PRACTICAL DESIGNING

OF

RETAINING WALLS,

WITH APPENDICES ON

STRESSES IN MASONRY DAMS

BY

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PREFACE TO SEVENTH EDITION

In the first six editions of this work, considerable space was given to the results of experiments on model retaining-walls and rotating retaining-boards. As this part of the subject has been fully discussed by the writer in a paper entitled "Experiments on Retaining-walls and Pressures on Tunnels,"* it was thought best to omit a detailed discussion of the experiments in this edition, particularly as an adequate interpretation requires the consideration of the theory of earth pressure when the earth is supposed endowed with both friction and cohesion. More important still, the omission gives space for a more adequate treatment of the designing of walls of various types.

The present work is divided into an Introduction, where the direction of the

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earth thrust receives careful attention, and four chapters, pertaining to reservoir walls and the theory of retaining-walls, developed both by the graphical and analytical methods and leading up, after a short discussion of experiments, to the practical designing of retaining walls.

The design of five different types of retaining-walls is given in detail not only for a horizontal earth surface but likewise for the earth surface at the angle of repose.

The tables, giving ratio of base to height, for the most familiar types of walls, should prove especially valuable to constructors.

In the brief discussion of dams, the occasion is taken to develop certain well-known elementary principles that are common to retaining-walls as well as dams. In subsequent chapters of this work a good deal of new matter is given for the first time; notably in the analytical theory of the retaining-wall, and in the graphical discussion of "the limiting plane" in Chapter II. The theory of the retaining-wall has been deduced, with the one assumption of a plane surface of rupture, from well-known mechanical laws; Coulomb's "wedge of maximum thrust" being incidentally proved in the course of the demonstration, but not assumed as a first principle.

Appendices I, II and III on Masonry Dams, have been added, leading to the computation of the "Stresses in a Masonry Dam" on any plane not too near the base. The results, especially when taken in connection with the experiments on rubber dams made in England by Messrs. Wilson and Gore, are thought to be of the highest importance.

The limits of this book preclude the consideration of the stresses due to temperature changes and "uplift" due to water pressure, subjects which are now engaging the serious attention of engineers.

WM. CAIN.

CHAPEL HILL, N. C., May 5, 1914.

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PRACTICAL DESIGNING

OF

RETAINING-WALLS.

INTRODUCTION.

1. THE retaining or revetment wall is generally a wall of masonry, intended to support the pressure of a mass of earth or other material possessing some frictional stability. In certain cases, however, as in dock-walls, the backing or filling — as the material behind the wall is called — is liable to become in part or wholly saturated with water, so that the subject of water-pressure has to be considered to complete the investigation. In cases where the filling is deposited behind the wall after it is built, the full pressure due to the pulverulent fresh earth or other backing is experienced; and the wall is designed to meet such pressure, with a certain factor of safety, as near as it can be ascertained. In time the earth becomes more or less consolidated by the settling due to gravity, vibrations, and rains, from the compressibility of the material, which thus brings into action those cohesive and chemical affinities which manufacture solid clays out of loosely aggregated materials, and often causes the bank eventually even to shrink away from the wall intended to support it, when, of course, there will be no pressure exerted against the wall.

2. Where a wall is built to support the face of a cutting, the pressure may be nothing at first, but it would be very unwise to make the wall much thinner than in the preceding case; for it is a well-known fact of observation, that incessant rains often saturate the ground of open cuttings to such an extent as to bring down masses of earth, whose surface of rupture is curved, being more or less vertical at the top and approaching a cycloid somewhat in section; the surface of sliding being so lubricated by the water that the pressure exerted horizontally by this sliding mass is even greater than for dry pulverulent materials. It is,

in fact, on this account, as well as from the force exerted by water in freezing, and from the disturbing influences caused by the passage of heavy trains, wagons, etc., which set up vibrations that lower the co-efficient of friction of the earth, and besides add considerably by their weight to the thrust of the backing, that a factor of safety against overturning and sliding of the wall is introduced, which factor in practice generally varies between two and three when the actual lateral pressure of the earth is considered.

3. It is stated that retaining-walls in Canada require a greater thickness at the top to resist the action of frost than farther south where the frost does not penetrate the ground to so great a depth. Again, if the strata in a cutting dip towards the wall, with thin beds of clay, etc., interposed that may act as lubricants when wet, the pressure against the wall may become enormous; or if fresh earth-filling is deposited upon an inclined surface of rock, or other impervious material that may become slippery when the water penetrates and accumulates at its sur-

face, the pressure may become much greater than that due to dry materials. It is found, too, that certain clays swell when exposed to the air with great force; others, again, remain unchanged. In all such exceptional cases the engineer must use his best judgment after a careful study of the material he has to deal with. The theory and methods used in this book will not deal with such exceptional cases, but simply with dry or moist earth-filling supported by good masonry upon a firm foundation; and it is believed the theory deduced will be of material assistance to any one who may have to deal with even very exceptional conditions, or, as in the case of military engineers, with the design of revetment-walls partly as a means of defence.

4. When a retaining-wall fails, it is not generally from not having sufficient section for dry backing properly laid (in layers horizontal or inclined downwards from the wall), but because the earth has been dumped in any fashion against the wall, and no "weep holes" have been provided to let off the water that is sure in time of rains to

saturate the bank. If to this is added bad masonry, and a yielding foundation, or one liable to be washed out, the final destruction of the wall can be pretty confidently counted on.

5. The following little table of weights and angles of repose of various materials used in construction may prove of assistance, but in any actual case the engineer should determine them by actual experiment: —

	Weight per Cubic Foot in Pounds.	Angle of Repose.
Water	62.4	0
Mud	102.	0–?
Shingle, gravel .	90–109–120	35°–48°
Clay	120	14°–45°
Gravel and earth,	126	–
Settled earth . .	120–137	21°–37°
Dry sand . . .	90	34°
Damp sand . .	120–128	35°–45°
Marl	100	–
Brick	90–135	–
Mortar	86–110	–
Brickwork . .	110	–
Masonry . . .	110–144	–
Sandstone . . .	130–157	–
Granite . . .	164–172	–

We may assume generally, as safe values for brickwork, 110 pounds per cubic foot;

and for walls, one-half ashlar and one-half rubble backing, of granite 142 pounds, and of sandstone 120 pounds per cubic foot, though the last two values are generally exceeded. For ordinary earth or sand filling the angle of repose can be taken at one and one-half base to one rise, or a slope of 33°42′ with weights per cubic foot varying from 100 to 130.

It is always advisable, where practicable, to put a layer of shingle next the wall, and to consolidate the layers of the filling by punning or other means, so as to reduce the natural slope as much as possible.

With a well-built wall, designed after methods to be given; having a good foundation-course, larger than the body of the wall, to better distribute the pressure, and resist sliding, and backed as described; with weeping holes near the bottom at intervals, — there should be no fear of failure under ordinary conditions.

6. It would take us too far to enter into the history of the theory of the retaining-wall. On this point see an interesting article by Professor A. J. DuBois in the "Journal

of the Franklin Institute" for December, 1879, on "A New Theory of the Retaining-Wall." In this work, three methods will be developed: the first, a graphical method that will make clear the foundations on which all the theory rests; the second, a purely analytical method, and the third, a graphical solution founded on it. Only the two graphical methods are available where the earth surface is not plane.

7. In case a wall moves forward, however little, or there is settling of the earth behind it, the earth generally rubs against the back of the wall, thus developing friction. There are, however, certain inclinations of the back of the wall that will be specially examined in articles 28–31, for which the earth sooner breaks along some interior plane, in its mass, than along the wall, so that a certain wedge of earth will move with the wall as it overturns or tends to move. For all other cases, which include nearly all the cases in practice, there will be rubbing of the earth against the wall, so that the earth-thrust against

the wall must be assumed to make, with the normal to the wall, an angle equal to the co-efficient of friction of earth on wall, unless this is greater than for earth on earth, in which case any slight motion of the wall forward will carry with it a thin layer of earth, so that the rubbing surfaces are those of earth on earth.

8. These suppositions are found to agree with experiments. The old theory that assumed the earth-thrust as normal to the back of the wall, or, as in Rankine's theory, always parallel to the top slope, does not so agree, and, in fact, often gives, for walls at the limit of stability, the computed thrust as double that actually experienced. The true theory, therefore, includes all the friction at the back of the wall that is capable of being exerted. This friction, combined with the normal component of the thrust, gives the resultant earth-thrust inclined below the normal to the back of the wall at the angle of friction to this normal.[1]

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[1] In Annales des Ponts et Chaussées for April, 1887, M. Siégler has given the results of some simple experiments proving the existence of a vertical component of the earth-

9. Rankine's assumption that the direction of the earth-thrust is *always* parallel to the top slope applies only to the case of an imaginary *incompressible* earth, homogeneous, made up of little grains, possessing the resistance to sliding over each other called friction, but without cohesion; of indefinite extent, the top surface being plane; the earth resting on an incompressible foundation, or one uniformly compressible, and

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thrust against the movable side of a box filled with sand, by actually measuring the increased friction at the bottom of the movable board, held in place, caused by this vertical component. The box was one foot square at the base; and for successive heights of sand of one-third, two-thirds, and one foot, the vertical components of the thrust for earth level at top were 0.66 pound, 1.76 pounds, and 3.97 pounds, respectively. Similarly for a box, 0.5 × 0.8 feet, filled with sand, but having a movable bottom supported firmly on iron blocks, the force necessary to move the blocks under the sides and under the bottom was measured; and from this the relative weights of sand supported by the bottom and sides of the box was found to be as one to one, nearly, for a height of sand of 0.6 foot, and about two to one for a height of 1.18 foot, the total weights ascertained by the friction apparatus also checking out with the actual to within five per cent. Other experimenters have actually weighed the amounts held up by the sides and bottom, respectively. See Engineering News for May 15 and 29, 1886, also the issue for March 3, 1883, on "A Study of the Movement of Sand;" also see article 60 following.

being *subjected to no external force but its own weight.*

For such a material, the only pressure which any portion of a plane parallel to the top slope of greatest declivity can have to sustain is the weight of material directly above it; so that the pressure on the plane is everywhere uniform and vertical. If we now suppose a parallelopipedical particle, whose upper and lower surfaces are planes parallel to the top slope, and bounded on the other four sides by vertical planes, we see that the pressures on the upper and lower surfaces are vertical, and their difference is equal, opposite to, and balanced by the weight of the particle. It follows that the pressures on the opposite vertical faces of the particle must balance each other independently, which can only happen when they act parallel to the top surface, in which case only are they directly opposed. The pressures, therefore, on the two vertical faces parallel to the line of greatest declivity will be horizontal; and on the other two faces, parallel to the line of greatest declivity. This is Rankine's reasoning, and it is sound for the material and conditions assumed. It is likewise applicable to a material of the same kind, only *compressible*, provided we suppose it deposited, as snow falls, everywhere to the same depth, on an absolutely incompressible, or a uniformly compressible, plane foundation, parallel to the ultimate top slope of the earth; for then the compression is uniform throughout the mass, and does not affect the reasoning. But if we suppose, as usually happens, that the foundation is not uniform in compressibility, then the earth will tend to sink where it is most yielding. This sinking is resisted to a certain extent by the friction resulting from the thrust of the earth surrounding the falling mass, so that much of its weight is transmitted to the sides, as actually happens in the case of fresh earth deposited over drains, culverts, or tunnel linings which settle appreciably. In the case of a tunnel driven through old ground, most if not all the weight of the mass above it is transmitted to the sides; at least, at first, before the timbering or masonry is got in. Again, if the mass of earth is of variable depth, even on a firm foundation, the mass of greatest depth will sink most, thus transmitting some of its weight to the sides, so that throughout the entire mass the pressure is nowhere the same at the same depth as assumed. The vertical pressure over a drain or small culvert crossing an ordinary road embankment is less, too, for another reason, where the embankment is highest. The earth-thrust on a vertical plane, parallel to the line of road, is horizontal for a symmetrical section when the plane bisects that section. On combining this thrust with the weight of the material on either side, we see that the resultant load on the culvert is removed farther from the centre than if there was no horizontal thrust. It is on account of this tendency to equalize pressure by aid of the friction resulting from the earth-thrust, that sand, when it can be confined, is one of the best foundations, whether in mass or in the form of sand piles.

10. In the case of earth deposited behind a retaining-wall on a good foundation, the settling of the earth will generally be greater than that of the wall, so that the earth rubs

against the wall, giving generally the direction of the thrust no longer inclined, even approximately parallel to the top slope (except when the latter is at the angle of repose), but making with the normal to the back of the wall an angle downwards equal to the angle of friction. If the wall should settle more than the filling, the thrust would at first have a tendency to be raised above the normal. But if such a thrust, when combined with the weight of the wall, passes outside of the centre of the base of the wall, the top of the wall will move over slightly, the earth will get a grip on the wall in the other direction; so that it is plainly impossible for the wall (for usual batters at least) to overturn or slide on its base, without this full friction, acting downwards at the back of the wall, being exerted. Hence the theory which supposes it is safe; for although it is possible that the earth may make the effort at times to exert the full thrust given by Rankine's formula, yet this effort is suppressed *instanter* by the external force now introduced by the wall friction, which force was expressly excluded from the Rankine theory. The exceptions to this rule will be noted in article 31.

11. Weyranch's objections to taking the thrust inclined at the angle ϕ' of friction to the normal are easily met. He says, Take a tunnel-arch; and if we suppose the pressure, as we go up from either side, to make always the angle ϕ' with the normal, we shall have at the crown two differently directed pressures: similarly for a horizontal wall with level-topped earth resting on it. If there is no relative motion, or tendency to motion, the thrust in the latter case is of course vertical, and in the former is probably vertical at the crown and inclined elsewhere; but if the arch or wall moves, and there is rubbing of the earth on the masonry, there is necessarily friction exerted; so that the thrust at any point can have but one direction, making the angle ϕ' with the normal.

12. Mr. Benjamin Baker, in his paper before the Institution of Civil Engineers, on the "Actual Lateral Pressure of Earthwork" (republished by Van Nostrand as "Science Series," No. 56), tested an old

theory (where the earth-thrust was assumed to act normal to the wall) by the results of experiments, and found the theoretical pressure often double the actual. In the discussion which followed, not a single engineer so much as alluded to a truer theory which assumes the true direction of the earth-thrust, and has been known and used, just across the channel, since the time of Poncelet.

The writer tested this theory by many of the experiments recorded by Baker and some others, and found it to agree, within certain limits, remarkably well (see "Van Nostrand's Magazine" for February, 1882). These results have been carefully revised, and new experiments included, in the table given farther on, from which the reader can form a fair estimate of the theory as a working theory within certain limits that will be indicated.

The reader is referred, however, to Mr. Baker's essay, not only for experiences under ordinary conditions, but for those exceptional cases which seem to defy all mathematical analysis. In fact, the engineer almost invariably has to assume the weights of earth and masonry, and angle of repose of the earth. Where there is water, the conditions one day may be very different from what they are the next. especially if the foundation is bad, as often happens; in which case the wall will move over simply on account of the compressibility of the foundation, so that it has perhaps nothing like the estimated stability. For all such cases an allowance must be made over the results given for a firm foundation, etc., as to which no rule can be given.

As water often saturates the filling, and perhaps gets under the wall, we must consider, in certain cases, water-pressure in connection with the thrust of the backing. Therefore, a short chapter on reservoir-walls, or dams, follows, in which many of the principles that must likewise apply to retaining-walls proper are given.

CHAPTER I.

RESERVOIR-WALLS.

13. THE design of reservoir-walls is a subject that has received the attention of many engineers and mathematicians; but they are by no means agreed, except in a general way, upon the precise profile that is best to satisfy, as uniformly as possible, the requirements of strength and stability.

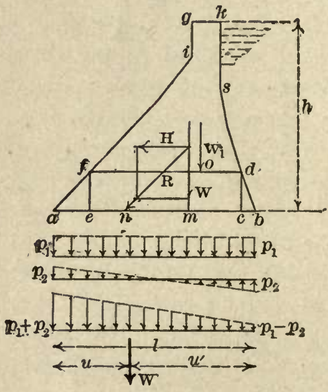
We shall very briefly, and by the shortest means, point out the main principles of design of a dam that resists overturning or sliding by its weight alone, and is called a *gravity dam*, in contradistinction to one built on a curve that requires the aid of *arch* action to render it stable.

Let Fig. 1 represent a slice of the dam contained between two vertical parallel planes one foot apart, and perpendicular to the faces.

When the dam is large, a roadway is

generally built on top, so that the faces *ks* and *gi* are vertical or nearly so for some distance down; after which the profile is designed to meet certain requirements, to be given presently. Let us suppose that the

Fig. 1



dam has been properly designed down to the horizontal joint *df*, and that the weight of the portion above *df* equals W_1, regarding the weight of a cubic foot of masonry as 1, and that its resultant cuts the joint *df* at the point *o*.

To design the part *fabd* below *df* by a rapid though tentative method, we must first assume the slopes *db* and *fa* corresponding to the depth *dc;* then compute the areas of the triangles *bcd* and *afe*, and of the rectangle *fecd*. The distances of the centres of gravity of these areas (which represent volumes) from the point *b* are respectively $\frac{2}{3}\overline{bc}$, $\overline{be} + \frac{1}{3}\overline{ae}$, and $\overline{bc} + \frac{1}{2}\overline{ce}$. On multiplying each area by its corresponding arm from *b*, adding the products to $W_1(\overline{bc} + \overline{do})$, and dividing by the sum of W_1 (which equals the area of *gkdf*) and the portion added *fabd*, we find the horizontal distance $\overline{bm}$ from *b* to where the resultant of the weight above joint $\overline{ab}$ cuts this joint. Its amount *W* is equal to the sum of the areas $(W_1 + abdf)$, and we have only to combine *W* acting along the vertical through *m*, with the horizontal thrust *H* of the water acting on the face *ksdb*, to find the resultant *R* on the joint, and the point *n* where it cuts that joint.

There is a vertical pressure of the water on the part *sdb;* but, as it adds to the stability, it is generally neglected, particularly as the inner face is generally nearly vertical.

14. The horizontal pressure of the water H for the height h, by known laws of mechanics, is equal to the area $h \times 1$ multiplied by the depth of its centre of gravity $\frac{h}{2}$ below the surface of the water, and by the weight of a cubic foot of water w, where a cubic foot of masonry is taken as the unit. This pressure acts horizontally at $\frac{1}{3}h$ above the joint $\overline{ab}$, so that its moment about the point n where the resultant R cuts the base $\overline{ab}$ is $h \cdot \frac{h}{2} \cdot w \cdot \frac{h}{3} = \frac{h^3 w}{6}$. The moment of W about the same point is $W \times \overline{mn}$. As these two moments must be equal, we find the distance between the resultant pressures on joint $\overline{ab}$ for reservoir empty and reservoir full,

$$\overline{mn} = \frac{h^3 w}{6W}.$$

The above is substantially one of the methods adopted by Consulting Engineer A. Fteley in the design of the proposed Quaker Bridge Dam. See his interesting report, and that of B. S. Church, chief engineer, with many diagrams of existing dams of large proportions, in "Engineering News" for 1888, Jan. 7, 14, Feb. 4, 11; also the discussions by the editor in the numbers for Feb. 4 and 25, and March 3.

15. There are three well-known conditions, that must hold at any joint if the profiles $\overline{fa}$ and $\overline{db}$ have been designed correctly: —

1st, The points m and n where the resultants for reservoir empty or full cut the base $\overline{ab}$ must lie within the middle third of the joint or base $\overline{ab}$.

2d, The unit pressures of the masonry at the points a or b must not exceed a certain safe limit.

3d, No sliding must occur at any point.

16. The last condition is evident, and requires that $H < Wf$ where f is the coefficient of friction of masonry on masonry, the adhesion of the mortar being neglected. If ϕ is the angle of repose of masonry on masonry, $f = \tan\phi$, and we must always have,

$$\frac{H}{W} < \tan\phi\,;$$

that is, the resultant R must never make with the normal to the joint an angle greater than the angle of friction. In fact, in practice, we should employ some factor of safety as 2 or 3, so that $2H$ or $3H$ should always be less than Wf. This third condition is of supreme importance at the foundation joints of dock-walls, which fail (when they fail at all) by sliding from the insufficient friction afforded by the wet foundation. For ordinary retaining-walls, too, the foundation should, when practicable, be inclined, so that R shall make a small angle with the normal to the base. In all cases, deep foundations are to be preferred, as the earth in front of the wall resists the tendency to slide appreciably.

17. We shall now proceed to give a reason for the first condition above, and likewise deduce a formula to ascertain the unit stresses at the points a and b.

If we decompose the resultant R at the point n, distant $u = \overline{an}$ from a (Fig. 1), into its two components H and W, the former is resisted by the friction of the joint, and will be neglected in computing the stresses at a and b, though it doubtless affects them in some unknown manner. The remaining force W, acting vertically at n, must necessarily cause greater pressure at the nearest edge than elsewhere on the joint, at least when the angle at a is not too acute, and the dam is a monolithic structure. For large dams built of stones in cement, it is likely that there will be greater pressure at the middle of the base than in a monolithic structure where the resistance to shearing or sliding along *vertical* planes is much greater than in a wall made up of many blocks, particularly if they are laid dry. But it is probably best, until experiment can speak more decisively on the point, to assume the pressure greatest at the toe nearest the resultant, and as given by the following theory: —

Call l = length of joint ab

$u = \overline{an}$ = distance from R to nearest toe;

then if we suppose applied at the centre of the joint two vertical opposed forces, each equal to W, it does not affect equilibrium. We can now suppose the force W acting downwards at the centre to be the resultant of a uniformly distributed stress $p_1 = \frac{W}{l}$, shown by the little arrows just below joint $\overline{ab}$; and that the remaining forces W, one at the centre and one at n, acting in opposite directions, and constituting a couple, whose moment is $W\,(\frac{1}{2}l - u)$, cause a uniformly increasing stress, as in ordinary flexure (shown by the little arrows below the first), whose intensity at a or b is by known laws,

$$p_2 = \frac{My}{\mathrm{I}} = W\left(\tfrac{1}{2}l - u\right)\frac{\frac{1}{2}l}{\frac{1}{12}l^3} = W\frac{3l - 6u}{l^2}.$$

The total stress p at the nearest toe a is therefore the sum of p_1 and p_2, and is compressive.

$$\therefore\ p = 2\left(2 - \frac{3u}{l}\right)\frac{W}{l}. \qquad . \quad . \quad . \quad (1)$$

The stress at b is of course $p_1 - p_2$, where this is not minus indicating tension, unless the joint can stand the tension required. If we call u' the distance from n to the farthest toe, i.e. $u' = \overline{nb}$, we have the moment of the two weights $W = W(u' - \frac{1}{2}l)$. On substituting this value for $W(\frac{1}{2}l - u)$ in the value for p_2 above, we find for the unit stress at b the identical equation (1) above, provided we replace u by u'; so that the equation is general, and applies to either toe, if we only substitute for u the distance of the resultant from that toe. The stress is distributed, as shown by the lower set of arrows in Fig. 1, where there is only compression on the joint as should always obtain. The stress is thus uniformly increasing from the right to the left. If the limit of elasticity is nowhere exceeded, it follows that a plane joint before strain will remain a plane joint after strain, as must undoubtedly be the rule for single rectangular blocks.

Referring to equation (1), we see that if we replace u by $u' = \frac{2}{3}l$, that the stress at b is zero, from which point it increases uniformly to a, where its intensity, for $u = \frac{1}{3}l$, is $p = 2\frac{W}{l}$, or twice the mean. For greater values of u' than $\frac{2}{3}l$, the stress at b becomes tensile, which is not desirable; hence the reason for condition 1 above, that the resultant should lie within the middle third of the joint.

If the joint cannot resist tension at all, and R strikes outside the middle third, the joint will bear compression only over a length $3u$, and the maximum intensity at a is now $2\frac{W}{3u}$. This is evident, if we treat $3u = l'$ as the length of joint, and substitute this value for l in formula (1). There is now no pressure at the distance $3u = l'$ from the left toe by the previous reasoning for the original joint l, and to the right of that point the joint will open, or tend to open. It is evident for full security that the resultant should strike within the middle third some distance to allow for contingencies.

18. Having computed the unit pressures at the nearest toes for reservoir full or empty, condition 2 requires that these pressures do not exceed certain limits: in case they do, the lower profiles have to be revised, and the computation above repeated, until all the conditions are satisfied.

In the proposed design for Quaker Bridge Dam, maximum pressures per square foot at the toes, at the base, were limited to 30,828 lbs. at the back, and 33,266 lbs. at the face; these pressures diminishing gradually to one-half to within about 100 feet from the top, the total height of dam from the foundation being 265 feet; the argument being that the lower parts could stand more pressure than the upper parts shortly after construction, on account of the cement there attaining a greater strength. Besides, for this unprecedented height of dam, to keep the lower pressures within more usual limits "it would be necessary to spread the lower parts in an impracticable manner, and to incline the slopes to an extent incompatible with strength."

It is evident that by this method of design there is no fixed rule by which any two computers could arrive at the same profile, having given the upper part empirically, sufficient in section to carry a roadway, and to resist the additional stresses due to the shock of waves and ice, at a time, too, when the mortar is not fully set.

Such a rule is most easily introduced by requiring a certain factor of safety against overturning, and, moreover, that the factor of safety against sliding along any plane shall not fall below a certain amount. It is suggested, however, that the factors of safety should increase from the foundation upwards, to make the section equally strong everywhere against overturning, when allowance is made for the effects of wind and wave action, floating bodies, the expansive force of ice, or perhaps the malicious use of dynamite. If this is admitted, it would add one more condition (4) to the three previously stated, and would secure greater uniformity in design. See Appendix.

As to the unit pressure test (condition 2), it must be observed, that we know little or nothing as to what limit to impose; for not only is the stress all dead load (which would allow of higher unit stresses), but the unit resistance of masonry in great bulk is undoubtedly much greater than in small masses (not to speak of tests on small specimens as a criterion), since the shearing off which follows, or is an incident to, crushing can hardly occur in the interior of a large mass of masonry.

19. We shall find in the end, that, for different forms of retaining-walls to sustain earth, that a factor of safety of about 2.5 against overturning is highly desirable, and that it will generally satisfy the middle third limit. In such walls this factor must be introduced to provide against an actual increase of the earth-thrust, due to water, freezing, accidental loads, and above all to the tremors caused by passing trains or

vehicles (if these are not considered separately), which it is well-known have caused, by increased weight, and the increased pressure due to lowering the natural slope, a gradual leaning and destruction of walls of considerable stability for usual loads.

In a very high dam this is different: the pressure rarely changes but little, except on the upper portions; so that, if such conditions were to hold indefinitely, the limiting unit stresses should control the lower profile more than a factor of safety against overturning. But, as pointed out by the editor of "Engineering News" (in the issues above referred to), a dam on which the fate of a city may ultimately depend should be designed, as far as possible, to resist earthquakes also. For that contingency, there is a reason for the factor of safety against overturning and sliding being as great as possible throughout; and by putting the gravity dam in the arch form, convex up stream, the resistance to earthquake and other shocks is enormously increased.

20. We have now given the general prin-

ciples that should guide in the design of dams, which likewise apply in the design of retaining-walls proper, where, however, the height is rarely sufficient to call for much, if any, change of profile, and the maximum pressures are usually far within safe limits when a proper factor of safety against overturning or sliding has been introduced, which satisfies likewise the condition that the resultant shall cut the base within the middle third. We of course have, as stated before, the direction of the earth-thrust inclined below the normal to the wall at the angle of friction ; otherwise, the methods above are applicable when the value of that earth-thrust has been determined. For dock or river walls, saturated with water, The buoyant and lubricating effect of the water must be considered.

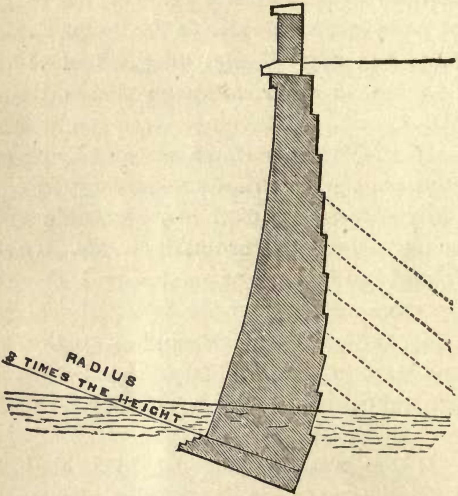
If we suppose the filling of gravel, the water surrounding each stone allows freedom of motion ; but the weight of the solid stones of the filling must now be taken less than when in air, by the weight of an equal volume of water, or at the rate of 62.4 lbs. per cubic foot (or say 64 for salt water), and the earth-thrust then found for the angle of repose of stone lubricated with water. Thus, if the weight of the solid stone be 150.4 lbs. per cubic foot, and the voids are thirty per cent, the weight of solid stone in water is 88 lbs. per cubic foot, and that of the filling $88 \times .70 = 61.6$ lbs. in water, although it was 105 in air.

If the wall is founded on a porous stratum, the weight of the masonry is similarly reduced by 62.4 lbs. per cubic foot, or say one-half ordinarily; but if the foundation is rock or good clay, "there is no more reason why the water should get under the wall than it should creep through any stratum of a well-constructed masonry or puddle-dam," as Mr. Baker has observed.

If the water cannot get in behind the wall, the water in front only assists the stability.

It has been previously observed that sliding is principally to be guarded against in dock-walls and others similarly situated, which can only be done by a sufficient weight of masonry irrespective of its shape, unless the foundation is inclined, which even in the case of piling has been effected

Fig. 2



by driving the piles obliquely, of course as nearly at right angles to the resultant pressure as is practicable.

Fig. 2 represents a wall with a curved batter, in brickwork with radiating courses, that might be used for a quay or river-wall, or a sea-wall, as ships can come closer to the brink than in the case of a straight batter; besides, for sea-walls it resists the action of the waves better. The centre of gravity can be found by dividing the cross section up into approximate rectilinear figures, and proceeding as in finding the position of *W* in Fig. 1. Its position is a little farther back than for a straight batter, which adds to its stability. But it is difficult to construct, the joints at the back are often thicker than is advisable, and there is probably no ultimate economy in its use.

CHAPTER II.

THEORY OF RETAINING-WALLS.

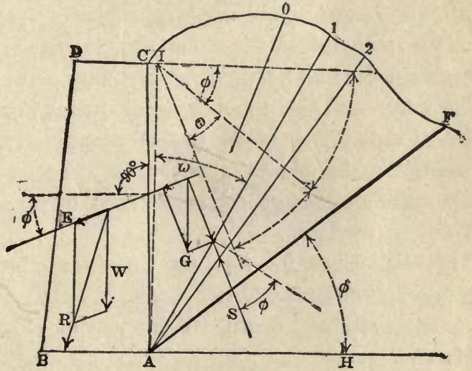
Graphical Method.

21. In the theory of earth-pressure that follows, we shall consider the earth as a homogeneous, compressible mass, made up of particles possessing the resistance to sliding over each other called friction, but without cohesion. This is a much simpler definition than the one that Rankine's theory calls for (see Art. 9), and is more true to nature; the only approximation, in fact, consisting in neglecting cohesion, if we consider a homogeneous earth like dry sand.

Let Fig. 3 represent a vertical section of a retaining-wall *ABCD*, backed by earth, whose length perpendicular to the plane of the paper is unity.

Assumption. We assume that the earth behind the wall, whether the top surface is a plane or not, has a tendency to slide along some *plane* surface of rupture as *A*1, *A*2, ...

Fig. 3



No proof is given of this assumption, so that it can only be tested by experiment; but for the present we shall adopt it.

In connection with the hypothesis of a plane surface of rupture, we shall use only one principle of mechanics relative to the stability of a granular mass, first stated by Rankine as follows: —

It is necessary to the stability of a granular mass, that the direction of the pressure between the portions into which it is divided by any plane should not, at any point, make with the normal to that plane an angle exceeding the angle of repose.

This principle will alone enable us to ascertain the earth-thrust against any plane without resorting to a special principle, like Coulomb's "wedge of maximum thrust," which last, however, will be incidentally demonstrated as a consequence of the above law.

22. In Fig. 3, let us consider the triangular prisms $CA0$, $CA1$, . . . , as regards sliding down their bases $A0$, $A1$, ...

If AF is the natural slope of the earth, the tendency of the prism CAF to slide along AF is exactly balanced by friction, as is well known. But if we consider other possible planes of rupture, lying above AF, as $A0$, $A1$, . . . , we see, unless the wall offers a resistance, that sliding along some one of these planes must occur: so that the earth exerts an *active* thrust against the wall, which must be resisted by it; otherwise, overturning or sliding would ensue.

In case the wall is subjected to a thrust from left to right, as from earth, water, etc., acting on BD, and this thrust is sufficient to more than counterbalance the *active* thrust of the earth to the right of the wall, it will bring in the *passive* resistance of the earth to sliding up some plane as $\overline{A2}$, and the surface of rupture will now resist motion upwards, in place of downwards as hitherto.

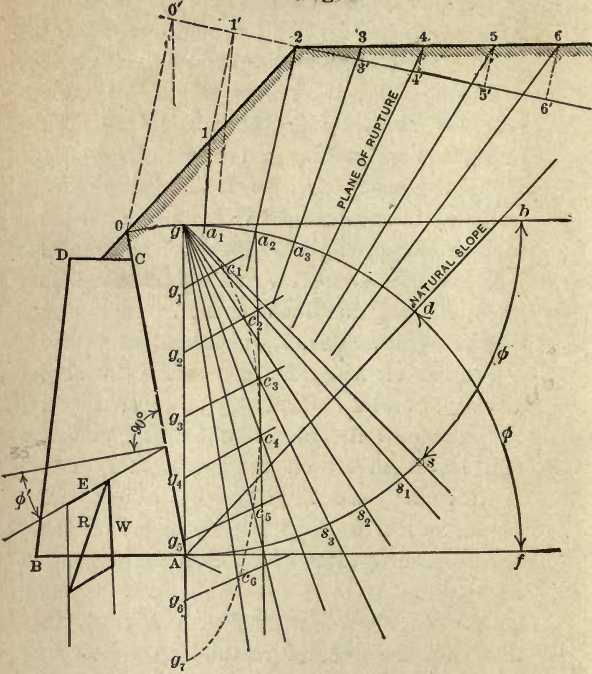
In the first case, of active thrust, where the prism is just on the point of moving down the plane, we know by mechanics that the resultant pressure on the plane of rupture makes an angle ϕ of friction of earth on earth with the normal to that plane and directed *below* the normal; in the second case, of *passive* thrust, the direction of the pressure lies above or nearer the horizontal than the normal, and makes the angle ϕ with the latter.

23. In the first case, where the wall receives only the active thrust of the prism of maximum thrust, let us call G (Fig. 3) the weight in pounds of this prism; S the resultant pressure on the surface of rupture, making an angle ϕ with the normal to that plane below the normal; and E the resultant earth-pressure on the wall, which (except for cases to be noted in Art. 31) makes an angle ϕ' of friction of earth on wall with the normal to the wall below the normal, unless $\phi' > \phi$, in which case a thin layer of earth will go with the wall, in case of relative motion, and this layer rubbing against the remaining earth will only cause the friction of earth on earth, and E will only be directed at an angle ϕ below the normal; supposing always that the tendency to relative motion corresponds to the earth moving *down* along the back of the wall AC, as in settling from its compressibility, or as in case of an incipient rotation of the wall forward, from a greater pressure on the outer toe or a slight unequal compression of the foundation.

It remains to find the position of the true plane of rupture. As preliminary to this, we note from Fig. 3 an expeditious way of finding the direction of S on any trial plane of rupture, as $A1$. Thus calling ω the angle that $A1$ makes with the vertical AI, the straight line making an angle $(\phi + \omega)$ with any horizontal, as DCI, below that horizontal, is parallel to S, since any line inclined at an angle ω below the horizontal is perpendicular to $A1$, and S is inclined at an angle ϕ below that normal. In laying off the equal angles, it is convenient to use a common radius, AH, to describe the arcs having A and I respectively as centres, and to take chord distances of the arcs ϕ and ω, and lay them off on the arc with I as a centre, as shown. For any other trial plane, as $A2$, we have simply to lay off the corresponding value of ω below the angle ϕ as before.

24. We shall now refer to Fig. 4, to illustrate the general method to follow to find the earth-thrust E in pounds. Here the wall, one foot long perpendicular to the plane of the paper, is shown in section $BACD$, the earth sloping at an angle from

Fig. 4



some point on the top of the wall to the point marked 2, where it is horizontal. This is called a *surcharged* wall, the earth lying above the horizontal plane of the top of the wall being called the *surcharge*.

Extend the line $\overline{AC}$ of the inner face to 0, where it intersects the top slope of the earth; the possible prisms of rupture are then $A01$, $A02$, $A03$, . . . , and we shall now proceed to reduce these areas to equivalent triangles having the same base $A2$. Draw the parallels $00'$, $\overline{11'}$, $33'$, . . . , to line $\overline{A2}$ to intersection with a perpendicular to $\overline{A2}$, passing through the point 2. Then the triangle $A02$ is equivalent to the triangle $A0'2$, and $A12$ to $A1'2$, so that triangle $A0'1'$ is equivalent to $A01$. Similarly $A26$ is equivalent to triangle $A26'A$, having the same base, $A2$, and vertices in a line parallel to this base, giving the same altitude. Thus the area $A026A$ is replaced by $A0'6'A$; and the weight of the corresponding prism, if we call e the weight per cubic foot of earth, is $\frac{1}{2}\overline{A2} \times 0'6' \times e$. Similarly the weight of $A024$ is $\frac{1}{2}\overline{A2} \times 0'4' \times e$; so that if we use $0'1'$, $0'2$, $0'3'$, . . . , to represent the weights of the successive prisms $A01$, $A02$, $A03$, . . . , on the force diagram given below, we have simply to multiply the value of E, given by construction, by $\frac{1}{2}e, \overline{A2}$ to find its true value in pounds.

We next lay off the successive values of $(\phi + \omega)$, as in Fig. 3. Thus, with any convenient radius, as $\overline{A0}$, we describe an arc, *ogdf*, and call the intersections with $\overline{A1}$, $\overline{A2}$, . . . , a_1, a_2, a_3, . . . , respectively. Next, through point g on the arc in the vertical through A, draw vertical and horizontal lines, and describe an arc, bss_1, . . . with the same radius; then draw $\overline{gs}$, making the angle ϕ below the horizontal $\overline{gb}$ (by making chord $\overline{bs}$ = chord $\overline{fd}$), and lay off with dividers, chords ss_1, ss_2, ss_3, . . . , equal to chords ga_1, ga_2, ga_3, . . . It is evident now that lines gs_1, gs_2, gs_3, . . . , make the angles ϕ with the normals to the successive planes $A1$, $A2$, $A3$, . . . , and thus give the direction of the S's corresponding to those planes.

We now lay off with dividers on the vertical line $\overline{gA}$ the distances gg_1, gg_2, . . . equal respectively to $0'1'$, $0'2$, $0'3'$, . . . , and draw through the points g_1, g_2, g_3, parallels to the direction of E (drawn as before explained) to intersection with the lines $\overline{gs_1}$, $\overline{gs_2}$, $\overline{gs_3}$, . . . , which intersections call c_1, c_2, c_3, . . . , respectively.

25. It follows that the lines g_1c_1, g_2c_2, g_3c_3, . . . , represent the thrusts E due to the successive prisms of rupture $A01$, $A02$, . . . , and we shall now prove that the greatest of these lines, which is found to be $\overline{g_4c_4}$, represents the actual active thrust upon any stable wall.[1] This follows from the simple fact, that if we regard any other thrust than the maximum as the true one, on combining this lesser thrust, taken as acting to the right, with the weight of the wedge of rupture corresponding to the maximum thrust, we necessarily find that the resultant falls below the position first assumed ; so that it makes an angle with the normal to the corresponding plane of rupture greater than the angle of repose, which, by the principle of Art. 21, is inconsistent with stability. Thus, in Fig. 4, if we choose to assert that *any* trial thrust, as $\overline{g_2c_2}$, less than the maximum $\overline{g_4c_4}$, is the true one, on

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[1] This method of laying off the trial thrusts, so that the maximum could readily be obtained, was first given by Professor Eddy, in New Constructions in Graphical Statics.

shortening the lengths $\overline{g_3c_3}$, $\overline{g_4c_4}$, . . . , representing superior thrusts, to the common length $\overline{g_2c_2}$, and drawing through the new positions of c_3, c_4, . . . , straight lines to g, which thus represent the resultant thrusts on the planes $A3$, $A4$, . . . , we see that the new directions fall below the first assumed positions, and therefore make angles with the normals to the planes greater than ϕ, which is absolutely inconsistent with equilibrium. It follows that any thrust less than the maximum, as determined by the construction above, is impossible; and that this maximum thrust thus found is the actual *active* thrust exerted against the wall. In this consists what is known as Coulomb's "wedge of maximum thrust," which is here established by aid of the single mechanical principle enunciated in Art. 21.

The prism of rupture in this case is $A024A$, the plane $\overline{A4}$ being the *surface of rupture.*

To find the resultant thrusts on all the other assumed planes, we combine the actual thrust found with the weight of earth lying above the plane. Thus, *extending* g_1c_1, g_2c_2, . . . to a common length g_4c_4, or to the vertical tangent to the dotted curve, the lines drawn from g through the corresponding intersections with this vertical will represent the thrusts on the planes $A1$, $A2$, . . . , which are thus inclined nearer the horizontal than the old trial values, and thus make less angles than ϕ with the normals to their corresponding planes ; so that the conditions of stability are all satisfied, and, if the wall gives, sliding will only occur down the plane of rupture $A4$.

In the analytical method followed by Weyrauch, E is assumed to be constant, and to equal the actual thrust on the wall ; and the real surface of rupture is taken to be that plane for which the angle that S (Fig. 3) makes with the normal is the greatest (ϕ) consistent with equilibrium, which is in agreement with what we have just proved

Winkler adopts the same method, in preference to the Coulomb method. In fact. he asserts that "no author, from Coulomb down, has given any direct satisfactory proof of Coulomb's principle." It is hoped that the above demonstration will prove complete and satisfactory. The method evidently gives the least thrust, for the assumed direction of E, that will keep the mass from sliding down the surface of rupture.

The earth can *resist* a much greater pressure from the wall side, since a continuously increasing pressure from the left causes all the resultants on planes $A1$, $A2$, . . . to *approach* the normals, then to pass them, and finally to lie above them, with the sole condition that none of them must make angles greater than ϕ with their corresponding normals (see Art. 34).

26. *To find the thrust E in pounds*, we multiply g_4c_4 to scale by $\frac{1}{2}\overline{A2}.e$. Finally, if we know the position of E, we combine it with the weight of the wall in pounds, acting along the vertical through its centre of gravity, to get the resultant on the base. If the upper surface of the earth is level, or with a uniform slope from the point 0 (Fig. 4), then the sections of the prisms of rupture for various heights of the wall, or

for any values of $\overline{A0}$, are similar triangles, so that the thrusts E, which vary directly with the weight of the corresponding prisms, will also vary directly as the areas of these triangles, or as the squares of the homologous lines $\overline{A0}$, or as the squares of the height of point 0 from the base AB. It follows, as in the case of water-pressure, that for these cases the resultant E of the earth-thrust acting along the face $A0$ is found at a point $\frac{1}{3}\overline{A0}$ along $\overline{A0}$ from the base AB.

For the surcharged wall it is possibly higher; in fact, Scheffler takes it in constructing his tables, for all cases, at $\frac{4}{10}\overline{A0}$ along $\overline{A0}$. But experiment indicates either that the thrust is overestimated for surcharged walls, or that it acts not higher than at one-third the height of 0 above the base; so that it will prove safe in practice to take the latter limit if we use the theoretical thrust. As to the latter, it is evident that cohesion (which we have neglected) has a greater area to act upon along the surface of rupture for any kind of surcharged wall, than for earth either level or sloping down from the top of the wall; so that we should expect the thrust to be somewhat overestimated when we neglect cohesion altogether, since the resistance to sliding down any plane due to it is directly as the area of the surface of separation.

27. In case the earth is level with the top of the wall, the construction of Fig. 4 again applies, only the line $\overline{0'6'}$ now coincides with the horizontal through C, and the reduction of areas to equivalent triangles is omitted, since now all the triangles have the same altitude, equal to the height of the wall.

If, however, the earth slopes uniformly from the top of the wall, at a less angle than the angle of repose, we can assume any point as 2, on this slope, and effect the construction of Fig. 4 as before; or, better, we can divide this slope into a number of parts at 1, 2, . . . , and treat $\overline{01}$, $\overline{02}$, . . . , successively as the bases and the perpendicular from A upon $\overline{02}$, produced as the common altitude; so that, using $\overline{01}$, $\overline{02}$, . . . , as representing the weights of the corresponding prisms on the load line gg, we have finally to multiply the value of gc, corresponding to the greatest thrust, by $\frac{1}{2}e$, multiplied by this perpendicular, to get the maximum thrust E in pounds.

In case the surface of the earth slopes indefinitely *at the angle of repose*, the graphical method fails to find the surface of rupture, which analysis shows, in this case, to approach indefinitely to the plane of natural slope passing through the point A, though practically it may be shown that planes of rupture slightly above the latter will give almost identically the same earth-thrust, so that they can safely be used. In fact, it is well to state here, that, for earth-level at top, the surface of rupture, as observed in experiments with every kind of backing, agrees very well with theory; but, as the surcharge grows higher, the actual surface of rupture lies nearer the vertical than the theoretical, and the thrust is correspondingly less, particularly for walls leaning backwards at top, which, for a high surcharge, actually receive much less thrust than the simple theory after Coulomb's hypothesis, neglecting cohesion, calls for; and it is not surprising that it is so. But we shall defer the comparison of numerical results till later.

28. *Case where E does not make the angle ϕ or ϕ' with the normal to the wall.*

In Fig. 5, let $\overline{AC}$ represent the inner face of the wall, backed by earth sloping upwards from C in the direction $C - 10$. *There are certain positions for the wall AC lying to the left of the vertical $\overline{Ag}$, for which the true thrust on it is found by ascertaining the thrust on the vertical plane $\overline{A0}$, extending from the foot of the wall A to where it intersects the top slope C — 10, having assumed the direction of the thrust on $\overline{A0}$, after Rankine, as parallel to the top slope, and combining this thrust, acting at $\frac{1}{3}\overline{A0}$ above A, with the weight of the mass of earth, A0C, lying between $\overline{A0}$ and $\overline{AC}$, acting along the vertical through its centre of gravity.* The thrust on $\overline{A0}$ is thus combined with the weight of $A0C$, at a point on $\overline{AC}$, one-third of its length going from A to C.

This direction of the thrust on $\overline{A0}$ parallel to the top slope is in agreement with Rankine's principle for the case of an unlimited mass of earth of the same depth everywhere, on an uniformly compressible foundation (Art. 9), and doubtless agrees very nearly with the direction and amount of the earth-thrust in ordinary cases, except near comparatively rigid retaining-walls, or other bodies, where the direction is generally changed, as previously pointed out. Let us ascertain the limiting position of AC, below which the true thrust must be ascertained in the manner just stated. To do this, we first assume the thrust on $\overline{A0}$ as acting parallel to the top slope, and find its intensity corresponding by previous methods; and afterwards prove, for positions of $\overline{AC}$ below the limit, to be found by construction, that no thrust on $\overline{A0}$ having a less inclination to the vertical is consistent with equilibrium.

The construction necessary to find the thrust on $\overline{A0}$, from the earth on the right, is similar to that given for Fig. 4, except that the top slope is now uniform, and will only be briefly indicated. Thus, divide the top slope $0 - 10$, to the right of $\overline{Ag}$, into a number of parts, made equal for convenience, and draw through the points of division lines from A produced on to meet the arc described with $\overline{Ag}$ as a radius at the points $a_1, a_2, \ldots$ Then with g as a centre, and $\overline{gA}$ as a radius, describe a semicircle as shown; also draw $\overline{gb}$ horizontal, and lay off arc bs equal to ϕ, the angle of repose, and from s lay off arcs ss_1, ss_2, . . . , equal to ga_1, ga_2, and draw the lines gs_1, gs_2, . . . , from g through the extremities of these arcs to represent the directions of the resultants on the successive planes of rupture, which are thus inclined below the normals to those planes at the angle ϕ respectively. Next, on the vertical $\overline{gA}$, lay off gg_1, gg_2, . . . , equal to the bases 01, 02, of the supposed prisms of rupture lying to the right of Ag, and through their extremities draw $g_1 1'$, $g_2 2'$, . . . , parallel to top slope to intersection $1'$, $2'$, . . . , with the directions of the resultants first found. The greatest of these lines cc', to scale, represents the actual thrust on $A0$; and we have only to multiply it by $\frac{1}{2}ep$, where p is the perpendicular let fall from A on the top slope $0-10$ produced, to scale, to get the pressure in pounds, if desired.

Now, if the direction of the pressure on the wall AC cannot be taken as usual, inclined below the normal to AC, at an angle ϕ, it is (Art. 7) because, in case of motion, the earth does not rub against the wall sufficiently to develop the required friction, whence it must follow that the earth breaks along some plane as $A4$, $A5$, . . . , to the left of Ag, where the thrust is inclined at the angle ϕ to its normal ; so that this plane is a veritable *plane of rupture*, and its position can be found as usual on assuming the direction of the thrust on $\overline{A0}$ as parallel to the top slope.

In case such a plane exists between $\overline{A0}$ and $\overline{AC}$, the earth below it, if the wall moves, will go with the wall ; further, it is evident that the thrust against the vertical plane $\overline{A0}$, due to the wedge of rupture on the left, must exactly equal the thrust first found corresponding to the wedge of rupture on the right, otherwise equilibrium will be impossible.

To ascertain the position of this plane of rupture on the left, that we shall hereafter call the *limiting plane*, most accurately, it is well to magnify the lines representing the forces as much as the limits of the drawing will admit of. We have consequently divided the top surface, $\overline{0C}$, into a number of equal parts, of which the first eight are only one-fourth the length of the corresponding parts to the right of $\overline{A0}$. By laying off the loads gg_1, gg_2, . . . , however, to a scale four times as large as just used, we have the lengths gg_1, gg_2, . . . , exactly four times the lengths 01, 02, . . . , along the surface to the left of 0, so that the old lettering applies again.

We now produce the lines $A1$, $A2$, . . . , to intersection n_1, n_2, . . . , with the arc gn (it is obvious that the top slope, $\overline{0C}$, should best be drawn, in the first instance, through g, for accurately fixing the positions of n_1, n_2, . . .) ; then lay off, below the horizontal, the angle $dgm = \phi$; and from m, the intersection of gm with the semicircle dAb, lay off the arcs mm_1, mm_2, . . . , equal to gn_1, gn_2, . . . : so that the lines gm_1, gm_2, . . . , all make angles equal to ϕ with the normals to the corresponding planes $A1$, $A2$, . . .

Next, on drawing through g_1, g_2, . . . , lines parallel to the assumed direction of the thrust on $A0$, to intersection with the corresponding lines gm_1, gm_2, . . . , the greatest of the intercepts (g_5t_5 nearly) represents, to the scale of loads, the thrust on the plane $A0$; and this length should exactly equal four times the length $\overline{cc'}$ representing the thrust from the right, as we find to be the case. The plane of rupture to the left of the vertical through A thus coincides nearly with $A5$, which is marked "limit" on the drawing. [On a larger drawing, for $\phi = 33° \, 42'$ and the top sloping at 25°, the limiting plane was found to make an angle of 15° to 16° (see a more accurate determination in Art. 41) to the left of the vertical Ag, and to lie slightly below $A5$, as this drawing would indicate.]

If we lay off along the lines parallel to top slope, through g_1, g_2, . . . , the true thrusts, g_1t_1, g_2t_2, . . . , g_7t_7, . . . , the directions of gt_1, gt_2, . . . , gt_7, . . . , of the true thrusts on the planes $A1$, $A2$, . . . , $A7$, . . . , all necessarily lie above the first assumed directions; so that the actual thrusts on all planes other than $A5$ (which we shall regard as the plane of rupture, for convenience), lying above or below $A5$, make less angles with the normals to those planes than the angle of friction, just as we found in Art. 25.

The conditions of stability of Art. 21 are thus satisfied in the present case; but it is evident that this is no longer so if we lower the direction of the thrust on $\overline{A0}$, which lessens the horizontal component of the thrust from the right, since intersections like $6'$, $7'$, in the right diagram move towards the vertical $\overline{Ag}$, though the reverse obtains for the diagram to the left, which of itself indicates some absurdity. If, now, we combine the new thrust on $\overline{A0}$ from the right (which has a less horizontal component than before) with the wedges of earth lying to the left of $\overline{A0}$, it is readily seen that the directions of some of the resultants, as gt_5, ... , will fall below their first positions, and will thus make greater angles with the normals to their planes than the laws of stability will admit of; so that any lowering of the first assumed position, parallel to the top slope, of the thrust on $\overline{A0}$, is impossible.

We thus reduce to an absurdity every other case but the one assumed, which is therefore true; so that the proposition enunciated at the beginning of this article is demonstrated.

We see, therefore, that we cannot, as before, assume the direction of the thrust on the wall, $\overline{AC}$, as having the direction gm_c, making the angle ϕ with the normal to $\overline{AC}$, and find the wedge of maximum thrust corresponding; but that its true direction, gt_c, is found by combining the thrust found on $\overline{A0}$, acting parallel to top slope, with the weight of the wedge of earth, $0AC$, between the wall and the vertical plane $A0$; otherwise, if the left diagram is constructed, we find its direction and amount in a similar manner to that used in finding the direction, etc., of gt_7, . . . , by laying off on $\overline{gA}$ (produced if necessary) $\overline{0C} \times 4$; from the end of this line we draw a parallel to the top slope $\overline{0C}$ to intersection t_c, with the vertical through t_5. The line $\overline{gt_c}$ to the last scale used multiplied by $\frac{1}{2}ep$ (where p is the perpendicular from A on $\overline{0C}$ to the scale used in laying off $\overline{0C}$) gives the thrust E against the wall in pounds. It is laid off in position by drawing a line parallel to gt_c through a point on $\overline{AC}$, $\frac{1}{3}\overline{AC}$ above A, as previously enunciated.

29 ($\phi' < \phi$). In case this construction gives a thrust on the wall which makes a greater angle with its normal than the co-efficient of friction, ϕ' of wall on earth, ϕ' being less than ϕ, then it is correct to assume the direction of E as making this angle ϕ' with the normal, and proceed as in Fig. 4 to find the thrust. In the preceding article, no trial-thrust on the vertical plane $\overline{A0}$ was assumed to lie nearer the horizontal than the top slope, as there was no reason for considering such exceptions to the usual direction in a mass of unlimited extent. Now, however, the wall requires the thrust on $\overline{A0}$ to lie nearer the horizontal than $\overline{0C}$ does, in which case the horizontal component will be increased (since intersections like 7′, 8′, move away from the vertical $\overline{Ag}$), and the thrusts on all planes $A1$, $A2$, . . . , lying to the left of $\overline{Ag}$, will be raised above

their previous positions, gt_1, gt_2, . . . ; so that the thrusts on all the planes now make less angles than ϕ with the normals to those planes, so that the conditions for stability of "the granular mass" are assured.

30. The "limiting plane," corresponding to the plane of rupture on the left, can be found by a different construction from that given above. Thus, having found the line $\overline{cc'}$ representing the maximum thrust from the earth to the right of $\overline{A0}$, multiply by 4, say, and combine with the successive wedges of earth lying to the left of $A0$, on magnifying the lines $\overline{01}$, $\overline{02}$, . . . , in the same proportion, thus giving the lines gt_1, gt_2, . . . , for the direction of the thrusts on the planes $A1$, $A2$, . . . ; these all lie above the directions gm_1, gm_2, . . . , making the angles ϕ with the normals to the planes, except for the limiting plane, where gt_5 and gm_5 nearly coincide, as they should exactly if $A5$ was the limiting plane. The lowest relative position of $\overline{gt}$ with respect to $\overline{gm}$ is, of course, the one selected. It is evident, though, that the construction for the wedge

of greatest thrust to the left of Ag gives a more accurate evaluation of the thrust than the one to the right; so that we can preferably use the left construction not only for getting the limiting plane, but for finding the thrust on any wall lying below the limiting plane.

It is evident, from what precedes, that the double construction of Art. 28 applies *only* when the thrust on $A0$ is parallel to the top slope; for the moment it is lowered, there results several planes of rupture to the left of $A0$, which is impossible. Even if we attempt the left construction, we have seen besides that the resulting thrust on $A0$ is greater than by the construction on the right.

In case the face of the wall, AC, lies above the "limiting plane," as found before, we evaluate the thrust on it, as in Fig. 4, by assuming its direction to make an angle with the normal equal to ϕ or to ϕ' when $\phi' < \phi$. Thus, if the inner face of the wall had the position $A2$, to the left of $A0$, the direction of the thrust on it would now be gm_2, in place of gt_2 as before, and the conditions of stability of the granular mass will be found to be everywhere verified as in Fig. 4 (see Art. 25).

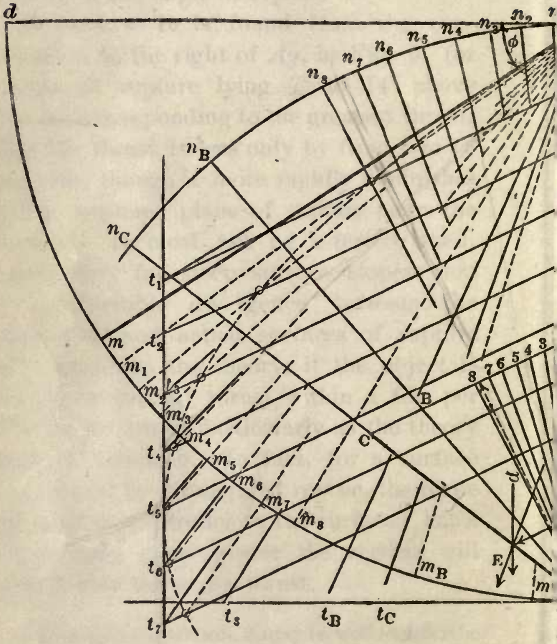
31. *Summary.* — For all cases of top slope, when the inner face of the wall is battered, we first find the limiting plane by the construction of Art. 28; then when the inner face of the wall makes a less angle with the vertical than the limiting plane does (as is nearly always the case in practice, unless the surface of the earth slopes at or near the angle of repose, in which case the limiting plane is at or very near the vertical), we assume the direction of the thrust on it as making the angle ϕ or ϕ' (for $\phi' < \phi$) with the normal, and proceed as in Art. 23, *et seq.*; but, if the face of the wall lies below the limiting plane, we proceed as in Art. 28, or if $\phi' < \phi$ we may have to proceed as in Art. 29, to find the true thrust.

If the wall leans backward, there is no need to find the limiting plane, as the usual construction applies.

For earth level at top, the limiting plane is inclined to the left of the vertical equally with the plane of rupture to the right; as the top slope increases, it approaches the vertical, and coincides with it for the surface sloping at the angle of repose.

Remark. — It is found from the construction to the right of $\overline{Ag}$, in Fig. 5, for planes of rupture lying 7° to 14° above the one corresponding to the greatest thrust, that the thrust is less only by from 6 to 16 per cent, though it more rapidly diminishes as the assumed plane of rupture nears the vertical. It must not be inferred, then, particularly for steep surface slopes, that a considerable divergence between the theoretical and actual surfaces of rupture will invalidate the theory, if the object is simply to get the thrust within a few per cent of the truth, particularly as the theory neglects cohesion. In fact, for a surface slope equal to the angle of repose, the plane of rupture is parallel to the surface; but a plane lying much nearer the vertical will give nearly the same thrust.

32. In this connection, it may be well to describe an experiment made by Lieut.-Col. Audé in 1848, and repeated subsequently by Gen. Ardant, M. Curie, and M. Gobin, on a peculiar retaining-wall


d
n_2
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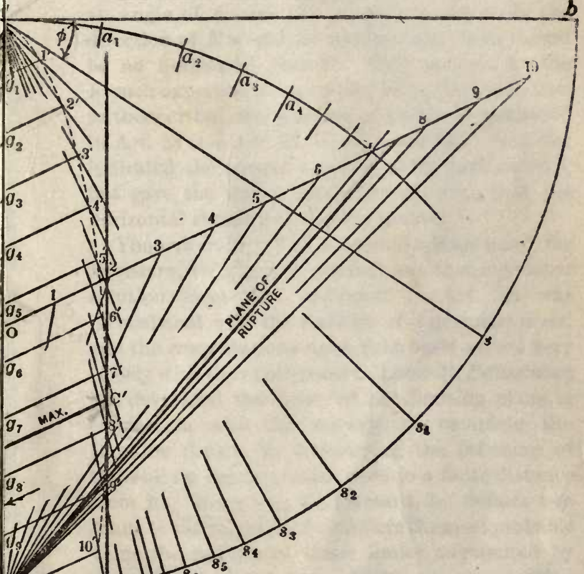

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g_1
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O
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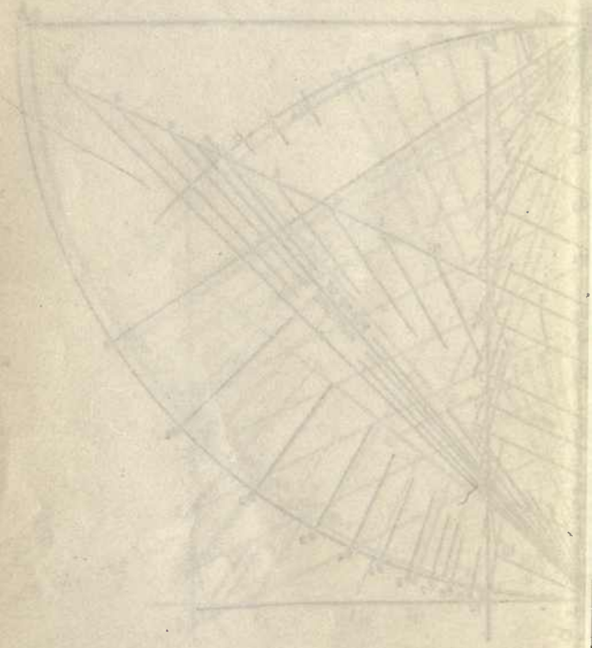
g_7
MAX.
g_8
g_9
g_{10}
C'
PLANE OF
RUPTURE
s_1
s_2
s_3
s_4
s_5
s_6
s_7
s_8
s_9

g: 5



made of a triangular block or frame, in which the inner face was inclined to the horizontal at the angle of repose of the sand backing, when, of course, by the usual assumption as to E making an angle of ϕ with the normal to the wall, the direction of E would be vertical, and there should be no horizontal thrust! This seemed, to the French experimenters, to offer a puzzling objection to theory; but the solution is clearly as indicated in Art. 28 (see Art. 67, Exps. 9 and 10). Scheffler indicated the correct solution as far back as 1857, but gave the wrong reason for it; viz., that the horizontal thrust was thereby greater.

The writer, in "Van Nostrand's Magazine" for February, 1882 (p. 99), pointed out that any other solution than that indicated in Art. 28 was inconsistent with the stability of a granular mass, and the computations upon that basis agreed very closely with the experiments. Later M. Boussinesq has developed the theory of the limiting plane in connection with the attempt to complete the Rankine theory, by considering the influence of the wall on the pressures even to a finite distance from it. According to Flamant, he defines two limits to the thrust, and considers the most probable value the smaller of these limits augmented by $\frac{9}{22}$ of their difference. From an examination of the numerical values computed by Flamant ("Annales des Ponts et Chausses," April, 1885), the results do not differ greatly from those given by the simple theory alone used in this work.

33. The disturbing influence of the wall in changing the normal character of the stresses can be illustrated as follows: If the thrust on the vertical plane $\overline{AO}$ (Fig. 5) acts parallel to the surface $\overline{0-10}$, it meets the plane of rupture at one-third of its length above A, through which point the weight of the prism of rupture acts also; so that the resultant on this plane acts at this point, which corresponds to a pressure on the plane of rupture uniformly increasing from the surface downwards. If, however, the wall causes the thrust on $\overline{AO}$ to make a $\left\{\begin{matrix}\text{less}\\ \text{greater}\end{matrix}\right\}$ angle with the horizontal, the resultant on the corresponding plane of rupture on the right acts $\left\{\begin{matrix}\text{below}\\ \text{above}\end{matrix}\right\}$ the point situated at one-third of the length of the plane above A, so that the pressures on it are no longer uniformly increasing. This abnormal state, doubtless, does not extend far into the mass before the usual direction of the thrust in a large mass of earth is attained; but the fact throws doubt upon the assumption of a *plane* surface of rupture for all cases where the direction of the thrust on the vertical plane does not act parallel to the upper surface.

It appears reasonable to suppose, if the line through the centres of pressure on all sections of a retaining-wall passes through their centres of gravity, that no rotation of the wall occurs; further, if it was possible for the masonry and earth or rock backing to settle together the same amount, — the backing, say, having been carefully deposited in horizontal layers, — then, even for a level-topped bank, the maximum thrust, as given by Rankine's formula, will be exerted, and there will be no friction at the back of the wall to change the usual direction of the earth-thrust in a large bank. If the wall has not the stability, or the settling is not as assumed, the top will move over, friction at the back of the wall is exerted, and the horizontal thrust becomes smaller than before, corresponding to a different prism of thrust, as we ascertain by the construction of Fig. 4, for the two cases of E horizontal and E inclined downwards. The excess of the horizontal thrust in one case over that in the other must necessarily be resisted by the ground-surface, on which the filling rests by friction, which it is generally capable of doing. If not, then the Rankine thrust will be exerted. Similarly, if we consider any road embankment, whose sides slope at the angle of repose, the horizontal thrust on some longitudinal plane in the interior must be finally resisted by the ground to one side under the embankment. If, however, the weight of earth above, multiplied by the co-efficient of friction of earth on ground-surface, is less than the horizontal thrust, the earth must slide, and the slope become flatter, until equilibrium obtains from a less horizontal thrust. Scheffler computes for an embankment of triangular section where $\phi = 45°$, and the angle of friction on the ground-surface is only 5°, that the slope of the embankment would change to 32° 15′. For the ground-friction angle = 7° 7′ 20″ there would be exact equilibrium; so that, generally, there need be no fear from spreading of embankments due to this cause, as the amount of friction required is very small.

34. We have now given methods for finding the thrust against a retaining-wall, which simply resists this *active* thrust of the earth, for the usual cases of a surcharged wall and earth-level at the top, to which may be added the case of earth sloping downwards from the top of the wall to the rear, for which the construction is evident. It now remains to find *the passive resistance of the earth to sliding up some inclined plane due to an active thrust of the wall* from left to right (Fig. 4), caused by water, earth, or any other agency acting against the wall on the left. Now (Art. 22) we lay off the angle *bgs* (Fig. 4) above $\overline{bg}$, and then, from the new position of *s*, lay off arcs ss_1, ss_2, . . . , below *s* equal to ga_1, ga_2, . . . , as before, giving the direction of gs_1, gs_2, . . . , inclined at angle ϕ above the normal to the

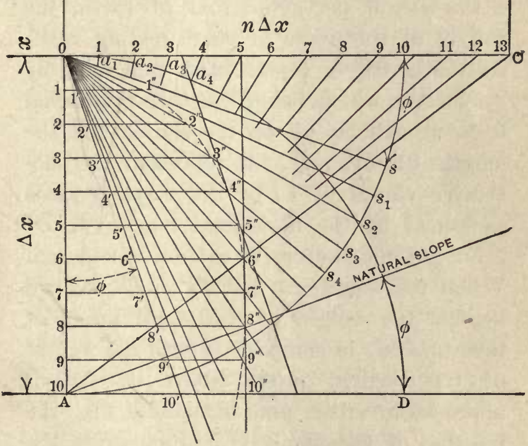
corresponding planes $A1$, $A2$, . . . The construction then proceeds as before, only it is now the least of the resistances, $\overline{gc}$, that represents the *passive* resistance of the earth to sliding up the plane of rupture corresponding; for any increase over this causes the thrust on some planes to make greater angles than ϕ with the normals, as is easily shown. Let us call N the component normal to AC of the resistance and suppose a slight movement of the wall horizontally to the right; then since the earth moves *upwards* along the plane of rupture and the plane AC, the friction of the earth along AC, $N \tan \phi$, acts upwards and the resistance of the *wall* downwards.

The thrust E is now inclined at the angle ϕ above the normal to AC and not below as formerly. The active thrust is of course the only one exerted, unless the wall tends to slide, so that the consideration of the passive resistance is of small practical value. In case of a heavy structure resting on a foundation, we can replace the total weight by that of earth, and estimate the active thrust exerted against a vertical plane just below the foundation, for the full weight of the supposed earth, by the method to be given in the next article. The earth to one side of this vertical plane can be conceived to exert a passive thrust, which may be estimated as explained, and should exceed the active thrust for a stable foundation. This method, though, of estimating the stability of a foundation, while doubtless on the safe side, is otherwise illusory, as any one who has seen a heavy locomotive move at great speed along a narrow embankment must admit. The mass, by its friction, rapidly and safely transmits and distributes the weight over the ground, without exerting any horizontal thrust at the side slopes, which are perfectly stable.

35. *Underground Pressures.* — To find the unit pressure at a depth x below the surface of a large mass of earth, level at top, of indefinite extent, and resting upon a uniformly compressible foundation, everywhere at the same depth (see Art. 9), we proceed as follows: Let Fig. 6 represent a slice of the earth contained between two

vertical planes one unit apart, and bounded on one side by the horizontal plane $\overline{0C}$, at a depth x below the surface, on the left by the vertical plane $A0$, whose depth is

Fig. 6



Δx, and below by the plane AC; the planes $\overline{A0}$, $\overline{0C}$, and $\overline{AC}$ being supposed perpendicular to the plane of the paper. Let the length $\overline{A0} = \Delta x$, and the length $\overline{0C} = n.\Delta x$. The plane $\overline{AC}$ will be considered to take successively the positions $A1$, $A2$, . . . ; so that if we divide $A0 = \Delta x$ into ten equal parts, as shown, and lay off similar equal parts along $\overline{0C}$, as $\overline{AC}$ varies in position, n will take the successive values 0.1, 0.2, . . . Calling e the weight per cubic foot of earth, the weight of the prism of earth resting vertically over $\overline{0C}$ is represented generally by $e \,.\, x \,.\, n \,.\, \Delta x$, which, being directly proportional to n, we can lay off on the vertical $\overline{0A}$ the lengths 01, 02, . . . , to represent the successive values of n, or the vertical loads sustained by the horizontal bases $\overline{01}$, $\overline{02}$, . . . , of the successive prisms considered. When the length Δx is very small, we can neglect the weight of the small prism of thrust, $A0C$, in comparison with the weight of the vertical prism above it, without appreciable error, and ultimately find the position of the plane $\overline{AC}$, which gives the true thrust against $\overline{A0}$, by previous methods.

Thus, draw the quadrants shown with A and 0 as centres, and $\overline{A0}$ as a radius ; note the intersections a_1, a_2, . . . , of the lines $A1$, $A2$, . . . , with the arc $0D$; next, construct angle $Cos = \phi$ the angle of repose of the earth. and arcs $ss_1 = 0a_1$, $ss_2 = 0a_2, \ldots$; so that each of the lines $0s_1, 0s_2, \ldots$, next drawn, make the angle ϕ with the normals to the corresponding planes $A1$, $A2, \ldots$, and thus represent the direction of the resistances offered by these planes in turn regarded as planes of rupture. On drawing horizontals through the points of division $1, 2, \ldots$, on $\overline{A0}$ to intersection $1''$, $2''$, . . . , with the corresponding directions $0s_1$, $0s_2$, . . . , we note, that, if the thrust on $A0$ is taken as horizontal (Art. 9), the lines $11''$, $22''$, . . . , represent the horizontal thrusts caused by the weights resting on the successive prisms $A01$, $A02$, . . . , treated as successive wedges of rupture. The greatest of these $\overline{77''}$ represents the actual thrust on $A0$; for if we assert that *any* other, as $\overline{44''}$, represents the actual thrust, to get the corresponding thrusts on all the planes $A1$, $A2$, . . . , in direction and amount, we must lay off a length equal to $\overline{44''}$ along each of the horizontals $11''$, $22''$, . . . , produced if necessary, and through the extremities draw lines to 0, which thus represent in amount and direction the thrusts on the corresponding planes. But since $\overline{44''}$ is less than $\overline{77''}$, this construction will give a thrust on the plane $\overline{A7}$, lying below the position $\overline{07''}$, and thus making a greater angle than ϕ with its normal, which is inconsistent with the laws of stability of a granular mass. Hence, any other thrust than the maximum, as given by the above construction, is impossible.

The length of $\overline{77''}$ to scale is 0.52, which we must now multiply by $ex\Delta x$ to get the total horizontal thrust on the plane $\overline{A0}$ in pounds. On dividing this thrust (0.52 $ex\Delta x$) by the area pressed $= 1 \times \Delta x$, we get the unit pressure on a vertical plane at a depth x below the surface equal to $0.52e.x$, which is called "the intensity of pressure," at a depth x. As we neglected the weight of the prism $A0C$, we must conceive Δx to diminish indefinitely, so that the error tends indefinitely towards zero, and the approximate intensity of pressure on $\overline{A0} = \Delta x$ approaches indefinitely that at the point 0.

By analysis we shall show hereafter that the plane of rupture, $\overline{A7}$ in this case, bisects the angle between the natural slope and the vertical.

In this case we have taken $\phi = 18° \ 26'$, and the resulting intensity ($0.52ex$) is found to be exactly that given by the usual formula, $ex \tan^2\left(45° - \frac{\phi}{2}\right)$. The intensity at any point of a vertical plane thus varies directly with x. The total amount on a vertical plane of depth x from the surface is then $\int_o^x Cxdx = \frac{Cx^2}{2}$ (where $C = 0.52e$ in the present case), and its resultant is at a depth z equal to the limit of the sum of the moments of the pressures ($Cxdx$) on the elementary areas $dx \times 1$, taken about the top surface, divided by the total pressure, or

$$z = \int_o^x Cx^2dx \div \frac{Cx^2}{2} = \frac{2}{3}x.$$

Also, $\frac{Cx^2}{2} = \frac{ex}{2} \times 0.52x = \frac{ex}{2} \times$ line representing thrust, if old construction is used. These are precisely the conclusions derived from previous constructions.

In case the top surface is sloping, a similar construction applies, only $\overline{0C}$ must now be drawn parallel to the top slope, and the pressure on $\overline{0A}$ must be assumed to act parallel to this direction. The construction is similar to that given for Fig. 5 (on neglecting the weight of the wedge of thrust as above), either to the right or left of the vertical $\overline{Ag}$, only as the weight of the prisms vertically above $\overline{01}$, $\overline{02}$, ... (Fig. 5) is now represented by $ex\ n \triangle x \times \cos i$ (where i is the inclination of the top slope to the horizontal), we must multiply the length of the line $\overline{cc'}$ (Fig. 5) to scale, by $ex \cos i$, to get the intensity of the pressure at the depth x, since the lengths n alone were laid off to represent the loads gg_1, gg_2 . . . , as in Fig. 6, and the resulting thrust $\overline{cc'}$ must now be magnified $ex \triangle x \cos i$ times to get the thrust in pounds on the plane $\triangle x \times 1$. As $\triangle x$ approaches zero indefinitely, the approximate intensity $\dfrac{ex\ \triangle x \cos i}{\triangle x}\ \overline{cc'}$, on the area $\triangle x \times 1$, approaches that at the depth x ($= ex \cos i . \overline{cc'}$) as near as we please. It must be observed that $\overline{A0}$ in Fig. 5 must be taken equal to unity in this construction, and the same scale used in laying off the distances along the top slope 0 — 10.

36. If the earth to the right of $\overline{A0}$, in Fig. 6, does not experience the similar *active* thrust of earth to the left of $\overline{A0}$, but only the *passive* resistance of a tunnel lining, etc., of an underground structure, the conditions are changed if this lining *gives* in consequence of its elasticity; for the wedge of thrust, $A0C$, cannot move to the left without developing friction along the surface $\overline{0C}$, therefore the pressure on this surface must no longer be taken as vertical, but as inclined at a direction $0—1C'$, making an angle ϕ with the vertical (Fig. 6). The load on any supposed wedge of thrust, as $A04$, is now represented by $\overline{04'}$, the thrust on $\overline{A0}$ by $\overline{4'4''}$, and the pressure on the plane $\overline{A4}$ by $\overline{04''}$. The greatest of the lines, $1'1''$, $2'2''$, . . . , now represents the true thrust, and it is readily found to be $4'4'' = .33$ to scale; so that the intensity of the thrust on a square foot at the depth x is now $0.33ex$, or one-third the intensity on the horizontal plane $\overline{0C}$. Mr. Baker ("Science Series," No. 56) found for a heading, driven for the Campden-Hill Tunnel, at a depth of 44 feet from the

surface,—the angle of repose of the overlying clay, sand, and ballast, heavily charged with water, being only 18° 26′ as assumed above,—that the relative deflections of the timbering in the roof and sides indicated that the vertical and horizontal intensities of pressure were in the ratio of 3.5 to 1, which is very near what we obtain by the last construction. The first construction indicates a ratio of only 2 to 1.

In most cases, a portion of the weight of the earth abovo the tunnel is transferred to the sides (Art. 9), though here it was thought that "the full weight of the ground took effect upon the settings."

We have now carefully examined the conditions of interior equilibrium of a mass of earth, and ascertained the thrusts exerted, whether in the interior or against a retaining-wall; and we see that the graphical method is capable of handling, with equal ease, any case that ordinarily presents itself. The results, of course, agree with the analytical method, founded on the same hypotheses; but as it is often more convenient to calculate the thrust, even when a graphical method is afterwards used for testing the stability of the wall, we shall now proceed to deduce formulas for evaluating it.

CHAPTER III.

THEORY OF RETAINING-WALLS.

Analytical Method.

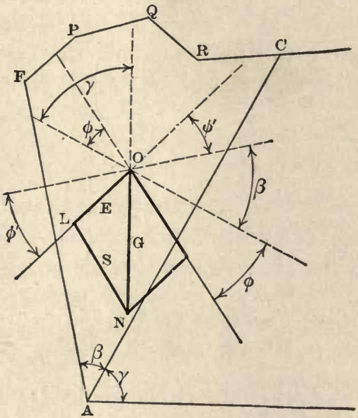
37. As in the preceding chapter, we shall assume a *plane* surface of rupture, and regard the mass as subject only to the laws of gravity and frictional stability stated in Art. 21.

In Fig. 7, let $AFPQRC$ represent a cross-section of the earth-filling, taken at right angles to the inner face of the wall AF. We shall consider the conditions of equilibrium of a prism of this earth contained between two parallel planes, perpendicular to the inner face of the wall, and one unit apart, regarding the wall AF as resisting the tendency of the earth to slide *down* some plane, as AC, passing through its inner toe.

Call G the weight of the prism of earth

AFPQRC in pounds, directed vertically; *E*, the earth-thrust against the wall *AF*, directed at an angle ϕ' of friction of earth on wall when $\phi' < \phi$, or of ϕ when $\phi' > \phi$,

Fig. 7



below the normal to the inner face of the wall (Art. 7); and *S* the reaction of the plane *AC*, inclined at an angle ϕ (the angle of repose of earth) below the corresponding normal, since the prism is supposed to be on the point of moving *down* the plane

AC. These three forces are in equilibrium when E and S act towards O and G acts downwards.

Call the angle that AC makes with the horizontal γ, and the angle FAC, β. On drawing the parallelogram of forces as shown, we have, since E and G are proportional to the sines of the opposite angles in the triangle ONL,

$$\frac{E}{G} = \frac{\sin ONL}{\sin NLO}.$$

It is easily seen from the figure that $ONL = \gamma - \phi$, and that $NLO = \phi + \beta + \phi'$; hence the above general relation becomes,

$$\frac{E}{G} = \frac{\sin(\gamma - \phi)}{\sin(\phi + \phi' + \beta)} \quad . \; . \; . \; (1).$$

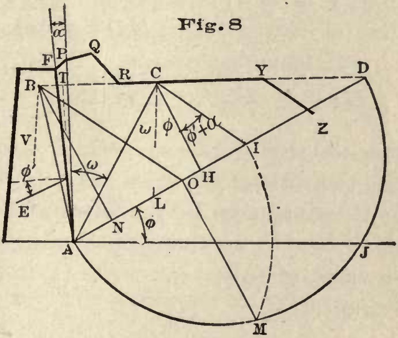
Now, if we conceive the plane AC, always passing through the point A, to vary its position, that value of E, corresponding to the greatest value obtained by the construction above, is the thrust actually exerted against the wall; for, if AC is the plane of rupture corresponding to this greatest trial thrust, any less value of the resistance of the wall E will cause S to make an angle greater than ϕ with the normal to AC, which (Art. 21) is inconsistent with the law of stability of a granular mass (also see Art. 25): hence the least thrust consistent with equilibrium corresponds to the greatest value of E thus obtained; and this is the actual *active* thrust exerted against the wall, when the wall simply *resists* the tendency to overturning or sliding on its base, caused by the tendency of the prism of rupture to descend. If there is a thrust exerted on the wall towards the earth, from any external force acting on the left of the wall; from left to right; then, if this be supposed to increase gradually, the *active* thrust of the earth on the right is first overcome; then, as the external force increases, the directions of S, on all planes as AC, approach the normals to those planes, pass them, and finally the full *passive resistance* of some prism of earth to sliding upwards along its base is brought into play. The greatest force E, as regards sliding up the base of some prism, which can be exerted is that corresponding to the *least* of the trial forces, E, obtained by supposing the position of the plane AC to vary, for S lying above the normal to AC at an angle ϕ for each plane; for if we suppose AC to represent the corresponding plane of rupture, if the external force, equal to E, and acting from left to right, is increased, it necessarily causes the direction of S to make a greater angle than ϕ with the corresponding normal, which is inconsistent with equilibrium (Art. 21).

In this chapter we shall only consider the passive resistance of the wall to overturning or sliding caused by the *active* thrust of the earth tending to descend, which is all that is required in estimating the stability of retaining-walls.

38. We shall now express the value of G for the earth-profile shown in Fig. 8, taken to represent the general case, and proceed to find the maximum value of E, for different trial-planes, which represents the actual thrust exerted against a stable wall. We shall suppose the true plane of rupture to intersect the part RY of the profile; the line RY is then produced to

B, so that the area of the triangle *ABC* is equal to that of the polygon *AFPQRC*, which can be effected by ordinary geometrical means. The point *B* therefore does not change, as we suppose the position of *C* to vary between *R* and *Y*.

Fig. 8



Let us drop the perpendicular *AT* from *A* upon *BY*, and designating by *e* the weight per cubic foot of earth, we have $G = \frac{1}{2}e.AT.BC$.

For future convenience we have designated, in Fig. 8, the angle that *AC* makes with the vertical ω, and the angle that the inner face of the wall *AF* makes with the vertical α; so that the angle β of (1) is now replaced by $(\omega + \alpha)$ if the wall leans forward, or by $(\omega - \alpha)$ if the wall leans backwards.

In Fig. 8, let us draw the line CI, making the angle $ACI = (\phi + \phi' + \beta) = (\phi + \phi' + \omega + \alpha)$ to intersection I, with the line of natural slope AD through A. If the wall leans backwards,

$$ACI = (\phi + \phi' + \omega - \alpha).$$

Since the angle $(\gamma - \varphi) = CAI$, we can replace the ratio of the sines in (1) by that of the sides opposite in the triangle ACI, or of CI to AI; so that, substituting the above value of G, we can write (1) in the following form:—

$$E = \tfrac{1}{2}\, e \, . \, AT \, . \, BC \, . \, \frac{CI}{AI} \; . \; . \; . \; (2).$$

On drawing BO *parallel to* CI to intersection O with AD, we have, from this relation and the similar triangles, BOD and CID.

$$BC = BD\,\frac{OI}{OD}\,;\; CI = ID.\frac{BO}{OD}\,;$$

which substituted in (2) gives,

$$E = \tfrac{1}{2}e\left(\frac{AT.BD.BO}{OD^2}\right).\ \frac{OI.ID}{AI} \quad \ldots \ (3).$$

The terms in the ( ) remain constant as we vary the position of AC. For brevity, call $AI = x$, $AD = a$, $AO = b$; then we can write the variable term,

$$\frac{OI.ID}{AI} = \frac{(x-b)\,(a-x)}{x} = a + b - \frac{ab}{x} - x,$$

which is a maximum for $x = \sqrt{ab}$, as we find by placing its first derivative equal to zero. This value of x substituted in the variable term gives,

$$a + b - 2\sqrt{ab} = \frac{(a - \sqrt{ab})^2}{a};$$

so that the *actual* thrust E on the wall can be written,—

$$E = \tfrac{1}{2}e\left(\frac{AT.BD.BO}{OD^2}\right)\frac{(a-\sqrt{ab})^2}{a} \quad \ldots \ (4).$$

Now, drawing the perpendiculars BN and CH from B and C upon AD, we observe that since the angle $ACH = \omega + \phi$ (AC makes the angle ω with a vertical at C, and CH makes the angle ϕ with this same vertical, since the sides are respectively perpendicular to those of the angle $DAJ = \phi$), and the whole angle $ACI = (\omega + \phi + \phi' + \alpha)$, it follows that the angle $HCI = NBO = (\phi' + \alpha)$ as marked, if the wall leans forwards; otherwise $HCI = NBO = (\phi' - \alpha)$, since ACI is then equal to $(\omega + \phi + \phi' - \alpha)$, as previously observed.

To reduce (4) to a simpler form, we remark that $AT.BD$ represents double the area of the triangle ABD, and can be replaced by $AD.BN = AD.BO \cos OBN$; which gives

$$\frac{AT\,BD.BO}{\overline{OD}^2} = a \cos OBN \left(\frac{BO}{OD}\right)^2;$$

$$\therefore E = \tfrac{1}{2} e. \cos OBN \left(\frac{BO}{OD}\right)^2$$

$$(a - \sqrt{ab})^2 \quad . \; . \; . \; (5).$$

Now, from similar triangles, BOD, CID, we have $\dfrac{BO}{OD} = \dfrac{CI}{ID}$, which, substituted in the above expression, we have, noting that $(a - \sqrt{ab}) = (a - x) = ID$, the very simple formula,

$$E = \tfrac{1}{2}e.\cos(\phi' + \alpha)\,\overline{CI}^2 \ldots (6).$$

It is to be remarked, that, if the wall leans backwards, $\cos(\phi' + \alpha)$ is to be replaced in this formula by $\cos(\phi' - \alpha)$; further, if we lay off $IL = IC$ on the line IA, and draw a line from L to C, the thrust E is exactly represented by the area of the triangle ICL multiplied by e, the weight per cubic foot of the earth.

39. This simple conclusion has been previously reached, in an entirely different manner, by Weyrauch (see "Van Nostrand's Magazine" for April, 1880, p. 270), who states that Rebhahn in 1871 found a similar result, assuming, however, that $\phi' = 0$. or $\phi' = \phi$ (for the special cases of earth-level at top, or sloping at the angle of repose, I infer).

Recurring now to the fact, that for the true plane of rupture we found

$$x = AI = \sqrt{ab} = \sqrt{AD.AO},$$

and that angle $NBO = (\phi' + \alpha)$ or $(\phi' - \alpha)$, according as the wall leans forwards or backwards, we have the following simple construction to find the plane of rupture and earth-thrust E, as given by Weyrauch in 1878, for a uniform slope and wall leaning forward.

Having found the point B on the prolongation of the line RY, which it is thought will be intersected by the plane of rupture, so that area $ABR =$ area $AFPQR$, we next draw BO, making the angle NBO with the normal to the line of natural slope AD, equal to $(\phi' + \alpha)$ or $(\phi' - \alpha)$, according as the inner face of the wall lies to the left or to the right of the vertical through A (replace ϕ' by ϕ whenever $\phi' > \phi$); then erect a perpendicular at O to AD to intersection M, with the semicircle described upon AD as a diameter, and lay off $AI =$ chord AM, since $AI = \sqrt{AO.AD}$; next, draw IC parallel to OB to intersection C with the top slope, whence AC will be the plane of rupture if the point C falls upon RY as assumed; otherwise another plane, as YZ, will have to be assumed as containing the point C, and the construction effected as before.

Having found C in this manner, E can be computed from (1), since $G = \frac{1}{2}\ AT.BC$

is now known: or by measuring CI to scale, E can be found directly from (6).

This graphical construction is more rapid and accurate in working than the methods of the preceding chapter, and is superior to Poncelet's construction, in taking less space to effect.

In surcharged walls, the point B will generally lie to the right of AF. Thus, in Fig. 4 the upper line $\overline{26}$ is extended to the left; from 0 a line is then drawn parallel to $A2$ to intersection $0'$ with the line $\overline{26}$ extended. The point $0'$ thus found corresponds to the point B of Fig. 8.

40. The construction is true whether the earth-surface slopes upwards or downwards from the top of the wall.

In the latter case, if the surface, say BD, falls upon the line BO, the construction fails; but a formula given farther on gives the value of E.

If the surface BD falls below BO, it is easily seen, on drawing a figure, that all the previous equations hold, and we reach the same conclusion as before, $AI = \sqrt{AD.AO}$; only as AO now is larger than AD, the semicircle must be described upon AO as a diameter, and the perpendicular to the point M erected at D; or AI can be calculated if preferred. If the points O, I, and D are near together, it will be best to compute BC from $BC = BD.\frac{OI}{OD}$, since the terms in the right member can be measured to scale.

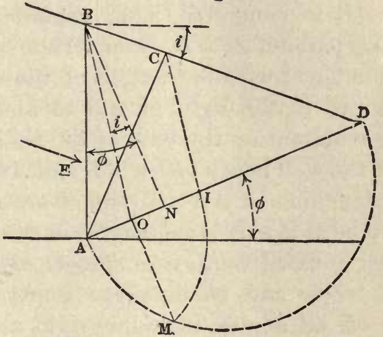
41. *Position of the Limiting Plane.*—In Fig. 9, let BD represent the earth-surface, uniformly sloping at the angle i to the horizontal, of an unlimited mass of earth (Art. 9), in which the pressure on a vertical plane, AB, can be taken as parallel to the surface BD. Let AD represent the line of natural slope; it is required to find the position of the plane of rupture AC, corresponding to the thrust E, acting above the horizontal at the angle i, and of course balancing the opposed thrust of the earth to the left of AB.

On referring to Fig. 7, it is seen that equation (1) holds on replacing the denominator of the right member by sin $(\beta+\phi-i)$. Therefore, in Fig. 8, the angle

ACI must now be laid off equal to $(\beta+\phi-i)$, whence the line CI falls below CH, and BO below BN, both being inclined to these normals at the same angle, $\phi'+\alpha=i+0=i$.

With this exception, the above demonstration holds throughout, and we reach

Fig. 9



the following construction to find the point C. From B draw BO, making the angle i below the normal BN to AD, or preferably making the angle $(\phi - i)$ with the vertical AB, to intersection O with AD. From O draw OM perpendicular to AD to intersection M, with the semicircle described upon

AD as a diameter; lay off *AI* along *AD*, equal to chord *AM*, and from *I* draw a parallel to *BO* to intersection *C* with the top slope *BD*. The plane *A C* is the plane of rupture, or the *limiting plane* of Art. 28, which see.

If the inner face of the wall lies below *AC*, then (Art. 28) the thrust$=\frac{1}{2}e.\cos i.\overline{CI}^2$ on *AB* is computed, and, regarded as acting parallel to *BD*, from left to right, is combined with the weight of the earth and wall to the right of *AB* to find the true resultant on the base of the wall.

If the wall lies between *AB* and *A C*, the constructions of Arts. 37 and 38 are used.

To be as accurate as possible in these, as in all constructions, true *straight edges* on both ruler and triangle are imperative. Lay off all angles, including right angles, by aid of a beam compass to a large radius, say ten inches, using a table of chords (except for the right angle) and an accurate linear scale. With all care, the angles *BAC* thus found can scarcely be counted on to nearer than ten minutes, which, however, is sufficiently accurate.

In the table below will be found, for various inclinations i, the values of the angle BAC that the limiting plane makes with the vertical; also the co-efficient K (see Art. 42), or the thrust on $AB = \frac{1}{2}e \cos i \, \overline{CI}^2$, when AB and e are both taken as unity, made out for earth which naturally takes a slope of one and a half to one, or whose angle of repose is $33° \, 42'$.

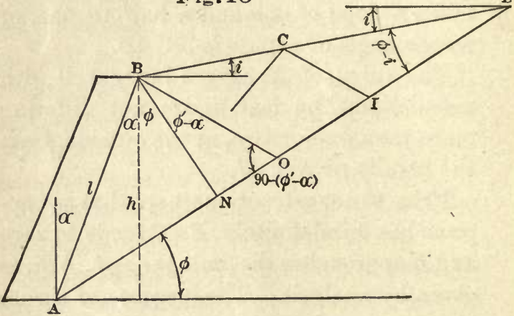
The value of K agrees fairly well with calculation, the last figure not differing more than one or two, at the outside, from the results of Art. 47.

From the construction we see that as i approaches ϕ indefinitely, BAC tends to zero and E approaches the limit $\frac{1}{2}e \cos \phi . \overline{AB}^2$, as given by analysis. The increase of thrust is very rapid from $i = 30°$ to $i = \phi = 33° \, 42'$.

i	0°	5°	10°	15°	20°	25°	26°34	30°	33°42′
BAC	28°09′	26°	24°	21° 50′	19°10′	16°	14°40′	11° 10′	0°
E	.143	.145	.149	.157	.172	.194	.207	.244	.416

42. *Uniform Top Slope; Formula for Earth-thrust.*—When the upper surface of the earth slopes uniformly at the angle i to the horizontal, it is easy to deduce from what precedes a general formula for the thrust exerted by it. Fig. 10 represents a

Fig. 10



retaining-wall leaning towards the earth. We shall first deduce a formula for this case, when it will be observed, as we proceed, that the same formula holds, when the wall leans forward, on simply changing α to $(-\alpha)$.

In this case, we note from Fig. 10 the following values for the angles.—

$$
\begin{aligned}
NBO &= \phi' - \alpha, \\
ABO &= \phi + \phi', \\
AOB &= 90 - (\phi' - \alpha), \\
ADB &= \phi - i, \\
ABD &= 90 + \alpha + i, \\
BAO &= 90 - (\phi + \alpha).
\end{aligned}
$$

Finally, designate by l the length AB from the inner toe to where the inner face of the wall intersects the top slope, and by h its corresponding vertical projection.

From formula (5) we deduce, remembering that $OD = (a - b)$,

$$E = \tfrac{1}{2}e.\,\overline{BO}^2 \left[\frac{a - \sqrt{ab}}{a - b}\right]^2 .\cos OBN \ldots (7).$$

We can now write the [ ] as follows :—

$$\frac{a - \sqrt{ab}}{a - b} = \frac{1 - \sqrt{\frac{b}{a}}}{1 - \frac{b}{a}} = \frac{1}{1 + \sqrt{\frac{b}{a}}}$$

Place $n = \sqrt{\frac{b}{a}}$; to find its value in terms of the functions of known angles, we have from the triangles AOB and ABD by the law of sines,

$$\frac{AO}{AB} = \frac{\sin(\phi + \phi')}{\cos(\phi' - \alpha)}, \quad \frac{AB}{AD} = \frac{\sin(\phi - i)}{\cos(\alpha + i)}.$$

On multiplying these two equations together, and extracting the square root, we find,

$$n = \sqrt{\frac{AO}{AD}} = \sqrt{\frac{\sin(\phi+\phi')\sin(\phi-i)}{\cos(\phi'-\alpha)\cos(\alpha+i)}}, \quad (8),$$

Again, from the triangle BOA, we have,

$$BO = \frac{\cos(\phi + \alpha)}{\cos(\phi' - \alpha)} l.$$

Substituting these values in (7), and putting $\cos OBN = \cos(\phi' - \alpha)$ for this case, and we have finally,

$$E = \left(\frac{\cos(\phi+\alpha)}{n+1}\right)^2 \frac{el^2}{2\cos(\phi' - \alpha)} \; . \; . \; . \; (9).$$

Or, since $h = l \cos \alpha$, we likewise have,

$$E = \left(\frac{\cos(\phi + \alpha)}{(n+1)\cos\alpha}\right)^2 \frac{eh^2}{2\cos(\phi' - \alpha)} \; . \; . \; . \; (10).$$

If we term the co-efficient of eh^2 in (10), K, we can write this formula,

$$E = Keh^2 \; . \; . \; . \; (11)$$

in which, for *walls leaning backwards,* as in Fig. 10,

$$K = \left(\frac{\cos(\phi + \alpha)}{(n+1)\cos\alpha}\right)^2 \frac{1}{2\cos(\phi' - \alpha)} \ldots (12),$$

where n has the value given in (8).

For *walls leaning forwards,* we easily note the changes in the angles used, and can verify that formula (11) obtains; but now,

$$K = \left(\frac{\cos(\phi - \alpha)}{(n+1)\cos\alpha}\right)^2 \frac{1}{2\cos(\phi' + \alpha)} \ldots (13);$$

and,

$$n = \sqrt{\frac{\sin(\phi + \phi')\sin(\phi - i)}{\cos(\phi' + \alpha)\cos(\alpha - i)}} \ldots (14);$$

which we obtain from the old values by simply changing α to $(-\alpha)$.

It is to be observed, for all cases, when $\phi' > \phi$ that we must replace ϕ' in all the formulas by ϕ.

These formulæ are identical with those of Bresse ("Cours de Mécanique Appliquée," Vol. I. 3d ed.) and Weyrauch, for the case of the wall leaning forward, the only cases examined by them. Bresse uses the

Poncelet method for the general case, which leads to Poncelet's celebrated construction. The routes pursued by these authors is different from that given above, the method of Weyrauch, in particular, being much more complicated; still, all three methods lead to precisely the same formula, so that it must be considered as established beyond question.

Weyrauch, too, in subsequent reductions, follows Rankine as to the direction of the earth-thrust against the wall, whereas Bresse takes it as above. The case of the "limiting plane" is not considered by either.

43. The case where the top surface slopes downwards to the rear is very rarely met with in practice. The previous formulæ apply though directly on simply changing i to $(-i)$, since it is seen that angle $ADB = (\phi + i)$ and angle $ABD = 90 + (\alpha - i)$, and the ratio $\frac{AB}{AD}$ is now equal to $\frac{\sin(\phi + i)}{\cos(\alpha - i)}$.

44. *Earth Level at Top; Back of Wall Vertical.*—For the earth level at top, back of wall vertical, and $\phi' = \phi$ as usually taken, the formula (11) takes a very simple

form. Here we have $\alpha = 0, \phi' = \phi$, $i = 0$, whence,

$$n = \sqrt{\frac{\sin 2\phi \sin \phi}{\cos \phi}} = \sin \phi \sqrt{2},$$

and

$$E = \frac{\cos \phi}{2 (1 + \sin \phi \sqrt{2})^2} . eh^2 \ldots (15).$$

For $\phi' = 0$, which corresponds to *a perfectly smooth wall*, or otherwise may refer to the direction of the pressure on a vertical plane in a mass of earth of indefinite extent, level on top (Art. 9), we have, when $\alpha = 0$ and $i = 0$, $n = \sin \phi$ and,

$$E = \frac{1 - \sin \phi}{1 + \sin \phi} \cdot \frac{eh^2}{2}$$

$$= \tan^2 \left(45° - \frac{\phi}{2}\right) \cdot \frac{eh^2}{2} \ldots (16).$$

The equality of the two co-efficients of $\frac{eh^2}{2}$ in (16) is easily verified from the known formula,

$$\tan^2 \tfrac{1}{2} (x) = \frac{1 - \cos x}{1 + \cos x}$$

by putting $(90 - \phi)$ for x in both members.

Referring to Fig. 7, and regarding AF

vertical, the top surface horizontal, and $\phi' = 0$, we note that $G = \frac{e}{2} . h^2 \tan \beta$ and $E = \frac{e}{2} h^2 \tan \beta \tan (\gamma - \phi)$, in which $\gamma = 90 - \beta$. Now, this result must agree with the right member of (16), which is only possible when $\beta = \left(45 - \frac{\phi}{2}\right)$ or $2\beta = (90 - \phi)$; whence it follows that for $\phi' = 0$, $\alpha = 0$, $i = 0$, as assumed, *the plane of rupture bisects the angle between the vertical and the line of natural slope.*

45. *Earth sloping at the Angle of Repose.*—For this case we shall assume $\alpha = 0$ and $\phi' = \phi$ in addition to $i = \phi$, whence $n = 0$ and,

$$E = \frac{\cos \phi}{2} . eh^2 \quad . \; . \; . \quad (17),$$

as found in a different manner in Art. 41. This simple formula can likewise be deduced directly from equation (1) of Art. 37, referring to Fig. 7,

$$\frac{E}{G} = \frac{\sin (\gamma - \phi)}{\sin (\phi + \phi' + \beta)} = \frac{\cos (\beta + \phi)}{\sin (2 \phi + \beta)}.$$

On substituting the value of G, which is easily found for this case to be,

$$G = \frac{1}{\cot \beta - \tan \phi} \frac{eh^2}{2} = \frac{\sin \beta \cos \phi}{\cos (\beta + \phi)} \frac{eh^2}{2},$$

we find for the trial thrust

$$E = \frac{\sin \beta \cos \phi}{\sin (2 \phi + \beta)} \frac{eh^2}{2}$$

$$= \frac{\cos \phi}{\sin 2 \phi \cot \beta + \cos 2 \phi} \frac{eh^2}{2}.$$

Now, by the reasoning of Art. 37, the true thrust is the greatest value the above expression can have, as β varies, and its greatest value corresponds to $\beta = 90 - \phi$; for then $\cot \beta$ is least, and E greatest, since $\cot \beta$ is in the denominator. On substituting this value a simple reduction gives $E = \frac{1}{2} \cos \phi \,.\, eh^2$ as found above in (17). Since we have just found, for this case, that $\beta = 90 - \phi$, it follows that *the surface of rupture coincides with the natural slope.* The value of E from equation (1) in this case assumes the form $0 \times \infty$, since G becomes infinite for an indefinitely sloping surface; but on reducing to the form above we easily see the limit that E approaches indefinitely, which is its true value. The construction of Art. 39 fails for this case, but the one of Art. 41 leads directly to (17).

46. *Pressure of Fluids.*—The general formula (9) above is true, no matter how small the angle of repose ϕ becomes, and must approach indefinitely the expression for the pressure of liquids, as ϕ and ϕ' tend towards zero; so that at the limit, for $\phi = \phi' = i = 0$, we have the normal thrust of a liquid whose weight per cubic foot is e,

$$E = \tfrac{1}{2}\, el^2 \cos \alpha = \tfrac{1}{2}\, eh^2 \sec \alpha \quad . \ . \ . \quad (18),$$

a well-known formula. By Art. 44 we see that for $\phi = 0$, $2\beta = 90$, or the plane of rupture approaches an inclination of 45° as ϕ approaches zero indefinitely.

47. *Rankine's Formula for the Earth-thrust on a Vertical Plane, in an Indefinite Mass, sloping uniformly.* In Art. 9 we have stated the conditions that such a mass must satisfy in order that the pressure on a vertical plane, whose intersection with the top slope is a horizontal line, may be parallel to the line of greatest declivity.

Also in Art. 28 we have seen, that, when the wall face lies below the limiting plane, this direction of the thrust is the true one on a vertical plane, passing through the inner toe of the wall.

We have $\alpha = 0, \phi' = i$, and $l = h$, which gives in formula (9),

$$E = \left(\frac{\cos \phi}{n+1}\right)^2 \frac{eh^2}{2 \cos i}$$

where,

$$n = \sqrt{\frac{\sin(\phi + i) \sin(\phi - i)}{\cos^2 i}}$$

$$= \frac{\sqrt{\sin^2 \phi \cos^2 i - \cos^2 \phi \sin^2 i}}{\cos i}$$

$$= \frac{\sqrt{\cos^2 i - \cos^2 \phi}}{\cos i}.$$

Whence,

$$E = \frac{\cos^2 \phi \cos i}{(\cos i + \sqrt{\cos^2 i - \cos^2 \phi})^2} \cdot \frac{eh^2}{2}.$$

Now, since we can write,

$$\cos^2 \phi = (\cos i + \sqrt{\cos^2 i - \cos^2 \phi})$$
$$\times (\cos i - \sqrt{\cos^2 i - \cos^2 \phi})$$

the above value becomes, on striking out the common factor, $(\cos i + \sqrt{\cos^2 i - \cos^2 \phi})$

$$E = \frac{eh^2}{2} \cos i \frac{\cos i - \sqrt{\cos^2 i - \cos^2 \phi}}{\cos i + \sqrt{\cos^2 i - \cos^2 \phi}}, \quad (19),$$

which is Rankine's well-known formula for earth pressure.

Now since Rankine's formula was framed without the use of any assumption, as that of a plane of rupture, and is accepted as correct for the case in question, it follows, that, when the pressure is assumed to be parallel to the surface, the assumption that the surface of rupture is a *plane* will give correct results, and can be safely used in the graphical method which is absolutely dependent on it.

It will be observed that formulæ (16) and (17) can be deduced directly from (19) by making $i = 0$ and $i = \phi$ respectively. Rankine has given a simple graphical construction of the last fraction in (19) in his "Civil Engineering," which saves labor in computing.

48. *Unit Pressures on a Vertical Plane at Depth x below a uniformly Sloping Surface, the Direction of the Pressure being*

taken Parallel to the Line of Greatest Declivity.—As in Art. 35 we shall consider a wedge of thrust of infinitesimal dimensions, of which the left face AB (Fig. 10) is vertical, and the upper surface parallel to the top slope. The weight of the vertical prism that rests upon any trial base as $\overline{BC}$ is, $e \,.\, BC \,.\, \cos i \,.\, x \,.= AT \,.\, BC \,.\, ex/AB$ (Fig. 8); so that neglecting the weight of the infinitely small wedge ABC we get the value of E from equation (1) of Art. 37 by simply replacing G by this value. Equation (2) of Art. 38 is thus replaced by

$$E = \frac{ex}{AB} \,.\, AT \,.\, BC \frac{CI}{AI'}$$

which is exactly that given in Art. 38 multiplied by the constant $2x/AB$. All the subsequent reductions, therefore, hold if we simply put $h = AB$ in the final equations, and multiply the result by $2x/AB$. Hence divide (19) by $AB = h$ and change the coefficient $eh/2$ to ex, to find the intensity of the pressure, $E \div AB$, at a depth x; and on integrating this expression, multiplied by dx, between the limits o and h, we are at once conducted to (19), which is thus proved true by the method of integration of the effects of earth particles, which is independent of the assumption of a plane surface of rupture extending to the surface.

Precisely the same conclusions hold for a vertical wall, or one leaning forwards, when E is assumed to make the angle ϕ' or ϕ with the normal to the wall, since G is simply replaced as before by the weight of the vertical prism for a *uniform top slope*, and ultimately we replace h^2 by $2x$ in the general formula (11) to get the intensity of pressure in the direction given, at the depth x from the surface, so that on integrating as before we deduce (11) without the necessity of considering the surface of rupture as extending to the surface. The graphical method, using this hypothesis, should again give good results. It is possible though, in this case, that the influence of the wall friction may have some effect in deflecting the weights of the vertical prisms from a vertical line; for, when it is so transmitted, the usual direction of the pressure is parallel to the surface (Art. 9). For *walls leaning backwards* the prisms do not rest vertically over the bases of the prisms of thrust, and the theory would seem to be inapplicable; so that the formulæ for this case, (8) and (9), have to rest upon the unproved hypothesis of a plane surface of rupture extending to the surface, and may depart considerably from the truth. We conclude, that, except for the cases for which Rankine's formula is applicable, the plane surface of rupture is still an unproved hypothesis.

49. *Point of Application of the Thrust; Uniform Slope.*—We have the normal component of the thrust on a wall, by (9) whether the wall inclines forward or backward or is vertical, expressed by the relation,

$$E_1 = (9) \times \cos \phi' = cl^2,$$

c being constant; whence the thrust on the area $dl \times 1$ is nearly

$$dE_1 = 2cldl,$$

and the distance from where the inner face of the wall interesects the top slope to the centre of pressure is equal to the limit of the sum of the elementary pressures multiplied by their arms. divided by the total pressure, or,

$$\frac{\int_0^l 2cl^2 dl}{cl^2} = \frac{2}{3} l;$$

hence the centre of pressure on the wall is $\frac{1}{3}h$ vertically above the base.

50. *Surcharge uniformly distributed.*—If the filling of height h has a horizontal surface upon which a uniform load of any kind rests, replace its weight by that of an equivalent quantity of earth, giving the total load the same, and call the height of the reduced load h'. The total pressure on the vertical wall of height h is now by (11), $E = Ke\,((h + h')^2 - h'^2) = Keh\,(h + 2h')$, whence,

$$dE = Ke.2\,(h + h')\,dh;$$

and the distance of the centre of pressure from the top of the wall downwards is,

$$\frac{2\int_0^h (h + h')\,hdh}{h\,(h + 2h')} = \frac{h}{3}\left(\frac{2h + 3h'}{h + 2h'}\right);$$

or from the base of the wall upwards,

$$h - \frac{2h^2 + 3hh'}{3h + 6h'} = \left(1 + \frac{h'}{h + 2h'}\right)\frac{h}{3}.$$

It is more than probable that the theory for this case will prove illusory in practice, and will give a large excess of pressure; so that, most frequently, such surcharged loads are ordinarily allowed for by a large factor of safety, particularly where the earth is bound by cross-ties, stringers, etc., or the surcharge is not free to move laterally as well as vertically.

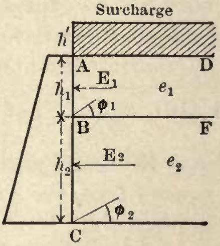


Fig. 10 (a).

In the case of *sea walls*, the backing is saturated with water at high tide, up to a certain level BF, fig. 10 (a), so that it is well to ignore the friction at the back of the wall on BC and for additional safety it will be neglected on the portion AB.

Call the weight of the backing per cubic foot above BF, e_1 and the angle of repose ϕ_1. The corresponding quantities for the saturated backing below BF will be designated by e_2 and ϕ_2. The value of ϕ_2 should be found by experiment and e_2 computed as explained at p. 31. If the water, at high tide, is at the same height on the front and back faces of the wall, the water pressures on those faces will balance and need not be considered.

Let $AB = h_1$ and $E_1 =$ earth thrust on AB; $BC = h_2$ and $E_2 =$ earth thrust on BC.

By Art. 44 (16), $K = \frac{1}{2} \operatorname{Tan}^2 \left(45° - \frac{\phi}{2}\right)$;

therefore if $h'e_1 = W =$ surcharge or load in pounds per square foot, on AD; by the analysis just given,

$$E_1 = Ke_1h_1 \left(h_1 + 2h^1\right) = \tan^2 \left(45° - \frac{\phi_1}{2}\right).$$

$$\left[\frac{e_1h_1^2}{2} + h_1W\right]$$ and E_1 acts above B, a distance,

$$\left(1 + \frac{h^1}{h_1 + 2h^1}\right)\frac{h_1}{3}.$$

Next, assuming that the load on the horizontal plane BF is uniform and W_0 lbs. pr. sq. ft. and calling the height of this load reduced to the specific gravity of the earth below BF, h_0,

$$\therefore h_0\, e_2 = W_0 = W + h_1\, e_1.$$

Hence as before,

$$E_2 = \tan^2\left(45^\circ - \frac{\phi_2}{2}\right).\ \left[\frac{e_2\, h_2{}^2}{2} + h_2 W_0\right]$$

and E_2 acts above C, a distance,

$$\left(1 + \frac{h_0}{h_2 + 2h_0}\right)\frac{h_2}{3}$$

If no surcharge is considered, the formulas apply on making,

$$W = o,\ h' = o\ \therefore\ W_0 = h_1\, e_1,\ h_0 = h_1\, e_1 \div e_2.$$

In either case having found E_1 and E_2 in magnitude and position, the position of the resultant $E_1 + E_2$, can be found by taking moments about C.

The above formulas will be found to reduce to those given by Mr. D. C. Serber, in Engineering News, Aug. 23, 1906. It is stated there, that the Department of Docks of New York City specify a surcharge of 1,000 lbs. pr. sq. ft., acting as a vertical load.

51. *Moments of the Thrust about the Inner Toe of the Wall.*—Let us decompose the thrust E against the wall into two components, E_1 and E_2, respectively normal to and acting along the inner face of the wall. If E makes the angle ϕ' with the normal to the wall, we have, from $E = Keh^2$,

$$E_1 = E \cos \phi' = K \cos \phi' . eh^2 \cdot$$

or putting, $K_1 = K \cos \phi'$
we have, $E_1 = K_1 \, eh^2$;
also, $E_2 = E \sin \phi' = E_1 \tan \phi'$.

It is understood in these formulæ, that, when $\phi' > \phi$, we must replace ϕ' by ϕ.

If the inner face of the wall makes an angle α with the vertical, we have the thrust acting at a distance $cl = ch \sec \alpha$ from the inner toe of the wall, where $c = \frac{1}{3}$ by theory for a uniform slope; therefore, *the moment M of the thrust about the inner toe of the wall* is $E_1 \, cl$, since the moment of E_2 is zero; or putting for abbreviation,

$$m = c\,K_1 \sec \alpha$$

we have,

$$M = E_1\,ch \sec \alpha = c\,K_1 \sec \alpha\,.\,eh^3 = meh^3.$$

In subsequent investigations it is well to recall that h represents the vertical height from the inner toe of the wall to where the line of the inner face pierces the top surface of the earth backing, and that e represents the weight per cubic foot of earth.

CHAPTER IV

EXPERIMENTAL METHODS. COMPARISON WITH THEORY. THE PRACTICAL DESIGNING OF RETAINING WALLS

52. Many experiments have been recorded pertaining both to retaining-walls proper and to rotating retaining-boards. Where the backing is of dry sand, possessing little or no cohesion, the results, for the retaining-walls proper, agree fairly well with the theory advanced in this book, which includes all the wall friction that can be exerted, especially where the walls were several feet in height; but they do not agree with the Rankine theory, in which the direction of the pressure on a vertical plane is always assumed parallel to the earth surface.

53. The results for some of the experiments on model walls at the limit of stability are given in the adjoining table.

No.	Authority.	h feet.	t feet.	$\frac{e}{w}$	α	i	ϕ	ϕ'	q	q_0
1	Baker........	4.000	1.000	2.20	0	0	39° 48′	22°	−.06	− .55
2	Lt. Hope.....	10.000	1.920	0.95	0	0	36° 53′	ϕ	+.04	− .58
3	Trautwine....	0.500	0.175	3.18	0	0	33° 41′	22°	−.03	− .75
4	Curie........	0.558	0.159	3.29	0	0	35°	ϕ	+.02	−1.32
5	Curie........	2.952	0.755	0.62	0	45°	45°	35°	+.03	
6	Curie........	2.910	1.476	0.62	−27° 31′	0	35°	ϕ	+.11	
7	Curie........	1.880	1.790	0.62	−55°	0	33° 30′	ϕ	+.02	

h =height and t =thickne ss of base of wall in feet; e =weight of earth, w =weight of wall, per cubic foot; ϕ =angle of repose of earth, ϕ' =angle of friction of earth on wall; i =angle with the horizontal of the surface of the surcharge. q =ratio of the distance from the center of pressure on the base of the wall to the outer toe, to the thickness t of the wall, using the method including all the wall friction. q_0 =ditto, using Rankine's method. α =angle the inner face of the wall makes with the vertical, counted as + or −, according as the wall leans towards or from the earth. The thrust is assumed to act at one-third the height from the base to the surface of the filling.

The walls were all vertical walls of rectangular cross-section, except the last two, which were peculiar wooden triangular frames whose inner faces made angles 27° 30′ and 55° respectively with the vertical.

In No. 6, the face coincided with the "limiting plane" (Arts. 28 and 41) and in No. 7 was below it. In either case, the thrust was first found on the vertical plane through the foot of the inner face and this was combined with the weight of the earth over the face and the weight of the frame to find the resultant on the base (see Art. 32).

Wall No. 1 was of pitch-pine blocks, backed by macadam screenings, the level surface of which was 3 inches below the top of the wall. Wall No. 5, of brick in Portland cement, was a surcharged one; the level upper surface of the surcharge being 4.26 ft. above the top of the wall, the surcharge extending entirely over the top of the wall at 45° to the horizontal. In the other walls, the earth surface was level with the top of the wall. Wall No. 2 was of bricks laid in wet sand; No. 3, of wood, and No. 4 was of wood coated on the back with sand.

54. Elaborate experiments on rotating retaining-boards, backed by sand, have been made by Leygue ("Annales des Ponts et Chaussées," Nov., 1885), Darwin and others, which, in the earlier editions of this work, were given in detail. They are omitted here, since they have been

discussed by the writer very fully in an article entitled "Experiments on Retaining-walls and Pressures on Tunnels."* The conclusion was drawn that the results can be harmonized with theory by including the influence of cohesion. The discussion involved a complete theory, mainly graphical, of earth pressure, where the earth is supposed endowed with both friction and cohesion.

As regards the experiments of Leygue on rotating-boards, it was found that, assuming an adhesion or cohesion, of only about 1 lb. per sq.ft., for the dry sand used, the experiments were in harmony with theory; but that the results differ essentially from the usual theory where cohesion is neglected. The discrepancies were proved to be due entirely to the small size of the models used and it is suggested that in future, walls of 6 feet and upwards in height be experimented on, where the influence of a cohesion of only 1 lb. per sq.ft. is very small and can be neglected in the analysis.

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* Transactions Am. Soc. C.E., Vol. LXXII, p. 403 (1911).

55. *Center of Pressure.* Leygue, in the experiments on retaining-boards, found the moment of the earth thrust about the toe and also determined the surface of rupture. Using the corresponding wedge of rupture, the writer computed the thrust and its normal component. On dividing the moment given by the latter, the quotient gives the distance of the center of pressure of the earth thrust from the base. It was found to lie, as an average for all the experiments, at 0.34 height of the board in contact with the filling for dry sand and 0.405 height for millet seed. For sand, the values varied, for a vertical wall from 0.319 per earth surface horizontal to 0.346 for the surface sloping at the angle of repose. For boards leaning towards the earth, when *tan* α (Fig. 10, p. 94) was $+\frac{1}{3}$, the variation was from 0.296 to 0.337; for tan $\alpha = +\frac{2}{3}$ from 0.325 to 0.375. For boards leaning from the earth, tan $\alpha = -\frac{1}{3}$, the variation was from 0.352 to 0.363.

These results are approximate, for although the exact prisms of rupture were used, the chord of necessity replaced the true curved line of rupture in the construction of Fig. 3, p. 35, and cohesion was neglected. The effect of cohesion is to lower the center of pressure; so that doubtless for sand absolutely devoid of cohesion, the ratios should be larger. However, from lack of more complete observations on large models, the theoretical value $\frac{1}{3}$ will be used in the computations below.

56. The center of pressure for a surcharged wall of the type shown by Fig. 4, p. 40, only with the back vertical and the each surface extending from C, the top of the inner face, at the angle of repose, $\phi = 33°41'$, to the level surface 2–6, has been found by the writer * for various ratios of h' to h, where h = height of wall, h' = height of surcharge above the top of wall and c = vertical distance from foot

$\frac{h'}{h}$	c	$\frac{h'}{h}$	c
∞	0.333	1.00	0.364
2.00	0.353	0.75	0.364
1.50	0.356	0.50	0.364
1.25	0.360	....	.....
1.11	0.362	0.00	0.333

* Trans. Am. Soc. C.E., Vol. LXXII, p. 410.

of wall to the center of pressure, divided by the height of the wall.

In finding the values of c, h' was taken as 10 feet and the earth thrusts on walls of heights, 1, 2, 3, . . ., 20 ft., were found by the construction of Fig. 8 (see pp. 88–89). By subtraction, the earth thrust on each foot of wall was obtained, and by taking moments about convenient points, the centers of pressure for heights of wall varying from 5 to 20 feet were easily obtained and c computed, as given in the table.

57. From a discussion of all the experiments, the conclusion was drawn that the sliding-wedge theory, involving wall friction, is a practical one for the *design* of walls backed by granular materials and subjected to a static load. Often, however, in practical design, vibration due to a moving load has to be allowed for; also the effect of heavy rains. Both these influences tend generally to lower the coefficient of friction and add to the weight of the filling. To allow for these influences, in designing, the normal component E_1 of the earth thrust will alone be

multiplied by a factor of safety σ, the friction $E_1 \tan \phi'$, exerted downwards along the back of the wall, remaining unchanged. This allows very materially for a decrease in ϕ' due to rains and vibration, as well as for an increase in the thrust. A factor of safety $\sigma = 3.5$ is suggested for walls 6 ft. high, decreasing to 3 for walls 10 ft. and upwards. For walls 50 ft. high and upwards, or for lower walls with a high surcharge, this factor may possibly be still further decreased, since before the embankment is finished, the cohesive and chemical actions in the earth have doubtless consolidated it to such an extent that the actual thrust is much less than the computed one when cohesion is neglected.

In any case, the true thrust E (not multiplied by any factor) when combined with the weight of the wall, must give a resultant that will pierce the base within its middle third, since it is desirable that pressure should be exerted over the whole base. If this does not obtain for a certain type of wall, the base should be made wider.

If heavy loads, as railway trains, pass

over the surface of the filling, near a retaining-wall, the weight of the load should be replaced by an equal weight of earth and the earth thrust determined as in Art. 50 or by aid of the construction of Fig. 8, p. 83, or that of Fig. 4, p. 40.

With an earth foundation, a footing of masonry, projecting beyond the wall, should be built of such width that the true resultant on the base should pass near its center. This should totally prevent the increased leaning with time sometimes observed. Lastly, to ensure against sliding, the base should be inclined.

58. *General Formula for Stability of Retaining-walls against Overturning.* Let Fig. 11 represent a wall *ABCD*, whose length perpendicular to the plane of the paper is unity and whose exterior and interior faces and diagonal *AC*, make angles with the vertical equal to β, α and ω respectively. Let W denote the weight of the wall and g the horizontal distance from its line of action to the outer toe A; also call σ the factor by which it is necessary to multiply the normal thrust K_1eh^2, leaving the friction fK_1eh^2 at the back

of the wall constant, in order that the resultant on the base may pass through the outer toe. Here $f = \tan \phi'$ (when

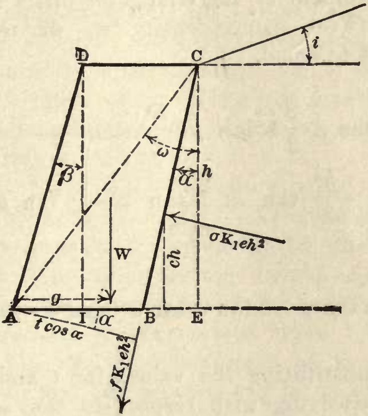

Fig. 11.

$\phi' > \phi$, replace ϕ' by ϕ) and the quantities h, t, e, w, i, ϕ and ϕ' have the meanings given in Art. 52.

Taking moments around A, we have,

$$Wg + fK_1eh^2\, t \cos \alpha = \sigma K_1 eh^2(ch \sec \alpha + t \sin \alpha).$$

We find also, $t = h(\tan \omega - \tan \alpha)$; and since the moment Wg is equal to the sum of the moments of the triangular prism ADI and the rectangular prism $IDCE$, minus the moment of the triangular prism BCE, all of the same density w, we readily find it to equal,

$$\left[\frac{h^2}{2} \tan \beta \cdot \frac{2}{3} h \tan \beta + \frac{h^3}{2} (\tan^2 \omega - \tan^2 \beta) - \frac{h^2}{2} \tan \alpha .\ h(\tan \omega - \frac{1}{3} \tan \alpha)\right] w;$$

or,

$$\frac{wh^3}{6}(3 \tan^2\omega - 3 \tan \omega \tan \alpha + \tan^2\alpha - \tan^2\beta).$$

On substituting the values for t and Wg and resolving with respect to $\tan \omega$, we find,

$$\tan^2 \omega + \tan \omega \left[2\frac{e}{w}K_1(f \cos \alpha - \sigma \sin \alpha) - \tan \alpha\right] =$$

$$\frac{e}{w}2K_1 [\sigma c \sec \alpha + \tan \alpha(f \cos \alpha - \sigma \sin \sigma)] - \frac{1}{3}(\tan^2 \alpha - \tan^2 \beta).$$

This formula equally applies when the inner face of the wall leans away from the earth, or B falls to the right of E, on simply replacing *sin* α and *tan* α by $(-\sin\alpha)$ and $(-\tan\alpha)$ throughout. As this formula is independent of h, it is true for all values of h. When h is given, tan ω is found from the formula, whence, $t = h\ (\tan\omega - \tan\alpha)$.

59. Since $t/h = (\tan\omega - \tan\alpha)$, if we take $h = 1$, the value of $t = AB$ corresponding, represents the ratio of the thickness of the base to h for any height of wall. Hence, for simplicity in the following applications to the various types 1, 2, 3, 4, 5, Fig. 12, the thicknesses at top and bottom and the volume will be computed

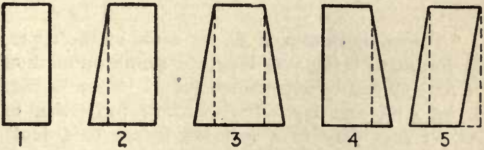

Fig. 12

for $h = 1$. The natural slope will be taken at 3 base to 2 rise or $\phi = 33°\ 41'$ and it will be assumed that $\phi' = \phi$, (whence $f = \tan \phi' = \frac{2}{3}$), $c = \frac{1}{3}$ and $\sigma = 3$, which refers to walls 10 feet high and upwards.

The tables given below are computed for two ratios of specific weight of earth to wall: $e/w = \frac{2}{3}$ and $e/w = \frac{4}{5}$, corresponding, perhaps, to concrete and good brick walls respectively.

60. *Type* 1. *Vertical Rectangular Wall.*

$$\alpha = 0,\ \beta = 0,\ t = \tan \omega.$$

The general formula of Art. 58, reduces to

$$t^2 + \frac{e}{w} 2K_1 ft = \frac{e}{w} 2K_1.$$

When $i = 0$, from p. 99, we have, since $K_1 = K \cos \phi' = K \cos \phi$*

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* The computation of K, for some of the types, by formulas, being very long, the graphical method of Art. 39 can be substituted for it. Thus in Fig. 8, let $e = 1$ and lay off h = vertical projection of $AF = 1$ foot (say to a scale 10 inches to 1 foot) and draw from B, now coinciding with F, a horizontal line to represent the earth surface; then exactly as indicated on p. 88, locate the points O, I, C, H. The thrust $E = Keh^2 = K = \frac{1}{2} CI.CH \therefore K_1 = K \cos \phi$. When $i \doteq \phi$, $AD \doteq \infty$, $AI \doteq \infty$ and

$$K_1 = \frac{1}{2}\frac{\cos^2\phi}{(1+\sqrt{2}\sin\phi)^2} = 0.109.$$

$$\therefore \text{ for } e/w = \tfrac{2}{3},\ t = 0.334,$$

$$e/w = \tfrac{4}{5},\ t = 0.363.$$

When $i = \phi$, page 100, $K_1 = \frac{1}{2}\cos^2\phi = 0.346$. Whence, for $e/w = \frac{2}{3}$, $t = 0.541$; $e/w = \frac{4}{5}$, $t = 0.583$.

61. *Type 2. Vertical Back. Front face battered at 2 inches to the foot.*
$\alpha = 0$, $\tan\beta = \frac{1}{6}$, $\beta = 9°\ 28'$. The formula reduces to,

$$t^2 + \frac{e}{w}2K_1ft = \frac{e}{w}2K_1 + \frac{1}{3}\tan^2\beta.$$

$AC \doteq AD$; hence the point I can be taken anywhere on AD. With $i = \phi$ and C located as before $\therefore$ as above, $K_1 = \frac{1}{2}CI.CH\cos\phi$.

In type 5, the earth pressure on the wall was taken as making the angle ϕ with its normal. The assumption was only intended for usual batters of leaning walls, say $\alpha < 10°$, for which it is practically correct. For large values of α, the assumption is not to be made, the error increasing with the angle α.

For $i=0$, as above, $K_1=0.109$.

$\therefore e/w=\frac{2}{3}$, $t^2+0.097t=0.144+0.0093$,

$$\therefore t=0.346.$$

$e/w=\frac{4}{5}$, $t^2+0.116t=0.174+0.0093$,

$$\therefore t=0.374.$$

For $i=\phi$, $K_1=0.346$, Art. 60.

$\therefore e/w=\frac{2}{3}$, $t=0.548$; $e/w=\frac{4}{5}$, $t=0.589$.

62. *Type* 3. *Both faces battered* 2 *inches to the Foot.* On replacing *sin* α by $(-\textit{sin}\ \alpha)$, *tan* α by $(-\textit{tan}\ \alpha)$ in the general formula, and noting that here, $\tan^2\alpha=\tan^2\beta$,

$$\tan^2\omega+\left[\frac{e}{w}2K_1(f\cos\alpha+\sigma\sin\alpha)+\tan\alpha\right]$$

$$\tan\omega=\frac{e}{w}2K_1\left[\sigma\tfrac{1}{3}\sec\alpha-\tan\alpha(f\cos\alpha+\sigma\sin\alpha)\right].$$

Formulas (13) and (14) p. 97, give, for $\phi=\phi'=33°\ 41'$, $i=0$, $\alpha=9°\ 28'$; $n=0.8434$, $K_1=0.143$.

For $e/w=\frac{2}{3}$, $\tan^2\omega+0.387$, $\tan\omega=0.157$.

$\therefore \tan\omega=0.247 \therefore t=\tan\omega+\tan\alpha=0.414$.

For $e/w=\frac{4}{5}$, $\tan^2\omega+0.430$, $\tan\ \omega=0.189$.

$\therefore\ \tan\ \omega=0.270 \therefore\ t=0.437$.

63. *Type* 3 *continued.* Let $i=\phi$. In this case, the "limiting plane" of Art. 28 concides with the vertical AO of Fig. 5, p. 50. Since the inner face of the wall AB, Fig. 5, lies below it, the thrust on $AO=T=\frac{1}{2}e\cos\phi$, $\overline{AO}^2$ (acting parallel to BO) must now be combined with the weight of the earth ABO to find the resultant on AB. Taking, as before, the vertical height h of $AB=1$, we find $AO=1.111$ and for $e=1$, $T=0.513$. On combining graphically, this thrust on AO, making the angle ϕ with the horizontal with the weight of $ABO(e=1)$, we find the resultant thrust on $AB=0.570$ and that it makes an angle 32° 03′ with the normal to AB. We have to substitute in the general formula $f=\tan\ 32°\ 03'=0.626$; also the normal component of the thrust $=0.570\times\cos\ 32°\ 03'=0.483$. As this corresponds to the assumed height of $AB=h=1$ and $e=1$, it is the value of K_1. Whence substituting $K_1=0.483$, $f=0.626$*

* Formulas have been derived by the writer for

in the formula of Art. 62, we have,

for $e/w=\frac{2}{3}$, $\tan^2\omega+0.881\tan\omega=0.535$.

$$\therefore \tan\omega=0.414,\ t=0.580.$$

For $e/w=\frac{4}{5}$, $\tan^2\omega+1.0024\tan\omega=0.642$.

$$\therefore \tan\omega=0.439.\quad \therefore t=0.606.$$

64. *Type 4. Front Face Vertical, Inner Face Battered 2 Inches to the Foot.* The moment formula differs from that of Art. 62 only in the addition of the term $(-\frac{1}{3}\tan^2\alpha)$ to the right member. Hence,

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K_1 and f for any value of i, but the work is too long to be given here. The results for $\iota=\phi$ will be stated.

From the formula,

$$\tan\left[\epsilon-\left(45°-\frac{\phi}{2}\right)\right]$$

$$=\tan\left(45°+\frac{\phi}{2}-\alpha\right)\tan^2\left(45-\frac{\phi}{2}\right),$$

compute ϵ. In this instance, $\epsilon=48°\ 34'$. The thrust on the wall AB makes with the normal to the wall, the angle,

$$\gamma=90°-(\epsilon+\alpha)=90°-58°\ 02'=31°\ 58';$$

whence $\quad f=\tan 31°\ 58'=0.624.$

The value of K_1 is now given by the formula,

$$K_1=\frac{\cos\gamma\cos(\phi-\alpha)\tan\alpha}{2\quad\cos(\phi+\epsilon)\cos\alpha},$$

which for $\gamma=31°\ 58'$; $\epsilon=48°\ 34'$, $\phi=33°\ 41'$, $\alpha=9°28'$, reduces to $K_1=0.483$, as found graphically.

we at once derive, when $i=0{:}K_1=0.143$ (Art. 62), for $e/w=\frac{2}{3}$, $\tan^2\omega+0.387\tan\omega=0.148$.

$\therefore\ \tan\omega=0.237\ \therefore\ t=\tan\omega+\tan\alpha=0.403$; for $e/w=\frac{4}{5}$, $\tan^2\omega+0.430\tan\omega=0.180$.

$\therefore\ \tan\omega=0.260\ \therefore\ t=0.260+0.167=0.427$. When $i=\phi$, as in Art. 63, $K_1=0.483$, $f=0.626$, and the moment formula just quoted reduces to:

$$e/w=\tfrac{2}{3},\ \tan^2\omega+0.881\tan\omega=0.526;$$
$$\therefore\ \tan\omega=0.408,\ t=\tan\omega+\tan\alpha=0.574.$$
$$e/w=\tfrac{4}{5},\tan^2\omega+1.024\tan\omega=0.633.$$
$$\therefore\ \tan\omega=0.434,\ t=0.601.$$

65. *Type 5. Leaning Wall. Front Face Battered 2 Inches to the Foot, Rear Face Parallel to the Front Face.* The formulas of p. 96 are now applicable for computing $K_1=K\cos\phi$. For $\phi=\phi'=33°\ 41'$, $i=0$, $\alpha=9°\ 28'$, we derive $n=0.7544$, $K_1=0.081$. Putting $\alpha=\beta=9°\ 28'$, the moment formula is,

$$\tan^2\omega+\left[\frac{e}{w}2K_1(f\cos\alpha-\sigma\sin\alpha)-\tan\alpha\right]\tan\omega$$
$$=\frac{e}{w}2K_1\left[\frac{\sigma}{3}\sec\alpha+\tan\alpha\,(f\cos\alpha-\sigma\sin\alpha)\right].$$

Taking as before, $\sigma=3$, $f=\frac{2}{3}$, $\alpha=9^\circ\ 28'=\beta$, when $e/w=\frac{2}{3}$, $\tan^2\omega-0.149\tan\omega=0.112$. $\therefore \tan\omega=0.416$. $\therefore t=\tan\omega-\tan\alpha=0.249$. For $e/w=\frac{4}{5}$, $\tan^2\omega-0.144\tan\omega=0.135$. $\therefore \tan\omega=0.446$. $\therefore t=0.446-0.167=0.279$. Assuming $i=\phi$, the formulas of p. 96, give,

$$n=0,\ K_1=K\cos\phi=\frac{1}{2}\,\frac{\cos^2(\phi+\alpha)\cos\phi}{\cos^2\alpha\cos(\phi-\alpha)};$$

whence $K_1=0.250$.

$\therefore e/w=\frac{2}{3}$, $\tan^2\omega-0.112\tan\omega=0.347$;

$\therefore \tan\omega=0.648$. $\therefore t=0.481$;

$e/w=\frac{4}{5}$, $\tan^2\omega-0.101\tan\omega=0.416$.

$\therefore \tan\omega=0.697$, $t=0.530$.

66. As a check on the computations, the values of K_1, for all the cases discussed, were likewise found by the graphical construction of Fig. 8, p. 83. Then, Fig. 11, the resultant of the components $3K_1eh^2$ and fK_1eh^2, for $h=1$, was combined with the weight W of the wall, acting through its center of gravity, to find the resultant on the base. In every instance, it passed nearly or exactly through the outer toe.

The next step was, assuming $\sigma=1$, to combine K_1eh^2 and fK_1eh^2 ($h=1$), Fig. 11, to find the true resultant on BC, which was then combined with W to find the true center of pressure on the base AB of the wall. Call a the distance from this center of pressure to the center of the base AB; then the ratio a/t, was computed and inserted in the following table, which contains the results of the above computations. When $a/t < 0.167$, the true center of pressure on the base is within the middle third limit, so that the whole base is in compression; when $a/t = 0.167$ there is no stress at the inner toe, and when $a/t > 0.167$, part of the base only is in bearing. The ratio a/t will be counted positive or negative according as the resultant on the base meets it to the left or to the right of its center.

It will be observed, for cases one and three of type 4, that $a/t > 0.167$. In the first case, increase t from 0.403 to 0.417; in the second case, from 0.427 to 0.432. These values are inserted in the table in parentheses. The resultant on the base, in each case, will then cut the base $\frac{1}{3}t$ from the outer toe.*

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* If the resultant on the base of the wall for the actual thrust ($\sigma = 1$) is to pass $\frac{1}{3}$ base from the outer toe, then for the leaning wall shown

67. In the last column of the table, is given the angle that the true resultant on the base makes with the normal to the base. This should not exceed the angle ϕ' of friction of masonry on earth or sliding will occur. The factor of safety against sliding will be at least, $\frac{\tan \phi'}{\tan \theta}$ and if possible this factor should not exceed two. The average angles of friction of masonry on dry clay, dry earth and firm sand or gravel, are 27°, 30°, 35° respectively, but on wet clay, 11° to 18° has been given. Hence it is not always possible, for reasonable thicknesses of wall, to ensure a factor of safety of 2 against sliding. In such cases, the base should be inclined, so that the resultant on it, should make an angle with

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in Fig. 11, the moment formula, deduced in a similar manner to that of Art. 58, is as follows:

$$\tan^2 \omega + \left[\frac{e}{w} 4K_1(f \cos \alpha - \sin \alpha) + \tan \beta\right] \tan \omega$$

$$= \frac{e}{w} 2K_1[3c \sec \alpha + 2 \tan \alpha (f \cos \alpha - \sin \alpha)] + \tan \beta (\tan \alpha + \tan \beta).$$

The formula is adapted to the case where the inner face of the wall leans away from the earth, by replacing *sin* α by $(-\sin \alpha)$ and *tan* α by $(-\tan \alpha)$.

its normal, much less than the probable angle of friction. In the tabular thicknesses, no foundation slab was assumed, though one is always desirable and it should be constructed with the toe projecting beyond the front face of the wall sufficiently to allow the resultant on the base to pass as near its center as is practicable and thus distribute the pressure on the base more uniformly.

For an actual wall, the unit pressure on the base (the "soil pressure") should be computed by (1), Art. 15 and if too large, the foundation slab must be widened, so as not to subject the soil to a greater pressure than is accepted as safe.

If a value of t is desired, for a value of e/w intermediate between $\frac{2}{3}$ and $\frac{4}{5}$, it can be found with substantial accuracy, by ordinary interpolation from the tabular values, assuming a linear variation.

68. On referring to the column of volumes (or areas of cross-sections for a length of wall unity) it will be observed that for level-topped earth, the types are economical in the order,

3, 5, 2, 4, 1.

Type	$\tan\beta$	$\tan\alpha$	$\tan i$	K_1	$\frac{e}{w}$	t	t'	vol.	$\frac{a}{t}$	θ
1	0	0	0	.109	2/3	.334	.334	.334	.126	11°
			2/3	.346	2/3	.541	.541	.541	.163	18°
			0	.109	4/5	.363	.363	.363	.119	12°
			2/3	.346	4/5	.583	.583	.583	.161	20°
2	$\frac{1}{6}$	0	0	.109	2/3	.346	.180	.263	.064	13°
			2/3	.346	2/3	.548	.381	.464	.051	20°
			0	.109	4/5	.374	.207	.292	.051	14°
			2/3	.346	4/5	.589	.422	.506	.049	22°
3	$\frac{1}{6}$	$-\frac{1}{6}$	0	.143	2/3	.413	.080	.247	.121	14°
			2/3	.483	2/3	.580	.248	.414	.084	23°
			0	.143	4/5	.437	.103	.270	.119	16°
			2/3	.483	4/5	.606	.272	.439	.076	24°
4	0	$-\frac{1}{6}$	0	.143	2/3	.403	.237	.320	.174	12°
						(.417)	(.250)	(.333)	(.167)	
			2/3	.483	2/3	.574	.408	.491	.124	21°
			0	.143	4/5	.427	.260	.344	.169	13°
						(.432)	(.265)	(.348)	(.167)	
			2/3	.483	4/5	.601	.434	.517	.117	23°

Type	$\tan\beta$	$\tan\alpha$	$\tan i$	K_1	$\frac{e}{w}$	t	t'	vol.	$\frac{a}{t}$	θ
5	$\frac{1}{6}$	$\frac{1}{6}$	0	.081	2/3	.249	.249	.249	−.089	12°
			2/3	.250	2/3	.481	.481	.481	+.013	18°
			0	.081	4/5	.279	.279	.279	−.071	13°
			2/3	.250	4/5	.530	.530	.530	+.019	19°

Various types of retaining-walls, designed for $\phi = \phi' = 33° 41'$, $\sigma = 3$, $c = \frac{1}{3}$.

Types 3 and 5 are nearly equal in volume, but the pressure on the base is better distributed in number 5.

When the earth surface slopes at the angle of repose, the volumes increase in the order 3, 2, 4, 5, 1.

The value of t is the width of the wall at the base, t', the width at top, both for $h=1$. They likewise represent the ratios t/h, t'/h for any height of wall h. Thus for $h=10$ ft., type 2, $i=0$, $e/w=\frac{2}{3}$, width at base $=3.46$ ft., width at top $=1.80$ ft.

69. Walls with projections at intervals, on the exterior or interior, are known as *buttressed* or *counterforted* walls respectively. Fig. 13 shows a good form of buttressed

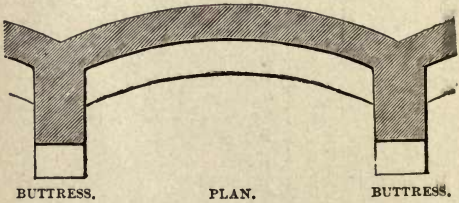

Fig. 13

wall, with the face in the form of arches, convex from the earth side. In designing such walls, moments are taken about the outer toes of the buttresses. The great objection to counterforted walls in masonry not reinforced, is that the counterforts are apt to break away from the face wall; so that they have not found favor in America, in spite of the large economy shown. When reinforced concrete is used, they present a very effective type of wall.

APPENDIX I.

DESIGN FOR A VERY HIGH MASONRY DAM.

ENGINEERS are by no means agreed upon the proper profile to give high-masonry dams; although the three conditions, that *there shall be no tension at any horizontal joint, safe unit stresses everywhere, and no possible sliding along any plane joint*, seem to be generally accepted as essential to a good design.

The writer suggests one more condition, that *the factors of safety against overturning about any joint on the outer face shall increase gradually as we proceed upwards from the base*, to allow for the proportionately greater influence, on the higher joints, of the effects of wind and wave action, ice, floating bodies, dynamite, or other accidental forces. The exact amount of increase must be largely a matter of judgment; but, if the principle is accepted, it can only result in making stromger dams.

The accompanying sketch of a dam 258 feet high to the surface of water (see also "Engineering News" for June 23, 1888) satisfies the four condi-

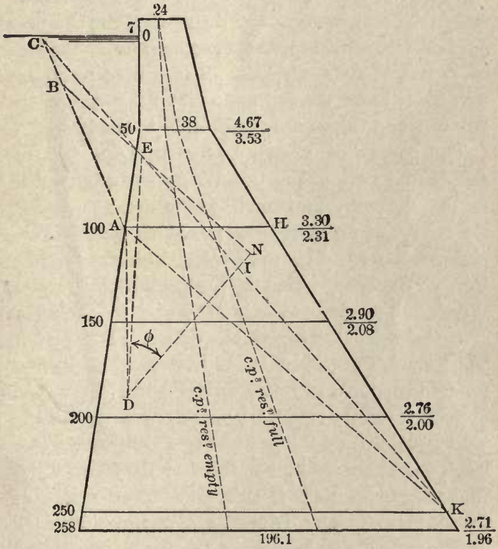

Fig. 14

tions named, and will be briefly described. The dam is of the same total height (265 feet) and volume (nearly) as the proposed Quaker-Bridge dam, and, for ease of comparison, is designed, as was that dam, for masonry weighing $2\frac{1}{2}$ times as much as water. The dam is 24 feet wide at top, 38 feet wide, 50 feet below the surface of water (7 feet below the top), and 196.1 feet wide at the base. The up-stream face is vertical for the first 57 feet from the top, and then batters at the rate of 30 feet in 200 to the base. The outer face slopes uniformly from the top to 50 feet below the water surface, and then slopes uniformly to the base.

The curves of pressure, for reservoir full or empty (the lines connecting the centres of pressure on the different horizontol joints are here styled the curves of pressure), are found as hitherto explained, and are seen to lie well within the middle third of the base, so that the horizontal joints under the static pressure are only subjected to compression throughout their whole extent. Further, it was found by construction, that if a horizontal force be assumed as acting at the surface of water, of such intensity (29,375 pounds) as to cause the total resultant, on the joint 50 feet below the water level, to cut the joint one-third of its width from the outer face; then if this same force, acting at the surface of water, is combined in turn with each of the other resultants on the lower horizontal joints, the new centres of pressure will still lie well within the middle third for the lower joints. To secure uniformity of results for all the joints, the width at the 50 feet level should be increased, although it is now much greater than ordinarily constructed. If, however, the effects of earthquake vibrations are to be guarded against, we cannot replace them by the action of a single force acting at the surface, so that the increased width of the upper joints must be largely a matter of judgment.[1]

The numbers to the right of the figure, in the form of a fraction, give for the corresponding joints, for the upper numbers, the factor against overturning, or the factor by which it is necessary to multiply the static horizontal thrust of the water to cause the total resultant to pass through the outer edge of the joint considered; and for the lower numbers, the ratio of the weight of masonry above a joint to the static thrust of water against it; which is, in a certain sense, a factor of safety against sliding on a horizontal joint. These factors are seen to increase from the base upwards, so that the suggested fourth condition is satisfied.

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1 It is stated in Engineering News for June 30, 1888, on the authority of Mr. Thomas C. Keefer, President American Society of Civil Engineers, that "an ice bridge of about 90 feet span, between two fixed abutments, expanded so from a rise of temperature, as to rise 3 feet in the centre." If we regard the arch thus formed as free to turn at the abutments and at the crown, we easily find for ice one foot thick, the horizontal thrust H exerted at the abutments, from the equation, $3H = \frac{62.5}{2} \times \overline{45}^2$, to be in pounds per square foot $H = 21{,}094$ pounds. Much higher pressures may possibly be experienced sometimes near the top of high dams in northern latitudes, and it seems only proper to include such contingencies in their design.

The unit stresses, in pounds per square foot, at the outer edges of the joints for reservoir full, and at the inner edges for reservoir empty, are given in columns 4 and 5 of the following table, being computed from the formula

$p = \left(4 - \frac{6u}{l}\right) \frac{W}{l}$ of Chap. I.

Depth of Joint below Water Level. 1.	Water Pressure. 2.	Weights of Masonry. 3.	Pressure at outer edge. 4.	Pressure at inner edge. 5.
feet.			lbs.	lbs.
50	1,250	4,417	8,860	10,460
100	5,000	11,540	13,480	16,130
150	11,250	23,420	20,410	21,440
200	20,000	40,040	27,330	27,170
250	31,250	61,420	34,350	33,130
258	33,282	65,270	35,360	34,120

The numbers of columns 2 and 3 for one foot in length of the wall are expressed in weights of cubic feet of water, and must be multiplied by 62.5 to reduce to pounds.

The unit pressures, although necessarily high, are still permissible. By spreading the lower part of the dam still more, these unit stresses would be theoretically diminished, though it is likely that in reality the pressures at the positions of the old toes would not be very materially altered; but the masonry being surrounded with other masonry could, most probably, stand a higher pressure.

The unit pressures p given in columns 4 and 5 are not the maximum normal pressures at the faces. In Appendix III (*e*), it is proved that the maximum normal stress at a face acts parallel to that face on a plane at right angles to it and that its intensity is given by the formula, $f = p \sec^2\phi$, where ϕ is the angle the face makes with the vertical. In this example, where $\phi = 31° 23'$ for the outer face and $8° 32'$ for the inner face, the

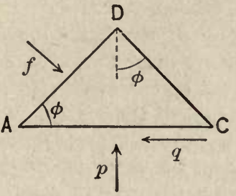

Fig. 15.

values of f at the outer and inner faces are found by multiplying the numbers given in columns 4 and 5 by 1.37 and 1.02 respectively.

The first derivation of the important formula, $f = p \sec^2\phi$, has been credited to Levy by Dr. Unwin,[1] who likewise states that in several old dams which have lasted for centuries, the values of p, ranged from $11\frac{1}{2}$ to 14 tons per square foot, giving the maximum compressive stresses f from 15 to 20 tons per square foot (234 to 311 lbs. per square inch).

[1] *Minutes of Proceedings*, Inst. C. E., Vol. CLXXII, Part II, p. 134.

The so-called factors against overturning are not true ones, for a computation shows that if the water pressure down to the joints 50, 100, and 150 feet below the surface should become 2, $1\frac{1}{2}$, $1\frac{1}{3}$ times the original, respectively, that tension would just begin to be exerted at the inner face. This would happen for lower joints for thrusts about $1\frac{1}{4}$ to $1\frac{1}{5}$ times the original. If, from any cause, as accidental forces at the top, earthquakes, etc., the thrusts should be increased over these amounts, causing tension at the inner edges beyond the capacity of the mortar to withstand, the joints would crack and open, water would get in, diminishing the weight of the masonry materially, the centres of pressure would move outwards, and the unit pressures at the outer toes would very much increase, leading perhaps ultimately to the destruction of the dam through sliding, overturning, or crushing at the down-stream face.

We shall now consider the capacity of resistance of the dam to *sliding along any oblique joint* as AK.[1] Let AB represent, in magnitude and direction, the resultant of the water pressure and weight of masonry on the horizontal joint AH, and let the vertical AD represent the weight of the triangular mass AHK, all for one foot in length of the wall. Draw $DN \perp AK$ and $BN \parallel AK$ to intersection N;

[1] See Annales des Ponts et Chaussées for May, 1887.

then DN = component of BD normal to plane AK, and $DN \times \tan \phi$ (where $\tan \phi$ = co-efficient of friction of masonry on masonry) is the total friction that can be exerted by the plane AK. If we lay off angle $NDE = \phi$ (taken as 35° here) to intersection E with the parallel component BN, we have $DN \tan \phi = EN$, so that BE must be resisted by cohesion; and the unit-shearing stress along the plane $AK = \frac{BE}{AK}$. If, now, we produce KE on to intersection C, with AB produced, we have the unit shear represented by $\frac{BC}{AC}$, which is a maximum, for various planes passing through A, when C is farthest removed from B.

On effecting this construction, then, for a series of planes passing through A, we quickly find the plane which will have to supply the maximum intensity of shear, or the plane of rupture, to lie near AK (there is very little difference for a series of planes lying near each other); and the shear per square foot required to resist sliding, in addition to the frictional resistance, to be about twenty-seven hundred and fifty pounds. To offer the greatest resistance to sliding, there should be no regular courses, and the stones should break joint vertically as well as horizontally, or the courses near the outer face should be curved so as to be approximately normal to that face. For a retaining-wall of dry rubble, carelessly laid, we see that there is every probability of failure by sliding along some inclined plane. Here the stones must be carefully interlocked to prevent sliding. For the reservoir-wall, where the best cement is used, and the joints are broken, there should be no fear of sliding when sufficient thickness is given to avoid tension. In the Habra dam, a hundred and sixteen feet high, this was not done; and the dam broke along a plane, passing through the outer toe nearly, and making the angle of friction ϕ of masonry on masonry with the horizontal.

It is well to note, too, that friction alone will not prevent sliding along planes inclined not far from the horizontal as well as those below, so that a proper resistance to shear must be provided for in every dam. Possibly the weak point of many dams is in this very particular.

The capacity of the dam in question to resist rotation about the toe of an inclined base may next be tried, and it will be found to be stable; for the weight of masonry, as well as its arm, increases to counterbalance the increase of arm of the water-thrust. The dam thus satisfies all the conditions of stability; and, although some of its dimensions may be changed with advantage perhaps, it yet suffices very well to point out the principles of design.

See Engineering News for January 12, 1893 and May 9, 1907 for effects of expansion of ice.

APPENDIX II.

STRESSES IN MASONRY DAMS.[1]

THE object of this investigation is to determine the amounts and distribution of the stresses in a masonry dam, at points not too near the foundations, having assumed the usual "law of the trapezoid," that vertical unit pressures on horizontal planes vary uniformly from face to face.

Experiment indicates that such vertical stresses increase pretty regularly in going from the inner to the outer face, for reservoir full, until we near the down-stream or outer face, where the stress gradually changes to a decreasing one, which decrease continues to the end of the horizontal section. The law of the trapezoid is thus only approximately true over part of the section, but, as it gives an excess pressure where it attains a maximum, it errs on the safe side.

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[1] What follows in Appendices II and III was first given by the author in Trans. Am. Soc. C.E., Vol. LXIV, p. 208.

The profile of the dam selected is of the triangular type, with some additions at the top, but the method used in determining the stresses is general and will apply to any type of profile. The final equations will give, at any (interior or exterior) point of the horizontal section considered, the vertical unit stress on the horizontal section, the normal stress on a vertical plane, and the unit shear on either horizontal or vertical planes. From these stresses, the maximum and minimum normal stresses, and the planes on which they act, can be determined, and ultimately, if desired, the stress on any assumed plane can be ascertained.

The solution presented is approximate, which is justifiable, in view of the approximation involved in "the law of the trapezoid" used. The results, however, are practically correct, as will be evident from the checks applied, resulting from the exact theory given in Appendix III. The theory used, being simple, should be easily followed.

Let Fig. 16 represent a slice of the dam contained between two vertical parallel planes, 1 ft. apart and perpendicular to the faces. The batter of OB is $\frac{130}{200}=\frac{0.65}{1}$; that of OE being $\frac{4}{200}=\frac{0.02}{1}$. The batter of the inner face was found by trial, so that the centers of pressure on horizontal sections, for reservoir empty, should nowhere pass more than a fraction of a foot outside the middle third of the section. The simple type of profile shown was adopted for ease of computation.

For convenience in subsequent computations, the breadths, $b=EB$, of horizontal sections, corresponding to various depths, h, below the surface of the water in the reservoir, are given, all dimensions being in feet:

$$h = 199.0, \quad b = 133.330;$$
$$h = 199.5, \quad b = 133.665;$$
$$h = 200.0, \quad b = 134.000;$$
$$h = 200.5, \quad b = 134.335;$$
$$h = 201.0, \quad b = 134.670.$$

Take the weight of 1 cu. ft. of masonry equal to 1; then the weight of masonry above any

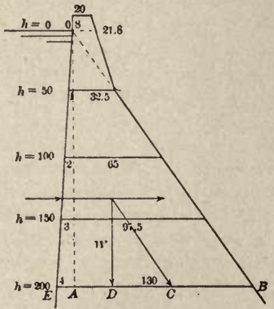

FIG. 16.

section is equal to the corresponding area in Fig. 16 above that section. The area of the portion above EOB is readily found to be 712, and its moment about the vertical, AO, is 11,603, the unit of length being the foot. In Fig. 16, D is where the vertical through the center of gravity of the dam above the joint, EB, cuts that joint, and C is the center of pressure on that joint when the water pressure on EO is combined with the weight of masonry, W, above EB.

As h varies, suppose each horizontal joint, in turn, marked similarly to the joint at $h=200$, with the letters E, A, D, C, B; then, for any joint, on taking moments of the triangles, AOB, AOE, and the area above OB about A, we find

$$AD=\frac{\frac{AO}{6}(AB^2-EA^2)+11{,}603}{W}.$$

Assuming that the masonry weighs $2\frac{1}{2}$ times the water per cubic unit, then the weight of a cubic foot of water is $\frac{2}{5}$. It would entail but little extra trouble here, where the inner face has a uniform batter throughout, to include the vertical component of the water pressure on the face, EO; but it will be neglected, as usual.

The horizontal water pressure for the height, h, is thus, $\frac{2}{5}\times\frac{h^2}{2}=\frac{1}{5}h^2$, and its moment about C is $\frac{1}{5}h^2\times\frac{1}{3}h=\frac{1}{15}h^3$.

Taking moments of W and water pressure about C, we have at once,

$$DC=\frac{1}{15}\times\frac{h^3}{W}.$$

From the last two formulas, we derive the following results:

h	W	AD	DC
199	13978.335	40.49141	37.58483
200	14112.000	40.70316	37.79289
201	14246.335	40.91488	38.00089

A seven-place logarithmic table was used throughout, the aim in the computations being to get the seventh significant figure correct within one or two units. The necessity for this accuracy will be seen later.

The distances EC and CB are now readily derived.

For $h=199$, $EC=82.05624$, $CB=51.27376$;
$h=200$, $EC=82.49605$, $CB=51.50395$;
$h=201$, $EC=82.93577$, $CB=51.73423$.

On any plane, EB, the vertical unit pressure

$$\text{at } B=p_1=\frac{4b-6CB}{b^2}W,$$

$$\text{at } E=p_2=\frac{4b-6EC}{b^2}W;$$

where $b=EB$, and W is the weight of masonry above the plane. This follows from the assumed "law of the trapezoid."

From these formulas we derive:

At $h=199$, $p_1=177.45483$, $p_2=32.22542$;
$h=200$, $p_1=178.3855$, $p_2=32.24139$;
$h=201$, $p_1=179.3160$, $p_2=32.25798$.

Call p the vertical unit stress at a distance, x', from E; then

$$p = p_2 + \frac{p_1 - p_2}{b} x';$$

and the total stress on the base, x', is

$$P = \frac{1}{2}[p_2 + p]x' = p_2 x' + \frac{p_1 - p_2}{2b} x'^2. \quad . \quad . \quad (1)$$

To find the unit shear on vertical or horizontal planes,[1] consider a slice of the dam, bounded by

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[1] The writer desires here to acknowledge his indebtedness to a recent paper on "Stresses in Masonry Dams," by Ernest Prescot Hill, M. Inst. C.E., published in *Minutes of Proceedings*, Inst. C.E., Vol. CLXXII, p. 134. Mr. Hill considers the case of a dam with a vertical inner face. By the aid of the calculus, he effects an exact solution, which leads to general formulas for shear and normal pressures on vertical planes.

The principles at the base of his method, though somewhat disguised by the calculus notation, are essentially the same as those used by the author.

Mr. Hill ascribes to Professor W. C. Unwin the suggestion, "that the shearing stress at any point may be found by considering the difference between the total net vertical reactions [between that point and either face] along two horizontal planes at unit distance apart," and states that Prof. Unwin "has applied the principle to a triangular dam by the use of algebraical methods."

Dr. Unwin states (Proc. Inst. C.E., Vol. CLXXII, Part II, p. 161) that he ascertained after his papers were written, that by a different method, Levy had previously arrived at the same conclusions.

horizontal planes at $h=199$ and $h=200$, the water face and a vertical plane, at a distance x, from the inner face (Fig. 17), in equilibrium under the water pressure acting horizontally on its left face and the forces exerted by the other parts of the dam on the slice. These forces consist of the uniformly increasing stress, P', on top, acting down; the uniformly increasing stress, P, on the bottom, acting up; a shear acting on the vertical plane

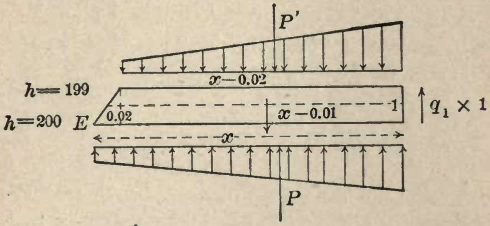

Fig. 17.

at the right, of average intensity q_1 per square foot, the weight of the body $(x-0.01)$, besides the horizontal forces to be given later. The vertical component of the water pressure is here neglected, as usual. The origin for x is taken, here and in all subsequent work, at the level, $h=200$, at the inner face.

For equilibrium, the sum of the vertical components must be zero.

Therefore,

$$q_1=(x-0.01)+P'-P. \quad . \quad . \quad . \quad (2)$$

To find P', substitute in Equation (1), $x' = x - 0.02$, $p_2 = 32.22542$, $p_1 - p_2 = 145.22941$, $b = 133.330$, giving $P' = 32.20364x + 0.5446238x^2 - 0.6442906$. For P, $x' = x$, $p_2 = 32.24139$, $p_1 - p_2 = 146.1441$, and $b = 134$; therefore,

$$P = 32.24139x + 0.5453138x^2.$$

Substituting in Equation (2), we derive the average unit shear,

$$q_1 = -0.6542906 - 0.96225x - 0.0006900x^2. \quad . \quad (3)$$

This value of q_1 is strictly correct when $x \geq 0.02$. It is slightly in error when $0 < x < 0.02$.

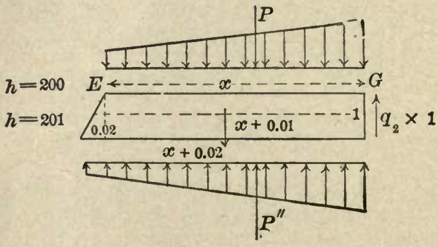

FIG. 18.

A similar investigation holds to obtain the average unit shear, q_2 (Fig. 18), on a vertical plane, at a distance, x, from E, extending from the level, $h = 200$, to the level, $h = 201$.

We have, for equilibrium,

$$q_2 = (x + 0.01) + P - P''. \quad . \quad . \quad (4)$$

We find P'' by substituting in Equation (1), $x'=(x+0.02)$, $p_2=32.25798$, $p_1-p_2=147.05802$, and $b=134.67$. $P''=32.27982x+0.5459941x^2+0.6453780$. Substituting this, and the value previously found for P, in Equation (4), we derive,

$$q_2=-0.6353780+0.96157x-0.0006803x^2. \quad (5)$$

This is strictly correct only when $x \geq 0$.

The mean, $\frac{1}{2}(q_1+q_2)$, of these average shears will be assumed as approximately equal to the intensity of shear at the point, $G(x=EG)$, at the level, $h=200$. Call q this intensity of shear on a vertical plane at G; therefore,

$$q=-0.6448343+0.96191x-0.0006856x^2. \quad (6)$$

Checks.—By Appendix III (*b*) and (*d*), the exact value of q, at either face, $=p \tan \phi$, where $p=$ vertical unit normal stress at the face and ϕ is the angle the face makes with the vertical. Thus, at the inner face, $q=-32.24139\times 0.02=-0.6448278$, whereas Equation (6) gives for $x=0$, $q=-0.6448343$.

At the outer face, the exact value is, $178.3855\times 0.65=115.9506$, whereas Equation (6) gives, for $x=134$, $q=115.9405$.

A still more searching test can be devised. It is a well-known principle that the intensity of shear at a point, on vertical or horizontal planes, is the same [Appendix III (*a*)]. Therefore, regarding Equation (6) as giving the horizontal unit shear, at the level, $h=200$, where $b=134$ ft.; the total shear, from face to face, on this level, is

$$\int_{x=0}^{x=134} q dx = 7999.75.$$

This should equal the total water pressure down to the same level, $\frac{1}{5}(200)^2=8000$. Formula (6) thus gives practically exact results.

In order to find the normal unit stress on a vertical plane, we shall assume that q_1, given by Equation (3), equals the intensity of shear on a vertical or horizontal plane at the point, x, at $h=199.5$; and that q_2, given by Equation (5), gives the shear intensity at x at $h=200.5$. This evidently supposes that the shear intensity increases uniformly, vertically, from $h=199$ to $h=201$.

Consider a portion of the dam, Fig. 19, bounded by the water face; the plane, FM, at the level, $h=199.5$, on which the total shear is Q', the plane EN, at the level 200.5, on which the total shear is Q, and the vertical plane, MN, 1 sq. ft. in area, on which the average normal stress is p'. The water pressure on EF will be supposed to be exerted horizontally. It is equal to 80 units. Assuming, as stated, that q_1 = intensity of horizontal shear at M, and q_2 = the corresponding intensity at N, we have, taking the origin as before at O,

$$Q' = \int_{0.01}^{x} q_1 \, dx; \quad Q = \int_{-0.01}^{x} q_2 \, dx;$$

or,

$$Q'=0.006494794-0.6542906x+0.481125x^2 \\ -0.00023x^3;$$

$$Q=-0.00640186-0.6353780x+0.480785x^2 \\ -0.0006803\frac{x^3}{3}.$$

Checks.—The total water pressure for $h=199.5$ is $\frac{1}{5}(199.5)^2=7960.05$ and for $h=200.5$, $\frac{1}{5}(200.5)^2=8040.05$. The first should equal Q', for $x=133.665$, or 7959.22; the second should equal Q,

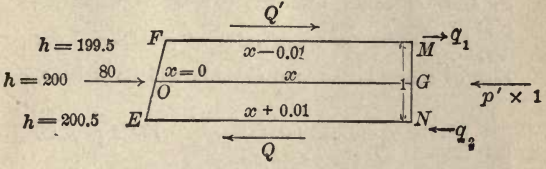

FIG. 19.

for $x=134.335$, or 8041.12. The slight differences tend to give confidence in the results.

For equilibrium, the sum of the horizontal forces acting on $EFMN$, Fig. 19, must be zero; therefore,

$$p'=80+Q'-Q\,, \quad . \quad . \quad . \quad . \quad (7)$$

$$p'=80.01-0.0189x+0.00034x^2-0.00000323x^3.$$

This average stress will now be assumed to be the intensity of the horizontal unit stress on vertical planes at $h=200$.

It will now be perceived why a seven-place table was necessary in the computations, the coefficients of x^2 and x^3 having only two or three significant figures in the final result. If the planes originally had been taken 0.1 ft. apart vertically, a ten-place table would have been required.

Checks.—The value of p', for $x=0$, $p'=80.012896$, is the same as that given by Appendix III (*d*), $80+0.6448\times0.02$. When $x=134$, the formula gives $p'=75.81$, whereas the exact theory, Appendix III (*b*), gives $p'=m^2p=(0.65)^2\times178.39=75.37$. The difference is 0.44 at the outer face. For any other point, it might be assumed to vary with x, so that it could be corrected by substracting $\frac{0.44}{134}x=0.0033x$ from the value of p' above. For ease of computation, the formula will be written,

$$p'=80.01-0.02x+0.00034x^2-0.00001\frac{x^3}{3}. \qquad (8)$$

The first coefficient of x^3 cannot be counted on to the last two figures, hence we are permitted to change 323 to 333 in that coefficient. When $x=134$, Equation (8) gives $p'=75.41$, nearly the exact value.

The three formulas for p, q, and p', at the level $h=200$, are thus as follows:

$$p=32.24+1.09063x;$$

$$q=-0.64+0.962x-0.000686x^2;$$

$$p'=80.01-0.02x+0.00034x^2-0.00001\frac{x^3}{3}.$$

Since the weight per cubic foot of masonry was assumed as two and one-half times that of water, we must multiply the stresses given in Table I by $\frac{5}{2}(62.5) = 156.25$, to reduce to pounds per square foot; or by 1.085, to reduce to pounds per square inch.

TABLE I.

x	0	10	25	50
p	32.24	43.15	59.50	86.77
q	−0.64	8.91	22.98	45.75
p'	80.01	80.02	79.66	79.11
Max. f	80.02	82.06	94.67	128.85
Min. f	32.23	41.11	44.48	37.03
θ for max. f . .	90° 46′	77° 06′	56° 50′	42° 36

x	75	100	134
p	114.04	141.30	178.39
q	67.65	88.70	115.95
p'	79.01	78.08	75.37
Max. f	166.40	203.85	253.71
Min. f	26.64	15.52	0
θ for max. f . .	37° 44′	35° 12′	33° 01′

In Table 1 the stresses are those experienced at the level, $h = 206$.

p = vertical unit stress on a horizontal plane;
q = shearing unit stress on horizontal or vertical planes;
p' = horizontal unit stress on vertical planes;
Max. f = maximum normal stress acting on a plane inclined to the horizontal at the angle, θ, given on the last line;

Min. f = minimum normal stress acting on a plane perpendicular to the last.

From max. f and min. f, with θ, the ellipse of stress can be drawn, and the stress in any direction, with the plane on which it acts, can be ascertained.

It will be observed that there is no tension exerted anywhere, and that the maximum compression is 253.71, or 275 lbs. per square inch, which is exerted at the outer face, parallel to that face, upon a plane at right angles to the face.

In Appendix III (e), the important formula, for the maximum normal intensity at the outer face, acting parallel to that face,

$$f=\frac{p}{\cos^2\phi},$$

is proved. In this instance, $p=178.39$, $\tan\phi=0.65$, therefore $\phi=33°\ 01'$, whence $f=253.71$.

This stress is unaccompanied with any conjugate stress, perpendicular to the face. In the interior of the dam, where conjugate stresses prevail, the masonry is perhaps better able to withstand a certain compressive stress than at the face. The distribution of stresses, at the level, $h=200$, is shown in Fig. 20, on the supposition that the base of the dam is a little below that level. The connection with the foundation materially modifies this distribution; but Fig. 20 shows the distribution for sections, say, from 10 to 20 ft. above the base, up to the level $h=100$, fairly well, on the basis of the trapezoid law. As has been mentioned before, this law gives a pressure greater than the actual at the outer face.

Since the batter of the inner face is very small, the results of Table I should agree approximately, except near the inner face, with those found by Mr. Hill in the paper referred to in the foot note.

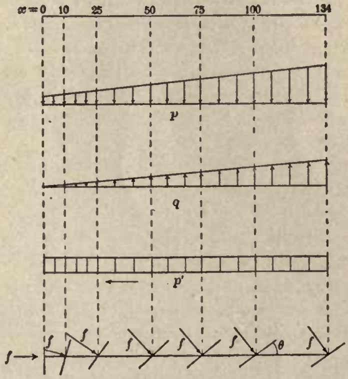

FIG. 20.

Substituting numerical values, Mr. Hill's formulas, for $h=200$, reduce to

$$q=0.9426x-0.0005768x^2,$$

$$p'=80-0.0001289x^2-0.0000009615x^3;$$

giving:

x	0	10	25	50	75	100	134
q	0	9.36	23.20	45.69	67.45	88.49	115.95
p'	80	79.99	79.90	79.56	78.87	77.75	75.38

On comparnig these formulas with those of the writer, it will be observed that the absolute term in the value of q and a consequent term of the first degree in x, in the value of p', are lacking in Mr. Hill's formulas. This results from taking the inner face as vertical. Although the coefficients also differ, it is seen that the numerical values are very nearly the same.

In Fig. 21 are shown, on a drawing of the dam, to scale, the lines of the centers of pressure for reservoir full and empty.

To the right, and under the word "factors," are certain numbers, written in the form of fractions. For any joint, the upper number gives the factor against overturning, or the number by which it is necessary to multiply the water pressure down to the joint, to cause the total resultant to pass through the outer edge of the joint considered. The lower numbers give the ratio of the weight of masonry above a joint to the water pressure corresponding.

It is believed that these "factors" should increase from the base upward, to allow somewhat for earthquakes, expansion of ice in freezing, etc., since the effects of such accidental forces is proportionately greater on the upper joints.

Stresses due to water infiltration are not included here; neither are stresses due to temperature changes.

The unit stresses, f, in pounds per square inch, acting parallel to the adjacent face, are as follows,

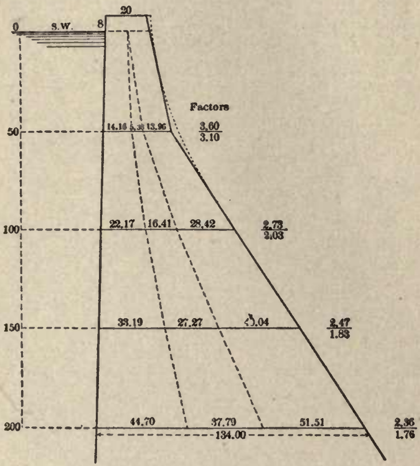

Fig. 21.

and refer to the outer edges of the joints, for reservoir full, and to the inner edges for reservoir empty:

h	f at Outer Edge,	at Inner Edge.
50	85	58
100	136	133
150	204	180
200	275	228

The stresses, f, are normal pressures on planes perpendicular to the respective faces, and are the greatest stresses that can be experienced in the dam. In fact, they are greater than the true stresses, since the trapezoid law is not exact, particularly near the base, as before remarked. It would then seem that the dam, thus far, is safe, since the maximum unit stress is less than concrete, even, is subjected to daily, in good practice.

For an actual construction, the outer face should be curved, from near $h=50$ to the top, as shown by the curved dotted line in Fig. 21.

The subject of the stresses in masonry dams has caused a great deal of discussion among British engineers in the last two or three years. The subject was reopened by Mr. L. W. Atcherly and Professor Karl Pearson,[1] who gave the results of certain experiments which seemed to indicate considerable tension across vertical planes near the outer toe. The late Sir Benjamin Baker, Hon. M. Am. Soc. C. E., also published [2] the results of experiments on a model dam of stiff jelly, and very recently, the "Experimental Investigations" of Sir J. W. Ottley and Mr. A. W. Brightmore [3] on elastic dams of "plasticine" (a kind of modeling clay) and the experiments of Messrs. J. S. Wilson and W. Gore [4] on "India Rubber Models" have been presented.

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[1] Minutes of Proceedings, Inst. C. E., Vol. CLXII, p. 456.

[2] Ibid., Vol. CLXII, p. 123.

[3] Ibid., Vol. CLXXII, p. 89.

[4] Ibid., Vol. CLXXII, p. 107.

It is not the object of this paper to discuss these later experiments; but it may be remarked that they show very plainly that no tension exists near the outer toe, but that tension does exist at the inner toe, where the dam is joined to the foundation, and it has become a serious matter how to deal with it. The influence of the foundation in modifying the distribution of the stresses at the base of the dam was found to be very great, causing the shear there to be more uniform than higher up, where the parabolic law, nearly as given by the formulas above, was found to hold. Also, above some undertermined plane, a small distance above the base, the usual "law of the trapezoid" was found to be approximately correct, leading to stresses on the safe side at the outer toe. This law leads to stresses at the outer toe of the base considerably in excess of the true ones.

It was found, from the rubber models particularly, as theory indicates, that the greatest normal pressures are exerted at the down-stream face, for reservoir full, and they act in a direction parallel to that face.

APPENDIX III.

RELATIONS BETWEEN STRESSES AT ANY POINT OF A DAM.

(a) Consider a cube of masonry, Fig. 22, the edge of which has the length, a, bounded by vertical and horizontal planes and subjected to normal and shearing forces, caused by the action of the other parts of the dam. Since a will be supposed to diminish indefinitely, the weight of the cube, which is proportional to a^3, is an infinitesimal of the third order, and can be neglected in comparison with the normal forces, which vary as a^2 and are thus of the second order.

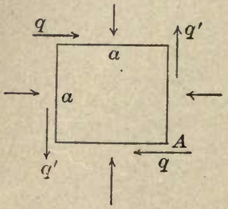

FIG. 22.

Similarly, the average unit stresses exerted on the faces can be treated from the first as the unit stresses at any point, A, of the cube. As a diminishes indefinitely, the oppositely directed normal forces approach equality and balance independently; hence the couples formed by the shears on opposite faces must likewise approach equality; the one being right-handed, the other left-handed; therefore $qa \times a = q'a \times a$, or $q = q'$; hence, the intensities of shear at a point on two planes at right angles are equal. The relative directions of the shears on two planes at right angles are determined, as above, from the consideration that one resulting couple must be right-handed and the other left-handed. This applies also to Figs. 23 to 26.

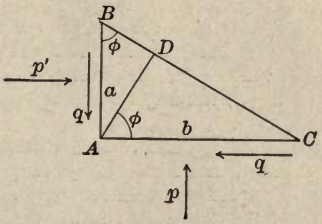

FIG. 23.

(*b*) In Fig. 23, *ABC* is the right section of a prism at the outer face, with lateral faces one unit in length, perpendicular to the plane of the paper.

Let *AB* be vertical; $\tan \phi = m$, a constant;

p = normal intensity on a horizontal plane at C;

p' = normal intensity on a vertical plane at C;

q = shear intensity on horizontal or vertical planes at C.

The weight of the prism is $\frac{1}{2}ab$.

Balancing vertical as well as horizontal components, we have, when $a=AB$ and $b=AC$ are very small,

$$pb=qa+\tfrac{1}{2}ab, \text{ nearly};$$

$$p'a=qb.$$

Dividing the first equation by b, the second by a, the limit, as a and b approach zero, gives exactly,

$$p=q\cot\phi, \quad \text{therefore} \quad q=mp;$$

$$p'=q\tan\phi, \quad \text{therefore} \quad p'=m^2p, \quad pp'=q^2.$$

These equations give the relations between p, q, and p' at the outer face. The same relations hold at the inner face, for reservoir empty, on replacing ϕ by ϕ', the angle the inner face makes with the vertical.

For the remaining cases, the final limits will be written at once, since the complete process of deriving them is evident from the above. In fact, the weight of the prism, $\frac{1}{2}ab$, being of the second order, can be neglected in comparison, with qa, etc.

(c) For reservoir full, calling $w=\frac{2}{5}h$, the intensity of water pressure, horizontally or vertically, at C, we have at the inner face, putting $\tan\phi'=n$, Fig. 24,

$$pb=qa+wb; \quad p'a=qb+wa;$$

therefore $$p=\frac{1}{n}q+w; \quad p'=qn+w.$$

(*d*) If the vertical component of the water pressure is neglected, these equations reduce to

$$p=\frac{1}{n}q;\quad p'=qn+w;$$

therefore $\qquad q=pn;\quad p'=n^2p+w.$

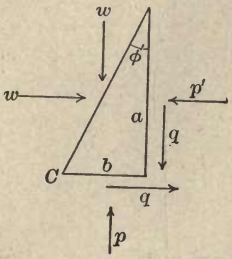

FIG. 24.

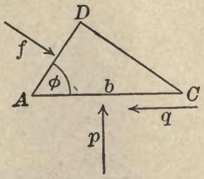

FIG. 25.

(*e*) Since the shear on the outer face is zero, therefore, by (*a*), the shear on a plane, AD, Fig. 25, perpendicular to the outer face, is also zero, or the stress on AD is normal.

Call f the intensity of such a stress at C. The total pressure on $AD=f\times AD=fb\cos\phi$, and its vertical component is $fb\cos^2\phi$, therefore balancing the vertical components,

$$pb=fb\cos^2\phi;$$

therefore $$f=\frac{p}{\cos^2\phi}=p\sec^2\phi.$$

This is a most important formula for finding the maximum normal intensity at the outer face. It applies equally to the inner face for reservoir empty, on changing ϕ to ϕ', the angle the inner face makes with the vertical. For either face, p is the vertical normal unit stress at the face considered.

(f) Principal Normal Stresses at Any Point in the Dam and the Planes on which they Act.—In the prism, ABC, Fig. 26, let AB be one of the planes

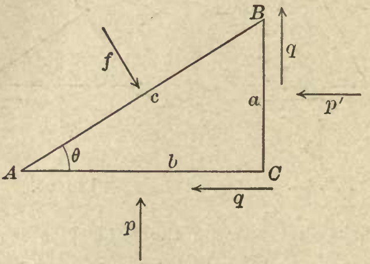

FIG. 26.

on which the stress is normal. Let f be its intensity. The stress on the plane, AB, of unit length perpendicular to the plane of the paper, is thus fc; its vertical component is $fc \cos\theta = fb$, and its horizontal component is $fc \sin\theta = fa$, θ being the angle that AB makes with the horizontal

Place the sum of the vertical forces acting on ABC equal to zero; also place the sum of horizontal forces equal to zero.

$$fb = pb + qa, \quad \text{therefore} \quad f - p = q \tan \theta,$$

$$fa = qb + p'a, \quad \text{therefore} \quad f - p' = q \cot \theta.$$

The difference of the last two equations gives

$$p - p' = q(\cot \theta - \tan \theta) = q \frac{1 - \tan^2 \theta}{\tan \theta},$$

therefore $$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2q}{p - p'}.$$

The angles, θ (differing by 90°), computed from this equation, give the directions of the planes, AB, on which the stress is entirely normal.

From an equation above, we likewise have

$$\tan \theta = \frac{f - p}{q}.$$

This gives directly the plane on which a given f acts.

To deduce a formula for f, take the product of two equations above:

$$(f - p)(f - p') = q^2,$$

therefore $$f = \frac{1}{2}[p + p' \pm \sqrt{(p + p')^2 - 4(pp' - q^2)}].$$

This equation gives the two values of f corresponding to the two planes mentioned; compressive when f is positive, tensile when negative There can be no tension when $pp' \geqq q^2$.

A better form for computation is,

$$f = \frac{1}{2}[p + p' \pm \sqrt{(p-p')^2 + 4q^2}].$$

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