PRACTICAL CURVE TRACING

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WITH CHAPTERS ON

DIFFERENTIATION AND INTEGRATION

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WITH DIAGRAMS

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PREFACE

DURING recent years the use of squared paper in the solution of various problems, for the representation of the results of experiments and for the deduction of formulæ to express those results has developed very rapidly, and is now taught to a greater or less extent to the great majority of students. At the same time, it has become more and more recognized that while the student, and especially the student of Engineering, needs a good knowledge of various branches of Mathematics, and, above all, the ability to make use of his knowledge in the solution of practical problems, it is not necessary to burden him with a large amount of purely academic Mathematics, of the kind which has been aptly called "Mental Gymnastics." Thus, while it is necessary that he should be able to recognize the nature of the curve represented by a given equation, or to find the equation corresponding to a given curve, and should be perfectly familiar with the chief characteristics of those curves which he is constantly meeting in practice, it is, as a rule, undesirable that he should spend a considerable amount of time in making a complete study of Co-ordinate Geometry from the purely mathematical standpoint. This book is an attempt to present the methods of curve plotting in an orderly sequence, and at the same time to give the student that knowledge of the properties of the chief families of curves which is essential for him. The author's experience as a teacher has shown him that by far the best method of introducing the Calculus to the student who requires it as a tool in his mental workshop is from the graphical standpoint, following upon some such treatment of curves as is given here. He has, therefore, added chapters upon Differentiation and Integration, which he hopes may prove of use by giving to the student a general idea of the principles involved, and especially of their real meaning, before he proceeds to a somewhat fuller treatment of the subject.

The diagrams in the book are in every case reduced from the original full-size drawings from which the measurements quoted in the text were taken; but it is essential that the student, when working through the book, should plot every curve for himself, and check all the measurements of slope, etc., from his own drawings. In the reproductions the one-tenth inch squares have been omitted for clearness, and the squares shown are in all cases those of one inch side on the original drawing.

The method of measuring the slope of a curve by actually drawing the tangent is sometimes objected to on the ground of inaccuracy; but the author's experience shows that by good and careful workmanship it is possible to rely on the results so obtained to a degree of accuracy which is sufficient for all practical purposes.

The author wishes to express his gratitude to Prof. J. Goodman for much valuable help and advice in the preparation and publication of the book; and also to Messrs. G. E. Edson and G. Calverley, for assistance in the preparation of the drawings, and in other ways.

R. HOWARD DUNCAN.

THE UNIVERSITY OF LEEDS,
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CHAPTER I

INTRODUCTORY

Co-ordinates.—If we wish to fix the position of a certain point on a sheet of paper or, say, in a field, the simplest method is to measure its perpendicular distances from two adjacent sides of the sheet of paper or field. The same idea lies at the root of the use of squared paper, the position of points being fixed by their perpendicular distances from two fixed lines at right angles to each other drawn upon the sheet of paper. These lines are called the *axes*, and the perpendicular distances of a point from these axes are called its *co-ördinates*. In general work it is usual to represent the co-ordinates by the letters x and y. The co-ordinate x is then understood to be measured *horizontally* from the vertical axis, which is known as the "axis of Y," and the co-ordinate y is measured *vertically* from the horizontal axis, which is known as the "axis of X." Thus, according to the usual convention, the axis of X and the x co-ordinate are both horizontal, while the axis of Y and the y co-ordinate are both vertical. The student should, however, not confine himself to the use of the letters x and y to represent the co-ordinates, as in many cases it is more convenient to use letters which suggest the quantities represented; thus, in plotting a curve to represent the relative variation of the pressure and the volume in an expanding fluid, it is much more convenient to use the letters p and v to represent these quantities respectively. If the co-ordinates of a point be x and y respectively, the point is represented by the symbol (x, y). Thus, in Fig. 1 the point A, which is distant 2 units from the axis of Y and 3 from the axis of X, is written (2, 3).

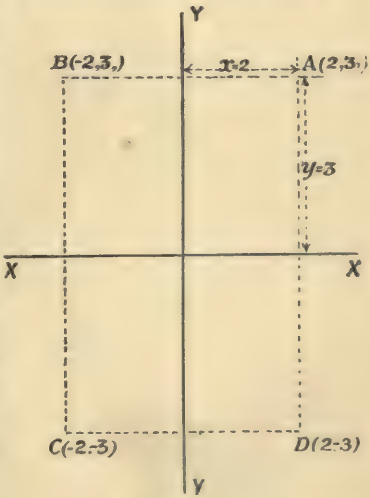


Fig. 1.

B

The Use of Signs.—A point may be either to the right or to the left of the axis of Y, and either above or below the axis of X. It becomes necessary, therefore, to adopt some convention in order to distinguish the direction in which the co-ordinate is to be measured. In order to do this we make use of the signs + and −, as shown in the following table.

Co-ordinate of x

measured to the *right* of OY is +
,, ,, *left* ,, −

Co-ordinate of y

measured *above* OX is +
,, *below* ,, −

Thus, referring to Fig. 1, the point B is represented by (− 2, 3), the point C by (−2, −3), and the point D by (2, −3). The two axes divide the sheet into four quadrants, which are numbered consecutively the first, second, third, and fourth quadrants. The quadrant in which a point falls can be determined at once from the signs of its co-ordinates (see Fig. 2).

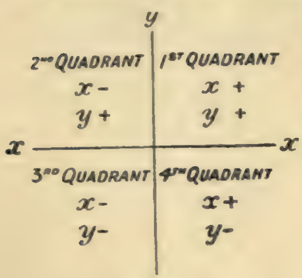


Fig. 2.

Constants and Variables.—All quantities may be divided into two classes, *constants* and *variables*. A constant quantity is one which maintains a certain fixed value. A variable is a quantity whose value changes. Variables may themselves be subdivided into two other classes, *independent* and *dependent* variables. An independent variable is one to which any suitable value may be assigned at will. A dependent variable is one which depends for its value upon the value assigned to some independent variable, and which varies with the latter according to some fixed law. Thus, for example, sin A is a variable quantity which depends for its value upon that of the independent variable A, and is therefore termed a dependent variable. It should be noted, however, that of two variables connected with each other in this way, either may be taken as the independent variable, the other then being dependent upon it. Thus, the relation $x = \sin A$ may be written in the form $A = \sin^{-1}x$, x in this latter case being taken as the independent variable, upon whose value A is dependent.

Function of a Variable.—If a variable y depends for its value upon another variable x, so that when the value of x is known that of y is also known, and such that any change in x produces a corresponding change in y, then y is said to be a "*function*" of x, and x is sometimes spoken of as the "*argument*" of y. For example, (1) any algebraic expression containing x is a function of x; (2) the trigonometrical ratios are functions of the angle; (3) the weight of a sphere of known material is a function of its radius. It is sometimes convenient to have a symbol to represent *any* function of x. This is then written in one of the forms $f(x)$; $F(x)$; $\phi(x)$, or $\psi(x)$.

Graphical Representation of a Function.—The variation of a function relatively to its independent variable may be represented usefully by a curve, obtained by assigning a series of convenient values to the independent variable and calculating the corresponding values of the function. Each pair of values of the independent and its function are then taken as the co-ordinates of a point, and a smooth curve drawn through the points so obtained. It will be found that a regular variation is always represented by a regular curve, and that similar functions are always represented by curves of the same nature. A set of curves of the same general form, and therefore representing similar functions, are said to belong to the same "family." It is the object of this book to show the form of curve which represents a given function, and conversely to show how a function may be determined from its representative curve or "graph."

CHAPTER II

CURVE PLOTTING FROM GIVEN DATA AND FROM AN EQUATION

THE simplest use of squared paper is in the mere graphical representation of natural observations or the results of experiments. Every one is more or less familiar with the curves published in the daily papers showing the rise and fall of the barometer, and with the statistical diagrams published in the magazines. The student should be able to plot such curves quickly and accurately, and should be able to read a curve with perfect ease, that is, to be able to tell at a glance the sort of relation existing between the co-ordinates plotted.

It cannot be too strongly insisted upon that good results can only be attained by careful and accurate work in this, as in all graphical methods. Neatness of execution is the first essential towards successful work.

Choice of Scales.—The accuracy of the result is of course limited by the scales to which the curves are plotted, the larger the scale the greater the degree of accuracy attainable. But this statement is only true to a certain extent. If the scales are taken so large that the points of observation are widely separated from each other, it becomes difficult to determine the form of the curve connecting them; and, further, if the scales are so large that they can be read to a greater degree of accuracy than that to which the figures plotted were determined, any experimental errors are magnified and appear to be more important than they really are. In this way a set of figures which really approximate to "the straight line law" may appear, if plotted to too large a scale, to give only a curve of bewildering irregularity. In choosing the scales to which to plot any given set of figures, these two considerations should be borne in mind: (1) that the scales should be large enough to give the required degree of accuracy, and (2) that they should not be so large as to separate the points of observation by inconveniently great distances, or to represent a degree of accuracy greater than that of the original figures. For the sake of greater freedom in choosing the scales, the paper should be purchased, not in small sheets of uniform size, but in large sheets or rolls, which can be cut to the required shape and size. It is perhaps necessary to point out that the paper must be ruled in inches and tenths of an inch, *not* in eighths or twelfths of an inch, as some makers supply it. With paper so ruled, it is possible by good workmanship to plot fairly accurately to the $\frac{1}{100}$ part of an inch. This dimension, then, should be taken as commensurate with the degree of accuracy of the experimental figures. Thus, if the figures to be plotted are accurate to within 0·1, say, 0·1 should be represented on the paper by not less than 0·01 or greater than 0·1 inch. This will ensure the curve being of as great accuracy as the figures from which it is obtained, without the scales being so great as to unduly emphasize experimental errors.

The convenience of the scales is another point to which attention should be paid. Thus, no practical man would choose such a scale as 3 units to the inch, or 1 unit to 3 inches, this obviously leading to difficulty in plotting a curve, or in reading it when plotted. The best scales to use are $\frac{1}{2}$, 1, or 2 inches to represent 1 unit or any integral power of 10 or of 0·1 units. Other scales which are fairly convenient but not so simple to use as the former, are $\frac{1}{4}$ inch or 4 inches to 1, 10, etc., units.

The scales should be clearly marked upon the axes. It is undesirable to mark the figures opposite each point of observation as is sometimes done, but each inch should have its scale value written opposite to it. Excepting under special conditions—as, for example, when all the figures are far distant from zero as compared with their differences between each other—the scales should both begin at zero at the intersection of the axes or "origin."

Points of Observation.—All the points of observation, that is those points representing the figures obtained by experiment or otherwise determined for the purpose of plotting the curve, should be clearly marked. By far the best mark to use for this purpose is a simple cross formed of a vertical and a horizontal line, thus $+$, so showing clearly the horizontal and vertical co-ordinates. The $\times$ cross should never be used, as it detracts from the neatness and therefore from the accuracy of the work. When two or more curves are to be plotted on the same sheet, they should be drawn in different colours, each set of points of observation being, of course, of the same colour as the curve to which they belong. It is sometimes useful also to write out the corresponding sets of figures in the same colours as the curves which represent them. Where it is impossible to make use of various colours—as, for example, when copies of the curves are to be made by a printing out process—the curves should be distinguished by varying types of lines: thick and thin plain lines, dotted lines, and so forth. In this case it will be necessary to use different marks for the points of observation belonging to each curve, such as $\oplus$ and $\diamondplus$.

A Typical Curve.—We will now proceed to plot a curve from data experimentally determined, and will take as our example the variation in the volume of 1 lb. of dry saturated steam as its pressure varies from that of the atmosphere to 80 lbs. per square inch absolute. The figures are given in the table.

Pressure	14·7	20	30	40	50	60	70	80	lbs. per sq. in.
Volume	26·4	19·7	13·5	10·3	8·4	7·0	6·1	5·4	cubic feet.

Here the pressure is given to 0·1 lb. per square inch, and the figures given are separated in most cases by 10 lbs. per square inch. We therefore choose as our scale of pressures 1 inch to 10 lbs. per square inch. The volumes are also given to 0·1 cubic foot; but successive figures are nearer together than in the pressure scale, it will therefore be convenient to take a somewhat larger scale, say 1 inch to 5 cubic feet. Then, plotting each pair of figures in succession, as previously described, we obtain the curve shown in Fig. 3. In this, as in other diagrams in the book, the $\frac{1}{10}$-inch lines have been omitted for the sake of clearness.

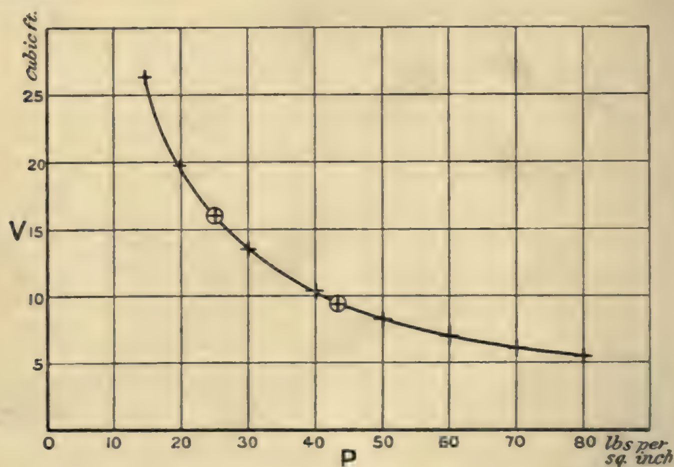


Fig. 3.

From the shape of the curve we see at once that the rate of variation of the volume relatively to the pressure is not uniform, but is greater at low pressures, that is to say, that at low pressures a given change in the pressure causes a greater change of volume than is produced by the same pressure change at a higher pressure. The observation of such points as this is what has been referred to as "reading the curve."

Interpolation.—We can use such a curve as this to determine the volume at any other pressure, or *vice versâ*, within the range of the curve. For example, suppose it is desired to determine the volume at a pressure of 25 lbs. per square inch, then, marking the point at which the curve cuts the horizontal line drawn through 25 lbs. per square inch on the pressure scale and projecting down vertically on to the volume scale, we find that the corresponding volume is 16·0 cubic feet. Conversely, if we wish to know at what pressure the volume of 1 lb. of steam is 9·5 cubic feet, proceeding similarly, we find that it is 43·4 lbs. per square inch. These points are marked on the curve by the symbol ⊕. This process of determining intermediate values from the curve is known as "Interpolation," and is always legitimate when the curve is regular, the values so found being just as accurate as the drawing of the curve will allow.

Extrapolation.—The method of obtaining extra values from the curve described in the last paragraph, may in some cases be extended to determine values lying outside the range covered by the original data. It is then known as "Extrapolation." The method of extrapolation must, however, be used with extreme caution. In those cases where the curve is of such a nature that it can easily be produced with accuracy (the only curve which can really be so produced being the straight line), and where it is known that there is no change in the law connecting the variables outside the original range, the method may be safely used. In all other cases it is obviously unsound. To illustrate this, in the case where the original law gives us a straight line, thereby fulfilling the first condition, we will take the case of a tension test of a bar of steel, plotting the load (horizontal) with the resulting extension (vertical) up to the elastic limit. The figures from such a test are given below.

Load	0	1	2	3	4	5	6
Extension . . .	0	0·0009	0·0018	0·0027	0·0037	0·0046	0·0055
Load	7	8	9	10	11	12	tons.
Extension . . .	0·0064	0·0074	0·0083	0·0092	0·0101	0·0110	inches.

This curve is plotted in Fig. 4, the scales taken being 1 inch to 1 ton, and 1 inch to 0·001 inch respectively. Allowing for experimental errors, the points lie on a straight line, and the mean straight line has been drawn through them. Now, if we attempt to extrapolate in order to determine the extension for a load of, say, 15 tons, we should obtain a value of 0·01375 inch, as shown by the dotted produced line and the point marked ⊕. But the experimental figures for loads greater than 12 tons are

Load	13	14	15	16	tons.
Extension . .	0·0230	0·18	0·23	0·30	inches.

These figures show that the curve has not continued in a straight line, but has curved rapidly upwards as indicated in the figure, and

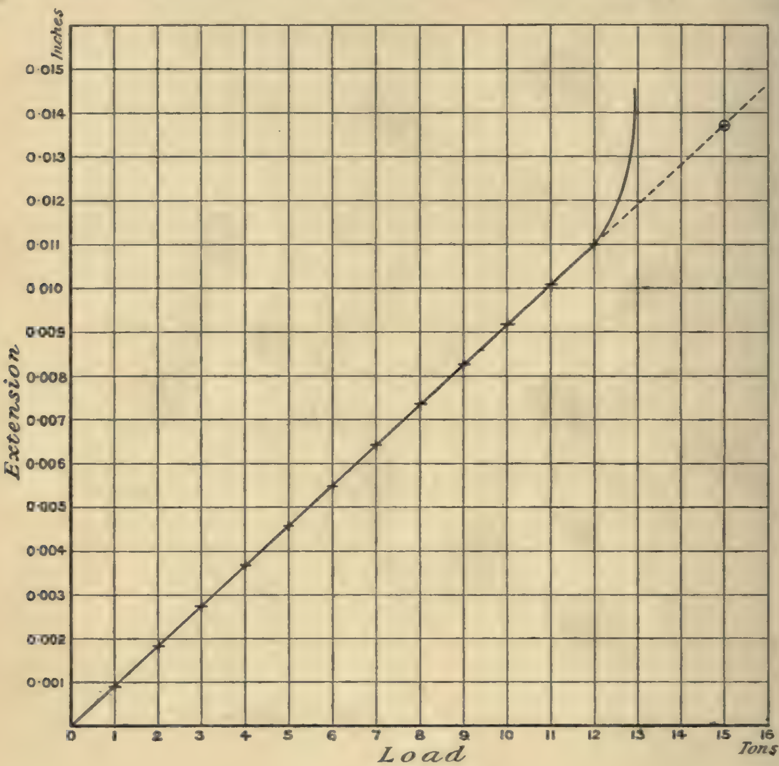


Fig. 4.

that instead of the extrapolated value of 0·01375 inch, the true extension for 15 tons is 0·23 inch, the extrapolated value being therefore entirely wrong and altogether misleading.

Graphical Representation of an Equation.—Any equation involving two unknowns, x and y, has an infinite number of solutions, that is, pairs of values of x and y which satisfy it. These pairs of values when plotted, however, all lie upon a curve, which is the graphical representation or graph of the equation. If it is required to draw a curve which represents a given equation, it is necessary to find by calculation a number of these pairs of values within the required range, and, using them as the co-ordinates of a series of points, plot them and draw a smooth curve through them. Thus, supposing that it is required to draw the graph of the equation

$$y = x^3 - 5x^2 + 20$$

between the values $x = 0$ and $x = 5$, we must assign a series of values to x which lie between the stated values (in this case taking values of x differing successively by 0·5) and calculate the corresponding values of y. It is best to arrange the work in tabular form, thus—

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0
x^3	0	0·12	1·0	3·4	8·0	15·6	27·0	42·9	64·0	91·3	125·0
x^2	0	0·25	1·0	2·25	4·0	6·25	9·0	12·25	16·0	20·25	25·0
$5x^2$	0	1·3	5·0	11·3	20·0	31·3	45·0	61·3	80·0	101·3	125·0
$y = x^3 - 5x^2 + 20$	20·0	18·9	16·0	12·1	8·0	4·3	2·0	1·6	4·0	10·0	20·0

Then, since x varies between 0 and 5, a convenient scale for the X-axis will be 1 inch to 1. y varies between 1 and 20, and it is

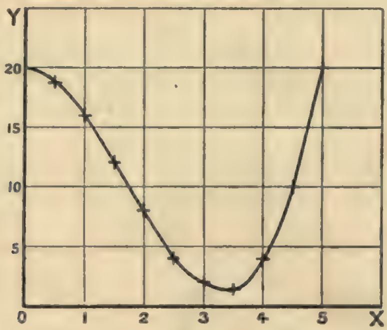


Fig. 5.

necessary to be able to represent 0·1. A convenient scale for the Y-axis will therefore be 1 inch to 5. Then, plotting x and y in the usual way, we obtain the curve shown in Fig. 5. Sometimes, having obtained a series of points such as the above, it is desirable, at certain critical points in the curve, to obtain some extra points at closer intervals, and sometimes, also, it is useful to obtain some points conversely, that is, by assuming values for y and calculating the corresponding values of x.

CHAPTER III

THE SIMPLE EQUATION OF THE FIRST DEGREE—THE STRAIGHT LINE

The General Equation of the First Degree and its Graphical Representative.—Any simple equation of the first degree in x and y may be reduced to the form

$$y = ax + b$$

where a and b are constants. As has been stated in Chapter I., all equations of the same general form have curves of the same nature as their graphical representatives. It is required to find the type of curve which represents the above equation. In order to do this we will take a particular case of the equation and plot its graph. Let the equation be

$$y = 2{\cdot}5x + 3$$

Then, arranging the calculations in tabular form as before, we obtain the following series of values for x and y :—

x	-3	-2	-1	0	$+1$	2	3
$2{\cdot}5x$ . . .	$-7{\cdot}5$	$-5{\cdot}0$	$-2{\cdot}5$	0	$+2{\cdot}5$	$5{\cdot}0$	$7{\cdot}5$
$y = 2{\cdot}5x + 3$	$-4{\cdot}5$	$-2{\cdot}0$	$+0{\cdot}5$	$3{\cdot}0$	$5{\cdot}5$	$8{\cdot}0$	$10{\cdot}5$

Plotting these figures, we obtain the *straight line* shown by the thick continuous line in Fig. 6.

Again, consider the equation,

$$y = -3x + 2{\cdot}8$$

We obtain the values for x and y given below

x	-3	-2	-1	0	$+1$	2	3
$-3x$ . . .	$+9$	$+6$	$+3$	0	-3	-6	-9
$y = -3x + 2{\cdot}8$	$11{\cdot}8$	$8{\cdot}8$	$5{\cdot}8$	$2{\cdot}8$	$-0{\cdot}2$	$-3{\cdot}2$	$-6{\cdot}2$

which plotted give us again a *straight line* represented in the same figure by the thinner line.

From these and other examples, which the student should plot for himself, it will be seen that every equation of the first degree in x and y is represented by a straight line.* From this fact the relation

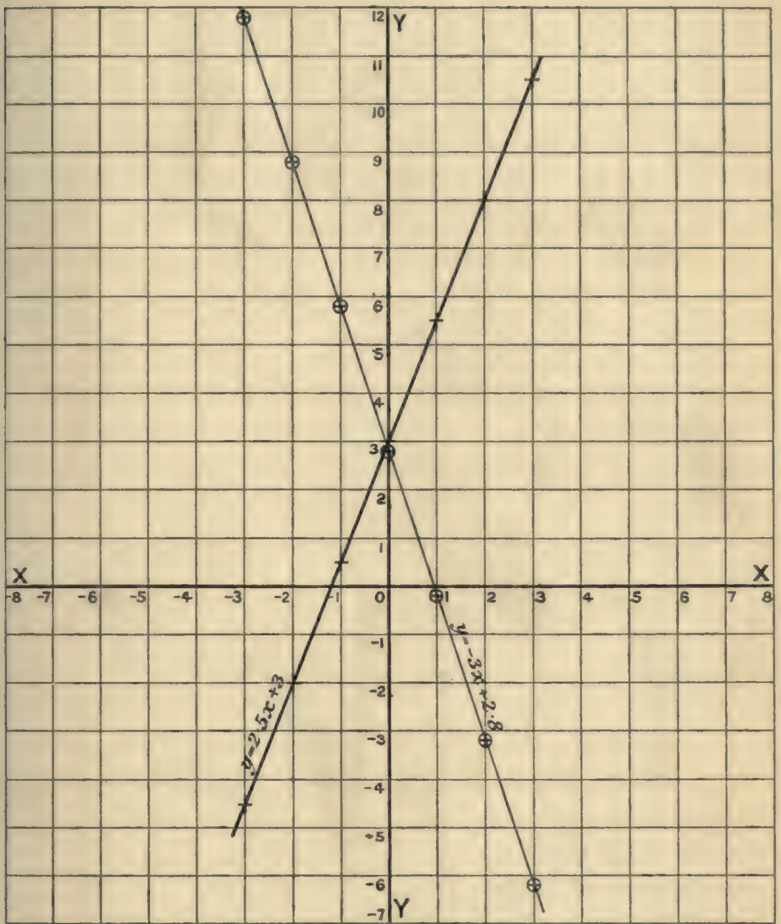


Fig. 6.

between x and y given by the equation $y = ax + b$, is known as *the straight-line law.*

Graphical Interpretation of the Constants.—It remains to determine the meaning of the constants a and b. To do this, we shall take a series of equations having the same value for a, but different values

* Rigid proofs of this and other similar statements may be found in books on Co-ordinate Geometry, but lie outside the province of the present work.

for b, and another series having different values for a but the same value for b, and examine the resulting lines.

First, consider the equations

$$(1)\ y = 2x, \qquad (2)\ y = 2x + 5,$$
$$(3)\ y = 2x + 12, \qquad (4)\ y = 2x - 5,$$

in all of which the value of a is 2, but the values of b are respectively 0, 5, 12, and -5. Knowing that such equations give us straight lines, it would, of course, be sufficient to find two pairs of values of x and y from each equation in order to fix the corresponding line; but in order to further illustrate the truth of this statement, we will determine several points on each line.

x	-10	-5	0	$+5$	10	15	
$2x$	-20	-10	0	$+10$	20	30	Thick continuous.
$2x + 5$ . . .	-15	-5	$+5$	15	25	35	Thin continuous.
$2x + 12$ . . .	-8	$+2$	12	22	32	47	Dotted.
$2x - 5$ . . .	-25	-15	-5	$+5$	15	25	Dot and dash.

These lines are plotted in Fig. 7, the type of line used for each being given at the end of the corresponding row of figures above. It is seen at once that *all these lines are parallel*, that is, that they have the same *slope*. The usual way of measuring the slope of a line is by the ratio of the vertical rise to the corresponding horizontal increase, that is, by the ratio of the change in y to the corresponding change in x. Then, taking any two points, P and Q, on one of the lines, and completing the right-angled triangle PQR, the slope $= \frac{PR}{QR}$. In the figure P and Q have been taken on the line $y = 2x - 5$,

$$\text{then } PR = 20 - 5 = 15,\ QR = 12{\cdot}5 - 5 = 7{\cdot}5$$

and the slope $= \frac{15}{7{\cdot}5} = 2$, which it should be noted is the value for a in the equation of the line. The student should find the slope of each line in the same way.

Again, consider the series of equations

$$(1)\ y = x + 5, \qquad (2)\ y = 2x + 5,$$
$$(3)\ y = 3x + 5, \qquad (4)\ y = -2x + 5,$$

in all of which the value of b is 5, but in which the values for a are respectively 1, 2, 3, and -2. We calculate a number of points on each.

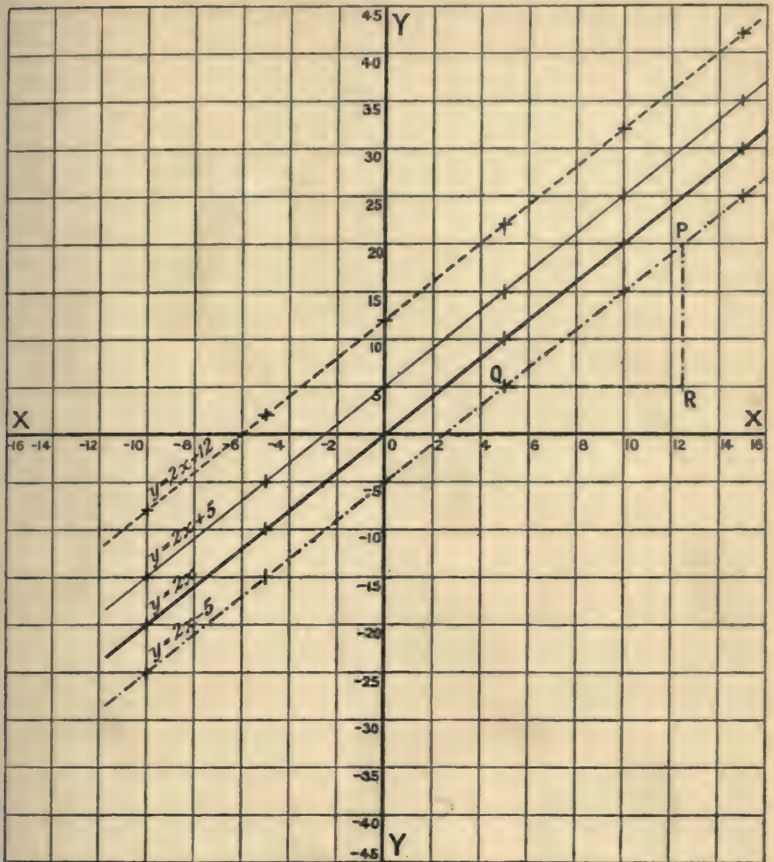


Fig. 7.

x	-15	-10	-5	0	$+5$	10	15	
$x+5$ . .	-10	-5	0	$+5$	10	15	20	Thick continuous.
$2x$ . . .		-20	-10	0	$+10$	20	30	
$2x+5$ . .		-15	-5	$+5$	15	25	35	Thin continuous.
$3x$ . . .		-30	-15	0	$+15$	30	45	
$3x+5$.		-25	-10	$+5$	20	35	50	Dotted.
$-2x$ . .		$+20$	$+10$	0	-10	-20	-30	
$-2x+5$.		25	15	5	-5	-15	-25	Dot and dash.

These are plotted in Fig. 8. It is seen again that the whole series have a common property, namely, that they all intersect the axis of Y at the same point. Further, this point is at a distance 5 from the origin. The distance intersected upon the axis of Y between a given

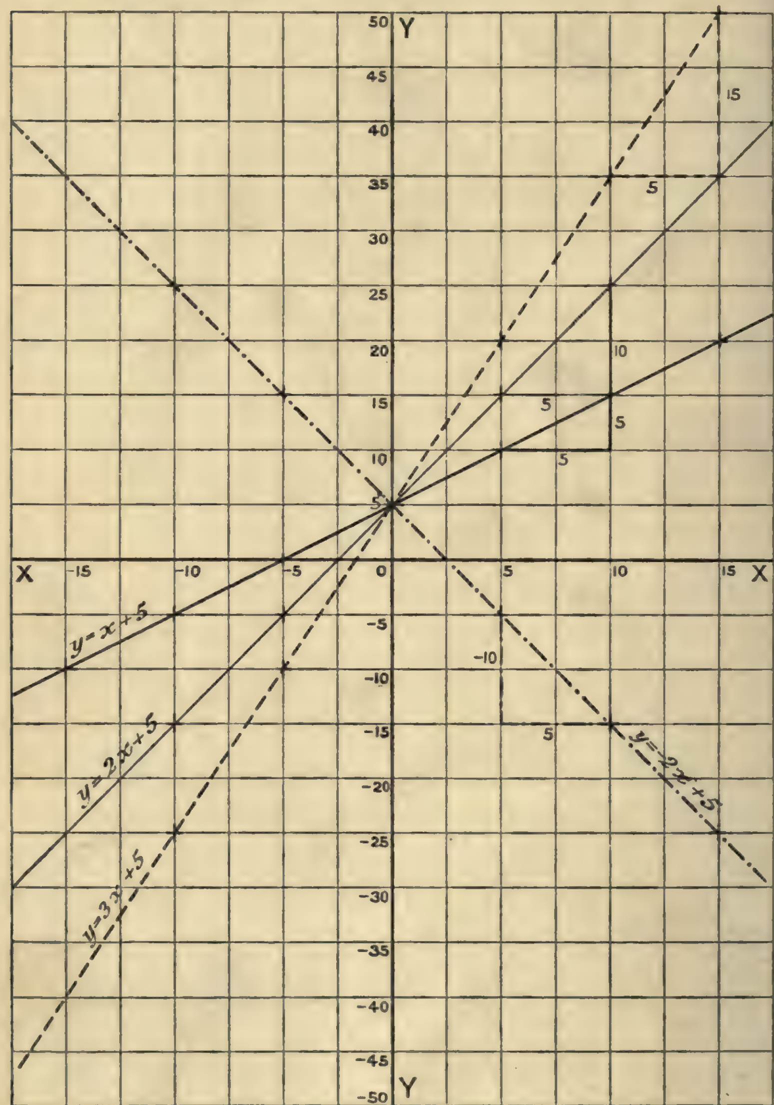


Fig. 8.

curve and the origin is called "The intercept upon the axis of Y" of that curve. In this case, then, the intercept of all the lines upon the axis of Y is equal to 5, which is the value for b in their equations.

Refer again to Fig. 7. It is seen that the intercept on the axis of Y for the line $y = 2x$ is 0, that for the line $y = 2x + 12$ is 12, for $y = 2x - 5$ it is -5. Hence we draw the conclusion that b *is equal to the intercept on the axis of* Y. This conclusion may be easily verified by putting $x = 0$ in the equation $y = ax + b$, when we obtain the value $y = b$. But when $x = 0$ the line cuts the axis of Y. Hence b is the intercept on that axis.

Looking again at Fig. 8, we see that all the lines are of different slope. Drawing a right-angled triangle for each line, and measuring the slope by it, we find that the slope of the line $y = x + 5$ is 1, that of $y = 2x + 5$ is 2, that of $y = 3x + 5$ is 3. The line $y = -2x + 5$ slopes in the opposite direction to the others, y getting less as x gets greater. This gives us *negative slope*, and measuring it we find it to be -2. It is evident from this, then, that *a gives us the slope of the line.*

Summing up our conclusions, then, we have shown that

(1) Every equation of the form $y = ax + b$ is represented by a straight line.

(2) All lines having the same value for a are parallel.

(3) All lines having the same value for b have the same intercept on the axis of Y.

(4) a is equal to the slope of the line.

(5) b is equal to the intercept of the line on the axis of Y.

We can verify these conclusions by looking at the matter in another way. Consider the equation $y = ax$, or $\frac{y}{x} = a$. This means that the ratio of y to x is constant and is equal to a; and therefore it is evident that the equation will be represented by a straight line passing through the origin, and such that the ratio of its vertical rise to its horizontal increment is equal to a, that is, that the slope of the line is a. Now, if we introduce a term b, so that the equation becomes $y = ax + b$, we have increased every value of y by the same amount b, or in other words, we have moved the line bodily through a vertical distance equal to b. But the line $y = ax$ passed through the origin, therefore the line $y = ax + b$ will intersect the axis of Y at a height b.

Lines of No Slope and of Infinite Slope.—Putting $a = 0$ in the general equation, we derive the equation $y = b$. In this case, y is constant and equal to b for all values of x—that is, the equation is represented by a straight line parallel to the axis of X, and distant b from it.

Putting $a = \infty$, we obtain the equation—

$$y = \infty x + b,$$

$$\text{or } x = \frac{y}{\infty} - \frac{b}{\infty} = 0 - 0 = 0$$

unless b is also infinite, when the equation becomes $x = c$, giving a line parallel to the axis of Y, and distant c from it. These lines are shown in Fig. 9.

Fig. 9.

From the above it follows that the equations of the axes X and Y are respectively, $y = 0$ and $x = 0$.

To Draw the Line directly from its Equation.—We are now in a position to draw a straight line at once from its equation without actually plotting a series of points upon it. It is known that the equation $y = ax + b$ gives us a straight line whose slope is a, and whose intercept on the axis of Y is b. To draw the line, then, measure first a distance b along the axis of Y, and through this point draw a line whose slope is a. To take a concrete example, draw the line whose equation is $y = 3x - 2$. In Fig. 10, P is the point on the axis of Y for which $y = -2$. Then, draw PR parallel to the axis of X making PR = 1, from R draw RQ parallel to the axis of Y making RQ = 3, and join PQ, which is the line required. Of course, PR and RQ must be measured on the respective scales of x and y, and it is better to draw RQ vertically until it cuts the horizontal line through $3 \times 1 - 2 = 1$, than to actually measure its length equal to 3. If the scales are small it is more accurate to make PR = 10, or any other convenient number, and correspondingly to make RQ $= a \times 10$.

The Equation of a given Straight Line.—It is now possible for us to determine the equation corresponding to any given straight line. In Fig. 11 we have the straight line PQ, and it is required to find its equation. It is known that this will be of the form $y = ax + b$, where a is the slope of the line, and b the intercept upon the axis of Y. Produce the line until it cuts the axis of Y at the point R. The y co-ordinate of R is 2·3, therefore $b = 2{\cdot}3$. Draw RM parallel to the axis of X, and PM perpendicular to it. Then

$$a = \frac{\text{PM}}{\text{RM}} = \frac{5{\cdot}6 - 2{\cdot}3}{4{\cdot}5 - 0} = \frac{3{\cdot}3}{4{\cdot}5} = 0{\cdot}733$$

and the required equation is $y = 0{\cdot}733x + 2{\cdot}3$.

In some cases this method is inconvenient, as, for example, when the line does not cut the axis of Y within the limits of the sheet of paper. Such a line is shown in Fig. 12. Here a different method must be adopted. Take any two points on the line, such as A = (5, 20) and B = (12, 3), and substitute their co-ordinates in the general

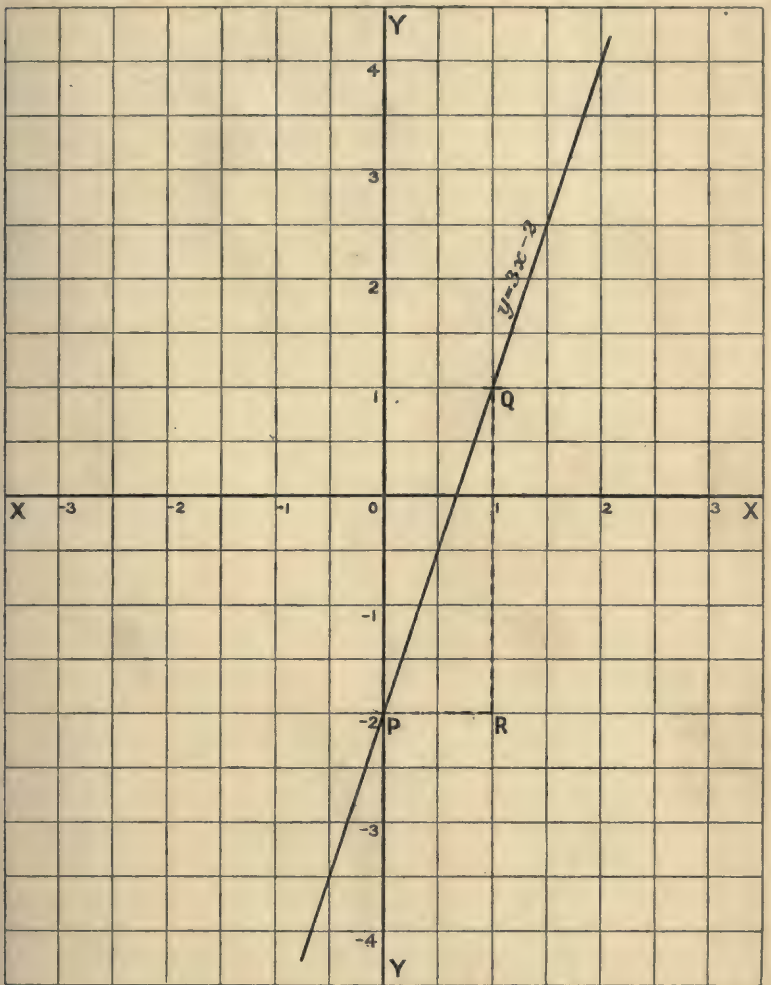


Fig. 10.

equation of the straight line, thus obtaining the two simultaneous equations in a and b—

$$20 = 5a + b$$

$$3 = 12a + b$$

Subtracting, $$17 = -7a$$

or $$a = -\frac{17}{7} = -2{\cdot}43$$

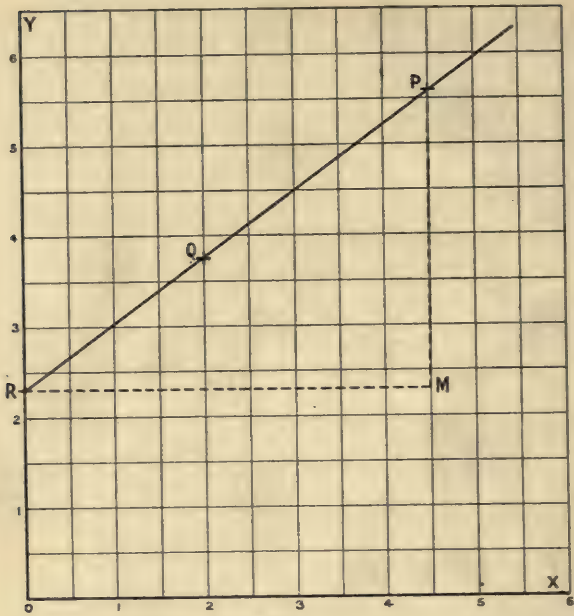


Fig. 11.

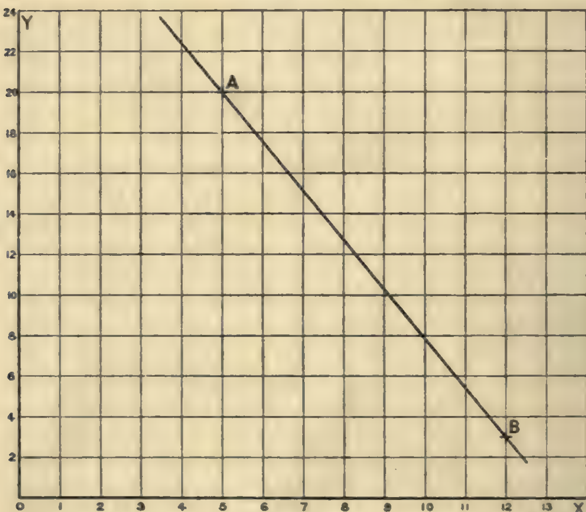


Fig. 12.

Substituting this value for a in the first equation—

$$20 = 5 \times (-2{\cdot}43) + b$$
$$= -12{\cdot}15 + b$$

or
$$b = 32{\cdot}15$$

Then the required equation is

$$y = -2{\cdot}43x + 32{\cdot}15$$

The Mean Straight Line.—The line whose equation is to be determined is usually obtained as the result of a series of experiments. The points so obtained only very rarely lie quite accurately in a dead straight line, owing to errors of observation. It is then necessary to draw a straight line which lies as evenly as possible amongst the points and approximates as closely as possible to them. This is known as "The Mean Straight Line." The best method of drawing it is by means of a stretched piece of thread which can be moved about amongst the points until the best position is obtained, the sum of the distances of the

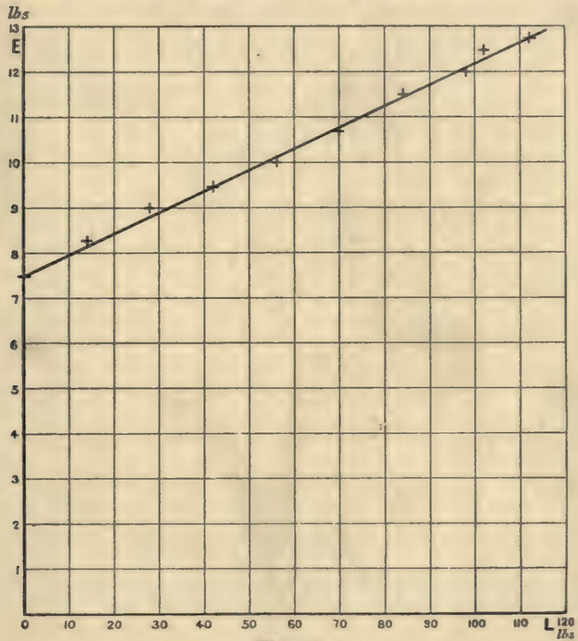


Fig. 13.

points from the line on either side being about the same. The ends of the threads are then marked, and subsequently joined by a straight line. The edge of a transparent set-square or straight-edge may be used instead of the thread, but on the whole the former is the more satisfactory method.

We will conclude with an example :—In a test of a Weston pulley block, it was found that an effort E lbs. was required to lift a load of L lbs. Find the equation connecting L and E.

L . .	0	14	28	42	56	70	84	98	112 lbs.
E . .	7·5	8·25	9·0	9·5	10·0	10·75	11·5	12·0	12·75 lbs.

Taking the axis of L horizontal and that of E vertical, we obtain on plotting the points shown in Fig. 13. These points, it is seen, do not lie absolutely on a straight line, but do so approximately. The mean straight line is then drawn among them by means of a stretched string as already described. It is required to find the equation of this line, which will be of the form

$$E = aL + b$$

The intercept on the axis of E is 7·52, therefore $b = 7{\cdot}52$. The E co-ordinate of the point for which L is 100 is 12·20. Then, drawing the vertical and horizontal line through these two points, we obtain the slope of the line, that is,

$$a = \frac{12{\cdot}20 - 7{\cdot}52}{100 - 0} = \frac{4{\cdot}68}{100} = 0{\cdot}0468$$

Therefore the relation between L and E is given by the equation

$$E = 0{\cdot}0468\,L + 7{\cdot}52$$

or

$$L = 21{\cdot}35\,E - 160{\cdot}5$$

CHAPTER IV

$y = ax^n + bx + c$.—THE PARABOLIC FAMILY

THE general equation $y = ax^n + bx + c$ represents an important family of curves. In order to find the graphical meaning of each of the four constants a, b, c, and n, we will consider each of them in turn, and by suitably varying them determine the effect such variation in each has upon the shape of the curve.

The Meaning of n.—The fundamental curves, of which those given by the full equation above are variations, are given by the simpler general equation $y = x^n$, which we will first examine fully. In this case we have put $a = 1$, and $b = c = 0$, and by varying n we shall be enabled to arrive at its graphical meaning.

If $n = 1$ the equation becomes $y = x$, which, as has been shown in the previous chapter, represents a straight line of unit slope through the origin.

$n = 2$ or $y = x^2$.—A preliminary examination of this equation will give us some information as to the nature of the curve before the latter is actually plotted. The equation may be written in the form $x = \pm\sqrt{y}$. From this it is evident that if y is negative there can be no real values of x, since the square root of a negative quantity is imaginary. Hence, *the curve lies wholly above the axis of* X. Again, for every value of y there will be two numerically equal values of x, one positive and the other negative ; that is to say, for each value of y there will be two points on the curve equidistant from the axis of Y—one on the right and the other on the left. Hence, the axis of Y bisects every horizontal chord of the curve, or *the curve is symmetrical about the axis of* Y. Proceeding to plot the curve the following set of figures is obtained :—

x . .	0	±0·5	±1·0	±1·5	±2·0	±2·5	±3·0	±3·5	±4·0	±4·5	±5·0	±5·5	±6·0	±6·5	±7
$y=x^2$	0	0·25	1·0	2·25	4·0	6·25	9·0	12·25	16·0	20·25	25·0	30·25	36·0	42·25	49·0

These, when plotted, give us the curve shown in Fig. 14 by the heavy continuous line. It should be noted that the curve has the two characteristics formerly determined, namely, that it lies wholly above the axis of X, and is symmetrical about the axis of Y. It consists of one infinite branch opening out in width as y increases. This curve is known as the **Parabola.**

$n = 3$ or $y = x^3$.—Since x and x^3 are always of the same sign, it follows that, x and y being of the same sign, *the curve lies wholly in the first and third quadrants.* Further, since $(-x)^3$ is numerically equal to $(+x)^3$, the portion of the curve in the third quadrant must be of the same shape as that in the first quadrant, so that the reflection of one portion in a mirror placed along the axis of X would be symmetrical about the axis of Y with the other portion. Two curves, or portions of curves, with this relationship, may be said to be "invertly symmetrical" with each other. Hence, *this curve is invertly symmetrical about the axis of* Y. A series of values of x and y satisfying this equation are given :—

x . .	−3·5	−3·0	−2·5	−2·0	−1·5	−1·0	−0·5	0	+0·5	1·0	1·5	2·0	2·5	3·0	3·5
$y = x^3$	−42·9	−27·0	−15·6	−8·0	−3·4	−1·0	−0·12	0	+0·12	1·0	3·4	8·0	15·6	27·0	42·9

This curve is shown in Fig. 14 by the thinner continuous line. As has been already shown, it lies wholly in the first and third quadrants, and is invertly symmetrical about the axis of Y. This curve is known as the **Cubic Parabola.**

$n = 4$ or $y = x^4$.—Since n is even, the same lines of argument will apply to this curve as were used in examining the curve $y = x^2$; that is, the curve will lie wholly above the axis of X and be symmetrical about the axis of Y.

x	0	±0·5	±1·0	±1·5	±1·8	±2·0	±2·3	±2·5	±2·6
$y = x^4$ . . .	0	0·06	1·0	5·1	10·5	16·0	28·2	39·0	45·5

This curve is shown in Fig. 14 by the dotted line.

$n = 5$ or $y = x^5$.—n being odd, this curve will lie wholly in the first and third quadrants, and be invertly symmetrical about the axis of Y.

x . . .	−2·1	−2·0	−1·8	−1·6	−1·4	−1·0	−0·5	0	+0·5	1·0	1·4	1·6	1·8	2·0	2·1
$y = x^5$.	−41·0	−32·0	−18·9	−10·5	−5·4	−1·0	−0·03	0	+0·03	1·0	5·4	10·5	18·9	32·0	41·0

This curve is shown in Fig. 14 by the dot-and-dash line.

Comparison of the Curves.—From an examination of these curves the characteristics of the family may now be obtained, and the graphical meaning of n determined. Note the following points :—

(1) The curves fall into two classes :

(*a*) Those for which n is even, which lie wholly in the first and second quadrants, that is above the axis of X, and are symmetrical about the axis of Y. We shall refer to these as the "even curves."

(*b*) Those for which n is odd, which lie wholly in the first and third quadrants, and are invertly symmetrical about the axis of Y, having a point of inflection at the origin. We shall refer to these as the "odd curves."

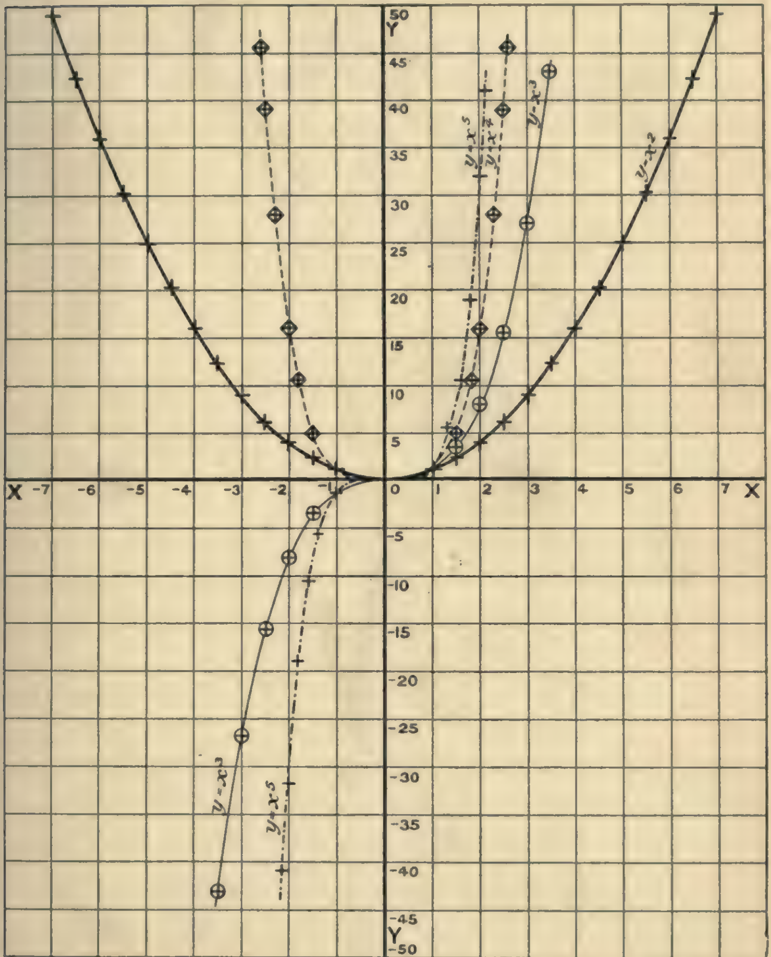


Fig. 14.

(2) All the curves pass through two common points, namely, the origin and the point (1, 1). All the even curves also pass through the point $(-1, 1)$ and all the odd curves through the point $(-1, -1)$.

(3) The curves cut out one another at all these points, excepting at the origin, where they are tangential to each other.

(4) All the curves cut the axis of Y at right angles.

(5) Any horizontal line for which y is greater than 1 intersects the curves in the order of their indices, a curve of greater index being nearer to the axis of Y than one of lesser index.

(6) Any horizontal line for which y is positive and less than 1 intersects the curves in the order of their indices, a curve of *smaller* index being nearer to the axis of Y than one of greater index.

(7) Hence, of any two curves that of greater index lies within that of smaller index above the common point (1, 1), but outside it between that point and the origin; or the curve of greater index is of greater slope above the point (1, 1), but changes its slope more rapidly below that point, so forming a blunter apex than that of smaller index.

The relative positions and forms of the curves between the origin

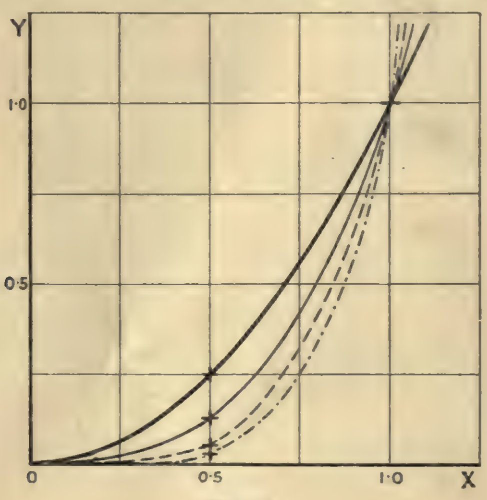


Fig. 15.

and the point (1, 1) are shown more clearly in Fig. 15, which gives this portion of the curves on a larger scale.

From the above conclusions it is obvious that the constant n determines the *form* of the curve, or is the "*form constant.*"

n an Improper Fraction.—It will only be necessary to examine this case for positive values of x. Taking as an example the equation $y = x^{2\cdot5}$, we obtain the corresponding values of x and y given in the table below:—

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
$y = x^{2\cdot5}$ . .	0	0·18	1·0	2·76	5·66	9·86	15·60	23·0	32·0

This curve is plotted in Fig. 16, being represented by the continuous line. The curves $y = x^2$ and $y = x^3$ have also been drawn on this figure for purposes of comparison, being represented by broken lines. It is seen that the curve is of precisely the same type as those already

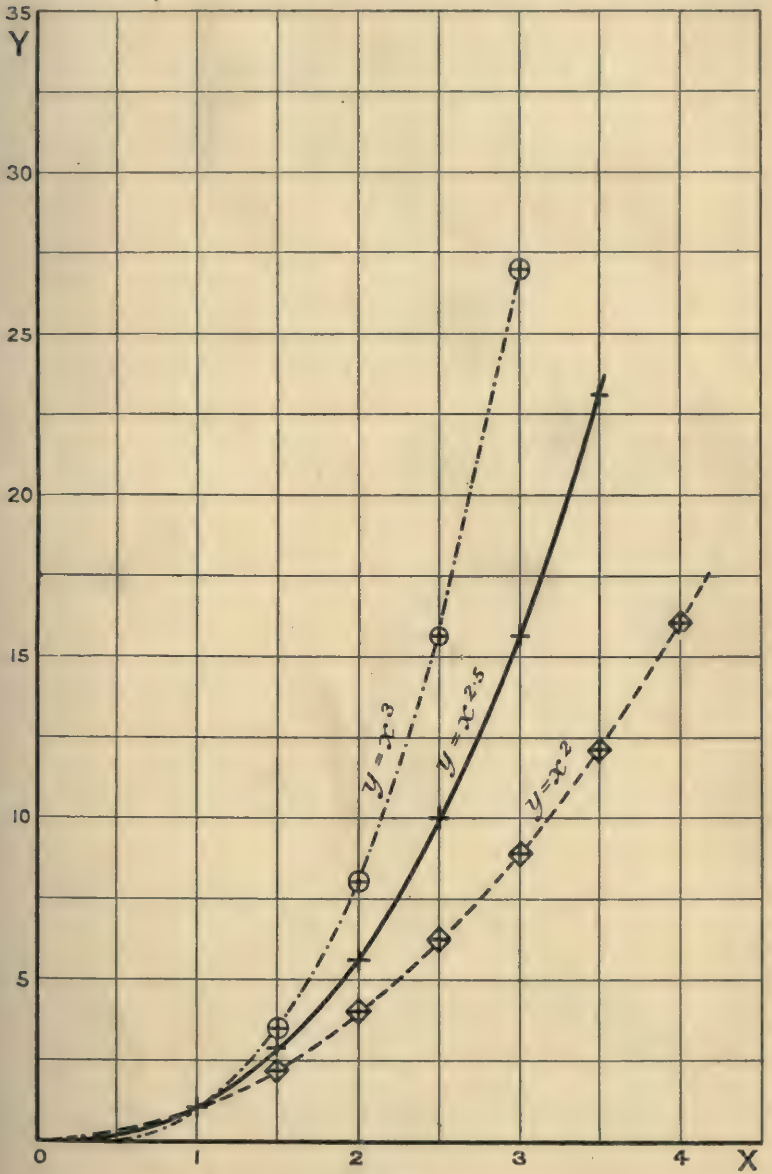


Fig. 16.

discussed, and lies between the two nearest whole number curves, $y = x^2$ and $y = x^3$.

n a Positive Number less than 1.—If n is less than 1, let it be equal to $\frac{1}{m}$, where m is a number greater than 1. Then the equation $y = x^n$ becomes $y = x^{\frac{1}{m}}$ or $x = y^m$. This is of the same form as $y = x^m$, with the variables x and y interchanged with each other. We should, then, expect such an equation to give us a parabolic curve, having its axis of symmetry in the direction of the axis of X instead of in that of the axis of Y. As an illustration of this, we will plot the curve $y = x^{\frac{1}{2}}$ or $x = y^2$ (see Fig. 17).

x	0	1	2	3	4	5	6
$y = x^{\frac{1}{2}}$ . .	0	$\pm 1{\cdot}00$	$\pm 1{\cdot}41$	$\pm 1{\cdot}73$	$\pm 2{\cdot}00$	$\pm 2{\cdot}24$	$\pm 2{\cdot}45$

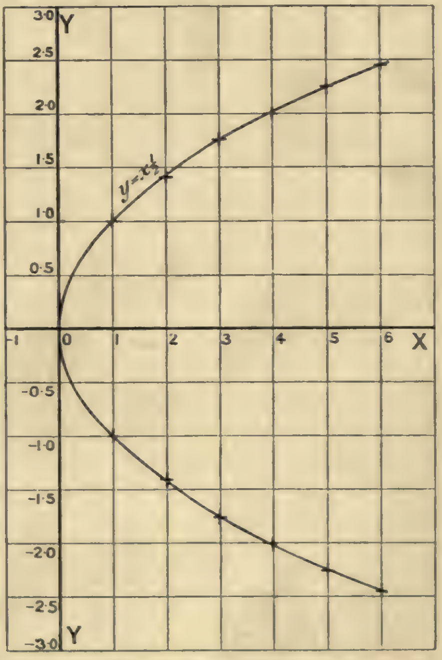


Fig. 17.

n a Negative Number.—If n is negative, let it be equal to $-m$, where m is positive. Then the equation $y = x^n$ may be written $y = x^{-m}$ or $y = \frac{1}{x^m}$ or $yx^m = 1$. This equation gives us the general equation of another family of curves, which will be considered in the next chapter.

The Constant a.—Keeping the values of b and c zero, we will now introduce the constant a, so obtaining the equation $y = ax^n$. It is evident that the result of this in any curve will be to increase each value of y, a times, without otherwise altering the form of the curve. The same result precisely could be obtained by re-plotting the curve $y = x^n$, retaining the same scale for x as before, but increasing the scale of y, a times. To further illustrate the effect of this constant, Fig. 18 shows the curves given by the equations $y = x^2$ (thick line), $y = \frac{1}{2}x^2$ (thinner line), and $y = 2x^2$ (dotted line).

x. . . .	0	± 1	± 2	± 3	± 4
x^2 . . .	0	1·0	4·0	9·0	16·0
$\frac{1}{2}x^2$ . . .	0	0·5	2·0	4·5	8·0
$2x^2$ . . .	0	2·0	8·0	18·0	32·0

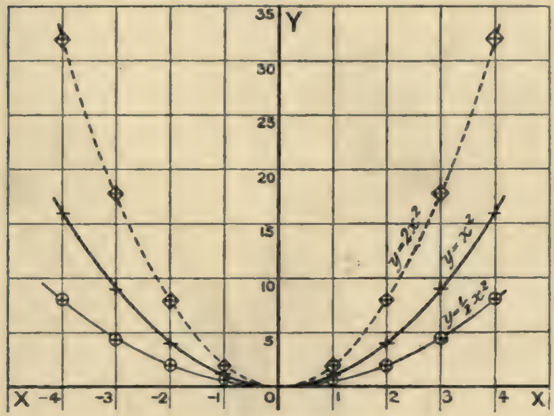


Fig. 18.

The Constant c.—Omitting for the present the term containing x, we will now introduce a constant term, c, thus obtaining a general equation of the form $y = ax^n + c$. If now we put $x = 0$, we have $y = c$, that is the curve cuts the axis of Y, not at the origin as hitherto, but at a distance from the origin equal to the constant c. In other words, c is the intercept on the axis of Y. Further, it is obvious that for any given value of x the value of y is increased by a constant amount, c, over that given by the equation $y = ax^n$. That is, the curve is of the same form as $y = ax^n$, but is raised bodily through a vertical distance, c. In Fig. 19 are plotted three cubic parabolas, or rather one cubic parabola in three positions, given respectively by the equations $y = x^3$ (thick continuous line); $y = x^3 + 10$ (thinner continuous line); and $y = x^3 - 15$ (dotted line). The figures for plotting these curves are given below.

x . . .	−3·0	−2·5	−2·0	−1·5	−1·0	−0·5	0	+0·5	1·0	1·5	2·0	2·5	3·0
x^3 . . .	−27·0	−15·6	−8·0	−3·4	−1·0	−0·1	0	+0·1	1·0	3·4	8·0	15·6	27·0
$x^3 + 10$.	−17·0	−5·6	+2·0	6·6	9·0	9·9	10·0	10·1	11·0	13·4	18·0	25·6	37·0
$x^3 - 15$.	−42·0	−30·6	−23·0	−18·4	−16·0	−15·1	−15·0	−14·9	−14·0	−11·6	−7·0	+0·6	12·0

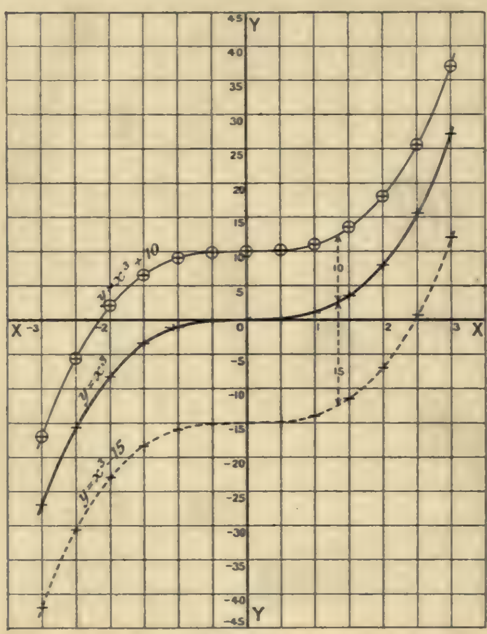


Fig. 19.

From the diagram it is seen that if any vertical line be drawn cutting the three curves, the length of the line intercepted between the first two is equal to 10, and that between the first and the last is equal to 15, showing that the curves are identical in form, but that the second ($c = 10$) has been raised a distance 10 above the first, and the third ($c = -15$) has been lowered to a distance 15 *below* the first ($c = 0$).

The Constant b **when** $n = 2$.—In determining the effect upon the curve of introducing a term bx into the equation, so obtaining the general form $y = ax^n + bx + c$, we will first examine the case when $n = 2$. Consider the equation—

$$y = x^2 + 2kx + k^2$$

This can be reduced to the form

$$y = (x + k)^2$$

If, now we write X for $(x + k)$ we obtain the equation of a simple parabola $y = X^2$ referred to axes $y = 0$ and $X = 0$, that is, $x + k = 0$

or $x = -k$. That is, the curve represented by this equation will be identical in form with the curve $y = x^2$, but moved horizontally until its axis of symmetry coincides with the line $x = -k$; that is to say, moved through a horizontal distance, k, to the left. But in this equation $2k$ is equal to b in the general equation, or $k = \frac{b}{2}$. Again, c in the general equation is here equal to k^2, that is, to $\frac{b^2}{4}$. Hence, if this relation between b and c holds good in any equation of the form $y = x^2 + bx + c$, the effect is merely to move the curve $y = x^2$ horizontally to the left a distance equal to $\frac{b}{2}$. As an example of this, the curve $y = x^2 + 6x + 9$ is plotted in Fig. 20, being represented by the thicker continuous line. The fundamental curve, $y = x^2$, is represented in the same figure by the dotted line for purposes of comparison. The calculations for plotting the curve $y = x^2 + 6x + 9$, that is, $y = (x+3)^2$, are tabulated below.

x	-8	-7	-6	-5	-4	-3	-2	-1	0	$+1$	2
$x + 3$ . .	-5	-4	-3	-2	-1	0	$+1$	2	3	4	5
$y = (x+3)^2$	25	16	9	4	1	0	1	4	9	16	25

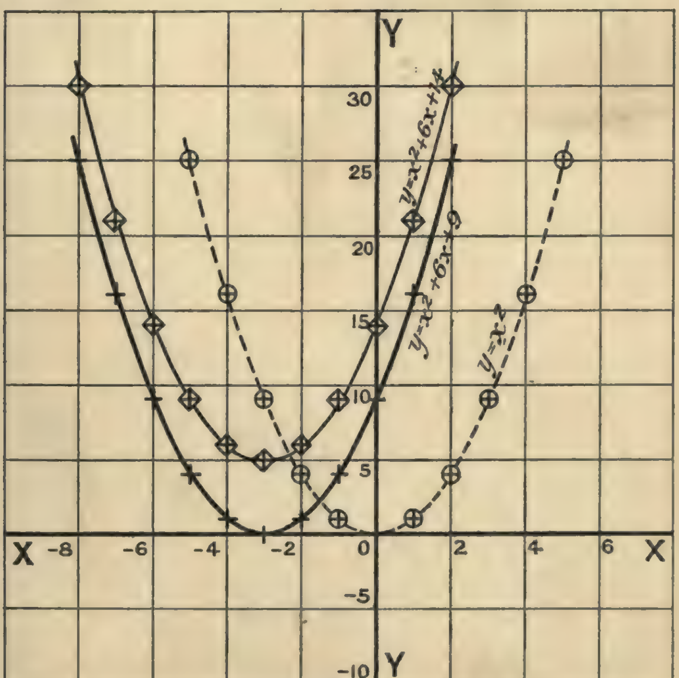


Fig. 20.

Notice that the form of this curve is identical with that of $y = x^2$, but that it is moved horizontally to the left a distance equal to 3, that is $\frac{1}{2}b$. The student should test this by transferring the curve $y = x^2$ to tracing paper, and then applying his tracing to the curve $y = x^2 + 6x + 9$, to see that the two are really identical in form. It should be noted, also, that this curve differs from all those previously plotted in that it does not cut the axis of Y at right angles.

In considering the above the constant a has been taken as unity. We have already seen, however, that this constant affects only the *scale* to which the curve is plotted. If a is not unity, we must first divide the right-hand side of the equation throughout by a. Then if in the equation $y = ax^2 + bx + c$, that is, $y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$, we have the relation $\frac{c}{a} = \left(\frac{b}{2a}\right)^2$, the result is to move the curve a horizontal distance $\frac{b}{2a}$ to the left.

If b is negative, the result will, of course, be that the curve is shifted to the *right* instead of to the left.

We now proceed to the consideration of the case in which the right-hand side of the equation is not a perfect square. For the sake of simplicity we will again assume that the value of a is unity; this constant, as we have seen, only affecting the scale to which the curve is plotted. Then, let $c = \left(\frac{b}{2}\right)^2 + d$, so that the equation $y = x^2 + bx + c$ becomes

$$y = x^2 + bx + \left(\frac{b}{2}\right)^2 + d$$
$$= \left(x + \frac{b}{2}\right)^2 + d$$

This curve, then, is identical with the parabola $y = \left(x + \frac{b}{2}\right)^2$ raised vertically through a distance, d, that is, $c - \left(\frac{b}{2}\right)^2$. Hence, any equation of the form $y = x^2 + bx + c$ gives a parabola identical in form with the parabola $y = x^2$, moved horizontally a distance $\frac{b}{2}$ to the left, and raised vertically through a height $c - \left(\frac{b}{2}\right)^2$, the intercept upon the axis of Y being, of course, equal to c. As an illustration of this, the curve $y = x^2 + 6x + 14$, represented by the thinner continuous line, has been added to Fig. 20.

x	-8	-7	-6	-5	-4	-3	-2	-1	0	$+1$	2
$y = x^2 + 6x + 14$ $= (x+3)^2 + 5$	30	21	14	9	6	5	6	9	14	21	30

Finally, putting this into a perfectly general form by reintroducing the constant a, we have that the equation

$$y = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

represents a curve of precisely the same form as the parabola $y = ax^2$, but moved horizontally through a distance $\frac{b}{2a}$ to the left, and raised vertically through a distance $a\left\{\frac{c}{a} - \left(\frac{b}{2a}\right)^2\right\} = \frac{4ac - b^2}{4a}$. The factor a in the last expression is introduced because all vertical dimensions are multiplied by the vertical scale-constant a.

The Constant b in other Cases.—The effect of adding a term bx to the general equation $y = ax^n + c$, when n is not equal to 2, is by no means so simple as in the case just considered. It will be sufficient for the purpose of this book, however, to indicate the effect it has in general. In the case of the cubic parabola, for example, the effect is quite different from that upon the ordinary parabola. It produces no movement of the parabola as a whole, but increases the slope at every point by a constant amount, b. The cubic parabola is *always* invertly symmetrical about the axis of Y unless its equation contains a term involving the second or some fractional power of x. This can be seen from the fact that x^3 and x are always necessarily of the same sign. Hence, $x^3 + bx$ is always numerically equal to $(-x)^3 + b(-x)$. If, then, c be zero, equal and opposite values of x always produce equal and opposite values of y, that is, the curve always has invert symmetry relatively to the axis of Y. The effect of adding a constant term c is, as has been already shown, merely to raise the curve vertically through a distance equal to c, which does not affect the symmetry relatively to the axis of Y. In Fig. 21 the curves $y = x^3$ (continuous line) and $y = x^3 + 5x$ (broken line) are plotted to illustrate this point, the co-ordinates of the points of observation being tabulated below.

x	−4	−3	−2	−1	0	+1	2	3	4
x^3	−64	−27	−8	−1	0	+1	8	27	64
$5x$	−20	−15	−10	−5	0	+5	10	15	20
$x^3 + 5x$ . . .	−84	−42	−18	−6	0	+6	18	42	84

Notice that the latter curve does not cut the axis of Y at right angles.

If a and b are of opposite sign, the form of the cubic parabola in the neighbourhood of the origin is materially altered. If a is positive the slope of the primary curve $y = ax^3$ to the right of the origin is positive also, becoming less as x approaches 0; if b is negative, the curve, as has been pointed out, has its slope *diminished* by an amount numerically equal to b. Hence for some distance to the right of the origin the slope of the curve $y = ax^3 - bx$ will become negative, and

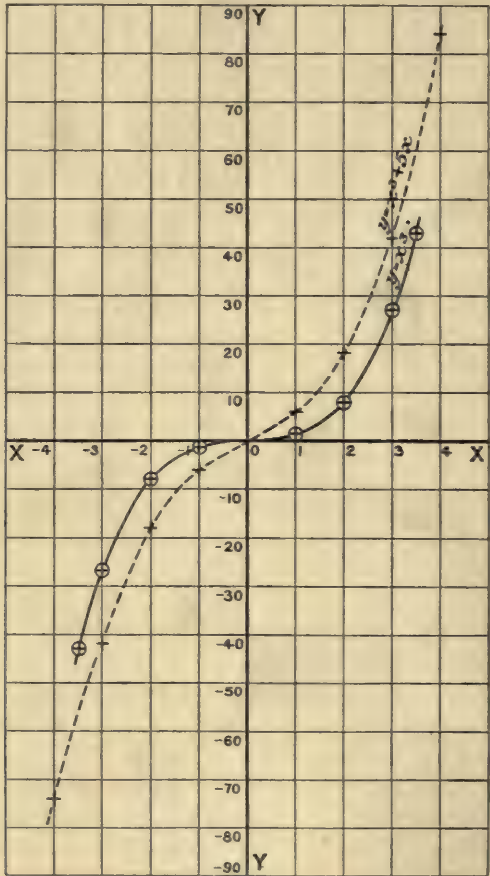


Fig. 21.

the curve is depressed below the axis of X for a distance given by the relation $ax^3 = bx$, or $x = \sqrt{\dfrac{b}{a}}$. Similarly, immediately to the left of the origin the slope becomes negative, and is raised above the axis of X for the same distance. Thus, we get a curve containing a maximum (at A) and a minimum (at B), and a point of inflexion (C). This is shown in the curve plotted in Fig. 22, whose equation is $y = x^3 - 4x$.

x . .	$-4{\cdot}0$	$-3{\cdot}5$	$-3{\cdot}0$	$-2{\cdot}0$	$-1{\cdot}5$	$-1{\cdot}0$	$-0{\cdot}5$	0	$+0{\cdot}5$	$1{\cdot}0$	$1{\cdot}5$	$2{\cdot}0$	$3{\cdot}0$	$3{\cdot}5$	$4{\cdot}0$
x^3 .	$-64{\cdot}0$	$-42{\cdot}6$	$-27{\cdot}0$	$-8{\cdot}0$	$-3{\cdot}4$	$-1{\cdot}0$	$-0{\cdot}1$	0	$+0{\cdot}1$	$1{\cdot}0$	$3{\cdot}4$	$8{\cdot}0$	$27{\cdot}0$	$42{\cdot}6$	$64{\cdot}0$
$-4x$.	$+16{\cdot}0$	$14{\cdot}0$	$12{\cdot}0$	$8{\cdot}0$	$6{\cdot}0$	$4{\cdot}0$	$2{\cdot}0$	0	$-2{\cdot}0$	$-4{\cdot}0$	$-6{\cdot}0$	$-8{\cdot}0$	$-12{\cdot}0$	$-14{\cdot}0$	$-16{\cdot}0$
x^3-4x	$-48{\cdot}0$	$-28{\cdot}6$	$-15{\cdot}0$	0	$+2{\cdot}6$	$+3{\cdot}0$	$+1{\cdot}9$	0	$-1{\cdot}9$	$-3{\cdot}0$	$-2{\cdot}6$	0	$+15{\cdot}0$	$+28{\cdot}6$	$+48{\cdot}0$

The reasons for the above statements will be seen more clearly in the chapter dealing more particularly with the slope of curves in general (see p. 106).

When $n = 4$ the introduction of the bx term has a still different effect. Let the student plot for himself the curves $y = x^4$ and, say, $y = x^4 + 10x$. He will see (1) that the apex is moved both laterally *and* vertically, (2) that the curve is not symmetrical about the axis of

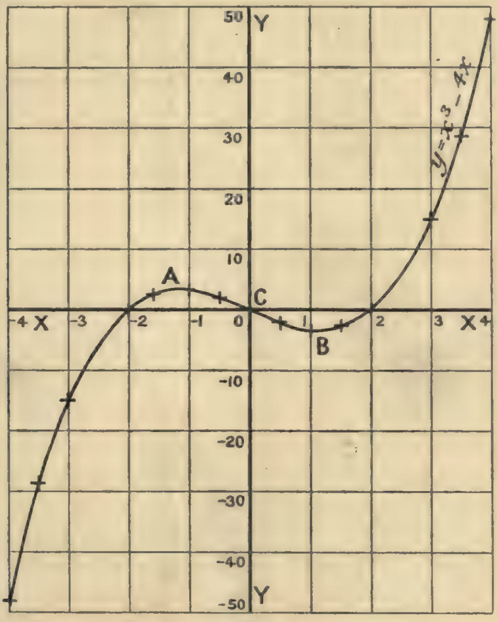


Fig. 22.

Y, and (3) that the curve is not symmetrical about *any* vertical line, but appears to be tilted.

The really salient point to be observed in all these cases is that which is common to all, namely, that *when b is zero the curve cuts the axis of Y at right angles, but when b is not zero the curve does not cut the axis of Y at right angles.* In the chapter on "The Slope of Curves" it will be shown, in fact, that the slope of the curve as it cuts the axis of Y—that is, when $x = 0$—is equal to b. The curves, however, still retain the general shape typical of the parabolic family.

Other Terms in the General Equation.—It would not be complete to leave this subject without pointing out that when n is greater than 2, a term containing x^2 may be introduced into the equation without affecting the parabolic character of the curve; when x is greater than

3, a term in x^3, and so forth. Thus, the complete general equation of the parabolic family of curves is—

$$y = ax^n + ex^{n-1} + fx^{n-2} \ldots + bx + c$$

We need not, however, discuss the meaning of such additional terms in detail.

The Equation of a Given Parabola.—The only curve whose form can be exactly tested directly is the straight line. In order to find the equation of a parabola, then, it is necessary first to obtain from it a straight line which bears some close relationship to it. We can do this very simply by reducing the parabola to its logarithmic analogue. Consider first the simple parabolic curve given by the equation—

$$y = ax^n$$

Take logarithms throughout, then—

$$\log y = n \,.\, \log x + \log a$$

This is now a simple equation of the first degree in $\log x$ and $\log y$, and therefore represents a straight line connecting these two variables. In other words, if $\log x$ and $\log y$ be plotted, the result will be a straight line whose slope is n, and whose intercept on the axis of $\log Y$ is $\log a$. Hence, n and a can be directly determined from this line, so fixing the equation of the original curve. An example will, perhaps, make the method clearer. The values of x and y, given in the first two lines of the table below, are connected by an equation of the form $y = ax^n$. (In the case of the curve itself being given, these values would be obtained by direct measurement from it.) In the third and fourth lines of the table the values of $\log x$ and $\log y$ for each pair of values of x and y are given, as taken directly from a table of logarithms.

x	0	1	2	3	4	5	6	7	8	9
y	0	2·1	7·1	14·3	23·7	35·3	48·2	63·4	80·4	98·8
Log x . .	—	0·000	0·301	0·477	0·602	0·699	0·778	0·846	0·903	0·954
Log y . .	—	0·322	0·852	1·156	1·375	1·548	1·683	1·802	1·906	1·999

In Fig. 23, $\log x$ and $\log y$ are plotted as co-ordinates, and it is seen that a straight line results. Then—

$$\log a = \text{the intercept on the axis of Y} = 0·322$$

$$\therefore a = 2·1$$

$$\text{Also } n = \text{the slope of the line} = \frac{\text{PM}}{\text{RM}} = \frac{1·55 - 0·5}{0·7 - 0·1} = \frac{1·05}{0·6} = 1·75$$

$$\therefore y = 2·1\, x^{1·75}$$

Here it was known that the equation of the curve was of the form $y = ax^n$. If this is not known, the curve itself must first be plotted, when if it appears (1) to be of the general parabolic form, (2) to pass through the origin, and (3) to cut the axis of Y at right angles, it is probable that an equation of this form may be found to fit it with a fair degree of accuracy.

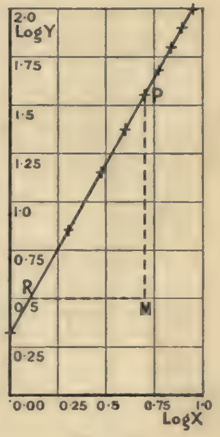


Fig. 23.

The Use of Logarithmic Squared Paper.—The trouble of actually looking out the logarithms of x and y from the tables may be obviated by the use of logarithmically ruled paper. In this, instead of the rulings being at equal intervals of inches and tenths of an inch as in ordinary squared paper, the paper is divided up in both directions in the proportion of the logarithms of the numbers from 1 to 10, the rulings being therefore of exactly the same nature as those on an ordinary slide rule. Each sheet of paper only contains a range of values of x and y equivalent to one characteristic in their logarithms. If more than this is desired, however, two or more sheets of paper must be joined together, care being taken that the join is made accurately. Of course, on this paper no choice of scales is possible; the length of each sheet representing *either* from 1 to 10, *or* from 10 to 100, and so on. In Fig. 24 the set of values of x and y given in the last paragraph have been plotted on logarithmic paper, two sheets being necessary in the direction of Y, as the values range from 2·1 to 98·8, that is, over a range involving two characteristics for the logarithms. The slope is measured by dividing the increase in log y by the corresponding increase in log x, as was seen in the last paragraph. Hence, on this paper we must measure the vertical and horizontal sides of the slope triangle, *not* on the scales to which the line is plotted, but on any scale of equal parts (the same in both directions). That is, PM and QM are both measured, say, in inches. Then we have—

$$n = \frac{\text{PM}}{\text{QM}} = \frac{10{\cdot}32}{5{\cdot}90} = 1{\cdot}75.$$

a is the value of y when $x = 1$, and is therefore the intercept on the axis of Y on logarithmic paper, in this case 2·1 as before, being read directly from the scale of Y.

It is sometimes not necessary to use two sheets of paper, even though the numbers do vary through a range represented by two characteristics in their logarithms. If we divide all the values of either x or y by any, the same, constant, the logarithmic line, when plotted, will be of the same slope as before, but will be moved bodily parallel to itself through a distance to the left or down equal to the logarithm of the dividing constant. That is to say, the value of n is unaltered, although that of a is. This follows at once from the fact that

$$\log\frac{x}{d} = \log x - \log d$$

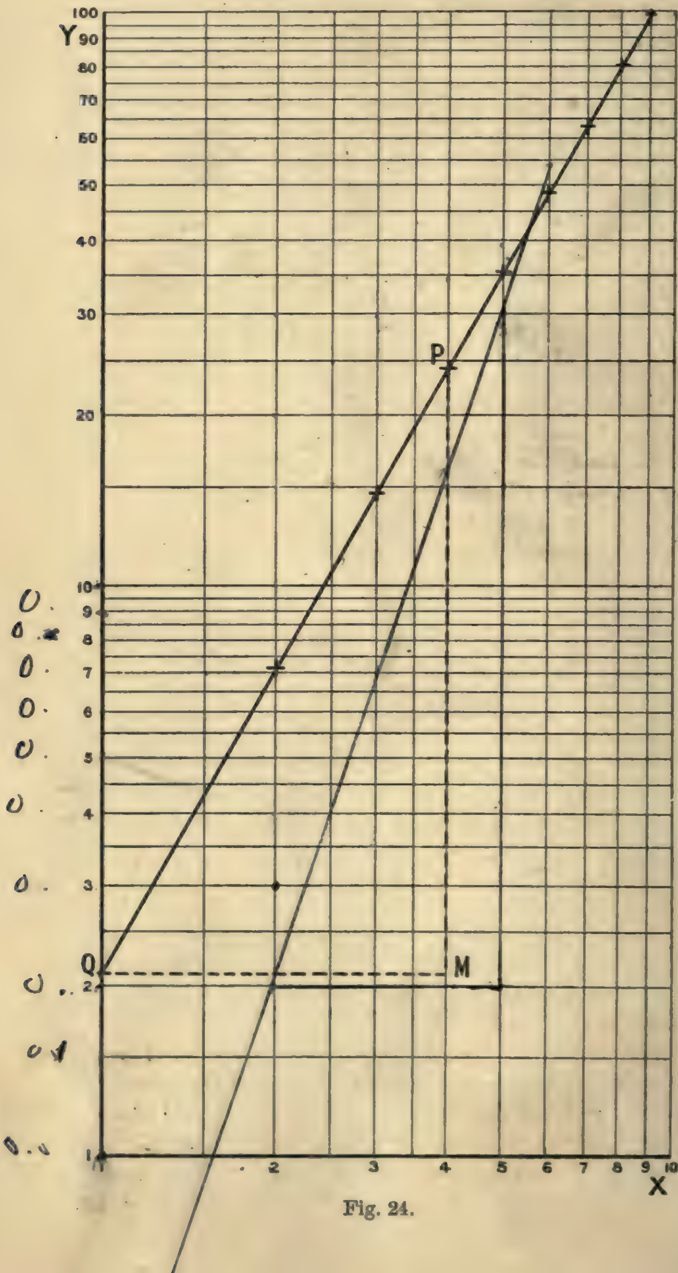


Fig. 24.

the value of each logarithm being therefore diminished by the constant quantity log d. In the values of x and y given below, those of x come within the range of one logarithmic characteristic, but those of y range over two. Upon dividing the values of y by 2, however, they all fall within the range of one characteristic. Plotting x and $\frac{1}{2}y$ on the logarithmic paper, we get the straight line shown in Fig. 25. For purposes of comparison, part of the line obtained by plotting x and y is also shown by a thicker line. It is seen that the two are parallel, that is, of the same slope, but that one is lower than the other.

x . . .	0	1·0	1·5	2·0	2·5	3·0	3·5	4·0
y . . .	0	2·10	3·86	5·93	8·30	10·90	13·75	16·80
$\frac{1}{2}y$ . .	0	1·05	1·93	2·97	4·15	5·45	6·88	8·40

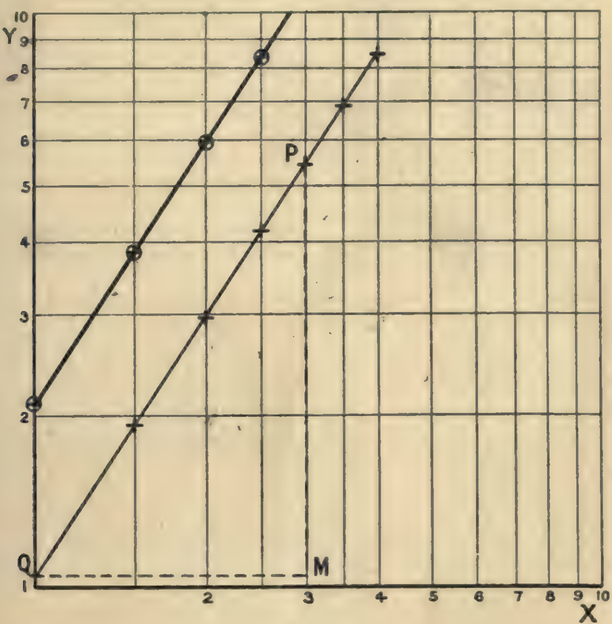


Fig. 25.

Then, measuring the slope,

$$n = \frac{PM}{QM} = \frac{7 \cdot 04}{4 \cdot 70} = 1 \cdot 5$$

To obtain a we must multiply the intercept on the axis of Y by our vertical divisor—namely, 2—since all vertical dimensions were reduced in that ratio.

Instead of using logarithmic paper the co-ordinates may be measured by a pair of dividers from the scale of an ordinary slide rule; but this method is more cumbersome and tedious than that of using a table of logarithms and ordinary squared paper.

To Find an Equation of the Type $y = ax^n + c$.—If the curve does not pass through the origin, but still cuts the axis of Y at right angles, we have seen that its equation is of the above form, containing a constant term, c, but no term of the first degree in x. The value of this constant c is equal to the intercept of the curve on the axis of Y. This constant can therefore be determined at once by inspection. Then, subtracting the value of c from each value of y, we shall reduce the equation to the simpler type already discussed.

For let $$y - c = y_1$$
Then $$y_1 = y - c = ax^n$$

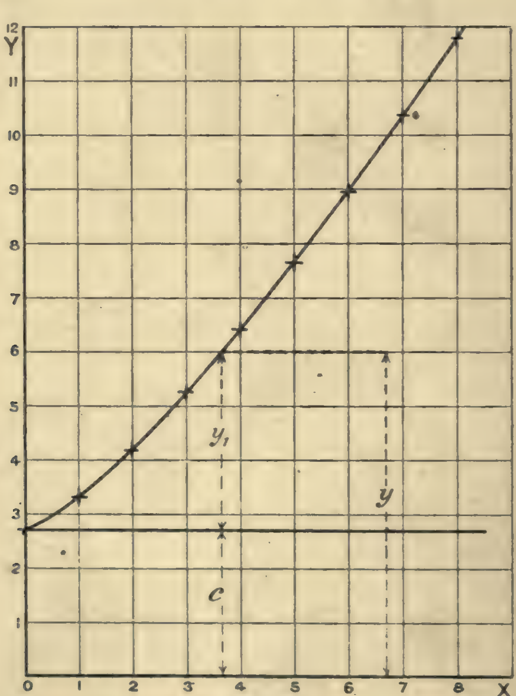


Fig. 26.

It will be sufficient to add an example of this case. The values of x and y, given below, are plotted to give the curve shown in Fig. 26. This is of general parabolic form, and appears to cut the axis of Y at right angles (hence there will be no term of the first degree in x in the equation). It, however, does not pass through the origin, but makes an intercept on the axis of Y equal to 2·71 ($= c$). Hence $y_1 = y - 2{\cdot}71$. In the third line of the table the values of y_1, obtained by subtracting 2·71 from each value of y, are entered.

x	0	1	2	3	4	5	6	7	8
y	2·71	3·32	4·21	5·25	6·40	7·65	8·96	10·36	11·81
$y_1 = y - c$ . .	0·0	0·61	1·50	2·54	3·69	4·94	6·25	7·65	9·10

The values of x and y_1 are then plotted on logarithmic paper, or their logarithms found and plotted on squared paper. The former method has been adopted here, and the straight line (Fig. 27) results.

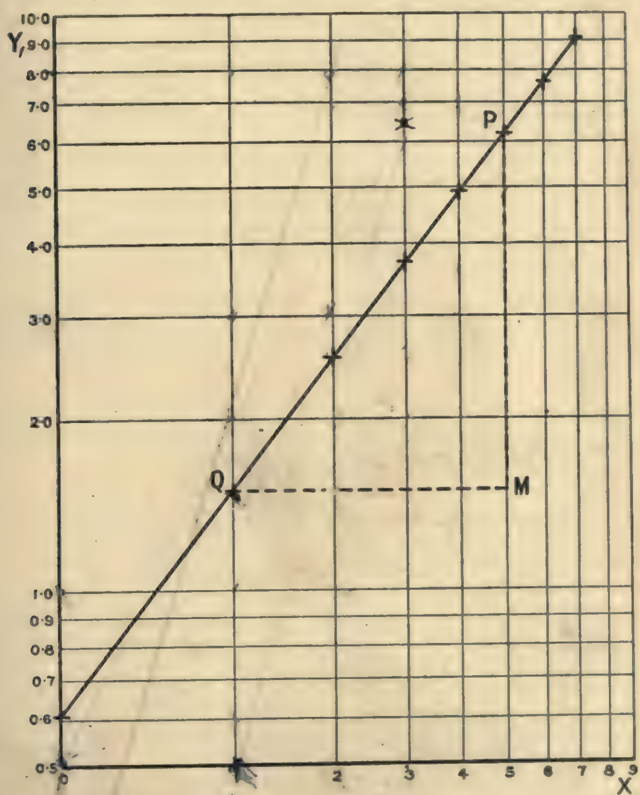


Fig. 27.

Then—

$$a_1 = \text{the intercept on the axis of } Y_1 = 0{\cdot}61$$

$$n = \text{slope} = \frac{PM}{QM} = \frac{6{\cdot}10}{4{\cdot}70} = 1{\cdot}3$$

Hence the complete equation of the curve is

$$y = 0{\cdot}61x^{1{\cdot}3} + 2{\cdot}71$$

To Find an Equation of the Type $y = ax^n + bx + c$.—We now come to the case in which the curve, whose equation it is required to find, does not cut the axis of Y at right angles. We have seen that this means that a term of the first degree in x occurs in the equation. In the general equation of such a curve, as given above, put

$$y_1 = ax^n$$

and $$y_2 = bx + c$$

Then $$y = y_1 + y_2 = ax^n + bx + c.$$

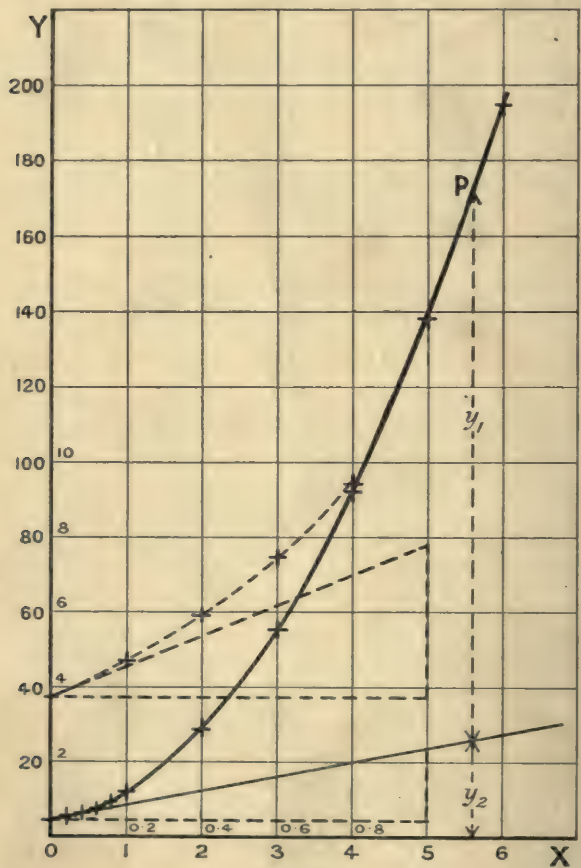


Fig. 28.

The second of these equations gives us a *straight line* of slope b and making an intercept on the axis of Y equal to that of the original curve, namely c. This straight line is, in fact, *the tangent to the curve* at its point of intersection with the axis of Y. This will be seen more clearly in Chapter IX. The first equation gives us the equation of a parabolic curve referred to this tangent as the axis of X. In Fig. 28 we have an example of a curve of this type, the actual graphical meanings of y_1 and y_2 being shown for the point P. Fig. 29 shows the corresponding curve $y_1 = ax^n$ reduced to the ordinary axis of X, that is, the vertical heights, y_1, of the original curve above the tangent $y_2 = bx + c$, have been re-plotted with the corresponding values of x, above the ordinary axis of X, giving a simple parabolic curve. We have, then, a method by which the four constants a, b, c, and n can be determined. Having plotted the curve, the tangent at the point

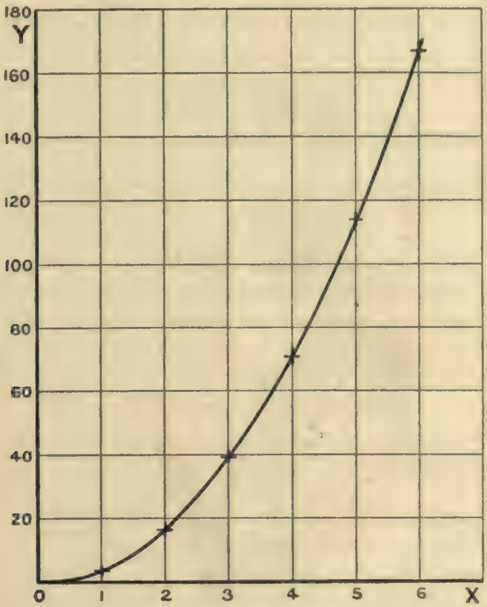


Fig. 29.

where it intersects the axis of Y must be drawn by eye. If the curve is carefully drawn by a clear, smooth, fine line it is not a difficult matter to do this with a fair amount of accuracy. Two methods may be recommended. Perhaps the most reliable is to use the edge of a transparent set-square, and first adjust it as nearly as possible as a tangent to the curve at a point a little distant, say about an inch, from the axis of Y. Then the set-square is gradually brought down to the point of intersection, rolling it, as it were, on the curve, and ultimately marking its position when touching the curve at the axis of Y. If this be done several times and a fair average taken among the results, a close approximation to the true tangent will be obtained.

The other method is to use a stretched thread, either in the same way as the straight-edge, or, holding one end at the point where it is required to draw the tangent, to oscillate the other backwards and forwards until the true tangent position is obtained. Having obtained this tangent, the value of b (its slope) may be at once determined. The value of c is of course the intercept of the curve on the axis of Y It is sometimes advisable to plot the portion of the curve in the neighbourhood of the axis of Y to a larger scale, in order to draw the tangent more accurately. Its slope may then be measured on this enlarged diagram, and a line of the same slope drawn as a tangent to the original curve. The values of y_1 must next be obtained, either by reading off directly from the curve its vertical distance above the tangent for each value of x, or by calculating the different values of $bx + c$ and subtracting them from the corresponding values of y. Then, plotting y_1 and x on logarithmic paper or obtaining their logarithms and plotting them on ordinary paper, the values of a and n may be determined as already explained.

In the example worked out below, several extra pairs of co-ordinates for small values of x are given in order to define the form of the curve near to the axis of Y more exactly, so enabling the student to draw the tangent with a greater degree of accuracy. The curve connecting the values of x and y given is plotted in Fig. 28, and the portion near the axis of Y is shown on the same figure by the dotted curve, the scales for this portion being figured above the axis of X and to the right of the axis of Y respectively.

The value of c is seen to be 3·7 and we have

$$b = \text{the slope of the tangent} = \frac{7{\cdot}8 - 3{\cdot}7}{1{\cdot}0 - 0} = \frac{4{\cdot}1}{1{\cdot}0} = 4{\cdot}1.$$

Hence the equation for the tangent is—

$$y_2 = 4{\cdot}1x + 3{\cdot}7$$

In the fourth line of the table the values of y_2 for each value of x are given, those of $4{\cdot}1x$ being entered in the third line. Then in the fifth line are given the corresponding values of y_1, obtained by subtracting the values of y_2 given in the fourth line from those of y in the second. Fig 29 shows the value of y_1 plotted with x giving the simple parabolic curve

$$y_1 = ax^n$$

In practice, however, it is not necessary to plot this curve. As the values of y_1 range over three logarithmic characteristics, we will not use logarithmic paper in this example. The values of $\log x$ and $\log y_1$ are therefore found and entered in the table.

x	0·0	0·2	0·4	0·6	0·8	1·0	2·0	3·0	4·0	5·0	6·0
y	3·70	4·65	5·91	7·48	9·42	11·7	28·6	55·0	91·6	138·5	195·3
$4{\cdot}1x$ . . .	0·0	—	—	—	—	4·1	8·2	12·3	16·4	20·5	24·6
$y_2 = 4{\cdot}1x + 3{\cdot}7$	3·7	—	—	—	—	7·8	11·9	16·0	20·1	24·2	28·3
$y_1 = y - y_2$.	0	—	—	—	—	3·9	16·7	39·0	71·5	114·3	167·0
Log x . .	—	—	—	—	—	0·000	0·301	0·477	0·602	0·699	0·778
Log y_1 . .	—	—	—	—	—	0·591	1·222	1·591	1·854	2·059	2·223

Log x and log y_1 are plotted in Fig. 30, giving the straight line shown. Then, as before—

$$\log a = 0{\cdot}591$$
$$\therefore a = 3{\cdot}9$$
$$\text{and } n = \text{slope} = \frac{\text{PM}}{\text{RM}} = \frac{2{\cdot}059 - 0{\cdot}591}{0{\cdot}699 - 0{\cdot}000} = \frac{1{\cdot}468}{0{\cdot}699} = 2{\cdot}1$$

Hence the complete equation of the original curve is

$$y = 3{\cdot}9x^{2{\cdot}1} + 4{\cdot}1x + 3{\cdot}7$$

Parabolic Approximation to any Curve.—The general equation of any parabolic curve as given on p. 34 may be written in a slightly different form thus—

$$y = a + bx + cx^2 + dx^3 + ex^4 + \ldots$$

If we have any number, say k, points, one of which $(0, a)$ is on the axis of Y, a curve of this type may be drawn through them, whose equation will involve k constants (a, b, c, etc.), that is, its right-hand side will consist of an ascending series in x, whose highest term is of the $(k - 1)$th degree, for we can substitute the co-ordinates of each of our given points in the general equation above, so obtaining k simultaneous equations in the constants a, b, c, etc.

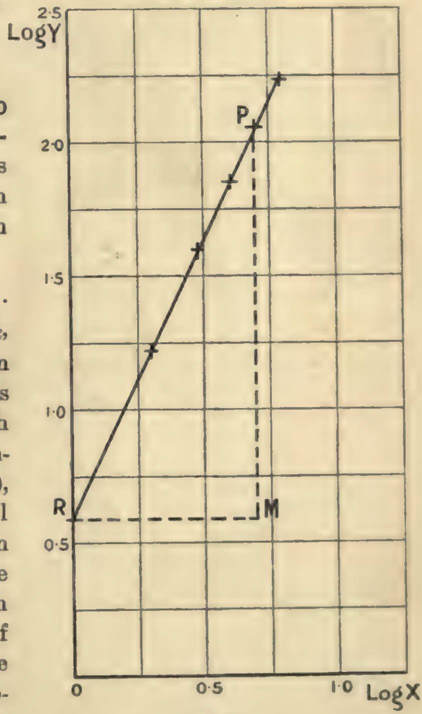


Fig. 30.

Solving these, we then determine those values of the constants which will give us an equation for a parábolic curve passing through the k given points. This enables us to obtain an approximate equation to any desired degree of accuracy for *any* given continuous curve; for we can choose any suitable number of points upon the curve—the greater the number taken the greater being the accuracy of the approximation—and find an equation of the above type for the parabolic curve passing through them. In choosing the points it is well to take them at fairly even distances along the curve, but at places where the curvature is changing rapidly, to take one or two extra points, and also to take any points which may seem to be critical, such as maxima and minima. Usually in practice we only desire an approximate equation to represent a small portion of a curve, in which case only three or four terms of the equation need be found, that is, only three or four points need be taken. For a greater number the algebraic solution of the resulting equations becomes long and tedious. The student should draw for himself any curve at random, and then proceed to find first the nearest cubic equation to fit it, then the nearest quartic, and so on, noticing that as the number of points taken be increased the curve actually represented by the equation found (which should itself be plotted for more points) becomes more and more near to the original curve—unless the latter happens to be a true parabolic curve of, say, the nth degree, when terms higher than the nth should vanish.

Position of the Vertex of a Parabola.—We have seen that the ordinary parabola $y = ax^2 + c$ is symmetrical about the axis of Y, having its vertex at a height c above the origin. The axis of the parabola then coincides with the axis of Y. Upon introducing a term bx into the equation, however, this axis of symmetry is shifted a distance $\frac{b}{2a}$ to the left, remaining vertical. The vertex, of course, is always on this axis of symmetry. We sometimes wish to find the horizontal position of the vertex without actually plotting the curve, as in the case considered in the next paragraph. As stated above, we see at once that the x co-ordinate of the vertex is $-\frac{b}{2a}$, or we may proceed independently in the manner now to be described. Since the parabola is symmetrical about the axis, any horizontal chord is necessarily bisected by the axis. Hence if we take any horizontal line ($y =$ a constant) which cuts the curve,* and determine the x co-ordinates of the two points of intersection, the x co-ordinate of the

* Any straight line cuts a curve of the nth degree in n points, real, coincident, or imaginary, since any equation of the nth degree has n roots. When we ordinarily speak of a line cutting a curve, we mean that these points are real. For a tangent two or more are coincident, that is, the corresponding roots of the equation are equal. If an even number are coincident we have an ordinary tangent, if an odd number, a tangent which crosses the curve, as at a point of inflexion.

vertex will be their mean. For instance, suppose we wish to find the position of the vertex of the parabola $y = 3x^2 + 2x + 4$. First, we will find its points of intersection with the horizontal straight line $y = 10$, say.

Then for these points we have—

$$3x^2 + 2x + 4 = 10$$

or

$$3x^2 + 2x - 6 = 0$$

$$\therefore x = \frac{-2 \pm \sqrt{4 + 72}}{6}$$

$$= \frac{-2 \pm \sqrt{76}}{6} = \frac{-2 \pm 8{\cdot}72}{6}$$

$$= \frac{6{\cdot}72}{6} \text{ or } \frac{-10{\cdot}72}{6} = 1{\cdot}12 \text{ or } -1{\cdot}79$$

Then the x co-ordinate of the vertex will be the mean of these, that is—

$$\bar{x} = \frac{1{\cdot}12 - 1{\cdot}79}{2} = \frac{-0{\cdot}67}{2} = -0{\cdot}33$$

Note that this is the same result as we should have obtained by simple substitution in the expression, $-\frac{b}{2a}$. For—

$$\bar{x} = -\frac{b}{2a} = -\frac{2}{2 \times 3} = -0{\cdot}33$$

And for the y co-ordinate we have—

$$\bar{y} = 3\bar{x}^2 + 2\bar{x} + 4$$
$$= 3\,(-0{\cdot}33)^2 + 2\,(-0{\cdot}33) + 4$$
$$= 0{\cdot}33 - 0{\cdot}66 + 4$$
$$= 3{\cdot}67$$

This is illustrated graphically in Fig. 31.

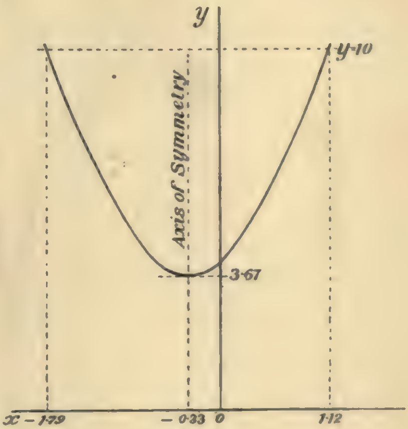


Fig. 31.

Application to Bending Moment Diagrams.—The curve of bending moment for any beam which is loaded with a uniformly distributed load or loads is either a single parabola or a series of arcs of parabolas for those portions of the beam under these loads, the axes of such parabolas being vertical. The maximum bending moment is therefore represented by the vertex of one of the parabolas if the vertex occurs in that portion of the curve making up the bending moment diagram. This gives us a method for determining the maximum bending moment, which is sometimes useful as an alternative to the usual method of finding first the position of zero shearing force. As an illustration of this method we will work out the following example:—

A bridge, weighing 1 ton per foot run, is 100 feet long and carries a uniformly distributed load of $1\frac{1}{2}$ tons per foot run for a distance of 80 feet from the left-hand abutment. Find the position and magnitude of the maximum bending moment.

First, to find the supporting forces, R_A and R_B.

Take moments about A (see Fig. 32). Then—

$$100\,R_B = 100 \times 50 + 120 \times 40$$

or
$$R_B = 50 + 48 = 98 \text{ tons}$$

and
$$R_A = 220 - 98 = 122 \text{ tons}$$

Next, to find the equations of bending moments.

Let M_K = the bending moment at any point K, distant x feet from A. Then—

$$M_K = 122x - 2{\cdot}5x \times \frac{x}{2}$$
$$= -1{\cdot}25x^2 + 122\,x \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the equation of a parabola whose axis is vertical, and which opens *downwards* (since a is negative), though for convenience it is usual to invert the diagram as in Fig. 32.

This equation gives us the curve of bending moment only up to the end of the load of $1\frac{1}{2}$ tons per foot run. Beyond that point we have—

$$M_K = 122x - 120(x - 40) - x \times \frac{x}{2}$$
$$= 122x - 120x + 4800 - 0{\cdot}5x^2$$
$$= -0{\cdot}5x^2 + 2x + 4800 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation is that of a parabola similar to that given in equation (1). The vertex of one or other of these parabolas will be the point of maximum bending moment, whichever falls upon the curve of bending moment. We proceed to find the position of both. In this case we will take the line of no bending moment, that is $M_K = 0$ as our horizontal line. Then for the intersection of this line with the first curve we have—

$$-1{\cdot}25x^2 + 122x = 0$$
$$\textit{i.e. } x = 0 \text{ or } x = \frac{122}{1{\cdot}25} = 97{\cdot}7$$

Then the vertex of this parabola is determined by

$$x = \frac{97{\cdot}7}{2} = 48{\cdot}8 \text{ feet to the right of A}$$

Or, adopting the other method, we have—

$$\bar{x} = -\frac{b}{2a} = -\frac{122}{2(-1{\cdot}25)} = 48{\cdot}8 \text{ feet, as before.}$$

For the second parabola we have—

$$-0{\cdot}5x^2 + 2x + 4800 = 0$$

or
$$x^2 - 4x - 9600 = 0$$

i.e.
$$(x - 100)(x + 96) = 0$$

i.e.
$$x = 100 \text{ or } x = -96$$

Then the vertex of this parabola is determined by

$$\bar{x} = \frac{100 - 96}{2} = 2 \text{ feet to the right of A}$$

or, by the simpler method,

$$\bar{x} = -\frac{b}{2a} = -\frac{-4}{2} = 2 \text{ feet}$$

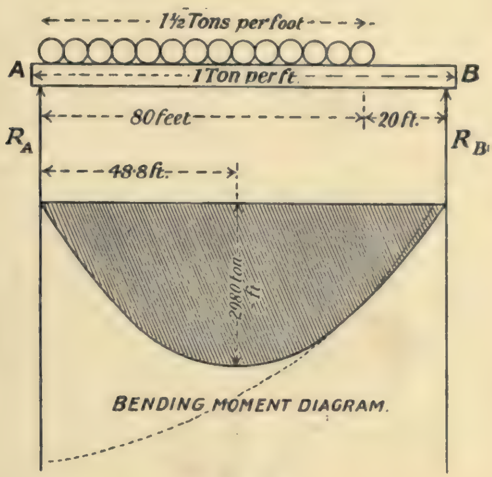


Fig. 32.

Of these two vertices the first falls within the length of the beam governed by its own parabola, and is therefore the one required, that is, the maximum bending moment is at a distance 48·8 feet to the right of the left-hand abutment.

Then the maximum bending moment

$$\begin{aligned} &= -1{\cdot}25x^2 + 122x \\ &= -1{\cdot}25\,(48{\cdot}8)^2 + 122 \times 48{\cdot}8 \\ &= -2970 + 5950 \\ &= 2980 \textit{ ton-feet} \end{aligned}$$

In the diagram the two parabolas have been continued by dotted lines over the portion of the beam not governed by them. It is possible that neither vertex may actually fall upon the bending moment curve, in which case the intersection of the two parabolas will be the point of maximum bending moment.

CHAPTER V

$yx^n = a$.—THE HYPERBOLIC FAMILY

It was stated in the last chapter that when the constant n in the equation $y = ax^n$ was negative, the equation, which could then be written in the form $yx^n = a$ (n then being positive), gave rise to another family of curves. It is this family which we are about to consider in the present chapter.

Fundamental Curve $yx = a$.—If we put n equal to 1 in the equation $yx^n = a$, we deduce the simpler equation $yx = a$, which we will now proceed to consider. If a is positive, x and y must obviously be of the same sign—that is, they must either be both positive or both negative. Hence the curve will lie wholly in the first and third quadrants. If a is negative, x and y must always be of opposite sign, or the curve must lie wholly in the second and fourth quadrants. Again, if either x or y were zero, the other co-ordinate must be infinity, hence the curve never touches either axis within a finite distance of the origin. We have, then, a curve which lies wholly within two alternate quadrants and which never touches either axis; hence the curve consists of two distinct and separate branches, one in each of the two quadrants. Moreover, the equation is symmetrical with respect to x and y, that is, these symbols occur in it in precisely the same way; therefore any pair of values of x and y may be interchanged, the truth of the equation still being satisfied. Hence the curve will be symmetrical about the line $x = y$, that is, about a straight line of unit slope passing through the origin. It will also be symmetrical about a line at right angles to this namely, $y = -x$. Further, for any real value of x, however small or however great, there corresponds a real value of y, namely $\frac{a}{x}$, and *vice versâ*. Hence, both branches of the curve will extend to infinity, both in the direction of X and in that of Y. The curve consists, therefore, of two equal and similar infinite branches, in alternate quadrants, symmetrical about the lines of unit slope passing through the origin.

We will now proceed to plot such a curve, taking any suitable value for a, say 100, the equation then being—

$$yx = 100.$$

For the first quadrant branch we have the following values for the co-ordinates of a series of points on the curve :—

x	2·0	2·5	3·3	5·0	10	20	30	40	50
y	50	40	30	20	10	5·0	3·3	2·5	2·0

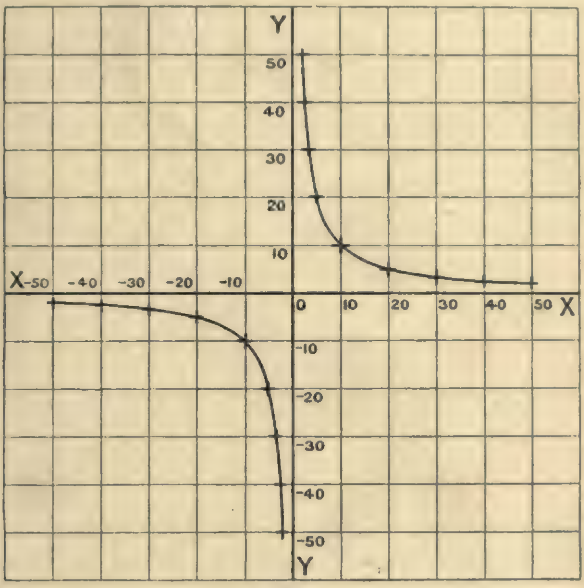


Fig. 33.

For the branch in the third quadrant the same values of x and y satisfy the equation, with their signs changed. Upon plotting these points, we obtain the curve shown in Fig. 33. This is known as the **Rectangular Hyperbola.**

Asymptotes.—If we proceed to take larger values of x than $x = 50$ in the above equation, the corresponding values of y become smaller and smaller. Similarly, if the values of y are increased, those of x become continuously smaller. Hence, the curve continues to get closer and closer to the axes of X and Y, but, as has been already stated, it never touches them within a finite distance. Straight lines which bear this relation to a curve, that is, which continually approach nearer and nearer to it, but never touch it, are called *asymptotes* to the curve, and the curve is said to be *asymptotic* to the straight lines. If either co-ordinate be put equal to zero in the equation, the corresponding value of the other is seen to be infinity. Hence, the asymptote does touch the curve at infinity, and may in fact be regarded as the *tangent to the curve at infinity.* In the curve we have just considered the two asymptotes are at right angles, being the axes of X and Y, hence the name "rectangular" hyperbolas. Other hyperbolas may be drawn whose asymptotes are not at right angles, but they are of small practical importance from our present standpoint. The other curves to be considered in this chapter are not true hyperbolas, but bear the same sort of relation to the rectangular hyperbola which the cubic parabola does to the true parabola—that is, their equations are of the same general type, and the curves of the same general form—and are, therefore, said to belong to the hyperbolic family, or to be hyperbolic curves.

Isothermal Expansion of a Gas.—If a gas expands or contracts at constant temperature, its volume (v) varies inversely as its pressure (p), or $pv = a$ *constant* (Boyle's Law). This expansion at constant temperature is known as isothermal expansion, and the curve connecting the pressure and the volume is obviously of the type we have just considered, that is, it is a rectangular hyperbola. From this fact, isothermal expansion is sometimes called "hyperbolic" expansion.

Expansion of a Gas under other Conditions.—The curves of expansion of gases under other conditions than that of constant temperature are found to be such that the relationship between the pressure and the volume of gas, which they indicate, can be very closely represented by equations of the type

$$pv^n = \text{a constant},$$

that is, by the general equation which is being considered in this chapter. Since the expansion curves furnish by far the most important and numerous examples of this family, we will substitute the letter p for y and the letter v for x as our co-ordinate symbols, p of course being plotted vertically and v horizontally.

The Meaning of the Constant n.—It is only necessary to study in detail the positive branch of the curves, as negative values of p and v do not occur in practice. Following our usual method, we will assign certain values to the constant n and proceed to plot the corresponding curves. For the sake of convenience, however, we will not keep the values of the constant a (which it will be seen later is merely a scale factor) the same for all the curves, but will so vary it as to make all the curves pass through one point. Taking for our fundamental isothermal curve the value of a to be 100 as before, we will choose the values of this constant for the other curves, so that they all pass through the point $p = 20$, $v = 5$. The values of p and v for the curve $pv = 100$ will be the same as those given in the table on p. 49. This curve is represented in Fig. 34 by the thick continuous line.

Next, let $n = 2$. Then we have—

$$a = pv^n = pv^2 = 20 \times 5^2 = 20 \times 25 = 500$$

since the point (5, 20) is to lie on the curve. Then for the equation of the second curve we have—

$$pv^2 = 500$$

or,

$$p = \frac{500}{v^2}$$

Calculating the values of p and v from this equation, we have—

v	3	4	5	10	20	30	40	50
v^2	9	16	25	100	400	900	1600	2500
$p = \frac{500}{v^2}$ . .	56·0	31·0	20·0	5·0	1·25	0·56	0·31	0·20

This curve is represented in Fig. 34 by the thinner continuous line.

We will take one other curve for a value of n greater than 1, namely—

$$pv^{1\cdot5} = a$$

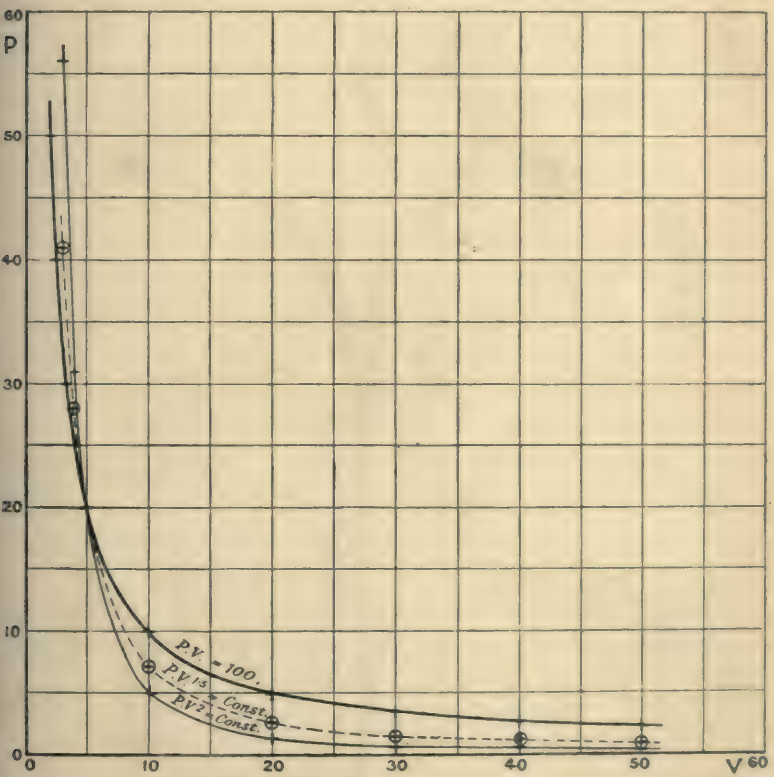


Fig. 34.

Then $$a = pv^{1\cdot5} = 20 \times 5^{1\cdot5} = 20 \times 11\cdot2 = 224$$

or we have for the equation for the third curve—

$$pv^{1\cdot5} = 224$$

or $$p = \frac{224}{v^{1\cdot5}}$$

from which we obtain the corresponding values of p and v given below.

v	3	4	5	10	20	30	40	50
$v^{1\cdot5}$ . . .	5·20	8·00	11·20	31·60	89·5	164·2	252·5	354·0
$p = \frac{224}{v^{1\cdot5}}$ . .	43·1	28·0	20·0	7·1	2·5	1·4	0·9	0·6

This curve is represented in Fig. 34 by the dotted line.

Finally, we will take the value of n to be 0·5. Then—

$$a = pv^{0\cdot5} = 20 \times \sqrt{5} = 20 \times 2\cdot24 = 44\cdot8$$

Then the equation for this curve is—

$$pv^{0\cdot5} = 44\cdot8$$

or $$p = \frac{44\cdot8}{v^{0\cdot5}}$$

$$i.e.\ p = \frac{44\cdot8}{\sqrt{v}}$$

and our values p and v are as given below.

v . . .	1	2	3	4	5	10	20	30	40	50
$\sqrt{v}$ . .	1·00	1·414	1·732	2·00	2·24	3·16	4·47	5·48	6·32	7·07
$p = \frac{44\cdot8}{\sqrt{v}}$	44·8	31·8	25·9	22·4	20·0	14·2	10·3	8·2	7·1	6·3

This curve is not represented in Fig. 34.

Whatever the value of n may be in the general equation $pv^n = a$, the value $p = 0$ gives $v = \infty$, and the value $v = 0$ gives $p = \infty$. Hence all these curves are asymptotic to the axes of X and Y. But while this is so for all the curves, it is obvious that it is only for the value $n = 1$ that the curve is symmetrical about the line $p = v$. That is to say, that the only member of the family which has the line $p = v$ as an axis of symmetry is the fundamental rectangular hyperbola, or isothermal curve. Further, it is evident that any member of the family intersects every other member at one point, but at one point only. For consider any two curves, $pv^n = a$ and $pv^m = b$. For any points of intersection the same values of p and v will satisfy both equations. By division we have for these common points—

$$\frac{v^m}{v^n} = \frac{b}{a}$$

or

$$v^{m-n} = \frac{b}{a}$$

$$i.e.\ v = \sqrt[m-n]{\frac{b}{a}}$$

Now it is, of course, possible for $\sqrt[m-n]{\frac{b}{a}}$ to have two values, but in that case one will be positive and the other negative. One will therefore refer to the first quadrant branches of the curves, and the other to the third quadrant branches. That is to say, that for the branches of the curves lying within the first quadrant, which is the only case we are considering, there is only one pair of values of p and v common to the two curves—that is, they intersect at one point only. Further, since the condition of the curves having branches in the first quadrant is that a and b are both positive,* $\frac{b}{a}$ is positive also, and therefore $\sqrt[m-n]{\frac{b}{a}}$ has always one real positive value—that is, that any curve of the family intersects every other curve in one point.

We are now in a position to see the effect of the value of n in the form of the curve. Referring to Fig. 33, we see that the curve of greater n is lower than a curve of lesser n below their point of intersection. Perhaps the best way of expressing the relationship between the various curves, however, is to express it in the form that *at the point of intersection of any two curves of the type $pv^n = a$, that of greater n has a greater (negative) slope than that of lesser n.*

The Constant a.—We can write the equation $pv^n = a$ in the form $p = \frac{a}{v^n}$. If, therefore, we change the value of a, we change the value of p corresponding to any given value of v in the same ratio; or a, as in the case of the parabolic curves, is merely a scale constant. It should be noted also that if a is positive, the values of p and v are always of the same sign, that is, that the curve lies wholly in the first and third quadrants; while if a is negative, the curve lies in the second and fourth quadrants, p and v being then of opposite sign.

To find the Equation of a given Hyperbolic Curve.—In order to do this we proceed exactly as in the case of the parabolic curves, plotting

* See next paragraph.

the logarithms of p and v (or plotting p and v on logarithmic paper), and so obtaining a straight line. Thus if

$$pv^n = a$$

then $$\log p + n \log v = \log a$$

or $$\log p = -n . \log v + \log a$$

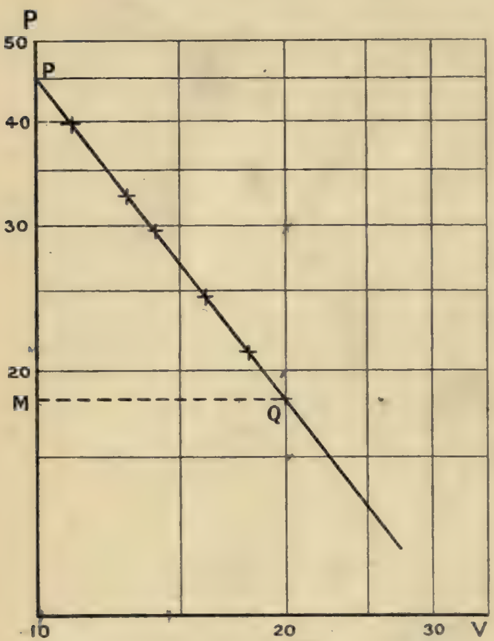


Fig. 35.

This is the equation of a straight line of slope equal to $-n$. It should be particularly noticed that the straight line obtained is of negative slope, but the value of n is positive, being numerically equal to the slope of the logarithmic straight line, but of opposite sign to it.

It should be noticed that the *scales* to which p and v are measured affect the value of the constant a only, and do not change that of n. Hence it is quite sufficient to read off the values of p and v from an indicator diagram in actual inches, or any other convenient units, without converting the readings into the respective actual values in pressure and volume units.

EXAMPLE.—The following values of p and v were measured on the compression curve of a gas-engine indicator diagram. It is required to find the value of n for the compression process.

v . .	20	18	16	14	13	12	11
p . .	18·5	21·0	24·7	29·7	32·2	35·7	39·7

In this case, since the values of p and v both fall within the range represented by one logarithmic characteristic, they may be conveniently plotted directly on logarithmic paper. This has been done in Fig. 35, so obtaining the straight line shown.

Then $$n = -\text{slope} = -\frac{PM}{QM}$$

$$= \frac{3·86}{2·95} = 1·306$$

CHAPTER VI

$y = a \,.\, e^{bx}$.—THE EXPONENTIAL FAMILY
$y = a \,.\, \log_e bx$.—THE LOGARITHMIC FAMILY

HITHERTO, in all the equations with which we have been concerned, x, the independent variable, has appeared raised to some constant power or powers. We now pass on to consider the type of equation in which x appears as the *exponent* or index of the power to which some constant quantity is raised, the corresponding curves being hence called the *exponential curves*. The general equation of the family is given by

$$y = a \,.\, e^{bx}$$

where a and b are constants, and e the base of the Napierian system of logarithms. To this equation might be added a constant term, c, the effect of which, of course, would be, as in previous cases, to raise the curve vertically through a distance c. This equation may be written in another form, viz.—

$$\log_e y = \log_e a + bx$$

or

$$bx = \log_e y - \log_e a$$

i.e.

$$x = \frac{1}{b} \log_e \frac{y}{a}.$$

Another series of curves known as the *logarithmic family* are derived from the general equation

$$y = a \,.\, \log_e bx.$$

This last equation, however, is of identically the same *form* as the former, with the variables x and y interchanged. Hence it is obvious that the logarithmic curves are of the same form as the exponential curves, *with the axes interchanged.*

The Constant a.—Considering now the general equation $y = a \,.\, e^{bx}$, if we put $x = 0$, e^{bx} is equal to unity, whatever finite value may be assigned to b. Hence, when $x = 0$, $y = a$, or a is the intercept of the curve on the axis of Y. Again, for any other value of x, y varies directly as a. Hence a is again a vertical scale constant, affecting the scale to which y is plotted, but in no other way altering the form of the curve.

The Constant b.—For any given value of x, e^{bx} becomes greater as b becomes greater. Hence, the greater the value assigned to b, the steeper is the curve. That is, the greater the value of b, the nearer is the curve to the axis of Y at any given height above the axis of X. Again, if b is positive, e^{bx} is greater than unity for all positive values of x, but is less than unity for all negative values of x, and is always positive. Hence the curve lies wholly above the axis of X, that is, in the first and second quadrants. (If a were negative, the curve would lie wholly *below* the axis of x.) Again, putting $y = 0$, we have $e^{bx} = 0$, or $x = -\infty$. That is, the curve is asymptotic to the negative direction of the axis of X. Hence we see that the curve consists of a single branch, asymptotic to the negative direction of the axis of X, rising steadily, until when $x = 0$, $y = a$, and thence continuing to rise evenly until when $x = \infty$, $y = \infty$ also.

Next, consider the equation when b is negative. The most convenient method of doing this will be to compare the two curves $y = a \,.\, e^{bx_1}$ and $y = a \,.\, e^{-bx_2}$. For the same value of y in each case we have

$$a \,.\, e^{bx_1} = a \,.\, e^{-bx_2}$$

where x_1 and x_2 are the horizontal co-ordinates of the respective points on the two curves for which y has the same value.

Then $$bx_1 = -bx_2$$

or $$x_1 = -x_2$$

That is, that at any given height above the axis of X, the curves are at the same distance from the axis of Y, but at opposite sides thereof. In other words, the curves are symmetrical with each other about the axis of Y. That is to say, that they are identical in form, but the one is completely reversed with respect to the other. Hence the curve $y = ae^{-bx}$ will be asymptotic to the *positive* direction of the axis of X, and will then steadily rise as x *diminishes*, intersecting the axis of Y when $y = a$, and giving a value $y = \infty$ when $x = -\infty$.

It is sometimes convenient to write the equation in a somewhat different form. We may put e^b equal to f^g, where f and g are *any* constants. Then the equation

$$y = a \,.\, e^{bx}$$

becomes $$y = a \,.\, f^{gx}.$$

We are now in a position to plot a few examples of curves belonging to this family. We will assume that $a = 1$ in each case. Remembering that $\log_e 10 = 2{\cdot}303$, that is, that $e^{2{\cdot}303} = 10$, we will for convenience take the values of b to be successively $2{\cdot}303$, $4{\cdot}606$, and $-2{\cdot}303$, thus obtaining the equations—

$$y = e^{2{\cdot}303x} \quad \text{or } y = 10^x$$

$$y = e^{4{\cdot}606x} \qquad \text{or } y = 10^{2x} \qquad \text{i.e. } y = 100^x$$

and $$y = e^{-2{\cdot}303x} \qquad \text{or } y = 10^{-x} \qquad \text{i.e. } y = \frac{1}{10^x}$$

The easiest way of obtaining our series of points on the curves is to assume values for y, and then determine the corresponding values of x for the three curves

$$x = \log_{10} y$$
$$x = \tfrac{1}{2}\log_{10} y$$
and
$$x = -\log_{10} y$$

respectively. The figures so obtained are tabulated below.

y	0·2	0·4	0·6	0·8	1·0	2·0	3·0	4·0	5·0	6·0
$x = \log y$ . .	−0·699*	−0·398	−0·222	−0·097	0	+0·301	0·477	0·602	0·699	0·778
$x = \frac{1}{2}\log y$ . .	−0·350	−0·199	−0·111	−0·048	0	+0·150	0·238	0·301	0·350	0·389
$x = -\log y$.	+0·699	+0·398	+0·222	+0·097	0	−0·301	−0·477	−0·602	−0·699	−0·778

These curves are plotted in Fig. 36, being represented respectively by a continuous line, a dotted line, and a dot-and-dash line. It is readily

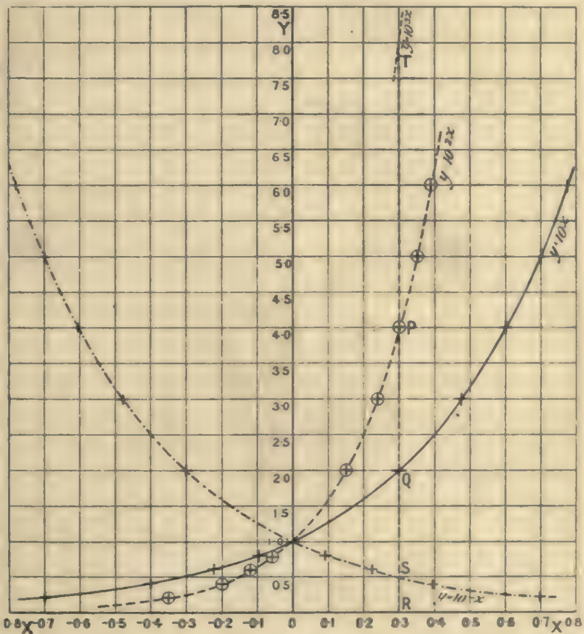


Fig. 36.

seen that these curves are of the character already described, and that the two curves

$$y = e^{2\cdot303x} \text{ and } y = e^{-2\cdot303x}$$

* For purposes of plotting, the logarithms must be obtained wholly of one sign, not with a negative characteristic and a positive mantissa, as is usual for purposes of calculation. Thus—

$$\log 0\cdot2 = \bar{1}\cdot301 = -1\cdot000 + 0\cdot301 = -0\cdot699$$

are identical in form, but reversed in position. Since the last curve, that in which b is negative, is of considerable importance, as will be seen in the next chapter, the positive portion has been replotted to a larger scale of y in Fig. 36A.

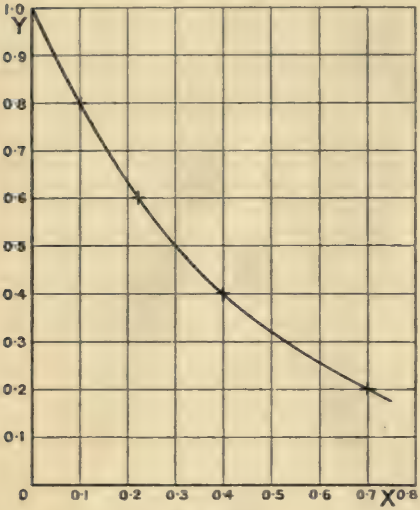


Fig. 36A.

The Curves as a Table of Powers.—It is readily seen that a series of curves of this type furnishes a graphical table of powers and roots of numbers. For example, if, in Fig. 36, a vertical straight line PQR be drawn, intersecting the curve $y = 10^{2x}$ at the point P, the curve $y = 10^{x}$ at Q, and the axis of X at R, then

$$\overline{PR} = \overline{QR}^2$$

or

$$\overline{QR} = \sqrt{\overline{PR}}$$

Similarly, the curves $y = 10^{x}$ and $y = 10^{-x}$ give a table of reciprocals. If the line QR intersects the curve $y = 10^{-x}$ in S, then

$$\overline{QR} = \frac{1}{\overline{SR}}$$

Hence also

$$\overline{SR} = \frac{1}{\sqrt{\overline{PR}}}$$

If a fourth curve, $y = 10^{3x}$, were drawn, intersecting the line PR in T, then we should have—

$$\overline{TR} = \overline{QR}^3 = \overline{PR}^{\frac{3}{2}} \qquad \text{or } \overline{PR} = \overline{TR}^{\frac{2}{3}}$$

A small portion of this curve is indicated in the figure.

To find the Equation for a Given Curve.—As in the case of the parabolic and hyperbolic families, the curve must be reduced to an equivalent straight line before the values of the constants a and b can be determined. The logarithmic form of the equation

$$y = ae^{bx}$$

is

$$\log y = \log a + bx \log e$$
$$= \log a + 0{\cdot}434bx.$$

Hence, if the values of x be plotted with those of $\log y$, a straight line will result. It must be carefully noted that the logarithms of y *only* are taken. Then the slope of this line is $0{\cdot}434b$, and the intercept on the axis of log Y is log a. If the equation is required in the form

$$y = a \,.\, 10^{gx}$$

then the slope of the line is g.

The figures given below lie on a curve of this form, as is seen in Fig. 37, where they are plotted. The values of log y are found as entered in the third line of the table.

x	0·1	0·2	0·3	0·4	0·5
y	1·84	2·59	3·66	5·18	7·33
Log y . . .	0·265	0·413	0·564	0·714	0·865

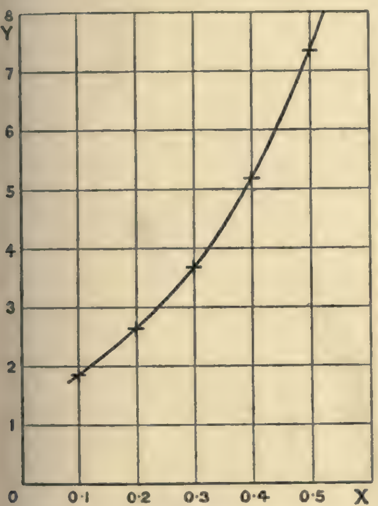


Fig. 37.

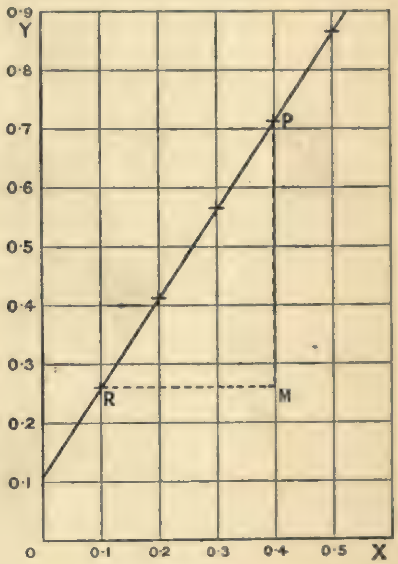


Fig. 38.

Fig. 38 shows the straight line obtained by plotting x with log y.

Then, log a = the intercept on the axis of log Y = 0·114
Therefore, $a = 1·3$

Also, if we wish to obtain the equation in the form

$$y = a \,.\, 10^{gx}$$

Then, $$g = \text{the slope of the line} = \frac{0·715 - 0·264}{0·4 - 0·1}$$

$$= \frac{0·451}{0·3} = 1·5$$

Therefore the equation is—

$$y = 1{\cdot}3 \times (10)^{1{\cdot}5x}$$

or

$$\begin{aligned} y &= 1{\cdot}3 \times (e^{2{\cdot}303})^{1{\cdot}5x} \\ &= 1{\cdot}3e^{3{\cdot}46x} \end{aligned}$$

The Use of Semi-Logarithmic Paper.—In finding the equations of curves belonging to the parabolic and hyperbolic families, in which the logarithms of *both* the co-ordinates had to be plotted with each other, we saw that the labour could be minimized by using paper already ruled logarithmically in *both* directions. Similarly, in the present case, we may minimize the labour by using paper ruled in the direction of X into equal parts, and logarithmically in the

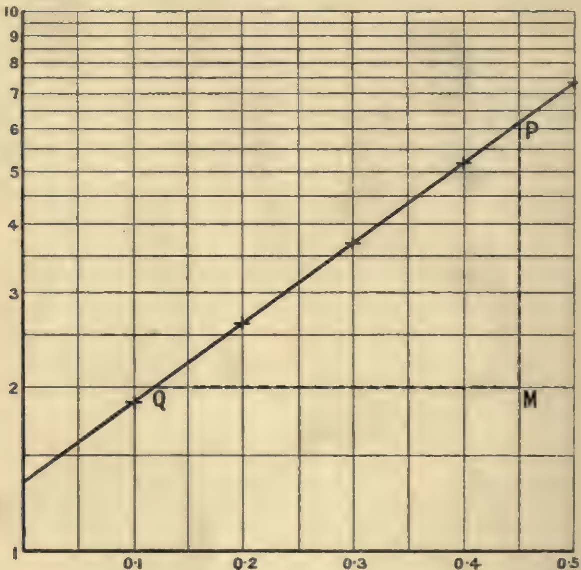


Fig. 39.

direction of Y only. This is known as "Semi-Logarithmic" paper. Since the X scale is one of equal parts, the values of x may be plotted to any convenient scale on it, as when using ordinary squared paper, but the values of y must be plotted to the scale printed on the paper, or multiples of it, by powers of 10, as in the case of logarithmic paper. The values of x and y for the curve given in the last paragraph are plotted on semi-logarithmic paper in Fig. 39. It is seen that a straight line results, and the value of a is found at once by the intercept on the axis of Y, being equal to 1·3 as before. The measurement of the slope offers some slight degree of difficulty. If the scale of X is chosen such that the scales of x and of log y are equal—that is, if the length of one sheet of the paper (which is square) in the X direction is taken equal to unity (*i.e.* to log 10)—the slope may be measured, as in the case of logarithmic paper, by measuring the vertical and horizontal distances to any, the same, scale of equal parts. If, however, any other scale is taken for x, the slope obtained by measuring the vertical and horizontal dimensions to the same scale of equal parts must be divided by the scale value of the length of one sheet of paper. Thus, in Fig. 39, the scale value of the X axis on one sheet is 0·5. Therefore measuring PM and QM to any scale of equal parts we have—

$$b = \frac{\text{PM}}{\text{RM} \times 0{\cdot}5} = \frac{4{\cdot}80}{6{\cdot}41 \times 0{\cdot}5} = 1{\cdot}5$$

which gives the same equation for the curve as was derived by the other method in the last paragraph.

The Logarithmic Curves.—At the beginning of the chapter we referred to the logarithmic family of curves, given by the general equation

$$y = a \,.\, \log bx,$$

and showed that they were of precisely the same nature as the exponential curves, with the axes interchanged; that is, the curves will be asymptotic to the negative direction of the axis of Y, and will rise steadily as x increases, being contained wholly in the fourth and first quadrants (a being positive), intersecting the axis of X at the point $x = \frac{1}{b}$, $y = 0$, *whatever the value of* a. This may be seen by putting $y = 0$ in the equation above, when we have—

$$a \,.\, \log bx = 0$$

i.e. $\log bx = 0$

or $bx = 1$

i.e. $x = \frac{1}{b}$

As an example of this family we will plot the curve

$$y = 3 \log_{10} 2x$$

The necessary points are calculated in the table below, and the curve is plotted in Fig. 40.

x	0·2	0·4	0·5	0·6	0·8	1·0	2·0	3·0	4·0	5·0
$2x$ . . .	0·4	0·8	1·0	1·2	1·6	2·0	4·0	6·0	8·0	10·0
Log $2x$ . .	−0·398	−0·097	0·000	+0·079	0·204	0·301	0·602	0·778	0·903	1·000
$y = 3 \log 2x$	−1·19	−0·29	0·00	+0·24	0·61	0·90	1·91	2·33	2·71	3·00

To find the equation of a curve of this type it is obvious that y

must be plotted with log x, or x and y must be plotted on semi-logarithmic paper, the logarithmic axis being horizontal. A straight

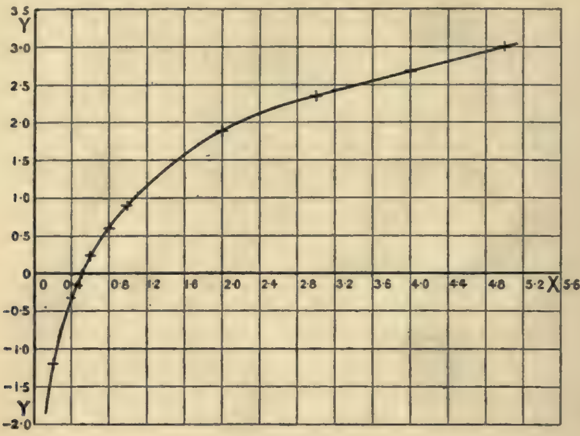


Fig. 40.

line will then result, whose slope is a, and whose intercept on the axis of Y is equal to $a \,.\, \log b$. This is seen by writing the equation in the form

$$y = a \,.\, \log x + a \,.\, \log b.$$

CHAPTER VII

THE SINE CURVE

The simple fundamental Sine Curve, given by the equation

$$y = \sin x$$

is familiar to all students of elementary trigonometry and mechanics. Since the trigonometrical functions of $(360^\circ + \theta)$ are in all respects identical with those of θ, the curve, in common with all trigonometrical curves, repeats itself exactly while the angle varies through the amounts represented by successive complete revolutions of the generating line. A curve of this type which exactly repeats itself during successive equal intervals, or cycles, is termed a "*cyclic curve.*" In Fig. 41 is plotted one cycle of the simple sine curve whose equation is given above, x being measured, for simplicity, in degrees. The figures for plotting this curve are given below, being obtained directly from the tables.

x . . .	0	20°	40°	60°	80°	90°	100°	120°	140°	160°	
$y = \sin x$	0·000	0·342	0·643	0·866	0·985	1·000	0·985	0·866	0·643	0·342	
x . . .	180°	200°	220°	240°	260°	270°	280°	300°	320°	340°	360°
$y = \sin x$	0·000	−0·342	−0·643	−0·866	−0·985	−1·000	−0·985	−0·866	−0·643	−0·342	0·000

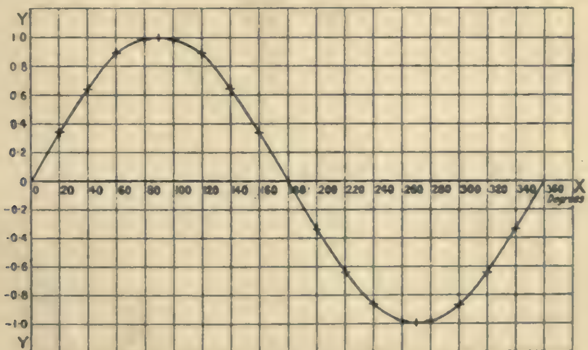


Fig. 41.

The Graphical Representation of a Vibration.—The simplest form of a vibratory motion is that known as a "simple harmonic motion."

This may be defined as the motion of a particle Q along the diameter AB of a circle, such that Q always coincides with the projection upon AB of another particle, P (which may be real or imaginary), which moves with uniform angular motion round the circumference of the circle whose diameter is AB. (See Fig. 42.)

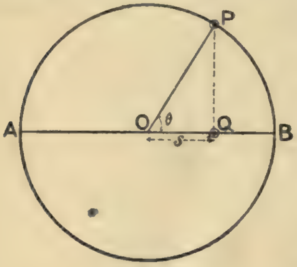


Fig. 42.

After any given time, t, from zero position, which we will assume to be when P and Q both coincide with B, let the angle BOP be equal to θ. Then, since P is assumed to be moving with uniform angular velocity, $\theta = k \,.\, t$, where k is some constant. Let the corresponding displacement of Q from its mid-position, O, be s, and the radius of the circle, that is the greatest displacement of Q from its mid-position, be r. This maximum displacement from mid-position, r, is known as the amplitude of the vibration. Then

$$\frac{s}{r} = \cos\theta = \cos kt$$

or

$$s = r \,.\, \cos kt = r \,.\, \sin(90^\circ - kt)$$

Hence, if the displacement of Q from its mid-position, s, be plotted with t, the time from some fixed position of Q, usually taken as either the extreme end or as the middle of the swing, the resulting curve will be a sine curve. That is, the sine curve represents a simple vibration. It is from this standpoint in particular that we shall regard it.

The General Equation of the Sine Curves.—The equation obtained above to represent a vibration is not in the simple form $y = \sin x$, but involves three constants, represented in that case by r, 90°, and $-k$. Introducing three constants to represent these, we obtain the general equation of the family of sine curves in the form

$$y = a \,.\, \sin(bx + c).$$

The graphical meaning of these three constants, a, b, and c, must now be deduced. At present, for the sake of simplicity, we shall assume that x is measured in degrees.

The Constant a.—In order to determine the meaning of the constant a, we will assign to the constants b and c the values 1 and 0 respectively, so obtaining the equation

$$y = a \,.\, \sin x.$$

Now, as in former cases, this constant might be called simply a vertical scale constant, but in this case it will be well to examine this statement rather more fully. The greatest possible value of $\sin x$ is unity. Hence the greatest possible value of y is a. That is, in the case of a vibration, a is the greatest displacement from mid-position, or the *amplitude*. Hence we may call a the *amplitude constant*. We will now proceed to plot the curves represented by equations for which the values of a are respectively 1 and 2, that is, the two curves given by the equations

$$y = \sin x$$

and

$$y = 2 \sin x$$

represented in Fig. 43 by the continuous and the broken lines respectively. Since

$$\sin x = \sin (180° - x) = -\sin (180° + x) = -\sin (360° - x)$$

it will be only necessary to tabulate the values of $\sin x$ for values of x between 0 and 90°, as the values then repeat themselves numerically as shown in the table of values of $\sin x$ on p. 63.

x	0	20°	40°	60°	80°	90°
Sin x . . .	0	0·342	0·643	0·866	0·985	1·000
2 sin x . . .	0	0·684	1·286	1·732	1·970	2·000

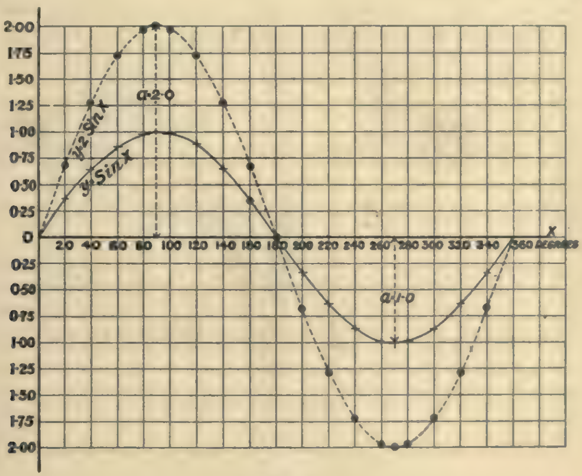


Fig. 43.

It is seen that a is the greatest distance of the curve above or below the axis of X, or the amplitude of the vibration, as already stated.

The Constant b.—Proceeding now to investigate the meaning of the constant b, we will assume the amplitude a to be unity, and the constant c zero. Then we obtain the equation

$$y = \sin bx$$

This curve will trace out one complete cycle while bx changes through an amount corresponding to one revolution, or 360°. Hence, the cycle length of the curve—that is, the change in the value of x during which the curve makes one complete cycle—is $\frac{1}{b}$ of the cycle length of the fundamental curve, $y = \sin x$. Another method of expressing this is to say that the curve makes b complete cycles, while x changes by an amount corresponding to one revolution, 360°, or 2π radians. Expressing this in terms of a vibration, we should state that the *period* (or in the case of a wave motion, the *wave length*) of the vibration is $\frac{1}{b}$th that of the primary or fundamental vibration $y = \sin x$, or that, calling the *frequency* of the fundamental vibration unity, the *frequency* of the vibration given by $y = \sin bx$ is equal to the value of the constant b. The frequency of a vibration is defined to be the number of *complete swings* (to *and* fro) made in any fixed interval of time. Usually this fixed interval of time is taken to be one second or one minute, but here we have assumed it to be the period of one complete cycle of the fundamental vibration. (This would be equal to 2π seconds, the second being taken as the unit of time, since the time units must be supposed to be angles expressed in radians for the purpose of determining the values of $\sin x$.) Or we might state the point in a slightly different way by saying that b was the number of complete vibrations of the curve $y = \sin bx$ in any given time to the number of complete vibrations of the curve $y = \sin x$ in the same time.

To illustrate this *frequency* of the vibrations we will plot the two curves

$$y = \sin x$$

and

$$y = \sin 2x$$

represented respectively by the continuous line and the broken line in Fig. 44. The necessary series of values of x and y for the two curves are calculated in the table below for a range of x corresponding to one complete cycle of the fundamental curve.

x.	$y_1 = \sin x$.	$2x$.	$y_2 = \sin 2x$.	x.	$y_1 = \sin x$.	$2x$.	$y_2 = \sin 2x$.
0	0·000	0	0·000	180°	0·000	360°	0·000
20°	0·342	40°	0·643	200°	−0·342	400°	+0·643
40°	0·643	80°	0·985	220°	−0·643	440°	0·985
45°	—	90°	1·000	225°	—	450°	1·000
60°	0·866	120°	0·866	240°	−0·866	480°	0·866
80°	0·985	160°	0·342	260°	−0·985	520°	0·342
90°	1·000	180°	0·000	270°	−1·000	540°	0·000
100°	0·985	200°	−0·342	280°	−0·985	560°	−0·342
120°	0·866	240°	−0·866	300°	−0·866	600°	−0·866
135°	—	270°	−1·000	315°	—	630°	−1·000
140°	0·643	280°	−0·985	320°	−0·643	640°	−0·985
160°	0·342	320°	−0·643	340°	−0·342	680°	−0·643
				360°	0·000	720°	0·000

Here we see that the curve $y = \sin 2x$ makes two complete cycles, while the fundamental curve $y = \sin x$ makes one, that is, that the frequency of the former is 2, that of the latter being unity; also, the cycle-length, 180°, of the former curve is exactly one-half that of the latter, 360°.

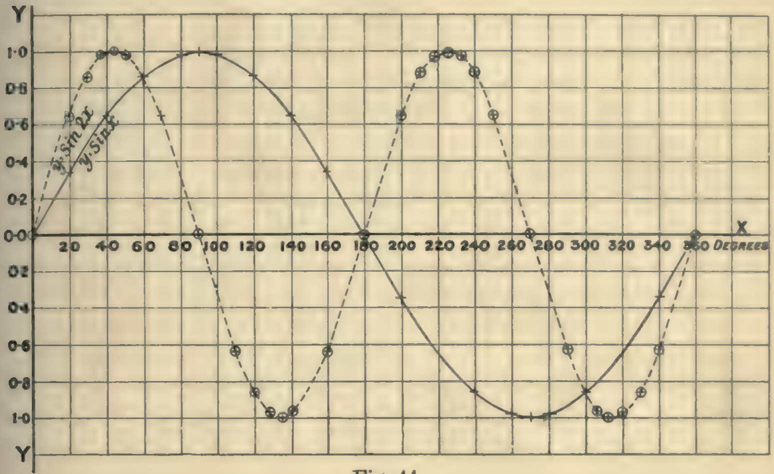


Fig. 44.

The Constant *c*.—We now pass to the consideration of the constant c. Assign to each of the constants a and b the value unity. Then the equation becomes—

$$y = \sin (x + c)$$

The simplest way of examining this equation to determine the effect of the constant c will be to proceed directly to plot three curves having c respectively equal to 0, $+30°$, and $-45°$, that is, whose equations are—

$$y = \sin x$$
$$y = \sin (x + 30°)$$
and
$$y = \sin (x - 45°)$$

The necessary figures are tabulated on p. 68.

These curves are plotted in Fig. 45, being represented respectively by the thick continuous line, the thinner continuous line, and the broken line.

We see at once that the curves are identical in form, amplitude, and frequency. They are, however, moved horizontally relatively to each other. If any horizontal line, ABC, is drawn cutting the corresponding portions of the curves in A (on $y = \sin (x + 30°)$), B (on $y = \sin x$), and C (on $y = \sin (x - 45°)$), it will be seen that the distance AB is equal to 30°, and the distance BC is equal to 45°. Hence c gives the relative horizontal displacement of the curve with respect to

x.	Sin x.	$x+30°$.	Sin $(x+30°)$.	$x-45°$.	Sin $(x-45°)$.	x.	Sin x.	$x+30°$.	Sin $(x+30°)$.	$x-45°$.	Sin $(x-45°)$.
0	0·00	30°	0·50	−45°	−0·71	180°	0·00	210°	−0·50	135°	0·71
20°	0·34	50°	0·77	−25°	−0·42	200°	−0·34	230°	−0·77	155°	0·42
40°	0·64	70°	0·94	−5°	−0·09	220°	−0·64	250°	−0·94	175°	0·09
45°	—	—	—	0	0·00	225°	—	—	—	180°	0·00
60°	0·87	90°	1·00	+15°	+0·26	240°	−0·87	270°	−1·00	195°	−0·26
80°	0·98	110°	0·94	35°	0·57	260°	−0·98	290°	−0·94	215°	−0·57
90°	1·00	—	—	—	—	270°	−1·00	—	—	—	—
100°	0·98	130°	0·77	55°	0·82	280°	−0·98	310°	−0·77	235°	−0·82
120°	0·87	150°	0·50	75°	0·97	300°	−0·87	330°	−0·50	255°	−0·97
135°	—	—	—	90°	1·00	315°	—	—	—	270°	−1·00
140°	0·64	170°	0·17	95°	0·99	320°	−0·64	350°	−0·17	275°	−0·99
150°	—	180°	0·00	—	—	330°	—	360°	0·00	—	—
160°	0·34	190°	−0·17	115°	0·91	340°	−0·34	370°	+0·17	295°	−0·91
						360°	0·00	390°	0·50	315°	−0·71

the fundamental curve $y = \sin x$. It should be noticed that when c is positive, the displacement is to the *left*, and when c is negative, the

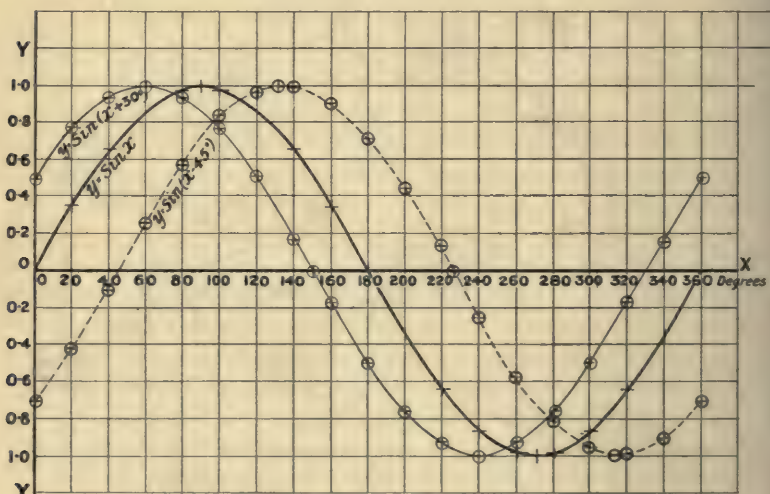


Fig. 45.

displacement is to the *right*. This corresponds to a difference of "*phase*" between two or more vibrations. Hence *c is equal to the phase difference.* The meaning of this phase difference may be readily seen by reference to the conception of a simple harmonic motion as being derived from the projection upon a diameter of a particle which is itself moving uniformly in a circle. Let there be two particles, P and R, moving uniformly round the circumference of a circle (Fig. 46), their projections Q and S on the diameter then moving with simple harmonic motion along the diameter. Then the phase difference between the vibrations of Q and S is represented by the angle POR. If P and R are moving in the direction of the arrow, so that R is ahead of P, the phase of S will be in advance of that of Q. Now, in Fig. 45, the thin line corresponds to a positive value for c, the phase constant. Does this represent a phase in advance of the fundamental vibration, or behind it? In Fig. 46, S is in advance of Q, because it passes any given point in its line of motion before Q. In Fig. 45, the displacement is represented vertically, and it is seen that the curve $y = \sin(x + 30°)$ reaches any given point in its swing—for example, the point of maximum displacement—earlier than the fundamental curve $y = \sin x$ reaches the same point. That is, the former curve represents a vibration whose phase is in advance of the latter, although the curve seems apparently to be lagging behind. Hence a positive value of c gives a positive phase difference; that is, of two curves, that for which c is greater represents a vibration in advance of that for which c is less.

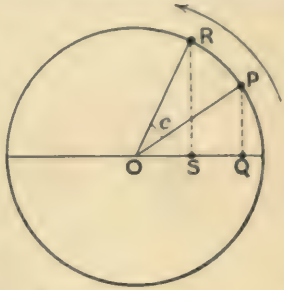


Fig. 46.

It should be noted that strictly we can only speak of a phase difference between two vibrations of the same frequency. Otherwise the phase difference will be constantly changing. For, referring again to Fig. 46, if P and R are not moving with the same angular velocity, in which case Q and S will not have the same frequency, the angle POR, which measures the phase difference, varies continuously. In this case we can only speak of the phase difference at a given instant, and c measures its *initial* value, that is when $x = 0$.

Hence we see, in conclusion, that the three fundamental properties of a vibration—namely, the amplitude, frequency, and phase—are represented exactly by the three constants a, b, and c, respectively, in the general equation of the sine curve

$$y = a \,.\, \sin(bx + c)$$

To determine the Equation for a Given Sine Curve.—Any sine curve being plotted, the constants a, b, and c in the general equation may be determined by inspection. The amplitude of the curve is the maximum displacement from mid-position, *i.e.* is one-half the distance between the extreme ends of the swing. Hence, assuming that the mean line, or line of no displacement, which we have hitherto assumed to be the axis of X, is not given, its position must first be determined. (If this line were not the axis of X, the equation would be increased by a constant term, d, as is easily seen from the effect of such a constant term in the equations of the other families of curves already considered. The general equation would then be of the form

$$y = a \, . \sin (bx + c) + d$$

Draw two lines, AB and CD, touching the curve at the points of extreme displacement. Then the amplitude is equal to half the distance between them, *i.e.* the distance between either of the lines AB or CD and the mean line EF. For the curve shown in Fig. 47, AB is the horizontal line for which $y = 7 \cdot 2$, and CD is the line $y = -3 \cdot 4$. Hence the amplitude—

$$a = \frac{7 \cdot 2 - (-3 \cdot 4)}{2} = \frac{10 \cdot 6}{2} = 5 \cdot 3$$

and d the height of the mean line, or line of no displacement, above the axis of X is $(7 \cdot 2 - 5 \cdot 3) = 1 \cdot 9$.

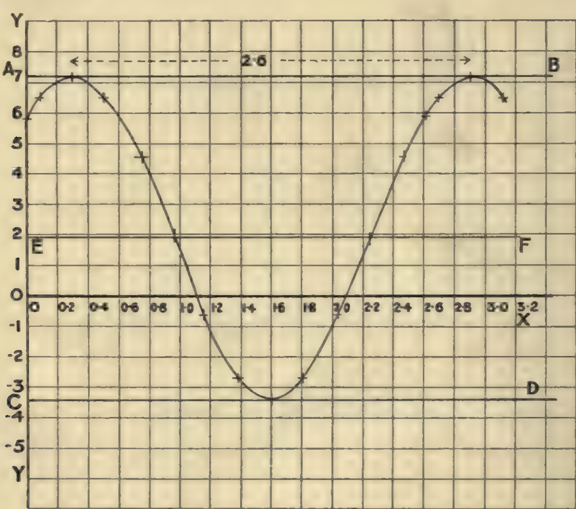


Fig. 47.

Next to determine the frequency constant, b. It has been shown that b expresses the number of complete swings in one cycle-length of the fundamental sine curve $y = \sin x$, b being unity if one complete swing corresponds to a change in the value of x of $360°$, or of 2π, that is, if the time of one complete vibration is 2π seconds.* Now, the length of one complete swing of the curve in Fig. 47 is equal to $2 \cdot 6$. Therefore we have—

$$b = \frac{2\pi}{2 \cdot 6} = \frac{6 \cdot 28}{2 \cdot 6} = 2 \cdot 41$$

* We have hitherto spoken of x as being measured in degrees, for the sake of simplicity, c being therefore measured in degrees also. But in dealing with vibrations the natural unit of angular measurement, the radian, must be taken, and therefore both x and c must be measured in radians.

Lastly, c expresses the phase of the curve, that is the angle the generating point has moved through from the dead centre position when $x = 0$. Putting $x = 0$ in the equation $y = a \,.\, \sin(bx + c) + d$, we have—

$$y - d = a \,.\, \sin c$$

or

$$\sin c = \frac{y - d}{a}$$

For the curve in Fig. 47, when $x = 0$, $y = 5{\cdot}9$,

then

$$\sin c = \frac{5{\cdot}9 - 1{\cdot}9}{5{\cdot}3} = \frac{4{\cdot}0}{5{\cdot}3} = 0{\cdot}755$$

Therefore

$$c = 49^\circ = 0{\cdot}854 \text{ radians}$$

Hence the complete equation of the curve is—

$$y = 5{\cdot}3 \sin(2{\cdot}41x + 0{\cdot}854) + 1{\cdot}9$$

Compound Sine Curves.—Two or more simple harmonic motions may be simultaneously given to the same particle. If these are not in the same straight line, a great variety of curves may be traced out by the particle under their combined influence. These are drawn mechanically by the various forms of harmonigraph, but are of small practical importance. The simplest case is that in which a particle is given two simple harmonic motions of the same amplitude and frequency in directions at right angles to each other, differing in phase by a right angle, that is, one is at its point of maximum displacement when the other is in mid-position. It is readily seen that the resulting motion of the particle will be circular, of radius equal to the common amplitude, making one revolution during the time taken for each complete swing of either simple vibration. Such motions, however, do not partake of the cyclic nature, and are not really vibrations. We confine ourselves, therefore, to the case in which the superimposed simple harmonic motions take place in the same straight line, giving rise to a cyclic vibration of a more complicated character. Since each simple vibration may be represented by a sine curve, the sum of the separate simple motions will be represented by the sum of the corresponding sine curves, giving as the general equation

$$y = a \,.\, \sin(bx + c) + f \,.\, \sin(gx + h) + \ .\ .\ .$$

The curves derived from this equation are known as "compound sine curves." We will proceed to plot an example of the series, namely, the curve

$$y = \sin(2x + 30^\circ) + 2 \sin 3x$$

This is composed of two simple sine curves, one of which is of double the amplitude and $1\frac{1}{2}$ times the frequency of the other, the initial difference of phase being 30°. These two constituent curves will, of course, be represented by the equations

$$y_1 = \sin(2x + 30^\circ)$$

and

$$y_2 = 2 \sin 3x$$

While the first of these is tracing out two complete cycles, the second traces out three, and at the end of this period each will be in identically the same state as it was at the beginning. That is, after two cycles of the first curve and three of the second, the compound curve will have completed one cycle. To put this in another form, the cycle-length of the first will be $\frac{360^\circ}{2} = 180^\circ$, and that of the second will be $\frac{360^\circ}{3} = 120^\circ$. It is obvious that the cycle-length of the compound curve will be the least common multiple of the cycle-lengths of its constituents, that is 360°. Hence it will be sufficient to obtain the values of x and y while x varies from 0 to 360°, and since each quarter-cycle of the second constituent occupies only 30° we must take the values of x at intervals of not less than 10°. The figures so obtained are tabulated below. Angles greater than 360° are tabulated as the equivalent angle for ease in working. Thus $2 \times 220^\circ$ is tabulated as $360^\circ + 80^\circ$.

x°.	$2x^\circ$.	$(2x+30)^\circ$.	$y_1 = \sin(2x+30)$.	$3x^\circ$.	Sin $3x^\circ$.	$y_2 = 2 \sin 3x$.	$y = y_1 + y_2$.
0	0	30	0·500	0	0·000	0·000	0·50
10	20	50	0·766	30	0·500	1·000	1·77
20	40	70	0·940	60	0·866	1·732	2·67
30	60	90	1·000	90	1·000	2·000	3·00
40	80	110	0·940	120	0·866	1·732	2·67
50	100	130	0·766	150	0·500	1·000	1·77
60	120	150	0·500	180	0·000	0·000	0·50
70	140	170	0·174	210	−0·500	−1·000	−0·83
80	160	190	−0·174	240	−0·866	−1·732	−1·91
90	180	210	−0·500	270	−1·000	−2·000	−2·50
100	200	230	−0·766	300	−0·866	−1·732	−2·50
110	220	250	−0·940	330	−0·500	−1·000	−1·94
120	240	270	−1·000	360	0·000	0·000	−1·00
130	260	290	−0·940	360 + 30	+0·500	+1·000	+0·06
140	280	310	−0·766	60	0·866	1·732	0·97
150	300	330	−0·500	90	1·000	2·000	1·50
160	320	350	−0·174	120	0·866	1·732	1·56
170	340	360 + 10	+0·174	150	0·500	1·000	1·17
180	360	30	0·500	180	0·000	0·000	0·50
190	360 + 20	50	0·766	210	−0·500	−1·000	−0·23
200	40	70	0·940	240	−0·866	−1·732	−0·79
210	60	90	1·000	270	−1·000	−2·000	−1·00
220	80	110	0·940	300	−0·866	−1·732	−0·79
230	100	130	0·766	330	−0·500	−1·000	−0·23
240	120	150	0·500	360	0·000	0·000	+0·50
250	140	170	0·174	720 + 30	+0·500	+1·000	1·17
260	160	190	−0·174	60	0·866	1·732	1·56
270	180	210	−0·500	90	1·000	2·000	1·50
280	200	230	−0·766	120	0·866	1·732	0·97
290	220	250	−0·940	150	0·500	1·000	0·06
300	240	270	−1·000	180	0·000	0·000	−1·00
310	260	290	−0·940	210	−0·500	−1·000	−1·94
320	280	310	−0·766	240	−0·866	−1·732	−2·50
330	300	330	−0·500	270	−1·000	−2·000	−2·50
340	320	350	−0·174	300	−0·866	−1·732	−1·91
350	340	720 + 10	+0·174	330	−0·500	−1·000	−0·83
360	360	30	0·500	360	0·000	0·000	+0·50

This curve is plotted in Fig. 48, represented by the continuous line. The two constituent curves are also shown by the two broken

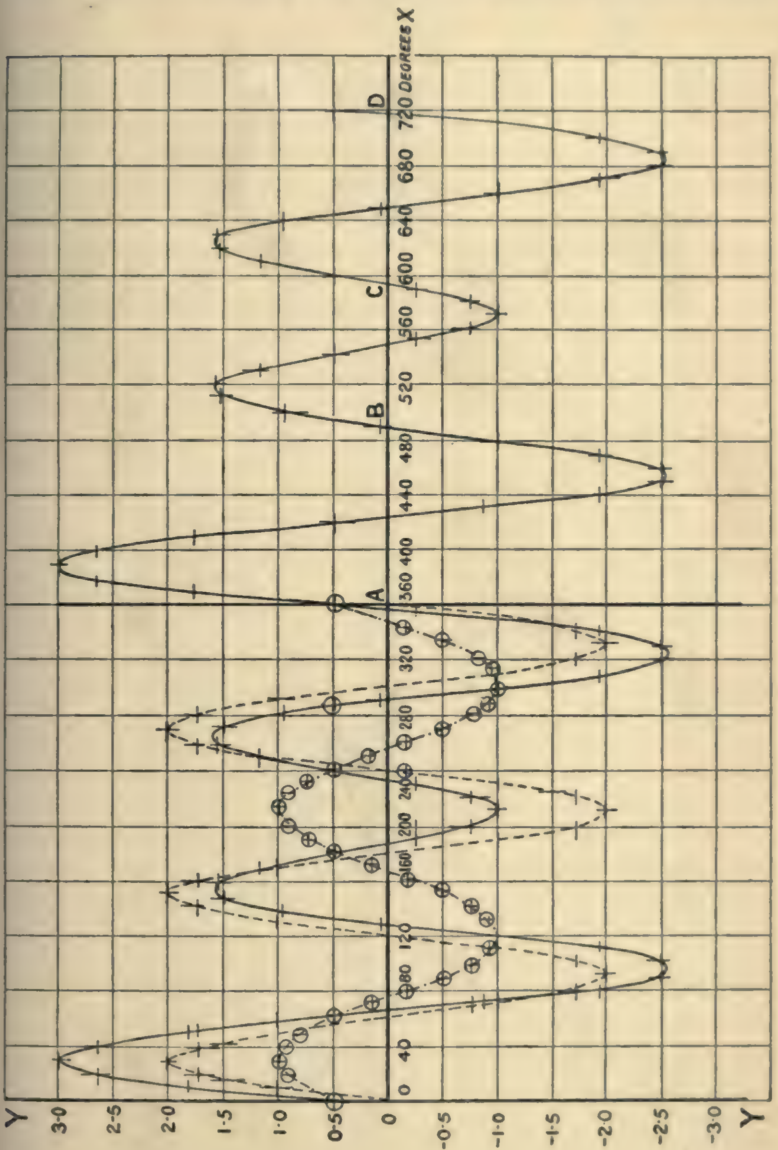


Fig. 48.

lines. Of course the vertical co-ordinate of the compound curve for any given value of x is equal to the algebraic sum of the vertical co-ordinates of the constituent curves for the same value of x. The curve is plotted over a range of values of x equal to 720°, that is, equal to two cycle-lengths of the compound curve. It is seen at once that the curve is a cyclic oscillatory curve, but it is of a much more complicated character than the simple sine curve. In one cycle it has three maxima, one for which y is equal to 3·0, and the other two for which y is equal to about 1·58, and three minima, for two of which y is equal to about $-2·55$, and for the third y is equal to $-1·0$. There are three complete swings during each cycle, but in only one of them does the vibrating particle reach its extreme displacement, and then only on one side of the zero position. The wave-length of each of the three inner swings measured on the axis of X is different from each of the others; thus AB = 134°, BC = 104°, and CD = 122°. An infinite variety of these curves may be formed by taking various values for the relative amplitudes, frequencies, and phases, and by introducing three or more constituent curves; but this example will suffice for our present purpose. In the examples on p. 130 the student will find various other cases, all of which he should plot for himself. These curves are of special importance in electrical work, there representing the variation in voltage of two- and three-phase alternating currents.

The determination of the equation of a compound sine curve involves its analysis into its constituent or component simple sine curves. This may be done by Fourier's analysis, but as this method involves a considerable knowledge of "higher" mathematics, it lies outside the scope of this book.

The Damped Sine Curve.—A most interesting curve is obtained by combining a simple sine curve with an exponential curve, for which the coefficient of x is negative. Its general equation is given by

$$y = a \,.\, e^{-bx} \,.\, \sin (cx + d)$$

This is known as the "Damped Sine Curve." We will proceed to plot such a curve at once. Since x appears not only as the angle whose sine is to be determined, but also in the exponent of e, it must, of course, be expressed in natural units, that is, in radians. In order to lessen our calculations we will assign to the constant a the value unity, and to d the value zero. Then, putting $b = \frac{1}{3}$ and $c = 3$, we obtain the equation—

$$y = e^{-\frac{1}{3}x} \,.\, \sin 3x$$

The cycle-length of the fundamental sine curve $y = \sin 3x$ is $\frac{2\pi}{3}$, hence by plotting the curve over a range of values of x equal to 2π we shall obtain three cycle-lengths of the fundamental curve, which will be sufficient for our purpose. The necessary figures for plotting the curve are obtained in the table below. The column headed "equivalent angle" is introduced for convenience in using the table of sines, giving the angle in the previous column reduced to the corresponding first revolution angle, and showing the corresponding angle in the first quadrant. The values of e are most easily obtained thus—

$$\text{Log}_{10} e^{\frac{x}{3}} = \frac{x}{3} \times 0{\cdot}434 = 0{\cdot}145x$$

These values are tabulated in the second column, their antilogarithms —that is, the values of $e^{\frac{x}{3}}$—in the third, then the values of $e^{-\frac{x}{3}}$ are the reciprocals of these.

x.	$\text{Log}_{10} e^{\frac{x}{3}}$ $= 0{\cdot}145x$.	$e^{\frac{x}{3}}$	$u = e^{-\frac{x}{3}}$	$3x$ radians.	$3x \times 57{\cdot}3$ degrees.	Equivalent angle.	$v = \sin 3x$.	$y = u \times v$.
0·0	0·000	1·000	1·000	0·0	0	—	0·000	0·000
0·2	0·029	1·069	0·936	0·6	34	34	0·566	0·529
0·4	0·058	1·143	0·875	1·2	69	69	0·934	0·817
0·6	0·087	1·221	0·819	1·8	103	180 − 77	0·974	0·797
0·8	0·116	1·306	0·766	2·4	138	180 − 42	0·669	0·513
1·0	0·145	1·396	0·717	3·0	172	180 − 8	0·139	0·100
1·05	—	—	—	3·14 (π)	180	—	0·000	0·000
1·2	0·174	1·493	0·670	3·6	206	180 + 26	−0·438	−0·294
1·4	0·203	1·596	0·628	4·2	241	180 + 61	−0·875	−0·548
1·6	0·232	1·706	0·587	4·8	277	360 − 83	−0·922	−0·582
2·0	0·290	1·950	0·513	6·0	344	360 − 16	−0·276	−0·141
2·09	—	—	—	6·28 (2π)	360	—	0·000	0·000
2·2	0·318	2·080	0·481	6·6	378	18	+0·309	+0·149
2·4	0·348	2·228	0·449	7·2	413	53	0·799	0·359
2·6	0·376	2·377	0·421	7·8	447	87	0·999	0·420
2·8	0·405	2·541	0·394	8·4	482	180 − 58	0·848	0·334
3·0	0·434	2·716	0·369	9·0	517	180 − 23	0·391	0·144
3·14	—	—	—	9·42 (3π)	540	180	0·000	0·000
3·2	0·462	2·904	0·344	9·6	551	180 + 11	−0·191	−0·067
3·4	0·493	3·105	0·322	10·2	585	180 + 45	−0·707	−0·228
3·6	0·521	3·319	0·302	10·8	620	180 + 80	−0·985	−0·298
3·8	0·550	3·548	0·282	11·4	654	360 − 66	−0·914	−0·257
4·0	0·580	3·802	0·263	12·0	688	360 − 32	−0·530	−0·139
4·18	—	—	—	12·56 (4π)	720	360	0·000	0·000
4·2	0·608	4·055	0·246	12·6	722	2	+0·035	+0·008
4·4	0·637	4·335	0·231	13·2	757	37	0·602	0·139
4·6	0·666	4·634	0·216	13·8	791	71	0·946	0·204
4·8	0·695	4·955	0·203	14·4	826	180 − 74	0·961	0·194
5·0	0·723	5·284	0·189	15·0	860	180 − 40	0·643	0·120
5·2	0·753	5·662	0·177	15·6	895	180 − 5	0·087	0·015
5·23	—	—	—	15·70 (5π)	900	180	0·000	0·000
5·4	0·781	6·039	0·166	16·2	929	180 + 29	−0·485	−0·080
5·6	0·810	6·437	0·155	16·8	963	180 + 63	−0·891	−0·138
5·8	0·840	6·918	0·145	17·4	998	360 − 82	−0·990	−0·143
6·0	0·868	7·379	0·134	18·0	1032	360 − 48	−0·743	−0·101
6·2	0·887	7·709	0·130	18·6	1065	360 − 15	−0·256	−0·033
6·28	—	—	—	18·84 (6π)	1080	360	0·000	0·000

The curve is plotted in Fig. 49, the broken lines showing the two curves $y = e^{-\frac{1}{3}x}$ and $y = -e^{-\frac{1}{3}x}$. The following points are at once noticeable :—

(1) The *amplitude* of the curve constantly diminishes, the curve always being contained between the two curves $y = \pm e^{-\frac{1}{3}x}$, and being tangential to them *near* its maximum and minimum points.

(2) The *frequency* of the curve is constant, that is, the length of each "cycle" is the same as that of every other "cycle."

(3) In any portion of the curve, that is between any two consecutive points for which $y = 0$, the earlier portion *before* the maximum or minimum point is steeper than the latter portion *after* that point.

(4) The curve will *never* coincide with the axis of X, for the two curves $y = \pm e^{-\frac{1}{3}x}$ are asymptotic to that axis, and the curve oscillates between them.

Hence we have an infinite oscillatory curve whose frequency remains constant, but whose amplitude constantly diminishes. This curve represents a vibration which is gradually dying out, or being "damped," hence the name "Damped Sine Curve." In such a

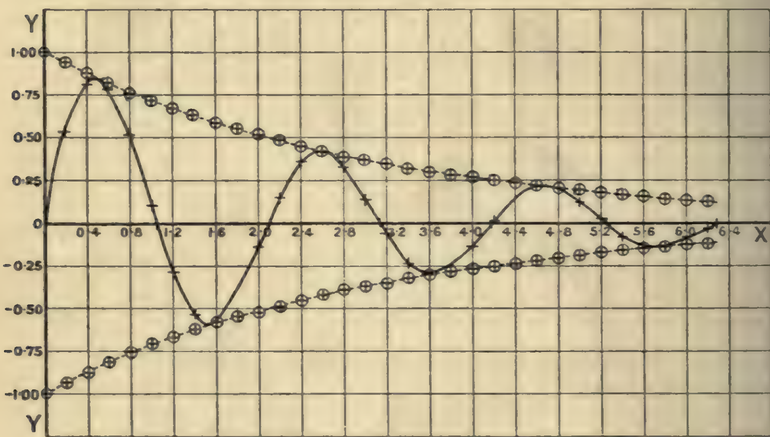


Fig. 49.

vibration—for example, that of a pendulum which is not receiving fresh impulses and whose support is not perfectly frictionless—the time of the successive vibrations is the same, but the amplitude, or length of swing, gradually diminishes until the oscillation becomes imperceptibly small.

The equation of a damped sine curve may be readily obtained, for its frequency is known, its initial phase is known, and its amplitude constant, a, may be easily obtained. Its frequency is known, for the "cycle"-length may be measured directly. Its initial phase is known, for when $x = 0$, $e^{-bx} = 1$, and therefore the phase constant, $d = \sin^{-1}\frac{y}{a}$. The amplitude constant may be approximately determined by drawing a smooth curve touching the damped sine curve near its maximum points and producing it to the axis of Y, when its intercept on that axis is equal to a.

CHAPTER VIII

THE GRAPHICAL SOLUTION OF EQUATIONS

The graphical representation of functions gives us a ready method of solving any equation, even such as cannot be solved by the ordinary methods of algebra and trigonometry. There are two ways of applying the method, which will be most clearly shown by their application to the solution of simple equations.

First Method.—Let the equation whose solution is required be reduced to the form

$$ax + b = 0$$

Then, if the corresponding equation

$$y = ax + b$$

be plotted, a straight line results. The value of x for which $y = 0$ in the second equation will, of course, give the solution of the first. That is, the solution of the equation $ax + b = 0$ is the x co-ordinate of the point at which the straight line $y = ax + b$ cuts the axis of X.

As an example, consider the equation

$$3{\cdot}1x - 4{\cdot}5 = 2{\cdot}72 - 1{\cdot}5x$$

This reduces to

$$4{\cdot}6x - 7{\cdot}22 = 0$$

Then put $\qquad y = 4{\cdot}6x - 7{\cdot}22$

This represents the straight line shown in Fig. 50, whose intercept upon the axis of Y is $-7{\cdot}22$, and whose slope is $4{\cdot}6$. This cuts the axis of X at the point for which $x = 1{\cdot}57$, which is therefore the required solution of the equation, as is easily verified by the ordinary algebraic method.

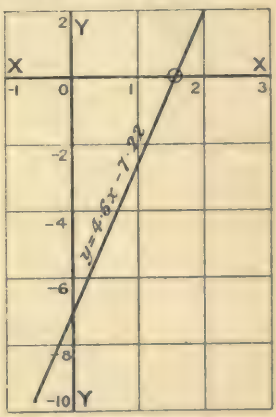


Fig. 50.

Second Method.—The second method of applying the method of curve plotting to the solution of equations is most readily illustrated by using it to obtain the solution of a pair of simultaneous simple equations. Let the equations be

$$y = ax + b$$

and $$y = cx + d$$

The values of x and y which satisfy both these equations constitute the solution required. Each of the equations when plotted will give a straight line. Then the only pair of values of x and y which satisfy both equations will be the co-ordinates of the point of intersection of these two lines. That is, the solution of the equations is given by the values of the co-ordinates of their common point.

Consider, for example, the two equations

$$y = 3x - 2$$

and $$y = 2x + 3.$$

The first of these is of slope 3, and makes an intercept on the axis of Y equal to -2; the second is of slope 2, and has an intercept 3. These two lines are plotted in Fig. 51. They intersect at the point for which $x = 5$ and $y = 13$, which therefore gives the solution required.

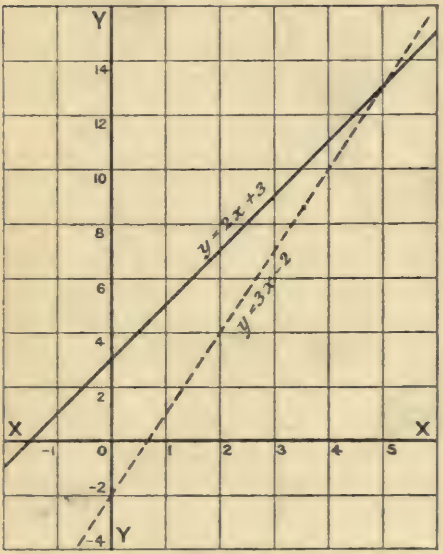


Fig. 51.

Of course no one would use the graphical method for the solution of a simple equation, but the method is most clearly understood by reference to the straight-line case.

In order to be able to read off the co-ordinates of the point of intersection as accurately as possible, the scales should be chosen so that the lines cut one another at an angle as nearly approximating to a right angle as possible. Obviously, if the angle between the lines is very acute the exact point of intersection is difficult to define. A little practice enables the scales to be chosen almost instinctively, but if the angle between the lines is found, upon plotting, to be very acute, the curves should be replotted to a more suitable scale. By enlarging the scales and replotting the portion of the curves near to their point of intersection, the result may be obtained to any desired degree of accuracy, as will be seen in the further examples given below.

General Statement of the Methods.—We may now proceed to state in general terms the two methods.

1. Arrange the equation in the form

$$f(x) = 0$$

Form a second equation by equating the left-hand side of this to y, so obtaining

$$y = f(x)$$

Plot this second equation, then the x co-ordinates of its points of intersection with the axis of X give the required roots of the first equation.

2. The alternative method, corresponding to the solution of the simultaneous equations given above, is very often more convenient than the first method. It may be described thus—

Arrange the equation which is to be solved in the form

$$f(x) = \mathrm{F}(x)$$

choosing such a distribution of the terms over the two sides of the equation as may make the necessary calculations of the co-ordinates for plotting as simple as possible.

Equate each of these functions x in turn to y, so obtaining a pair of simultaneous equations represented by

$$y = f(x)$$

and $$y = \mathrm{F}(x)$$

Plot these two curves separately on the same sheet of paper. Then the x co-ordinates of their points of intersection give the required roots of the original equation.

Whichever method be adopted, the curves should be plotted on a fairly small scale at first, in order to determine the points of intersection approximately, and then the portions of the curves near these points should be plotted on a large scale in order to find the actual values of the roots, as described above. If still greater accuracy be required, they may be still further enlarged, as may be required.

Examples.—A few examples should suffice to make the methods perfectly clear.

Example 1.—It is required to solve the equation—

$$x^3 + 4 = 3x$$

We will solve this equation by each method in turn, in order to show more clearly the difference of procedure between them.

Method 1.—By transposition of the terms we obtain

$$x^3 - 3x + 4 = 0$$

Then it is necessary to plot the equation

$$y = x^3 - 3x + 4$$

To obtain the points for plotting we have the following table :—

x	-4	-3	-2	-1	0	$+1$	2	3	4
x^3	-64	-27	-8	-1	0	$+1$	8	27	64
$-3x$	$+12$	$+9$	$+6$	$+3$	0	-3	-6	-9	-12
$y = x^3 - 3x + 4$	-48	-14	$+2$	6	4	2	6	22	56

We see from this table that one point of intersection with the axis of X will be between $x = -3$ and $x = -2$, since the value of y changes sign between these two values of x. But we know that for a cubic equation there will be *three* roots, all of which *may* be real, or *two* of which may be imaginary. Are there other two points of intersection of this curve with the axis of X? From our knowledge of the form of a cubic parabola we know that there will certainly be no such points outside the range which we have calculated. But we notice that there is a downward tendency of the curve, followed by an upward return between the values of x, $x = -1$ and $x = +2$. There may be, then, a small portion of this curve dropping below the axis of X again; but

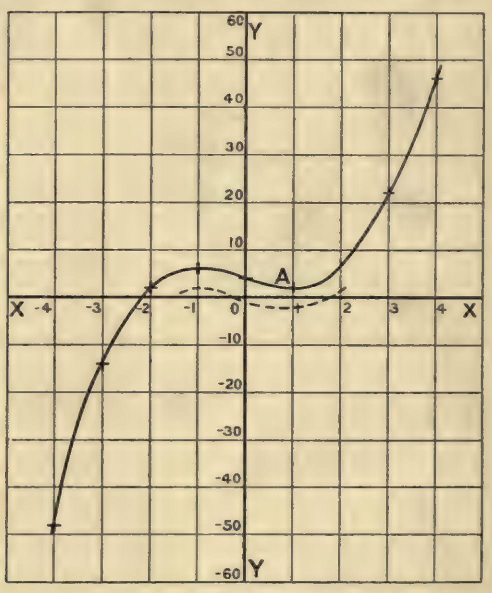


Fig. 52.

judging from the general run of the figures, it appears to be improbable. We may now plot the curve, as shown in Fig. 52. From this we see clearly that there is only one real root of the equation, which lies between $-2{\cdot}3$ and $-2{\cdot}1$.

We may notice in passing that this result throws considerable light on a point which frequently causes difficulty to the student of algebra, that is, the meaning of "imaginary roots" of an equation. If we imagine the minimum point of the curve A to become lower until it is below the axis of X, say as shown by the dotted curve (this, of course, would involve an alteration in the equation), it is obvious that the curve would then intersect the axis of X in two other points, thus giving in all three real roots to the equation. If the curve were tangential to the axis of X, these two points would coincide, that is, the two roots would be equal; while in the actual position of the curve it just fails to cut the axis of X, and these two roots are therefore imaginary. It is readily seen, moreover, that in order to make these two roots real, the constant term of the equation must be less, that is, the curve as a whole must be lowered. The greatest value of this term, in order that the roots may be real, would be $4 - 1{\cdot}8 = 2{\cdot}2$, that is, the equation of the curve would be

$$y = x^3 - 3x + 2{\cdot}2$$

or the equation whose roots would be found would have been

$$x^3 - 3x + 2{\cdot}2 = 0$$

in which case the roots would be equal, while if this term were still smaller the roots would be unequal, that is, there would be three real roots in all.

Returning now to our one real root, we must plot the portion of the curve lying between $x = -2{\cdot}3$ and $x = -2{\cdot}1$ on a larger scale, in order to find the roots accurately.

x	−2·30	−2·28	−2·26	−2·24	−2·22	−2·20	−2·18	−2·16	−2·14	−2·12	−2·10
x^3	−12·18	−11·88	−11·54	−11·23	−10·95	−10·66	−10·38	−10·09	−9·80	−9·54	−9·26
$-3x$	+6·90	+6·84	+6·78	+6·72	+6·66	+6·60	+6·54	+6·48	+6·42	+6·36	+6·30
$y = x^3 - 3x + 4$	−1·28	−1·04	−0·76	−0·51	−0·29	−0·06	+0·16	0·39	0·62	0·82	1·04

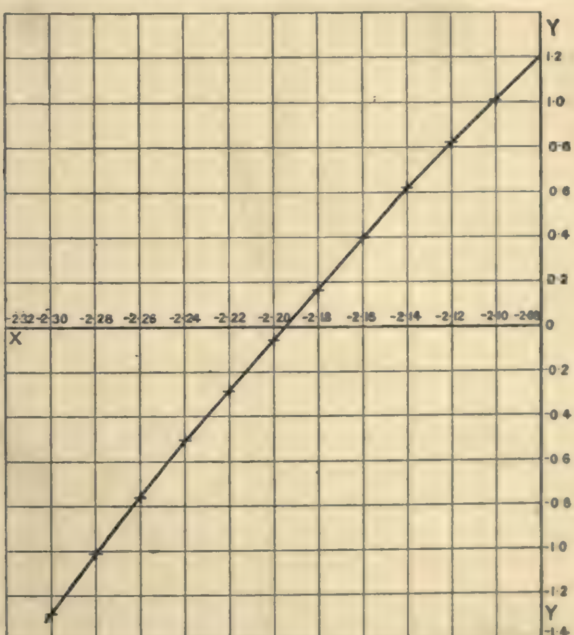


Fig. 53.

We see from these values of the co-ordinates that the required root is between $-2{\cdot}20$ and $-2{\cdot}18$. It is, however, necessary to plot a few points on each side of these values in order to guide us as to the shape of the curve. This portion of the curve is plotted on a large scale in Fig. 53, giving us as our required root the value $x = -2{\cdot}195$. If greater accuracy were required, the method could be extended by plotting on a still larger scale the part of the curve lying between $x = -2{\cdot}194$ and $-2{\cdot}196$, but this is very rarely necessary.

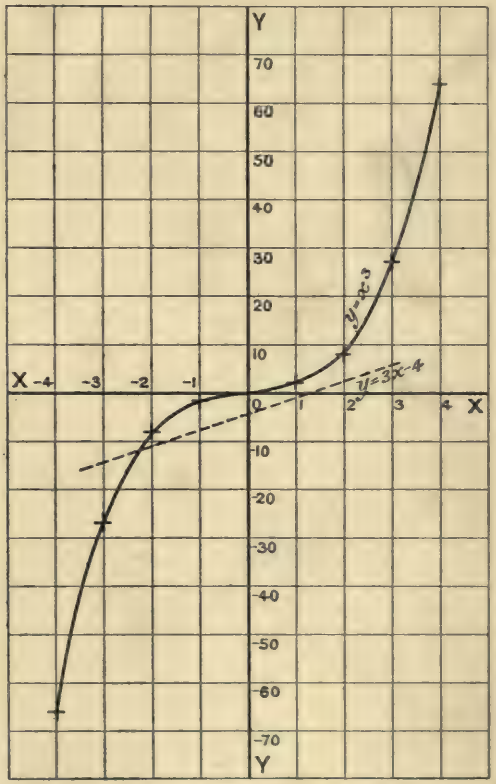


Fig. 54.

Method 2.—To solve the same equation by the second method we will arrange it in the form

$$x^3 = 3x - 4$$

Then forming the two simultaneous equations we obtain

$$y = x^3$$

and $$y = 3x - 4$$

The first of these is the equation of the simple cubic parabola, the figures for plotting which will be found on p. 22, and the second represents a straight line making an intercept equal to -4 on the axis of Y and of slope 3. These two are plotted in Fig. 54, whence it is seen as before that their only real point of intersection, that is the only real root of the original equation, lies between $x = -2 \cdot 1$ and $x = -2 \cdot 3$. In order to obtain the required root more exactly, this portion of both curves must be replotted on a larger scale, as shown in Fig. 55. The values for x^3 will be found in the second line of the table on p. 81, from which the portion of the cubic parabola is

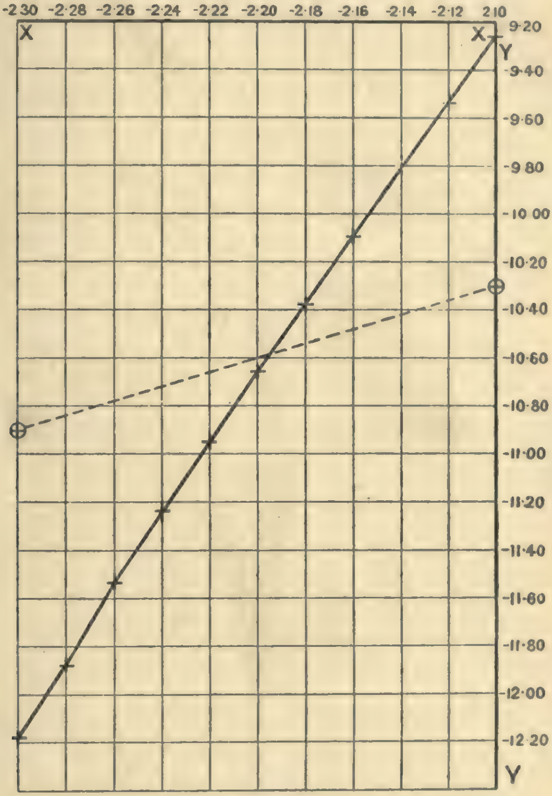


Fig. 55.

plotted; while in order to draw the straight line we must obtain two points upon it, say when $x = -2 \cdot 1$ and when $x = -2 \cdot 3$ respectively; then for these two points we have

$$y = 3x - 4 = -6 \cdot 3 - 4 = -10 \cdot 3$$

and

$$y = 3x - 4 = -6 \cdot 9 - 4 = -10 \cdot 9$$

respectively. By comparison of these extreme values of y for the straight line with those of x^3 we see that it is only necessary to plot the cubic parabola between $x = -2 \cdot 24$ and $x = -2 \cdot 16$. Then the point of intersection of the two curves is seen to be when $x = -2{\cdot}195$, which is the required root as before.

Example 2.—Taking one more example of a rather more complicated nature, we will solve the equation—

$$3x^2 - 20 \log_{10}x - 7{\cdot}077 = 0$$

The second method will be the most convenient in this case, plot-

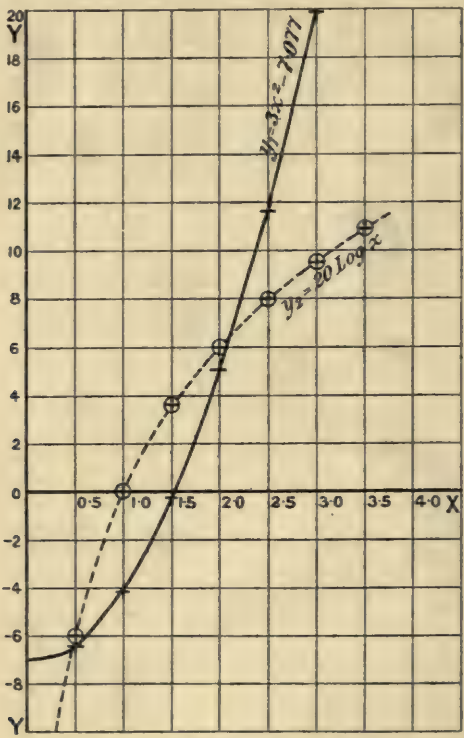


Fig. 56.

ting a parabola and a logarithmic curve. The equation is arranged in the form

$$3x^2 - 7{\cdot}077 = 20 \log_{10}x$$

Then the equations of the two curves to be plotted will be

$$y_1 = 3x^2 - 7{\cdot}077$$

and

$$y_2 = 20 \log_{10}x$$

Since the logarithmic curve given by the second equation lies wholly on the positive side of the axis of Y, it is only necessary to determine points on both curves for which x is positive. For the first small-scale approximation, we will increase x by successive increments of $0{\cdot}5$, obtaining the values of the co-ordinates for the two curves given in the table below.

x	0·0	0·5	1·0	1·5	2·0	2·5	3·0	3·5
x^2	0·00	0·25	1·00	2·25	4·00	6·25	9·00	12·25
$3x^2$	0·00	0·75	3·00	6·75	12·00	18·75	27·00	36·75
$y_1 = 3x^2 - 7{\cdot}077$	−7·077	−6·327	−4·077	−0·327	+4·923	11·673	19·923	29·673
$\text{Log}_{10}x$ . .	−∞	−0·3010	0·0000	+0·1761	0·3010	0·3979	0·4771	0·5441
$y_2 = 20 \log_{10}x$	−∞	−6·020	0·000	+3·522	6·020	7·958	9·542	10·882

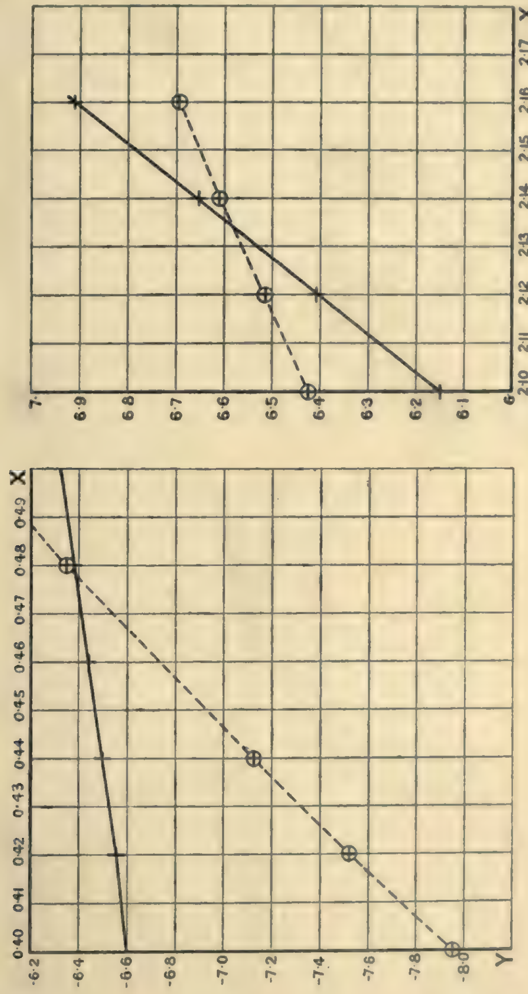


Fig. 57.

From this table, comparing the values of y_1 with those of y_2, it is seen that there will be two points of intersection of the curves when they are plotted, that is, there will be two values of x which will satisfy the original equation. These two values of x will be respectively between 0 and 0·5, and between 2·0 and 2·5. Upon plotting the two curves, as in Fig. 56, these values of x are obtained with a greater degree of accuracy, as the points of intersection are found to lie respectively between $x = 0·4$ and $x = 0·5$, and between $x = 2·1$ and $x = 2·2$. It becomes necessary, then, to replot the two portions of the curves lying between these values of x on a larger scale. The necessary figures are tabulated below.

x . . .	0·40	0·42	0·44	0·46	0·48	0·50	2·10	2·12	2·14	2·16	2·18	2·20
x^2 . .	0·1600	0·1764	0·1936	0·2116	0·2304	0·2500	4·4100	4·4944	4·5796	4·6656	4·7524	4·8400
$3x^2$ . .	0·480	0·529	0·581	0·635	0·691	0·750	13·230	13·483	13·739	13·997	14·257	14·520
$y_1 = 3x^2 - 7·077$	−6·597	−6·548	−6·496	−6·442	−6·386	−6·327	+6·153	6·406	6·662	6·920	7·180	14·443
$\log_{10} x$	−0·3979	−0·3768	−0·3565	−0·3372	−0·3188	−0·3010	+0·3222	0·3263	0·3304	0·3345	0·3385	0·3424
$y_2 = 20 \log_{10} x$	−7·958	−7·536	−7·130	−6·744	−6·376	−6·020	+6·444	6·526	6·608	6·690	6·770	6·848

Upon plotting these values, as in the two portions of Fig. 57, the values of x at the points of intersection, that is the roots of the original equation, may be read off to three places of decimals, and are found to be respectively

$$x = 0·479 \quad \text{and} \quad x = 2·134.$$

CHAPTER IX

THE SLOPE OF A CURVE—DIFFERENTIATION

We must now proceed to study the questions involved in the consideration of the measurement and meaning of the slope of a curve. In Chapter III it was shown that the slope of a straight line was measured by the tangent of the angle it made with the positive direction of the axis of X. That is, referring to Fig. 58, the slope of the line PQ is equal to $\tan\theta$, that is to $\frac{PM}{QM}$. If the co-ordinates of P be (x_1, y_1) and those of Q be (x_2, y_2), then

$$PM = y_1 - y_2 \qquad \text{and} \qquad QM = x_1 - x_2$$

or, the slope of the line $= \frac{y_1 - y_2}{x_1 - x_2}$.

It is convenient to express a change in the value of x by the symbol δx, and the corresponding change in y by δy, so that $\delta x = x_1 - x_2$ and $\delta y = y_1 - y_2$. Then we have the slope of the line $= \frac{\delta y}{\delta x}$.

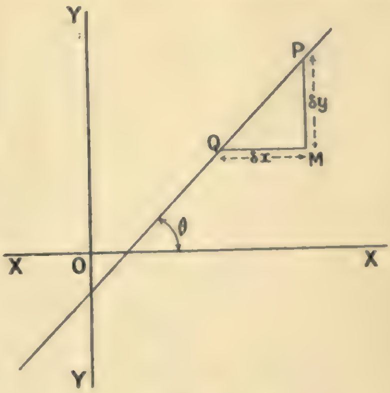


Fig. 58.

It must be very carefully borne in mind that the symbol δx does *not* mean some quantity represented by δ multiplied by x. The symbol is one complete entity, and simply represents, as stated above, a change, usually a small change, in the value of x.

Measurement of the Slope of a Curve.—The slope of a straight line is constant, that of a curve is constantly changing as the curve is traced out. It becomes necessary, then, to see how the slope of a curve at any given point may be determined. If two points, P and Q, be taken near together upon a curve (Fig. 59), the slope of the chord or straight line joining them will obviously give a measure of the *average* slope of the curve between these two points. Hence

$$\text{the average slope of the arc PQ} = \frac{PM}{QM} = \frac{\delta y}{\delta x}.$$

Now imagine the point Q gradually to travel along the curve towards P. The ratio $\frac{\delta y}{\delta x}$ will change as it does so, and the arc PQ becoming smaller, the average slope of the arc will become more and more nearly equal to the actual slope at the point P. The two increments δy and δx themselves become smaller, but their ratio $\frac{\delta y}{\delta x}$ may change very little, and may become greater, as it obviously does in the case of the curve in Fig. 59. Now imagine Q to coincide with P. Then the chord PQ (produced) becomes the tangent to the curve at the point P. But its slope still measures the average slope of the arc PQ, which is now the actual slope of the curve at the point P, the arc PQ having become infinitely small. Hence, *the slope of a curve at any point is measured by the slope of the tangent drawn to the curve at that point.* Now, in this limiting case, when P and Q are coincident, δy and δx have both become infinitely small. But the ratio $\frac{\delta y}{\delta x}$ measures the slope of the line PQ, hence

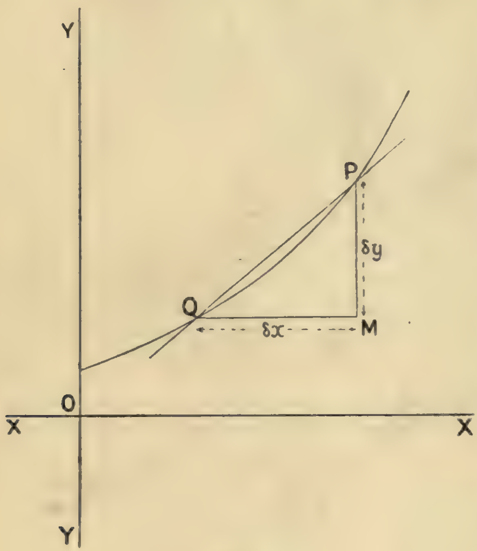


Fig. 59.

the limiting value of the ratio $\frac{\delta y}{\delta x}$ when δy and δx have themselves become infinitely small has a *real finite value*, namely, the slope of the tangent to the curve at the point P. This limiting value of the ratio $\frac{\delta y}{\delta x}$ when δx (and therefore δy also) is infinitely small is written $\frac{dy}{dx}$ and is called the "*Differential Coefficient of y relatively to x.*" Note, that just as δx is one complete symbol representing a change in x and cannot be split up into separate parts, so the symbol $\frac{dy}{dx}$ is also one complete symbol representing one definite thing, namely, the limiting value of the ratio $\frac{\delta y}{\delta x}$, or the differential coefficient of y relatively to x, and must *not* be thought of as the quotient of two separate quantities, dy and dx. The process of evaluating the quantity $\frac{dy}{dx}$ is known as "*differentiating*" y relatively to x.

Physical Meaning of the Slope of a Curve.—In either of the cases shown in Figs. 58 and 59, the quantity δy, that is PM, measures the change in y while x is changing by an amount δx, that is QM. Hence the ratio $\frac{\delta y}{\delta x}$ gives the ratio of the change in y to that in x, or it measures the average rate of change of y relatively to x. But the quantity $\frac{dy}{dx}$ is only a limiting value of the ratio $\frac{\delta y}{\delta x}$. Hence, $\frac{dy}{dx}$ *measures the rate of change of y relatively to x.* That is to say, the slope of a curve at any point, measured by the value of the differential coefficient of y relatively to x, $\frac{dy}{dx}$, measures the rate of change of y relatively to x.

A mechanical example will perhaps serve to make this clear. The velocity of a particle is defined to be its time-rate of change of position, and is measured by dividing the distance covered by the time taken to cover it, that is

$$v = \frac{\delta s}{\delta t}$$

where v = the velocity, s = the displacement from some fixed point, and t = the time elapsed from some given instant. If the velocity be not uniform, this measures the average velocity during the interval of time δt. The velocity at a given instant is the limiting value of this average velocity when the interval of time δt becomes infinitely small. Hence, we have that the velocity at any given instant—

$$v = \frac{ds}{dt}$$

Let the curve in Fig. 60 represent the relation between the distance covered, s feet, in t seconds from rest. Then the velocity at any given instant may be determined by drawing a tangent to the curve at the corresponding point and measuring its slope. For example, the velocity after 5 seconds is found by drawing a tangent at the point P (for which $t = 5$). The slope of this tangent is given by—

$$\text{slope} = \frac{\text{RM}}{\text{QM}} = \frac{6{\cdot}05 - 4{\cdot}25}{1{\cdot}0} = 1{\cdot}80$$

or the velocity after 5 seconds is 1·80 feet per second. For the case represented by the curve in Fig. 60, the velocity has been found at the end of every second, and the results are tabulated below, the table also, for convenience, showing the distances travelled up to the end of each second.

t . .	0	1	2	3	4	5	6	7	seconds.
s . .	0·0	2·55	3·50	3·75	4·05	5·15	7·5	11·3	feet.
$v=\frac{ds}{dt}$	5·20	1·20	0·58	0·00	0·60	1·80	3·00	4·80	feet per second

The velocities so found are plotted in Fig. 61 with the times at which they occur, so obtaining a velocity-time curve.

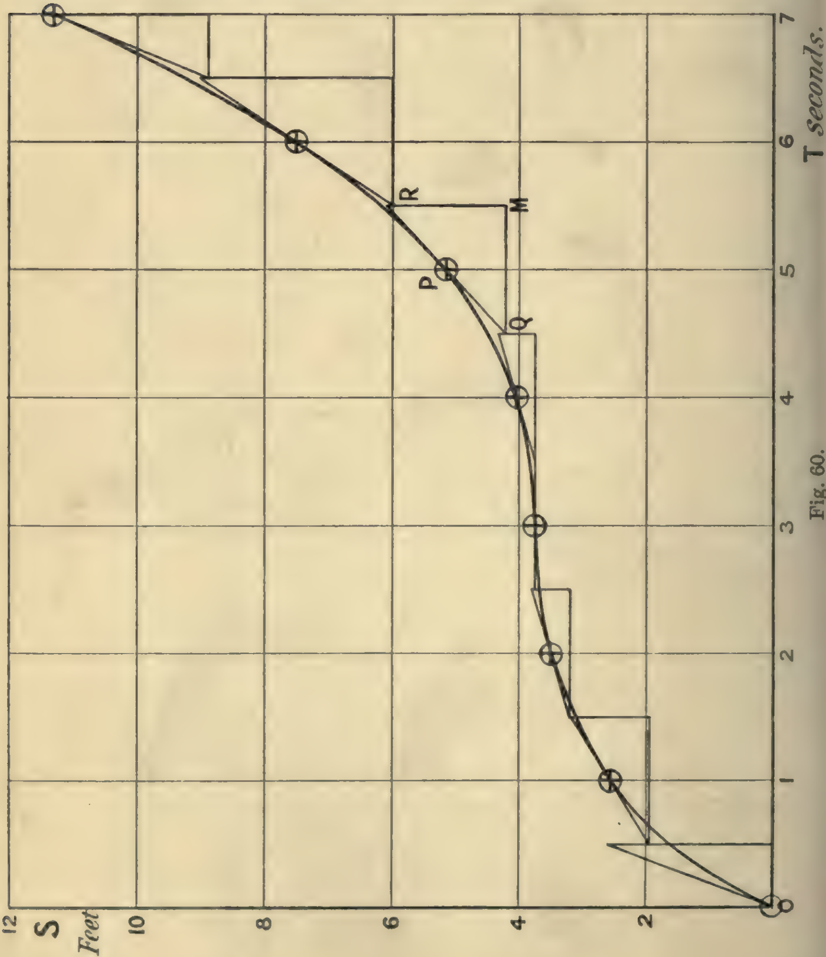


Fig. 60.

Successive Differentiation.—The slope of this curve (Fig. 61) measures the rate of change of velocity, that is, the acceleration, f. Hence we have—

$$f = \frac{dv}{dt}$$

But $$v = \frac{ds}{dt}$$

Hence $$f = \frac{d\left(\frac{ds}{dt}\right)}{dt}$$

which is usually written as

$$f = \frac{d^2s}{dt^2}$$

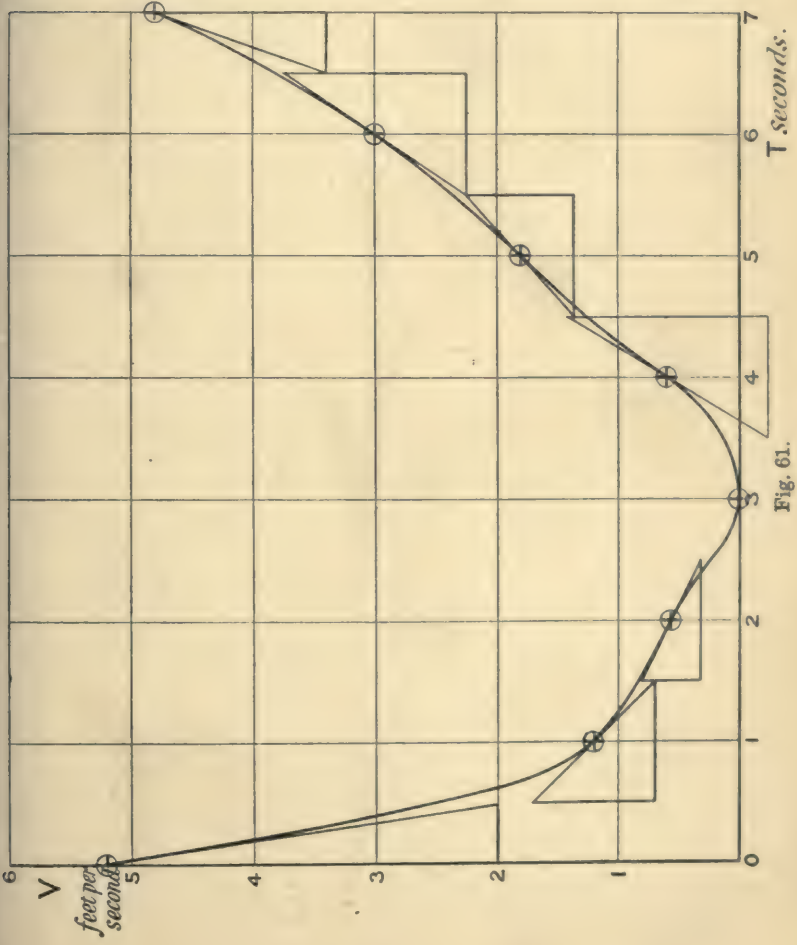


Fig. 61.

This is, of course, one complete symbol, and represents that s is to be differentiated *twice* with respect to t, and is called the "second differential coefficient of s relatively to t." The value of $\frac{d^2s}{dt^2}$, that is, of the acceleration, is found at the end of each second by drawing the tangent to the velocity-time curve and measuring its slope as before, the results being given in the table below.

t . .	0	1	2	3	4	5	6	7	seconds.
$f=\frac{d^2s}{dt^2}$	−6·40	−1·00	−0·50	0·00	+1·66	0·90	1·48	2·80	feet per second per second.

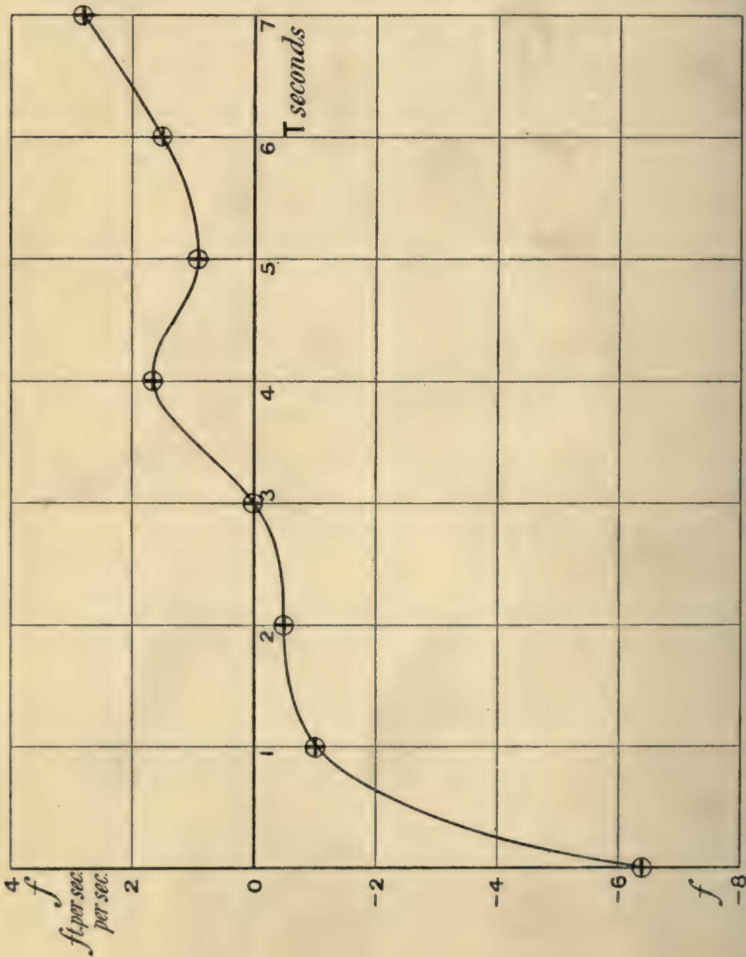


Fig. 62.

From these values the acceleration-time curve shown in Fig. 62 is plotted. The slope of this would measure the rate of change of acceleration and would be expressed by the third differential coefficient of s relatively to t, which would be written $\frac{d^3s}{dt^3}$.

Methods of Differentiation.—We have seen above how to proceed graphically to find the value of $\frac{dy}{dx}$ for any given function of x, which we have called y; that is, by plotting the curve connecting x and y, drawing the tangent to the curve at the required point, and measuring its slope. This process may be called "Graphical Differentiation."

If, however, y is a regular function of x, another method is available. If the form of $f(x)$, or y, is known in terms of x, the general form of $\frac{dy}{dx}$ may be determined, and its value ascertained for any given value of x by substitution of that value in the general expression. We proceed to deduce, by the graphical process, the form of $\frac{dy}{dx}$ for a few simple forms of the function y. Rigid proofs of these results will be found in any elementary book on the Differential Calculus.

Differentiation of x^n.—(1) When $n = 1$, we have $y = x$, which is the equation of a straight line of unit slope. Hence

$$\frac{dy}{dx} = 1$$

which may be written
$$\frac{dy}{dx} = 1 \times x^0$$

(2) When $n = 2$, we have $y = x^2$, which is the equation of a parabola. This parabola is plotted in Fig. 63 from the values of x and y on p. 21. Tangents are drawn to the curve at various points, and their slopes, that is the values of $\frac{dy}{dx}$ at those points, determined, the results being tabulated below.

x	-4	-3	-2	-1	0	$+1$	2	3	4
$\frac{dy}{dx}$	-8	-6	-4	-2	0	$+2$	4	6	8

Upon plotting $\frac{dy}{dx}$ with x we obtain a straight line, whose equation is readily seen to be

$$\frac{dy}{dx} = 2x$$

(3) When $n = 3$, we have $y = x^3$, which is the equation of the cubic parabola plotted in Fig. 64 (continuous line) from the values of x and

y on p. 22. Proceeding in the same manner as in the last case, we obtain the following values of $\frac{dy}{dx}$:—

x	-3	-2	-1	0	$+1$	2	3
$\frac{dy}{dx}$	27	12	3	0	3	12	27

Upon plotting these values with x we obtain the curve shown in the same figure by the broken line, which appears to be a parabola.

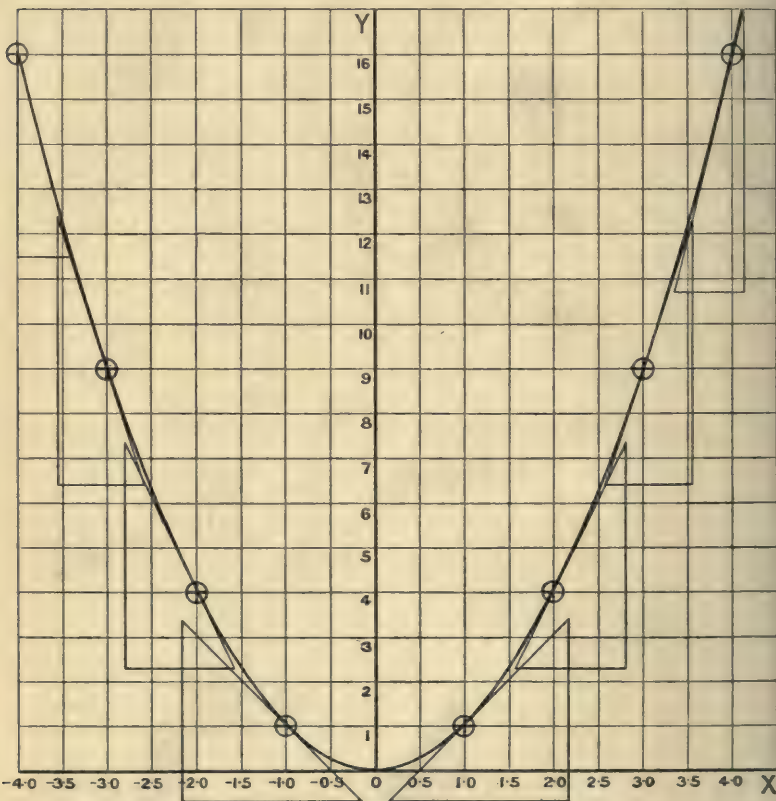


Fig. 63.

The equation of this parabola may be determined by plotting log x with log $\left(\frac{dy}{dx}\right)$. This is, in this case, however, scarcely necessary, as obviously for any value of x,

$$\frac{dy}{dx} = 3x^2$$

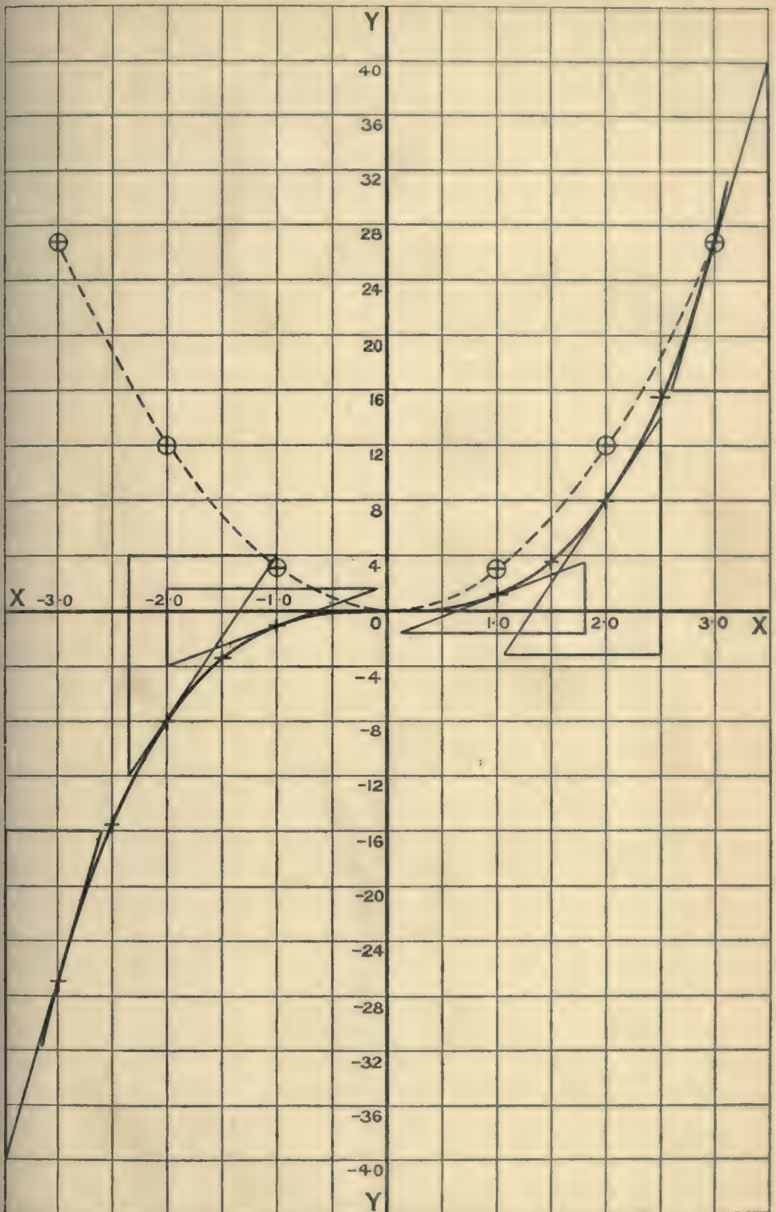


Fig. 64.

The student should proceed in this way for the further cases when n is equal to 4, 5, etc., in each case plotting the curve $y = x^n$, drawing tangents at several points upon it, measuring their slope, and re-plotting the values of $\frac{dy}{dx}$ so obtained with those of x. In each case a parabolic curve will result, whose equation must then be obtained by plotting logarithmically. He will find that when—

$$y = x, \qquad \frac{dy}{dx} = 1x^0$$

$$y = x^2, \qquad \frac{dy}{dx} = 2x^1$$

$$y = x^3, \qquad \frac{dy}{dx} = 3x^2$$

$$y = x^4, \qquad \frac{dy}{dx} = 4x^3$$

$$y = x^5, \qquad \frac{dy}{dx} = 5x^4, \text{ etc.}$$

From these the general result may be deduced as

$$\text{when } y = x^n, \qquad \frac{dy}{dx} = nx^{n-1}$$

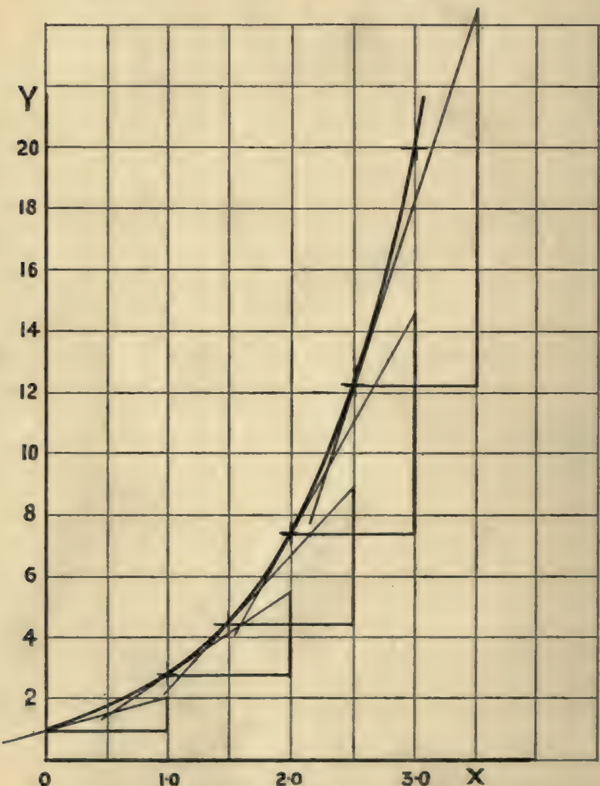


Fig. 65.

When n is negative, the curves obtained by plotting are, of course, hyperbolic instead of parabolic. The student should test whether the same result holds good in this case, proceeding in the same manner.

Differentiation of e^x.—In Fig. 65 the curve $y = e^x$ is plotted from the figures below.

x	0·0	1·0	1·5	2·0	2·5	3·0
e^x. . . .	1·00	2·72	4·48	7·35	12·2	20·0

Tangents are drawn to it, and their slopes measured. It is found that the slope of the curve at any point is exactly equal to the value of y at that point. That is, if the values of $\frac{dy}{dx}$ be plotted with the corresponding values of x, the resultant curve is identical with the original curve $y = e^x$. Hence,

$$\frac{dy}{dx} = y = e^x$$

Differentiation of Log$_e$ x.—Proceeding as in the former cases, the curve $y = \log_e x$ is plotted (Fig. 66). The values of x and y for this

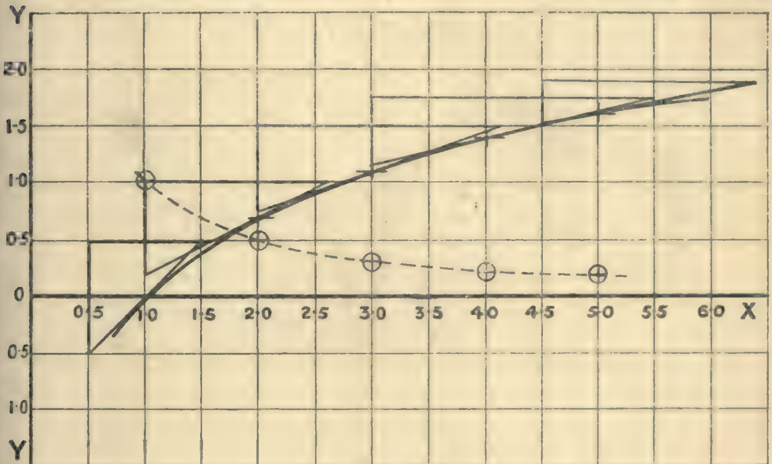


Fig. 66.

curve are given below. The slopes of the tangents at various points are measured, the results being given in the third line below.

x	1·0	2·0	3·0	4·0	5·0
$y = \log_e x = 2{\cdot}303 \log_{10} x$	0·000	0·694	1·100	1·388	1·610
$\frac{dy}{dx}$	1·00	0·50	0·33	0·25	0·20

On plotting these values of $\frac{dy}{dx}$ with those of x, the curve represented in Fig. 66 by the broken line results. This appears to belong to the hyperbolic family. To test this, the values are plotted on logarithmic paper in Fig. 67, when it is seen that this is so, for a straight line results. The slope of this line is found on measurement to be unity, and is negative. Hence the equation of the curve is

$$x \times \frac{dy}{dx} = 1$$

or

$$\frac{dy}{dx} = \frac{1}{x}$$

Differentiation of Sin x and Cos x.—When differentiating trigono-

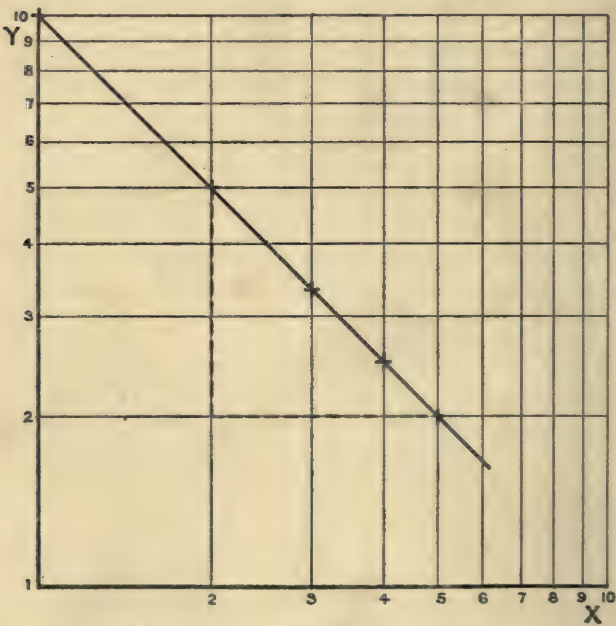


Fig. 67.

metrical functions the angle must always be understood to be measured in natural units, that is, in radians. Proceeding to plot the function $y = \sin x$, we obtain the curve shown by a continuous line in Fig. 68, the necessary figures for plotting it being given below. For convenience in finding the values of $\sin x$, the points have been taken at intervals of $\frac{\pi}{6} = 0{\cdot}523$, that is, $30°$.

x radians . . .	0·00	0·52	1·05	1·57	2·09	2·62	
Equivalent angle in degrees	0	30	60	90	180 − 60	180 − 30	
$y = \sin x$ . . .	0·00	0·50	0·87	1·00	0·87	0·50	
x radians . . .	3·14	3·66	4·29	4·71	5·23	5·75	6·28
Equivalent angle in degrees	180	180 + 30	180 + 60	270	360 − 60	360 − 30	360
$y = \sin x$ . . .	0·00	−0·50	−0·87	−1·00	−0·87	−0·50	0·00

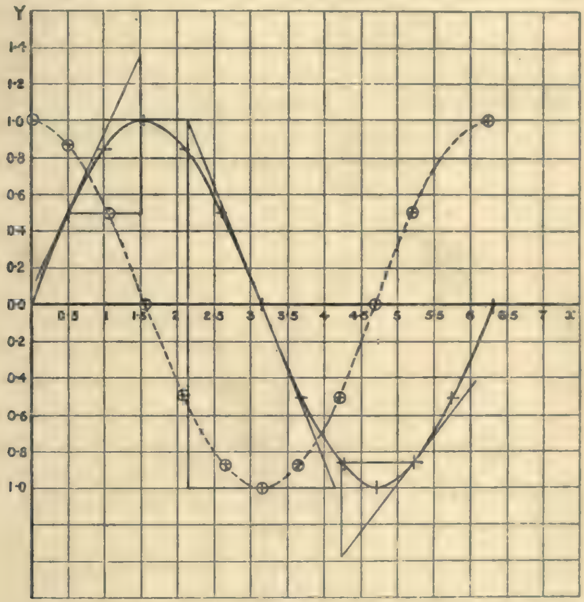


Fig. 68.

Tangents are drawn to the curve as usual, and their slopes measured. The results obtained are tabulated below.

x . . .	0·00	0·52	1·05	1·57	2·09	2·62	3·14	3·66	4·29	4·71	5·23	5·75	6·28
$\frac{dy}{dx}$ . . .	1·00	0·87	0·50	0·00	−0·50	−0·87	−1·00	−0·87	−0·50	−0·00	+0·50	+0·87	1·00

These values of $\frac{dy}{dx}$ being plotted with those of x give the curve shown in the same figure by the dotted line, which is obviously another sine curve of the same frequency and amplitude as the first one, but differing from it in phase by 90° or $\frac{\pi}{2}$. Its equation is therefore

$$\frac{dy}{dx} = \sin\left(x + \frac{\pi}{2}\right) = \cos x$$

Similarly, if this curve be graphically differentiated, another sine curve will result, whose phase will be 90° ahead of this one. That is

$$\frac{d(\cos x)}{dx} = \sin(x + \pi) = -\sin x$$

The student should, of course, actually work through this in order to test the truth of the statement.

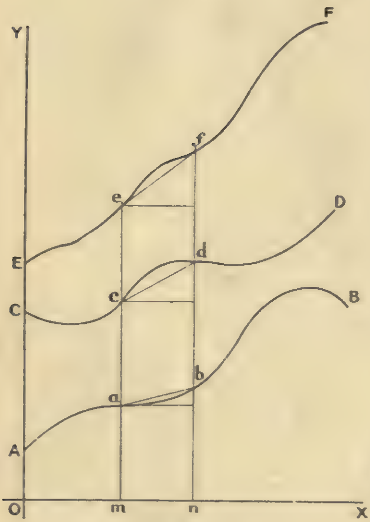


Fig. 69.

Differentiation of $a \cdot f(x)$.—It has been seen already that the only effect of introducing a constant a into any equation of the form $y = f(x)$, so obtaining the equation $y = a \cdot f(x)$, is to increase, as it were, the vertical scale to which the original curve was plotted a times. Hence the slope at any point will also be increased a times, or

$$\frac{d(ay)}{dx} = a \cdot \frac{dy}{dx}$$

Differentiation of a Sum.—If two curves AB and CD (Fig. 69) be combined by adding their ordinates, so obtaining the curve EF, the slope of the latter curve at any given distance from the axis of Y (that is, for a certain value of x) will be the sum of the slopes of the two former curves at the corresponding points. For consider the chords of the curves ab, cd, and ef. We have

$$fn = dn + bn$$

and $$em = cm + am$$

Therefore $$fn - em = dn + bn - cm - am$$
$$= (dn - cm) + (bn - am)$$

And therefore—

$$\frac{fn - em}{mn} = \frac{dn - cm}{mn} + \frac{bn - am}{mn}$$

But $\frac{fn - em}{mn}$ measures the slope of ef

$\frac{dn - cm}{mn}$ " " cd

$\frac{bn - am}{mn}$ " " ab

Therefore the slope of ef is the sum of the slopes of cd and ab. Hence, also, for the limiting values of the chords when e becomes coincident with f, c with d, and a with b, the slope of the curve EF at f is equal to the sum of the slopes of CD at d, and AB at b. Thus we see that the differential coefficient of a sum is equal to the sum of the differential coefficients of the terms; or, if

$$y = u + v - w$$

Then
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$$

Examples.—(1) Differentiate the expression $3x^{4\cdot1} - 4x^{2\cdot5}$

Here
$$y = 3x^{4\cdot1} - 4x^{2\cdot5}$$
$$\therefore \frac{dy}{dx} = 3 \times \frac{d(x^{4\cdot1})}{dx} - 4 \times \frac{d(x^{2\cdot5})}{dx}$$
$$= 3 \times 4\cdot1 \times x^{4\cdot1-1} - 4 \times 2\cdot5 \times x^{2\cdot5-1}$$
$$= 12\cdot3\,x^{3\cdot1} - 10x^{1\cdot5}$$

(2) A body moves so that its distance, s feet, from a fixed point, t seconds after it was at that point, is given by $s = 3 \sin t$. (Notice that this is a simple harmonic motion of amplitude 3 feet, frequency 1, and of the same phase as the fundamental.) Find its velocity and acceleration after 0·2 second.

We have
$$v = \frac{ds}{dt} = 3\,\frac{d(\sin t)}{dt} = 3 \cos t$$

When $t = 0\cdot2$
$$v = 3 \cos (0\cdot2)$$
$$= 3 \cos (0\cdot2 \times 57\cdot73)^0$$
$$= 3 \cos 11^0\cdot46$$
$$= 3 \times 0\cdot980 = 2\cdot64 \text{ feet per sec.}$$

Again
$$f = \frac{dv}{dt} = 3\,\frac{d(\cos t)}{dt}$$
$$= -3 \sin t$$
$$= -3 \sin 11^0\cdot46$$
$$= -3 \times 0\cdot198 = -0\cdot594 \text{ feet per sec. per sec.}$$

Some interesting points may be noticed which incidentally follow from this latter example. In the first place, if either the displacement, velocity, or acceleration of a body moving with simple harmonic motion be plotted as a function of the time, a sine curve results. In the second place, we have that

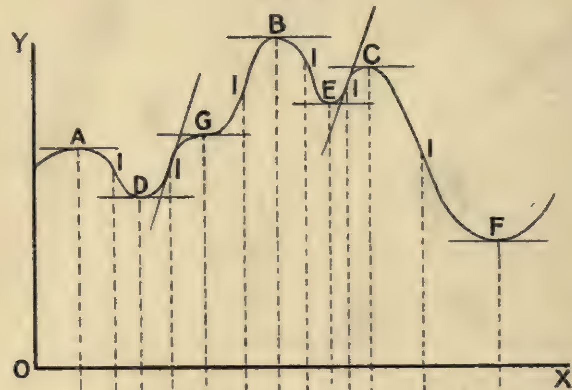
$$s = 3 \sin t$$

and
$$f = -3 \sin t,$$

that is, the acceleration is here numerically equal to the displacement, but of opposite sign. This is not necessarily the case, but it is true that the acceleration varies directly as the displacement from mid-position.

Maxima and Minima.—If y, a function of x, increases as x increases up to a certain value, and then begins to decrease, that value is said to be a *maximum value* of the function, and, similarly, if it decreases as x increases down to a certain value, and then begins to increase, that value is said to be a *minimum value.* Thus, in Fig. 70, the values of y at A, B, and C are maximum values, and those at D, E, and F are minimum values. It should be noticed that the maximum values are

Fig. 70.



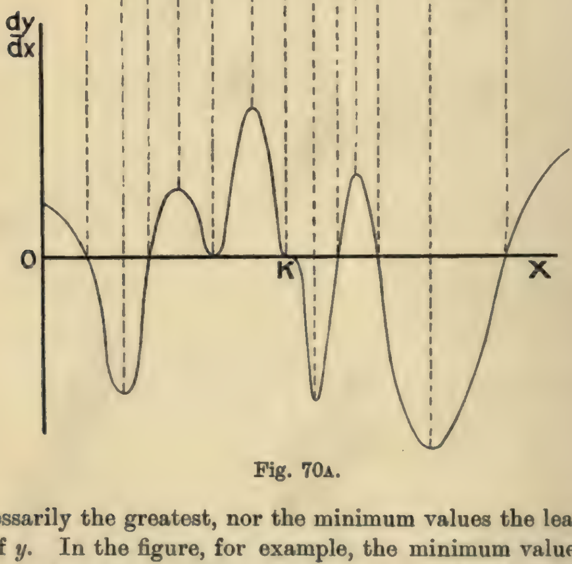


Fig. 70A.

not necessarily the greatest, nor the minimum values the least possible values of y. In the figure, for example, the minimum value of y at E is greater than the maximum value at A. Again, in the cubic parabola in Fig. 22, on p. 33, there is a maximum value at A and a minimum at B, but the curve rises to infinity in the first quadrant, and falls to minus infinity in the third, without any further maximum or minimum points, so that the maximum and minimum values at A and B respectively are by no means the greatest and least values possible. Obviously, at the point up to which y has been increasing and at which it begins to decrease, the tangent to the curve is horizontal, for the slope, that is $\frac{dy}{dx}$, changes its sign at that point from + to −, and therefore is equal to zero at the maximum. Similarly, the tangent is horizontal or $\frac{dy}{dx}=0$ at a minimum. This is illustrated in Fig. 70, in which the tangents at the maximum and minimum points are drawn. Hence, $\frac{dy}{dx}=0$ for any value of x which makes y a maximum or a minimum. It does not, however, follow that the converse statement is true, that is, that every value of x which causes $\frac{dy}{dx}$ to vanish necessarily corresponds to a maximum or minimum value of y. At the point G, for example, in Fig. 70, the tangent is horizontal, that is $\frac{dy}{dx}=0$, yet G is neither a maximum nor a minimum point. The value of y has increased up to G, ceases to increase further, and then begins to increase again, instead of to decrease as it would do were G a maximum. This is known as a *point of inflexion* at which the tangent is horizontal, or a horizontal point of inflexion. A point of inflexion in general may be defined as a point at which the curve changes its curvature from being convex towards one side to being concave towards that side. The tangent at a point of inflexion will obviously pass from one side of the curve to the other. In Fig. 70 there are points of inflexion at each of the points marked I, at one or two of which tangents have been drawn. The curve is steepest at the points of inflexion, unless, as at G, the slope is zero, in other words, the slope of the curve is a maximum or a minimum at a point of inflexion. These points may be seen more clearly by drawing the corresponding curve of $\frac{dy}{dx}$, as in Fig. 70A. From this it is seen that when y is a maximum or minimum, $\frac{dy}{dx}$ is zero, while when the curve of y is passing through a point of inflexion the value of $\frac{dy}{dx}$ is either a maximum or a minimum, that is, the slope of the $\frac{dy}{dx}$ curve, measured by $\frac{d^2y}{dx^2}$, is zero. When there is a horizontal point of inflexion (G in Fig. 70), the tangent at the corresponding maximum or minimum on the $\frac{dy}{dx}$ curve coincides with the axis of X. Hence, when $\frac{dy}{dx}$

is zero, the value of y is either a maximum or a minimum, *or* the curve of y is passing through a horizontal point of inflexion. If $\frac{d^2y}{dx^2}$ is zero, the curve of y is passing through a point of inflexion. Hence, if $\frac{dy}{dx}$ vanishes for any given value of x but $\frac{d^2y}{dx^2}$ does not vanish, the corresponding value of y is either a maximum or a minimum, but if *both* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ vanish, then the value of y is neither a maximum nor a minimum, but the curve is passing through a horizontal point of inflexion. This method of reasoning might be extended further, for if on the curve of $\frac{dy}{dx}$ there was a horizontal point of inflexion, at which the tangent was the axis of X, that is when $\frac{dy}{dx} = 0$, as at K in Fig. 70A, the value of $\frac{dy}{dx}$ would be neither a maximum nor a minimum, that is, this would not correspond to a point of inflexion on the curve of y, but to a maximum or minimum. We may express our results in general in the following form, a rigid proof of which will be found in any book on the differential calculus :—

If, for any value of x the successive differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc., are all equal to zero, the first differential coefficient which does *not* vanish being $\frac{d^ny}{dx^n}$, then if n is even, the value of y is either a maximum or a minimum, while if n is odd the value of y is neither a maximum nor a minimum, but corresponds to a horizontal point of inflexion on the curve of y.

It should be noticed that between every two consecutive maxima, there is always one minimum, and between every two consecutive minima there is always one maximum, or the maximum and minimum points on a curve occur alternately. Further, between any maximum and the next minimum, or *vice versâ*, there is *at least one* point of inflexion.

It remains now to determine a method of deciding whether any value of x which causes $\frac{dy}{dx}$ to vanish, but which does not cause $\frac{d^2y}{dx^2}$ to vanish, corresponds to a maximum or to a minimum value of y. At any maximum point the value of y, after increasing, begins to decrease, that is, the slope of the curve, or $\frac{dy}{dx}$ changes from positive to negative. Hence $\frac{dy}{dx}$ is diminishing, or $\frac{d^2y}{dx^2}$ is *negative*. Similarly, for any minimum value of y, $\frac{d^2y}{dx^2}$ is *positive*. Hence, we may sum up our conclusions thus :—

When $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is $-$, then y is a *maximum*.

When $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is $+$, then y is a *minimum*.

When $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, then y is neither a maximum nor a minimum, unless $\frac{d^3y}{dx^3}$ is also equal to 0.

An extension of this argument to the general case given above will show that the same result applies there also. That is, that if $\frac{d^ny}{dx^n}$ is the first differential coefficient which does not vanish, n being even, then y is a maximum or a minimum according as $\frac{d^ny}{dx^n}$ is negative or positive. In most practical cases, however, the conditions of the problem will make it obvious whether any value of y, corresponding to $\frac{dy}{dx} = 0$, is a maximum or a minimum.

Examples on Maxima and Minima.—(1) Find the values of x which give maximum and minimum values of the expression—

$$x^3 + x^2 - 8x + 15$$

Put
$$y = x^3 + x^2 - 8x + 15$$

Then
$$\frac{dy}{dx} = 3x^2 + 2x - 8$$

Then, for a maximum or minimum value of y,

$$3x^2 + 2x - 8 = 0$$

or
$$(3x - 4)(x + 2) = 0$$

i.e.
$$x = \tfrac{4}{3} \text{ or } -2$$

To see which of these values of x gives a maximum value of y, and which a minimum, we have, differentiating again—

$$\frac{d^2y}{dx^2} = 6x + 2$$

Then when $x = \frac{4}{3}$,

$$\frac{d^2y}{xd^2} = 6 \times \frac{4}{3} + 2 = 10$$

which, being positive, gives y a minimum value.

Again, when $x = -2$,

$$\frac{d^2y}{dx^2} = -12 + 2 = -10$$

which, being negative, gives y a maximum.

Hence, y is a maximum when $x = -2$, and a minimum when $x = \frac{4}{3}$.

(2) The strength of a beam of rectangular section varies directly as the breadth and the square of the depth. Find the dimensions of the strongest rectangular beam which can be cut from a cylindrical log of wood 1 foot in diameter.

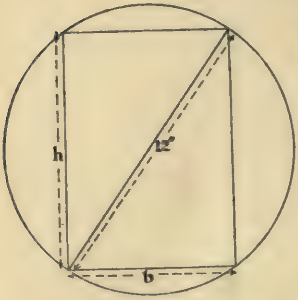


Fig. 71.

Here let Z = the strength modulus

then $\qquad Z = kbh^2$

where b is the breadth, and h the depth of the section.

But we have $\quad b^2 + h^2 = 144$ (see Fig. 71)

or $\qquad h^2 = 144 - b^2$

Hence, $\qquad Z = kb(144 - b^2)$

$$= 144kb - kb^3$$

then $\qquad \frac{dZ}{db} = 144k - 3kb^2$

Equating to zero for a maximum,

$$144k - 3kb^2 = 0$$

or $\qquad b^2 = \frac{144}{3} = 48$

i.e. $\qquad b = \sqrt{48} = 6{\cdot}94$ inches

and $\qquad h = \sqrt{144 - b^2} = \sqrt{144 - 48}$

$$= \sqrt{96} = 9{\cdot}80 \text{ inches.}$$

The above is the only possible value of b which causes $\frac{dZ}{db}$ to vanish, hence there is no doubt that this is the solution required to give the strongest possible beam, as the least value of the strength would be zero, when either b or h was made equal to zero.

Applications of the Method of Differentiation to Previous Work.— Some points in previous chapters may be shown very clearly by the method of differentiation. A few of these cases are further considered below.

In the comparison of the different curves represented by the general equation $y = x^n$, it was stated that a curve of greater index is steeper than one of less index for values of x greater than unity, that is above the common point (1,1) (see p. 24). Differentiating y relatively to x, we have

$$\frac{dy}{dx} = nx^{n-1}$$

Now, if x is greater than 1, x^{n-1} is greater the greater the value of n, hence $\frac{dy}{dx}$, or the slope of the curve is also greater, for a value of x greater than 1, for the greater value of n.

Again, on p. 32 it was shown that in the cubic parabola given by

$$y = ax^3 + bx + c$$

there were both a maximum and a minimum point if a and b were of opposite sign, but neither a maximum nor a minimum if they were of the same sign. Here

$$\frac{dy}{dx} = 3ax^2 + b$$

Equating this to zero for any maximum or minimum points,

$$3ax^2 + b = 0$$

or $$x^2 = -\frac{b}{3a}$$

i.e. $$x = \pm\sqrt{-\frac{b}{3a}}$$

Hence, x can only have real values to satisfy this equation when a and b are of opposite sign, that is, there can only be maximum and minimum points on the curve if a and b are of opposite sign. Further, there will be maximum or minimum values of y for each of these two values of x, for

$$\frac{d^2y}{dx^2} = 6ax$$

and this expression cannot be zero unless either a or x be zero, that is, the same value of x does not cause both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to vanish. If a and b are of opposite sign, then there are two values of x which give maximum or minimum values of y, and since maxima and minima always occur alternately, there must be one of each, the maximum occurring when x has the negative sign, and the minimum when x is positive.

The method of determining an equation of the type

$$y = ax^n + bx - c$$

(p. 40) may be shown to be correct by this method. It will be remembered that a tangent is drawn to the curve at its point of intersection with the axis of Y, and it was stated that the slope of this tangent would be equal to b, that is, the slope of the curve itself at this point is equal to b.

For the slope of the curve $= \frac{dy}{dx} = nax^{n-1} + b$

then, when $x = 0$, $\qquad \frac{dy}{dx} = b$

Again, in the hyperbolic family, it was stated on p. 53 that in the curves represented by the equation

$$pv^n = a$$

that of greater n is steeper than that of lesser n at their point of intersection.

Here $$p = av^{-n}$$

therefore $$\frac{dp}{dv} = a(-n)\,(v^{-n-1})$$
$$= -nav^{-n-1}$$

But $$a = pv^n$$

therefore $$\frac{dp}{dv} = -npv^n v^{-n-1}$$
$$= -npv^{n-n-1}$$
$$= -n\,.\,\frac{p}{v}$$

But at the point of intersection both p and v have the same values for both curves, therefore the slope is negative and varies directly as the value of n.

CHAPTER X

THE AREA OF A CURVE—INTEGRATION

Measurement of the Area of a Curve.—The area of a curve which does not itself form a closed figure, is usually understood as the area of the figure bounded by the curve, the axis of X, and two ordinates or straight lines parallel to the axis of Y. The area of such a figure may be obtained approximately by the "method of mean ordinates." By this method the figure is divided up into any convenient number of vertical strips of equal width, and the mid-ordinate of each strip drawn. Then, considering each strip as being approximately a rectangle, its area is the length of the mid-ordinate multiplied by the width of the strip, and the area of the whole figure is equal to the sum of the areas of the several strips, that is, to the sum of the products obtained by multiplying the length of each mid-ordinate by the width of one strip, or to the product of the mean of the mid-ordinates and the whole length of the figure. Thus, in Fig. 72, let the width of each strip be represented by δx, and the heights of the mid-ordinates be respectively y_1, y_2, y_3, etc., the area of the portion of the curve which is required being that lying between the values of x equal to X_1 and X_2 respectively. The area of, say, the third strip is given by $y_3 . \delta x$, and

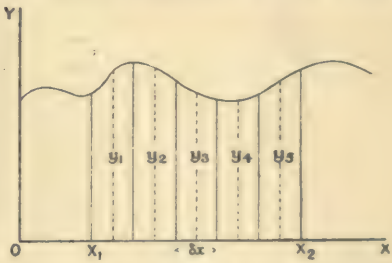


Fig. 72.

$$\text{the area of the whole figure} = y_1 . \delta x + y_2 . \delta x + y_3 . \delta x + \ . \ . \ .$$

Using the ordinary notation to express a sum of several terms of the same general form, this becomes

$$\text{area of the figure} = \Sigma_{X_1}^{X_2} y . \delta x$$

This last symbol is understood to mean "the sum of all terms of the form $y . \delta x$ lying between the values of x equal to X_1 and X_2, *i.e.* for which y is equal to $f(X_1)$ and $f(X_2)$ respectively, where $y = f(x)$." It is, of course, evaluated by determining the value of each term and adding the results together by the ordinary process of arithmetical addition.

Now, it is evident that this approximate method of determining the area of the curve may be made more nearly true by using a larger number of strips, the width of each strip being then, of course, proportionally smaller. The larger the number of strips taken, and the narrower they become, the more close does the approximation become to the true area of the curve. It is not possible, however, to obtain the mathematically exact area of the curve by any *finite* number of strips, however great. If, however, we imagine the number of the strips to become *infinitely* great, and the width of each, therefore, infinitely small, the method would then give us the true area of the curve with mathematical exactitude. We will now represent the infinitely small width of one strip by the symbol dx, keeping the symbol δx to mean, as in the last chapter, a small *finite* difference in x. For the sign of summation Σ, which is used in the case of finite quantities, we will now substitute the sign $\int$, which is used in exactly the same way as the former when the terms to be added together are infinitely small. Then we have

$$\text{the area of the figure} = \int_{X_1}^{X_2} y \, . \, dx.$$

This symbol, $\int_{X_1}^{X_2} y \, . \, dx$, means simply the sum of an infinite number of infinitely small terms of the form $y \, . \, dx$, lying between the values of x equal to X_1 and X_2, *i.e.* for which $y = f(X_1)$ and $f(X_2)$ respectively. The result of this summation, represented by $\int_{X_1}^{X_2} y \, . \, dx$, is termed the "*integral* of y relatively to x between the limits $x = X_1$ and $x = X_2$." Since this represents the sum of an infinite number of infinitely small things, it is obvious that it cannot be evaluated by the ordinary methods of arithmetic. The process of evaluating this sum, that is of determining the value of $\int_{X_1}^{X_2} y \, . \, dx$, is termed "integration." It should be quite clearly grasped that integration is simply and solely a special method of addition, applicable to a special case, namely, when the quantities to be added together are individually infinitely small and when the number of them is infinitely great.

Integration the Reverse of Differentiation.—Let the curve in Fig. 73 represent the variation in the velocity of a body relatively to the time which has elapsed from its starting-point. Then the area of this curve from $t = 0$ to $t = T_2$ will be measured by $\int_0^{T_2} v \, . \, dt$. What is the physical meaning of the area of this curve? The mean height of the curve $\bar{v}$ would represent the mean velocity of the body during this interval of time. Now, from the very conception of mean height, the area $= \bar{v} \cdot T_2$; but the product of the mean velocity of a body and the interval of time over which the mean velocity is measured is equal to the distance covered by the body in that interval, for

$$\bar{v} = \frac{s}{t}$$

or

$$s = \bar{v} \cdot t$$

Hence, we have that the area of the curve, that is, $\int_0^{T_2} v \,.\, dt$, measures the distance covered, or

$$s = \int_0^{T_2} v \,.\, dt$$

But in the last chapter we saw that

$$v = \frac{ds}{dt}.$$

This example illustrates the truth of the fundamental fact that if

$$z = \frac{dy}{dx} \qquad \text{then } y = \int z \,.\, dx$$

or that *integration is the reverse of differentiation.*

Method of Integration.—The above statement provides us with the basis of the method of integration. It may be said at once that there is no direct method of integration, the only method being, in effect, to guess the result, and then test the truth of the guess by differentiating back again into the original form. This, however, need form no real difficulty, for an exactly similar statement might be made with perfect truth about the arithmetical process of division, thus: There is no direct method of division, the only method being to guess the result and then test the truth of the guess by multiplying back again into the original quantity. A moment's thought upon the ordinary method of division will show the truth of this statement. But in the case of division experience comes to our aid, for having tested the truth of the statement that, say, $6 \div 3 = 2$, by doing the reverse multiplication, $3 \times 2 = 6$, a few times, experience subsequently tells us that $6 \div 3 = 2$ with such certainty that there is in future no further need to test it in the same way. Just in the same manner in the case of integration, we may once for all reverse the process of differentiation in the case of a few simple functions, so obtaining a series of "standard forms of integration" which may subsequently be utilized at all times. We will proceed to deduce a few of these standard forms.

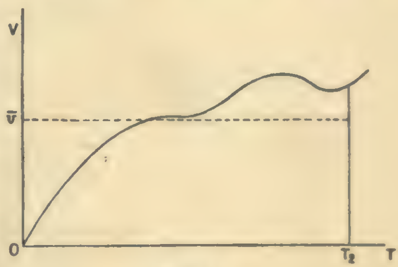


Fig. 73.

Integration of x^n.—If we differentiate x^m, we have—

$$\frac{dx^m}{dx} = m \,.\, x^{m-1}$$

Hence, from the statement above (p. 111)

$$x^m = \int m \,.\, x^{m-1} \,.\, dx$$
$$= m \,.\, \int x^{m-1} \,.\, dx$$

since m is a constant. Hence

$$\int x^{m-1} \,.\, dx = \frac{x^m}{m}$$

Now, put $m - 1 = n$, so that $m = n + 1$, then

$$\int x^n \,.\, dx = \frac{x^{n+1}}{n+1}$$

Integration of e^x.—Since

$$\frac{de^x}{dx} = e^x$$

therefore also $$\int e^x \,.\, dx = e^x$$

Integration of $\frac{1}{x}$.—In this case, if we attempt to apply the standard form for the integration of x^n, given above, we have

$$\int \frac{1}{x} \,.\, dx = \int x^{-1} \,.\, dx = \frac{x^0}{0} = \infty$$

Upon substitution of the required limits in the manner shown hereafter, this becomes $\infty - \infty$, which is indeterminate. We have, however, an alternative method, for

$$\frac{d(\log_e x)}{dx} = \frac{1}{x}$$

and therefore $$\int \frac{1}{x} \,.\, dx = \log_e x$$

Integration of Sin x and Cos x.—We have—

$$\frac{d(\sin x)}{dx} = \cos x$$

and therefore $$\int \cos x \,.\, dx = \sin x$$

Again, $$\frac{d(\cos x)}{dx} = -\sin x$$

and therefore $$\int \sin x \,.\, dx = -\cos x$$

Integration of a Constant.—Since

$$\frac{d(ax)}{dx} = a$$

therefore also $$\int a \,.\, dx = ax$$

Integration of a Sum.—Since the differential coefficient of the sum of several functions is equal to the sum of their several differential coefficients, so also the integral of the sum of several functions must be equal to the sum of their several integrals. Thus, if

$$y = u + v - w$$

then $$\int y \,.\, dx = \int u \,.\, dx + \int v \,.\, dx - \int w \,.\, dx$$

Definite Integration.—The above standard forms give simply the form of the general integral corresponding to a given function, or the "indefinite integral," as it is called. Referring again to Fig. 72, the indefinite integral gives the general equation of the curve which would be obtained by plotting the areas of the given curve up to various ordinates with the corresponding values of x. If the value X_1 were substituted for x in this expression, the result would express the area of the curve between the axis of Y and the vertical straight line $x = X_1$. Similarly, if X_2 were substituted for x, the result would give the area of the curve between the axis of Y and the vertical straight line $x = X_2$. The area of the curve between the vertical straight lines $x = X_1$ and $x = X_2$ would obviously be given by the difference between the two results thus obtained. But this area would be that given by the definite integral $\int_{X_1}^{X_2} y \,.\, dx$. Hence to convert an indefinite integral into the corresponding definite integral between the limits X_1 and X_2, these limits must in turn be substituted in the indefinite integral, and the results so obtained subtracted from each other. That is, expressing the method symbolically, if

$$\int y \,.\, dx = F(x)$$

then $$\int_{X_1}^{X_2} y \,.\, dx = F(X_2) - F(X_1)$$

Examples on Integration.—(1) Evaluate the expression—

$$\int_2^5 (x^2 - 2x + \sqrt{x}) \,.\, dx$$

Here the general, or indefinite integral—

$$\begin{aligned}\int (x^2 - 2x + \sqrt{x}) \,.\, dx &= \int x^2 \,.\, dx - 2\int x \,.\, dx + \int x^{\frac{1}{2}} \,.\, dx \\ &= \frac{x^3}{3} - \frac{2x^2}{2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \\ &= \frac{x^3}{3} - x^2 + \frac{2}{3}x^{\frac{3}{2}}\end{aligned}$$

Then, substituting the limiting values of x, namely 2 and 5, we have—

$$\begin{aligned}\int_2^5 (x^2 - 2x + \sqrt{x}) \,.\, dx &= \left\{\frac{5^3}{3} - 5^2 + \frac{2}{3} \times (5)^{\frac{3}{2}}\right\} - \left\{\frac{2^3}{3} - 2^2 + \frac{2}{3} \times (2)^{\frac{3}{2}}\right\} \\ &= \frac{125}{3} - 25 + \frac{2}{3} \times 11{\cdot}2 - \frac{8}{3} + 4 - \frac{2}{3} \times 2{\cdot}83 \\ &= 41{\cdot}67 - 25{\cdot}00 + 7{\cdot}46 - 2{\cdot}67 + 4{\cdot}00 - 1{\cdot}89 \\ &= 23{\cdot}57\end{aligned}$$

which is the required result.

(2) Prove that the area of a parabola is two-thirds of that of the circumscribing rectangle.

Consider the parabola given by the equation $y = x^{\frac{1}{2}}$ (Fig. 74). When $x = \mathrm{L}$, let $y = \mathrm{H}$, so that $\mathrm{H} = \sqrt{\mathrm{L}}$.

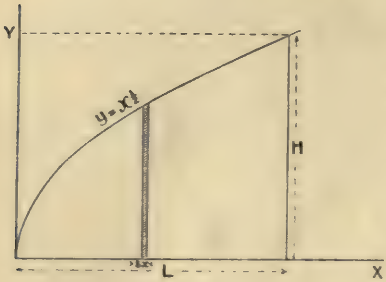


Fig. 74.

Imagine that the figure is divided up into a number of strips each of width δx, one of which is shown in the figure, the mean height of any one being represented by y. Then

$$\text{the area of the figure} = \Sigma_0^{\mathrm{L}} y \,.\, \delta x$$

or, in the limit when the number of strips becomes infinitely large,

$$\text{area} = \int_0^{\mathrm{L}} y \,.\, dx$$

But $$y = x^{\frac{1}{2}}$$

Therefore $$\text{area} = \int_0^{\mathrm{L}} x^{\frac{1}{2}} \,.\, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\mathrm{L}}$$

$$= \frac{2}{3}\mathrm{L}^{\frac{3}{2}} = \frac{2}{3}\mathrm{L} \,.\, \sqrt{\mathrm{L}} = \frac{2}{3}\mathrm{L} \,.\, \mathrm{H}$$

which proves the required statement. Note, that after integration the general form of the indefinite integral is written between square brackets, and the limits written outside the bracket to the right.

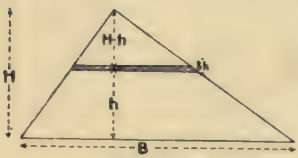


Fig. 75.

(3) Find an expression for the Second Moment, or "Moment of Inertia" (I), of a triangle about its base.

Let the length of the base of the triangle be B, and its vertical height be H (Fig. 75).

Consider the shaded strip of thickness δh and length b, drawn parallel to the base. Its area is equal to $b \,.\, \delta h$.

But $$\frac{b}{\mathrm{H} - h} = \frac{\mathrm{B}}{\mathrm{H}} \qquad \text{or } b = \frac{\mathrm{B}(\mathrm{H} - h)}{\mathrm{H}}$$

Then the area of the strip $= \dfrac{\mathrm{B}(\mathrm{H} - h)}{\mathrm{H}} \,.\, \delta h.$

The moment of inertia of this strip about the base is its area multiplied by the square of its distance from the base.

Then $$\mathrm{I}_{\text{strip}} = \frac{\mathrm{B}(\mathrm{H} - h)}{\mathrm{H}} \,.\, \delta h \,.\, h^2$$

But the moment of inertia of the whole triangle is equal to the sum of the moments of inertia of all the strips of which it is supposed to be formed.

Then $$\mathrm{I} = \Sigma_0^{\mathrm{H}} \frac{\mathrm{B}(\mathrm{H} - h) \,.\, h^2}{\mathrm{H}} \,.\, \delta h$$

Then, in the limit, when δh, the thickness of the strip, becomes infinitely small, this expression becomes—

$$\begin{aligned}
I &= \int_0^H \frac{B(H-h) \,.\, h^2}{H} \,.\, dh \\
&= \int_0^H \left\{ Bh^2 - \frac{Bh^3}{H} \right\} . \, dh \\
&= B\int_0^H h^2 \,.\, dh - \frac{B}{H}\int_0^H h^3 \,.\, dh \\
&= B\left[\frac{h^3}{3}\right]_0^H - \frac{B}{H}\left[\frac{h^4}{4}\right]_0^H \\
&= \frac{BH^3}{3} - 0 - \frac{BH^4}{4H} + 0 \\
&= \frac{BH^3}{3} - \frac{BH^3}{4} \\
&= \frac{BH^3}{12}
\end{aligned}$$

(4) The rate of flow of water through an orifice in the bottom of a tank, Q cubic feet per second, is given by the expression—

$$Q = K \,.\, a \,.\, \sqrt{2gh}$$

where a is the area of the orifice in square feet;

h is the head of water above the orifice at any instant;

K is a coefficient of discharge.

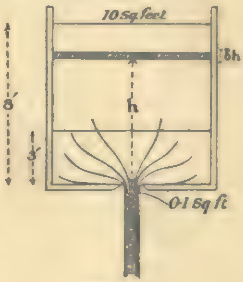


Fig. 76.

Find the time required to lower the level of water in a vertical sided tank, of area 10 square feet, from 8 feet to 3 feet, through an orifice in the bottom of the tank, whose area is 0·1 of a square foot, K being 0·62.

Let the level of the water be lowered by an amount δh feet in δt seconds (Fig. 76). Then a volume of water equal to $10\delta h$ cubic feet flows away in δt seconds, the area of the surface being 10 square feet, or the rate of flow,

$$Q = \frac{10 \,.\, \delta h}{\delta t}$$

But

$$\begin{aligned}
Q &= K \,.\, a \,.\, \sqrt{2gh} = 0{\cdot}62 \times 0{\cdot}1 \times \sqrt{64{\cdot}4\,h} \\
&= 0{\cdot}062 \times 8{\cdot}02 \times h^{\frac{1}{2}} \\
&= 0{\cdot}497h^{\frac{1}{2}}
\end{aligned}$$

Equating these two expressions for Q, we have—

$$0{\cdot}497h^{\frac{1}{2}} = \frac{10 \,.\, \delta h}{\delta t}$$

or

$$\begin{aligned}
\delta t &= \frac{10}{0{\cdot}497} \,.\, h^{-\frac{1}{2}} \,.\, \delta h \\
&= 20{\cdot}15h^{-\frac{1}{2}} \,.\, \delta h
\end{aligned}$$

Then the total time to lower the level of the water is given by—

$$T = \Sigma\delta t = 20{\cdot}15\,\Sigma_3^8 h^{-\frac{1}{2}} \,.\, \delta h$$

Upon proceeding to the limit when the quantity δh becomes infinitely small, we have—

$$\begin{aligned}
T &= 20{\cdot}15 \int_3^8 h^{-\frac{1}{2}} \, . \, dh \\
&= 20{\cdot}15 \left[\frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_3^8 \\
&= 20{\cdot}15 \left[\frac{h^{\frac{1}{2}}}{\frac{1}{2}} \right]_3^8 \\
&= 20{\cdot}15 \times 2(\sqrt{8} - \sqrt{3}) \\
&= 40{\cdot}3 \times (2{\cdot}828 - 1{\cdot}732) \\
&= 40{\cdot}3 \times 1{\cdot}096 \\
&= 44{\cdot}2 \text{ seconds.}
\end{aligned}$$

TABLES

TABLE OF FOUR-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	25	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

TABLE OF FOUR-FIGURE LOGARITHMS—*continued*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE OF FOUR-FIGURE ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

TABLE OF FOUR-FIGURE ANTILOGARITHMS—*continued*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE OF SINES

	0′	5′	10′	15′	20′	25′	30′	35′	40′	45′	50′	55′	1	2	3	4
0°	0000	0015	0029	0044	0058	0073	0087	0102	0116	0131	0145	0160	3	6	9	12
1	0175	0189	0204	0218	0233	0247	0262	0276	0291	0305	0320	0334	3	6	9	12
2	0349	0364	0378	0393	0407	0422	0436	0451	0465	0480	0494	0509	3	6	9	12
3	0523	0538	0552	0567	0581	0596	0610	0625	0640	0654	0669	0683	3	6	9	12
4	0698	0712	0727	0741	0756	0770	0785	0799	0814	0828	0843	0857	3	6	9	12
5	0872	0886	0901	0915	0929	0944	0958	0973	0987	1002	1016	1031	3	6	9	12
6	1045	1060	1074	1089	1103	1118	1132	1146	1161	1175	1190	1204	3	6	9	12
7	1219	1233	1248	1262	1276	1291	1305	1320	1334	1349	1363	1377	3	6	9	12
8	1392	1406	1421	1435	1449	1464	1478	1492	1507	1521	1536	1550	3	6	9	12
9	1564	1579	1593	1607	1622	1636	1650	1665	1679	1693	1708	1722	3	6	9	12
10	1736	1751	1765	1779	1794	1808	1822	1837	1851	1865	1880	1893	3	6	9	12
11	1908	1922	1937	1951	1965	1979	1994	2008	2022	2036	2051	2063	3	6	9	11
12	2079	2093	2108	2122	2136	2150	2164	2179	2193	2207	2221	2235	3	6	9	11
13	2250	2264	2278	2292	2306	2320	2334	2349	2363	2377	2391	2405	3	6	8	11
14	2419	2433	2447	2462	2476	2490	2504	2518	2532	2546	2560	2574	3	6	8	11
15	2588	2602	2616	2630	2644	2658	2672	2686	2700	2714	2728	2742	3	6	8	11
16	2756	2770	2784	2798	2812	2826	2840	2854	2868	2882	2896	2910	3	6	8	11
17	2924	2938	2952	2965	2979	2993	3007	3021	3035	3049	3062	3076	3	6	8	11
18	3090	3104	3118	3132	3145	3159	3173	3187	3201	3214	3228	3242	3	6	8	11
19	3256	3269	3283	3297	3311	3324	3338	3351	3365	3379	3393	3406	3	5	8	11
20	3420	3434	3448	3461	3475	3488	3502	3516	3529	3543	3557	3570	3	5	8	11
21	3584	3597	3611	3624	3638	3651	3665	3679	3692	3706	3719	3732	3	5	8	11
22	3746	3760	3773	3786	3800	3813	3827	3840	3854	3867	3881	3894	3	5	8	11
23	3907	3921	3934	3947	3961	3974	3987	4001	4014	4027	4041	4054	3	5	8	11
24	4067	4081	4094	4107	4120	4134	4147	4160	4173	4187	4200	4213	3	5	8	11
25	4226	4239	4253	4266	4279	4292	4305	4318	4331	4344	4357	4370	3	5	8	11
26	4384	4397	4410	4423	4436	4449	4462	4475	4488	4501	4514	4527	3	5	8	10
27	4540	4553	4566	4579	4592	4605	4617	4630	4643	4656	4669	4681	3	5	8	10
28	4695	4708	4720	4733	4746	4759	4772	4784	4797	4810	4823	4835	3	5	8	10
29	4848	4861	4874	4886	4899	4912	4924	4937	4950	4962	4975	4987	3	5	8	10
30	5000	5013	5025	5038	5050	5063	5075	5088	5100	5113	5125	5138	3	5	8	10
31	5150	5163	5175	5188	5200	5213	5225	5237	5250	5262	5275	5287	2	5	7	10
32	5299	5312	5324	5336	5348	5361	5373	5385	5398	5410	5422	5434	2	5	7	10
33	5446	5459	5471	5483	5495	5507	5519	5531	5544	5556	5568	5580	2	5	7	10
34	5592	5604	5616	5628	5640	5652	5664	5676	5688	5700	5712	5724	2	5	7	10
35	5736	5748	5760	5771	5783	5795	5807	5819	5831	5842	5854	5866	2	5	7	9
36	5878	5890	5901	5913	5925	5937	5948	5960	5971	5983	5995	6007	2	5	7	9
37	6018	6030	6041	6053	6065	6076	6088	6099	6111	6122	6134	6145	2	5	7	9
38	6157	6168	6180	6191	6202	6214	6225	6237	6248	6259	6271	6282	2	5	7	9
39	6293	6305	6316	6327	6338	6350	6361	6372	6383	6394	6406	6417	2	4	7	9
40	6428	6439	6450	6461	6472	6483	6494	6506	6517	6528	6539	6550	2	4	7	9
41	6561	6572	6583	6593	6604	6615	6626	6637	6648	6659	6670	6681	2	4	7	9
42	6691	6702	6713	6724	6734	6745	6756	6767	6777	6788	6799	6809	2	4	6	9
43	6820	6831	6841	6852	6862	6873	6884	6894	6905	6915	6926	6936	2	4	6	9
44	6947	6957	6967	6978	6988	6999	7009	7019	7030	7040	7050	7061	2	4	6	8

TABLE OF SINES—*continued*

	0′	5′	10′	15′	20′	25′	30′	35′	40′	45′	50′	55′	1	2	3	4
45°	7071	7081	7092	7102	7112	7122	7133	7143	7153	7163	7173	7183	2	4	6	8
46	7193	7203	7214	7224	7234	7244	7254	7264	7274	7284	7294	7304	2	4	6	8
47	7314	7323	7333	7343	7353	7363	7373	7383	7392	7402	7412	7422	2	4	6	8
48	7431	7441	7451	7461	7470	7480	7490	7499	7509	7518	7528	7538	2	4	6	8
49	7547	7557	7566	7576	7585	7595	7604	7613	7623	7632	7642	7651	2	4	6	8
50	7660	7670	7679	7688	7698	7707	7716	7725	7735	7744	7753	7762	2	4	6	7
51	7771	7781	7790	7799	7808	7817	7726	7735	7744	7753	7862	7871	2	4	5	7
52	7880	7789	7898	7907	7916	7925	7934	7942	7951	7960	7969	7978	2	4	5	7
53	7986	7995	8004	8013	8021	8030	8039	8047	8056	8064	8073	8082	2	3	5	7
54	8090	8099	8107	8116	8124	8133	8141	8150	8158	8166	8175	8183	2	3	5	7
55	8192	8200	8208	8216	8225	8233	8241	8249	8258	8266	8274	8282	2	3	5	7
56	8290	8299	8307	8315	8323	8331	8339	8347	8355	8363	8371	8379	2	3	5	6
57	8387	8395	8403	8410	8418	8426	8434	8442	8450	8457	8465	8473	2	3	5	6
58	8480	8488	8496	8504	8511	8519	8526	8534	8542	8549	8557	8564	2	3	5	6
59	8572	8579	8587	8594	8601	8609	8616	8624	8631	8638	8646	8653	1	3	4	6
60	8660	8668	8675	8682	8689	8696	8704	8711	8718	8725	8732	8739	1	3	4	6
61	8746	8753	8760	8767	8774	8781	8788	8795	8802	8809	8816	8823	1	3	4	6
62	8829	8836	8843	8850	8857	8863	8870	8877	8884	8890	8897	8903	1	3	4	5
63	8910	8917	8923	8930	8936	8943	8949	8956	8962	8969	8975	8982	1	3	4	5
64	8988	8994	9001	9007	9013	9020	9026	9032	9038	9045	9051	9057	1	3	4	5
65	9063	9069	9075	9081	9088	9094	9100	9106	9112	9118	9124	9130	1	2	4	5
66	9135	9141	9147	9153	9159	9165	9171	9176	9182	9188	9194	9199	1	2	3	4
67	9205	9210	9216	9222	9228	9233	9239	9244	9250	9255	9261	9266	1	2	3	4
68	9272	9277	9283	9288	9293	9299	9304	9309	9315	9320	9325	9331	1	2	3	4
69	9336	9341	9346	9351	9356	9362	9367	9372	9377	9382	9387	9392	1	2	3	4
70	9397	9402	9407	9412	9417	9422	9426	9431	9436	9441	9446	9450	1	2	3	4
71	9455	9460	9465	9469	9474	9479	9483	9488	9492	9497	9502	9506	1	2	3	4
72	9511	9515	9520	9524	9528	9533	9537	9542	9546	9550	9555	9559	1	2	3	4
73	9563	9567	9571	9576	9580	9584	9588	9592	9596	9600	9605	9609	1	2	2	3
74	9613	9617	9621	9625	9628	9632	9636	9640	9644	9648	9652	9655	1	2	2	3
75	9659	9663	9667	9670	9674	9678	9681	9685	9689	9692	9696	9699	1	1	2	3
76	9703	9706	9710	9713	9717	9720	9724	9727	9730	9734	9737	9740	1	1	2	3
77	9744	9747	9750	9753	9757	9760	9763	9766	9769	9772	9775	9778	1	1	2	3
78	9781	9784	9787	9790	9793	9796	9799	9802	9805	9808	9811	9813	1	1	2	2
79	9816	9819	9822	9825	9827	9830	9833	9835	9838	9840	9843	9846	1	1	2	2
80	9848	9851	9853	9856	9858	9860	9863	9865	9868	9870	9872	9875	0	1	1	2
81	9877	9879	9881	9884	9886	9888	9890	9892	9894	9897	9899	9901	0	1	1	2
82	9903	9905	9907	9909	9911	9913	9914	9916	9918	9920	9922	9924	0	1	1	2
83	9925	9927	9929	9931	9932	9934	9936	9937	9939	9941	9942	9944	0	1	1	1
84	9945	9947	9948	9950	9951	9953	9954	9955	9957	9958	9959	9961	0	1	1	1
85	9962	9963	9964	9966	9967	9968	9969	9970	9971	9973	9974	9975	0	0	1	1
86	9976	9977	9978	9979	9980	9980	9981	9982	9983	9984	9985	9986	0	0	1	1
87	9986	9987	9988	9988	9989	9990	9990	9991	9992	9992	9993	9993	0	0	0	1
88	9994	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	9998	0	0	0	0
89	9998	9999	9999	9999	9999	9999	10000	10000	10000	10000	10000	10000	0	0	0	0

STANDARD FORMS

y	$\frac{dy}{dx}$	$\int y \, . \, dx$
x^n	$n \, . \, x^{n-1}$	$\frac{x^{n+1}}{n+1}$
e^x	e^x	e^x
a^x	$a^x \, . \, \log_e a$	$\frac{a^x}{\log_e a}$
$\log_e x$	$\frac{1}{x}$	
$\log_a x$	$\frac{1}{x} \, . \, \log_e a$	
$\frac{1}{x}$		$\log_e x$
$\sin x$	$\cos x$	$-\cos x$
$\cos x$	$-\sin x$	$\sin x$
$\tan x$	$\sec^2 x$	$\log_e (\sec x)$
$\cot x$	$-\operatorname{cosec}^2 x$	$\log_e (\sin x)$
$\sec x$	$\sec x \, . \, \tan x$	
$\operatorname{cosec} x$	$-\operatorname{cosec} x \, . \, \cot x$	

EXAMPLES

CHAPTER I

1. Plot the following points:—(5, 3); (−3, 4); (2, −6); (0·5, 2·3); (−4·7, −3·6); (3·5, −2·7); (4·2, 5·9); (0, −2·8); (−7·1, 5·3); (−3·2, −3·2); (5·1, 0).

2. Two points, A (5·7, 2·3) and B (−2·5, −4·8), are joined by a straight line. Find the co-ordinates of the middle point of this line.

3. Plot the three points, A (−5·2, 4·3), B (6·2, 0), and C (0·4, 4·5), and from A draw a straight line perpendicular to BC, and meeting it at D. Find the co-ordinates of the point D.

4. With the point (5, 3) as centre, and radius equal to 7, describe a circle. What are the co-ordinates of the points at which this circle intersects the axes of X and of Y?

5. The two variables x and y are connected by the equation

$$x^2 + 3xy - 5y^2 = 0$$

Express this relationship in the form

$$y = f(x)$$

CHAPTER II

1. In an experiment on a three-sheave pulley-block, an effort, E lbs., was found to be necessary to lift a load of L lbs. Plot the curve connecting L and E, and find by interpolation the efforts necessary to lift loads of 150 and 350 lbs. respectively.

L . .	0	56	112	214	319	424	536	lbs.
E . .	8	21	36	61	86	116	140	lbs.

2. A tension test of a specimen of brass gave the extensions shown in the table below for the corresponding loads. Plot the curve connecting load and extension, and deduce the probable extensions for loads of 1500 and 2500 lbs. respectively.

Load	0	950	1200	1500	1800	2100	2400	2670	lbs.
Extension . .	0·00	0·03	0·17	0·42	0·82	1·39	2·19	3·18	ins.

3. The strength of struts of the same section and material depends upon their length. A series of struts were tested and gave the following results.

Plot a curve showing the relation between the length of a strut and its strength, and find by interpolation the probable strength of similar struts 18 and 43 feet long respectively.

Length in feet .	10	15	20	25	30	35	40	45	50
Strength in tons	100	44·5	25·0	16·0	11·1	8·2	6·3	4·9	4·0

4. The population of the United Kingdom at each census for the last eighty years is given in the table below. Plot a curve showing the variation of the population, and from it obtain the probable population in the years 1847 and 1883.

Year	1821	1831	1841	1851	1861	1871	1881	1891	1901
Population in millions	20·89	24·03	26·71	27·37	28·93	31·48	34·48	37·73	41·46

5. Plot the curves $y = 3x^2 - 4$ and $y = 2x + 1$, and find their points of intersection.

Plot the following curves:—

6. $y = 3{\cdot}1\sqrt[3]{x}$.
7. $y^2 + 2xy - 3x^2 = 0$.
8. $y = (2{\cdot}5)^{3x}$.
9. $x^2 + y^2 = 16$.

CHAPTER III

Plot the values of x and y given in the tables below, and measure the slopes of the resulting lines and their intercepts upon the axis of Y.

1.

x . .	0	2	4	6	8	10
y . .	0	4·2	8·4	12·6	16·8	21·0

2.

x . .	3	5	7	9	11
y . .	0·2	0·8	1·4	2·0	2·6

3.

x . .	10	20	30	40	50
y . .	400	650	900	1150	1400

4.

x . .	0·5	1·0	1·5	2·0	2·5
y . .	5·25	4·50	3·75	3·00	2·25

Draw, without actual plotting, the straight lines given by the equations—

5. $y = 3x + 7$.
6. $y = 18 - 12x$.
7. $y = 0{\cdot}12x - 3{\cdot}14$.
8. $2x + 3y - 5 = 0$.
9. $x = 0{\cdot}75y + 2{\cdot}16$.
10. $x - 2y = 3$.

11. The figures in Example 1 in Chapter II are connected by a law of the form

$$L = aE + b$$

Find this law.

12. H is the total heat of evaporation of steam at a temperature t° F. Plot the values of H and t given below, and from your curve deduce an expression for H in terms of t.

t . .	102	153	202	250	299	350	401	°F.
H . .	1113	1129	1143	1158	1173	1189	1204	B.T.U.

13. The elastic extensions, e'', of a specimen of mild steel corresponding to loads of W tons are given in the table below. Determine the relationship between the load and the extension it causes, and hence calculate the probable extension for a load of 9·3 tons.

Load, L tons . . .	0	1	2	3	4	5
Extension, e'' . . .	0·0009	0·0020	0·0029	0·0039	0·0050	0·0060
Load, L tons . . .	6	7	8	9	10	11
Extension, e'' . . .	0·0069	0·0081	0·0090	0·0100	0·0111	0·0121

14. The weight of steam used per minute by a certain engine when tested under varying load is given in the table below. Plot a curve connecting the steam consumption and the indicated horse-power, and find an equation expressing the relationship between these two quantities.

I.H.P. . . .	17·2	15·0	13·3	11·9	8·5	5·1	2·2	
Weight of steam per minute	10·93	10·00	9·00	8·33	6·57	5·03	3·20	lbs.

Also plot the curve connecting the consumption per I.H.P. hour with the power, and deduce its equation.

CHAPTER IV

1. Plot the following quartic parabolas:—

(a) $y = 2x^4$.
(b) $y = 2x^4 + 100$.
(c) $y = 2x^4 + 50x + 100$.
(d) $y = 2x^4 + 20x^2 + 50x + 100$.

Plot the parabolic curves given by the following equations:—

2. $y = -2{\cdot}1x^3$.
3. $y = 0{\cdot}01x^6 - 300$.
4. $y = x^{0{\cdot}3}$.

Find the equations of the parabolic curves obtained by plotting the following sets of values of x and y:—

5.

x . . .	0	1	2	3	4	5
y . . .	0	3·40	11·83	24·51	41·05	61·55

6.

x . . .	0	1·00	1·25	1·50	1·75	2·00
y . . .	0	121	267	499	860	1360

7.

x . . .	0	2·00	6·50	12·94	21·08	30·82
y . . .	0	1·00	2·00	3·00	4·00	5·00

8.

x . . .	0	1	2	3	4	5
y . . .	3·10	2·60	0·85	1·91	6·02	11·60

9.

x . . .	0	2	4	6	8	10
y . . .	24·3	34·7	62·1	91·6	134·3	184·4

10.

x . .	0·0	0·3	0·6	1·0	2·0	3·0	4·0	5·0
y . .	8·30	11·58	15·22	20·50	35·44	52·50	71·45	92·21

11. Determine, without plotting, the co-ordinates of the vertex of the parabola

$$y = 2x^2 - 4x - 6$$

12. A beam 100 feet long rests on two supports 60 feet apart, placed symmetrically. The central span carries a uniformly distributed load of 2 tons per foot run, the left-hand outer span a uniformly distributed load of $1\frac{1}{2}$ tons per foot run, and the right-hand outer span one of 1 ton per foot run. Determine the position and magnitude of the maximum bending moment on the middle span.

CHAPTER V

1. A gas expands from a pressure of 150 lbs. per square inch, absolute, to atmospheric pressure, according to the law

$$pv^{1·31} = \text{a constant}$$

Plot the curve of expansion, the initial volume being 1 cubic foot.

2. The values of the pressure and volume during the expansion stroke of the gas-engine, the curve of compression for which is given on p. 54, are given in the table below. Plot the two strokes as completely as the figures allow, and determine the value of n during the expansion.

v .	10	10·4	10·6	10·8	11	12	13	14	16	18	20	23
p .	45·2	123·2	157·7	181·7	188·2	166·2	146·2	129·7	105·7	87·2	74·2	58·7

3. In an air-compressor trial the following values of the pressure and volume during the compression process were obtained from the indicator card. Determine the value of n in the equation $PV^n = constant$, for the process, using logarithmically squared paper.

P. .	68·8	59·2	48·6	39·2	33·2	26·8	22·0	18·0	lbs. per sq. inch.
V. .	0·62	0·71	0·87	1·05	1·22	1·51	1·81	2·22	

4. Obtain an indicator diagram from any engine, the clearance for which is known to you, and, making your own measurements of pressure and volume, determine the laws for the compression and expansion curves thereon.

CHAPTER VI

1. Construct a diagram from which may be read off at once the 5th, 4th, $\frac{5}{4}$th, and the $\frac{4}{5}$th powers of any number between 1 and 20.

2. In an experiment to determine the coefficient of friction, μ, for a belt passing round a pulley, a load W lbs. was hung from one end of the belt, and a pull P lbs. applied to the other end in order to raise W. It is known that the quantities are connected by a law of the form $P = We^{\mu a}$ where a is the angle of contact between the belt and pulley *measured in radians*. The following values of P corresponding to various angles of contact were obtained. Determine the value of the coefficient of friction. What was the amount of the load W, which was kept constant throughout?

a degrees . . .	90	120	150	180	210	240	270	300	330
P lbs.	5·62	6·93	8·52	10·50	12·90	15·96	19·68	24·24	29·94

3. The change of entropy (ϕ) in raising 1 lb. of water from 32° F. to a temperature t° F. is given in the table below. Plot the curve connecting the entropy with the temperature, and deduce a formula for determining the entropy from the absolute temperature, which will be of the form

$$\phi = a \log_e T - b$$

(The absolute temperature $T = t° F. + 461$). You may use either squared paper or semi-logarithmic paper in your solution of this question.

t° F. .	102	153	213	228	250	281	312	341	381
ϕ . .	0·136	0·221	0·313	0·334	0·368	0·411	0·452	0·491	0·542

CHAPTER VII

1. A particle moves round the circumference of a circle 3 feet in diameter with uniform angular velocity, making one complete revolution every 2 seconds. Find the position of its projection upon a diameter of the circle 0·1, 0·2, 0·3, 0·4, and 0·5 seconds respectively after the particle was at the end of that diameter.

Plot the curves:—

2. $y = 2 \sin (3x - 30°)$.
3. $y = 1{\cdot}5 \cos (0{\cdot}5x + 0{\cdot}7)$.
4. $y = 2 \sin x \,.\, \cos x$.
5. The following values of x and y lie on a curve of the form

$$y = a \,.\, \sin (bx + c)$$

x degrees	0	6·66	10	20	30	36·66	40	50	60
y . . .	0·657	0·700	0·689	0·536	0·239	0·000	−0·121	−0·450	−0·657
x degrees	66·66	70	80	90	96·66	100	110	120	
y . . .	−0·700	−0·689	−0·536	−0·239	0·000	+0·121	0·450	0·657	

Plot the curve and find the values of the constants a, b, and c. Plot a second sine curve of double the amplitude and one-third the frequency of that given, and differing from it in phase by 50 degrees.

6. In the mechanism shown in Fig. 77, the horizontal rod is constrained to move horizontally between guides, and derives its motion from the crank OP, the pin P of which slides in the vertical slot carried by the rod. If OP

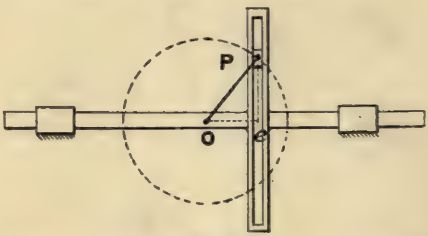


Fig. 77.
(*From Goodman's "Mechanics applied to Engineering."*)

is 6 inches long and makes one revolution every two seconds, moving uniformly, write down an equation expressing the displacement, s, of any point in the rod from its midposition, t seconds after P is vertically above O. Also plot the curve showing the same relationship.

Plot each of the compound sine curves given below over one complete cycle.

7. $y = \sin x + \sin (3x - 25)^0$.
8. $y = 2 \sin 0{\cdot}5x + \cos x$.
9. $y = 3{\cdot}5 \sin (0{\cdot}2x + 45)^0 - 2{\cdot}1 \sin (0{\cdot}4x + 60)^0$.
10. $y = 1{\cdot}9 \sin (x - 0{\cdot}2) + 2{\cdot}5 \sin (4x + 1{\cdot}57)$.
11. $y = \sin x + \cos x$.
12. $y = 2 \sin 4x \,.\, \cos x$.
13. A compound sine curve is made up of two components, A and B. A is of amplitude 1·2 and frequency 1·5, B is of amplitude 1·8 and frequency 4·5, the initial difference of phase between them being 90°. Write down the equation of the compound curve, and plot it for one complete cycle.
14. Plot the damped sine curve given by the equation

$$y = 2e^{-0{\cdot}2x} \sin 5x$$

over four complete "vibration-lengths."

CHAPTER VIII

Solve the following equations graphically :—

1. $3{\cdot}1x - 4{\cdot}2 = 2x + 0{\cdot}76$.
2. $4x - 3y = 6x - 2y + 4{\cdot}1 = 7{\cdot}8$.
3. $3x^2 - 2x - 6{\cdot}7 = 0$.
4. $\begin{cases} 2{\cdot}1x^2 - 43x + y - 5{\cdot}6 = 0 \\ 3x + 2y = 3{\cdot}8. \end{cases}$
5. $3x^3 - 2{\cdot}1x + 7{\cdot}3 = 0$.
6. $(2{\cdot}1)^x + 6{\cdot}8x = (3{\cdot}5)^{x^2} - 4{\cdot}1$.
7. $3x^2 - 20 \log_{10} x - 7{\cdot}077 = 0$.
8. $x^{1{\cdot}8} - \sin 2x - 2 = 0$.
9. $\sqrt{x^3} - x = \sqrt[3]{x^5} - 35{\cdot}8$.
10. $0{\cdot}5x^{2{\cdot}1} + 4x - 12 = 0$.
11. $\mathrm{L} \sin x = \sqrt{x} + 7{\cdot}95$.
12. $\log_e x = 2 \tan\left(x - \frac{\pi}{6}\right)$.

CHAPTER IX

1. The distance s feet covered in t seconds by a certain body is given in the table below. Plot the space-time curve, and from it obtain the velocity at the end of each second. Now plot the velocity-time curve, and from it obtain the acceleration-time curve.

t . . .	0	1	2	3	4	5	seconds.
s . . .	0·00	2·37	6·48	10·56	12·32	13·04	feet.

Differentiate the following functions of x :—

2. $x^{3{\cdot}1}$.
3. $2x^{0{\cdot}5}$.
4. $7{\cdot}3x^{5{\cdot}6}$.
5. $0{\cdot}72\sqrt[3]{x^2}$.
6. $\dfrac{1}{x^{2{\cdot}7}}$.
7. $\dfrac{1}{x}$.
8. $2{\cdot}7e^x$.
9. $(3{\cdot}72)^x$.
10. $\log_{10} x$.
11. $\log_2 x$.
12. $\sin (x - 0{\cdot}3)$.
13. $3 \cos (x + 2{\cdot}5)$.
14. $3{\cdot}1x^{0{\cdot}72} + 2{\cdot}5x^{3{\cdot}7}$.
15. $2{\cdot}52\sqrt{x} - 3{\cdot}1\sqrt[3]{x}$.
16. $e^x - x^e$.
17. $\sin x - 2 \cos x$.
18. $2 \tan x - 3 \log_e x$.

Evaluate the following expressions :—

19. $\dfrac{d^2}{dx^2}(3x^3)$.
20. $\dfrac{d^2}{dv^2}(20{\cdot}2v^{-1{\cdot}2})$.
21. $\dfrac{d^3}{dt^3}(t^4 - 5{\cdot}7t^3 + 2t)$.
22. $\dfrac{d^4}{dx^4}(x^7)$.
23. $\dfrac{d^2}{dx^2}(\sin x - \cos x)$.

24. A particle moves in a straight line so that its distance, s feet, from its starting-point after t seconds from rest is given by the equation

$$s = 2t^3 - 0{\cdot}4t^2$$

Find its velocity after 1, 2, and 3 seconds respectively.

25. Draw the part of the curve $y = 2 \sin 4x$ between $x = 0{\cdot}174$ radians and $x = 0{\cdot}261$ radians.

Measure the slope at three points and compare with the values of $\dfrac{dy}{dx}$.

26. The temperature of saturated steam depends upon its absolute pressure according to the equation

$$t = 117{\cdot}9p^{0{\cdot}222}$$

Find the rate of change of the temperature relatively to the pressure, at atmospheric pressure, and at pressures of 100 and 200 lbs. per square inch above the atmosphere.

Find the maximum and minimum values of the following expressions:—

27. $3x^2 - 2x + 4$.
28. $2x^3 - 3x + 2$.
29. $5x^4 - 16x^2 - 8$.
30. $x^{1\cdot5} - 3\cdot2x + 7$.
31. $3 + s - 2s^2$.
32. $h - h^3 + 2h^2$.

33. A rectangular tank with a square base is to be constructed to hold 10,000 gallons of water. Find the lengths of its sides in order that its weight may be as small as possible.

34. Find the least area of sheet metal which can be used to construct a cylindrical vessel to hold 20,000,000 cubic feet of gas, one end being open.

35. The Post Office regulations state that the combined length and girth of a parcel must not exceed 6 feet. Find the greatest volume of a cylindrical parcel which may be sent by post.

36. Find the velocity and acceleration of the projection of the particle in Question 1, Chapter VII., at each instant named therein.

CHAPTER X

Obtain expressions for the following indefinite integrals:—

1. $\int x^3 . dx$.
2. $\int dx$.
3. $\int h^{\frac{1}{2}} . dh$.
4. $\int 5t^4 . dt$.
5. $\int v^{-1\cdot4} . dv$.
6. $\int \frac{6}{y} . dy$.
7. $\int 2 \sin \theta . d\theta$.
8. $\int (x^2 - 3x + 2) . dx$.
9. $\int (p^3 - 2p + p^{-1}) . dp$.

Evaluate the following definite integrals:—

10. $\int_2^5 x^2 . dx$.
11. $\int_0^2 6t^{2\cdot5} . dt$.
12. $\int_1^3 (2x^4 - 3x^2) . dx$.
13. $\int_{1\cdot5}^{4\cdot5} h^{-\frac{1}{2}} . dh$.
14. $\int_0^R (2Rh^{\frac{1}{2}} - h^{\frac{3}{2}}) . dh$.
15. $\int_{700}^{1500} \frac{dT}{T}$.
16. $\int_{V_1}^{V_2} \frac{dv}{V^n}$.
17. $\int_0^{\frac{\pi}{2}} 2 \cos \theta . d\theta$.

18. Find an expression for the second moment or "moment of inertia" of a rectangle about an axis through its centre parallel to one of its sides.

19. Find the area of the figure bounded by the cubic parabola $y = x^3$, the axis of X, and the vertical straight lines $x = 2$ and $x = 5$.

20. Find the work done (which is measured by the area of the curve) during the isothermal expansion of 1 cubic foot of air from a pressure of 5000 lbs. per square foot to atmospheric pressure.

21. Find the value of the second polar moment (polar moment of inertia) of a circle, 5 feet in diameter, about an axis drawn through its centre, at right angles to the plane of the circle.

22. A vessel in the form of half a cylinder, with its axis vertical, is 12 feet long and 4 feet in diameter, and is full of water. Find the time taken to reduce the depth of water in it to 1 foot, through a sharp-edged circular orifice in the bottom, 3 inches in diameter. Coefficient of discharge = 0·62.

23. A quantity of steam expands so as to satisfy the law $pv^{1\cdot13} = 8000$. Find the average value of the pressure p as the steam expands from a volume $v = 4$ cubic feet to $v = 10$ cubic feet.

ANSWERS

I. **2.** (1·6, −1·25). **3.** (−4·23, 6·51). **4.** (−1·34, 0); (11·28, 0); (0, −1·9); (0, 7·88). **5.** $y = 0{\cdot}84x$ *or* $y = -0{\cdot}24x$.

II. **1.** 44·8 lbs.; 93·5 lbs. **2.** 0·42″; 2·57″. **3.** 30·9; 5·4 tons. **4.** 27·2. 35·1 millions. **5.** (1·67, 4·33); (−1, −1).

III. **1.** 2·1; 0. **2.** 0·3; −0·7. **3.** 25, 150. **4.** −1·5; 6. **11.** L = 4·04E − 30. **12.** H = 1082 + 0·305t. **13.** e = 0·0010L; 0·0093″. **14.** W = 0·51 P + 2·21; $w = 30{\cdot}6 + \frac{132{\cdot}6}{P}$.

IV. **5.** $y = 3{\cdot}4x^{1{\cdot}8}$. **6.** $y = 121x^{3{\cdot}5}$. **7.** $x = 2y^{1{\cdot}7}$. **8.** $y = 0{\cdot}5x^{2{\cdot}1} - 3{\cdot}1$. **9.** $y = 3{\cdot}2x^{1{\cdot}7} + 24{\cdot}3$. **10.** $y = 2{\cdot}2x^{1{\cdot}7} + 10x + 8{\cdot}3$. **11.** (1, −8). **12.** 30·83 feet from the left-hand support; 650 ton feet.

V. **2.** 1·544. **3.** 1·06.

VI. **2.** μ = 0·4; W = 3 lbs. **3.** $\phi = \log_e T - 6{\cdot}21$.

VII. **1.** 1·43; 1·21; 0·885; 0·465; 0 feet from the centre. **5.** $y = 0{\cdot}7 \sin (3x + 70°)$; $y = 1{\cdot}4 \sin (x + 20°)$. **6.** $s = 6 \sin (3{\cdot}14t)$. **13.** $y = 1{\cdot}2 \sin 1{\cdot}5x + 1{\cdot}8 \sin (4{\cdot}5x \pm 90°)$
or $y = 1{\cdot}2 \sin (1{\cdot}5x \pm 90°) + 1{\cdot}8 \sin 4{\cdot}5x$.

VIII. **1.** 4·51. **2.** $x = -0{\cdot}45$; $y = -3{\cdot}2$. **3.** 1·26; −0·59. **4.** $x = 3{\cdot}32$ *or* −0·56; $y = -3{\cdot}07$ *or* 2·74. **5.** −1·516. **6.** 1·507. **7.** 2·134. **8.** 1·515. **9.** 12·53. **10.** 2·288. **11.** 2·73 *or* 0·0114. **12.** 4·295.

IX. **2.** $3{\cdot}1x^{2{\cdot}1}$. **3.** $x^{-0{\cdot}5}$. **4.** $40{\cdot}8x^{4{\cdot}6}$. **5.** $0{\cdot}48x^{-\frac{1}{3}}$. **6.** $-\frac{2{\cdot}7}{x^{3{\cdot}7}}$. **7.** $-\frac{1}{x^{2}}$. **8.** $2{\cdot}7e^x$. **9.** $1{\cdot}315 \times (3{\cdot}72)^x$. **10.** $\frac{0{\cdot}434}{x}$. **11.** $\frac{1{\cdot}44}{x}$. **12.** $\cos (x - 0{\cdot}3)$. **13.** $-3 \cos (x + 2{\cdot}5)$. **14.** $2{\cdot}24x^{-0{\cdot}28} + 9{\cdot}25x^{2{\cdot}7}$. **15.** $1{\cdot}26x^{-\frac{1}{4}} - 1{\cdot}03x^{-\frac{5}{4}}$. **16.** $e^x - e(x)^{e-1}$. **17.** $\cos x + 2 \sin x$. **18.** $2 \sec^2 x - \frac{3}{x}$. **19.** $18x$. **20.** $53{\cdot}3v^{-3{\cdot}2}$. **21.** $24t - 34{\cdot}2$. **22.** $840x^3$. **23.** $\cos x - \sin x$. **24.** 5·2; 22·4; 51·6 feet per second. **26.** 0·658; 0·407 degrees per lb. per. sq. inch. **27.** Min. 3·67. **28.** Max. 3·413; min. 0·587. **29.** Two max. each −20·8; min. −8. **30.** Max. 4·085. **31.** Max. 3·125. **32.** Max. 0·618; min. −1·64. **33.** 14·8 × 14·8 × 7·4 feet. **34.** 323,000 sq. feet. **35.** 2·55 cub. feet. **36.** Velocity: 1·46; 2·77; 3·80; 4·48; 4·71 feet per second. Acceleration: 14·01; 11·91; 8·65; 4·55; 0 feet per second per second.

X. **1.** $\frac{x^4}{4}$. **2.** x. **3.** $\frac{2}{3}h^{\frac{3}{2}}$. **4.** t^5. **5.** $-\frac{1}{0{\cdot}4v^{0{\cdot}4}}$. **6.** $6 \log_e y$. **7.** $-2 \cos \theta$. **8.** $\frac{x^3}{3} - \frac{3x^2}{2} + 2x$. **9.** $\frac{p^4}{4} - p^2 + \log_e p$. **10.** 38·9. **11.** 19·3. **12.** 70·8. **13.** 1·78. **14.** $\frac{14}{15}R^{\frac{5}{2}}$. **15.** 0·761. **16.** $\frac{V_2^{1-n} - V_1^{1-n}}{1-n}$. **17.** 2. **18.** $\frac{BH^3}{12}$. **19.** 152·2. **20.** 4370 foot lbs. **21.** 61·2 feet^4 units. **22.** 4 mins. 13 secs. **23.** 950.

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THE END

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