PLANE TRIGONOMETRY

AND

APPLICATIONS

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PREFACE

The characteristic features of this book may be summarized as follows:

1. *The method of presentation is thoroughly heuristic.* This enables the student to get a firm grasp of the subject by teaching him to recognize the fundamental ideas which underlie and unify the separate steps of the mathematical argument, instead of confusing him by a disconnected dogmatic statement of isolated facts.

2. *The book is divided into two parts.* The first part is devoted to the theoretical and numerical solution of triangles. The second part treats of the functions of the general angle, their addition theorems and other properties, together with applications to simple harmonic curves, simple harmonic and wave motion, and harmonic analysis. Part One is also published separately and is well adapted for use in secondary schools. The complete book is intended for the freshman course in colleges.

3. *The discussion of the solution of triangles,* in Part One, is not interrupted by any digressions about coördinate systems, addition theorems, and the like. It has been thought desirable to postpone to the second part the consideration of all of these matters, which are indeed important but unnecessary for the solution of triangles.

4. *The definitions for the functions of an obtuse angle* have been made to grow organically out of the needs of the problem of solving triangles, in a way which seems both simple and natural, and which at the same time illustrates an important principle of mathematical procedure.

5. *The whole theory of triangles has been unified* by giving a central position to the area problem. As a consequence, almost all of the necessary equations present themselves spontaneously and in a connected fashion. The law of tangents is the only one which causes any difficulty in this respect. But the law of tangents also has been made to submit to a heuristic treatment, by

introducing the notion of the *form-ratio* of a triangle, and combining this notion with a direct geometric proof of the formulæ for $\sin A + \sin B$ and $\sin A - \sin B$.

6. *The numerical aspect of the work has been discussed very fully.* Directions for computation are given in great detail; most of the common sources of error are pointed out; and methods for detecting and correcting them are indicated. After a thorough discussion of the significance of the number of decimal places needed in a computation, the student is urged to train and use his judgment on this matter. He is given an opportunity to do this by supplying him with complete five- and three-place tables and a partial set of four-place tables.

7. *The slide rule is explained* with considerable detail and its use recommended. A number of other labor-saving devices are discussed.

8. *The examples have been selected with great care.* Examples without real significance have been avoided, and the numbers have been chosen so as not to lead to five-place calculations when such a show of accuracy would be absurd. Special efforts have been made to word the examples in such a way as to avoid ambiguity.

9. *The applications cover a wider field than usual,* and include problems in heights and distances, surveying, navigation, engineering, astronomy, and physics. But the examples involving such applications are not, as in most texts, introduced at random and without previous explanation. Every notion which is required for the solution of any example in the book is fully explained on the spot or in some earlier portion of the text.

10. *The use of a few new terms,* such as the *standard position* of an angle, odd and even *cardinal angles,* has helped to simplify materially the statement of a number of important results.

11. *The addition formulæ are presented in two different ways.* The first, more elementary method, is made to yield the general result by the help of mathematical induction. The second method, based on the notions of directed lines, line-segments, and angles, appears here in a very simple and elegant form.

12. *The articles on harmonic and wave motion* tend to show the student that Trigonometry has other applications besides the solution of triangles.

13. *A considerable amount of historical matter has been introduced,* not in the form of detached historical notes, but organically connected with the topic under discussion. Most of this matter was gathered from BRAUNMÜHL's *Vorlesungen über die Geschichte der Trigonometrie.* Professors CAJORI and KARPINSKI have kindly answered some questions of a historical nature about which we were in doubt.

14. *The type and the manner of spacing used in the tables* are the results of a number of experiments, the object being to produce a set of tables which should be as pleasant to the eye as possible. The tables are bound separately for various reasons. In order to make them easily legible, a certain size of page was necessary, and it was thought undesirable to use so large a page for the text itself. In the second place, it is a great advantage for the student if he can have his text and his tables open before him at the same time. In the third place, it is often desirable, in examinations, to allow students to use their tables without their books. Finally, a separation of the tables and text makes it easy to use this text with other tables, or these tables with other texts, thus providing a maximum of elasticity in organizing a course.

Many of the older texts on Trigonometry have been consulted during the preparation of this book, and the attempt has been made to learn from all of them. The works of SERRET, LÜBSEN, WIEGAND, CROCKETT, MORITZ, HALL and FRINK have been especially helpful. A few purely numerical examples have actually been taken from these and other texts without change, so as to reduce somewhat the task of computing the answers, and at the same time to make the answers more trustworthy. Most of the examples, however, are new; many of them are new in kind.

In conclusion, the author and editor wish to acknowledge their indebtedness to their colleagues at the University of Chicago for various helpful suggestions and criticisms.

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PLANE TRIGONOMETRY

AND

APPLICATIONS

PART ONE

SOLUTION OF TRIANGLES

CHAPTER I

THE OBJECT OF TRIGONOMETRY

1. Direct measurement of lines. One of the most common operations of practical geometry is that of measuring the distance between two points. In its simplest form this consists merely in the repeated application of some unit of measurement to the required distance.

The units of measurement most frequently used for this purpose are a foot rule, a yardstick, a surveyor's chain, tape lines of definite length, etc. Fractional parts of the unit are usually read from a graduated *scale*, engraved or stamped on the standard used. A familiar illustration of this is the scale of inches on an ordinary foot rule.

In spite of the apparent simplicity of this process, it is a matter of great practical difficulty to carry out such measurements with a high degree of precision. The sources of error are numerous and, in part, unavoidable. No instrument made by man is absolutely accurate. Thus, if we use a yardstick, it will not be absolutely straight, and it may be a trifle too long or too short. It will be very difficult to make sure that we are laying off the second yard of our distance exactly where the first yard ends. Consecutive positions of the yardstick will form angles with each other, which are not exactly equal to 180°. In fact, it is almost impossible to run a straight line of considerable length, with any degree of accuracy, without the help of more complicated instruments, such as the transit described in Art. 2.

The graduated scales, used for measuring fractional parts of the unit of length, are also affected by various sources of inaccuracy, and it will be difficult for the observer to estimate accurately a fractional part of the smallest visible division on the scale.

Enough has been said to indicate just a few of the many difficulties encountered in the, apparently so simple, operation of measuring the length of a line, and to emphasize the fact that we must always regard the result of such a measurement as an *approximation*, even if the most refined instruments known to Science have been used.

The difference between rough and fine measurements is one of degree only. The more refined the method, the smaller will be the "probable error" and the closer the approach to the truth. But we can never be sure that a quantity has been measured with *absolute* precision.

EXERCISE I

1. What are some of the sources of inaccuracy in measuring the length of a table?

2. If you wish to measure the distance *diagonally* across a table, by means of a foot rule, what additional sources of inaccuracy will appear? Would a stretched cord be of some use in this connection?

3. How would you measure the distance diagonally across a room from one of the floor corners to the opposite corner of the ceiling? Do you know of any other method by which this distance might be obtained, except that of direct measurement?

4. How would you join two given points by a straight line (say for the purpose of constructing a fence), over a level piece of ground, if the distance is too great to enable you to stretch a cord? Do you know of any property of the line of sight which might help in the solution of such a problem?

5. If you attempt to measure two different distances, of which one is about ten times as great as the other, using the same foot rule and the same method of measurement in both cases, which of these two distances will probably be obtained with greater accuracy? Why?

6. What difficulties arise, and how would you attempt to meet them, if you were asked to measure the horizontal distance between two points on an uneven piece of ground?

7. Suppose you have measured a distance (say the edge of a table) with great care and have found it to be equal to 4 feet and $9\frac{3}{8}$ inches. Is this the *exact* length of the table, or may it be a small fraction of an inch greater or less?

8. If you were to repeat the measurement with still greater care, making use of a more perfect standard of length, is it likely that you would find exactly the same result as before?

9. In any such measurement can you ever attain *absolute* accuracy? If not, why not?

10. Is there any way of *knowing* whether a measurement is *absolutely* accurate? Is there any way of knowing whether it is accurate to within a certain limit of accuracy, say one tenth of an inch?

11. If a distance has been measured by a process which may be relied upon to give a result accurate only to one one-hundredth of an inch, is it desirable, proper or honest, to give the result expressed in decimal parts of an inch to more than two decimal places? Why not? In performing calculations based upon such measurements, how many decimal places should we ordinarily carry?

12. In measuring distances by means of metal rods, when great accuracy is required, changes of temperature must be taken into account. Why?

13. With what units of length are you familiar?

14. Gather from an encyclopedia what you can concerning the "standard yard" kept at Washington.

15. What is the metric system? What are the relations to each other of the units called millimeter, centimeter, decimeter, and meter? What is the length of a meter in inches?

16. Can you see any reason why the metric system should be preferable to the English system of weights and measures?

2. Direct measurement of angles. The operation of measuring angles is scarcely less important than that of measuring distances. A **protractor** is an instrument used for this purpose. In its simplest form, a protractor consists of an arc of a circle graduated to degrees (Fig. 1). A

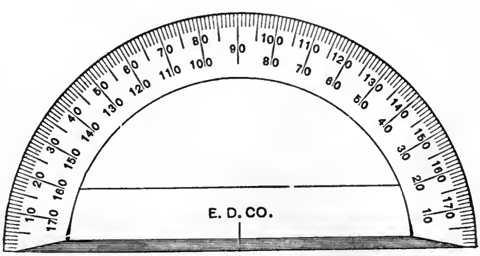


FIG. 1

mere inspection of the instrument will enable the student to see how angles may be measured and constructed by means of it.

The unit usually employed in measuring angles is the ninetieth part of a right angle, and is called a *degree.* A degree is divided into sixty equal parts, each of which is called a *minute*, and each minute is subdivided into sixty *seconds.* Thus

$$60 \text{ seconds } (60'') = \text{one minute } (1'),$$
$$60 \text{ minutes } (60') = \text{one degree } (1°),$$
$$90 \text{ degrees } (90°) = \text{one right angle.}$$

Very frequently, the angles smaller than one degree are expressed as decimal parts of a degree instead of in minutes and seconds.

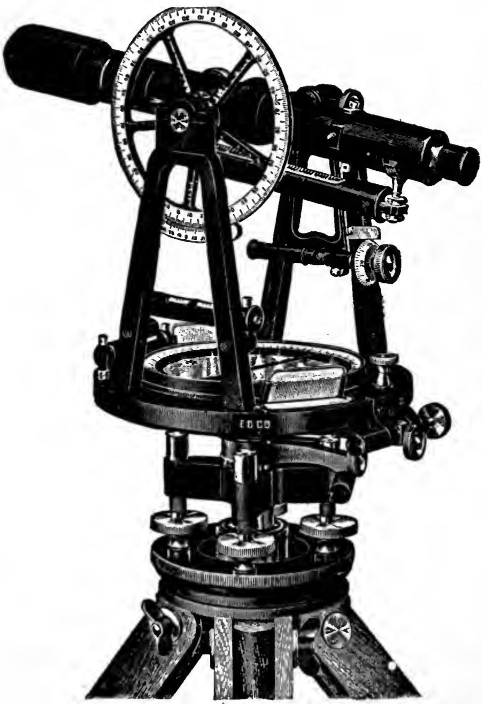


Fig. 2

For the purpose of measuring angles in the field, surveyors make use of an instrument called a **transit** or **theodolite.** The essential parts of this instrument are (cf. Fig. 2):

1. a horizontal graduated circle;

2. a movable circular plate adjusted so as to be capable of rotation around the center of the horizontal graduated circle;

3. an index attached to the movable plate in such a way as to enable the ob-

server to read off the amount of its rotation with reference to the fixed horizontal circle ;

4. two standards attached to the movable plate and carrying a horizontal axis to which is attached a telescope and also, in a complete instrument, a vertical graduated circle, used for measuring angles whose sides lie in a vertical plane.

The transit is usually supported on a tripod. If we wish to measure the angle between two horizontal lines, the tripod is placed over the vertex of the angle and the telescope is pointed toward some point on one of the sides of the angle. The index will then point to a definite division on the horizontal circle. The operation of ascertaining the division of the circle toward which the index is pointing, is known as "reading the circle." After reading the circle and noting the result, the telescope is directed toward a point on the other side of the angle. The difference between the two readings of the horizontal circle will give the magnitude of the angle.

In a similar way the vertical circle, which is attached to the axis of the telescope, makes possible the measurement of angles whose sides lie in a vertical plane.

Both circles are usually graduated to whole degrees. The index, in most instruments, is not a simple pointer, but a so-called **vernier**, an ingenious device which enables the observer to determine the reading of the circle to within a small fraction of a degree, even if the circle is graduated only to whole degrees. In the most accurate instruments, the vernier is replaced by a **reading microscope**.

It is obviously very important that the horizontal circle be exactly horizontal, and that its center be exactly over the vertex of the angle which is to be measured. To help in making these adjustments, the surveyor uses a **spirit level** and a **plumb line**.

Most transits are also supplied with a **compass**, which enables the observer to determine the absolute directions of the lines which he is surveying.

EXERCISE II

1. Use a protractor to draw angles of 10°, 20°, 31°, 47° 30′, 67°, 78°, 86°.

2. Draw five angles at random and measure them as accurately as possible with your protractor.

3. Draw a triangle at random, measure its angles and find their sum. What should this sum be? If you have obtained a different result for the sum, what are the reasons?

4. Construct, out of cardboard, an instrument embodying the principle of the transit, substituting for the telescope some other method of taking a sight.

5. Why is it important that the horizontal circle of a transit should be truly horizontal? How does the spirit level enable us to make it so? Study the article on the spirit level in some encyclopedia or in a treatise on surveying.

6. Study the articles on *vernier*, *micrometer*, *reading microscope*, *compass*, in an encyclopedia or in some appropriate treatise, and write an abstract of the same.

7. Describe the sources of inaccuracy which you think may arise in the measurement of an angle by means of a protractor or theodolite.

8. Deduce rules for converting minutes and seconds into decimal parts of a degree and *vice versa.*

9. Apply this rule to the angles

$$30^\circ\ 20',\ 10^\circ\ 45',\ 8^\circ\ 40'\ 20'',\ 3^\circ\ 8'\ 2''.$$

10. Express the following angles in degrees, minutes, and seconds:

$$23^\circ.14,\ 18^\circ.25,\ 46^\circ.235.$$

3. The impossibility of finding all distances by direct measurement. We have discussed briefly the direct methods for measuring distances and angles. It is clear from our discussion that it is very *difficult* to measure great distances in that way. But there are many cases in which it is altogether *impossible* to apply such direct methods. How, for example, should we proceed to find the distance through a mountain or across an extensive valley? How shall we find the distance from New York to London, or from the earth to the moon, by direct measurement?

It is clear that, if we wish to answer such questions at all, we shall have to devise some method different from that of direct measurement.

The attempt to solve, by indirect methods, problems whose direct solution is inconvenient or impossible usually leads to great advances in Science. The most important theorems of elementary geometry were probably first discovered by the Ancients in their attempts to devise convenient and practical methods for the measurements which they found necessary for the purposes of their everyday life. It is generally believed, for instance, that the Egyptians became expert geometricians and surveyors at an early period of their history, because it was so important for them to be able to reëstablish the boundary lines of their lands after the effacement of their marks by the annual inundations of the Nile.

The earliest Greek philosophers and mathematicians were pupils of the Egyptians, and some of their first achievements were connected with problems of the particular kind which we are now discussing. Thus it is reported that THALES OF MILETUS (about 600 B.C.) measured the height of a pyramid by measuring the length of its shadow, at the instant when the shadow of a vertical stick by its side was exactly as long as the stick itself. The height of the pyramid would then be equal to the length of its shadow at that moment.

This method has the inconvenient feature of compelling the observer to wait (many hours perhaps) for the right moment. According to a report by PLUTARCH, THALES also devised a second method which avoided this inconvenience. This method involves the use of the simplest properties of similar triangles and may be illustrated by means of the following example:

We place a stick 5 feet high into the ground near a building whose height we wish to find. At any convenient moment we measure the

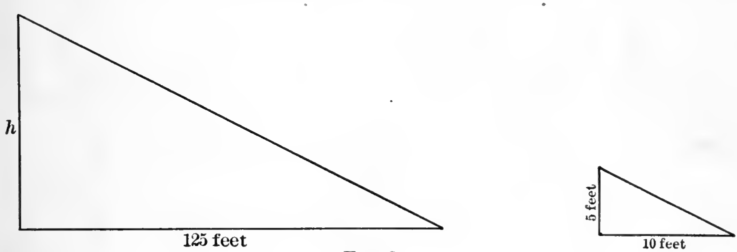


FIG. 3

length of the shadows of the stick and of the building. Suppose we find in this way that the shadow of the building is 125 feet long at the moment when the shadow of the stick is 10 feet long. If h denotes the height of the building, we shall have (Fig. 3)

$$h : 125 = 5 : 10,$$

whence

$$h = 62.5 \text{ feet.}$$

According to EUDEMUS (about 300 B.C.), one of the earliest writers on the history of mathematics, THALES also invented a method for measuring the distance from the shore to a ship at sea. Although Eudemus does not describe Thales's method in detail, he says enough to lead us to conclude that it was essentially as follows: Let BL be a tower, say a lighthouse on the shore, and

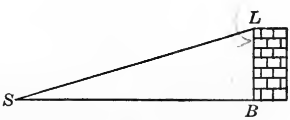


FIG. 4

let S be the ship at sea. Measure the angle BLS and the height of the tower. Construct a similar triangle $B'L'S'$ on the drawing board and measure $B'S'$ and $B'L'$. Then BS can be found from the proportion

$$BS : BL = B'S' : B'L'.$$

4. The graphic method. The examples discussed in Art. 3 show how we may solve many problems of practical geometry by indirect measurements. We did not measure directly the quantity which we were seeking, but some other related quantities, and ultimately, by means of these relations, we determined the desired quantity itself.

But these same examples may serve to illustrate another point. The solutions which we gave are examples of the **graphic method**, that is, of a process which makes use of drawing instruments, of accurate geometric constructions and measurements, for the purpose of obtaining the values of the unknown quantities. Such graphical methods are often extremely valuable and have been developed in recent times, in connection with other parts of mathematics, so as to give rise to very important results.

The graphic solution of any problem about triangles will finally reduce to an application of certain theorems of geometry which state that a triangle is determined and may be constructed when certain ones of its six parts (sides and angles) are given. It is clear, then, that these theorems are particularly important for the graphic method. Some of the following questions have been chosen for the purpose of aiding the student to refresh his memory in regard to these matters.

EXERCISE III

In the following examples and throughout the book we shall usually denote the angles of a triangle by A, B, C and the sides opposite to these angles by a, b, c, respectively. The student should be provided with a protractor, a pair of compasses, and a ruler divided decimally, say into centimeters and millimeters. All constructions and measurements should be made as carefully as possible.

1. Given $a = 3.72$, $b = 4.91$, $c = 2.56$. Find the angles.

2. Given $a = 4.27$, $B = 35°$, $C = 69°$. Find the remaining parts of the triangle.

3. Given $b = 5.63$, $c = 6.71$, $A = 27°$. Find the remaining parts of the triangle.

4. Given $a = 4.23$, $b = 5.16$, $A = 55°$. Find the remaining parts of the triangle.

5. When a, b, c are given at random, can we *always* find a triangle of which a, b, c are the sides, or is there some restriction on the possible values of a, b, c?

6. If a, b, A are given, there may be one or two solutions, or no solution. Discuss these cases.

7. Is a triangle determined when we know the magnitudes of its three angles? Why? When the three angles of a triangle are given, have we obtained essentially more information than if only two of them are given? Why? Is the third angle of a triangle independent of the other two?

8. What do you mean by similar triangles?

9. Under what conditions are two triangles similar?

10. Use the shadow method of Thales (Art. 3) to measure the height of some building in your neighborhood.

11. By means of a transit, or else by means of the home-made instrument of cardboard suggested in Ex. 4 of Exercise II, find the distance of some object situated on the opposite side of the street from your home, without crossing the street. Afterward check your result by direct measurement.

12. How may a person on board ship find his distance from a building on the shore if he knows its height?

5. The desirability of an arithmetical method for solving triangles. We have seen that it is a rather simple matter to solve a triangle by the graphic method. But we can hardly feel altogether satisfied with the graphic solution, for it will clearly not permit us to reach any great degree of accuracy. To be sure, by making our drawings on a very large scale, we might lay off distances with considerable precision, but we should still encounter the difficulty of accurately plotting angles. Clearly it would require extraordinary skill and exceedingly fine instruments to enable us to draw an angle so accurately that its error should not exceed one minute of arc. Many other circumstances combine to make a graphical solution unsatisfactory if a high degree of precision is required.

The main value of the graphic method lies in furnishing a solution whose approximate correctness is apt to be apparent to *the eye, and which may therefore, in almost all cases, serve as a check on the more complete solution obtained in some other way.*

But the lack of accuracy is only one of the defects of the graphic method, although perhaps the most important one from a practical point of view. The other defect which we wish to emphasize is more of a theoretical nature. The parts of a triangle (its sides and angles) are usually given as *numbers* (so many feet, so many degrees). It seems natural, therefore, to suppose that there must exist some *arithmetical process* for the solution of triangles.

Trigonometry enables us to find the unknown parts of a triangle by arithmetical processes.

This statement must *not* be regarded as a complete definition of trigonometry. We shall see later that the solution of triangles by arithmetical methods constitutes only a part, although an important part, of trigonometry.

EXERCISE IV

1. What are the principal sources of inaccuracy in solving a triangle by the graphic method?

2. We wish to construct an isosceles triangle, given the angle at the vertex and the altitude. Suppose that the error made in constructing the angle is 5′. Will the effect of this error on the base of the triangle, as obtained from the construction, be different for different altitudes? Will it be greater or less for the higher triangles?

3. If you wish to find a point as the intersection of two straight lines, are you likely to obtain a more accurate result if the two lines are at right angles, or if they are nearly parallel?

4. A side and two adjacent angles of a triangle are given, and its other parts are to be found graphically. Is such a graphic solution likely to be accurate if both of the angles are very small? If the sum of the given angles is very close to 180°?

5. Formulate, in your own words, the distinction between an arithmetical and a graphical process.

CHAPTER II

THE TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

6. The necessity of introducing new ideas. At the close of Chapter I we formulated the problem of devising arithmetical methods for the solution of triangles. But, if we think of the many shapes which a triangle may assume, this problem appears to be a most formidable one. We shall, therefore, for the present, confine our attention to the comparatively simple case of a *right triangle.* We have good reason to suppose that, if we succeed in solving our problem for all right triangles, we shall be able to deal later with the general case also, since every triangle may be decomposed into two right triangles.

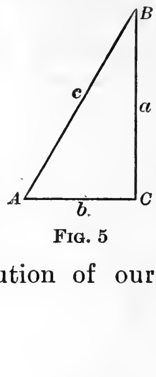


FIG. 5

Let us then consider a triangle ABC, right-angled at C (Fig. 5), and let us denote the sides opposite the angles A, B, C by a, b, c, respectively. We are already acquainted with two relations which will assist us in the solution of our problem, namely:

$$(1) \qquad A + B = 90^\circ$$

and the theorem of PYTHAGORAS,

$$(2) \qquad a^2 + b^2 = c^2.$$

The first of these equations enables us to calculate one of the acute angles if the other one is given. The second provides a method for calculating any one of the three sides if the other two are given.

But we have nothing as yet which will enable us to find the angle A if two of the sides (say a and b) are given, although the triangle is clearly determined by these sides and might be constructed by geometry. The equations (1) and (2), unaided by other relations, are *obviously* inadequate for this purpose, since (1) is a relation between the *angles* A and B alone, while (2) involves only the *sides* of the triangle. Now, clearly, a statement which is only concerned with the *angles* of a triangle cannot convey any positive information about the *sides*, and *vice versa*.

In order that we may be able to solve a right triangle by arithmetical processes, there must, therefore, be added to equations (1) *and* (2) *certain other relations, in which the sides and angles shall not be separated, but in which they shall occur simultaneously.*

The discovery of such relations and their adequate formulation is the foundation upon which all of trigonometry must finally rest.

A simple illustration will make clear the nature of these new relations. The numerical measure for the steepness of an inclined plane, say a mountain road, may be given in two ways. We may say that the road makes a certain angle A (say 5°) with the horizontal plane, or we may say that it rises a certain number of feet (say 87.5 feet) in a horizontal distance of 1000 feet.

The quotient, $\frac{87.5}{1000}$, is technically known as the *slope*, *grade*, or *gradient* of the road. The gradient is clearly connected with the angle A in such a way that, if A should vary, to every value of the angle there corresponds a definite value of the gradient and *vice versa*. Thus, if AB (Fig. 6) represents the road, and if a and b are both expressed in feet, the gradient of AB is equal to $\frac{a}{b}$. The value of this quotient changes with the angle A, so that for different angles we find different values for the gradient.

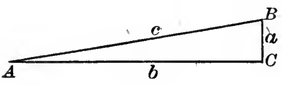


FIG. 6

The general question before us may be formulated as follows. If the acute angle A of the right triangle ABC becomes larger or smaller, what effect will such a change have upon the sides? And, conversely, if the sides of a triangle change, what effect will this have on the angles?

Now, since similar triangles have their corresponding angles equal, it is clear that there are *some* changes in the lengths of the sides which produce *no* change in the angles. In fact, if all of the sides of a triangle are magnified in the same ratio, the angles are not changed at all.

In the two right triangles ABC and $A'B'C'$ (Fig. 7) we have

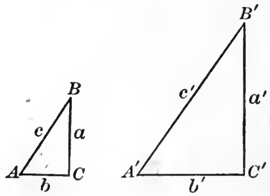


FIG. 7

$$\frac{a}{c}=\frac{a'}{c'},\ \frac{b}{c}=\frac{b'}{c'},\ \frac{a}{b}=\frac{a'}{b'},\ \text{etc.,}$$

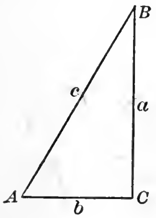
if the angle at A' is equal to that at A. But if the angle A' is different from A, the ratios $\frac{a'}{c'}$, $\frac{b'}{c'}$, $\frac{a'}{b'}$, etc., will not be equal to $\frac{a}{c}$, $\frac{b}{c}$, $\frac{a}{b}$, etc., respectively. For if they were, corresponding pairs of sides of the two triangles would have the same ratio, the triangles would be similar, and, contrary to our hypothesis, angle A' would have to be equal to angle A.

Consequently, while the *lengths* of the individual sides of a right triangle have nothing to do with the size of its angles, the *ratios* of these lengths *are* connected with the magnitude of the angles in a very intimate fashion. In fact, so close is this relation that the values of the ratios $\frac{a}{c}$, $\frac{b}{c}$, $\frac{a}{b}$, etc., may be determined (by construction) as soon as the acute angle A has been chosen, and conversely; if one of these ratios is given, we can find (by construction) one, and only one, corresponding acute angle A.

7. Definitions of the trigonometric functions of an acute angle. We have seen that the values of the ratios $\frac{a}{c}$, $\frac{b}{c}$, etc., of the sides of a right triangle are closely bound up with the magnitude of the angle A. If the angle changes, each of these ratios changes and *vice versa*.

Now, *one variable quantity is called a* **function** *of another, if they are so related that any change in the latter produces a corresponding change in the former.*

Consequently, each of the six ratios $\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{a}, \frac{c}{b}$, determined by the sides of the right triangle ABC, is a *function* of the angle A, because any change in A produces a corresponding change in each of those ratios. Each of these functions has received a name and a symbol. The reason for choosing these names will appear later (see Arts. 10 and 70), and cannot be discussed with profit at the present moment.



We proceed to give the formal definitions of the six trigonometric functions of an acute angle A.

Construct any right triangle (cf. the Fig.), *one of whose acute angles is equal to the given angle A.** *Of the two legs of this right triangle, that one which passes through the vertex of the angle A is said to be* ***adjacent*** *to A. The other leg is said to be* ***opposite*** *to A, and the third side of the triangle is called its* ***hypotenuse***.

The **sine of** A is the ratio of the **opposite side** to the **hypotenuse.**

The **cosine of** A is the ratio of the **adjacent side** to the **hypotenuse.**

The **tangent of** A is the ratio of the **opposite side** to the **adjacent side.**

The **cotangent of** A is the ratio of the **adjacent side** to the **opposite side.**

The **secant of** A is the ratio of the **hypotenuse** to the **adjacent side.**

The **cosecant of** A is the ratio of the **hypotenuse** to the **opposite side.**

---

* We have seen in Art. 6 that the size of this right triangle is absolutely of no consequence, since any two triangles of this kind are similar, so that the corresponding ratios for the two triangles will be equal.

In symbols we may write these definitions as follows:

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b},$$

(1)

$$\csc A = \frac{c}{a}, \quad \sec A = \frac{c}{b}, \quad \cot A = \frac{b}{a}.$$

These symbols are written abbreviations of the names of the functions. In speaking, the symbols are pronounced as though the name of the function had been written out in full. Thus, tan A is pronounced *tangent of A* or *tangent A*; csc A is pronounced *cosecant of A* or *cosecant A*, etc.

In defining the trigonometric functions, we made use of the numerical measures of certain line-segments, namely, of the sides of a right triangle. Now, the numerical measure of a line-segment changes if the unit of measurement is changed. One might, therefore, expect the values of the trigonometric functions of an acute angle to change with every change of the unit of length. But this is *not* the case. Owing to the fact that only *ratios* of these line-segments appear in equations (1), the functions sin A, cos A, etc., are found to have the same value whether the line-segments a, b, c are measured in feet, inches, or in terms of any other unit of length.

Consider, for instance, a right triangle for which $a = 3$ feet, $b = 4$ feet, $c = 5$ feet. According to (1) we have $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, etc. Let us now introduce the inch as unit of length instead of the foot. The numerical measures of the sides of the triangle will now be $a = 36$, $b = 48$, $c = 60$. According to (1) we shall *now* find $\sin A = \frac{36}{60}$, $\cos A = \frac{48}{60}$, etc. But $\frac{36}{60} = \frac{3}{5}$, $\frac{48}{60} = \frac{4}{5}$, etc., so that we obtain precisely the same values for sin A, cos A, etc., whether the sides of the triangle be expressed in feet or inches.

If the trigonometric functions were *concrete* numbers, that is, if they were the numerical measures of some kind of concrete quantity such as a length, an area, or a volume, their values would change every time that a change is made from one unit of length to another. We have just seen that this is not the case. Therefore, *the trigonometric functions are pure or abstract numbers.**

* This same fact is sometimes (somewhat inadequately) expressed by the statement that *the trigonometric functions are ratios.*

8. The practical need of tables giving the values of the trigonometric functions. The trigonometric functions just defined will enable us to find the unknown parts of a right triangle when certain parts are given, provided only that we can devise a practical method for actually obtaining the numerical values of these functions for any given acute angle. Now, the values of the functions have been calculated by mathematicians and the results have been collected in tabular form for convenient use. From the practical point of view, therefore, it only remains to explain the arrangement and the use of the tables.

To the more scientifically inclined student, however, the question will immediately suggest itself as to how these useful tables were actually computed. We shall reserve the answer to this question for the second part of the book. The following examples, however, will show how a table of trigonometric functions may be prepared by the graphic method, provided that no very high degree of accuracy be required.

EXERCISE V

1. Find the functions of 40° by the graphic method.

Solution. With the help of a protractor construct the angle PAQ (Fig. 8) equal to 40°. Choose any point B on AQ and drop a perpendicular BC from B to AP. Measure the distances BC, AC, and AB. Then will

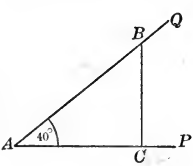


FIG. 8

(1) $$\sin 40° = \frac{CB}{AB}, \quad \cos 40° = \frac{AC}{AB}.$$

Although the point B might be chosen *anywhere* on AQ, it will be especially convenient to make AB equal either to one unit, ten units, or one hundred units. For, as equations (1) show, we have to divide by AB, and if AB is equal to 1, 10, or 100, we avoid the long division which would otherwise be necessary.

Let us, therefore, make $AB = 10$ centimeters. We should then find by measurement,

$$CB = 6.4 \text{ cm.}, \quad AC = 7.7 \text{ cm.}$$

According to (1), therefore,

$$\sin 40° = \frac{6.4}{10} = 0.64, \quad \cos 40° = \frac{7.7}{10} = 0.77.$$

Further we find

$$\tan 40° = \frac{6.4}{7.7} = 0.83, \quad \cot 40° = \frac{7.7}{6.4} = 1.20,$$

$$\sec 40° = \frac{10}{7.7} = 1.30, \quad \csc 40° = \frac{10}{6.4} = 1.56.$$

We should obtain a more accurate result for tan 40°, and more conveniently, if we were to use another triangle, making this time $AC = 10$ cm. Measurement would then give $BC = 8.4$ cm., and

$$\tan 40° = \frac{8.4}{10} = 0.84.$$

2. Find the functions of the following angles by the graphic method:

(*a*) 10°. (*b*) 15°. (*c*) 20°. (*d*) 70°.

Construct carefully each of the following right triangles, measure the angles, and find the six functions of each acute angle.

3. $a = 3, \ b = 4, \ c = 5.$

4. $a = 5, \ b = 12, \ c = 13.$

5. $a = 8, \ b = 15, \ c = 17.$

6. Construct and measure an acute angle whose sine is equal to $\frac{1}{3}$.

7. Construct and measure an acute angle whose tangent is equal to $\frac{3}{5}$.

8. Construct and measure an acute angle whose cosine is equal to $\frac{4}{5}$.

9. Can you think of two right triangles (Fig. 7) with different angles A and A', for which the sides a and a' are nevertheless equal?

10. Can you conceive of two right triangles (Fig. 7) with different angles A and A', for which the ratios $\frac{a}{b}$ and $\frac{a'}{b'}$ are nevertheless equal?

11. Why, then, do we speak of the ratio $\frac{a}{b}$ as a function of A? Why do we not introduce a or ab as a function of A?

12. Assuming that we have access to a table of the values of the trigonometric functions, show how we might solve each of the following problems. To find the remaining parts of a right triangle when the following parts are given.

I. $a, \ b, \ C = 90°$. III. $a, \ c, \ C = 90°$.
II. $a, \ A, \ C = 90°$. IV. $c, \ A, \ C = 90°$.

13. Show that neither the sine nor the cosine of an acute angle can ever be greater than unity.

14. Show that the tangent of an acute angle may have any positive value whatever. Similarly for the cotangent.

15. What restrictions, if any, are there on the values which the secant and cosecant of an acute angle may assume?

9. Relations between the six trigonometric functions of an acute angle. The preceding discussion suffices to indicate the importance of constructing a table of the values of the trigonometric functions. The task of computing these tables may be abbreviated very considerably by noting that the six functions are not independent of each other. In fact, we have (Fig. 9)

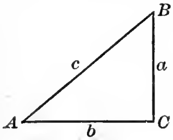


FIG. 9

$$(1)\qquad \sin A = \frac{a}{c},\quad \csc A = \frac{c}{a},$$
$$\cos A = \frac{b}{c},\quad \sec A = \frac{c}{b},$$
$$\tan A = \frac{a}{b},\quad \cot A = \frac{b}{a},$$

so that we obtain at once the relations

$$\csc A = \frac{1}{\sin A},\quad \sec A = \frac{1}{\cos A},\quad \cot A = \frac{1}{\tan A},$$
$$\sin A = \frac{1}{\csc A},\quad \cos A = \frac{1}{\sec A},\quad \tan A = \frac{1}{\cot A},$$

or,

$$(2)\qquad \mathbf{\sin A \csc A = 1,\quad \cos A \sec A = 1,\quad \tan A \cot A = 1.}^*$$

But, two numbers whose product is equal to unity are called *reciprocals* of each other. Therefore, equations (2) are equivalent to the following statement:

The sine and cosecant, the cosine and secant, and finally the tangent and cotangent, of an acute angle are reciprocals.

Clearly, knowledge of this fact reduces greatly the labor of computing tables of the functions. For, having found the values of the three functions, sine, cosine, and tangent, the values of the remaining three can be obtained from these by computing their reciprocals.

But there are other relations besides (2) which enable us to reduce still further the labor involved in constructing a table of the trigonometric functions. We have

$$\tan A = \frac{a}{b}.$$

---

* Note that $\sin A \csc A$ is written for $\sin A \times \csc A$ just as ab is written for $a \times b$.

If we divide both numerator and denominator of the fraction $\frac{a}{b}$ by c, we find

$$\tan A = \frac{a/c}{b/c}.$$

But we have, by definition (cf. equations (1))

$$\frac{a}{c} = \sin A,\ \frac{b}{c} = \cos A,$$

and therefore

(3) $$\mathbf{\tan A = \frac{\sin A}{\cos A}.}$$

Of course, since cot A is the reciprocal of tan A, we also have

(4) $$\mathbf{\cot A = \frac{\cos A}{\sin A}.}$$

The relations (2), (3), (4) enable us to calculate the values of all six functions when the sine and cosine are known. But it actually suffices to know the sine. For, if we divide both members of the familiar equation

(5) $$a^2 + b^2 = c^2$$

by c^2, we find

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

But, by definition, we have

$$\frac{a}{c} = \sin A,\ \frac{b}{c} = \cos A,$$

so that (5) becomes

(6) $$\mathbf{\sin^2 A + \cos^2 A = 1,}$$

where $\sin^2 A$ and $\cos^2 A$ have been written, as is customary, for $(\sin A)^2$ and $(\cos A)^2$, respectively.

It follows from relations (2), (3), (4), and (6) that, if we know the value of a single one of the six trigonometric functions of an acute angle, the values of the remaining five may be computed. The detailed proof of this statement will be left to the student in some of the examples given below.

EXERCISE VI

In each of the following twelve examples, the value of one function of the acute angle A is given. Find the values of the remaining functions.

1. $\sin A = \frac{3}{5}$.	**4.** $\cot A = 2$.	**7.** $\sin A = x$.	**10.** $\cot A = x$.
2. $\cos A = \frac{5}{13}$.	**5.** $\sec A = \frac{13}{12}$.	**8.** $\cos A = x$.	**11.** $\sec A = x$.
3. $\tan A = 1$.	**6.** $\csc A = \frac{17}{8}$.	**9.** $\tan A = x$.	**12.** $\csc A = x$.

13. Construct an isosceles right triangle and make use of this figure for the purpose of computing the functions of 45°.

14. Divide an equilateral triangle into two right triangles by dropping a perpendicular from one of its vertices to the opposite side. Make use of this figure for the purpose of computing the functions of 30° and 60°.

15. Collect in a table the results of Exs. 13 and 14.

16. Prove the formula $\mathbf{sec^2 \mathit{A} = 1 + tan^2 \mathit{A}}$.

17. Prove the formula $\mathbf{csc^2 \mathit{A} = 1 + cot^2 \mathit{A}}$.

18. Prove that the sine, tangent, and secant of an angle increase when the angle grows from 0° to 90°.

19. Prove that the cosine, cotangent, and cosecant of an angle decrease when the angle grows from 0° to 90°.

20. Prove that $\tan A < 1$, $\cot A > 1$ if $A < 45°$, and that $\tan A > 1$, $\cot A < 1$ if $A > 45°$.

21. Show that each of the six functions may be expressed either as a product or as a quotient of two of the others.

10. Relations between the functions of complementary angles. As a result of the relations discussed in the preceding article, the problem of computing the values of the six trigonometric functions for every angle between 0° and 90° has been reduced to that of computing these values for a single one of these functions. But we may reduce the problem still further by observing the relations between the functions of the two acute angles of the same right triangle.

We have (cf. Fig. 10)

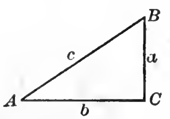


FIG. 10

$$\sin A = \frac{a}{c}, \quad \cos B = \frac{a}{c},$$

$$\cos A = \frac{b}{c}, \quad \sin B = \frac{b}{c},$$

$$\tan A = \frac{a}{b}, \quad \cot B = \frac{a}{b},$$

$$\cot A = \frac{b}{a}, \quad \tan B = \frac{b}{a},$$

$$\sec A = \frac{c}{b}, \quad \csc B = \frac{c}{b},$$

$$\csc A = \frac{c}{a}, \quad \sec B = \frac{c}{a},$$

and therefore

$$(1) \quad \begin{array}{lll} \sin A = \cos B, & \tan A = \cot B, & \sec A = \csc B, \\ \cos A = \sin B, & \cot A = \tan B, & \csc A = \sec B. \end{array}$$

Since the angles A and B are complementary, we may write these equations as follows:

$$(2) \quad \begin{array}{ll} \sin (90° - A) = \cos A, & \cot (90° - A) = \tan A, \\ \cos (90° - A) = \sin A, & \sec (90° - A) = \csc A, \\ \tan (90° - A) = \cot A, & \csc (90° - A) = \sec A. \end{array}$$

An easy way to remember these formulæ is as follows: Let the six functions be grouped into three pairs: sine and cosine, tangent and cotangent, secant and cosecant. Let us speak of either function of one of these pairs as the *cofunction* of the other. Then, the six formulæ (2) are all included in the following statement.

Any trigonometric function of the complement of an angle A is equal to the cofunction of A.

It is apparent that this theorem will enable us to find the trigonometric functions of any acute angle greater than 45°, if we know the functions of all angles less than 45°. Thus, for instance, tan 75° is equal to cot 15°, sin 82° is equal to cos 8°, etc. As a consequence of this fact it is possible to reduce the space occupied by the tables of the functions to exactly half of what would otherwise be necessary.

The relation between the functions of complementary angles is also important in another respect. It is this relation which has given rise to the words cosine, cotangent, and cosecant. The cosine is the sine of the complement. At a time when Latin was still the universal language of the scientific world, the cosine was therefore called *complementi sinus*.

This was later (in the seventeenth century) contracted to *cosinus*. The words cotangent and cosecant originated in the same manner.

EXERCISE VII

1. Express as functions of the complementary angles

$$\sin 37°, \cos 62°, \tan 13°, \cot 75, \sec 12° 15', \csc 55° 37'.$$

2. If the table of values of the functions is so arranged as to give only the functions of angles less than 45°, how may we obtain the values of

$$\sin 57°, \cos 63° 15', \tan 75° 12', \cot 67° 18'?$$

3. What acute angle is that whose sine is equal to the sine of its complement?

4. Find an acute angle for which $\tan A = \cot (45° + A)$.

HINT. Substitute for $\tan A$ its equal $\cot (90° - A)$ and note that two acute angles with the same cotangent are equal to each other.

5. Find an acute angle for which $\sin 2A = \cos (45° - A)$.

6. Find an acute angle for which $\cot 3A = \tan 2A$.

7. Find an acute angle for which $\cos A = \sin 6A$.

8. Find an acute angle for which $\sec 2A = \csc 7A$.

11. The values of the functions of 0°, 30°, 45°, 60°, 90°. While we have decided to postpone the general question of the arithmetical calculation of the trigonometric functions, we have already performed this calculation for a few special angles, viz.: 30°, 45°, 60° (cf. Exs. 13, 14 of Exercise VI). The figures there suggested, and the results are as follows:

$$(1)\quad\begin{cases}\sin 45° = \cos 45° = \dfrac{a}{a\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2},\\ \tan 45° = \cot 45° = 1,\\ \sec 45° = \csc 45° = \sqrt{2}.\end{cases}$$

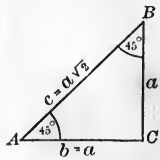


FIG. 11

$$(2)\quad\begin{cases}\sin 30° = \cos 60° = \dfrac{\frac{1}{2}c}{c} = \dfrac{1}{2},\\ \cos 30° = \sin 60° = \tfrac{1}{2}\sqrt{3},\\ \tan 30° = \cot 60° = \dfrac{\frac{1}{2}c}{\frac{1}{2}c\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3},\\ \cot 30° = \tan 60° = \sqrt{3},\\ \sec 30° = \csc 60° = \dfrac{c}{\frac{1}{2}c\sqrt{3}} = \dfrac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3},\\ \csc 30° = \sec 60° = 2.\end{cases}$$

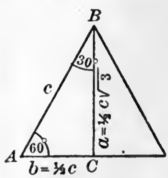


FIG. 12

An acute angle of a right triangle can never be 0° or 90°, so that the definitions of Art. 7 are not applicable to such angles. The acute angle A may, however, *approach* either 0° or 90° *as a limit*, and if it does, its functions in some cases approach definite finite limits. By sin 0°, cos 0°, sin 90°, cos 90°, etc., we mean such limits whenever they exist.

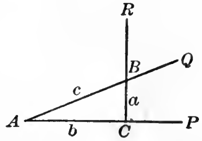


FIG. 13

In Fig. 13, let the angle $A = PAQ$ be thought of as decreasing toward the limit zero as a result of the rotation of the side AQ around A as a center, while the side AP remains fixed. Through any point C of AP draw CR perpendicular to AP, and denote by B the intersection of CR with the rotating line AQ.

As the angle A approaches zero, the side a approaches zero and c approaches b. Consequently

$$\sin A = \frac{a}{c} \text{ approaches } \frac{0}{b} \text{ or } 0, \quad \cos A = \frac{b}{c} \text{ approaches } \frac{b}{b} \text{ or } 1,$$

$$\tan A = \frac{a}{b} \text{ approaches } \frac{0}{b} \text{ or } 0, \quad \sec A = \frac{c}{b} \text{ approaches } \frac{b}{b} \text{ or } 1.$$

In this sense we may say that

$$\sin 0° = 0, \ \cos 0° = 1, \ \tan 0° = 0, \ \sec 0° = 1.$$

The function $\cot A = \frac{b}{a}$ *has no limit when A approaches zero.* For, the ratio $\frac{b}{a}$ grows larger and larger as A approaches zero, since b remains fixed while a grows smaller and smaller. Clearly, this ratio may be made larger than any number however great, by choosing A and, hence, a small enough. This is expressed in symbolic language as follows:

$$\cot 0° = \infty,$$

or in words: *The cotangent of an acute angle increases without bound when the angle approaches zero as a limit.*

A similar argument holds for $\csc A = \frac{c}{a}$, since c approaches

b and a approaches zero. Hence, in symbolic language, we have

$$(3) \qquad \begin{cases} \sin 0^\circ = 0, & \tan 0^\circ = 0, & \sec 0^\circ = 1, \\ \cos 0^\circ = 1, & \cot 0^\circ = \infty, & \csc 0^\circ = \infty. \end{cases}$$

By a similar argument the student may deduce the following results:

$$(4) \qquad \begin{cases} \sin 90^\circ = 1, & \tan 90^\circ = \infty, & \sec 90^\circ = \infty, \\ \cos 90^\circ = 0, & \cot 90^\circ = 0, & \csc 90^\circ = 1, \end{cases}$$

the exact meaning of each of which should be expressed in words as in the cases which have just been treated *in extenso*.

CHAPTER III

SOLUTION OF RIGHT TRIANGLES BY NATURAL FUNCTIONS

12. Arrangement and use of the table of natural functions. The numerical values of the trigonometric functions are usually called the **natural functions** to distinguish them from the *logarithms* of these functions which we shall study later.

Table * V gives the numerical values of the sine, cosine, tangent, and cotangent to four decimal places for every tenth of a degree from 0° to 90°. The values of the secant and cosecant are omitted because they are not used very frequently. They may of course be calculated, whenever necessary, by the formulæ

$$\sec A = \frac{1}{\cos A}, \quad \csc A = \frac{1}{\sin A}. \qquad \text{(Art. 9)}$$

The following is a sample portion of the table. Only this part of the table will be required for the following illustrative examples.

ANGLE	N SIN	d	N TAN	d	N COT	d	N COS	d	
35°.0	0.5736	14	0.7002	26	1.4281	52	0.8192	11	55°.0
.1	0.5750	14	0.7028	26	1.4229	53	0.8181	10	54°.9
.2	0.5764	15	0.7054	26	1.4176	52	0.8171	10	.8
.3	0.5779	14	0.7080	27	1.4124	53	0.8161	10	.7
.4	. . .		. . .		. . .		. . .		.6
.5	. . .		. . .		. . .		. . .		.5
	. . .		. . .		. . .		. . .		
	N COS	d	N COT	d	N TAN	d	N SIN	d	ANGLE

* See *Logarithmic and Trigonometric Tables*, compiled by E. J. Wilczynski and H. E. Slaught.

PROBLEM 1. Find the functions of 35°.2.

Solution. In the left-hand column find 35°.2. The four numbers which are printed in the horizontal row to the *right* of 35°.2 are, *from left to right*, the sine, tangent, cotangent, and cosine of 35°.2, as indicated by the name printed at the *head* of each of these columns. Therefore

$$\sin 35°.2 = 0.5764, \tan 35°.2 = 0.7054, \cot 35°.2 = 1.4176,$$
$$\cos 35°.2 = 0.8171.$$

PROBLEM 2. Find the functions of 54°.8.

Solution. In the right-hand column find 54°.8. The four numbers which are printed in the horizontal row to the *left* of 54°.8 are, *from right to left*, the sine, tangent, cotangent, and cosine of 54°.8, as indicated by the name printed at the *foot* of each of these columns. Therefore

$$\sin 54°.8 = 0.8171, \tan 54°.8 = 1.4176, \cot 54°.8 = 0.7054,$$
$$\cos 54°.8 = 0.5764.$$

Thus every number of the table does double duty. For example, 0.5764 is both the sine of 35°.2 and the cosine of 54°.8, as it should be. (See Art. 10, equations (2).)

Angles less than 45° are given in the left-hand column of the table, and the names of the corresponding functions are found at the *top* of the page. Angles greater than 45° are given in the right-hand column with the names of the functions at the *bottom* of the page.

The table gives the values of the functions only for every tenth of a degree. If the given angle contains fractional parts of this unit, its functions cannot be read directly from the table. In such cases we make use of the process of **interpolation**, the nature of which will become apparent from the following examples.

PROBLEM 3. Find the sine of 35°.17.

Solution. This angle lies between 35°.1 and 35°.2. More precisely, it lies $\frac{7}{10}$ of the way from the former toward the latter. We conclude that its sine will be $\frac{7}{10}$ of the way from

$$\sin 35°.1 = 0.5750 \text{ toward } \sin 35°.2 = 0.5764.$$

But the difference d between these last two numbers is 0.0014, seven tenths of which is equal to 0.0010 (reduced to four decimal places). Therefore

$$\sin 35°.17 = 0.5750 + 0.0010 = 0.5760.$$

PROBLEM 4. Find the cotangent of 35°.17.

Solution. From the table we find

$$\cot 35^\circ.1 = \quad 1.4229$$
$$\cot 35^\circ.2 = \quad 1.4176$$
$$d = \cot 35^\circ.2 - \cot 35^\circ.1 = -0.0053 \qquad \text{(tabular difference)}$$

We must add $\frac{7}{10}$ of d to cot 35°.1. But $\frac{7}{10}d = -0.0037$.
Therefore
$$\cot 35^\circ.17 = 1.4192.$$

We observe that in problem 3 the correction was positive, while in problem 4 it was negative.

If we always interpolate from the smaller toward the larger angle, the correction will be positive in the case of sine and tangent, negative in the case of cosine and cotangent. For, the former two functions *increase* with the angle, while the latter two *decrease*.

There will never be any serious danger of giving the wrong sign to the correction, if we cultivate the habit of running through the numbers of the table near the place we are using, so as to see in which direction they are growing.

EXERCISE VIII

1. Find all of the functions of 15°.3, 28°.7, 63°.4, 82°.1.

2. Find sin 37°.24, cos 62°.19, tan 53°.42, cot 27°.13.

3. Formulate in words the principle upon which the method of interpolation is based. Is this principle absolutely exact, or is it in the nature of an approximation?

4. What are the numbers in the four narrow columns of the table headed d, and what purpose do they serve?

5. In the arrangement of the table as explained, what use has been made of the results of Art. 10?

We have shown how to find the functions when the angle is given. It remains to show how to find an angle when one of its functions is known. The general method will be apparent from the following examples.

PROBLEM 5. The tangent of an unknown acute angle A is equal to 0.7046. Find the angle A.

Solution. In the specimen table on page 25 we observe that the number 0.7046 does not occur in the tangent column. However, we find there the two numbers 0.7028 and 0.7054 between which 0.7046 lies. Thus we have

$$\tan 35^\circ.1 = 0.7028,$$
$$\tan A = 0.7046,$$
$$\tan 35^\circ.2 = 0.7054.$$

Between tan 35°.1 and tan A, the difference is 0.0018.

Between tan 35°.1 and tan 35°.2, the difference is 0.0026.

Therefore, tan A is $\frac{18}{26}$ of the way from tan 35°.1 toward tan 35°.2, and consequently

$$A = 35^\circ.1 + \tfrac{18}{26} \text{ of one tenth of a degree,}$$

or

$$A = 35^\circ.1 + 0^\circ.07 = 35^\circ.17,$$

reducing to the nearest hundredth of a degree.

PROBLEM 6. Find the acute angle A whose cosine is 0.5772.

Solution. We have

$$\cos 54^\circ.7 = 0.5779,$$
$$\cos A = 0.5772,$$
$$\cos 54^\circ.8 = 0.5764,$$

whence

$$\cos A - \cos 54^\circ.7 = -0.0007,$$
$$\cos 54^\circ.8 - \cos 54^\circ.7 = -0.0015.$$

Therefore, cos A is $\frac{7}{15}$ of the way from cos 54°.7 toward cos 54°.8. Hence

$$A = 54^\circ.7 + \tfrac{7}{15} \text{ of one tenth of a degree}$$

or

$$A = 54^\circ.7 + 0^\circ.05 = 54^\circ.75.$$

With a little practice, the student will soon become sufficiently expert in the process of interpolation to enable him to perform this operation mentally. He should train himself with that end in view.

If we wish to find, by means of our table, the natural functions of an angle which is expressed in degrees, minutes, and seconds, the minutes and seconds should first be converted into decimal parts of a degree. This may easily be done, remembering that $1' = \frac{1}{60}$ of a degree and $1'' = \frac{1}{60}$ of a minute. Table VII may be used to save time in making this transformation.

EXERCISE IX

1. Find the values of

$$\sin 18^\circ 12', \cos 67^\circ 23', \tan 58^\circ 34', \cot 64^\circ 16'.$$

2. Find the acute angles for which

$$\sin A = 0.5673, \qquad \tan C = 1.7328,$$
$$\cos B = 0.2791, \qquad \cot D = 0.8924.$$

13. Solution of right triangles by means of the table of natural functions. A right triangle is determined by any two of its parts (not counting the right angle) provided that at least one of these parts is a side. The relations, which we have found between the angles and sides, enable us to compute the remaining parts of a right triangle when any two such parts are given. In order to make sure of this fact, let us see what cases may present themselves.

If both of the given parts are sides, there are two possibilities. Either the two given sides include the right angle (Case 1), or else one of the given sides is the hypotenuse (Case 2). If one of the given parts is a side and one is an angle, we have again two possibilities, viz.: given the hypotenuse and one acute angle (Case 3); or, given one leg and one acute angle (Case 4). We proceed to discuss these cases in order.

CASE 1. *Given the two sides of the triangle which include the right angle (the two legs).*

In our previous notation this means that a and b are given. In this case we may first compute

$$\tan A = \frac{a}{b}$$

and then find A from the table. We may then find c from

$$c = \frac{a}{\sin A} = \frac{b}{\cos A}$$

in two ways (affording a check), and B from

$$B = 90° - A.$$

CASE 2. *Given one leg and the hypotenuse.*

We may denote the given leg by a. Then a and c are given. The solution is accomplished by means of the formulæ

$$\sin A = \frac{a}{c}, \quad b = a \cot A = c \cos A, \quad B = 90° - A.$$

CASE 3. *Given the hypotenuse and one acute angle.*

We may denote the given acute angle by A. Then A and c are given. The solution is given by the equations

$$a = c \sin A, \; b = c \cos A, \; B = 90° - A.$$

CASE 4. *Given one leg and one acute angle.*

Since the knowledge of one acute angle implies that of the other, we may assume that a and A are the given parts. To find the remaining parts, we use the formulæ

$$b = a \cot A, \; c = \frac{a}{\sin A} = \frac{b}{\cos A}, \; B = 90° - A.$$

Our discussion has shown that the methods at our disposal suffice to find the remaining parts of a right triangle when two independent parts of the triangle (not counting the right angle) are given. To establish this fact was the purpose of the above classification. It is not necessary, nor even desirable, when solving a numerical problem of this sort, to find out first under what case it falls. In practice it is better not to refer to this classification at all, but to pick out and solve those among the four equations

(1) $$\sin A = \frac{a}{c}, \; \cos A = \frac{b}{c}, \; \tan A = \frac{a}{b}, \; A + B = 90°$$

which contain *only one unknown* quantity each. The remaining equations may then, in most cases, serve as a check.

A more complete check is given by the equation

(2) $$a^2 + b^2 = c^2.$$

In order to avoid the inconvenience of forming the quantities a^2, b^2, c^2 by actual multiplication, we have supplied a table of squares (Table VI). The arrangement and use of this table will be apparent from the following examples.

EXAMPLE 1. Find the squares of 0.324 and of 3.24.

Solution. In the left-hand column of Table VI we find 0.32. In the same horizontal row with this number, and in the column headed 4, we find 0.1050. Therefore

$$(0.324)^2 = 0.1050, \quad (3.24)^2 = 10.50.$$

EXAMPLE 2. Find the squares of 0.3243, of 3.243, and of 32.43.

Solution. From the table we find

$$(0.324)^2 = 0.1050, \quad (0.325)^2 = 0.1056.$$

The difference between the two squares is 0.0006. The number 0.3243 is three tenths of the way from 0.324 toward 0.325. Therefore, its square will be three tenths of the way from 0.1050 toward 0.1056. That is

$$(0.3243)^2 = 0.1050 + \tfrac{3}{10} \text{ of } 0.0006 = 0.1050 + 0.0002 = 0.1052,$$

and

$$(3.243)^2 = 10.52, \quad (32.43)^2 = 1052.$$

EXAMPLE 3. Find the square root of 0.5520.

Solution. We find from the table that this number is the square of 0.743. Therefore

$$\sqrt{0.5520} = 0.743.$$

EXAMPLE 4. Find the square root of 0.5525.

Solution. The table gives

$$\sqrt{0.5520} = 0.743, \quad \sqrt{0.5535} = 0.744.$$

Therefore, by interpolation

$$\sqrt{0.5525} = 0.743 + \tfrac{5}{15} \text{ of } 0.001 = 0.743 + 0.0003 = 0.7433.$$

In engineering practice, equation (2) is used very extensively in connection with such tables of squares, not merely for checking, but for the purpose of performing the original calculation. The tables of INSKIP and SMOLEY are particularly convenient for this purpose, if the distances are expressed in feet, inches, and thirty-seconds of an inch.

The following examples illustrate the methods for solving right triangles.

EXAMPLE 5. In a right triangle, right angled at C, given $a = 3.479$, $b = 2.321$. Compute the remaining parts of the triangle.

Solution. We find first

$$\tan A = \frac{a}{b} = \frac{3.479}{2.321} = 1.4989.$$

The table of tangents gives

$$A = 56°.2 + \tfrac{51}{56} \text{ of } 0°.1 = 56°.30,$$

reducing to the nearest hundredth of a degree, as usual. The tables of sines and cosines give

$$\sin A = 0.8320, \quad \cos A = 0.5548.$$

We compute next

$$c = \frac{a}{\sin A} = \frac{3.479}{0.8320} = 4.181,$$

and use the equation $c \cos A = b$ as a check. We find

$$c \cos A = 2.320, \quad b = 2.321.$$

The two members of the check equation do not agree exactly, but we have no right to expect absolute agreement. All of the numbers used in the calculation are merely approximations, giving us the values of the functions to the *nearest* unit of the fourth decimal place. In combining several such approximate numbers, the error may occasionally exceed two or three units of the last decimal place.

Finally we find

$$B = 90° - A = 33°.70.$$

We may exhibit this solution more compactly as follows. The figures in parentheses indicate the order in which the various results are obtained.

Formulæ. $\tan A = \frac{a}{b}, \quad c = \frac{a}{\sin A}, \quad B = 90° - A.$

Check. $c \cos A = b.$

Given $a = 3.479$ (1)	$\sin A = 0.8320$ (5)	$c = 4.181$ (7)
Given $b = 2.321$ (2)	$\cos A = 0.5546$ (6)	$c \cos A = 2.320$ (8) Check.
$\tan A = 1.4989$ (3)	$A = 56°.30$ (4)	$B = 33°.70$ (9).

Example 2. In a right triangle, right angled at C, given $c = 5.783$, $A = 42°.39$. Compute the remaining parts of the triangle.

Solution. Formulæ. $a = c \sin A, \quad b = c \cos A, \quad B = 90° - A.$

Check. $a^2 + b^2 = c^2.$

Given $c = 5.783$ (1)	$a = 3.899$ (6)
Given $A = 42°.39$ (2)	$b = 4.271$ (7)
$B = 47°.61$ (3)	$a^2 = 15.20$ (8)
$\sin A = 0.6742$ (4)	$b^2 = 18.24$ (9)
$\cos A = 0.7386$ (5)	$a^2 + b^2 = 33.44$ (10) Check.
	$c^2 = 33.44$ (11) Check.

Remark. The quantities a^2, b^2, c^2 required for the check were obtained from the table of squares. When no such table is available, it is usually desirable to write the check equation in the form

$$a^2 = c^2 - b^2 = (c - b)(c + b),$$

since this form of the equation reduces by one the number of multiplications required.

In this example, the check computation would then yield

$$a^2 = 15.202, \qquad c - b = 1.512,$$
$$c + b = 10.054,$$
$$c^2 - b^2 = 15.202 \quad \text{Check.}$$

EXERCISE X

In each of the following right triangles, right angled at C, two parts are given. Compute the remaining parts and check. Also check by means of a graphic solution to provide against gross errors.

1. $a = 27, \quad A = 25°.1.$

2. $a = 34.5, \quad C = 52.8.$

3. $a = 2.781, \quad b = 3.056.$

4. $b = 87.95, \quad A = 55°.36.$

5. $c = 604.5, \quad A = 47°.53.$

6. $a = 8.695, \quad b = 7.321.$

7. $b = 62.78, \quad c = 81.39.$

8. $B = 29°.58, \quad c = 2354.$

9. A gravel roof slopes three fourths of an inch per horizontal foot. What angle does it make with a horizontal plane?

10. The pitch of a gable roof is the quotient obtained by dividing the height of the ridge-pole above the garret floor by the width of the garret. What is the pitch of a gable roof covering a garret 38 feet wide, if the ridge-pole is 15 feet above the garret floor, and what angle does the roof make with a horizontal plane?

11. At a time when the sun was 55° above the horizon, the shadow of a certain building was found to be 112 feet long. How high is the building?

12. The side of a regular decagon is 3.471 feet. Find the radii of the inscribed and circumscribed circles.

13. The side of a regular polygon of n sides is equal to a. Find formulæ for the radii of the inscribed and circumscribed circles.

CHAPTER IV

DISCUSSION OF SOME DEVICES FOR REDUCING THE LABOR INVOLVED IN NUMERICAL COMPUTATIONS

14. The number of decimal places. As we attempted to point out in Chapter I, every number obtained as a result of measurement is really an approximation. If we measure the distance between two dots on our drawing board, by means of a carefully constructed scale which reads to the fiftieth part of an inch, we may still estimate half of this smallest scale unit with the naked eye. Let us assume that the divisions of the scale are reliable, that the ruler is very nearly straight, that the dots are very small, and that we are using the greatest of care in our measurement. We may then concede that the result of such a measurement (say 5.34 inches) is accurate to the nearest $\frac{1}{100}$ of an inch. This means that no number, with two figures to the right of the decimal point, is as close to the true value as 5.34. It means that 5.33 is certainly too small and that 5.35 is certainly too large. It does not mean that the true value is exactly 5.34 inches, but that the true value lies between 5.335 and 5.345 inches.

When we record the result of such a measurement, the number of decimal places which we write (two in this example) is an indication of the degree of precision which we claim for the result. In this connection, let us note emphatically that a zero, when obtained as the last digit of the measure of a quantity, should never be suppressed. Suppose, for instance, that in the above example we had obtained 5.30 inches as the result of our measurement. This means that we are certain that the true value of the distance lies somewhere between 5.295 and 5.305 inches.

If we were to record this result as 5.3 inches, suppressing the final zero, we should be giving the erroneous impression that we had measured the distance only to the nearest tenth of an inch and that it might have any value between 5.25 and 5.35 inches.

Thus, *the number of decimal places, which we use in recording the result of a measurement, is an indication of the degree of precision which we attribute to this result.*

This being so, we are guilty of negligence whenever we express such a result by a decimal with fewer places than we are able to guarantee. For we are thus throwing away knowledge which we have actually had in our possession. But it would be dishonest to express our result with *more* decimal places than we can guarantee. For we should then be tending to mislead others into thinking our measurements more accurate than they really are.

We may estimate the degree of precision of a number, as we have just done, on an *absolute* scale. But clearly it will usually be more reasonable to adopt as a measure of precision the ratio of the "probable error" of the measurement to the total magnitude of the quantity measured. If we do this, an error of one foot in a thousand is to be regarded as of no greater importance than an error of $\frac{1}{1000}$ of an inch in an inch. In either case we may say that it is an error of $\frac{1}{10}$ of one per cent.

Whenever we properly record the result of a measurement by a number consisting of four digits, no matter where the decimal point may be placed, this means, in the light of our preceding discussion, that the error of the last digit is guaranteed to be less than half a unit of the last decimal place. Now, the smallest number expressed by four digits is 1000. Let us suppose that the exact value of our unknown quantity is $x = 1000$ units, but that as a result of our measurement we have found $x = 999.5$ units. Then our error is $\frac{1}{2000}$, or $\frac{1}{20}$ of one per cent, of the total magnitude measured. The largest number expressible by four digits is 9999. If the exact value of x is 9999 and our error of measurement is half a unit, the error will be to the total magnitude measured as $\frac{1}{2}$ is to 9999, or as 1 is to 19998. It will be an error of about $\frac{1}{200}$ of one per cent of the whole. Thus, four digits are certainly sufficient to express the result of a measurement if its degree of accuracy does not exceed $\frac{1}{20}$ of one per cent. Now, most of the ordinary operations of surveying fall well within this limit, so that four decimal places are usually sufficient to express the results obtained by the surveyor.

15. The accuracy of a sum, difference, product, or quotient of two numbers obtained by measurement. It is clear that the sum or difference of two numbers can have no greater precision than the less accurate of the two numbers. Consequently, it is useless and misleading to retain more decimal places in one term of a sum or difference than in the others.

In forming a product we are apt to do a great deal of useless work if we fail to remember that the factors, and therefore the product, are mere approximations. Suppose we have measured the sides a and b of a rectangular field to the nearest hundredth of a foot and have found $a = 35.67$ ft., $b = 86.72$ ft. To find the area of the field we form the product ab. The ordinary method (shown in the margin) gives the result 3093.3024 square feet. But if we allow the result to stand in this form, we shall exhibit either our ignorance or our desire to create a false impression. For this would seem to indicate a result precise to the nearest $\frac{1}{10000}$ of a square foot, whereas it is uncertain by more than a whole square foot as we shall now show.

```
    86.72
    35.67
---------
    60704
   52032
  43360
 26016
---------
3093.3024
```

In fact, the equations $a = 35.67$ ft., $b = 86.72$ ft. merely mean that a and b are between the limits

$$35.665 < a < 35.675, \quad 86.715 < b < 86.725$$

respectively, so that the area must be between the limits

$$86.715 \times 35.665 < ab < 86.725 \times 35.675$$

or

$$3092.685475 < ab < 3093.919375.$$

The uncertainty in the value of ab is therefore so great that no digit to the right of the decimal point has any real significance. Even the last digit preceding the decimal point is not certain, so that we are even slightly overstating our accuracy, if we write simply

$$ab = 3093 \text{ square feet.}$$

Thus, we see that the product of two approximate four-place numbers is to be regarded as accurate to no more than four places. Consequently, it is a waste of time and labor to actually work out all of the partial products in the multiplication. We might abbreviate the process as follows:

86.72	or better, by	86.72
35.67	writing the	35.67
6.	more important	2602.
52.	partial products	434.
434.	first;	52.
2602.		6.
3094.		3094.

The fact that we find 3094, instead of 3093, need not disturb us, since we have seen from our above considerations that the last digit is actually uncertain to the extent of one unit.

This process, which is sometimes called *abbreviated multiplication*, is obviously preferable to the ordinary method, since it does away with the labor of first finding numbers which must afterwards be discarded.

Similar remarks may be made for division. More generally, whenever we are dealing with numbers whose first four digits only can be regarded as certain, it is wise to abbreviate all of our calculations correspondingly. It should be remarked, however, that, in certain exceptional cases, very exact results may be obtained from inaccurate data and *vice versa*. But this is not the place for a discussion of such cases.

16. Labor-saving devices. We have seen that extensive calculations, even with four-place numbers, are apt to be troublesome and laborious. In many problems it is necessary to use five- six- or seven-place numbers. In such cases, the amount of labor required becomes excessive, even if we make use of the abbreviated method of multiplication and division.

A very important aid in performing such calculations is furnished by certain tables, such as tables of squares (of which Table VI is an example), tables of cubes, and reciprocals of numbers, etc. CRELLE'S Tables, which are merely a systematically arranged and extensive set of multiplication tables, are particularly valuable.

A second great aid to numerical calculation is furnished by calculating machines, which are now being used very extensively in commercial as well as scientific work. The ordinary cash register is one of the simplest of these machines.

Graphic methods for solving numerical problems constitute a third great class of aids to calculation. These methods have, in recent times, been modified and extended, so as to become capable of greater accuracy, and for many problems no other method of solution is known. The slide rule, which may be classed either with the graphic methods or among the calculating machines, has become so popular among engineers as to exclude, in their work, almost all other methods of calculation. (See Arts. 29 and 30.)

But the most important of all of these labor-saving devices, without which the slide rule and many similar contrivances cannot even be thoroughly understood, is the method of calculation by logarithms.

17. Definition of logarithms. It is apparent, from what has been said, that the real cause of the laboriousness of extensive calculations lies in the operations of multiplication and division. Addition and subtraction, even of numbers with many places, are comparatively easy. It was this fact which caused JOHN NAPIER (1550–1617) and JOBST BÜRGI

(1552–1632)* to consider the possibility of devising a method, by means of which addition and subtraction might be made to do the work of multiplication and division. The method which they invented for this purpose is essentially equivalent to the one which we shall now explain, though very different from it in form. We must remember that the notations of modern algebra, to which we are accustomed and which are of the greatest assistance to us in our mathematical arguments, are the results of a century-long process of development. This process was far from complete in Napier's and Bürgi's time. The greatness of the achievement of these men can only be properly appreciated when judged from the standpoint of the mathematical knowledge of those days.†

From our present point of view the possibility of reducing the operations of multiplication to that of addition is an immediate consequence of a familiar fact of algebra. This fact, embodied in the formula

$$a^x a^y = a^{x+y},$$

states that the product of two powers of the same base is itself a power of that base, whose exponent is equal to the sum of the exponents of the two original powers. We may express this fact by the statement that to the *multiplication* of two powers of the same base corresponds the *addition* of their exponents. By this simple remark, multiplication is actually converted into addition for all such numbers as are known powers of a common base.

We shall assume that the fixed number a, the base of our system, is positive and different from unity. If the exponent x is a positive integer, there will be comparatively few

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* NAPIER was of Scotch and BÜRGI of Swiss nationality. Bürgi's discovery of logarithms was unquestionably independent of Napier's and was made at about the same time. But Napier's book "*Mirifici Logarithmorum canonis descriptio,*" containing an account of his method was published in 1614, six years earlier than Bürgi's "*Arithmetische und Geometrische Progress-Tabuln.*"

† For an excellent account of the history of logarithms see CAJORI in *The American Mathematical Monthly*, Vol. 20 (1913).

numbers which can be regarded as powers of a. Suppose, for example, that $a = 2$. Then 2, 4, 8, 16, 32, etc., are integral powers of 2, but 1, 3, 5, 6, 7, 9, 10, etc., are not. But, from our previous study of algebra, we are acquainted with the fact that the symbol a^x may be defined, not merely for the case when the exponent is a positive integer, but also when the exponent is any positive or negative rational number of the form $\pm\frac{p}{q}$ (p and q being integers), or zero. These definitions are as follows:

If x is a positive integer ($x = p$),

I. $a^x = a^p = a \cdot a \cdot a \cdots$ (a product of p factors each equal to a).

If x is a positive rational fraction $\left(x = \frac{p}{q}\right)$,

II. $$a^x = a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p,$$

where $\sqrt[q]{a}$ means the positive qth root of a.

If x is a negative rational fraction $\left(x = -\frac{p}{q}\right)$,

III. $$a^x = a^{-p/q} = \frac{1}{a^{p/q}} = \frac{1}{\sqrt[q]{a^p}}.$$

Finally, if $x = 0$,

IV. $$a^x = a^0 = 1.$$

Now, there is nothing remarkable about the fact that we have been able to define the symbol a^x in all of these cases. We might have done that in many different ways. But it *is* remarkable that, if we adopt these particular definitions, the formulæ

(1) $$a^x \cdot a^y = a^{x+y},$$

(2) $$\frac{a^x}{a^y} = a^{x-y},$$

(3) $$(a^x)^y = a^{xy}$$

turn out to be true, not only when the exponents are positive integers but in all of the four cases for which we have defined the symbol a^x. That this should be so, is of course not merely a fortunate coincidence. It is due to the fact that the definitions II, III, IV were deliberately chosen in such a way as to insure the universal validity of formulæ (1), (2), and (3). These formulæ, which are collectively known as the three *index laws*, are fundamental for the following discussion.

Equations I, II, III, IV suffice to define the symbol a^x whenever x is a rational number. Now, every irrational number can be approximated, as closely as we may desire, by means of a decimal fraction; and this decimal fraction (which is a rational number) will take the place of the original irrational number in all numerical calculations. We may define a^x, when x is irrational, as follows:

Let x_1 be the closest approximation, to the irrational number x, which is possible by means of a decimal fraction with only one figure to the right of the decimal point. Let x_2 be the closest approximation possible by means of a decimal fraction with only two figures to the right of the decimal point. Let $x_3, x_4, \cdots x_n, \cdots$ be similar approximations with $3, 4, \cdots n$ figures after the decimal point. The sequence of rational numbers

$$x_1, x_2, x_3, \cdots x_n, \cdots$$

has the irrational number x as a limit. Then a^x is defined to be the limit of the second sequence of numbers

$$a^{x_1}, a^{x_2}, a^{x_3}, \cdots a^{x_n}, \cdots.*$$

As an example, consider $a = 10$, $x = \sqrt{2}$. We have

$$x_1 = 1.4,\ x_2 = 1.41,\ x_3 = 1.414,\ x_4 = 1.4142, \text{ etc.},$$

$$a^{x_1} = 10^{1.4},\ a^{x_2} = 10^{1.41},\ a^{x_3} = 10^{1.414},\ a^{x_4} = 10^{1.4142}, \text{ etc.}$$

By $10^{\sqrt{2}}$ we mean the limit approached by the numbers of this second sequence.

For practical purposes, however, $\sqrt{2}$ is replaced by one of the approximations 1.4, 1.41, 1.414, etc., namely, the first one which is sufficiently accurate for the particular problem

* These limits exist, but it would carry us too far to prove this fact.

under consideration; and $10^{\sqrt{2}}$ is replaced by the first one of the numbers $10^{1.4}$, $10^{1.41}$, etc., which is sufficiently close to the true value for the purposes of the problem in question.

It may be shown that, if the above definition of a^x for irrational values of x be adopted, *the index laws will hold also for irrational exponents.*

We may now state, without formal proof, a theorem which is fundamental in the theory of logarithms, in so far as the very existence of logarithms depends upon it. This theorem is as follows:

If **a** *is a positive number different from unity, there exists one and only one exponent* **x** *(positive, negative, or zero), such that*

$$a^x = N,$$

where **N** *is any positive number.*

Although we state this theorem without proof, the student may easily convince himself of its great plausibility by a process which, if carried out to its logical conclusion, would constitute a proof. Suppose, for instance, that $a = 2$ and that $N = 1000$. We have

$$2^5 = 32,\ 2^6 = 64,\ 2^7 = 128,\ 2^8 = 256,\ 2^9 = 512,\ 2^{10} = 1024.$$

We conclude that the value of x for which

$$2^x = 1000$$

must be between 9 and 10. Now $2^{1/2} = \sqrt{2} = 1.4142 \cdots$. Therefore

$$2^{9.5} = 2^9 . 2^{1/2} = 512 \times 1.4142 = 724.1.$$

But

$$2^{10} = 1024,$$

so that x must lie between 9.5 and 10. We may obtain closer and closer limits between which x must lie by continuing this process, and thus ultimately show that there exists a number x (as a limit of a sequence), for which $2^x = 1000$. This argument at the same time indicates a process by means of which the exponent x may be calculated to any desired number of decimal places.

We are now ready to define a logarithm.

The **logarithm** *of any positive number* **N**, *with respect to the base* **a**, *is the exponent of the power to which the base* **a** *must be raised in order to obtain the number* **N**.*

In other words, if $a^x = N$,

we say that x (the exponent) is the logarithm of N with respect to the base a. In symbols we write this same statement as follows:

$$x = \log_a N.$$

EXAMPLE. Since $5^3 = 125$, we have $\log_5 125 = 3$.

EXERCISE XI

1. What are the logarithms of 2, 4, 8, 16, 32, 64, 128 with respect to the base 2? Write out each of these results in symbols; thus, $\log_2 4 = 2$.

2. What are the logarithms of 3, 9, 27, 81, 243 with respect to the base 3?

3. What are the logarithms of 10, 100, 1000, 10,000 with respect to the base 10?

4. What are the logarithms of 3, 9, 27, 81, 243 with respect to the base 27?

5. What are the logarithms of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, $\frac{1}{243}$ with respect to the base 3?

6. What are the values of 2^x, 3^x, 4^x, 10^x when x is equal to zero? What, then, is the logarithm of 1 with respect to each of the bases 2, 3, 4, 10?

7. What is the logarithm of 1 with respect to any base a?

8. What is the logarithm of any number with respect to itself as base?

9. Find, approximately to two decimal places, the number whose logarithm, with respect to the base 2, is equal to 1.5.

18. The properties of logarithms. Those properties of logarithms which are of importance for the purposes of numerical calculation, are immediate consequences of the index laws and of the definition of logarithms. In fact, we

* The word logarithm is derived from the Greek λόγος or *logos*, meaning proportion or ratio, and ἀριθμός or *arithmos*, meaning number. The reason for choosing this name will be apparent from the theorem stated in Exercise XII, Ex. 4.

may write each of two positive numbers, M and N, in the form

(1) $$M = a^x, \quad N = a^y,$$

so that, in accordance with the definition of logarithms,

(2) $$x = \log_a M, \quad y = \log_a N.$$

According to the first index law (Art. 17, equation (1)), the product of M and N is equal to

$$MN = a^x \cdot a^y = a^{x+y},$$

whence, by the definition of logarithms,

$$\log_a (MN) = x + y = \log_a M + \log_a N.$$

The theorem expressed by this formula may obviously be extended to any number of factors. Hence,

I. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

From (1) we obtain by division,

$$\frac{M}{N} = a^{x-y},$$

making use of the second index law (Art. 17, equation (2)). Therefore, by the definition of logarithms,

$$\log_a \left(\frac{M}{N}\right) = x - y = \log_a M - \log_a N,$$

a result which may be formulated as follows :

II. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

The same fact may, of course, be stated in the equivalent form: *the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

According to the third index law (Art. 17, equation (3)), we have

$$(a^x)^p = a^{px}.$$

Therefore, we find from (1)

$$M^p = a^{px},$$

or, by the definition of logarithms,

$$\log_a M^p = px = p \log_a M.$$

Consequently,

III. *The logarithm of the p^{th} power of a number M is obtained by multiplying the logarithm of M by p.*

Since the third index law is true whether p be an integer or a fraction, the last theorem has the following corollary, obtained by putting p equal to $\frac{1}{n}$:

IV. *The logarithm of the n^{th} root of a number M is obtained by dividing the logarithm of M by n.*

The content of the two equations

$$a^1 = a, \quad a^0 = 1$$

may be stated as follows :

V. *The logarithm of any number, with respect to itself as base, is equal to unity.*

VI. *The logarithm of unity, with respect to any base, is equal to zero.*

It is clear now how logarithms will serve to reduce the operations of multiplication and division to those of addition and subtraction. Suppose that we have at our disposal a *table* of logarithms. To multiply M by N we look up the logarithms of these numbers from the table and add them together. We then find the number whose logarithm is equal to this sum by again referring to the table ; this number is the product MN. To divide M by N we proceed in the same way, except that in this case we form the difference $\log M - \log N$ instead of the sum.

EXERCISE XII

1. If $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, find $\log_{10} 12$, $\log_{10} (\frac{3}{2})$, $\log_{10} (\frac{27}{4})$, $\log_{10} \sqrt[3]{6}$.

2. Express, in terms of $\log_a p$ and $\log_a q$, the following quantities:

$$\log_a (p^2 q^3),\ \log_a \left(\frac{p^3}{q^2}\right),\ \log_a \sqrt{\frac{p^{-4}}{q^{-7}}}.$$

3. Prove the truth of the following statement. If $\log_{10} x$ is expressed as a decimal fraction (x being a positive number greater than unity), the logarithm of $10^k x$ (k being a positive integer) will differ from $\log_{10} x$ only in its integral part.

4. Prove the theorem: If the numbers a_1, a_2, a_3, ... are in geometrical progression, their logarithms are in arithmetical progression.

5. Prove the equation

$$\log_a \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = 2 \log_a (x + \sqrt{x^2 - 1}).$$

CHAPTER V

CALCULATION BY LOGARITHMS

19. Common logarithms. With scarcely an exception, the civilized nations of all times have made use of the decimal system for expressing numbers, both in the spoken and in the written language.* For this reason, the number 10 is especially well adapted to serve as base for a system of logarithms. Logarithms with respect to the base 10 are usually known as *common logarithms;* they are also sometimes called *Briggsian* logarithms, in honor of HENRY BRIGGS (1556–1630),† who constructed the first table of common logarithms.

In this book we shall have very little occasion, hereafter, to speak of any except the common logarithms. We shall therefore agree to abbreviate the symbol $\log_{10} N$ to $\log N$, the base 10 being understood when no other base is mentioned explicitly.

The positive integral powers of 10, such as 10, 100, 1000, etc., the negative integral powers of 10, such as 0.1, 0.01, 0.001, etc., and the zero power of 10, which is equal to 1, are the only numbers whose common logarithms are integers. The logarithms of all other numbers have an integral and a fractional part.

The fractional part of the logarithm is called the **mantissa,** *while the integral part of the logarithm is known as its* **characteristic.**

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* It is usually admitted that the predominance of the decimal system over all others is due to the fact that the normal human being has ten fingers. This opinion has certainly been generally held since the time of ARISTOTLE.

† BRIGGS was the first Savilian Professor of Geometry at Oxford. According to BALL (see Ball's *Primer of the History of Mathematics*), Briggs was also the first to make systematic use of the decimal notation in working with fractions.

20. Properties of the mantissa. We consider the mantissa and the characteristic separately because, in practice, the method for finding the characteristic of a logarithm is entirely different from that employed for finding its mantissa. The reason for this will appear from the following discussion.

Let us consider an example. From the definition of a common logarithm, we know that

(1) $\log \sqrt[4]{10} = \log 10^{\frac{1}{4}} = \frac{1}{4} \log 10 = \frac{1}{4} \cdot 1 = 0.25000.$

Now it is not difficult to compute $\sqrt[4]{10}$ by elementary methods. We may, for instance, first compute $\sqrt{10}$ to as many decimal places as we desire, and then extract the square root of the result. We find, in this way, to five decimal places,

(2) $$\sqrt[4]{10} = 1.77828$$

or, if we combine (1) and (2),

(3) $$\log 1.77828 = 0.25000.$$

From the theorem about the logarithm of a product, we conclude

$$\log\ 17.7828 = \log\ (1.77828 \times 10) = \log\ 1.77828 + \log\ 10$$
$$= 0.25000 + 1 = 1.25000,$$
$$\log\ 177.828 = \log\ (1.77828 \times 100) = \log\ 1.77828 + \log\ 100$$
$$= 0.25000 + 2 = 2.25000,$$

. . . .

We observe that the numbers 1.77828, 17.7828, 177.828, etc., contain the same succession of digits, and differ from each other only in the position of the decimal point. Their logarithms, on the other hand, whose values we have just calculated, differ from each other only in the value of the characteristic.

Again, if we make use of the theorem about the logarithm of a quotient, we find from (3)

$$\log 0.177828 = \log \frac{1.77828}{10} = 0.25000 - 1,$$

$$\log 0.0177828 = \log \frac{1.77828}{100} = 0.25000 - 2,$$

. . . .

Now the negative quantities, which appear on the right members of these equations, are not written in the form which we ordinarily use for negative quantities. Thus, for instance, we have found the value of log 0.0177828 to be 0.25000 − 2, a result which we should ordinarily write in the form − 1.75000, to which it is obviously equal. If we agree to write every negative logarithm in this unusual form, as a difference between a *positive proper fraction* and an integer, thus making its fractional part positive, we gain the advantage that the mantissas will be the same for any two numbers which contain the same succession of digits, even if none of these digits appear to the left of the decimal point. We avoid, in this way, the necessity of using two different tables of mantissas, one for numbers greater, and one for numbers less, than unity.

Let us recapitulate the result of our discussion in two formal statements.

I. *We agree to express the logarithm of any positive number N in such a form that its mantissa shall be positive.*

This can be done whether log N is positive or negative, that is, whether N be greater or less than unity. In the latter case, the negativeness of log N is brought about entirely by means of the negative characteristic.

As a consequence of this agreement, the following statement will be true in all cases.

II. *If two numbers contain the same succession of digits, that is, if they differ only in the position of the decimal point, their logarithms will have the same mantissa and will differ only in the value of the characteristic.*

It is for this reason that the tables give only the *mantissas* of the logarithms and that, in looking up the mantissas, we pay no attention to the position of the decimal point in the given number.

21. Determination of the characteristic. The characteristic of a logarithm is easily determined by inspection. Its value depends merely on the position of the decimal point. Since we have

$$10^0 = 1,\ 10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000, \text{ etc.},$$

or

$$\log 1 = 0,\ \log 10 = 1,\ \log 100 = 2,\ \log 1000 = 3, \text{ etc.},$$

we draw the following conclusions:

If $1 < N < 10$, then $0 < \log N < 1$. $\therefore$ log N has the characteristic 0.

If $10 < N < 100$, then $1 < \log N < 2$. $\therefore$ log N has the characteristic 1.

If $100 < N < 1000$, then $2 < \log N < 3$. $\therefore$ log N has the characteristic 2.

. . . .

If $10^k < N < 10^{k+1}$, then $k < \log N < k+1$. $\therefore$ log N has the characteristic k.

We may formulate these results as follows:

III. *If k is a positive integer, and if the number N lies between 10^k and 10^{k+1}, the characteristic of log N is equal to k.*

Since such a number N has $k+1$ digits to the left of the decimal point, we obtain the following rule:

IV. *If N is any number greater than* 1, *the characteristic of its logarithm is one less than the number of digits in its integral part.*

The student is advised to make but little use of this rule on account of its mechanical character. Statement III provides a better method (less mechanical and easier to remember) for determining the characteristic.

It remains to show how to find the characteristic of log N when $N < 1$.

If $.1 < N < 1$, then $-1 < \log N < 0$. $\therefore$ log N has the characteristic -1.

If $.01 < N < .1$, then $-2 < \log N < -1$. $\therefore$ log N has the characteristic -2.

If $.001 < N < .01$, then $-3 < \log N < -2$. $\therefore \log N$ has the characteristic -3.

. . . .

If $\frac{1}{10^{k+1}} < N < \frac{1}{10^k}$, then $-(k+1) < \log N < -k$. $\therefore \log N$ has the characteristic $-(k+1)$.

Examination of this table leads to the following two statements, either of which may be used to determine the characteristic of $\log N$ when $N < 1$.

If k is a positive integer, and if the number N lies between $\frac{1}{10^k}$ and $\frac{1}{10^{k+1}}$, the characteristic of log N is $-(k+1)$.

If N is less than 1, *and is expressed as a decimal fraction having k zeros between the decimal point and the first significant figure, then the characteristic of the logarithm of N is $-(k+1)$.*

In one of our illustrations we had found

$$\log 0.0177828 = 0.25000-2.$$

We must never write this in the form

$$\log 0.0177828 = -2.25000,$$

since only the characteristic is negative and not the fractional part. Some computers use the notation

$$\log 0.0177828 = \bar{2}.25000;$$

but for most purposes it is preferable to write

$$\log 0.0177828 = 8.25000-10,$$

and similarly

$$\log 0.177828 = 9.25000-10.$$

In other words, *in actual practice, we write a positive characteristic* $10-k$ *in place of the negative characteristic* $-k$, *and then subtract* 10 *from the whole logarithm.*

22. Arrangement and use of the table of logarithms. We have already mentioned the fact that the table of logarithms gives only the mantissas. The characteristics must be supplied by the computer by the methods of Art 21.

The table which we shall ordinarily use (Table I) gives the mantissa, for every number from 1 to 9999, to five decimal places.

In order to explain the arrangement of this table, we shall reprint a small portion of it, and solve a number of typical examples, chosen in such a way as to require only this part of the table for their solution.

N	0	1	2	3	4	5	6	7	8	9	PP		
												19	20
—	—	—	—	—	—	—	—	—	—	—	1	1.9	2.0
—	—	—	—	—	—	—	—	—	—	—	2	3.8	4.0
—	—	—	—	—	—	—	—	—	—	—	3	5.7	6.0
220	34242	262	282	301	321	341	361	380	400	420	4	7.6	8.0
221	439	459	479	498	518	537	557	577	596	616	5	9.5	10.0
222	635	655	674	694	713	733	753	772	792	811	6	11.4	12.0
223	830	850	869	889	908	928	947	967	986	*005	7	13.3	14.0
—	—	—	—	—	—	—	—	—	—	—	8	15.2	16.0
—	—	—	—	—	—	—	—	—	—	—	9	17.1	18.0
—	—	—	—	—	—	—	—	—	—	—			

PROBLEM 1. Find the logarithm of 221.4.

Solution. To find the mantissa we ignore the decimal point. We read down the left-hand column of the table (headed N) until we find the first three digits of our number, viz., 221. The numbers printed in the same horizontal row with 221 are, in order, the mantissas of the logarithms of 2210, 2211, 2212, ..., 2219, as indicated by the number at the head of each of the next ten columns. To save space, however, the first two digits of the mantissa are never printed more than once in each row. In our case we find the mantissa, from the column headed 4, to be .34518. Since 221.4 is between $100 = 10^2$ and $1000 = 10^3$, the characteristic is 2. Therefore

$$\log 221.4 = 2.34518.$$

PROBLEM 2. Find log 22.39.

Solution. Looking for the mantissa as before, we find *005. The asterisk indicates that the first two digits of the mantissa are not 34, as one might suppose, but 35. The reason for this appears clearly from the table. Therefore

$$\log 22.39 = 1.35005.$$

If the number N contains more than four digits, its logarithm cannot be read directly from the table. But it may be found by *interpolation.* We illustrate this process by an example.

Problem 3. Find log 222.73.

Solution. From the table we find, supplying the characteristics ourselves,

$$\log 222.70 = 2.34772$$
$$\log 222.80 = 2.34792$$
$$\text{Tabular difference} = 0.00020 = 20 \text{ units of the fifth decimal place.}$$

Since 222.73 is $\frac{3}{10}$ of the way from 222.70 toward 222.80, we add $\frac{3}{10}$ of the tabular difference to log 222.70. Therefore

$$\log 222.73 = 2.34772 + \tfrac{3}{10} \text{ of } 0.00020,$$

or

$$\log 222.73 = 2.34772 + 0.00006 = 2.34778.$$

The auxiliary tables in the margin, headed P P (abbreviation for proportional parts), facilitate the process of interpolation.

Thus, in problem 3, we refer to the auxiliary table with 20 (the tabular difference) at its head. In the third row we find $\frac{3}{10}$ of 20 or 6,0.

It remains to show how to find the number when its logarithm is given.

Problem 4. Given $\log N = 9.34857 - 10$. Find the value of N to five significant figures.

Solution. The characteristic of $\log N$ is $9 - 10$ or -1. Therefore, the number N must be between $10^{-1} = 0.1$ and $10^0 = 1$. Consequently, the decimal point will precede the first significant figure of N.

The mantissa 34857 does not occur in the table, but it falls between the two tabular mantissas 34850 and 34869.

Thus we have:

$$9.34850 - 10 = \log 0.22310 \text{ (from the table)},$$
$$9.34857 - 10 = \log N,$$
$$9.34869 - 10 = \log 0.22320 \text{ (from the table)},$$

so that N lies between 0.22310 and 0.22320.

We observe that $\log N$ lies $\frac{7}{19}$ of the way from $\log 0.22310$ toward $\log 0.22320$. Therefore, N lies $\frac{7}{19}$ of the way from 0.22310 toward 0.22320. That is,

$$N = 0.22310 + \tfrac{7}{19} \text{ of 10 units of the fifth decimal place.}$$

But

$$\tfrac{7}{19} \text{ of 10 units} = \tfrac{70}{19} \text{ units} = 3\tfrac{13}{19} \text{ units} = 4 \text{ units.}$$

Therefore

$$N = 0.22310 + 0.00004 = 0.22314.$$

Also in this inverse problem (to find the number when its logarithm is given) interpolation is aided by the auxiliary tables in the margin.

Thus, in problem 4, the tabular difference is 19. The difference between log N and the smaller of the two tabular logarithms, between which log N lies, is 7. The auxiliary table with 19 at its head, shows that, among the tenths of 19, that one, which comes closest to the value 7, is the fourth. Consequently, N is $\frac{4}{10}$ of the way from 0.22310 toward 0.22320. Therefore, up to five decimal places, $N = 0.22310 + 0.00004 = 0.22314$.

23. Cologarithms. Since we obtain the same result whether we divide N by M, or multiply N by $\frac{1}{M}$, we may, in a logarithmic calculation, add the logarithm of $\frac{1}{M}$ instead of subtracting log M. *The logarithm of* $\frac{1}{M}$ *is called the* **cologarithm** *of* M. Therefore

$$\text{colog}\, M = \log \frac{1}{M} = \log 1 - \log M = -\log M,$$

since log 1 is equal to zero.

Cologarithms, like logarithms, are written with positive mantissas. Consequently, the cologarithm of a number is most easily found by subtracting its logarithm from zero, written in the form $10.00000 - 10$, as in the following example.

PROBLEM 5. Find the cologarithm of 222.73.

Solution.

$$\begin{array}{r} 10.00000 - 10 \\ \underline{\log 222.73 = 2.34778 } \\ \text{colog}\, 222.73 = 7.65222 - 10 \end{array}$$

It is easy to perform this operation of subtraction from $10.00000 - 10$ mentally. There is no gain, however, from the use of cologarithms when we are dealing with a quotient of only two numbers. A real advantage is gained by the introduction of cologarithms, when more than two logarithms are to be combined by addition and subtraction. For the logarithms which are to be subtracted we then substitute cologarithms, enabling us to complete the operation by a single addition.

It often happens, just as in the case of forming a cologarithm, that we wish to subtract a logarithm from another smaller one. In all such cases we change the form of the minuend by adding and subtracting 10, or some convenient multiple of 10, as in the following example.

PROBLEM 6. Compute $\frac{32.34}{472.3}$.

Solution. We find from Table I,

$$\log 32.34 = 1.50974,$$
$$\log 472.3 = 2.67422.$$

In order to subtract the latter logarithm from the former, we write

$$\log 32.34 = 11.50974 - 10,^*$$
$$\log 472.3 = 2.67422$$
$$\log \frac{32.34}{472.3} = 8.83552 - 10$$

Hence, from the table, $\frac{32.34}{472.3} = 0.068473.$

24. Extraction of roots by means of logarithms. Since

$$\log \sqrt[p]{x} = \log x^{1/p} = \frac{1}{p} \log x,$$

it is easy to extract roots of any order by means of logarithms. If the characteristic of $\log x$ is not negative, no further remark is necessary. If $\log x$ is negative, we proceed as in the following example:

PROBLEM 7. Compute by logarithms: $\sqrt{.53760}$, $\sqrt[3]{.53760}$, and $\sqrt[5]{.53760}$.

Solution. $\log 0.53760 = 9.73046 - 10.$

$$\log \sqrt{.53760} = \tfrac{1}{2} \log 0.53760 = \tfrac{1}{2}\ (19.73046 - 20) = 9.86523 - 10.$$
$$\log \sqrt[3]{.53760} = \tfrac{1}{3} \log 0.53760 = \tfrac{1}{3}\ (29.73046 - 30) = 9.91015 - 10.$$
$$\log \sqrt[5]{.53760} = \tfrac{1}{5} \log 0.53760 = \tfrac{1}{5}\ (49.73046 - 50) = 9.94609 - 10.$$

Therefore, from Table I,

$$\sqrt{.53760} = .73322,\quad \sqrt[3]{.53760} = .81312,\quad \sqrt[5]{.53760} = .88326.$$

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* A computer with some experience will refrain from actually writing the logarithm in the form 11.50974 − 10. It is easy for him to carry out the calculation *as though* it were so written.

25. Logarithmic calculations which involve negative numbers. We have only defined the logarithms of positive numbers. But this suffices for our purposes. Clearly, when we compute a product or quotient, its numerical value may be found first, without paying any attention to the signs of the various factors. Afterwards, the proper sign (+ or −) may be prefixed to the result according as there is an even or an odd number of negative factors.

The easiest way to keep a count of the negative factors is to use the method, introduced by GAUSS,* of writing the letter n immediately after a logarithm which corresponds to a negative number. In forming a sum or difference of logarithms, we write an n after the result only if an *odd* number of the separate logarithms is affected by an n.

EXAMPLE. If $N = -222.73$, we write

$$\log N = 2.34778\ n.$$

EXERCISE XIII

1. Making use of the tables, find log 3726, log 67.43, log 729800, log 0.3896, log 0.008527.

2. Making use of the tables, find log 32653, log 76.431, log 879450, log 0.045723, log 0.0059426.

3. By means of the tables, find the numbers whose logarithms are 3.84522, 1.68079, 8.89064 − 10, 7.12548 − 10, 2.27068.

4. By means of the tables, find the numbers whose logarithms are 3.89067, 9.24110 − 10, 1.52195.

5. Given $a = 3.1572$, $b = 7.2916$, $c = 45.731$. Compute by logarithms the values of ab, bc, ca.

6. With the same values of a, b, c compute $\frac{ab}{c}$.

7. With the same values of a, b, c compute $\sqrt[3]{\frac{a^2b}{c^5}}$.

8. Compute the volume of a hemispherical dome if its diameter is 150.32 feet. (Volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.)

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* C. F. GAUSS (1777–1855) was without question one of the greatest and most versatile mathematicians of all times. He was director of the observatory and professor of astronomy at Göttingen from 1807 to the end of his life.

9. If a sum of money P (the principal) is earning interest at the rate of $r\%$ a year, and if the interest is added to the principal at the end of each year, show that the amount, at the end of n years, will be

$$A = P\left(1 + \frac{r}{100}\right)^n.$$

In this case the interest is said to be compounded annually.

10. Find the amount on $157.38 for 7 years at $3\frac{1}{2}\%$ compound interest.

11. How much money must I put into the bank at 3% compound interest, so that the amount may be $ 500 at the end of five years?

26. The logarithms of the trigonometric functions. In solving right triangles by means of logarithms, we frequently have to find the logarithm of a trigonometric function of an angle. It would be very burdensome if we had to look up first, in the table of natural functions, the value of the function and then, from the table of mantissas, find its logarithm. In order to avoid this complication, there has been constructed an additional table which enables us to find directly the values of the logarithms of the sine, cosine, tangent, and cotangent for every angle between 0° and 90°. These quantities are denoted by the symbols log sin or L. Sin, log cos or L. Cos, etc., and are pronounced log sine, log cosine, log tangent, and log cotangent.

In the tables which accompany this book, Table II gives the values of the logarithms of the trigonometric functions directly, to five decimal places, for every minute of arc. If the angle contains fractional parts of a minute, we obtain its functions from the table by interpolation.

The arrangement of this table resembles that of the table of natural functions so closely, that it is unnecessary to describe it in detail. It should be noted, however, that in this table the characteristics of the logarithms are also given. But since the natural sines and cosines of all acute angles, and the tangents of all angles less than 45°, are proper fractions, these characteristics are negative and have been expressed in the form $9 - 10$, $8 - 10$, etc. *The continually recurring* -10 *has not been printed*, and should be supplied by the computer. It is understood, once for all, that 10 is to be subtracted from all of the logarithms in the first, second, and fourth columns of the table, while the logarithms printed in the third column are provided with their correct characteristics and require no such modification.

The process of interpolation may be applied to the table of logarithms of the trigonometric functions in the same way as to the table of natural functions or to the table of logarithms of numbers.

The following examples are intended to illustrate the application of Table II.

EXAMPLE 1. Find log sin, log cos, log tan, log cot of 41° 15′ 35″.

Solution. For convenience in interpolation convert 35″ into decimal parts of a minute. Then 41° 15′ 35″ = 41° 15′.58.

We find, from the table, the following material:

41°

′	L Sin	D	L Tan	C D	L Cot	L Cos	D		P P	
—	—		—		—	—		—		15
—	—		—		—	—		—	1	1.5
15	9.81911	15	9.94299	25	0.05701	9.87613	12	45	2	3.0
16	9.81926		9.94324		0.05676	9.87601		44	3	4.5
—	—		—		—	—		—	4	6.0
—	—		—		—	—		—	5	7.5
—	—		—		—	—		—	6	9.0
—	—		—		—	—		—	7	10.5
—	—		—		—	—		—	8	12.0
—	—		—		—	—		—	9	13.5
	L Cos	D	L Cot	C D	L Tan	L Sin	D	′		

48°

We conclude:

log sin 41° 15′.58 = 9.81911 + .58 of 15 units of the 5th decimal place.
log tan 41° 15′.58 = 9.94299 + .58 of 25 units of the 5th decimal place.
log cot 41° 15′.58 = 0.05701 − .58 of 25 units of the 5th decimal place.
log cos 41° 15′.58 = 9.87613 − .58 of 12 units of the 5th decimal place.

We may use the marginal tables of proportional parts to complete the interpolation. Thus, the table headed 15, shows that .5 of 15 is 7.5 and

.08 of 15 is 1.2, and consequently .58 of 15 is 8.7 or 9 units of the fifth decimal place. Therefore

$$\log \sin 41° \, 15'.58 = 9.81920 - 10.$$

In the same way we find

$$\log \tan 41° \, 15'.58 = 9.94314 - 10, \ \log \cot 41° \, 15'.58 = 0.05686,$$
$$\log \cos 41° \, 15'.58 = 9.87606 - 10.$$

EXAMPLE 2. Find the logarithms of the functions of 48° 44'.42.

Solution. This angle is the complement of that of Example 1. Hence each of its functions is equal to the corresponding cofunction of 41° 15'.58, and the values obtained are the same as in Example 1 with the name of each function changed to the corresponding cofunction.

Just as in the table of natural functions, these values, for angles greater than 45°, may be obtained directly from the table by reading the degrees of the angle at the *foot* of the page, the minutes in the right-hand column, and the name of the function at the foot of each of the four columns. We find, in this way,

$$\log \sin 48° \, 44'.42 = 9.87606 - 10, \ \log \cot 48° \, 44'.42 = 9.94314 - 10,$$
$$\log \tan 48° \, 44'.42 = 0.05686, \qquad \log \cos 48° \, 44'.42 = 9.81919 - 10.$$

EXAMPLE 3. Given $\log \tan A = 0.53219$. Find A.

Solution. The given logarithm does not appear anywhere in the column at the foot of which is printed the name L Tan. But we do find in this column

$$\log \tan 73° \, 38' = 0.53212,$$
$$\log \tan 73° \, 39' = 0.53259.$$
$$\text{Tabular difference for } 1' = 0.00047.$$

The given value of $\log \tan A$ is $\frac{7}{47}$, or $\frac{15}{100}$, of the way from the first toward the second of these tabular logarithms. Therefore

$$A = 73° \, 38'.15$$

27. The accuracy of five-place tables. As we have said repeatedly, the number of decimal places used in stating the result of a measurement is to be regarded as an indication of its degree of precision. We shall ordinarily wish to make all calculations, based upon such measurements, with a sufficient number of decimal places to avoid introducing inaccuracies which might have an appreciable influence upon the results, *i.e.* an influence comparable with that produced by the unavoidable errors of observation.

Five-place tables are quite accurate enough to satisfy this condition in almost all problems of engineering and natural science. In fact, in most problems of this kind, the distances are not measured so accurately as to exclude an error of one twentieth of one per cent of their value, and the angles are read to the nearest minute only. But five-place tables are far more accurate than this. In fact, a distance expressed by a five-place number presupposes an accuracy of at least $\frac{1}{200}\%$, and an angle may usually be determined from five-place logarithms of its functions with an error of not more than 2 or 3 seconds of arc.

We must not forget, however, that the degree of accuracy of a table is not the same in all parts of the table, and that we must use our judgment in the selection of the formulæ which we wish to use in solving a problem.

28. The trigonometric functions of angles near 0° or 90°. Thus, for instance, if we wish to determine an angle for which $\log \cos A = 9.99998 - 10$, our table cannot furnish an accurate result. We find, by referring to the table, that A may have any value between $0° \, 29'$ and $0° \, 36'$.

A small angle cannot be determined, with any degree of accuracy, from the value of its cosine.

In the same way, we see that *an angle very close to* 90° *cannot be determined accurately from the value of its sine.*

In most cases we shall be able to modify the formula, which we are using, in such a way as to avoid this difficulty. If, for instance, the angle A (known to be very small) is to be determined from the value of its cosine, we shall seek some other formula as a solution of the same problem by means of which the angle A can be determined from the value of its sine or tangent. The problem then reduces to that of finding a small angle when its sine or tangent is given. If we again refer to our table, we find that this problem also gives rise to a difficulty. The method of interpolation, which we ordinarily use, becomes both cumbersome and inexact in the case of such small angles, because the tabular differences are very large and change very rapidly from one place in the table to another.

In order to meet this difficulty, we have provided a separate table (Table III), giving the values of the logarithmic functions for every second of arc from $0° 0'$ to $0° 3'$ and from $89° 57'$ to $90°$, and for every ten seconds from $0°$ to $2°$ and from $88°$ to $90°$.

Another method of meeting this difficulty, preferable in some respects, will be explained in the second part of this book (Art. 85); it involves the auxiliaries S and T (Table IV).

EXERCISE XIV

1. Find log sin $15° 6'$, log cos $35° 13'$, log tan $57° 28'$, log cot $76° 44'$.

2. Find log sin $18° 23'.35$, log tan $41° 46'.27$, log cos $64° 17' 43''$, log cot $25° 12' 38''$.

3. Find the angles for which

$$\log\sin A = 9.42553 - 10, \qquad \log\cos B = 9.60618 - 10$$
$$\log\cot C = 9.68497 - 10, \qquad \log\tan D = 0.14193.$$

4. Find the angles for which

$$\log\cot A = 0.11157, \qquad \log\tan B = 9.75465 - 10,$$
$$\log\cos C = 9.68334 - 10, \qquad \log\sin D = 9.56652 - 10.$$

5. Find log sin $0° 2' 15''$, log tan $1° 10' 22''$.

6. Find the angles for which

$$\log\sin A = 5.83170 - 10, \qquad \log\tan B = 8.32313 - 10.$$

7. Find, by logarithms, the angle A, if $\tan A = a/b$ and $a = 1.2291$, $b = 14.950$.

29. The logarithmic or Gunter scale. No graphical process is more familiar than the addition and subtraction of line-segments, and this process may evidently be used as a substitute for addition and subtraction of numbers. Since addition of logarithms corresponds to multiplication of numbers, we may find the logarithm of a product graphically by adding line-segments, whose lengths are equal to the logarithms of the factors.

In order to do this, we must have some means for actually finding a line-segment whose length shall be equal to the logarithm of a given number.

Let us take a line-segment of convenient length, say 10 centimeters, as unit of length. In terms of this unit, the whole distance (10 centimeters = 100 millimeters) represents log 10, since the logarithm of 10 is equal to unity. If we count all distances from the left-hand end of the line, we may label the right-hand end 10 to indicate that this distance represents log 10. The left-hand end will then be labeled 1, because $\log 1 = 0$.

From the table of logarithms we have, to two decimal places,

$$\begin{aligned} &\log 1 = 0.00, \quad \log 2 = 0.30, \quad \log 3 = 0.48, \quad \log 4 = 0.60,\\ (1)\quad &\log 5 = 0.70, \quad \log 6 = 0.78, \quad \log 7 = 0.85, \quad \log 8 = 0.90,\\ &\log 9 = 0.95, \quad \log 10 = 1.00. \end{aligned}$$

We mark the points on our line-segment whose distances from the left-hand end, measured in terms of the whole line as unit, are in order equal to log 2, log 3, log 4, ... log 9, and label them 2, 3, 4, ... 9, respectively. If the whole line-segment is 10 centimeters long, these points will, on account of (1), be at distances 30, 48, 60, 70, 78, 85, 90, 95 millimeters, respectively, from the left-hand end of the line-segment (cf. Fig. 14).

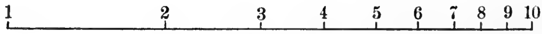

FIG. 14

A scale constructed in this way is called a **logarithmic scale**, and its usefulness for purposes of calculation was first pointed out by EDMUND GUNTER* in 1620. It enables us to find a line-segment equal in length to the logarithm of any number between 1 and 10. It is easy to see how, by means of such a scale and a pair of dividers, multiplication and division may be reduced to the simple graphical processes of adding and subtracting line-segments.

30. The slide rule. Some years before 1630, WILLIAM OUGHTRED † noticed that the use of the dividers might be avoided by constructing two equal logarithmic scales (Scales

* Professor of astronomy in Gresham College, London (1581–1626).

† OUGHTRED (1575–1660) was a fellow of King's College, Cambridge.

A and *B* of Fig. 15), capable of sliding by each other, as indicated in the figure.*

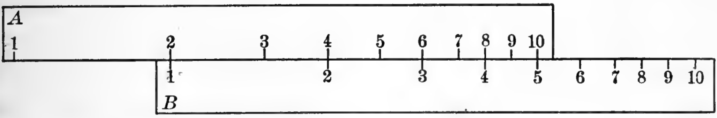

FIG. 15

The use of this simple bit of apparatus for the purpose of multiplication and division will be apparent from the following examples:

To multiply 2 by 3. Place scale *B* in such a way that its left-hand index (*i.e.* the division marked 1) falls directly under the division marked 2 on scale *A*. Directly above the division marked 3 on scale *B*, we shall find, on scale *A*, the product which (of course) is 6. To justify this process it suffices to note that it is equivalent to adding the logarithm of 3 to that of 2.

Fig. 15 shows scales *A* and *B* in the proper position for the purposes of this example.

To divide 6 by 3. Under the division 6 of scale *A*, place division 3 of scale *B*. Over the division 1 of scale *B* we shall find the quotient ($\frac{6}{3} = 2$) on scale *A* (cf. Fig. 15).

The instrument actually in use, the MANNHEIM slide rule, is a slight amplification of the one just described (cf. Fig. 16). It has four scales, usually denoted by *A*, *B*, *C*, *D*, respectively, the scales *A* and *D* being on the rule, and *B* and *C* on the slide.

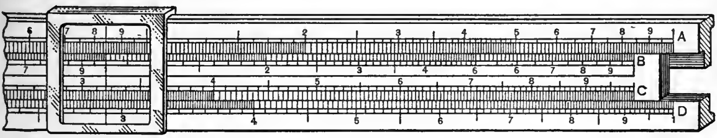

FIG. 16

The scale *A* is composed of two logarithmic scales such as that of Fig. 14, so that its right-hand end might be labeled 100, since log 100 = 2. On most slide rules, however, the first principal division on scale *A* after 9 is not labeled 10,

* Oughtred's instruments were described in publications of WILLIAM FOSTER, one of his pupils, in 1632 and 1633.

as in Fig. 14, but 1, the next one is not labeled 20, but 2, and so on to the last one, which is again labeled 1 instead of 100 or 10. Thus, the two halves of scale *A* are exact copies of one another. This is done for precisely the same reason that the mantissas only are printed in our tables of logarithms. The slide rule also makes use of the mantissas only. The characteristics, or what amounts to the same thing, the position of the decimal point in the result, must be obtained by inspection or by special rules.

Scale *B* is on the upper edge of the slide, in direct contact with scale *A* on the rule, and is an exact copy of scale *A*. These two scales together may be used for multiplication and division as explained above.

Scale *D* is on the lower part of the rule. It is a single logarithmic scale, from 1 to 10, of the same length as the combined two scales of A. The logarithm of any number is therefore represented, on scale *D*, by a distance twice as great as that which represents the logarithm of the same number on scale *A*. It follows from this that the number which is found on scale *A*, vertically above any number of scale *D*, is the square of the latter. Any number on scale *D*, on the other hand, is the square root of the number vertically above it on scale *A*.

Scale *C* is on the lower edge of the slide, in direct contact with slide *D* on the rule. It is an exact copy of scale *D*. These two scales together may be used for multiplication and division, according to the same rules which hold for scales *A* and *B*.

Besides these four scales, the slide rule is supplied with a *runner* (cf. Fig. 16), which is useful in performing compound operations, and also in comparing two scales (such as *A* and *D*), which are not in direct contact with each other. The runner was made a permanent feature of the slide rule by MANNHEIM in 1851.*

* AMÉDÉE MANNHEIM (1831–1906), a distinguished geometer of recent times. The runner had however been used occasionally, long before Mannheim, by a number of English mathematicians.

It often happens, in manipulating the slide rule, that the result is to be sought opposite a number of the slide which falls outside of the scale on the rule. In such cases, we may *shift* the slide, bringing the right-hand index to the place which the left-hand index occupied previously, and read off the result as before. For, such a shift has no influence on the mantissa, since it merely amounts to dividing the result by 10. On the Mannheim rule, this shifting of the slide may be avoided by working with scales A and B rather than with C and D. Scales C and D, however, have the advantage of greater accuracy.

If the slide be withdrawn entirely, it will be found to have three other scales on its reverse side, two of which are labeled S and T. These are scales of logarithmic sines and tangents, respectively, and may be used for calculating such products as

$$c \sin A, \quad c \tan A.$$

The middle scale on the reverse side is used for finding the value of the logarithm of a number, and is important if we wish to compute a power of a number with a complicated fractional exponent.

For more complete information concerning the slide rule, we must refer to the manuals which are usually presented to the purchaser of such an instrument.* Cheap slide rules, especially constructed for the beginner, may now be obtained of all dealers under the name Student's or College Slide Rule. Engineers and computers use the slide rule so extensively that the student will find it advisable to make himself familiar with the instrument by actual use.

The Mannheim slide rule, which we have described, admits of three-figure accuracy. In some (exceptional) cases, results correct to four decimal places may be obtained by its use. The THACHER and FULLER slide rules, more complicated instruments, but constructed on essentially the same

* See also RAYMOND's Plane Surveying.

principles, admit of far greater accuracy. The EICHHORN Trigonometric Slide Rule was invented for the purpose of solving triangles, and is especially adapted for this work. But, of course, it has not the wide range of usefulness of the ordinary slide rule.

CHAPTER VI

APPLICATION OF LOGARITHMS TO THE SOLUTION OF RIGHT TRIANGLES

31. The general method. We have shown, in Chapter III, how to solve right triangles by means of the natural functions, and we have become acquainted with the theory and use of logarithms in Chapter V. To solve a right triangle by logarithms, it suffices to combine the results of these two chapters. We use the same formulæ as in Chapter III, but perform the multiplications and divisions by means of logarithms, using the table of logarithmic sines, cosines, etc., in place of the table of natural functions.

In order to illustrate the various practical questions which arise in such a calculation, we shall give a rather extended discussion of the following example:

EXAMPLE. The legs of a right triangle were found to be $a = 527.38$ feet and $b = 621.24$ feet. Calculate the hypotenuse and the acute angles A and B.

32. The preliminary graphic solution.

We first make a drawing, approximately to scale, making

$$a = 5.3 \text{ centimeters},$$
$$b = 6.2 \text{ centimeters}.$$

We find by measurement

$$c = 8.1 \text{ centimeters},$$
$$A = 40°.5,\quad B = 49°.5.$$

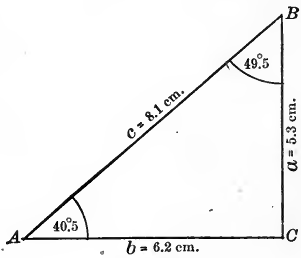

FIG. 17

This figure and these measurements serve two purposes. In the first place, the figure helps us pick out the formulæ which we shall need for the trigonometric solution of our triangle. In the second place, comparison of the approximate values of the unknown quantities obtained graphically, with the final results as obtained by calculation, constitutes a valuable check. If the results obtained by the two methods should differ by more than can be accounted for by the inaccuracies of the graphic solution, we must look for a mistake in our calculation.

33. The gross errors. Mistakes, which are large enough to be detected by means of a graphic check, are known as *gross errors* and are usually due to one of the following causes:

1. The use of a wrong formula.
2. A misplaced decimal point or, what amounts to the same thing, an erroneous characteristic.
3. The use of a number taken from a wrong column in the tables, resulting, for instance, in erroneously using the log cos of an angle in place of the log sin.
4. Addition of two logarithms when subtraction is required or *vice versa*.
5. Purely arithmetical errors of addition and subtraction.

The errors of the first four classes can be avoided by the exercise of a sufficient amount of care. The student should not attempt to gain speed in calculation until he has first learned to be accurate. The errors of the last class are quite unavoidable, but they will usually be detected almost as soon as made if the ordinary arithmetical checks for addition and subtraction be applied every time that one of these operations is used.

34. Selection of formulæ and checks. After completing the approximate graphical solution of a triangle, we pick out the formulæ which we wish to use in the computation.

In our example, these are the following:

(1) $$\tan A = \frac{a}{b},\ c = \frac{a}{\sin A} = \frac{b}{\cos A},\ B = 90° - A,$$

or, in logarithmic form,

(2) $$\log \tan A = \log a - \log b, \qquad B = 90° - A,$$
$$\log c = \log a - \log \sin A = \log b - \log \cos A.$$

We have two formulæ for c, and in most cases it makes little difference which one we decide to use. If we were to use both, the agreement of the two results would constitute a partial check on the accuracy of our work. It would not be a total check however; a mistake in the logarithm of a or b could not be detected by means of it.* It is hardly worth while therefore to use both formulæ for c. That one is to be preferred which has the greater denominator, as the result obtained from it is likely to be the more accurate.

For a complete check, we may make use of the equation

$$a^2 + b^2 = c^2,$$

which, however, we prefer to write in the form

(3) $$a = \sqrt{c^2 - b^2} = \sqrt{(c + b)\,(c - b)},$$

which is more convenient for logarithmic computation.

It should not be necessary to write the formulæ in the logarithmic form (2). Form (1) is shorter and more directly connected with the geometry of the problem. Moreover, it contains all of the information that is necessary for the solution of the problem, for anybody who has studied logarithms.

35. The framework or skeleton form. Having selected the formulæ, we proceed to plan the details of the computation by providing a definite, properly marked place for every number which will be needed in the course of the work. Moreover, we shall plan these details in such a way that those numbers which are to be combined by addition or subtraction will have their places in the same vertical column next to each other.

In our particular example we may adopt the following framework:

Given	$a=$	(1)		$\log b=$	(4)
	$b=$	(2)		$\log \cos A=$	(7)
	$\log a=$	(3)		$\log c=$	(4) − (7) = (8)
	$\log b=$	(4)		$c=$	(9)
	$\log \tan A =$	(3) − (4) = (5)		$b=$	(2)
Results	$A=$	(6)		$c-b=$	(9) − (2) = (10)
	$B=$	(16)		$c+b=$	(9) + (2) = (11)
	$c=$	(9)		$\log (c-b)=$	(12)
				$\log (c+b)=$	(13)
				$\log (c^2-b^2)=$	(12) + (13) = (14)
			Check	$\log \sqrt{(c^2-b^2)}=$	$\frac{1}{2}$ (14) = (15)
				$\log a=$	(3)

* Since such a mistake could be interpreted as leading to the correct solution of a triangle different from the given one, namely, that one whose sides a' and b' have as logarithms the values which were, by mistake, assigned to a and b.

The numbers in parenthesis merely indicate the order in which this skeleton form may be filled in, and how some of the results are obtained. The student should use these numbers only to aid him in understanding the construction of the framework and the plan of the computation. They should not be used in writing out the actual calculation.

36. The computation. We are now prepared to carry out the computation. This should be done on paper ruled into squares of such a size that each figure may conveniently occupy one square. We obtain the following results:

$$\text{Given}\begin{cases} a = 527.38 \\ b = 621.24 \end{cases}$$

$$\log a = 2.72212$$
$$\log b = 2.79326$$
$$\log \tan A = 9.92886 - 10$$

$$\text{Results}\begin{cases} A = 40^\circ\ 19'.69 \\ B = 49^\circ\ 40'.31 \\ c = 814.92 \end{cases}$$

$$\log b = 2.79326$$
$$\log \cos A = 9.88215 - 10$$
$$\log c = 2.91111$$
$$c = 814.92$$
$$b = 621.24$$
$$c - b = 193.68$$
$$c + b = 1436.16$$
$$\log (c - b) = 2.28709$$
$$\log (c + b) = 3.15720$$
$$\log (c^2 - b^2) = 5.44429$$
$$\left.\begin{matrix} \log \sqrt{c^2 - b^2} = 2.72215 \\ \log a = 2.72212 \end{matrix}\right\}\text{Check.}$$

Remark. We observe that the check is not absolute. The agreement is as close, however, as we should expect. The inevitable inaccuracies, arising from the neglected higher decimal places, often manifest themselves by discrepancies amounting to several units of the fifth decimal place. Consequently, we may declare the check to be satisfactory.*

37. Revision of the computation when the check is unsatisfactory. If, in the solution of such an example, the results fail to check satisfactorily, the magnitude of the discrepancy will help us to locate the error. If the discrepancy is very great, the error must be one of the *gross* kind which we have discussed in Art. 33. In case of a comparatively small discrepancy, our error is probably due to one of the following causes:

1. Purely arithmetical errors of addition and subtraction in the last few decimal places.

* If a and b differ considerably, use as check $a = \sqrt{(c-b)(c+b)}$ or $b = \sqrt{(c-a)(c+a)}$, according as $b < a$ or $b > a$.

2. Inexact interpolation, which would ordinarily affect only the last decimal place.

3. Addition of the correction obtained by interpolation when it should be subtracted, or *vice versa*.

This last mistake may be avoided by carefully inspecting the table *after* the interpolation has been completed, so as to make sure that the quantity calculated actually lies between the two numbers of the table between which it *should* fall.

EXERCISE XV

In each of the following examples (1–10), two parts of a right triangle are given in the usual notation. Find the other parts:

1. $c = 627$, $A = 23° 30'$.

2. $c = 934$, $B = 76° 25'$.

3. $a = 637$, $A = 4° 35'$.

4. $b = 48.532$, $B = 36° 44'.00$.

5. $a = 38.313$, $b = 19.522$.

6. $a = 13.690$, $b = 16.926$.

7. $a = 67.291$, $c = 110.970$.

8. $b = 618.42$, $c = 1843.70$.

9. $a = 965.24$, $A = 75° 15'.2$.

10. $a = 7.3298$, $b = 6.1032$.

An isosceles triangle may be divided into two equal right triangles by dropping a perpendicular from the vertex to the base. Using the notations of Fig. 18, find the missing sides and angles of the following isosceles triangles. (Exs. 11–13.)

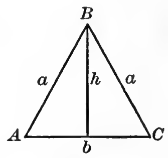

FIG. 18

11. $b = 2.1452$, $B = 121° 14'.60$.

12. $A = 52° 10'.2$, $a = 600.20$.

13. $h = 7.447$, $A = 76° 14'.00$.

14. Prove that the area S of a right triangle is

$$S = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ac \cos A.$$

15. Prove that the area S of a right triangle is

$$S = \tfrac{1}{2} c^2 \sin A \cos A.$$

16–25. Find the area of each of the right triangles in Exs. 1–10.

26–30. Find formulæ for the area of the isosceles triangle of Fig. 18, in terms of b and h; a and b; a and h; a and B; a and A.

31–33. Apply the results of Exs. 26–30 to find the areas of the isosceles triangles of Exs. 11–13.

34. Show that the perimeter p of a regular polygon of n sides inscribed in a circle of radius R is

$$p = 2\,nR \sin \frac{180°}{n},$$

that the radius r of the inscribed circle is

$$r = R\cos\frac{180^\circ}{n},$$

and that the area S of the polygon is

$$S = nR^2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}.$$

35. Find the radius of the inscribed circle, the perimeter and the area of a regular pentagon, if the radius of the circumscribed circle is 12 feet.

36. Find the perimeter, the length of one side, the radii of the inscribed and circumscribed circles of a regular octagon whose area is 24 square feet.

37. Since the polygon of Ex. 33 approaches the circle of radius R as limit when n grows beyond all bound, what limits do $n\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}$ and $2n\sin\frac{180^\circ}{n}$, respectively, approach?

38. Applications to simple problems of surveying, navigation, and geography. The connection between surveying and trigonometry is so obvious as to require no further explanation. Moreover, we have already discussed this relation in Chapter 1.

Many of the following examples are concerned with simple problems of surveying, and most of the technical terms which occur in them are self-explanatory. Nevertheless, we shall give a brief discussion of these terms, so as to make the applications seem more concrete and vivid. The student who wishes to know more about the subject should consult a treatise on surveying.*

A **plumb line** is a cord to one end of which is attached a weight. If such a plumb line be suspended, by fastening the other end of the cord to a fixed support, it will oscillate to and fro, and finally come to rest in its position of equilibrium, which is called the **vertical line** of the place of observation.

Since the earth is approximately spherical in shape and the plumb line points toward the earth's center, the vertical lines of different places are not parallel. But the angle between

* For instance, RAYMOND's Plane Surveying.

the vertical lines of two stations which are not very far apart (say ten miles), is so small that for most purposes these lines may be regarded as parallel. When a tract of land (to be surveyed) is comparatively small, it is therefore legitimate to neglect the effect of the earth's curvature, and the problem becomes one of **plane surveying.** The more difficult problems connected with a *geodetic survey*, in which the earth's spherical form is taken into account, require knowledge of the methods of *spherical trigonometry.*

We are concerned with plane surveying only, so that we shall regard the vertical lines of all places which occur in such a survey as parallel.

A **vertical plane** is one which contains a vertical line.

A **horizontal line** or **plane** is one which is perpendicular to a vertical line.

An inclined line or **plane** is one which is neither vertical nor horizontal.

An angle is said to be a **horizontal or vertical angle**, according as the plane of its sides is a horizontal or vertical plane. Both horizontal and vertical angles may be measured by means of the transit (Art. 2). Inclined angles may be measured by means of an instrument known as a *sextant.*

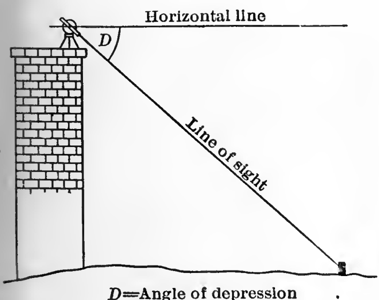

FIG. 19

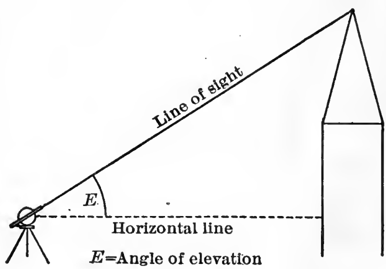

FIG. 20

The angle which the line of sight from the observer to an object makes with a horizontal line, in the same vertical plane, is called the **angle of elevation** or the **angle of depres-**

sion, according as the object is above or below the horizontal plane of the observer (cf. Figs. 19 and 20).

The **angle subtended by a line** is that which is obtained by joining the extremities of the line to the eye of the observer.

The direction or **bearing** of any horizontal line is usually described by means of the angle which it makes with the north-south line or **meridian**, the latter being located approximately with the help of a surveyor's **compass**. Surveyors always measure the bearing of a line as an acute angle from the north or south end of the meridian toward the east or west point, as the case may be. Thus, in Fig. 21, the bearing of OA is N 45° E, that of OB is S 10° W, that of OC is N 30° *W*.

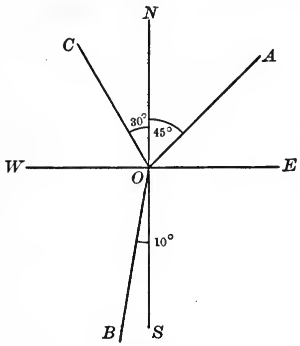

Fig. 21

Surveyors usually measure distances by means of a Gunter's chain, which is 4 rods or 66 feet long, and is divided into 100 links. For this reason, the operation of measuring the length of a line in the field is frequently called **chaining**.

In order to measure the difference of level between two places, A and B (cf. Fig. 22), the observer at O first makes his telescope point in a horizontal direction by means of a spirit level attached to the telescope. An assistant holds a graduated rod, R, in a vertical position at A, and the observer at O reads, by means of the telescope, the division on the graduated rod where it is struck by the horizontal line of sight.

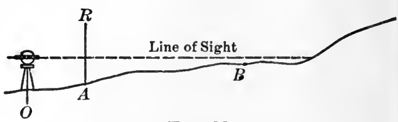

Fig. 22

He repeats this operation with the rod at B. The difference between the two readings gives the difference of level between A and B. Of course, if the difference of level between A and B exceeds the length of the rod, intermediate stations must be introduced. This operation is known technically as **leveling**.

The **navigator** does not always express bearings in the same language as the surveyor. He divides the circumference into 32 equal parts, called points of the compass. Thus one point of the compass is an angle of $11\frac{1}{4}$ degrees. The division points are named, as indicated in Fig. 23, with obvious reference to the four cardinal points of the compass,—north, south, east, and west.

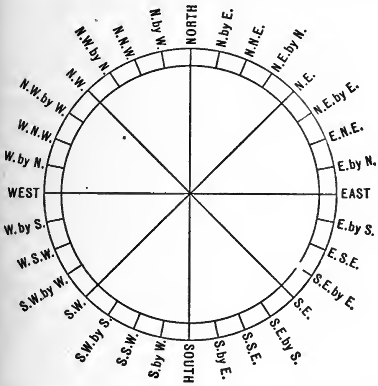

FIG. 23.—The points of the Mariner's Compass

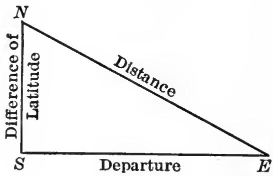

FIG. 24

The navigator also makes use of the terms **departure** (to denote the east and west component of a course), and **difference in latitude** (to denote the north and south component). These terms are illustrated in Fig. 24.

EXERCISE XVI

In solving the following problems, the student should exercise his judgment in regard to the number of decimal places to be used in the calculation. (See Chap. I, Arts. 1, 2; Chap. IV, Arts. 14, 15.) Many of these problems may be solved by means of three place tables (Tables IX, X, and XI of our collection), or by means of the slide rule. (See Chap. V, Arts. 29, 30.) In all of the examples the slide rule may be used as a check.

1. At a point 180.00 feet away from the base of a tower and in the same horizontal plane with it, the angle of elevation of the top was found to be 65° 40′.5. Find the height of the tower.

2. From the top of a cliff 120 feet above the level of a lake, the angle of depression of a boat was found to be 27° 40′. What is the air line distance from the top of the cliff to the boat?

3. In order to measure the width of a river, a base line AC is measured along one bank 215.6 feet long. By means of a transit, a point B is located on the opposite bank such that ACB is a right angle. The angle BAC is found to be $55° \, 16'.2$. What is the width BC of the river?

4. From the top of a mountain 2653 feet above the floor of the valley, the angles of depression of two farmhouses in the level valley beneath, both of which were due east of the observer, were found to be 25° and 56°. What is the horizontal distance between the two houses?

5. From the top of a hill, the angles of depression of two consecutive milestones on a straight level road, running due south from the observer, were found to be $22° \, 31'$ and $48° \, 15'$. How high is the hill?

HINT. Treat this problem as one involving two unknowns: 1st, the height of the hill; 2d, the horizontal distance from one of the milestones to the foot of the perpendicular dropped from the top of the hill to the horizontal plane of the road.

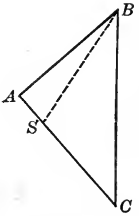

FIG. 25

6. Three lighthouses A, B, C, are situated as in Fig. 25, the triangle ABC being right-angled at A. At the moment when a ship S is crossing the line AC, the angle ASB is found to be 75°. If the distance between the lighthouses A and B is 12 miles, what are the distances from S to A and B?

7. A light on a certain steamer is known to be 35 feet above the water. An observer on the shore, whose instrument is 5 feet above the water, finds the angle of elevation of this light to be 5°. What is the distance from the observer to the steamer?

8. What angle does a mountain slope make with a horizontal plane, if it rises 200 feet in a horizontal distance of one tenth of a mile?

9. The cable of a captive balloon is 835 feet long. Assuming the cable to be straight, how high is the balloon when all of the cable is out if, owing to the wind, the cable makes an angle of 25° with a vertical line?

10. A ship is sailing due west at the rate of 8.9 miles per hour. A lighthouse is observed due south at 10 P.M. The bearing of the same lighthouse at 11:55 P.M. was S. 34° E. Find the distance from the lighthouse to the ship at the time of the second observation.

HINT. In Fig. 26, S and S' represent the two positions of the ship and L represents the lighthouse. Angle $SS'L = 90° - 34°$.

FIG. 26

11. Find the area of the tract of land corresponding to the following description. From A the boundary line runs N. 24° E. 20 chains to B, thence N. 85° W. 35.67 chains to C, and thence back to A.

12. The shadow of a chimney, 50 feet high, is 60 feet long. What is the altitude (or angle of elevation) of the sun at that instant?

13. The last row of seats in a circular tent is 20 feet away from the central pole, which is 18 feet high, and which is to be fastened by ropes from its top to stakes driven in the ground. How long must these ropes be in order that they may be 6 feet above the ground over the last row of seats, and at what distance from the center must the stakes be driven?

14. How long must a ladder be to reach a window 45 feet high, directly above a porch 15 feet high, if the porch projects 10 feet from the building?

15. A building 125 feet high, with a flat roof, faces north on a boulevard. The distance from this building to the one directly opposite is 180 feet. How far back from the edge of the roof should a chimney 6 feet high be placed, so as to be invisible from any point on the boulevard due north of the chimney?

16. A cylindrical pipe 36 inches in diameter is to be joined to a second cylindrical pipe 18 inches in diameter. The axes of the two cylinders are pieces of the same horizontal line and their ends are 6 feet apart. The joining piece is to be in the form of a frustum of a cone. Draw a sectional view of the joining piece and compute the length of the slanting side and the angles.

17. We wish to construct a house with a gable roof. If the house is 25 feet wide, if the height under the eaves is 27 feet and the height to the ridge pole 35 feet, how long must the rafters be so that their ends may be at a horizontal distance of 2 feet from the side of the house?

18. The angle of elevation of the center of a spherical balloon 20 feet in diameter was found to be 65°. The angle which it subtended at the same time was 2° 30′. What is the height of the balloon above the horizontal plane of the observer?

19. Two stations, A and B, are to be connected by a railroad. Both stations are in the same horizontal plane, and B is 35 miles northeast of A. The two stations are separated by a lake, which terminates at a point C, 12 miles north and 29 miles east of A. Find the lengths and bearings of the two portions of the road from A to C and from C to B.

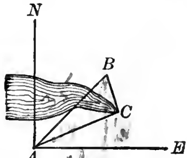

FIG. 27

20. A lecture room, 50 feet long and 18 feet high, is to be supplied with a sloping floor. The front part of the floor, for the first ten feet, is to be horizontal so as to admit of the placing of lecture tables and apparatus. The highest part of the sloping floor, at the back of the room, is to be 8 feet from the ceiling. What length of sloping timbers is required for this construction? Each of these timbers is to be supported at both ends and by six intermediate uprights placed at equal horizontal distances from each other and from the end supports. How far should each of these eight supports project above the horizontal part of the floor?

21. In order to determine the height of a mountain above a level plane, we may measure a horizontal base line of length b in the same vertical plane with the summit of the mountain and observe the angles of elevation, A and B, of the summit from the two ends of the base line. Find a formula for the vertical height h of the mountain above the level of the plane.

22. Apply the formula of Ex. 21 to the case $b = 100$ feet, $A = 30°$, $B = 35°$.

23. A flagstaff, known to be h feet in length, stands on top of a cliff. An observer, in the same horizontal plane with the base of the cliff, finds the angles of elevation of the top and bottom of the flagstaff to be A and B respectively. Find a formula for the height of the cliff.

24. Apply the formula of Ex. 23 to the case $h = 25$, $A = 40° 25'$, $B = 37° 10'$.

25. The angle of elevation of the top of a spire from the third floor of a building was 35° 10′. The angle of elevation from a point directly above, on the fifth floor of the same building, was 25° 33′. What is the height of the spire and its horizontal distance from the place of observation, if the distance between consecutive floors is 12 feet and the first floor rests on a basement 5 feet above the level of the street?

26. Prove the following statement. If R is the radius of the earth regarded as a sphere, the radius of the parallel of latitude which passes through a place P of latitude L, is

$$r = R \cos L.$$

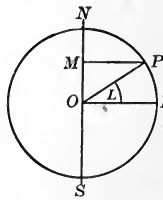

FIG. 28

HINT. Use Fig. 28, where O denotes the earth's center, N and S the north and south poles, $OE = R$ the radius of the equator, $\angle EOP = L$ the latitude of the place P, and $r = MP$ the radius of the parallel of latitude which passes through P.

27. Let d be the length, in miles, of a degree of longitude at the equator. Show that the length of a degree of longitude, at latitude L, will be $d \cos L$.

28. Show that the radius r (in feet) of the horizon of an observer, h feet above the earth's surface, is given by the formula

$$r = \frac{R}{R + h}\sqrt{h\ (2\ R + h)},$$

if R denotes the earth's radius expressed in feet.

HINT. Use Fig. 29, where $OQ = OM = R$, $MP = h$, $QN = r$, using the angle $NOQ = A$ as auxiliary.

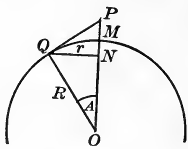

FIG. 29

29. A micrometer screw is to be cut from a cylindrical steel rod, 5 millimeters in diameter, in such a way that one complete revolution of the screw will move the wire W, attached to the movable frame F (Fig. 30), through a distance of one millimeter. What angle will the thread of the screw make with a plane perpendicular to the axis of the screw?

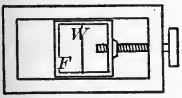

FIG. 30

30. A street railway track is d feet from the curbstone. In passing a corner (Fig. 31), where the street is deflected through an angle of $K°$, it is desired to have the rail pass at a distance of d' feet from the corner. Show that the radius of the circular curve ACB must be

$$r = \frac{d - d' \cos \frac{K}{2}}{1 - \cos \frac{K}{2}}.$$

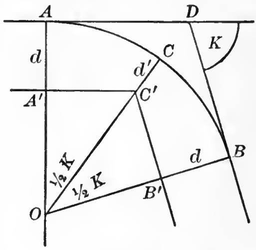

FIG. 31

31. Solve the problem of Ex. 30 numerically for the cases $d = 10$ feet, $d' = 4$ feet, $K = 90°$; and $d = 9$ feet, $d' = 3.5$ feet, $K = 60°$.

32. In order to find the horizontal distance between the points A and F which are at different levels and situated on opposite sides of a rolling valley (see Fig. 32), the distances $AB = d_1$, $BC = d_2$, ... $EF = d_5$ are measured along the ground by chaining. A transit is placed at O, and $A'B'C'$ ... F' represents the line of sight of the instrument. This line of sight is made horizontal by means of a spirit-level attached to the telescope. The vertical distances

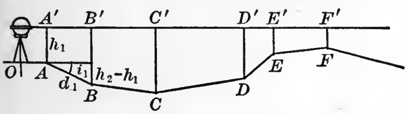

FIG. 32

$$AA' = h_1,\ BB' = h_2,\ CC' = h_3,\ DD' = h_4,\ EE' = h_5,\ FF' = h_6$$

are measured by a rod. (Cf. Fig. 22 for method of using rod.)

Show that the distance

$$A'F' = d_1 \cos i_1 + d_2 \cos i_2 + d_3 \cos i_3 + d_4 \cos i_4 + d_5 \cos i_5,$$

where i_1, i_2, i_3, i_4, i_5 are the angles of inclination of AB, BC, CD, DE, EF, respectively. The angle i_1 is determined by the equation

$$\sin i_1 = \frac{h_2 - h_1}{d_1}.$$

The angles $i_2 \cdots i_5$ are determined in similar fashion.

33. In constructing a telegraph line across a hill $ABC, \cdots I$ (Fig. 33), posts were set at $A, B, C, \cdots I$, these points being determined by *level chaining* * in such a way that the horizontal distance between any two of them $A'B' = B'C' = C'D' = \cdots = H'I' = d$ feet. By leveling, the elevations of the points A, B, C, etc., were found to be

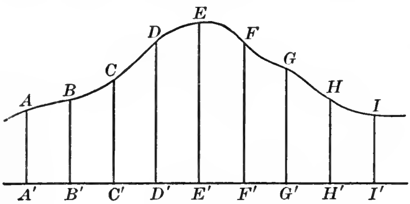

FIG. 33

$$AA' = h_1,\ BB' = h_2,$$
$$CC' = h_3, \cdots II' = h_9.$$

Find a method for computing the amount of wire required between A and I, assuming that the telegraph poles are vertical and of the same height, and making no allowance for sag.

39. Right triangles of unfavorable dimensions. If the hypotenuse and one side (say b and c) are given, we have the equation

$$\text{(1)} \qquad \cos A = \frac{b}{c}$$

to determine the angle A. But, if b differs very little from c, the value of $\cos A$ will be very close to unity and, as we observed in Art. 28, it will be impossible to determine A with any degree of accuracy from this equation.

A surveyor will usually (not always) be in a position to avoid this difficulty. For he has a certain amount of liberty in the choice of his triangles. But, in many problems of astronomy and mathematical geography, no such choice is possible, so that it becomes a matter of practical importance to find a formula for determining the angle A, which shall not be liable to the same objection as (1).

* In level chaining, the surveyor's chain or tape is held in a horizontal position, so as to measure the *horizontal* distance between two points and not the distance along the slope.

Such a formula may be obtained as follows. In Fig. 34, draw AD, the bisector of the angle A, and also BE perpendicular to AD. Then

$$\tfrac{1}{2}A = \angle CAD = \angle CBE$$

(both angles being complementary to $\angle AEB$), and

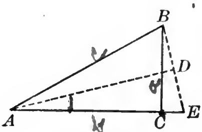

FIG. 34

$$AE = AB = c, \quad CE = c - b.$$

Consequently we find, from the right triangle BCE,

$$(2) \qquad \tan \tfrac{1}{2} A = \frac{CE}{BC} = \frac{c-b}{a}.$$

But

$$a = \sqrt{c^2 - b^2} = \sqrt{(c-b)(c+b)},$$

so that we may write, in place of (2),

$$(3) \qquad \tan \tfrac{1}{2} A = \sqrt{\frac{c-b}{c+b}}.$$

This is the desired formula, which should be applied instead of (1), whenever the value of b is very close to that of c, *i.e.* whenever the angle A is very small.

Some of the following examples will illustrate the usefulness of this formula as well as the application of Table III for the functions of small angles.

EXERCISE XVII

1. At what distance may a mountain 14,000 feet high be seen at sea, if the earth's radius is 3963 miles?

2. How high above the earth's surface must a balloon rise, in order to enable an observer to see a point 50 miles away?

3. If the moon's parallax (the angle which the earth's radius subtends as seen from the moon) is 57′, and if the earth's radius is 3963 miles, what is the moon's distance from the earth?

4. If the angular diameter of the moon, as seen from the earth, is 31′ 20″ and the distance from the earth to the moon is 239,100 miles, what is the moon's diameter in miles?

5. If the distance from the earth to the sun is 92,000,000 miles, and the angular diameter of the sun as seen from the earth is 32′, what is the diameter of the sun in miles?

CHAPTER VII

THEORY OF OBLIQUE TRIANGLES

40. The area of an oblique triangle in terms of two of its sides and the included angle.

Let the sides b, c of a triangle and the angle A be given.

If we consider the side $AB = c$ as the base, then the altitude $CD = h$ is the length of the perpendicular dropped from C to AB. The foot D of this perpendicular may fall on the

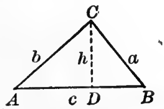

FIG. 35

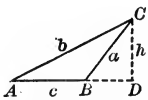

FIG. 36

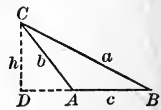

FIG. 37

line segment AB (Fig. 35), to the right of B (Fig. 36), or to the left of A (Fig. 37). In all of these cases we have,

$$(1) \qquad S = \tfrac{1}{2}\, ch,$$

if S denotes the area of the triangle.

Now h can be expressed in terms of the given quantities, b, c, and A, as follows.

In Figs. 35 and 36, in which A is an acute angle, we have

$$\frac{h}{b} = \sin A \text{ or } h = b \sin A,$$

and therefore, by substituting this value of h in (1),

$$(2) \qquad S = \tfrac{1}{2}\, bc \sin A.$$

If A is an obtuse angle (Fig. 37), we have

$$\frac{h}{b} = \sin \angle DAC = \sin (180° - A), \text{ or } h = b \sin (180° - A),$$

and consequently, by substituting this value of h in (1),

$$(3) \qquad S = \tfrac{1}{2}\, bc \sin (180° - A).$$

Thus, the area of a triangle is equal to

$$\tfrac{1}{2}\, bc \sin A \quad \text{or} \quad \tfrac{1}{2}\, bc \sin (180° - A)$$

according as A is an acute or an obtuse angle.

While we have found a complete solution of our problem, the result is not quite as convenient as it might be, since we have two different formulæ for S according as A is an acute or an obtuse angle. Is there any way in which we might avoid the distinction between these two cases?

The expression

$$\tfrac{1}{2}\, bc \sin A$$

is meaningless, from our present point of view, if A is an obtuse angle. For we have, as yet, given no definition for the sine of an obtuse angle, the definitions of Art. 7 being applicable to acute angles only. Clearly, however, it will be desirable to attach a meaning to the symbol $\sin A$, also in the case when A is an obtuse angle, now that we are dealing with oblique triangles, some of whose angles may be obtuse.

We may define the sine of an obtuse angle in any way we choose, *so long as it is not inconsistent with the definitions already agreed upon*, and we naturally choose our definitions and notations in such a way as to reduce to a minimum the number of formulæ and theorems which must be remembered.

Now we can make a single formula do the work of both (2) and (3), by adopting the following **definition for the sine of an obtuse angle.**

The sine of an obtuse angle A is equal to the sine of the acute angle 180° − A, *which is supplementary to A;* or in symbols,

(4) $\quad$ **$\sin A = \sin (180° - A)$, A being obtuse.**

As a consequence of this definition, equation (3), which gives the expression for the area of the triangle when A is obtuse, reduces to

$$S = \tfrac{1}{2}\, bc \sin A,$$

so that formula (2) may be used whether A be acute or obtuse. The same formula is obviously true when A is a right angle; for in that case $\sin A = 1$, and the formula reduces to

$$S = \tfrac{1}{2}\, bc$$

where c is the base and b the altitude.

We therefore have a single formula

$$\boldsymbol{S = \tfrac{1}{2}\, bc \sin A}$$

for the area of a triangle in terms of two sides and the included angle, whether the latter be acute, right, or obtuse.

41. The law of sines. Since the triangle has three angles and three pairs of including sides, we may write three different expressions for the area of the same triangle, viz.:

$$S = \tfrac{1}{2}\, bc \sin A = \tfrac{1}{2}\, ca \sin B = \tfrac{1}{2}\, ab \sin C.$$

The equality of these three expressions is a very important fact, since it gives rise to the following relations between the sides and angles of any triangle:

$$bc \sin A = ca \sin B = ab \sin C.$$

We may write these relations in a somewhat simpler form, by dividing all three members of the continued equation by the product abc. We find in this way

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

or

(1) $$\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}},$$

whence

(2) $$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \frac{a}{c} = \frac{\sin A}{\sin C}, \quad \frac{b}{c} = \frac{\sin B}{\sin C}.$$

These formulæ contain the so-called **law of sines,** which may be expressed in words as follows: *any two sides of a triangle are to each other as the sines of the opposite angles.*

The first explicit statement and proof of the law of sines, known at the present day, is to be found in a treatise on trigonometry by the Persian, NASĪR ADDĪN, or NASIR EDDIN (1201–1247 A.D.). Nasīr Addīn's treatise may also be regarded as the first in which trigonometry was treated as a separate science, independent of its applications to astronomy.

EXERCISE XVIII

1. What becomes of the law of sines when one of the angles (say C) is a right angle?

2. Prove the law of sines directly from Figs. 34, 35, 36, by computing the value of h in each of the two right triangles into which ABC is divided by the altitude.

3. Show that the law of sines may be used to solve the following problem: Given two angles of a triangle and one of its sides; to find the other sides and the remaining angle.

4. The formula,
$$\sin(180° - A) = \sin A,$$
holds when A is an obtuse angle, as a consequence of the definition adopted for the sine of an obtuse angle. Show that the same equation is also true if A is an acute or right angle.

42. The law of cosines. A generalization of the theorem of Pythagoras. We have learned to recognize the importance of the theorem of Pythagoras in the theory of right triangles, and the question naturally arises: what takes the place of this theorem in the case of an oblique triangle?

Most students will remember that the answer to this question is contained in the following two propositions of geometry:

Theorem 1. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides *diminished* by twice the product of one of those sides and the projection of the other upon that side.

Theorem 2. In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides *increased* by twice the product of one of those sides and the projection of the other upon that side.

The proof of Theorem 1 (repeated from Geometry) is as follows: Let A be an acute angle. The triangle ABC will have the form represented in Figs. 38 or 39, according as the angle B is acute or obtuse. In either case we put

$$BC = a, \quad CA = b, \quad AB = c,$$

and

$$AD = m,$$

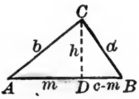

FIG. 38

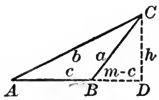

FIG. 39

so that m is the projection of b upon c, or upon c produced. The right triangle BCD gives

$$a^2 = h^2 + \overline{BD}^2$$

in both cases. Now $BD = c - m$ in Fig. 38, and $BD = m - c$ in Fig. 39. Therefore we find, in either case,

$$a^2 = h^2 + c^2 - 2\,cm + m^2.$$

The right triangle ACD gives

$$h^2 = b^2 - m^2$$

in both cases. If this value of h^2 be substituted in the equation above, we find

(1) $$a^2 = b^2 + c^2 - 2\ cm \ (A \textit{ being an acute angle}),$$

which proves Theorem 1.

To prove Theorem 2, we refer to Fig. 40. We have, in this case,

$$a^2 = h^2 + \overline{BD}^2 = h^2 + (c + m)^2$$

or

$$a^2 = h^2 + c^2 + 2\ cm + m^2,$$

and

$$h^2 = b^2 - m^2,$$

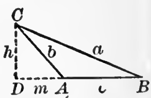

FIG. 40

whence

(2) $$a^2 = b^2 + c^2 + 2\ cm \ (A \textit{ being an obtuse angle}),$$

which proves Theorem 2.

From either Fig. 38 or 39 we obtain, by observing the right triangle ACD,

$$m = AD = b \cos A.$$

In Fig. 40 we have instead,

$$\angle\ CAD = 180° - A, \quad m = b \cos CAD = b \cos(180° - A).$$

If we substitute these values in (1) and (2), we find

(3) $$a^2 = b^2 + c^2 - 2\ bc \cos A \ (\textit{if } A \textit{ is acute}),$$

(4) $$a^2 = b^2 + c^2 + 2\ bc \cos(180° - A) \ (\textit{if } A \textit{ is obtuse}).$$

Just as in Art. 41, we have found two different theorems and two different formulæ for the two cases when A is an acute or an obtuse angle. Can we again find a *single* theorem and a *single* formula to do the work of both?

Equations (3) and (4) show that this may indeed be done, provided that *we define the cosine of an obtuse angle to be a negative number, numerically equal to the cosine of its supplement* (which is of course an acute angle). For, with this definition, we shall have

$$(5)\qquad \cos A = -\cos(180^\circ - A)\ (\textit{if } A \textit{ is an obtuse angle}),$$

so that (4), as a consequence of (5), assumes the same form as (3). But formula (3) holds also when A is a right angle, for in that case $\cos A = 0$, and the formula reduces to the theorem of Pythagoras. Thus, *one and the same formula* (3)

$$a^2 = b^2 + c^2 - 2\,bc\cos A$$

will be applicable to all cases if it be understood, in accordance with our definitions, that $\cos A$ *is positive, zero, or negative according as the angle* A *is acute, right, or obtuse.*

Equation (3) is generally known as the **law of cosines**, and completely replaces Theorems 1 and 2 of this Article. The law of cosines obviously enables us to compute the third side of an oblique triangle when two sides and the included angle are given. But it also enables us to find the angles of a triangle when its three sides are given. For, we find from (3), by transposition,

$$2\,bc\cos A = b^2 + c^2 - a^2,$$

and therefore

$$(6)\qquad \cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

The two geometric theorems (Theorems 1 and 2 of this article) to which the law of cosines is equivalent, were well known to the Ancients; and the problem of finding the angles of a triangle, when its three sides are given, was solved by PTOLEMY (2d century A.D.) of Alexandria in his *Almagest* by means of these theorems. The explicit formulation of the law of cosines, however, seems to be due to the great French mathematician, FRANÇOIS VIÈTE (also known as VIETA), (1540–1603).

EXERCISE XIX

Solve the following triangles, using the tables of squares and natural functions:

1. $a = 2, \quad b = 3, \quad C = 30°.$

2. $b = 3.5, \; c = 2.4, \; A = 52°.$

3. $c = 2.34, \; a = 4.31, \; B = 116°.$

4. $a = 3, \quad b = 6, \quad c = 8.$

5. $a = 1.0, \quad b = 2.0, \quad c = 1.5.$

6. The relation $\cos A = -\cos(180° - A)$ is true for all obtuse angles A as a consequence of the definition of the cosine of an obtuse angle. Prove that this formula is also true if A is any acute angle or a right angle.

7. Show that the relation $\sin^2 A + \cos^2 A = 1$ holds for obtuse as well as for acute angles.

8. If A is an acute angle,

$$\tan A = \frac{\sin A}{\cos A}, \; \cot A = \frac{\cos A}{\sin A}, \; \sec A = \frac{1}{\cos A}, \; \csc A = \frac{1}{\sin A}.$$

Let us *define* $\tan A$, $\cot A$, $\sec A$, $\csc A$ by means of these same equations when A is an obtuse angle. Show that, as a consequence of these definitions, we have

$$\tan A = -\tan(180° - A), \quad \cot A = -\cot(180° - A),$$
$$\sec A = -\sec(180° - A), \quad \csc A = \csc(180° - A),$$

for any obtuse angle A.

9. Show that the equations of Ex. 8 are valid also if the angle A is acute.

10. The law of cosines gives three equations for any triangle. Equation (3) of Art. 42 is one of these. Write the other two.

11. Write the equations for $\cos A$, $\cos B$, and $\cos C$ in terms of the three sides of the triangle.

12. Show that in any triangle,

$$a^2 + b^2 + c^2 - 2\,bc\cos A - 2\,ca\cos B - 2\,ab\cos C = 0.$$

43. Properties of the functions of an obtuse angle.* We have defined in Art. 40 the sine, in Art. 42, the cosine, and in Ex. 8, Exercise XIX, the remaining functions of an obtuse angle. As a consequence of these definitions we have the following system of equations :

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* Some instructors may prefer to change the order of topics by passing to the discussion of the general angle, Art. 60 *et seq.*, and returning later to Art. 44. There is no reason why this should not be done.

$$(1)\quad \begin{cases} \sin(180° - A) = \sin A, & \cos(180° - A) = -\cos A, \\ \tan(180° - A) = -\tan A, & \cot(180° - A) = -\cot A, \\ \sec(180° - A) = -\sec A, & \csc(180° - A) = \csc A, \end{cases}$$

which are valid whether the angle A be acute, right, or obtuse. (Cf. Exercise XVIII, Ex. 4, Exercise XIX, Exs. 6, 8, and 9.)

But an obtuse angle B may be written either in the form

$$B = 180° - A, \text{ or } B = 90° + A',$$

both A and A' being acute angles. Moreover, from

$$B = 180° - A = 90° + A'$$

follows

$$A' = 90° - A, \text{ or } A = 90° - A',$$

that is, the angles A and A' are complementary. Let us substitute $A = 90° - A'$ in (1). We find that the left member of the first equation of (1) becomes

$$\sin(180° - A) = \sin[180° - (90° - A')] = \sin(90° + A').$$

The right member of the same equation assumes the form

$$\sin A = \sin(90° - A') = \cos A'. \qquad \text{(Art. 10.)}$$

Consequently, this first equation of system (1) becomes

$$\sin(90° + A') = \cos A'.$$

In the same way we may prove the remaining equations of the following system:

$$(2)\quad \begin{cases} \sin(90° + A') = \cos A', & \cos(90° + A') = -\sin A', \\ \tan(90° + A') = -\cot A', & \cot(90° + A') = -\tan A', \\ \sec(90° + A') = -\csc A', & \csc(90° + A') = \sec A'. \end{cases}$$

The student should carry out the details of these substitutions; and note the close resemblance between these formulæ and the formulæ of Art. 10, for $\sin(90° - A)$, $\cos(90° - A)$, etc.

This resemblance is so close as to make it easy to remember equations (2). Their right members differ from the right members of the corresponding equations for $\sin(90° - A)$, $\cos(90° - A)$, etc., at most in sign. This remark suffices to help us remember that $\sin(90° + A)$

is either equal to $+\cos A$ or to $-\cos A$, that $\cos(90^\circ + A)$ is either equal to $+\sin A$ or $-\sin A$, etc. In order to choose between these alternatives we may argue as follows. Let A be an acute angle. Then $\sin A$ and $\cos A$ are both positive. But since $90^\circ + A$ will then be obtuse, $\sin(90^\circ + A)$ is positive and $\cos(90 + A)$ is negative. Therefore, of the two alternatives

$$\sin(90^\circ + A) = \cos A, \text{ or } \sin(90^\circ + A) = -\cos A,$$

we must discard the latter, since its left member is positive and its right member is negative. Similarly, of the two alternatives

$$\cos(90^\circ + A) = \sin A, \text{ or } \cos(90 + A) = -\sin A,$$

we must discard the former. For it involves the contradiction that the negative number $\cos(90^\circ + A)$ should be equal to the positive number $\sin A$. This argument fixes the two formulæ

$$\sin(90^\circ + A) = \cos A, \ \cos(90^\circ + A) = -\sin A$$

in our memory. The remaining formulæ of system (2) may be remembered in similar fashion.

The first two formulæ of system (1), namely,

$$\sin(180^\circ - A) = \sin A, \ \cos(180^\circ - A) = -\cos A,$$

are easy to remember, since it was upon these equations that we based our definitions of the sine and cosine of an obtuse angle. The remaining formulæ of system (1) follow directly from these two.

EXERCISE XX

1. Express the following quantities as functions of acute angles; $\sin 100^\circ$, $\cos 115^\circ$, $\tan 162^\circ$, $\cot 99^\circ$, $\sec 120^\circ$, $\csc 175^\circ$.

2. By means of the table of natural functions, find $\sin 98^\circ.5$, $\cos 176^\circ.3$, $\tan 124^\circ.7$, $\cot 134^\circ.6$.

3. Explain how equations (1) and (2) of Art. 43 provide two different methods of finding the functions of obtuse angles from the table for acute angles.

44. Other formulæ for the area of an oblique triangle. The methods of Art. 40 suffice to determine the area of a triangle if one side and the corresponding altitude (c and h), or if two sides and the included angle (b, c, and A) are given.

Let us suppose now that one side and two adjacent angles

(c, A, B) are given. We have in the first place from Art. 40

(1) $$S = \frac{1}{2} bc \sin A,$$

and it only remains to modify this formula by expressing b in terms of c, A, and B. The law of sines, in the form

(2) $$\frac{b}{c} = \frac{\sin B}{\sin C},$$

and the equation

(3) $$A + B + C = 180°$$

enable us to do this. For we find from (2) and (3)

$$b = \frac{c \sin B}{\sin C} = \frac{c \sin B}{\sin [180° - (A+B)]} = \frac{c \sin B}{\sin (A + B)}$$

since, according to Art. 43, the sine of any acute or obtuse angle is equal to the sine of its supplement.

If we substitute this value of b in (1), we find

(4) $$S = \frac{c^2 \sin A \sin B}{2 \sin (A + B)},$$

the desired formula for S in terms of c, A, and B.

If one side c and two angles, not both adjacent to c, are given, we may first find the third angle from (3) and then use formula (4).

Let us, finally, obtain a formula for S in terms of the three sides a, b, c. We start again from equation (1) which already contains the sides b and c of the triangle, and which will give the desired formula if we can express $\sin A$ in terms of a, b, and c. This may be done by making use of the law of cosines (which gives $\cos A$ in terms of a, b, and c), and the relation

(5) $$\sin^2 A + \cos^2 A = 1,$$

which holds for obtuse as well as for acute angles. (See Ex. 7, Exercise XIX.)

To avoid the inconvenience of writing cumbersome square root signs, let us square both members of (1). We find, making use of (5),

$$S^2 = \tfrac{1}{4} b^2c^2 \sin^2 A = \tfrac{1}{4} b^2c^2 (1 - \cos^2 A)$$

or, factoring the binomial on the right,

$$S^2 = \tfrac{1}{4} b^2c^2 (1 + \cos A)(1 - \cos A).$$

By the law of cosines (Equation (6), Art. 42), this becomes

$$\begin{aligned} S^2 &= \tfrac{1}{4} b^2c^2 \left(1 + \frac{b^2 + c^2 - a^2}{2\,bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2\,bc}\right) \\ &= \frac{b^2c^2}{4} \cdot \frac{2\,bc + b^2 + c^2 - a^2}{2\,bc} \cdot \frac{2\,bc - b^2 - c^2 + a^2}{2\,bc} \\ &= \tfrac{1}{16} [(b + c)^2 - a^2][a^2 - (b - c)^2]. \end{aligned}$$

Each of the factors on the right member may be factored, giving the equation

(6) $S^2 = \tfrac{1}{16} (b + c + a)(b + c - a)(a + b - c)(a - b + c).$

The first factor on the right member of (6) is the perimeter of the triangle, and the formula becomes especially simple if we denote half of this perimeter by s, so that

(7) $$a + b + c = 2\,s.$$

The other three factors of the right member of (6) will then become

(8) $$\begin{aligned} -a + b + c &= 2\,s - 2\,a = 2\,(s - a), \\ a - b + c &= 2\,s - 2\,b = 2\,(s - b), \\ a + b - c &= 2\,s - 2\,c = 2\,(s - c), \end{aligned}$$

so that (6) reduces to

$$S^2 = s\,(s - a)(s - b)(s - c),$$

whence finally, since S is positive,

(9) $$\boldsymbol{S = \sqrt{s\,(s - a)(s - b)(s - c)}},$$

a famous equation generally known as **Hero's formula**, after Hero of Alexandria, who lived about 120 B.C., and wrote a famous textbook on surveying.

EXERCISE XXI

Find the area of the following triangles to four significant figures.

1. $a = 2, \quad b = 3, \quad C = 30°.$

2. $b = 3.5, c = 2.4, A = 52°.$

3. $c = 5, \ A = 30°, B = 75°.$

4. $a = 3, \ A = 30°, B = 75°.$

5. $a = 3, \quad b = 6, \quad c = 8.$

6. $a = 1.0, b = 2.0, \ c = 1.5.$

45. The radius and center of the inscribed circle. The fundamental basis for all of the different formulæ which we have obtained for the area of a triangle, so far, was the equation

$$S = \tfrac{1}{2}\, ch,$$

from which all of the others were derived.

But this formula is unsymmetrical, since it singles out one of the sides of the triangle and subjects it to a treatment different from that accorded to the other two. We may avoid this lack of symmetry by picking out a point *M* anywhere inside of the triangle *ABC* (see Fig. 41), and joining *M* to the three vertices. The area of the given triangle will then appear as the sum of the areas of the three triangles *BMC*, *CMA*, *AMB*, whose altitudes are respectively equal to *MD*, *ME*, and *MF*.

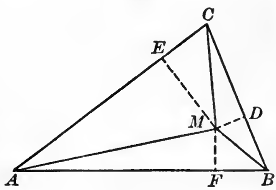

FIG. 41

Clearly, there is one position of the point *M* which is better adapted for this purpose than any other, namely the center *O* of the inscribed circle. For the distances from *O* to the three sides of the triangle are equal to each other, so that the three triangles *BOC*, *COA*, and *AOB* will have equal altitudes.

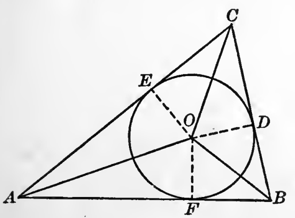

FIG. 42

Let *O* (Fig. 42) be the center of the inscribed circle and let *r* be the radius of this circle.

Then the area of the triangle is

$$\begin{aligned} S &= BOC + COA + AOB \\ &= \tfrac{1}{2}\, ar + \tfrac{1}{2}\, br + \tfrac{1}{2}\, cr \\ &= \tfrac{1}{2}\,(a + b + c)\, r, \end{aligned}$$

or finally

(1) $$S = sr,$$

where s denotes the half-perimeter of the triangle as in Art. 44.

Since we also have

$$S = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{(Equation (9), Art. 44)},$$

we find from (1), substituting this value of S and dividing both members of the resulting equation by s,

(2) $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

which enables us to compute the radius of the inscribed circle when the sides of the triangle are given.

If we wish to know, not merely the *radius* of the inscribed circle, but also the position of its center, we must compute the lengths of the line segments AF, BF, etc. in Fig. 42. Now we know from Geometry that

(3) $$AF = AE, \ BD = BF, \ CE = CD.$$

Moreover, the sum of these six line-segments is equal to the perimeter $2s$ of the triangle. Therefore the sum of three of these segments, one chosen from each of the three equations (3), will be equal to half of the perimeter. That is,

$$AF + BD + CD = s.$$

But

$$BD + CD = a,$$

and therefore

(4) $$AF = AE = s - a.$$

In the same way we find

(5) $$\begin{cases} BD = BF = s - b, \\ CE = CD = s - c. \end{cases}$$

The three equations, (4) and (5), enable us to locate the points D, E, F in which the inscribed circle touches the sides of the triangle and consequently determine completely the position of the center O of the circle.

46. The half-angle formulæ. Since the center O of the inscribed circle is the point of intersection of the three angle bisectors of the triangle (see Fig. 42), the angle FAO is equal to $\frac{1}{2}A$. In the right triangle FAO, we have therefore

$$\tan \tfrac{1}{2} A = \frac{FO}{AF}.$$

But FO is the radius r of the inscribed circle, and AF we have just found to be equal to $s - a$. (Equation (4), Art. 45.) We find therefore

(1) $$\tan \tfrac{1}{2} A = \frac{r}{s-a},$$

and in the same way

(2) $$\tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c},$$

the value of r being given by equation (2) of Art. 45.

These three equations, usually known as **the half-angle formulæ**, are very important in providing a second method for computing the angles of a triangle when its sides are given. They are far more convenient for this purpose than the law of cosines, if logarithms are to be used. In fact, the law of cosines is so cumbersome from the point of view of logarithmic calculation, that we shall seek to find a substitute for it also in the only other case in which we have proposed to make any use of it, namely in the solution of a triangle when two sides and the included angle are given. (See Art. 42, p. 96.)

We should not neglect to note, however, that the law of cosines has in recent times again come into practical use, especially in engineering practice, the calculations being performed not with logarithms, but with the help of tables of squares and products or with a calculating machine. The Eichhorn Trigonometric slide rule, mentioned on page 66, is based entirely upon the law of cosines.

47. The circumscribed circle. The center O of the circumscribed circle is the point of intersection of the three perpendicular bisectors of the sides of the triangle. Therefore, if N is the middle point of the side AB of the triangle (see Fig. 43), the line ON will be perpendicular to AB.

Now angle AOB is measured by the arc AB, and the angle ACB is measured by half this arc (being an inscribed angle). Therefore

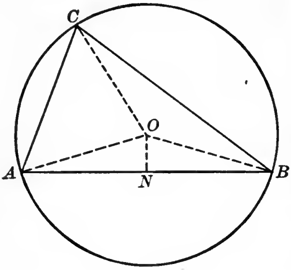

FIG. 43

$$\angle AON = \angle ACB = C$$

since each of these angles is equal to half of $\angle AOB$. Consequently we find, making use of the right triangle AON,

$$\sin C = \sin AON = \frac{AN}{AO}.$$

If we denote AO, the radius of the circumscribed circle, by R, this becomes

$$\sin C = \frac{\frac{1}{2}c}{R} = \frac{c}{2R}, \text{ or } 2R = \frac{c}{\sin C}$$

whence, making use of the law of sines (Equation (1), Art. 41),

(1) $$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

thus completing the law of sines by providing a simple geometrical interpretation for the common value of the three equal ratios

$$\frac{a}{\sin A}, \frac{b}{\sin B}, \frac{c}{\sin C};$$

namely, the diameter of the circumscribed circle.

Since the area of the triangle is

$$S = \tfrac{1}{2} bc \sin A,$$

and since we find, from (1),

$$\sin A = \frac{a}{2R},$$

we obtain the following remarkable formula

(2) $$S = \frac{abc}{4R}$$

for the area of the triangle.

EXERCISE XXII

1. Compute the radius of the inscribed circle and the lengths of the line-segments into which this circle divides the sides of the triangle whose sides are $a = 3$, $b = 6$, $c = 8$.

2. What relation will there be between the sides of a triangle if the inscribed circle bisects one of its sides? If it touches one of the sides at a trisection point?

3. Find formulæ for the distances from the center of the inscribed circle to the three vertices of the triangle.

4. Making use of the results of Ex. 3, show that

$$\sin \tfrac{1}{2} A = \sqrt{(s-b)(s-c)/bc} \text{ and } \cos \tfrac{1}{2} A = \sqrt{s(s-a)/bc}.$$

5. Find a formula for the area of the inscribed circle, and for the ratio of this area to that of the triangle itself.

6. By means of the formulæ of Ex. 5, show that

$$\pi \sqrt{\frac{(s-a)\ (s-b)\ (s-c)}{s^3}} < 1.$$

7. Show that

$$r = s \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C.$$

8. A circle is said to be *escribed to a triangle*, or inscribed externally, if it touches one of the sides externally and the prolongations of the other two sides (Fig. 44). There are three such circles for every triangle. Let r_a be the radius of that one which touches the side a externally. Show that

$$AM = AN = s$$

and

$$r_a = s \tan \tfrac{1}{2} A = \frac{S}{s-a} = \frac{rs}{s-a}.$$

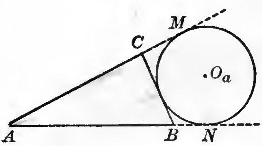

FIG. 44

9. If r_a, r_b, r_c, are the radii of the three escribed circles, and r denotes the radius of the inscribed circle, show that

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

10. Show that

$$r_a + r_b + r_c - r = 4R,$$

if R is the radius of the circumscribed circle.

48. The form ratios of a triangle. If two sides of a triangle are equal, the triangle is said to be *isosceles;* if no two sides are equal, it is usually called a *scalene* triangle. But if the difference between two sides of a triangle is small, *as compared with the combined length of these two sides*, the triangle will differ but little from the isosceles form. Consequently, the ratio of the difference between two sides of a triangle to their combined length may be taken as a numerical measure for the *departure of the triangle from the isosceles* form, or as its **form ratio** with respect to those two sides.

There are three such form ratios for every triangle. If we assume that the notation be so chosen that

$$a \geqq b \geqq c,$$

these three form ratios are

$$\frac{a-b}{a+b}, \quad \frac{a-c}{a+c}, \quad \frac{b-c}{b+c}.$$

Clearly, one of them will be equal to zero if the triangle is isosceles, and all three will vanish for an equilateral triangle.

By means of the law of sines, each of these form ratios may be expressed in terms of the angles of the triangle. In fact, according to Art. 47, equation (1), we have

$$a = 2R\sin A, \quad b = 2R\sin B, \quad c = 2R\sin C,$$

so that we obtain the expression

$$(1) \qquad \frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

for the form ratio with respect to the sides a and b. Similar expressions hold, of course, for the other form ratios; $\frac{a-c}{a+c}$ and $\frac{b-c}{b+c}$.

49. The formulæ for the sum and difference of two sines. The fraction which occurs in the right-hand member of equation (1), Art. 48, is not adapted to logarithmic calculation. But we shall show that both numerator and denominator of this fraction, namely, the expressions $\sin A - \sin B$, and $\sin A + \sin B$, may be written in the form of products, thus making it easy to compute their values by logarithms.

To discover the product form of these expressions, we must construct a figure in which the angles A and B, the sines of these angles, and the sum and difference of their sines, shall appear.

We construct first (Fig. 45) the angles

$$\angle XOP = A \text{ and } \angle XOQ = B.$$

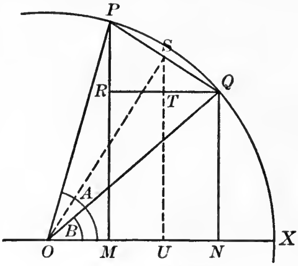

FIG. 45

With their common vertex O as center, and any convenient radius r, we draw a circle whose intersections with the sides of the angles we denote by X, P, and Q, respectively.

Let PM and QN be perpendicular, and QR parallel to OX. Then

(1) $$r \sin A = MP, \text{ and } r \sin B = NQ.$$

But MP and NQ are the two bases of the trapezoid $MPQN$. If S is the middle point of the chord PQ, and US be drawn perpendicular to OX, US is the median of this trapezoid, and we shall have,

$$US = \tfrac{1}{2}(MP + NQ),$$

whence, according to (1),

(2) $$r(\sin A + \sin B) = MP + NQ = 2\, US.$$

On the other hand,

$$TS = \tfrac{1}{2} PR = \tfrac{1}{2}(MP - NQ),$$

and hence,

(3) $$r(\sin A - \sin B) = MP - NQ = 2\, TS.$$

In order to express the right members of (2) and (3) in terms of r, A and B, we consider the right triangles OSU and SQT in which US and TS occur. It is apparent from the figure that

(4) $$\begin{cases} \angle SOQ = \tfrac{1}{2}\, QOP = \tfrac{1}{2}(A - B), \\ \angle UOS = B + \tfrac{1}{2}(A - B) = \tfrac{1}{2}(A + B), \end{cases}$$

and

(5) $$\angle TSQ = \angle UOS = \tfrac{1}{2}(A + B),$$

since each of these angles is complementary to the angle USO.

Hence we have from the right triangles OSU and SQT,

(6) $$\begin{aligned} US &= OS \sin UOS = OS \sin \tfrac{1}{2}(A + B), \\ TS &= SQ \cos TSQ = SQ \cos \tfrac{1}{2}(A + B), \end{aligned}$$

and from the triangle OQS,

(7) $$\begin{aligned} OS &= OQ \cos SOQ = r \cos \tfrac{1}{2}(A - B), \\ SQ &= OQ \sin SOQ = r \sin \tfrac{1}{2}(A - B). \end{aligned}$$

Substituting (6) and (7) in (2) and (3) and dividing by r, we find

(8) $$\begin{aligned} \sin A + \sin B &= 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B), \\ \sin A - \sin B &= 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B), \end{aligned}$$

which are the desired formulæ.

The student should observe that our proof of equations (8), based on Fig. 45, remains valid even if A is an obtuse angle, as long as MP is greater than or equal to NQ; that is, as long as $A + B$ does not exceed 180°. We may therefore apply equations (8) whenever A and B are two angles of the same triangle, since the sum of two angles of a triangle can never exceed 180°.

50. The law of tangents. Let us now return to the expression (1) of Art. 48 for the form ratio of the triangle with respect to the sides a and b, of which sides we shall assume a to be the greater. We had found

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

If now we make use of equations (8) of Art 49, we find

$$\frac{a - b}{a + b} = \frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)},$$

whence

$$\frac{a-b}{a+b} = \cot \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B),$$

since we have, for any angle M,

$$\frac{\sin M}{\cos M} = \tan M, \quad \frac{\cos M}{\sin M} = \cot M.$$

But the tangent and cotangent of the same angle are reciprocals, so that we may write finally

$$(1) \qquad \frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)},$$

a very important formula, generally known as **the law of tangents.** We find in the same way:

$$(2) \qquad \begin{cases} \dfrac{a-c}{a+c} = \dfrac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)}, \\ \dfrac{b-c}{b+c} = \dfrac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}. \end{cases}$$

The law of tangents seems to have been expressed in this form for the first time by VIETA, to whom we also owe the modern form of the law of cosines. It had however been stated, in a more complicated but equivalent form, about ten years earlier by the Dutch mathematician THOMAS FINK or FINCHIUS (1561–1656) in his *Geometria rotundi.* It was also Fink who first introduced the names tangent and secant for the functions which we now call by these names. Many previous authors had used the name *umbra* = shadow for the function which we now call tangent, on account of the relation of this function to the shadow cast by a vertical stick. (Compare Art. 3 and solve the problem there attributed to Thales by trigonometry.)

Fink not only discovered the law of tangents, but pointed out its principal application; namely, to aid in solving a triangle when two sides and the included angle are given. The possibility of such an application will appear from the following

Illustrative Example. Given a, b, and C. To find A, B, and C.

Solution. The law of tangents (Equation (1)) gives

$$(3) \qquad \tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A+B),$$

an equation in which the right member is completely known, since a, b, and C are given, and since

(4) $$\tfrac{1}{2}(A+B)=\tfrac{1}{2}(180^\circ - C) = 90^\circ - \tfrac{1}{2}C.$$

Thus we can compute $\tan\frac{1}{2}(A-B)$ by means of (3) and then find $\frac{1}{2}(A-B)$ from the table. Hence, knowing $\frac{1}{2}(A-B)$ and $\frac{1}{2}(A+B)$ we can find

(5) $$A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B),$$
$$B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B).$$

The sine law now enables us to find c, since

(6) $$\frac{c}{a}=\frac{\sin C}{\sin A} \text{ gives } c=\frac{a\sin C}{\sin A}.$$

If we wish to solve the same problem by the law of cosines, we first compute c from the equation

(7) $$c^2=a^2+b^2-2ab\cos C$$

and afterward determine the angles A and B by using the law of sines.

The first method has the advantage over the second that formulæ (3), (4), (5), (6) are in a convenient form for logarithmic computation, while equation (7) is not.

51. A second proof of the law of tangents and Mollweide's equations. The law of tangents may also be proved directly from a figure without making use of the formulæ for

$$\sin A+\sin B$$

and

$$\sin A-\sin B.$$

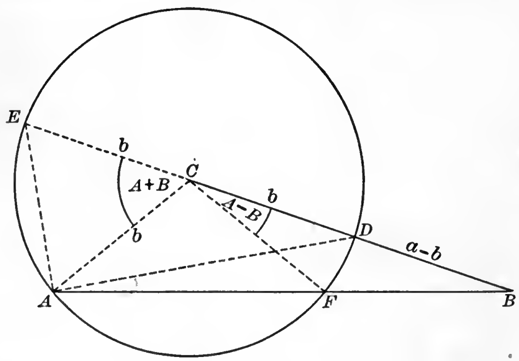

Fig. 46

Let $BC=a$ and $CA=b$ (Fig. 46) be two sides of the triangle ABC, and let $a>b$. Draw a circumference with C as center and radius equal to b, and let D and E be the points in which this circumference meets BC and BC produced. Let F be the point of intersection of the circumference with AB. Then we have

(1) $$BD = a - b, \quad BE = a + b.$$

Since $\angle ECA$ is an exterior angle of the triangle ABC, the opposite interior angles being A and B, we have

$$\angle ECA = A + B,$$

so that

(2) $$\angle EDA = \tfrac{1}{2}(A + B)$$

since the latter angle, being inscribed in the circumference, is measured by half the arc which it subtends.

It remains to find an angle in Fig. 46, equal to $\frac{1}{2}(A - B)$. In order to do this, let us draw CF. Since ACF is an isosceles triangle, we have

$$\angle AFC = \angle FAC = A$$

and, since $\angle AFC$ is an exterior angle of the triangle BFC,

$$A = \angle FCB + B,$$

whence

$$\angle FCB = A - B.$$

Since $\angle DAF$ is an inscribed angle subtending the same arc,

(3) $$\angle DAB = \angle DAF = \tfrac{1}{2}(A - B).$$

Let us apply the law of sines to the two triangles ABD and ABE. We find, from the first triangle,

$$\frac{a-b}{c} = \frac{\sin DAB}{\sin ADB} = \frac{\sin \frac{1}{2}(A-B)}{\sin (180^\circ - EDA)} = \frac{\sin \frac{1}{2}(A-B)}{\sin EDA}$$

or finally

(4) $$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}.$$

Similarly the triangle ABE gives

$$\frac{a+b}{c} = \frac{\sin EAB}{\sin AEB}.$$

But

$$\angle EAB = \angle EAD + \angle DAB = 90^\circ + \tfrac{1}{2}(A - B),$$
$$\angle AEB = 90^\circ - \angle EDA = 90^\circ - \tfrac{1}{2}(A + B),$$

owing to (2) and (3) and the further fact that $\angle EAD$ is a right angle, since it is inscribed in a semi-circumference.

Therefore

$$\frac{a+b}{c}=\frac{\sin\,[90^\circ+\frac{1}{2}(A-B)]}{\sin\,[90^\circ-\frac{1}{2}(A+B)]},$$

which may be written

$$(5)\qquad \frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)},$$

as a consequence of equations (2), Art. 43, and of Art. 10.

If now we divide equations (4) and (5), member by member, we find

$$\frac{a-b}{a+b}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\;\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)};$$

that is, the law of tangents.

Incidentally we have found two new formulæ, (4) and (5). They assume a somewhat more serviceable form by means of the relation $A+B+C=180^\circ$, which gives

$$\tfrac{1}{2}(A+B)=90^\circ-\tfrac{1}{2}C,$$

and therefore

$$\sin\tfrac{1}{2}(A+B)=\cos\tfrac{1}{2}C,\quad \cos\tfrac{1}{2}(A+B)=\sin\tfrac{1}{2}C.$$

If we introduce these values in equations (4) and (5), they become

$$(6)\qquad \frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C},\quad \frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}.$$

These formulæ, known as the **Mollweide equations**, *are particularly convenient for the purpose of checking the accuracy of the numerical solution of a triangle.* For each of these equations contains all of the six parts of the triangle, so that an error in any one of these parts would be likely to make itself felt by a lack of agreement between the two members of one of these equations.

Of course there are two other pairs of equations of the form (6), the left members of which are $\frac{a-c}{b}$, $\frac{a+c}{b}$, and $\frac{b-c}{a}$, $\frac{b+c}{a}$, respectively.

It is not justifiable historically to call equations (6) Mollweide's equations. The formula for $\frac{a+b}{c}$ is to be found in NEWTON'S *Arithmetica Universalis.* Both equations (6) are given in SIMPSON'S *Trigonometry, Plane and Spherical* (1748), and also in F. W. VON OPPEL'S *Analysis Triangulorum* (1746). All of these works antedate considerably the publication of these equations by MOLLWEIDE in 1808.

EXERCISE XXIII

1. Show that the law of tangents may also be obtained from Fig. 46 by drawing through D a line parallel to AE and meeting the side AB in G, and then computing $\tan \frac{1}{2}(A-B)$ and $\tan \frac{1}{2}(A+B)$ from the right triangles ADG and AED.

2. Still another proof of the law of tangents proceeds as follows. In Fig. 47, draw CD, the bisector of angle C, and drop perpendiculars, AL and BM, from A and B to CD. Let N be the point of intersection of AB with CD. Then

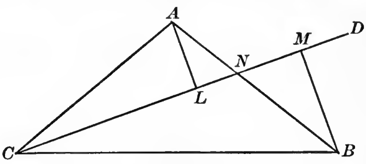

FIG. 47

$$\angle BCD = \angle DCA = \tfrac{1}{2} C$$

and

(1) $$\angle LAN = \angle NBM = 90° - \angle BNM.$$

But $\angle BNM$ is an exterior angle of the triangle BCN, so that

$$\angle BNM = B + \tfrac{1}{2} C.$$

Moreover,

$$90° = \tfrac{1}{2}(A + B + C)$$

and therefore, by substitution in (1),

$$\angle LAN = \tfrac{1}{2}(A + B + C) - (B + \tfrac{1}{2} C) = \tfrac{1}{2}(A - B).$$

Now, from the right triangles ANL and BNM, we see that

$$\tan \tfrac{1}{2}(A-B) = \frac{LN}{LA} = \frac{MN}{MB},$$

and therefore also

(2) $$\tan \tfrac{1}{2}(A-B) = \frac{LN+MN}{LA+MB}.$$

But

$$LN + MN = CM - CL = a \cos \tfrac{1}{2} C - b \cos \tfrac{1}{2} C = (a - b) \cos \tfrac{1}{2} C,$$

$$LA + MB = b \sin \tfrac{1}{2} C + a \sin \tfrac{1}{2} C = (a + b) \sin \tfrac{1}{2} C,$$

which gives, on substitution in (2),

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \cot \tfrac{1}{2} C,$$

and this is equivalent to the law of tangents, since

$$C = 180^\circ - (A + B).$$

CHAPTER VIII

SOLUTION OF OBLIQUE TRIANGLES

52. The fundamental problem. Each of the laws found in Chapter VII contains four of the six parts of a triangle, and thus suggests the possibility of computing any one of these four parts when the other three parts which occur in that law are given.

On the other hand, the relation

(1) $$A + B + C = 180°,$$

which is true for every triangle, contains only three of its parts. The three angles of a triangle are therefore *not independent* of each other, as is the case with the three sides. The familiar fact, that a triangle is not determined by its three angles, is to be regarded as a consequence of this. For, on account of relation (1), the three angles of a triangle are not independent data.

Now *there exists no other equation which, like* (1), *contains no more than three parts of the triangle and which is true for all triangles.* For, if there were such an equation, for instance, between a, b, and c, it would be impossible to find a triangle in which more than two of these quantities have arbitrarily assigned values since two of the three quantities would then determine the third. But this is contrary to well-known facts of geometry; since a triangle may be constructed for which all three sides a, b, and c have arbitrarily assigned values, provided only that $a + b > c$. The values of a and b do *not* determine the value of c; while the values of A and B *do* determine the value of C.

Our illustration shows incidentally that there *may*, and, in fact, *do* exist, besides the *equation* (1), certain *inequalities* between three parts of a triangle which must be satisfied by all triangles. The inequality $a + b > c$ is an instance.

Any three parts of a triangle, unless all three of them are angles, may therefore be regarded as independent data. Consequently there arises the following fundamental problem:

To find the remaining parts of a triangle when any three independent parts are given.

The discussion of this problem leads to a division into four cases.

Case I. One side and two angles are given.

Case II. Two sides and the included angle are given.

Case III. Two sides and the angle opposite to one of them are given.

Case IV. All three sides are given.

53. Case I. Given one side and two angles. Let c, A, B be the given quantities. We may use the formulæ

$$(1)\qquad \begin{cases} C = 180^\circ - (A+B), \\ a = \dfrac{c \sin A}{\sin C}, \quad b = \dfrac{c \sin B}{\sin C}, \end{cases}$$

to compute C, a, and b.

The most reliable and convenient check is furnished by one of Mollweide's equations. (See Equations (6), Art. 51.)

$$(2)\qquad a - b = \frac{c \sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C}.$$

The law of tangents may also serve as a check instead of (2), but it is not quite as convenient. In calculating the check, it is convenient to think of A as larger than B, so that $A - B$ may be positive. If A is less than B, interchange A and B, and also a and b in equation (2).

EXERCISE XXIV

EXAMPLE 1. Given $c = 327.85$, $A = 110^\circ\ 52'.9$, $B = 40^\circ\ 31'.7$. Find C, a, and b. Check the results.

Solution.

$$\text{Given}\begin{cases} c = 327.85 & (1)\\ A = 110^\circ\,52'.9 & (2)\\ B = 40^\circ\,31'.7 & (3)\end{cases}$$

$$A + B = 151^\circ\,24'.6 \quad (4)$$
$$C = 28^\circ\,35'.4 \quad (5)$$

$\log c$	$= 2.51568$	(6)
$\log \sin A$	$= 9.97049-10$	(7) *
$\text{colog} \sin C$	$= 0.32008$	(9) †
$\log a$	$= 2.80625$	(10)
a	$= 640.10$	(12)
$\log c$	$= 2.51568$	(6)
$\log \sin B$	$= 9.81280-10$	(8)
$\text{colog} \sin C$	$= 0.32008$	(9)
$\log b$	$= 2.64856$	(11)
b	$= 445.20$	(13)

Check.

$$A - B = 70^\circ\,21'.2 \quad (14)$$
$$\tfrac{1}{2}(A - B) = 35^\circ\,10'.6 \quad (15)$$
$$\tfrac{1}{2}C = 14^\circ\,17'.7 \quad (16)$$

$\log c$	$= 2.51568$	(6)
$\log \sin \frac{1}{2}(A - B)$	$= 9.76050-10$	(17)
$\text{colog} \cos \frac{1}{2} C$	$= 0.01366$	(18) †
$\log (a - b)$	$= 2.28984$	(19)
$a - b$	$= 194.90$	(12)-(13)
$a - b$	$= 194.91$	(From (19))

Remarks. The numbers (1), (2), etc., indicate the order in which the separate results are written down and are meant to assist the student in understanding the arrangement of the computation. These numbers should not appear in the student's own work.

Solve and check the following triangles.

2. $a = 467.00$, $A = 56^\circ\,28'.0$, $B = 69^\circ\,14'$.
3. $a = 24.31$, $A = 45^\circ\,18'$, $B = 22^\circ\,11'$.
4. $a = 148.30$, $A = 37^\circ\,24'.0$, $C = 76^\circ\,48'.5$.
5. $A = 71^\circ\,13'\,30''$, $B = 40^\circ\,34'\,15''$, $c = 236.54$.
6. $a = 3.4356$, $A = 17^\circ\,43'.4$, $C = 60^\circ\,35'.7$.
7. $A = 47^\circ\,13'.2$, $B = 65^\circ\,24'.5$, $a = 43.176$.
8. $B = 100^\circ\,21'\,10''$, $C = 58^\circ\,17'\,20''$, $a = 31.656$.
9. $a = 52.780$, $A = 37^\circ\,41'\,15''$, $B = 77^\circ\,29'\,40''$.
10. $A = 57^\circ\,23'\,12''$, $C = 68^\circ\,15'\,30''$, $c = 832.56$.

54. Case II. Given two sides and the included angle. Let a, b, and C be the given parts. If $a > b$, we use the formula

$$(1) \qquad \tfrac{1}{2}(A + B) = 90^\circ - \tfrac{1}{2}C$$

* Remember that $\sin 110^\circ\,52'.9 = \sin(90^\circ + 20^\circ\,52'.9) = \cos 20^\circ\,52'.9$. (Art. 43.)

† Remember the rule for finding a cologarithm. (Art. 23.)

to find $\frac{1}{2}(A+B)$, and the law of tangents,

$$(2)\qquad \tan\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B)=\frac{a-b}{a+b}\cot\tfrac{1}{2}C,$$

to find $\frac{1}{2}(A-B)$. We then find A and B from

$$(3)\qquad \begin{aligned}A&=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B),\\ B&=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B),\end{aligned}$$

and apply the law of sines to find c, giving

$$(4)\qquad c=\frac{a\sin C}{\sin A}.$$

As checks we use the relations

$$(5)\qquad A+B+C=180^\circ,$$

and one of the Mollweide equations, in the form

$$(6)\qquad \frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}.$$

The law of sines furnishes another relation which might be used as a check instead of (6). But (6) is more reliable.

If $a<b$, we interchange the letters a and b, and also A and B in all of the above formulæ, to avoid the appearance of negative angles.

EXERCISE XXV

Example 1. Given $a = 469.71$, $b = 264.37$, $C = 96°\,57'.6$. Find A, B, and c and check the results.

Solution.

Given $C = 96°\,57'.6$	(1)		$\frac{1}{2}C = 48°\,28'.8$	(6)
Given $a = 469.71$	(2)		$\log\cot\frac{1}{2}C = 9.94711-10$	(8)
Given $b = 264.37$	(3)		$\log(a-b) = 2.31247$	(10)
$a+b = 734.08$	(4)		$\text{colog}(a+b) = 7.13425-10$	(11)
$a-b = 205.34$	(5)		$\log\tan\frac{1}{2}(A-B) = 9.39383-10$	(12)
			$\frac{1}{2}(A+B) = 41°\,31'.2$	(7)
$\log a = 2.67183$	(18)		$\frac{1}{2}(A-B) = 13°\,54'.6$	(13)
$\log\sin C = 9.99679-10$	(19)		$A = 55°\,25'.8$	(15)
$\text{colog}\sin A = 0.08437$	(20)		$B = 27°\,36'.6$	(16)
$\log c = 2.75299$	(21)		$C = 96°\,57'.6$	(1)
$c = 566.23$				

Checks.

$\log(a-b) = 2.31247$	(10)	$\log\sin\frac{1}{2}(A-B) = 9.38091 - 10$	(14)
$\log c = 2.75299$	(21)	$\log\cos\frac{1}{2}C = 9.82144 - 10$	(9)
$\log\dfrac{a-b}{c} = 9.55948 - 10$	(22)	$\log\dfrac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C} = 9.55947 - 10$	(23)
		$A + B + C = 180^\circ\ 0'.0$	(17)

The checks consist in the result (17) and the close agreement of (22) and (23).

Solve and check the following triangles:

2. $b = 472$, $c = 324$, $A = 78^\circ\ 40'$.
3. $a = 748$, $b = 375$, $C = 63^\circ\ 35'.5$.
4. $a = 42.38$, $b = 35.00$, $C = 43^\circ\ 14'\ 40''$.
5. $b = 0.941$, $c = 1.256$, $A = 35^\circ\ 17'\ 28''$.
6. $a = 12.3460$, $b = 5.7213$, $C = 65^\circ\ 30'\ 10''$.
7. $a = 25.384$, $c = 52.925$, $B = 28^\circ\ 32'\ 20''$.
8. $b = 0.14367$, $c = 0.11412$, $A = 42^\circ\ 14'.6$.
9. $a = 138.65$, $b = 226.19$, $C = 59^\circ\ 12'\ 54''$.
10. $b = 1436.7$, $c = 1141.2$, $A = 42^\circ\ 14'\ 35''$.

55. Case III. Given two sides and the angle opposite to one of them.

Geometric discussion. Let A, a, b be the given parts. We construct the angle XAY equal to the given angle A, and lay off AC, on AY, equal to the given side b. With C as center, and a radius equal to the given side a, we strike an arc. If this arc intersects AX in B and B', one or both of the triangles ABC and $AB'C$ *may* be solutions of the problem.

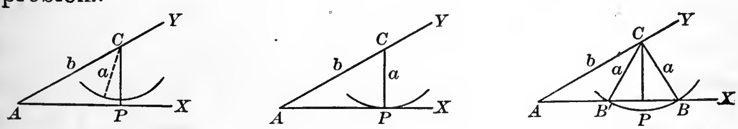

FIG. 48 FIG. 49 FIG. 50

The following cases may present themselves if A is an *acute* angle:

I. $a < CP$, i.e. $a < b \sin A$. (Fig. 48.)

The triangle is impossible in this case, since the arc of radius a, with C as center, will not intersect AX.

II. $a = CP = b \sin A$. (Fig. 49.)

The solution is the right triangle APC.

III. $b > a > b \sin A$. (Fig. 50.)

There are two solutions in this case. The triangles ABC and $AB'C$ *both* satisfy all of the requirements of the problem.

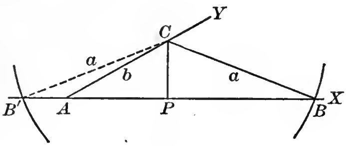

FIG. 51

IV. $a \geqq b$. (Fig. 51.)

If $a > b$, there is one solution only; namely, ABC. The triangle $AB'C$ does not contain the given angle A, but its supplement. If $a = b$, ABC is isosceles, and $AB'C$ reduces to a triangle of zero area. Thus we may say that there is one solution only if $a \geqq b$.

The following additional cases may present themselves if A is an *obtuse* angle.

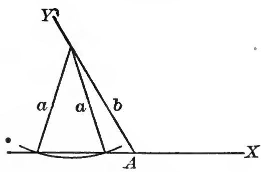

FIG. 52

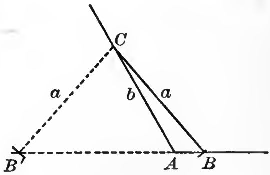

FIG. 53

V. $a \leqq b$. (Fig. 52.)

The triangle is impossible if $a < b$ and of zero area if $a = b$.

VI. $a > b$. (Fig. 53.)

There is one solution; namely, the triangle ABC. The triangle $AB'C$ does not contain the given angle A, but its supplement.

Trigonometric discussion. How do the equations of trigonometry bring into evidence these various cases?

The law of sines gives

(1) $$\sin B = \frac{b \sin A}{a}.$$

If A, a, b are given, we may compute the right member of this equation. Suppose first that A is an *acute* angle.

If the right member of (1) is greater than unity, the triangle is impossible, since the sine of an angle is never greater than unity. This is Case I of the geometric discussion.

If $\frac{b \sin A}{a} = 1$, B must be a right angle. (Case II.)

If the given values of a, b, and the acute angle A, are such that

$$(2) \qquad \frac{b \sin A}{a} < 1,$$

we can find not merely *one* angle to satisfy equation (1) but *two* such angles, which are supplementary to each other, one acute and one obtuse. For if K is the acute angle which satisfies equation (1), the obtuse angle $180° - K$, which has the same sine as K (See Art. 40), will also satisfy equation (1).

Thus we find in the first place, from (1), the two possibilities

$$(3) \quad B = K \text{ (acute angle)}, \quad B = 180° - K \text{ (obtuse angle)}.$$

We must remember, however, that $A + B$ must be less than 180°. If, as we have supposed, A is an acute angle, $A + K$ is certainly less than 180°. But $A + 180° - K$ is less than 180° if and only if $A < K$; that is (see Fig. 50, where $\angle K = \angle ABC$), if and only if $a < b$. Only in this case then will there be two solutions. That is, we have two solutions if A is acute and if $b > a > b \sin A$, in accordance with Case III of the geometric discussion.

If A is acute, but if $A + 180° - K \geqq 180°$; that is, if $A \geqq K$ and therefore $a \geqq b$, the obtuse angle solution $180° - K$ for B becomes inadmissible and the problem has only one solution, in agreement with Case IV of the geometric discussion.

If A is an obtuse angle, the obtuse angle solution $180° - K$ for B is never admissible, since a triangle can contain at most one obtuse angle. The acute angle solution $B = K$ is admissible if and only if $A + K$ is less than 180°. This distinction gives rise to the two remaining Cases, V and VI, of the geometric discussion.

In the trigonometric solution of a numerical problem of this kind, it is essential to remember the following facts:

1. *If, on computing the sine of an angle, we find its value to be greater than unity, the triangle is impossible.*

2. *If the sine of an angle is found to be a positive proper fraction, there are* **two** *possibilities for the corresponding angle. One of these angles is acute and the other, the supplementary angle, is obtuse.*

3. *The sum of any two angles of a triangle must be less than* 180°.

The sine of B having been found from the law of sines, as indicated above, it will become apparent from the corresponding values of B whether the number of solutions is 0, 1, or 2. If there is one solution, we find C from

$$A + B + C = 180°$$

and c from the law of sines. We may check by one of Mollweide's equations or by the law of tangents. If both values of B are admissible, we use each of them in succession, so as to find the remaining parts of the two triangles which are solutions of the problem.

EXERCISE XXVI

Example 1. Given $A = 15° 32'.7$, $a = 103.21$, $b = 152.37$. Find the remaining parts of the triangle or triangles determined by these data.

Solution.

Formulæ: $\sin B = \dfrac{b \sin A}{a}$, $C = 180° - (A + B)$, $c = \dfrac{a \sin C}{\sin A}$.

Check: $$b - a = \frac{c \sin \frac{1}{2}(B - A)}{\cos \frac{1}{2} C}.$$

Given $\begin{cases} A = 15° 32'.7 & (1) \\ a = 103.21 & (2) \\ b = 152.37 & (3) \end{cases}$ Results $\begin{cases} B = 23° 18'.4 & B' = 156° 41.6 \\ C = 141° 8'.9 & C' = 7° 45.7 \\ c = 241.58 & c' = 52.01 \end{cases}$

Computation.

$\log b = 2.18290$	(4)	$\log a = 2.01372$	(5)
$\log \sin A = 9.42813$	(7)	$\log \sin C = 9.79748{-}10$	(15)
$\text{colog}\, a = 7.98628{-}10$	(6)	$\text{colog} \sin A = 0.57187$	(17)*
$\log \sin B = 9.59731{-}10$	(8)	$\log c = 2.38307$	(18)
$B = 23° 18'.4$	(9)	$c = 241.58$	(20)
$B' = 156° 41'.6$	(10)	$\log a = 2.01372$	(5)
$B + A = 38° 51'.1$	(11)	$\log \sin C' = 9.13050{-}10$	(16)
$B' + A = 172° 14'.3$	(12)	$\text{colog} \sin A = 0.57187$	(17)*
$C = 141° 8'.9$	(13)	$\log c' = 1.71609$	(19)
$C' = 7° 45'.7$	(14)	$c' = 52.01$	(21)

Check.

$$B - A = 7° 45'.7 \quad (22)$$
$$B' - A = 141° 8'.9 \quad (23)$$
$$\tfrac{1}{2}(B - A) = 3° 52'.9 \quad (24)$$
$$\tfrac{1}{2}(B' - A) = 70° 34'.5 \quad (25)$$

$\log c = 2.38307$	(18)	$\log c' = 1.71609$	(19)
$\log \sin \tfrac{1}{2}(B - A) = 8.83056{-}10$	(26)	$\log \sin \tfrac{1}{2}(B' - A) = 9.97455{-}10$	(27)
$\text{colog} \cos \tfrac{1}{2} C = 0.47811$	(28)	$\text{colog} \cos \tfrac{1}{2} C' = 0.00100$	(29)
$\log (b - a) = 1.69174$	(30)	$\log (b - a) = 1.69164$	(31)
$b - a = 49.17$	(32)†	$b - a = 49.16$	(33)‡
$b - a = 49.16$	(34)‡		

Example 2. Given $A = 15° 32'.7$, $a = 10.321$, $b = 152.37$. Find the remaining parts of the triangle or triangles determined by these data.

Solution.

Given $\begin{cases} A = 15° 32'.7 \\ a = 10.321 \\ b = 152.37 \end{cases}$

$$\begin{aligned} \log b &= 2.18290 \\ \log \sin A &= 9.42813{-}10 \\ \text{colog}\, a &= 8.98628{-}10 \\ \hline \log \sin B &= 0.59731 \end{aligned}$$

Since $\log \sin B$ has the characteristic zero, $\sin B$ is greater than unity. Therefore, the triangle is impossible.

Example 3. Given $A = 15° 32'.7$, $a = 167.38$, $b = 152.37$. Find the remaining parts of the triangle or triangles determined by these data.

Solution.

Given $\begin{cases} A = 15° 32'.7 \\ a = 167.38 \\ b = 152.37 \end{cases}$ Results $\begin{cases} B = 14° 7'.2 \\ C = 150° 20'.1 \\ c = 309.11 \end{cases}$

* Obtained from (7). † Obtained from the logarithm above.

‡ Obtained by subtraction from (2) and (3).

Computation.

$$\begin{aligned} \log b &= 2.18290 \\ \log\sin A &= 9.42813\text{–}10 \\ \text{colog}\, a &= 7.77629\text{–}10 \\ \hline \log\sin B &= 9.38732\text{–}10 \\ B &= 14^\circ\ 7'.2 \\ B' &= 165^\circ\ 52'.8 \\ B + A &= 29^\circ\ 39'.9 \\ B' + A &= 181^\circ\ 25'.5^* \\ C &= 150^\circ\ 20'.1 \end{aligned} \qquad \begin{aligned} \log a &= 2.22371 \\ \log\sin C &= 9.69454\text{–}10 \\ \text{colog}\sin A &= 0.57187 \\ \hline \log c &= 2.49012 \\ c &= 309.11 \end{aligned}$$

Check.

$$\begin{aligned} A - B &= 1^\circ\ 25'.5 \\ \tfrac{1}{2}(A - B) &= 0^\circ\ 42'.8 \\ \tfrac{1}{2} C &= 75^\circ\ 10'.1 \\ a - b &= 15.01 \end{aligned} \qquad \begin{aligned} \log c &= 2.49012 \\ \log\sin\tfrac{1}{2}(A - B) &= 8.09516\text{–}10 \\ \text{colog}\cos\tfrac{1}{2} C &= 0.59180 \\ \hline \log(a - b) &= 1.17708 \\ a - b &= 15.03 \end{aligned}$$

Find out whether the triangles corresponding to the following data are possible, how many solutions there are, and what are the values of the missing parts.

4. $a = 98,\quad b = 100,\quad A = 120^\circ.$

5. $a = 767,\quad b = 242,\quad A = 36^\circ\ 53'\ 2''.$

6. $a = 3541,\quad b = 4017,\quad A = 61^\circ\ 27'.$

7. $a = 67.53,\quad b = 56.82,\quad A = 77^\circ\ 14'\ 19''.$

8. $a = 9.4672,\quad c = 14.433,\quad A = 11^\circ\ 14'.3.$

9. $a = 413.28,\quad b = 378.19,\quad B = 50^\circ\ 16'\ 25''.$

10. $a = 345.46,\quad b = 531.75,\quad A = 26^\circ\ 47'\ 32''.$

56. Case IV. Given the three sides of the triangle. We use the formulæ

$$s = \tfrac{1}{2}(a + b + c),\quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\tan\tfrac{1}{2} A = \frac{r}{s-a},\quad \tan\tfrac{1}{2} B = \frac{r}{s-b},\quad \tan\tfrac{1}{2} C = \frac{r}{s-c},$$

and the check

$$A + B + C = 180^\circ.$$

* Therefore B' is inadmissible. There is only one solution, as might have been foreseen, since $a > b$.

EXERCISE XXVII

EXAMPLE 1. Given $a = 34.278$, $b = 25.691$, $c = 30.175$. Find the angles A, B, C.

Solution.

Given	$a = 34.278$	Results	$A = 75° \; 13'.0$
	$b = 25.691$		$B = 46° \; 26'.6$
	$c = 30.175$		$C = 58° \; 20'.2$
	$2s = 90.144$	*Check.*	$A + B + C = 179° \; 59'.8$
	$s = 45.072$		$\text{colog}\, s = 8.34609 - 10$
	$s - a = 10.794$		$\log (s - a) = 1.03318$
	$s - b = 19.381$		$\log (s - b) = 1.28737$
	$s - c = 14.897$		$\log (s - c) = 1.17310$
	$2s = 90.144$ *		$\log r^2 = 1.83974$ †

$\log r = 0.91987$	$\log r = 0.91987$	$\log r = 0.91987$ ‡
$\log (s - a) = 1.03318$	$\log (s - b) = 1.28737$	$\log (s - c) = 1.17310$
$\log \tan \frac{1}{2} A = 9.88669 - 10$	$\log \tan \frac{1}{2} B = 9.63250 - 10$	$\log \tan \frac{1}{2} C = 9.74677 - 10$
$\frac{1}{2} A = 37° \; 36'.5$	$\frac{1}{2} B = 23° \; 13'.3$	$\frac{1}{2} C = 29° \; 10'.1$

Remark. In this problem some time may be saved by writing $\log r$ on the lower margin of a slip of paper and placing it above $\log (s - a)$ to find $\log \tan \frac{1}{2} A$, above $\log (s - b)$ to find $\log \tan \frac{1}{2} B$, and above $\log (s - c)$ to find $\log \tan \frac{1}{2} C$. A similar device is often useful in similar cases. Most computers also save time by omitting the -10 attached to logarithms with negative characteristics. This omission can never give rise to serious misunderstanding.

Find the angles of the triangles whose sides have the following values:

2. $a = 79.3$, $b = 94.2$, $c = 66.9$.

3. $a = 0.785$, $b = 0.850$, $c = 0.633$.

4. $a = 312$, $b = 423$, $c = 342$.

5. $a = 25.17$, $b = 34.06$, $c = 22.17$.

6. $a = 93146$, $b = 176530$, $c = 95768$.

7. $a = 12.653$, $b = 17.213$, $c = 23.106$.

If only *one* of the three angles is to be calculated, it may be more convenient to make use of the formulæ for $\sin \frac{1}{2} A$ or $\cos \frac{1}{2} A$, which may be found from the indications given in Example 4, Exercise XXII.

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* Obtained by adding s, $s - a$, $s - b$, $s - c$, as a check on the additions and subtractions required to find these quantities.

† Obtained by adding the four logarithms above $\log r^2$, since

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

‡ Obtained by taking one half of $\log r^2$.

57. Problems in heights and distances, plane surveying, and plane sailing. Some of the following problems are direct applications of the methods which have just been explained. In others, it will be necessary to consider several triangles in succession or simultaneously.

It will always be advisable to draw a figure, approximately to scale, to denote the known as well as the unknown sides and angles of the figure by properly chosen letters, and to write down the formulæ to be used in their general form, leaving the substitution of the numerical values to the last. Much blundering and much unnecessary work may be avoided by adopting this plan.

The student should use his judgment in regard to the number of decimal places used in the numerical part of the work. Use three-place tables whenever possible. Many of the problems may be solved, wholly or in part, by the slide-rule.

EXERCISE XXVIII

1. In order to find the height of a tower, the angles of elevation of its top are measured from two stations, A and B, in the same horizontal line with its base, and on the same side of the tower. If the angles of elevation of the tower from A and B are 32° and 65° respectively, and if the distance AB is 500 feet, find the height of the tower.

2. Find the distances from the two stations of Ex. 1 to the foot of the tower.

3. If the angles of elevation of the tower from A and B, in Ex. 1, are L and M respectively, and if the distance AB is equal to d feet, show that the height of the tower is

$$h = \frac{d \sin L \sin M}{\sin (M - L)}.$$

4. An obstacle (a house) was found to interfere with the running of a straight line from A in the direction AB. (See Fig. 54.) An angle ABE was turned at B, equal to 123°, and the distance BE was measured equal to 150 feet. The angle BEC was made equal to 63°. How long must the distance EC be, and what angle must be turned at C, in order that CD may be the prolongation of AB?

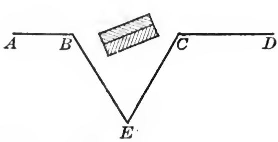

FIG. 54

5. How may angles at B and E be chosen, in Ex. 4, so as to avoid all computation?

6. Two straight railroad tracks intersect at an angle of 75°. What will be the distance, at the end of 20 minutes, between two trains which start from the crossing at the same instant, their speeds being 30 and 40 miles per hour respectively?

7. Each of two battleships passing each other fired a salute. A person on shore observed that the interval between the flash and the report of the gun was 4 seconds for one ship and 6 seconds for the other. The angle, at his eye, subtended by the two ships, just as the salute was fired, was 55°. The velocity of sound is about 1140 feet per second. Find the distance between the ships.

8. Two lighthouses are 2.789 miles apart, and a certain rock is known to be 4.325 miles from one of them. The angle subtended by the two lighthouses at the rock is 16° 13′. How far is the rock from the other lighthouse? How many solutions are there to this problem? Can we make a choice between these solutions if we know which of the two lighthouses is nearer to the rock?

9. Find the radius of the largest cylindrical gas tank which can be constructed on a triangular lot whose sides measure 73, 82, 91 feet respectively, and locate the center of its circular base.

10. An observer measures the angle of elevation of a cloud due south of him at the moment when the sun also is due south (at apparent noon). The angle of elevation of the sun was 65°, that of the cloud 75°. If the shadow of the cloud falls 550 feet north of the observer, how high is the cloud?

11. At 9 P.M. two lights, known to be 8 miles apart, are observed to be due east from a certain vessel. At 10 P.M. one of these lights bears N.E. and the other N.N.E. If the course of the ship was due south, what was its rate?

12. A tower is situated on top of a conical hill whose sides make an angle of 15° with the horizontal plane. At a distance of 120 feet from the foot of the tower (the distance being measured along the slope) the tower subtends an angle of 20°. Find the height of the tower.

13. If, in Ex. 12, the side of the hill makes an angle I with the horizontal plane, and if the angle subtended by the tower, at a distance of d feet from its foot, is A, show that the height of the tower is

$$h = \frac{d \sin A}{\cos (A + I)}.$$

14. A tower 54 feet high, situated on top of a conical hill, subtends an angle of $15° 30'$ at a point 120 feet from the foot of the tower (the distance being measured along the slope). What angle does the side of the hill make with the horizontal plane?

15. To find the slope of a railroad embankment, one end of a pole 12 feet long was placed on the level ground 6 feet from the foot of the embankment, and the other end was found to fall at a point 7.5 feet up its face. What angle does the embankment make with a horizontal plane?

Remark. A transit cannot conveniently be used to measure an angle formed by two walls, the angle formed by an embankment or buttress with a horizontal plane, etc. In such cases, as in this example, it is more convenient to measure distances and determine the angles by calculation.

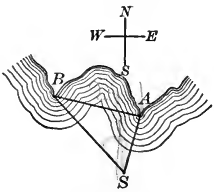

FIG. 55

16. Two capes, A and B (Fig. 55), were observed from a ship at sea; one of them bore N.N.E. and the other N.W. It was found from the chart that the second cape bore W. by N. from the first and was 25.3 miles distant from it. What was the distance of the ship from each of the two capes?

17. A battleship leaves port A, on a due easterly course, at the rate of 16 miles per hour. A dispatch boat starts from B at the same moment. The port B bears S.S.W. of port A and is 25 miles distant from it. If the dispatch boat has a rate of 22 miles per hour, what should be the direction of its course so that it may meet the battleship, if neither ship alters its rate or course? At what time will they meet?

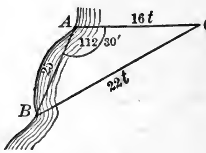

FIG. 56

HINT. In Fig. 56 we have $AC = 16\,t$, $BC = 22\,t$, if t denotes the time (in hours) which passes between the time of sailing and the moment of meeting, and if C represents the place of meeting.

18. The angle of elevation of the top of a tower, at a point in the same horizontal plane with its base, is equal to A. At a point h feet directly above the first the angle of depression of the foot of the tower was found to be equal to B. Prove that the height of the tower is equal to $h \tan A \cot B$.

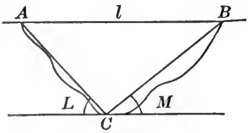

FIG. 57

19. A valley has the cross section shown in Fig. 57, the angles L and M and the distance $AB = l$ having been obtained by a survey. It is planned to connect the points A and B by a bridge, supported by a pier at C. How high must this pier be made?

20. Show that the area of a quadrilateral is $\frac{1}{2}dd' \sin A$, if d and d' are the lengths of its diagonals and if A is one of the angles which the diagonals make with each other.

21. Two sides of a parallelogram are 3.41 and 2.60 feet long and the shorter diagonal is 1.58 feet long. Find the length of the other diagonal.

22. The sides of a field $ABCD$ are: $AB = 57$ feet, $BC = 43$ feet, $CD = 45$ feet, $DA = 47$ feet; and the distance from A to C is 50 feet. Find the area of the field.

23. Two streets intersect at an angle of 75°. The corner lot has frontages of 150 feet and 115 feet on the two streets, and the remaining two boundary lines of the lot are perpendicular to the two streets. What is their length, and what is the area of the lot?

24. In order to measure the distance between two pumping stations, A and B, in Lake Michigan, a base line $CD = 17.7$ chains was measured along the shore. (See Fig. 58.) The following angles were measured:

$$ACD = C_1 = 132° \, 29',$$
$$ACB = 82° \, 20',$$
$$CDA = D_1 = 45° \, 59',$$
$$CDB = D_2 = 124° \, 48.$$

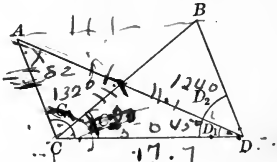

FIG. 58

Compute the distance AB.

(Fig. 58 is not drawn to scale.)

25. Devise a plan for finding the distance between any two inaccessible points, A and B, in the same horizontal plane if two points, C and D, can be found in the same plane, from both of which A and B are visible.

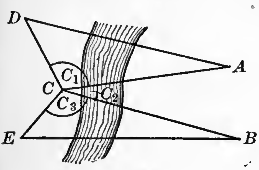

FIG. 59

26. Devise a plan for finding the distance between two inaccessible points, A and B, if both are visible from only *one* accessible point C.

HINT. In Fig. 59, select a point D from which A and C are visible, and a point E from which B and C are visible. Measure CD, CE, and the angles D, C_1, C_2, C_3, E.

27. To compute the distance between two accessible points, A and B, if no point can be found from which both A and B can be seen. (For instance, if A and B are points on opposite sides of an inaccessible mountain.) Take a point C from which A may be seen and a point D from which B is visible. If C is visible from D, measure the angles ACD and CDB and the distances AC, CD, DB. Show how to calculate the distance AB from these data.

28. If the points A and B of Ex. 27 are inaccessible, the distances AC and BD cannot be found by direct measurement. In such a case (see Fig. 60), select points C, D, E, F so that A, C, E shall be visible from D, and D, F, B from E. Measure the angles C, D_1, D_2, E_1, E_2, F, and the distances CD, DE, and EF. Show how to find the distance AB from these measurements.

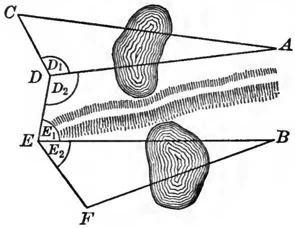

FIG. 60

29. A tower is situated on top of a conical hill as in Ex. 12. Two points A and A' are chosen on the side of the hill at distances $d = 43$ feet and $d' = 98$ feet respectively from the top, the point A' being the lower one and the distances d and d' being measured along the slope. The angles subtended by the tower at A and A' were $A = 42°$, $A' = 23°$ respectively. Find the height of the tower and the angle of inclination of the side of the hill.

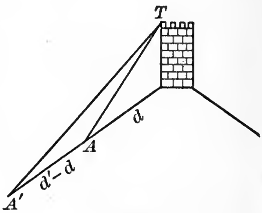

FIG. 61

30. Two observers, A and B, are 3 miles apart, A being due west of B and in the same horizontal plane. Both observers measure, at the same instant, the bearing and angle of elevation of a balloon. A finds that the balloon bears N. 47° E. and that its angle of elevation is 23°. B finds that the balloon bears N. 35° W. and that its angle of elevation is 19°.5. Find the height of the balloon above the horizontal plane of A and B.

31. To find the horizontal distance AD and the vertical distance DC from A to an inaccessible point C (Fig. 62) when it is not convenient to measure a base line in the same vertical plane with C.

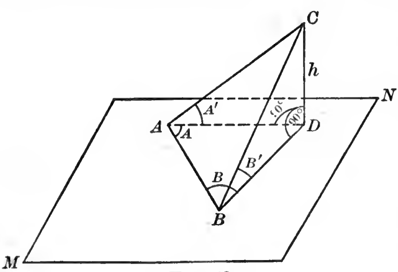

FIG. 62

Measure a horizontal base line AB, k feet long, in any direction through A. Let D be the foot of the perpendicular from C to the horizontal plane, MN, of AB. Measure the horizontal angles $BAD = A$ and $ABD = B$, and the vertical angles $DAC = A'$ and $DBC = B'$.

Show that
$$AD = \frac{k \sin B}{\sin (A + B)}, \quad BD = \frac{k \sin A}{\sin (A + B)},$$
$$h = \frac{k \sin B \tan A'}{\sin (A + B)} = \frac{k \sin A \tan B'}{\sin (A + B)}.$$

The two formulæ for h should give the same result in any numerical problem of this kind. Lack of agreement indicates inaccuracy in one of the observed angles or in the computation.

32. Two points, A and B, in the same horizontal plane and separated by a ridge, are to be connected by a straight level tunnel. In order to find the distance between them, the surveyors measured the inclined angle, BCA, subtended by AB from the top C of a neighboring hill, whose height, $CD = h$, above the plane of A and B is known. They also measured the angles of depression of A and B from C. Devise a method for computing the length of the tunnel AB.

33. Prove, by means of the trigonometric formulæ, that the angles of a triangle can be found when the *ratios* of the three sides of the triangle are given, even if the absolute values of the three sides are not known. Prove the same fact by geometry.

34. Let A, B, C be three points of a horizontal line, whose mutual distances, $AB = b$, $AC = c$, $BC = b + c$, are known. (See Fig. 63.) To find the horizontal distances, p, q, r, of an inaccessible point, E, from these three points, and the height h, it suffices to measure the three angles of elevation, A, B, C, of E from A, B, C respectively.

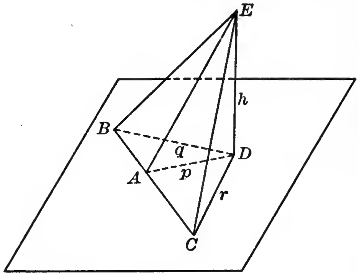

FIG. 63

For the figure gives

(1) $p = h \cot A, \qquad q = h \cot B,$
$r = h \cot C,$

and also

$$q^2 = b^2 + p^2 - 2\,bp \cos BAD,$$
$$r^2 = c^2 + p^2 + 2\,cp \cos BAD;$$

whence

$$\frac{b^2 + p^2 - q^2}{b} = \frac{r^2 - c^2 - p^2}{c}.$$

Substitute the values of p, q, r from (1) and solve the resulting equation for h^2. We find

$$(2) \qquad h^2 = \frac{bc(b + c)}{(\cot^2 C - \cot^2 A)b + (\cot^2 B - \cot^2 A)c}.$$

After computing h from (2), p, q, and r may be found from (1).

35. Given the mutual distances of three points A, B, C, to find the distances AD, BD, CD from a fourth point D in the plane of ABC to each of the given three points, when the angles are given which the lines AB, BC, CA subtend at D.

This problem, usually called *Pothenot's* problem, may be solved as follows:

In Fig. 64, let

$$BC = a, \quad CA = b, \quad AB = c$$

be the given mutual distances of A, B, and C. Of course the angles of the triangle ABC may then also be regarded as known.

Let $\angle CDB = L, \quad \angle CDA = M, \quad \angle BDA = N$

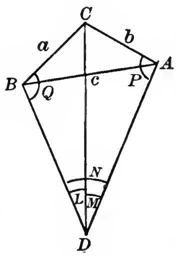

FIG. 64

be the angles subtended by the sides of the triangle from D. These angles we regard as known by measurement and, of course, $N = M + L$.

The problem of finding the distances AD, BD, CD may evidently be regarded as solved, if we can find the two angles

$$\angle CAD = P \text{ and } \angle CBD = Q.$$

We shall show how to compute these angles.

Applying the law of sines to the triangles ACD and BCD, we find

(1) $$CD = \frac{b \sin P}{\sin M} = \frac{a \sin Q}{\sin L},$$

whence $$\frac{\sin P}{\sin Q} = \frac{a \sin M}{b \sin L}.$$

Hence, by the theory of proportion,

$$\frac{\sin P - \sin Q}{\sin P + \sin Q} = \frac{a \sin M - b \sin L}{a \sin M + b \sin L},$$

which gives (Equations (8), Art. 49),

(2) $$\frac{\tan \frac{1}{2}(P - Q)}{\tan \frac{1}{2}(P + Q)} = \frac{1 - \dfrac{b \sin L}{a \sin M}}{1 + \dfrac{b \sin L}{a \sin M}}.$$

On the other hand we have

$$P + Q + C + N = 360°,$$

and therefore

(3) $$\tfrac{1}{2}(P + Q) = 180° - \tfrac{1}{2}(C + N).$$

Since the angles C and N are known, (3) gives the value of $\frac{1}{2}(P + Q)$. If this value be substituted in (2), we may obtain from (2) the value of $\frac{1}{2}(P - Q)$. From $\frac{1}{2}(P + Q)$ and $\frac{1}{2}(P - Q)$, we find P and Q themselves by addition and subtraction. The distance CD may then be computed by (1) in two ways, providing a check for the correctness of the work.

Remark 1. The problem obviously becomes indeterminate, if the point D should happen to be on the circumference of the circle determined by the three points A, B, C. For, as the point D moves along this circumference, the angles L and M do not change, so that the position of the point D on the circumference is not determined by the value of these angles. It is evident, then, that if the point D is close to the circumference of the circle circumscribed about the triangle ABC, its position cannot be determined by this method with any considerable degree of accuracy.

Remark 2. The name POTHENOT's problem cannot be justified historically. A complete solution of this problem was given by SNELLIUS in his *Doctrinæ triangulorum canonicæ libri quatuor*, which appeared in 1627, almost seventy years before Pothenot presented his solution of the same problem to the Paris Academy of Science.

36. Show that the problem of Pothenot may also be solved by drawing a circle through the three points A, B, D, and by making use of the triangle ABE, where E is the second point of intersection of CD with this circle.

37. In surveying a harbor, a submerged rock was located, for charting purposes, by sighting three known objects A, B, C on land from a boat immediately above the rock. The known distances were $BC = a = 312$ feet, $CA = b = 520$ feet, and the angle C was known to be $65° 27'$. The angles obtained by observation were $L = CDB = 23° 25'$ and $M = CDA = 32° 52'$. Find and check the distance from the rock to C and the angle which this line makes with the side AC of the known triangle.

In Exs. 38 to 45, we use the notations of Chapter VII; A, B, C for the angles, a, b, c for the sides, $2s$ for the perimeter, r and R for the radii of the inscribed and circumscribed circle respectively, and S for the area of the triangle. Show how to find all of the sides and angles of the triangles determined by the following data.

38. $a + b, c, A - B$ are given.

39. $a - b, c, A - B$ are given.

40. R, a, b are given.

41. R, A, B are given.

42. r, A, B are given.

43. S, A, B are given.

44. S, a, b are given.

45. s, R, a are given.

46. A furnace maker receives an order for a number of furnaces, some 40 inches and some 42 inches in diameter. These furnaces are to be fitted on the outside with an iron casting whose inside length, measured along the arc, is 26 inches. In order to avoid the necessity of making two different castings, the manufacturer considers the possibility of making a single casting, to fit *exactly* a furnace 41 inches in diameter but, of course, not fitting exactly either of the sizes ordered. His experience tells him that such a casting will serve the purpose, if no point of its inner surface is more than a quarter of an inch from the outer surface of the furnace after being placed in position. Will it be necessary to make separate castings for the two different sizes?

58. Displacements, velocities, and forces. If a body is transported from one place in the plane to another, and we wish to describe its change of position or *displacement*, it is clearly not sufficient to state *how far* the body has been moved. We must also include in our description a statement concerning the *direction* of the displacement.

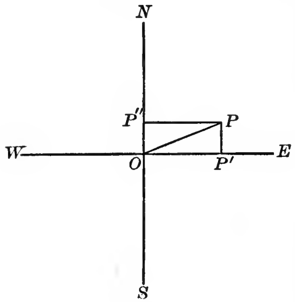

FIG. 65

Thus, the displacement from O to P (Fig. 65) may be described completely by stating its *magnitude* (the length of the line-segment OP), and its *direction* (either the bearing NOP of the line OP or the angle EOP or some other angle which fixes the direction of OP).

If the displacement is in a horizontal plane and the directions from O to N, S, E, W in Fig. 65 represent north, south, east, and west respectively, the projections OP' and OP'' of OP on OE and ON are called the easterly and northerly *components* of the displacement. If the displacement is not horizontal, we define, in a similar manner, its horizontal and vertical components.

Let us suppose that a point M is displaced from O to P. (See Fig. 66.) The displacement may be represented in magnitude and direction by the directed line-segment OP. (The arrowhead indicates that the displacement is from O toward P, and not from P toward O.) Let OQ represent a second displacement. If the point M, originally at O, be made to undergo both of these displacements in succession (in either order), it will ultimately arrive at R, where R is the fourth vertex of the parallelogram determined by OP and OQ. For this reason, the displacement OR is said to be the *resultant* of the displacements OP and OQ.

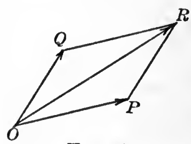

FIG. 66

We may think of the two displacements OP and OQ as taking place simultaneously. An instance of this sort is furnished by a passenger shifting his position on board of a moving ship. His total displacement, in space, is the resultant of that which is due to the motion of the ship and of that caused by his own muscular efforts.

The *velocity* of a train, of a bullet, or of any uniformly moving object, is measured by the distance which it describes in a unit of time, that is, by a displacement. Therefore a velocity, like a displacement, has direction as well as magnitude. The case of a passenger's stroll on board a moving ship suffices to illustrate the phrase: *resultant of two velocities.* Since the velocity of a uniformly moving body is its displacement in a unit of time, the resultant of two velocities is found by the same method as a resultant of two displacements, *i.e.* by the parallelogram construction illustrated in Fig. 66.

It is one of the fundamental facts of mechanics (proved by countless experiments) that *two forces acting upon the same material point combine into a single* **resultant** *force according to this same parallelogram law.* This fact is generally known as the *law of the parallelogram of forces.*

Owing to the fact that displacements, velocities, and forces are directed quantities which combine according to the parallelogram law, these three classes of things have many properties in common. There are many other instances of quantities of this same character and, on account of their importance, they have received a special name.

Directed quantities, which combine in accordance with the parallelogram law, are called **vectors**.*

By a proper choice of the units, every vector may be represented by a directed line-segment or, what amounts to the same thing, by a displacement.

Thus, the line-segments of Fig. 66 may be interpreted as forces, if the directions of these line-segments coincide with the directions of the forces, and if each segment is made to contain as many length units as there are force units in the corresponding force.

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* From the Latin *vector*, meaning one who carries or conveys.

EXERCISE XXIX

1. A steamer is moving N.N.E. with a velocity of 16 miles per hour. Find the northerly and easterly components of its velocity.

2. A horizontal force of 10 lb. and a vertical force of 24 lb. are acting simultaneously on a point. Find the magnitude and direction of the resultant.

3. A schooner is sailing due west at the rate of 6 miles per hour. A sailor is crossing the deck, from south to north, at the rate of 3 miles per hour. What is the magnitude and direction of his velocity in space?

4. A force of 250 lb. is acting on a body in a direction which makes an angle of 17° with the horizontal plane. How much of this force tends to lift the body, and what part of it tends to move the body in a horizontal plane?

5. Two forces, of magnitudes 350 lb. and 510 lb., respectively, act upon the same point, in directions which make an angle of 35° with each other. Find the magnitude of the resultant, and the angles which it makes with each of the component forces.

6. A force of 216 lb. is resolved into two components, which make angles of 27° and 32° respectively, with the direction of the original force. Find the magnitude of each component.

7. A man wishes to reach a point on the opposite side of the river, 250 yards upstream. The velocity of the current is 2.5 miles per hour and the width of the river is 300 yards. If the man's rate of rowing (in still water) is 4 miles per hour, in what direction must he point the head of the boat in order that his course may be a straight line?

59. Reflection and refraction of light. The path of light, in a homogeneous medium like air, is rectilinear. But if a ray of light meets the polished surface of a sheet of metal or glass, its direction is changed in accordance with the law that *the angle of incidence is equal to the angle of reflection.*

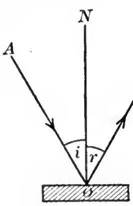

FIG. 67

This law is illustrated in Fig. 67, where the ray AO strikes the reflecting surface of the mirror at O and is reflected in the direction OB. The line ON, perpendicular to the reflecting surface at O, is called its normal. The angle NOA, or i, is called the *angle of incidence*, and the angle NOB, or r, is the *angle of reflection.* According to the law of reflection of light (verified by thousands of experiments) these two angles are equal.

When a ray of light AO, after passing through air, meets the bounding surface of a second transparent medium like glass (see Fig. 68), only a part of the light is reflected. Another portion of the light enters the second medium and continues on its way in a path OB, which makes a certain angle with the direction AO of the original ray.

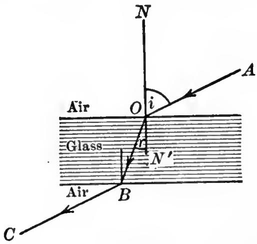

FIG. 68

If NN' is the perpendicular or normal to the bounding surface at O, the angle NOA, or i, is called the *angle of incidence* and $N'OB$, or r, is the *angle of refraction.*

As a result of numerous experiments, it has been found that the quotient

$$\frac{\sin i}{\sin r}$$

will have the same value, for a given kind of glass, for all different values of i. In other words, as the angle of incidence changes the angle of refraction also changes, but in such a way as to leave the quotient

$$\frac{\sin i}{\sin r} = n \qquad (1)$$

unchanged. This quotient n is called the **index of refraction** of the glass with respect to air. Its value is different for different kinds of glass. For ordinary crown glass n is about equal to 1.5.

If the ray of light again emerges into air, as indicated in Fig. 68, after having passed through a sheet of glass with exactly parallel sides, careful measurements show that the ray BC is parallel to the original ray AO. In other words, the direction of a ray of light is not changed by passing through a sheet of plate glass whose two faces are exactly parallel to each other. Therefore: *the index of refraction of* *air with respect to glass is the reciprocal of the index of refraction of glass with respect to air.*

The law of the refraction of light was first discovered by SNELLIUS in 1618. The simple formula (1) was given by DESCARTES in 1637.

EXERCISE XXX

1. A light is placed on the line perpendicular to a plane circular mirror at its middle point. The distance from the light to the mirror is 15.76 inches, and the mirror is 8.32 inches in diameter. Consider two rays which strike the mirror in the extremities of one of its diameters. What angle will they make with each other after reflection?

2. In an experiment, a source of light is to be placed at A, a mirror at B, and a photographic plate at C. The distances are: $AB = 5.367$ meters, $BC = 6.329$ meters, $CA = 7.361$ meters. What angle must the mirror at B make with the line AB so that the light, reflected at B, may pass through C?

3. Two billiard balls, A and B, have been placed at distances a and b inches respectively from the same cushion. The line joining them makes an angle L with the cushion. Let K be the angle at which the first ball must strike the cushion, so as to hit the second after rebounding. Show that

$$\tan K = \frac{b+a}{b-a} \tan L.$$

Remark. Billiard balls not endowed with a lateral rotation (without "English") rebound in accordance with the law of reflected light.

4. Find the index of refraction from the following observations.
(Wüllner)

Angles of incidence, i . . .	40°	60°	80°
Angles of refraction, r . . .	24° 24′	33° 38′	38° 57′

The three values obtained for n will disagree slightly owing to inaccuracies in the measurements.

5. A ray of light strikes a plate of crown glass at an angle of incidence of 37°. Find the angle between the reflected and the refracted ray, if the index of refraction is 1.559.

6. A ray of light $ABCD$ passes through a glass prism whose cross section (see Fig. 69) is an equilateral triangle. If the index of refraction is 1.559, what must be the angle of incidence in order that the path of the light in the prism may be parallel to one of its faces? What angle will the ray CD make with its original direction AB, after emerging from the prism?

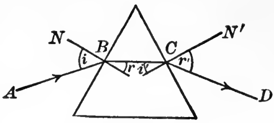

FIG. 69

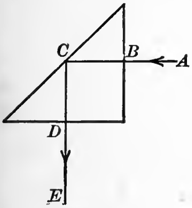

FIG. 70

7. A ray of light, ABC, etc., enters a glass prism, whose cross section is an isosceles right triangle and whose index of refraction is 1.5, at the point B (Fig. 70), at right angles to the face of the prism. At C no part of the ray can be refracted (Why?), and all of it is reflected in the direction CE. Such a prism is called a total reflecting prism.

THE GREEK ALPHABET

α,	Α	Alpha	ν,	Ν	Nu
β,	Β	Beta	ξ,	Ξ	Xi
γ,	Γ	Gamma	ο,	Ο	Omicron
δ,	Δ	Delta	π,	Π	Pi
ε,	Ε	Epsilon	ρ,	Ρ	Rho
ζ,	Ζ	Zeta	σ, ς,	Σ	Sigma
η,	Η	Eta	τ,	Τ	Tau
θ, ϑ,	Θ	Theta	υ,	Υ	Upsilon
ι,	Ι	Iota	φ,	Φ	Phi
κ,	Κ	Kappa	χ,	Χ	Chi
λ,	Λ	Lambda	ψ,	Ψ	Psi
μ,	Μ	Mu	ω,	Ω	Omega

PART TWO

PROPERTIES OF THE TRIGONOMETRIC FUNCTIONS

CHAPTER IX

THE GENERAL ANGLE AND ITS TRIGONOMETRIC FUNCTIONS

60. The notion of the general angle. In elementary geometry we usually think of an angle as ready-made. We there think of two lines as given and understand by the angle between them a measure of their difference of direction. But many reasons urge us to oppose to this *static* idea what might be called the *dynamic* concept of angle, which presents an angle, not as the ready-made difference of direction between two fixed lines, but as something which is *generated* by the rotation of a straight line around a fixed point as pivot.

Thus, for instance, we shall say that the minute hand of a clock describes, or generates, an angle of 90° in fifteen minutes, an angle of 360° in one hour, an angle of 1890° in five hours and a quarter. Although the minute hand points to the same place on the face of the clock after any number of complete revolutions, we are not likely to make the error of ignoring these complete revolutions.* If we did, we should be ignoring the distinction between 1 o'clock, 2 o'clock, 3 o'clock, etc. If we were to say that an angle of 360° is the same as one of 0°, or that an angle of 450° is equal to one of 90°, we should be committing the same error.

We see that, while an angle in the sense of elementary geometry can never be greater than 180°, our new concept of angle permits us to speak of angles of *any* magnitude.

* It is the purpose of the hour hand to record the number of complete revolutions.

Our notions will be enriched in still another way, if we adopt the dynamic instead of the static concept of angle. The line, whose rotation generates the angle, may revolve in either of two opposite directions, clockwise or counterclockwise; and we must distinguish between these two kinds of rotation, just as we distinguish between two motions in opposite directions on a straight line. This distinction may be made by ascribing to every angle, not merely a *magnitude*, but also a *sign* depending upon the direction of the rotation by which the angle is generated.

It is customary to speak of counterclockwise rotations as positive, and of clockwise rotations as negative.

The reason for this convention * will appear later (Art. 63).

EXERCISE XXXI

✓ Using a protractor, combine the following angles graphically and check the results arithmetically.

1. $30° + 60°$, $50° - 30°$, $30° - 60°$, $50° + (-30°)$, $30° + (-60°)$.

2. $25° + 15° - 35°$, $135° - (-25°) + 150°$.

3. $225° + 345° - 185°$, $30° + (3 \times 15°)$.

4. What angle does the minute hand of a clock describe in 3 hours and 25 minutes? in 5 hours and 13 minutes?

5. Suppose that the dial of a clock is transparent so that it may be read from both sides. Each of two persons, stationed on opposite sides of the dial, observes the motion of the minute hand for fifteen minutes. Upon comparing notes, they find that they do not agree in regard to the angle described by the minute hand during this period of time. In what respect do they differ?

6. What is the magnitude of the angle described by a spoke of a carriage wheel, 3 feet in diameter, when the carriage travels a distance of 500 feet?

Note. Think of the wheel as if it were turning on the axle while the carriage is standing still.

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* The word *convention* is here used in a special sense, meaning an *arbitrary agreement.*

7. The earth describes an approximately circular orbit about the sun as center in 365 days. What angle will the line joining the sun to the earth (the earth's *radius vector*) describe in 415 days?

8. Two wheels, A and B, are joined by a belt as in Fig. 71. The diameter of A is twice that of B, and A is moving in counterclockwise direction. What angle will a spoke of B describe while A rotates through an angle of 300°?

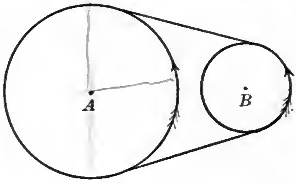

FIG. 71

9. If the two wheels of Ex. 8 are joined by a crossed belt, what angle will a spoke of B describe when A rotates through an angle of 300°?

10. If n wheels are connected by gears, what kind of a number must n be in order that the first and last wheel may rotate in the same direction? in opposite directions?

61. Initial and terminal side. Standard position of an angle. Our new concept of an angle, as a measure for the amount of rotation of a line, leads us to distinguish between the **initial** and **terminal** sides of an angle.

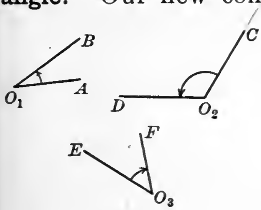

FIG. 72

Thus, in Fig. 72, we have three angles whose senses of rotation are indicated by curved arrows. The angles AO_1B and CO_2D are positive, for the rotation is counterclockwise. Angle EO_3F is negative. The *initial* sides of these angles are AO_1, CO_2, EO_3, and their *terminal* sides are BO_1, DO_2, and FO_3 respectively.

If an angle is thought of as generated by the rotation of a straight line, the initial and final positions of this line are called the initial and terminal sides of the angle respectively.

If we wish to compare two parallel directed line-segments in regard to magnitude and sign, we usually think of one of them as being moved, until its initial point coincides with the initial point of the other. In the same way, in order to compare two angles we usually place them so that their vertices and initial sides shall coincide.

It is customary, for purposes of comparison, to place all

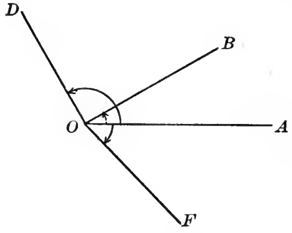

FIG. 73

angles in such a position that their initial sides are on a horizontal line and pointed toward the right. An angle placed in this way shall be said to be in its **standard position.**

Thus, Fig. 73 represents the three angles of Fig. 72 in their standard positions; the angles AOB, AOD, and AOF of Fig. 73 being equal to the three angles AO_1B, CO_2D, and EO_3F of Fig. 72 respectively.

EXERCISE XXXII

Place the following angles in their standard positions:

1. 15°, 225°, 415°, 768°.

2. – 25°, – 275°, – 615°, – 365°.

3. If two angles differ by an integral multiple of 360° and both angles are placed in standard position, how will the terminal sides of the two angles be situated with respect to each other?

4. If two angles, placed in standard position, have the same terminal side, what is the relation between them?

5. If the sum of two angles is an integral multiple of 360°, how will the terminal sides of the two angles be situated with respect to each other, both angles being placed in standard position?

62. The notion of the trigonometric functions of a general angle. Having formulated the notion of a *general angle*, it becomes necessary to revise our definitions of the trigonometric functions, since our original definitions are applicable to acute angles only. To be sure, we have already made some progress in this direction by defining the functions of an obtuse angle. (See Arts. 40 and 42.) But those definitions were provisional, and it will be advisable to reopen the whole question, so as to gain a larger and more adequate point of view.

Our new concept of an angle, as a measure for the amount of rotation of a line, practically forces upon us the following considerations which automatically suggest the new definitions of the trigonometric functions.

Let us draw a positive acute angle θ (see the Greek alphabet on page 132) in its standard position (Fig. 74 *a*). Let us choose a point P anywhere on its terminal side and from P drop a perpendicular PM to the initial side. Then, in accordance with the definitions of the functions of an acute angle, we have

$$(1) \qquad \sin\theta = \frac{MP}{OP}, \quad \cos\theta = \frac{OM}{OP}.$$

Let us now think of the angle θ as growing. Nothing remarkable happens until θ reaches 90°. At that moment, and as the motion continues, our original definitions cease to be applicable, because the right triangle POM, of which θ is an *interior* angle, ceases to exist. But we may think of PMQ (in Fig. 74 *a*) as a plumb line attached to a point P on the moving terminal side of the angle. If there is no obstacle at O, this line, remaining always vertical, will pass from the right to the left of O as θ grows from an acute into an obtuse angle (see Fig. 74 *b*), and the line-segment OM will change its direction. To indicate this change of direction of OM, we represent OM by a positive number in Fig. 74 *a* and by a negative number in Fig. 74 *b*. If we count the distance OP (which does not change) as positive, in all positions of the moving line, and if we retain equations (1) as definitions for $\sin\theta$ and $\cos\theta$, we see that $\cos\theta$ becomes *negative* when θ becomes obtuse. The line-segment PM does not change its direction until θ grows beyond 180°. Therefore the sine of an obtuse angle, like that of an acute angle, is positive. The sine of an angle, however, becomes negative when the angle lies between 180° and 360°.

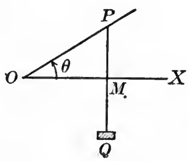

FIG. 74 *a*

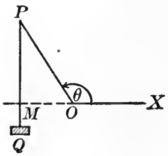

FIG. 74 *b*

63. Rectangular coördinates. All of these things may be stated more briefly by the introduction of *rectangular coördi-*

nates, a notion of utmost importance, not merely in trigonometry, but in other branches of mathematics.

Let us draw two lines, unbounded in length and perpendicular to each other. We shall usually think of one of them as horizontal and call it the **x-axis**, and call the other, which is vertical, the **y-axis**. The point O, in which the two axes intersect, is called the **origin of coördinates.**

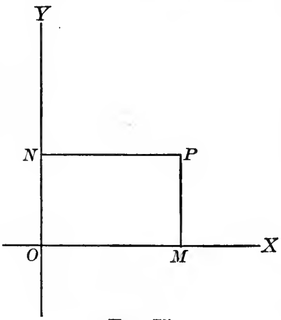

FIG. 75

We adopt a unit of length, and denote the distances from any point P to these two axes by x and y respectively. In Fig. 75 we have

$$NP = OM = x, \quad MP = ON = y,$$

where the notation is chosen in such a way that x is measured on or parallel to the x-axis, and y on or parallel to the y-axis.

We call x the **abscissa** and y the **ordinate** of the point P. Both numbers together are called the **coördinates** of P.

If we take into account only the magnitudes and not the directions of the lines OM, ON, etc., that is, if x and y are regarded as numbers without sign, there will be four points which have the same coördinates.

For instance, the points P, P', P'', P''', in Fig. 76, would all correspond to $x = 3$, $y = 2$.

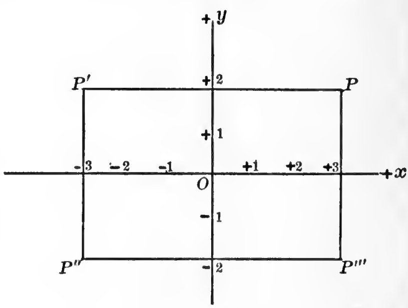

FIG. 76

In order to avoid this inconvenience, we introduce the convention that the abscissas of all points to the right of the y-axis shall be positive, and of those to the left negative; that the ordinates of all points above the x-axis shall be positive, and of those below negative.

The coördinates of the four points in Fig. 76 are now different from each other.

The coördinates of P are $x = +3$, $y = +2$,
The coördinates of P' are $x = -3$, $y = +2$,
The coördinates of P'' are $x = -3$, $y = -2$,
The coördinates of P''' are $x = +3$, $y = -2$.

The positive directions of the x- and y-axes, which have now been defined, *will hereafter be indicated by a plus sign* (as in Fig. 76).

The distance from the origin O to any point P is usually denoted by r and is called the **radius vector** of that point. We shall always regard the radius vector as *positive*, and clearly we shall always have (see Fig. 77),

$$r = +\sqrt{x^2 + y^2}.$$

Let us think of OP (Fig. 77) as rotating around O as a center. If OP originally coincides with the positive x-axis, it will require a counterclockwise rotation of 90°, or a clockwise rotation of 270°, to bring it into coincidence with the positive y-axis. We naturally think of the numerically smaller angle first, and define the *positive sense of rotation* to be *that one* which enables us to turn the *positive* x-axis into the position of the *positive* y-axis by means of a rotation of *only 90°*. But this implies that the *positive direction of rotation is counterclockwise*. (Cf. Art. 60.)

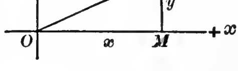

FIG. 77

EXERCISE XXXIII

Plot the points whose coördinates are given in Exs. 1 to 5. Find, by *measurement*, to the nearest degree for the angles and to the nearest tenth of a unit for the distances, the radius vector of each point and the positive angle which it makes with the positive x-axis. Find the same results by *calculation*, making use of three-place tables.

1. $x = +3$, $y = +4$.

2. $x = +12$, $y = -5$.

3. $x = -2.4$, $y = +5.5$.

4. $x = -2.6$, $y = -2.1$.

5. $x = +1.27$, $y = -2.18$.

In the following examples, r denotes the radius vector of a point P, and θ the positive angle which this radius vector makes with the positive direction of the x-axis. Plot the points. Find, by *measurement* to the nearest tenth of a unit, the abscissas and ordinates of these points. Find the same results by *calculation* with three-place tables.

6. $r = 2, \theta = 30°$.

7. $r = 5, \theta = 135°$.

8. $r = 3, \theta = 210°$.

9. $r = 4, \theta = 285°$.

10. $r = 2.56, \theta = 310° 20'$.

64. Definition of the trigonometric functions of a general angle. We are now in a position to give the definitions of the functions of a general angle in a compact manner.

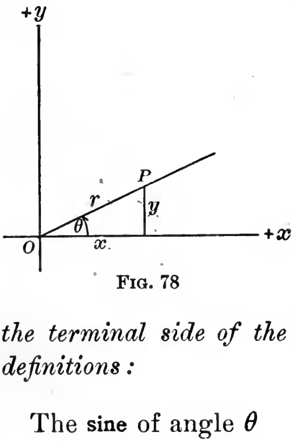

FIG. 78

Place the angle θ in its standard position; that is, with its vertex on the origin and its initial side on the positive x-axis of a system of rectangular coördinates. (See Fig. 78, where θ is a positive acute angle.) *Pick out a point P, different from the origin, anywhere on the terminal side of the angle. Then we adopt the following definitions:*

The **sine** of angle θ $= \dfrac{\text{ordinate of } P}{\text{radius vector of } P}$, $\quad \sin\theta = \dfrac{y}{r}$.

The **cosine** of angle θ $= \dfrac{\text{abscissa of } P}{\text{radius vector of } P}$, $\quad \cos\theta = \dfrac{x}{r}$.

The **tangent** of angle θ $= \dfrac{\text{ordinate of } P}{\text{abscissa of } P}$, $\quad \tan\theta = \dfrac{y}{x}$.

The **cotangent** of angle θ $= \dfrac{\text{abscissa of } P}{\text{ordinate of } P}$, $\quad \cot\theta = \dfrac{x}{y}$.

The **secant** of angle θ $= \dfrac{\text{radius vector of } P}{\text{abscissa of } P}$, $\quad \sec\theta = \dfrac{r}{x}$.

The **cosecant** of angle θ $= \dfrac{\text{radius vector of } P}{\text{ordinate of } P}$, $\quad \csc\theta = \dfrac{r}{y}$.

As Fig. 78 shows, these definitions reduce to the familiar definitions of Art. 7 if θ is an acute angle. In the case of an obtuse angle, they give results which agree with the definitions of Arts. 40, 42, and 43. But in our present definitions, θ is not restricted either in magnitude or sign; it may be a positive or negative angle of any magnitude. The quantities x and y may be positive, zero, or negative, but r is always positive.

EXERCISE XXXIV

Construct carefully the following angles and, by measurement, find approximate values of the six trigonometric functions, correct to two significant places, paying particular attention to their signs.

1. 25°. **2.** 320°. **3.** 110°. **4.** − 130°. **5.** +.725°. **6.** − 10°.

7. In Art. 11, the exact values of the functions of 30°, 45°, and 60° were expressed by means of radicals. In a similar way find the values of the functions of the following angles:

120°, 135°, 150°, 210°, 225,° 240°, 300°, 315°, 330°.

8. What are the signs of the trigonometric functions of the following angles:

150°, 320°, 1000°, − 625°.

Find, by construction and measurement to the nearest degree, the values of the angles for which

9. $\sin\theta = -\frac{4}{5},\quad \cos\theta = +\frac{3}{5}.$

10. $\tan\theta = 1,\quad \cos\theta = -\dfrac{1}{\sqrt{2}}.$

11. Show that the values of the trigonometric functions of a general angle, as given by the definitions of Art. 64, will not be changed if, instead of the point P, any other point P' on the terminal side of the angle be chosen.

65. Discussion of the exceptional cases. Each of the trigonometric functions is defined in Art. 64 *formally*, as a quotient of two numbers. This formal definition will have a real significance whenever the two numbers actually *have* a quotient. Now we know from Algebra that two numbers, D (the dividend) and d (the divisor), always have a unique quotient q if the divisor is *different from zero.* That is, there exists a number q such that

(1) $$\frac{D}{d} = q,$$

or what amounts to the same thing, such that

(2) $$D = dq,$$

whenever d is different from zero.

Now, let us discuss the case where d (the divisor) is equal to zero, while D (the dividend) is not. In this case there exists *no* number q which satisfies equation (2). For this equation now becomes

(3) $$D = 0 \cdot q,$$

and its right member is equal to zero no matter what number we substitute for q, while its left member is, by hypothesis, different from zero. Consequently it involves a contradiction to assume that a number has been obtained by dividing another by zero, and **the operation of dividing by zero is therefore excluded from Algebra.**

The formal definitions of Art. 64 involve divisions by x, y, and r, and therefore lose their significance in any case in which one of these divisors is equal to zero. Now $r = OP$ is the radius vector of a point P which may be chosen anywhere on the terminal side of the angle except at O. (Compare the wording of the definitions in Art. 64.) Therefore r is never equal to zero. From this fact and the equation

(4) $$x^2 + y^2 = r^2,$$

we conclude further that x and y cannot both be equal to zero at the same time.

It will now be clear that the formal definitions of Art. 64 fail to provide the symbols

(5) $$\begin{matrix} \tan 90°, & \sec 90°, & \tan 270°, & \sec 270°, \\ \cot 0°, & \csc 0°, & \cot 180°, & \csc 180°, \end{matrix}$$

with any actual meaning. But they *do* define each of the six trigonometric functions of all positive angles less than 360° with the eight exceptions just mentioned.

If the angle θ is greater than 360°, or if it is negative, other exceptional cases appear. But their relation to the eight exceptional cases (5) is so simple that we may leave it to the student to complete this discussion.

Although the tangent of 90° is not defined, our definitions are clearly applicable to the tangent of an angle θ which differs from 90° by the slightest conceivable amount. Let us see how the tangent of an angle θ behaves when θ approaches 90° as a limit. We have, in Fig. 79,

$$\tan\theta = \frac{y}{x} = \frac{MP}{OM},$$

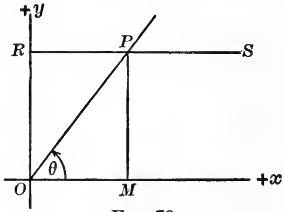

FIG. 79

where P may be any point different from O on the terminal side of the angle. For our present purpose it will be convenient to select the point P in the following manner. Draw a line RS parallel to the x-axis at any convenient distance from OM, and let P be the intersection of the terminal side of the angle θ with this fixed line. Then, as θ approaches 90°, $MP = y$ will remain constant, while $OM = x$ approaches the limit zero. The quotient $\frac{y}{x} = \tan\theta$ will therefore become larger and larger. Since $OM = x$ may be made as *small* as we please by taking the angle $\theta = MOP$ close enough to 90°, while the ordinate MP always remains the same, we see that the quotient can be made as *large* as we please. In other words, the angle θ can be made to differ so little from 90° that its tangent will become larger than any number whatsoever. This is what is meant by the statement that $\tan\theta$ *becomes infinite* when θ approaches 90° as a limit. We sometimes express this same statement by writing

(6) $$\tan 90° = \infty,$$

a symbolic equation which should be interpreted as a shorthand account of the situation which has just been described. It is *not* a definition of tan 90°. For ∞ is not a number, and the symbolic equation (6) is not at all concerned with what happens to tan θ when θ is equal to 90°. It merely tells us, in symbolic form, what happens when θ *approaches* 90° as a limit; namely, that tan θ then increases *without bound.*

In the preceding discussion we considered a variable angle MOP which approached 90° as a limit. In order to see what happened to its tangent, we chose the point P on the terminal side of the angle in such a way that its distance y from the x-axis remained constant.

We may obtain the same result in a slightly different way, which may, to some students, appear more conclusive. Let the angle MOP approach 90° as before. Then (Fig. 80),

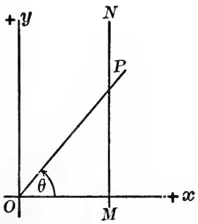

Fig. 80

$$\tan MOP = \frac{MP}{OM} = \frac{y}{x}.$$

Let us, this time, choose the point P on the terminal side of the angle in such a way that its distance x from the y–axis remains unchanged while θ approaches 90° as a limit. It is evident from the figure that $MP = y$ will then increase without bound. Therefore, we obtain again the result that tan $\theta = y/x$ *becomes infinite* when θ approaches 90° as a limit.

So far we have tacitly assumed that θ is an *acute* angle *increasing* toward 90° as a limit. What happens when θ starts as an *obtuse* angle to *decrease* toward 90° as a limit?

Since the tangent of any angle between 90° and 180° is negative, an argument precisely similar to that just carried out shows that the numerical value of tan θ again grows beyond all bounds when θ approaches 90°, remaining, however, always negative. We see therefore that the following statements are both true:

1. When θ is *acute* and *increases* toward 90° as a limit, tan θ, remaining always *positive*, grows numerically beyond bound.

2. When θ is *obtuse* and *decreases* toward 90° as a limit, tan θ, remaining always *negative*, grows numerically beyond bound.

These two statements are frequently summed up in the symbolic formula

$$\tan 90° = \pm\infty. \tag{7}$$

A precisely similar discussion will show that $\tan\theta$ again becomes infinite when θ approaches 270°, that $\sec\theta$ becomes infinite when θ approaches 90° or 270°, and that $\cot\theta$ and $\csc\theta$ become infinite when θ approaches either 0° or 180°. The functions $\sin\theta$ and $\cos\theta$ are always finite.

66. The four quadrants. The x- and y-axes divide the plane into four portions called *quadrants*. The quadrant bounded by the positive x- and y-axes is usually called the *first quadrant*. If we start from the first quadrant and describe a path around the origin in the counterclockwise direction, we traverse in order the 1st, 2d, 3d, and 4th quadrants.

An angle is said to be in the first, second, third, or fourth quadrant according to the quadrant in which its terminal side falls when the angle is in its standard position, that is, with its initial side upon the positive x-axis.

The *cardinal angles* 0°, 90°, 180°, 270°, etc., may be regarded as belonging to either one of the two quadrants upon whose boundaries they lie.

The following table gives the signs of the trigonometric functions of an angle in the various quadrants:

	I	II	III	IV
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent . . .	+	−	+	−
Cotangent . . .	+	−	+	−
Secant	+	−	−	+
Cosecant . . .	+	+	−	−

EXERCISE XXXV

1. Prove each of the following symbolic statements and explain its significance in words.

$$\cot 0^\circ = \pm\infty, \quad \tan 90^\circ = \pm\infty, \quad \cot 180^\circ = \pm\infty, \quad \tan 270^\circ = \pm\infty,$$
$$\csc 0^\circ = \pm\infty, \quad \sec 90^\circ = \pm\infty, \quad \csc 180^\circ = \pm\infty, \quad \sec 270^\circ = \pm\infty.$$

2. Show that the numerical value of the sine or cosine of an angle can never exceed unity.

3. Show that D/d may have any value whatever if D and d are both equal to zero, and hence that the symbol $\frac{0}{0}$ is wholly indeterminate. Why can no one of the trigonometric ratios ever have this form?

4. Determine the quadrants of the following angles and the signs of their trigonometric functions.

$$325^\circ, 710^\circ, 1045^\circ, 609^\circ, 412^\circ, -52^\circ.$$

5. In what quadrant is an angle if its sine and cosine are both positive? If its sine is positive and its tangent negative? If its secant and tangent are both positive?

6. If we know that the sine and cosine of an angle have the same sign, what can we say about the quadrant of the angle?

7. Is there an angle whose tangent is positive and whose cotangent is negative?

8. If we are told that the tangent and cotangent of an angle are both positive, does this enable us to determine the quadrant of the angle?

67. General character of the trigonometric functions. Their periodicity. We are now in a position to understand how the functions change with the angle. For the purposes of this discussion it will be convenient to think of the radius vector r as constant. This means that the point P, which according to our definition must be selected on the terminal side of the angle, describes the circumference of a circle as θ changes from 0° to 360°. (See Fig. 81.) It is easy to verify the following statements by reference to the figure:

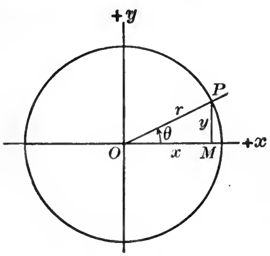

FIG. 81

As θ increases from $0°$ to $90°$, $\sin\theta$ increases from 0 to 1.
As θ increases from $90°$ to $180°$, $\sin\theta$ decreases from 1 to 0.
As θ increases from $180°$ to $270°$, $\sin\theta$ decreases from 0 to -1.
As θ increases from $270°$ to $360°$, $\sin\theta$ increases from -1 to 0.

It is also evident that the function $\sin\theta$ repeats its values in exactly the same order if P moves around the circumference a second, third, $\cdots$ nth time. The same thing is true of the other trigonometric functions, a very important fact which may be expressed as follows:

Each of the six trigonometric functions is **periodic** *and its period is equal to* 360°. *That is, each of these functions repeats its values at intervals of* 360°, *so that*

$$\sin(\theta + n\cdot 360°) = \sin\theta,\quad \cos(\theta + n\cdot 360°) = \cos\theta, \text{ etc.,}$$

where n is any positive or negative integer or zero.

The behavior of each of the six functions in the neighborhood of each of the four cardinal angles may be recapitulated for convenience of reference in the following table:

	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
0°	0	$+1$	0	$\mp\infty$	$+1$	$\mp\infty$
90°	$+1$	0	$\pm\infty$	0	$\pm\infty$	$+1$
180°	0	-1	0	$\mp\infty$	-1	$\pm\infty$
270°	-1	0	$\pm\infty$	0	$\mp\infty$	-1

The student should use this table to describe in words the variation of each of the six functions as θ changes from 0° to 360°.

EXERCISE XXXVI

Discuss the variation of the following functions as θ varies from 0° to 360°.

1. $\cos\theta$.
2. $\tan\theta$.
3. $\cot\theta$.
4. $\sec\theta$.
5. $\csc\theta$.
6. $\sin 2\theta$.
7. $2\sin\theta$.
8. $\sin 3\theta$.
9. $\sin 4\theta$.
10. $\tan 4\theta$.
11. $\sin(-\theta)$.
12. $\cos(-\theta)$.
13. $\sin\frac{1}{2}\theta$.
14. $\sin\frac{1}{3}\theta$.
15. $\sin(\theta + 25°)$.

68. Relations between the trigonometric functions of a general angle. The relations which we found in Art. 9, between the functions of an acute angle, still hold without alteration for an angle of any magnitude. In fact the equation between the abscissa, ordinate, and radius vector of a point P, that is,

$$x^2 + y^2 = r^2,$$

is true, no matter in what quadrant the point P may be situated.

This is due to the fact that only the squares of x, y, and r occur in this relation.

If we divide both members of the above equation by r^2, we find

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$

But, by the definitions of Art. 64, we have, in all quadrants,

$$\frac{x}{r} = \cos\theta, \quad \frac{y}{r} = \sin\theta,$$

so that the preceding equation becomes

$$\textbf{(1)} \qquad \sin^2\theta + \cos^2\theta = 1.$$

Since we have, by definition,

$$\sin\theta = \frac{y}{r}, \qquad \csc\theta = \frac{r}{y},$$

$$\cos\theta = \frac{x}{r}, \qquad \sec\theta = \frac{r}{x},$$

$$\tan\theta = \frac{y}{x}, \qquad \cot\theta = \frac{x}{y},$$

we find at once

(2) $\sin\theta\csc\theta = 1, \quad \cos\theta\sec\theta = 1, \quad \tan\theta\cot\theta = 1.$

We have also
$$\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta},$$

which, combined with (2), gives the further relations

(3) $$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

If we divide both members of (1) by $\cos^2\theta$ and make use of (2) and (3), we find

(4) $$1 + \tan^2\theta = \sec^2\theta.$$

In a similar fashion, if we divide both members of (1) by $\sin^2 \theta$, we see that

(5) $$\mathbf{1 + \cot^2 \theta = \csc^2 \theta.}$$

When the angle θ is acute, all of its functions are positive. Consequently if one of its functions is given, all of the others may be found without ambiguity by means of the above relations. (Cf. Art. 9 and Exercise VI, Exs. 7–12.)

But if we do not know in what quadrant an angle lies and are given the value of merely one of its functions, the angle itself and its other functions are not determined uniquely.

If we are told, for instance, that $\sin \theta = \frac{1}{2}$, equation (1) only tells us that

$$\cos^2 \theta = 1 - (\tfrac{1}{2})^2 = \tfrac{3}{4};$$

so that

$$\cos \theta = \pm \tfrac{1}{2}\sqrt{3},$$

where either sign may be taken. In fact there *are* two angles between 0° and 360° whose sines are equal to $\frac{1}{2}$; namely, 30° and 150°. We may distinguish between them by stating whether the cosine is positive or negative.

EXERCISE XXXVII

Find the other functions of the angle θ as determined by each of the following conditions:

1. $\sin \theta = -\frac{1}{2}$ and θ is in the third quadrant.

2. $\sin \theta = -\frac{1}{2}$ and θ is in the fourth quadrant.

3. $\tan \theta = +2$ and θ is in the third quadrant.

4. $\cot \theta = -3$ and $\sin \theta$ is positive.

5. $\sec \theta = +2$ and $\tan \theta$ is negative.

6. Find the values of the other functions if $\sin \theta = a$. Are all possible values of a admissible? State a reason for your answer.

7. If $\tan \theta = m$, find the values of the other functions.

8. If $\sec \theta = k$, find the values of the other functions. Are all values of k admissible in this problem? State a reason for your answer.

69. Trigonometric identities which involve functions of a single angle. By means of the relations of the preceding article, an expression which involves the trigonometric functions of an angle θ may be written in a great many different forms. It is often important to be able to recognize that two trigonometric expressions, although different in form, are really identical. This may frequently be done by inspection. In more complicated cases it is advisable to express each of the two quantities, whose identity we wish to establish, in terms of some one of the six functions (the sine, for example). It will then become evident as a mere matter of algebra whether or not the two quantities are really identical.

EXERCISE XXXVIII

1. Show that $\sec\theta - \tan\theta\sin\theta = \cos\theta$, for all values of θ for which $\tan\theta$ and $\sec\theta$ are defined. (See p. 142.)

Solution. We have, for all values of θ, for which $\tan\theta$ and $\sec\theta$ are defined,

$$\sec\theta - \tan\theta\sin\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\sin\theta = \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta,$$

which proves the truth of the original assertion.

2. Prove that $\tan A + \cot A = \sec A \csc A$ is an identity.*

Solution. Denote the quantity on the left member by L and that on the right member by R. Then

$$L = \tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A\cos A} = \frac{1}{\sin A\cos A},$$

$$R = \sec A\csc A = \frac{1}{\cos A}\,\frac{1}{\sin A} = \frac{1}{\sin A\cos A}.$$

Therefore $L = R$; that is,

$$\tan A + \cot A = \sec A\csc A. \qquad \text{Q. E. D.}$$

Prove that the following statements are identities:

3. $\cos\theta\tan\theta = \sin\theta$.

4. $\sin\phi\cot\phi = \cos\phi$.

5. $\sin^2\theta + \sin^2\theta\tan^2\theta = \tan^2\theta$.

6. $\cos^2 A - \sin^2 A = 1 - 2\sin^2 A$.

7. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1$.

8. $(\csc^2\theta - 1)\sin^2\theta = \cos^2\theta$.

9. $1 + \tan^2\theta = \dfrac{1}{1-\sin^2\theta}$.

10. $\cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$.

11. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$.

12. $\dfrac{\sec\theta\cot\theta - \csc\theta\tan\theta}{\cos\theta - \sin\theta} = \csc\theta\sec\theta$.

13. $\dfrac{\cos\theta\cot\theta - \sin\theta\tan\theta}{\csc\theta - \sec\theta} = 1 + \sin\theta\cos\theta$.

14. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$, if θ is an acute angle.

*In other words, show that the left member is equal to the right member for all values of A for which the functions $\tan A$, $\cot A$, $\sec A$, $\csc A$ have been defined; that is, for all values of A except $A = 0°, 90°, 180°, 360°$, etc.

CHAPTER X

GRAPHIC REPRESENTATIONS OF THE TRIGONOMETRIC FUNCTIONS

70. Line representation of the trigonometric functions. The trigonometric functions were *defined* as *ratios* or *abstract* numbers, not as lines. (See Art. 7.) This does not, however, preclude the possibility of *representing* them as lines. *An abstract or concrete quantity of any kind may be represented as a line-segment, by choosing arbitrarily a certain line-segment to represent a unit of the same kind.*

Thus we may represent as lines the populations of the various states of the Union, taking a line-segment one inch long to represent a population of 1,000,000. The populations of New York and Illinois will then be represented by line-segments 9.11 and 5.64 inches in length respectively. Thus, although a population obviously is not a line-segment, it may be *represented* by a line-segment. In the same way we may represent the values of the trigonometric functions by lines, although they are not lines, but abstract numbers.

The following is a convenient method for obtaining a representation of the values of the trigonometric functions as line-segments.

We construct a circle with the origin of coördinates as center. An angle whose vertex is at the center of this circle will subtend an arc whose numerical measure, in degrees, minutes, and seconds, is equal to that of the angle. We may therefore speak indifferently either about the functions of the *angle* or of the functions of the *arc*. The point in which the initial side of the angle meets the circle is called the *origin of the arc.* The point in which the terminal side of the angle meets the circle is called the *terminus, or the end of the arc.*

If an angle is placed in its standard position, the origin of the subtended arc will be at A, the point in which the positive x-axis meets the circle. We shall call this point the *primary origin of arcs.* The point B, in which the positive y-axis intersects the circle is called the *secondary origin of arcs.*

Let us choose any convenient unit of length, say an inch, and let us agree to measure all distances in terms of this unit. We then construct the circle, the so-called *unit circle*, whose center is at the origin of coördinates and whose radius is equal to the unit of length.

In Fig. 82, let $AOQ = \theta$ be any angle in its standard position, and AP the arc which it subtends on the unit circle. Then

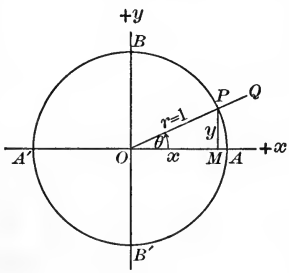

FIG. 82

$$(1) \qquad \sin\theta = \frac{y}{r} = \frac{y}{1} = y = MP,$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x = OM,$$

since the distance $r = OP$ is equal to the unit of length, so that $r = 1$. Now $x = OM$ and $y = MP$ are the coördinates of P, the terminus of the arc AP. Consequently we may express our result as follows:

If any angle θ is placed in its standard position, the value of its sine is equal, in magnitude and sign, to the ordinate of the terminus of the arc which the angle subtends on the unit circle. The value of its cosine is equal to the abscissa of the terminus of this arc.

In order to find a line representation for $\tan\theta$ and $\cot\theta$, we draw tangents to the unit circle at A and B and denote by T and T' the points in which the terminal side of the angle intersects these two tangents. (See Fig. 83.)

Then we find

$$\tan\theta = \frac{AT}{OA} = \frac{AT}{1} = AT,$$

$$\cot\theta = \tan BOT' = \frac{BT'}{OB} = \frac{BT'}{1} = BT',$$

since the radius of the circle

$$OA = OB = 1.$$

If θ is an obtuse angle, OP will have to be prolonged backward in order to intersect the tangent at A. Moreover the point of intersection T will then be below A. Now the tangents BT' and AT are parallel to the x- and y-axes respectively. Let us agree to give signs to the line-segments measured on these two tangents as though they were abscissas or ordinates of a point. That is, let AT be positive or negative according as T is above or below A, and let BT' be positive or negative according as T' is to the right or left of B. This convention is indicated in Fig. 83 by the two + signs at the ends of the two tangents. The student may now verify that the equations

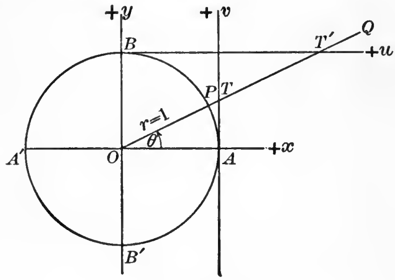

FIG. 83

(2) $$\tan\theta = AT,\ \cot\theta = BT',$$

which we have obtained from Fig. 83 in the case of an acute angle, will give correct results in *magnitude* and *sign*, no matter in what quadrant the angle θ may happen to fall.

We may formulate our results as follows:

If any angle θ is placed in its standard position, the value of its tangent is equal, in magnitude and sign, to the ordinate of the point in which the terminal side of the angle, prolonged backward if necessary, intersects the tangent to the unit circle at the primary origin of arcs.

The cotangent of the angle is equal, in magnitude and sign, to the abscissa of the point in which the terminal side of the angle, prolonged backward if necessary, intersects the tangent to the unit circle at the secondary origin of arcs.

Referring once more to Fig. 83, we have

$$\text{(3)}\quad \begin{aligned} \sec\theta &= \frac{OT}{OA} = \frac{OT}{1} = OT, \\ \csc\theta = \sec BOT'' &= \frac{OT''}{OB} = \frac{OT''}{1} = OT''. \end{aligned}$$

These line representations for the secant and cosecant will hold, in magnitude and sign, not merely for acute angles, but for angles in any quadrant, if we agree to make the following conventions in regard to sign. OT shall be positive if T is on the same side of O as P, *i.e.* if T is on the terminal side of the angle θ. OT shall be negative if T is on the terminal side of the angle θ prolonged backward. OT' shall be positive or negative according as T' falls on the terminal side of the angle θ or on the terminal side prolonged backward.

We leave it to the student as an exercise to verify these statements in detail and to formulate the contents of equations (3) in words.

Figures 84 to 87 illustrate the line representation of the

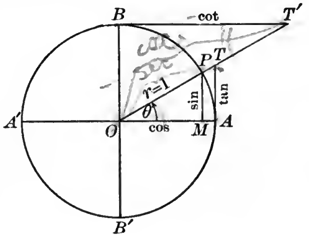

FIG. 84

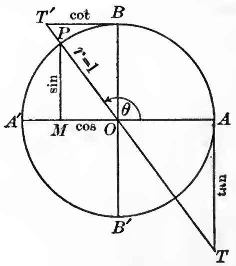

FIG 85

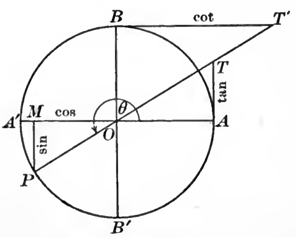

FIG. 86

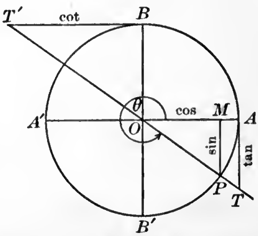

FIG. 87

trigonometric functions for an angle in each one of the four quadrants. In each of these figures

$$MP = \sin\theta,\quad AT = \tan\theta,\quad OT = \sec\theta,$$
$$OM = \cos\theta,\quad BT' = \cot\theta,\quad OT' = \csc\theta.$$

These line representations of tan θ and sec θ suffice to explain why the names tangent and secant were chosen for these functions. The word sine is not capable of such a simple explanation and has a long and complicated history.

The Greeks did not use the six functions which we have introduced. In place of the sine of an angle they made use of the chord PQ, subtended by the angle POQ, on a circle of known radius. (See Fig. 88.) If the circle has a unit radius, Fig. 88 shows that this chord PQ is equal to twice QR, or

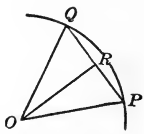

FIG. 88

$$PQ = 2\sin\tfrac{1}{2}POQ.$$

Thus the chord, used by the Greeks, is essentially twice the sine of half the angle.

ĀRYABHATA, a famous Hindoo mathematician (born 476 A.D.), was apparently the first to introduce the sine of the angle in place of the chord, and he, quite naturally, spoke of it as the half-chord or *jyā-ardhā*, where *jyā* is the Sanskrit for chord or bowstring and *ardhā* for one half. For the sake of brevity the adjective *ardhā* was soon omitted and the sine was called simply *jyā*.

The *Arabs*, who far more than any other people cultivated the sciences during the Middle Ages, took over this word from the Hindoos, but changed its spelling to *jība*, so as to make the spelling accord with the pronunciation in the sense of their own language.

But in written Arabic the consonants only are represented by definite characters, the vowels being merely indicated by dots which are frequently omitted altogether. As a consequence of this practice, the Hindoo word *jība* was soon corrupted into *jaīb*, a genuine Arabic word meaning *bosom*, *heart*, or *pocket* according to the context.

In the twelfth century, when the Arabic texts were translated into Latin, the word *jaīb* was translated literally by the Latin word *sinus* meaning bosom.

Thus, a foreign word was first converted by the Arabs into a word of their own language having a similar sound but an entirely different meaning, and later this Arabic word was translated literally into Latin. Of course the derivation of the English word *sine* from *sinus* is obvious.

71. Graphs of functions, a number of whose numerical values are given. There is a second way of representing graphically the values of the trigonometric functions, which is even more important than that which has just been discussed. For it gives us, in a still more vivid fashion, a picture of all the most essential properties of these functions; and it has the further advantage of being applicable, not merely to the trigonometric functions, but to all of the other functions which naturally arise in pure and applied mathematics. In order to lead the student to appreciate fully the power of this new method, we shall first illustrate it by a number of examples taken from fields other than trigonometry.

x = Date	y = Population
1790	3,929,214
1800	5,308,483
1810	7,239,881
1820	9,638,453
1830	12,860,692
1840	17,063,353
1850	23,191,876
1860	31,443,321
1870	38,558,371
1880	50,155,783
1890	62,947,714
1900	75,994,575
1910	91,972,266

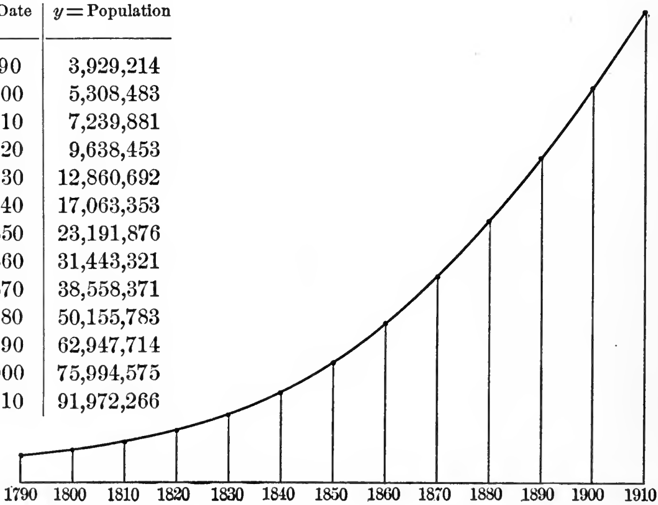

Fig. 89

The population of the United States is determined every ten years by a national census. The table in the margin gives the results of these censuses. It is customary to represent the contents of this table graphically by laying off the dates horizontally (*i.e.* as abscissas), and erecting for each of these dates a vertical line (ordinate), which shall give by its length in terms of an appropriately chosen unit the population at that time. Figure 89 gives such a graphic representation of the facts contained in this population table and presents these facts in a more easily intelligible form than the table itself. Moreover if we join the endpoints of the ordinates by a smooth curve (the population curve), we may draw some fairly reliable conclusions as to the state of the population in the years 1805, 1815, etc., in which no census was taken.

If we plot the population curves of two or more countries upon the same sheet, a great many interesting matters may be brought out by comparison.

Clearly we may adopt such a graphic method, whenever we have a table giving a relation between two variables; that is, a table which shows that to certain numerical values of a first quantity x there correspond certain numerical values of a second quantity y.

EXERCISE XXXIX

1. The following table gives the population of the cities of New York, Chicago, and Philadelphia for the years named:

	1850	1860	1870	1880	1890	1900	1910
New York .	515,547	805,651	942,292	1,206,299	1,515,301	3,437,202	4,766,883
Chicago . .	28,269	109,206	298,977	503,298	1,099,850	1,698,572	2,185,283
Philadelphia	340,045	585,529	674,022	847,170	1,046,964	1,293,697	1,549,008

Make a graph illustrating this information,* and from the graph find the probable population of each of these cities in 1908.

2. Make a population table and a population curve for the city and state in which you live.

3. Let the student provide himself with a railroad time-table, giving the names of the various stations, their distances from the starting point, and the times at which a certain train leaves these stations. Draw a distance-time diagram for one or several trains, plotting the times as abscissas and the distances as ordinates.

4. On April 3, 1912, the following temperatures were observed in Chicago:

3 A.M.	32°	11 A.M.	38°	7 P.M.	34°
4 A.M.	32°	12 M.	39°	8 P.M.	33°
5 A.M.	32°	1 P.M.	39°	9 P.M.	34°
6 A.M.	31°	2 P.M.	38°	10 P.M.	34°
7 A.M.	32°	3 P.M.	37°	11 P.M.	33°
8 A.M.	33°	4 P.M.	36°	12 P.M.	33°
9 A.M.	34°	5 P.M.	35°	1 A.M.	32°
10 A.M.	36°	6 P.M.	35°	2 A.M.	31°

Represent graphically.

* In all such work, involving plotting of curves, it is advisable to use cross-section paper.

72. Graphs of simple algebraic functions. The relation between the variables x and y, instead of being given by a table as in the previous examples, may be given by an equation. We may then use the equation for the purpose of constructing a table, and then draw a graph as before.

EXAMPLE 1. Find the graph of $y = 2x - 5$.

x	y
-2	-9
-1	-7
0	-5
$+1$	-3
$+2$	-1
$+3$	$+1$
$+4$	$+3$
$+5$	$+5$

Solution. If we substitute $x = 0$ in the given equation, we find $y = -5$; for $x = 1$, we find $y = -3$; etc. We construct in this way the table printed in the margin. If we plot the points $x = -2$, $y = -9$; $x = -1$, $y = -7$; etc., obtained in this way, we find the points marked in Fig. 90 with a little cross. All of these points are found to be on a straight line.

This observation makes it seem likely that *all* of the points whose coördinates satisfy the equation $y = 2x - 5$, not merely those which we happened to compute, are on this same straight line. It is not difficult to prove that this is so, but the proof will not be given here.

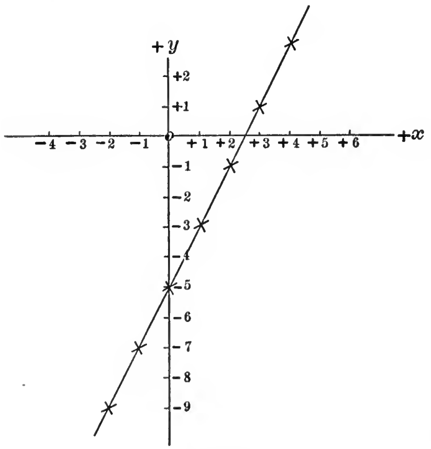

FIG. 90

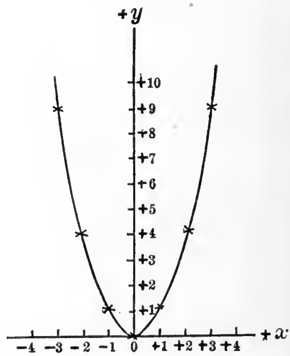

FIG. 91

EXAMPLE 2. Find the graph of $y = x^2$.

Solution. As before, we construct the table in the margin by computing the values of y which correspond to the values $x = -3, -2, -1, 0,$

x	y
-3	9
-2	4
-1	1
0	0
$+1$	1
$+2$	4
$+3$	9

$+1$, $+2$, $+3$. We then plot the points obtained in this way, the points marked with a little cross in Fig. 91, and unite them by a smooth curve.

The principle involved in these examples, that to every equation between two variables x and y there corresponds a curve and *vice versa*, is at the foundation of *Analytic Geometry*, which is one of the most important developments of modern mathematics. The great merit of having introduced this idea into mathematics is due to DESCARTES (1596–1650) and FERMAT (1601–1665).

EXERCISE XL

Find the graphs of the following equations:

1. $y = x$.

2. $y = -x$.

3. $y = -2x + 1$.

4. $y = 2x^2$.

5. $y = -x^2$.

6. $y = x^2 - 5$.

7. $y = x^2 - 5x$.

8. $y = 2x^2 - 3x + 1$.

9. $y = x^3$.

10. $y = \frac{1}{3}x^3 - 1$.

11. $y = \dfrac{1}{x}$.

12. $y = \dfrac{1}{x} - 1$.

73. Graphs of the trigonometric functions. We now proceed to apply this method to the relation

$$y = \sin x.$$

We choose an arbitrary line-segment on the x-axis to represent one degree and another arbitrary line-segment on the y-axis to represent the unit value of the sine, that is, the abstract number 1. If we are using millimeter paper, or a metric scale, it will be convenient to make a distance of one millimeter on the x-axis stand for one degree, and to measure the ordinates in terms of a unit 10 centimeters or 100 millimeters long. As this is a rather large scale it will probably be necessary to paste several sheets together in order to be able to construct the whole curve.

x	$y = \sin x$
0	0
10	17
20	34
30	50
40	64
50	76
60	87
70	94
80	98
90	100

From the table of natural sines we obtain the table in the margin, in which the values of the angle x as well as the corresponding values of $\sin x$ are expressed in millimetres in accordance with the adopted scale, which makes 1 mm. on the x-axis stand for 1°, and 100 mm. on the y-axis stand for the unit value of the sine. This table enables us to plot ten points of our curve representing ten values of the function $\sin x$ in the first quadrant. (See Fig. 92, which is a reduced copy of such a curve.)

The definition of the sine of a general angle (Art. 64) and the line representation of the sine in the unit circle (Art. 70), both show very clearly that two angles like 80° and 100°, or 70° and 110°, which

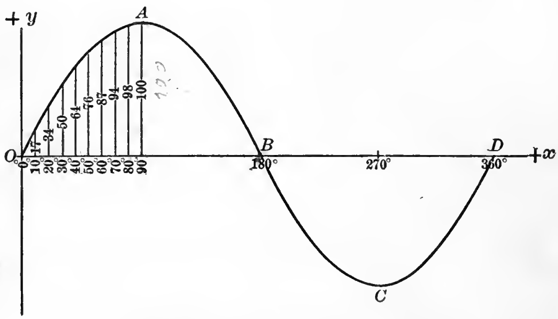

FIG. 92. — The Sine Curve

differ by the same amount from 90° but in opposite directions have the same sine. Consequently that portion AB of our curve, which represents the values of $\sin x$ for angles in the second quadrant, will be a symmetric counterpart of the first portion OA, which corresponds to angles in the first quadrant. (See Fig. 92.)

The unit circle also makes it evident that the sines of two angles which differ by 180° are numerically equal but opposite in sign. Consequently that portion BCD of our curve, which corresponds to angles in the third and fourth quadrants and all of whose ordinates are negative, may be obtained easily from the known part OAB. The parts OAB

and BCD of the curve are in fact congruent, but are situated on opposite sides of the x-axis.

We have already noted that the sine function repeats its values at intervals of 360°. It is a periodic function with a period of 360° (Art. 67). This manifests itself in the graph

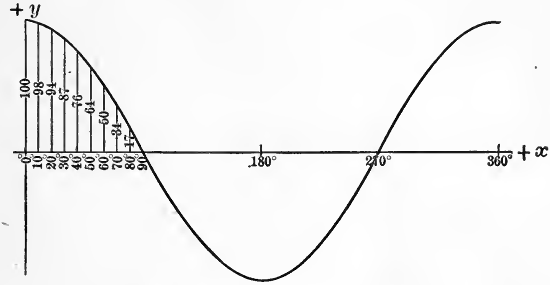

FIG. 93. — The Cosine Curve

by the fact that the picture in each of the intervals from 360° to 720°, etc., from − 360° to 0°, etc., is an exact copy of that piece of the curve which lies between 0° and 360°.

The curve obtained in this way from the sine function is called the **sine curve.**

The **cosine curve,** which is the graph of the function

$$y = \cos x,$$

is of the same general character as the sine curve. Its form is given in Fig. 93, and may be obtained by applying to the cosine function an argument exactly similar to that which has just been carried out for the sine.

We may apply the same method to the function

$$y = \tan x.$$

However, the graph obtained in this way differs very essentially from the sine and cosine curves.

In fact we know that when x approaches 90° from below, that is, if x assumes a succession of values like

$$89°,\ 89°.9,\ 89°.99,\ 89°.999, \text{ etc.},$$

the tangent of x, remaining always positive, will grow numerically beyond all bound. If on the other hand x approaches $90°$ from above, through a sequence of values like

$$91°,\ 90°.1,\ 90°.01,\ 90°.001, \text{ etc.},$$

the tangent of x, remaining always negative, again grows numerically beyond all bound. (Cf. Art. 65.) We see, therefore, that the difference between the values of

$$\tan(90° + h) \text{ and } \tan(90° - h)$$

grows larger and larger as the angles

$$90° + h \text{ and } 90° - h$$

themselves come closer and closer together.

We express this by saying that *the function* $\tan x$ *is discontinuous for* $x = 90°$. The corresponding property of the graph is an interruption or break in the otherwise continuous curve. There are no such breaks in the sine or cosine curves. We can think of a material point (say the point of a lead pencil) as actually describing a sine curve without interruption. If we were to attempt to do the same thing for the tangent curve, we should have to interrupt the path of the point at $x = 90°$, at $x = 270°$, etc.

We meet here *the important distinction between continuous and discontinuous functions*, the precise formulation of which must be left to a later point in the student's career.

The tangent is, of course, a periodic function and repeats its values at intervals of $360°$. But we may now observe that, unlike the sine or cosine, it repeats its values after the shorter interval of $180°$. To recapitulate: *the tangent is a periodic function of period* $180°$, *and is discontinuous for* $x = 90°$ *and for all values of* x *which differ from* $90°$ *by integral multiples of* $180°$.

Figure 94 shows the form of the tangent curve.

EXERCISE XLI

1. Plot the curves $y = 2 \sin x$, $y = 3 \sin x$, $y = 4 \sin x$.

2. Plot the curves $y = \sin 2x$, $y = \sin 3x$, $y = \sin 4x$.

3. How are the curves of Exs. 1 and 2 related to the curve $y = \sin x$?

4. Show that the curve $y = \cot x$ is discontinuous for $x = 0°$, $180°$, $360°$, etc., and has the form indicated in Fig. 95.

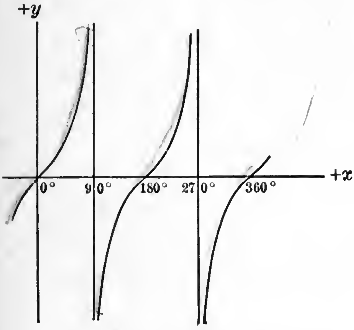

Fig. 94. — The Tangent Curve

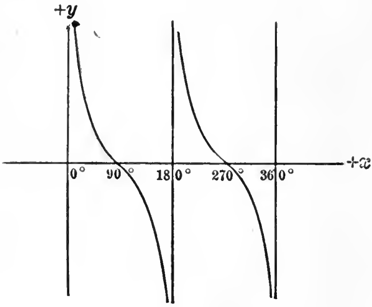

Fig. 95. — The Cotangent Curve

5. Show that the curves $y = \sec x$ and $y = \csc x$ have the forms indicated in Figs. 96 and 97.

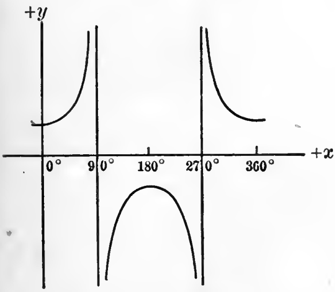

Fig. 96. — The Secant Curve

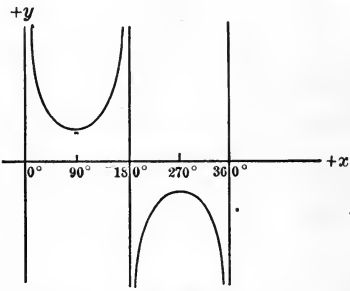

Fig. 97. — The Cosecant Curve

74. The natural unit of circular measurement. Definition of a radian. In constructing the graphs of the trigonometric functions, the student may have observed that the units of measurement on the x- and y-axes were *both* chosen arbitrarily, and might have been selected in infinitely many different ways, thus altering materially the appearance of the resulting curve.

This mutual independence of the two scales, on the two coördinate axes, is a natural consequence of the fact that the two quantities x and y were regarded as different in kind. One of them was regarded as an angle measured in degrees, and the other as an abstract number. Having chosen a certain horizontal line-segment as representative of the unit of angles (1°), it was still admissible to choose arbitrarily another (vertical) line-segment to represent the unit of abstract numbers. Whenever x and y represent two quantities of different kind, the x- and y-scales are, in the nature of things, independent of each other.

Even if x and y are quantities of the same kind, it is often more convenient to choose the lengths of the units different on the two scales. If this were not done, the resulting curve might fail utterly to serve the purposes for which it was intended.

Thus, when we draw a profile map of an extensive country (showing the elevations of various points in a certain vertical cross section), the vertical scale must be chosen much larger than the horizontal scale. Otherwise the differences of elevation, as depicted on the map, would become so small as to be unnoticeable.

Nevertheless it will usually be desirable to choose the horizontal and vertical units equal to each other whenever x and y may be regarded as quantities of the same kind, provided that the resulting curve does not thereby lose its usefulness as it would in the example just quoted.

Now the considerations of Art. 70 show that, from a certain point of view, the quantities x and y which occur in such an equation as

$$y = \sin x$$

may be regarded as quantities of the same kind.

In fact, we observed in Art. 70 that we might think of the number x as the measure of the *arc AP* (Fig. 82) instead of as the measure of the corresponding *angle AOP*. We saw further that certain line-segments could be constructed whose lengths, in terms of the radius of the circle as unit, were equal to the values of $\sin x$, $\cos x$, etc. If then we measure the length of the arc x in terms of the radius of the circle as unit, instead of in degrees, we shall have the arc and its trigonometric functions expressed in terms of the same unit.

This unit of arc measure, an arc of a circle whose length is equal to the radius of the circle, is called a **radian.**

One advantage gained by measuring arcs in radians is this: the arc and its trigonometric functions will then be expressed in terms of the same unit.

If r is the radius of a circle, the length of its circumference is equal to $2\pi r$. Therefore a circumference may be said to contain 2π radians. Since it also contains 360 degrees, we have

(1) $$2\pi \text{ radians} = 360°,$$

whence

$$1 \text{ radian} = \frac{360°}{2\pi} = \frac{180°}{\pi}.$$

Since $\pi = 3.14159265$, we find, to seven decimal places,

(2) $$1 \text{ radian} = 57°.2957795.$$

On the other hand we find from (1)

(3) $$1° = \frac{\pi}{180} \text{ radians},$$

or

(4) $$1° = 0.0174533 \text{ radian}.$$

These equations make it easy to find the number of degrees in an angle or arc when its measure is given in radians, or *vice versa.*

Clearly it follows, from the definition of a radian, that the length of the arc which an angle of one radian intercepts on the circumference of a circle of radius r is itself equal to r. Then, an angle of half a radian at the center will intercept an arc on the circumference whose length is equal to $\frac{1}{2}r$.

In general: *an angle of θ radians at the center of a circle of radius r intercepts an arc s upon the circumference whose length is*

(5) $$s = r\theta.$$

In the simplicity of this formula lies a second great advantage of measuring angles in radians rather than in degrees.

EXERCISE XLII

Convert into radians the following angles:

1. $90°$. **3.** $30°$. **5.** $+693° 20'$.

2. $270°$. **4.** $+25° 15'$. **6.** $-1030° 0'$.

Convert into degrees the following angles which are given in radians, state their quadrants and the signs of their trigonometric functions:

7. $\frac{\pi}{2}$. **9.** $\frac{\pi}{16}$. **11.** 0.7691.

8. $\frac{\pi}{4}$. **10.** 3.14159. **12.** 5.3214.

13. Prove that the area of a sector of a circle of radius r is equal to $\frac{1}{2}r^2\theta$ if θ, the angle at the center, is measured in radians.

14. Prove that a segment of a circle of radius r, whose arc is equal to θ radians, has the area

$$\tfrac{1}{2}r^2(\theta - \sin\theta).$$

15. Compute the area of a circular segment of radius 11 feet, if its arc is equal to 52°.

16. How will the formulæ of Exs. 13 and 14 be modified, if the angle θ is expressed in degrees?

17. A cord is stretched around two wheels, a large one of radius r and a smaller one of radius r' feet, the distance between the centers of the wheels being d feet. If the cord is not crossed and if θ is the angle of inclination, expressed in radians, of the free part of the cord to the line of centers of the wheels, show that

$$\sin\theta = \frac{r - r'}{d}, \quad l = 2\left[d\cos\theta + \left(\frac{\pi}{2} + \theta\right)r + \left(\frac{\pi}{2} - \theta\right)r'\right],$$

where l is the entire length of the cord.

18. If the cord in Ex. 17 is crossed, show that its length l may be found by means of the formulæ

$$\sin\theta = \frac{r + r'}{d}, \; l = 2\left[d\cos\theta + \left(\frac{\pi}{2} + \theta\right)(r + r')\right].$$

19. Find the length of a belt which is to be stretched around two wheels 3 and 2 feet in diameter respectively, if the distance between the centers of the two wheels is 5 feet: (*a*) if the belt is crossed, (*b*) if it is not crossed.

20. Draw the graphs of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, if x is expressed in radians, using the same unit of length for x distances and y distances.

75. Relations between the functions of two symmetrical angles. Let us consider two angles, like $90°-\theta$ and $90°+\theta$, which differ from one of the cardinal angles by the same amount but in opposite directions. If we place two such angles in the standard position, their terminal sides will be symmetrically situated with respect to one of the two coördinate axes, so that this axis will bisect the angle between them. As a consequence of this fact the trigonometric functions of the two angles are related to each other in a very simple fashion.

In order to obtain these relations we shall consider each of the four cardinal angles separately, making use of the line representation of the functions given in Art. 70.

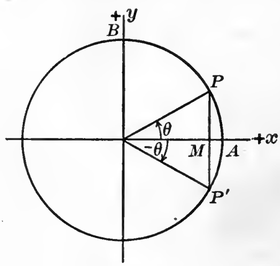

FIG. 98

Let us begin with the cardinal angle 0° or 0 radians. Figure 98 represents the unit circle and the two angles

$$\theta = \angle AOP \text{ and } -\theta = \angle AOP',$$

each in its standard position. The arcs AP and AP' subtended by these angles on the unit circle are symmetrical with respect to A. Therefore the ordinates of P and P' (the termini of these arcs) are numerically equal and opposite in sign, while their abscissas are equal in magnitude and sign.

But the ordinate and abscissa of the terminus of the arc AP are respectively equal to the sine and cosine of θ. The ordinate and abscissa of P' are respectively equal to the sine and cosine of $-\theta$. (See Art. 70.)

Consequently we have

(1) $$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta.$$

Consider next the case of two angles $90° - \theta$ and $90° + \theta$ symmetric with respect to the cardinal angle $90°$. In this case (see Fig. 99) P and P' have the same ordinate, while their abscissas are numerically equal but opposite in sign. Consequently

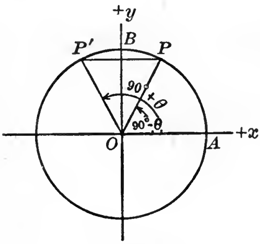

FIG. 99

$$(2\,a)\qquad \begin{aligned}\sin(90° - \theta) &= \sin(90° + \theta),\\ \cos(90° - \theta) &= -\cos(90° + \theta),\end{aligned}$$

or, if the angles are measured in radians,

$$(2\,b)\qquad \sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right),\ \cos\left(\frac{\pi}{2} - \theta\right) = -\cos\left(\frac{\pi}{2} + \theta\right).$$

By a precisely similar argument, the details of which we leave to the student, we find

$$(3\,a)\qquad \begin{aligned}\sin(180° - \theta) &= -\sin(180° + \theta),\\ \cos(180° - \theta) &= \cos(180° + \theta);\end{aligned}$$

or,

$$(3\,b)\qquad \begin{aligned}\sin(\pi - \theta) &= -\sin(\pi + \theta),\\ \cos(\pi - \theta) &= \cos(\pi + \theta),\end{aligned}$$

according as the angle is measured in degrees or radians, and also

$$(4\,a)\qquad \begin{aligned}\sin(270° - \theta) &= \sin(270° + \theta),\\ \cos(270° - \theta) &= -\cos(270° + \theta);\end{aligned}$$

or,

$$(4\,b)\qquad \begin{aligned}\sin\left(\frac{3\,\pi}{2} - \theta\right) &= \sin\left(\frac{3\,\pi}{2} + \theta\right),\\ \cos\left(\frac{3\,\pi}{2} - \theta\right) &= -\cos\left(\frac{3\,\pi}{2} + \theta\right).\end{aligned}$$

Of course we shall also have

$$\begin{aligned}\sin(360° - \theta) &= -\sin(360° + \theta),\\ \cos(360° - \theta) &= \cos(360° + \theta).\end{aligned}$$

But these equations are really repetitions of (1) if we remember that, on account of the periodicity of the sine and cosine,

$$\sin(360° + \theta) = \sin\theta,\ \sin(360° - \theta) = \sin(-\theta),$$
$$\cos(360° + \theta) = \cos\theta,\ \cos(360° - \theta) = \cos(-\theta).$$

The relations, which correspond to (1), (2), (3), (4) for the remaining trigonometric functions, may easily be obtained by expressing the tangent, cotangent, secant, and cosecant in terms of the sine and cosine. (See Art. 68.) Thus, for instance,

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

The student should actually work out the sixteen equations obtainable in this way and combine them in tabular form with the eight equations (1) to (4).

The student should also observe that, although we have constructed the figures for the case when θ is a positive acute angle, our proof of formulæ (1), (2), (3), (4) remains valid word for word, if θ is a positive or negative angle of any magnitude. A good way to convince one's self of this fact is to think of the angle θ as variable and to follow out mentally the changes which would take place in such a figure as Fig. 98 or Fig. 99 when the angle θ increases or decreases. The fact that equations (1) to (4) are universally valid will thus be rendered intuitive.

76. Relations between functions of two angles whose sum or difference is a right angle. If θ is an acute angle, we know, from Art. 10, that

(1) $$\sin(90° - \theta) = \cos\theta,\ \cos(90° - \theta) = \sin\theta.$$

If we combine these equations with (2 a) of Art. 75, we find further

(2) $$\sin(90° + \theta) = \cos\theta,\ \cos(90° + \theta) = -\sin\theta.$$

We wish to show that the four equations (1) and (2) are true, not merely when θ is an acute angle, but when θ is a positive or negative angle of any magnitude. This may be done by the *method of mathematical induction.*

We begin by proving the following theorem. *If equations* (1) *and* (2) *are true for a certain angle* θ, *they are also true for the angle* $\theta' = 90° + \theta$.

PROOF. By hypothesis, equations (1) and (2) are true for the angle θ. Therefore we have

$$\sin \theta' = \sin (90° + \theta) = \cos \theta, \quad \cos \theta' = \cos (90° + \theta) = -\sin \theta.$$

But

$$\sin (90° - \theta') = \sin [90° - (90° + \theta)] = \sin(-\theta) = -\sin \theta,$$

and

$$\cos (90° - \theta') = \cos [90° - (90° + \theta)] = \cos (-\theta) = \cos \theta,$$

(Art. 75, equations (1)),

which proves that

$$(3) \qquad \sin (90° - \theta') = \cos \theta', \quad \cos (90° - \theta') = \sin \theta',$$

since both members of the first equation are equal to $-\sin \theta$, and both members of the second are equal to $\cos \theta$.

Since equations (2 *a*) of Art. 75 are true for all angles, we now find

$$(4) \qquad \begin{aligned} \sin (90° + \theta') &= \sin (90° - \theta') = \cos \theta', \\ \cos (90° + \theta') &= -\cos (90° - \theta') = -\sin \theta'. \end{aligned}$$

Since equations (3) and (4) are the same as (1) and (2), with θ' in place of θ, we have actually proved our theorem; namely, *if* equations (1) and (2) are true for the angle θ, they are also true for the angle $\theta' = 90° + \theta$.

We know that equations (1) and (2) *are* true for all positive acute angles. As a consequence of the theorem just proved, they are successively seen to be true for all positive angles in the second, third, or fourth quadrant, and consequently for all positive angles whatever.

But they are also true for all negative angles. For let θ be a negative angle. Let $n \cdot 360°$, where n is a positive integer, be the lowest integral multiple of 360° which makes

$$\theta' = \theta + n \cdot 360°$$

a positive angle. Then, on account of the periodic character of the sine and cosine, we shall have

$$(5) \qquad \begin{aligned} \sin\theta' &= \sin(\theta + n \cdot 360^\circ) = \sin\theta, \\ \cos\theta' &= \cos(\theta + n \cdot 360^\circ) = \cos\theta, \end{aligned}$$

and similarly

$$(6) \qquad \begin{aligned} &\sin(90^\circ - \theta') = \sin(90^\circ - \theta), \quad \cos(90^\circ - \theta') = \cos(90^\circ - \theta), \\ &\sin(90^\circ + \theta') = \sin(90^\circ + \theta), \quad \cos(90^\circ + \theta') = \cos(90^\circ + \theta). \end{aligned}$$

Since θ' is a positive angle, we have

$$\begin{aligned} &\sin(90^\circ - \theta') = \cos\theta', \quad \cos(90^\circ - \theta') = \sin\theta', \\ &\sin(90^\circ + \theta') = \cos\theta', \quad \cos(90^\circ + \theta') = -\sin\theta'. \end{aligned}$$

If in these equations we substitute the values (5) and (6), we find

$$\begin{aligned} &\sin(90^\circ - \theta) = \cos\theta, \quad \cos(90^\circ - \theta) = \sin\theta, \\ &\sin(90^\circ + \theta) = \cos\theta, \quad \cos(90^\circ + \theta) = -\sin\theta. \end{aligned}$$

Therefore *equations* (1) *and* (2) *are true for positive and negative angles of any magnitude.*

The formulæ for $\tan(90^\circ + \theta)$, $\sec(90^\circ + \theta)$, etc., may be found by expressing these functions of $90^\circ + \theta$ in terms of $\sin(90^\circ + \theta)$ and $\cos(90^\circ + \theta)$ and making use of (2); for instance, we find

$$\tan(90^\circ + \theta) = \frac{\sin(90^\circ + \theta)}{\cos(90^\circ + \theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

77. The quadrantal formulæ. If we unite equations (1) of Art. 76 with the corresponding formulæ for the remaining four functions, we obtain the following system of equations:

$$(1) \qquad \begin{cases} \sin(90^\circ - \theta) = \cos\theta, & \cos(90^\circ - \theta) = \sin\theta, \\ \tan(90^\circ - \theta) = \cot\theta, & \cot(90^\circ - \theta) = \tan\theta, \\ \sec(90^\circ - \theta) = \csc\theta, & \csc(90^\circ - \theta) = \sec\theta. \end{cases}$$

In the same way we find, from equations (2) of Art. 76, the system:

$$(2) \qquad \begin{cases} \sin(90^\circ + \theta) = \cos\theta, & \cos(90^\circ + \theta) = -\sin\theta, \\ \tan(90^\circ + \theta) = -\cot\theta, & \cot(90^\circ + \theta) = -\tan\theta, \\ \sec(90^\circ + \theta) = -\csc\theta, & \csc(90^\circ + \theta) = +\sec\theta. \end{cases}$$

Since these equations are true for angles of any magnitude and not merely for acute angles, we conclude from (1) and (2) that

$$\sin(180° - \theta) = \sin(90° + \overline{90° - \theta}) = \cos(90° - \theta) = \sin\theta,$$
$$\cos(180° - \theta) = \cos(90° + \overline{90° - \theta}) = -\sin(90° - \theta) = -\cos\theta,$$

etc., giving rise to the further system of equations:

$$(3)\quad \begin{cases} \sin(180° - \theta) = \sin\theta, & \cos(180° - \theta) = -\cos\theta, \\ \tan(180° - \theta) = -\tan\theta, & \cot(180° - \theta) = -\cot\theta, \\ \sec(180° - \theta) = -\sec\theta, & \csc(180° - \theta) = \csc\theta. \end{cases}$$

In similar fashion we find

$$\sin(180° + \theta) = \sin(90° + \overline{90° + \theta}) = \cos(90° + \theta) = -\sin\theta,$$
$$\cos(180° + \theta) = \cos(90° + \overline{90° + \theta}) = -\sin(90° + \theta)$$
$$= -\cos\theta, \text{ etc.,}$$

so that we obtain

$$(4)\quad \begin{cases} \sin(180° + \theta) = -\sin\theta, & \cos(180° + \theta) = -\cos\theta, \\ \tan(180° + \theta) = \tan\theta, & \cot(180° + \theta) = \cot\theta, \\ \sec(180° + \theta) = -\sec\theta, & \csc(180° + \theta) = -\csc\theta. \end{cases}$$

Again we have

$$\sin(270° - \theta) = \sin(90° + \overline{180° - \theta}) = \cos(180° - \theta)$$
$$= -\cos\theta, \text{ etc.,}$$

whence

$$(5)\quad \begin{cases} \sin(270° - \theta) = -\cos\theta, & \cos(270° - \theta) = -\sin\theta, \\ \tan(270° - \theta) = \cot\theta, & \cot(270° - \theta) = \tan\theta, \\ \sec(270° - \theta) = -\csc\theta, & \csc(270° - \theta) = -\sec\theta, \end{cases}$$

and similarly

$$(6)\quad \begin{cases} \sin(270° + \theta) = -\cos\theta, & \cos(270° + \theta) = \sin\theta, \\ \tan(270° + \theta) = -\cot\theta, & \cot(270° + \theta) = -\tan\theta, \\ \sec(270° + \theta) = \csc\theta, & \csc(270° + \theta) = -\sec\theta. \end{cases}$$

Finally we find the equations

$$(7)\quad \begin{cases} \sin(360° - \theta) = -\sin\theta, & \cos(360° - \theta) = \cos\theta, \\ \tan(360° - \theta) = -\tan\theta, & \cot(360° - \theta) = -\cot\theta, \\ \sec(360° - \theta) = \sec\theta, & \csc(360° - \theta) = -\csc\theta, \end{cases}$$

and the system

$$(8)\quad \begin{cases} \sin(360^\circ+\theta)=\sin\theta, & \cos(360^\circ+\theta)=\cos\theta,\\ \tan(360^\circ+\theta)=\tan\theta, & \cot(360^\circ+\theta)=\cot\theta,\\ \sec(360^\circ+\theta)=\sec\theta, & \csc(360^\circ+\theta)=\csc\theta,\end{cases}$$

which latter equations merely express the periodic character of the trigonometric functions.

Since, on account of the periodicity of the functions, we have

$$\sin(360^\circ-\theta)=\sin(-\theta+360^\circ)=\sin(-\theta), \text{ etc.,}$$

we may also write, in place of (7),

$$(9)\quad \begin{cases} \sin(-\theta)=-\sin\theta, & \cos(-\theta)=\cos\theta,\\ \tan(-\theta)=-\tan\theta, & \cot(-\theta)=-\cot\theta,\\ \sec(-\theta)=\sec\theta, & \csc(-\theta)=-\csc\theta.\end{cases}$$

The 48 formulæ (1) to (8), the so-called **quadrantal formulæ**, have a very important practical application. *They serve the purpose of finding the values of the functions of angles not situated in the first quadrant.*

For instance, if we wish to find the sine and cosine of 310°, we may use equations (6), which give

$$\sin 310^\circ=\sin(270^\circ+40^\circ)=-\cos 40^\circ,$$
$$\cos 310^\circ=\cos(270^\circ+40^\circ)=\sin 40^\circ.$$

The numerical values of cos 40° and sin 40° may, of course, be taken from the tables.

On account of the practical application just mentioned, it is important to be able to remember the 48 quadrantal formulæ. This is not at all difficult if we impress upon our minds some of their peculiarities.

We observe in the first place that all of the angles which appear in the left members of these equations are of the form

a multiple of $90^\circ \pm \theta$, that is, *a cardinal angle* $\pm\,\theta$,

while the angle which appears in the right member is always simply θ.

Let us speak of those cardinal angles, like 90° and 270°, which are odd multiples of 90° as *odd cardinal angles*, while the *even cardinal angles*, like 0° and 180°, are even multiples of 90°.

If now we look through our list of quadrantal formulæ, we observe that they are all included in one of the two forms:

$$(10)\quad \begin{cases} \textit{function of } (\textbf{even } \textit{cardinal angle } \pm \theta) \\ \qquad\qquad = \pm\, \textbf{same } \textit{function of } \theta, \\ \textit{function of } (\textbf{odd } \textit{cardinal angle } \pm \theta) \\ \qquad\qquad = \pm\, \textit{corresponding } \textbf{co-}\textit{function of } \theta, \end{cases}$$

it being understood, as in Art. 10, that the six functions are arranged in three pairs, sine and cosine, tangent and cotangent, secant and cosecant, each member of each pair being regarded as the co-function of the other.

We have observed, then, that *the same function occurs in both members of a quadrantal formula whenever the corresponding cardinal angle is even. If the cardinal angle is odd, the function which appears in the right member is the co-function of that one which appears on the left.*

It remains to describe a method for remembering which of the two signs, + or −, should be used in any one of these formulæ. We may determine this sign by thinking of the particular case when θ is a positive acute angle. The quadrant of the angle in the left member of the equation will then be evident by inspection, and therefore also the sign of the left member. The ambiguous sign ±, on the right member, must then be chosen in such a manner as to make the two members of the equation agree in sign.

An example will make this clear. We wish to find the formula for $\sin(270° + \theta)$. Since 270° is an odd cardinal angle, we have in the first place

$$\sin(270° + \theta) = \pm \cos\theta.$$

To determine the sign on the right member, we think of the case when θ is a positive acute angle. Then $270° + \theta$ is in the fourth quadrant, and $\sin(270° + \theta)$ is negative, while $\cos\theta$, being the cosine of a positive acute angle, is positive. Therefore we must choose the − sign, so that

$$\sin(270° + \theta) = -\cos\theta,$$

since choice of the + sign would lead to the absurdity of equating a positive to a negative number.

EXERCISE XLIII

Express the following as functions of positive acute angles :

1. $\cos(-75°)$. **3.** $\tan 517°$. **5.** $\cot 175°$.

2. $\sin 325°$. **4.** $\csc(-412°)$. **6.** $\sec 1562°$.

7. Express the functions of Exs. 1–6 as functions of positive acute angles less than 45°.

8. Compute the value of the expression $2\sin(3\theta + 10°)$ for the values of $\theta = 25°, 50°, 75°, 100°$.

Find the values of the following expressions :

9. $\cos 60° \cos 120° - \sin 60° \sin 120°$.

10. $\sin 30° \cos 300° + \cos 30° \sin 300°$.

11. $\tan \frac{17\pi}{6} \tan \frac{14\pi}{3} + \cot\left(\frac{-11\pi}{6}\right) \cot\left(\frac{-4\pi}{3}\right)$.

12. $\cos 315° \sin 11° - \tan 293° \sec 25°$.

Simplify the following expressions :

13. $a^2 + b^2 + 2ab\cos(180° - x)$.

14. $(a - b)\tan(90° + A) + (a + b)\cot(-A)$.

15. $a\sin\left(\frac{3\pi}{2} - \theta\right) + b\cos(\pi - \theta)$.

16. $\tan\theta + \tan(\pi - \theta)$.

State for what values of θ each of the following expressions is positive, and for what values of θ it is negative :

17. $\sin\theta - \cos\theta$. **19.** $\sin^2\theta - \cos^2\theta$.

18. $\sin\theta + \cos\theta$. **20.** $\tan\theta - \cot\theta$.

21. Find the formulæ for the functions of $\theta - \frac{\pi}{2}$ in terms of the functions of θ, the angle θ being measured in radians.

22. Find formulæ for the functions of $\theta - \pi$ in terms of the functions of θ.

78. Properties of the sine and cosine curves. The properties of the sine and cosine which were discussed in Arts. 75, 76, 77 manifest themselves very clearly if we make use of the graphs of these functions as obtained in Art. 73. We shall slightly modify these graphs, however, by thinking of the angle x as being measured in radians (see Art. 74) rather than in degrees, and by choosing the same unit of length to

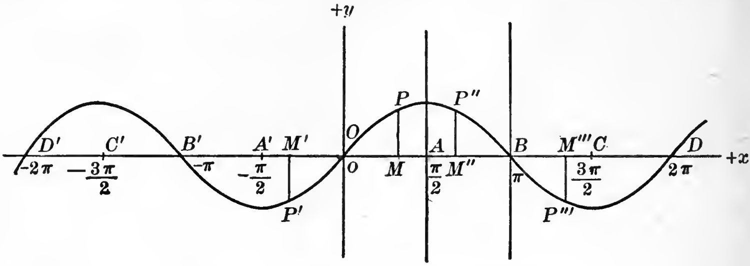

FIG. 100. — The Sine Curve. Natural Scale

represent one radian on the x-axis as that which represents the abstract number 1 on the y-axis.

Figure 100 represents the sine curve

$$y = \sin x$$

constructed in accordance with this choice of units.

Let us consider two points of this curve which are at the same distance from the y-axis but on opposite sides, such as P and P'. The ordinates of these points are obviously numerically equal but opposite in sign. Denote the abscissa of P by z; then, that of P' will be $-z$, and we find that

(1). $$\sin(-z) = -\sin z.$$

If we draw a line parallel to the y-axis through the point A for which $x = \frac{\pi}{2}$, this line clearly divides the curve into two symmetrical portions. Consequently two points, such as P and P'', at the same distance from this line but on opposite sides of it, will have equal ordinates. That is,

$$\sin OM = \sin OM''.$$

But if we denote MA by z, we have

$$OM = \frac{\pi}{2} - z, \quad OM'' = \frac{\pi}{2} + z,$$

so that

(2). $$\sin\left(\frac{\pi}{2} - z\right) = \sin\left(\frac{\pi}{2} + z\right).$$

Let the curve be divided into two portions by a line parallel to OY through the point B whose abscissa is equal to π. Points of the curve at equal distances from this line but on opposite sides of it, such as P'' and P''', have ordinates numerically equal but opposite in sign. Therefore

$$(3)_s \qquad \sin(\pi - z) = -\sin(\pi + z).$$

In similar fashion we find

$$(4)_s \qquad \sin\left(\frac{3\pi}{2} - z\right) = \sin\left(\frac{3\pi}{2} + z\right),$$

$$(5)_s \qquad \sin(2\pi - z) = -\sin(2\pi + z).$$

The points M and M'' were equidistant from A. Consequently the distances OM and $M''B$ are equal. Since the ordinates MP and $M''P''$ are equal, we shall have

$$\sin OM'' = \sin OM.$$

If we put $OM = z$, we shall have $M''B = z$, and therefore

$$OM'' = \pi - z,$$

so that the preceding equation becomes

$$(6)_s \qquad \sin(\pi - z) = \sin z.$$

By similar considerations in connection with the cosine curve, we find a system of relations which correspond completely to the above equations $(1)_s$ to $(6)_s$. They are

$$(1)_c \qquad \cos(-z) = \cos z,$$

$$(2)_c \qquad \cos\left(\frac{\pi}{2} - z\right) = -\cos\left(\frac{\pi}{2} + z\right),$$

$$(3)_c \qquad \cos(\pi - z) = \cos(\pi + z),$$

$$(4)_c \qquad \cos\left(\frac{3\pi}{2} - z\right) = -\cos\left(\frac{3\pi}{2} + z\right),$$

$$(5)_c \qquad \cos(2\pi - z) = \cos(2\pi + z),$$

$$(6)_c \qquad \cos(\pi - z) = -\cos z.$$

The truth of all these equations which are thus suggested by the curves has been established, in a slightly different notation, in Arts. 75–77.

We can hardly fail to notice the striking similarity between the sine and cosine curves. In order to put into evidence the relations between them, we construct Fig. 101 which contains them both.

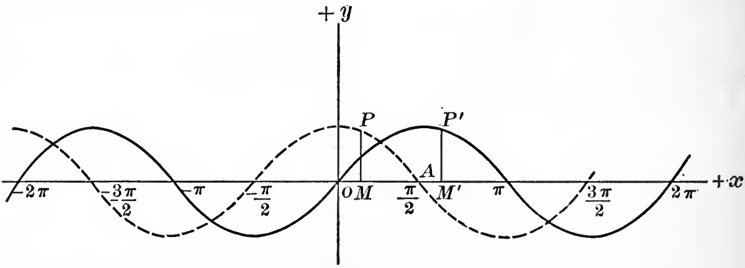

FIG. 101

This figure suggests that, if we displace the cosine curve toward the right through a distance of $\frac{\pi}{2}$ units, it will coincide with the sine curve. If this is true, a point P of the cosine curve whose coördinates are

$$(a)\qquad OM = z,\ MP = \cos z,$$

and which by our displacement will be brought into coincidence with a point P' whose coördinates are

$$(b)\qquad OM' = z + \frac{\pi}{2},\ M'P' = MP,$$

should, in its new position, be a point on the sine curve. Therefore the coördinates, OM' and $M'P'$, of this point P' should satisfy the relation $y = \sin x$ which is satisfied by the coördinates of all points of the sine curve. This gives

$$(c)\qquad M'P' = \sin\left(z + \frac{\pi}{2}\right),$$

and, on combination with (a) and (b),

$$(7).\qquad \sin\left(z + \frac{\pi}{2}\right) = M'P' = MP = \cos z.$$

Thus, equation $(7)_8$ must be true if the geometric relation between the sine and cosine curves suggested by Fig. 101 is actually based on fact. But this equation coincides with formula (2) of Art. 76, except for the notation, so that its validity is no longer open to question.

Consequently, *the sine and cosine curve differ only in position and may be brought into coincidence by a displacement of* $\frac{\pi}{2}$ *units parallel to the x-axis.*

EXERCISE XLIV

1. What geometric property of Fig. 101 corresponds to the relations

$$\sin\left(\frac{\pi}{2} - z\right) = \cos z,\quad \cos\left(\frac{\pi}{2} - z\right) = \sin z?$$

2. How are the relations

$$\sin\left(\frac{3\pi}{2} + z\right) = -\cos z,\quad \cos\left(\frac{3\pi}{2} + z\right) = \sin z$$

to be obtained from Fig. 101?

3. Plot the tangent and cotangent curves and discuss these graphs in a fashion analogous (so far as possible) to the discussion of Art. 78.

CHAPTER XI

RELATIONS BETWEEN THE FUNCTIONS OF MORE THAN ONE ANGLE

79. The addition theorems for sine and cosine. In Art. 77 we expressed $\sin\left(\theta + \frac{\pi}{2}\right)$, $\cos\left(\theta + \frac{\pi}{2}\right)$, etc., in terms of $\sin\theta$ and $\cos\theta$. The angle θ was an angle of any magnitude, but the angle added to it was always an integral multiple of $\frac{\pi}{2}$ radians or 90°. The question now arises whether it is possible to find similar formulæ for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, where both α and β are angles of any magnitude.

Let us assume, to begin with, that α and β are both positive acute angles whose sum $\alpha + \beta$ is also acute. We place the acute angle α in its standard position xOA. (See Fig. 102.) We then place the angle β with its initial side upon OA (the terminal side of the angle α), so as to make $\angle AOB$ equal to β. Then

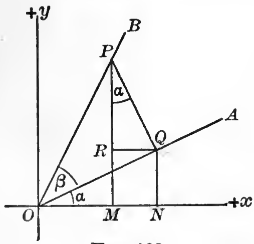

FIG. 102

$$\angle xOB = \alpha + \beta,$$

and moreover this angle is in its standard position. Therefore if we take any point P, different from O, on its terminal side OB and drop a perpendicular PM from P to the x-axis, we shall have

$$(1) \qquad \sin(\alpha + \beta) = \frac{MP}{OP}, \quad \cos(\alpha + \beta) = \frac{OM}{OP}.$$

Let us drop perpendiculars PQ, QN, and QR from P to OA, from Q to the x-axis, and from Q to MP. Then we have

$$(2) \qquad \sin(\alpha + \beta) = \frac{MP}{OP} = \frac{NQ + RP}{OP} = \frac{NQ}{OP} + \frac{RP}{OP}.$$

Now $\frac{NQ}{OP}$ is a ratio of sides of two *different right triangles*, namely, ONQ and OPQ. But these triangles have the side OQ in common, and this common side may be used to transform $\frac{NQ}{OP}$ into a product of two ratios, each of which contains two sides of the *same* right triangle and is therefore a trigonometric function of the acute angles of this triangle. In fact we find

(3) $$\frac{NQ}{OP} = \frac{NQ}{OQ} \cdot \frac{OQ}{OP} = \sin \alpha \cos \beta.$$

In the same way we find for the second term of (2)

$$\frac{RP}{OP} = \frac{RP}{PQ} \cdot \frac{PQ}{OP}.$$

But

$$\frac{PQ}{OP} = \sin \beta, \text{ and } \frac{RP}{PQ} = \cos RPQ = \cos \alpha,$$

since the sides of the angle RPQ are respectively perpendicular to those of α. Consequently we find

(4) $$\frac{RP}{OP} = \cos \alpha \sin \beta.$$

If (3) and (4) be substituted in (2), we obtain the important formula

(5) $$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

Referring once more to Fig. 102, we have

(6) $$\cos (\alpha + \beta) = \frac{OM}{OP} = \frac{ON - RQ}{OP} = \frac{ON}{OP} - \frac{RQ}{OP}.$$

If we again transform each of these ratios into a product of two others, we find

$$\frac{ON}{OP} = \frac{ON}{OQ} \cdot \frac{OQ}{OP} = \cos \alpha \cos \beta,$$

$$\frac{RQ}{OP} = \frac{RQ}{PQ} \cdot \frac{PQ}{OP} = \sin \alpha \sin \beta,$$

so that

$$(7) \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

It is easy to see that equations (5) and (7) remain valid if α and β are acute angles, even if their sum is greater than a right angle. In that case we shall have the situation represented in Fig. 103. If we make precisely the same constructions as before, the proof of the formula for $\sin(\alpha+\beta)$ will remain applicable word for word. But since $\alpha+\beta$ is now in the second quadrant, its cosine is negative; that is,

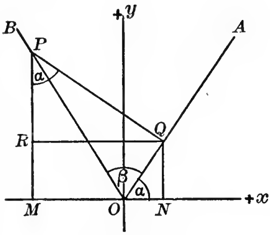

FIG. 103

$$\cos(\alpha+\beta) = -\frac{MO}{OP},$$

where by MO we mean merely the positive number expressing the length of the line-segment MO, so that $-MO$ is a negative number.

Now the figure shows that

$$MO = MN - ON = RQ - ON,$$

so that

$$\cos(\alpha+\beta) = -\frac{MO}{OP} = \frac{ON-RQ}{OP} = \frac{ON}{OP} - \frac{RQ}{OP}.$$

From this point on, the proof proceeds exactly as in the previous case, beginning from equation (6).

Thus, we have proved that equations (5) and (7) are certainly true if α and β are positive acute angles, even if their sum is greater than 90°.

We may now show that these formulæ are true for two angles in any quadrant. In order to do this, we first prove the following theorem. *If the formulæ*

$$(8) \qquad \begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta, \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \end{aligned}$$

are true for two angles α and β, they remain true if either of these angles be increased by 90°.

PROOF. Let us assume that equations (8) are true for a certain pair of angles α and β. Put

$$\alpha' = 90° + \alpha,$$

so that

$$\alpha' + \beta = (90° + \alpha) + \beta.$$

We shall then have (Art. 77, equations (2)),

$$(9) \quad \begin{aligned} \sin(\alpha' + \beta) &= \sin(90° + \alpha + \beta) = \cos(\alpha + \beta) \\ &= \cos\alpha \cos\beta - \sin\alpha \sin\beta, \\ \cos(\alpha' + \beta) &= \cos(90° + \alpha + \beta) = -\sin(\alpha + \beta) \\ &= -\sin\alpha \cos\beta - \cos\alpha \sin\beta, \end{aligned}$$

and also

$$\sin\alpha' = \sin(90° + \alpha) = \cos\alpha, \quad \cos\alpha' = \cos(90° + \alpha) = -\sin\alpha,$$

whence

$$\sin\alpha = -\cos\alpha', \qquad \cos\alpha = \sin\alpha'.$$

If we substitute these values in (9), we find

$$\sin(\alpha' + \beta) = \sin\alpha' \cos\beta + \cos\alpha' \sin\beta,$$
$$\cos(\alpha' + \beta) = \cos\alpha' \cos\beta - \sin\alpha' \sin\beta.$$

But these formulæ are of the same form as equations (8), with $\alpha' = \alpha + 90°$ in place of α. The same process would show that equations (8) would still be satisfied if we replaced β by $\beta' = 90° + \beta$. Consequently our theorem is proved.

We know already that equations (8) *are* true if α and β are any two positive acute angles. On account of the theorem just proved they will still be true if α is in the second quadrant and β in the first, and hence also if both α and β are in the second quadrant, and hence if one of these angles is in the third quadrant, etc. Therefore equations (8) are true for all positive angles.

But they are also true if either or both angles are negative. For instance, let α be a negative angle while β is positive. By adding to α a sufficient number n of complete positive revolutions, we shall obtain

$$\alpha' = \alpha + n \cdot 360°,$$

a positive angle. But then

$$\sin \alpha' = \sin \alpha, \qquad \cos \alpha' = \cos \alpha,$$

$$\sin (\alpha' + \beta) = \sin (\alpha + \beta), \ \cos (\alpha' + \beta) = \cos (\alpha + \beta),$$

so that

$$\sin (\alpha + \beta) = \sin (\alpha' + \beta) = \sin \alpha' \cos \beta + \cos \alpha' \sin \beta$$
$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta,$$

and similarly for $\cos (\alpha + \beta)$.

If β is negative, we may proceed in the same manner. Consequently we obtain the following important theorem:

If α and β are two positive or negative angles of any magnitude, the sine and cosine of their sum are always given by the formulæ

$$\textbf{(8)} \quad \begin{cases} \sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta, \\ \cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \end{cases}$$

The *method of mathematical induction* employed for proving the general validity of these formulæ is of fundamental importance in all parts of mathematics. We have already made use of it in Art. 76.

Equations (8) are usually known as the *addition theorems* for sine and cosine.

EXERCISE XLV

1. Compute the sine and cosine of 75°.

Solution. We have $75° = 30° + 45°$, $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$. Therefore

$$\sin 75° = \sin 30° \cos 45° + \cos 30° \sin 45° = ?$$
$$\cos 75° = \cos 30° \cos 45° - \sin 30° \sin 45° = ?$$

Compute the sines and cosines of the following angles:

2. 120°. **4.** 210°. **6.** 195°.

3. 150°. **5.** 105°. **7.** 165°.

8. Find formulæ for $\sin (45° + \theta)$ and $\cos (45° + \theta)$.

9. Find formulæ for $\sin (30° + \theta)$ and $\cos (30° + \theta)$.

10. Find formulæ for $\sin(60° + \theta)$ and $\cos(60° + \theta)$.

11. Show that those of the quadrantal formulæ of Art. 77, which involve the sine and cosine, are special cases of the addition theorems for the sine and cosine.

Prove that the following equations are true for all values of the angles which appear in them. That is, prove that they are *identities.*

12. $\sin(\alpha + \beta)\cos\beta + \cos(\alpha + \beta)\sin\beta = \sin(\alpha + 2\beta)$.

13. $\cos(\alpha + \beta)\cos\beta - \sin(\alpha + \beta)\sin\beta = \cos(\alpha + 2\beta)$.

14. Show how $\sin(\alpha + \beta + \gamma)$ and $\cos(\alpha + \beta + \gamma)$ may be expressed in terms of the functions of α, β, and γ.

Simplify the following expressions:

15. $\sin(1 + n)\theta\cos(1 - n)\theta + \cos(1 + n)\theta\sin(1 - n)\theta$.

16. $\cos(1 + n)\theta\cos(1 - n)\theta - \sin(1 + n)\theta\sin(1 - n)\theta$.

80. The addition theorems for tangent and cotangent. The addition formulæ for the tangent and cotangent may be obtained as consequences of those for the sine and cosine. We have

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

The fraction in the right member of this equation may be expressed in terms of $\tan\alpha$ and $\tan\beta$ by dividing both numerator and denominator by $\cos\alpha\cos\beta$. We obtain in this way

$$\tan(\alpha + \beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{1 - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}},$$

or, finally,

$$(1) \qquad \boldsymbol{\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}.$$

By a similar process we find

$$(2) \qquad \cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}.$$

EXERCISE XLVI

Find the tangents and cotangents of the following angles:

1. 75°.
3. 210°.
2. 105°.
4. 150°.

Prove that the following equations are identities:

5. $\tan(45° + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.

6. $\cot(45° + \theta) = \dfrac{\cot\theta - 1}{\cot\theta + 1}$.

7. Find formulæ for $\tan(30° + \theta)$ and $\cot(30° + \theta)$.

8. Find formulæ for $\tan(60° + \theta)$ and $\cot(60° + \theta)$.

81. The subtraction formulæ. The formulæ for $\sin(\alpha - \beta)$, $\cos(\alpha - \beta)$, etc., may be obtained from those for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, etc. For, since the addition formulæ are known to be true for all positive and negative values of α and β, we may write

$$\sin(\alpha - \beta) = \sin[\alpha + (-\beta)]$$
$$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta),$$
$$\cos(\alpha - \beta) = \cos[\alpha + (-\beta)]$$
$$= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta).$$

But, on account of equations (9), Art. 77, we have

$$\sin(-\beta) = -\sin\beta, \quad \cos(-\beta) = \cos\beta,$$

so that we find

$$(1) \qquad \begin{aligned} \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta, \\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta. \end{aligned}$$

In the same way we find

$$(2) \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta},$$

and

$$(3) \qquad \cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}.$$

Of course equations (2) and (3) may also be obtained from (1) by division.

EXERCISE XLVII

1. Let the student devise a geometric proof for equations (1), in the case when α and β are positive acute angles, α being the larger of the two.

Note. If Fig. 102 be described in *words*, but if the angle there denoted by β be instead regarded and constructed as a negative angle, the same description will apply to both figures and the same method of transforming the ratios which was used in Art. 79 will be effective in the present example.

82. Formulæ for converting products of trigonometric functions into sums, and vice versa. Before the invention of logarithms the calculations of products and quotients was a very laborious process. In those days, then, it was considered a great simplification if a formula whose numerical evaluation required multiplication could be transformed into another requiring only addition or subtraction.* The addition and subtraction formulæ will enable us to accomplish this for a product of two sines, of two cosines, or of a sine and cosine.

In fact, we have the two equations

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

from which we derive, by addition and subtraction,

$$(1)\qquad \begin{aligned}\sin(\alpha+\beta) + \sin(\alpha-\beta) &= 2\sin\alpha\cos\beta,\\ \sin(\alpha+\beta) - \sin(\alpha-\beta) &= 2\cos\alpha\sin\beta.\end{aligned}$$

In the same way, from

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

we find

$$(2)\qquad \begin{aligned}\cos(\alpha+\beta) + \cos(\alpha-\beta) &= 2\cos\alpha\cos\beta,\\ \cos(\alpha+\beta) - \cos(\alpha-\beta) &= -2\sin\alpha\sin\beta.\end{aligned}$$

From these last equations we find, by transposition and division by 2,

$$(3)\qquad \begin{aligned}\cos\alpha\cos\beta &= \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)],\\ \sin\alpha\sin\beta &= \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)],\end{aligned}$$

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* This was the so-called *prosthaphæretic method.*

whereas, from the first equation of (1), we find

(4) $$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)].$$

The result obtained from the second equation of (1) is the same as (4) except for an interchange of the letters α and β.

Equations (3) *and* (4) *are important in that they enable us to transform a product of two sines, two cosines, or of a sine and a cosine into a sum or difference.*

For many purposes this is very important even nowadays,* although not for the purposes of numerical calculation. In fact, at the present time, for numerical work we prefer formulæ which involve multiplication to those involving addition, because the former process is more easily performed by logarithms.

The above formulæ, slightly modified, may also be used for the purpose of converting sums and differences of sines and cosines into products. Let us put

$$\alpha+\beta = A, \qquad \alpha-\beta = B,$$

so that $$\alpha = \tfrac{1}{2}(A+B), \qquad \beta = \tfrac{1}{2}(A-B).$$

Then, equations (1) and (2) yield the following four formulæ:

(5) $$\begin{cases} \sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B), \\ \sin A - \sin B = 2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B), \\ \cos A + \cos B = 2\cos\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B), \\ \cos A - \cos B = -2\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B). \end{cases}$$

We have seen the first two of these equations before, having derived them directly from a figure (in Art. 49) for the purpose of proving the law of tangents. Our proof, at that time, did not permit us to affirm that the formulæ were true for all angles A and B. That such is the case, however, has now been made evident, since the present proof was obtained without placing any restrictions on the values of the angles A and B.

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* For instance, in harmonic analysis (see Arts. 111 and 112) and in the integral calculus.

EXERCISE XLVIII

Prove the following equations:

1. $\sin 3\theta + \sin \theta = 2 \sin 2\theta \cos \theta.$

2. $\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = 2 \sin \frac{\pi}{4} \cos x = \sqrt{2} \cos x.$

3. $\dfrac{\sin 6\alpha + \sin 4\alpha}{\cos 6\alpha + \cos 4\alpha} = \tan 5\alpha.$

4. $\dfrac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \dfrac{\tan \frac{\alpha - \beta}{2}}{\tan \frac{\alpha + \beta}{2}}.$

5. $\cos \alpha - \cos 3\alpha = 2 \sin \alpha \sin 2\alpha.$

6. $\sin 3\alpha + \cos \alpha = \sin 3\alpha + \sin (90^\circ - \alpha) = ?$

7. By generalizing the process observed in Ex. 6, derive a formula for $\sin \alpha + \cos \beta$.

8. Derive a formula for $\sin \alpha - \cos \beta$.

Reduce the following products to sums or differences:

9. $\sin 4\alpha \cos 2\alpha.$

10. $\sin 6\theta \sin 4\theta.$

11. $\cos 2\beta \sin 4\beta.$

12. $\cos 2\theta \cos 8\theta.$

13. $\sin 5\alpha \cos 3\alpha.$

14. $\sin^2\theta \cos \theta.$

15. Making use of the formulæ (5) of Art. 82, show how to derive the law of tangents from the law of sines.

83. Functions of double angles. If we put $\beta = \alpha$ in equations (8) of Art. 79, we find

(1) $$\sin 2\alpha = 2 \sin \alpha \cos \alpha,$$

and

(2) $$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha.$$

On account of the relation

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

the latter equation may also be written in either of the following two forms:

(3) $$\cos 2\alpha = 1 - 2 \sin^2 \alpha,$$

or

(4) $$\cos 2\alpha = 2 \cos^2 \alpha - 1.$$

$\cos 2a + 1 = 2\cos^2 a$

If we put $\beta = \alpha$ in equations (1) and (2) of Art. 80, we find

(5) $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha},$$

(6) $$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha},$$

which equations may, of course, also be derived from (1) and (2) by division.

84. Functions of half angles. In Art. 83 we regard the functions of α as known, and we learn how to compute the functions of 2α. We shall now invert the problem by regarding as known the functions of 2α, the problem being to calculate the functions of α, the half angle. To put the character of the problem more clearly into evidence, we shall put

$$2\alpha = \theta, \qquad \alpha = \tfrac{1}{2}\theta,$$

which merely amounts to thinking of any angle θ as a double angle; namely, as double its half.

With this change of notation, equations (3) and (4) of Art. 83 become

(1) $$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}, \qquad \cos\theta = 2\cos^2\frac{\theta}{2} - 1.$$

If we solve the first of these equations for $2\sin^2\frac{\theta}{2}$ and the second for $2\cos^2\frac{\theta}{2}$, we find

(2) $$2\sin^2\frac{\theta}{2} = 1 - \cos\theta, \qquad 2\cos^2\frac{\theta}{2} = 1 + \cos\theta,$$

whence

(3) $$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}, \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}.$$

The ambiguous signs on the right members are determined by the quadrant of the angle $\frac{\theta}{2}$. If θ is a positive angle not greater than 180°, $\frac{\theta}{2}$ is in the first quadrant and the + sign must be chosen in both of the equations (3).

From (3) we find, by division,

$$(4)\qquad \tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, \qquad \cot\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}},$$

the appropriate sign being again determined by the quadrant of the angle $\frac{1}{2}\theta$.

According to (1), Art. 83, we have

$$(5)\qquad 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta.$$

Let us divide each member of the first equation of (2) by the corresponding member of (5). We find

$$(6)\qquad \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}.$$

By a similar process we obtain, from the second equation of (2),

$$(7)\qquad \cot\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta}.$$

These last two formulæ might also have been derived from (4). But the proof we have given is preferable since it avoids the necessity of discussing the ambiguous sign, a discussion which would be necessary if we had followed the other method.

The equations for $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, etc., are of very great importance, because *they may be used for the purpose of computing a table of trigonometric functions*. In fact, we have already shown how to calculate the values of the functions of 0°, 30°, 45°, 60°, and 90° (Art. 11). By means of the addition and subtraction formulæ (Arts. 79, 81) we are therefore in a position to find also the functions of 15° and 75°. We can now calculate the values of the functions of one half of 15° or 7°.5, of one half of 7°.5, or 3°.75, etc. By continuing this process of bisection and combining the results by means of the addition theorem, we may obviously compute the values of the functions for a set of angles between 0° and 90° as close together as we please. By interpolation we may then find the functions of 1°, 2°, 3°, etc.

The method, of which we have just given an outline, is essentially the same as that employed by PTOLEMY (second century A.D.).* Ptolemy, however, also made use of the inscribed pentagon (cf. Exercise XLIX, Ex. 12), and his table was a table of chords, not of sines. (See Art. 70.) His table gives the values of the chord for each half degree of arc with a degree of accuracy somewhat greater than that which would correspond to a modern five-place table. The earlier tables of HIPPARCHUS and MENELAUS are not extant.

The Hindus followed the method which we have outlined even more closely. In fact, the table given by Āryabhata (born 476 A.D.) gives the values of the sine at intervals of 3° 45′. As we have seen, this is precisely the interval which would arise as a result of continued bisection of 30°.

Essentially the same method was used in subsequent improvements and enlargements of these tables, especially by RHETICUS (1514–1574) and PITISCUS (1561–1613). Other far more powerful methods have since been developed, based essentially on the notions of the calculus and the theory of infinite series.

EXERCISE XLIX

1. From the functions of 30° find those of 60°.

2. From the functions of 60° find those of 120°.

3. From the functions of 90° find those of 45°.

4. From the functions of 30° find those of 15°.

5. From the functions of 15° find those of 7°.5.

6. Find formulæ for $\sin 3\alpha$, $\cos 3\alpha$, $\tan 3\alpha$.

HINT. Put $3\alpha = 2\alpha + \alpha$.

7. Find formulæ for $\sin 4\alpha$, $\cos 4\alpha$, $\tan 4\alpha$.

8. Find formulæ for $\sin 5\alpha$, $\cos 5\alpha$, $\tan 5\alpha$.

9. Prove formula (1), Art. 83, by means of a figure.

10. Given $\tan\theta = \frac{3}{4}$, θ being in the first quadrant. Find the functions of 2θ and $\frac{1}{2}\theta$.

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* Ptolemy's great work on Astronomy, usually known as the *Almagest*, remained in undisputed authority until the time of *Copernicus*. The so-called Ptolemaic system of astronomy, as opposed to the more modern Copernican system, was named after him for this reason.

11. If θ is in the third quadrant and $\sin\theta = \frac{-2}{\sqrt{5}}$, find the functions of 2θ.

12. In a circle of radius 1, inscribe a regular pentagon. Show that, by means of this construction the trigonometric functions of 72° and 18° may be computed. In particular, show that

$$\sin 18^\circ = \tfrac{1}{4}(\sqrt{5} - 1).$$

13. Making use of the results of Ex. 12, compute the functions of 12°.

14. Making use of the results of Ex. 13, compute the functions of 6°.

Assuming the truth of the law of cosines, and setting $s = \frac{1}{2}(a + b + c)$, prove the following formulæ for the functions of the half angles of a triangle.

15. $\sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$

16. $\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}.$

17. $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$

18. Prove formulæ (6) and (7) of Art. 84 by means of equations (4).

85 *a*. The limit $\frac{\sin\theta}{\theta}$ and related limits. Although the method sketched in Art. 84 for calculating a table of the values of the trigonometric functions is adequate, it involves far more labor than is actually necessary. The following theorem, whose truth is almost self-evident, is of great importance in this connection as it enables us to calculate the sine of a very small angle with a minimum of effort.

If an angle or arc is expressed in radians, the quotient $\frac{\sin\theta}{\theta}$ *approaches the limit* 1 *when the angle itself approaches zero as a limit. In symbols*

$$\lim_{\theta \doteq 0} \frac{\sin\theta}{\theta} = 1.$$

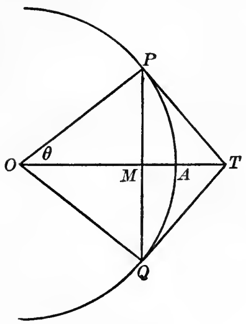

FIG. 104

In order to prove this theorem, let us draw an acute angle $AOP = \theta$ as in Fig. 104, and symmetrically the angle AOQ also equal to θ. With the vertex O as center and any convenient radius r, draw the circular arc PAQ and join PQ, intersecting OA in M. The tangents PT and QT, at P and Q, will meet in a point T of OA prolonged.

Obviously we shall have

(1) $PQ < \text{arc } PAQ < PT + TQ.$

But

$$PQ = 2\,PM = 2\,r\sin\theta,$$
$$\text{arc } PAQ = 2\text{ arc } AP = 2\,r\theta \quad \text{(Art. 74, equation (5))},$$
$$PT + TQ = 2\,PT = 2\,r\tan\theta,$$

where the truth of the second equation depends essentially upon our assumption that θ is expressed in radians. If these values be substituted in (1), we find

$$2\,r\sin\theta < 2\,r\,\theta < 2\,r\tan\theta,$$

or, after division by the positive quantity $2\,r$,

(2) $$\sin\theta < \theta < \tan\theta.$$

If we divide all three members of this inequality by the positive number $\sin\theta$, we find

(3) $$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

We know that $\cos\theta$ approaches the limit 1 when θ approaches zero. Since the value of $\frac{\theta}{\sin\theta}$, according to (3), lies between 1 and $\frac{1}{\cos\theta}$, which latter quantity itself approaches 1, $\frac{\theta}{\sin\theta}$ must also have 1 as its limit. That is,

(4) $$\lim_{\theta \doteq 0} \frac{\theta}{\sin\theta} = 1.$$

From (3) we have further

$$1 > \frac{\sin\theta}{\theta} > \cos\theta,$$

so that, by a similar argument, we find

(5) $$\lim_{\theta \doteq 0} \frac{\sin\theta}{\theta} = 1.$$

If we divide all members of (2) by the positive quantity $\tan\theta$, we find

$$\cos\theta < \frac{\theta}{\tan\theta} < 1,$$

so that

(6) $$\lim_{\theta \doteq 0} \frac{\theta}{\tan\theta} = \lim_{\theta \doteq 0} \frac{\tan\theta}{\theta} = 1.$$

Since

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta},$$

$$\frac{\tan(-\theta)}{-\theta} = \frac{-\tan\theta}{-\theta} = \frac{\tan\theta}{\theta},$$

equations (4), (5), and (6) will still remain true if θ approaches zero through negative instead of positive values.

These formulæ have many important applications. For the present, we shall mention only the one indicated at the beginning of this article. The content of equation (5) may be formulated as follows. If we write

(7) $$\frac{\sin\theta}{\theta} = 1 - \delta,$$

the angle θ (expressed in radians) may be chosen so small that δ $\left(\text{the difference between 1 and } \frac{\sin\theta}{\theta}\right)$ will become less than any previously assigned quantity. Since we find from (7)

$$\sin\theta - \theta = -\delta\theta,$$

we see that we can make the angle θ (expressed in radians) so small that the difference between θ and $\sin\theta$ becomes less than any previously assigned small fraction of θ.

Suppose, for instance, that we wish to compute the sine of a small angle to 5 decimal places, that is, with an error which shall be less than 5 units of the sixth decimal place, or .000005. We now know that the angle may be chosen so small that its sine may be equated to the radian measure of the angle itself with an error of less than 5 units of the sixth decimal place. In other words, the equation

$$(8) \qquad \sin\theta = \theta \text{ (in radians)}$$

will be true up to five decimal places for all angles which are sufficiently small.

Of course, our method does not inform us just *how* small θ must be in order that equation (8) may be true up to five decimal places. It would take us too far afield to investigate this question, a complete answer to which is beyond the scope of this book. The student may convince himself, however, by actual comparison with the tables, that equation (8) is true to five decimal places for all angles less than 2°. In all numerical work, then, involving such small angles, no error noticeable in five-place calculations is introduced by putting $\sin\theta = \theta$ (in radians).

Since we have

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \text{ (Art. 84, equation (1)),}$$

we may put

$$(9) \qquad \cos\theta = 1 - 2\left(\frac{\theta}{2}\right)^2 = 1 - \frac{1}{2}\theta^2,$$

a formula which will certainly be true up to the fifth decimal place for all angles less than 2°. In fact equation (9) holds to five decimal places even for angles much larger than this and may serve for the purpose of computing the cosines of such angles. Since θ^2 is small as compared with θ, if θ itself is small, we shall even be justified in equating $\cos\theta$ to 1 for very small angles. Our tables show that no error is introduced in five-place calculations by putting $\cos\theta = 1$, if θ is less than 0° 16′.

Since (8) is true to five decimal places if $\theta < 2°$, we see that we shall have

$$\sin 2\theta = 2\theta$$

with the same degree of approximation if $\theta < 1°$. More generally the formula

$$\sin n\theta = n\theta \ (\theta \text{ in radians}) \tag{10}$$

is correct to five decimal places if $n\theta$ is less than 2°.

The results deduced in this article make it very easy to compute the functions of very small angles. By combining these results with the methods of Art. 83 an extensive table of the trigonometric functions may be constructed with comparative ease.

EXERCISE L

Compute the values of the following functions of small angles to five decimal places by the method of Art. 85 *a* and compare with the values obtained from the table:

1. $\sin 12'$. **3.** $\sin 1°$. **5.** $\tan 1°$.

2. $\tan 15'$. **4.** $\cos 1°$. **6.** $\cot 1°$.

7. What will be the angle subtended by a lamp-post 10 feet high at a distance of one mile?

8. In order to find the distance from the earth to the moon, the following plan may be adopted. Two astronomers stationed at A and B respectively (Fig. 105) observe at the same instant the angular distance of the moon's center M from their respective zeniths (their overhead points), Z and Z'. This gives the angles

$$\alpha = ZAM \text{ and } \beta = Z'BM.$$

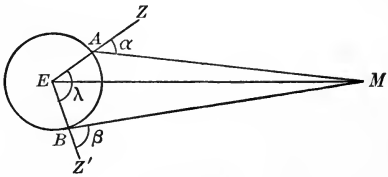

FIG. 105

For the sake of simplicity assume that both stations A and B are on the equator, that the moon is in the plane of the equator, and let E be the center of the earth. Then $\angle AEB = \lambda$ is equal to the difference between the longitudes of the two stations and may be regarded as known. We may now compute

$$\angle EAM = 180° - \alpha,\ \angle EBM = 180° - \beta,\ \angle AEB = \lambda.$$

Now the sum of the four angles of the quadrilateral $AEBM$ is four right angles; that is,

$$\angle M + 360° - \alpha - \beta + \lambda = 360°,$$

whence

$$\angle M = \alpha + \beta - \lambda.$$

From the value of M obtained in this way, it is easy to compute the angle subtended by the earth's radius at the center of the moon. This angle is called the *moon's parallax.*

Find the moon's distance from the earth if the moon's parallax is 57′ and if the earth's radius is 4000 miles.

9. The apparent diameter of the moon as seen from the earth is about 31′. Making use of the result of Ex. 8, what is the moon's diameter in miles?

10. The sun's parallax is about 8″.8. Assuming 4000 miles as the length of the earth's radius, find the distance from the earth to the sun.

85 *b*. The auxiliary quantities S and T. We have seen in Art. 28 that the ordinary tables of sines and tangents become inconvenient for very small angles. To avoid this inconvenience, we constructed an additional table (Table III), giving the values of the sines and tangents of such small angles directly for every second of arc. But we may accomplish the same purpose in another way, by means of an auxiliary table occupying far less space than the additional table just mentioned. This second method is based on the fact that the quotients

$$\frac{\sin\theta}{\theta} \text{ and } \frac{\tan\theta}{\theta}$$

change very slowly if θ is a small angle.

We have just seen that each of these quotients has unity as its limit when θ approaches zero, provided that the angle is measured in radians. Let us instead express θ in minutes of arc. Let θ' denote the number of minutes and $\theta^{(R)}$ the number of radians contained in the angle θ. Then, according to Art. 85 *a*,

$$(1) \qquad \underset{\theta \doteq 0}{\text{limit}} \frac{\sin\theta}{\theta^{(R)}} = \underset{\theta \doteq 0}{\text{limit}} \frac{\tan\theta}{\theta^{(R)}} = 1.$$

Since (Art. 74)

$$1^\circ = \frac{\pi}{180} \text{ radians} = 0.0174533 \text{ radian},$$

and therefore

$$1' = 0.0002909 \text{ radian},$$

the angle θ, which contains θ' minutes, will contain

$$\theta^{(R)} = 0.0002909\,\theta' \text{ radian}.$$

Consequently we find

$$\frac{\sin\theta}{\theta'} = \sin\theta \div \frac{\theta^{(R)}}{0.0002909} = 0.0002909\,\frac{\sin\theta}{\theta^{(R)}},$$

and therefore, on account of (1),

(2) $$\lim_{\theta \doteq 0} \frac{\sin\theta}{\theta'} = \lim_{\theta \doteq 0} \frac{\tan\theta}{\theta'} = 0.0002909.$$

In other words, if θ' is an angle expressed in minutes, the common limit toward which $\frac{\sin\theta}{\theta'}$ and $\frac{\tan\theta}{\theta'}$ tend, when θ' approaches zero, is a number whose first seven decimal places are given by 0.0002909.

Let us write

(3) $$s = \frac{\sin\theta}{\theta'}, \qquad t = \frac{\tan\theta}{\theta'}.$$

These quantities change their values very slowly for small values of θ. In fact we have just seen that, for angles which are small enough, we shall have

$$\log s = \log t = \log 0.0002909 = 6.46373 - 10.$$

For $\theta = 2^\circ = 120'$ we have

$$s = \frac{\sin 2^\circ}{120}, \qquad t = \frac{\tan 2^\circ}{120},$$

which gives, if we look up the logarithms from the tables,

$$\begin{array}{rl} \log \sin 2^\circ = & 8.54282 - 10 \\ \log 120 = & 2.07918 \\ \hline \log s = & 6.46364 - 10 \end{array} \qquad \begin{array}{rl} \log \tan 2^\circ = & 8.54308 - 10 \\ \log 120 = & 2.07918 \\ \hline \log t = & 6.46390 - 10 \end{array}$$

Therefore, while θ changes from 0° to 2°, $\log s$ changes only by 9 units and $\log t$ by 17 units of the fifth decimal place.

Table IV enables us to find the values of $S = \log s$ and $T = \log t$ for every angle between 0° and 2°. To find the logarithm of the sine or tangent of such an angle we have the formulæ (immediate consequences of (3))

$$\begin{aligned} \log\ \sin\theta &= \log\theta' + \log s = \log\theta' + S, \\ \log\tan\theta &= \log\theta' + \log t = \log\theta' + T. \end{aligned} \tag{4}$$

If the log sin or log tan of a small angle is given, to find the angle, we may do this by means of one of the equations

$$\begin{aligned} \log\theta' &= \log\sin\theta - S, \\ \log\theta' &= \log\tan\theta - T, \end{aligned} \tag{5}$$

obtained from (4) by transposition.

Of course the quantities S and T are available, not only for sines and tangents of small angles, but also for cosines and cotangents of angles close to 90°.

EXERCISE LI

1. Find the sine and tangent of 1° 13′.21 by using the auxiliaries S and T.

Solution. Since $\theta = 1°\ 13'.21$, we have $\theta' = 73'.21$.

$$\begin{array}{rl} \log\theta' &= 1.86457 \\ S &= 6.46369 - 10 \\ \hline \log\sin\theta &= 8.32826 - 10 \end{array} \qquad \begin{array}{rl} \log\theta' &= 1.86457 \\ T &= 6.46379 - 10 \\ \hline \log\tan\theta &= 8.32836 - 10 \end{array}$$

2. Given $\log\sin\theta = 8.24798 - 10$. Find θ.

Solution. We find from Table IV corresponding to $\log\sin\theta = 8.24798 - 10$, $S = 6.46370$. Formula (5) leads to the calculation

$$\begin{array}{rl} \log\sin\theta &= 8.24798 - 10 \\ S &= 6.46370 - 10 \\ \hline \log\theta' &= 1.78428 \end{array} \qquad \therefore \theta' = 60'.85 = 1°\ 0'.85.$$

Find the values of the logarithms of the following functions by means of the auxiliaries S and T:

3. sin 1° 21′.63.

4. tan 0° 32′.61.

5. cos 89° 13′.21.

6. cot 88° 21′.75.

Find the angles determined by the following functions by means of S and T:

7. $\log\sin\theta = 7.76345 - 10$.

8. $\log\tan\theta = 8.50731 - 10$.

9. $\log\cos\theta = 8.42371 - 10$.

10. $\log\cot\theta = 8.53729 - 10$.

CHAPTER XII

DIRECTED LINES AND DIRECTED LINE-SEGMENTS *

86. Plan of another proof for the addition formulæ. When we proved the addition theorem in Art. 79, we found it necessary to divide the proof into a number of cases according as the angles were in the first, second, third, or fourth quadrants. To be sure, by making use of the method of mathematical induction we found it a fairly simple matter to make an exhaustive discussion covering all cases. Nevertheless we feel that it must be possible to devise a method enabling us to prove this theorem at one stroke for angles of any magnitude. The key to the solution of this problem is found to be a careful formulation of the notions of a *directed line* and a *directed line-segment.* Since these notions are of very great importance, not only in this connection, but in many other parts of pure and applied mathematics, we shall find it worth our while to speak of them, even if they are not absolutely indispensable for the proof of the addition theorem.

87. Directed lines and segments. A straight line is infinite in extent and is determined by any two distinct points upon it. We may, however, think of one and the same straight line as having either of two opposite directions, in which case we speak of it as **a directed line.** Since we can never draw more than a finite portion of a line, we may indicate the direction of a directed line by placing a + sign near one end of that portion which actually appears in the figure. In Fig. 106 we have thus indicated the direction of

* This chapter may be omitted in a first course if the time is insufficient.

the directed line l which is to be thought of as pointing toward the upper right-hand corner of the page.

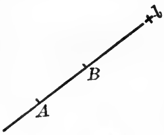

FIG. 106

This method of indicating the direction of a directed line has already been used in this book to indicate the positive direction of the x-axis and y-axis of a system of rectangular coördinates. (See Art. 63.) These are directed lines.

A **line-segment** is a finite portion of a line and may be described by naming its end points, such as AB in Fig. 106. But again, we may think of it as a **directed line-segment**, thus distinguishing between AB and BA.

When a directed line-segment lies upon a directed line, its direction or **sense** may be the same as that of the line or else opposite to it. If a directed line-segment on l is 5 units long and if its direction is the same as that of l, we may represent it by the number $+5$. A line-segment of the same length and opposite direction will be represented by -5. In general, *a directed line-segment, which lies on a directed line, shall be counted positive or negative according as it has the same or the opposite direction as the directed line.* For such line-segments, we always have $BA = -AB$, or $AB + BA = 0$.

We are now ready to prove the following theorem. *If A, B, C, are any three points on a directed line, then*

$$(1) \qquad AB + BC = AC,$$

where AB, BC, and AC are directed line-segments.

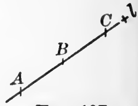

FIG. 107

PROOF. 1. Let AC be positive and let B be between A and C. Then AB and BC are also positive and the truth of the theorem, in this case, is obvious. (See Fig. 107.)

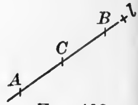

FIG. 108

2. Let AC be positive, but let C be between A and B. Then AB is positive, but BC is negative and equal to $-CB$. Thus

$AB + BC = AB - CB = AC.$ (See Fig. 108.)

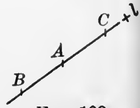

FIG. 109

3. Let AC still be positive, but let A be between B and C. Then (Fig. 109),

$$AB + BC = -BA + BC = BC - BA = AC.$$

If AC is negative, there are again three cases according as the order of the three points is CBA, BCA, or CAB. But the above relation (1) will be found to be true in all cases. These last three cases may, of course, also be reduced to the former three by reversing the direction of the line l.

The following is a simple corollary of the above theorem. *If A, B, C, D, are any four points of a directed line, we have the relation*

$$(2) \qquad AB + BC + CD = AD$$

between the directed line-segments AB, BC, CD, and AD.

In fact, by the theorem just proved, we have

$$AB + BC = AC, \qquad AC + CD = AD,$$

so that we find by addition

$$AB + BC + AC + CD = AC + AD,$$

which reduces to (2) if we subtract AC from both members.

It may now be proved by induction that, in general, *if A, B, C, $\cdots$ M, N are any finite number of points on a directed line, then*

$$(3) \qquad AB + BC + CD + \cdots + MN = AN.$$

Equations (1), (2), (3) may also be proved by algebra. On the directed line l, let us introduce a point O as origin or zero point of a scale, whose positive readings are on that side of O which corresponds to the positive direction of the line l. This is precisely what we did when we established scales upon the x-axis and y-axis of a coördinate system (Art. 63). Denote by l_A the reading of the scale which corresponds to the point A, by l_B that which corresponds to B. The difference $l_B - l_A$ will give the length of the line-segment AB, affected with a plus or minus sign according as AB is a positive or negative line-segment in the sense of our previous definition.

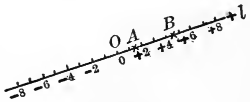

FIG. 110

If then we have any three points A, B, C on the directed line, we shall have

$$AB = l_B - l_A, \qquad BC = l_C - l_B, \qquad AC = l_C - l_A,$$

and therefore

$$AB + BC = l_B - l_A + l_C - l_B = l_C - l_A = AC,$$

which is the same as (1). In the same way we may also prove equations (2) and (3).

88. Angles between directed lines. Let l and m be two directed lines, and let us denote the angle between their positive directions by (l, m) or (m, l) according as we think of l or m as the initial side of the angle. To be perfectly specific, we understand by (l, m) the angle, less than 360°, through which it would be necessary to rotate the directed line l in the counter-clockwise direction, in order to make its positive direction coincide with the positive direction of the directed line m. In Fig. 111, the angle (l, m) is marked θ. Similarly (m, l) is the angle through which m would have to be turned in the positive (counter-clockwise) direction in order to make the positive direction of m coincide with that of l. In Fig. 111 (m, l) is marked θ'.

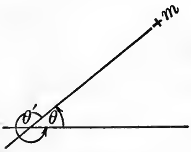

FIG. 111

We see that we shall always have

(1) $$(l, m) + (m, l) = 360°,$$

so that

$$(m, l) = 360° - (l, m)$$

and therefore (see Art. 77, equations (7)),

(2) $$\sin (m, l) = -\sin (l, m), \qquad \cos (m, l) = \cos (l, m).$$

89. Projections. The projection of a point P on a line l is the foot of the perpendicular dropped from the point to the line. The projection of a line-segment AB on a line l (see Fig. 112) is the line-segment $A'B'$ of l bounded by the projections of A and B. If AB is a *directed* line-segment, so is $A'B'$.

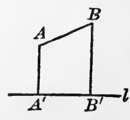

FIG. 112

We wish to solve the following problem. *Given a directed line-segment AB on a directed line p; to find the magnitude and sign of its projection upon any second directed line l.*

The line-segment AB may have the same sense as the directed line p or else the opposite sense; it will be represented by a positive number in the first case and by a negative number in the second (Art. 87). If we denote by $|AB|$ the *positive* number which represents merely the length (regardless of direction) of the line-segment AB, we shall have

(1) $$AB = |AB|, \text{ read } AB = \textit{length } AB,$$

or

(2) $$AB = -|AB|, \text{ read } AB = \textit{minus length } AB,$$

according as the direction of AB agrees with that of the directed line p or not.

Let us consider first the case (1) in which AB is positive. (See Figs. 113 and 114.) Let OM be the projection of AB on l. Choose O as origin and l as the x-axis of a system of

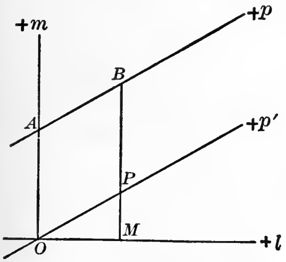

FIG. 113

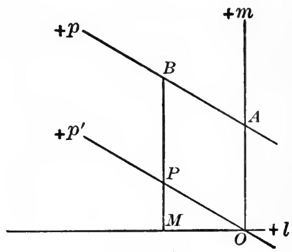

FIG. 114

coördinates, so that the positive x-axis coincides with the positive direction of the line l. The line m through O, perpendicular to l and with its positive direction as indicated in the figures, will then be the y-axis. Through O draw a directed line p' parallel to p and let P be its intersection with the line MB through B perpendicular to l.* The line-

* The word *parallel* in this theory means, not only that the lines are parallel in the ordinary sense, but that their positive directions are the same. The word *anti-parallel* is sometimes used for two parallel directed lines whose positive directions are opposite.

segments AB and OP have the same length and direction, and the same directed line-segment OM on l as projection. But, by the definition of the cosine of a general angle (Art. 64),

$$\cos(l, p) = \cos(l, p') = \frac{OM}{OP},$$

since OM is the abscissa and OP the radius vector of a point P on the terminal side of the angle (l, p'), this angle being in its standard position and $OP = AB$ being positive in the case under consideration. Consequently we have

$$OM = OP \cos(l, p) = AB \cos(l, p).$$

Since OM is the projection of AB on l, we may write this as follows:

(3) $\quad OM = \text{proj}_l\, AB = AB \cos(l, p) = AB \cos(p, l);$*

for, according to equation (2), Art. 88, the angles (l, p) and (p, l) have the same cosine.

Thus the projection upon the directed line l, of the positive line-segment AB of the directed line p, is equal to AB multiplied by the cosine of the angle between l and p. The projection is a directed line-segment on l, positive if (l, p) is in the first or fourth, negative if (l, p) is in the second or third quadrant.

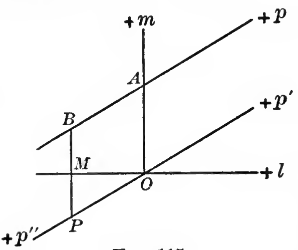

Fig. 115

We have proved equation (3) when AB is positive. Let us consider now the case when AB is a negative line-segment of p. (See Fig. 115.) The projection of AB, as well as that of OP, is OM. We may think of OP, which is a negative line-segment on p', as a positive line-segment of a directed line p'', coincident with p' but having the opposite direction. But this latter directed line makes an angle with l just 180° greater than (l, p'); that is,

(4) $$(l, p'') = (l, p') + 180°.$$

* The symbol $\text{proj}_l\, AB$ is read "projection of AB upon l."

For if we rotate l in the counterclockwise direction around O as center, it will take 180° more to make $+l$ coincide with $+p''$ than with $+p'$. According to formula (3), which we know to be valid for all positive line-segments, we have therefore

$$\text{proj}_l AB = |AB| \cos(l, p''),$$

where $|AB|$ denotes the *length* of the line-segment AB taken as a positive number. But according to (4) and equations (4) of Art. 77, we have

$$\cos(l, p'') = -\cos(l, p') = -\cos(l, p) = -\cos(p, l),$$

so that

$$\text{proj}_l AB = -|AB| \cos(l, p) = -|AB| \cos(p, l).$$

Since in our case AB was a negative line-segment, we had (cf. equation (2))

$$AB = -|AB|,$$

so that we may write finally

$$(3) \qquad \text{proj}_l AB = AB \cos(l, p) = AB \cos(p, l).$$

In other words, *formula* (3) *for the projection upon a directed line l, of a directed line-segment AB of a second directed line p, is true for both positive and negative line-segments.*

If we think of the line-segments as mere lengths, not endowed with direction, formula (3) may be simplified to

$$A'B' = \text{proj}_l AB = AB \cos\theta,$$

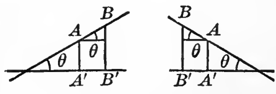

FIG. 116 FIG. 117

where θ is the angle between AB and the line l, this angle being understood in the sense of elementary geometry without any reference to a positive sense of rotation. Consequently θ may be regarded as an acute angle, so that its cosine will always be positive. (See Figs. 116 and 117.)

90. Projection of a broken line. Let us connect two points A and C by a directed line-segment AC. We may think of this line-segment as describing the shortest path from A to C in magnitude and direction. Let ABC be a second (longer) path from A to C made up of two directed line-segments AB and BC (Fig. 118).

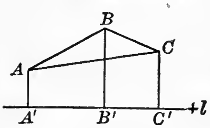

FIG. 118

Let us project these three line-segments on any directed line l, so that

(1) $A'B' = \text{proj}_l\, AB,\ B'C' = \text{proj}_l\, BC,\ A'C' = \text{proj}_l\, AC.$

Since $A'B'$, $B'C'$ and $A'C'$ are three directed line-segments of a directed line, we have (Art. 87, equation (1))

$$A'C' = A'B' + B'C';$$

whence, substituting the values (1),

(2) $$\text{proj}_l\, AC = \text{proj}_l\, AB + \text{proj}_l\, BC.$$

By means of the more general relation (3) of Art. 87, we may prove the following general theorem, of which (2) expresses the simplest special case.

Let us connect two points by two paths, one of which is a single directed line-segment, while the other is made up of a finite number of directed line-segments. Then the projection, upon any directed line, of the first path is equal to the sum of the projections of the various line-segments which make up the second path.

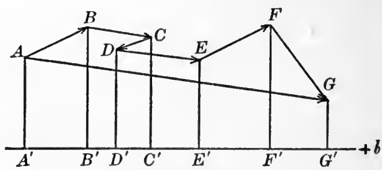

FIG. 119

Figure 119, in which the various line-segments are marked with arrowheads to indicate their directions, illustrates this theorem. Since the positive direction of l is toward the right, the projections $A'G'$, $A'B'$, $B'C'$, $D'E'$, $E'F'$, $F'G'$ are all positive, while the projection $C'D'$ of CD is negative. Consequently

$$\text{proj}_l\, AB + \text{proj}_l\, BC + \text{proj}_l\, CD$$
$$= A'B' + B'C' + C'D' = A'B' + B'C' - D'C' = A'D'.$$

Let the line l in Fig. 118 coincide with AC, and denote by a, b, c the lengths of the three line-segments BC, AC, and

AB respectively. (See Fig. 120.) According to (2) and Art. 89, equation (3), we shall have

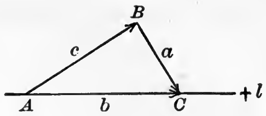

FIG. 120

(3) $b = c \cos A + a \cos C,$

a relation which may be used to advantage in many problems concerning triangles. Of course the two equations similar to (3)

$$(4) \qquad \begin{aligned} c &= a \cos B + b \cos A, \\ a &= b \cos C + c \cos B, \end{aligned}$$

may be proved in the same fashion.

91. Direction cosines of a line. Let l' be a directed line through the origin of coördinates. The angles (x, l') and (l', y), which its positive direction makes with the positive x-axis and y-axis, are called its *direction angles.* The direction angles (x, l) and (l, y) of a line l which does not pass through the origin are defined to be the same as those of a parallel line l' which does pass through the origin. Observe that the angle (x, l) has the x-axis as initial side, while the initial side of the second direction angle (l, y) is not the y-axis, but the line l.

If the angle (x, l) is in the first quadrant (Fig. 121), we clearly have

$$(1) \qquad (x, l) + (l, y) = 90°.$$

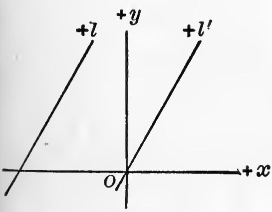

FIG. 121

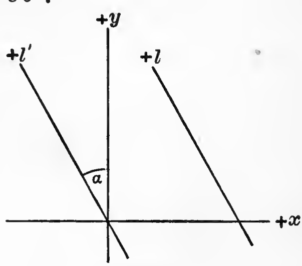

FIG. 122

If the angle (x, l) is in the second quadrant (Fig. 122), we have

$$(x, l) = (x, l') = 90° + \alpha,$$
$$(l, y) = (l', y) = 360° - \alpha.$$

For clearly it requires a positive (counterclockwise) rotation of $360° - \alpha$ to make $+l'$ coincide with $+y$. Therefore we have, in this case,

(2) $$(x, l)+(l, y)=360°+90°,$$

and the same relation holds if (x, l) is in the third or fourth quadrant, as may be verified easily. Thus we always have either $(l, y)=90°-(x, l)$ or $(l, y)=360°+90°-(x, l)$, so that in all cases

(3) $$\cos(y, l)=\cos(l, y)=\sin(x, l), \ \sin(l, y)=\cos(x, l).$$

The two quantities $\cos(x, l)$ and $\cos(y, l)$ are called the *direction cosines of the directed line l.*

Since we have, for any angle,

$$\cos^2(x, l)+\sin^2(x, l)=1,$$

we find from (3) *the following simple relation between the direction cosines of a line l;*

(2) $$\mathbf{\cos^2(x, l)+\cos^2(y, l)=1.}$$

92. Formula for the cosine of the angle between two lines whose direction cosines are given. Let us consider two directed lines l and m whose direction cosines are $\cos(x, l)$, $\cos(y, l)$, and $\cos(x, m)$, $\cos(y, m)$ respectively. We wish to find a formula for the cosine of the angle between the two lines.

We may assume that the lines l and m pass through the origin. If they do not, we may first solve the problem for two lines l', m', parallel to l, m respectively, which do pass through the origin. Since l', m' have the same direction cosines as l and m, and since the angle between l', m' is equal to that between l, m, the two problems are clearly equivalent.

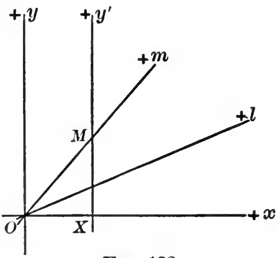

FIG. 123

Let us choose a point M (see Fig. 123), on the line m, such that the line-segment OM is positive. Let X be the projection of M on the x-axis. Then the projection of the broken line OXM on l will be equal to that of OM (Art. 90, equation (2)). That is,

(1) $$\text{proj}_l\, OM = \text{proj}_l\, OX + \text{proj}_l\, XM.$$

On account of Art. 89, equation (3), we have

(2) $$\text{proj}_l\, OM = OM \cos(m, l),$$
$$\text{proj}_l\, OX = OX \cos(x, l), \quad \text{proj}_l\, XM = XM \cos(y, l),$$

since XM is a directed line-segment on a directed line y' parallel to the y-axis.*

Substitution of (2) in (1) gives

(3) $$OM \cos(m, l) = OX \cos(x, l) + XM \cos(y, l).$$

But
$$OX = \text{proj}_x OM = OM \cos(x, m),$$
$$XM = \text{proj}_{y'} OM = \text{proj}_y OM = OM \cos(y, m),$$

since the projections of OM, on the two parallel directed lines y and y', are equal. If we substitute these values in (3), we find

$$OM \cos(m, l)$$
$$= OM \cos(x, l) \cos(x, m) + OM \cos(y, l) \cos(y, m)$$

or, upon division by OM,

(4) $$\cos(m, l) = \cos(x, l) \cos(x, m) + \cos(y, l) \cos(y, m),$$

the desired formula.

The proof which we have given of this formula is perfectly general; that is, it is applicable, no matter how large or small the angles (l, m), (x, l), etc., may be, in what quadrants they happen to lie, or whether they are positive or negative. In fact, the figure (Fig. 123) has not really been used in the demonstration except for the purpose of suggesting the order of the various steps of the argument. Every step of this proof can be justified by quoting a previously demonstrated general theorem.

It will be a good exercise for the student to repeat the argument with a different figure in which some or all of the angles concerned are not acute.

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* In Fig. 123 we have chosen the positive direction of y' to correspond to that of y. This is not essential, but it is convenient.

93. New proof for the addition and subtraction formulæ. Let us denote the angles (x, l) and (x, m) by α and β respectively. Then

$$\cos(l, m) = \cos(\alpha - \beta),$$

and (Art. 91, equations (3)),

$$\cos(y, l) = \sin(x, l) = \sin\alpha, \quad \cos(y, m) = \sin(x, m) = \sin\beta.$$

Consequently we find, from equation (4) of Art. 92,

$$(1) \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

the subtraction formula for the cosine. We shall leave it as an exercise for the student to deduce from this the subtraction formula for the sine and the addition formulæ for both functions. (See Exercise LII.)

94. The generalized law of sines. If we attribute a definite direction to every line of the plane, and define angles between them, as in Art. 88, with reference to a positive direction of rotation, the angle between any two such directed lines may be greater than 180° and the trigonometric functions of such angles will have definite signs as well as numerical magnitude. Moreover, every line-segment will then also have a definite sign.

It may be shown that the law of sines, when written in the form

$$(1) \qquad \frac{BC}{\sin(b, c)} = \frac{CA}{\sin(c, a)} = \frac{AB}{\sin(a, b)}$$

will be true of any triangle, not merely with regard to the magnitudes of the quantities involved, but also with regard to their signs.

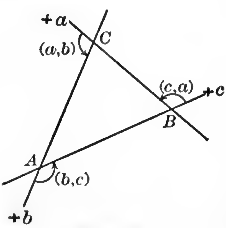

FIG. 124

In order to explain this statement, consider first the case that the positive directions of the three directed lines a, b, c (the sides of the triangle) are from B toward C, from C toward A, and from A toward B respectively. (Cf. Fig. 124.) Then BC, CA, and AB are all positive, and the three denominators in (1) are either all positive or all negative.

In Fig. 124 they are all positive. But they would all be negative if the clockwise rotation had been chosen as positive direction of rotation. They would all be negative, even without any change in the choice of the positive direction of rotation, if the points named A and B in Fig. 124 were interchanged.

Consequently equations (1) are true in this case, not merely numerically, but also with regard to sign.

Let us now invert the positive direction of a single one of the lines, that of a, for instance. Then BC becomes negative; the angle (b, c) and the segments CA and AB remain unaltered; (c, a) and (a, b) each change by 180°, so that their sines change sign. Consequently equations (1) will still be verified.

By combining several such changes we easily arrive at the conclusion that equations (1) will always be true, in magnitude and sign, no matter how the positive directions of the three lines a, b, c may have been selected or which of the two opposite kinds of rotation be regarded as positive.

This generalization of the law of sines is due to the great geometer Möbius (1790–1868), and is of great importance in many applications, especially in projective geometry.

EXERCISE LII

1. Show that the principal theorem of Art. 90 may be enunciated as follows. If a finite number of directed line-segments form a closed polygon, the sum of their projections upon any directed line is equal to zero.

2. Show that the x-component of the resultant of two forces (cf. Art. 58) is equal to the sum of the x-components of the two original forces. Similarly for the y-components.

3. From formula (1) of Art. 93 deduce the formula for $\sin(\alpha - \beta)$.

4. From formula (1) of Art. 93 deduce the formulæ for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$.

5. Generalize the law of tangents from the standpoint of directed line-segments in a way which shall be analogous to the generalization of the law of sines carried out in Art. 94.

Hint. The law of tangents may be deduced from the law of sines.

6. Generalize in similar fashion the projection formulæ (3) and (4) of Art. 90.

CHAPTER XIII

THE INVERSE TRIGONOMETRIC FUNCTIONS AND TRIGONOMETRIC EQUATIONS

95. The problem of inverting the trigonometric functions. In the light of our recent studies we may say, with a considerable degree of propriety, that *trigonometry is a discussion of the properties of the trigonometric functions.* However, our discussion of these functions, so far, has been somewhat one-sided. With the exception of a few practical applications in the first part of the book, we have always looked upon these functions from the point of view of what may be called the *direct problem;* that is, given the angle, to find the function. We now propose to discuss, somewhat more fully than has been done so far, the *inverse problem;* that is, given the value of one of the functions, to find the corresponding angles.

We notice at once a fundamental difference between these two problems, which may be illustrated by the relation between an angle and its sine. Every angle has only one sine, but there are many angles which have the same sine. Consequently while *the direct problem* (to find the sine of a given angle) *has only one solution, the inverse problem* (to find an angle with a given sine) *has many solutions.*

96. Determination of all of the angles which correspond to a given value of one of the functions. When we are solving triangles, the angles are necessarily either in the first or second quadrant, so that in most cases we experience little difficulty in finding a unique angle as an answer to a given problem of this kind. But, even in this restricted field, we found that a problem may have *two* solutions, owing to the fact that there exists two angles θ, one acute and one obtuse, corresponding to a given positive value of $\sin\theta$. (Cf. Art. 55.)

If the sine of an angle is given, and no particular quadrant or set of quadrants is prescribed for the angle, the number of values which the angle may have is unlimited.

We may prove this and find out how all of these angles are related to each other as follows:

Let the given value of the sine be denoted by s. If s is numerically greater than unity, the problem has no solution, since no angle has a sine numerically greater than 1. If s is numerically less than 1 and positive, there exists an acute angle θ and an obtuse angle $180° - \theta$, such that

$$\sin\theta = \sin(180° - \theta) = s.$$

But, on account of the periodic character of the sine, we also have

$$\sin(\theta + n \cdot 360°) = \sin\theta = s,$$
$$\sin(180° - \theta + n \cdot 360°) = \sin(180° - \theta) = s,$$

where n is any positive or negative integer or zero. We see therefore that all angles, such as

(1) $$n \cdot 360° + \theta = 2n \cdot 180° + \theta,$$

or

(2) $$n \cdot 360° + 180° - \theta = (2n + 1) \cdot 180° - \theta,$$

have the same sine as the angle θ. This may be expressed as follows. All of those angles which can be obtained by adding a given angle to any even multiple of 180° or else by subtracting the given angle from any odd multiple of 180°, have the same sine.

That these are the only angles which have the same sine follows easily from the fact that two distinct angles in the *same quadrant* cannot have the same sine.

If the given value of s is negative, nothing essential is changed in the above argument, except that θ will then be in the third or fourth quadrant instead of being an acute angle. It will still be true that all of the angles given by formulæ (1) and (2), and no others, have the same sine as θ.

Now we may include all of the angles (1) and (2) in the single expression

(3) $$m \cdot 180^\circ \pm \theta,$$

where m may be any integer, even or odd, and where the + or − sign is to be used according as m is even or odd. We may avoid this awkward distinction by writing in place of (3)

(4) $$m \cdot 180^\circ + (-1)^m \theta,$$

since $(-1)^m$ is equal to $+1$ or -1 according as m is even or odd.

Thus, *if θ is one angle whose sine is equal to a given number s, the most general angle which has the same sine is*

$$m \cdot 180^\circ + (-1)^m \theta,$$

where m is any positive or negative integer or zero.

A brief way of indicating this fact is given by the following equations:

(5) $$\sin [m \cdot 180^\circ + (-1)^m \theta] = \sin \theta$$

or

$$\sin [m\pi + (-1)^m \theta] = \sin \theta,$$

where we use the first or the second form according as θ is expressed in degrees or in radians. Since the cosecant of an angle is the reciprocal of its sine, we have also

(6) $$\csc [m \cdot 180^\circ + (-1)^m \theta] = \csc \theta$$

or

$$\csc [m\pi + (-1)^m \theta] = \csc \theta.$$

We find, by precisely similar considerations,

(7) $$\begin{aligned} \cos (2\,m \cdot 180^\circ \pm \theta) &= \cos \theta, \\ \sec (2\,m \cdot 180^\circ \pm \theta) &= \sec \theta, \end{aligned} \quad \text{or} \quad \begin{aligned} \cos (2\,m\pi \pm \theta) &= \cos \theta, \\ \sec (2\,m\pi \pm \theta) &= \sec \theta, \end{aligned}$$

and

(8) $$\begin{aligned} \tan (m \cdot 180^\circ + \theta) &= \tan \theta, \\ \cot (m \cdot 180^\circ + \theta) &= \cot \theta, \end{aligned} \quad \text{or} \quad \begin{aligned} \tan (m\pi + \theta) &= \tan \theta, \\ \cot (m\pi + \theta) &= \cot \theta, \end{aligned}$$

according as the angles are expressed in degrees or in radians.

EXERCISE LIII

Without making use of the trigonometric tables, find all of the angles which satisfy the following conditions:

1. $\sin\theta = \frac{1}{2}$.
2. $\cos\theta = -\frac{1}{2}$.
3. $\tan\theta = 1$.
4. $\sec\theta = \sqrt{2}$.
5. $\cot\theta = \sqrt{3}$.
6. $\csc\theta = 1$.
7. $\tan\theta = \infty$.
8. $\sin\theta = 0$.

Making use of the tables, find all of the angles which satisfy the following equations:

9. $\sin\theta = -.4721$.
10. $\cos\theta = +.3216$.
11. $\tan\theta = 1.7269$.
12. $\sec\theta = 2.7213$.

97. The inverse trigonometric functions. In the equation

(1) $$x = \sin y,$$

we now propose to regard y, the angle or arc, as a function of x, the sine. We may express this new way of looking at the relation (1), by saying that

y is an angle whose sine is equal to x

or

(2) *y is an* arc *whose* sine *is equal to x,*

a statement which is usually written in the contracted form

(3) $$y = \textbf{arc sin } x.$$

It should be noted that (1) and (3) are merely different ways of expressing the same relation between x and y. They differ only in one respect. In (1) y is regarded as given and x is to be found, while in (3) x is regarded as given and y is to be found. The relation between the functions (1) and (3), that of being *inverses* of each other, is of the same kind as in the more familiar case of the function $x = y^2$, which has as its inverse $y = \pm\sqrt{x}$.

In the same way we define the equation

$$y = \textbf{arc cos } x$$

to mean that y is an arc whose cosine is equal to x. Therefore this equation is equivalent to

$$x = \cos y.$$

Similarly, if $x = \tan y$, we write

$$\boldsymbol{y = \textbf{arc tan}\ x,}$$

and in the same way we define the symbols

$$\textbf{arc cot}\ x, \qquad \textbf{arc sec}\ x, \qquad \textbf{arc csc}\ x.$$

Let us return to equations (1) and (3). We know that the sine of an angle is never numerically greater than 1. Consequently we can find no angle whose sine is equal to x if x is numerically greater than 1. We may express this as follows:

The function arc sin x is defined only for those values of x which are not numerically greater than 1.

If x is numerically less than unity, there exists not merely *one* angle whose sine is equal to x, but the number of such angles is unlimited. (See Art. 96.) If x is positive, one of the corresponding angles is a positive acute angle. If x is negative, the smallest corresponding *positive* angle is in the third quadrant. But in this case there is a *negative* acute angle whose sine is equal to the given negative value of x.

Let us use the symbol

$$\textbf{Arc sin}\ x,$$

distinguished from arc sin x by the use of the capital letter A, to indicate the numerically smallest angle or arc whose sine is equal to x.

The function Arc sin x, like arc sin x, is defined only for the values of x between -1 and $+1$; that is, for those values of x for which

$$-1 \leqq x \leqq +1.$$

But for every such value of x, Arc sin x has only one value, while arc sin x has infinitely many values. For positive values of x, Arc sin x is a positive acute angle, and for negative values of x it is a negative acute angle. No value of Arc sin x ever exceeds the limits $\pm 90°$ or $\pm \frac{\pi}{2}$ radians, so that we shall always have

$$-\frac{\pi}{2} \leqq \text{Arc}\sin x \leqq \frac{\pi}{2}. \tag{5}$$

We shall henceforth speak of Arc sin x as the **principal value** *of arc sin x*, and we know from equation (5) of Art. 96 that

$$\text{(6)} \qquad \text{arc}\sin x = m\pi + (-1)^m \text{Arc}\sin x,$$

where m is any positive or negative integer or zero.

In many books the symbol arc sin x is used in the restricted sense which we have given to Arc sin x. For some purposes the distinction between the two functions arc sin x and Arc sin x is not important. But for certain other questions, a careful discussion of the principal value is the only way to avoid hopeless confusion.

This whole matter will become very clear if we make a graph of the function

$$\text{(3)} \qquad y = \text{arc}\sin x.$$

Since this equation between x and y has the same significance as

$$\text{(1)} \qquad x = \sin y,$$

we may plot the latter relation instead of (3). But we have already studied the graph of the similar equation

$$\text{(7)} \qquad y = \sin x \text{ (see Arts. 73 and 78)},$$

and clearly the graph of (1) may be obtained from that of (7) by interchanging x and y. In other words the graph of (1), or what amounts to the same thing, the graph of (3), is a sine curve placed in a vertical position. (See Fig. 125.)

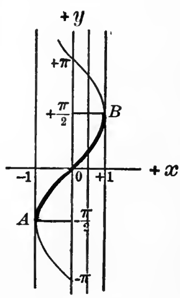

FIG. 125

The graph shows clearly that the function

$$y = \text{arc}\sin x$$

is not defined for values of x which are numerically greater than unity. It also shows that for every admissible value of x there are an infinite number of values of y, viz., the ordinates of all of the points P_1, P_2, P_3, etc., in which a line parallel to the y-axis, at a distance x, intersects the curve.

But the curve which corresponds to the principal value of arc sin x,

$$y = \text{Arc}\sin x,$$

consists merely of that portion of the graph of arc sin x which lies between the points A and B. This part of the curve is indicated in Fig. 125 by a heavier line.*

In a similar way we define a principal value Arc cos x for the function arc cos x. *The function arc cos x is defined for all values of x which are not numerically greater than* 1, *and has infinitely many values for every admissible value of x. It is convenient to define as its principal value Arc cos x, the smallest positive angle whose cosine is equal to x, so that Arc cos x is subject to the following inequalities*

$$0 \leqq \text{Arc}\cos x \leqq \pi.$$

The detailed discussion of these statements is left to the student as an exercise.

It should be remarked that the notations $\sin^{-1} x$, $\cos^{-1} x$, etc., are also in use for arc sin x, arc cos x, etc. This second notation, which is frequently used in other branches of mathematics, has the advantage of emphasizing the fact that $\sin x$ and $\sin^{-1} x$ are inverse functions of each other. But it has the disadvantage of colliding with the customary notation for exponents, and therefore tends to create confusion. Thus we usually write $\sin^2 x$ for $(\sin x)^2$, and a^{-1} for $1/a$. Thus the symbol $\sin^{-1} x$ might consistently be interpreted to mean

$$\frac{1}{\sin x} = \csc x,$$

which is something entirely different from arc sin x.

* If we had chosen as principal value of arc sin x the smallest positive angle whose sine is equal to x, the principal value would not be represented by a continuous (unbroken) curve. One part of this curve would be OB, and the other part (corresponding to negative values of x) would be between $y = \pi$ and $y = 3\pi/2$.

EXERCISE LIV

1. Show that the function $y = \text{arc}\tan x$ is defined for all values of x. Draw the graph of the function and show that its principal value may be selected subject to the conditions

$$0 \leqq \text{Arc}\tan x \leqq \pi.$$

2. Investigate the function $\text{arc}\cot x$ in the same way.

3. Are the functions $\text{arc}\sec x$ and $\text{arc}\csc x$ defined for all values of x? Draw their graphs and choose principal values for these functions.

Find the values of the following expressions:

4. $\text{Arc}\sin \frac{1}{2}$.

5. $\text{Arc}\cos \frac{1}{2}$.

6. $\text{Arc}\tan 1$, $\text{arc}\tan 1$.

7. $\text{Arc}\cos(\frac{1}{2}\sqrt{3})$, $\text{arc}\cos(\frac{1}{2}\sqrt{3})$.

By using the table of natural functions compute the values of the following quantities:

8. $\text{Arc}\tan 1.3722 + \text{Arc}\cos 0.4321$.

9. $\text{Arc}\sin 0.3425 + \text{Arc}\cot 1.7264$.

10. Obtain the value of $\sin(\text{Arc}\sin \frac{3}{5} + \text{Arc}\sin \frac{1}{2})$.

11. Obtain the value of $\sin(\text{Arc}\sin \frac{3}{5} + \text{Arc}\cos \frac{1}{2})$.

12. If x and y are positive numbers, less than unity, show that

$$\sin(\text{Arc}\sin x + \text{Arc}\sin y) = x\sqrt{1-y^2} + y\sqrt{1-x^2}.$$

Solution. For abbreviation put

$$\text{Arc}\sin x = u, \quad \text{Arc}\sin y = v.$$

Since x and y are positive numbers, less than unity, u and v will be positive acute angles such that

(1) $$\sin u = x, \qquad \sin v = y,$$

and therefore

(2) $$\cos u = \sqrt{1-x^2}, \qquad \cos v = \sqrt{1-y^2}.$$

We now find

$$\sin(\text{Arc}\sin x + \text{Arc}\sin y) = \sin(u+v) = \sin u \cos v + \cos u \sin v,$$

and therefore, on account of (1) and (2),

$$\sin(\text{Arc}\sin x + \text{Arc}\sin y) = x\sqrt{1-y^2} + y\sqrt{1-x^2}.$$

13. Show that $\cos(\text{Arc}\sin x - \text{Arc}\sin y) = \sqrt{1-x^2}\sqrt{1-y^2} + xy$ if $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

14. Show that $\text{Arc}\sin x + \text{Arc}\cos x = \frac{\pi}{2}$ if $-1 \leqq x \leqq +1$.

15. Prove the formula $\text{Arc}\tan x - \text{Arc}\tan y = \text{Arc}\tan \dfrac{x-y}{1+xy}$.

16. Do the equations of Exs. 12 and 13 undergo any modifications if negative values of x and y are admitted? Discuss in order the cases $x<0, y>0$; $x>0, y<0$; $x<0, y<0$.

98. Trigonometric equations. It often happens that angles are to be determined by means of equations which they must satisfy. Such equations usually contain the trigonometric functions of the unknown angle and are then known as *trigonometric equations*.

Let us confine our attention to the case where *one unknown angle* is to be found as a solution of *one equation*. Such an equation may have one of the following forms.

I. It contains θ algebraically, but does not contain the trigonometric functions of θ.

EXAMPLE. $$\theta^2 - \frac{\pi}{2} = 0.$$

II. It contains the trigonometric functions of the angle θ in algebraic combinations, but does not contain the angle θ itself explicitly.

EXAMPLES. $\sin^2\theta - \cos\theta = 0,\quad 2\tan\theta + \cot\theta = -3.$

III. It contains the angle θ and its trigonometric functions simultaneously.

EXAMPLE. $$\theta - \frac{1}{2}\sin\theta = \frac{\pi}{2}.$$

Clearly the equations of the first type are merely algebraic equations, and their discussion properly belongs to a treatise on algebra.

It may be shown that the equations of the second type can also be reduced to algebraic equations, although it is often easier to effect their solution without so reducing them.

The equations of the third type will not be considered in this book. The solution of such equations is a difficult matter and can be accomplished in a satisfactory manner only in a few cases. It should be mentioned, however, that the method of graphs usually provides an approximate solution for such equations.

We shall now discuss a few simple examples of equations of the second type, in such a way as to illustrate the fact that their solution may be reduced to that of algebraic equations.

EXERCISE LV

1. Solve the equation $2\cos^2 x - 5 + 7\sin x = 0$.

Solution. We have $\cos^2 x = 1 - \sin^2 x$. Therefore the given equation becomes

$$-3 - 2\sin^2 x + 7\sin x = 0.$$

Consequently, if we put $\sin x = s$, we obtain the quadratic equation for s,

$$2s^2 - 7s + 3 = 0.$$

The solution of this equation gives

$$s = 3 \text{ or } s = \tfrac{1}{2}.$$

The first solution must be discarded, since $s = \sin x$ cannot be numerically greater than unity. The second solution tells us that

$$\sin x = \tfrac{1}{2},$$

so that $x = 30°$ and $x = 150°$ are the only positive angles smaller than 360° which satisfy the equation.

In examples 2 to 5, find all positive angles less than 360° which satisfy the given equations. This may be done without using any tables.

2. $\cos 2\theta + \cos\theta = -1$.

3. $\cot 2\theta + \tan\theta = -\frac{2}{3}\sqrt{3}$.

4. $\tan 2\theta = -2\sin\theta$.

5. $\sec^2\theta + \csc^2\theta = 4$.

In solving the following equations, make use of the tables:

6. $\sin x \tan x = -\frac{9}{20}$.

7. $\cos x \cot x = -\frac{5}{6}$.

8. Solve the equation $\cos 3x = \sin 2x$.

99. The equation $a\sin\theta + b\cos\theta = c$. The equations of the form

$$(1) \qquad a\sin\theta + b\cos\theta = c$$

may be solved by the method of the preceding article. But, unless the numbers a, b, c are especially simple, it is far more convenient to proceed as follows:

We introduce two auxiliary quantities l and L, such that

$$(2) \qquad \begin{aligned} l\sin L &= a, \\ l\cos L &= b, \end{aligned}$$

where, moreover, l is assumed to be positive. That it is always possible to find a positive number l and an angle L which satisfy equations (2) becomes obvious if we think of a and b as the rectangular coördinates of a point P. (See Fig. 126.) Equations (2) show that l is the radius vector OP of P and that L is the angle which OP makes with the positive direction of the x-axis. Our figure also shows us that

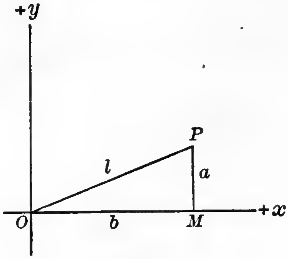

Fig. 126

$$\tan L = \frac{a}{b}, \quad l = +\sqrt{a^2 + b^2}. \tag{3}$$

Of course these same equations also follow directly from (2) without the use of geometry.

If now we substitute the values (2) for a and b in (1), we find

$$l(\sin L \sin \theta + \cos L \cos \theta) = c,$$

or by Art. 79, equation (8),

$$l \cos(\theta - L) = c. \tag{4}$$

Therefore, in order to solve (1) we may first determine l and L from (2) and then find θ from (4).

EXERCISE LVI

1. Solve the equation $2.1346 \sin \theta - 3.0526 \cos \theta = 0.9875$.

Solution.

$a = +2.1346$	(1)
$b = -3.0526$	(2)
$c = +0.9875$	(3)
$\log a = \log(l \sin L) = 0.32932$	(4)
$\log b = \log(l \cos L) = 0.48467\,n$	(5)
$\log \tan L = 9.84465\,n$	(7) = (4) − (5)
$L = 145°\ 2'.15$	(8)
$\log \sin L = 9.75820$	(9)
$\log \cos L = 9.91355\,n$	(10)
$\log l = 0.57112$	(11) = (4) − (9) = (5) − (10)
$\log c = 9.99454$	(6)
$\log \cos(\theta - L) = 9.42342$	(12) = (6) − (11)

$$\begin{aligned}
\theta_1 - L &= 74° \, 37'.61 \quad (13)\\
\theta_2 - L &= 285° \, 22'.39 \quad (14)\\
L &= 145° \quad 2'.15 \quad (8)\\
\theta_1 &= 219° \, 39'.76 \quad (15) = (13) + (8)\\
\theta_2 &= 70° \, 24'.54 \quad (16) = (14) + (8)
\end{aligned}$$

Remarks. The numbers in parentheses indicate the order in which the results are written down and how some of them are obtained. The -10 has not been written down in the case of negative characteristics. The n which follows the logarithms of b, $\tan L$ and $\cos L$ indicates that the corresponding numbers are negative. (See Art. 25.) L is chosen in the second quadrant because, l being positive, $\sin L$ is positive and $\cos L$ is negative. The values $\theta_1 - L$ and $\theta_2 - L$, both obtained from (12), are the two values, less than 360°, which $\theta - L$ may have so as to correspond to the given value (12) of $\log \cos(\theta - L)$,

Check. By substitution in the original equation,

$$\begin{aligned}
\log a &= 0.32932 & \log b &= 0.48467\, n\\
\log \sin \theta_1 &= 9.80500\, n & \log \cos \theta_1 &= 9.88639\, n\\
\hline
\log (a \sin \theta_1) &= 0.13432\, n & \log (b \cos \theta_1) &= 0.37106
\end{aligned}$$

$$\begin{aligned}
a \sin \theta_1 &= -1.3624\\
b \cos \theta_1 &= +2.3500\\
\hline
a \sin \theta_1 + b \cos \theta_1 &= +0.9876\\
c &= +0.9875
\end{aligned}$$

This calculation checks θ_1 and therefore also L. The relation between θ_1, θ_2, and L is so simple as to make unnecessary a separate check for θ_2.

2. Solve the equation

$$-3.2471 \sin \theta + 5.7469 \cos \theta = -6.3271.$$

3. Solve the equation

$$2.1725 \sin \theta + 3.2749 \cos \theta = 5.7216.$$

CHAPTER XIV

APPLICATIONS TO THE THEORY OF WAVE MOTION

100. Simple harmonic motion. When we began to study trigonometry, it was for a very practical purpose. We wished to find an arithmetical method for solving triangles. We accomplished this purpose by means of the trigonometric functions and by using tables of the numerical values of these functions. Later we generalized the notion of the trigonometric functions more than was strictly necessary for the simple problem of solving triangles, and we found it to be an interesting task to investigate these trigonometric functions and their various properties for their own sake. We shall now find that these properties, aside from their theoretical interest, have in their turn most important applications.

The fundamental reason for the great importance of the trigonometric functions lies in their *periodicity*. (Cf. Art. 67.) Many natural phenomena are periodic in character, and whenever the attempt is made to represent such a phenomenon by a mathematical expression, the trigonometric functions are found to be indispensable.

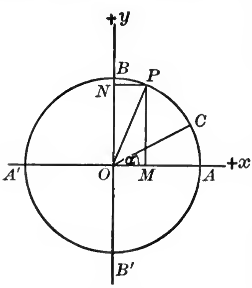

FIG. 127

The simplest periodically recurring motions are connected with uniformly rotating bodies. Let the point P (Fig. 127) describe a circular path of radius a around the point O as center, and let us assume that it is moving with a constant angular velocity of ω radians per second. Let us assume further that P starts its motion at the time $t = 0$ from the point C, which is so located that $\angle AOC$ is equal to α radians.

If P is the position of the point at the time t, that is, t seconds after the motion has begun, we shall have

$$\theta = \omega t + \alpha, \tag{1}$$

where θ denotes the angle AOP expressed in radians. The point will describe its circular path in counterclockwise or clockwise fashion according as ω is positive or negative.

While the point P is moving in its circular path, its projection M upon the x-axis will oscillate to and fro between the points A and A', reaching its greatest speed when at O, gradually slowing down until it reaches A', when it reverses the direction of its motion, returns with gradually increasing speed to O, after reaching which point it slows down again until it reaches A and again reverses its motion.

To find an analytic expression for the motion of the point M, we observe that

$$OM = x, \quad OP = a, \quad AOP = \theta, \quad OM = OP \cos AOP,$$

whence, making use of (1),

$$x = a \cos (\omega t + \alpha). \tag{2}$$

In the same way we see that N, the projection of P upon the y-axis, moves in accordance with the equation

$$y = a \sin (\omega t + \alpha). \tag{3}$$

Equations (2) and (3) are so closely related that it will suffice to study one of them. In fact, if we put in (3) $\alpha = \frac{\pi}{2} + \alpha'$, it becomes (cf. Art. 77, equations (2)),

$$y = a \sin \left(\omega t + \alpha' + \frac{\pi}{2}\right) = a \cos (\omega t + \alpha'),$$

which is of the same form as (2). We shall therefore confine our attention to equation (3).

Since any diameter of the circle may be chosen as y-axis, we may express our result as follows. *If a point describes a circular path with uniform velocity, its projection upon any fixed diameter of the circle moves in accordance with an equation of the form* (3). Such a motion is called a **simple harmonic motion.** The quantity a which measures the maximum distance of the point N from its mean position O is called the **amplitude**.

101. The period and phase constant. When the angle θ has increased from its initial value α by 2π radians, the point P will have described a complete circumference and the motion of the point N will have passed through all of its phases. The time T, which is required to accomplish this, is called the **period** of the simple harmonic motion. The period is determined by the condition that the angle ωT described by the point P in the time T must be equal to 2π. Therefore we find

$$(1) \qquad \omega T = 2\pi, \quad T = \frac{2\pi}{\omega}.$$

If we wish to put the period into evidence in the equation of a simple harmonic motion, we observe that (1) gives

$$\omega = \frac{2\pi}{T},$$

so that we may write, in place of (3), Art. 100,

$$(2) \qquad y = a \sin\left(\frac{2\pi t}{T} + \alpha\right).$$

This equation represents a simple harmonic motion of period T and of amplitude a. The quantity α is called the **phase constant.** The phase constant is an angle and enables us to calculate the distance from O to the position occupied by the moving point N at the time $t = 0$ when the motion began.

Thus, if $\alpha = 0$, the point N starts from O as its initial position and begins to move upward; if $\alpha = \frac{\pi}{2}$, the initial position of N is at B, etc.

Let us think of two different points oscillating up and down along the line BB', each in accordance with an equation of the form (2), the amplitudes and periods of the two motions being the same while the phase constants are different. Then these two points will move in exactly the same way, only that one will always be ahead of the other. The second point will appear to be imitating the motion of the first, lagging behind it in a perfectly definite fashion. This is what is meant by saying that the phases of the two motions are different, the **phase** of the motion (2) at the time t being equal to

$$\frac{2\pi t}{T}+\alpha.$$

The use of the word *phase* in this connection is not merely accidental. The appearance of the moon at a given instant (its phase) depends upon the place in its orbit around the earth which it happens to occupy at that time. By analogy we speak of a periodic phenomenon as passing through all of its phases in the course of a period, and of course two otherwise identical periodic phenomena may have their corresponding phases occur at different times. It is this difference which manifests itself in the different values of the phase constant.

We have already observed that the substitution $\alpha=\frac{\pi}{2}+\alpha'$ converts (3), Art. 100, into an equation of form (2), Art. 100. We may now express this fact as follows:

The two equations

$$y_1=a\sin\frac{2\pi t}{T},\quad y_2=a\cos\frac{2\pi t}{T}=a\sin\left(\frac{2\pi t}{T}+\frac{\pi}{2}\right)$$

represent two simple harmonic motions of amplitude a and period T, which differ only in phase, the phase difference being equal to $\frac{\pi}{2}$ *radians or* 90°.

The time interval which elapses between corresponding phases of these two motions is $\frac{1}{4}T$, that is, a quarter period.

102. Some illustrations of simple harmonic motion. The notion of simple harmonic motion is of fundamental importance in many problems of applied mathematics. The motion of a simple pendulum, the vibrations of a tuning fork, and many of the motions of elastic bodies may be described conveniently in terms of simple harmonic motion. The vertical motion of a particle of a water wave is approximately of the same type, and the whole theory of sound and light is based on the idea of harmonic motion.

EXERCISE LVII

In Exs. 1–5 the unit of time is one second and the unit of length one inch. Describe completely the simple harmonic motion given by each of these equations; that is, determine their amplitudes, periods, and phase constants, assuming equation (2) of Art. 101 as the standard form for the equation of such a motion.

1. $$y = 2.3745 \sin \frac{2\pi t}{5}.$$

2. $$y = 3.7216 \sin \left(4t + \frac{\pi}{12}\right).$$

3. $$y = 1.4712 \sin (2.7215 t + 1.7291).$$

4. $$y = 11.7261 \cos (7t).$$

5. $$y = 2.7268 \sin \left(\frac{2\pi t}{16.7214} + 13^\circ\right).$$

6. A certain pendulum has a period of oscillation of 5 seconds. Its motion is started by displacing the bob from its position of equilibrium three inches toward the right and then releasing it. Write the equation of its motion. What will be the corresponding equation for a point halfway between the lower end of the pendulum and its point of support?

Hint. Since the amplitude of the oscillation is small as compared with the length of the pendulum, the motion may be regarded as taking place approximately in a straight line. Take this line as x-axis, positive toward the right, and choose as origin the point of equilibrium. Let the time t be measured in seconds from the moment in which the pendulum is released. Then the required equation for the lower end of the pendulum will be

$$x = 3 \sin \left(\frac{2\pi t}{5} + \frac{\pi}{2}\right).$$

7. A cork is bobbing up and down owing to the passing water waves. These waves are 4 inches high (*i.e.* the difference of level between the crest and trough is 4 inches). If seventy of these waves pass in a minute, and we start to count time from one of the instants when the cork has reached its highest position, what equation will describe the motion of the cork approximately?

8. Show that the following mechanism enables us to convert uniform circular motion into simple harmonic motion. RR' (Fig. 128) is a rod which may slide back and forth in its own direction, its motion being limited by the guides A, B, A', B'. Attached to the rod there is a slot S perpendicular to the rod. A crank C, moving with uniform velocity in a circle around the point O, is made to fit accurately into the slot S. Every point of the rod RR' will then describe a simple harmonic motion.

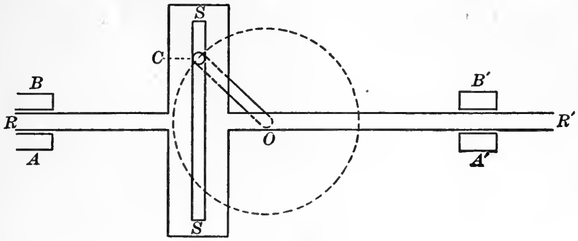

FIG. 128

9. Show that a point on the piston rod of a steam engine will move to and fro approximately according to the law of simple harmonic motion.

103. Simple harmonic curves. In the equation characteristic of a simple harmonic motion, namely,

$$y = a \sin(\omega t + \alpha),$$

let us substitute x in place of t and interpret x and y as the rectangular coördinates of a point in a plane. The curves obtained as a result of plotting such equations,

$$y = a \sin(\omega x + \alpha),$$

are called **simple harmonic curves.**

We may also think of the relation between simple harmonic motion and simple harmonic curves in the following more concrete fashion. Attach a light pin P (Fig. 129) to one of the prongs of a vibrating tuning fork, and allow it to press lightly against a strip of smoked glass. If this strip of glass is at rest, the pin will make a short straight line upon it. But if the strip be moved with a constant velocity in a direction perpendicular to that of the vibration of the tuning fork, the point P will describe a wavelike curve upon the slide. This curve may be regarded as a *record* of the motion of the tuning fork. It is easy to see that the record produced in this way by any simple harmonic motion is a simple harmonic curve.

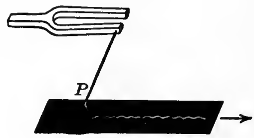

FIG. 129

The simplest case of a simple harmonic curve is that of the sine curve

$$y = \sin x,$$

whose form has, by this time, become familiar to the student. (Compare the middle curve in Fig. 130.) It has the form of a wave line with *nodes* at the points $x=0$, $x=\pm\pi$, $x=\pm 2\pi$, etc., with *crests* or *maxima* one unit high above the points

$$x=\frac{\pi}{2},\ x=\frac{5\pi}{2},\ x=\frac{9\pi}{2},\ \text{etc.},$$

of the x-axis, and with *troughs* or *minima* one unit deep below the points

$$x=\frac{3\pi}{2},\ x=\frac{7\pi}{2},\ x=\frac{11\pi}{2},\ \text{etc.},$$

of the x-axis. The length of one complete undulation of the curve is equal to 2π.

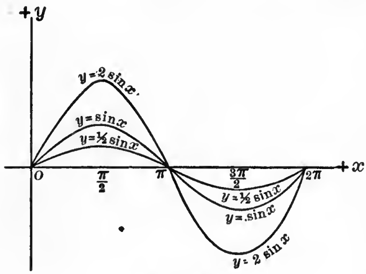

FIG. 130

104. Amplitude. It is clear that the curve

(1) $$y=a\sin x,$$

where a is any fixed positive number, is of the same general form as the sine curve. It has the same nodes (points of intersection with the x-axis), and each of its undulations has the same length as that of the sine curve. Its maxima are above the same points of the x-axis as those of the sine curve, but they are higher or lower according as a is greater or less than unity. a is called the **amplitude** of the curve. Figure 130 shows three such curves of amplitude, $\frac{1}{2}$, 1, and 2. It is clear, then, that the depth of the wavelike curve (1) is dependent upon the value of the amplitude.

105. Wave length. The two curves

$$y=\sin x \text{ and } y=\sin 2x$$

are shown together in Fig. 131. The latter curve has the same general form as the former, but each of its undulations (its wave length) is only half as long.

Similarly, the curve

(1) $$y=\sin nx,$$

where n is a positive integer, is found to be a wave line of the same height as the sine curve, but having n complete undulations between $x = 0$ and $x = 2\pi$. Consequently its wave length λ is given by

$$(2) \qquad \lambda = \frac{2\pi}{n}.$$

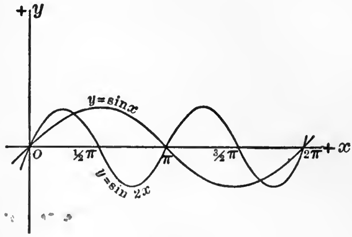

FIG. 131

Equation (1) represents a simple harmonic curve of wave-length $\lambda = \frac{2\pi}{n}$, even if n is not an integer. For the function $\sin nx$ will pass through all of its values just once, while nx changes from 0 to 2π; that is, while x changes from 0 to $\frac{2\pi}{n}$.

We may put the wave length into evidence in the equation of the curve, by solving (2) for n and substituting the resulting value of n in (1). We find $n = \frac{2\pi}{\lambda}$, and therefore

$$(3) \qquad y = \sin\frac{2\pi x}{\lambda}$$

as the equation of a simple harmonic curve of wave length λ. If we combine this result with that of Art. 104, we see that

$$(4) \qquad y = a\sin\frac{2\pi x}{\lambda}$$

represents a simple harmonic curve of wave length λ and of amplitude a.

106. Phase constant. If we compare the curves

$$y = \sin x \text{ and } y = \cos x,$$

we observe that they are identical in form. That is, both are simple harmonic curves of the same wave length and amplitude. They differ only in position. We can, in fact, slide one of these curves along the x-axis in such a way as to make it coincide with the other. This, as we have observed before (Art. 78), is the geometrical significance of the equation

$$\sin\left(\frac{\pi}{2}+x\right)=\cos x.$$

We may therefore dispense with the cosine curve altogether and regard it as a displaced sine curve. More generally, the same thing is true of the curve

(1) $$y=\sin(x+\alpha),$$

which coincides with the sine curve if $\alpha=0$ and with the cosine curve if $\alpha=\frac{\pi}{2}$. *Equation* (1) *represents a sine curve displaced toward the left through a distance of* α *units.* The quantity α is called the **phase constant** of this curve.

If we combine the results of Arts. 104 and 105 with our latest remark, we see that

(2) $$y=a\sin\left(\frac{2\pi x}{\lambda}+\alpha\right)$$

represents a simple harmonic curve of wave length λ, *amplitude* a, *and phase constant* α.

This equation may be put into a different form by making use of the addition formula of the sine. (See Art. 79.) For we have

$$y=a\left[\sin\frac{2\pi x}{\lambda}\cos\alpha+\cos\frac{2\pi x}{\lambda}\sin\alpha\right].$$

Consequently, if we put

(3) $$m=a\cos\alpha,\quad n=a\sin\alpha,$$

we shall find

(4) $$y=m\sin\frac{2\pi x}{\lambda}+n\cos\frac{2\pi x}{\lambda}.$$

Conversely, any equation of the form (4) *has a simple harmonic curve as its graph.*

For if m and n are given, we may compute a and α from (3), thus obtaining the value of amplitude and phase constant. Since a is positive, we have, from (3),

$$(5) \qquad a = +\sqrt{m^2+n^2}, \quad \tan\alpha = \frac{n}{m},$$

the quadrant of α being determined from the sign of its sine and cosine as given by (3).

EXERCISE LVIII

Draw the simple harmonic curves which correspond to the equations given in Exs. 1 to 5:

1. $y = 2\sin\left(3x + \frac{\pi}{4}\right)$.

2. $y = 3\cos\left(2x - \frac{\pi}{6}\right)$.

3. $y = 3.1\sin(7.6x + 6.2)$.

4. $y = 2.8\sin\left(\frac{2\pi x}{3.8} + 72°\right)$.

5. $y = \sin(x + \pi)$, $y = -\sin x$.

6. In the general theory of simple harmonic curves we assumed a to be a positive quantity. Show that this assumption does not, after all, really exclude from consideration those cases in which a is negative. In other words show that a simple harmonic curve, for which a is negative, coincides with another one for which a is positive and whose phase constant differs from that of the first curve by π radians.

7. Write the equations of the curves of Exs. 1 to 5 in the form (4) of Art. 106.

8. A simple harmonic curve is given by the equation

$$y = 2.75\sin\frac{2\pi x}{5.76} + 3.76\cos\frac{2\pi x}{5.76}.$$

Determine its wave length, amplitude, and phase constant.

9. Discuss in the same way the equation

$$y = 3.72\cos(7.52x) - 2.67\sin(7.52x).$$

107. Wave motion. Our use of the word *wave*, in connection with simple harmonic curves, is not quite in accordance with the accepted meaning of this term. Ordinarily when we speak of a wave, a water wave, for example, we mean a peculiar kind of motion. If the cross section of the surface of the water at a given instant is a simple harmonic curve, we should properly speak of this curve, not as the wave, but as the instantaneous *profile of the wave.** It is characteristic of a wave that this profile is in motion.

We shall obtain an excellent idea of wave motion by allowing a simple harmonic curve to glide along the x-axis with a uniform velocity v. We shall call such a wave a *simple harmonic wave*, and v its *velocity of propagation.*

Let t denote the time (expressed in seconds), and let us assume that v, the velocity of propagation (expressed in feet per second), is positive, so that the wave advances in the direction of the positive x-axis. Let the simple harmonic curve

(1) $$y = a \sin\left(\frac{2\pi x}{\lambda} + \alpha\right)$$

be the wave profile at the time t. The profile of the wave at the time $t = 0$ (t seconds earlier) was a curve of the same form as (1), but situated farther toward the left. Therefore its equation can differ from (1) only in the value of the phase constant. Consequently we may assume that

(2) $$y = a \sin\left(\frac{2\pi x}{\lambda} + \alpha_0\right)$$

is the equation of the wave profile at the time $t = 0$. We wish to find the relation between α, α_0, v, λ, and t.

The nodes of the wave profile at the time $t = 0$, that is, its intersections with the x-axis, are obtained from (2) by equating y to zero. These nodes (see Fig. 132), the points M_0, M_0', M_0'', etc., are infinite in number, and the distance between two consecutive ones is equal to $\frac{1}{2}\lambda$ or one half of the wave length. One of these nodes, M_0 say, will be obtained by equating $\frac{2\pi x}{\lambda} + \alpha_0$ to zero. Since OM_0 is the abscissa of the point M_0 and since for this point

$$\frac{2\pi x}{\lambda} + \alpha_0 = 0,$$

* The wave profile is the cross section which one would obtain of the surface of the water if it were to freeze suddenly while a wave is passing.

we shall have

(3) $$OM_0 = -\frac{\lambda\alpha_0}{2\pi}.$$

After t seconds, the wave profile has moved from its original position $M_0A_0M_0' \cdots$ to $MAM' \cdots$. The equation of the wave profile is now given by (1). The nodes of this profile are the points of the curve for which

$$\frac{2\pi x}{\lambda} + \alpha = k\pi,$$

where k is either equal to zero or to a positive or negative integer. Consequently these nodes are the points of the x-axis whose abscissas have the values

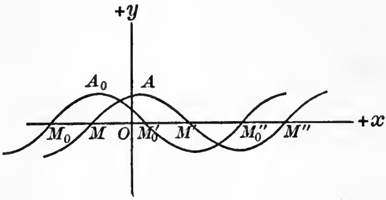

FIG. 132

$$x = -\frac{\lambda\alpha}{2\pi} + \tfrac{1}{2}k\lambda, \text{ where } k = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

One of these points is the new position M occupied by the point M_0 as a consequence of the motion of the wave profile from $M_0A_0M_0' \cdots$ to $MAM' \cdots$. Since OM is the abscissa of M, we shall therefore have

(4) $$OM = -\frac{\lambda\alpha}{2\pi} + \tfrac{1}{2}k\lambda,$$

where k is a definite positive or negative integer or zero whose precise value remains to be determined.

But if v is the velocity of propagation of the wave in feet per second, every point on it moves through a distance of vt feet in t seconds. Therefore

(5) $$OM - OM_0 = vt.$$

If we substitute in this equation the values (3) and (4) for OM and OM_0, we find

(6) $$\frac{\lambda\alpha_0}{2\pi} - \frac{\lambda\alpha}{2\pi} + \tfrac{1}{2}k\lambda = vt.$$

This equation must be true for all values of t. In particular it must be true for $t=0$. But for $t=0$ we have $\alpha=\alpha_0$, so that the equation involves a contradiction unless $k=0$. Consequently the integer k which appears in equations (4) and (6) must be equal to zero, and we have

$$\frac{\lambda\alpha_0}{2\pi}-\frac{\lambda\alpha}{2\pi}=vt, \text{ or } \frac{\lambda}{2\pi}(\alpha_0-\alpha)=vt;$$

whence

(7) $$\alpha=\alpha_0-\frac{2\pi vt}{\lambda}.$$

If we substitute this value of α in (1), we find

(8) $$y=a\sin\left[\frac{2\pi}{\lambda}(x-vt)+\alpha_0\right]$$

as the general equation of a simple harmonic wave of amplitude a and wave length λ, *whose velocity of propagation is equal to v.* We may still speak of α_0 as the phase constant. It is the phase constant of the wave profile at the time $t=0$. The phase constant of the profile curve at any other instant may be computed from (7).

If in (8) we assign a fixed value to t, we obtain the equation of the wave profile at that instant. Let us instead assign a fixed value to x, so that y becomes a function of t alone. In the case of a water wave this would amount to a study of the upward and downward oscillations of a cork floating upon the water. We shall naturally inquire as to the length of time which is required to complete such an oscillation. This time is called the *period* and may be denoted by T. Clearly T is the time which must be added to t so as to change the argument of the sine function in (8) by $\pm 2\pi$. Now if we increase t by T without changing x, this argument changes by $-\frac{2\pi vT}{\lambda}$, and this will be equal to -2π if

$$\frac{vT}{\lambda}=1, \text{ or } T=\frac{\lambda}{v}.$$

Therefore, *if v is the velocity of propagation, λ the wave length, and T the period, we have the relation*

$$\lambda = Tv. \tag{9}$$

Instead of (8) we may now write

$$y = a \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) + \alpha_0\right] \tag{10}$$

as the general equation of a wave of length λ, period T, amplitude a, and phase constant α_0.

We see finally that *a simple harmonic wave has for its profile at any moment a simple harmonic curve, and that every point upon it oscillates up and down in accordance with the law of simple harmonic motion.*

EXERCISE LIX

If the units of length and time are feet and seconds respectively, compute the wave length, period, amplitude, and phase constant of each of the waves represented by the following equations:

1. $y = 3 \sin\left[2\pi\left(\frac{x}{5} - \frac{t}{15}\right) + \frac{\pi}{2}\right].$
2. $y = 2 \sin\left(3x - 7t + \frac{\pi}{4}\right).$
3. $y = a \sin(bx + ct + d).$

108. General harmonic motion. A simple harmonic motion is the simplest case of an oscillatory phenomenon, and many natural periodic motions may be adequately described as simple harmonic motions. We have already given some examples of such cases. But very frequently the motion, although of an oscillatory character, is more complicated. In the case of a water wave, for instance, we see that the profile of the wave is not a simple harmonic curve, but that there are smaller waves (ripples so to speak) running along the backs of the larger ones, thus complicating the motion. Simple experiments show that the sound waves produced by a tuning fork are very approximately represented by simple harmonic motion; but other musical instruments, such as the violin, the piano, the human voice, produce sound waves which resemble the more complicated water waves.

A tuning fork which makes 129 oscillations in a second causes a certain simple harmonic motion of the air particles whose period is $\frac{1}{129}$th of a second and which produces a certain tone usually denoted by C. If the same note is struck on the piano, it is found that the principal part of the motion of the air particles again has $\frac{1}{129}$th of a second as its period. But the motion is not simply harmonic. It is a combination of this fundamental motion with one twice as fast, with another three times as fast, and so on. In other words, the motion of the air particles is given by an equation of the form

$$(1) \quad y = a_1 \sin\left(\frac{2\pi}{T}t + \alpha_1\right) + a_2 \sin\left(\frac{4\pi}{T}t + \alpha_2\right) + a_3 \sin\left(\frac{6\pi}{T}t + \alpha_3\right) + \cdots,$$

where the period of the first and principal term is T, that of the second $\frac{1}{2}T$, that of the third $\frac{1}{3}T$, etc.

This is not the place to discuss details of the theory of sound. Our purpose in entering upon this theory at all was merely to explain one of the many instances in which sums of simple harmonic functions of the form (1) present themselves as indispensable.

We wish to learn how the various terms in (1) combine. For that purpose the length of the period T makes but little difference. We shall therefore put

$$T = 2\pi$$

since the formulæ will then assume a somewhat simpler appearance. Then (1) reduces to

$$(2) \quad y = a_1 \sin(t + \alpha_1) + a_2 \sin(2t + \alpha_2) + a_3 \sin(3t + \alpha_3) + \cdots.$$

Now each of these terms may be expanded in accordance with the addition theorem (Art. 79), so that

$$a_1 \sin(t + \alpha_1) = a_1 \cos\alpha_1 \sin t + a_1 \sin\alpha_1 \cos t,$$
$$a_2 \sin(2t + \alpha_2) = a_2 \cos\alpha_2 \sin 2t + a_2 \sin\alpha_2 \cos 2t, \text{ etc.}$$

Consequently, if we introduce new constants $A_1, A_2, \cdots, B_1, B_2, \cdots$ by putting

$$A_1 = a_1 \sin \alpha_1, \quad A_2 = a_2 \sin \alpha_2, \cdots,$$
$$B_1 = a_1 \cos \alpha_1, \quad B_2 = a_2 \cos \alpha_2, \cdots,$$

equation (2) becomes

$$(3) \qquad \begin{aligned} y = A_1 \cos t + A_2 \cos 2t + A_3 \cos 3t + \cdots \\ + B_1 \sin t + B_2 \sin 2t + B_3 \sin 3t + \cdots. \end{aligned}$$

Let us call the fixed point, with which the moving point would tend to coincide if the amplitudes a_1, a_2, etc., of all the simple harmonic motions of (1) were to approach the limit zero, the *center of oscillation.* We have tacitly assumed so far that the center of oscillation was the origin of coördinates. Let us drop this specialization, and let $\frac{1}{2} A_0$ be the fixed value to which y would reduce if all of the oscillations were to disappear; that is, let $\frac{1}{2} A_0$ be the ordinate of the center of oscillation.* Then we must add $\frac{1}{2} A_0$ to the right member of (3), so that we obtain finally

$$(4) \qquad \begin{aligned} y = \tfrac{1}{2} A_0 + A_1 \cos t + A_2 \cos 2t + A_3 \cos 3t + \cdots \\ + B_1 \sin t + B_2 \sin 2t + B_3 \sin 3t + \cdots \end{aligned}$$

as the typical equation of a general harmonic motion.

As in the case of a *simple* harmonic motion, we may make a graphical *record* of this motion. Suppose, for instance, that the motion to be investigated is the vibration of a metallic wire. We attach a light pen to the wire so as to enable it to write upon a strip of smoked glass. If the glass be left at rest while the wire is caused to vibrate, the pen will merely describe a straight line upon the smoked glass. If, however, the glass be moved with a rapid uniform motion at right angles to the direction of vibration of the wire, there will appear as record a wavelike curve. This curve will belong to the class considered in the next article.

109. General harmonic curves. If we put x in place of t in equation (4) of Art. 108, we find

$$(1) \qquad \begin{aligned} y = \tfrac{1}{2} A_0 + A_1 \cos x + A_2 \cos 2x + A_3 \cos 3x + \cdots \\ + B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \cdots. \end{aligned}$$

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* The reason for denoting this quantity by $\frac{1}{2} A_0$ rather than A_0 will appear later. (See Art. 112.)

The curves which are obtained as a result of plotting an equation of this form are called *general harmonic curves* and are capable of an extraordinary variety of forms. In fact it can be shown, by methods involving the integral calculus, that an *infinite* series of the form (1) may be found to represent almost *any* continuous curve, and even extensive classes of discontinuous curves*. In this book, however, we are concerned only with sums of the form (1) involving a *finite* number of terms and the curves represented by them. The name *harmonic curves* will be understood to apply only to such curves.

We proceed to discuss an example. Let us plot the curve whose equation is

(2) $$y = \sin x + \sin 2x.$$

We begin by drawing the two familiar curves

(3) $$y_1 = \sin x \text{ and } y_2 = \sin 2x,$$

the two dotted curves of Fig. 133. From these curves it is easy to construct the curve (2). For we see from (2) and (3) that

$$y = y_1 + y_2$$

for every value of x. If then we find the ordinate of each of the two dotted curves for a given value of x, their algebraic sum will be the ordinate of a point on the required curve. The resulting curve is indicated in Fig. 133 by a full line. A few points of this curve may easily

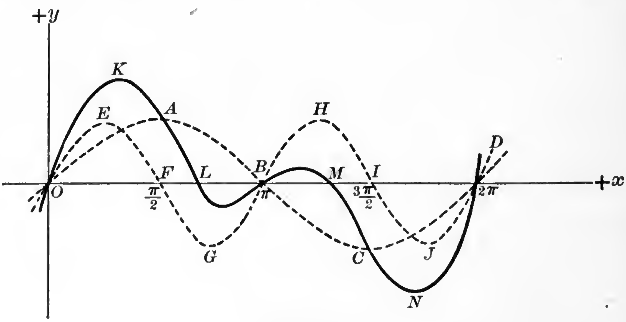

FIG. 133

* Such series are usually called FOURIER's series in honor of the great mathematical physicist who first stated, and in part proved, the above theorem. The first rigorous proof was furnished much later by DIRICHLET.

be obtained by inspection. For $x = 0$, y_1 and y_2 are both zero, and therefore also $y = y_1 + y_2 = 0$. Consequently the point O is on the curve. For $x = \pi/2$, $y_1 = 1$ and $y_2 = 0$, so that $y = 1$ and the curve passes through the point A. For values of x between $\pi/2$ and π, y_2 is negative so that $y_1 + y_2$ will be less than y_1. Consequently the full line curve in this interval lies below the corresponding portion AB of the curve $y_1 = \sin x$. For a certain value of x in this interval (determined by the equation $\sin x + \sin 2x = 0$) y_1 and y_2 will be numerically equal but opposite in sign, so that at that point the curve $y = \sin x + \sin 2x$ will cross the x-axis. This is the point L of Fig. 133. For any value of x we may obtain y_1 and y_2 by measurement from the two dotted curves. If we form the sum of these two quantities with due regard to sign, we find the corresponding ordinate of the required curve.*

EXERCISE LX

Plot the following harmonic curves:

1. $y = 2 \sin x + \sin 2x$.

2. $y = \sin x + \frac{1}{2} \sin 2x$.

3. $y = \sin x + \cos 2x$.

4. $y = 5 \sin x + \sin 4x$.

110. Harmonic analysis or trigonometric interpolation. We have seen how a number of simple harmonic curves may be compounded into a single general harmonic curve. It often happens that a curve is given, actually drawn out on paper, as for instance in the case of a self-recording barometer or thermometer. If the curve is of a periodic character, the question arises whether it may be regarded as a harmonic curve, that is, whether it is possible to compound it out of a number of simple harmonic curves by the method of Art. 109. And if so, the problem presents itself to actually find the component simple harmonic curves. The process of solving this problem is known as *harmonic analysis* and is of great importance in many branches of pure and applied mathematics.

Let us suppose that the given curve is periodic, so that it

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* Michelson and Stratton have devised a machine for performing mechanically the operation of combining a number of simple harmonic curves. This machine is also capable of performing the inverse operation discussed in Art. 110. For this reason it has been called a *harmonic analyzer*.

consists of an infinite number of equal pieces, and let the length of one of these pieces, the wave length, be equal to λ. If λ is less than 2π, we may, by a process of magnification or stretching, replace the given curve by another one similar to it whose wave length is just equal to 2π. If we can solve our problem for this second curve of wave length 2π, it will be easy to solve the corresponding problem for the original curve. If λ is greater than 2π, a process of compression enables us to reduce the problem again to the case of a curve of wave length 2π. We may therefore assume that the wave length of the given periodic curve is equal to 2π without reducing, by this assumption, in any essential fashion the general applicability of our results. The object of this assumption is merely to simplify the resulting formulæ.

Let $P_0P_1P_2 \cdots$ (Fig. 134) be the given curve of wave length $OA = 2\pi$, and let us divide OA into an odd number,

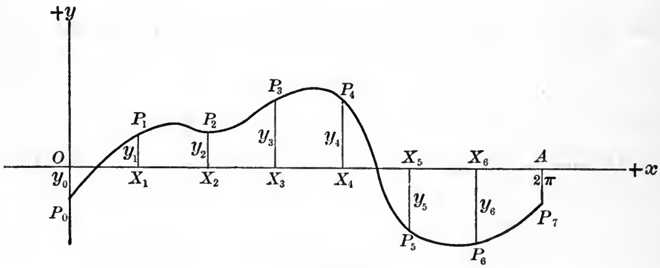

FIG. 134

say $2m+1$, of equal parts. Not counting A, there will then be $2m+1$ points of division; namely, O, X_1, X_2, $\cdots X_{2m}$. In Fig. 134 we have made $2m+1=7$.

At these points of division we construct the ordinates

$$OP_0, X_1P_1, X_2P_2, \cdots$$

of the curve. Let $y_0, y_1, y_2, \cdots y_{2m}$ be these ordinates, each with its proper sign prefixed. On account of the periodic character of the curve, the ordinate at A will be the same as that at O. This is the reason that we did not count A as one of the points of division. If we had included A, we should really have been counting O twice.

We can always find a harmonic curve involving terms in x, $2x$, $3x$, $\cdots mx$, which passes through the $2m+1$ points P_0, P_1, P_2, $\cdots P_{2m}$. For the general equation of such a harmonic curve is

$$(1) \qquad \begin{aligned} y = \tfrac{1}{2} A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_m \cos mx \\ + B_1 \sin x + B_2 \sin 2x + \cdots + B_m \sin mx, \end{aligned}$$

and therefore contains $2m+1$ coefficients A_0, A_1, $\cdots A_m$, B_1, $\cdots B_m$, which may be determined in such a way as to make the corresponding curve (1) pass through the $2m+1$ given points. In fact, the curve (1) will pass through the point P_0, if the value of y obtained from (1), for $x = 0$, is equal to the ordinate y_0 of the given point P_0; that is, if

$$(2) \qquad y_0 = \tfrac{1}{2} A_0 + A_1 + A_2 + \cdots + A_m.$$

The abscissa of P_1 is $OX_1 = \dfrac{2\pi}{2m+1}$. Therefore the curve (1) will pass through P_1, if the value of y obtained from (1), for $x = \dfrac{2\pi}{2m+1}$, is equal to the ordinate y_1 of the given point P_1; that is, if

$$(3) \qquad \begin{aligned} y_1 = \tfrac{1}{2} A_0 + A_1 \cos\frac{2\pi}{2m+1} + A_2 \cos\frac{2\cdot 2\pi}{2m+1} \\ + \cdots + A_m \cos\frac{m\cdot 2\pi}{2m+1} \\ + B_1 \sin\frac{2\pi}{2m+1} + B_2 \sin\frac{2\cdot 2\pi}{2m+1} \\ + \cdots + B_m \sin\frac{m\cdot 2\pi}{2m+1}. \end{aligned}$$

In the same way we find that the curve (1) will pass through the points P_2, P_3, $\cdots P_{2m}$ if the following additional equations are satisfied;

$$(4)\quad\begin{cases} y_2 = \frac{1}{2}A_0 + A_1\cos\frac{4\pi}{2m+1} + A_2\cos\frac{2\cdot 4\pi}{2m+1} + \cdots \\ \quad + A_m\cos\frac{m\cdot 4\pi}{2m+1} + B_1\sin\frac{4\pi}{2m+1} + B_2\sin\frac{2\cdot 4\pi}{2m+1} \\ \quad + \cdots + B_m\sin\frac{m\cdot 4\pi}{2m+1}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ y_{2m} = \frac{1}{2}A_0 + A_1\cos\frac{2m\pi}{2m+1} + A_2\cos\frac{2\cdot 2m\pi}{2m+1} + \cdots \\ \quad + A_m\cos\frac{m\cdot 2m\pi}{2m+1} + B_1\sin\frac{2m\pi}{2m+1} + B_2\sin\frac{2\cdot 2m\pi}{2m+1} \\ \quad + \cdots + B_m\sin\frac{m\cdot 2m\pi}{2m+1}. \end{cases}$$

Now the equations (2), (3), and (4) are together $2m+1$ in number, and involve $2m+1$ unknown quantities, viz., the quantities, A_0, A_1, $\cdots$ A_m, B_1, $\cdots$ B_m. The coefficients of these quantities (sines and cosines of known angles) are known numbers, and the quantities y_0, y_1, $\cdots$ y_{2m}, the ordinates of the given points P_0, P_1, $\cdots$ P_{2m}, are known. We may therefore, in general, solve these $2m+1$ equations for the $2m+1$ unknown quantities and then substitute the values obtained for A_0, A_1, $\cdots$ A_m, B_1, $\cdots$ B_m in (1). The harmonic curve obtained by plotting (1) will then pass through the given points.

If the given curve is fairly smooth, and if the points P_0, P_1, $\cdots$ P_{2m} be taken close enough together, that is, if m be chosen large enough, we may find by the above method a harmonic curve which may be regarded as replacing the original curve everywhere with any desired degree of approximation.*

The following example will illustrate this method. Let y be a periodic function of x, of period 2π, such that the values of y which correspond to

$$x = 0°,\ 72°,\ 144°,\ 216°,\ 288°$$

* A more precise formulation of this statement must be left for a much later stage of the student's mathematical career.

are respectively

$$y = 0, \ +2, \ +\tfrac{1}{2}, \ -\tfrac{1}{2}, \ -2.$$

Our general theory tells us that we may find an expression of the form

$$y = \tfrac{1}{2} A_0 + A_1 \cos x + A_2 \cos 2x + B_1 \sin x + B_2 \sin 2x,$$

which assumes the five values assigned to y for the five given values of x. Moreover we have the following five equations for the five unknown coefficients A_0, A_1, A_2, B_1, B_2:

$$\begin{cases} 0 = \tfrac{1}{2} A_0 + A_1 + A_2, \\ 2 = \tfrac{1}{2} A_0 + A_1 \cos 72° + A_2 \cos 144° + B_1 \sin 72° + B_2 \sin 144°, \\ \tfrac{1}{2} = \tfrac{1}{2} A_0 + A_1 \cos 144° + A_2 \cos 288° + B_1 \sin 144° + B_2 \sin 288°, \\ -\tfrac{1}{2} = \tfrac{1}{2} A_0 + A_1 \cos 216° + A_2 \cos 72° + B_1 \sin 216° + B_2 \sin 72°, \\ -2 = \tfrac{1}{2} A_0 + A_1 \cos 288° + A_2 \cos 216° + B_1 \sin 288° + B_2 \sin 216°. \end{cases}$$

Since we have

$$\begin{aligned} \sin 216° &= \sin (360° - 144°) = - \sin 144°, \\ \cos 216° &= \cos (360° - 144°) = \cos 144°, \\ \sin 288° &= \sin (360° - 72°) = - \sin 72°, \\ \cos 288° &= \cos (360° - 72°) = \cos 72°, \end{aligned}$$

the above equations may also be written as follows:

$$\begin{aligned} &(1)\ \tfrac{1}{2} A_0 + A_1 + A_2 = 0, \\ &(2)\ \tfrac{1}{2} A_0 + A_1 \cos 72° + A_2 \cos 144 + B_1 \sin 72° + B_2 \sin 144° = 2, \\ &(3)\ \tfrac{1}{2} A_0 + A_1 \cos 144° + A_2 \cos 72° + B_1 \sin 144° - B_2 \sin 72° = \tfrac{1}{2}, \\ &(4)\ \tfrac{1}{2} A_0 + A_1 \cos 144° + A_2 \cos 72° - B_1 \sin 144° + B_2 \sin 72° = -\tfrac{1}{2}, \\ &(5)\ \tfrac{1}{2} A_0 + A_1 \cos 72° + A_2 \cos 144° - B_1 \sin 72° - B_2 \sin 144° = -2. \end{aligned}$$

From (2) and (5) we find by addition

$$(6) \qquad \tfrac{1}{2} A_0 + A_1 \cos 72° + A_2 \cos 144° = 0,$$

and similarly from (3) and (4),

$$(7) \qquad \tfrac{1}{2} A_0 + A_1 \cos 144° + A_2 \cos 72° = 0.$$

From (1) we have

$$\tfrac{1}{2} A_0 = - A_1 - A_2$$

which, substituted in (6) and (5), gives

$$(8) \qquad \begin{aligned} A_1 (\cos 72° - 1) + A_2 (\cos 144° - 1) &= 0, \\ A_1 (\cos 144° - 1) + A_2 (\cos 72° - 1) &= 0. \end{aligned}$$

If we multiply both members of the first of these equations by $\cos 72° - 1$, those of the second by $-(\cos 144° - 1)$, and add, we find

$$(9) \qquad A_1 [(\cos 72° - 1)^2 - (\cos 144° - 1)^2] = 0.$$

From the table of natural functions, we find to two decimal places

$$\cos 72° = 0.31, \ \cos 144° = - \cos 36° = - 0.81.$$

Therefore

$$\cos 72^\circ - 1 = -0.69, \quad \cos 144^\circ - 1 = -1.81,$$

so that the coefficient of A_1 in (9) is not equal to zero. Consequently we conclude from (9) that $A_1 = 0$. According to (8) and (1), we must then have also $A_2 = 0$, $A_0 = 0$.

If now we put $A_0 = A_1 = A_2 = 0$, in (1) to (5), these five equations reduce to the following two:

(10)
$$B_1 \sin 72^\circ + B_2 \sin 144^\circ = 2,$$
$$B_1 \sin 144^\circ - B_2 \sin 72^\circ = \tfrac{1}{2}.$$

From these equations we eliminate first B_2 and then B_1, giving

(11)
$$(\sin^2 72^\circ + \sin^2 144^\circ)B_1 = 2\sin 72^\circ + \tfrac{1}{2}\sin 144^\circ,$$
$$(\sin^2 72^\circ + \sin^2 144^\circ)B_2 = 2\sin 144^\circ - \tfrac{1}{2}\sin 72^\circ.$$

From the table of natural sines we find, correct to two decimal places,

$$\sin 72^\circ = 0.95, \quad \sin 144^\circ = \sin 36^\circ = 0.59,$$

so that

$$\sin^2 72^\circ + \sin^2 144^\circ = 0.90 + 0.35 = 1.25.$$

Consequently, equations (11) become

$$1.25\, B_1 = 1.90 + 0.30 = 2.20,$$

$$1.25\, B_2 = 1.18 - 0.48 = 0.70;$$

whence finally

(12)
$$B_1 = 1.76, \quad B_2 = 0.56.$$

Since we have already found $A_0 = A_1 = A_2 = 0$, the function which we were seeking is

(13)
$$y = 1.76 \sin x + 0.56 \sin 2x.$$

In order to check our result we may substitute the five given values of x in (13) and verify that the corresponding values of y are actually those which were originally given. That equation (13) gives $y = 0$ for $x = 0$ is obvious. For $x = 72^\circ$ and for $x = 144^\circ$, we find from (13)

$$y = 1.76 \times 0.95 + 0.56 \times 0.59 = 1.67 + 0.33 = 2.00,$$

and

$$y = 1.76 \times 0.59 + 0.56 \times (-0.95) = 1.04 - 0.53 = 0.51,$$

respectively, checking to within one unit of the last decimal place computed. To check the other two pairs of values requires no additional computation.

In this example, the values of y for $x = 144^\circ = 180^\circ - 36^\circ$ and for $x = 216^\circ = 180^\circ + 36^\circ$ were numerically equal but opposite in sign. The values of y for $x = 72^\circ = 180^\circ - 108^\circ$ and for $x = 288^\circ = 180^\circ + 108^\circ$ were also numerically equal but opposite in sign. It is owing to this circumstance that A_0, A_1, and A_2 turned out to be all three equal to zero. Having become aware of this fact, we may abbreviate our work very much by at once equating A_0, A_1, A_2, etc., to zero, whenever the given values of the function are numerically equal but opposite in sign for those among the given angles which differ numerically by the same amount from 180°.

Similarly, if the given values of the function y are equal numerically *and* in sign for all of those among the given angles x which differ numerically by the same amount from 180°, the expression for y will contain no sine terms; that is, B_1, B_2, B_3, etc., will all be equal to zero.

EXERCISE LXI

1. Find a periodic function, involving only the angles x and $2x$, which assumes the values

$$y = 0,\ +2,\ +1,\ -1,\ -2,$$

for

$$x = 0°,\ 72°,\ 144°,\ 216°,\ 288°,$$

respectively. Compute the coefficients to two decimal places.

2. Find a periodic function, involving only the angles x and $2x$, which assumes the values

$$y = +2,\ +1,\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ +1,$$

for

$$x = 0°,\ 72°,\ 144°,\ 216°,\ 288°,$$

respectively. Compute the coefficients to two decimal places.

111. Theorems leading to the general solution of the problem of trigonometric interpolation. We have shown in Art. 110 that the problem of trigonometric interpolation may be reduced to that of solving a system of $2m+1$ equations of the first degree with $2m+1$ unknowns. But we can accomplish much more than this. We shall derive elegant and convenient formulæ for the solutions of these equations, enabling us to find the values of the coefficients A_k and B_k by a direct and simple process. But before we can do this, we must prepare the way by proving some theorems necessary for this purpose.

We begin by proving that the following formula

$$(1)\quad \sin a + \sin(a+t) + \sin(a+2t) + \cdots + \sin(a+mt) = \frac{\sin\left(a+\frac{mt}{2}\right)\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}},$$

published by EULER * in 1743, is true for all values of a and t, provided that $\sin\frac{t}{2}$ is not equal to zero.

PROOF. Let us denote the sum in the left-hand member of (1) by s_m. If we multiply s_m by $2\sin\frac{t}{2}$, we shall have

$$2s_m\sin\frac{t}{2} = 2\sin a\sin\frac{t}{2} + 2\sin(a+t)\sin\frac{t}{2} + 2\sin(a+2t)\sin\frac{t}{2} + \cdots + 2\sin(a+mt)\sin\frac{t}{2}.$$

Every term in the right member of this equation contains a product of two sines, and may therefore be expressed as a difference of two cosines by means of formula (4) of Art. 82; that is,

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)].$$

We find, in this way,

$$\begin{aligned}2s_m\sin\frac{t}{2} = {} & \cos(a-\tfrac{1}{2}t) - \cos(a+\tfrac{1}{2}t)\\ & + \cos(a+\tfrac{1}{2}t) - \cos(a+\tfrac{3}{2}t)\\ & + \cos(a+\tfrac{3}{2}t) - \cos(a+\tfrac{5}{2}t)\\ & + \cdots\cdots\cdots\cdots\\ & + \cos\{a+(m-\tfrac{1}{2})t\} - \cos\{a+(m+\tfrac{1}{2})t\}.\end{aligned}$$

* EULER (1707–1783) was born in Switzerland, but spent most of the years of his scientific career in St. Petersburg and Berlin. His work was of fundamental importance in all parts of pure and applied mathematics. Although absolutely blind during the latter part of his life, he continued to labor and to make important contributions up to the end.

Clearly all of the terms in the right member except the first and last will destroy each other, so that we are left with the equation

$$2\,s_m \sin\frac{t}{2} = \cos(a - \tfrac{1}{2}t) - \cos\{a + (m + \tfrac{1}{2})t\}.$$

But we have the formula (see Art. 82, equations (5)),

$$\cos A - \cos B = -2\sin\frac{A-B}{2}\cos\frac{A+B}{2},$$

so that

$$2\,s_m \sin\frac{t}{2} = -2\sin\frac{1}{2}(-t - mt)\sin\frac{1}{2}(2a + mt)$$

$$= 2\sin\left(a + \frac{mt}{2}\right)\sin\frac{(m+1)t}{2},$$

whence, if $\sin\frac{t}{2}$ is not equal to zero,

$$s_m = \frac{\sin\left(a + \frac{mt}{2}\right)\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}}.$$

This is the same as equation (1), the formula which we wished to prove.

Let us rewrite formula (1) with a' in place of a, and then put

$$a' = a + \frac{\pi}{2}.$$

Since
$$\sin a' = \sin\left(a + \frac{\pi}{2}\right) = \cos a,$$

we then find

$$(2)\quad c_m = \cos a + \cos(a + t) + \cos(a + 2t) + \cdots + \cos(a + mt)$$

$$= \frac{\cos\left(a + \frac{mt}{2}\right)\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}},$$

a formula which may also be obtained directly by a process strictly analogous to the one employed for the proof of equation (1). Formula (2) is also due to EULER.

If in equations (1) and (2) we put $a=0$, we find

$$(3)\quad \sin t+\sin 2\,t+\sin 3\,t+\cdots+\sin mt=\frac{\sin\dfrac{mt}{2}\sin\dfrac{(m+1)t}{2}}{\sin\dfrac{t}{2}}$$

and

$$(4)\quad 1+\cos t+\cos 2t+\cos 3t+\cdots+\cos mt=\frac{\cos\dfrac{mt}{2}\sin\dfrac{(m+1)t}{2}}{\sin\dfrac{t}{2}}.$$

Let us subtract $\frac{1}{2}$ from both members of (4). We obtain the formula

$$\frac{1}{2}+\cos t+\cos 2\,t+\cdots+\cos mt=\frac{\cos\dfrac{mt}{2}\sin\dfrac{(m+1)t}{2}}{\sin\dfrac{t}{2}}-\frac{1}{2}$$

$$=\frac{2\cos\dfrac{mt}{2}\sin\dfrac{(m+1)t}{2}-\sin\dfrac{t}{2}}{2\sin\dfrac{t}{2}}=\frac{\sin\left(m+\dfrac{1}{2}\right)t+\sin\dfrac{t}{2}-\sin\dfrac{t}{2}}{2\sin\dfrac{t}{2}},$$

where we have made use of one of the equations (4) of Art. 82. The numerator of the last fraction obviously reduces to its first term, so that we find finally

$$(5)\quad \frac{1}{2}+\cos t+\cos 2\,t+\cos 3\,t+\cdots+\cos mt=\frac{\sin\dfrac{(2\,m+1)t}{2}}{2\sin\dfrac{t}{2}},$$

a formula which was known to SNELLIUS in 1627.

The angle t which occurs in all of these equations may have any value whatever excepting only those values for which $\sin\frac{t}{2}$ is equal to zero; that is, excepting the values $2\,k\pi$, where k is an integer. We shall now apply these formulæ to the particular case when t is a commensurable fractional part of the entire circumference, so that

$$t=\frac{k}{n}\,2\,\pi,$$

where both k and n are positive integers and where

$$n>1.$$

We shall further put $\qquad m=n-1.$

We may express these assumptions more concretely as follows. Let us divide the circumference of the circle into n equal parts, where $n>1$. Then $\frac{2\,\pi}{n}$ will be the angle subtended by one of these parts. The smallest multiple of this angle which is equal to a complete circumference is of course the nth. Therefore the angles

$$\frac{2\,\pi}{n},\ 2\frac{2\,\pi}{n},\ 3\frac{2\,\pi}{n},\cdots(n-1)\frac{2\,\pi}{n}$$

are all distinct, and we propose to find the sum of their sines as well as of their cosines. We consider next the angle $2\frac{2\,\pi}{n}=\frac{4\,\pi}{n}$ and all of its multiples up to $(n-1)\,\frac{4\,\pi}{n}$ and calculate the sum of their sines and of their cosines. In general, we consider the angle $k\,\frac{2\,\pi}{n}=\frac{2\,k\pi}{n}$ and its multiples $2\frac{2\,k\pi}{n}$, $3\frac{2\,k\pi}{n}\cdots\,(n-1)\,\frac{2\,k\pi}{n}$ and compute the sum of their sines and of their cosines.

According to (3) we have (putting $t=\frac{2\,k\pi}{n}$ and $m=n-1$)

$$\sin\frac{2\,k\pi}{n}+\sin 2\frac{2\,k\pi}{n}+\sin 3\frac{2\,k\pi}{n}+\,\cdots\,+\sin\,(n-1)\frac{2\,k\pi}{n}$$

$$=\frac{\sin\,\dfrac{(n-1)k\pi}{n}\,\sin\,k\pi}{\sin\,\dfrac{k\pi}{n}},$$

a formula which will be valid unless $\sin\,\frac{k\pi}{n}$ is equal to zero;

that is, unless k is an integral multiple of n. Excluding this case for a moment, we see that the right member is equal to zero since $\sin k\pi$ occurs as a factor in the numerator and since the sine of any integral multiple of π is equal to zero. Consider now the excluded case when k is a multiple of n. Then every term on the left member is individually equal to zero. We see therefore that

$$(6)\quad S_k = \sin\frac{2\,k\pi}{n} + \sin 2\frac{2\,k\pi}{n} + \sin 3\frac{2\,k\pi}{n} + \cdots + \sin(n-1)\frac{2\,k\pi}{n} = 0,$$

for every value of k, whether k is divisible by n or not.

We may prove in the same way, making use of equation (4), that

$$(7)\quad C_k = 1 + \cos\frac{2\,k\pi}{n} + \cos 2\frac{2\,k\pi}{n} + \cos 3\frac{2\,k\pi}{n} + \cdots + \cos(n-1)\frac{2\,k\pi}{n} = 0,$$

if k is not a multiple of n.

If k is a multiple of n, all of the angles $\frac{2\,k\pi}{n}$, $2\frac{2\,k\pi}{n}$, etc., are integral multiples of $2\,\pi$, so that each of the cosines which appears in C_k is equal to unity. There are n terms in C_k. Therefore

$$(8)\quad C_k = 1 + \cos\frac{2\,k\pi}{n} + \cos 2\frac{2\,k\pi}{n} + \cdots + \cos(n-1)\frac{2\,k\pi}{n} = n,$$

if k is a multiple of n.

Let us now consider two angles of the form $\frac{2\,k\pi}{n}$ and $\frac{2\,l\pi}{n}$, and denote by C_{kl} the sum

$$(9)\quad C_{kl} = 1 + \cos\frac{2\,k\pi}{n}\cos\frac{2\,l\pi}{n} + \cos 2\frac{2\,k\pi}{n}\cos 2\frac{2\,l\pi}{n} + \cdots + \cos(n-1)\frac{2\,k\pi}{n}\cos(n-1)\frac{2\,l\pi}{n}.$$

A product of two cosines may be expressed as a sum by means of formula (3) of Art. 82; that is,

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)].$$

Therefore we may write, if we apply this formula to each term of C_{kl},

$$\begin{aligned} C_{kl} = {} & \tfrac{1}{2}[1+1] \\ & + \tfrac{1}{2}\left[\cos\frac{2(k-l)\pi}{n} + \cos\frac{2(k+l)\pi}{n}\right] \\ & + \tfrac{1}{2}\left[\cos 2\frac{2(k-l)\pi}{n} + \cos 2\frac{2(k+l)\pi}{n}\right] \\ & + \cdots\cdots\cdots\cdots \\ & + \tfrac{1}{2}\left[\cos(n-1)\frac{2(k-l)\pi}{n} + \cos(n-1)\frac{2(k+l)\pi}{n}\right]. \end{aligned}$$

Now the sum of all of the terms in the first column may be equated to $\frac{1}{2}C_{k-l}$ if we again make use of the notation C_k defined by equations (7) and (8). Similarly we observe that the sum of the terms in the second column is $\frac{1}{2}C_{k+l}$. Therefore

$$C_{kl} = \tfrac{1}{2}(C_{k-l} + C_{k+l}). \tag{10}$$

Consider now the expression, analogous to C_{kl},

$$S_{kl} = \sin\frac{2k\pi}{n}\sin\frac{2l\pi}{n} + \sin 2\frac{2k\pi}{n}\sin 2\frac{2l\pi}{n} + \cdots + \sin(n-1)\frac{2k\pi}{n}\sin(n-1)\frac{2l\pi}{n}. \tag{11}$$

Since we have (see Art. 82)

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)],$$

we find

$$\begin{aligned} S_{kl} = {} & \tfrac{1}{2}\left[\cos\frac{2(k-l)\pi}{n} - \cos\frac{2(k+l)\pi}{n}\right] \\ & + \tfrac{1}{2}\left[\cos 2\frac{2(k-l)\pi}{n} - \cos 2\frac{2(k+l)\pi}{n}\right] \\ & + \cdots\cdots\cdots\cdots \\ & + \tfrac{1}{2}\left[\cos(n-1)\frac{2(k-l)\pi}{n} - \cos(n-1)\frac{2(k+l)\pi}{n}\right., \end{aligned}$$

so that

(12) $$S_{kl} = \tfrac{1}{2}(C_{k-l} - C_{k+l}).$$

Finally let us put

(13) $$(S_k, C_l) = \sin\frac{2\,k\pi}{n}\cos\frac{2\,l\pi}{n} + \sin 2\,\frac{2\,k\pi}{n}\cos 2\,\frac{2\,l\pi}{n} + \cdots + \sin(n-1)\,\frac{2\,k\pi}{n}\cos(n-1)\frac{2\,l\pi}{n}.$$

Since we have (see Art. 82)

$$\sin\alpha\cos\beta = \tfrac{1}{2}\,[\sin(\alpha-\beta) + \sin(\alpha+\beta)],$$

we find by a repetition of the above method

(14) $$(S_k, C_l) = \tfrac{1}{2}(S_{k-l} + S_{k+l}).$$

We have already shown (cf. equations (6), (7), and (8)) that

(15) $$\begin{cases} S_k = 0 \text{ for all values of } k, \\ C_k = 0 \text{ for all values of } k \text{ which are } \textit{not} \text{ divisible by } n, \\ C_k = n \text{ for all values of } k \text{ which } \textit{are} \text{ divisible by } n. \end{cases}$$

If we make use of these facts, equations (10), (12), and (14) now teach us that the following statements are true.

(16) $$\begin{cases} C_{kl} = 0 \text{ if } \textit{neither } k-l \textit{ nor } k+l \text{ is divisible by } n, \\ C_{kl} = \dfrac{n}{2} \text{ if } \textit{either } k-l \textit{ or } k+l \text{ is divisible by } n, \text{ but } \textit{not both}, \\ C_{kl} = n \text{ if } \textit{both } k-l \textit{ and } k+l \text{ are divisible by } n. \end{cases}$$

(17) $$\begin{cases} S_{kl} = 0 \text{ if } \textit{neither } k-l \textit{ nor } k+l \text{ is divisible by } n, \\ S_{kl} = +\dfrac{n}{2} \text{ if } k-l \textit{ is} \text{ divisible by } n \text{ and } k+l \textit{ is not} \text{ divisible by } n, \\ S_{kl} = -\dfrac{n}{2} \text{ if } k-l \textit{ is not} \text{ divisible by } n \text{ and } k+l \textit{ is} \text{ divisible by } n, \\ S_{kl} = 0 \text{ if } k-l \text{ and } k+l \text{ are } \textit{both} \text{ divisible by } n. \\ (S_k, C_l) = 0 \text{ for all values of } k \text{ and } l. \end{cases}$$

112. General solution of the problem of trigonometric interpolation. We are now ready for the solution of the problem of trigonometric interpolation. We divide the circumference into $2m+1$ equal parts by means of the angles

$$x = 0, \quad \frac{2\pi}{2m+1}, \quad \frac{4\pi}{2m+1}, \quad \frac{6\pi}{2m+1}, \quad \cdots, \quad \frac{4m\pi}{2m+1},$$

and denote the corresponding values of the function y by

$$y_0, \; y_1, \; y_2, \; y_3, \cdots y_{2m}$$

respectively. The function

$$\begin{aligned} y = \tfrac{1}{2} A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_m \cos mx \\ + B_1 \sin x + B_2 \sin 2x + \cdots + B_m \sin mx \end{aligned}$$

will actually assume these values, if the $2m+1$ quantities $A_0, A_1, \cdots, A_m, B_1, B_2, \cdots, B_m$ are chosen so as to satisfy the $2m+1$ equations

$$(1) \quad \begin{cases} y_0 = \tfrac{1}{2} A_0 + A_1 + A_2 + \cdots + A_m, \\ y_1 = \tfrac{1}{2} A_0 + A_1 \cos \dfrac{2\pi}{2m+1} + \cdots + A_m \cos m \dfrac{2\pi}{2m+1} \\ \qquad + B_1 \sin \dfrac{2\pi}{2m+1} + \cdots + B_m \sin m \dfrac{2\pi}{2m+1}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_{2m} = \tfrac{1}{2} A_0 + A_1 \cos \dfrac{4m\pi}{2m+1} + \cdots + A_m \cos m \dfrac{4m\pi}{2m+1} \\ \qquad + B_1 \sin \dfrac{4m\pi}{2m+1} + \cdots + B_m \sin m \dfrac{4m\pi}{2m+1}, \end{cases}$$

all of which may be expressed by the single equation

$$(2) \quad \begin{aligned} y_\nu = \tfrac{1}{2} A_0 + A_1 \cos \frac{2\nu\pi}{2m+1} + \cdots + A_m \cos m \frac{2\nu\pi}{2m+1} \\ + B_1 \sin \frac{2\nu\pi}{2m+1} + \cdots + B_m \sin m \frac{2\nu\pi}{2m+1}, \end{aligned}$$

if we think of ν as assuming in succession the $2m+1$ values

$$\nu = 0, 1, 2, 3, \cdots, 2m.$$

Let r be any integer between 0 and m, let us multiply y_0 by 1, y_1 by $\cos \dfrac{2r\pi}{m+1}$, y_2 by $\cos 2 \dfrac{2r\pi}{m+1}$, $\cdots$, y_ν by

$\cos \nu \dfrac{2r\pi}{m+1}$, $\cdots$, y_{2m} by $\cos 2m \dfrac{2r\pi}{2m+1}$, and add. If we make use of equations (1) and the notations introduced in Art. 111, we find

$$(3)\quad y_0 + y_1 \cos\frac{2r\pi}{2m+1} + \cdots + y_{2m}\cos 2m\frac{2r\pi}{2m+1}$$
$$= \frac{1}{2}A_0 C_r + A_0 C_{1r} + A_2 C_{2r} + \cdots + A_m C_{mr}$$
$$+ B_1(S_1, C_r) + B_2(S_2, C_r) + \cdots + B_m(S_m, C_r).$$

From Art. 111, (17), we know that all of the quantities (S_k, C_l) are equal to zero. Since r was chosen as an integer between 0 and m, r can be divisible by $2m+1$ (which number corresponds to the n of equations (15), (16), (17) of Art. 111), only if $r = 0$. In that case we shall have, according to (15) of Art. 111,

$$C_r = C_0 = n = 2m + 1,$$

and according to (16), Art. 111,

$$C_{10} = C_{20} = \cdots = C_{m0} = 0,$$

since none of the numbers $1, 2, \cdots, m$ are divisible by $2m+1$. Consequently, equation (3) reduces to

$$(4)\qquad y_0 + y_1 + y_2 + \cdots + y_{2m} = \tfrac{1}{2}A_0(2m+1)$$

in the case $r = 0$.

If $r > 0$, $C_r = 0$. Moreover, C_{kr} will be zero for all values of k for which neither $k - r$ nor $k + r$ is divisible by $2m+1$. But k as well as r can at most be equal to m, so that neither $k - r$ nor $k + r$ is ever as large as $2m+1$. The only case therefore in which one of these numbers can be divisible by $2m+1$ is when $k = r$. Thus we have, according to (16), Art. 111,

$$C_{kr} = 0 \text{ for } k \text{ different from } r,$$
$$C_{rr} = \frac{2m+1}{2}.$$

Consequently equation (3) gives, for $r > 0$,

$$(5)\quad y_0 + y_1 \cos\frac{2r\pi}{2m+1} + y_2 \cos 2\frac{2r\pi}{2m+1} + \cdots + y_{2m}\cos 2m\frac{2r\pi}{2m+1} = \frac{2m+1}{2}A_r,$$

and we notice that equation (4) may be thought of as included in (5) for $r = 0$.* We find therefore the formula

$$(6)\quad A_r = \frac{2}{2m+1}\left[y_0 + y_1\cos\frac{2r\pi}{2m+1} + y_2\cos 2\frac{2r\pi}{2m+1} + \cdots + y_{2m}\cos 2m\frac{2r\pi}{2m+1}\right]$$

$$(r = 0, 1, 2, \cdots, m),$$

enabling us to compute $A_0, A_1, \cdots, A_m$ in terms of the given quantities $y_0, y_1, \cdots, y_{2m}$.

It remains to find a corresponding formula for B_r. In order to do this we return to equations (1), multiply them in order by $0, \sin\frac{2r\pi}{2m+1}, \sin 2\frac{2r\pi}{2m+1}, \cdots, \sin 2m\frac{2r\pi}{2m+1}$, and add. This gives

$$y_1\sin\frac{2r\pi}{2m+1} + y_2\sin 2\frac{2r\pi}{2m+1} + \cdots + y_{2m}\sin 2m\frac{2r\pi}{2m+1}$$
$$= \tfrac{1}{2}A_0S_r + A_1(S_r, C_1) + A_2(S_r, C_2) + \cdots + A_m(S_r, C_m) + B_1S_{1r} + B_2S_{2r} + \cdots + B_mS_{mr}.$$

All of the terms in the right member of this equation reduce to zero except B_rS_{rr}, which, according to Art. 111, (17), becomes equal to $B_r\frac{2m+1}{2}$. Consequently we find

$$(7)\quad B_r = \frac{2}{2m+1}\left[y_1\sin\frac{2r\pi}{2m+1} + y_2\sin 2\frac{2r\pi}{2m+1} + \cdots + y_{2m}\sin 2m\frac{2r\pi}{2m+1}\right]$$

$$(r = 1, 2, 3, \cdots, m).$$

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* It was for this reason that the notation $\frac{1}{2}A_0$, rather than A_0, was chosen in Art. 108 for the constant term of the harmonic function.

Equations (6) and (7) furnish the complete solution of the problem of trigonometric interpolation. *The coefficients of the harmonic function*

$$y = \tfrac{1}{2}A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_m \cos mx$$
$$+ B_1 \sin x + B_2 \sin 2x + \cdots + B_m \sin mx$$

which assumes the arbitrarily assigned values

$$y = y_0, y_1, y_2, \cdots, y_{2m}$$

for the $2m+1$ *equidistant values of the argument*

$$x = 0, \quad \frac{2\pi}{2m+1}, \quad \frac{4\pi}{2m+1}, \cdots\cdots, \frac{4m\pi}{2m+1},$$

are given by equations (6) *and* (7).

EXERCISE LXII

1. Solve the problem treated in detail in Art. 110 by the method of Art. 112.

2. Solve the problems of Exercise LXI by the general method of Art. 112.

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