 27° 45' 16''$.

6. $a = 3$, $b = 5$, $c = 7$.

7. $a = 23$, $c = 17$, $C = 42° 23'$.

Determine the area and the radii of the inscribed and circumscribed circles in each of the triangles indicated in Exs. 8–10:

8. $a = 256$, $C = 17° 13'$, $B = 45° 16'$.

9. $b = 2.25$, $c = 1.75$, $A = 54°$.

10. $a = 15$, $b = 17$, $c = 25$.

11. The ratio of the lengths of two sides of a triangle is 5 : 8, the included angle is 35°. Determine the other angles of the triangle.

12. A wireless tower is built on the edge of a cliff. From a boat at sea, the angle of elevation of the top of the tower is 30°. After rowing towards the shore for a distance of 400 feet it is found that the angles of elevation of the bottom and top of the tower are 45° and 57° respectively. How high is the cliff and how high is the tower?

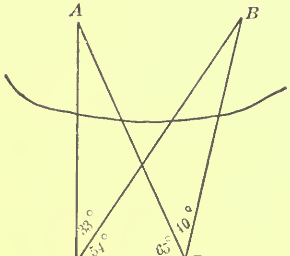


Fig. 47

13. To determine the distance between two points A and B, situated on the surface of a lake, two points C and D are selected in such a manner that both A and B are visible from C and D (see Fig. 47). It is found that CD

= 200 feet and that $\angle ACB = 33°$, $\angle BCD = 54°$, $\angle ADB = 40°$ and $\angle CDA = 65°$. Show that $AC = \frac{200 \sin 65°}{\sin 28°}$, and that $BC = \frac{200 \sin 75°}{\sin 21°}$. Then calculate the distance AB.

14. A vessel is sailing due east at the rate of 20 miles per hour. At 10 A.M. a lighthouse L is bearing N 10° W, while a second lighthouse M is bearing N 40° E; at 2 P.M. the bearings of L and M are N 50° W and N 5° E respectively. How far are L and M apart?

15. Determine also from the observations recorded in Problem 14 the direction of the line from L to M.

16. To measure the height of a mountain AF, above a horizontal plane P, we measure the distance between two stations B and C (see Fig. 48), so selected

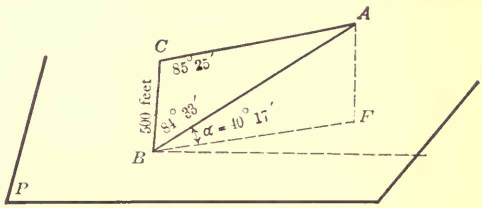


FIG. 48

that at least one of them, say B, lies in the plane P, that each is visible from the other and that A is visible from both. The angles ACB, ABC and the angle of elevation α of A as seen from B are measured. It is found that $BC = 500$ feet, $\angle ACB = 85° 25'$, $\angle ABC = 84° 33'$ and $\alpha = 40° 17'$. Determine the height of the mountain above the plane P.

17. The angle of elevation of a church steeple T from a point R is 17° 25'. At a point S, 250 feet from R, the line TR subtends an angle of 73° 47', while from R the line TS subtends an angle of 65° 8'. Determine the height of T above the horizontal plane through R.

18. A flag staff on top of a monument subtends an angle of 3° at a point P, 400 feet above the ground and at a horizontal distance of 300 feet from the foot of the monument. From the same point P the monument itself subtends an angle of 43°. Determine the height of the flag staff and the height of the monument.

19. One side of a triangle is 75 feet, and the angle opposite this side is 34°. The sum of the other sides is 125 feet. Determine all the sides and angles of this triangle.

20. A building 20 feet high is surmounted by a steeple 30 feet high. How far from the foot of the building must an observer stand in order that the building and steeple may subtend the same angle at his eye, which is 5 feet above the ground?

21. A pole 10 feet tall is divided into two parts, a lower part of 6 feet and an upper part of 4 feet. From a point P, 7 feet above the bottom of the pole, these two parts subtend the same angle, the pole being held vertically. How far is the pole from P?

22. The angle of elevation of a church steeple from a point A, due south of it, is 27°; and from a point B, due west of the steeple, and in the same horizontal plane as A, the angle of elevation is 35°. Moreover the distance AB is 150 yards. Determine the height of the steeple above the plane AB.

23. From a point A the angle of elevation of the top T of a flagpole which stands on top of a building is 37° 47′. From a point B, 100 feet nearer to the building and lying in the vertical plane through T and A, the angle of elevation of T is 47° 32′. How high is T above the ground?

24. The angle of elevation of the bottom of the flagpole described in Problem 23, as seen from B, is 45° 25′. Determine the height of the pole.

25. To determine the height XY of a wireless-tower X above a horizontal plane P two points A and B are selected in this plane P (see Fig. 49). The

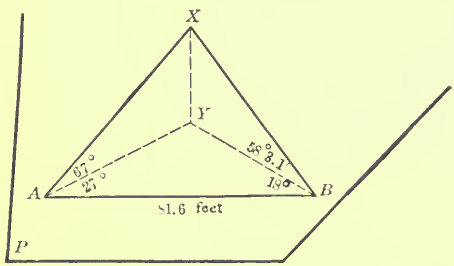


FIG. 49

angles XAY, XBY, YAB and ABY and the distance AB are measured. It is found that $\angle XAY = 67°$, $\angle XBY = 58° 3'.1$, $\angle YAB = 27°$, $\angle YBA = 18°$ and $AB = 81.6$ feet. Determine the height XY and check the calculations.

CHAPTER VIII

INVERSE TRIGONOMETRIC FUNCTIONS. TRIGONOMETRIC EQUATIONS

96. Inverse functions. The concept *function* which we have used in preceding chapters will now be studied in somewhat greater detail. The concept may be defined for our purpose in the following way:

DEFINITION I. **If a relation between two variables, x and y, is given in such a way that to every value of either there correspond one or more values of the other, then x is a function of y, and y is a function of x.**

If these two functional dependences of y upon x and of x upon y are written down explicitly, we obtain two functions, one in terms of x, yielding y, the other in terms of y, yielding x. In the first of these functions x is the **independent** variable and y the **dependent** variable; in the second y is the independent variable and x the dependent variable. Two such functions which result from the same relation between the two variables are called **inverse** with respect to each other. This may be expressed as follows:

DEFINITION II. **If a relation between two variables, x and y, be solved in turn for x in terms of y and for y in terms of x, we obtain two functions which form a pair of inverse functions.**

The important connections between them are recognized more readily if one letter be used for the independent variable in both functions and another letter for the dependent variable in both functions.

Example 1. The relation between the variables x and y determined by the equation

$$3x + 2y - 7 = 0 \quad (1)$$

gives rise to the functions

$$y = \frac{7 - 3x}{2} \quad (2) \quad \text{and} \quad x = \frac{7 - 2y}{3} \quad (3).$$

The graphical representation of the relation between x and y can readily be obtained from equations (2) or (3); in either case we will obtain the straight line of Figure 50.

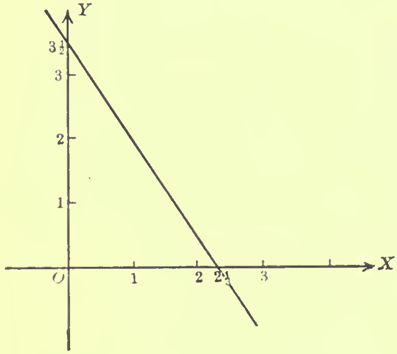


FIG. 50

Denoting the independent variable by t and using u to denote the dependent variable in both cases, equations (2) and (3) will become

$$u = \frac{7 - 3t}{2} \quad \text{and} \quad u = \frac{7 - 2t}{3}.$$

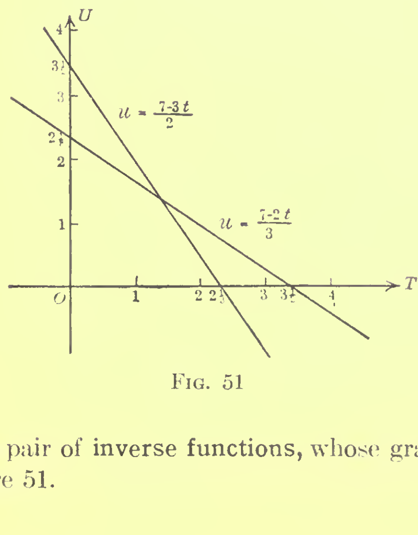


FIG. 51

Here we have a pair of inverse functions, whose graphs are represented in Figure 51.

Example 2. The relation $y^2 - x = 0$ gives rise to the pair of inverse functions

$$u = t^2 \quad \text{and} \quad u = \sqrt{t},$$

whose graphs are given in Figure 52.

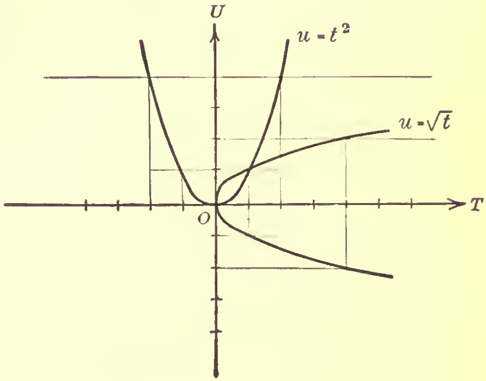


FIG. 52

In the first function of this pair, *one positive value of u corresponds to every value of t*; but in the other function *two values of u correspond to every positive value of t*, and no value of u to negative values of t. The function $u = t^2$ is a **single-valued** function of t, defined for all values of t; the inverse function $u = \sqrt{t}$ is a **two-valued** function of t defined for positive values of t only. It may happen that the inverse function of a single-valued function is 3-valued, or, in general, **multiple-valued**.

97. Graphs of inverse functions. Let the functions $u = f(t)$ and $u = g(t)$ be a pair of inverse functions whose graphs are the curves drawn in Figure 53. If the point $P(a, b)$ belongs to the graph of $u = f(t)$, then the point $Q(b, a)$ must be on the graph of $u = g(t)$, because the two functions may be thought of as having been obtained from one and the same equation between x and y, x and y being replaced by t and u in one case, and by u and t in the other. Now it is readily seen that two such points $P(a, b)$ and $Q(b, a)$ are symmetrically situated with respect to the 45° line MN (see Fig. 54), i.e., that MN is the perpendicular bisector of the line PQ. For $\triangle$s ROP and QOS are congruent. (Why?)

Hence $PO = QO$ and $\angle ROP = \angle SOQ$. Consequently $\angle POT = \angle QOT$ (Why?) and $\triangle POT$ and $\triangle QOT$ are congruent. (Why?) Therefore $PT = QT$ and $\angle PTO = \angle QTO = 90°$, which

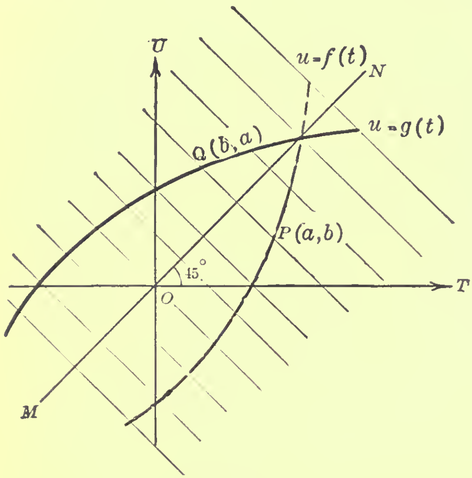


Fig. 53

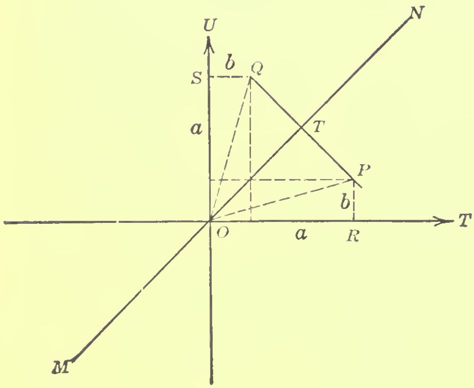


Fig. 54

was to be proved. Consequently from points A, B on the graph of any function, points A', B', . . . on the graph of the inverse function may be obtained by determining A', B', . . . as points which are symmetrically situated with A, B, . . . with respect to the 45° line MN, i.e., by reflecting the points A, B, . . . in

MN as a mirror*. These results may be summarized in the following theorem:

THEOREM I. **If $u = f(t)$ and $u = g(t)$ are a pair of inverse functions the graph of either function may be obtained by reflecting the graph of the other function in the 45° line as a mirror.**

98. Exercises.

Determine the inverse function associated with each of the following functions:

1. $u = 4t - 3$. **2.** $u = t^2/3$. **3.** $u = \log_{10} t$. **4.** $u = t^3$.

Construct the graphs of the following pairs of inverse functions:

5. $u = \sqrt{t-4}, \quad u = t^2 + 4$.

6. $u = t/3, \quad u = 3t$.

7. $u = \sqrt{9 - t^2}, \quad u = \sqrt{9 - t^2}$.

8. $u = \dfrac{3t-7}{5t-4}, \quad u = \dfrac{4t-7}{5t-3}$.

99. The inverse sine function. The equation $y = \sin x$ establishes a relation between the variables x and y, represented graphically by the sine curve (see dashed curve in Fig. 55). For

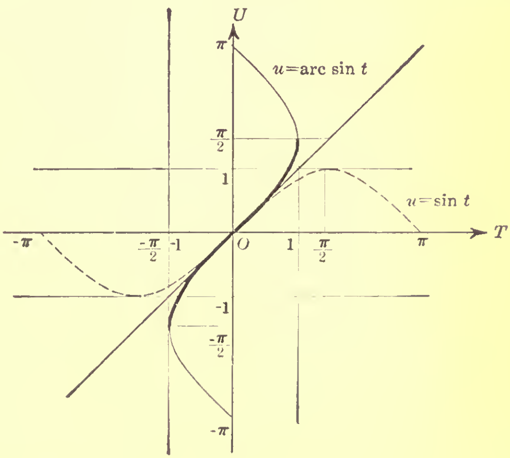


FIG. 55

every value of x there is one corresponding value of y. If the equation be solved for x in terms of y, and x and y be replaced by u and t respectively, we obtain the inverse sine function; it is expressed in the form $u = \text{arc} \sin t$, which is read "u is the angle

* See footnote on page 45.

whose sine is t." The graph of the inverse sine function of t, i.e., of arc sin t, may be obtained, in virtue of Theorem I, by reflecting the sine curve in the 45° line as a mirror. In this way we obtain the full-drawn curve of Figure 55.

The inverse sine function of t cannot be expressed in a simple manner in terms of algebraic or trigonometric functions of t. It must be regarded as an entirely new function, defined as the inverse of the sine function. From this definition its properties will be derived by the aid of its graph.

We observe that the function $u = \text{arc sin } t$ gives an indefinitely large number of values of u for every value of t, which lies between -1 and 1, and that it gives no value of u at all for values of t which lie outside the range $(-1, 1)$. The inverse sine function is an infinitely multiple-valued function, defined only for values of t between -1 and 1.

So, e.g., arc sin $\frac{1}{2} = 30°, 150°, 510°, -210°, \ldots$ just as sin 30 $= \sin 150° = \sin 510° = \sin(-210°) = \cdots = \frac{1}{2}$.

DEFINITION III. **The symbol Arc sin t, called the principal value of arc sin t, is used to designate the numerically smallest angle whose sine is equal to t.**

For example Arc sin $\frac{1}{2} = 30° = \pi/6$.

It is clear that Arc sin t is a single-valued function of t, defined for values of t between -1 and $+1$. Its graph is given by the heavily drawn portion of the graph of arc sin t in Figure 55. When t is between 0 and 1, Arc sin t will be between 0 and $\frac{\pi}{2}$; when t is between -1 and 0, Arc sin t will be between $-\frac{\pi}{2}$ and 0.

100. The other inverse trigonometric functions. In a similar manner the equations $u = \text{arc cos } t$, $u = \text{arc tan } t$, $u = \text{arc cot } t$, $u = \text{arc sec } t$ and $u = \text{arc cosec } t$ represent the inverse functions associated with the remaining trigonometric functions cos t, tan t, cot t, sec t, and cosec t respectively. These equations are read "u is the angle whose cosine is equal to t," etc. The graphs of the inverse trigonometric functions are obtained from the graphs of the associated trigonometric functions by the use of Theorem I, i.e., by reflecting the latter in the 45° line as a mirror. In Figures 56 and 57, the graphs of the functions arc cos t and arc tan t are indicated by the full-drawn curves. We observe that both of these functions are infinitely multiple-valued for those

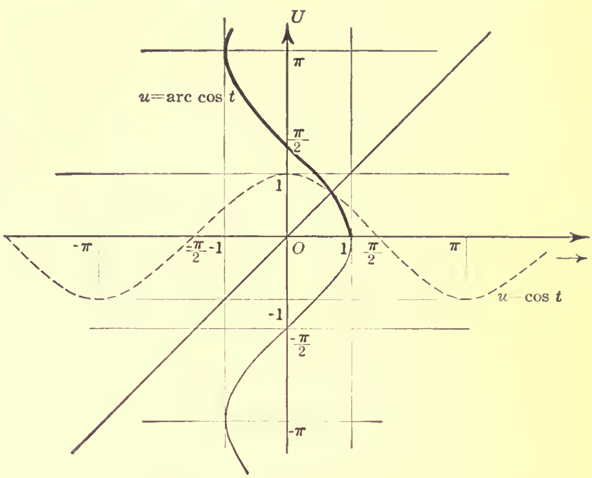

U
π
u=arc cos t
π/2
1
T
-π
-π/2 -1
O
1 π/2
π
u = cos t
-1
-π/2
-π

Fig. 56

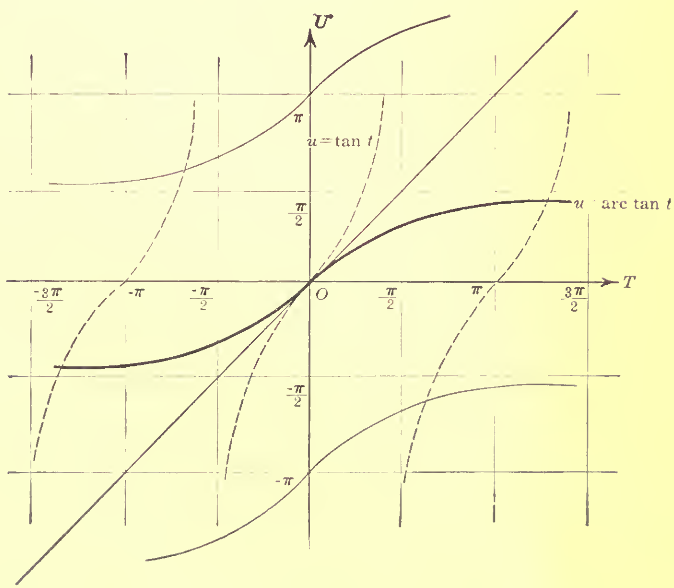

U
π
u=tan t
π/2
u = arc tan t
T
-3π/2
-π
-π/2
O
π/2
π
3π/2
-π/2
-π

Fig. 57

values of the independent variable for which they exist at all; i.e., for *every value of t between* -1 *and* 1 there is an infinitely large number of values of arc cos t; while for *every* value of t there is an infinitely large number of values of arc tan t.

If cot $u = t$, then tan $u = 1/t$; hence arc cot t = arc tan $1/t$, i.e., an angle whose cotangent is equal to t is equal to an angle whose tangent is equal to $1/t$. For an analogous reason we have arc sec t = arc cos $1/t$ and arc cosec t = arc sin $1/t$. Since the functions arc cot t, arc sec t, and arc cosec t are so readily expressible by means of the functions arc tan t, arc cos t, and arc sin t respectively, we shall limit our study to the latter three functions.

As in the case of arc sin t, we define single-valued functions corresponding to the multiple-valued functions arc cos t and arc tan t.

Definition IV. **The symbol Arc cos t, called the principal value of arc cos t, designates the least positive angle whose cosine is equal to t.**

Definition V. **The symbol Arc tan t, called the principal value of arc tan t, designates the numerically smallest angle whose tangent is equal to t.**

The graphs of the single-valued functions Arc cos t and Arc tan t are given by the heavily drawn portions of Figures 56 and 57 respectively.

When t lies between 0 and 1, Arc cos t is between 0 and $\pi/2$;
when t lies between -1 and 0, Arc cos t is between $\pi/2$ and π;
when t is positive, Arc tan t is between 0 and $\pi/2$;
when t is negative, Arc tan t is between $-\pi/2$ and 0.

101. Exercises.

1. Determine arc cos $\frac{1}{2}$; Arc tan 1; Arc cot $\sqrt{3}$; Arc cos 1; arc sin 0; Arc cos $1/\sqrt{2}$.

2. Evaluate sin (arc sin $\frac{1}{3}$); cos (arc cos $\frac{2}{5}$); tan (arc tan $\sqrt{7}$).

3. Determine: cos (arc tan $1/\sqrt{3}$); tan (arc sin $\sqrt{\frac{1}{2}}$); sec (arc cos $\frac{1}{4}$); sin (arc cos .4321); cos (Arc sin 0); cot (Arc tan $\sqrt{3}$).

4. Evaluate: cos (Arc cos $\frac{1}{2}$ + Arc sin $\frac{1}{2}$); tan (Arc sin $1/\sqrt{2}$ − Arc tan 1); sin (Arc tan $\sqrt{3}$ + Arc cos $1/\sqrt{2}$).

5. Construct the graphs of the functions u = arc sin $2t$; u = arc sin $t/3$; u = arc cos $t/3$.

102. Relations between multiple-valued and single-valued inverse functions. There is a simple algebraic relation between the multiple-valued function arc sin t and the single-valued function Arc sin t. If u = Arc sin t, we know that

$t = \sin u = \sin(n \cdot 180° + u)$, when n is an even integer, and that $t = \sin u = \sin(n \cdot 180° - u)$, when n is an odd integer (see Chapters V and VI).

Hence $\quad \text{arc sin } t = n \cdot 180° + \text{Arc sin } t$, when n is even,

and $\quad \text{arc sin } t = n \cdot 180° - \text{Arc sin } t$, when n is odd.

These two equalities, which may be derived directly from the graph of arc sin t (see Fig. 55), are combined in the single formula:

$$\textbf{arc sin } t = n \cdot 180° + (-1)^n \textbf{ Arc sin } t.$$

In entirely analogous manner we obtain the further formulae:

$$\textbf{arc cos } t = n \cdot 360° \pm \textbf{Arc cos } t,$$
$$\textbf{arc tan } t = n \cdot 180° + \textbf{Arc tan } t.$$

103. Trigonometric equations. Let us consider the following problem: To construct a rectangle whose perimeter is 10″ and whose diagonal is 4″.

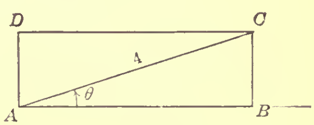


FIG. 58

If $ABCD$ be the required rectangle and θ the angle between the diagonal AC and the side AB (see Fig. 58), we have:

$$AB = AC \cos \theta = 4 \cos \theta$$

and

$$BC = AC \sin \theta = 4 \sin \theta.$$

The conditions of the problem are therefore expressed by the equation

$$8 \cos \theta + 8 \sin \theta = 10.$$

This equation involves trigonometric functions of the unknown θ; it is called a **trigonometric equation.** Equations of this sort occur in various branches of mathematics and in its applications. We consider their solution.

104. Solution of trigonometric equations. "To solve a trigonometric equation" means to determine all possible angles which satisfy the equation. It is important to notice the difference between a trigonometric identity, which is true for *all angles* for which it has a meaning, and a trigonometric equation, which is satisfied by certain angles only. The determination of these special angles constitutes the "solution of the trigonometric equation."

The work of solving a trigonometric equation may be divided into three parts, viz.:

(1) To express all the trigonometric ratios occurring in the equation in terms of a single ratio of a single angle,

(2) To solve the resulting equation obtained in (1), as an algebraic equation for that ratio, and

(3) To determine all the angles corresponding to the values for that ratio found in (2).

Returning to the example of **103**, we proceed as follows:

(1) We express $\cos\theta$ in terms of $\sin\theta$ by means of the formula: $\cos\theta = \sqrt{1 - \sin^2\theta}$. Substituting this in the equation derived in **103**, we obtain the equation

$$\sin\theta + \sqrt{1 - \sin^2\theta} = \tfrac{5}{4},$$

or

$$\sqrt{1 - \sin^2\theta} = \tfrac{5}{4} - \sin\theta.$$

(2) We square both sides of this equation, collect terms and clear of fractions. In this way we obtain the following quadratic equation in terms of $\sin\theta$:

$$32\sin^2\theta - 40\sin\theta + 9 = 0.$$

This equation we solve by means of the quadratic formula, which gives us:

$$\sin\theta = \frac{40 \pm \sqrt{1600 - 1152}}{64} = \frac{40 \pm \sqrt{448}}{64} = \frac{40 \pm 21.1660}{64};$$

$$\therefore \quad \sin\theta = 61.1660/64 \quad \text{or} \quad 18.8340/64.$$

(3) From this we conclude:

$$\theta = \text{arc}\sin 61.1660/64 \quad \text{or} \quad \text{arc}\sin 18.8340/64.$$

By means of the tables, we find

$$\text{Arc sin } 61.1660/64 = 72° \, 53' \text{ and Arc sin } 18.8340/64 = 17° \, 7'.$$

$$\therefore \quad \theta = n \cdot 180° + (-1)^n \, 72° \, 53' \text{ or } n \cdot 180° + (-1)^n \, 17° \, 7'.$$

The nature of the problem is such that only acute angles θ are applicable, so that we need only consider the principal values of θ. Using these, we find:

$$AB = 4 \cos \theta = 1.177 \text{ or } 3.823,$$

and $$BD = 4 \sin \theta = 3.823 \text{ or } 1.177;$$

we verify that $$2\,AB + 2\,BD = 10.$$

105. Further examples.

1. *To solve the equation*

$$\tan \theta \cdot \tan 2\,\theta + \cot \theta + 2 = 0,$$

we use the formulae:

(1) $$\tan 2\,\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \quad \text{and} \quad \cot \theta = \frac{1}{\tan \theta}$$ (see **74**, 9).

Substituting these expressions in the original equation, it becomes:

$$\frac{2 \tan^2 \theta}{1 - \tan^2 \theta} + \frac{1}{\tan \theta} + 2 = 0.$$

(2) This is an algebraic equation in terms of $\tan \theta$; we clear the equation of fractions and collect terms. This leads us to the following quadratic equation:

$$\tan^2 \theta - 2 \tan \theta - 1 = 0.$$

If we solve this equation for $\tan \theta$ by means of the quadratic formula, we find:

$$\tan \theta = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2} = 2.4142 \text{ or } -.4142.$$

(3) It remains to determine arc tan 2.4142 and arc tan $-.4142$. From the tables, we find: Arc tan .4142 = 22° 30′, and therefore Arc tan $(-.4142) = -22° \, 30'$, whence follows arc tan $(-.4142)$ $= n \cdot 180° - 22° \, 30'$.

Furthermore Arc tan 2.4142 = $67° \, 30'$, and therefore arc tan 2.4142 = $n \cdot 180° + 67° \, 30'$.

2. *To solve the equation*

$$3 \sin^2 \theta - 4 \sin \theta - 4 = 0.$$

In this problem the object of step (1) has already been accomplished, since the equation in its original form is an algebraic equation in terms of $\sin \theta$.

Solving it by means of the quadratic formula, we find:

$$\sin \theta = 2 \text{ or } -\tfrac{2}{3} = -.6667.$$

There are no angles whose sine equals 2; we have only to determine therefore arc sin $(-.6667)$. From the tables, we find that Arc sin $.6667 = 41° \, 49'$; hence Arc sin $(-.6667) = -41° \, 49'$, from which we obtain the final result:

$$\theta = \text{arc sin } (-.6667) = n \cdot 180° + (-1)^n \cdot (-41° \, 49').$$

For: $n = 1$, we find $\theta = 180° + 41° \, 49' = 221° \, 49'$;
$n = 2$, $\theta = 360° - 41° \, 49' = 318° \, 11'$;
$n = -1$, $\theta = -180° + 41° \, 49' = -138° \, 11'$;
$n = -2$, $\theta = -360' - 41° \, 49' = -401° \, 49'$;
$n = 0$, $\theta = -41° \, 49'$; etc.

The method outlined above for the solution of trigonometric equations is not always the shortest or the most convenient one. Frequently other special methods enable us to obtain a solution more quickly or in better form. The student will do well to bear this in mind, particularly in cases in which the algebraic equation, to which the method described above leads, is such that he cannot solve it. One such special method is illustrated by the following example.

3. *To solve the equation*

$$\cos 2x - \sin 2x = 1.$$

Since $\tan 45° = 1$, we may write this equation in the form

$$\tan 45° \cos 2x - \sin 2x = 1,$$

from which we derive, by multiplying both sides of the equation by the corresponding sides of the equality $\cos 45° = 1/\sqrt{2}$,

$$\sin 45° \cos 2x - \cos 45° \sin 2x = 1/\sqrt{2},$$

or

$$\sin (45° - 2x) = 1/\sqrt{2}.$$

From this we conclude that we must have

$$45° - 2x = \text{arc sin } 1/\sqrt{2} = n \cdot 180° + (-1)^n \cdot 45°,$$

i.e.,
$$2x = 45° - n \cdot 180° - (-1)^n \cdot 45°,$$

or finally:
$$x = \frac{45° - (-1)^n \cdot 45°}{2} - n \cdot 90°.$$

For $n = 0, 1, -1, 2, -2$, we obtain the special values $x = 0°$, $x = -45°$, $x = 135°$, $x = -180°$, $x = 180°$ respectively.

The same method may be applied in solving any equation of the form

$$a \cos m\,\theta + b \sin m\,\theta = c,$$

provided

$$|c| \leqq \sqrt{a^2+b^2}.$$

We divide both sides of the equation by b and determine the angle α by the formula $\alpha = \text{Arc tan } a/b$. Then $\tan \alpha = a/b$, and $\cos \alpha = \frac{b}{\sqrt{a^2+b^2}}$, the square root being taken with the sign of b; then the equation will take the following form:

$$\tan \alpha \cos m\,\theta + \sin m\,\theta = c/b.$$

We now multiply both sides of the equation by $\cos \alpha$ and make use of the addition formula for the sine; in this manner we find

$$\sin (\alpha + m\,\theta) = \frac{c \cos \alpha}{b} = \frac{c}{\sqrt{a^2+b^2}}.$$

Since now $|c| \leqq \sqrt{a^2+b^2}$, the right hand side of this equation is always numerically less than 1. It may therefore be solved for $\alpha + m\,\theta$, so that we obtain the final result:

$$\theta = \frac{1}{m}\left[-\alpha + \text{arc sin } \frac{c}{\sqrt{a^2+b^2}}\right],$$

where $\alpha = \text{Arc tan } a/b$, and where the square root is to be taken with the sign of b.

106. Exercises.

1. Solve the equation: $\tan x + 3 \cot x = 4$.

2. Solve the equation: $\sin \theta + \cos \theta = \sqrt{2}/2$.

3. Solve the equation: $\tan \theta - 3 \cos \theta + \sec \theta = 0$.

4. Solve the equation: $\cos 2x = 3 \sin x + 2$.

5. Solve the equation: $\sin \theta = \tan \theta/2$.

6. Solve the equation: $\cos 2A = 2 \sin^2 A$.

7. Solve the equation: $4 \sec^2 \theta = 3 (\tan \theta + 2)$.

8. Determine all the angles whose tangent is the reciprocal of their sine.

9. Determine all the angles whose secant is the reciprocal of their tangent.

10. The graphs of the functions $\sin \theta$ and $\tan \theta$ are drawn with respect to one set of axes and units. Determine all those values of θ for which the ordinate on the tangent curve is twice as long as the corresponding ordinate on the sine curve.

11. Construct an isosceles triangle whose perimeter is 20 inches and whose altitude is 6 inches. (HINT. Take the base angle of the triangle as the unknown.)

12. Construct a right triangle of which the perimeter is 36 inches and one leg is 12 inches.

13. Determine a point P on the circumference of a circle of radius 5 inches, the sum of whose distances from the extremities of a fixed diameter AB is equal to 14 inches.

14. Construct a triangle ABC such that the perimeter is equal to 30 inches, the side c is equal to 10 inches and the angle B is equal to 60°.

15. Resolve a force of 100 pounds into two mutually perpendicular components, whose sum is equal to 120 pounds.

16. Construct a triangle ABC of which the perimeter is 60 feet the side a is equal to 20 feet and the angle B is equal to 45°.

ANSWERS

Art. 4, Page 3

2. $(3,3\sqrt{3})$. **3.** $(2,-2)$. **6.** $(0,0)$. **7.** $(3,5)$; $(5,0)$; $(1,1)$. **8.** $5, \sqrt{41}, \sqrt{52}, 5\sqrt{2}$. **9.** ± 4; ± 5; impossible; $\pm\sqrt{3}$.

Art. 9, Page 6

2. 65°, 12°, 113°, −64°, −127°, 202°, −235°, −337°, 598°. **3.** 91°, 53°, 392°, −15°, 495°, −107°, −333°, 639°.

Art. 11, Page 8

1. $\pi/4$, $\pi/6$, $-5\pi/4$, $11\pi/6$, 3π, $-5\pi/6$, $\pi/3$, $-3\pi/2$, $-9\pi/2$, $7\pi/6$, $8\pi/3$, $-65\pi/18$. **2.** 270°, −240°, 72°, 225°, −90°, $270°/\pi$, $414°/\pi$. **5.** 300π. **6.** 10 feet. **7.** $144°/\pi$. **8.** 2 feet. **9.** 6000π. **10.** .84 feet.

Art. 22, Page 14

2*a*. 1. **2*b*.** $-\sqrt{3}/2$. **2*c*.** 0. **2*d*.** $(-3+\sqrt{3})/4$. **3*a*.** $-7\sqrt{2}/12$. **3*b*.** $3/2$. **3*c*.** $(-2+\sqrt{3})/3$, **3*d*.** $46/9$.

Art. 25, Page 17

3*a*. $-13/4$. **3*b*.** 4. **3*c*.** $-1/\sqrt{3}$. **3*d*.** -6.

Art. 28, Page 19

2. $\sin\theta = \pm 4/\sqrt{17}$, $\cos\theta = \pm 1/\sqrt{17}$, $\cot\theta = 1/4$, $\sec\theta = \pm\sqrt{17}$, $\operatorname{cosec}\theta = \pm\sqrt{17}/4$.

Art. 32, Page 22

1. 3, −3, 1/3, 0, 3/5, 1. **4.** 2/3, 3/2, 3/2, −2/3, −1/4.

Art. 34, Page 23

8. .46831. **9.** .28452. **10.** 2.21638. **11.** .60061. **12.** −.93404.

Art. 39, Page 30

1. 3.1335. **2.** 1.9851. **3.** .0052026. **4.** .5712. **5.** .63014. **6.** 4.2266. **7.** 583.35. **8.** 8.524. **9.** 39.478. **10.** .31365. **11.** 2.0550. **12.** 1.3425. **13.** 4.0198. **14.** 7360.5. **15.** .38638. **16.** 30.628. **17.** .0093403. **18.** .060285. **19.** .34697. **20.** 5.3321.

Art. 44, Page 35

1. $A = 35° \ 20' \ 28''$, $B = 54° \ 39' \ 32''$, $c = 644.83$. **2.** $A = 62° \ 10' \ 8''$, $B = 27° \ 49' \ 52''$, $c = .16193$. **3.** $A = 30°$, $B = 60°$, $b = 3.717$. **4.** $A = 48° \ 10' \ 51''$, $B = 41° \ 49' \ 9''$, $a = 14.609$. **5.** $B = 21° \ 35'$, $a = 22{,}073$, $c = 23{,}737$. **6.** $A = 62° \ 47'$, $a = .12279$, $c = .13808$. **7.** $B = 5° \ 25'$, $a = 376.32$, $b = 35.682$. **8.** $B = 45° \ 25'$, $a = 265.33$, $b = 269.22$. **9.** $B = 44° \ 57'$, $b = .08497$, $c = .12027$. **10.** $A = 49° \ 56' \ 5''$, $B = 40° \ 3' \ 55''$, $b = .057956$. **11.** $A = 16° \ 14' \ 26''$, $B = 73° \ 45' \ 34''$, $c = 14.252$. **12.** $A = 84° \ 58'$, $a = 1.0020$, $b = .088254$. **13.** $V = 65° \ 34'$, $a = 24.751$, $h = 20.809$. **14.** $P = 25° \ 37'$, $a = 19.940$, $h = 8.621$. **15.** $P = 41° \ 51' \ 32''$, $V = 96° \ 16' \ 56''$, $b = 17.950$. **16.** $b = .84246$, $a = 1.1038$, $P = 67° \ 34'$. **17.** $P = 53° \ 34'$, $V = 72° \ 52'$, $a = 20.368$. **18.** $b = .64065$, $a = .32559$, $V = 159° \ 22'$. **19.** $P = 62° \ 49' \ 45''$, $V = 54° \ 20' \ 30''$, $h = 8.8480$. **20.** $b = 12.286$, $h = 2.9235$, $V = 129° \ 6'$. **21.** $b = 23{,}454$, $h = 11{,}727$, $P = 45°$.

Art. 47, Page 38

1. 4.3301 feet. **2.** 2.5 feet. **3.** 0 feet; 7 feet.

Art. 49, Page 39

1. 4,504.8 feet; 4,247.7 feet. **2.** 272.04 feet. **3.** 2,012.2 feet. **4.** 2,641.4 feet. **5.** 27.155 feet. **6.** 52.781 miles; 72.705 miles. **7.** 53.868 feet; 23.246 feet. **8.** 1.3313 miles. **9.** 566.09 feet; 504.38 feet. **10.** 90.21 feet; 125.30 feet. **11.** 7.980 miles. **12.** .059 miles. **13.** 320.43 feet; 640.86 feet. **14.** $8\frac{3}{4}$ feet; 28 feet. **15.** 67.127 miles; 78.142 miles; 104.43 miles.

Art. 67, Page 57

1. $(\sqrt{6} + \sqrt{2})/4$, $(\sqrt{6} - \sqrt{2})/4$.

Art. 69, Page 59

1. $2 + \sqrt{3}$, $2 - \sqrt{3}$.

Art. 71, Page 61

3. $\sqrt{8 - 2\sqrt{6} - 2\sqrt{2}}/4$, $\sqrt{8 + 2\sqrt{6} + 2\sqrt{2}}/4$, $(\sqrt{2} - 1)(\sqrt{3} - \sqrt{2})$.

Art. 73, Page 62

1. $2 \sin 60° \cos 5°$. **2.** $-2 \sin 45° \sin 30°$. **3.** $2 \sin 4° \ 30' \cos 22° \ 30'$. **4.** $2 \cos 64° \ 30' \cos 22° \ 30'$. **5.** $2 \cos 12° \sin 30°$. **6.** $-2 \sin 45° \cos 12°$. **7.** .22490. **8.** 2.8834. **9.** $-$ .24629. **10.** .99726.

Art. 75, Page 63

1*a*. $(\sqrt{6} - \sqrt{2})/4, -(\sqrt{6} + \sqrt{2})/4, \sqrt{3} - 2$. **1*b*.** $\sqrt{2 - \sqrt{3}}/2, -\sqrt{2 + \sqrt{3}}/2, \sqrt{3} - 2$. **4.** .14762.

Art. 78, Page 67

1. $a = .29015$, $c = .23995$, $C = 31° 42'$. **2.** $a = 173.95$, $b = 237.62$, $B = 105°$. **3.** $a = 12.736$, $b = 6.6607$, $A = 62° 55' 25''$. **4.** $b = 170.27$, $c = 145.80$, $B = 84° 20' 50''$. **5.** 559.40 feet. **7.** 137.18 feet.

Art. 80, Page 73

1. $c = 72.612$, $B = 51° 40' 30''$, $C = 92° 44' 30''$; $c' = 20.149$, $B' = 128° 19' 30''$, $C' = 16° 5' 30''$. **2.** $a = 1187.5$, $A = 140° 23' 56''$, $C = 15° 49' 4''$. **3.** No solution. **4.** $b = 5.1662$, $A = 16° 7' 27''$, $B = 26° 34' 33''$. **5.** $c = 79.735$, $B = 26° 15' 36''$, $C = 134° 19' 24''$; $c' = 13.280$, $B' = 153° 44' 24''$, $C' = 6° 50' 36''$. **6.** $a = .082650$, $A = 122° 58'$, $B = 28° 31'$. **7.** 100.91 miles. **8.** 203.87 feet or 195.72 feet. **9.** 61.671 miles.

Art. 82, Page 76

1. $c = 137.48$, $A = 49° 6' 22''$, $B = 70° 53' 36''$. **2.** $P = 53° 7' 50''$, $Q = 59° 29' 25''$, $R = 67° 22' 50''$. **3.** $A = 26° 44'$, $B = 23° 23' 12''$, $C = 129° 52' 48''$. **4.** $t = .56081$, $R = 34° 34' 6''$, $S = 100° 26'$. **5.** 399 feet. **6.** $84° 15' 37''$.

Art. 88, Page 80

1. $a = 97.988$, $B = 52° 31' 18''$, $C = 14° 44' 42''$. **2.** $b = 5247.1$, $A = 36° 54' 20''$, $C = 54° 38' 40''$. **3.** $r = .04343$, $P = 47° 16' 5''$, $Q = 85° 7' 55''$. **4.** $y = 1.2136$, $X = 35° 28' 24''$, $Z = 24° 31' 36''$. **5.** 211.40 feet. **6.** 92.685 million miles.

Art. 91, Page 82

1. $A = 61° 6' 29''$, $B = 58° 46' 4''$, $C = 60° 7' 24''$. **2.** $A = 132° 45' 4''$, $B = 10° 28' 36''$, $C = 36° 46' 26''$. **3.** $X = 35° 39' 50''$, $Y = 81° 56' 20''$, $Z = 62° 23' 50''$. **4.** $A = 44° 24' 50''$, $B = 57° 7' 20''$, $C = 78° 27' 50''$. **5.** Angles should be $85° 50' 24''$, $68° 7' 4''$, $26° 2' 28''$. **6.** N $42° 50' 40''$ E or S $42° 50' 40''$ E.

Art. 94, Page 86

5. $\Delta = .055898$, $R = .3033$. **6.** $\Delta = 3.3797$, $R = 1.6544$. **7.** $\Delta = 697.4$, $R = 35.648$. **8.** $\Delta = 90{,}218$, $R = 316.89$. **9.** $\Delta = 426.6$, $r = 8.6172$. **10.** $\Delta = .16379$, $r = .15166$. **11.** $\Delta = 157{,}375$, $r = 128.71$. **12.** $\Delta = .0016056$, $r = .016573$.

Art. 95, Pages 86–89

1. 1,387.1 yards, 1,106.7 yards, 121° 16′ 7″, 15° 43′ 53″. **2.** 5.1818 miles; 16.6 min. **3.** 2,722.7 feet. **4.** 71.393 feet. **5.** $c = 75.527$, $B = 20° 38′ 55″$, $C = 131° 35′ 49″$, $\Delta = 626.3$. **6.** $A = 21° 47′ 12″$, $B = 38° 12′ 46″$, $C = 120°$, $\Delta = 6.4951$. **7.** $b = 23.962$, $A = 65° 47′ 0″$, $B = 71° 50′ 0″$, $\Delta = 185.75$; $b′ = 10.016$, $A′ = 114° 13′ 0″$, $B′ = 23° 24′ 0″$, $\Delta′ = 77.642$. **8.** $\Delta = 7{,}768.8$, $r = 28.433$, $R = 144.33$. **9.** $\Delta = 1.5928$, $r = .54269$, $R = 1.1556$. **10.** $\Delta = 124.44$, $r = 4.3665$, $R = 12.807$. **11.** 36° 17′ 58″, 108° 42′ 2″. **12.** 239.93 feet, 129.53 feet. **13.** 300.92 feet. **14.** 106.84 miles. **15.** N 75° E. **16.** 1,849.6 feet. **17.** 109.34 feet. **18.** 16.07 feet, 346.40 feet. **19.** 96.742 feet, 28.258 feet, 133° 50′ 14″, 12° 9′ 46″. **20.** $15\sqrt{7}$ feet. **21.** $\sqrt{23}$ feet. **22.** 61.699 feet. **23.** 266.88 feet. **24.** 19.04 feet. **25.** 84.01 feet.

Art. 98, Page 94

1. $u = (t + 3)/4$. **2.** $u = \sqrt{3t}$. **3.** $u = 10^t$. **4.** $u = \sqrt[3]{t}$.

Art. 101, Page 97

1. ±60°, ±420°, etc.; 45°, 225°, etc.; 30°; 0°; 0°; 180°; etc.; 45°. **2.** 1/3; 2/5; $\sqrt{7}$. **3.** $\pm\sqrt{3}/2$; ±1; 4; ±.9018; 1; $1/\sqrt{3}$. **4.** 0; 0; $(\sqrt{3} + 1)/2\sqrt{2}$.

Art. 106, Page 103

1. $45° + n \cdot 180°$, $71° 33′ 54″ + n \cdot 180°$. **2.** $n\pi + (-1)^n \pi/6 - \pi/4$. **3.** $n \cdot 180° + (-1)^n 41° 48′ 39″$. **4.** $n\pi + (-1)^{n+1}\pi/6$, $n\pi + (-1)^{n+1}\pi/2$. **5.** $2n\pi$, $(2n + 1)\pi/2$. **6.** $n\pi \pm (-1)^n\pi/6$. **7.** $n \cdot 180° + 49° 36′ 34″$, $n \cdot 180° - 23° 2′ 39″$. **8.** $n \cdot 360° \pm 51° 49′ 30″$. **9.** $n \cdot 180° + (-1)^n 38° 10′ 30″$. **10.** $n\pi$, $2n\pi \pm \pi/3$, $2n\pi \pm 2\pi/3$. **11.** Base angle = 61° 55′ 40″; base = 6.4. **12.** Sides are 15 and 9; angles are 36° 52′ 12″ and 53° 7′ 48″. **13.** Chords are 8 and 6. **14.** $A = 60°$, $a = b = 10$. **15.** 97.42 and 22.58 pounds. **16.** $b = 16.7964$, $c = 23.204$, $C = 77° 39′$.

INDEX

(The Numbers refer to Articles)