The Poles of a Right Line

WITH

RESPECT TO A CURVE OF ORDER n

A THESIS

Presented to the Faculty of Philosophy of the
University of Pennsylvania

By

ROXANA HAYWARD VIVIAN

In Partial Fulfilment of the Requirements
for the degree
Doctor of Philosophy

PHILADELPHIA
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PRESS OF
THE NEW ERA PRINTING COMPANY,
LANCASTER, PA.

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The short article, "Allgemeine Eigenschaften der Algebraischen Curven," published by Steiner in Crelle's Journal, vol. XLVII, pp. 1–6, has been the starting point for many investigations in the theory of polar curves and envelopes. His theorems, stated without proof, are given with reference to the general curve of order n. Salmon in "Higher Plane Curves,"[1] gives a résumé of Steiner's theorems with reference to the general curve, and a more specific discussion of the cubic. He uses a method partly analytic and partly geometric. The most complete treatment of the subject of poles and polars is that of Cremona in his "Introduction to the Study of Plane Curves," published in 1865. He bases the theory entirely upon the properties of loci of harmonic means; and his purpose, stated in the preface, is to give a satisfactory geometrical proof of the theorems enunciated by Steiner and other writers on the subject. These three, Salmon, Cremona, and Steiner, have considered the question of poles and polars from the standpoint of the general curve of order n, and with important articles by Clebsch and others, to which references will be made later, constitute the chief sources for the general theory of polar curves.

By no one of these writers has a detailed study been made of the poles of a right line, or the $(n-1)^2$ intersections of the first polar curves of points in a right line, with respect to a curve of order n. Steiner in general considers such points as the envelope of first polar curves of points on a given locus. In case the directrix is a right line, the envelope, by his formula, reduces to order zero. Cremona studies them from the standpoint of base points of a pencil of curves of order $n-1$, while both Salmon and Cremona call them specifically "poles" of the line, and give some limitations to their position in the case of cubics.

[1] Pp. 357–368.

It is proposed in this paper to investigate, by analysis as far as possible, the character and position of the poles of a right line $L = \xi x + \eta y + \zeta z = 0$ in the different relations it may have with respect to a base curve $U = 0$, whose equation is homogeneous and of order n, and certain curves derived from U. The cases for any of these loci of singularities of higher order than a triple point formed by three ordinary branches, or a cusp with an ordinary branch passing through it, will not be considered except in discussing the relation of tangents to first polar curves at certain points of higher order of multiplicity.

§1.

The Pencil of Curves of Which the Poles are Base Points.

The system of first polar curves with respect to $U = 0$ is of the form $xU_1 + yU_2 + zU_3 = 0$, where (x, y, z) is any point in the plane. Taking the polars of all points in a line determined by two fixed points, the system reduces to a pencil of curves projectively related to points on that line, and their $(n-1)^2$ intersections are the poles of the line. These curves determine on the line an involution of degree $n-1$, and the $2(n-2)$ double points of this involution are the points where curves of the pencil touch the line.[1] Any curve of the pencil is completely determined when one point in addition to the $(n-1)^2$ base points is known. If this point is taken infinitely near a base point it determines there the tangent to a single curve of the pencil, and the pencil of curves and the pencil of tangents are projectively related and have a $(1, 1)$ correspondence with each other and with the points of the line. When two of the polars touch, the two pencils of tangents belonging to the two coinciding base points reduce to a common tangent to all the curves except the one which has there a double point; if a base point is a multiple point of order r for all curves of the pencil and these have r common tangents at the point, one curve of the

[1] Clebsch, *Crelle*, vol. LVIII, p. 280.

pencil has the base point for a multiple point of order $r+1$; and in general all the properties which hold for a pencil of curves will be true for these.[1]

The coördinates of the poles of $\xi x + \eta y + \zeta z = 0$, or the base points of the pencil of first polars of the points in the line, are given by the intersections of

$$\frac{U_1}{\xi} = \frac{U_2}{\eta} = \frac{U_3}{\zeta}.$$

§ 2.

The Related Curves.

Closely connected with the theory of polar curves are the Hessian and the Steinerian of the base curve, and through them the Cayleyan. In general, that is when U is non-singular, they are of orders $3(n-2)$, $3(n-2)^2$, and $3(n-2)(5n-11)$ respectively, and the Hessian has no double points.[2] The Steinerian and the Cayleyan have a $(1,1)$ correspondence with the Hessian. In addition to their ordinary definitions it will be convenient to characterize the Hessian as the locus of coincident, or double, poles of a line, and the Steinerian as the envelope of line polars of points on the Hessian.[3] The corresponding points on the two curves are in the same relation as those which Professor Cayley calls "conjugate poles" for the cubic,[4] and the Cayleyan is the envelope of lines joining conjugate poles. There is also another locus upon which lie all the inflexions of first polar curves for the pencil belonging to $L=0$, and which has the base points of the pencil for triple points. The equation and certain important characteristics of this curve will be developed later.

The equations of the Hessian and the Steinerian are deduced from the condition that any polar may have a double point.

[1] Cremona, *Introduction*, § 10.
[2] Salmon, *Higher Plane Curves*, p. 363.
[3] Cremona, § 20, No. 118a.
[4] *Memoir on Curves of the Third Order, Collected Papers*, vol. II, p. 382.

Let $V_1 = x_1 U_1 + y_1 U_2 + z_1 U_3 = 0$ be the polar of the point (x_1, y_1, z_1) on $L = 0$. If this has a double point at (x', y', z'), we have the identical relations

$$\Delta V_1 = 0,$$
$$\Delta^2 V_1 = (\alpha_1 x + \alpha_2 y + \alpha_3 z)(\beta_1 x + \beta_2 y + \beta_3 z),$$
$$\alpha_1 x' + \alpha_2 y' + \alpha_3 z' = 0, \quad \beta_1 x' + \beta_2 y' + \beta_3 z' = 0,$$
$$x_1 U'_{111} + y_1 U'_{112} + z_1 U'_{113} = 2\alpha_1\beta_1,$$
$$x_1 U'_{112} + y_1 U'_{122} + z_1 U'_{123} = \alpha_1\beta_2 + \alpha_2\beta_1,$$
$$x_1 U'_{113} + y_1 U'_{123} + z_1 U'_{133} = \alpha_1\beta_3 + \alpha_3\beta_1,$$
$$x_1 U'_{122} + y_1 U'_{222} + z_1 U'_{223} = 2\alpha_2\beta_2,$$
$$x_1 U'_{123} + y_1 U'_{223} + z_1 U'_{233} = \alpha_2\beta_3 + \alpha_3\beta_2,$$
$$x_1 U'_{133} + y_1 U'_{233} + z_1 U'_{333} = 2\alpha_3\beta_3.$$

Eliminate the α's, β's, and x_1, y_1, z_1, and the Hessian is obtained

$$\begin{vmatrix} U'_{11} & U'_{12} & U'_{13} \\ U'_{12} & U'_{22} & U'_{23} \\ U'_{13} & U'_{23} & U'_{33} \end{vmatrix} = 0,$$

or the locus of double points on first polar curves. The Steinerian may be found by eliminating the α's, β's and x', y', z'; although to express it in algebraic form is practically impossible for curves of higher order than the fourth. The intersections of L with the Hessian and the Steinerian give respectively the number of double points on first polar curves which lie on L, and the number of points whose first polars have a double point for the pencil of curves belonging to L.

All points which may be cusps on first polar curves lie on the Hessian, and also on a curve of order $4(n-3)$, which is obtained by using the condition $\alpha = \beta$ and eliminating α and x_1, y_1,

z_1. The corresponding points (x', y', z') are cusps on the Steinerian, and are $12(n-2)(n-3)$ in number.[1]

The first polars V_1 and V_2 of two points which determine a line will in general have no common intersections which lie on either L, U, or the curves just mentioned; and poles which do not lie on L, U, or the Hessian may be conveniently termed "free" poles.[2] Poles which lie on the Steinerian and the Cayleyan are included among the free poles, since poles on these loci need not satisfy conditions of the particular kind which govern the others. Any pole of a line will be a triple point of the general inflexion locus, which is not fixed as are the Hessian, Steinerian, and Cayleyan, by the base curve, but varies with the line. The number of free poles will be diminished and their characteristics will be changed as the line is defined by special relations to U and the Steinerian, or as singularities are introduced into the base curve. In no case, however, can the number of free poles, depending only on the base curve, be less than $n-1$ while U is a proper curve with none of the complex singularities; for at the multiple point of highest order, $n-1$, the first polars have $(n-2)^2$ intersections, and these, with $n-2$ additional common points if all the branches are cuspidal, leave $n-1$ free poles.

§ 3.

Poles when U has no Double Points or Other Singularities.

Under this hypothesis the Hessian has no singularity.

I. A line which has only ordinary intersections with U and the Steinerian can have only free poles, as is evident from the conditions which exist when a pole lies upon either of these loci.

[1] Clebsch, *Ueber Curven vierter Ordnung*, *Crelle*, Vol. LIX, p. 130. Cf. also, Steiner's article in *Crelle*, Vol. XLVII, pp. 1–6, and Henrici's, *Proc. Lon. Math. Soc.*, Vol. II, p. 112.

[2] Salmon in his discussion of the cubic omits from the number of poles those which occur at a double point or cusp; but it seems better to include these in the total number, since they come under the definition, and have all the characteristics of poles.

(*a*) If a pole lies on L it must be either a point of tangency of the line with U, or a double point on U.

Let (x', y', z'), a pole of the line, satisfy the equation $L = 0$. The coördinates of any point in $L = 0$ satisfy the relation

$$(1) \qquad xU_1' + yU_2' + zU_3' = 0,$$

including those of the pole itself, so that

$$x'U_1' + y'U_2' + z'U_3' = 0;$$

hence (x', y', z') is on the curve U. For (1) to hold for every point in L the line must be tangent to U at (x', y', z'), or else U_1', U_2', U_3' must vanish identically and (x', y', z') is a double point on U. Thus L must have two points in common with U at (x', y', z'), which may be consecutive or coincident, the two conditions having the same value here.[1]

(*b*) If a pole lies on U by a similar method it can be shown to be a point of tangency of L with the curve, or else a double point on the curve.

(*c*) If a pole lies on the Hessian L is tangent to the Steinerian at the corresponding point.

Let (x', y', z') be a pole of the line and a point in the Hessian. It is then a double point on a first polar curve, for example, the first polar V_1 of (x_1, y_1, z_1) which is an intersection of L and the Stei nian. Now

$$V_1 = x_1U_1 + y_1U_2 + z_1U_3 = 0;$$

and since it has a double point at (x', y', z')

$$x_1U_{11}' + y_1U_{12}' + z_1U_{13}' = 0,$$
$$x_1U_{12}' + y_1U_{22}' + z_1U_{23}' = 0,$$
$$x_1U_{13}' + y_1U_{23}' + z_1U_{33}' = 0,$$

[1] The two conditions are equivalent algebraically and geometrically here, though it may happen, as in the case when the line passes through a double point on the Steinerian, that they are only equivalent algebraically. Cf. Jonquières, *Sur les problemes de contact des courbes algebriques*, *Crelle*, Vol. LXVI, p. 291.

If (x_2, y_2, z_2) is a second point on L, any other point on the line has coördinates $(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2)$; and examining the tangent at (x', y', z') to the first polar of any point (x, y, z), we have

$$\begin{aligned} xU'_{11} + yU'_{12} + zU'_{13} &= \lambda(x_2U'_{11} + y_2U'_{12} + z_2U'_{13}), \\ xU'_{12} + yU'_{22} + zU'_{23} &= \lambda(x_2U'_{12} + y_2U'_{22} + z_2U'_{23}), \\ xU'_{13} + yU'_{23} + zU'_{33} &= \lambda(x_2U'_{13} + y_2U'_{23} + z_2U'_{33}); \end{aligned}$$

hence the tangents to all first polars of the pencil at (x', y', z') coincide with

$$x[x_2U'_{11} + y_2U'_{12} + z_2U'_{13}] + y[x_2U'_{12} + y_2U'_{22} + z_2U'_{23}] \\ + z[x_2U'_{13} + y_2U'_{23} + z_2U'_{33}] = 0.$$

A pole on the Hessian is therefore in general composed of two coincident poles.[1] Such a pole corresponds also to a double point on the polar envelope of a curve, but for a directrix of the first degree the envelope consists simply of the $(n-1)^2$ poles of the line and is of order zero. The consecutive curve to V_1 will have common tangents with it and therefore a double point at (x', y', z'). The consecutive point on L is on the Steinerian and the line is tangent to the Steinerian at the point (x_1, y_1, z_1) which corresponds to (x', y', z').[2] It might seem as in a preceding case that a line through a double point on the Steinerian would satisfy the same condition, but it will be shown later that a pole on the Hessian will result only in a special case. The above analysis is in agreement with Cremona's theorems[3]: If two curves of a pencil touch the point of tangency counts for two double points and to it corresponds the point where the line touches the Steinerian.

At any pole, as has been stated, the tangents to first polars form a pencil of lines, and when two curves of the pencil touch it is geometrically evident that the two pencils belonging to the

[1] These may be called poles of order 2.
[2] The tangents to the Steinerian are thus line polars of points on the Hessian.
[3] *Introduction,* § 14, No. 88a, and § 19, No. 112a.

two coincident poles reduce to a single line, the common tangent. The condition for coincident poles is that the poles lie on the Hessian, and it is interesting to see that the analytic condition for the reduction of the pencil is the same. Using a method similar to that used by Clebsch,[1] L may be defined by two points, (x_1, y_1, z_1) and (x_2, y_2, z_2), such that the tangents to their respective polars at the pole (x', y', z') cut L in those points. For let the line through (x_1, y_1, z_1) and (x', y', z') be $\alpha x + \beta y + \gamma z = 0$; then

$$\alpha x' + \beta y' + \gamma z' = 0,$$
$$\alpha x_1 + \beta y_1 + \gamma z_1 = 0;$$

and if this line coincides with the tangent to the polar of

$$(x_1, y_1, z_1) \quad \text{at} \quad (x', y', z')$$
$$x_1 U'_{11} + y_1 U'_{12} + z_1 U'_{13} = \lambda\alpha,$$
$$x_1 U'_{12} + y_1 U'_{22} + z_1 U'_{23} = \lambda\beta,$$
$$x_1 U'_{13} + y_1 U'_{23} + z_1 U'_{33} = \lambda\gamma;$$

hence eliminating x_1, y_1 and z_1,

$$\begin{vmatrix} U'_{11} & U'_{12} & U'_{13} & \alpha \\ U'_{12} & U'_{22} & U'_{23} & \beta \\ U'_{13} & U'_{23} & U'_{33} & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix} = 0.$$

There are then two lines (α, β, γ) through (x', y', z') which contain points whose first polars touch them at (x', y', z'). It is also evident that there are $2(n - 2)$ points in any line where first polars of points in the line may touch it.

$L = 0$ may then be defined by the pair of points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the two lines through (x', y', z') whose respective polars touch the lines there; and the condition that the lines coincide is

[1] *Crelle*, Vol. LVIII, pp. 279–280. Cf. also Salmon, *Higher Plane Curves*, pp. 343–344.

$$\begin{vmatrix} U'_{11} & U'_{12} & U'_{13} \\ U'_{12} & U'_{22} & U'_{23} \\ U'_{13} & U'_{23} & U'_{33} \end{vmatrix} = 0,$$

so that (x', y', z') is on the Hessian.

These pairs of lines are what Cremona calls the "indicatrices" of a point—the pair of tangents which can be drawn from it to its conic polar.[1]

(*d*) If a pole lies on the Steinerian it is a double point on a polar conic. This is simply the definition of the Steinerian, and is all that can be affirmed in regard to such a pole; for the reciprocal relation between the Hessian and the Steinerian does not lead to the reciprocal characteristic that the tangents to the Hessian are line polars of points on the Steinerian, except in the case of cubics when the two loci coincide. Corresponding to the points on L, there is a pencil of first polars, $V_1 + \lambda V_2$, which represents all the first polar curves of points on the line, and these all pass through the $(n-1)^2$ poles. The case is different for the polar conics of points on a line, which are of the form

$$x^2 U_{11} + y^2 U_{22} + z^2 U_{33} + 2yz\, U_{23} + 2zx\, U_{31} + 2xy\, U_{12} = 0,$$

where U_{11}, U_{22}, etc., are of degree $n-2$ in the coördinates of the point and the system is not a pencil. The conics do not all pass through the pole on the Steinerian and the preceding analysis cannot be applied.

If the line is tangent to the Hessian at (x', y', z') the polar conics of the two intersections have double points at the corresponding point on the Steinerian. If this is a pole of the line it is not to be distinguished from the case where the line simply intersects the Hessian, for the number of polar conics which may have a double point at the pole is not taken into account.

A pole on the Steinerian, therefore, is the result of the perfectly general condition that the line considered is the line polar

[1] Cremona, *Introduction*, §14, No. 90c, and §19, No. 112. The Hessian with the base curve forms the locus of points for which the indicatrices reduce to a single line.

of the point, and it is numbered among the free poles. The same is true for the Cayleyan.

II. It follows immediately from I that when L has points of tangency with U and the Steinerian the number of free poles is diminished.

If L has only ordinary points of contact with U each point of tangency is a pole and lies on both the loci. The case of asymptotic tangents is a special case which gives a pole at infinity. The maximum number of such poles which may lie on U is $[n/2]$, leaving $(2n-1)(n-2)/2$ possible free poles when n is even, and $(n-1)(2n-3)/2$ when n is odd.

If L is tangent to U at an inflexion, it is tangent to the Steinerian as well, and has a pole on the Hessian, the point of inflexion; for all first polars of its points touch the line at the point of inflexion. Such a pole is of order 2 and is on the three loci, L, U, and the Hessian. The number of double poles which may arise from inflexional tangency cannot be greater than $[n/3]$.

If L is tangent to the Steinerian, but not an inflexional tangent to U, there is again a pole on the Hessian at the corresponding point and necessarily a double pole. The maximum number of double poles is $[3(n-2)^2/2]$.

Combinations of simple and inflexional tangency with U give corresponding combinations of single and double poles. A line which has more than one simple point of tangency with the Steinerian[1] also presents no difficulty in classifying the poles fixed by the given conditions. Combinations of tangency, however, between U and the Steinerian, depend upon the number of conditions L can satisfy with respect to these curves. If a method could be devised for finding the maximum number of times a line may be tangent to these curves simultaneously, the maximum number and character of the poles defined by these conditions would follow immediately. But the investigation for a special and simple case is rendered practically impossible on account of the difficulty of obtaining the Steinerian in a suitable form.

[1] Cremona, *Introduction*, § 20, No. 119. If a line is a double tangent to the Steinerian all first polars touch at the two corresponding points on the Hessian.

§ 4.

THE INFLEXION LOCUS.

The position of inflexions on polar curves does not in any way limit the number of free poles of a line, but a consideration of the inflexions serves to define somewhat the character of the poles, and before turning to the case where U has double points or other singularities it will be convenient to discuss the general inflexion locus, and the inflexion cubic for any point in the plane and in particular for any pole of the line L.

Referring to §2, if the polar V_1 of (x_1, y_1, z_1) has an inflexion at (x', y', z'), the six equations involving U_{111}, U_{112}, etc., must exist, and also

$$\varDelta V_1 = a_1 x' + a_2 y' + a_3 z' = 0,$$

$$\varDelta^2 V_1 = (a_1 x' + a_2 y' + a_3 z')(\beta_1 x + \beta_2 y + \beta_3 z) = 0.$$

Eliminating the a's and β's, the point (x_1, y_1, z_1) must satisfy the two conditions

$$x_1 U'_1 + y_1 U'_2 + z_1 U'_3 = 0,$$

$$\begin{vmatrix} x_1 U'_{111} + y_1 U'_{112} + z_1 U'_{113} & x_1 U'_{112} + y_1 U'_{122} + z_1 U'_{123} & x_1 U'_{113} + y_1 U'_{123} + z_1 U'_{133} \\ x_1 U'_{112} + y_1 U'_{122} + z_1 U'_{123} & x_1 U'_{122} + y_1 U'_{222} + z_1 U'_{223} & x_1 U'_{123} + y_1 U'_{223} + z_1 U'_{233} \\ x_1 U'_{113} + y_1 U'_{123} + z_1 U'_{133} & x_1 U'_{123} + y_1 U'_{223} + z_1 U'_{233} & x_1 U'_{133} + y_1 U'_{233} + z_1 U'_{333} \end{vmatrix} = 0;$$

hence there are three points whose first polars satisfy the given condition. Clebsch deduces from this the theorem: "There are always three different poles whose polars have an inflexion at a given point";[1] but an examination of the conditions under which the determinant cubic is derived, as well as a study of special cases, shows that, strictly speaking, an inflexion will not always result. The locus evidently includes all points whose first polars have three consecutive or coincident points at (x', y', z') in the same straight line. It will however be convenient to adopt a broader meaning of the term "inflexion," as it is used by Clebsch and other writers, to include all the cases which fol-

[1] *Crelle*, Vol. LIX, p. 127.

low, and should be so understood throughout this section. The lines thus related to first polar curves may be inflexional tangents in the ordinary sense, tangents at a double point, any line through a triple point, or a straight line through the point which forms part of a degenerate polar. One point will always have corresponding to it a real line fulfilling the given conditions, while the other two lines may be conjugate imaginaries.

The locus of all inflexions of polar curves of points on L is obtained by eliminating (x_1, y_1, z_1) from

$$\xi x_1 + \eta y_1 + \zeta z_1 = 0,$$
$$x_1 U_1' + y_1 U_2' + z_1 U_3' = 0,$$

and the determinant cubic above. The resulting equation is of degree $6(n-2)$ and has the base points of the pencil for triple points. It is evident that points on this locus which are double points on polar curves are also on the Hessian, so that the two curves are closely connected.

When the point (x_1, y_1, z_1) whose polar has an inflexion at (x', y', z') describes the line L, there is a third condition for it to satisfy, and in general the three loci will have no common intersection, or one. But at an ordinary pole of the line

$$L = \xi x_1 + \eta y_1 + \zeta z_1 = \lambda(x_1 U_1' + y_1 U_2' + z_1 U_3') = 0,$$

and any line has, corresponding to each pole, a set of three points whose first polars have an inflexion at the pole. These are the intersections of the line with the cubic determinant belonging to that pole, and the sets of three are $(n-1)^2$ in number. It follows from this property of the poles that they are triple points of the inflexion locus of the line.

As a special case a tangent to $U = 0$ contains three points whose first polars have an inflexion at the point of tangency.

The term "inflexion cubic" for any point will be used to designate the determinant cubic when the coördinates of the point have been substituted in U_{111}, U_{112}, etc. These curves, the inflexion cubic and the inflexion locus, are both derived from U, but they are evidently not dependent upon U alone, as are the

Hessian, Steinerian and Cayleyan, since the former varies for every point in the plane and the latter with every line. It is necessary to examine the inflexion cubic corresponding to a double point or cusp on the base curve.

If U has a double point by a suitable choice of axes and coordinates its equation may be written in the form

$$a_0xyz^{n-2} + (ax^3 + bx^2y + cxy^2 + dy^3)z^{n-3} \\ + (a'x^4 + b'x^3y + c'x^2y^2 + d'x^3y + e'y^4)z^{n-4} + \cdots = 0,$$

where $(0, 0, 1)$ is the double point with tangents $x = 0$ and $y = 0$. Thence we obtain

$$\begin{aligned}
U_{111} &= 6az^{n-3} + (24a'x + 6b'y)z^{n-4} + \cdots,\\
U_{112} &= 2bz^{n-3} + (6b'x + 4c'y)z^{n-4} + \cdots,\\
U_{113} &= (6ax + 2by)z^{n-4} + \cdots,\\
U_{122} &= 2cz^{n-3} + (4c'x + 6d'y)z^{n-4} + \cdots,\\
U_{123} &= a_0(n-2)z^{n-3} + (n-3)(2bx + 2cy)z^{n-4} + \cdots,\\
U_{223} &= (n-3)(2cx + 6dy)z^{n-4} + \cdots,\\
U_{133} &= (n-2)(n-3)a_0yz^{n-4} + \cdots,\\
U_{233} &= (n-2)(n-3)a_0xz^{n-4} + \cdots,\\
U_{222} &= 6dz^{n-3} + (6d'x + 24e'y)z^{n-4} + \cdots,\\
U_{333} &= a_0(n-2)(n-3)(n-4)xyz^{n-5} + \cdots.
\end{aligned}$$

Evaluating these expressions for $(0, 0, 1)$ and substituting in the determinant, the locus of points whose first polars have an inflexion at $(0, 0, 1)$ is

$$\begin{vmatrix} 6ax + 2by & 2bx+2cy+a_0(n-2)z & a_0(n-2)y \\ 2bx+2cy+a_0(n-2)z & 2cx + 6dy & a_0(n-2)x \\ a_0(n-2)y & a_0(n-2)x & 0 \end{vmatrix} = 0,$$

or

$$3(ax^3 + dy^3) - (bx^2y + cxy^2) - a_0(n-2)xyz = 0.$$

Thus the inflexion cubic has a double point at $(0, 0, 1)$ with the tangents of U, $x = 0$ and $y = 0$.

For a cusp U takes the form

$$a_0x^2z^{n-2} + (ax^3 + bx^2y + cxy^2 + dy^3)z^{n-3} + \cdots = 0.$$

The quantities U_{111}, U_{112}, etc., are the same as before except the following:

$$\begin{aligned}
U_{113} &= 2a_0(n-2)z^{n-3} + (n-3)(6ax + 2by)z^{n-4} + \cdots,\\
U_{123} &= (n-3)(2bx + 2cy)z^{n-4} + \cdots,\\
U_{133} &= 2a_0(n-2)(n-3)xz^{n-4} + \cdots,\\
U_{233} &= (n-3)(n-4)bx^2z^{n-5} + \cdots,\\
U_{333} &= a_0(n-2)(n-3)(n-4)x^2z^{n-5} + \cdots;
\end{aligned}$$

and the inflexion cubic is

$$\begin{vmatrix} 6ax + 2by + 2a_0(n-2)z & 2bx + 2cy & 2a_0(n-2)x \\ 2bx + 2cy & 2cx + 6dy & 0 \\ 2a_0(n-2)x & 0 & 0 \end{vmatrix} = 0,$$

or

$$x^2(cx + 3dy) = 0;$$

so that the cubic reduces to three straight lines, two of which coincide with the cuspidal tangent.

If the point (x', y', z'), whose inflexion cubic is considered, is a double point of U, every line in the plane intersects the cubic in three points which satisfy $xU_1' + yU_2' + zU_3' = 0$; or at a double point of U three poles of any pencil have an inflexion. If the line pass through a double point of U, since, as we have seen, the inflexion cubic has there a double point, two of the three intersections are represented by the double point. The tangents to the first polar of the double point meet it in three points which are coincident, though not on the same branch. They thus satisfy the algebraic conditions and count for two of the tangents required. Corresponding to the third intersection is an inflexional tangent, strictly speaking, or else a line forming part of a degenerate polar.[1] If the line is tangent to U at the double point its three intersections with the inflexion cubic coincide. Two of these as before are at the double point and have the same effect, while the third, which is consecutive to the double point, gives the line itself as inflexional tangent to the polar of a point in it and at the point itself.

Passing to the case where (x', y', z') is a cusp on U, any line which does not go through the cusp meets the inflexion cubic in two coincident points on the cuspidal tangent, and in a third point on the other right line which makes up the cubic. The oint in the cuspidal tangent is one whose first polar has a dou ble point at (x', y', z'), and the tangents at this double point count for two of the required lines. The third point is the one whose first polar has the cuspidal tangent for inflexion tangent. An ordinary line through the cusp has there three intersections with the inflexion cubic. The cuspidal tangent counted twice corresponds to two of the intersections as cuspidal tangent to their first polar curves, and as inflexional tangent corresponds to the third. Finally, the cuspidal tangent has three intersections with the inflexion cubic at the cusp, two represented by the cuspidal tangent as before, and the third giving the line itself as inflexion tangent to the polar of one of its points at that point.

An examination of the quartic

$$U = x^2y^2 + y^2z^2 + z^2x^2 + \tfrac{5}{2}xyz^2 - 2yzx^2 - 2zxy^2 = 0$$

will serve to illustrate these different cases. No simpler example is possible, for this is the lowest order of curve for which the Hessian and the Steinerian are distinct, and all but one of the poles which could be fixed by means of cusps and double points are at the vertices of the triangle of reference. The cuspidal tangents are $z - x = 0$, $y - z = 0$; and those at the double point are $x + 2y = 0$, and $2x + y = 0$. Differentiating

[1] This third intersection agrees with Cremona's statement, *Introduction*, § 10, No. 47, that one curve of a pencil will have an inflexion at a double pole, though he approaches the question from an entirely different standpoint and does not consider whether or not the point is a true inflexion.

$$U_1 = 2xy^2 + 2xz^2 + \tfrac{5}{2}yz^2 - 4xyz - 2zy^2,$$
$$U_2 = 2x^2y + 2yz^2 + \tfrac{5}{2}xz^2 - 2zx^2 - 4xyz,$$
$$U_3 = 2y^2z + 2x^2z + 5xyz - 2yx^2 - 2xy^2;$$
$$U_{11} = 2y^2 + 2z^2 - 4yz, \qquad U_{22} = 2x^2 + 2z^2 - 4xz,$$
$$U_{12} = 4xy + \tfrac{5}{2}z^2 - 4zx - 4yz, \qquad U_{23} = 4yz + 5zx - 2x^2 - 4xy,$$
$$U_{13} = 4xz + 5yz - 4xy - 2y^2, \qquad U_{33} = 2y^2 + 2x^2 + 5xy;$$
$$U_{111} = U_{222} = U_{333} = 0.$$

Forming the partial derivatives of the third order and evaluating for the points $(0, 0, 1)$, $(1, 0, 0)$, etc.,

	(0, 0, 1)	(1, 0, 0)	(1, 1, 0)	(1, 1, 1)
$U_{112} = 4y - 4z$	−4	0	4	0
$U_{113} = 4z - 4y$	4	0	−4	0
$U_{122} = 4x - 4z$	−4	4	4	0
$U_{123} = 5z - 4x - 4y$	5	−4	−8	−3
$U_{223} = 4z - 4x$	4	−4	−4	0
$U_{233} = 4y + 5x$	0	5	9	9
$U_{133} = 4x + 5y$	0	4	9	9

The inflexion cubic for the double point $(0, 0, 1)$ is

$$(A) \qquad \begin{vmatrix} 4(z-y) & 5z - 4(x+y) & 4x + 5y \\ 5z - 4(x+y) & 4(z-x) & 5x + 4y \\ 4x + 5y & 5x + 4y & 0 \end{vmatrix} = 0,$$

which reduces to

$$16(x^3 + y^3) - 6z(x^2 + y^2) + 38xy(x + y) - 15xyz = 0;$$

and for the cusp $(1, 0, 0)$

$$(B) \qquad \begin{vmatrix} 0 & 4(y-z) & 4(z-y) \\ 4(y-z) & 4(x-z) & 5z - 4(x+y) \\ 4(z-y) & 5z - 4(x+y) & 4x + 5y \end{vmatrix} = 0,$$

or

$$(y - z)^2(y - 2z) = 0.$$

(A) has at $(0, 0, 1)$ a double point with tangents $x + 2y = 0$ and $2x + y = 0$; and at the two cusps the inflexion cubics reduce to the cuspidal tangent counted twice and a third straight line through the cusp, according to the general theory.

I. When the line has a pole at a point not on the curve.

Example 1. The point $(1, 1, 1)$ has the inflexion cubic

$$(D) \qquad \begin{vmatrix} 0 & -3z & 9z-3y \\ -3z & 0 & 9z-3x \\ 9z-3y & 9z-3x & 9x+9y \end{vmatrix} = 0,$$

or

$$27(18z^3 - 3yz^2 - 3xz^2 + 2xyz) = 0.$$

The polar of a point on any line will have one inflexion at $(1, 1, 1)$ if there is a common intersection of L, (D), and the first polar of the point in which U_1, U_2, and U_3 have been evaluated for $(1, 1, 1)$; and there will be three simultaneous intersections if $(1, 1, 1)$ is a pole of the line selected. The line $x + y + 10z = 0$ has a pole at $(1, 1, 1)$ and its three intersections with (D) are $(1, -1, 0)$, $(1, -6, \frac{1}{2})$ and $(1, -\frac{1}{6}, -\frac{1}{12})$. The respective polars of these points are

$$(x-y)(4zx + 4zy - 4xy - z^2) = 0,$$
$$(z-x)(2y^2 + 19yz + 26xz - 26xy) = 0,$$
$$(y-z)(26xy - 26yz - 19xz - 2x^2) = 0;$$

showing that the polars are degenerate, and the straight lines which pass through the pole correspond to the inflexional tangents or tangents at a double point on a proper curve.

Example 2. The line $x + y - 2z = 0$ has a pole at $(1, 1, 0)$ for which the inflexion cubic is

$$(E) \qquad \begin{vmatrix} 4y-4z & 4x+4y-8z & 9z-4x-8y \\ 4x+4y-8z & 4x-4z & 9z-8x-4y \\ 9z-4x-8y & 9z-8x-4y & 9x+9y \end{vmatrix} = 0,$$

which reduces to

$$4(x^3+y^3)+8(x^2y+xy^2)+63(yz^2+xz^2)$$
$$-28(y^2z+x^2z)-40xyz-54z^3=0.$$

The intersections of the line with the cubic (E) are

$$x=1,1,\infty;\ y=1,1,\infty;\ z=1,1,1;$$

and the line is tangent to the cubic at $(1, 1, 1)$, while the third intersection is at infinity. The polar of $(1, 1, 1)$ is

$$z(3yz+3xz-2xy)=0,$$

a degenerate cubic which is to be counted twice on account of the condition of tangency.

II. Every line has a single pole at the double point. (See § 5, I.)

Example 3. The line $176x+149y+15z=0$ has intersections with (A) at $(6, -9, 19)$, $(45, -60, 68)$, and $(-35, 20, 212)$, with corresponding polars

$$56x^2z-56x^2y-26xy^2+26y^2z+107xyz-3yz^2-\tfrac{21}{2}xz^2=0,$$
$$46zy^2-46xy^2-256x^2y+256x^2z$$
$$+400xyz-60xz^2-\tfrac{15}{2}yz^2=0,$$
$$494y^2z-494xy^2-20yz^2-\tfrac{95}{2}yz^2$$
$$+1120xyz-384x^2y+384x^2z=0.$$

Changing to Cartesian coördinates, and making, for each curve, the tangent at the origin the y-axis, these equations reduce to

$$x(12y^2-16xy+16x+278y-21)+40y^3=0,$$
$$x(42y-4xy-15y^2+4x-\tfrac{15}{2})+19y^3=0,$$
$$x(665y^2-352y-10)+3971y^3=0.$$

This shows that there exist three real points in the line $176x+149y+15z=0$ whose first polars have an inflexion in the strict sense of the word at the pole of the line $(0, 0, 1)$.

Example 4. The line $x-2y=0$ passes through the double point and its intersections with (A) are

$$x=0,\,0,\,\tfrac{10}{31}\,;\ y=0,\,0,\,\tfrac{5}{31}\,;\ z=1,\,1,\,1.$$

The double point counted twice has corresponding to it the tangents to its first polar at $(0,0,1)$; and the polar of $(10,5,31)$ is

$$42\,(y^2z-xy^2)+52\,(x^2z-x^2y)+95xyz+35yz^2+\tfrac{65}{2}\,zx^2=0.$$

This equation transformed as before gives

$$x\,(65-8xy-34y+8x+140y^2)-392y^3=0,$$

which shows an inflexion at $(0,\,0,\,1)$.

Example 5. The line $z=0$ bears no special relation to the inflexion cubic for $(0,\,0,\,1)$, but has a pole at that point, and by choosing suitable coördinates any line may be taken for $z=0$. We may then find the points whose first polars have an inflexion at $(0,\,0,\,1)$ by the following general method.

Let $x-ky=0$ be the equation of the line joining $(0,\,0,\,1)$ to an intersection of (A) with $L=0$; then the tangent to the first polar of the point is $x+ky=0$ (see § 5, I), and k may be so determined that $x+ky=0$ is tangent at an inflexion. The polar of $(k,\,1,\,0)$ is

$$V_k=k\,(2xy^2+2xz^2+\tfrac{5}{2}\,yz^2-4xyz-2y^2z)+2x^2y+2yz^2$$
$$+\tfrac{5}{2}\,xz^2-2zx^2-4xyz=0\,;$$

and the tangent to V_k at $(0,0,1)$ is

$$x\,(4k+5)+y\,(5k+4)=0.$$

This must meet V_k in three points at $(0,\,0,\,1)$. Substituting

$$x=-y\left(\frac{5k+4}{4k+5}\right),\ \text{and letting } z=1,$$

$$2y^3\left[\frac{4+k-5k^2}{4k+5}\right]$$
$$+\,2y^2\left[\frac{10k^2+18k+8}{4k+5}-k-\left(\frac{5k+4}{4k+5}\right)^2\right]=0.$$

The coefficient of y vanishes identically since the line is tangent by hypothesis. For three values of y to be equal to zero the coefficient of y^2 must vanish, and

$$(k+1)(8k^2+11k+8)=0,$$

which gives one real and two imaginary values for k. When $k=-1$ the polar of $(-1, 1, 0)$ is

$$(x-y)(z^2+4xy-4xz-4yz)=0,$$

another degenerate polar; while the two remaining points in $z=0$ whose first polars have an inflexion at $(0, 0, 1)$ are imaginary.[1]

III. Every line has a double pole at the cusp (see § 5).

Example 6. Intersecting (B) by the line $x=0$, we obtain $(0, 1, 1)$ twice, and $(0, 2, 1)$. The tangent to the polar of $(0, 1, 1)$ is indeterminate, and this point corresponds to the polar which has a double point at the pole. The polar of $(0, 2, 1)$ is

$$2x^2y-2xy^2-3xyz-2x^2z+4yz^2+2y^2z+5xz^2=0,$$

and its tangent at $(1, 0, 0)$ is $y-z=0$, the cuspidal tangent. Transforming so that $y-z=0$, becomes $z=0$ we have

$$z(2-7y+5z-10y^2+4yz)+6y^3=0;$$

and $y-z=0$ is therefore tangent at an inflexion.

Cases where the line passes through a cusp or coincides with the tangent there, and other special positions of the line may readily be examined by similar methods.

[1] Writing the polar conic

$$2x^2+2ky^2-yz(5k+4)-zx(4k+5)+4xy(k+1)=0,$$

dividing by the tangent

$$x(4k+5)+y(5k+4)=0,$$

and equating the coefficients in the remainder to zero, leads to the same values of k.

§ 5.

Poles when the Base Curve has Double Points and Cusps.

A double point on U is a pole for every line in the plane and presents several peculiar characteristics. It is a double point on its own first polar curve, and therefore corresponds to itself as a point on the Hessian and the Steinerian. It is a double point on the Hessian[1] which represents always two of the points which can be double points for the pencil of curves belonging to a line through it, and is therefore a double point on the Steinerian. This is a particular case of the following theorem proved by Henrici[2] by a very elegant analysis: "A point whose first polar has a cusp is a cusp on the Steinerian, and one whose first polar has two double points is a double point on the Steinerian."[3] The tangents to the Hessian at the double point are the same[1] as those of U, and also are tangents to the Steinerian, since they are line polars of the corresponding point on the Hessian.

This pole lies on U, the Hessian and the Steinerian simultaneously, irrespective of any condition introduced by the position of the line itself, and it is in general a single pole, contrary to the usual character of a pole on the Hessian; but the Hessian includes all points which are simply double points on U as well as those where all first polars touch. A cusp is a double pole for all lines in the plane since all first polars touch there. Thus there is a lower limit for the number of single and double poles found on U, for any line in the plane, depending upon the number of double points and cusps which U has. One of these single poles may change into a double pole, and a double pole into one of higher order, for certain lines in the plane—namely, lines through the singular points and tangents at those points. This lowest number is increased by the points of tangency of the

[1] Salmon, *Higher Plane Curves*, p. 60.

[2] *Proc. Lond. Math. Soc.*, Vol. II, p. 112.

[3] Cf. Cremona, *Introduction*, § 20, No. 120. If a first polar has two double points p and p', the pole o is a double point on the Steinerian and the tangents at o are line polars of p and p'.

line with U. Any pole which is common to L, U, and the Hessian may be either a double point or an inflexion : if it is on the Steinerian as well it must be a double point.

I. When L does not pass through the double point the pole is single, and the tangent to the first polar of a point in L is the harmonic conjugate, with respect to the tangents of U at the double point, to the line joining the point whose polar is taken to the double point.[1]

U may be written

$$a_0xyz^{n-2} + (ax^3 + bx^2y + cxy^2 + dy^3)z^{n-3} + \cdots = 0.$$

Let L be the line $z = 0$ for simplicity. Then any point $(x_1, y_1, 0)$ on L has for polar

$$V_1 = x_1U_1 + y_1U_2 = 0.$$

The tangent to V_1 at $(0, 0, 1)$ is $xy_1 + yx_1 = 0$; and the line through $(x_1, y_1, 0)$ and $(0, 0, 1)$ is $xy_1 - yx_1 = 0$, the harmonic conjugate of the tangent.

In particular the point where the line intersects one of the tangents at the double point has a polar which touches the other tangent at the double point.

II. When L passes through the double point it is evident that the harmonic conjugates reduce to a single line conjugate to L which is the common tangent to all first polars, and there are two coincident poles. This is one of the special cases referred to in §3 (c), for L passes through a double point on the Steinerian and has a pole on the Hessian. The two points composing the double pole must be regarded as lying on the conjugate to L, while L simply intersects the polars at the double point, having two points in common only with the polar of the double point, which has a double point at the pole. It should be noted that we have here another condition for a double pole.

III. If L is tangent to U at a double point, it is tangent to all first polars of its points, and to the Hessian and the Steinerian, and has three consecutive points in common with U, the

[1] Cf. Cremona's theorem, *Introduction*, § 13, No. 74, based on his geometrical proof.

Hessian, and the Steinerian. The three intersections with the Hessian count for three double points of the pencil, and the double pole lies on the line itself.

IV. When the double point is a cusp, the two tangents coincide and become the common tangent to U and to all first polars. This is also a condition for a double pole. The point where any line intersects the cuspidal tangent is the one whose first polar has a double point, counting for two double points of the pencil as in the ordinary case where polar curves have a common tangent.

V. Any line through the cusp has three points in common with the Hessian, since a cusp on U is a triple point on the Hessian[1] consisting of a cusp with a simple branch through it. This pole then counts for three double points of the pencil and the polar of the cusp itself has a cusp there.[2] Any line through the cusp has at least two poles on the cuspidal tangent at the point of tangency, but this double pole as in a former case cannot be regarded as lying on the line. The pencil at a cusp differs from a pencil at an ordinary point, when the line passes through it, only in having a cusp at the pole instead of a double point. The cusp on U should be a cusp on the Steinerian by Henrici's theorem already quoted in this section, [3] but this is a special case since the cuspidal tangent forms a part of the Steinerian.

VI. If L is tangent to U at a cusp, every point in it is one whose first polar curve has a double point at the cusp.[4] All first polars have four intersections at the cusp, making a pole of order 4. The pairs of tangents to the polars, or their polar conics, form a quadratic involution with the vertex at the cusp in which one of the two double elements is the cuspidal tangent itself.

[1] Salmon, *Higher Plane Curves*, p. 61.

[2] Cf. Cremona, § 14, 88b: If a base point of a pencil is a cusp for one it counts for three double points.

[3] Also compare Cremona, *Introduction*, § 20, No. 121: If a first polar have a cusp p, the pole is a cusp on the Steinerian and has the line polar of p for cuspidal tangent.

[4] Cremona, *Introduction*, §14, 88d.

Let

$$V_1 = x_1 U_1 + y_1 U_2 + z_1 U_3 = 0,$$
$$V_2 = x_2 U_1 + y_2 U_2 + z_2 U_3 = 0,$$

be the first polars of any two points on the cuspidal tangent. The pairs of tangents at the cusp (x', y', z') are given by

$$C_1 = x^2[x_1 U'_{111} + y_1 U'_{112} + z_1 U'_{113}] + y^2[x_1 U'_{122} + y_1 U'_{222} + z_1 U'_{223}] + z^2[x_1 U'_{133} + y_1 U'_{233} + z_1 U'_{333}] + 2yz[x_1 U'_{123} + y_1 U'_{223} + z_1 U'_{233}] + 2zx[x_1 U'_{113} + y_1 U'_{123} + z_1 U'_{133}] + 2xy[x_1 U'_{112} + y_1 U'_{122} + z_1 U'_{123}] = 0,$$

and a similar expression C_2. Any other point on the cuspidal tangent is $(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2)$, and the tangents to its first polar are given by $C_1 + \lambda C_2 = 0$. Two curves of the system will have a cusp corresponding to the double elements of the involution. These are given by the values of λ which make $C_1 + \lambda C_2$ a perfect square; and, since one of the pairs is the cuspidal tangent and therefore real, the involution is hyperbolic or non-overlapping.[1]

§ 6.

Intersections of Higher Order with the Steinerian.

The character of the poles conditioned by the line having ordinary contact with the Steinerian has already been discussed in §3, and it was shown that parallel theorems for the Hessian cannot be deduced. The condition of passing through a double point on the Steinerian is not equivalent to the condition of tangency, but a line may pass through a double point on the Steinerian and have no pole at the corresponding point on the Hessian. A line through a double point on U however, since it is a double point on both these loci, has a pole on each, the double point itself. This is a special case depending on the relation of the line to the base curve.

When the line has three intersections with the Steinerian at any point the following distinctions arise:

I. The three points may be consecutive on the same branch and the line is tangent at a point of inflexion.

[1] Scott, *Analytical Geometry*, p. 162.

Regarding this as the limiting position of two points of tangency with the Steinerian which unite to form an inflexion, the corresponding points on the Hessian unite, while the common tangents to the polar curves move to coincidence in the same way that the points of tangency on the Steinerian do, giving three consecutive points in the same straight line for all polars.[1] The common tangent is an ordinary tangent to the Hessian, since only two points have moved to coincidence; and the line has here a pole of order 3, which counts for three double points of the pencil.

II. Two points may be consecutive and the third in another branch, so that the line is tangent to the Steinerian at a double point.

The condition of tangency gives a pole on the Hessian of order 2. The polar of the point has three double points, two of which coincide at the pole as in the ordinary case but still count for two.

III. If the Steinerian admits a triple point, the three intersections may be on three different branches.

Corresponding to a triple point with three simple branches is a polar curve with three double points, but no condition is necessarily imposed on any pole. A line tangent to the Steinerian at the triple point meets it in four points there, and from the condition of tangency the corresponding point on the Hessian is a pole of order 2, at which the double point counts for two.

A triple point on the Steinerian formed by a simple branch through a cusp, has a polar with two double points and a cusp. No pole is defined for a line passing through the point by this condition. If, however, the line is tangent to the simple branch at the double point, the corresponding point on the Hessian is a pole of order 2. The polar of the triple point has a double point at the pole, counting for two, and a cusp elsewhere. If the line is a cuspidal tangent, the cusp is at the pole and counts for three double points; the double point elsewhere counts for a single double point of the pencil.

Any multiple point which the Steinerian may have with dis-

[1] Cf. Cremona, *Introduction*, § 20, No. 119.

tinct or coincident tangents to the several branches will give analogous results.

§ 7.

The Base Curve with Triple Points and Multiple Points of Higher Orders.

The first polars of any two points in the plane pass through a triple point on U twice, and the four intersections compose a pole of order 4 for any line in the plane. The polar of the triple point has there a triple point with the tangents of U for its three tangents. A pencil of first polars has ordinarily $3(n-2)^2$ curves which may have a double point, and these correspond to the intersections of the line with the Steinerian; but in the case considered the first polar of every point in the plane has a double point at the triple point of U. Thus the ordinary Steinerian is indeterminate when the base curve has a triple point, but there is only a finite number of points whose second polars have a double point, and the locus for these can be found, at least theoretically. Adopting the notation suggested by Salmon,[1] this locus is the second-Steinerian of order $6(n-3)^2$; and the corresponding second-Hessian is the locus of such double points, among which are the triple points of U, and is of order $12(n-3)$. In general the ϑ-Steinerian and ϑ-Hessian are of orders $3\vartheta(n-\vartheta-1)^2$ and $3\vartheta^2(n-\vartheta-1)$ respectively.

It has been shown that the inflexion cubic for a cusp is composed of the cuspidal tangent taken twice and another line through the cusp; and it, as well as the Steinerian, is therefore degenerate when U has a cusp. If U has a triple point, the inflexion cubic for that point is indeterminate, as may readily be proved by evaluating the determinant for such a point. It is evident that the Steinerian is connected with the inflexion cubic for any point as the Hessian is with the inflexion locus for any line.

At a multiple point of order k, first polar curves will have a multiple point of order $k-1$ in general, but the polar of the

[1] *Higher Plane Curves*, p. 365.

multiple point itself will have precisely the same multiple point that U has. The number of poles at the point may be increased, as we have seen, by the position of the line or by coincidence of tangents.

There are certain harmonic relations which govern the tangents to first polar curves at a multiple point of U in consequence of the harmonic properties of poles and polars, and we shall conclude this paper with an outline of these relations.

A. Triple Points.

I. U has an ordinary triple point at $(0, 0, 1)$ and is of the form

$$a_0xy(x-y)z^{n-3}+(ax^4+bx^3y+cx^2y^2+dxy^3+ey^4)z^{n-4}+\cdots=0.$$

(*a*) L does not pass through the triple point.
The polar of any point (x_1, y_1, z_1) in L is

$$x_1[a_0y(2x-y)z^{n-3}+\cdots]+y_1[a_0x(x-2y)z^{n-3}+\cdots]$$
$$+z_1[a_0(n-3)xy(x-y)z^{n-4}+\cdots]=0,$$

with tangents at $(0, 0, 1)$ given by

$$x_1y(2x-y)+y_1x(x-2y)=0.$$

There are two double rays in this quadratic involution and therefore two polar curves have a cusp at the triple point. The two factors of the first term are y and the harmonic conjugate of y with respect to the other two tangents; and the second term is the corresponding expression for x. This relation of tangents is similar to the case for a double point.

(*b*) L passes through the triple point, or $L=x-ky=0$.
In this case the tangents are contained in

$$ky(2x-y)+x(x-2y)=0,$$

and are the same for all curves of the pencil, except for $(0, 0, 1)$ which has the tangents of U.[1] This again is an extension of the

[1] Cf. Cremona, *Introduction*, § 10, No. 48: If A and A' are tangent to all curves of a pencil at a, a curve may be found which has a for a triple point. If A and A' coincide all curves have a cusp.

case for a double point, where all polar curves touch the harmonic conjugate of the line with respect to the two tangents of U at the double point.

(c) L is tangent to U at the triple point.

Let L be $x-y=0$, and the tangents are $x^2-y^2=0$. Thus all polars have two common tangents, the line itself and its harmonic conjugate with respect to the other two.

II. U has a cuspidal branch at $(0, 0, 1)$, or its lowest term is $a_0x^2yz^{n-3}$. The tangents to first polars are given by

$$x(2x_1y+y_1x)=0.$$

(a) A line not passing through the triple point will have for the pairs of tangents to its first polar curves the cuspidal tangent with a pencil of lines through the triple point. One polar of the pencil with have a cusp there.

(b) A line $L=x-ky=0$ through the triple point has tangents given by $x(2ky+x)=0$, and they are the same for the polars of all its points. The polar of the triple point has a triple point and only this one polar of the pencil has a cusp.

(c) L is tangent at the triple point.

Let $L=x=0$, the cuspidal tangent, and the tangents to first polars are given by $x^2=0$; and all first polars of the pencil have a cusp at the triple point.

On the other hand if $L=y=0$, the tangents are $xy=0$, and only the polar of the triple point has a cusp.

B. *Quadruple Points.*

I. U has a quadruple point with four ordinary branches through it, or

$$U=xy(x-y)(x+ky)z^{n-4}+(ax^5+\cdots)z^{n-5}+\cdots=0.$$

The polar of (x_1, y_1, z_1) has tangents at $(0, 0, 1)$ given by

$$x_1[3x^2y+2(k-1)xy^2-ky^3]$$
$$+y_1[x^3+2(k-1)xy^2-3kxy^2]=0,$$

and the four double elements of the involution are the tangents to first polars which have a cuspidal branch at (0, 0, 1). As in the preceding cases the line $x - k'y = 0$ has the same tangents for all polars except that of the quadruple point. If the line is the tangent $x - y = 0$, we have

$$(x-y)[x^2 + 2(k+1)xy + ky^2] = 0,$$

which may be written

$$(x-y)\,[x(x+\overline{2k+1}y) - xy + y(2x+ky)] = 0;$$

where $x + \overline{2k+1}y$ is the harmonic conjugate of $x - y$ with respect to y and $x + ky$, and $2x + ky$ is the harmonic conjugate of $x - y$ with respect to x and $x + ky$. This may be written in other forms,

$$(x-y)\,[x(x+\overline{2k+1}y)$$
$$- x(x+ky) + (x+ky)(x+y)] = 0, \text{ etc.,}$$

showing other combinations of harmonic conjugates.

II. One of the branches is cuspidal, and U may be written $x^2(x-y)(x+ky)z^{n+4} + \cdots = 0$. The polar curve has tangents given by

$$x_1[4x^3 + 3(k-1)x^2y - 2kxy^2] + y_1[(k-1)x^3 - 2k^2xy] = 0.$$

For any line $x - k'y = 0$ the cuspidal tangent is a common tangent to all first polars as before. For the tangent $x - y = 0$ we obtain

$$x[(3+k)x^2 - (5k-1)xy - 2ky^2] = 0;$$

and for the cuspidal tangent, $x = 0$,

$$x^2[k(x-y) - (x+ky)] = 0,$$

giving combinations of harmonic conjugates.

III. Two of the branches are cuspidal and U has for its lowest term $x^2y^2z^{n-4}$. The tangents for any polar are

$$xy(x_1y + y_1x) = 0,$$

and any polar has a triple point with two fixed tangents, while the third belongs to a pencil through the point. If $L = x - ky = 0$, the tangents are all the same; while for one of the tangents to U, $x = 0$, we have $x^2y = 0$.

C. Quintuple Points.

I. The five branches are simple, and U has for its lowest term $xy(x-y)(x+ky)(x+ly)z^{n-5}$. The tangents to first polars are given by

$$x_1[4x^3y + 3(l+k-1)x^2y^2 - 2(k+l-kl)xy^3 - kly^4]$$
$$+ y_1[x^4 + 2(l+k-1)x^3y - 3(k+l-kl)x^2y^2 - 4kly^4] = 0,$$

and the six double elements of the involution are the cuspidal tangents to first polars at the multiple point. For $L = x - k'y = 0$ the tangents are the same for all first polars. The tangents to first polars of the pencil belonging to $x - y = 0$ are

$$(x-y)[xy(\overline{l+k+2}\,x + \overline{2kl+l+k}\,y)$$
$$+ (x+y)(x+ky)(x+ly)] = 0,$$

where the first term in the bracket is composed of the product of two tangents, x and y, and the harmonic conjugate of $x - y$ with respect to $x + ky$ and $x + ly$; and the second term is the product of $x + ky$ and $x + ly$, and the harmonic conjugate of $x - y$ with respect to x and y. By this selection the result is more simple on account of the symmetry of the pairs of tangents. Arranging the pairs in different order a third term appears; for instance

$$(x-y)[x(x+ly)(x+\overline{2k+1}\,y)$$
$$+ y(x+ky)(x+ly) + lxy(x+ky)] = 0.$$

Combinations of cuspidal branches give results corresponding to those obtained for lower orders of multiplicity, and these methods may be extended to higher orders where the branches are simple and cuspidal. The equations involving the tangents will allow various combinations of harmonic conjugates, and it is to be observed that the forms can be made more symmetric when the multiple point on U is of odd order.